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NOTES ON OPERATOR ALGEBRAS

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NOTES ON OPERATOR ALGEBRAS

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CONTENTS

PREFACE

This is the collection of notes which have been distributed during the lectures on operator algebras in the academic year 1992. The primary purpose of the lecture was to help the students to catch up current topics in C^*-algebras. It has been assumed that they have backgrounds on abstract measure theory and elementary functional analysis. After a brief introduction to the general theory of C^*-algebras and von Neumann algebras, we plunge into concrete examples of C^*-algebras such as AF algebras, free group C^*-algebras, irrational rotation algebras and Cuntz algebras, which have been studied during the seventies and early eighties. Through the discussion, we introduce the notions of tensor product, crossed product and K-theory for C^*-algebras. We refer to the Introduction at the beginning of each Chapter for more detailed contents of this note.

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CHAPTER 1

C^*–ALGEBRAS AND THEIR REPRESENTATIONS

A C^*-algebra is a Banach $*$-algebra with the norm condition $\|x^*x\| = \|x\|^2$ which relates the involution and the norm structures. Two typical examples, the C^*-algebra $C(X)$ of all continuous functions on a locally compact Hausdorff space X and the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space are presented in §1.1, together with the Banach $*$-algebra $L^1(G)$ of a locally compact group G with the convolution, which is not a C^*-algebra in general. Note that $\mathcal{B}(\mathcal{H})$ is nothing but the matrix algebra when $\mathcal{H}$ is finite-dimensional. One of the basic tools for studying Banach algebras is the notion of spectra, with which we can formulate the element $f(x) = \sum_{n=0}^{\infty} a_n x^n$ from an element x, for an *analytic* function $f(\lambda) = \sum a_n \lambda^n$. If the underlying algebra is a C^*-algebra then the *continuous* function calculus $f(x)$ is possible for a self-adjoint element x.

We note that the Fourier transform converts the convolution of $L^1(G)$ into the pointwise multiplication of $C(\widehat{G})$, which is much easier than the convolution. The Fourier transform may be generalized for arbitrary commutative Banach algebras to get the Gelfand transform, which turns out to be an isometric $*$-isomorphism for C^*-algebras. In this way, we see that a commutative C^*-algebra is nothing but $C(X)$ for a locally compact Hausdorff space X. Therefore, the study of commutative C^*-algebras is equivalent to that of locally compact Hausdorff spaces.

The involution naturally induces an order relation as in the cases of matrices. Elementary properties of this order structures are considered in §1.4, together with the interrelations between order, norm and algebraic structures. Most interesting is the notion of positive linear functionals, with which we construct a representation of a C^*-algebra on a Hilbert space. In other words, we realize an element of a C^*-algebra as a bounded linear operator on a Hilbert space. Employing the Hahn-Banach theorem, we see that there are sufficiently many positive linear functionals on a C^*-algebra, and so, every C^*-algebra can be considered as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. The correspondence between pure states and irreducible representations is explained in §1.6. We conclude this chapter with a brief introduction to the notions of liminal and postliminal C^*-algebras.

1.1. Definition and Examples

Definition. A Banach space A over the complex field with an associative multiplication is said to be a *Banach algebra* if

$$\|xy\| \le \|x\|\,\|y\|, \qquad x, y \in A. \tag{1.1.1}$$

It follows that the multiplication $(x, y) \mapsto xy$ is *jointly* continuous, and so the closure of a subalgebra (respectively an ideal) is again a subalgebra (respectively an ideal).

An *involution* is a map $x \mapsto x^*$ of A satisfying

$$(x+y)^* = x^* + y^*, \quad (\alpha x)^* = \overline{\alpha} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x$$

for $x, y \in A$ and $\alpha \in \mathbb{C}$. A Banach algebra A with an involution is said to be an *involutive Banach algebra* if

$$\|x^*\| = \|x\|, \tag{1.1.2}$$

and a C^*-*algebra* if

$$\|x^* x\| = \|x\|^2, \tag{1.1.3}$$

respectively. Note that the condition (1.1.3) implies (1.1.2). If A contains the multiplicative identity 1_A then we always assume that

$$\|1_A\| = 1.$$

Example 1.1.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ with respect to the operator norm. Endowing $\mathcal{B}(\mathcal{H})$ with the multiplication by the composition of operators, $\mathcal{B}(\mathcal{H})$ is a Banach algebra. For every $x \in \mathcal{B}(\mathcal{H})$, we use the correspondence between bounded sesquilinear forms on $\mathcal{H}$ and bounded linear operators on $\mathcal{H}$, to see that there exists a unique bounded linear operator x^* on $\mathcal{H}$ satisfying

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle, \quad \xi, \eta \in \mathcal{H}.$$

With the involution $x \mapsto x^*$, $\mathcal{B}(\mathcal{H})$ becomes a C^*-algebra. The relation (1.1.3) is actually a powerful tool to compute the operator norm of a non-selfadjoint operator even in the case of finite-dimensional Hilbert space, in which case $\mathcal{B}(\mathcal{H})$ is nothing but the matrix algebra M_n of all $n \times n$ matrices over the complex field. We will see later that every finite-dimensional C^*-algebra is the direct sum of matrix algebras.

We can also construct the matrix algebras over given algebras. Let A be an involutive algebra and denote by $M_n(A)$ the set of all $n \times n$ matrices with entries in A. Then $M_n(A)$ is again an involutive algebra with operations;

$$(1.1.4) \quad (ab)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad (a^*)_{ij} = a_{ji}{}^*, \qquad a = [a_{ij}], b = [b_{ij}] \in M_n(A).$$

In order to define a natural C^*-norm on $M_n(A)$ for a C^*-algebra A, we need the representation theory of C^*-algebras, which will be one of the main topic of this chapter.

Example 1.1.2. Let X be a locally compact Hausdorff space, and $C_0(X)$ the Banach space of all complex-valued continuous functions on X vanishing at infinity with respect to the uniform norm. Defining

$$(fg)(x) = f(x)g(x), \quad \overline{f}(x) = \overline{f(x)}, \quad x \in X,$$

$C_0(X)$ becomes a C^*-algebra. It is easy to see that $C_0(X)$ is unital if and only if X is compact. Even if X is not compact, $C_0(X)$ is considered as a maximal ideal of the unital C^*-algebra $C(X \cup \{\infty\})$, where $X \cup \{\infty\}$ is the one-point compactification of X. We can consider every element f of

$C_0(X)$ as a bounded linear operator M_f on the Hilbert space $L^2(X,\mu)$ by the multiplication;

$$M_f(\xi)(x) = f(x)\xi(x), \quad \xi \in L^2(X,\mu), x \in X, \tag{1.1.5}$$

where μ is a positive Borel measure on X. When X is compact and $\mu(X) = 1$, note that the following relation

$$\int_X f(x)d\mu(x) = \langle M_f(1_X), 1_X\rangle, \qquad f \in C(X) \tag{1.1.6}$$

holds.

Exercise 1.1.1. For which measure μ, is the map $f \mapsto M_f$ from $C_0(X)$ into $\mathcal{B}(L^2(X,\mu))$ an isomorphism or an isometry?

Exercise 1.1.2. Let μ be a positive measure on the real line $\mathbb{R}$. Determine whether or not there is a vector $\xi \in L^2(\mathbb{R},\mu)$ with the analogous relation as (1.1.6);

$$\int_{\mathbb{R}} g(x)d\mu(x) = \langle M_g(\xi), \xi\rangle, \qquad g \in C_0(\mathbb{R}). \tag{1.1.7}$$

Above two examples turn out to be typical among C^*-algebras. Every commutative C^*-algebra is $C_0(X)$ for a locally compact Hausdorff space X: Every C^*-algebra is a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. These two theorems will be the main topics of this chapter.

Example 1.1.3. For a C^*-algebra A and a locally compact space X, denote by $C_0(X,A)$ the space of all continuous functions from X into A vanishing at infinity. Define multiplication, involution and norm by

$$\begin{gathered}(fg)(x) = f(x)g(x), \quad f^*(x) = (f(x))^*,\\ \|f\| = \sup_{x\in X} \|f(x)\|,\end{gathered} \tag{1.1.8}$$

for $f \in C_0(X)$ and $x \in X$. Then $C_0(X,A)$ is a C^*-algebra.

Exercise 1.1.3. Show that there exists a $*$-isomorphism from $C_0(X, M_n)$ onto $M_n(C_0(X))$.

Example 1.1.4. Consider the Banach space $L^1(\mathbb{R})$ and define the multiplication and involution by

$$(f * g)(s) = \int_{\mathbb{R}} f(t)g(s-t)dt, \quad s \in \mathbb{R},$$
$$f^*(s) = \overline{f(-s)}, \quad s \in \mathbb{R}.$$

It is a standard fact that $L^1(\mathbb{R})$ becomes an involutive Banach algebra. Again, every f can be considered as a bounded linear operator on $L^2(\mathbb{R})$ as in Example 1.1.2;

$$\lambda_f(\xi)(s) = (f * \xi)(s), \quad \xi \in L^2(\mathbb{R}), s \in \mathbb{R}. \tag{1.1.9}$$

Note also that this algebra has no identity. But, we have an *approximate identity* $\{u_n\}$ in $L^1(\mathbb{R})$ in the following sense;

$$\|f * u_n - f\|_1 \to 0, \qquad f \in L^1(\mathbb{R}).$$

We define the Fourier transform $\widehat{f}$ of f as usual;

$$\widehat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-ist}dt, \quad s \in \mathbb{R}.$$

It is also a standard fact that the correspondence $f \mapsto \widehat{f}$ from $L^1(\mathbb{R})$ is norm-decreasing $*$-isomorphism into $C_0(\mathbb{R})$. It is easy to see that the following diagram commutes;

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\lambda_f} & L^2(\mathbb{R}) \\ \widehat{}\downarrow & & \downarrow\widehat{} \\ L^2(\mathbb{R}) & \xrightarrow{M_{\widehat{f}}} & L^2(\mathbb{R}) \end{array}$$

where $L^2(\mathbb{R}) \xrightarrow{\widehat{}} L^2(\mathbb{R})$ denotes the Plancherel transform.

Exercise 1.1.4. Show that $L^1(\mathbb{R})$ is not a C^*-algebra. Discuss the relations between norms $\|f\|_1$, $\|\widehat{f}\|_\infty$, $\|\lambda_f\|$ and $\|M_{\widehat{f}}\|$ for $f \in L^1(\mathbb{R})$.

If an involutive Banach algebra A has no identity, then we may embed A into a unital algebra A_I as was in Example 1.1.2. A_I is nothing but the direct sum of A and the complex field $\mathbb{C}$ endowed with

$$(x,a)(y,b) = (xy + bx + ay, ab),$$
$$(x,a)^* = (x^*, \overline{a}),$$
$$\|(x,a)\| = \|x\| + |a|.$$

Exercise 1.1.5. Show that A_I is an involutive Banach algebra with the identity $(0,1)$. Show also that A is an ideal of A_I under the identification of $x \mapsto (x,0)$. Give an example of a C^*-algebra A for which A_I is not a C^*-algebra.

For $x \in A_I$, define the left multiplication $L_x : A \to A$ by $L_x(y) = xy$. From the condition (1.1.3), we see that $\|x\| = \|L_x\|$ for each $x \in A$. We define

$$\|x\| = \|L_x\|, \qquad x \in A_I. \tag{1.1.10}$$

Proposition 1.1.1. *For a non-unital C^*-algebra A, A_I is a C^*-algebra under the norm given by* (1.1.10).

Note that we should first show that (1.1.10) defines a norm. Indeed, if $\|L_{(x,a)}\| = 0$ for $a \neq 0$ then it is easy to see that $-\frac{1}{a}x$ is the identity of A, to get a contradiction. Also, the condition (1.1.3) follows from the fact that A is an ideal of A_I [**T**, Proposition I.1.5].

Exercise 1.1.6. Show that there is an isometric $*$-isomorphism from the C^*-algebra $C_0(X)_I$ onto $C(X \cup \{\infty\})$ in Example 1.1.2.

1.2. Spectrum and Function Calculus

Definition. For an element x of a unital Banach algebra A, define the *spectrum* of x by

$$\mathrm{sp}_A(x) = \{\lambda \in \mathbb{C} : x - \lambda \text{ is not invertible in } A\}.$$

Also, we define the *spectral radius* $r_A(x)$ of x by

$$r_A(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(x)\}.$$

Note that the spectrum of a continuous function $f \in C(X)$ is just the range of f. For various interesting examples, we refer to [**K**, §3.2]. If $\|x\| < 1$ then the sequence $y_n = \sum_0^n x^k$ converges to an element $y \in A$ by the completeness, and the relation

$$y(1-x) = (1-x)y = 1 \tag{1.2.1}$$

holds. Especially, if $\|x\| < |\lambda|$ then $x - \lambda$ is invertible, and so $\lambda \notin \mathrm{sp}_A(x)$. This means that $\mathrm{sp}(x)$ is bounded and

$$r_A(x) \le \|x\|. \tag{1.2.2}$$

The relation (1.2.1) also means that the open ball $B(1_A, 1)$ centered at 1_A with radius 1 is contained in the group $G(A)$ of all invertible elements of A. For $x \in G(A)$, consider the left multiplication $L_x : A \to A$, which is a homeomorphism with the inverse $L_{x^{-1}}$. Because the image of $B(1_A, 1)$ under L_x is an open neighborhood of x contained in $G(A)$, we see that $G(A)$ is open. From this, it is easy to see that the complement of $\mathrm{sp}_A(x)$ is open. From the relation (1.2.1), we get

$$\|(1-x)^{-1} - 1^{-1}\| \le \sum_1^\infty \|x\|^k = \|x\|(1 - \|x\|)^{-1},$$

and we see that the map $x \mapsto x^{-1}$ is continuous at $1 \in G(A)$. By the similar argument using L_x, the map $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$.

We consider the function

$$\lambda \mapsto \rho((x-\lambda)^{-1}), \qquad \lambda \notin \mathrm{sp}_A(x), \tag{1.2.3}$$

for a continuous linear functional ρ on A. It is easy to see that this function is holomorphic and vanishing at ∞. If $\mathrm{sp}_A(x)$ is empty, this function vanishes everywhere by the Liouville theorem, and especially we have $\rho(x^{-1}) = 0$ for each continuous linear functional ρ on A. It follows that $x^{-1} = 0$ by the Hahn-Banach theorem, a contradiction. Summing up, we have the following:

Theorem 1.2.1. *The set* $\mathrm{sp}_A(x)$ *is a non-empty compact subset of the complex plane.*

If p is a polynomial in one variable then the element $p(x) \in A$ is well-defined for $x \in A$. With simple calculations, we see that

$$\mathrm{sp}(p(x)) = \{p(\lambda) : \lambda \in \mathrm{sp}(x)\}, \quad \mathrm{sp}(x^{-1}) = \{\lambda^{-1} : \lambda \in \mathrm{sp}(x)\}.$$

The main purpose of this section is to show that the similar construction of new elements from given one is possible and the same relation holds for holomorphic functions (respectively continuous functions) in Banach algebras (respectively C^*-algebras). By the way, if $\lambda \in \mathrm{sp}(x)$ then $\lambda^n \in \mathrm{sp}(x^n)$, and so $|\lambda^n| \leq \|x^n\|$. Hence, we have $|\lambda| \leq \|x^n\|^{\frac{1}{n}}$ for each $\lambda \in \mathrm{sp}(x)$, and so

$$r(x) \leq \liminf \|x^n\|^{\frac{1}{n}}. \tag{1.2.4}$$

We digress for a while to discuss vector-valued integration and differentiation. Let f be a function from a measure space (Ω, μ) into a Banach space X such that $\rho \circ f$ is integrable for each $\rho \in X^*$, the dual space of X. If there exists a vector $y \in X$ such that $\rho(y) = \int_\Omega (\rho \circ f) d\mu$ for each $\rho \in X^*$, then we define

$$y = \int_\Omega f d\mu.$$

Same reasoning is applied to differentiation: For a function $f : \Omega \to X$, where Ω is an open subset of $\mathbb{C}$, we say that f is holomorphic on Ω if the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for each $z \in \Omega$. This is, in fact, equivalent to say that $\rho \circ f$ is holomorphic for every $\rho \in X^*$, and the usual Taylor expansion theorem and the Cauchy integral theorem hold [**Ru2**, Theorem 3.31].

Considering the function in (1.2.3), we see that the function

$$f : \lambda \mapsto (1_A - \lambda x)^{-1}$$

is holomorphic on a neighborhood of 0, and has the Taylor expansion $\sum x^n \lambda^n$ whose radius of convergence is $(\limsup \|x^n\|^{\frac{1}{n}})^{-1}$. So, if $a < \limsup \|x^n\|^{\frac{1}{n}}$ then the series does not converge for $\lambda = \frac{1}{a}$, and so $1 - \frac{1}{a}x$ is not invertible, that is, $a \in \mathrm{sp}(x)$. Together with (1.2.4), we have the following:

Theorem 1.2.2. *For an element x in a unital Banach algebra A, we have*

$$r_A(x) = \lim \|x^n\|^{\frac{1}{n}}.$$

For an element x in a Banach algebra A, let f be a holomorphic function on a neighborhood Ω containing $\mathrm{sp}(x)$, γ a smooth closed path in Ω enclosing $\mathrm{sp}(x)$. We define

$$f(x) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda - x)^{-1} d\lambda. \tag{1.2.5}$$

Of course, it can be shown that the above formula is true for a polynomial f. The proof of the following theorem is easy by calculation.

Theorem 1.2.3 (Holomorphic Function Calculus). *Fix $x \in A$. The correspondence $f \mapsto f(x)$ from the algebra of all holomorphic functions on a neighborhood of $\mathrm{sp}(x)$ into A is a homomorphism with $f(1) = 1_A$ and $f(\iota) = x$, where $\iota(\lambda) = \lambda$. If f is represented by the Taylor series $\sum a_n\lambda^n$, then we have*

$$f(x) = \sum a_n x^n.$$

Furthermore, we have

$$\mathrm{sp}(f(x)) = \{f(\lambda) : \lambda \in \mathrm{sp}(x)\}. \tag{1.2.6}$$

If g is a holomorphic function on a neighborhood of $\mathrm{sp}(f(x))$, then we have

$$(g \circ f)(x) = f(g(x)). \tag{1.2.7}$$

If $\{f_n\}$ is a sequence of holomorphic functions on a neighborhood of $\mathrm{sp}(x)$ converging to f uniformly on compacta then we also have

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0. \tag{1.2.8}$$

Now, we turn our attention to special properties of spectra for the case of C^*-algebras. An element x of a C^*-algebra A is said to be *self-adjoint* if $x^* = x$, *normal* if $xx^* = x^*x$ and *unitary* if $x^*x = xx^* = 1$ when A is unital. We denote by A_h (respectively $\mathcal{U}(A)$) the real vector space of all self-adjoint elements (respectively the group of all unitary elements of A). Every element x of a C^*-algebra can be written as the linear combination of two self-adjoint elements:

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2i}. \tag{1.2.9}$$

Proposition 1.2.4. *Let A be a C^*-algebra. Then we have the following:*

(i) $\|x\| = r(x)$, *for a normal element* $x \in A$.

(ii) $\mathrm{sp}(x) \subseteq \mathbb{R}$, *for a self-adjoint element* $x \in A$.

(iii) $\mathrm{sp}(x) \subseteq \{\lambda : |\lambda| = 1\}$, *for a unitary element* $x \in A$.

The first property together with the C^*-norm condition (1.1.3) is a very powerful tool to compute the operator norm of a bounded linear operator on a Hilbert space even in the case of finite-dimensional space as was mentioned in Example 1.1.1, because $r(x)$ is nothing but the largest absolute value among eigenvalues of the matrix x, and x^*x is always self-adjoint.

Exercise 1.2.1. Let $A_t = \begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$ be the linear map between $\mathbb{C}^2$ for each $t \in \mathbb{R}$. Find the value of t for which the operator norm of A_t becomes smallest.

Exercise 1.2.2. Find a 2×2 matrix x for which $r(x) < \|x\|$.

The following theorem is a powerful tool when we exhibit an element of C^*-algebra with pre-assigned properties. It also explains why C^*-algebras are much more convenient to deal with than Banach algebras, in which function calculus is available only for holomorphic functions as was in Theorem 1.2.3. In the next chapter, we will see that even measurable function calculus is possible in so called von Neumann algebras. The proof is easy using the Stone-Weierstrass theorem on the interval $[-\|x\|, \|x\|]$ by Proposition 1.2.4.

Theorem 1.2.5 (Continuous Function Calculus). *Let x be a fixed self-adjoint element in a C^*-algebra A. Then there exists an isometric $*$-homomorphism $f \mapsto f(x)$ from $C(\mathrm{sp}(x))$ into A. Furthermore, we have*

$$\mathrm{sp}(f(x)) = \{f(\lambda) : \lambda \in \mathrm{sp}(x)\}. \tag{1.2.10}$$

Moreover, $f(x)$ is normal and if an element $y \in A$ commutes with x then it also commutes with $f(x)$.

If A is not unital, everything can be done in the identification A_I. Note that the resulting function calculus $f(x)$ belongs to A if $x \in A$. For $x \in A$ with $\|x\| \leq 1$, considering the function $f \in C(\mathrm{sp}(x))$ given by

$$f(t) = t + i\sqrt{1-t^2}, \quad t \in \mathrm{sp}(x) \subseteq [-1, 1],$$

we have the following:

Corollary 1.2.6. *Every element of a C^*-algebra is the linear combination of at most four unitaries.*

As an another important application of Theorem 1.2.5, we state the following proposition which says that the norm structure of a C^*-algebra is uniquely determined by the involution and multiplication. This proposition will play a rôle when we realize every C^*-algebra as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Proposition 1.2.7. *A $*$-homomorphism $\pi : A \to B$ from an involutive Banach algebra A into a C^*-algebra B is norm-decreasing. If π is a $*$-isomorphism between C^*-algebras then it is an isometry.*

1.3. Commutative C^*-algebras

In this section, we study the Gelfand transform of a commutative Banach algebra which turns out to be a generalization of the Fourier transform. Through this transform, we realize abstract commutative C^*-algebras as concrete ones $C_0(X)$ as in Example 1.1.2. Before going further, we recall the Fourier transform for locally compact abelian (LCA) group G. A *character* χ on G is a continuous homomorphism from G into the circle group $\mathbb{T}$. The set of all characters on G is denoted by $\widehat{G}$, which is again an LCA group with the pointwise multiplication and the compact-open topology. For example, we have

$\widehat{\mathbb{T}} = \mathbb{Z}$: Every character on $\mathbb{T}$ is of the form $e^{it} \mapsto e^{int}$, for some $n \in \mathbb{Z}$.
$\widehat{\mathbb{Z}} = \mathbb{T}$: Every character on $\mathbb{Z}$ is of the form $n \mapsto e^{int}$, for some $e^{it} \in \mathbb{T}$.
$\widehat{\mathbb{R}} = \mathbb{R}$: Every character on $\mathbb{R}$ is of the form $e^{it} \mapsto e^{ist}$, for some $s \in \mathbb{R}$.

In general, the dual group of a discrete group is compact, and the dual group of a compact group is discrete. Note that there is a natural homomorphism $e : G \to \widehat{\widehat{G}}$, *the evaluation map*, defined by

$$e(x)(\chi) = \chi(x), \qquad x \in G, \quad \chi \in \widehat{G}. \tag{1.3.1}$$

The Pontryagin-van Kampen duality theorem says that the evaluation map e is a topological isomorphism from G onto $\widehat{\widehat{G}}$ for every LCA group G. We also recall that there exists a unique (up to constant multiplication) positive regular Borel measure, said to be the *Haar measure*, on G which is invariant under translations. We denote by f_y the translation of f by y, that is, $f_y(x) = f(x-y)$.

Now, we endow $L^1(G)$ with the operations

$$(f * g)(x) = \int_G f(x-y)g(y)dy, \quad f, g \in L^1(G),$$
$$f^*(x) = \overline{f(-x)}, \quad f \in L^1(G).$$

Then $L^1(G)$ becomes an involutive Banach algebra as in Example 1.1.4.

For $f \in L^1(G)$, we define the *Fourier transform* $\widehat{f}$ by

$$\widehat{f}(\chi) = (f * \chi)(0) = \int_G f(x)\chi(-x)dx, \qquad \chi \in \widehat{G}. \tag{1.3.2}$$

This transform converts multiplication by a character into translation and vice versa;

$$\widehat{f\chi} = \widehat{f}_\chi, \qquad \widehat{f_y}(\chi) = \widehat{f}(\chi)\chi(-y). \tag{1.3.3}$$

Now, we show that every $\widehat{f}$ is a continuous function on $\widehat{G}$. In fact, this property characterizes the compact-open topology as follows:

Theorem 1.3.1. *Let $\{\chi_n\}$ be a sequence of characters of G. Then the following are equivalent:*

i) *$\widehat{f}(\chi_n) \to \widehat{f}(\chi)$ for every $f \in L^1(G)$.*
ii) *$\{\chi_n\}$ converges to χ uniformly on compacta.*

The direction ii) $\Rightarrow$ i) follows from the Lebesgue's convergence theorem. For the converse direction, observe that $y \mapsto f_y$ is a uniformly continuous map from G into $L^1(G)$. Take f with $\widehat{f}(\chi) \neq 0$. Use the second relation of (1.3.3) and 3ϵ-technique to show that $\{\widehat{f}(\chi_n)(\chi_n(x))\}$ converges uniformly to $\{\widehat{f}(\chi)(\chi(x))\}$ on a neighborhood of every point of G. A usual compactness argument completes the proof.

The above theorem says that every $\widehat{f}$ is a continuous function on $\widehat{G}$ with respect to the compact-open topology. We show that $\widehat{f}$ vanishes at infinity. Note that every character χ gives rise a continuous complex homomorphism of $L^1(G)$ via $f \to \widehat{f}(\chi)$;

$$\widehat{f * g}(\chi) = \widehat{f}(\chi)\,\widehat{g}(\chi). \tag{1.3.4}$$

Lemma 1.3.2. *Every nonzero continuous complex homomorphism is obtained in this way.*

For the proof, note that every continuous homomorphism h is a bounded linear functional on $L^1(G)$ of norm 1, and so it is represented by an L^∞-function.

Now, $\widehat{G}\cup\{0\}$ is the set of all continuous homomorphisms of $L^1(G)$. Hence, it follows from the Banach-Alaoglu theorem (or equivalently, Tychonoff theorem) that $\widehat{G}\cup\{0\}$ is compact. From this, it is clear that every $\widehat{f}$ vanishes at infinity. It is easy to see that

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

We denote by $A(\widehat{G})$, the *Fourier algebra* of $\widehat{G}$, the image of $L^1(G)$ under the Fourier transform. Then, $A(\widehat{G})$ is a separating self-adjoint subalgebra of $C_0(\widehat{G})$. It follows from the Stone-Weierstrass theorem that $A(\widehat{G})$ is dense in $C_0(\widehat{G})$. Summarizing, we have the following:

Theorem 1.3.3. *The Fourier transform $f \mapsto \widehat{f}$ is a norm-decreasing homomorphism from $L^1(G)$ onto the dense subalgebra $A(\widehat{G})$ of $C_0(\widehat{G})$.*

Denote by Δ the set of all complex homomorphisms of a commutative unital Banach algebra A. For each $x \in A$, we define the *Gelfand transform* $\widehat{x}$ by

$$\widehat{x}(h) = h(x), \quad h \in \Delta. \tag{1.3.5}$$

We endow Δ with the smallest topology that makes every $\widehat{x}$ continuous on Δ as was in Theorem 1.3.1. The space Δ is said to be the *maximal ideal space*, because every complex homomorphism corresponds to a maximal ideal of A. In fact, it is clear that $\operatorname{Ker} h$ is a maximal ideal of A for each complex homomorphism h of A. Conversely, if I is a maximal ideal of A then it is easy to see that I is closed and every nonzero element of the quotient algebra A/I is invertible. Using Theorem 1.2.1, it is also easy to see that A/I is nothing but the complex field.

Using the Tychonoff theorem or Banach-Alaoglu theorem, it is easy to see that Δ is compact Hausdorff. In fact, the formula $h(x - h(x)1) = 0$ shows that $x - h(x)1$ is not invertible, and so $|h(x)| \leq \|x\|$ by (1.2.2). This means that

$$\Delta \subseteq \prod_{x \in A} B(0, \|x\|),$$

where $B(0, \|x\|)$ is the closed ball on the complex plane. The topology on Δ is nothing but the subspace topology of the Tychonoff topology on the product space. Now, the Gelfand transform

$$x \mapsto \widehat{x} \tag{1.3.6}$$

is a homomorphism from A into the C^*-algebra $C(\Delta)$, whose range will be denoted by $\widehat{A}$. A Banach algebra is said to be *semi-simple* if the Gelfand transform is an isomorphism. The inversion theorem for the Fourier transform says that $L^1(G)$ is semi-simple. We refer to [**Ru2**, §11.13] for more concrete examples of the Gelfand transforms.

Using the above correspondence between maximal ideals and complex homomorphisms, it can be also shown that the range of the function $\widehat{x}$ is just $\mathrm{sp}(x)$, and so we have

$$\|\widehat{x}\|_\infty = r(x) \leq \|x\|. \tag{1.3.7}$$

If A is a commutative C^*-algebra then every element of A is normal, and so the equality holds in (1.3.7) by Proposition 1.2.4 (i). Hence, we have the following:

Theorem 1.3.4. *If A is a unital C^*-algebra then the Gelfand transform is an isometric $*$-isomorphism whose range is the whole algebra $C(\Delta)$.*

Note that the meaning of "$*$-preserving" is $\widehat{x^*} = \overline{\widehat{x}}$, which is a consequence of Proposition 1.2.4 (ii). From this, we see that the range is self-adjoint, and so it is dense by the Stone-Weierstrass theorem. Hence, the range is the whole algebra $C(\Delta)$ by the completeness of A.

Example 1.3.1. Denote by $C_b(X)$ the set of all continuous bounded functions on a non-compact, locally compact Hausdorff space X. With the same operations as in Example 1.1.2, $C_b(X)$ becomes a unital C^*-algebra whose maximal ideal space will be denoted by βX.

Exercise 1.3.1. Let h_x be a complex homomorphism on $C_b(X)$ defined by the formula $h_x(f) = f(x)$, for $f \in C_b(X)$. Show that the mapping $x \mapsto h_x$ is a homeomorphism from X onto a dense subspace of βX.

Now, we look at the case when a C^*-algebra A has no identity. We denote by Δ_I the maximal ideal space of A_I.

Exercise 1.3.2. Show that there is a unique $h_0 \in \Delta_I$ such that $h_0(x) = 0$ for each $x \in A$. Show also that A can be identified as the ideal $\{x \in C(\Delta_I) : x(h_0) = 0\}$ of $C(\Delta_I)$.

1.4. Order Structures of C^*-algebras

One of the main technical advantages of involutive algebras is that there is a natural order structure; we say that elements of the forms x^*x are *positive*, as in the cases of $C_0(X)$ and matrix algebras. In the case of C^*-algebras, we use continuous function calculus (Theorem 1.2.5) to show the following:

Theorem 1.4.1. *For a self-adjoint element x of a C^*-algebra A, the following are equivalent:*

(i) $\mathrm{sp}_{A_I}(x) \subseteq \mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$,

(ii) $x = h^2$ *for some* $h \in A_h$,

(iii) $x = y^*y$ *for some* $y \in A$.

If A is a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, then the above conditions are also equivalent to

(iv) $\langle x\xi, \xi\rangle \geq 0$ *for each* $\xi \in \mathcal{H}$.

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial. For the proof of (iii) $\Longrightarrow$ (i), first note that the condition (i) is equivalent to

(v) $\|x - a1_A\| \leq a$ for some $a \geq \|x\|$.

Using this condition, we see that the relation

$$x = x^*,\ \mathrm{sp}(x) \subseteq \mathbb{R}^+,\ \mathrm{sp}(y) \subseteq \mathbb{R}^+ \quad \Longrightarrow \quad \mathrm{sp}(x+y) \subseteq \mathbb{R}^+. \tag{1.4.1}$$

holds in C^*-algebras. Also, the equality $(1-yx)(y(1-xy)^{-1}x + 1) = 1$ shows that the following simple relation

$$\mathrm{sp}(xy) \subseteq \mathrm{sp}(yx) \cup \{0\} \tag{1.4.2}$$

also holds in Banach algebras. We use the decomposition (1.2.9) and the above two relations (1.4.1) and (1.4.2) to get the following:

Lemma 1.4.2. *If* $\mathrm{sp}(x^*x) \subseteq (-\infty, 0]$ *then* $x = 0$.

Now, we assume that $x = y^*y$ for some $y \in A$ and consider the functions

$$u^+(t) = \max\{t, 0\}, \qquad u^-(t) = -\min\{t, 0\}, \quad t \in \mathbb{R}, \tag{1.4.3}$$

and put $p = u^+(x)$, $q = u^-(x)$. Then it follows that $x = p - q$ and $pq = 0$, and so

$$(yq)^*(yq) = q^*y^*yq = q^*(p-q)q = -q^3.$$

Considering the range of the function $-(u^-)^3$, we see that $yq = 0$ by Lemma 1.4.2. Hence, we have $q^3 = 0$ and $\|q\| = 0$ by Theorem 1.2.2 and Proposition 1.2.4 (i), and this completes the proof of (iii) $\Longrightarrow$ (i).

For the proof of (iv) $\Longrightarrow$ (i), note that if every normal non-invertible operator $x \in \mathcal{B}(\mathcal{H})$ has an approximate eigenvectors $\{\xi_n\}$. If $\lambda \in \mathrm{sp}(x)$ then we have $\lim_n \langle (x - \lambda)\xi_n, \xi_n \rangle = 0$, and so $\lambda \geq 0$. In the course of the proof, we have shown the following:

Proposition 1.4.3. *Every element of a* C^**-algebra is the linear combination of at most four positive elements.*

The set of all positive elements of A is denoted by A_+, and we write $x \leq y$ if $y - x \in A_+$. The set A_+ is a norm-closed cone of the real vector space A_h satisfying $A_+ \cap (-A_+) = \{0\}$. If A has the identity 1_A then it is easy to express how the notions of positivity, multiplication and norm are related. Indeed, from the first condition of the above Theorem 1.4.1 and continuous function calculus, it is easy to see that

$$A_+ = \{x \in A_h : \|x - \|x\|1_A\| \leq \|x\|\}, \tag{1.4.4}$$

$$\|x\| = \inf\{a \in \mathbb{R}^+ : -a1_A \leq x \leq a1_A\}, \tag{1.4.4}$$

$$-y \leq x \leq y \Longrightarrow \|x\| \leq \|y\|, \tag{1.4.6}$$

$$x \in A_+ \cap G(A) \Longleftrightarrow x \geq t1_A \text{ for some } t > 0, \tag{1.4.7}$$

which are trivial in the C^*-algebra $C_0(X)$.

The element h in Theorem 1.4.1 (ii) is determined uniquely among positive elements, and denoted by $x^{\frac{1}{2}}$. It can be shown (see [**P**, Proposition 4.2.8] or [**M**, Theorem 2.2.6]) that

$$x \leq y \Longrightarrow x^{\frac{1}{2}} \leq y^{\frac{1}{2}}, \tag{1.4.8}$$

which will be useful to construct approximate identities. Note that even if f is an increasing function in the plain sense, the relation $x \leq y \Longrightarrow f(x) \leq f(y)$ does not hold in general.

Now, let B be a C^*-subalgebra of a unital C^*-algebra A, and denote

$$\Lambda = \{u \in B \cap A_+ : \|u\| < 1\}.$$

We can show that the set Λ is directed, that is, for $u_1, u_2 \in \Lambda$, there is $u \in \Lambda$ such that $u_1 \leq u$ and $u_2 \leq u$. To do this, choose $\delta > 0$ such that $\|(1+\delta)u_i\| \leq 1, i = 1, 2$. Put

$$z_n = \frac{1}{1+\delta}(1+\delta)^{\frac{1}{n}}(u_1 + u_2)^{\frac{1}{n}}.$$

By using (1.4.8), it is easy to see that $u_i \leq z_n$ for each $i = 1, 2$ and $n = 2^k, k = 1, 2, \ldots$, and $z_n \in \Lambda$ for sufficiently large n. It turns out that the set $\Lambda = \{u_\lambda\}$ plays the role of an *approximate identity* for B, in the following sense;

$$\lim_\lambda \|x - xu_\lambda\| = \lim_\lambda \|x - u_\lambda x\| = 0, \quad y \in B.$$

Theorem 1.4.4. *Every C^*-algebra A has an approximate identity. If A is separable, then the net may be chosen to be a sequence.*

For the proof, we may assume that $\|x\| < 1$. If $u_\lambda \geq (x^*x)^{\frac{1}{n}}$ then we have

$$0 \leq x(1 - u_\lambda)^2 x^* \leq x(1 - u_\lambda)x^* \leq x(1(x^*x)^{\frac{1}{n}})x^*,$$

and so

$$\|x - xu_\lambda\|^2 = \|x(1 - u_\lambda)^2 x^*\| \leq \|x(1 - (x^*x)^{\frac{1}{n}} x^*\|.$$

It is easy to see that the last quantity converges to 0 as $n \to \infty$ from the continuous function calculus. Note that $(x^*x)^{\frac{1}{n}} \in \Lambda$.

The concept of approximate identity is very useful to deal with ideals of C^*-algebras. It can be shown that every norm-closed two-sided ideal I of a C^*-algebra is self-adjoint, hence a C^*-subalgebra. If $\{u_\lambda\}$ is an approximate identity for I, then the quotient norm may be expressed in terms of $\{u_\lambda\}$ as follows:

$$\|x + I\| = \lim_\lambda \|x - xu_\lambda\| = \lim_\lambda \|x - u_\lambda x\|, \quad x \in A. \tag{1.4.9}$$

From this, it is easy to see that the quotient of a C^*-algebra is again a C^*-algebra.

A subcone of M of a C^*-algebra A is said to be *hereditary* if

$$0 \leq x \leq y, y \in M \Longrightarrow x \in M.$$

A C^*-subalgebra B is said to be *hereditary subalgebra* if B_+ is hereditary in A_+. For a subcone M, denote

$$L(M) = \{x \in A : x^*x \in M\}.$$

Theorem 1.4.5. *Let A be a C^*-algebra. The mappings $B \mapsto B_+$, $M \mapsto L(M)$ and $L \mapsto L \cap L^*$ are bijective, order-preserving correspondences between the sets of hereditary C^*-subalgebras of A, closed hereditary cones of A_+ and closed left ideals of A.*

Following proposition provides useful methods to characterize and construct hereditary C^*-subalgebras.

Proposition 1.4.6. *Let A be a C^*-algebra. Then we have the following:*

(i) *A C^*-subalgebra B of A is hereditary if and only if $x \in A$ and $y, y' \in B$ implies $yxy' \in B$.*
(ii) *For $x \in A_+$, $(xAx)^-$ is the hereditary C^*-subalgebra of A generated by x.*
(iii) *Conversely, every separable hereditary C^*-subalgebra of A is of the form $(xAx)^-$ for some $x \in A_+$.*

Exercise 1.4.1. Describe all left ideals and hereditary C^*-subalgebras of the matrix algebra M_n in a concrete way, and explain the correspondences given in Theorem 1.4.5. Show that M_n is simple.

Exercise 1.4.2. Find all hereditary C^*-subalgebras and left ideals of the C^*-algebra $C(X)$ for a compact Hausdorff space X.

1.5. Representations of C^*-algebras

Let A be a unital C^*-algebra. A linear functional ϕ on A is said to be *positive* if

$$\phi(x^*x) \geq 0, \qquad x \in A.$$

Every positive linear functional ϕ induces a sesquilinear form on A by

$$\langle x, y\rangle = \phi(y^*x), \quad x, y \in A,$$

especially, we have

$$\phi(y^*x) = \overline{\phi(x^*y)}, \qquad x, y \in A, \tag{1.5.1}$$

$$|\phi(y^*x)|^2 \leq |\phi(x^*x)\phi(y^*y)|, \qquad x, y \in A. \tag{1.5.2}$$

This inner product induces a definite inner product on the quotient space A/L_ϕ, where

$$L_\phi = \{x \in A : \phi(x^*x) = 0\}.$$

We denote by $\mathcal{H}_\phi$ the Hilbert space obtained by the completion of A/L_ϕ. For every $x \in A$, denote by $\pi_\phi(x)$ the linear map on the pre-Hilbert space A/L_ϕ induced by the multiplication;

$$\pi_\phi(x)(y + L_\phi) = xy + L_\phi, \qquad y \in A, \tag{1.5.3}$$

which is well-defined by (1.5.2). The relation $x^*x \leq \|x\|^2 1_A$ implies

$$\phi((xy)^*xy) = \phi(y^*x^*xy) \leq \|x\|^2\phi(y^*y), \tag{1.5.4}$$

and so $\pi_\phi(x)$ extends to a bounded linear map on $\mathcal{H}_\phi$. Also, if we denote by ξ_ϕ the vector in $\mathcal{H}_\phi$ represented by the identity 1 of A, then we have

$$\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi\rangle, \quad x \in A. \tag{1.5.5}$$

A *representation* of an involutive algebra A is a pair $\{\pi, \mathcal{H}\}$ of a Hilbert space $\mathcal{H}$ and a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. A set X of vectors in $\mathcal{H}$ is said to be *cyclic* if the set $\{\pi(x)\xi : x \in A, \xi \in X\}$ is dense in $\mathcal{H}$, and $\{\pi, \mathcal{H}\}$ a *cyclic representation* if there is a single cyclic vector.

Exercise 1.5.1. Let X be a cyclic set for a representation $\{\pi, \mathcal{H}\}$ of a C^*-algebra A. Define $\phi_\xi(x) = \langle \pi(x)\xi, \xi\rangle$ for $x \in A$ and $\xi \in \mathcal{H}$. Show that

$$\|\pi(x)\|^2 = \sup\left\{\frac{\phi_\xi(y^*x^*xy)}{\phi_\xi(y^*y)} : \xi \in X, y \in A, \phi_\xi(y^*y) \neq 0\right\}$$

holds for each $x \in A$.

Theorem 1.5.1. *Let A be a unital C^*-algebra. For every positive linear functional ϕ, there exists a unique, up to unitary equivalence, representation $\{\pi_\phi, \mathcal{H}_\phi\}$ and a cyclic vector ξ_ϕ with the relation* (1.5.5).

We say that two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ of A are *unitarily equivalent* if there is an isometry U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U\pi_1(x)U^* = \pi_2(x)$ for each $x \in A$. The uniqueness part of Theorem 1.5.1 can be seen easily from the density of the set $\{\pi_\phi(x)\xi_\phi; x \in A\}$ in $\mathcal{H}_\phi$.

Exercise 1.5.2. Let ϕ be the linear functional of the C^*-algebra $C[0,1]$ given by the Lebesgue measure on the unit interval. Describe the induced representation. Do the same question for the normalized trace of the matrix algebra.

A positive linear functional ϕ is said to be a *state* if $\phi(1_A) = 1$. The set of all states on A is denoted by $\mathcal{S}(A)$. This is equivalent to the condition $\|\phi\| = 1$ by the following proposition which explains together with (1.4.4) and (1.4.5) how the norm and order structures are related.

Proposition 1.5.2. *A linear functional ϕ on a unital C^*-algebra A is positive if and only if ϕ is bounded and $\|\phi\| = \phi(1_A)$.*

One direction is easy if we use the fact that every positive linear functional ϕ is *self-adjoint*, that is, $\phi(x^*) = \overline{\phi(x)}$, or equivalently $\phi(x) \in \mathbb{R}$ for each self-adjoint element $x \in A$. For the converse, we use the continuous function calculus. As an application of the Hahn-Banach extension theorem together with Proposition 1.5.2, we have the following essential result.

Lemma 1.5.3. *Let x be an element of a C^*-algebra A. Then $x = 0$ if and only if $\phi(x) = 0$ for each state ϕ on A.*

We consider the direct sum of all representations $\{\pi_\phi, \mathcal{H}_\phi\}$;

$$\pi(x) = \bigoplus_{\phi \in \mathcal{S}(A)} \pi_\phi(x), \qquad x \in A, \tag{1.5.6}$$

which acts on the Hilbert space $\bigoplus_{\phi \in \mathcal{S}(A)} \mathcal{H}_\phi$. By Lemma 1.5.3, we have the following:

Theorem 1.5.4. *If A is a C^*-algebra then the above representation π is a $*$-isomorphism.*

Combining with Proposition 1.2.7, we see that every C^*-algebra can be realized as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. If A is not unital then we may consider the representation $\{\pi, \mathcal{H}\}$ of the unital C^*-algebra A_I. By using an approximate identity for A, we see that the cyclic vector ξ for the representation $\{\pi, \mathcal{H}\}$ is still a cyclic vector for the representation $\{\pi|_A, \mathcal{H}\}$ of A. The representation in (1.5.6) is said to be the *universal representation* of A. By Zorn's lemma, it is easy to see that every representation is the sum of cyclic representations, and so it is a subrepresentation of the universal representation. If A has a *faithful* state ϕ, that is, $\phi(x^*x) = 0$ implies $x = 0$, then the induced representation $\{\pi_\phi, \mathcal{H}_\phi\}$ is already faithful, that is, isomorphic.

Exercise 1.5.3. Let A be a C^*-algebra. Show that there exists a unique norm on the involutive algebra $M_n(A)$ in Example 1.1.1 which makes $M_n(A)$ a C^*-algebra.

For a vector ξ in a Hilbert space $\mathcal{H}$, we define a positive linear functional ω_ξ on $\mathcal{B}(\mathcal{H})$ by the formula

$$\omega_\xi(x) = \langle x\xi, \xi\rangle, \qquad x \in \mathcal{B}(\mathcal{H}).$$

If ξ is a unit vector then ω_ξ is a state, called a *vector state*. The relation (1.5.5) says that ϕ is nothing but the composition of the vector state given by ξ_ϕ and the $*$-homomorphism π_ϕ. Conversely, if a representation $\{\pi, \mathcal{H}\}$ is given with a cyclic unit vector ξ and ϕ is a state given by

$$\phi(x) = \langle \pi(x)\xi, \xi\rangle, \quad x \in A,$$

then we see that the induced representation π_ϕ is unitarily equivalent to π from the uniqueness part of Theorem 1.5.1. If π is the universal representation given by (1.5.6) then every state of A is of the form $\omega_\eta \circ \pi$ for a unit vector η in $\bigoplus_{\phi\in\mathcal{S}(A)} \mathcal{H}_\phi$. In other word, every state of $\pi(A)$ is a vector state. If A acts on a Hilbert space then every state of A is the weak* limit of the finite sums of vector states. This follows from the following theorem which characterizes weak* dense subsets of the state space in terms of the order and norm structures.

Theorem 1.5.5. *Let A be a unital C^*-algebra and $\mathcal{S}_0$ a subset of $\mathcal{S}(A)$. Then the following are equivalent:*

(i) *If $x \in A_h$ and $\phi(x) \geq 0$ for each $\phi \in \mathcal{S}_0$ then $x \in A_+$.*
(ii) *For each $x \in A_h$, we have $\|x\| = \sup\{|\phi(x)| : \phi \in \mathcal{S}_0\}$.*
(iii) *The convex hull of $\mathcal{S}_0$ is weak* dense in $\mathcal{S}(A)$.*

With $a = \sup\{|\rho(x)| : \rho \in \mathcal{S}_0\}$, we see $\rho(a1_A \pm x) \geq 0$ for $\rho \in \mathcal{S}_0$. If we assume (i) then $-a1_A \leq x \leq a1_A$, and $\|x\| \leq a$ by (1.4.4). In order to show (ii) $\Longrightarrow$ (iii), we assume that the weak* closure $\overline{\text{co}}\mathcal{S}_0$ of the convex hull of $\mathcal{S}_0$ is a proper subset of $\mathcal{S}(A)$ and take $\rho_0 \in \mathcal{S}(A) \setminus \overline{\text{co}}\mathcal{S}_0$. By the Hahn-Banach theorem, there exist $x \in A$ and a real number α such that $\text{Re}\,\rho_0(x) > \alpha$ and $\text{Re}\,\rho(x) \leq \alpha$ for each $\rho \in \overline{\text{co}}\mathcal{S}_0$. If we denote by x_h the real part of x in (1.2.9) then

$$\alpha < \text{Re}\,\rho_0(x) = \rho_0(x_h) \leq \|x_h\| = \sup\{|\rho(x_h)| : \rho \in \mathcal{S}_0\} \leq \alpha,$$

a contradiction. The inclusion (iii) $\Longrightarrow$ (i) is an another application of the Hahn-Banach theorem together with the following decomposition of bounded linear functionals:

Theorem 1.5.6. *If ϕ is a bounded self-adjoint linear functional on a C^*-algebra A. Then $\phi = \phi_+ - \phi_-$ for positive linear functionals ϕ_+ and ϕ_- on A with $\|\phi\| = \|\phi_+\| + \|\phi_-\|$. Hence, every bounded linear functional is the sum of at most four positive linear functionals.*

When A is an involutive Banach algebra with a faithful representation, we define a new norm on A by

$$\|x\|_c = \sup\{\|\pi(x)\|\}, \tag{1.5.7}$$

where π runs through all representations. By Proposition 1.2.7, we have $\|x\|_c \leq \|x\|$ and the completion of A is a C^*-algebra, called the *enveloping C^*-algebra* of A. The construction of group algebra $L^1(G)$ in §1.3 can be carried out for every locally compact group G by the following modification;

$$\begin{aligned} (x * y)(t) &= \int_G x(s)y(s^{-1}t)ds, \\ x^*(t) &= \delta(t)^{-1}\overline{x(t^{-1})}, \end{aligned} \tag{1.5.8}$$

where δ is the *modular function* for the left invariant Haar measure. We define the *left regular representation* of $L^1(G)$ by

$$(1.5.9) \qquad (\lambda(x)\xi)(t) = \int_G x(s)\xi(s^{-1}t)ds, \quad \xi \in L^2(G), x \in L^1(G),$$

which is always faithful. The enveloping C^*-algebra of $L^1(G)$ is said to be the *group* C^**-algebra* and denoted by $C^*(G)$. The norm closure of the image $\lambda(L^1(G))$ in $\mathcal{B}(L^2(G))$ is said to be the *reduced group* C^**-algebra*, and denoted by $C^*_\lambda(G)$.

Exercise 1.5.4. For the cyclic group $\mathbb{Z}_n$, describe the C^*-algebra $C^*_\lambda(\mathbb{Z}_n)$ as a subalgebra of M_n. Do the same thing for the permutation group S_3 on three elements.

Exercise 1.5.5. Let G be an LCA group. Discuss the relation between the Fourier transform $f \mapsto \widehat{f}$ and the left regular representation $f \mapsto \lambda(f)$. Show that the extension of the left regular representation λ to $C^*(G)$ is faithful, and so $C^*(G)$ is $*$-isomorphic to $C^*_\lambda(G)$. Show also that these C^*-algebras are in fact $*$-isomorphic to the C^*-algebra $C_0(\widehat{G})$.

With the above Exercise 1.5.5, it is evident that every positive lineal functional ϕ on $C^*(G)$ is given by a positive measure on $\widehat{G}$ if G is commutative. The restriction of ϕ to $L^1(G)$ is again realized as a function f in $L^\infty(G)$, and the function arising in this manner is said to be a *positive definite* function on G.

Exercise 1.5.6. What is the positive definite function on G associated with the point mass on $\widehat{G}$?

Exercise 1.5.7. Let f be a function on $\mathbb{Z}$ given by

$$f(0) = 1, \quad f(1) = \alpha, \quad f(-1) = \overline{\alpha}, \quad f(n) = 0, \quad n = \pm 2, \pm 3, \dots.$$

Find the condition on α for which f is positive definite. Find the eigenvalues of the $n \times n$ matrix $[a_{ij}]$ given by

$$a_{ii} = 1, \quad a_{i,i+1} = \alpha, \quad a_{i,i-1} = \overline{\alpha}, \quad a_{ij} = 0 \text{ otherwise.}$$

Show that the above condition on α is equivalent to the positivity of the $n \times n$ matrix $[f(i-j)]^n_{i,j=1}$ for each $n = 1, 2, \dots$.

1.6. Pure States and Irreducible Representations

Let A be a unital C^*-algebra. We endow the set $\mathcal{S}(A)$ of all states of A with the smallest topology for which the function $\phi \mapsto \phi(x)$ is continuous for each $x \in A$ as for the case of maximal ideal space Δ in §1.3. This is the relative topology of the weak* topology of the dual space A^* or equivalently the relative topology of the product topology on $\mathbb{C}^A$. By the Banach-Alaoglu or Tychonoff theorem, $\mathcal{S}(A)$ is compact. Also, by the Krein-Milman theorem $\mathcal{S}(A)$ is the closed convex hull of its extreme points. Recall that a point x of a convex set S in a vector space is said to be an *extreme point* of S if x cannot be expressed as a convex combination of another points in S. An extreme point ϕ of $\mathcal{S}(A)$ is said to be a *pure state* of A and denote by $\mathcal{P}(A)$ the set of all pure states of A.

Lemma 1.6.1. *A state ϕ is pure if and only if every positive linear functional τ with $\tau \leq \phi$ is a scalar multiple of ϕ.*

Recall that every closed ideal of $C(X)$ is of the form

$$\{f \in C(X) : f(x) = 0 \text{ for each } x \in K\}$$

for a closed subset K of X. From this and Lemma 1.6.1, it is easy to characterize all pure states of commutative C^*-algebras. Note that the equivalence between (ii) and (iii) of Theorem 1.6.2 is already contained in Exercise 1.3.1 if we consider the C^*-algebra $C_b(X) = C(X)$ when X is compact.

Theorem 1.6.2. *Let ϕ be a positive linear functional on the C^*-algebra $C(X)$. Then the following are equivalent:*

(i) *ϕ is a pure state.*
(ii) *$\phi(f) = f(x)$ for some $x \in X$.*
(iii) *$\phi(fg) = \phi(f)\phi(g)$ for each $f, g \in C(X)$.*

Hence, if ϕ is a pure state of $C(X)$ then ϕ is in itself the induced representation π_ϕ, which acts on the one-dimension Hilbert space. In the following, we will study the non-commutative analogue. We begin with the following simple lemma, which will be used repeatedly.

Lemma 1.6.3. *Let p be the projection of $\mathcal{H}$ onto a closed subspace E of $\mathcal{H}$, and $x \in \mathcal{B}(\mathcal{H})$. Then we have the following:*

(i) *E is invariant under x if and only if $xp = pxp$.*

(ii) *E is invariant under x if and only if $E^\perp$ is invariant under x^*.*

(iii) *Both E and $E^\perp$ are invariant under x if and only if $xp = px$.*

Furthermore, E is invariant under a $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ if and only if $px = xp$ for each $x \in A$.

Let ϕ be a pure state of a C^*-algebra A and $\{\pi_\phi, \mathcal{H}_\phi\}$ the induced representation. Using Lemma 1.6.1 again, it is easy to see that there is no nontrivial closed subspace of $\mathcal{H}_\phi$ which is invariant under $\pi_\phi(x)$ for each $x \in A$. Indeed, if E is a closed subspace of $\mathcal{H}_\phi$ which is invariant under $\pi_\phi(A)$, denote by P the projection of $\mathcal{H}_\phi$ onto E. Then P commutes with $\pi_\phi(x)$ for each $x \in A$. Define a positive linear functional by

$$\psi(x) = \langle \pi_\phi(x)P\xi_\phi, P\xi_\phi\rangle, \qquad x \in A.$$

We see that

$$\langle \pi_\phi(x^*x)P\xi_\phi, P\xi_\phi\rangle = \|\pi(x)P\xi_\phi\|^2 = \|P\pi(x)\xi_\phi\|^2 \le \|\pi_\phi(x)\xi\|^2 = \phi(x^*x),$$

and so we have $\psi = \lambda\phi$ for a scalar λ by Lemma 1.6.1. It follows that

$$\langle P\pi_\phi(x)\xi_\phi, \xi_\phi\rangle = \langle \lambda\pi_\phi(x)\xi_\phi, \xi_\phi\rangle, \qquad x \in A.$$

Because ξ_ϕ is a cyclic vector, it follows that $P = \lambda 1$ and $P = 1$ or $P = 0$.

In general, a representation π of an involutive Banach algebra A on a Hilbert space $\mathcal{H}$ is said to be *irreducible* if there is no nontrivial $\pi(A)$-invariant closed subspace of $\mathcal{H}$.

Theorem 1.6.4. *Let $\{\pi_\phi, \mathcal{H}_\phi\}$ be the induced representation of a state ϕ on a C^*-algebra A. Then ϕ is a pure state if and only if $\{\pi_\phi, \mathcal{H}_\phi\}$ is an irreducible representation.*

For the converse, we need the following lemma which will be easy if we use the Spectral Resolution in the next chapter (see Remark 2.1.3).

Lemma 1.6.5. *A representation $\{\pi, \mathcal{H}\}$ of a C^*-algebra A is irreducible if and only if only scalar operators commute with $\pi(A)$.*

Note that one direction has been already proved during the discussion before Theorem 1.6.4. Now, we assume that π_ϕ is an irreducible representation and ψ is a positive linear functional on A such that $\psi \le \phi$. Using this condition, we may define a bounded sesquilinear form on $\mathcal{H}_\phi$ by

$$B(\pi_\phi(x)\xi_\phi, \pi_\phi(y)\xi_\phi) = \psi(y^*x), \qquad x, y \in A,$$

on the dense subspace $\{\pi_\phi(x)\xi_\phi; x \in A\}$. Hence, there exists a unique positive operator x_0 on $\mathcal{H}_\phi$ such that

$$\langle x_0\xi, \eta\rangle = B(\xi, \eta), \qquad \xi, \eta \in H_\phi.$$

It is easy to check that x_0 commutes with $\pi_\phi(x)$ for each $x \in A$, and so $x_0 = \lambda 1$ for a scalar λ by Lemma 1.6.5. From this, it is also easy to see that ψ is a scalar multiple of ϕ, and so ϕ is pure by Lemma 1.6.1.

Because $\mathcal{S}(A)$ is the closed convex hull of $\mathcal{P}(A)$, it is easy to see from Lemma 1.5.3 that $x = 0$ if and only if $\phi(x) = 0$ for each $\phi \in \mathcal{P}(A)$. Therefore, the representation given by

$$\pi(x) = \bigoplus_{\phi \in \mathcal{P}(A)} \pi_\phi(x) \tag{1.6.1}$$

is also a faithful representation of A together with (1.5.6).

Exercise 1.6.1. Show that every vector state ω_ξ on $\mathcal{B}(\mathcal{H})$ is a pure state.

Exercise 1.6.2. Find all pure states of the matrix algebra M_n. Show that the identity map $\iota : M_n \to \mathcal{B}(\mathcal{H}_n)$ is an irreducible representation, where $\mathcal{H}_n$ is the n-dimensional Hilbert space. Show that every irreducible representation of M_n is unitarily equivalent to the identity representation ι on $\mathcal{H}_n$.

The similar results as in the above Exercise 1.6.2 also hold for the C^*-algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$. The C^*-algebra $\mathcal{K}(\mathcal{H})$ is the norm closure of the ideal of $\mathcal{B}(\mathcal{H})$ consisting of all finite rank operators on $\mathcal{H}$. In the next chapter, we will study the dual space of $\mathcal{K}(\mathcal{H})$, and see that the double dual of $\mathcal{K}(\mathcal{H})$ is just $\mathcal{B}(\mathcal{H})$, as in the case of sequence spaces; ${c_0}^{**} = \ell^\infty$. In the course of discussion, we will see that every positive linear functional on $\mathcal{K}(\mathcal{H})$ is of the form

$$\sum_i \lambda_i \omega_{\xi_i}, \tag{1.6.2}$$

for some orthonormal system $\{\xi_i\}$ of $\mathcal{H}$ and a nonnegative real numbers with $\sum_i \lambda_i < \infty$ (see Exercise 2.2.1). From this, we see the following:

Theorem 1.6.6. *Every pure state of $\mathcal{K}(\mathcal{H})$ is a vector state and every non-trivial irreducible representation of $\mathcal{K}(\mathcal{H})$ is unitarily equivalent to the identity representation $\iota : \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. If $\{\pi, \mathcal{H}\}$ is an irreducible representation of a C^*-algebra A such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then we have $\pi(A) \supseteq \mathcal{K}(\mathcal{H})$.*

For the last assertion, we need the double commutant theorem which will be discussed in the next chapter (see Theorem 2.1.4).

A C^*-algebra A is said to be *liminal* or CCR (Completely Continuous Representation) if $\pi(A) = \mathcal{K}(\mathcal{H})$ for every irreducible representation π, and said to be *postliminal* or GCR (Generalization of CCR) if $\pi(A) \supseteq \mathcal{K}(\mathcal{H})$ for every irreducible representation π. Note that if a unital C^*-algebra is liminal then every irreducible representation is of finite dimensional. A C^*-algebra is said to be *n-homogeneous* if every irreducible representation is of n-dimensional, and *subhomogeneous* if every irreducible representation is of finite dimensional with bounded dimensions. Structures of liminal or postliminal C^*-algebras are relatively well known with their *spectra*, the set of all irreducible representations with a suitable topology as in the case of commutative C^*-algebras. Especially, the structures of liminal C^*-algebras with Hausdorff spectra are completely determined as *bundles* of matrix algebras or $\mathcal{K}(\mathcal{H})$'s over their spectra. We refer to Dixmier's book [**D**, Chapter 10] for the details.

On the other hand, a C^*-algebra A is said to be *antiliminal* if the zero ideal is the only liminal closed two-sided ideal. General structures of antiliminal C^*-algebras are still mysterious and our lecture will focus on several classes of these algebras.

We close this section by considering the extensions and restrictions of states and pure states.

Theorem 1.6.7. *Let B be a unital C^*-subalgebra of a unital C^*-algebra A. Then we have the following:*

(i) *Every state of B extends to a state of A.*
(ii) *Every pure state of B extends to a pure state of B.*
(iii) *If $\{\pi, \mathcal{H}\}$ is a representation of B then there is a representation $\{\rho, \mathcal{K}\}$ of A such that $\mathcal{H} \subseteq \mathcal{K}$, $\mathcal{H}$ is invariant under $\rho(B)$ and $\rho(x)|_{\mathcal{H}} = \pi(x)$ for $x \in B$. If π is irreducible then ρ may be chosen to be irreducible.*

In general, a restriction of a pure state need not to be pure. Actually, it is easy to find an example of a pure state ρ of M_2 and a C^*-subalgebra B of M_2 such that $\rho|_{M_2}$ is not pure.

Proposition 1.6.8. *Let ρ be a pure state on a unital C^*-algebra A with the center $\mathcal{Z}(A)$. Then the restriction $\rho|\mathcal{Z}(A)$ is also a pure state.*

NOTE

Every material in this chapter is standard. We usually have followed Kadison and Ringrose' book [**K**] for §1.2, and Rudin's book [**Ru1**] for the Fourier transforms in §1.3. The proofs of Theorems 1.4.1 and 1.4.4 were taken from [**K**, §4.2], and the detailed arguments for the latter parts of §1.4 can be found in Pedersen's book [**P**, §1.5] or Murphy's book [**M**, §3.1, §3.2]. The construction in Theorem 1.5.1 is called the Gelfand-Naimark-Segal construction, which is actually possible for any involutive Banach algebras with a bounded approximate identity [**T**, §I.9], [**D**, §2.4]. For the detailed argument in the last part of §1.6, we refer to [**D**, §4.1]. Following is a list of useful references for this chapter, although they are not mentioned explicitly.

1. W. Arveson, *An Invitation to C*-algebras*, Graduate Texts in Math., Vol. 39, Springer-Verlag, 1976.
2. S. K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Graduate Texts in Math., Vol. 15, Springer-Verlag, 1974.
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5. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Grundlehren Math. Wissen., Vol. 115, Springer-Verlag, 1963.
6. L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Univ. Series in Higher Math., Van Nostrand, 1953.
7. C. E. Rickart, *General Theory of Banach Algebras*, Univ. Series in Higher Math., Van Nostrand, 1960.

CHAPTER 2

VON NEUMANN ALGEBRAS

A von Neumann algebra is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which coincides with its double commutant. In §2.1, we introduce a topology on $\mathcal{B}(\mathcal{H})$, and show the von Neumann double commutant theorem: A unital $*$-subalgebra is a von Neumann algebra if and only if it is closed under this topology. Another topologies under which the closures of $*$-subalgebras coincide will be discussed. One of them is nothing but the weak* topology if we realize $\mathcal{B}(\mathcal{H})$ as the Banach space dual of the space of all trace class operators on $\mathcal{H}$. In the course of discussion, we show that the *measurable* function calculus is possible in von Neumann algebras, especially von Neumann algebras are abundant in projections. This fact enables us to study von Neumann algebras in terms of their projections, and leads us to classify von Neumann algebras into several types. The arguments are very similar as in the theory of equipotency in set theory. During the discussion, we will see that every finite-dimensional C^*-algebra is the direct sum of matrix algebras. In order to exhibit nontrivial examples of von Neumann algebras, we consider three types of constructions; group von Neumann algebras, tensor products and crossed products. These are main motivations for the further study of C^*-algebras.

2.1. Spectral Resolution and Double Commutant Theorem

We begin with the following fundamental result.

Proposition 2.1.1. *Let $\{x_\lambda : \lambda \in \Lambda\}$ be an increasing net of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ which is bounded by a scalar operator $c1$. Then there exists an operator $x \in \mathcal{B}(\mathcal{H})$ with $x = \sup_\lambda x_\lambda$. Moreover, we have*

$$\lim_\lambda \|x_\lambda \xi - x\xi\| = 0, \qquad \xi \in \mathcal{H}. \tag{2.1.1}$$

For each $\xi \in \mathcal{H}$, the net $\{\langle x_\lambda \xi, \xi\rangle : \lambda \in \Lambda\}$ of real numbers converges to a real number. By the polarization

$$(2.1.2) \qquad \langle x_\lambda \xi, \eta\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle x_\lambda(\xi + i^k\eta), (\xi + i^k\eta)\rangle,$$

we see that $\{\langle x_\lambda \xi, \eta\rangle : \lambda \in \Lambda\}$ converges to a complex number $B(\xi, \eta)$ in $\mathbb{C}$. It is easy to see that B is a self-adjoint bounded sesquilinear form, and so there exists a self-adjoint bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that

$$\langle x\xi, \eta\rangle = B(\xi, \eta), \qquad \xi, \eta \in \mathcal{H}.$$

It is clear that $\sup_\lambda x_\lambda = x$ from the relation $\lim_\lambda \langle x_\lambda \xi, \xi\rangle = \langle x\xi, \xi\rangle$. Now, we have

$$\|(x - x_\lambda)\xi\|^2 = \lim_\lambda \langle (x - x_\lambda)^2 \xi, \xi\rangle \le \|x - x_\lambda\| \langle (x - x_\lambda)\xi, \xi\rangle,$$

and the last quantity converges to 0.

Definition. We say that a net $\{x_\lambda : \lambda \in \Lambda\}$ in $\mathcal{B}(\mathcal{H})$ converges to $x \in \mathcal{B}(\mathcal{H})$ in the *strong operator topology* if (2.1.1) holds for each $\xi \in \mathcal{H}$, and in the *weak operator topology* if

$$(2.1.3) \qquad \lim_\lambda \langle x_\lambda \xi, \eta\rangle = 0, \qquad \xi, \eta \in \mathcal{H}.$$

A unital $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$ is said to be a *von Neumann algebra* if M is closed under the strong operator topology.

Note that every von Neumann algebra is a C^*-algebra and $\mathcal{B}(\mathcal{H})$ itself is a von Neumann algebra. The above proposition says that if we take a net $\{x_\lambda\}$ of self-adjoint elements in a von Neumann algebra M then $\sup_\lambda x_\lambda$ lies in M. For a locally compact Hausdorff space X, we denote by $\mathcal{B}_b(X)$ the commutative C^*-algebra of all bounded Borel functions on X.

Exercise 2.1.1. Let X be a compact Hausdorff space. Assume that whenever a family $\mathcal{F}$ in $C(X)$ has an upper bound, $\mathcal{F}$ has the least upper bound. Show that X is *extremely disconnected*, that is, the closure of each open set is open. Show that the maximal ideal space of $\mathcal{B}_b(X)$ is extremely disconnected.

Theorem 2.1.2. *For a given self-adjoint element x in a von Neumann algebra M, there is a $*$-homomorphism $f \mapsto f(x)$ from $\mathcal{B}_b(\mathrm{sp}(x))$ into M, which extends the continuous function calculus on $C(\mathrm{sp}(x))$. If $\{f_n\}$ is an increasing sequence in $\mathcal{B}_b(\mathrm{sp}(x))_h$ converging to f then $f(x) = \sup_n f_n(x)$ in M.*

For each $\xi \in \mathcal{H}$, define a positive linear functional μ_ξ on $C(\mathrm{sp}(x))$ by

$$\mu_\xi(f) = \langle f(x)\xi, \xi\rangle, \qquad f \in C(\mathrm{sp}(x)), \tag{2.1.4}$$

where $f(x)$ is defined by the continuous function calculus. Because $\mathrm{sp}(x)$ is compact, every element $f \in \mathcal{B}_b(\mathrm{sp}(x))$ is integrable with respect to the measure μ_ξ, and so μ_ξ extends to a positive (hence, bounded) linear functional on $\mathcal{B}_b(\mathrm{sp}(x))$. Define a bounded linear functional by

$$\mu_{\xi,\eta}(f) = \frac{1}{4}\sum_{1=0}^{3} i^k \mu_{\xi+i^k\eta}(f), \qquad f \in \mathcal{B}_b(\mathrm{sp}(x)). \tag{2.1.5}$$

Now, we fix $f \in \mathcal{B}_b(\mathrm{sp}(x))_h$. Then $B(\xi,\eta) = \mu_{\xi,\eta}(f)$ defines a bounded self-adjoint sesquilinear form on $\mathcal{H}$, and so there exists a unique bounded self-adjoint operator, denoted by $f(x)$, satisfying

$$\mu_{\xi,\eta}(f) = \langle f(x)\xi, \eta\rangle, \qquad f \in \mathcal{B}_b(\mathrm{sp}(x))_h. \tag{2.1.6}$$

If $f_n \nearrow f$ then we see that $\langle f_n(x)\xi,\xi\rangle \to \langle f(x)\xi,\xi\rangle$ for each $\xi \in \mathcal{H}$ by the Lebesgue convergence theorem, and so $\sup_n f_n(x) = f(x)$. Recall [**Pe**, Proposition 6.2.9] that every $f \in \mathcal{B}_b(\mathrm{sp}(x))$ is the limit of an increasing sequence $\{f_n\}$ in $C(\mathrm{sp}(x))$. Hence, we see that $f(x) \in M$ by Proposition 2.1.1. In order to show that $f \mapsto f(x)$ is a homomorphism, it suffices to show that $f^2(x) = (f(x))^2$ for $f \in \mathcal{B}_b(\mathrm{sp}(x))_+$. Again, we take $f_n \in C(\mathrm{sp}(x))_+$ with $f_n \nearrow f$. Then ${f_n}^2 \nearrow f^2$, and so we have

$$\begin{aligned}\langle f^2(x)\xi,\xi\rangle &= \lim_n \langle f_n^2(x)\xi,\xi\rangle = \lim_n \langle f_n(x)^2\xi,\xi\rangle \\ &= \lim_n \|f_n(x)\xi\|^2 = \|f(x)\xi\|^2 = \langle f(x)^2\xi,\xi\rangle,\end{aligned}$$

by Proposition 2.1.1. We extend the map $f \mapsto f(x)$ in (2.1.6) to the whole $\mathcal{B}_b(\mathrm{sp}(x))$ by the complexification.

For each Borel subset A of $\mathrm{sp}(x)$, we may define the projection $E(A)$ by

$$E(A) = \chi_A(x), \tag{2.1.7}$$

because $\chi_A \in \mathcal{B}_b(\mathrm{sp}(x))$. If we denote the complex Borel measure $\mu_{\xi,\eta}$ on $\mathrm{sp}(x)$ by

$$\mu_{\xi,\eta}(\chi_A) = \langle E(A)\xi, \eta\rangle,$$

for each $\xi, \eta \in \mathcal{H}$, then it is easy to see that

$$\langle f(x)\xi, \eta\rangle = \int_{\mathrm{sp}(x)} f(\lambda) d\mu_{\xi,\eta}(\lambda), \qquad \xi, \eta \in \mathcal{H}. \tag{2.1.8}$$

The projections given by (2.1.7) are said to be the *spectral projections* for the self-adjoint element $x \in \mathcal{B}(\mathcal{H})$. If an element $y \in \mathcal{B}(\mathcal{H})$ commutes with x then it also commutes with every spectral projection for x. The formula (2.1.8) is usually expressed as

$$f(x) = \int_{\mathrm{sp}(x)} f(\lambda) dE(\lambda),$$

which is said to be the *spectral resolution* of x, when $f(\lambda) = \lambda$.

Remark 2.1.3. Now, we are ready to prove Lemma 1.6.5. Assume that $x \in \mathcal{B}(\mathcal{H})$ commutes with $\pi(A)$. In order to show that $x = \lambda 1$ for some $\lambda \in \mathbb{C}$, we may assume that x is self-adjoint. Then every spectral projection of x commutes with $\pi(A)$, and so their ranges are all invariant under $\pi(A)$. It follows that every spectral projection is 0 or 1, and this completes the proof of Lemma 1.6.5.

For a subset S of $\mathcal{B}(\mathcal{H})$, we define the *commutant* S' by the set of all elements $x \in \mathcal{B}(\mathcal{H})$ which commute with every element of S. If $\{x_\lambda\}$ is a net in S and $x_\lambda \to x$ in the weak operator topology then we have

$$\begin{aligned}\langle xa\xi, \eta\rangle &= \lim_\lambda \langle x_\lambda a\xi, \eta\rangle = \lim_\lambda \langle ax_\lambda \xi, \eta\rangle \\ &= \lim_\lambda \langle x_\lambda \xi, a^*\eta\rangle = \langle x_\lambda \xi, a^*\eta\rangle = \langle ax\xi, \eta\rangle,\end{aligned}$$

for each $a \in S'$, and so $x \in (S')'$. The von Neumann's double commutant theorem says that the converse also holds for unital $*$-subalgebras of $\mathcal{B}(\mathcal{H})$.

Theorem 2.1.4. *Let A be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the double commutant A'' of A coincides with the strong operator and weak operator closures of A.*

Let $x \in A''$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$. In order to show that x is in the strong operator closure of A, it suffices to find $x_0 \in A$ such that

$$(2.1.9) \qquad \|(x - x_0)\xi_i\| < \epsilon, \qquad i = 1, 2, \ldots, n.$$

Denote by $\widetilde{\mathcal{H}}$ the direct sum of n-copies of $\mathcal{H}$, and $\xi = (\xi_1, \ldots, \xi_n) \in \widetilde{\mathcal{H}}$. We also denote by $\widetilde{y} \in \mathcal{B}(\widetilde{\mathcal{H}})$ the operator $(\xi_1, \ldots, \xi_n) \mapsto (y\xi_1, \ldots, y\xi_n)$. Then the projection P of $\widetilde{\mathcal{H}}$ onto the closure of the subspace

$$(2.1.10) \qquad \{\widetilde{y}\xi : y \in A\}$$

commutes with $\widetilde{A} = \{\widetilde{y} : y \in A\}$ because the range of P is invariant under $\widetilde{A}$. If we represent $\widetilde{x}$ as the $n \times n$ matrix whose diagonals are x then it is easy to see that $\widetilde{x} \in (\widetilde{A})''$ from the condition $x \in A''$. Hence, $\widetilde{x}$ commutes with P, and so the range of P is invariant under $\widetilde{x}$. Because A is unital, we see that $\widetilde{x}\xi = \widetilde{x}\widetilde{1}\xi$ is in this range, which is the closure of the subspace (2.1.10). Hence, we can find $x_0 \in A$ satisfying (2.1.9).

Hence, a unital $*$-subalgebra M is a von Neumann algebra if and only if $M = M''$. A von Neumann algebra R acting on $\mathcal{H}$ is said to be a *factor* if $R \cap R' = \mathbb{C}1$. It is easy to see that R is a factor if and only if R' is a factor, especially $\mathcal{B}(\mathcal{H})$ is a factor. The center $M \cap M'$ of M will be denoted by $\mathcal{Z}(M)$.

For an operator $x \in \mathcal{B}(\mathcal{H})$, we denote by $p = (x^*x)^{\frac{1}{2}}$. Then we have $\|p\xi\| = \|x\xi\|$ for each $\xi \in \mathcal{H}$, and so we have an isometry from $\overline{p(\mathcal{H})}$ onto $\overline{x(\mathcal{H})}$ which sends $p\xi$ to $x\xi$. Denote by v the extension of this isometry to the whole $\mathcal{H}$ by defining $v(\xi) = 0$ for ξ in the orthogonal complement of $\overline{p(\mathcal{H})}$. Then we have

$$(2.1.11) \qquad x = vp, \qquad \operatorname{Ker} x = \operatorname{Ker} v.$$

An operator v in $\mathcal{B}(\mathcal{H})$ is said to be a *partial isometry* if $\|v(\xi)\| = \|\xi\|$ for each $\xi \in (\operatorname{Ker} v)^{\perp}$. If w is an another partial isometry satisfying (2.1.11) then it is easy to see that $v = w$. We show that the above *polar decomposition* of an operator is possible in von Neumann algebras. Let x be an element of a von Neumann algebra M acting on $\mathcal{H}$. If u is a unitary operator in M' then we see that u^*vu is a partial isometry satisfying (2.1.11) and so $u^*vu = u$. Hence, v commutes with every unitary in M', and so $v \in M'' = M$. The positive part $p = (x^*x)^{\frac{1}{2}}$ in the decomposition (2.1.11) will be denoted by $|x|$.

We close this section with one more important approximation theorem, so called the Kaplansky density theorem:

Theorem 2.1.5. *Let A be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The unit ball $(\overline{A})_1$ of the strong operator closure of A lies in the strong operator closure $\overline{(A_1)}$. If $x \in (\overline{A})_1$ is self-adjoint then x is in the strong operator closure of $A_1 \cap A_h$.*

2.2. Preduals of von Neumann Algebras

In this section, we introduce another topologies on $\mathcal{B}(\mathcal{H})$ under which von Neumann algebras are also closed. Let $\{e_i : i \in I\}$ be an orthonormal basis of $\mathcal{H}$. We define the *trace* by

$$\mathrm{Tr}(x) = \sum_i \langle xe_i, e_i \rangle, \qquad x \in \mathcal{B}(\mathcal{H})_+.$$

The following relations are easy consequences of the definition. Note that the above definition of the trace is independent of the choice of the orthonormal basis by (2.2.2).

$$\text{(2.2.1)} \qquad \mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*), \qquad x \in \mathcal{B}(\mathcal{H}),$$

$$\text{(2.2.2)} \qquad \mathrm{Tr}(uxu^*) = \mathrm{Tr}(x), \qquad x \in \mathcal{B}(\mathcal{H})_+, u \in \mathcal{U}(\mathcal{B}(\mathcal{H})),$$

$$\text{(2.2.3)} \qquad \|x\| \le \mathrm{Tr}(x), \qquad x \in \mathcal{B}(\mathcal{H})_+,$$

$$\text{(2.2.4)} \qquad \mathrm{Tr}(y^*xy) \le \|yy^*\|\mathrm{Tr}(x), \qquad x \in \mathcal{B}(\mathcal{H})_+, y \in \mathcal{B}(\mathcal{H}).$$

We define the sets of *trace class* operators and *Hilbert-Schmidt* operators by

$$\mathcal{T}(\mathcal{H}) = \text{span of } \{x \in \mathcal{K}(\mathcal{H}) : x \ge 0, \mathrm{Tr}(x) < \infty\},$$
$$\mathcal{S}(\mathcal{H}) = \{x \in \mathcal{K}(\mathcal{H}) : \mathrm{Tr}(x^*x) < \infty\},$$

respectively. From the polarization $y^*x = \frac{1}{4}\sum i^k(x+i^ky)^*(x+i^ky)$ as in (2.1.2), we have

$$xy = x^{\frac{1}{2}}x^{\frac{1}{2}}y = \frac{1}{4}\sum i^k(y+i^k1)^*x(y+i^k1), \qquad x \in \mathcal{B}(\mathcal{H})_+.$$

Hence, we see by (2.2.4) that $\mathrm{Tr}(xy) < \infty$ if $x \ge 0$, $\mathrm{Tr}(x) < \infty$ and $y \in \mathcal{B}(\mathcal{H})$. Therefore, $\mathcal{T}(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$ because it is self-adjoint. From this and the polar decomposition, it is easy to see that

$$\mathcal{T}(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : \mathrm{Tr}(|x|) < \infty\}. \tag{2.2.5}$$

The following easy relation

$$(x+y)^*(x+y) \le 2(x^*x + y^*y) \tag{2.2.6}$$

shows that $\mathcal{S}(\mathcal{H})$ is a vector space. Because $\mathcal{T}(\mathcal{H})$ is an ideal, $\mathcal{S}(\mathcal{H})$ is also an ideal by its definition. If $x, y \in \mathcal{S}(\mathcal{H})$ then we see that $y^*x \in \mathcal{T}(\mathcal{H})$ by the polarization again, and so the formula

$$\langle x, y\rangle_{\mathrm{Tr}} = \mathrm{Tr}(y^*x), \qquad x, y \in \mathcal{S}(\mathcal{H}) \tag{2.2.7}$$

defines an inner product by (2.2.3). It turns out that $\mathcal{S}(\mathcal{H})$ is a Hilbert space under this inner product.

Now, take $x \in \mathcal{T}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{H})$. We would like to estimate the value $|\mathrm{Tr}(yx)|$ as follows: If $x = v|x|$ is the polar decomposition of x then $|x|^{\frac{1}{2}} \in \mathcal{S}(\mathcal{H})$ and so $(yv|x|^{\frac{1}{2}})^* \in \mathcal{S}(\mathcal{H})$. We use the Cauchy-Schwarz inequality for the inner product (2.2.7) to calculate

$$\begin{aligned}|\mathrm{Tr}(yx)|^2 &= |\mathrm{Tr}(yv|x|^{\frac{1}{2}}|x|^{\frac{1}{2}})|^2 = \langle |x|^{\frac{1}{2}}, (yv|x|^{\frac{1}{2}})^*\rangle_{\mathrm{Tr}} \\ &\le \mathrm{Tr}(|x|)\,\mathrm{Tr}(|x|^{\frac{1}{2}}v^*y^*yv|x|^{\frac{1}{2}}) \le \mathrm{Tr}(|x|)\,\|v^*y^*yv\|\,\mathrm{Tr}(|x|) \\ &= \|y\|^2(\mathrm{Tr}(|x|))^2.\end{aligned}$$

Hence, we have

$$|\mathrm{Tr}(yx)| \le \|y\|\,\mathrm{Tr}(|x|), \qquad x \in \mathcal{T}(\mathcal{H}), y \in \mathcal{B}(\mathcal{H}). \tag{2.2.8}$$

Using the above relation, it is easy to see that $\mathcal{T}(\mathcal{H})$ is a normed space under the norm $\|x\|_{\mathrm{Tr}} = \mathrm{Tr}(|x|)$. Now, we are ready to determine the dual space of $\mathcal{K}(\mathcal{H})$ as a Banach space. The relation (2.2.8) shows that every $x \in \mathcal{T}(\mathcal{H})$ gives rise to a bounded linear functional

$$\phi_x : y \mapsto \mathrm{Tr}(yx), \qquad y \in \mathcal{K}(\mathcal{H}) \tag{2.2.9}$$

on $\mathcal{K}(\mathcal{H})$ and $\|\phi_x\| \leq \mathrm{Tr}(|x|)$. Conversely, if ϕ is a bounded linear functional on $\mathcal{K}(\mathcal{H})$, then we have

$$|\phi(y)|^2 \leq \|\phi\|^2 \|y\|^2 \leq \|\phi\|^2 \langle y, y\rangle_{\mathrm{Tr}}, \qquad y \in \mathcal{S}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H}).$$

Hence, ϕ is a bounded linear functional on $\mathcal{S}(\mathcal{H})$, and so there exists a unique $x^* \in \mathcal{S}(\mathcal{H})$ such that

$$\phi(y) = \langle y, x^*\rangle_{\mathrm{Tr}} = \mathrm{Tr}(xy), \qquad y \in \mathcal{S}(\mathcal{H}),$$

because $\mathcal{S}(\mathcal{H})$ is a Hilbert space. Using the polarization again, we see that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ for $x, y \in \mathcal{S}(\mathcal{H})$, and so we have

$$\phi(y) = \mathrm{Tr}(yx), \qquad y \in \mathcal{S}(\mathcal{H}). \tag{2.2.10}$$

Now, for each projection $p \in \mathcal{B}(\mathcal{H})$ of finite rank, we have

$$|\mathrm{Tr}(p|x|)| = |\mathrm{Tr}(pv^*x)| = |\phi(pv^*)| \leq \|\phi\|.$$

Hence, we see that $|\mathrm{Tr}(|x|)| \leq \|\phi\|$ and $x \in \mathcal{T}(\mathcal{H})$. Therefore, the correspondence $x \mapsto \phi_x$ defines an isometry from $\mathcal{T}(\mathcal{H})$ onto the dual space of $\mathcal{K}(\mathcal{H})$ by the formulae (2.2.9) and (2.2.10). From this, we also see that $\mathcal{T}(\mathcal{H})$ is a Banach space.

Exercise 2.2.1. Show that x is a positive (respectively self-adjoint) element of $\mathcal{T}(\mathcal{H})$ if and only if ϕ_x is positive (respectively self-adjoint). Show also that every positive element of $\mathcal{T}(\mathcal{H})$ is of the form $x = \sum_i \lambda_i P_i$ with nonnegative real numbers $\{\lambda_i\}$ such that $\sum_i \lambda_i < \infty$ and orthogonal projections $\{P_i\}$. Show that every positive linear functional on $\mathcal{K}(\mathcal{H})$ is of the form (1.6.2).

Now, we show that the dual space of $\mathcal{T}(\mathcal{H})$ is just $\mathcal{B}(\mathcal{H})$. The relation (2.2.8) again shows that each $y \in \mathcal{B}(\mathcal{H})$ defines a bounded linear functional

$$\psi_y : x \mapsto \mathrm{Tr}(yx), \qquad x \in \mathcal{T}(\mathcal{H}), \tag{2.2.11}$$

and $\|\psi_y\| \le \|y\|$. In order to show that every bounded linear functional of $\mathcal{T}(\mathcal{H})$ is in this form, we introduce the rank one operator $x_{\xi,\eta}$ defined by

$$x_{\xi,\eta}(\zeta) = \langle \zeta, \eta \rangle \xi, \qquad \zeta \in \mathcal{H}, \tag{2.2.12}$$

which is determined by $\xi, \eta \in \mathcal{H}$. Then it is easy to see that

$$|\psi(x_{\xi,\eta})| \le \|\psi\| \, \|\xi\| \, \|\eta\|, \qquad \psi \in \mathcal{T}(\mathcal{H})^*,$$

and so there exists a unique operator $y \in \mathcal{B}(\mathcal{H})$ such that

$$\psi(x_{\xi,\eta}) = \langle y\xi, \eta \rangle, \qquad \|y\| \le \|\psi\|.$$

By a straightforward calculation, we have

$$\psi(x) = \mathrm{Tr}(yx) = \psi_y(x), \qquad x \in \mathcal{T}(\mathcal{H}),$$

and so $y \mapsto \psi_y$ is an isometry from $\mathcal{B}(\mathcal{H})$ onto the dual space of $\mathcal{T}(\mathcal{H})$. We summarize as follows:

Theorem 2.2.1. *The dual space of $\mathcal{K}(\mathcal{H})$ is $\mathcal{T}(\mathcal{H})$, and the dual space of $\mathcal{T}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$.*

The weak* topology on $\mathcal{B}(\mathcal{H})$ induced by the relation $\mathcal{T}(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ is said to be the σ-*weak operator* topology. By definition, this is the smallest topology on $\mathcal{B}(\mathcal{H})$ for which the map

$$y \mapsto \mathrm{Tr}(yx), \qquad y \in \mathcal{B}(\mathcal{H})$$

is continuous for each $x \in \mathcal{T}(\mathcal{H})$. Note that the weak operator topology is the smallest topology on $\mathcal{B}(\mathcal{H})$ for which the map $y \mapsto \mathrm{Tr}(yx)$ is continuous for each finite rank operator x. Hence, the σ-weak operator topology is strictly larger than the weak operator topology if $\mathcal{H}$ is infinite-dimensional. By Exercise 2.2.1, we see that every positive σ-weak operator continuous functional on $\mathcal{B}(\mathcal{H})$ is of the form

$$\phi(y) = \sum \langle y\xi_n, \xi_n \rangle, \qquad y \in \mathcal{B}(\mathcal{H}),$$

for some orthogonal system $\{\xi_n\}$ of $\mathcal{H}$ and $\|\phi\| = \sum \|\xi_n\|^2$. Similarly, we see that every σ-weak operator continuous linear functional on $\mathcal{B}(\mathcal{H})$ is of the form

$$y \mapsto \sum \langle y\xi_n, \eta_n \rangle, \qquad y \in \mathcal{B}(\mathcal{H}),$$

for some ℓ^2-summable orthogonal systems $\{\xi_n\}$ and $\{\eta_n\}$. Let A be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and $\{\xi_n\}$ an ℓ^2-summable sequence of vectors. By the same method of *infinite* amplification as in the proof of the double commutant theorem, we see that an element x in A'' can be approximated by elements of A in the sense

$$\sum \|(x - x_0)\xi_n\|^2 < \epsilon^2$$

for some $x_0 \in S$. Applying the Cauchy-Schwarz inequality, we see that A'' is also the σ-weak operator closure of A.

We say that a net $\{x_\lambda\}$ of $\mathcal{B}(\mathcal{H})$ converges to $x \in \mathcal{B}(\mathcal{H})$ in the *σ-strong operator topology* if

$$\lim_\lambda \sum_n \|(x_\lambda - x)\xi_n\|^2 \to 0, \qquad \sum \|\xi_n\|^2 < \infty.$$

We have shown in fact that A'' is the closure of A with respect to the σ-strong operator topology. If M is a von Neumann algebra acting on $\mathcal{H}$, we denote by $M^\perp$ the set of *annihilators* of M;

$$M^\perp = \{x \in \mathcal{T}(\mathcal{H}) = \mathcal{K}(\mathcal{H})^* : \mathrm{Tr}(yx) = 0 \text{ for each } y \in M\}.$$

From the general theory of the duality between annihilators and quotients, we see that the dual space of $\mathcal{T}(\mathcal{H})/M^\perp$ is just M. We summarize as follows:

Theorem 2.2.2. *A unital $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if M is σ-weak operator closed. For every von Neumann algebra M, there is a Banach space X such that X^* is isometrically isomorphic* *to M. The weak* topology on M is identical with the σ-weak topology in this isomorphism.*

The converse is also true. More precisely, if a C^*-algebra A is the dual space of a Banach space then there is a faithful representation $\{\pi, \mathcal{H}\}$ of A such that $\pi(A)$ is a von Neumann algebra acting on $\mathcal{H}$ (see [**T**, §III.3]). This gives an abstract characterization of von Neumann algebras which is free from Hilbert spaces.

Now, if $\{\pi, \mathcal{H}\}$ is the universal representation of a C^*-algebra A, then the von Neumann algebra $\pi(A)''$ generated by $\pi(A)$ is said to be the *enveloping von Neumann algebra* of A. As was noted in §1.5, every state of $\pi(A)$ is a vector state, and so extends to a state of $\pi(A)''$. In this way, we see that every $\phi \in A^*$ corresponds to a σ-weak operator continuous functional $\overline{\phi}$ of $\pi(A)''$ such that $\phi(x) = \overline{\phi}(\pi(x))$ for each $x \in A$. It is also easy to see that the map $\phi \mapsto \overline{\phi}$ is an isometry by Theorem 2.1.5, and so A^* is isometrically isomorphic with the predual of $\pi(A)''$. Taking the adjoints, we see that $\pi(A)''$ with the σ-weak operator topology is identified with A^{**} with the weak* topology.

2.3. Type Classification of Factors

From Theorem 2.1.2, we see that von Neumann algebras are abundant in projections. Actually, the spectral resolution shows that every element of a von Neumann algebra is the norm limit of finite linear combinations of projections.

Definition. Let M be a von Neumann algebra acting on $\mathcal{H}$. Two projections p and q are said to be *equivalent* and denoted by $p \sim q$ if there exists $v \in M$ such that

$$p = v^*v, \qquad q = vv^*. \tag{2.3.1}$$

Also, we say that p is *weaker* than q and denote by $p \precsim q$ if p is equivalent to a subprojection of q.

Note that the element v in (2.3.1) should be found in M. It is easy to see that such an element is a partial isometry from the range space of p onto the range space of q and $\sim$ is an equivalent relation. If $x = v|x| = v(x^*x)^{\frac{1}{2}}$ is the polar decomposition of x in M, then we have seen that v is the partial isometry from the range space of x onto the range space of $|x|$, which is identical with the range space of x^*. Hence, we have

$$\mathcal{R}(x) \sim \mathcal{R}(x^*) \tag{2.3.2}$$

for each $x \in M$, where $\mathcal{R}(x)$ denotes the *range projection* of x, more precisely, the projection onto the range space of x.

By a standard argument of Bernstein in set theory, one can show that $\precsim$ is a partial order. We show that this is a linear order in a factor.

Lemma 2.3.1. *Each pair of nonzero projections in a factor have equivalent nonzero subprojections.*

Let p be a nonzero projection in a factor R acting on $\mathcal{H}$ and denote by E the closed subspace of $\mathcal{H}$ spanned by $\{xp\xi : x \in R, \xi \in \mathcal{H}\}$. It is easy to check that E is invariant under R and R'. By Lemma 1.6.3, we see that the projection onto E is contained in the center of R, and so we see that $E = \mathcal{H}$. Now, if q is an another nonzero projection in R then the same argument is applied, and we find $x, y \in R$ and $\xi, \eta \in \mathcal{H}$ such that

$$0 \neq \langle xp\xi, yq\eta \rangle = \langle qy^*xp\xi, \eta \rangle = \langle \xi, px^*yq\eta \rangle.$$

By (2.3.2), we see that $\mathcal{R}(px^*yq) \sim \mathcal{R}(qy^*xp)$. Now, using the standard maximal argument, it is easy to see the following:

Proposition 2.3.2. *Any two projections in a factor are comparable each other.*

A projection p is said to be *infinite* if it is equivalent to a proper subprojection of p, and *finite* if it is not infinite. It should be noticed that the notion of finiteness depends heavily on the von Neumann algebra in which the projection p lies. It is clear that a minimal projection is finite.

Definition. A factor R is said to be

(i) of *type* I if R has a nonzero minimal projection;

(ii) of *type* II if R has no nonzero minimal projections, and has a nonzero finite projection;

(iii) of *type* III if R has no nonzero finite projection.

We proceed to determine the structures of type I factors. Let q be a nonzero projection of a type I factor $R \subseteq \mathcal{B}(\mathcal{H})$ with a minimal projection p_0. By Proposition 2.3.2, q has a subprojection q_0 which is equivalent to p_0. Because q_0 is also minimal, we see that every nonzero projection in R dominates a minimal projection. Considering a maximal family of orthogonal minimal projections, we see that $1_{\mathcal{H}}$ is the sum of n orthogonal minimal projections, where n is a cardinal number. We say that R is a factor of *type* I_n in this case. Let ξ be a unit vector in the range space of the minimal projection p_0, and denote by q the projection whose range $\mathcal{H}_0$ is the closure of $\{x\xi : x \in R\}$. Then, $q \in R'$ and $xq = qxq$ may be considered as an operator acting on $\mathcal{H}_0$. We show that correspondence

$$R \to \mathcal{B}(\mathcal{H}_0) : x \mapsto xq \tag{2.3.3}$$

is a $*$-isomorphism. This is an immediate consequence of the following:

Lemma 2.3.3. *Let R be a factor and $x \in R$, $y \in R'$ with $xy = 0$. Then we have $x = 0$ or $y = 0$.*

Assume that $y \neq 0$ and denote by $\widetilde{q}$ the supremum of all projections q in R' such that $xq = 0$. By Proposition 2.1.1, $\widetilde{q} \in R'$. If p is a projection in R' then we have $xp\widetilde{q} = px\widetilde{q} = 0$ and so $\mathcal{R}(p\widetilde{q}) \leq \widetilde{q}$. From this, we have

$$p\widetilde{q} = \widetilde{q}p\widetilde{q} = (\widetilde{q}p\widetilde{q})^* = (p\widetilde{q})^* = \widetilde{q}p,$$

and $\widetilde{q} \in R'' = R$ by the spectral resolution. Because R is a factor we see that $\widetilde{q} = 1$ and so $x = 0$.

If R is a factor of type I_n then it is clear that the Hilbert space $\mathcal{H}_0$ in the above discussion is the n-dimensional Hilbert space and every one-dimensional projection in $\mathcal{B}(\mathcal{H}_0)$ corresponds to a minimal projection in R in (2.3.3). This shows that the range of (2.3.3) is the whole algebra $\mathcal{B}(\mathcal{H}_0)$ because every one-dimensional projection is equivalent each other. We summarize as follows:

Theorem 2.3.4. *For each cardinal number n, there is only one factor of type I_n up to $*$-isomorphism. This factor is $*$-isomorphic to $\mathcal{B}(\mathcal{H}_n)$ with the n-dimensional Hilbert space $\mathcal{H}_n$.*

Let A be a finite-dimensional C^*-algebra. It is clear that A has a faithful representation on a finite-dimensional Hilbert space $\mathcal{H}$ and A can be considered as a von Neumann algebra acting on $\mathcal{H}$. Let $\{p_i : 1 = 1, 2, \dots, n\}$ be the set of all minimal projections in the center $\mathcal{Z}(A)$. The correspondence

$$x \mapsto xp_1 + xp_2 + \cdots + xp_n \tag{2.3.4}$$

defines a $*$-isomorphism from A onto the direct sum $Ap_1 \oplus Ap_2 \oplus \dots Ap_n$. It is easy to see that Ap_i is a factor acting on the range space of p_i for each p_i. In fact, if p is a minimal projection of $\mathcal{Z}(M)$ then Mp is a factor because each central projection in Mp is a subprojection of p in $\mathcal{Z}(M)$, and so it is 0 or p. Together with Theorem 2.3.4, we see the following:

Proposition 2.3.5. *Every finite-dimensional C^*-algebra is $*$-isomorphic to the finite direct sum of matrix algebras.*

In the remaining of this section, we discuss the type decomposition of von Neumann algebras. Actually, we have already encountered crucial arguments just before. Let x be an element of a von Neumann algebra M. The *central carrier* C_x is the greatest lower bound of all central projections q such that $qx = x$, or equivalently, the complement of the the least upper bound of all central projections p such that $px = 0$. The same argument as in the proof of Lemma 2.3.3 shows that $xy = 0$ if and only if $C_xC_y = 0$ for $x \in M$ and $y \in M'$. Also, if q is a projection in M' then

$$Mq \to MC_q : xq \mapsto xC_q \tag{2.3.5}$$

is a $*$-isomorphism as in (2.3.3). Note that Mq is a von Neumann algebra acting on the range space of q. It can be shown that $(Mq)' = qM'q$.

A projection p in a von Neumann algebra M is said to be *abelian* in M if the von Neumann algebra pMp is abelian. The notion of abelian projections corresponds to that of minimal projections in factors. More precisely, A projection p in M is abelian if and only if p is minimal in the class of projections in M with the same central carrier.

We say that a von Neumann algebra M is of *type* I if M has an abelian projection with the central carrier 1, of *type* II if M has no nonzero abelian projections but has a finite projection with the central carrier 1, and of *type* III if M has no nonzero finite projections. A von Neumann algebra M is said to be *finite* if 1 is a finite projection, and *properly infinite* if every nonzero central projection is infinite. A von Neumann algebra of type I is said to be of *type* I_n if 1 is the sum of n equivalent abelian projections for some cardinal number n. A von Neumann algebra of type II is said to be of *type* II_1 if it is finite and of *type* II_∞ if it is properly infinite.

Every von Neumann algebra M has mutually orthogonal central projections p_n $(n = 1, 2, \ldots, \infty)$, q, r and s, with sum 1, and maximal with respect to the properties that Mp_n, Mq, Mr and Ms are 0 or of type I_n, II_1, II_∞ and III, respectively. Furthermore, every von Neumann algebra of a type may be expressed as the "direct integral" of factors of the same types. Hence, the study of von Neumann algebras is reduced to that of factors in some sense. Von Neumann algebras of type I are closely related with postliminal C^*-algebras as follows:

Theorem 2.3.6. *Let A be a separable C^*-algebra. Then the following are equivalent:*

(i) *A is postliminal, that is, we have $\mathcal{K}(\mathcal{H}) \subseteq \pi(A)$ for every irreducible representation π.*
(ii) *Every representation π generates a von Neumann algebra of type I, that is, $\pi(A)''$ is of type I.*
(iii) *The universal representation of A generates a von Neumann algebra of type I.*
(iv) *Two representations of A with the same kernels are unitarily equivalent each other.*

2.4. Factors Arising from Discrete Groups

In this section, we present several examples of factors which is not of type I. Let G be a *discrete* group. We denote by $\mathcal{R}_\lambda(G)$ the von Neumann algebra on $\ell^2(G)$ generated by the reduced group C^*-algebra $C^*_\lambda(G)$. Recall that the product and involution in (1.5.8) is nothing but

$$\chi_s * \chi_t = \chi_{st}, \qquad {\chi_s}^* = \chi_{s^{-1}} \tag{2.4.1}$$

for the characteristic functions on singletons. Noticing that $\{\chi_t : t \in G\}$ is an orthonormal basis for $\ell^2(G)$, we see that $\mathcal{R}_\lambda(G)$ is the von Neumann algebra acting on $\ell^2(G)$ generated by the operators

$$L_s : \chi_t \mapsto \chi_s * \chi_t = \chi_{st}, \qquad \chi_t \in \ell^2(G).$$

The von Neumann algebra acting on $\ell^2(G)$ generated by the operators

$$R_s : \chi_t \mapsto \chi_t * \chi_{s^{-1}} = \chi_{ts^{-1}}, \qquad \chi_t \in \ell^2(G),$$

will be denoted by $\mathcal{R}_\rho(G)$. Although the operators L_ξ and R_ξ may be defined similarly for every $\xi, \eta \in \ell^2(G)$, it is not clear whether they define *bounded* linear operators on $\ell^2(G)$. But, it is easy to see the following:

Lemma 2.4.1. *If $T \in \mathcal{B}(\ell^2(G))$ and $\xi \in \ell^2(G)$ satisfy the relation $\langle T\chi_s, \chi_t\rangle = \langle \xi * \chi_s, \chi_t\rangle$ for each $s, t \in G$ then we have $T = L_\xi$.*

Now, we have concrete descriptions for the elements of $\mathcal{R}_\lambda(G)$ and $\mathcal{R}_\rho(G)$ as follows:

Proposition 2.4.2. *We have*

$$\mathcal{R}_\lambda(G) = \{L_\xi \in \mathcal{B}(\ell^2(G)) : \xi \in \ell^2(G)\},$$
$$\mathcal{R}_\rho(G) = \{R_\xi \in \mathcal{B}(\ell^2(G)) : \xi \in \ell^2(G)\},$$

and $\mathcal{R}_\lambda(G)' = \mathcal{R}_\rho(G)$.

Using Lemma 2.4.1, one can show that the sets in the proposition are $*$-subalgebras of $\mathcal{B}(\ell^2(G))$. Actually, if $\xi, \eta \in \ell^2(G)$ and L_ξ, L_y are bounded operators on $\ell^2(G)$ then we have

$$\begin{gathered} L_\xi + L_\eta = L_{\xi+\eta},\ aL_\xi = L_{a\xi},\ L_\xi L_\eta = L_{\xi*\eta},\ L_\xi{}^* = L_{\xi^*},\ L_e = 1, \\ L_\xi = L_\eta \Longrightarrow \xi = \eta. \end{gathered} \tag{2.4.2}$$

Furthermore, if R_η is also a bounded operator then we have

$$L_\xi R_\eta = R_\eta L_\xi, \tag{2.4.3}$$

from which we see that two sets in the proposition are von Neumann algebras and the last assertion follows.

Now, we show that $\mathcal{R}_\lambda(G)$ and $\mathcal{R}_\rho(G)$ are finite von Neumann algebras. To do this, first note that every characteristic function χ_s has the following *tracial* property;

$$\langle L_\xi L_\eta \chi_s, \chi_s\rangle = \langle L_\eta L_\xi \chi_s, \chi_s\rangle, \tag{2.4.4}$$

$$\langle T\chi_s, \chi_s\rangle = \langle T\chi_e, \chi_e\rangle, \qquad T \in \mathcal{R}_\lambda(G). \tag{2.4.5}$$

If v is a partial isometry in $\mathcal{R}_\lambda(G)$ with $vv^* = 1$ and $v^*v = p$ then we have

$$\langle p\chi_s, \chi_s\rangle = \langle v^*v\chi_s, \chi_s\rangle = \langle vv^*\chi_s, \chi_s\rangle = \langle \chi_s, \chi_s\rangle,$$

for each $s \in G$, and so, $p = 1$.

From the last relation of (2.4.2), it is easy to see that $\{L_s : s \in G\}$ is linearly independent set in $\mathcal{R}_\lambda(G)$. Therefore, $\mathcal{R}_\lambda(G)$ is infinite-dimensional if G is an infinite group. If $L_\xi \in \mathcal{R}_\lambda(G)$ commutes with L_s then we have $\xi * \chi_s = \chi_s * \xi$, and so

$$\xi(sts^{-1}) = (\xi * \chi_s)(st) = (\chi_s * \xi)(st) = \xi(t).$$

Hence, we see that if L_ξ is in the center of $\mathcal{R}_\lambda(G)$ then ξ is constant on every conjugacy class in G.

Now, we assume that G is an *i.c.c.* group, that is, every conjugacy class of a non-unital element is infinite. Under this assumption, it follows that $\xi = a\chi_e$ for some scalar a, for each $L_\xi \in \mathcal{Z}(\mathcal{R}_\lambda(G))$. We summarize as follows:

Theorem 2.4.3. *Let G be a discrete group and $\mathcal{R}_\lambda(G)$ the group von Neumann algebra acting on $\ell^2(G)$ generated by $\{L_s : s \in G\}$. Then $\mathcal{R}_\lambda(G)$ is a finite von Neumann algebra. If G is an i.c.c. group then $\mathcal{R}_\lambda(G)$ is a factor of type* II_1.

We have two typical examples of *i.c.c.* groups: It is well known that the free group F_n on n generators is an *i.c.c.* group. Another example is the group Π of all permutations on integers which fix all but finite integers. It is still a long-standing question whether $\mathcal{R}_\lambda(F_n)$ and $\mathcal{R}_\lambda(F_m)$ are $*$-isomorphic or not for $n \neq m$, although $\mathcal{R}_\lambda(F_n)$ and $\mathcal{R}_\lambda(\Pi)$ are not $*$-isomorphic each other

[**K**, Theorem 6.7.8]. The group Π has an another important property, *local finiteness.* The group Π may be expressed as

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n$$

where Π_n is the finite subgroup of Π consisting of elements which fix integers in $\{i \in \mathbb{Z} : |i| > n\}$. Note that $\{L_s : s \in \Pi_n\}$ generates a finite dimensional C^*-algebras, and so $\mathcal{R}_\lambda(\Pi)$ is the von Neumann algebra generated by an increasing sequence of finite dimensional $*$-subalgebras. Such a von Neumann algebra is said to be *hyperfinite*. It is known that there is only one hyperfinite II_1 factor on the separable Hilbert space up to $*$-isomorphism [**K**, §12.2].

In order to give an example of a factor of type II_∞, we introduce the notion of tensor products. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. We denote by $\mathcal{H}_1 \odot \mathcal{H}_2$ the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ as vector spaces. The *Hilbert space tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the unique inner product satisfying

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle, \qquad \xi_1, \xi_2 \in \mathcal{H}_1, \ \ \eta_1, \eta_2 \in \mathcal{H}_2. \tag{2.4.6}$$

This Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is characterized by the existence of a bilinear map $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ with the following property: For every *bounded* bilinear map $\phi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{K}$ into a Hilbert space $\mathcal{K}$, there exists a unique bounded linear map $\overline{\phi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$ such that $\phi = \overline{\phi} \circ p$. From this universal property, we may define the *tensor product* $x_1 \otimes x_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of $x_1 \in \mathcal{B}(\mathcal{H}_1)$ and $x_2 \in \mathcal{B}(\mathcal{H}_2)$ satisfying

$$(x_1 \otimes x_2)(\xi_1 \otimes \xi_2) = x_1\xi_1 \otimes x_2\xi_2, \qquad \xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2. \tag{2.4.7}$$

Exercise 2.4.1. Show that $\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|$ for $x_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$.

Let M_1 and M_2 be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The *von Neumann algebra tensor product* $M_1 \overline{\otimes} M_2$ is defined by the von Neumann algebra acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$, generated by $\{x_1 \otimes x_2 : x_1 \in M_1, x_2 \in M_2\}$. We proceed to deal with $M \overline{\otimes} \mathcal{B}(\mathcal{K})$ more concretely, where M is a von Neumann algebra acting on $\mathcal{H}$. Let $\{\eta_i : i \in I\}$

be an orthonormal basis of $\mathcal{K}$. Then $\mathcal{H} \otimes \mathcal{K}$ is nothing but the direct sum $\sum_{i\in I}^{\oplus} \mathcal{H}_i$, where $\mathcal{H}_i$ is a copy of $\mathcal{H}$, by the Hilbert space isomorphism

$$U : \sum_{i\in I}{}^{\oplus}\xi_i \mapsto \sum_{i\in I} \xi_i \otimes \eta_i : \sum_{i\in I}{}^{\oplus}\mathcal{H}_i \to \mathcal{H} \otimes \mathcal{K}. \tag{2.4.8}$$

If $x \in M$ and $y \in \mathcal{B}(\mathcal{K})$ then the operators $U^*(x \otimes 1_{\mathcal{K}})U$ and $U^*(1_{\mathcal{H}} \otimes y)U$ in $\mathcal{B}(\sum_{i\in I}^{\oplus} \mathcal{H}_i)$ have the matrix representations

$$\begin{pmatrix} x & 0 & \dots \\ 0 & x & \dots \\ & \dots & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_{11}1_{\mathcal{H}} & y_{12}1_{\mathcal{H}} & \dots \\ y_{21}1_{\mathcal{H}} & y_{22}1_{\mathcal{H}} & \dots \\ & \dots & \end{pmatrix},$$

respectively, where $[y_{ij}]$ is a numerical matrix representation of y with respect to the orthonormal basis $\{\eta_i\}$. So, we see that every entries of the matrix representing an element in $U^*(M\overline{\otimes}\mathcal{B}(\mathcal{K}))U \subseteq \mathcal{B}(\sum_{i\in I}^{\oplus} \mathcal{H}_i)$ lies in M.

Conversely, let T be an element of $\mathcal{B}(\sum_{i\in I}^{\oplus} \mathcal{H}_i)$ whose entries are in $M \subseteq \mathcal{B}(\mathcal{H})$. Using the matrix unit $\{e_{ij}\}$ of $\mathcal{B}(\mathcal{K})$, it is easy to see that T is in the strong operator limit of operators in $U^*(M\overline{\otimes}\mathcal{B}(\mathcal{K}))U$. Indeed, if the only nonzero entry of T is $T_{ij} \in M$ then $T = U^*(T_{ij} \otimes e_{ij})U \in U^*(M\overline{\otimes}\mathcal{B}(\mathcal{K}))U$, and so we have $T = \sum_{ij} U^*(T_{ij} \otimes e_{ij})U$ in the strong operator topology. Hence, $M\overline{\otimes}\mathcal{B}(\mathcal{K})$ is unitarily isomorphic with the von Neumann algebra consisting of operators in $\mathcal{B}(\sum_{i\in I}^{\oplus} \mathcal{H}_i)$, all of whose entries lie in M in the matrix representation. From this matrix representation, it is easy to see the following:

$$(M\overline{\otimes}\mathcal{B}(\mathcal{K}))' = M'\overline{\otimes}\mathbb{C}1_{\mathcal{K}}, \tag{2.4.9}$$

and so it follows that if M is a factor then $M\overline{\otimes}\mathcal{B}(\mathcal{K})$ is also a factor. Now, it is easy to see the following:

Theorem 2.4.4. *If R is a factor of type II_1 and $\mathcal{K}$ is an infinite dimensional Hilbert space then $R\overline{\otimes}\mathcal{B}(\mathcal{K})$ is a factor of type II_∞.*

Actually, every factor of type II_∞ may be obtained in this way [**K**, Theorem 6.7.10], and so the classification of type II factors is reduced to that of type II_1 factors. It has been known during sixties that there are uncountably many non-isomorphic factors of type II_1, although the complete classification of type II_1 factors is still far from being complete.

2.5. Factors Arising from Ergodic Theory

Let (X, μ) be a probability space and ϕ an invertible measure preserving transformation of X. We define the unitary map u and the bounded linear map m_f for each $f \in L^\infty(X)$, in the Hilbert space $L^2(X)$ by

$$(m_f\xi)(x) = f(x)\xi(x),$$
$$(u\xi)(x) = \xi(\phi^{-1}(x)),$$

for each $\xi \in L^2(X)$ and $x \in X$, respectively. We show that $\mathcal{A} = \{m_f : f \in L^\infty(X)\}$ is a *maximal* abelian von Neumann algebra acting on $L^2(X)$, that is, $\mathcal{A}' = \mathcal{A}$. First of all, note that if f is a measurable function on X and the multiplication map m_f defines a bounded linear map on $L^2(X)$ then $f \in L^\infty(X)$ with $\|f\|_\infty \le \|m_f\|$. Now, for $T \in \mathcal{A}'$, put $f = T(1_X)$. Then we have

$$m_f(g) = m_g(f) = m_gT(1_X) = Tm_g(1_X) = T(g), \qquad g \in L^\infty(X).$$

Because $L^\infty(X)$ is dense in $L^2(X)$, we see that $m_f = T$ as bounded linear operators on $L^2(X)$, and so it follows that $f \in L^\infty(X)$ and $T \in \mathcal{A}$.

From the above definitions, it is easy to see that

$$(2.5.1) \qquad um_fu^* = m_{f\circ\phi^{-1}}, \qquad f \in L^\infty(X).$$

Now, we put $\mathcal{H} = \sum_{n\in\mathbb{Z}}^{\oplus} \mathcal{H}_n$, where $\mathcal{H}_n$ is a copy of the Hilbert space $L^2(X)$ for each $n \in \mathbb{Z}$. Let M_f and U be the bounded operators in $\mathcal{B}(\mathcal{H})$ whose matrix representations are

$$(M_f)_{ij} = \delta_{ij}m_f, \qquad U_{ij} = \delta_{i,j+1}u,$$

respectively. In the identification $\mathcal{H} = L^2(X) \otimes \ell^2(\mathbb{Z})$, these are nothing but

$$M_f = m_f \otimes 1_{L^2(\mathbb{Z})}, \qquad U = u \otimes s,$$

where s denotes the right-shift operator on $\ell^2(\mathbb{Z})$. We denote by $M(X, \mu, \phi)$ (abbreviated by just M) the von Neumann algebra on $\mathcal{H}$ generated by the family $\{M_f : f \in L^\infty(X)\}$ and U.

In order to describe the elements of M in terms of matrices, we first find the commutants of $\mathcal{A}\otimes 1$ and $\{U\}$. Let T be an operator in M' whose matrix representation is $[T_{ij}]$. Then we see that $TU^n = U^nT$ for each $n \in \mathbb{Z}$ if and only if

$$T_{i,j+n}u^n = u^nT_{i-n,j}, \qquad i,j,n \in \mathbb{Z}. \tag{2.5.2}$$

Also we have $(\mathcal{A}\otimes \mathbb{C}1)' = \mathcal{A}'\otimes \mathcal{B}(L^2(X)) = \mathcal{A}\otimes \mathcal{B}(L^2(X))$ by (2.4.9), and so it follows that T commutes with $\mathcal{A}$ if and only if

$$T_{ij} \in \mathcal{A}, \qquad i,j \in \mathbb{Z}. \tag{2.5.3}$$

Hence, it follows that if $T \in M'$ then T_{ij} is of the form

$$T_{ij} = T_{i,(j-i)+i} = u^iT_{0,j-i}u^{-i} = u^im_{f_{j-i}}u^{-i}, \tag{2.5.4}$$

where $\{f_n : n \in \mathbb{Z}\}$ is taken from $L^\infty(X)$. Conversely, assume that the matrix representation of T is of the form in (2.5.4). By a calculation and the relation (2.5.1), we see that the conditions (2.5.2) and (2.5.4) are satisfied, and so $T \in M'$.

Now, we show that $T \in M$ if and only if its matrix representation is of the form

$$T_{ij} = u^{i-j}m_{g_{i-j}}, \tag{2.5.5}$$

where $\{g_n : n \in \mathbb{Z}\}$ is also taken from $L^\infty(X)$. First, we note that T_{ij} is of the form in (2.5.5) if and only if the following condition is satisfied:

$$T_{ij} = T_{i+n,j+n}, \qquad u^{j-i}T_{ij} \in \mathcal{A}, \qquad i,j,n \in \mathbb{Z}. \tag{2.5.6}$$

We denote by M_0 the linear span of $\{U^nM_f : n \in \mathbb{Z}, f \in L^\infty(X)\}$, and by S the linear subspace of $\mathcal{B}(\mathcal{H})$ consisting of operators T's whose matrix representations are of the forms in (2.5.5) or satisfying (2.5.6). Then we see that $M_0 \subseteq S$, because $(U^nM_f)_{ij} = \delta_{n,i-j}u^{i-j}m_f$ by the calculation. Furthermore, S is weak operator closed by (2.5.6) because the mapping $T \mapsto T_{ij}$ is a weak operator continuous map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(L^2(X))$. It follows that $M = \overline{M_0} \subseteq S$ because M_0 is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Conversely, it is also easy to see that $S \subseteq M'' = M$, using the relations (2.5.4) and (2.5.5).

From now on, we assume that (X, μ) is *countably separated*, that is, there is a countable family $\{E_n : n = 1, 2, \dots\}$ of nonempty measurable sets with the property: If $x, y \in X$ and $x \neq y$ then $x \in E_n$ and $y \notin E_n$ for some n. For example, the unit interval with the Lebesgue measure satisfies this condition with the family $\{[a, b) : a, b \in \mathbb{Q}\}$. Also, we assume that ϕ acts on X *freely*, that is, the fixed point set of ϕ^n is a null set for each nonzero $n \in \mathbb{Z}$. From these conditions it can be shown that

$$\mathcal{A} \cap u^n \mathcal{A} = \{0\}, \qquad n \in \mathbb{Z}, n \neq 0. \tag{2.5.7}$$

From (2.5.3), (2.5.5) and (2.5.7), we see that $\mathcal{A} \otimes 1_{L^2(X)}$ is a maximal abelian $*$-subalgebra of M. Because every maximal abelian $*$-subalgebra contains the center, it follows that

$$\mathcal{Z}(M) = \{m_f \otimes 1 : f \in L^\infty(X), u^n m_f u^{-n} = m_f \text{ for each } n \in \mathbb{Z}\}.$$

Hence, we see that M is a factor if and only if $m_{f \circ \phi^{-1}} = m_f$ implies that f is a constant function almost everywhere. This condition is actually equivalent to say that ϕ acts on X *ergodically*, that is, there is no nontrivial invariant measurable subset in X, where a trivial subset means of course a set of measure zero. We summarize as follows:

Theorem 2.5.1. *Let ϕ be an invertible measure preserving transformation on a countably separated probability space (X, μ) which acts freely. Then the associated von Neumann algebra $M(X, \mu, \phi)$ acting on the Hilbert space $L^2(X, \mu) \otimes \ell^2(\mathbb{Z})$ is a factor if and only if ϕ acts ergodically.*

In order to test the type of M, we introduce the notion of traces as follows: A unital positive linear functional τ on a factor R is said to be a faithful *trace* if

$$\tau(xy) = \tau(yx), \qquad x, y \in R, \tag{2.5.8}$$

$$\tau(x) > 0, \qquad x > 0. \tag{2.5.9}$$

It is easy to see that the condition (2.5.8) is equivalent to the condition

$$\tau(xx^*) = \tau(x^*x), \qquad x \in R, \tag{2.5.10}$$

as in the case of Tr on $\mathcal{B}(\mathcal{H})$ in §2.2. It is clear that if R has a trace then R is finite. Indeed, if $p \sim q \leq p$ and v is a partial isometry from p to q then $\tau(p-q) = \tau(v^*v) - \tau(vv^*) = 0$ and it follows that $p = q$ from (2.5.9). This is just the method we used in order to prove that the group von Neumann algebra is finite in the last section. Actually, it is easy to see that the formulae (2.4.4) and (2.4.5) defines a faithful trace

$$T \mapsto \langle T\chi_e, \chi_e \rangle, \qquad T \in \mathcal{R}_\lambda(G).$$

From now on, we assume that ϕ is an ergodic action on (X, μ) and so $R = M(X, \mu, \phi)$ is a factor. We define

$$\tau(T) = \int_X g_0 d\mu, \qquad T \in R,$$

in the matrix representation (2.5.5). We compute

$$(TT^*)_{00} = \sum_{n\in\mathbb{Z}} m_{|g_{-n}|^2 \circ \phi^n}, \qquad (T^*T)_{00} = \sum_{n\in\mathbb{Z}} m_{|g_n|^2},$$

using the rule of matrix multiplication and the relation (2.5.1). Because ϕ is a measure preserving transformation, we see that τ satisfies the condition (2.5.10).

If m_f is a minimal projection in $\mathcal{A}$ then it is easy to see that M_f is also a minimal projection in R, and so R is a factor of type I. One can also prove the converse, that is, if R is a factor of type I then there is an $f \in L^\infty(X)$ such that m_f is a minimal projection. It is also easy to see that $\mathcal{A}$ has a minimal projection if and only if $\mu(\{x\}) > 0$ for a point $x \in X$. We conclude as follows:

Theorem 2.5.2. *Let ϕ be an invertible measure preserving ergodic transformation on a countably separated probability space (X, μ) which acts freely. If $\mu(\{x\}) = 0$ for each $x \in X$ then $R = M(X, \mu, \phi)$ is a factor of type* II_1.

In order to construct a factor of type III, we should consider a transformation which preserves measurability and null sets but does not preserve measure. In this case, the operator u in the above construction is not a unitary any more, and should be replaced by another one. We refer the details to [**K**, §8.6]. In the above theorem, note that the range of projections under the trace is the closed unit interval $[0,1]$, whereas the range is discrete for the finite factors of type I. It can be shown that every finite factor has a unique trace up to a constant multiple, and the range of projections is the unit interval for each factor of type II_1. In this sense, a factor of type II_1 is said to be a *continuous* finite factor. We also refer to [**K**, Chapter 8] for the details.

NOTE

We have followed Pedersen's book [**Pe**, §4.5] for the proof of Theorem 2.1.2 and [**K**, §5.3.1] for the double commutant theorem. We have also followed [**Pe**, §3.4] for the proof of Theorem 2.2.1. We refer to [**T**, §II.1] for the proof which does not use the notion of traces. We also refer to Sakai's book [**S**] for the developments of von Neumann algebra theory free from Hilbert spaces, as was mentioned after Theorem 2.2.2. Basic reasoning in §2.3 was adapted from Kadison's book [**K**, Chapters 5 and 6]. For the proof of the equivalent statements for postliminal C^*-algebras, we refer to [**D**, Chapter 9] or [**P**, Chapter 6]. It is still a long-standing open question whether the last condition (iv) of Theorem 2.3.6 is equivalent to another conditions for non-separable C^*-algebras. We have followed Kadison's book [**K**, §6.7 and §8.6] for examples in §2.4 and §2.5. These examples of constructions will be main motivations for the later chapters. The following is a list of references for further studies on von Neumann algebras.

1. S. Anastasio and P. M. Willig, *The structure of factors*, Algorithmics Press, 1974.
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5. S. Stratila and L. Zsido, *Lectures on von Neumann Algebras*, Editura Academiei and Abacus Press, 1979.
6. V. S. Sunder, *An invitation to von Neumann algebras*, Universitext, Springer-Verlag, 1987.
7. D. M. Topping, *Lectures on von Neumann algebras*, Van Nostrand, 1971.

CHAPTER 3

APPROXIMATELY FINITE DIMENSIONAL C^*–ALGEBRAS

An approximately finite-dimensional C^*-algebra (AF algebra) is the inductive limit of an increasing sequence of finite-dimensional C^*-algebras. Every AF algebra has a *standard* system of finite dimensional C^*-algebras and $*$-homomorphisms. This system may be described in terms of the Bratteli diagram, which has every information of the corresponding AF algebra. There is an another invariant, the K_0-group as an ordered group, which classify AF algebras completely. We introduce K_0-groups for Banach algebras in §3.3. Because K_0 commutes with the inductive limits, it is a simple matter to describe K_0-groups of AF algebras in terms of inductive limit of ordered groups. Several examples of AF algebras are tested to get simple descriptions of their K_0-groups. Finally, we show in §3.5 that two unital AF algebras are $*$-isomorphic if and only if their K_0-groups are order isomorphic each other.

3.1. Bratteli Diagrams for AF C^*-algebras

A C^*-algebra A is said to be *approximately finite dimensional* or just AF, if there is an increasing sequence $\{A_n\}$ of finite dimensional C^*-subalgebras of A such that A is equal to the norm-closure of $\cup_n A_n$. Because the resulting C^*-algebra $\overline{\cup_n A_n}$ depends heavily on the embeddings $A_n \hookrightarrow A_{n+1}$, we need a more precise definition of *inductive limits* of C^*-algebras.

Let

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} A_n \xrightarrow{\phi_n} A_{n+1} \xrightarrow{\phi_{n+1}} \cdots \tag{3.1.1}$$

be a sequence of C^*-algebras and $*$-isomorphisms. We denote by $\mathcal{A}$ the $*$-algebra of all sequences in the product $\prod_n A_n$ of the form

$$x = (x_1, \dots, x_{n-1}, x_n, \phi_n(x_n), \phi_{n+1}\phi_n(x_n), \dots)$$

for some $n = 1, 2, \dots$. Then $\|x\|' = \lim_n \|x_n\|$ defines a C^*-seminorm on $\mathcal{A}$. The *inductive limit* of (A_n, ϕ_n) is the completion A of the quotient algebra $\mathcal{A}$ by the kernel of the seminorm $\|\cdot\|'$, and denoted by $A = \varinjlim(A_n, \phi_n)$. It is easy to see that the usual universal properties holds: For each n, there is a $*$-isomorphism $\iota_n : A_n \to A$ such that $\iota_n = \iota_{n+1}\phi_n$. Also, if there is an another C^*-algebra A' and a $*$-isomorphism $\iota'_n : A_n \to A'$ with $\iota'_n = \iota'_{n+1}\phi_n$ for each n then there is a $*$-isomorphism $\eta : A \to A'$ such that $\iota'_n = \eta\iota_n$ for each n. Given an another sequence $\{B_n, \psi_n\}$ of C^*-algebras and $*$-isomorphisms, assume that there are $*$-isomorphisms $\theta_n : A_n \to B_n$ such that the following diagram

$$
\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & \cdots & \longrightarrow & A_n & \xrightarrow{\phi_n} & A_{n+1} & \longrightarrow \cdots \\
\big\downarrow{\scriptstyle \theta_1} & & \big\downarrow{\scriptstyle \theta_2} & & & & \big\downarrow{\scriptstyle \theta_n} & & \big\downarrow{\scriptstyle \theta_{n+1}} & \\
B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & \cdots & \longrightarrow & B_n & \xrightarrow{\psi_n} & B_{n+1} & \longrightarrow \cdots
\end{array}
\tag{3.1.2}
$$

commutes. Then we have a $*$-isomorphism $\theta : \varinjlim A_n \to \varinjlim B_n$ from the above universal property. It is also easy to see that A is the norm-closure of $\bigcup_n \iota_n(A_n)$, and vice versa.

Hence, every *AF* C^*-algebra is the inductive limit of the system (3.1.1) with finite dimensional C^*-algebras A_n. We are going to find a standard system $\{B_n, \psi_n\}$ with the commuting diagram (3.1.2). Recall that every finite dimensional C^*-algebra is $*$-isomorphic with the finite direct sum of matrix algebras. So, we may assume that each A_n is in this form. For $\mathbf{p} = (p_1, p_2 \dots, p_r) \in \mathbb{N}^r$, we denote by $M(\mathbf{p}) = M_{p_1}(\mathbb{C}) \oplus M_{p_2}(\mathbb{C}) \oplus \cdots \oplus M_{p_r}(\mathbb{C})$. A $1 \times r$ matrix $D_1 = (d_1, d_2, \dots, d_r)$ whose entries are nonnegative integers determines a $*$-homomorphism $M(\mathbf{p}) \to M_q$ by

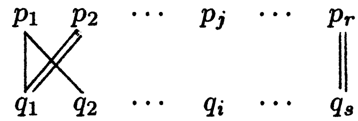
$$(x_1, x_2, \dots, x_r) \overset{D_1}{\longmapsto} \mathrm{Diag}\,(\overbrace{x_1, \dots, x_1}^{d_1 \text{ times}}, \overbrace{x_2, \dots, x_2}^{d_2 \text{ times}}, \dots, \overbrace{x_r, \dots, x_r}^{d_r \text{ times}}, \overbrace{0, \dots, 0}^{h \text{ times}}),$$

where $q = p_1d_1 + p_2d_2 + \cdots p_rd_r + h$. If $\mathbf{q} = (q_1, q_2, \dots, q_s) \in \mathbb{N}^s$ then an $s \times r$ matrix $D = [d_{ij}]$ whose rows are given by $D_1, \dots, D_s$ determines a *canonical* $*$-homomorphism $M(\mathbf{p}) \to M(\mathbf{q})$ by

$$(x_1, x_2, \dots, x_r) \overset{D}{\longmapsto} (D_1(x_1, x_2 \dots, x_r), \dots, D_s(x_1, x_2 \dots, x_r)).$$

This map may be expressed by the following diagram:

(3.1.3)



where there are d_{ij} lines between the points p_j and q_i. This diagram, said to be the *Bratteli diagram*, contains every informations on the $*$-homomorphism $M(\mathbf{p}) \to M(\mathbf{q})$ represented by the matrix D. A $*$-automorphism $\phi : A \to A$ of a C^*-algebra A is said to be an *inner automorphism* if there is a unitary $u \in A$ such that $\phi = \operatorname{Ad} u$, that is, $\phi(x) = uxu^*$ for each $x \in A$. The following is a key lemma.

Lemma 3.1.1. *Given a $*$-homomorphisms $\phi : M(\mathbf{p}) \to M(\mathbf{q})$, there is an inner automorphism σ of $M(\mathbf{q})$ such that $\sigma\phi$ is a canonical $*$-homomorphism.*

Proof. We may assume that $s = 1$ so that $M(\mathbf{q})$ is the matrix algebra acting on $\mathbb{C}^q$. Let $\{e^k_{ij} : 1 \le k \le r,\ 1 \le i, j \le p_k\}$ be the matrix units for $M(\mathbf{p}) = M_{p_1} \oplus \cdots M_{p_r}$, and put $e^k = \sum_i e^k_{ii}$. For each $k = 1, 2, \dots, r$, choose an orthonormal basis $\{x^k_\ell : \ell = 1, 2, \dots, d_k\}$ for the range of $\phi(e^k_{11})$ in $\mathbb{C}^q$, and put $x^k_{i\ell} = \phi(e^k_{i1})x^k_\ell$. Then $\{x^k_{i\ell} : 1 \le 1 \le p_k, 1 \le \ell \le d_k\}$ is an orthonormal basis for the range space of $\phi(e^k)$ for each k. We add more vectors $y_1, y_2, \dots, y_h$ to get the ordered orthonormal basis

$$X^1, X^2, \dots, X^r, y_1, y_2, \dots, y_h$$

of $\mathbb{C}^q$, where each X^k, $k = 1, 2, \dots, r$, is the ordered set consisting of the following $p_k d_k$ vectors;

$$\overbrace{x^k_{11}, x^k_{21}, \dots, x^k_{p_k 1}}^{p_k \text{ times}}, \overbrace{x^k_{12}, \dots, x^k_{p_k 2}}^{p_k \text{ times}}, \dots, \overbrace{x^k_{1 d_k}, \dots, x^k_{p_k d_k}}^{p_k \text{ times}}.$$

For $a = \sum a^k_{ij} x^k_{ij} \in M(\mathbf{p})$, we compute

$$\begin{aligned} \phi(a)x^\alpha_{\beta\ell} &= \sum_{ijk} a^k_{ij}\phi(e^k_{ij})\phi(e^\alpha_{\beta 1})x^\alpha_\ell \\ &= \sum_i a^\alpha_{i\beta}\phi(e^\alpha_{i1})x^\alpha_\ell \\ &= \sum_i a^\alpha_{i\beta} x^\alpha_{i\ell}. \end{aligned}$$

Let u be the unitary in M_q which send the above basis to the canonical basis of $\mathbb{C}^q$. From the above calculation, we see that $u\phi(a)u^*$ is a canonical map. $\square$

Corollary 3.1.2. *Every $*$-automorphism of a matrix algebra is inner.*

We put $B_n = A_n$ in the diagram (3.1.2) and let θ_1 be the identity map. Take an inner automorphism θ_2 of A_2 such that $\theta_2\phi_1$ is a canonical map. Also, we take an inner automorphism θ_3 of A_3 such that $\theta_3\phi_2\theta_2^{-1}$ is a canonical map. By induction, the bottom row of (3.1.2) becomes

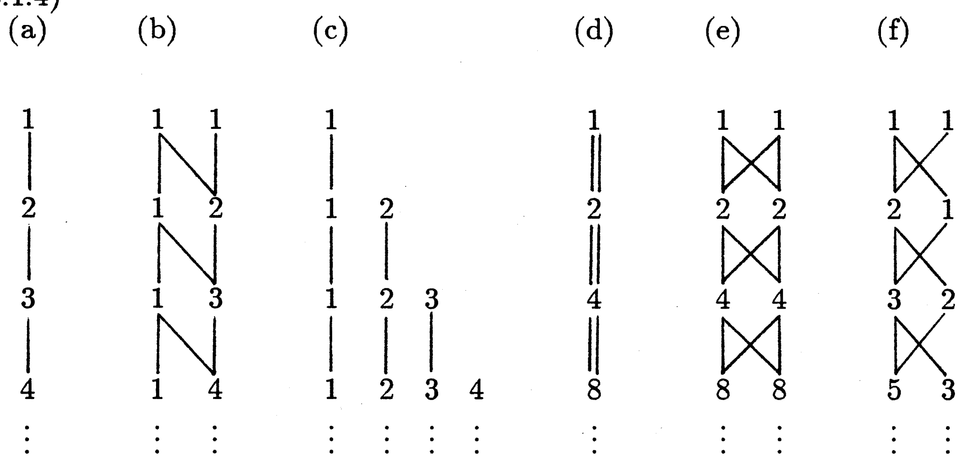
$$A_1 \xrightarrow{\theta_2\phi_1} A_2 \xrightarrow{\theta_3\phi_2\theta_2^{-1}} A_3 \longrightarrow \cdots \longrightarrow A_n \xrightarrow{\theta_{n+1}\phi_n\theta_n^{-1}} A_{n+1} \longrightarrow \cdots,$$

and so, we get the following theorem:

Theorem 3.1.3. *Every AF C^*-algebra A is $*$-isomorphic to the inductive limit of* (3.1.1), *where A_n is the finite direct sum of matrix algebras and ϕ_n is a canonical map, for each $n = 1, 2, \ldots$.*

Therefore, every AF algebra is represented by a tower of Bratteli diagram (3.1.3). Here are several examples:

(3.1.4)



Exercise 3.1.1. For each above diagram, describe the corresponding algebra A_n and the connecting map ϕ_n in terms of matrix D. Show that the three diagrams (a), (b) and (c) correspond to the C^*-algebras $\mathcal{K}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})_I$

and the c_0-sum $\bigoplus_{n=1}^{\infty} M_n$ of the matrix algebras, respectively. Show that the two diagrams (d) and (e) generate $*$-isomorphic AF algebras.

Exercise 3.1.2. Show that a compact metric space X is totally disconnected if and only if $C(X)$ is an AF algebra. Find the Bratteli diagram of $C(X)$ when X is the usual Cantor set.

If A_n is a matrix algebra and ϕ_n is unital for each $n = 1, 2, \ldots,$ the resulting AF algebra $A = \overline{\cup A_n}$ is said to be *uniformly hyperfinite*, or UHF C^*-algebra. The study of UHF algebras was initiated by Glimm [**Gl60**], and followed by Dixmier [**Di67**] who considered non-unital UHF algebras, in which the connecting maps ϕ_n's need not to be unital. After the work of Bratteli [**Br72**], the class of AF algebras was used to provide useful examples of C^*-algebras, and it is relatively easy to examine their structures via their Bratteli diagrams or K_0-groups, as we will study in this chapter. Nevertheless, it is very difficult to determine whether a given C^*-algebra is AF or not. We close this section with the following characterization of AF algebras [**Br72**, Theorem 2.2].

Proposition 3.1.4. *A C^*-algebra A is AF if and only if A is separable and the following condition is satisfied: Given $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$, there exists a finite dimensional C^*-subalgebra B of A and $y_1, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$ for each $i = 1, 2, \ldots, n$.*

3.2. Ideals and Representations of *AF* algebras

We begin with the following simple lemma.

Lemma 3.2.1. *Let I be a closed two-sided ideal of the inductive limit $A = \overline{\cup_n A_n}$ of C^*-algebras. Then, we have $I = \overline{\cup_n (I \cap A_n)}$.*

Proof. Note that I contains a closed ideal $J = \overline{\cup_n (I \cap A_n)}$, and so there is a surjective $*$-homomorphism $\phi : A/J \to A/I$ with $\phi(a + J) = a + I$. Because $A_n \cap J = A_n \cap I$, the restriction of ϕ to $(A_n + J)/J$ is decomposed by $*$-isomorphisms:

$$(A_n + J)/J \to A_n/(A_n \cap J) = A_n/(A_n \cap I) \to (A_n + I)/I \hookrightarrow A/I.$$

Since A/J is the closure of the union of $\{(A_n + J)/J\}$, we see that ϕ is an isometry, and it follows that $I = J$. □

From the above lemma, we see that every closed two-sided ideal of an AF algebra is again AF. It is also easy to see that the quotient of AF algebra is AF. As an application of K-theory, we know that the converse is also true: If I and A/I are AF algebras then A is also an AF algebra (see [**Ef**, Chapter 9]).

In order to investigate the structures of AF algebras in terms of their diagrams, we introduce several terminologies in [**LT80**]. Let $\mathcal{D}$ be the set of points $\{(n,i) : n = 1,2,\ldots,\ 1 \le i \le p_n\}$ in the diagram, where p_n is the number of direct summands in A_n. We say that the point (n,i) is a *descendant*, with *multiplicity* q, of the point (m,j) if $n > m$ and there are q paths from (m,j) to (n,i) with $q \ne 0$. For example, in (3.1.4.b), the point $(3,2)$ is a descendant of $(1,1)$ with multiplicity 2, whereas the point $(3,1)$ is not a descendant of $(2,1)$. A sequence $\{x_k : k = 1,2,\dots\}$ of $\mathcal{D}$ is said to be *connected* if x_{k+1} is a descendant of x_k for each $k = 1,2,\dots$.

Now, an ideal I of an AF algebra $A = \overline{\cup A_n}$ with the associated diagram $\mathcal{D}$ is of the form $I = \overline{\cup(I \cap A_n)}$. Because an ideal $I \cap A_n$ of A_n is again a subsum of matrix algebras in A_n, we see that I is an AF algebra represented by a subdiagram $\mathcal{K}$ of $\mathcal{D}$. In this way, it is easy to characterize ideals of AF algebras as follows:

Proposition 3.2.2. *Let A be an AF algebra with the associated diagram $\mathcal{D}$. A subdiagram $\mathcal{K}$ of $\mathcal{D}$ represents an ideal of A if and only if the following two conditions are satisfied:*

(i) *Every descendant of $x \in \mathcal{K}$ belongs to $\mathcal{K}$.*
(ii) *If every descendant of x at the next row belongs to $\mathcal{K}$ then $x \in \mathcal{K}$.*

Furthermore, the subdiagram $\mathcal{D} \setminus \mathcal{K}$ represents the quotient algebra A/I.

For example, the right column of (3.1.4.b) represents a closed two-sided ideal and the quotient by this ideal is represented by the left column.

Corollary 3.2.3. *An AF algebra A with the associated diagram $\mathcal{D}$ is simple if and only if for each x in n-th row in $\mathcal{D}$ there is $m > n$ such that every y in m-th row is a descendant of x.*

Now, we proceed to characterize several concepts mentioned at the end of §1.6. We say that an ideal I of a C^*-algebra is *primitive* if it is the kernel of an irreducible representation. It is easy to see that if I is a primitive ideal and I_1, I_2 are closed two-sided ideals with $I_1 I_2 \subseteq I$ then $I_1 \subseteq I$ or $I_2 \subseteq I$, that is, every primitive ideal is an *prime ideal*. It is known that the converse is also true for *separable* C^*-algebras [**P**, Proposition 4.3.6]. From this, it is easy to see that the associated diagram $\mathcal{K}$ of a primitive ideal of an AF algebra with the associated diagram $\mathcal{D}$ has the property:

(I_1) *Every two elements in* $\mathcal{D} \setminus \mathcal{K}$ *have common descendants.*

Note that (3.1.4.a) represents the C^*-algebra $\mathcal{K}(\mathcal{H})$. Conversely, it can be shown that an AF algebra with the associated diagram $\mathcal{D}$ is $*$-isomorphic to $\mathcal{K}(\mathcal{H})$ if and only if $\mathcal{D}$ satisfies the following two properties:

(I_2) *Every two elements in* $\mathcal{D}$ *have common descendants.*

(I_3) *For each connected sequence* $\{x_n\}$ *of* $\mathcal{D}$ *there are natural numbers* p, q *such that* x_m *is a descendant of* x_1 *with multiplicity* p *for all* $m \geq q$.

From the above considerations, the following characterizations should be plausible. We refer to [**LT80**] for the details.

Proposition 3.2.4. *Let* A *be an* AF *algebra with the associated diagram* $\mathcal{D}$. *Then we have the following:*

- (i) *A is liminal if and only if $\mathcal{D}$ has the property* (I_3).
- (ii) *A is postliminal if and only if for each connected sequence $\{x_n\}$ of $\mathcal{D}$, x_{n+1} is a descendant of x_n with multiplicity one for sufficiently large n.*
- (iii) *A is subhomogeneous if and only if the set of attached numbers (sizes of matrix algebras) are bounded above.*

Now, we focus our attention to the UHF algebra $A = \overline{\cup_n A_n}$ whose associated diagram is given by (3.1.4.d). We have another convenient description as follows: Each A_n is $*$-isomorphic to the C^*-algebra

$$\overbrace{M_2 \otimes M_2 \otimes \cdots \otimes M_2}^{n \text{ times}},$$

and the connection map is given by $x \mapsto x \otimes 1 : A_n \to A_{n+1}$ as was explained in §2.4. Recall that this isomorphism is given by

$$[x_{ij}]_{i,j=1}^k \mapsto \sum_{i,j=1}^k x_{ij} \otimes e_{ij} : M_k(M_\ell) = M_{k\ell} \to M_\ell \otimes M_k,$$

where $\{e_{ij}\}$ is the matrix unit for M_k. Therefore, A is the closure of the linear span of elements of the form

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots .$$

If $\{\phi_n\}$ is a sequence of states of M_2 then the *product state* $\phi = \otimes_{n=1}^\infty \phi_n$ of $\{\phi_n\}$ is defined by

$$\phi(x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots) = \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n).$$

It is easy to see that the product state of two pure states is also pure. Indeed, if ϕ_i is a vector state given by $\xi_i \in \mathbb{C}^2$ for each $i = 1, 2$, then $\phi_1 \otimes \phi_2$ is also a vector state given by $(\xi_1, \xi_2) \in \mathbb{C}^4$ (see Exercise 1.6.1 and 1.6.2). By induction, we see that if every state ϕ_n is pure then their product state ϕ is also pure. Now, if $\{\pi, \mathcal{H}\}$ is the G. N. S. construction associated with ϕ then $\pi(A)' = \mathbb{C}1_{\mathcal{H}}$ by Lemma 1.6.5. Hence, the von Neumann algebra generated by $\pi(A)$ is the whole algebra $\mathcal{B}(\mathcal{H})$, and $\pi(A)$ is $*$-isomorphic to A because A is simple by Corollary 3.2.3.

Next, we consider the case in which every ϕ_n is the normalized trace on M_2. In this case, the G. N. S. construction $\{\pi, \mathcal{H}, \xi\}$ of $\phi = \otimes_n \phi_n$ is also faithful. We denote by M the von Neumann generated by $\pi(A)$. Because ϕ satisfies the tracial property

$$\phi(xy) = \phi(yx), \qquad x, y \in A,$$

we see that the function

$$\tau : x \mapsto \langle x\xi, \xi\rangle, \qquad x \in M$$

satisfies the condition (2.5.8) because $\pi(A)$ is weak operator dense in M. To show the condition (2.5.9), suppose that $y \in M$ and $\tau(y^*y) = 0$, that is, $y\xi = 0$. For each $x \in A$, we have

$$\|y\pi(x)\xi\|^2 = \langle \pi(x^*)y^*y\pi(x)\xi, \xi\rangle = \langle \pi(x)\pi(x^*)y^*y\xi, \xi\rangle = 0,$$

by the tracial property, and so $y\pi(x)\xi = 0$ for each $x \in A$. Because ξ is a cyclic vector, it follows that $y = 0$. Now, we proceed to show that M is a factor. For a projection p in the center $M \cap M'$, we consider the another trace τ' on M given by $\tau'(x) = \tau(px)$. The restriction of τ' to the UHF algebra $\pi(A)$ is again a trace. By the uniqueness of the normalized trace for UHF algebra (see Exercise 3.2.1 below), we see that $\tau' = \tau(p)\tau$ on $\pi(A)$. It follows that $\tau(p)\tau(1-p) = \tau'(1-p) = 0$, and p is a trivial projection. Summing up, we have shown that M is a hyperfinite factor of type II_1.

Exercise 3.2.1. Show that there is a unique normalized trace on the matrix algebra. Show that every UHF algebra has a unique normalized trace.

Note that every state of a matrix algebra generates a factor. More generally, one can show that every product state of a UHF algebra generates a factor (see Powers' paper [**Po67**] or [**S**, Proposition 4.4.2]). Powers considered the product state $\phi_\lambda = \otimes\phi_n$, $0 < \lambda < 1$, given by

$$\phi_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\lambda + 1}(a + \lambda d), \qquad n = 1, 2, \ldots,$$

and showed that the von Neumann algebras R_λ's generated by ϕ_λ, $0 < \lambda < 1$, are non-isomorphic hyperfinite factors of type III. Also, Araki and Wood [**AW68**] showed that $R_{\lambda_1} \otimes R_{\lambda_2}$ is independent of the choice of (λ_1, λ_2) in most cases, and gives an example of hyperfinite III factor different from Powers' factors. There are another examples of hyperfinite factors of type III arising from ergodic theory which were considered by Krieger [**Kr76**]. After the pioneering work of Connes [**Co76**] with Tomita-Takesaki theory [**Ta70**], Haagerup [**Ha87**] showed that these are all possible hyperfinite factors of type III, and so we have now the complete list of hyperfinite factors.

3.3. K_0-groups for C^*-algebras

AF algebras are completely classified by their K_0-groups as scaled ordered groups. In this section, we introduce K_0-groups for general C^*-algebras. Because K-theory works for Banach algebras [**Tay75**] as well as for C^*-algebras, we will start with the definition of equivalence for idempotents. Two idempotents e and f are said to be *algebraically equivalent* if there are x, y in A such that $e = xy$ and $f = yx$ and denoted by $e \sim f$. This implies

$$(exf)(fye) = exfye = exyxye = e^4 = e, \qquad (fye)(exf) = f.$$

Hence, by replacing x and y by exf and fye respectively, we may assume that

$$xy = e,\ yx = f,\ x = ex = xf = exf,\ y = fy = ye = fye. \tag{3.3.1}$$

From this, it is also easy to see that $\sim$ is an equivalence relation and the following relation

$$e_1 \sim f_1,\ e_2 \sim f_2,\ e_1e_2 = f_1f_2 = 0 \implies e_1 + e_2 \sim f_1 + f_2 \tag{3.3.2}$$

holds

In a unital C^*-algebra, two projections p and q are said to be *unitarily equivalent* if there is a unitary $u \in A$ such that $upu^* = q$, and denoted by $p \sim_u q$. It is immediate to see that $p \sim_u q$ implies $p \sim q$.

Proposition 3.3.1. *Let A be a C^*-algebra. Then we have the following:*

(i) *For each idempotent e, there is a projection p such that $e \sim p$.*
(ii) *If p, q are projections and $p \sim q$ then there is $v \in A$ such that $p = v^*v$ and $q = vv^*$.*

If A is a unital C^*-algebra then we have

(iii) *If p, q are projections and $p \sim q$ then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.*
(iv) *If p, q are projections with $\|p - q\| < 1$ then we have $p \sim_u q$.*

Proof. (i) Note that $x = 1 + (e - e^*)(e^* - e)$ is invertible by Theorem 1.4.1, and $ex = ee^*e = xe$. From this, it is easy to check that $p = ee^*x^{-1}$ is a projection and the relations $pe = e$, $ep = p$ hold.

(ii) We write $p = xy$, $q = yx$ with the relation (3.3.1). Put $v = p(y^*y)^{\frac{1}{2}}x$. Because y^*y commutes with p and xx^*, we check that $vv^* = p$ and $v^*v = q$.

(iii) Write $P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, where $v^*v = p$ and $vv^* = q$. Then, $U = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}$ is a unitary in $M_2(A)$ and $UPU^* = Q$.

(iv) Put $v = 1 - p - q + 2qp$. Then $v^*v = 1 - (q-p)^2$ is invertible by the assumption. Because $v^*v = vv^*$, we see that $u = v(v^*v)^{-\frac{1}{2}}$ is a unitary. Since v^*v commutes with p, we also see that $up = uq$, that is, $upu^* = q$. □

From now on, we assume that A is a unital C^*-algebra and consider the sequence of C^*-algebras

$$M_1(A) \hookrightarrow M_2(A) \hookrightarrow \cdots \hookrightarrow M_n(A) \hookrightarrow M_{n+1}(A) \hookrightarrow \cdots, \tag{3.3.3}$$

where every embedding is defined by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ for $x \in M_n(A)$. The resulting direct limit C^*-algebra will be denoted by $A \otimes \mathcal{K}$. In the next chapter, we will see that this is actually the C^*-algebraic tensor product of A and $\mathcal{K}(\mathcal{H})$. Proposition 3.3.1 says that two equivalence relations $\sim$ and $\sim_u$ are same on the projections in $A \otimes \mathcal{K}$, and the resulting set of equivalence class will be denoted by $\mathcal{D}(A)$. By Proposition 3.3.1 (iv) and the following lemma, a representative of each equivalence class may be chosen among projections in $M_n(A)$ for some $n = 1, 2, \dots$.

Lemma 3.3.2. *Let p be a projection in the inductive limit $A = \varinjlim A_n$ of increasing sequence $\{A_n\}$. Then given $\varepsilon > 0$ there is a projection q in some A_n with $\|p - q\| < \varepsilon$.*

Proof. We take a self-adjoint element $x \in A_n$ with $\|p - x\|$ small enough. Then from the estimate

$$\|x - x^2\| \le \|x - p\| + \|p - x^2\| \le \|p - x\| + \|p - x\|\|p + x\| \le \|p - x\|(2 + \|x\|),$$

we see by (1.2.6) that $\mathrm{sp}(x)$ lies in the disjoint union of two interval containing 0 and 1. So, the function calculus $\chi_{(\frac{1}{2},\infty)}(x)$ by the characteristic function is the required projection. For more precise and simple estimate, we refer to [**Ef**, Lemma A8.1]. □

The following analogous lemma also will be useful.

Lemma 3.3.3. *Let u be a unitary in the inductive limit $A = \varinjlim A_n$ of increasing sequence $\{A_n\}$. Then given $\varepsilon > 0$ there is a unitary v in some A_n with $\|u - v\| < \varepsilon$.*

Proof. Choose $x \in A_n$ with $\|x\| \le 1$ and $\|x - u\| < \varepsilon$ with $\varepsilon < 1$. Because $\|u^*x - 1\| \le \|u^*(x - u)\| < 1$, we see that u^*x is invertible. Hence, x is invertible, and $v = x(x^*x)^{-\frac{1}{2}}$ is a unitary in A_n. Since $|\sqrt{t} - 1| \le |t - 1|$ on the unit interval, we see that

$$\|(x^*x)^{\frac{1}{2}} - 1\| \le \|x^*x - 1\| \le \|x^* - u^*\|\|x\| + \|u^*\|\|x - u\| < 2\varepsilon,$$

and so, $\|u - v\| \le \|u - x\| + \|x - v\| \le \|u - x\| + \|v\|\|(x^*x)^{\frac{1}{2}} - 1\| < 3\varepsilon$. □

Corollary 3.3.4. *If p and q are projections in $M_n(A)$ such that $p \sim q$ in $A \otimes \mathcal{K}$. Then we have $p \sim q$ in $M_k(A)$ for sufficiently large k.*

Proof. By Proposition 3.3.1 (iii), we may choose a unitary $u \in A \otimes \mathcal{K}$ such that $upu^* = q$. By Lemma 3.3.3, we choose a unitary $v \in M_k(A)$ with large k such that $\|u - v\| < \frac{1}{2}$. Since

$$\|vpv^* - q\| = \|vpv^* - upu^*\| \le \|vpv^* - vpu^*\| + \|vpu^* - upu^*\| < 1,$$

we see that $p \sim_u vpv^* \sim q$ in $M_k(A)$ by Proposition 3.3.1 (iv). □

For $[p]$ and $[q]$ in $\mathcal{D}(A)$, we define

$$[p] + [q] = [p \oplus q],$$

where $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$. This is well-defined by (3.3.2) and $\mathcal{D}(A)$ becomes a commutative semigroup with 0.

The Grothendieck construction of $\mathcal{D}(A)$ will be denoted by $K_0(A)$. Formally, it is the group of formal differences $[p] - [q]$, with the identification: $[p_1] - [q_1] = [p_2] - [q_2]$ if and only if there is $[r] \in \mathcal{D}(A)$ such that

$$[p_1] + [q_2] + [r] = [p_2] + [q_1] + [r].$$

It should be noted that $[p] = [q]$ in $K_0(A)$ does not imply $p \sim q$, but implies $p \oplus r \sim q \oplus r$ for a projection r in $M_n(A)$. If $\phi : A \to B$ is a $*$-homomorphism then we have the induced $*$-homomorphism $\phi_n : M_n(A) \to M_n(B)$ given by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. Hence, we have a $*$-homomorphism $\tilde{\phi} : A \otimes \mathcal{K} \to A \otimes \mathcal{K}$, and $[p] = [q]$ in $\mathcal{D}(A)$ implies that $[\tilde{\phi}(p)] = [\tilde{\phi}(q)]$ in $\mathcal{D}(B)$. Furthermore, the mapping $[p] \mapsto [\tilde{\phi}(p)]$ is a semigroup homomorphism, and so we get a group homomorphism $\phi_* : K_0(A) \to K_0(B)$. The following functorial properties are immediate:

$$(1_A)_* = 1_{K_0(A)}, \qquad (\psi\phi)_* = \psi_*\phi_*.$$

Example 3.3.1. $K_0(\mathbb{C}) = \mathbb{Z}$, $K_0(M_n) = \mathbb{Z}$, $K_0(\mathcal{B}(\mathcal{H})) = 0$.

We say that two C^*-algebras A and B are *stably isomorphic* if $A \otimes \mathcal{K}$ is $*$-isomorphic to $B \otimes \mathcal{K}$. It is clear that stably isomorphic C^*-algebras have the same K_0-groups.

Exercise 3.3.1. Show that A, $M_n(A)$ and $A \otimes \mathcal{K}$ are stably isomorphic each other for $n = 1, 2, \ldots$.

Now, we consider the order structures of K_0-groups. Recall that an *ordered group* is a pair (G, G_+) of an abelian group G and a submonoid G_+ of G with the properties:

$$G = G_+ - G_+, \qquad G_+ \cap (-G_+) = \{0\}. \tag{3.3.4}$$

The submonoid G_+ gives rise a partial order $x \leq y$ by $y - x \in G_+$. We write $x < y$ if $x \leq y$ and $x \neq y$. An element $u \in G_+$ is said to be an *order unit* if for each $x \in G$ there is a natural number n such that $x \leq nu$. A subgroup H of G is an *order ideal* if

$$y \in H, \ 0 \leq x \leq y \implies x \in H. \tag{3.3.5}$$

We also say that an ordered group G is *simple* if G has no proper order ideal. It is easy to see that G is simple if and only if every nonzero element in G_+ is an order unit. A *scaled ordered group* is a triple (G, G_+, u) with an ordered group (G, G_+) and a fixed order unit u. A group homomorphism $\phi : (G, G_+, u_G) \to (H, H_+, u_H)$ is said to be *positive* if $\phi(G_+) \subseteq H_+$, and *unital* if $\phi(u_G) = u_H$. The class of scaled ordered groups forms a category with unital positive homomorphisms.

We denote by $K_0(A)_+$ the set of all elements in $K_0(A)$ with the form $[p] - [0]$ for some $[p] \in \mathcal{D}(A)$. It is not true in general that $K_0(A)_+$ satisfies the condition (3.3.4) [**Cu81a**]. We say that a C^*-algebra A is *finite* if $p \leq q$, $p \sim q$ implies $p = q$ as in the case of von Neumann algebras. If A is unital then this is equivalent to say that $v^*v = 1$ implies $vv^* = 1$. We also say that a C^*-algebra is *stably finite* if $M_n(A)$ is finite for each $n = 1, 2, \ldots$.

Exercise 3.3.2. If $A = \varinjlim A_n$ then $M_k(A)$ is $*$-isomorphic to the inductive limit $\varinjlim_n M_k(A_n)$ for each $k = 1, 2, \ldots$.

Exercise 3.3.3. Show that every AF algebra is stably finite.

It is an open question whether every finite simple C^*-algebra is stably finite **[Bl88]**.

Proposition 3.3.5. *Let A be a stably finite C^*-algebra. Then the triple $(K_0(A), K_0(A)_+, [1_A])$ is a scaled ordered group.*

Proof. The first condition of (3.3.4) is clear. To show the second condition, let p be a projection in $M_n(A)$ and $[p] \in -K_0(A)_+$, that is, $[p] = -[q]$ for a projection $q \in M_k(A)$. Then there is a projection $r \in M_\ell(A)$ such that $p \oplus q \oplus r \sim r$ in $M_{n+k+\ell}(A)$ by Corollary 3.3.4. Because $M_{n+k+\ell}(A)$ is finite, we see that $p = q = 0$.

If $[p] \in K_0(A)_+$ with a projection p in $M_n(A)$ then it is easy to see that $p \oplus (1-p) \sim_u 1_n \oplus 0_n$ in $M_{2n}(A)$ with the unitary $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$, where 1_n and 0_n denote the identity and zero in $M_n(A)$. Hence, $[p] + [1-p] = [1_n] = n[1_A]$ and it follows that $n[1_A] - [p] = [1-p] \in G_0(A)_+$. □

Proposition 3.3.6. *If A is a stably finite simple C^*-algebra then $K_0(A)$ is a simple ordered group.*

Proof. Let $[p], [q] \in K_0(A)_+$ with projections in p, q in $M_n(A)$. Because A is simple, we see that there are $x_1, \dots, x_m, y_1, \dots, y_m$ in $M_n(A)$ with $p = \sum_{i=1}^m x_i q y_i$. Put

$$X = \begin{pmatrix} x_1 & \cdots & x_m \\ 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots \\ y_m & 0 & 0 & \end{pmatrix},$$

$P = \mathrm{Diag}(p, 0, \dots, 0)$ and $Q = \mathrm{Diag}(q, q, \dots, q)$ in $M_m(M_n(A))$. We check that $E = QYPXQ$ is an idempotent which is equivalent to P via PXQ and QYP. Because $E \le Q$, we have $[p] \le m[q]$, and so it follows that every element in $K_0(A)_+$ is an order unit. □

3.4. K_0-groups of AF algebras

In order to calculate K_0-groups of AF algebras, we first show that the functor K_0 commutes with the inductive limits as follows:

Theorem 3.4.1. *Let $A = \varinjlim A_n$ be the inductive limit of an increasing sequence $\{A_n\}$ of C^*-algebras containing 1_A with the inclusion maps $\iota_n : A_n \to A$. Let $\theta : \varinjlim K_0(A_n) \to K_0(A)$ be the induced map from $\theta_n = (\iota_n)_* : K_0(A_n) \to K_0(A)$. Then θ is an isomorphism preserving distinguished cones and order units.*

Proof. Let $[p] \in K_0(A)_+$ with a projection p in $M_k(A)$. By Exercise 3.3.2 together with Proposition 3.3.1 (iv) and Lemma 3.3.2, we have a projection q in $M_k(A_n)$ such that $p \sim q$ in $M_k(A)$. Hence, $[q] \in K_0(A_n)$ and $[p] = \theta_n([q])$. This shows that θ is surjective and preserves distinguished cones.

In order to show the injectivity, let $[p] - [q] \in K_0(A_n)$ with projections p, q in $M_k(A_n)$ and $\theta_n([p] - [q]) = 0$. It suffices to show that the image of $[p] - [q]$ is 0 in the induced map $K_0(A_n) \to K_0(A_m)$ for some $m > n$. The condition $\theta_n([p] - [q]) = 0$ says that $[p] - [q] = 0$ in $K_0(A)$, and so there is a projection r in $M_\ell(A)$ such that $p \oplus r \sim q \oplus r$, where we may assume that $\sim$ is made in $M_{k+\ell}(A)$ by Corollary 3.3.4. As before, we may also assume that $r \in M_\ell(A_m)$ for some large $m > n$. Because $p \oplus r$ and $q \oplus r$ lies in $M_{k+\ell}(A_m)$, we use the same argument as in Corollary 3.3.4 by Exercise 3.3.2, to see that $p \oplus r \sim q \oplus r$ in $M_{k+\ell}(A_m)$. Therefore, it follows that $[p] = [q]$ in $K_0(A_m)$. □

Let a unital canonical $*$-homomorphism (3.1.3) be given. We proceed to determine the induced unital positive homomorphism

$$K_0(M(\mathbf{p})) \to K_0(M(\mathbf{q})), \tag{3.4.1}$$

where $\mathbf{p} = (p_1, \ldots, p_r)$ and $\mathbf{q} = (q_1, \ldots q_s)$. With the same notations as in the proof of Lemma 3.1.1, it is easy to see that

$$(n_1, n_2 \ldots, n_r) \mapsto \sum_{k=1}^{r} n_k [e^k_{11}]$$

gives rise to a unital order isomorphism

$$(\mathbb{Z}^r, (\mathbb{Z}_+)^r, (p_1, \ldots, p_r)) \to (K_0(M(\mathbf{p})), K_0(M(\mathbf{p}))_+, [1_{M(\mathbf{p})}]),$$

and so the map (3.4.1) is determined by the the matrix D which represents the diagram (3.1.3). Hence, if A is an AF algebra with the associated diagram $\mathcal{D}$ then $K_0(A)$ is the inductive limit of the ordered groups:

$$\mathbb{Z}^{r(1)} \xrightarrow{D_1} \mathbb{Z}^{r(2)} \xrightarrow{D_2} \cdots \xrightarrow{D_{n-1}} \mathbb{Z}^{r(n)} \xrightarrow{D_n} \mathbb{Z}^{r(n+1)} \xrightarrow{D_{n+1}} \cdots, \tag{3.4.2}$$

where $r(n)$ is the number of points in the n-th row of $\mathcal{D}$ and D_n is the matrix which represents the connecting map $A_n \to A_{n+1}$. The ordered group obtained in this way is said to be a *dimension group*. Although there are more general methods to obtain concrete realizations of dimension groups [**Ef**, Chapter 4], we restrict ourselves to several simple cases in which we have a concrete realization of the K_0-group as an additive subgroup of $\mathbb{R}$ with the usual order.

First, we consider the UHF algebra generated by the sequence

$$M_{a_0} \xrightarrow{(a_1)} M_{a_0a_1} \xrightarrow{(a_2)} \cdots \xrightarrow{(a_{n-1})} M_{a_0a_1\cdots a_{n-1}} \xrightarrow{(a_n)} M_{a_0a_1\cdots a_n} \xrightarrow{(a_{n+1})} \cdots.$$

In this case, The K_0-group is determined by the following commuting diagram;

$$\begin{array}{ccccccccccc}
\mathbb{Z} & \xrightarrow{(a_1)} & \mathbb{Z} & \xrightarrow{(a_2)} & \cdots & \xrightarrow{(a_{n-1})} & \mathbb{Z} & \xrightarrow{(a_n)} & \mathbb{Z} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\iota_1} & & \downarrow{\scriptstyle\iota_2} & & & & \downarrow{\scriptstyle\iota_n} & & \downarrow{\scriptstyle\iota_{n+1}} & & \\
\mathbb{R} & = & \mathbb{R} & = & \cdots & = & \mathbb{R} & = & \mathbb{R} & = & \cdots
\end{array}$$

where $\iota_n(x) = \dfrac{x}{a_0a_1\cdots a_{n-1}}$. A *generalized natural number* $\mathfrak{n}$ is a map from the set $\{p_1, p_2, \dots\}$ of all prime numbers to $\{0, 1, \dots, \infty\}$ and we write $\mathfrak{n} = 2^{k_1}3^{k_2}5^{k_3}\cdots$ in an obvious way, where $k_i \in \{0, 1, \dots, \infty\}$. The set $\mathfrak{N}$ of all generalized natural numbers has a natural multiplication with the convention $n + \infty = \infty + n = \infty$. The set $\mathbb{N}$ of all natural numbers is identified with a subset of $\mathfrak{N}$ consisting of $\mathfrak{n}$ such that $\mathfrak{n}(p_i) < \infty$ for each i and $\mathfrak{n}(p_i) = 0$ for all but finitely many i. Each $\mathfrak{n} \in \mathfrak{N}$ determines a subgroup $G(\mathfrak{n})$ of $\mathbb{Q}$ by

$$G(\mathfrak{n}) = \{\frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{N},\ b \text{ divides } \mathfrak{n}\},$$

where we say that $\mathfrak{m}$ *divides* $\mathfrak{n}$ if $\mathfrak{m}(p_i) \le \mathfrak{n}(p_i)$ for each prime p_i. For example, we see that the AF algebra generated by the diagram (3.1.4.d) has the K_0-group $G(2^\infty)$.

Exercise 3.4.1. Show that $G(\mathfrak{m})$ is isomorphic to $G(\mathfrak{n})$ if and only if there are natural numbers p and q such that $p\mathfrak{n} = q\mathfrak{m}$.

Now, we consider AF algebras generated by systems in which every connecting map has the same diagram. Then the corresponding K_0-groups will be determined by the commuting diagram;

$$
\text{(3.4.3)}\qquad
\begin{array}{ccccccccccc}
\mathbb{Z}^r & \xrightarrow{D} & \mathbb{Z}^r & \xrightarrow{D} & \cdots & \xrightarrow{D} & \mathbb{Z}^r & \xrightarrow{D} & \mathbb{Z}^r & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & & & \downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \iota_{n+1}} & & \\
\mathbb{R} & = & \mathbb{R} & = & \cdots & = & \mathbb{R} & = & \mathbb{R} & = & \cdots
\end{array}
$$

where D is an $r \times r$ matrix whose entries are nonnegative integers, and each ι_n should be determined by a vector V_n in $\mathbb{R}^r$ whose entries are nonnegative real numbers with the property

$$
\text{(3.4.4)}\qquad \langle DX, V_n\rangle = \langle X, V_{n-1}\rangle, \qquad X \in \mathbb{Z}^r.
$$

We observe that if D is diagonal with the first diagonal entry d then $V_n = (\frac{1}{d^n}, 0, \ldots, 0)$ satisfies (3.4.4). This suggests us to consider eigenvectors of the transpose of D. We recall Perron-Frobenious theorem (see [**Ga**, §XIII.2] for the proof) as follows:

Theorem (Perron-Frobenious). *Let A be an irreducible square matrix whose entries are nonnegative. Then A has a positive eigenvalue r such that $|\lambda| \le r$ for each $\lambda \in \mathrm{sp}(A)$. For this eigenvalue r, there is a corresponding eigenvector V whose entries are positive.*

Recall that a matrix is *reducible* if it is of the form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ after *same* permutations of columns and rows. Now, we assume that D in (3.4.3) is irreducible, and let $r > 0$, $V = (v_1, v_2, \ldots, v_r)$ be the eigenvalue and eigenvector of the transpose of D with $v_i > 0$ and $\sum v_i = 1$. We see that $V_n = \dfrac{1}{r^{n-1}} V$ satisfies the condition (3.4.4).

Exercise 3.4.2. Show that the K_0-group of the AF algebra with the associated diagram (3.1.4.e) is isomorphic to the group $G(2^\infty)$.

For example, let A be the AF algebra with the associated Bratteli diagram (3.1.4.f). Then the eigenvalue of $D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is $\theta = \dfrac{1+\sqrt{5}}{2}$ and the corresponding eigenvector is $V = (\dfrac{1}{\theta}, \dfrac{1}{\theta^2})$ (note that $\dfrac{1}{\theta} = \theta - 1$). Considering the commuting diagram:

$$
\begin{array}{ccccccc}
\mathbb{Z}^2 & \xrightarrow{D} & \mathbb{Z}^2 & \xrightarrow{D} & \mathbb{Z}^2 & \xrightarrow{D} & \cdots \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle \frac{1}{\theta}V} & & \downarrow{\scriptstyle \frac{1}{\theta^2}V} & & \\
\mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & = & \cdots
\end{array} \tag{3.4.5}
$$

We see that K_0-group is $\mathbb{Z} + \mathbb{Z}\theta$, and A is not $*$-isomorphic to any UFH algebra.

Finally, we consider the AF algebra whose K_0-group is given by

$$
\mathbb{Z}^2 \xrightarrow{D_0} \mathbb{Z}^2 \xrightarrow{D_1} \cdots \xrightarrow{D_{n-1}} \mathbb{Z}^2 \xrightarrow{D_n} \mathbb{Z}^2 \xrightarrow{D_{n+1}} \cdots, \tag{3.4.6}
$$

where $D_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$ for a positive integer a_n, $n = 0, 1, \ldots$. We denote

$$
[a_0, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.
$$

Then we have $[a_0, \ldots, a_n] = \dfrac{p_n}{q_n}$, where p_n and q_n are defined by

$$
\begin{gathered}
p_{-1} = 1,\ q_{-1} = 0,\ p_0 = a_0,\ q_0 = 1, \\
p_n = a_n p_{n-1} + p_{n-2},\ q_n = a_n q_{n-1} + q_{n-2}, \quad n = 1, 2, \ldots,
\end{gathered} \tag{3.4.7}
$$

or equivalently,

$$
A_n = \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n = 1, 2, \ldots. \tag{3.4.8}
$$

It is well-known that the sequence $\{[a_0, \ldots, a_n] : n = 1, 2, \ldots\}$ converges to a number $\theta \in \mathbb{R}^+ \setminus \mathbb{Q}$. We consider the following commuting diagram;

$$
\begin{array}{ccccccccc}
\mathbb{Z}^2 & \xrightarrow{D_1 D_0} & \mathbb{Z}^2 & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{D_{2n-1}D_{2n-2}} & \mathbb{Z}^2 & \longrightarrow \cdots \\
\downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota_1} & & & & \downarrow{\scriptstyle \iota_{n-1}} & & \downarrow{\scriptstyle \iota_n} & \\
\mathbb{Z}^2 & = & \mathbb{Z}^2 & = & \cdots & = & \mathbb{Z}^2 & = & \mathbb{Z}^2 & = \cdots
\end{array}
$$

where $\iota_0 = 1$ and ι_n is given by the matrix $(A_{2n-1})^{-1} = \begin{pmatrix} q_{2n-2} & -q_{2n-1} \\ -p_{2n-2} & p_{2n-1} \end{pmatrix}$ whose determinant is 1. The relation (3.4.8) says that the diagram commutes.

Exercise 3.4.3. Show that $\bigcup_{n=0}^{\infty} \iota_n(\mathbb{Z}_+^2) = \{(m,n) \in \mathbb{Z}^2 : m\theta + n \geq 0\}$.

Hence, we see that the inductive limit of (3.4.6) is just $\mathbb{Z} + \mathbb{Z}\theta$. Note that if $a_n = 1$ for each $n = 1, 2, \dots$ then we have $\theta = \dfrac{1+\sqrt{5}}{2}$, as before.

For more interesting examples and further interplay with another fields of mathematics, we refer to **[Ef]** and references cited there.

3.5. Classification of AF algebras

In this section, we prove the following theorem due to Elliott **[El76]** which classify unital AF algebras. For nonunital AF algebras, the order unit should be replaced by the notion of *scales*. We refer to **[Ef]** for the details.

Theorem 3.5.1. *Let A and B be unital AF C^*-algebras and assume that there is a unital order isomorphism $\sigma : (K_0(A), [1_A]) \to (K_0(B), [1_B])$ between their K_0-groups. Then there is a $*$-isomorphism $\phi : A \to B$ with $\phi_* = \sigma$.*

We begin with the following lemma which proves the theorem for finite dimensional C^*-algebras. The proof is immediate from Lemma 3.1.1.

Lemma 3.5.2. *Let A and B be finite dimensional C^*-algebras. Then we have the following:*

(i) *If $\sigma : K_0(A) \to K_0(B)$ is a unital positive homomorphism then there is a unital $*$-homomorphism $\phi : A \to B$ such that $\phi_* = \sigma$.*
(ii) *If ϕ and ψ are unital $*$-homomorphism from A into B with $\phi_* = \psi_*$ then there is a unitary $u \in B$ such that $\psi(x) = u\phi(x)u^*$ for $x \in A$.*

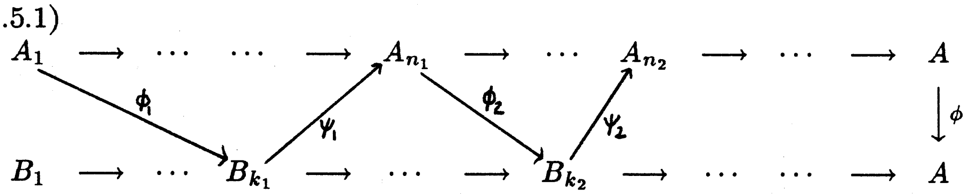
Lemma 3.5.3. *Let G be the unital dimension group obtained from the sequence (3.4.2) with the embedding $\iota_n : \mathbb{Z}^{r(n)} \to G$. If $\tau : \mathbb{Z}^k \to G$ is a unital positive homomorphism then there is a unital positive homomorphism $\tau_n : \mathbb{Z}^k \to \mathbb{Z}^{r(n)}$ with $\tau = \iota_n \tau_n$ for some $n = 1, 2, \dots$.*

Proof. For canonical generators $\{e_1, \dots, e_k\}$ of $\mathbb{Z}^k$, find the preimages of $\tau(e_i)$'s in some $\mathbb{Z}^{r(n)}$. $\square$

Proof of Theorem 3.5.1. Put $A = \varinjlim A_n$ and $B = \varinjlim B_k$, and denote by ι_{mn} (respectively $\eta_{k\ell}$) the connecting map from A_m to A_n (respectively from B_k to B_ℓ). Also, we denote by $\iota_n : A_n \to A$ and $\eta_k : B_k \to B$ the embeddings. By the above lemmas, there is a unital $*$-homomorphism $\phi_1 : A_1 \to B_{k_1}$ such that $\eta_{k_1 *}\phi_{1*} = \sigma\iota_{1*}$ for some k_1. Again, there is a unital positive homomorphism $\omega : B_{k_1} \to A_{n_1}$ such that $\iota_{n_1 *}\omega_* = \sigma^{-1}\eta_{k_1 *}$ for some n_1. This implies that $\iota_{n_1 *}\omega_*\phi_{1*} = \iota_{1*}$, and so we may assume that $\omega\phi_{1*} = \iota_{1n_1 *}$ since $K_0(A)$ is finitely generated. By Lemma 3.5.2 (ii), there is a unitary u in A_{n_1} such that $\iota_{1n_1}(x) = u(\omega(\phi_1(x))u^*$. We define $\psi_1(y) = u\omega(y)u^*$ for $y \in B_{k_1}$. Then we have

$$\iota_{1n_1} = \psi_1\phi_1 : A_1 \to A_n.$$

The following diagram illustrate the situation:
(3.5.1)



By the same way, we have a unital $*$-homomorphism $\phi_2 : A_{n_1} \to B_{k_2}$ such that $\phi_2\psi_1 = \eta_{k_1k_2}$. Continuing inductively, we have unital $*$-homomorphisms ϕ_n and ψ_n such that the diagram (3.5.1) commutes, and this gives the desired unital $*$-isomorphism ϕ. $\square$

Therefore, it follows that two *AF* C^*-algebras A and B are $*$-isomorphic each other if and only if there is a unital order isomorphism from $(K_0(A), [1_A])$ onto $(K_0(B), [1_B])$. A natural question is: Which ordered groups may be K_0-groups of *AF* algebras? In other words, we would like to characterize dimension groups among ordered groups. We say that an ordered group G is *unperforated* if $nx \geq 0$ for a natural number n implies $x \geq 0$. Because the group $\mathbb{Z}^r$ is unperforated, the same is true for dimension groups. Also, it is easy to see that every dimension group satisfies the *Riesz interpolation property*: If $x_i \leq y_j$ for $i, j = 1, 2$ then there is $z \in G$ such that $x_i \leq z \leq$

y_j for $i, j = 1, 2$. Because $\mathbb{Z}^r$ has this property, the same is also true for dimension groups. Effros, Handelman and Shen **[EHS80]** showed that these two conditions characterize dimension groups as follows:

Theorem 3.5.4. *An ordered group G is a dimension group if and only if G is countable, unperforated and has the Riesz interpolation property.*

Instead to prove this, we just mention one important application. Blackadar **[Bl81]** construct a simple unital C^*-algebra which has no nontrivial projections. It was a long-standing open question from fifties whether there is such a C^*-algebra.

The K_0-groups of AF algebras have another rich informations. Among them, we mention about tracial states of AF algebras. If τ is a tracial state on a stably finite unital C^*-algebra A, then τ induces a positive homomorphism $\tau_* : K_0(A) \to \mathbb{R}$ with $\tau([1_A]) = 1$ in the obvious way;

$$\tau_*([p] - [q]) = \tau(p) - \tau(q), \qquad [p] - [q] \in K_0(A).$$

Under certain conditions, every unital positive homomorphism from $K_0(A)$ into $\mathbb{R}$ arises in this way **[Bl**, §6.9]. Especially, this is the case for AF algebras. Indeed, if $A = \varinjlim A_n$ and $\phi : K_0(A) \to \mathbb{R}$ is a unital positive homomorphism then this gives a homomorphism $\phi_n : K_0(A_n) \to \mathbb{R}$. If $A_n = M_{p_1} \oplus \cdots M_{p_{r(n)}}$ then ϕ_n is defined by a vector $(\alpha_1, \dots, \alpha_{r(n)}) \in \mathbb{R}_+^{r(n)}$. It is easy to see that the map $\tau_n : A_n \to \mathbb{C}$ given by

$$\tau_n : (x_1, \dots, x_r) \mapsto \alpha_1 \mathrm{Tr}(x_1) + \cdots + \alpha_r \mathrm{Tr}(x_r), \qquad (x_1, \dots, x_r) \in A_n$$

defines a normalized tracial state τ on A with $\tau_* = \phi$, where Tr denotes the usual trace of matrix algebras.

For example, let $A = \varinjlim A_n$ be the AF algebra with the associated Bratteli diagram given by (3.1.4.f). If we define the sequence $\{q_n\}$ of natural numbers by

$$q_0 = 1, \quad q_1 = 1, \quad q_n = q_{n-2} + q_{n-1}, \qquad n = 2, 3, \dots,$$

then from the relation (3.4.5) we see that the unique normalized trace τ of A is given by

$$\tau(x, y) = \frac{1}{\theta^n} \mathrm{Tr}(x) + \frac{1}{\theta^{n+1}} \mathrm{Tr}(y), \quad (x, y) \in A_n = M_{q_n} \oplus M_{q_{n-1}},$$

where $\theta = \dfrac{1+\sqrt{5}}{2}$ as before.

Exercise 3.5.1. Show that the range of projections under the trace is $[0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta)$.

Exercise 3.5.2. Give the explicit formula for the normalized trace of the AF algebra whose associated diagram is given by (3.4.6).

In general, a unital positive homomorphism of an ordered group (G, u) is said to be a *state* of G and the set $S_u(G)$ of all states becomes a compact convex set with respect to the topology of pointwise convergence. If G is a simple dimension group which is not isomorphic to $\mathbb{Z}$ then $K = S_u(G)$ becomes a simplex and G may be realized as a dense subgroup of the additive group $\operatorname{Aff} K$ of all affine functions in $C(K, \mathbb{R})$, which is nothing but $\mathbb{R}^{r-1}$ if K is an r-simplex. In this way, we get concrete realizations of the K_0-groups for simple AF algebras. Note that examples in §3.4 were special cases with $r = 1$. We refer to **[Ef]** again for the details.

NOTE

General references for this chapter are **[Bl]**, **[Ef]** and **[Go]**. The latter part of §3.2 on representations of the UFH algebra was taken from **[M**, §6.2]. For a survey on the classification of hyperfinite factors of type III, we refer to **[Ha85a]**. We have followed Blackadar's book **[Bl]** for the definitions of equivalence relations in §3.3. Although our definition of K_0-groups may be applied for nonunital C^*-algebras, we should adjust the definition of K_0 in order to obtain the exactness of the sequence $K_0(I) \to K_0(A) \to K_0(A/I)$, and this will be done later together with the introduction of K_1-groups. Nevertheless, this adjusted definition will coincide with ours for unital C^*-algebras.

CHAPTER 4

TENSOR PRODUCTS OF C^*–ALGEBRAS AND NUCLEARITY

In this chapter, we will construct tensor products of two C^*-algebras. The main difficulty lies in the fact that there is no unique way to define a C^*-norm on the algebraic tensor product of C^*-algebras. In §4.1, we introduce the minimal and maximal tensor products of C^*-algebras, and examine their elementary properties. We also show that these two tensor products are distinct for the reduced group C^*-algebra of the free group on two generators. We say that a C^*-algebra A is *nuclear* if there is a unique C^*-norm on the algebraic tensor product $A \otimes B$ for every C^*-algebra B. Because the nuclearity passes to the inductive limits, every AF C^*-algebra is nuclear. During the seventies, a number of characterizations for the nuclearity has been developed. One of them is the approximation properties with completely positive linear maps with finite ranks. After discussion of elementary properties of completely positive linear maps in §4.2, we prove in §4.3 that the nuclearity is equivalent to the above approximation property. Another characterization is given in terms of its second dual: A C^*-algebra A is nuclear if and only if the enveloping von Neumann algebra A^{**} is injective. Because the complete proof of this theorem is beyond the scope of this note, we just discuss circumstances surrounding the injectivity of von Neumann algebras in §4.4.

A group C^*-algebra of a discrete group is nuclear if and only if the group is amenable. In §4.5, we discuss amenability of locally compact groups and prove this. More important is the fact that a group is amenable if and only if the full and reduced group C^*-algebras coincide. The free group is a typical example of a non-amenable group. Exploiting a special property, called the Powers' property, of the free group, we show in §4.6 that the reduced group C^*-algebra of the free group is a simple C^*-algebra with a unique trace. We also prove that the full group C^*-algebra of the free group is not exact, that is, the minimal tensoring with this C^*-algebra does not preserve the exactness of a short exact sequence. This depends on the residual finiteness of the free group. Note that the maximal tensoring always preserves the exactness. With this example of a non-exact C^*-algebra in hand, we produce many another examples of non-exact C^*-algebras in §4.7. The class of exact C^*-algebras form a very stable class of C^*-algebras: C^*-exactness is preserved under C^*-subalgebras, inductive limits and C^*-quotients.

4.1. Minimal and Maximal Tensor Products of C^*-algebras

Let A_1 and A_2 be C^*-algebras and denote by $A_1 \otimes A_2$ the algebraic tensor product of A_1 and A_2 as vector spaces. There are unique multiplication and involution on $A_1 \otimes A_2$ satisfying:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2,$$
$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*,$$

for $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. A norm γ on $A_1 \otimes A_2$ is said to be a *C^*-norm* if the C^*-norm conditions

$$\|xy\|_\gamma \leq \|x\|_\gamma \|y\|_\gamma, \qquad \|x^*x\|_\gamma = \|x\|_\gamma^2$$

hold for $x, y \in A_1 \otimes A_2$. It is clear that the completion $A_1 \otimes_\gamma A_2$ of $A_1 \otimes A_2$ with respect to the norm γ becomes a C^*-algebra. In this section, we give two important C^*-norms.

From now on, we assume that every C^*-algebra is *unital* for the simplicity, although everything below in this section is still true for non-unital C^*-algebras. For representations $\{\pi_i, \mathcal{H}_i\}$ of A_i, $i = 1, 2$, there is a unique representation, denoted by $\pi_1 \otimes \pi_2$, of $A_1 \otimes A_2$ satisfying

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2), \qquad a_i \in A_i.$$

For states $\phi_i \in \mathcal{S}(A_i)$, we also define a linear functional $\phi_1 \otimes \phi_2$ on $A_1 \otimes A_2$ by

$$(\phi_1 \otimes \phi_2)(a_1 \otimes a_2) = \phi_1(a_1)\phi_2(a_2), \qquad a_i \in A_i.$$

We first consider the case when A_i is a concrete C^*-algebra acting on a Hilbert space $\mathcal{H}_i$ for $i = 1, 2$. Then $A_1 \otimes A_2$ is a *normed* $*$-subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with the operator norm. For $\phi_i \in \mathcal{S}(A_i)$, we have

$$|(\phi_1 \otimes \phi_2)(x)| \le \|x\|, \qquad x \in A_1 \otimes A_2. \tag{4.1.1}$$

Indeed, one may apply Theorem 1.5.5 after showing (4.1.1) for finite sums of vector states. If $\{\pi_i, \mathcal{K}_i\}$ is a representation of A_i for $i = 1, 2$, then we also have

$$\|(\pi_1 \otimes \pi_2)(x)\| \le \|x\|, \qquad x \in A_1 \otimes A_2. \tag{4.1.2}$$

Because π_i is the sum of cyclic representations, we may assume that π_i has a cyclic vector ξ_i. Put

$$\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle, \qquad a \in A_i, \ i = 1, 2.$$

By (4.1.1), $\phi_1 \otimes \phi_2$ extends to a state ϕ on the C^*-subalgebra A of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by $A_1 \otimes A_2$. Because

$$(\phi_1 \otimes \phi_2)(x) = \langle (\pi_1 \otimes \pi_2)(x)(\xi_1 \otimes \xi_2), \xi_1 \otimes \xi_2 \rangle, \qquad x \in A_1 \otimes A_2, \tag{4.1.3}$$

there is a Hilbert space isomorphism from $\mathcal{H}_\phi$ onto $\mathcal{K}_1 \otimes \mathcal{K}_2$ which sends $\pi_\phi(x)\xi_\phi$ to $(\pi_1 \otimes \pi_2)(x)(\xi_1 \otimes \xi_2)$, where $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$ denotes the G. N. S. construction of A associated with ϕ. This shows that $\|(\pi_1 \otimes \pi_1)(x)\| = \|\pi_\phi(x)\|$ for $x \in A_1 \otimes A_2$, and completes the proof of (4.1.2).

Now, we define the *minimal* norm on $A_1 \otimes A_2$ by

$$\|x\|_{\min} = \sup \|(\pi_1 \otimes \pi_2)(x)\|, \qquad x \in A_1 \otimes A_2, \tag{4.1.4}$$

where π_i runs over all representations of A_i, $i = 1, 2$. From (4.1.2), it is easy to see that the supremum in (4.1.4) is taken if π_1 and π_2 are faithful. Note also that if π_1 and π_2 are faithful representations then $\pi_1 \otimes \pi_2$ is also faithful on $A_1 \otimes A_2$. Indeed, if $x = \sum_{k=1}^n a_k \otimes b_k$, where $\{b_k : k = 1, 2, \dots, n\}$ is linearly independent in A_2, and $(\pi_1 \otimes \pi_2)(x) = 0$, then $\{\pi_2(b_k)\}$ is also linearly independent and $\sum \pi_1(a_k) \otimes \pi_2(b_k) = 0$ as an operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. From this, it is easy to see that $\pi_1(a_k) = 0$ for each k. Because π_1 is faithful, we have $a_k = 0$ for each k, and so $x = 0$. Therefore, if A_i is a C^*-algebra acting faithfully on a Hilbert space $\mathcal{H}_i$, for $= 1, 2$, then the *minimal tensor product* $A_1 \otimes_{\min} A_2$ is $*$-isomorphic to the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by $\{a_1 \otimes a_2 : a_i \in A_i\}$, the *spatial tensor product* of A_1 and A_2. Compare with the definition of the tensor product of von Neumann algebras in §2.4. It is easy to see that every $*$-homomorphism $\sigma_i : A_i \to B_i$, $i = 1, 2$ gives rise to a unique $*$-homomorphism $\sigma_1 \otimes_{\min} \sigma_2 : A_1 \otimes_{\min} B_1 \to A_2 \otimes_{\min} B_2$, and

$$(4.1.5) \qquad A_1 \subseteq A_2,\ B_1 \subseteq B_2 \quad \Longrightarrow \quad A_1 \otimes_{\min} B_1 \subseteq A_2 \otimes_{\min} B_2.$$

Exercise 4.1.1. Show that $M_n \otimes_{\min} A$ is $*$-isomorphic to the C^*-algebra $M_n(A)$. Show that $C_0(X) \otimes_{\min} A$ is $*$-isomorphic to the C^*-algebra $C_0(X, A)$. Show also that the inductive limit of the sequence (3.3.3) is $*$-isomorphic to the C^*-algebra $\mathcal{K}(\mathcal{H}) \otimes_{\min} A$. Finally, if G_1 and G_2 are locally compact groups then show that $C^*_\lambda(G_1) \otimes_{\min} C^*_\lambda(G_2)$ is $*$-isomorphic to $C^*_\lambda(G_1 \times G_2)$.

If $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$ and $\{\pi_\psi, \mathcal{H}_\psi, \xi_\psi\}$ are the G. N. S. construction associated with states ϕ and ψ of A and B, respectively, then it is easy to see that

$$(4.1.6) \qquad (\phi \otimes \psi)(x) = \langle (\pi_\phi \otimes \pi_\psi)(x)(\xi_\phi \otimes \xi_\psi), \xi_\phi \otimes \xi_\psi \rangle, \qquad x \in A \otimes B.$$

as in (4.1.3). From this, it follows that $|(\phi \otimes \psi)(x)| \leq \|x\|_{\min}$ for $x \in A \otimes B$, and so $\phi \otimes \psi$ extends to a state $\phi \otimes_{\min} \psi$ of $A \otimes_{\min} B$. Considering the faithful representation given by (1.6.1), we see that

$$(4.1.7) \qquad \|x\|_{\min} = \sup\{\|(\pi_\phi \otimes \pi_\psi)(x)\| : \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B)\},$$

where $\mathcal{P}(A)$ denotes the set of all pure states on A as in §1.6. In the correspondence (1.5.3) between states and cyclic representations, we see that

$$\|\pi_\phi(a)\|^2 = \sup\{\frac{\phi(b^*a^*ab)}{\phi(b^*b)} : b \in A\},$$

because $\|b + L_\phi\|_{\mathcal{H}_\phi} = \phi(b^*b)$, or by Exercise 1.5.1. We apply this formula to (4.1.6) and (4.1.7) to get

$$(4.1.8) \quad \|x\|_{\min} = \sup\{\frac{(\phi \otimes \psi)(y^*x^*xy)}{(\phi \otimes \psi)(y^*y)} : y \in A \otimes B, \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B)\},$$

for each $x \in A \otimes B$. This gives an another definition of $\|\cdot\|_{\min}$ in terms of pure states of A and B. Of course, $\mathcal{P}(A)$ and $\mathcal{P}(B)$ may be replaced by the state spaces $\mathcal{S}(A)$ and $\mathcal{S}(B)$, respectively.

One of the main objects of this section is to show that $\|\cdot\|_{\min}$ is the smallest among all possible C^*-norms on $A \otimes B$. To do this, put

$$\mathcal{S}_\alpha = \{(\phi, \psi) \in \mathcal{P}(A) \times \mathcal{P}(B) : |(\phi \otimes \psi)(x)| \le \|x\|_\alpha,\ x \in A \otimes B\},$$

for an arbitrary C^*-norm α. It is easy to see that $\mathcal{S}_\alpha$ is a weak* closed subset of $\mathcal{P}(A) \times \mathcal{P}(B)$.

Theorem 4.1.1. *If α is a C^*-norm on $A \otimes B$ then we have $\|x\|_{\min} \le \|x\|_\alpha$ for each $x \in A \otimes B$.*

The proof is reduced to show the equality

$$\mathcal{S}_\alpha = \mathcal{P}(A) \times \mathcal{P}(B). \tag{4.1.9}$$

Indeed, for each $\phi \in \mathcal{P}(A)$ and $\psi \in \mathcal{P}(B)$ the relation (4.1.9) says that $\phi \otimes \psi$ extends to a state $\phi \otimes_\alpha \psi$ of $A \otimes_\alpha B$. For $y \in A \otimes B$ with $(\phi \otimes \psi)(y^* y) \neq 0$, the function

$$x \mapsto \eta(x) = \frac{(\phi \otimes_\alpha \psi)(y^* x y)}{(\phi \otimes_\alpha \psi)(y^* y)}$$

gives a state of $A \otimes_\alpha B$. Hence, for each $x \in A \otimes B$, we have

$$\frac{(\phi \otimes \psi)(y^* x^* x y)}{(\phi \otimes \psi)(y^* y)} = \eta(x^* x) \le \|x\|_\alpha^2,$$

and so the proof would be complete by (4.1.8).

We begin with abelian C^*-algebras. Recall that a state of an abelian C^*-algebra is pure if and only if it is multiplicative by Theorem 1.6.2.

Lemma 4.1.2. *Let A and B be abelian C^*-algebras then the relation* (4.1.9) *holds, and there is a unique C^*-norm on $A \otimes B$.*

Proof. If the relation (4.1.9) does not hold, then there are weak* open subsets U and V of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ such that

$$U \times V \subseteq \mathcal{P}(A) \times \mathcal{P}(B) \setminus \mathcal{S}_\alpha.$$

By Theorem 1.6.2, we see that there exist nonzero $a \in A$ and $b \in B$ such that

$$\phi(a) = \psi(b) = 0, \qquad \phi \in \mathcal{P}(A) \setminus U, \ \psi \in \mathcal{P}(B) \setminus V.$$

Therefore, we have

$$\begin{aligned} \|a \otimes b\|_\alpha &= \sup\{|\rho(a \otimes b)| : \rho \in \mathcal{P}(A \otimes_\alpha B)\} \\ &= \sup\{|(\phi \otimes_\alpha \psi)(a \otimes b)| : (\phi, \psi) \in \mathcal{S}_\alpha\} = 0, \end{aligned}$$

which is a contradiction because $a \otimes b \neq 0$, where the second equality follows from the obvious correspondence between $\rho \in \mathcal{P}(A \otimes_\alpha B)$ and $(\phi, \psi) \in \mathcal{S}_\alpha$ given by

$$\phi(a) = \rho_A(a) := \rho(a \otimes 1_B), \ \ a \in A, \qquad \psi(b) = \rho_B(b) := \rho(1_A \otimes b), \ \ b \in B.$$

The relation (4.1.9) also shows that

$$\|x\|_\alpha = \sup\{|(\phi \otimes_\alpha \psi)(x)| : \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B)\}, \qquad x \in A \otimes B,$$

but this is independent of α because $(\phi \otimes_\alpha \psi)(x) = (\phi \otimes \psi)(x)$ for $x \in A \otimes B$. □

Lemma 4.1.3. *If $\rho \in \mathcal{P}(A \otimes_\alpha B)$ and $\rho_A \in \mathcal{P}(A)$ then $\rho_B \in \mathcal{P}(B)$ and $\rho = \rho_A \otimes \rho_B$.*

Proof. For $a \in A$ and $b \in B$ with $0 \le b \le 1$, we have

$$\rho_A(a) = \rho(a \otimes 1) = \rho(a \otimes b) + \rho(a \otimes (1 - b)),$$

where $a \mapsto \rho(a \otimes b)$ and $a \mapsto \rho(a \otimes (1 - b))$ are positive linear functionals on A. So, it follows that $\rho(a \otimes b) = k\rho_A(a)$ for some scalar k, and $k = \rho(1 \otimes b)$. Hence, we have

$$\rho(a \otimes b) = \rho(1 \otimes b)\rho_A(a) = (\rho_A \otimes \rho_B)(a \otimes b), \qquad a \in A, \ b \in B.$$

The fact that $\rho_B \in \mathcal{P}(B)$ is easy. □

Exercise 4.1.2. Find an example of $\rho \in \mathcal{P}(M_2 \otimes M_2)$ such that ρ_{M_2} is not a pure state.

Lemma 4.1.4. *If A is abelian then the relation* (4.1.9) *holds.*

Proof. Let $\phi \in \mathcal{P}(A)$ be fixed, and put

$$\mathcal{P}_\alpha = \{\psi \in \mathcal{P}(B) : |(\phi \otimes \psi)(x)| \leq \|x\|_\alpha,\ x \in A \otimes B\}.$$

For $b \in B_h$, let C be the abelian C^*-subalgebra of B generated by 1_B and b. Then there exists $\psi_0 \in \mathcal{P}(C)$ such that $|\psi_0(b)| = \|b\|$. By Lemma 4.1.2, the closure of $A \otimes C$ in $A \otimes_\alpha B$ may be identified with $A \otimes_{\min} C$, and so the pure state $\phi \otimes_{\min} \psi_0$ extends to a pure state ρ of $A \otimes_\alpha B$. Because $\rho_A = \phi \in \mathcal{P}(A)$, we see that $\rho_B \in \mathcal{P}(B)$ by Lemma 4.1.3, and we have

$$|\rho_B(b)| = |\rho(1 \otimes b)| = |(\phi \otimes \psi_0)(1 \otimes b)| = |\psi_0(b)| = \|b\|.$$

By Theorem 1.5.5, we have $\mathcal{P}_\alpha = \mathcal{P}(B)$, and the proof is complete. □

Lemma 4.1.5. *If A is abelian then there is a unique C^*-norm on $A \otimes B$.*

Proof. We show that

$$\mathcal{P}(A \otimes_\alpha B) \subseteq \{\phi \otimes_\alpha \psi : \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B)\}.$$

Indeed, if $\rho \in \mathcal{P}(A \otimes_\alpha B)$ then ρ_A is pure by Proposition 1.6.8, because each element $a \otimes 1$ in $A \otimes_\alpha B$ is in the center. By Lemma 4.1.3, $\rho = \rho_A \otimes \rho_B$ with $\rho_B \in \mathcal{P}(B)$. Now, we have

$$\|x\|^2 = \|x^*x\|^2 = \sup\{(\phi \otimes_\alpha \psi)(x^*x) : \phi \in \mathcal{P}(A), \psi \in \mathcal{P}(B)\}, \quad x \in A \otimes B,$$

and the latter is independent of α as in the proof of Lemma 4.1.2. □

Proof of Theorem 4.1.1. As noted before, it suffices to show the relation (4.1.9). But, the exactly same argument as in the proof of Lemma 4.1.4 works, if we appeal to Lemma 4.1.5 instead of Lemma 4.1.2. □

Exercise 4.1.3. Show that

$$\|a \otimes b\|_\alpha = \|a\|\|b\|, \qquad a \in A,\ b \in B,$$

for every C^*-norm α on $A \otimes B$.

We have the obvious largest C^*-norm on $A \otimes B$ defined by

$$\|x\|_{\max} = \sup \|\pi(x)\|, \qquad x \in A \otimes B, \tag{4.1.10}$$

where π runs over all representations of $A \otimes B$. If π is a representation of $A \otimes B$ then we define the *restrictions* π_A and π_B by

$$\pi_A(a) = \pi(a \otimes 1_B), \qquad \pi_B(b) = \pi(1_A \otimes b), \qquad a \in A, \ b \in B.$$

Then the relation $\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$ holds for $a \in A$ and $b \in B$. It is clear that the identity map on $A \otimes B$ extends to a unique $*$-homomorphism $A \otimes_{\max} B \to A \otimes_{\min} B$. The maximal tensor products has the following universal property.

Proposition 4.1.6. *Let $\sigma : A \to C$ and $\tau : B \to C$ be $*$-homomorphisms such that $\sigma(A)$ and $\tau(B)$ commute each other. Then there exists a unique $*$-homomorphism $\pi : A \otimes_{\max} B \to C$ such that*

$$\pi(a \otimes b) = \sigma(a)\tau(b), \qquad a \in A, b \in B. \tag{4.1.11}$$

Proof. Note that the bilinear map $\pi : (a, b) \mapsto \sigma(a)\tau(b)$ induces a $*$-homomorphism with (4.1.11). It is easy to see that this homomorphism extends to $A \otimes_{\max} B$ because $x \mapsto \|\pi(x)\|$ defines a C^*-seminorm on $A \otimes B$. □

If $\sigma_i : A_i \to B_i$ be a $*$-homomorphism for $i = 1, 2$, then by the similar argument as above we see that there is a unique $*$-homomorphism $\sigma_1 \otimes_{\max} \sigma_2 : A_1 \otimes_{\max} B_1 \to A_2 \otimes_{\max} B_2$ which extends $\sigma_1 \otimes \sigma_2$.

Proposition 4.1.7. *If $\sigma_i : A_i \to B_i$ is a $*$-homomorphism onto B_i with kernel I_i for $i = 1, 2$, then the kernel of $\sigma_1 \otimes_{\max} \sigma_2$ is the closure of $I_1 \otimes A_2 + I_2 \otimes A_1$ in $A_1 \otimes_{\max} A_2$*

Proof. We denote by J the ideal described in the Proposition and τ the natural homomorphism onto $(A_1 \otimes_{\max} A_2)/J$. It suffices to show that

$$\|\tau(x)\| \le \|(\sigma_1 \otimes \sigma_2)(x)\|_{\max}, \qquad x \in A_1 \otimes A_2.$$

Note that $(\sigma_1(a_1), \sigma_2(a_2)) \mapsto \tau(a_1 \otimes a_2)$ is a well-defined bilinear map from $B_1 \times B_2$ and so there is a $*$-homomorphism $\rho : B_1 \otimes B_2 \to (A_1 \otimes_{\max} A_2)/J$ such that

$$\rho(\sigma_1 \otimes \sigma_2)(x) = \tau(x), \qquad x \in A_1 \otimes A_2.$$

It is easy to see that $y \mapsto \rho(y)$ is a C^*-seminorm on $B_1 \otimes B_2$, and so we have $\|\tau(x)\| = \|\rho(\sigma_1 \otimes \sigma_2)(x)\| \le \|(\sigma_1 \otimes \sigma_2)(x)\|_{\max}$. □

We know that if C is a C^*-subalgebra of A then every representation π of C extends to a representation of A in the sense of Theorem 1.6.7. If C is a norm-closed two-sided ideal of A then it is not so difficult to see that every representation π of C extends to a representation ρ of A on the same Hilbert space, and $\pi(C)$ and $\rho(A)$ have the same weak operator closures.

Corollary 4.1.8. *Let A and B be C^*-algebras and J a norm-closed two-sided ideal of B. Then the following short sequence*

$$0 \to A \otimes_{\max} J \to A \otimes_{\max} B \to A \otimes_{\max} (B/J) \to 0 \tag{4.1.12}$$

is exact.

Proof. We must show that the closure of $A \otimes J$ in $A \otimes_{\max} B$ is just $A \otimes_{\max} J$. To do this, it suffices to show that a representation π of $A \otimes J$ extends to a representation of $A \otimes B$. By the above discussion, π_J extends to a representation ρ of B so that $\pi_J(J)$ and $\rho(B)$ have the same weak operator closures. Hence, $\pi_A(A)$ and $\rho(B)$ commute each other, and so $\pi_A \otimes \rho$ is the required extension of π. □

We note that the above argument does not go well if J is just a C^*-subalgebra of B. In fact, we will see later the corresponding relation (4.1.5) for maximal tensor products is not true in general.

Exercise 4.1.4. Show that the full group C^*-algebra $C^*(G_1 \times G_2)$ is $*$-isomorphic to the maximal tensor product $C^*(G_1) \otimes_{\max} C^*(G_2)$, for locally compact groups G_1 and G_2.

We say that a C^*-algebra A is *nuclear* if there is only one C^*-norm on $A \otimes B$ for each C^*-algebra B, or equivalently, $A \otimes_{\min} B = A \otimes_{\max} B$ for each C^*-algebra B. Lemma 4.1.5 says that every abelian C^*-algebra is nuclear.

Note that every finite-dimensional C^*-algebra is nuclear because there is only one C^*-norm on $M_n \otimes A \cong M_n(A)$ by Exercise 1.5.3. Every approximately finite-dimensional C^*-algebra is also nuclear by the following easy proposition.

Proposition 4.1.9. *If $\{A_i\}$ is an increasing net of nuclear C^*-algebras, then the inductive limit $A = \overline{\cup_i A_i}$ is also nuclear.*

Now, we exhibit an example of a non-nuclear C^*-algebra. To do this, we need the following:

Proposition 4.1.10. *Let R be a factor on $\mathcal{H}$. If $\sum_{i=1}^n x_i y_i = 0$ with $x_i \in R$ and $y_i \in R'$, then there exists an $n \times n$ matrix $[c_{ij}]$ such that*

$$\sum_{j=1}^n c_{jk} x_j = 0, \quad k = 1, 2, \ldots, n, \quad \text{and} \quad \sum_{k=1}^n c_{jk} y_k = y_j, \quad j = 1, 2, \ldots, n.$$

Proof. The proof is a modification of Lemma 2.3.3 as follows: Let X be the operator in the matrix algebra $M_n(R)$ acting on the n-direct sum of $\mathcal{H}$ whose first row is $\{x_1, \ldots, x_n\}$ and another entries are zeros, and $\widetilde{Q} = [C_{jk}] \in M_n(R')$ the supremum of all projections Q in $M_n(R')$ with $XQ = 0$. For a projection $p \in R'$, we denote by P the projection in $M_n(R')$ whose diagonals are p. Then by the same argument as in the proof of Lemma 2.3.3, we see that $P\widetilde{Q} = \widetilde{Q}P$, and so each entry C_{jk} of $\widetilde{Q}$ commutes with p. Because R is a factor $C_{jk} = c_{jk}$ is a scalar as desired. The relation $X\widetilde{Q} = 0$ implies the first equations. If we denote by $Y \in M_n(R')$ whose first column is $\{y_1, \ldots, y_n\}$ and another entries are zeros then $XY = 0$ implies that $\mathcal{R}(Y) \leq \widetilde{Q}$, and so the second equations follow from $\widetilde{Q}Y = Y$. □

Corollary 4.1.11. *Let M be a factor acting on $\mathcal{H}$. Then the map*

$$\pi : x \otimes y \mapsto xy, \qquad x \in M,\ y \in M' \tag{4.1.13}$$

extends to a $*$-isomorphism of $M \otimes M'$ onto the $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ consisting of all finite sums of operators of the form xy, with $x \in M$ and $y \in M'$.

Corollary 4.1.12. *If A and B are simple C^*-algebras then $A \otimes_{\min} B$ is also simple.*

Proof. Let π be an irreducible representation of $A \otimes_{\min} B$ on $\mathcal{H}$. It suffices to show that π is faithful. Because $\pi_A(A)$ and $\pi_B(B)$ commute each other, we have

$$\begin{aligned}\pi_A(A)'' \cap \pi_A(A)' &\subseteq \pi_B(B)' \cap \pi_A(A)' \\ &= (\pi_A(A) \cup \pi_B(B))' = \pi(A \otimes_{\min} B)' = \mathbb{C}1_{B(\mathcal{H})},\end{aligned}$$

and so we see that $\pi_A(A)''$ is a factor. For $x = \sum a_i \otimes b_i \in A \otimes B$, if $\pi(x) = \sum \pi_A(a_i)\pi_B(b_i) = 0$ then $(\pi_A \otimes \pi_B)(x) = \sum \pi_A(a_i) \otimes \pi_B(b_i) = 0$ by Corollary 4.1.11. So, $x = 0$ since $\pi_A \otimes \pi_B$ is faithful. Therefore, $\|x\|_\alpha = \|\pi(x)\|$ gives a C^*-norm, and it follows that $\|x\|_{\min} \leq \|x\|_\alpha = \|\pi(x)\| \leq \|x\|_{\min}$ by Theorem 4.1.1 and Proposition 1.2.7. Hence, π is faithful. □

We also need the following elementary fact on inner product spaces:

Exercise 4.1.5. Let $\mathcal{H}$ be an inner product space with a unit vector ξ. Show that if $\|\eta_i\| \leq 1$ for $i = 1, 2$ and $\|\xi - \frac{1}{2}(\eta_1 + \eta_2)\| \leq \varepsilon$ then $\|\xi - \eta_i\| \leq 2\sqrt{\varepsilon}$ for $i = 1, 2$.

Example 4.1.1. The reduced group C^*-algebra $C^*_\lambda(F_2)$ of the free group F_2 on two generators is not nuclear.

Recall the factor construction in §2.4. We note that $C^*_\lambda(G)$ and $C^*_\rho(G)$ are the C^*-algebras acting on $\ell^2(G)$ generated by $\{L_s : s \in G\}$ and $\{R_s : s \in G\}$, respectively, where

$$\begin{aligned} L_s(\xi)(t) &= (\chi_s * \xi)(t) = \xi(s^{-1}t), \\ R_s(\xi)(t) &= (\xi * \chi_{s^{-1}})(t) = \xi(ts), \end{aligned}$$

for $\xi \in L^2(G)$. If we consider the free group F_2 on two generators then $C^*_\lambda(F_2)''$ and $C^*_\rho(F_2)''$ are factors. So, $\|x\|_\nu = \|\pi(x)\|$ with the $*$-isomorphism π in (4.1.13) gives a C^*-norm ν on $C^*_\lambda(F_2) \otimes C^*_\rho(F_2)$. We show that this norm is not equal to the minimal C^*-norm. By Corollary 4.1.11, $C^*_\lambda(F_2) \otimes_\nu C^*_\rho(F_2)$ is $*$-isomorphic to the C^*-subalgebra $\mathfrak{A}$ of $\mathcal{B}(\ell^2(F_2))$ generated by $\{L_s, R_s : s \in F_2\}$. We denote by a and b the generators of F_2. Let S be the subset of F_2 consisting of all reduced words beginning with a or a^{-1}. Then we observe the following two properties:

(I) $S \cup a^{-1}Sa = F_2 \setminus \{e\}$.

(II) The sets S, $b^{-1}Sb$ and bSb^{-1} are mutually disjoint.

Lemma 4.1.13. *The C^*-algebra $\mathfrak{A}$ contains a rank one projection onto the subspace $\mathbb{C}\chi_e$.*

Proof. Define a self-adjoint operator x in $\mathfrak{A}$ by

$$x = \frac{1}{4}(L_aR_a + L_a^*R_a^* + L_bR_b + L_b^*R_b^*).$$

Then x is a contraction and $x|_{\mathbb{C}\chi_e} = 1$. Let ξ be a unit vector in $(\mathbb{C}\chi_e)^\perp$ and put $\kappa = \|\xi - x\xi\|$. By Exercise 4.1.5, we see that

$$\|\xi - L_aR_a\xi\| < 2^{\frac{3}{2}}\kappa^{\frac{1}{4}} (:= \epsilon \text{ temporarily}).$$

For $M \subseteq F_2$, put $\lambda(M) = \sum_{s\in M} |\xi(s)|^2$. Then, we have

$$\begin{aligned}|\lambda(S) - \lambda(a^{-1}Sa)| &= |\lambda(S)^{\frac{1}{2}} + \lambda(a^{-1}Sa)^{\frac{1}{2}}| \cdot |\lambda(S)^{\frac{1}{2}} - \lambda(a^{-1}Sa)^{\frac{1}{2}}| \\ &\le 2\|\xi\| \cdot |(\sum_{s\in S} |\xi(s)|^2)^{\frac{1}{2}} - (\sum_{s\in S} |\xi(a^{-1}sa|^2)^{\frac{1}{2}}| \\ &\le 2\|\xi\|(\sum_{s\in S} |\xi(s) - \xi(a^{-1}sa)|^2)^{\frac{1}{2}} \\ &\le 2\|\xi\|\|\xi - L_aR_a\xi\| < 2\|\xi\|\epsilon,\end{aligned}$$

and similarly

$$|\lambda(S) - \lambda(b^{-1}Sb)| < 2\|\xi\|\epsilon, \qquad |\lambda(S) - \lambda(bSb^{-1})| < 2\|\xi\|\epsilon.$$

Now, we have $\lambda(a^{-1}Sa) < \lambda(S) + 2\|\xi\|\epsilon$, and so $\|\xi\|^2 = \lambda(a^{-1}Sa) + \lambda(S) < 2(\lambda(S) + \|\xi\|\epsilon)$ by the property (I). Hence, it follows that $\lambda(S) > \frac{1}{2}\|\xi\|^2 - \|\xi\|\epsilon$. Also, we have $\lambda(b^{-1}Sb) > \lambda(S) - 2\|\xi\|\epsilon > \frac{1}{2}\|\xi\|^2 - 3\|\xi\|\epsilon$, and similarly for $\lambda(bSb^{-1})$. By the property (II), we have

$$\|\xi\|^2 \ge \lambda(S) + \lambda(b^{-1}Sb) + \lambda(bSb^{-1}) > \frac{3}{2}\|\xi\|^2 - 7\|\xi\|\epsilon,$$

and this implies that

$$1 = \|\xi\| < 14\epsilon < 42\kappa^{\frac{1}{4}}. \tag{4.1.14}$$

Therefore, we have $\kappa \ge 42^{-4}$, and

$$\|(\lambda - x)\xi\| \ge \|\xi - x\xi\| - |\lambda - 1|\|\xi\| > |\kappa - (1 - \lambda)|\|\xi\|,$$

for each $\lambda \in (1-42^{-4}, 1)$ and $\xi \in (\mathbb{C}\chi_e)^\perp$ with $\|\xi\| = 1$. This shows that

$$\mathrm{sp}(x|(\mathbb{C}\chi_e)^\perp) \subseteq [-1, 1-42^{-4}], \tag{4.1.15}$$

because $|\kappa - (1-\lambda)| > 0$. Consider a continuous function $f : [-1,1] \to [0,1]$ such that $f \equiv 0$ on $[-1, 1-42^{-4}]$ and $f(1) = 1$. Then the function calculus $f(x)$ is the desired projection. □

Now, because $\mathfrak{A}' \subseteq \mathcal{R}_\lambda(F_2)' \cap \mathcal{R}_\rho(F_2)'$ consists of the scalar operators, we see that $\mathfrak{A}$ acts irreducibly on $\ell^2(F_2)$ by Lemma 1.6.5. By Theorem 1.6.6, we have the following:

Corollary 4.1.14. *The C^*-algebra $\mathfrak{A}$ contains the ideal $\mathcal{K}(\ell^2(F_2))$.*

Later, we will show that $C^*_\lambda(F_2)$ is simple. Assuming this, we see that $C^*_\lambda(F_2) \otimes_{\min} C^*_\rho(F_2)$ is simple by Corollary 4.1.12, and the proof of Example 4.1.1 would be complete. We have an alternative argument as follows: From Corollary 4.1.14, it is clear that there is no faithful tracial state on $\mathfrak{A}$. More precisely, note that $\{L_s P_e L_s^* : s \in F_2\}$, where P_e denotes the projection given by Lemma 4.1.13, is an orthogonal family of projections onto the one-dimensional subspace $\mathbb{C}\chi_s$. If τ is a tracial state of $\mathfrak{A}$, then

$$(\# M)\tau(P_e) = \tau(\sum_{s\in M} L_s P_e L_s^*) \le \tau(1)$$

for each finite subset M of F_2, and so $\tau(P_e) = 0$, where $\#M$ denotes the cardinality of M. On the other hand, we know from §2.4 that there is a faithful trace τ_1 and τ_2 of $C^*_\lambda(F_2)$ and $C^*_\rho(F_2)$ respectively. Hence, we see that $\tau_1 \otimes \tau_2$ extends to a faithful trace on $C^*_\lambda(F_2) \otimes_{\min} C^*_\rho(F_2)$, and this completes the proof of Example 4.1.1. □

NOTE

Tensor product of C^*-algebras was first introduced by Turumaru [**Tu52**] with the formula which is similar as in (4.1.8). Theorem 4.1.1 together with Corollary 4.1.12, Proposition 4.1.9 and Example 4.1.1 are due to Takesaki [**Ta64**], whereas the maximal C^*-norm was introduced by Guichardet [**Gu65**] in which Proposition 4.1.7 was proved. We have followed Kadison's book [**K**, §11.3] for the proof of Theorem 4.1.1. See also [**M**, §6.4], [**T**, §IV.4], [**S**, §1.22] or [**Gu**, §4]. Although the norm estimate (4.1.14) was taken from [**S**, Lemma 4.3.3], the argument using spectrum (4.1.15) in the proof of Lemma 4.1.13 is due to S. Wassermann [**Wa90**]. See also [**K**, Example 11.3.14]. Corollary 4.1.14 is due to Akemann and Ostrand [**AO75**], in which they also showed that the ideal $\mathcal{K}(\ell^2(F_2))$ is actually the unique norm-closed two-sided ideal of the C^*-algebra $\mathfrak{A}$. This shows that $\mathcal{K}(\ell^2(F_2))$ is $*$-isomorphic to the ideal of the homomorphism $C^*(F_2) \otimes_\nu C^*_\rho(F_2) \to C^*(F_2) \otimes_{\min} C^*_\rho(F_2)$. The survey note [**Tp81**] is a useful reference through this chapter.

4.2. Completely Positive Linear Maps

Let A and B be C^*-algebras and $\phi : A \to B$ a bounded linear map. We define the linear map $\phi_n : M_n(A) \to M_n(B)$ by

$$\phi_n([a_{ij}]_{i,j=1}^n) = [\phi(a_{ij})]_{i,j=1}^n, \qquad [a_{ij}]_{i,j=1}^n \in M_n(A). \tag{4.2.1}$$

Note that $\phi_n = \phi \otimes 1_n$ in the isomorphism $M_n(A) = A \otimes M_n$, where 1_n denotes the identity map on M_n. If we consider the transpose map τ between M_2, we have

$$\tau_2 : \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and so τ_2 is not positive although τ is a positive linear map. We say that a linear map ϕ is *n-positive* if the map ϕ_n is positive, and *completely positive* if ϕ_n is positive for each $n = 1, 2, \ldots$. It is clear that every $*$-homomorphism is completely positive. We show that if A or B is abelian then every positive linear map is already completely positive.

Proposition 4.2.1. *Let B be an abelian C^*-algebra. Then every positive linear map $\phi : A \to B$ is completely positive.*

Proof. Let ρ be a pure state of $B \otimes M_n$. It suffices to show that $\rho \circ \phi_n$ is positive. By Lemma 4.1.3 and Proposition 1.6.8, we see that ρ is of the form $\rho_1 \otimes \rho_2$ for some $\rho_1 \in \mathcal{P}(B)$ and $\rho_2 \in \mathcal{P}(M_n)$. Hence, we see that

$$\rho \circ \phi_n = (\rho_1 \otimes \rho_2) \circ (\phi \otimes 1_n) = (\rho_1 \circ \phi) \otimes \rho_2$$

is positive. □

Proposition 4.2.2. *If ϕ is a positive linear map from a commutative C^*-algebra $C_0(X)$ into a C^*-algebra B then ϕ is completely positive.*

Proof. We may assume that B acts on a Hilbert space $\mathcal{H}$. Note that $\mu_{ij}(f) = \langle \phi(f)\xi_i, \xi_j \rangle$ gives rise to a complex measure for vectors ξ_i, ξ_j in $\mathcal{H}$. Put $\mu = \sum_{i,j} |\mu_{ij}|$. Then there is $h_{ij} \in L^1(\mu)$ such that $\mu_{ij}(f) = \int_X f h_{ij} d\mu$

for each $f \in C_0(X)$. Now, for complex numbers $c_1, \ldots, c_n$, we have

$$\begin{aligned}\int_X f(\sum_{i,j} h_{ij} c_i \bar{c_j}) d\mu &= \sum_{i,j} \langle \phi(f)\xi_i, \xi_j \rangle c_i \bar{c_j} \\ &= \langle \phi(f)(\sum_i c_i \xi_i), (\sum_j c_j \xi_j) \rangle \geq 0,\end{aligned}$$

for each $f \geq 0$ in $C_0(X)$. Hence, we see that $\sum_{i,j} h_{ij}(x) c_i \bar{c_j} \geq 0$ for μ-almost all $x \in X$. This shows that

$$\sum_{i,j} \langle \phi(\bar{f_j} f_i)\xi_i, \xi_j \rangle = \int_X (\sum_{i,j} \overline{f_j(x)} f_i(x) h_{ij}(x)) d\mu(x) \geq 0,$$

for $f_1, \ldots, f_n$ in $M_n(C_0(X))$. Because every positive element of $M_n(A)$ is the sum of matrices of the form $[a_j^* a_i]$ with $a_i \in A$, the proof is complete. □

Here is an another important class of linear maps which are automatically completely positive. Let B be a unital C^*-subalgebra of a C^*-algebra A. A unital positive linear map $\phi : A \to B$ is said to be a *conditional expectation* if

$$\phi(cab) = c\phi(a)b, \qquad a \in A, \ b, c \in B. \tag{4.2.2}$$

Proposition 4.2.3. *Every conditional expectation $\Phi : A \to B \subseteq A$ is completely positive.*

Proof. First, we assume that B acts on a Hilbert space $\mathcal{H}$ with a cyclic vector ξ. For $\{b_i : i = 1, \ldots, n\}$ in B and $\{a_i : i = 1, \ldots, n\}$ in A, we have

$$\begin{aligned}\sum \langle \Phi(a_j^* a_i) b_i \xi, b_j \xi \rangle &= \sum_{i,j} \langle b_j^* \Phi(a_j^* a_i) b_i \xi, \xi \rangle \\ &= \sum_{i,j} \langle \Phi(b_j^* a_j^* a_i b_i) \xi, \xi \rangle \\ &= \langle \Phi(\sum_{i,j} (b_j^* a_j^* a_i b_i) \xi, \xi \rangle \geq 0,\end{aligned}$$

because $\sum_{i,j} b_j^* a_j^* a_i b_i \geq 0$, and Φ is positive by definition. Because ξ is cyclic, we have

$$\sum_{i,j} \langle \Phi(a_j^* a_i)\xi_i, \xi_j \rangle \geq 0, \qquad \xi_i, \xi_j \in \mathcal{H}.$$

For the general cases, we assume that B acts on a Hilbert space $\mathcal{H}$ universally. By the decomposition (1.5.6), we see that $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ and $\Phi(A)$ acts on $\mathcal{H}_\alpha$ with a cyclic vector. So, the map $\Phi_\alpha : A \to \mathcal{B}(\mathcal{H}_\alpha)$ given by $\Phi_\alpha(a) = \Phi(a)|_{\mathcal{H}_\alpha}$ is completely positive by above. From this, we see that Φ is completely positive. □

Now, we show the following fundamental theorem.

Theorem 4.2.4. *A linear map ϕ from a unital C^*-algebra A into $\mathcal{B}(\mathcal{H})$ is completely positive if and only if there exist a representation $\{\pi, \mathcal{K}\}$ of A and a bounded linear map from $\mathcal{H}$ to $\mathcal{K}$ such that*

$$\phi(a) = V^*\pi(a)V, \qquad a \in A. \tag{4.2.3}$$

Proof. For the sufficiency, let $[a_{ij}] \in M_n(A)$ be positive. Then we have

$$\sum_{i,j}\langle \phi(a_{ij})\xi_j, \xi_i\rangle = \sum_{i,j}\langle \pi(a_{ij})V\xi_j, V\xi_i\rangle \geq 0, \qquad \xi_i, \xi_j \in \mathcal{H},$$

and so we see that $[\phi(a_{ij})]$ is a positive operator on the n-fold direct sum of $\mathcal{H}$.

Proof of the necessity is a variant of G. N. S. construction as follows: Consider the algebraic tensor product $A \otimes \mathcal{H}$ and we define

$$\langle x, y\rangle_\phi = \sum_{i,j}\langle \phi(b_j^* a_i)\xi_i, \eta_j\rangle, \tag{4.2.4}$$

for $x = \sum_i a_i \otimes \xi_i$ and $y = \sum_j b_j \otimes \eta_j$ in $A \otimes \mathcal{H}$. Because ϕ is completely positive, we see that (4.2.4) gives rise to a positive bilinear form. Now, we also define the map π_0 from A to linear maps on $A \otimes \mathcal{H}$ by

$$\pi_0(a)(\sum_i a_i \otimes \xi_i) = \sum_i (aa_i) \otimes \xi_i, \qquad a \in A.$$

If we put $\rho(a) = \langle \pi_0(a)x, x\rangle_\phi$ then ρ is a positive linear functional on A, and so we have

$$\langle \pi_0(a)x, \pi_0(a)x\rangle_\phi = \langle \pi_0(a^*a)x, x\rangle_\phi = \rho(a^*a) \leq \|a^*a\|\rho(1_A) = \|a\|^2\langle x, x\rangle_\phi.$$

As in the G. N. S. construction in §1.5, we see that π_0 induces a representation on the Hilbert space $\mathcal{K} = \overline{(A \otimes \mathcal{H})/L_\phi}$. Define

$$V\xi = 1_A \otimes \xi + L_\phi, \qquad \xi \in \mathcal{H}.$$

Then V is a bounded linear map from $\mathcal{H}$ to $\mathcal{K}$, because $\|V\xi\|^2 = \langle \phi(1_A)\xi, \xi\rangle \leq \|\phi(1_A)\|\,\|\xi\|^2$. The required condition (4.2.3) is easily checked. □

Note that we have shown that

$$\|V\|^2 \leq \|\phi(1_A)\| \tag{4.2.5}$$

in the proof of Theorem 4.2.4. Using this, we obtain the following Schwarz inequality for completely positive linear maps.

Corollary 4.2.5. *If ϕ is a completely positive linear map from a unital C^*-algebra A into a C^*-algebra B then we have*

$$\phi(a)^*\phi(a) \leq \|\phi(1_A)\|\phi(a^*a), \qquad a \in A. \tag{4.2.6}$$

Exercise 4.2.1. Let ϕ_i be a completely positive linear map from a unital C^*-algebra A_i into a C^*-algebra B_i, for $i = 1, 2$. Show that the map $\phi_1 \otimes \phi_2 : A_1 \otimes B_1 \to A_1 \otimes B_2$ extends to a completely positive linear map from $A_1 \otimes_{\min} B_1$ into $A_2 \otimes_{\min} B_2$.

Now, we say that a linear map $\phi : A \to B$ is *n-bounded* if ϕ_n is a bounded linear map, and ϕ is *completely bounded* if each ϕ_n is bounded with

$$\|\phi\|_{cb} := \sup_n \{\|\phi_n\| : n = 1, 2, \dots\} < \infty.$$

We also say that ϕ is *completely contractive* if every ϕ_n is contractive. The following simple proposition will be useful later.

Proposition 4.2.6. *Let $\phi : A_1 \to A_2$ be a completely bounded linear map between C^*-algebras A_1 and A_2. For any C^*-algebra B, the linear map $\phi \otimes 1_B : A_1 \otimes B \to A_2 \otimes B$ extends to a bounded linear map $\phi \otimes_{\min} 1_B : A_1 \otimes_{\min} B \to A_2 \otimes_{\min} B$. Furthermore, we have $\|\phi \otimes_{\min} 1_B\| \leq \|\phi\|_{cb}$.*

Proof. We may assume that B is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For any finite dimensional projection p in $\mathcal{B}(\mathcal{H})$, we have

$$\|(1 \otimes p)((\phi \otimes 1)(x))(1 \otimes p)\| = \|\sum \phi(a_i) \otimes pb_ip\| \leq \|\phi\|_{cb}\|x\|,$$

for $x = \sum a_i \otimes b_i \in A \otimes \mathcal{B}(\mathcal{H})$. Because the net of finite dimensional projections in $\mathcal{B}(\mathcal{H})$ converges strongly to $1_{\mathcal{B}(\mathcal{H})}$, the conclusion follows. □

Now, we extend Proposition 1.5.2 to see how related positive linear maps and bounded linear maps are. Assume that $\phi : A \to \mathcal{B}(\mathcal{H})$ is a unital contraction and ξ is a unit vector in $\mathcal{H}$. Then $\rho(x) = \langle \phi(x)\xi, \xi \rangle$ defines a bounded linear functional on A with $\|\rho\| = 1 = \rho(1)$. By Proposition 1.5.2, we see that ρ is a positive linear functional, and so ϕ is also a positive linear map. The converse is also true as follows:

Theorem 4.2.7. *Let $\phi : A \to B$ be a unital linear map between C^*-algebras. Then ϕ is positive if and only if it is contractive.*

Proof. First, we consider the case when A is abelian. We may assume that B acts on a Hilbert space $\mathcal{H}$. Then ϕ is completely positive by Proposition 4.2.2, and $\phi(x) = V^*\pi(x)V$, where $(\pi, \mathcal{K})$ is a representation of A and V is a bounded linear map from $\mathcal{H}$ to $\mathcal{K}$. Therefore, it follows that

$$\|\phi(x)\| \leq \|V\|^2 \|\pi(x)\| \leq \|\phi(1_A)\| \|x\| = \|x\|,$$

by (4.2.5). The next proposition reduces the proof of the theorem to the case of abelian C^*-subalgebras generated by 1_A and unitaries. □

Proposition 4.2.8. *The unit ball of a unital C^*-algebra A is the closed convex hull of unitaries in A.*

Proof. For $x \in A$ with $\|x\| < 1$, the element

$$f(x, \lambda) = (1 - xx^*)^{-\frac{1}{2}}(1 + \lambda x)$$

exists and is invertible for each complex number λ with $|\lambda| = 1$. Using the power series expansion of $(1 - xx^*)^{-1}$, we calculate

$$f(x,\lambda)^* f(x,\lambda) + 1 = (1-xx^*)^{-1} + (1-x^*x)^{-1}\bar{\lambda}x^* + (1-xx^*)^{-1}\lambda x + (1-x^*x)^{-1},$$

and so, $f(x, \lambda)^* f(x, \lambda) = f(x^*, \bar{\lambda})^* f(x^*, \bar{\lambda})$. Therefore, the element

$$u_\lambda = f(x, \lambda) f(x^*, \bar{\lambda})^{-1}$$

is unitary for each λ with $|\lambda| = 1$. Noting that the function

$$u(\lambda) = (1 - xx^*)^{-\frac{1}{2}}(\lambda + x)(1 + \lambda x^*)^{-1}(1 - x^*x)^{\frac{1}{2}}$$

is holomorphic in a neighborhood of the unit disc, we see that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt = u(0) &= (1 - xx^*)^{-\frac{1}{2}}x(1 - x^*x)^{\frac{1}{2}} \\ &= (1 - xx^*)^{-\frac{1}{2}}(1 - xx^*)^{\frac{1}{2}}x = x.\end{aligned}$$

This completes the proof, because $u(\lambda) = \lambda u_{\bar{\lambda}}$ is unitary and the measure in the above integral can be approximated by convex combinations of point masses. □

Corollary 4.2.9. *Let $\phi : A \to B$ be a unital linear map between C^*-algebras. Then ϕ is completely positive if and only if ϕ is completely contractive.*

We close this section to see that the dual of a completely positive linear map is also completely positive, which will be used frequently in the next section. We identify $M_n(A^*)$ with $M_n(A)^*$ by

$$(4.2.7) \quad \phi(x) = \sum_{i,j=1}^{n} \phi_{ij}(x_{ij}), \qquad \phi = [\phi_{ij}] \in M_n(A^*),\ x = [x_{ij}] \in M_n(A).$$

Proposition 4.2.10. *If $\Phi : A \to B$ is a completely positive linear map then the adjoint map $\Phi^* : B^* \to A^*$ is also completely positive.*

Proof. For $x = [x_{ij}] \in M_n(A)$ and $\phi = [\phi_{ij}] \in M_n(B)^*$, we have

$$\begin{aligned}\langle x, (\Phi_n)^*(\phi)\rangle = \langle \Phi_n(x), \phi\rangle &= \sum_{i,j}\langle \Phi(x_{ij}), \phi_{ij}\rangle \\ &= \sum_{i,j}\langle x_{ij}, \Phi^*(\phi_{ij})\rangle = \langle x, (\Phi^*)_n(\phi)\rangle.\end{aligned}$$

Hence, it follows that $(\Phi_n)^* = (\Phi^*)_n$ with the identification (4.2.7), and the proof is complete because the adjoint of a positive map is also positive. □

NOTE

We refer to Effros' or Choi's article **[Ef78]** **[Ch82]** for the motivations why we should consider completely positive linear maps rather than just positive linear maps in dealing with the order structures of *non-commutative* operator algebras. Paulsen's monograph [**Pau**] is a useful reference for this section. Proposition 4.2.1 is taken from [**St63**, Lemma 6.1]. Proposition 4.2.2 together with Theorem 4.2.4 are due to Spinespring [**Ss55**]. See also [**Pau**, §4]. We also refer to [**Pau**, Theorem 7.4] for the similar result for completely bounded linear maps. The converse of Propositions 4.2.1 and 4.2.2 holds: If A and B are C^*-algebras such that every positive linear map from A to B is completely positive then either A or B is abelian [**To82**]. Choi [**Ch72**] was the first who exploited the differences between n-positivity and $(n+1)$-positivity. The Schwarz inequality, Corollary 4.2.4, is also valid for 2-positive linear maps [**Ch74**]. For further developments on this topic, we refer to the survey article [**Ky92**] and the references there. We refer to [**Ok70**] for an explicit example of a positive linear map for which Proposition 4.2.6 does not hold. Theorem 4.2.7 is due to Russo and Dye [**RD66**]. The proof of the essential part, Proposition 4.2.8, was taken from [**P**, Proposition 1.1.12]. See also [**Pau**, §2] for an another proof of Theorem 4.2.7.

4.3. Approximation Properties for C^*-algebras

The main purpose of this section is to prove the following characterization of nuclear C^*-algebras.

Theorem 4.3.1. *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) *A is nuclear.*

(ii) *For every representation $\{\pi, \mathcal{H}\}$ of A, the map*

$$\sum_i x_i \otimes y_i \mapsto \sum_i \pi(x_i)y_i, \qquad \sum_i x_i \otimes y_i \in A \otimes \pi(A)' \tag{4.3.1}$$

extends to a representation of $A \otimes_{\min} \pi(A)'$.

(iii) *The identity map $1_A : A \to A$ is approximated by (unital) completely positive linear maps of finite ranks in the point-weak (point-norm) topology.*

(iv) *The identity map $1_{A^*} : A^* \to A^*$ is approximated by completely positive contractions of finite ranks in the point-weak* topology.*

We say that a C^*-algebra A has the *completely positive approximation property* (CPAP) if A satisfies one of the above approximation properties. We need some preparations for the proof of Theorem 4.3.1. For a vector space V, we denote by V^d the algebraic dual of V. For unital C^*-algebras A and B, put

$$\mathcal{S}(A \otimes B) = \{\phi \in (A \otimes B)^d : \phi(1) = 1,\ \phi(x^*x) \geq 0 \text{ for each } x \in A \otimes B\},$$

where 1 denotes $1_A \otimes 1_B \in A \otimes B$. The only one obstacle to get the G. N. S. construction π_ϕ associated with $\phi \in \mathcal{S}(A \otimes B)$ satisfying (1.5.5) is the lack of the inequality (1.5.4). This may be overcome by the following:

Lemma 4.3.2. *As a real subspace of $A \otimes B$, we have $A_h \otimes B_h = (A \otimes B)_h$. If $x \in (A \otimes B)_h$ then there is a positive number α such that $x \leq \alpha 1$.*

Proof. Note that $A_h \otimes B_h \subseteq (A \otimes B)_h$ is clear. If $x = \sum_i a_i \otimes b_i$ with $x^* = x$, then we have

$$\begin{aligned} x &= \frac{1}{2}(x + x^*) \\ &= \frac{1}{4}\sum_i ((a_i + a_i^*) \otimes (b_i + b_i^*) - i(a_i - a_i^*) \otimes i(b_i - b_i^*)) \in A_h \otimes B_h. \end{aligned}$$

For the second statement, note that if $a = a_1 - a_2 \in A_h$ and $b = b_1 - b_2 \in B_h$ with $a_j \in A_+$ and $b_j \in B_+$, for $j = 1, 2$, then

$$a \otimes b = (a_1 - a_2) \otimes (b_1 - b_2) \leq a_1 \otimes b_1 + a_2 \otimes b_2 \leq 2\|a\|\|b\|1.$$

The general case follows from the first statement. □

Now, it is easy to see that

$$\begin{aligned} \|x\|_{\min} &= \sup\{\|\pi_\phi(x)\| : \phi \in \mathcal{S}(A \otimes B) \cap (A^* \otimes B^*)\}, \\ \|x\|_{\max} &= \sup\{\|\pi_\phi(x)\| : \phi \in \mathcal{S}(A \otimes B)\}, \end{aligned} \tag{4.3.2}$$

for $x \in A \otimes B$. One of the basic reasons to consider the tensor product of vector spaces is the following easy correspondence between a linear functional $\phi \in (A \otimes B)^d$ and a linear transformation T_ϕ from B to A^d given by

$$T_\phi(b)(a) = \phi(a \otimes b), \qquad a \in A,\ b \in B. \tag{4.3.3}$$

Considering the order structures on A and B, we get the following proposition which plays an important rôle in the proof of Theorem 4.3.1.

Proposition 4.3.3. *Let A and B be C^*-algebras and $\phi \in (A \otimes B)^d$. Then $\phi \in \mathcal{S}(A \otimes B)$ if and only if T_ϕ is a completely positive linear map from B into A^* and $T_\phi(1_B)$ is a state of A. Also, the map $\phi \mapsto T_\phi$ is a homeomorphism* *from $\mathcal{S}(A \otimes B)$ to the set $CP(B, A^*)$ of all completely positive linear maps from B into A^* with respect to the weak^d-topology on $\mathcal{S}(A \otimes B)$ and the topology of simple weak* convergence on $\mathcal{B}(B, A^*)$.*

Proof. First of all, we have $\phi(1_A \otimes 1_B) = 1$ if and only if $T_\phi(1_B)(1_A) = 1$. If $\phi \in \mathcal{S}(A \otimes B)$ then we see that $T_\phi(B_+) \subseteq (A^d)_+ = (A^*)_+$ by Proposition 1.5.2. Hence, T_ϕ maps B into A^*. Now, the following relation

$$\phi((\sum_i a_i \otimes b_i)^*(\sum_i a_i \otimes b_i)) = \sum_{i,j} T_\phi(b_i^* b_j)(a_i^* a_j)$$

completes the proof of the first statement by the correspondence (4.2.7). The second one is also easy if we note that a net $\{\phi_i\}$ in $\mathcal{S}(A \otimes B)$ converges to ϕ in weak^d topology if and only if $\phi_i(a \otimes b) \to \phi(a \otimes b)$ for each $a \in A$ and $b \in B$ if and only if $T_{\phi_i}(a)(b) \to T_\phi(a)(b)$. □

Exercise 4.3.1. Give an explicit example to show that $(M_2)_+ \otimes (M_2)_+ \subsetneqq (M_2 \otimes M_2)_+$.

Now, we proceed to see the relation between the dual space A^* and the commutant $\pi(A)'$, for a representation π of A. For $\phi \in \mathcal{S}(A)$, we denote by $[\phi] \subseteq A^*$ the complex linear span of the cone

$$C_\phi = \{\psi \in A^* : 0 \leq \psi \leq a\phi \text{ for some } a > 0\}.$$

We define

$$\tilde{\phi}(x \otimes y) = \langle y\pi_\phi(x)\xi_\phi, \xi_\phi\rangle, \qquad x \in A,\ y \in \pi_\phi(A)', \tag{4.3.4}$$

where $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$ is the G. N. S. construction associated with ϕ. Then we have $\tilde{\phi} \in \mathcal{S}(A \otimes \pi_\phi(A)')$.

Proposition 4.3.4. *Let A be a unital C^*-algebra and $\phi \in \mathcal{S}(A)$. Then we have the following:*

(i) *$T_{\tilde{\phi}} : \pi_\phi(A)' \to [\phi]$ is a bijection.*
(ii) *$T_{\tilde{\phi}}$ and $(T_{\tilde{\phi}})^{-1}$ are completely positive.*

Proof. First of all, we see that $T_{\tilde{\phi}}$ is completely positive by Proposition 4.3.3. For $y \in \pi_\phi(A)'$ with $y \geq 0$, we have

$$\begin{aligned} T_{\tilde{\phi}}(y)(x^*x) = \langle y\pi_\phi(x^*x)\xi_\phi, \xi_\phi\rangle &= \|y^{\frac{1}{2}}\pi_\phi(x)\xi_\phi\|^2 \\ &\leq \|y\|\|\pi_\phi(x)\xi_\phi\|^2 = \|y\|\phi(x^*x), \end{aligned}$$

for each $x \in A$, and so $T_{\tilde{\phi}}(y) \in C_\phi$. Conversely, if $\psi \in C_\phi$ then the sesquilinear form

$$(\pi_\phi(x_1)\xi_\phi, \pi_\phi(x_2)\xi_\phi) \mapsto \psi(x_2^*x_1), \qquad x_1, x_2 \in A$$

is positive and bounded on $\pi_\phi(A)\xi_\phi$. So, there is a positive operator y on $\mathcal{H}_\phi$ such that

$$\psi(x_2^*x_1) = \langle y\pi_\phi(x_1)\xi_\phi, \pi_\phi(x_2)\xi_\phi\rangle, \qquad x_1, x_2 \in A. \tag{4.3.5}$$

Now, it is straightforward to see that

$$\langle y\pi_\phi(x_1)\pi_\phi(x_2)\xi_\phi, \pi_\phi(x_3)\xi_\phi\rangle = \langle y\pi_\phi(x_2)\xi_\phi, \pi_\phi(x_1^*)\pi_\phi(x_3)\xi_\phi\rangle,$$

for $x_1, x_2, x_3 \in A$, and so $y \in \pi_\phi(A)'$. Also, since

$$T_{\tilde{\phi}}(y)(x) = \langle y\pi_\phi(x)\xi_\phi, \xi_\phi\rangle = \psi(x), \qquad x \in A,$$

we see that $T_{\tilde{\phi}}$ maps onto $[\phi]$. It is clear that $T_{\tilde{\phi}}$ is one-to-one.

It remains to show that $(T_{\tilde{\phi}})^{-1}$ is completely positive. Let $\psi = [\psi_{ij}] \in M_n([\phi])_+$ and

$$\xi = (\pi_\phi(x_1)\xi_\phi, \ldots, \pi_\phi(x_n)\xi_\phi) \in \mathcal{H}_\phi \oplus \cdots \oplus \mathcal{H}_\phi.$$

Then we have

$$\begin{aligned} \langle((T_{\tilde{\phi}})_n^{-1}\psi)\xi, \xi\rangle &= \sum_{i,j}\langle(T_{\tilde{\phi}})^{-1}(\psi_{ij})\pi_\phi(x_j)\xi_\phi, \pi_\phi(x_i)\xi_\phi\rangle \\ &= \sum_{i,j}\psi_{ij}(x_i^*x_j) \geq 0, \end{aligned}$$

by (4.3.5) and the identification (4.2.7). □

Proof of Theorem 4.3.1. Throughout the proof, we assume that A acts on a Hilbert space $\mathcal{H}$. The implication (i) $\Longrightarrow$ (ii) is clear by Proposition 4.1.6. For the proof of (ii) $\Longrightarrow$ (iii), we assume that $x_1, \dots, x_n \in A$, $\phi_1, \dots, \phi_n \in A^*$ and $\epsilon > 0$ are given. It suffices to show that there is a unital completely positive linear map $V : A \to A$ of finite rank such that

$$|\phi_i(x_i) - \phi_i(Vx_i)| < \epsilon, \qquad i = 1, 2, \dots, n. \tag{4.3.6}$$

By Theorem 1.5.6, we may assume that each ϕ_i is positive and $\phi = \sum_{i=1}^n \phi_i$ is a state. Let $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$ be the G. N. S. construction associated with ϕ. Because $\phi_i \le \phi$, we apply Proposition 4.3.4 (i) to see that there is $y_i \in \pi_\phi(A)'$ such that

$$\phi_i(x) = \langle y_i \pi_\phi(x)\xi_\phi, \xi_\phi\rangle, \qquad x \in A.$$

By the condition (ii), the functional $\tilde{\phi}$ in (4.3.4) extends to a bounded linear functional on $A \otimes_{\min} \pi_\phi(A)'$, which is also denoted by $\tilde{\phi}$. By Theorem 1.5.5, there are vectors $\eta_1, \dots \eta_m \in \mathcal{H} \otimes \mathcal{H}_\phi$ such that

$$|\tilde{\phi}(x_i \otimes y_i) - \sum_{j=1}^m \langle (x_i \otimes y_i)\eta_j, \eta_j\rangle| < \epsilon, \qquad i = 1, 2, \dots, n. \tag{4.3.7}$$

Because ξ_ϕ is a cyclic vector, we may assume that $\eta_j \in \mathcal{H} \odot \pi_\phi(A)\xi_\phi$, and write

$$\eta_j = \sum_{k=1}^{p_j} \xi_{j,k} \otimes \pi_\phi(x_{j,k})\xi_\phi, \qquad j = 1, 2, \dots, m,$$

with $\xi_{j,k} \in \mathcal{H}$ and $x_{j,k} \in A$. Define linear maps V_j and V of A by

$$V_j(x) = \sum_{k,\ell=1}^{p_j} \langle x\xi_{j,k}, \xi_{j,\ell}\rangle x_{j,\ell}^* x_{j,k}, \qquad x \in A,$$

$$V = \sum_{j=1}^m V_j.$$

Now, we show that each V_j is completely positive. For the brevity, we write $W(x) = \sum_{k,\ell=1}^p \langle x\xi_k, \xi_\ell\rangle x_\ell^* x_k$. Then we have

$$\begin{aligned}\sum_{i,j=1}^q \langle W(x_j^* x_i)\xi_i, \xi_j\rangle &= \sum_{i,j}\sum_{k,\ell} \langle x_i\xi_k, x_j\xi_\ell\rangle\langle x_k\xi_i, x_\ell\xi_j\rangle \\ &= \langle \sum_{i,k} x_i\xi_k \otimes x_k\xi_i, \sum_{j,\ell} x_j\xi_\ell \otimes x_\ell\xi_j\rangle \ge 0,\end{aligned}$$

for $x_i, x_j \in A$ and $\xi_i, \xi_j \in \mathcal{H}$. Hence, each V_j is completely positive, and so V is also a completely positive linear map of finite rank. Also, it is straightforward to see that the relation (4.3.7) is nothing but the required inequality (4.3.6), because

$$\phi_i(x_i) = \langle y_i \pi_\phi(x_i)\xi_\phi, \xi_\phi\rangle = \tilde{\phi}(x_i \otimes y_i),$$

and

$$\begin{aligned}\sum_{j=1}^{m}\langle (x_i \otimes y_i)\eta_j, \eta_j\rangle &= \sum_{j=1}^{m}\sum_{k,\ell=1}^{p_j}\langle x_i\xi_{j,k}\otimes y_i\pi_\phi(x_{j,k})\xi_\phi, \xi_{j,\ell}\otimes \pi_\phi(x_{j,\ell})\xi_\phi\rangle \\ &= \sum_{j=1}^{m}\phi_i(V_j x_i) = \phi_i(Vx_i).\end{aligned}$$

Two topologies for approximations are irrelevant, because the set of all completely positive linear maps with finite ranks is convex. To get an approximation by unital completely positive linear maps, let $1_A = \lim_i V_i$ in the point-norm topology. Then we may assume that $\|V_i(1) - 1\| < 1$ for each i, and the required net is given by $W_i(x) = (V_i(1))^{-\frac{1}{2}}V_i(x)(V_i(1))^{-\frac{1}{2}}$ for $x \in A$. The implication (iii) $\Longrightarrow$ (iv) is easy if we consider the adjoints with Proposition 4.2.10.

For the direction (iv) $\Longrightarrow$ (i), let B be a C^*-algebra acting on a Hilbert space $\mathcal{K}$, and J the kernel of the $*$-homomorphism $A \otimes_{\max} B \to A \otimes_{\min} B$. Put

$$\mathcal{S}_0 = \{\phi \in \mathcal{S}(A \otimes_{\max} B) : \phi(x) = 0 \text{ for } x \in J\}.$$

Then, $\mathcal{S}_0$ is weak* closed in $\mathcal{S}(A \otimes_{\max} B)$, and so it suffices to show that $\mathcal{S}_0$ is weak* dense in $\mathcal{S}(A \otimes_{\max} B)$. From the condition (iv), we see that the set of all completely positive linear maps of finite ranks is simple-weak* dense in $CP(B, A^*)$. Applying Proposition 4.3.3, it also suffices to show that if $\phi \in \mathcal{S}(A \otimes_{\max} B) = \mathcal{S}(A \otimes B)$ and T_ϕ is of finite rank then $\phi \in \mathcal{S}_0$. But, every finite rank operator in $\mathcal{B}(B, A^*)$ is of the form $b \mapsto \sum_{i=1}^n g_i(b)f_i$ for some $f_i \in A^*$ and $g_i \in B^*$, $i = 1, 2, \ldots, n$. The equation

$$T_\phi(b)(a) = \sum_{i=1}^{n} g_i(b)f_i(a), \qquad a \in A,\ b \in B$$

implies that $\phi = \sum_i f_i \otimes g_i$, and so ϕ extends to a bounded linear functional on $A \otimes_{\min} B$. This shows that ϕ vanishes on J, and completes the proof. □

Now, we turn our attention to the question what happen if we replace the minimal tensor product in (4.1.5) by the maximal one. This question is closely related with the characterization of nuclearity in terms of universal enveloping von Neumann algebras, which will be the topic of the next section. We say that a unital C^*-algebra A is *injective* if for any unital C^*-algebras $B \subseteq B_1$ with the common identities and a completely positive unital linear map $\phi : B \to A$, there exists a completely positive unital linear map $\Phi : B_1 \to A$ which extends ϕ.

Proposition 4.3.5. *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) *For any C^*-algebras $B \subseteq B_1$, we have*

$$A \otimes_{\max} B \subseteq A \otimes_{\max} B_1. \tag{4.3.8}$$

(ii) *For each $\phi \in \mathcal{S}(A)$, the von Neumann algebra $\pi_\phi(A)'$ is injective.*

Proof. Noting that $\mathcal{S}(A \otimes_{\max} B) = \mathcal{S}(A \otimes B)$, we see that (4.3.8) is the case if and only if every element in $\mathcal{S}(A \otimes B)$ extends to an element of $\mathcal{S}(A \otimes B_1)$. Note that if $\Phi : B \to \pi_\phi(A)'$ is a unital completely positive linear map then $T_{\tilde{\phi}} \circ \Phi \in CP(B, A^*)$ and $(T_{\tilde{\phi}} \circ \Phi)(1_B)$ is a state of A. Also, note that if $\Phi \in CP(B, A^*)$ and $\Phi(1) \in \mathcal{S}(A)$ then $\Phi(b) \in [\Phi(1)]$ for each $b \in B$. The correspondence in Proposition 4.3.3 together with Proposition 4.3.4 (ii) completes the proof. □

In the next section, we will see that the conditions in Proposition 4.3.5 are actually equivalent to the nuclearity. We consider the same question on (4.1.5) in another direction as follows:

Proposition 4.3.6. *Let A be a unital C^*-algebra. Then the following are equivalent:*

(i) *For any C^*-algebras A_1 and B with $A \subseteq A_1$, we have*

$$A \otimes_{\max} B \subseteq A_1 \otimes_{\max} B. \tag{4.3.9}$$

(ii) *For every faithful representation* $\{\pi, \mathcal{H}\}$ *of* A*, there exists a completely positive contraction* $\Phi : \mathcal{B}(\mathcal{H}) \to \pi(A)''$ *such that* $\Phi(\pi(a)) = \pi(a)$ *for each* $a \in A$*.*

Proof. The map $\sum_i x_i \otimes y_i \mapsto \sum x_i y_i$ extends to a representation of $\pi(A) \otimes_{\max} \pi(A)'$ by Proposition 4.1.6. Assuming (i), there is a representation σ of $\mathcal{B}(\mathcal{H}) \otimes_{\max} \pi(A)'$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that

$$xy\xi = \sigma(x \otimes y)\xi, \qquad x \in \pi(A),\ y \in \pi(A)',\ \xi \in \mathcal{H},$$

by Theorem 1.6.7. We define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\Phi(x)\xi = P\sigma(x \otimes 1)\xi, \qquad x \in \mathcal{B}(\mathcal{H}),\ \xi \in \mathcal{H},$$

where P denotes the projection of $\mathcal{K}$ onto $\mathcal{H}$. Noting that $\mathcal{H}$ is invariant under $\sigma(\pi(A) \otimes \pi(A)')$, we have

$$\begin{aligned}\Phi(x)y\xi = P\sigma(x \otimes 1)\sigma(1 \otimes y)\xi &= P\sigma(1 \otimes y)\sigma(x \otimes 1)\xi \\ &= \sigma(1 \otimes y)P\sigma(x \otimes 1)\xi = y\Phi(x)\xi,\end{aligned}$$

for $\xi \in \mathcal{H}$, $x \in \mathcal{B}(\mathcal{H})$ and $y \in \pi(A)'$. Therefore, we see that $\Phi(x) \in \pi(A)''$. It is also easy to see that $\phi(\pi(a)) = \pi(a)$ for $a \in A$.

For the converse, we assume that A_1 acts on a Hilbert space $\mathcal{H}$ universally, and so, A'' may be identified with A^{**}. By the assumption, there is a completely positive contraction $\Phi : \mathcal{B}(\mathcal{H}) \to A^{**}$ such that $\Phi(x) = x$ for $x \in A$. We define

$$d = (\Phi|_{A_1})^* \circ 1_{A^*} : A^* \subseteq A^{***} \to A_1^*.$$

Then we see that

$$d(\phi)|_A = \phi, \qquad \phi \in A^*. \tag{4.3.10}$$

If $\phi \in \mathcal{S}(A \otimes B)$ then $d \circ T_\phi \in CP(B, A_1^*)$ and $(d \circ T_\phi)(1)$ is a state of A_1. We see that the corresponding $\bar{\phi} \in \mathcal{S}(A_1 \otimes B)$ with $T_{\bar{\phi}} = d \circ T_\phi$ is an extension of ϕ by (4.3.10), and this completes the proof. □

We say that a C^*-algebra A has the *weak expectation property* (WEP) if A satisfies the conditions in Proposition 4.3.6. Later, we will see that WEP does not imply the nuclearity.

NOTE

The direction (iv) $\Longrightarrow$ (i) of Theorem 4.3.1 is due to by Lance [**La73**] in which the terminology "nuclear" was introduced. Takesaki [**Ta64**] called that *property* T. The proof of the other direction (ii) $\Longrightarrow$ (iii) of Theorem 4.3.1 was taken from [**Ki77**]. Choi and Effros [**CE78**] also proved this independently and showed that CPAP is also equivalent to the following conditions:

(i) The identity map $1_A : A \to A$ is approximated by completely positive contractions of the form $A \to M_n \to A$ in the point-weak (point-norm) topology.

(ii) The identity map $1_{A^*} : A^* \to A^*$ is approximated by completely positive contractions of the form $A^* \to M_n \to A^*$ in the point-weak (point-norm) topology.

Later, R. Smith [**Sm85**] showed that the following condition which is seemingly weaker than CPAP is actually equivalent to the CPAP.

(iii) The identity map $1_A : A \to A$ is approximated by complete contractions of the form $A \to M_n \to A$ in the point-weak (point-norm) topology.

Propositions 4.3.5 and 4.3.6 are taken from [**EL77**] and [**La73**], respectively. We refer to [**Ef78**] or [**La82**] for a survey on this topic.

4.4. Injective von Neumann Algebras

Recall that a unital C^*-algebra A is injective if for any unital C^*-algebras $B \subseteq B_1$ with the common identities and a completely positive unital linear map $\phi : B \to A$, there exists a completely positive unital linear map $\Phi : B_1 \to A$ which extends ϕ. First of all, we show Arveson's extension theorem which says that $\mathcal{B}(\mathcal{H})$ is injective. This is easy if $\mathcal{H}$ is finite dimensional from the correspondences (4.3.3) and (4.2.7), because every positive linear functional on $M_n \otimes B = M_n(B)$ extends to a positive linear functional on $M_n(B_1)$. In order to extend this to the infinite dimensional cases, we introduce a topology on the space $\mathcal{B}(B, \mathcal{H})$ of all bounded linear maps from a C^*-algebra B into $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{T}(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ under the relation (2.2.11).

Under the correspondence (4.3.3), every element $\mathcal{T}(\mathcal{H}) \otimes B$ defines a bounded linear functional on $\mathcal{B}(B, \mathcal{H})$ by the duality

$$\langle x \otimes y, \phi \rangle = \phi(y)(x) = \mathrm{Tr}(\phi(y)x), \tag{4.4.1}$$

for $x \in \mathcal{T}(\mathcal{H})$, $y \in B$ and $\phi \in \mathcal{B}(B, \mathcal{H})$, because

$$|\langle x \otimes y, \phi \rangle| \le |\mathrm{Tr}(\phi(y)x)| \le \|\phi(y)\| \|x\|_{\mathrm{Tr}} \le \|\phi\| \|y\| \|x\|_{\mathrm{Tr}}$$

by (2.2.8). If we denote by Z the closure of $\mathcal{T}(\mathcal{H}) \otimes B$ in $\mathcal{B}(B, \mathcal{H})^*$ then we see that the relation (4.4.1) also defines an isometric isomorphism between Z^* and $\mathcal{B}(B, \mathcal{H})$. Indeed, because Z is a separating subset of $\mathcal{B}(B, \mathcal{H})^*$, we have an isometric embedding $\mathcal{B}(B, \mathcal{H}) \subseteq Z^*$. It is also easy to see that this embedding is onto. We call the weak* topology on $\mathcal{B}(B, \mathcal{H})$ arising in this way the *BW-topology*. The following lemma justify this name.

Lemma 4.4.1. *A bounded net $\{\phi_i\}$ in $\mathcal{B}(B, \mathcal{H})$ converges to ϕ in the BW-topology if and only if $\lim_i \langle \phi_i(y)\xi, \eta \rangle = \langle \phi(y)\xi, \eta \rangle$ for each $y \in B$ and $\xi, \eta \in \mathcal{H}$.*

Proof. Recall that $\mathcal{T}(\mathcal{H})$ is the closure of the linear span of rank one operators $\{x_{\xi,\eta} : \xi, \eta \in \mathcal{H}\}$ in (2.2.12). The lemma follows from the relation $\mathrm{Tr}(yx_{\xi,\eta}) = \langle y\xi, \eta \rangle$ for $y \in \mathcal{B}(\mathcal{H})$. □

By the Banach-Alaoglu theorem, the unit ball $\mathcal{B}_1(B, \mathcal{H})$ is compact in the BW-topology. It is also easy to see that $CP_1(B, \mathcal{H}) = CP(B, \mathcal{H}) \cap \mathcal{B}_1(B, \mathcal{H})$ is closed in $\mathcal{B}_1(B, \mathcal{H})$ by Lemma 4.4.1, and so it is also compact in the BW-topology.

Theorem 4.4.2. *The C^*-algebra $\mathcal{B}(\mathcal{H})$ is injective.*

Proof. We denote by $\mathcal{P}$ the net of all finite dimensional projections in $\mathcal{B}(\mathcal{H})$. Let $\phi : B \to \mathcal{B}(\mathcal{H})$ be a completely positive unital linear map, and $B \subseteq B_1$. Then the map $\phi_P(\cdot) = P\phi(\cdot)P^*$ has a completely positive extension $\psi_P : B_1 \to \mathcal{B}(\mathcal{H})$ for each $P \in \mathcal{P}$ by the above discussion, because $P\mathcal{H}$ is finite dimensional. From the compactness of $CP_1(B, \mathcal{H})$, there is a subnet of $\{\psi_P\}$ which converges to ψ in $CP_1(B, \mathcal{H})$. We show that ψ is the required extension of ϕ. Indeed, for $y \in B$ and $\xi, \eta \in \mathcal{H}$, we denote by P the projection onto the subspace spanned by ξ and η. Then for any $Q \ge P$, we have $\langle \phi(y)\xi, \eta \rangle = \langle \psi_Q(y)\xi, \eta \rangle$, and so this completes the proof by Lemma 4.4.1. □

In order to get a more convenient way to characterize injectivity, we need the notion of projection of norm one.

Theorem 4.4.3. *Let $B \subseteq A$ be C^*-algebras and $\pi : A \to B$ a projection of norm one. Then π is a conditional expectation.*

Proof. Considering the enveloping von Neumann algebras and the double adjoint of π, we may assume that A and B are von Neumann algebras. Let e be a projection of B and $x \in A$. Because the set of linear spans of projections in a von Neumann algebra is dense, it suffices to show

$$e\pi(x) = \pi(ex), \qquad \pi(x)e = \pi(xe). \tag{4.4.2}$$

Put $f = 1_A - e$. Then $f\pi(ex) = \pi(ex) - e\pi(ex) \in B$, and so $f\pi(f\pi(ex)) = f\pi(ex)$. Hence, for any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}(\lambda+1)^2\|f\pi(ex)\|^2 &= \|f\pi(ex) + \lambda f\pi(ex)\|^2 \\ &= \|f\pi(ex + \lambda f\pi(ex))\|^2 \\ &\leq \|ex + \lambda f\pi(ex)\|^2 \\ &\leq \|ex\|^2 + |\lambda|^2\|f\pi(ex)\|^2,\end{aligned}$$

because e and f are orthogonal. This shows that $f\pi(ex) = 0$, and so $\pi(ex) = e\pi(ex)$. The above argument goes well if we interchange e and f, and so we also have $e\pi(fx) = 0$, that is, $e\pi(x) = e\pi(ex)$. This shows the first equality of (4.4.2). We proceed to show that π is positive. Note that $1_B = \pi(1_B) = \pi(1_B 1_A) = 1_B\pi(1_A) = \pi(1_A)$. For any $\phi \in \mathcal{S}(B)$, we have

$$\|\phi \circ \pi\| \leq \|\phi\| = \phi(1_B) = \phi \circ \pi(1_A) \leq \|\phi \circ \pi\|,$$

and so, $\phi\circ\pi$ is a positive linear functional on A by Proposition 1.5.2. Therefore, π is a positive linear map, and so it is self-adjoint. Taking adjoints in the first equality in (4.4.2), we also get the second equality. □

Therefore, every projection of norm one is completely positive by Proposition 4.2.3, and so the following corollary is immediate.

Corollary 4.4.4. *Let A be a unital C^*-algebra acting on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) *A is injective.*

(ii) *There is a norm one projection from $\mathcal{B}(\mathcal{H})$ onto A.*

Exercise 4.4.1. Show that if a separable C^*-algebra A is injective then it is finite dimensional.

Nuclear C^*-algebras have the following second dual characterization as for the case of postliminal C^*-algebras in Theorem 2.3.6.

Theorem 4.4.5. *Let A be a C^*-algebra. Then the following are equivalent:*

(i) *A is nuclear.*

(ii) *The enveloping von Neumann algebra A^{**} is injective.*

The complete proof of this theorem is beyond the scope of this note. Instead, we explain the circumstances surrounding a proof of this theorem. First of all, we need the notion of standard von Neumann algebras arising from Tomita-Takesaki modular theory [**Ta70**]. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A vector $\xi \in \mathcal{H}$ is said to be a *tracing vector* for M if

$$\langle xy\xi, \xi\rangle = \langle yx\xi, \xi\rangle, \qquad x, y \in M.$$

Every von Neumann algebra arising from a discrete group has a tracing cyclic vector by (2.4.4). If M has a tracing cyclic vector ξ then we see that the map $x\xi \mapsto x^*\xi$ extends to a conjugate-linear isometry $J : \mathcal{H} \to \mathcal{H}$ with $J = J^{-1} = J^*$.

Exercise 4.4.2. Under the above situation, show that the mapping $x \mapsto Jx^*J$ is a $*$-anti-isomorphism from M onto M'.

In general, a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ is said to be *standard* if there is a conjugate-linear isometry $J : \mathcal{H} \to \mathcal{H}$ with $J = J^{-1} = J^*$ such that the mapping $x \mapsto Jx^*J$ is a $*$-anti-isomorphism from M onto M'. The Tomita-Takesaki theory says that every von Neumann algebra is $*$-isomorphic to a standard von Neumann algebra (see [**SZ**, §10.15] for example).

Theorem 4.4.6. *A von Neumann algebra M on a Hilbert space $\mathcal{H}$ is injective if and only if the commutant M' is injective.*

Proof. We assume that $M \subseteq \mathcal{B}(\mathcal{H})$ is standard and π is a norm one projection from $\mathcal{B}(\mathcal{H})$ onto M. Then the map $x \mapsto J\pi(Jx^*J)^*J$ is the norm one projection onto M'. □

Now, we see that if A is nuclear then the von Neumann algebra $\pi_\phi(A)''$ is injective for each $\phi \in \mathcal{S}(A)$, by Proposition 4.3.5 and Theorem 4.4.6. From this, we see that A^{**} is injective, because the direct sum of injective von Neumann algebras is again injective.

The proof of the converse would be complete by the second statement of Theorem 4.3.1, if we prove the following theorem.

Theorem 4.4.7. *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be an injective von Neumann algebra. Then we have*

$$\|\sum_{i=1}^n x_i y_i\| \leq \|\sum_{i=1}^n x_i \otimes y_i\|_{\min}, \qquad x_i \in M,\ y_i \in M'.$$

The proof of this theorem is also beyond the scope of this note, because it involves much more machinery on von Neumann algebras than we have ever developed. For example, we need the Takesaki duality theorem [**Ta73**]. From this theorem, the proof of Theorem 4.4.7 is reduced to the case of finite von Neumann algebra, which has always a faithful trace. We close this section with the following important application of Theorem 4.4.5.

Proposition 4.4.8. *Let A be a C^*-algebra with a two-sided norm-closed ideal I. Then A is nuclear if and only if I and A/I is nuclear.*

The proof follows from the relation $A^{**} = I^{**} \oplus (A/I)^{**}$. We will see later that a C^*-subalgebra of a nuclear C^*-algebra need not to be nuclear by showing that $C^*_\lambda(F_2)$ may be embedded in a nuclear C^*-algebra (see §5.3).

NOTE

For the more general aspects of injectivity, we refer to [**CE77a**]. Theorem 4.4.2 is due to Arveson [**Ar69**] in which he also introduced the BW-topology. We have followed Paulsen's book [**Pau**, §6] for the proof. Note that a positive linear map may not have a positive extension [**Pau**, Example 2.13]. Theorem 4.4.3 is due to Tomiyama [**To57**], a slightly simpler proof here is taken from [**Str**, Theorem 9.1]. Von Neumann algebras with the second property of Corollary 4.4.4 have been studied in [**HT67**]. See also Tomiyama's lecture note [**To70**] in which Theorem 4.4.6 appears. Theorem 4.4.7 is due to a fundamental work by Connes [**Co76**], in which he actually showed that a factor acting on a separable Hilbert space is injective if and only if it is hyperfinite. For another simple proofs of this theorem and related topics, we refer to [**CE76a**], [**CE77b**], [**Co79**], [**Ha85b**], [**Pop86**] and [**Wa77a**]. Tomiyama [**To70**] studied C^*-algebras whose second dual satisfies the second condition of Corollary 4.4.4 and proved Proposition 4.4.8. See also [**CE76a**] and [**CE77b**]. From the discussion in this section, we see that the statements in Theorem 4.3.1, Proposition

4.3.5 and Theorem 4.4.5 are all equivalent, and they imply the WEP of Proposition 4.3.6. Clearly, every injective C^*-algebra has the WEP, but we will see in §4.7 that $\mathcal{B}(\mathcal{H})$ is not nuclear.

4.5. Amenable Groups

Let G be a locally compact topological group. Throughout this section, we always denote by ds the left invariant Haar measure. For each $s \in G$, we define $\lambda_s : L^\infty(G) \to L^\infty(G)$ by

$$\lambda_s f(t) = f(s^{-1}t), \qquad t \in G. \tag{4.5.1}$$

A state m on the C^*-algebra $L^\infty(G)$ is said to be a *mean* of G, and the set of all means of G will be denoted by $\mathfrak{M}(G)$. A locally compact group G is said to be *amenable* if there is a left invariant mean m in the sense that $m(f) = m(\lambda_s f)$ for each $s \in G$ and $f \in L^\infty(G)$. The proof of the following proposition suggests a method to produce a left invariant mean on a group.

Proposition 4.5.1. *Every abelian group G is amenable.*

Proof. Note that the group G acts on $\mathfrak{M}(G)$ by

$$(s \cdot m)(f) = m(\lambda_s f), \qquad s \in G,\ m \in \mathfrak{M}(G),\ f \in L^\infty(G). \tag{4.5.2}$$

Taking any $m \in \mathfrak{M}(G)$ and putting

$$m_n = \frac{1}{n+1} \sum_{r=0}^{n} s^r \cdot m, \qquad n = 1, 2, \dots, \tag{4.5.3}$$

we see that $\{m_n\}$ has a weak* limit point $m_0 \in \mathfrak{M}(G)$. For any $f \in L^\infty(G)$, we also have

$$|(s \cdot m_n)(f) - m_n(f)| = \frac{1}{n+1} |(s^{n+1} \cdot m)(f) - m(f)| \le \frac{2}{n+1} \|f\|_\infty,$$

and so we see that the fixed point set

$$F_s = \{m \in \mathfrak{M}(G) : m(f) = m(\lambda_s f),\ f \in L^\infty(G)\}$$

is nonempty for each $s \in G$. From the commutativity of G, we see that $t \cdot F_s \subseteq F_s$, and by the same argument as above, the transform $t|_{F_s}$ also has a fixed point. Hence, $F_t \cap F_s \neq \emptyset$, and so the family $\{F_s : s \in G\}$ has the finite intersection property. Therefore, any element of $\bigcap_{s\in G} F_s$ is a left invariant mean. □

Note that every left invariant mean m on G determines a left invariant finitely additive measure m (with the same notation) on the ds-measurable subsets of G by $m(E) = m(\chi_E)$ because $\chi_{sE} = \lambda_s \chi_E$. It is also easy to see that every left invariant finitely additive measure with $m(G) = 1$ arises in this way. Therefore, the Haar measure on a compact group is a left invariant mean in itself, and we see that every compact group is amenable.

Proposition 4.5.2. *The free group F_2 is not amenable.*

Proof. For each $x = a, a^{-1}, b, b^{-1}$, where a, b are the generators of F_2, we denote by E_x the set of all words beginning with x. Assume that there is a left invariant mean m on F_2, considered as a finitely additive measure. Then, we have $m(E_a) + m(E_{a^{-1}}) = m(E_a) + m(aE_{a^{-1}}) = m(F_2) = 1$, and similarly for b. Therefore, it follows that

$$1 = m(F_2) = m(\{e\}) + m(E_a) + m(E_{a^{-1}}) + m(E_b) + m(E_{b^{-1}}) \geq 2,$$

a contradiction. □

We put $\mathfrak{S}(G) = \{g \in L^1(G) : g \geq 0, \|g\|_1 = 1\}$. Then the convex set $\mathfrak{S}(G)$ may be considered as a weak* dense subset of $\mathfrak{M}(G)$ by the canonical embedding $L^1(G) \subseteq L^\infty(G)^*$. We also denote by $\mathfrak{K}(G)$ the set of all continuous functions on G with compact supports. We define the transformation $f \mapsto \widetilde{f}$ by

$$\widetilde{f}(s) = \overline{f(s^{-1})}, \qquad s \in G. \tag{4.5.4}$$

Compare this with the definition of f^* in (1.5.8). We note that

$$(f * \widetilde{g})(s) = \langle f, \lambda_s g\rangle = \overline{\langle \lambda_s g, f\rangle} = \langle \lambda_s \overline{g}, \overline{f}\rangle, \qquad s \in G,\ f, g \in L^2(G). \tag{4.5.5}$$

Theorem 4.5.3. *Let G be a locally compact group. Then the following are equivalent:*

(i) *G is amenable.*

(ii) *There is a net* $\{g_i\}$ *in* $\mathfrak{S}(G)$ *such that* $\|\lambda_s g_i - g_i\|_1 \to 0$ *for each* $s \in G$.

(iii) *There is a net* $\{f_i\}$ *in* $L^2(G)$ *with* $\|f_i\|_2 = 1$ *such that* $f_i * \widetilde{f_i} \to 1_G$ *pointwise.*

(iv) *There is a net* $\{k_i\}$ *in* $\mathfrak{K}(G)$ *with* $\|k_i\|_2 = 1$ *such that* $k_i * \widetilde{k_i} \to 1_G$ *pointwise.*

Proof. Let m be a left invariant mean of G. By the above discussion, there is a net $\{h_i\}$ in $\mathfrak{S}(G)$ such that $\langle f, h_i\rangle \to m(f)$ for each $f \in L^\infty(G)$. Because $\langle \lambda_{s^{-1}} f, h_i\rangle = \langle f, \lambda_s h_i\rangle$ we see that $\lambda_s h_i - h_i \to 0$ weakly. Hence, we can take a net $\{g_i\}$ in the convex combination of $\{h_i\}$ with the desired property in (ii). Conversely, if there is a net $\{g_i\}$ with $\|\lambda_s g_i - g_i\|_1 \to 0$ for each $s \in G$, then $\lambda_s g_i - g_i \to 0$ in $\mathfrak{M}(G)$ with respect to the weak* topology. Hence, any weak* limit point of $\{g_i\}$ is a left invariant mean.

For $g \in \mathfrak{S}(G)$, let $f \in L^2(G)$ be the pointwise square root of g. Then, we calculate

$$\begin{aligned}|1 - (f * \widetilde{f})(s)|^2 &= |(f * \widetilde{f})(e) - (f * \widetilde{f})(s)|^2 = |\langle f, f - \lambda_s f\rangle|^2 \\ &\le \|f - \lambda_s f\|_2^2 = \int |\sqrt{g(t)} - \sqrt{g(s^{-1}t)}|^2 dt \le \|g - \lambda_s g\|_1,\end{aligned}$$

for each $s \in G$. On the other hand, if $f \in L^2(G)$ with $\|f\|_2 = 1$ then $g = |f|^2 \in \mathfrak{S}(G)$, and for each $s \in G$ we also calculate

$$\begin{aligned}\|g - \lambda_s g\|_1 &= \langle |f| + \lambda_s|f|, |f| - \lambda_s|f|\rangle \\ &\le 2\|f - \lambda_s f\|_2 = 2(2 - 2\,\mathrm{Re}\,\langle f, \lambda_s f\rangle)^{\frac{1}{2}} \\ &\le 2\sqrt{2}\,|1 - \langle f, \lambda_s f\rangle|^{\frac{1}{2}} = 2\sqrt{2}\,|1 - (f * \widetilde{f})(s)|^{\frac{1}{2}}.\end{aligned}$$

These two inequalities give the proof of (ii) $\iff$ (iii).

It remains to show implication (iii) $\implies$ (iv). Note that every $f \in L^2(G)$ is approximated by $\{k_i\}$ in $\mathfrak{K}(G)$. From this, it is easy to see that $k_i * \widetilde{k_i} \to f * \widetilde{f}$ uniformly on G. $\square$

Note that the relation (4.5.1) defines a unitary representation $s \mapsto \lambda_s$ of G on the Hilbert space $L^2(G)$ which is continuous with respect to the strong operator topology. Note that a continuous unitary representation $\{\pi, \mathcal{H}\}$ of G

induces a non-degenerate representation π (we use the same notation) of the Banach algebra $L^1(G)$ by the relation

$$(4.5.6)\qquad \langle\pi(x)\xi,\eta\rangle = \int_G x(s)\langle\pi(s)\xi,\eta\rangle ds \qquad s\in G,\ x\in L^1(G),\ \xi,\eta\in\mathcal{H}.$$

Conversely, a non-degenerate representation π of $L^1(G)$ gives an unitary representation of G by

$$\pi(s):\pi(x)\xi\mapsto\pi(\lambda_s x)\xi,\qquad s\in G,\ x\in L^1(G).$$

The representation $s\mapsto\lambda_s$ of G corresponds to the left regular representation of $L^1(G)$ in (1.5.9) in this way.

Proposition 4.5.4. *Let $\phi\in L^\infty(G)$ be a continuous function. Then the following are equivalent:*

(i) $\langle\phi,x^* * x\rangle\geq 0$ *for each* $x\in L^1(G)$.

(ii) *There exist a continuous unitary representation* $\{\pi,\mathcal{H}\}$ *and a vector* $\xi\in\mathcal{H}$ *such that*

$$(4.5.7)\qquad \phi(s)=\langle\pi(s)\xi,\xi\rangle,\qquad s\in G.$$

Proof. The condition (i) says that ϕ is a positive linear functional on $L^1(G)$. Hence, with the G. N. S. construction $\{\pi,\mathcal{H},\xi\}$ of $L^1(G)$ associated with ϕ, we have

$$\int_G\phi(s)x(s)ds=\langle\phi,x\rangle=\langle\pi(x)\xi,\xi\rangle=\int_g x(s)\langle\pi(s)\xi,\xi\rangle ds,\qquad x\in L^1(G),$$

from which we infer the relation (4.5.7). Conversely, if ϕ is given by (4.5.7) then

$$\langle\phi,x^* * x\rangle=\int_G(x^* * x)(s)\langle\pi(s),\xi,\xi\rangle ds=\langle\pi(x^* * x)\xi,\xi\rangle\geq 0,$$

for each $x\in L^1(G)$. □

A continuous L^∞-function with the properties in Proposition 4.5.4 is said to be a *positive definite* function on G; we denoted by $\mathfrak{P}(G)$ the set of all continuous positive definite functions on G.

Exercise 4.5.1. Show that a continuous function ϕ is positive definite if and only if the $n \times n$ matrix $[\phi(s_i^{-1}s_j)]_{i,j=1}^n$ is positive semi-definite for each $n = 1, 2, \dots$ and $s_i, s_j \in G$. If ϕ, ψ are positive definite then the pointwise product $\phi\psi$ is also positive definite. Show that $\|\phi\|_\infty = \phi(e) = \|\xi\|_2^2$ in (4.5.7). Finally, show that $\widetilde{\phi} = \phi$ for $\phi \in \mathfrak{P}(G)$.

Every $g \in \mathfrak{K}(G)$ defines a bounded linear operator $\rho(g)$ on $L^2(G)$ by

$$\rho(g)h = h * g, \qquad h \in L^2(G). \tag{4.5.8}$$

Indeed, it is easy to see that $\|h * g\|_2 \le \|\delta^{-\frac{1}{2}} g\|_1 \|h\|_2$. From the following relation

$$\langle g, f^* * f\rangle = \langle f * g, f\rangle = \langle \rho(g)f, f\rangle, \qquad f, g \in \mathfrak{K}(G), \tag{4.5.9}$$

we see that $g \in \mathfrak{P}(G)$ if and only if $\rho(g)$ is a positive linear operator. The following relations

$$\rho(g)\rho(f) = \rho(f * g), \qquad \rho(g)^* = \rho(\widetilde{g}), \qquad f, g \in \mathfrak{K}(G) \tag{4.5.10}$$

are also easy, and from the first relation of these, we see $P(\rho(g)) = \rho(P(g))$ for a polynomial P, where we take the convolution as the multiplication in the definition of $P(g)$.

We will write $f \ll g$ if $g - f$ is positive definite. If $f, g \in \mathfrak{K}(G) \cap \mathfrak{P}(G)$ then from the relations in (4.5.10), we see that $f * \widetilde{g} = f * g$ is also a positive definite, and so it follows that

$$\langle f, g\rangle = \int_G f(s)\widetilde{g}(s^{-1})ds = (f * \widetilde{g})(e) \ge 0.$$

Therefore, if $0 \ll f \ll g$ in $\mathfrak{K}(G)$ then we have

$$\begin{aligned} \|g - f\|_2^2 &= \langle g, g\rangle + \langle f, f\rangle - 2\langle f, g\rangle \\ &= \langle g, g\rangle - 2\langle f, g - f\rangle - \langle f, f\rangle \le \|g\|_2^2 - \|f\|_2^2. \end{aligned} \tag{4.5.11}$$

Note that the relation (4.5.5) says that $\xi * \widetilde{\xi} \in \mathfrak{P}(G)$ for each $\xi \in L^2(G)$. The next proposition deals with the converse.

Proposition 4.5.5. *For each $g \in \mathfrak{K}(G) \cap \mathfrak{P}(G)$ there exists $\xi \in L^2(G)$ such that $g = \xi * \widetilde{\xi}$.*

Proof. We may assume that $0 \leq \rho(g) \leq 1$. Let $\{P_n\}$ be an increasing sequence of polynomials converging to the function $\sqrt{t}$ on $[0,1]$. We denote by $g_i = P_i(g)$, then $\rho(g_i) = P_i(\rho(g)) \nearrow \rho(g)^{\frac{1}{2}}$ in norm, by the continuous function calculus. Therefore, we see that $\{g_i\}$ is an increasing sequence with respect to the order $\ll$. From the relation $\rho(g_i)^2 \leq \rho(g)$, we also see that $g_i * g_i \ll g$, and so, $\|g_i\|_2 = g_i * g_i(e) \leq g(e)$ for each $i = 1, 2, \ldots$. By the relation (4.5.11), there exists $\xi \in L^2(G)$ such that $\|g_i - \xi\|_2 \to 0$. Now, we have

$$h * \xi = \lim_i h * g_i = \lim_i \rho(g_i)h = \rho(g)^{\frac{1}{2}} h,$$

in L^2-norm for each $h \in \mathfrak{K}(G)$, hence for each $h \in L^2(G)$. Applying this once more, we see that $h * \xi * \xi = \rho(g)h = h * g$ for each $h \in L^2(G)$. From this, we finally have $g = \xi * \xi = \xi * \widetilde{\xi}$, as was desired. □

In the above proof, we have shown that $\rho(\xi)$ may define a bounded linear operator on $L^2(G)$, even though ξ is not an element of $\mathfrak{K}(G)$.

Recall that the group C^*-algebra $C^*(G)$ is the completion of $L^1(G)$ with respect to the norm in (1.5.7);

$$\|x\|_c = \sup\{\|\pi(x)\|\}, \qquad x \in L^1(G),$$

where π runs through all representations of $L^1(G)$.

Proposition 4.5.6. *The restriction $\phi \mapsto \phi|_{L^1(G)}$ for $\phi \in C^*(G)^*$ defines an isometric isomorphism from $C^*(G)^*_+$ onto $L^1(G)^*_+$.*

Proof. It is clear that every positive linear functional of $L^1(G)$ extends to the whole $C^*(G)$. If $x \in L^1(G)$ is a nonnegative function, then considering the one dimensional trivial representation ι of G, we see that

$$\|x\|_1 = \int_G x(s)ds = \iota(x) \leq \|x\|_c \leq \|x\|_1.$$

Hence, it follows that the approximate identity $\{\epsilon_i\}$ of $L^1(G)$ is also an approximate identity for $C^*(G)$ with $\|\epsilon_i\|_1 = \|\epsilon_i\|_c = 1$. The conclusion follows from the general fact that if ϕ is a positive linear functional of an involutive Banach algebra with an approximate identity $\{\epsilon_i\}$ then

$$\|\phi\| = \lim_i \phi(\epsilon_i^* \epsilon_i). \quad \square \tag{4.5.12}$$

Therefore, we may identify $C^*(G)^*_+ = \mathfrak{P}(G)$. It is clear that the left regular representation λ of $L^1(G)$ extends to a representation of $C^*(G)$ whose image is the reduced group C^*-algebra $C^*_\lambda(G)$. From Theorem 1.5.5, it is easy to see that $C^*_\lambda(G)^*_+$ is identified with the subset $\mathfrak{P}_\lambda(G)$ of $\mathfrak{P}(G)$ consisting of the weak* limits of the positive cone generated by $\{\xi * \widetilde{\xi} : \xi \in \mathfrak{K}(G)\}$, which are positive definite functions associated with the left regular representation in the sense of (4.5.7). From Proposition 4.5.5, we see that $\mathfrak{P}_\lambda(G)$ is nothing but the weak* closure of $\{\xi * \widetilde{\xi} : \xi \in L^2(G)\}$ or $\{k * \widetilde{k} : k \in \mathfrak{K}(G)\}$. Now, we are ready to prove the main theorem of this section.

Theorem 4.5.7. *A locally compact group G is amenable if and only if $C^*(G) = C^*_\lambda(G)$.*

Proof. First, we assume that G is amenable and let ϕ be a state of $C^*(G)$. It suffices to show that $|\langle \phi, x\rangle| \le \|\lambda(x)\|$ for each $x \in L^1(G)$, from which we infer that $\|x\|_c \le \|\lambda(x)\|$. By Theorem 4.5.3, we choose a net $\{k_i\}$ in $\mathfrak{K}(G)$ with $\|k_i\|_2 = 1$ such that $k_i * \widetilde{k_i} \to 1$. Because $\phi \in \mathfrak{P}(G)$, there is $\xi_i \in L^2(G)$ such that $(k_i * \widetilde{k_i})\phi = \xi_i * \widetilde{\xi_i}$ by Proposition 4.5.5, and so we have $\xi_i * \widetilde{\xi_i} \to \phi$. Now, by the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} |\langle \phi, x\rangle| &= \lim_i \Big| \int_G x(s)(\xi_i * \widetilde{\xi_i})(s)ds \Big| \\ &= \lim_i \Big| \int_G x(s)\langle \lambda_s \overline{\xi_i}, \overline{\xi_i}\rangle ds \Big| = \lim_i |\langle \lambda(x)\overline{\xi_i}, \overline{\xi_i}\rangle| \le \|\lambda(x)\|. \end{aligned}$$

For the converse, we assume that $C^*(G) = C^*_\lambda(G)$. Note that $1_G \in L^\infty(G)$ is a positive definite function on G, that is, $1_G \in \mathfrak{P}(G) = \mathfrak{P}_\lambda(G)$. From the above discussion, there is a net $\{k_i\}$ in $\mathfrak{K}(G)$ such that $k_i * \widetilde{k_i} \to 1_G$ in the weak* topology. We may assume that $\|k_i\|_2 = (k_i * \widetilde{k_i})(e) = 1$. Note that

$$\begin{aligned} |(k_i * \widetilde{k_i})(s) - (k_i * \widetilde{k_i})(t^{-1}s)|^2 &= |\langle \lambda_s k_i, k_i - \lambda_t k_i\rangle|^2 \\ &\le \|k_i - \lambda_t k_i\|_2^2 = 2 - 2\,\mathrm{Re}\,(k_i * \widetilde{k_i})(t). \end{aligned}$$

Hence, for a compact neighborhood V of e with measure one, we have

$$\begin{aligned}|(\chi_V * k_i * \widetilde{k_i})(s) - (k_i * \widetilde{k_i})(s)| &= |\int_V ((k_i * \widetilde{k_i})(t^{-1}s) - (k_i * \widetilde{k_i})(s))dt| \\ &\leq \sqrt{2} \int_V (1 - \mathrm{Re}\,(k_i * \widetilde{k_i})(t))^{\frac{1}{2}} dt \\ &\leq \sqrt{2}\,(\mathrm{Re} \int_V (1 - (k_i * \widetilde{k_i})(t))dt)^{\frac{1}{2}} \to 0,\end{aligned}$$

as $i \to \infty$, from the weak* convergence. On the other hand, we also have

$$\begin{aligned}1 - (\chi_V * k_i * \widetilde{k_i})(s) &= \int_V (1 - (k_i * \widetilde{k_i})(t^{-1}s))dt \\ &= \int_V (1 - (k_i * \widetilde{k_i})(r))\delta(r)^{-1} dr \to 0,\end{aligned}$$

as $i \to \infty$. Therefore, $(k_i * \widetilde{k_i})(s) \to 1$ for each $s \in G$, and this completes the proof by Theorem 4.5.3. □

From now on, we restrict our attention to *discrete* groups, and relate the notion of amenability to that of nuclearity of the group C^*-algebras.

Theorem 4.5.8. *Let G be a discrete group. Then the following are equivalent:*

(i) *The group G is amenable*
(ii) *The C^*-algebra $C^*(G)$ is nuclear.*
(iii) *The C^*-algebra $C^*_\lambda(G)$ is nuclear.*
(iv) *The von Neumann algebra $\mathcal{R}_\lambda(G)$ is injective.*

Proof. Every $\phi \in \mathfrak{P}(G)$ defines a linear map $\rho \mapsto \phi\rho$ between $\mathfrak{P}(G)$ by Exercise 4.5.1, and this extends to a linear map $V_\phi : C^*(G)^* \to C^*(G)^*$. If $\phi(e) = 1$ then it is easy to see that the adjoint ${V_\phi}^* : C^*(G)^{**} \to C^*(G)^{**}$ is a unital positive linear map, and so $\|{V_\phi}^*\| = 1$ by Theorem 4.2.7, from which we infer that V_ϕ is a contraction. Also, it is easy to see that if ϕ has a finite support then V_ϕ is of finite rank, and $\phi_i \to 1_G$ implies $V_\phi \to 1_{C^*(G)^*}$ in the simple-weak* topology. Now, if G is amenable then there exists a net $\{\phi_i\}$ in $\mathfrak{P}(G) \cap \mathfrak{K}(G)$ with $\phi_i(e) = 1$ such that $\phi_i \to 1_G$ by Theorem 4.5.3. In order to show that $C^*(G)$ satisfies the condition (iv) of Theorem 4.3.1, it suffices to show that V_ϕ is completely positive.

Let $[\rho_{ij}] \in M_n(\mathfrak{P}(G))_+$. It suffices to show that $\sum_{i,j} V_\phi(\rho_{ij})(x_i^* x_j) \geq 0$ for $x_i \in C^*_\lambda(G)$. We may assume that $x_i = \sum_s a_{is}L_s$, where $\sum_s$ is a finite sum over $s \in G$, and so

$$x_i^* x_j = \sum_{s,t} \overline{a_{is}} a_{jt} L_{s^{-1}t}.$$

Note that $L_s \in C^*_\lambda(G)$ is nothing but χ_s as an element of $L^1(G)$ and $\langle \rho, \chi_s \rangle = \rho(s)$. Therefore, we have

$$\begin{aligned}\sum_{i,j} V_\phi(\rho_{ij})(x_i^* x_j) &= \sum_{i,j,s,t} \overline{a_{is}} a_{jt} \phi(s^{-1}t) \rho_{ij}(s^{-1}t) \\ &= \sum_{s,t} \phi(s^{-1}t)[\sum_{i,j} \overline{a_{is}} a_{jt} \rho_{ij}(s^{-1}t)] \geq 0,\end{aligned}$$

because $[\phi(s^{-1}t)]$ and $[\sum_{i,j} \overline{a_{is}} a_{jt} \rho_{ij}(s^{-1}t)]$ are positive semi-definite matrices by Exercise 4.5.1.

The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) follow from Proposition 4.4.8 and Proposition 4.3.5 together with Theorem 4.4.6.

It remains to show (iv) $\Longrightarrow$ (i). Let $\pi : \mathcal{B}(\ell^2(G)) \to \mathcal{R}_\lambda(G)$ be the norm one projection. Every $f \in \ell^\infty(G)$ defines a multiplication operator M_f in $\mathcal{B}(\ell^2(G))$ in the usual manner. Then we have

$$\pi(M_{\lambda_s f}) = \pi(L_s M_f L_s^*) = L_s \pi(M_f) L_s^*,$$

by Theorem 4.4.3. Now, we define a state m of $\ell^\infty(G)$ by

$$m(f) = \langle \pi(M_f)\chi_e, \chi_e \rangle, \qquad f \in \ell^\infty(G).$$

Then we have

$$m(\lambda_s f) = \langle \pi(M_f) L_s^* \chi_e, L_s^* \chi_e \rangle = \langle \pi(M_f)\chi_{s^{-1}}, \chi_{s^{-1}} \rangle = m(f),$$

by (2.4.5), and so m is a left invariant mean, as was desired. □

We conclude this section with an another characterization of amenability for finitely generated discrete group G, using the spectrum of an element of $C^*_\lambda(G)$ as in the proof of Lemma 4.1.13. Following is a generalization of Exercise 4.1.5.

Exercise 4.5.2. Let $\mathcal{H}$ be an inner product space with a unit vector ξ. Show that if $\|\eta_i\| \leq 1$ for $i = 1, 2, \ldots, n$ and $\|\xi - \frac{1}{n}(\eta_1 + \eta_2 + \cdots + \eta_n)\| \leq \varepsilon$ then $\|\xi - \eta_i\| \leq \sqrt{2n\varepsilon}$ for $i = 1, 2, \ldots, n$.

The following lemma is immediate from Theorem 4.5.3.

Lemma 4.5.9. *Let G be a discrete group. Then G is amenable if and only if given $\epsilon > 0$ and a finite subset F of G there is a unit vector $\xi \in \ell^2(G)$ such that*

$$\|L_s\xi - \xi\|_2 < \epsilon, \qquad s \in F.$$

Proposition 4.5.10. *Let G be a discrete group with a finite set A of generators with the symmetric condition: $a \in A \Longrightarrow a^{-1} \in A$. Let x be the self-adjoint element of $C^*_\lambda(G)$ given by*

$$x = \frac{1}{\#A}\sum_{a\in A} L_a. \tag{4.5.13}$$

Then G is amenable if and only if $1 \in \mathrm{sp}(x)$.

Proof. If G is amenable then there is $\xi \in \ell^2(G)$ with $\|\xi\| = 1$ such that $\|L_a\xi - \xi\| < \epsilon$ for each $a \in A$. Then we have

$$\|x\xi - \xi\| \le \frac{1}{\#A}\sum_{a\in A}\|L_a\xi - \xi\| < \epsilon,$$

and so $x - 1$ is singular. Conversely, if $x - 1$ is singular then there is a unit vector $\xi \in \ell^2(G)$ such that $\|x\xi - \xi\| < \epsilon$. By Exercise 4.5.2, we have

$$\|L_a\xi - \xi\| < \sqrt{2(\#A)\epsilon}, \qquad a \in A.$$

If $s = a_1a_2\dots a_n$ with $a_i \in A$, then we also have

$$\|L_s\xi - \xi\| \le \sum_{j=1}^{n}\|L_{a_1}\dots L_{a_{j-1}}(L_{a_j}\xi - \xi)\| < n\sqrt{2(\#A)\epsilon}.$$

Noting that the last number depends only on the length of $s = a_1\dots a_n$, the proof is complete by Lemma 4.5.9. □

NOTE

Every material in this section except Theorem 4.5.8 and Proposition 4.5.10 may be found in monographs such as [**D**], [**Grf**], [**Pat**], [**P**] or [**Pi**]. We refer to Paterson's book [**Pat**] for more general accounts of amenability and relations between another fields of mathematics. Through a close examination of the latter part of Theorem 4.5.7, we see that $\xi_i * \tilde{\xi}_i \to 1$ in the weak* topology implies, in fact, the convergence in the compact-open topology (see [**D**, Theorem 13.5.2]). Therefore, every convergences in Theorem 4.5.3 may be replaced by the compact-open topology. Proposition 4.5.6 was taken from [**Fe60**, Lemma 1.4]. Theorem 4.5.8 is due to Lance [**La73**]. Although the one direction (i) $\Longrightarrow$ (ii) holds for general locally compact groups (see **NOTE** of §5.1), the converse (ii) $\Longrightarrow$ (i) is not true in general. We refer to [**Pat88**] for more information in this direction. Proposition 4.5.10 was taken from [**dHRVa**], and will be useful in the next section.

4.6. Group C^*-algebras of the Free Groups

We have already met before several times the operator algebras associated with the free groups. First of all, the von Neumann algebra $\mathcal{R}_\lambda(F_2)$ is a factor of type II_1 with a faithful trace τ given by (2.4.5). Because F_2 is not amenable, we see that the natural quotient map $\lambda : C^*(F_2) \to C^*_\lambda(F_2)$, induced by the left regular representation, has a nontrivial kernel by Theorem 4.5.7. Finally, neither $C^*(F_2)$ nor $C^*_\lambda(F_2)$ are nuclear by Theorem 4.5.8, or by Example 4.1.1 together with Proposition 4.4.8. In this section, we will first show that $C^*_\lambda(F_2)$ is a simple C^*-algebra with a unique faithful trace. We denote by a and b the generators of F_2.

Let F be a finite subset of $F_2 \setminus \{e\}$. Then there exists an integer n such that the reduced word $b^n s b^{-n}$ begins and ends with a non-zero power of b for each $s \in F$. We denote by A the set of all reduced words in F_2 beginning with b^{-n} and put $B = F_2 \setminus A$. For an arbitrary integer N, we denote by

$$t_k = a^k b^n, \qquad k = 1, 2, \dots, N.$$

Then we have the following two crucial properties:

(I) $sA \cap A = \emptyset$ for each $s \in F$.

(II) $t_j B \cap t_k B = \emptyset$ for $j, k = 1, 2, \dots, N$ with $j \neq k$.

We say that a discrete group G is a *Powers group* if for a given finite subset F of $G \setminus \{e\}$ and an integer N there exists a partition $\{A, B\}$ of G and $t_1, \dots, t_N \in G$ satisfying the above properties (I) and (II).

Exercise 4.6.1. Show that every conjugacy class of a non-unital element in a Powers group is infinite, that is, every Powers group is an *i.c.c.* group. Show also that every Powers group is non-amenable.

Theorem 4.6.1. *Let G be a Powers group. Then $C^*_\lambda(G)$ is a simple C^*-algebra with a unique trace.*

From now on, we assume that G is a Powers group, and translate the above properties into the language of operators. We need the following elementary fact.

Exercise 4.6.2. Let $x \in \mathcal{B}(\mathcal{H})$ and p be a projection in $\mathcal{B}(\mathcal{H})$ satisfying $(1-p)x(1-p) = 0$. Show that $|\langle x\xi, \xi\rangle| \leq 2\|x\|\|p\xi\|$ for each unit vector $\xi \in \mathcal{H}$.

Lemma 4.6.2. *Let $x \in C^*_\lambda(G)$ be a self-adjoint element of the form*

$$x = \sum_{s\in F} a_s L_s,$$

where F is a finite subset in $G \setminus \{e\}$. Then there are $t_1, \dots, t_5 \in G$ such that

$$\|\frac{1}{5}\sum_{k=1}^{5} L_{t_k} x L_{t_k^{-1}}\| \leq \frac{2}{\sqrt{5}}\|x\|. \tag{4.6.1}$$

Proof. Let A, B and $t_1, \dots, t_5$ be with the properties (I) and (II). Let P be the projection P of $\ell^2(G)$ onto $\ell^2(A)$, and put $P_k = L_{t_k}(1-P)L_{t_k^{-1}}$ for $k = 1, 2, \dots, 5$. Then the property (II) says that the projections $P_1, P_2, \dots, P_5$ are pairwise orthorgonal. On the other hand, the property (I) says that $PL_sP = 0$ for each $s \in F$, from which we see that

$$(1 - P_k)L_{t_k} x L_{t_k^{-1}}(1 - P_k) = 0, \qquad k = 1, 2, \dots, 5.$$

For each $\xi \in \ell^2(G)$ with $\|\xi\| = 1$, we have

$$|\langle(\frac{1}{5}\sum_{k=1}^{5} L_{t_k} x L_{t_k^{-1}})\xi, \xi\rangle| \leq \frac{1}{5}\sum_{k=1}^{5} |\langle L_{t_k} x L_{t_k^{-1}}\xi, \xi\rangle| \leq \frac{2}{5}\sum_{k=1}^{5} \|x\|\|P_k\xi\|,$$

by Exercise 4.6.2. The conclusion follows from the orthogonality of $\{P_k\}$ and the easy inequality $(\sum_{k=1}^{n} \alpha_k)^2 \leq n\sum_{k=1}^{n} \alpha_k^2$ for $\alpha_k \in \mathbb{R}$. □

Proof of Theorem 4.6.1. Recall that the trace τ on $C^*_\lambda(G)$ is given by

$$\tau(x) = a_e, \qquad \text{for } x = \sum_{s\in G} a_s L_s.$$

By applying the above lemma, we see that $\tau(x)1$ lies in the closed convex hull of the unitary orbit $\{uxu^* : u \in A_u\}$ for each $x \in C^*_\lambda(G)$. Let I be a two-sided ideal in $C^*_\lambda(G)$ and $x \in I$ with $x \neq 0$. Put $y = \tau(x^*x)^{-1}x^*x \in I$. Then $\tau(y) = 1$, and so, given $\epsilon > 0$ there are unitaries u_i such that

$$\|1 - \frac{1}{n}\sum_{i=1}^{n} u_i y u_i^*\| < \epsilon.$$

This shows that $z = \frac{1}{n}\sum_{i=1}^{n} u_i y u_i^* \in I$ is invertible, and $I = C^*_\lambda(G)$. Finally, if σ is an another trace then $|\sigma(1-z)| < \epsilon$, and $\sigma(y) = \sigma(z) = 1 = \tau(y)$. □

Next, we will show that the following short sequence

(4.6.2)
$$0 \to C^*(F_2) \otimes_{\min} J \to C^*(F_2) \otimes_{\min} C^*(F_2) \to C^*(F_2) \otimes_{\min} C^*_\lambda(F_2) \to 0$$

is not exact, where J is the kernel of the quotient map $\lambda : C^*(F_2) \to C^*_\lambda(F_2)$ induced by the left regular representation of F_2. This shows that Corollary 4.1.8 does not hold for minimal tensor products. Before going further, we investigate more general situations. For a locally compact group G, we consider the unitary representation θ_G of $G \times G$ on the space $L^2(G)$ defined by

$$\theta_G(s_1, s_2)\xi(t) = \xi(s_1^{-1} t s_2), \qquad s_1, s_2, t \in G,\ \xi \in L^2(G). \tag{4.6.3}$$

In other word, $\theta_G(s_1, s_2) = L_{s_1} R_{s_2}$ for discrete groups. This extends to

$$\theta_G : C^*(G) \otimes C^*(G) \to \mathcal{B}(\ell^2(G)). \tag{4.6.4}$$

We say that a locally compact group G has the *factorization property* if the map θ_G in (4.6.4) extends to a representation of $C^*(G) \otimes_{\min} C^*(G)$. When π_1 and π_2 are unitary representations of G, we denote by $\pi_1 \otimes \pi_2$ the representation of G given by $(\pi_1 \otimes \pi_2)(s) = \pi_1(s) \otimes \pi_2(s)$ for $s \in G$.

Lemma 4.6.3. *If $\{\pi, \mathcal{H}\}$ is a unitary representation of G then $\lambda \otimes \pi$ is unitarily equivalent to a multiple of λ.*

Proof. Consider the unitary isomorphism $U : L^2(G) \otimes \mathcal{H} \to L^2(G, \mathcal{H})$ defined by $U(f \otimes \xi)(s) = f(s)\pi(s^{-1})\xi$. □

Theorem 4.6.4. *Let G be a finitely generated non-amenable group which has the factorization property. Then the the sequence*

$$0 \to C^*(G) \otimes_{\min} J \to C^*(G) \otimes_{\min} C^*(G) \to C^*(G) \otimes_{\min} C^*_\rho(G) \to 0 \tag{4.6.5}$$

is not exact in the middle, where J is the kernel of the map $C^*(G) \to C^*_\rho(G)$.

Proof. Let A be a symmetric set of generators and put

$$h = \frac{1}{\#A} \sum_{a \in A} \pi_u(a) \in C^*(G),$$

where π_u denotes the universal representation of G. If we denote by π_0 the one-dimensional trivial representation, then $\pi_0(1-h)=0$ and so $1 \in \mathrm{sp}(h)$. On the other hand, we see that $\mathrm{sp}(\rho(h)) \subseteq [-1, 1-\epsilon]$ for some $\epsilon > 0$, by Proposition 4.5.10. We take a function $f : [-1,1] \to [0,1]$ such that $f(1) = 1$ and $([-1, 1-\epsilon]) = 0$, and put $x = f(h) \in C^*(G)$ then $x \neq 0$. Now, we consider the following diagram

$$\begin{array}{ccccc} C^*(G) & \xrightarrow{\Delta} & C^*(G) \otimes_{\min} C^*(G) & \xrightarrow{\theta_G} & \mathcal{B}(\ell^2(G)) \\ & & \Big\downarrow {\scriptstyle 1\otimes\rho} & & \\ & & C^*(G) \otimes_{\min} C^*_\rho(G) & & \end{array}$$

where Δ is the homomorphism induced by the unitary representation $\pi_u \otimes \pi_u$. Note that $(1 \otimes \rho) \circ \Delta$ is associated with the representation $\pi_u \otimes \rho$, which is a multiple of ρ by Lemma 4.6.3. So, we have

$$\mathrm{sp}(((1 \otimes \rho)\Delta)(h)) = \mathrm{sp}(\rho(h)) \subseteq [-1, 1-\epsilon],$$

from which we have $\Delta(x) \in \mathrm{Ker}\,(1 \otimes \rho)$ by the choice of f.

Noting that $C^*(G) \otimes_{\min} J \subseteq \mathrm{Ker}\,\theta_G$, it suffices to show that $\theta_G(\Delta(x)) \neq 0$ in order to complete the proof. The homomorphism $\theta_G \circ \Delta$ is associated with the unitary representation α of G on $\ell^2(G)$ given by

$$\alpha(s)\xi(t) = \xi(s^{-1}ts), \qquad s, t \in G,\ \xi \in \ell^2(G).$$

Because the vector χ_e is fixed by $\alpha(G)$, we see that $\theta_G(\Delta(x))(\chi_e) = \chi_e$. □

In order to apply Theorem 4.6.4, we proceed to show that F_2 has the factorization property. This depends on the residual finiteness of the free group F_2. We say that a countable group G is *residually finite* if there is a decreasing sequence $\{N_k : k = 1, 2, \dots\}$ of normal subgroups of finite indices such that $\bigcap_k N_k = \{e\}$.

Exercise 4.6.3. Show that a countable group G is residually finite if and only if for each $s \in G$ with $s \neq e$ there is a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(s) \neq e$.

Considering the homomorphism

$$\phi_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_n)$$

induced from the quotient map $\mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, we see that $\mathrm{SL}_2(\mathbb{Z})$ is residually finite. It is well-known that the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate the free group F_2. Because $\phi_n|_{F_2}$ is nontrivial for $n \geq 3$, we see that F_2 is also residually finite.

In order to prove that every residually finite group has the factorization property, we need some preliminaries. For two unitary representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ of a group G, we define the unitary representation $\pi_1 \boxtimes \pi_2$, said to be the *exterior tensor product* of π_1 and π_2, of the group $G \times G$ on the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by

$$(\pi_1 \boxtimes \pi_2)(s,t) = \pi_1(s) \otimes \pi_2(t), \qquad (s,t) \in G \times t. \tag{4.6.6}$$

For a unitary representation $\{\pi, \mathcal{H}\}$ of a group G, we also define the *conjugate representation* $\overline{\pi}$ of G on the dual space $\mathcal{H}^*$ by

$$\langle \overline{\pi}(s)\overline{\xi}, \eta \rangle = \langle \overline{\xi}, \pi(s)^{-1}\eta \rangle, \qquad s \in G,\ \overline{\xi} \in \mathcal{H}^*,\ \eta \in \mathcal{H}. \tag{4.6.7}$$

Lemma 4.6.5. *Let G be a finite group. Then we have*

$$\theta_G \cong \bigoplus_{\sigma \in \widehat{G}} \overline{\sigma} \boxtimes \sigma, \tag{4.6.8}$$

where $\widehat{G}$ denote the set of all irreducible representations up to unitary equivalence.

Proof. For each representation $\{\pi, \mathcal{H}\}$ of G, we define the linear map $V_\pi : \mathcal{H}^* \otimes \mathcal{H} \to \ell^2(G)$ by

$$V_\pi(\overline{\xi} \otimes \eta)(s) = \langle \overline{\xi}, \pi(s)\eta \rangle, \qquad s \in G,\ \overline{\xi} \in \mathcal{H}^*,\ \eta \in \mathcal{H}.$$

Then, by a calculation, we see that $\theta_G(s,t) \circ V_\pi = V_\pi \circ (\overline{\pi} \boxtimes \pi)(s,t)$ for each $s, t \in G$. Furthermore, it is easy to see that V_π is an isomorphism if and only if π is an irreducible representation. Comparing the dimensions in both sides of (4.6.8), the proof is complete. □

Let $\{\sigma, \mathcal{K}\}$ be a subrepresentation of a representation $\{\pi, \mathcal{H}\}$ of a C^*-algebra A, that, is, $\mathcal{K} \subseteq \mathcal{H}$ and $\sigma(x) = \pi(x)|_{\mathcal{K}}$ for each $x \in A$. Then every state associated with $\{\sigma, \mathcal{K}\}$ is also associated with $\{\pi, \mathcal{H}\}$. We say that $\{\sigma, \mathcal{K}\}$ is *weakly contained* in $\{\pi, \mathcal{H}\}$ if every state associated with $\{\sigma, \mathcal{K}\}$ is the weak* limit of states which are sums of positive linear functionals associated with $\{\pi, \mathcal{H}\}$. We use the same terminology for continuous unitary representations of a locally compact group G, considering them as representations of $C^*(G)$. Employing the correspondence between $C^*(G)^*_+$ and $\mathfrak{P}(G)$ in the last section, the following is immediate.

Proposition 4.6.6. *Let σ and π be unitary representations of a discrete group G. Then the following are equivalent:*

(i) *σ is weakly contained in π.*

(ii) *Every positive definite function associated with σ is the limit of sums of positive definite functions associated with π.*

If σ admits a cyclic vector ξ, then the following condition is also equivalent:

(iii) *the function $s \mapsto \langle \sigma(s)\xi, \xi\rangle$ is the limit of sums of positive definite functions associated with π.*

Theorem 4.6.7. *Every residually finite group G has the factorization property.*

Proof. We denote by π_u the universal representation of G. It suffices to show that θ_G is weakly contained in $\pi_u \boxtimes \pi_u$. Let $\{N_k\}$ be a decreasing sequence of normal subgroups of finite indices so that $\bigcap_k N_k = \{e\}$. We denote by $G_k = G/N_k$ and $q_k : G \times G \to G_k \times G_k$ the product of the quotient maps for each $k = 1, 2, \ldots$. By Lemma 4.6.5, $\theta_{G_k} \circ q_k$ is contained in $\pi_u \boxtimes \pi_u$, and so, $\bigoplus_{k=1}^{\infty} \theta_{G_k} \circ q_k$ is also contained in $\pi_u \boxtimes \pi_u$. Therefore, it suffices to show that θ_G is weakly contained in $\bigoplus_{k=1}^{\infty} \theta_{G_k} \circ q_k$.

Note that the representation θ_G admits a cyclic vector χ_e and the map $(s,t) \mapsto \langle \theta_G(s,t)\chi_e, \chi_e\rangle$ is nothing but χ_D, the characteristic function on the diagonal D of $G \times G$. Similarly, we see that $\chi_{N_k \times N_k}$ is the the positive definite function associated with the representation $\theta_{G_k} \circ q_k$ and the vector $\chi_{N_k} \in \ell^2(G_k)$, for each $k = 1, 2, \ldots$. From the condition $\bigcap_k N_k = \{e\}$, we have

$$\chi_D = \lim_{k \to \infty} \chi_{N_k \times N_k},$$

and so the proof is complete by Proposition 4.6.6. □

Corollary 4.6.8. *Let G be a finitely generated non-amenable group which is residually finite. Then the sequence (4.6.5) is not exact.*

In the remainder of this section, we just mention another interesting properties of the C^*-algebras $C^*(F_2)$ and $C^*_\lambda(F_2)$. Using the universal property, Choi [**Ch80**] showed that $C^*(F_2)$ has no nontrivial projections, has a faithful irreducible representation, and has sufficiently many finite-dimensional representations. The last property has been extended to another groups such as the free products of free groups and finite groups [**GM90**]. See also [**EL92**]. It is also known that, for a large class of discrete torsion free groups, the full group C^*-algebras do not have non-trivial projections [**JP90**]. See also [**Va!**] for full group C^*-algebras with non-trivial projections. Non-exactness of the sequence (4.6.2) has an interesting implication on the lifting problem: There is no completely positive linear map $\tilde{\phi} : C^*_\lambda(F_2) \to C^*(F_2)$ such that $\pi \circ \tilde{\phi} = \mathrm{id} : C^*_\lambda(F_2) \to C^*(F_2)/J$, where $\pi : C^*(F_2) \to C^*(F_2)/J$ is the quotient. In other word, $\mathrm{id} : C^*_\lambda(F_2) \to C^*(F_2)/J$ has no completely positive lifting [**Wa77b**]. See also [**CE76b**], [**CE77c**], [**An78**] and [**EHa85**] for lifting problems.

We know that $C^*_\lambda(F_2)$ does not satisfy CPAP because it is not nuclear. It was shown by de Canniere and Haagerup [**dCH85**] that $C^*_\lambda(F_2)$ has the positive approximation property. From this, it can be shown that there exists a *positive* unital linear map $\phi : C^*_\lambda(F_2) \to \mathcal{B}(\mathcal{H})$ which is not extendable to $\mathcal{B}(\ell^2(F_2))$ [**Ro86**]. Compare with Theorem 4.4.2. A positive linear map from a nuclear C^*-algebra into $\mathcal{B}(\mathcal{H})$ is always extendable [**St86**]. We will note in the next chapter that $C^*_\lambda(F_2)$ has no nontrivial projections, and so give an example of a simple C^*-algebra with no nontrivial projections. We will also see that $C^*_\lambda(F_2)$ can be embedded in a nuclear C^*-algebra. Therefore, the nuclearity does not pass to the C^*-subalgebras.

NOTE

The simplicity of $C^*_\lambda(F_2)$ is due to powers [**Po75**] exploiting the properties (I) and (II). See also [**AO76**] and [**FP**, Chapter 2]. The proof of Lemma 4.6.2 here is a variant of the argument in [**PS79a**]. For a sharper estimate than (4.6.1) using only three unitaries L_s, we refer to [**dHS86**]. We also refer to [**dH83**] for various examples of Powers groups. For further results in this direction, we refer to [**BN88**], [**NB90**], [**Be91**], [**Be**] and [**NT**]. A

C^*-algebra A is said to have the *Dixmier property* if the convex closure of the unitary orbit intersects the center for each $x \in A$. In the proof of Theorem 4.6.1, we have shown that $C^*_\lambda(G)$ has the Dixmier property. This property has close relations with the simplicity and the uniqueness of a trace (see **[Rd82a]**, **[HZ84]**). Non-exactness of the sequence (4.6.2) was first proved by S. Wassermann **[Wa76b]** where the residual finiteness of the free group has been exploited. See also **[Wa90]**, **[Wa91]** for a simpler proof and the further results in this direction. The proof of residual finiteness of F_2 is taken from **[Va?]**. We refer to **[Ma69]** for a survey on residual properties. We have followed **[dHRVb]** for Theorems 4.6.4 and 4.6.7. We note that Lemma 4.6.5 holds for compact groups (see **[Wl**, Theorem 2.8.2] for example). For a certain class of discrete groups (satisfying Kazhdan's property T), Kirchberg **[Ki?]** recently announced that the converse of Theorem 4.6.7 is also valid. Note that the free group does not satisfy the property T.

4.7. Exact C^*-algebras

From now on throughout the remainder of this note, the minimal tensor product $A \otimes_{\min} B$ will be denoted by simply $A \otimes B$, and the algebraic tensor products $A \otimes B$ by $A \odot B$.

In the last section, we have seen that the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0 \tag{4.7.1}$$

need not to be exact. We say that a C^*-algebra A is *exact* if the sequence (4.7.1) is exact for every C^*-algebra B and its norm-closed two-sided ideal J. Corollary 4.6.8 says that the group C^*-algebra $C^*(G)$ is not exact if G is a finitely generated non-amenable group which is residually finite. In order to deal with the sequence (4.7.1), it is convenient to introduce the notion of slice maps.

Proposition 4.7.1. *For a functional $\phi \in A^*$, there is a unique bounded linear map $R_\phi : A \otimes B \to B$ satisfying*

$$R_\phi(a \otimes b) = \phi(a)b, \qquad a \in A,\ b \in B. \tag{4.7.2}$$

Furthermore, the map R_ϕ satisfies the following:

(i) $\|R_\phi\| = \|\phi\|$.
(ii) $\langle R_\phi(x), \psi\rangle = \langle x, \phi \otimes \psi\rangle$ *for each* $\psi \in B^*$.
(iii) *If* $x \in A \otimes B$ *and* $R_\phi(x) = 0$ *for each* $\phi \in A^*$ *then* $x = 0$.

Proof. For $x = \sum a_i \otimes b_i \in A \odot B$, we have

$$\begin{aligned}\|R_\phi(x)\| &= \sup\{|\langle R_\phi(x), \psi\rangle| : \|\psi\| \le 1\}\\ &= \sup\{|\langle x, \phi \otimes \psi\rangle| : \|\psi\| \le 1\} \le \|\phi\|\|x\|_{\min},\end{aligned}$$

from which we see that R_ϕ extends to a bounded linear map on $A \otimes B$ with $\|R_\phi\| \le \|\phi\|$. The remaining statements are easy. □

The linear map $R_\phi : A \otimes B \to B$ in the above proposition is said to be the *right slice map* associated with ϕ. The *left slice map* $L_\psi : A \otimes B \to A$ associated with $\psi \in B^*$ is also defined analogously. When C and D are C^*-subalgebras of A and B respectively, the *Fubini product* $F(C, D, A \otimes B)$ of C and D with respect to $A \otimes B$ is defined by

$$F(C, D, A \otimes B) = \{x \in A \otimes B : R_\phi(x) \in D, L_\psi(x) \in C \text{ for each } \phi \in A^* \text{ and } \psi \in B^*\}. \tag{4.7.3}$$

Although the Fubini product of C and D depends on $A \otimes B$ as well as C and D, we will denote just by $F(C, D)$ if no confusion arise. It is clear that $C \otimes D \subseteq F(C, D)$. When D is a C^*-subalgebra of B, we say that the triple (A, B, D) satisfies the *slice map conjecture* if $F(A, D) = A \otimes D$. The following proposition is immediate from Proposition 4.7.1.

Proposition 4.7.2. *The Fubini product $F(A, J)$ is just the kernel of the homomorphism $A \otimes B \to A \otimes (B/J)$ in the sequence* (4.7.1).

Therefore, we see that the sequence (4.7.1) is exact if and only if the triple (A, B, J) satisfies the slice map conjecture. For a pair (B, D) of C^*-algebras with $D \subseteq B$, we introduce the following condition:

(4.7.4) There exists a net $\{\pi_\lambda\}$ of completely bounded linear maps from B into D such that $\sup_i \|\pi_\lambda\|_{cb} < \infty$ and $\lim_i \|\pi_\lambda(x) - x\| = 0$ for $x \in D$.

If D is a nuclear C^*-subalgebra of a C^*-algebra B then the pair (B, D) satisfies the condition (4.7.4) by the first condition in the NOTE of §4.3.

Exercise 4.7.1. Show that the pair (B, D) satisfies the condition (4.7.4) if D is a closed ideal or a hereditary C^*-subalgebra of B.

Lemma 4.7.3. *Let (B, D) be a pair of C^*-algebras satisfying the condition (4.7.4). Then we have*

$$C \otimes D = (A \otimes D) \cap (C \otimes B), \tag{4.7.5}$$

for any C^*-algebras C and A with $C \subseteq A$.

Proof. Let $x \in C \otimes B$ and $\epsilon > 0$ be given. Then there is $y = \sum_{i=1}^n a_i \otimes b_i \in C \odot B$ such that $\|x - y\| < \epsilon$. Note that $\pi_\lambda \otimes 1 : A \otimes B \to C \otimes B$ is a bounded linear map with $\|\pi_\lambda \otimes 1\| \leq M$ by Proposition 4.2.6, where $M = \sup_i \|\pi_\lambda\|_{cb}$. Hence, we have

$$\|(\pi_\lambda \otimes 1)(x) - x\| \leq \|\pi_\lambda \otimes 1\| \|x - y\| + \sum_{i=1}^n \|\pi_\lambda(a_i) - a_i\| \|b_i\| + \|y - x\|,$$

and so, $x = \lim_\lambda (\pi_\lambda \otimes 1)(x)$ for $x \in C \otimes B$. Now, if $x \in (A \otimes D) \cap (C \otimes B)$ then $(\pi_\lambda \otimes 1)(x) \in C \otimes D$, and so we see that

$$x = \lim_\lambda (\pi_\lambda \otimes 1)(x) \in C \otimes D. \quad \square$$

Proposition 4.7.4. *Every C^*-subalgebra C of an exact C^*-algebra A is also exact.*

Proof. The proposition follows from

$$F(C, J, C \otimes B) \subseteq F(A, J, A \otimes B) \cap (C \otimes B) = (A \otimes J) \cap (C \otimes B) = C \otimes J$$

by Lemma 4.7.3, for an ideal J of a C^*-algebra B. $\square$

From this proposition, it follows that every C^*-subalgebra of a nuclear C^*-algebra is exact. Especially, we see that $\mathcal{B}(\mathcal{H})$ with an infinite dimensional Hilbert space $\mathcal{H}$ is not nuclear.

Exercise 4.7.2. If $\{A_i\}$ is an increasing sequence of exact C^*-algebras and $A = \overline{\bigcup_i A_i}$ then show that A is also exact.

From now on, we fix a pair (B, D) of C^*-algebras satisfying the condition (4.7.4) with the required net $\{\pi_\lambda : \lambda \in \Lambda\}$. Denote by $\Lambda(B)$ (respectively $\Lambda_0(B)$) the C^*-algebra of all bounded functions (respectively functions converging to 0) from Λ to B. Also, define $\delta : B \to \Lambda(B)$ by

$$\delta(b)_\lambda = b - \pi_\lambda(b), \qquad b \in B, \ \lambda \in \Lambda. \tag{4.7.6}$$

Finally, we denote by p_λ the λ-th projection from $\Lambda(B)$ into B.

Lemma 4.7.5. *Let A be a C^*-algebra. Then for each $x \in A \otimes B$ we have the following:*

(i) $x \in A \otimes D \iff (1_A \otimes \delta)(x) \in A \otimes \Lambda_0(B)$.
(ii) $x \in F(A, D, A \otimes B) \Longrightarrow (1_A \otimes \delta)(x) \in F(A, \Lambda_0(B), A \otimes \Lambda(B))$.

Proof. Note that if $z \in A \otimes \Lambda_0(B)$ then $(1 \otimes p_\lambda)(z) \in \Lambda_0(A \otimes B)$. Hence, if $(1 \otimes \delta)(x) \in A \otimes \Lambda_0(B)$ then we have

$$x - (1 \otimes \pi_\lambda)(x) = (1 \otimes p_\lambda)(1 \otimes \delta)(x) \in \Lambda_0(A \otimes B),$$

from which it follows that $x = \lim_\lambda (1 \otimes \pi_\lambda)(x) \in A \otimes D$. This shows the direction ($\Longleftarrow$) of (i). The remaining statements are clear. □

Now, for each natural number $n = 1, 2, \ldots,$ we introduce the following condition on a C^*-algebra A as follows:

(C_n) For any $\epsilon > 0$, there are completely positive contractions $\phi_n : M_n \to A$ and $\phi_n : A \to M_n$ such that $\|\psi_n \phi_n - 1\| < \epsilon$.

Lemma 4.7.6. *Let E_n be a fixed C^*-algebra with the condition (C_n) for each $n = 1, 2, \ldots.$ Then for $x \in A \otimes B$ we have*

$$\|x\| = \sup \|(1_A \otimes V)(x)\|,$$

where V runs through the set of all completely positive contractions from B into E_n, for $n = 1, 2, \ldots.$

Proof. It is clear that $\|(1 \otimes V)(x)\| \le \|x\|$. For the reverse inequality, we assume that A and B act on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. For $\xi, \eta \in \mathcal{H} \odot \mathcal{K}$ with $\|\xi\| = \|\eta\| = 1$, choose a finite dimensional projection $p \in \mathcal{B}(\mathcal{K})$ such that $(1 \otimes p)(\xi) = \xi$ and $(1 \otimes p)(\eta) = \eta$. Define $V : B \to \mathcal{B}(p\mathcal{K}) = M_n$ by $V(b) = pbp|_{p\mathcal{K}}$ for $b \in B$. Now, given $\epsilon > 0$, choose a completely positive contractions $W_1 : M_n \to E_n$ and $W_2 : E_n \to M_n$ such that $\|W_2 W_1 - 1\|_{cb} \le \epsilon / \|(1 \otimes V)(x)\|$. Then we have

$$\begin{aligned}|\langle x\xi, \eta\rangle| &= |\langle x(1 \otimes p)\xi, (1 \otimes p)\eta\rangle| = |\langle (1 \otimes V)(x)\xi, \eta\rangle| \le \|(1 \otimes V)(x)\| \\ &\le \|(1 \otimes W_2)(1 \otimes W_1)(1 \otimes V)(x)\| + \epsilon \le \|(1 \otimes W_1 V)(x)\| + \epsilon,\end{aligned}$$

and so the proof is complete, because $W_1V : B \to E_n$ is a completely positive contraction. □

We denote by $E = \prod_n E_n$ (respectively $E_0 = \bigoplus_n E_n$) the ℓ^∞-sum (respectively c_0-sum) of $\{E_n : n = 1, 2, \dots\}$, where E_n is a C^*-algebra satisfying the condition (C_n) for each $n = 1, 2, \dots$. We also denote by $\mathcal{S}$ the set of all completely positive contractions V from $\Lambda(B)$ into E such that $V(\Lambda_0(B)) \subseteq E_0$.

Lemma 4.7.7. *For $x \in A \otimes \Lambda(B)$, we have the following:*

(i) $x \in A \otimes \Lambda_0(B) \Longleftrightarrow (1 \otimes V)(x) \in A \otimes E_0$ *for each* $V \in \mathcal{S}$.

(ii) $x \in F(A, \Lambda_0(B), A \otimes \Lambda(B)) \Longrightarrow (1 \otimes V)(x) \in F(A, E_0, A \otimes E)$ *for each* $V \in \mathcal{S}$.

Proof. Every statement except $(\Longleftarrow)$ of (i) is trivial as in Lemma 4.7.5. Assume that $x \in A \otimes \Lambda(B) \setminus A \otimes \Lambda_0(B)$. Then there is a sequence $\{\lambda(k) : K = 1, 2, \dots\}$ of Λ such that $\|(1 \otimes p_{\lambda(k)})(x)\| > 2\epsilon$ for each $k = 1, 2, \dots$. By Lemma 4.7.6, there exists a completely positive contraction $V_k : B \to E_{n(k)}$ such that $\|(1 \otimes V_k)(1 \otimes p_{\lambda(k)})(x)\| > \epsilon$ for each $k = 1, 2, \dots$. We define $V : \Lambda(B) \to E$ by

$$V(b_\lambda) = (0, \dots, V_1(b_{\lambda(1)}), 0, \dots, 0, V_2(b_{\lambda(2)}), 0 \dots),$$

where each $V_k(b_{\lambda(k)})$ is at the $n(k)$-th position. Then $V \in \mathcal{S}$. If we denote by $q_n : E \to E_n$ the projection onto the n-th component, then we have

$$\|(1 \otimes q_{n(k)})(1 \otimes V)(x)\| = \|(1 \otimes V_k)(1 \otimes p_{\lambda(k)})(x)\| \geq \epsilon,$$

for each $k = 1, 2, \dots$. This shows that $(1 \otimes V)(x) \in A \otimes E \setminus A \otimes E_0$, and completes the proof. □

By Lemmas 4.7.5 and 4.7.7, we have the following:

Proposition 4.7.8. *Let (B, D) be a pair of C^*-algebras satisfying the condition* (4.7.4) *and E, E_0 as above. Then we have*

$$F(A, E_0, A \otimes E) = A \otimes E_0 \Longrightarrow F(A, D, A \otimes B) = A \otimes D.$$

Theorem 4.7.9. *Let E_n be a C^*-algebra satisfying the condition* (C_n) *for each $n = 1, 2, \dots$, and $E = \prod_n E_n$, $E_0 = \bigoplus_n E_n$. We also denote by $M = \prod_n M_n$ and $M_0 = \bigoplus_n M_n$. Then, for a C^*-algebra A, the following are equivalent:*

(i) *(A, B, D) satisfies the slice map conjecture for every pair (B, D) with the condition* (4.7.4).

(ii) *(A, B, D) satisfies the slice map conjecture for every hereditary C^*-subalgebra D of a C^*-algebra B.*

(iii) *A is an exact C^*-algebra.*

(iv) *$(A, \mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}))$ satisfies the slice map conjecture.*

(v) *(A, E, E_0) satisfies the slice map conjecture.*

(vi) *(A, M, M_0) satisfies the slice map conjecture.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv), (iii) $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (vi) are clear. Two implications (v) $\Longrightarrow$ (i) and (vi) $\Longrightarrow$ (i) follows from Proposition 4.7.8. Hence, it remains to show that (iv) implies (v). We denote by $\mathcal{H}_n$ the n-dimensional Hilbert space for each $n = 1, 2, \dots$, and put $\mathcal{H} = \bigoplus_n \mathcal{H}_n$. Then, under the embedding $M \subseteq \mathcal{B}(\mathcal{H})$ with $M_n = \mathcal{B}(\mathcal{H}_n)$, we have $M_0 = \mathcal{B}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$. It is also easy to see that $A \otimes M_0 = (A \otimes M) \cap (A \otimes \mathcal{K}(\mathcal{H}))$. Hence, we have

$$\begin{aligned} F(A, M_0, A \otimes M) &\subseteq (A \otimes M) \cap F(A, \mathcal{K}(\mathcal{H}), A \otimes \mathcal{B}(\mathcal{H})) \\ &= (A \otimes M) \cap (A \otimes \mathcal{K}(\mathcal{H})) = A \otimes M_0. \quad \square \end{aligned}$$

From the non-exactness of $\mathcal{B}(\mathcal{H})$, we see that the sequence

$$0 \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes (\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})) \to 0 \tag{4.7.7}$$

is not exact. We also know that $(\mathcal{B}(\mathcal{H}), M, M_0)$ does not satisfy the slice map conjecture. From this, it is not so difficult to see that the sequence

$$0 \to M \otimes M_0 \to M \otimes M \to M \otimes (M/M_0) \to 0 \tag{4.7.8}$$

is also non-exact.

In the remainder of this section, we show that how approximation properties are used to infer C^*-exactness. There are weaker notions of approximation properties than in §4.3: A C^*-algebra A is said to have the *completely contractive approximation property* (CCAP) if the identity map of A is approximated by complete contractions of finite ranks in the topology of point-norm convergences. Also, A is said to have the *completely bounded approximation property* (CBAP) if the identity map is approximated by a net $\{\pi_\lambda\}$ of completely bounded linear maps of finite ranks with $\sup_i \|\pi_\lambda\|_{cb} < \infty$. Compare with the third condition in the NOTE of §4.3. It was shown that the reduced group C^*-algebra $C^*_\lambda(F_2)$ of the free group satisfies the CCAP by de Canniere and Haagerup [**dCH85**]. Hence, CCAP is strictly weaker than nuclearity or CPAP.

Proposition 4.7.10. *Let A be a C^*-algebra with CBAP. Then the triple (A, B, D) satisfies the slice map conjecture for every C^*-algebra B and a C^*-subalgebra D of B.*

Proof. If $\{\pi_\lambda\}$ is the above net in the definition of CBAP then it is easy to see that $\lim_\lambda \|(\pi_\lambda \otimes 1_B)(x) - x\| = 0$ for each $x \in A \otimes B$. Suppose that $x \in A \otimes B$ and $R_\phi(x) \in D$ for every $\phi \in A^*$, and we fix λ. Then $(\pi_\lambda \otimes 1_B)(x) = \sum_{i=1}^n a_i \otimes b_i$, where $\{a_i\}$ is a finite set of linearly independent elements in $\pi_\lambda(A)$. For each $i = 1, 2, \ldots, n$, choose $\phi_i \in A^*$ such that $\phi_i(a_j) = \delta_{ij}$. Then we have

$$b_i = \sum_{j=1}^n \langle \phi_i, a_j \rangle b_j = R_{\phi_i}(\pi_\lambda \otimes 1_B)(x) = R_{\phi_i \circ \pi_\lambda}(x) \in D.$$

Therefore, it follows that $(\pi_\lambda \otimes 1_B)(x) = \sum_i a_i \otimes b_i \in A \otimes D$, and so we have $x = \lim_\lambda (\pi_\lambda \otimes 1_B)(x) \in A \otimes D$. □

Hence, we see that the C^*-algebra $C^*_\lambda(F_2)$ is exact. This also follows from Proposition 4.7.4 together with the fact that $C^*_\lambda(F_2)$ is a C^*-subalgebra of a nuclear C^*-algebra as was mentioned at the ends of §4.4 and §4.6.

A C^*-algebra A is said to be *nuclearly embeddable* if there is an isomorphic embedding $\iota : A \to B$ such that $\tau_\lambda \sigma_\lambda \to \iota$ in the point-norm topology for some nets $\sigma_\lambda : A \to M_{n_\lambda}$ and $\tau_\lambda : M_{n_\lambda} \to B$ of complete contractions. The similar argument as in Proposition 4.7.10 may be applied by Lemma 4.7.3, if A is a nuclearly embeddable C^*-algebra and D is a norm-closed two-sided ideal of A [**Wa90**]. Therefore, every nuclearly embeddable C^*-algebra is C^*-exact.

The converse is also true [**Ki!**]. For a large class of discrete groups G, it is now known that $C^*_\lambda(G)$ is an exact C^*-algebra by a work of A. Connes. See also [**QS92**]. Some of them do not satisfy CBAP [**Ha**].

NOTE

The notion of C^*-exactness was introduced by Kirchberg [**Ki78**], where Proposition 4.7.4 and Exercise 4.7.2 were announced. The notions of slice maps and Fubini products for C^*-algebras were introduced by Tomiyama [**To67**, **To75**] (see also [**Wa76a**]), where he showed that if C is subhomogeneous then $F(C, D, A \otimes B) = C \otimes D$. The converse is also true [**Ky85**, **HK88**]. We refer to Huruya's note [**Hu83**] or [**Ky88b**] for a survey on this topic. Lemma 4.7.3 was taken from [**Ky86**]. The essential arguments for Proposition 4.7.8 are contained in [**Ki83**, **Ky88a**], and used in [**HK88**] for the proof of Theorem 4.7.9. The condition (C_n) appears in [**HT83**] or [**Sm83**]. Equivalences between (ii) (iii) and (iv) of Theorem 4.7.9 are due to Kirchberg [**Ki83**], whereas the direction (iii) $\Longrightarrow$ (i) is found in [**Ac82**]. Non-exactness of the sequences (4.7.7) and (4.7.8) are due to Wassermann [**Wa78**] and Huruya [**Hu80**], respectively. Theorem 4.7.9 shows that pairs such as $(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}))$ and (M, M_0) play rôles of *testing pairs* for the C^*-exactness. For more examples of testing pairs, we refer to [**HK91**]. Recently, Kirchberg [**Ki!**] showed that the exactness is preserved under the C^*-quotient. The proof of proposition 4.7.10 is taken from [**Wa78**]. We refer to [**AB80**] and [**EHa85**] for another conditions related with C^*-exactness.

CHAPTER 5

CROSSED PRODUCTS OF C^*–PRODUCTS

The crossed product is one of the main tools to construct simple C^*-algebras, together with AF C^*-algebras and group C^*-algebras as we have seen before. After examining general properties of full and reduced crossed products in §5.1, we present two examples arising in this way. One of them is so called the irrational rotation C^*-algebra A_θ, which is obtained from the rotation on the circle by an irrational number θ. We classify these algebras by showing that the range of the projections under the trace is $[0,1]\cap(\mathbb{Z}+\mathbb{Z}\theta)$ as in the case of AF algebras in §3.4. We consider in §5.3 another simple C^*-algebras, called the *Cuntz algebra*, generated by isometries with orthogonal ranges. It turns out that these C^*-algebras may be considered as the crossed products by non-amenable discrete groups. In this way, we get an example of a non-nuclear C^*-subalgebra of a nuclear C^*-algebra. In §5.4, we give a brief survey on K-theory for C^*-algebras which enjoys the six-term exact sequence. Using this machinery, we calculate K-groups of irrational rotation C^*-algebras and Cuntz algebras in §5.5.

5.1. Full and Reduced Crossed Products of C^*-algebras

A *C^*-dynamical system* is a triple $\{A, G, \alpha\}$ of a C^*-algebra A, a locally compact group G and a homomorphism α from G into the group $\mathrm{Aut}\,(A)$ of all $*$-automorphisms of A such that $t \mapsto \alpha_t(a)$ is continuous for each $a \in A$. We denote by $\mathfrak{K}(G, A)$ the space of all continuous functions from G into A with compact supports. For $x, y \in \mathfrak{K}(G, A)$, we define

$$(5.1.1)\qquad \begin{aligned} x^*(t) &= \delta(t)^{-1}\alpha_t(y(t^{-1})^*), \\ (x*y)(t) &= \int_G x(s)\alpha_s(y(s^{-1}t))ds, \end{aligned}$$

for $t \in G$. By a straightforward calculation, we see that $\mathfrak{K}(G, A)$ is a normed $*$-algebra with respect to the above operations and the norm defined by

$$\|x\|_1 = \int_G \|x(t)\| dt, \qquad x \in \mathfrak{K}(G, A). \tag{5.1.2}$$

We denote by $L^1(G, A)$ the Banach $*$-algebra obtained by the completion of $\mathfrak{K}(G, A)$ with respect to the norm in (5.1.2). The *full crossed product* $G \ltimes_\alpha A$ is defined by the enveloping C^*-algebra of $L^1(G, A)$.

A *covariant representation* of a C^*-dynamical system $\{A, G, \alpha\}$ is a triple $\{\pi, u, \mathcal{H}\}$, where $\{\pi, \mathcal{H}\}$ is a representation of A and $\{u, \mathcal{H}\}$ is a unitary representation of G satisfying the condition

$$\pi(\alpha_t(a)) = u(t)\pi(a)u(t)^*, \qquad t \in G,\ a \in A. \tag{5.1.3}$$

For a covariant representation $\{\pi, u, \mathcal{H}\}$ of a dynamical system $\{A, G, \alpha\}$, we define

$$(\pi \times u)(x)\xi = \int_G \pi(x(t))u(t)\xi dt, \qquad x \in \mathfrak{K}(G, A),\ \xi \in \mathcal{H}. \tag{5.1.4}$$

By a direct calculation, we see that $\pi \times u$ is a $*$-representation of $\mathfrak{K}(G, A)$ on the Hilbert space $\mathcal{H}$. Because

$$\|(\pi \times u)(x)\| \leq \int_G \|\pi(x(t))u(t)\| dt = \|x\|_1, \qquad x \in \mathfrak{K}(G, A),$$

$\pi \times u$ extends to a $*$-representation of $L^1(G, A)$, hence to a representation of $G \ltimes_\alpha A$. Conversely, if $\{\rho, \mathcal{H}\}$ is a non-degenerate representation of $L^1(G, A)$ then the following pair (π, u) defined by

$$\begin{aligned} &\pi(a) = \text{s-}\lim_i \rho(\epsilon_i \otimes a), \qquad a \in A, \\ &u(s) : \rho(f \otimes a)\xi \mapsto \rho(\lambda_s f \otimes \alpha_s(a))\xi, \qquad s \in G,\ f \otimes a \in L^1(G, A),\ \xi \in \mathcal{H}, \end{aligned}$$

gives rise to a covariant representation satisfying (5.1.4), where $\{\epsilon_i\}$ is an approximate identity of $L^1(G)$ and $f \otimes a$ denotes the element of $L^1(G, A)$ defined by $t \mapsto f(t)a$ for $t \in G$.

For a representation $\{\pi, \mathcal{H}\}$ of A, we define the representation $\widetilde{\pi}$ of A on the Hilbert space $L^2(G, \mathcal{H})$ by

$$(\widetilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t), \qquad a \in A,\ t \in G. \tag{5.1.5}$$

We also denote by $\widetilde{\lambda}_s = \lambda_s \otimes 1_{\mathcal{H}}$ the representation of G on the space $L^2(G)\otimes\mathcal{H}$, or equivalently

$$(\widetilde{\lambda}_s\xi)(t) = \xi(s^{-1}t), \qquad s, t \in G,\ \xi \in L^2(G, \mathcal{H}). \tag{5.1.6}$$

It is easy to see that $\{\widetilde{\pi}, \widetilde{\lambda}, L^2(G, \mathcal{H})\}$ is a covariant representation of $\{A, G, \alpha\}$. We denote by $\mathrm{Ind}\pi = \widetilde{\pi} \times \widetilde{\lambda}$. By (5.1.4), we have

$$((\mathrm{Ind}\pi)(x)\xi)(t) = \int_G \pi(\alpha_{t^{-1}}(x(s)))\xi(s^{-1}t)ds, \tag{5.1.7}$$

for $x \in \mathfrak{K}(G, A)$ and $\xi \in L^2(G, \mathcal{H})$.

The *reduced crossed product* $G \ltimes_{\alpha r} A$ is the completion of $L^1(G, A)$ with respect to the C^*-norm given by

$$\|x\|_r = \sup\{\|(\mathrm{Ind}\pi)(x)\|\},$$

where π runs through all representations of A.

For a state $\phi \in \mathcal{S}(A)$ and $f \in \mathfrak{K}(G)$, we define a positive linear functional ϕ_f on $L^1(G, A)$ by

$$\phi_f(x) = \langle(\mathrm{Ind}\pi_\phi)(x)(f \otimes \xi_\phi), f \otimes \xi_\phi\rangle, \qquad x \in L^1(G, A), \tag{5.1.8}$$

where $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$ is the G. N. S. construction associated with ϕ, and $f \otimes \xi_\phi$ is an element of $L^2(G, \mathcal{H}_\phi)$ given by $(f \otimes \xi_\phi)(t) = f(t)\xi_\phi$, for $t \in G$. Then it is easy to see that

$$\phi_f(x) = \iint f(s^{-1}t)\overline{f(t)}\,\phi(\alpha_{t^{-1}}(x(s)))dsdt, \qquad x \in L^1(G, A). \tag{5.1.9}$$

Let $\{\sigma, \mathcal{H}\}$ be a faithful representation of A, and ϕ_i a state given by $\phi_i(a) = \langle\sigma(a)\xi_i, \xi_i\rangle$ for $a \in A$ and $\xi_i \in \mathcal{H}$. If $\phi \in \mathcal{S}(A)$ is defined by

$$\phi = \sum \lambda_i\phi_i, \qquad \lambda_i > 0,\ \sum \lambda_i = 1, \tag{5.1.10}$$

then we have

$$\begin{aligned}\phi_f(y^*x^*xy) &= \sum \lambda_i(\phi_i)_f(y^*x^*xy) \\ &\leq \sum \lambda_i\|(\mathrm{Ind}\sigma)(x)\|^2(\phi_i)_f(y^*y) = \phi_f(y^*y)\|(\mathrm{Ind}\sigma)(x)\|^2,\end{aligned} \tag{5.1.11}$$

for each $x, y \in L^1(G, A)$, whenever $f \in \mathfrak{K}(A)$. Because σ is faithful, every state is the weak* limit of ϕ's of the form in (5.1.10), and so we see that (5.1.11) holds for each $\phi \in \mathcal{S}(A)$. It is easy to see that $\{f \otimes \xi_\phi : f \in \mathfrak{K}(G)\}$ is cyclic for the representation $\mathrm{Ind}\pi_\phi$. Because $\mathrm{Ind}\pi$ is the direct sum of $\mathrm{Ind}\pi_\phi$'s, we have $\|(\mathrm{Ind}\pi)(x)\| \leq \|(\mathrm{Ind}\sigma)(x)\|$ for each representation π of A, by Exercise 1.5.1. Therefore, we have

$$\|x\|_r = \|(\mathrm{Ind}\sigma)(x)\|, \qquad x \in L^1(G, A), \tag{5.1.12}$$

whenever σ is a faithful representation of A.

Exercise 5.1.1. Let α be the trivial action of G on a C^*-algebra A. Show that $G \ltimes_\alpha A = C^*(G) \otimes_{\max} A$ and $G \ltimes_{\alpha r} A = C^*_\lambda(G) \otimes_{\min} A$.

Theorem 5.1.1. *If G is an amenable group then we have*

$$G \ltimes_\alpha A = G \ltimes_{\alpha r} A.$$

Proof. Let ϕ be a state of $G \ltimes_\alpha A$. It suffices to show that

$$\phi(x^*x) \leq \|x^*x\|_r, \qquad x \in L^1(G, A). \tag{5.1.13}$$

Let $\{\pi, u\}$ be the covariant representation of $\{A, G, \alpha\}$ associated with the G. N. S. construction $\{\pi_\phi, \mathcal{H}_\phi, \xi_\phi\}$. For $f \in \mathfrak{K}(G)$ with $\|f\|_2 = 1$, put

$$\phi_f(x) = \int_G (f * \tilde{f})(s)\langle \pi(x(s))u(s)\xi_\phi, \xi_\phi\rangle ds, \qquad x \in L^1(G, A).$$

Using the relation (5.1.3), we see that

$$\begin{aligned}\phi_f(x) &= \iint f(t)\overline{f(s^{-1}t)}\langle \pi(\alpha_{t^{-1}}(x(s))u(t^{-1}s)\xi_\phi, u(t^{-1})\xi_\phi\rangle ds dt \\ &= \langle (\mathrm{Ind}\pi)(x)\xi, \xi\rangle,\end{aligned}$$

where $\xi \in L^2(G, \mathcal{H}_\phi)$ is defined by $\xi(s) = \overline{f(s)}u(s^{-1})\xi_\phi$. Hence, it follows that ϕ_f satisfies the condition (5.1.13). Now, if G is amenable then 1_G is approximated by $f \in \mathfrak{K}(G)$ with $\|f\|_2 = 1$, and so ϕ itself approximated by ϕ_f's weakly on $\mathfrak{K}(G, A)$. Therefore, we see that the condition (5.1.13) holds for $x \in \mathfrak{K}(G, A)$, and this completes the proof. □

Following two exercises correspond to the relation (4.1.5) and Corollary 4.1.8.

Exercise 5.1.2. Let B be a C^*-subalgebra of a C^*-algebra A which is invariant under the action α on A. Show that $G \ltimes_{\alpha r} B$ is a C^*-subalgebra of $G \ltimes_{\alpha r} A$. If B is a closed ideal then so is $G \ltimes_{\alpha r} B$.

Exercise 5.1.3. Let I be a norm-closed two-sided ideal of A which is invariant under α. Show that the following sequence

$$0 \to G \ltimes_\alpha I \to G \ltimes_\alpha A \to G \ltimes_\alpha A/I \to 0$$

is exact.

Exercise 5.1.4. Let α be an action of G on a C^*-algebra A. Define the action β on $A \otimes_{\max} B$ or $A \otimes_{\min} B$ by $\beta_t(a \otimes b) = \alpha_t(a) \otimes b$ for $a \otimes b \in A \otimes B$. Show that

$$G \ltimes_\beta (A \otimes_{\max} B) = (G \ltimes_\alpha A) \otimes_{\max} B,$$
$$G \ltimes_{\beta r} (A \otimes_{\min} B) = (G \ltimes_{\alpha r} A) \otimes_{\min} B.$$

Therefore, if G is amenable and A is a nuclear C^*-algebra then $G \ltimes_{\alpha r} A$ is also nuclear.

From now on, we will concern exclusively on *discrete* actions, and so G will always denote a discrete group.

Let A be a concrete C^*-algebra acting on a Hilbert space $\mathcal{H}$, and α a discrete action of G on the C^*-algebra A. Then each element $s \in G$ gives rise to the unitary operator $\widetilde{\lambda}_s$ on $\ell^2(G, \mathcal{H})$ by (5.1.6), which will be denoted by U_s. Also, each $a \in A$ defines an operator on $\ell^2(G, \mathcal{H})$ by the relation (5.1.5). Then the reduced crossed product $G \ltimes_{\alpha r} A$ is the C^*-algebra acting on $\ell^2(G, \mathcal{H})$ generated by $\{a, U_s : a \in A, s \in G\}$. This is independent of the choice of $\mathcal{H}$ by (5.1.12). The covariant relation (5.1.3) is written by

$$\alpha_s(a)U_s = U_s a, \qquad a \in A,\ s \in G. \tag{5.1.14}$$

From this, we see that the relations

$$(\sum_s a_s U_s)^* = \sum_s \alpha_s((a_{s^{-1}})^*)U_s,$$

$$(\sum_s a_s U_s)(\sum_t b_t U_t) = \sum_s (\sum_t a_t \alpha_t(b_{t^{-1}s}))U_s$$

hold, where the summations above are finite. Furthermore, the set of all elements of the form

$$\sum_s a_s U_s, \qquad a_s \in A$$

is a dense $*$-subalgebra of $G \ltimes_{\alpha r} A$.

Note that every homeomorphism σ of a compact Hausdorff space X induces a $*$-automorphism α of the C^*-algebra $C(X)$ by $\alpha(f)(x) = f(\sigma^{-1}x)$. In this way, a topological dynamical system $\{X, G, \alpha\}$ induces a C^*-dynamical system $\{C(X), G, \alpha\}$.

Exercise 5.1.5. Let the cyclic group $\mathbb{Z}_n$ act on itself by the translation. Show that the resulting crossed product is $*$-isomorphic to the C^*-algebra M_n of $n \times n$ matrices.

Exercise 5.1.6. Let the permutation group S_3 act on the space $X = \{x_1, x_2, x_3\}$. Show that $S_3 \ltimes_{\alpha r} C(X)$ is not simple.

Proposition 5.1.2. *Let G be a discrete group and $s \in G$. Then the map*

$$x \mapsto x(s) : \ell^1(G, A) \to A$$

extends to a bounded linear map from $G \ltimes_{\alpha r} A$ to A.

Proof. It suffices to show that

$$\|x(s)\| \le \|x\|_r, \qquad x \in \ell^1(G, A).$$

Let $\epsilon > 0$ be given and $\{\pi, \mathcal{H}\}$ a representation of A. Choose a unit vector $\xi_0 \in H$ such that $\|\pi(x(s))\xi_0\| \geq \|\pi(x(s))\| - \epsilon$. If we define $\xi \in \ell^2(G, \mathcal{H})$ by $\xi(s^{-1}) = \xi_0$ and $\xi(t) = 0$ for $t \neq s^{-1}$, then $\|\xi\| = 1$ and

$$\|(\mathrm{Ind}\pi)(x)\xi\| \geq \|((\mathrm{Ind}\pi)(x)\xi)(e)\| = \|\pi(x(s))\xi_0\| \geq \|\pi(x(s))\| - \epsilon.$$

Hence, it follows that $\|(\mathrm{Ind}\pi)(x)\| \geq \|\pi(x(s))\| - \epsilon$. Because π and ϵ were arbitrary, this completes the proof. □

From the above proposition, every $x \in G \ltimes_{\alpha r} A$ corresponds to the set $\{x_s \in A : s \in G\}$, which is said to be the *Fourier coefficients* of x. Indeed, it is easy to see that $x = 0$ if and only if $x_s = 0$ for each $s \in G$. It is also easy to see that the map

$$E : x \mapsto x_e, \qquad x \in G \ltimes_{\alpha r} A \tag{5.1.15}$$

is a positive faithful norm one projection from $G \ltimes_{\alpha r} A$ onto A.

Now, we will consider the question when the reduced crossed product $G \ltimes_{\alpha r} A$ becomes a simple C^*-algebra. We say that a C^*-dynamical system $\{A, G, \alpha\}$ is *G-simple* if there is no nontrivial closed two-sided ideal of A which is invariant under the action α. By Exercise 5.1.2, we see that this is a necessary condition for the simplicity of $G \ltimes_{\alpha r} A$. Exercise 5.1.6 shows that this is not sufficient.

Theorem 5.1.3. *Let G be a discrete group and a C^*-dynamical system $\{A, G, \alpha\}$ satisfy the following condition: Given $\{s_i \in G \backslash \{e\} : i = 1, 2, \ldots, n\}$, $\{a_i \in A : i = 0, 1, \ldots, n\}$ with $a_0 \geq 0$ and $\epsilon > 0$, there is an element $a \in A_+$ with $\|a\| = 1$ such that*

$$\|aa_0a\| > \|a_0\| - \epsilon, \qquad \|aa_i\alpha_{s_i}(a)\| < \epsilon, \quad i = 1, 2, \ldots, n.$$

If A is G-simple then $G \ltimes_{\alpha r} A$ is simple.

Proof. Let π be a $*$-homomorphism then we see that $\pi|_A$ is faithful because $(\mathrm{Ker}\,\pi) \cap A = 0$, from the condition of G-simplicity. Let $x = \sum a_s U_s$ be a positive element in $\mathfrak{K}(G, A)$ and $a \in A$ be the element given by the assumption with $a_0 = a_e$. Then we have $\|\pi(aa_ea)\| = \|aa_ea\|$, and

$$\|\pi(aa_sU_sa)\| \leq \|aa_sU_sa\| = \|aa_s\alpha_s(a)U_s\| \leq \|aa_s\alpha_s(a)\|, \qquad s \neq e.$$

Therefore, we have

$$\begin{aligned}\|\pi(x)\| &\geq \|\pi(axa)\| \geq \|\pi(aa_e a)\| - \sum_{s\neq e} \|\pi(aa_s U_s a)\| \\ &\geq \|aa_e a\| - \sum_{s\neq e} \|aa_s\alpha_s(a)\| \geq \|a_e\| - (n+1)\epsilon = \|E(x)\| - (n+1)\epsilon,\end{aligned}$$

where n is the number of $s \in G\setminus\{e\}$ with $a_s \neq 0$. Because $\mathfrak{K}(G,A)$ is dense in $G \ltimes_{\alpha r} A$, it follows that $\|\pi(x)\| \geq \|E(x)\|$ for each $x \in (G \ltimes_{\alpha r} A)_+$. Because E is faithful we see that π is also faithful, and this completes the proof. □

We say that a topological dynamical system $\{X, G, \sigma\}$ is *minimal* if every *orbit* $\mathcal{O}(x) = \{\sigma_s(x) \in X : s \in G\}$ of $x \in X$ is dense in X.

Exercise 5.1.7. Show that a topological dynamical system $\{X, G, \sigma\}$ is minimal if and only if there is no nontrivial σ-invariant closed subset of X. Also show that $\{X, G, \sigma\}$ is minimal if and only if the corresponding C^*-dynamical system $\{C(X), G, \alpha\}$ is G-simple.

Let σ be a homeomorphism of a compact Hausdorff space X whose fixed point set $X^\sigma = \{x \in X : \sigma x = x\}$ is nowhere dense. For $f_0 \in C(X)_+$ and $\epsilon > 0$, it is easy to find $f \in C(X)_+$ such that $\|ff_0\| \geq \|f_0\| - \epsilon$ and $|f(x)f(\sigma^{-1}x)| < \epsilon$ for $x \in X$. The following corollary is an immediate consequence of Theorem 5.1.3 in this way.

Corollary 5.1.4. *Let $\{X, G, \sigma\}$ be a topological dynamical system with a discrete group G. Then $G \ltimes_{\alpha r} C(X)$ is simple if the action is minimal and every fixed point set $X^s = \{x \in X : \sigma_s(x) = x\}$ is nowhere dense for $s \neq e$.*

If σ is a $\mathbb{Z}$-action on X consisting of infinitely many points, then every point $x \in X$ is aperiodic. Therefore, we get the following:

Corollary 5.1.5. *Let X consists of infinitely many points. Then the C^*-algebra $\mathbb{Z} \ltimes_{\alpha r} C(X)$ is simple if and only if the action is minimal.*

Topological dynamical systems are one of the main source to construct simple C^*-algebras together with reduced group C^*-algebras and AF algebras. The irrational rotation on the circle is a typical example, which will be considered in the next section.

NOTE

General theory for the crossed products in this section is standard, and may be found in monograph [**P**] or papers [**Tka75**], [**Zm68**]. See [**Ku88**] for related results with Exercise 5.1.2. A stronger result than Exercise 5.1.4 is in [**Gr78**, Proposition 14]. See also [**Ra92**] for related topics. It should be noted that the converse of Theorem 5.1.1 does not hold. Indeed, there is a C^*-dynamical system $\{A, G, \alpha\}$ with a non-amenable group such that $G \ltimes_\alpha A = G \ltimes_{\alpha r} A$ is simple [**Sp91**], [**QS92**] (see also §5.3). Also, note that Theorem 5.1.1 together with Exercise 5.1.1 says that $C^*(G)$ is nuclear for an arbitrary locally compact amenable group G. Proposition 5.1.2 is taken from [**Zm68**, Théorème 4.12]. We refer to [**To87**, §3.2] or [**To92**, Theorem 1.3] for an another approach. Theorem 5.1.3 is found in [**OP82**], where it is also shown that the assumption is satisfied if α_s is an *properly outer* action for each $s \neq e$. The reduced group C^*-algebra $C^*_\lambda(F_2)$ shows that the converse of Corollary 5.1.4 is not true. If G is an amenable discrete group then the condition on the fixed point set is also necessary [**KT90**]. See also [**AS**]. A proof of Corollary 5.1.5 is found in [**Pow78**]. Another important notion for simplicity is that of *central shift*, for which the full crossed product is simple [**JL**]. We refer to [**EH67**], [**To87**] or [**To92**] for more informations on the C^*-algebras arising from topological dynamical systems.

5.2. Irrational Rotation Algebras

We denote by $\mathbb{T}$ the quotient space $\mathbb{R}/\mathbb{Z}$. For an irrational number θ in the unit interval, we also denote by σ_θ the homeomorphism of $\mathbb{T}$ given by the rotation $\sigma_\theta : x \mapsto x + \theta$. This gives rise to a $\mathbb{Z}$-action on the commutative C^*-algebra $C(\mathbb{T})$, and we denote by A_θ the C^*-crossed product defined by the associated C^*-dynamical system $\{C(\mathbb{T}), \mathbb{Z}, \alpha\}$. It is a standard fact that this action is minimal, and so A_θ is a simple C^*-algebra. Also, note that A_θ is nuclear by Exercise 5.1.4. Let $\{\pi, \mathcal{H}\}$ be a representation of $C(\mathbb{T})$ and U a unitary of $\mathcal{B}(\mathcal{H})$ satisfying the covariant relation

$$\pi(\alpha(f)) = U\pi(f)U^*, \qquad f \in C(\mathbb{T}), \tag{5.2.1}$$

then this gives rise to a faithful representation of A_θ, because A_θ is simple. For each fixed $x \in \mathbb{T}$, we denote by π_x the representation of $C(\mathbb{T})$ on $\ell^2(\mathbb{Z})$ defined by

$$\pi_x(f)(e_n) = f(\sigma^n x)e_n, \qquad n \in \mathbb{Z},$$

where $\{e_n : n \in \mathbb{Z}\}$ denotes the usual orthonormal basis of $\ell^2(\mathbb{Z})$. Then, π_x together with the bilateral shift $S : e_n \mapsto e_{n+1}$ on $\ell^2(\mathbb{Z})$ satisfies the condition (5.2.1), and so we get a faithful representation $\widetilde{\pi}_x$ of A_θ on $\ell^2(\mathbb{Z})$.

Exercise 5.2.1. Show that $\widetilde{\pi}_x$ is an irreducible representation for each $x \in \mathbb{T}$. Show that $\widetilde{\pi}_x$ and $\widetilde{\pi}_y$ are unitarily equivalent if and only if $x = 1 - y$. Show also that A_θ is not postliminal.

Let $C(\mathbb{T})$ act on $L^2(\mathbb{T})$ by the multiplications M_f for $f \in C(\mathbb{T})$. We also denote by U the unitary of $L^2(\mathbb{T})$ given by

(5.2.2) $$U\xi(t) = \xi(\sigma^{-1}t) = \xi(t-\theta), \qquad \xi \in L^2(\mathbb{T}),\ t \in \mathbb{T}.$$

Then, we see that A_θ is the C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by $\{M_f : f \in C(\mathbb{T})\}$ and U. The finite sums of the form $\sum M_{f_n} U^n$ form a dense $*$-subalgebra of A_θ. We also note that the formula

(5.2.3) $$\tau(\sum M_{f_n} U^n) = \int_0^1 f_0(t)dt$$

defines a faithful trace, where dt is the usual Lebesgue measure.

Exercise 5.2.2. Show that τ is the unique trace on A_θ.

Theorem 5.2.1. *For each $\beta \in [0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta)$, there is a projection $P \in A_\theta$ such that $\tau(P) = \beta$*

Proof. We may assume that $0 < \beta < \frac{1}{2}$. Consider the following element

$$P = M_{\alpha^{-m}(\overline{g})} U^{-m} + M_f + M_g U^m,$$

where f is self-adjoint, and m is a nonzero integer. Then, we see that P is a projection if and only if the following conditions are satisfied:

(i) $g(t)g(t-m\theta) = 0$,
(ii) $g(t)\{1 - f(t) - f(t-m\theta)\} = 0$,
(iii) $f(t)(1-f(t)) = |g(t)|^2 + |g(t+m\theta)|^2$.

Choose $\epsilon > 0$ with $0 < \epsilon < m\theta < m\theta + \epsilon < \frac{1}{2}$. Let f be the piece-wise linear function on $[0,1]$ connecting the points $(0,0), (\epsilon,1), (m\theta,1), (m\theta+\epsilon,0)$ and $(1,0)$. Also, let g be a continuous function whose support lies in the interval $[m\theta, m\theta+\epsilon]$ where $g(t) = \sqrt{f(t)(1-f(t))}$. Then, we see that f and g satisfy the above conditions and $\tau(P) = \int_0^1 f(t)dt = m\theta$. $\square$

We denote by V the multiplication by the function $t \mapsto e^{2\pi i t}$. Then we see that

(5.2.4) $$e^{2\pi i\theta} UV = VU,$$

where U is the unitary in (5.2.2), and A_θ is generated by U and V.

Exercise 5.2.3. Show that any C^*-algebra generated by two unitaries with the relation (5.2.4) is isomorphic to A_θ.

Recall that every irrational number θ in $[0,1]$ has a unique continued fraction expansion $[a_1, a_2, \ldots]$ as in (3.4.7). Let B_θ be the unital AF algebra whose K_0-group is $\mathbb{Z}+\mathbb{Z}\theta$. Note that the associated Bratteli diagram of B_θ is determined by the sequence (3.4.6). More explicitly, $B_\theta = \varinjlim(B_n, \phi_n)$, where $B_n = M_{q_n} \oplus M_{q_{n-1}}$ and

$$\phi_n : M_{q_n} \oplus M_{q_{n-1}} \to M_{q_{n+1}} \oplus M_{q_n} : (x,y) \mapsto (\mathrm{Diag}\,(\overbrace{x, \ldots, x}^{a_n \text{ times}}, y), x).$$

The figure (3.1.4.f) represents the Bratteli diagram for B_θ in the case $a_n = 1$ for each $n = 1, 2, \ldots$. In the following, we find unitaries U and V in B_θ satisfying the relation (5.2.4). This would give an embedding of A_θ into B_θ. We denote by $\{e_j : j \in \mathbb{Z}_{q_n}\}$ the usual orthonormal basis for $\mathbb{C}^{q_n}$. Let U_n and V_n be the unitaries in M_{q_n} given by

$$U_n : e_j \mapsto e_{j+1}, \qquad V_n : e_j \mapsto \lambda_n^j e_j, \qquad j \in \mathbb{Z}_{q_n},$$

where $\lambda_n = \exp(2\pi i \frac{p_n}{q_n})$. We will find a unitary W_n in M_{q_n} such that

$$\|U_n - W_n(\mathrm{Diag}\,(\overbrace{U_{n-1}, \ldots, U_{n-1}}^{a_n \text{ times}}, U_{n-2}))W_n^*\| < \frac{4\pi}{q_{n-2}}, \tag{5.2.5}$$

$$\|V_n - W_n(\mathrm{Diag}\,(\overbrace{V_{n-1}, \ldots, V_{n-1}}^{a_n \text{ times}}, V_{n-2}))W_n^*\| < \frac{\pi}{q_{n-2}} + \frac{4\pi}{q_{n-1}}. \tag{5.2.6}$$

Now, we define the map

$$\psi_n : B_n \to B_{n+1} : (x,y) \mapsto (W_n(\mathrm{Diag}\,(\overbrace{x, \ldots, x}^{a_n \text{ times}}, y))W_n^*, x).$$

Since $\sum \frac{1}{q_n} < \infty$, the sequences $\{(U_n, U_{n-1}) : n = 1, 2, \ldots\}$ and $\{(V_n, V_{n-1}) : n = 1, 2, \ldots\}$ converge to the unitaries U and V, respectively, in the inductive limit C^*-algebra $\varinjlim(B_n, \psi_n)$, which is $*$-isomorphic to B_θ by the discussion in §3.1. The easy relation

$$V_n U_n = \exp\left(2\pi i \frac{p_n}{q_n}\right) U_n V_n, \quad n = 1, 2, \ldots$$

would give the required relation (5.2.4), because $\lim_n \frac{p_n}{q_n} = \theta$. In order to find W_n with (5.2.5) and (5.2.6), we need the following lemma:

Lemma 5.2.2. *Let T be an operator in $\mathcal{B}(\mathcal{H})$ and $\{e_i, f_i : i = 0, 1, \ldots, n\}$ an orthonormal set of $\mathcal{H}$ with $Te_i = e_{i+1}, Tf_i = f_{i+1}$ for $i = 0, 1, \ldots, n-1$. Then there is an operator $S \in \mathcal{B}(\mathcal{H})$ such that*

(i) $S\xi = T\xi$ *for* $x \in \{e_i, f_i : i = 0, 1, \ldots, n-1\}^\perp$,
(ii) $S^n e_0 = f_n$ *and* $S^n f_0 = e_n$,
(iii) $\|S - T\| < \frac{\pi}{n}$.

Proof. Let M_j be the subspace spanned by $\{e_j, f_j\}$ for $j = 0, 1, \ldots, n$. Let U_j be the unitary from M_j onto M_{j+1} defined by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with respect to the given bases. Define

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/n} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

We define S on M_j to be the unitary VU_j for $j = 0, 1, \ldots, n$. From the relations $V^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\|V - 1\| = |e^{\pi i/n} - 1| = 2\sin\frac{\pi}{2n} < \frac{\pi}{n}$, it is easy to see that the required properties hold. □

For the brevity, we only consider the case when $a_n = 1$ for each $n = 1, 2, \ldots$, because the general case is similar. We fix a natural number s with $\frac{q_{n-2}}{4} \le s \le \frac{q_{n-2}}{2}$. This is possible if $n \ge 5$. We also denote by b the integral part of $\frac{q_{n-1}}{2}$ and put $c = q_{n-1} - b$. We apply Lemma 5.2.2 to the unitary U_n and the bases $\{e_{-b}, e_{-b+1}, \ldots, e_{-b+s}\}$ and $\{e_c, e_{c+1}, \ldots, e_{c+s}\}$, to get U_n' which send the subspace M_j spanned by $\{e_{-b+j}, e_{c+j}\}$ onto the subspace M_{j+1} spanned by $\{e_{-b+j+1}, e_{c+j+1}\}$, for $j = 0, 1, \ldots, s-1$. We also denote by $\{e_i' : i \in \mathbb{Z}_{q_{n-1}}\}$ and $\{e_i'' : i \in \mathbb{Z}_{q_{n-2}}\}$ for the usual orthonormal bases for $\mathbb{C}^{q_{n-1}}$ and $\mathbb{C}^{q_{n-2}}$, respectively. Define the unitary $W_n : \mathbb{C}^{q_{n-1}} \oplus \mathbb{C}^{q_{n-2}} \to \mathbb{C}^{q_n}$ by

$$W_n(e_i') = f_{i+b} \mod q_{n-1}, \qquad W_n(e_i'') = g_{i+d} \mod q_{n-2},$$

where

$$f_k = \begin{cases} (U_n')^k(e_c), & 0 \le k \le s-1, \\ e_{-b+k}, & s \le j < q_{n-1}, \end{cases}$$
$$g_\ell = \begin{cases} (U_n')^\ell(e_{-b}), & 0 \le \ell \le s-1, \\ e_{c+\ell}, & s \le \ell < q_{n-2}, \end{cases}$$

and d is chosen so that $|\lambda_n^c - \lambda_{n-2}^{-d}| < \dfrac{\pi}{q_{n-2}}$. We see that

$$U_n' = W_n(\mathrm{Diag}\,(U_{n-1}, U_{n-2}))W_n^*,$$

and get the inequality (5.2.5) because $\dfrac{\pi}{s} \le \dfrac{4\pi}{q_{n-2}}$. Put

$$V_n' = W_n(\mathrm{Diag}\,(V_{n-1}, V_{n-2}))W_n^*.$$

It remains to show that

$$\|V_n - V_n'\| < \frac{\pi}{q_{n-2}} + \frac{4\pi}{q_{n-1}}. \tag{5.2.7}$$

Recall the following basic properties for the sequences $\{p_n\}$ and $\{q_n\}$;

$$p_n q_{n-1} - q_n p_{n-1} = \pm 1,$$
$$\left|\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right| \le \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right| = \frac{1}{q_{n-2}q_{n-1}}.$$

For k with $s \le k < q_{n-1}$, we see that

$$\begin{aligned} \|(V_n - V_n')(e_{-b+k})\| &= |\lambda_n^{-b+k} - \lambda_{n-1}^{-b+k}| \\ &= |\exp\ 2\pi i(-b+k)\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) - 1| \\ &= |\exp\ \frac{2\pi i(-b+k)}{q_n q_{n-1}} - 1| < \frac{2\pi}{q_{n-1}}. \end{aligned}$$

For ℓ with $s \le \ell < q_{n-2}$, we also have

$$\begin{aligned} \|(V_n - V_n')(e_{c+\ell})\| &= |\lambda_n^{c+\ell} - \lambda_{n-2}^{-d+\ell}| = |\lambda_n^{c+\ell} - \lambda_{n-2}^{-d}\lambda_n^\ell + \lambda_{n-2}^{-d}\lambda_n^\ell - \lambda_{n-2}^{-d+\ell}| \\ &\le |\lambda_n^c - \lambda_{n-2}^{-d}| + |\lambda_n^\ell - \lambda_{n-2}^\ell| \\ &< \frac{\pi}{q_{n-2}} + |\exp\ 2\pi i\ell\left(\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right) - 1| \\ &\le \frac{\pi}{q_{n-2}} + |\exp\ \frac{2\pi i\ell}{q_{n-1}q_{n-2}} - 1| \le \frac{\pi}{q_{n-2}} + \frac{2\pi}{q_{n-1}}. \end{aligned}$$

Finally, it remains to estimate $\|V_n - V_n'\|$ on the subspace M_j, which is invariant under V_n and V_n', for $j = 0, 1, \ldots, s-1$. Considering the vectors $f_j = (U_n')^j(e_c)$ and $g_j = (U_n')^j(e_{-b})$ which lies in M_j, we see that $V_n'|_{M_j}$ has eigenvalues $\lambda_{n-1}^{-b+j} (= \lambda_{n-1}^{c+j})$ and λ_{n-2}^{-d+j}. Also, note that $V_n|_{M_j}$ has eigenvalues λ_n^{-b+j} and λ_n^{c+j}. Now, we have

$$\begin{aligned}
\|V_n'|_{M_j} - V_n|_{M_j}\| &\le \|V_n'|_{M_j} - \lambda_n^{c+j}1_{M_j}\| + \|\lambda_n^{c+j}1_{M_j} - V_n|_{M_j}\| \\
&\le \max\{|\lambda_{n-1}^{c+j} - \lambda_n^{c+j}|, |\lambda_{n-2}^{-d+j} - \lambda_n^{c+j}|\} + |\lambda_n^{-b+j} - \lambda_n^{c+j}| \\
&\le \max\left\{\frac{2\pi}{q_{n-1}}, \frac{\pi}{q_{n-2}} + \frac{2\pi}{q_{n-1}}\right\} + \frac{2\pi}{q_{n-1}} \\
&= \frac{\pi}{q_{n-2}} + \frac{4\pi}{q_{n-1}},
\end{aligned}$$

where the third inequality follows from the former calculations and

$$|\lambda_n^{-b+j} - \lambda_n^{c+j}| = |\lambda_n^{q_{n-1}} - 1| = |\exp\frac{2\pi i}{q_n} - 1| < \frac{2\pi}{q_n}.$$

This completes the proof of (5.2.7), and we get the following:

Theorem 5.2.3. *There is a $*$-isomorphism ρ from A_θ into B_θ.*

Combining Theorems 5.2.1 and 5.2.3, we see that the range of the projections in A_θ under the trace is exactly $[0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta)$. It is easy to see that $[0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta) = [0,1] \cap (\mathbb{Z} + \mathbb{Z}\theta')$ if and only if $\theta = \theta'$ or $\theta = 1 - \theta'$. Because A_θ and $A_{1-\theta}$ is $*$-isomorphic each other, we obtain the following classification.

Theorem 5.2.4. *Let θ and θ' be two irrational numbers in the unit interval. Then A_θ and $A_{\theta'}$ is $*$-isomorphic if and only if $\theta = \theta'$ or $\theta = 1 - \theta'$.*

Now, we turn our attention to the cases of rational rotations. Let $\theta = \frac{p}{q}$ be a rational number in the unit interval with $(p, q) = 1$. Then the rotation σ by θ defines a $\mathbb{Z}_q$-action on the torus $\mathbb{T}$. For a complex number λ with $|\lambda| = 1$ and $x \in \mathbb{T}$, the unitary U_λ in $M_q(\mathbb{C})$ given by

$$U_\lambda(e_{q-1}) = \lambda e_0, \qquad U(e_i) = e_{i+1}, \qquad i = 0, 1, \ldots, q-2,$$

and the q-dimensional representation of $C(\mathbb{T})$

$$\pi_x(f) : e_i \mapsto f(\sigma^i x)e_i, \qquad i = 0, 1, \ldots, q-1$$

satisfy the covariant relation $\pi_x(\alpha(f)) = U_\lambda \pi_x(f) U_\lambda^*$, and they give rise to the irreducible representation $\pi_x \times U_\lambda$. It can be shown that every irreducible representation arises in this way and two irreducible representations $\pi_x \times U_\lambda$ and $\pi_y \times U_\mu$ are unitarily equivalent if and only if $\lambda = \mu$ and the orbits of x and y are identical. In this way, we see that A_θ is a q-homogeneous C^*-algebra with the spectrum $\mathbb{T}^2$ [**To87**, Theorem 4.2.1]. Actually, every homogeneous C^*-algebra with the spectrum $\mathbb{T}^2$ is $*$-isomorphic to $A_\theta \otimes M_n(\mathbb{C})$ for a rational number θ in the unit interval and a natural number n [**DR85**]. The rational rotation C^*-algebras are also classified as in Theorem 5.2.4.

We also consider more general construction which includes the rotation algebras. Let G be a discrete group then a character χ of G induces a unique $*$-automorphism α_χ on the C^*-algebra $C_\lambda^*(G)$ by

$$\alpha_\chi(L_s) = \chi(s) L_s, \qquad s \in G.$$

In this way, we get a $\mathbb{Z}$-action on $C_\lambda^*(G)$ and consider the C^*-crossed product $\mathbb{Z} \ltimes_{\alpha_\chi} C_\lambda^*(G)$. These C^*-algebras are completely classified when G is a free group with finite generators, a finite group, or the free abelian group with two generators [**Yi90**]. If $G = \mathbb{Z}$ then the every character χ is determined by a complex number $e^{2\pi i\theta}$, for a number $\theta \in [0,1]$. The induced action α_χ on $C^*(\mathbb{Z}) = C(\mathbb{T})$ is nothing but the action induced from the rotation by θ, and gives rise to the rotation algebra A_θ. If G is a discrete subgroup of $\mathbb{T}$ then the inclusion map $\iota : G \to \mathbb{T}$ is a character in itself. It is easy to see that this character defines a minimal action on $\widehat{G}$, and so we get a simple C^*-algebra $A_G = \mathbb{Z} \ltimes_{\alpha_\iota} C(\widehat{G})$. The irrational rotation algebra A_θ corresponds to an infinite cyclic subgroup of $\mathbb{T}$. If G is an infinite torsion subgroup of $\mathbb{T}$ then A_G is the *weighted shift algebra* considered in [**BD75**]. For countable subgroups G_1 and G_2 of $\mathbb{T}$, two C^*-algebras A_{G_1} and A_{G_2} are $*$-isomorphic if and only if $G_1 = G_2$ [**Rd82b**]. The similar situation has been considered in [**dBZ84**] for an arbitrary discrete abelian group G.

We conclude this section to mention another interesting properties of the irrational rotation algebras A_θ. First of all, the set of all invertible elements of A_θ is dense [**Rd85**], [**AP89**], [**Pu90**]. This is equivalent to say that the *topological stable rank* of A_θ is 1 [**Rf83**]. There is an another notion of rank. A unital C^*-algebra is said to be of *real rank zero* if the set of invertible self-adjoint elements are dense in set A_h of all self-adjoint elements [**BP91**]. There are several equivalent conditions, and we just mention two of them: A has the property (FS) if the self-adjoint elements with finite spectra are dense in A_h; A has the property (HP) if every hereditary C^*-subalgebra of A has an approximate identity consisting of projections. It is known that every irrational rotation algebra A_θ is of real rank zero [**CEl90**], [**BKR92**], [**EE**]. From Proposition 4.2.8, we see that every element x in the unit ball A_1 of a unital C^*-algebra A is the limit of the convex combinations of unitaries in A. It was shown that if the invertible elements are dense in A then every $x \in A_1$ is the convex combination of two or three unitaries [**KP85**]. In the case of irrational rotation algebras, two unitaries are not sufficient [**PR88**].

NOTE

For more informations on the representations of the irrational rotation algebra A_θ, we refer to [**Bk84**]. For the conditions for crossed products to be postliminal, we also refer to [**Zm68**], [**AT**]. Theorems 5.2.1 and 5.2.3 are due to Rieffel [**Rf81**] and Pimsner-Voiculescu [**PV80a**], respectively. We have followed [**Da84**] for the proof of Theorem 5.2.3. See also [**Lo91**] for more systematic approach. The embeddability of $G \ltimes_\alpha C(X)$ into an AF algebra is characterized by Pimsner [**Pm83**]. See also [**Kj81**], [**Kj84**] for another relations between A_θ and B_θ. The rational rotation algebras are classified in [**HkS81**]. See also [**Yi86**] and the references cited there for another ways to classify rational rotation algebras. There are another homeomorphisms of $\mathbb{T}$, called *Denjoy homeomorphisms*. Crossed products arising from these homeomorphisms are classified in [**PSS86**].

5.3. C^*-algebras Generated by Isometries

Let $S_1, S_2, \ldots, S_n$ $(n \geq 2)$ be isometries on an infinite dimensional Hilbert space $\mathcal{H}$ satisfying the relation

$$S_1S_1^* + S_2S_2^* + \cdots + S_nS_n^* = 1. \tag{5.3.1}$$

We will denote by $C^*(S_i)$ the C^*-algebra generated by $S_1, \ldots, S_n$. We also denote by U the bilateral shift on $\ell^2(\mathbb{Z})$ which sends e_i onto e_{i+1}, where $\{e_i : i \in \mathbb{Z}\}$ is the usual orthonormal basis of $\ell^2(\mathbb{Z})$. If we put $T_i = S_i \otimes U$ for $i = 1, 2, \ldots, n$, then $\{T_i : i = 1, 2, \ldots, n\}$ is also a family of isometries satisfying the relation (5.3.1). We also denote by $C^*(T_i)$ the C^*-algebra generated by $T_1, \ldots, T_n$.

For $k = 1, 2, \ldots$, let Γ_k be the set of all ordered sets $p = (p_1, p_2, \ldots, p_k)$ whose elements come from $\{1, 2, \ldots, n\}$, and put $V_p = T_{p_1}T_{p_2}\ldots T_{p_k}$ for each

$p \in \Gamma_k$. We use the convention $\Gamma_0 = \{0\}$ and $V_0 = 1$. It is easy to see that the operators $\{V_p V_q^* : p, q \in \Gamma_k\}$ form a matrix unit, and so the linear combinations of them form a C^*-subalgebra B_k of $C^*(T_i)$ which is $*$-isomorphic to $M_{n^k}(\mathbb{C})$. Because $B_k \subseteq B_{k+1}$ and $1 \in B_k$ for $k = 1, 2, \ldots$, we see that $B = \overline{\cup_k B_k}$ becomes a UHF C^*-subalgebra of $C^*(T_i)$. Now, we define a linear map $\phi : C^*(T_i) \to \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$ by

$$\phi(x) = \sum_{i=-\infty}^{+\infty} E_i x E_i, \tag{5.3.2}$$

where $E_i = 1_{\mathcal{H}} \otimes p_i$, and $p_i \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is the one-dimensional projection onto $\mathbb{C}e_i$, for $i \in \mathbb{Z}$. Note that the right side of (5.3.2) converges to a bounded linear operator on $\mathcal{H} \otimes \ell^2(\mathbb{Z})$ by Proposition 2.1.1. For any $x \in C^*(T_i)$, we have

$$\phi(x)^*\phi(x) = \sum_i E_i x^* E_i x E_i \leq \sum_i E_i x^* x E_i \leq \|x^* x\| \sum_i E_i = \|x\|^2 1,$$

and so ϕ is a contraction. It is also easy to see that $\phi(V_p V_q^*) = V_p V_q^*$ if $p, q \in \Gamma_k$, whereas $\phi(V_p V_q^*) = 0$ if $p \in \Gamma_k$ and $q \in \Gamma_\ell$ with $k \neq \ell$. Because the linear combinations of $V_p V_q^*$ form a dense $*$-subalgebra of $C^*(T_i)$, we see that ϕ in (5.3.2) defines a conditional expectation of $C^*(T_i)$ onto B by Theorem 4.4.3. It is also easy to see that ϕ is faithful.

Lemma 5.3.1. *Put $R_k = T_2 T_1^k$, and $U_k = \sum_{p \in \Gamma_k} V_p R_k V_p^*$ for $k = 1, 2, \ldots$. Then we have $\phi(x) = \lim_{k\to\infty} U_k^* x U_k$ for each $x \in C^*(T_i)$, in the norm topology.*

Proof. Because $U_k^* U_k = \sum_{p \in \Gamma_k} V_p V_p^* = 1$, each U_k is an isometry and $\{U_k : k = 1, 2, \ldots\}$ is uniformly bounded. Hence, it suffices to show the equality for $x = V_p V_q^*$. If $p, q \in \Gamma_\ell$ then we have $U_\ell^* V_p V_q^* U_\ell = V_p V_q^*$ by a calculation. From the relation $B_\ell \subseteq B_k$ for $\ell \leq k$, we see that $\lim_k U_\ell^* x U_k = x$ for each $x \in B_\ell$ and $\ell = 1, 2, \ldots$. The equality follows from the relation $\phi(x) = x$ in this case. Now, we assume that $p \in \Gamma_k$ and $q \in \Gamma_\ell$ with $\ell < k$. Then we have

$$U_k^* V_p V_q^* U_k = \sum_{j \in \Gamma_k} V_p R_k^* V_q^* V_j R_k V_j^* = \sum_{j \in \Gamma_{k-\ell}} V_p R_k^* V_j R_k V_j^* V_q^*.$$

Note that $R_k^* V_j \neq 0$ only if $V_j = T_2 T_1^{k-\ell-1}$, in which case $R_k^* V_j = (T_1^*)^{\ell+1}$. Therefore, we see that $U_k^* V_p V_q^* U_k = 0$ since $T_1^* T_2 = 0$. For $m > k$ the relation $U_m^* V_p V_q^* U_m = 0$ also follows similarly. $\square$

Let I be a non-zero closed two-sided ideal of $C^*(T_i)$. Because ϕ is a faithful conditional expectation, we see that $\phi(I)$ is also a nonzero ideal of B, which is closed. Because every UHF algebra is simple by Corollary 3.2.3, we see that $\phi(I) = B$. By Lemma 5.3.1, $1 \in \phi(I) \subseteq I$, and so $C^*(T_i)$ is a simple C^*-algebra.

Theorem 5.3.2. *The C^*-algebra $C^*(S_i)$ is simple. If $S_1', \dots, S_n'$ are another isometries with the relation* (5.3.1) *then there is a $*$-isomorphism of $C^*(S_i)$ onto $C^*(S_i')$ which sends S_i to S_i' for each $i = 1, 2, \dots, n$.*

Proof. Note that $C(\mathbb{T})$ is the C^*-algebra generated by U. Let $\psi : C(\mathbb{T}) \to \mathbb{C}$ be the $*$-homomorphism given by the evaluation map at $1 \in \mathbb{T}$. Then $1 \otimes \psi$ is a $*$-homomorphism from $C^*(T_i)$ onto $C^*(S_i)$, which is an isomorphism from the simplicity of $C^*(T_i)$. For the second assertion, put $S_i'' = S_i \oplus S_i'$. Then the map $x \oplus y \mapsto x$ gives rise to a $*$-isomorphism from $C^*(S_i'')$ onto $C^*(S_i)$ as before. $\square$

We denote by $\mathcal{O}_n$ the C^*-algebra generated by n isometries $S_1, S_2 \dots, S_n$ with the relation (5.3.1). This is said to be the *Cuntz algebra*. The Cuntz algebra arises naturally from the crossed product constructions.

Let G be the free product $\mathbb{Z}_2 * \mathbb{Z}_{n+1}$ $(n \geq 2)$ of the two cyclic groups, that is, G is generated by a and b with the relations $a^2 = e$ and $b^{n+1} = e$. We denote by X the set of all sequences $x : \mathbb{N} \to \{a, b, b^2, \dots b^n\}$ such that $x_k = a$ if and only if $x_{k+1} = b^i$ for some $i = 1, 2, \dots, n$. Then X is a compact Hausdorff space with respect to the Tychonoff topology. We define two maps σ_a and σ_b from X into X by

$$\sigma_a(x_0, x_1, \dots) = \begin{cases} (x_1, x_2 \dots), & x_0 = a, \\ (a, x_0, x_1, \dots), & x_0 = b^i, \ i = 1, 2, \dots, n, \end{cases}$$

$$\sigma_b(x_0, x_1, \dots) = \begin{cases} (b, x_0, x_1, \dots), & x_0 = a, \\ (x_1, x_2, \dots), & x_0 = b^n, \\ (b^{i+1}, x_1, x_2, \dots), & x_0 = b^i, \ i = 1, 2, \dots, n-1. \end{cases}$$

It is easy to see that σ_a and σ_b are homeomorphisms of X satisfying the relations $(\sigma_a)^2 = (\sigma_b)^{n+1} = 1_X$. Therefore, we have the topological dynamical system $\{X, G, \sigma\}$.

It is easy to see that two points x and y in X are in the same orbit if and only if there is a natural number m such that $x_n = y_{n+m}$ for sufficiently large n, and so we see that the system $\{X, G, \sigma\}$ is minimal. Next, we show that the fixed point set X^s is nowhere dense for each $s \in G$. Because $X^{tst^{-1}} = \sigma_t(X^s)$, it suffices to consider conjugacy classes. It is easy to see that the set of conjugacy classes is represented by

$$\{a\} \cup \{b^i : i = 1, 2, \ldots, n\} \cup \Delta_a \cup \Delta_b,$$

where Δ_a (respectively Δ_b) is the set of all reduced words which begin with a power of b (respectively a) and end with a (respectively a power of b). We also note that $s \in G$ is of finite order if and only if it is conjugate to a or b^i for some $i = 1, 2, \ldots n$, for which the fixed point set X^s is empty. For $s \in \Delta_a \cup \Delta_b$, we define two points $x_+(s)$ and $x_-(s)$ in X to be the periodic sequences with periods given by the letters in the reduced words of s and s^{-1}, respectively. It is easy to see that X^s consists of the two points $x_+(s)$ and $x_-(s)$. From Corollary 5.1.4, we have the following:

Proposition 5.3.3. *The C^*-algebra $G \ltimes_{\alpha r} C(X)$ is simple.*

Let p be the characteristic function on the subset $X_0 = \{x \in X : x_0 = a\}$ of X which is closed and open. If we identify G with its image in $G \ltimes_\alpha C(X)$ then we have the following relations:

$$\begin{aligned} p + apa &= 1, \\ p + bpb^{-1} + \cdots + b^n pb^{-n} &= 1. \end{aligned} \tag{5.3.3}$$

Therefore, every covariant representation of $\{C(X), G, \alpha\}$ gives rise to a unitary representation of G and a self-adjoint projection p with the relations (5.3.3). Conversely, every covariant representation of $\{C(X), G, \alpha\}$ arises in this way. To show this, let $\{u, \mathcal{H}\}$ be a unitary representation of G with a projection p with the relations (5.3.3). Because the family $\{\sigma_s(X_0) : s \in G\}$ forms a base of X consisting of clopen subsets, the characteristic functions

$\{\chi_{\sigma_s(X_0)} : s \in G\}$ generate the C^*-algebra $C(X)$. We see that the mapping $\chi_{\sigma_s(X_0)} \mapsto u(s)pu(s)^*$ defines a representation of $C(X)$ by the relations (5.3.3). It is clear that this representation together with u is covariant.

Proposition 5.3.4. *The C^*-algebra $G \ltimes_\alpha C(X)$ is $*$-isomorphic to the C^*-algebra $M_2(\mathcal{O}_n)$.*

Proof. With the generators a, b and p of $G \ltimes_\alpha C(X)$ satisfying the relations (5.3.3), define generators of $M_2(\mathcal{O}_n)$ by

$$\begin{pmatrix} S_k & 0 \\ 0 & 0 \end{pmatrix} = ab^k p, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = ap,$$

for $k = 1, 2, \ldots, n$. Conversely, for the generators $S_1, \ldots, S_n$ of $\mathcal{O}_n$, put

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & S_n^* \\ S_1 & S_2 S_1^* + \cdots + S_n S_{n-1}^* \end{pmatrix}.$$

Then, it is easy to see that a, b and p satisfy the relation (5.3.3). □

By Theorem 5.3.2 and Proposition 5.3.4, we see that the full crossed product $G \ltimes_\alpha C(X)$ is simple, and so we have

$$G \ltimes_\alpha C(X) = G \ltimes_{\alpha r} C(X). \tag{5.3.4}$$

This gives an another proof of Proposition 5.3.3. Note that the group G contains a copy of the free group F_2 on two generators. For example, two elements $abab$ and ab^2ab^2 generate a copy of F_2. By Proposition 4.5.2, G is not amenable. Therefore, we see that the converse of Theorem 5.1.1 does not hold. We will also show that $G \ltimes_{\alpha r} C(X)$ is nuclear, from which we infer that the converse of Exercise 5.1.4 also fails. Because $C^*_\lambda(G)$ is a C^*-subalgebra of $G \ltimes_{\alpha r} C(X)$, this also gives an example of non-nuclear C^*-subalgebra of a nuclear C^*-algebra as was mentioned at the end of §4.4. To do this, we consider the another crossed product construction as follows:

Let $\mathcal{F}^n$ be the UHF algebra generated by the sequence

$$M_n \hookrightarrow M_{n^2} \hookrightarrow \cdots \hookrightarrow M_{n^k} \hookrightarrow \cdots.$$

Then $\mathcal{F}^n$ may be described as an infinite tensor product of copies N_j's of M_n, as was explained in §3.2. We denote by $A_j = \otimes_{i=j}^{\infty} N_i$, and consider the embeddings

$$A_0 \hookrightarrow A_{-1} \hookrightarrow \dots A_{-j} \hookrightarrow \dots$$

given by $x \mapsto e_{11} \otimes x : A_j \to A_{j-1}$. The resulting inductive limit $\mathcal{C}_n$ is $*$-isomorphic to the C^*-algebra $\mathcal{K}(\mathcal{H}) \otimes \mathcal{F}^n$. Because we may continue the above embeddings to the left-side for positive integers and all A_j are isomorphic, we may consider the $*$-automorphism α induced by the shift to the left. Then the crossed product $\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n$ has a unitary u with the relation $\alpha(x) = uxu^*$ for $x \in \mathcal{C}_n$.

Let P be the identity of A_0 sitting in the C^*-algebra $\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n$. Because $uPu^* = e_{11} \otimes P \in M_n \otimes A_1$, we have the relation $uP = PuP$. If we put $v = uP$, then we see that

$$P(\sum_{i<0} u^i x_i + x_0 + \sum_{i>0} x_i u^i)P = \sum_{i<0} v^i P x_i P + P x_0 P + \sum_{i>0} P x_i P v^i,$$

for $x_i \in \mathcal{C}_n$. This says that $\mathcal{G}_n = P(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P$ is generated by $P\mathcal{C}_n P = A_0$ and v. If we put $S_i = (e_{i1} \otimes P)v$ for $i = 1, 2, \dots, n$, then we have the relations

$$S_i^* S_i = P, \qquad S_1 S_1^* + \dots + S_n S_n^* = P.$$

For k-tuples $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ in Γ_k, we see that

$$(S_{p_1} \dots S_{p_k})(S_{q_1} \dots S_{q_k})^* = e_{p_1 q_1} \otimes \dots \otimes e_{p_k q_k} \otimes P \in A_0 = M_n \otimes \dots \otimes M_n \otimes A_k.$$

Therefore, it follows that the C^*-algebra A_0 is generated by the elements of the form $(S_{p_1} \dots S_{p_k})(S_{q_1} \dots S_{q_k})^*$, and so $\mathcal{G}_n$ is generated by $S_1, \dots, S_n$. Hence, $\mathcal{G}_n$ is $*$-isomorphic to $\mathcal{O}_n$. If we denote by P_k the identity of $A_k \subseteq \mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n$ for $k = 0, -1, -2, \dots$, then $\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n$ is the inductive limit of the sequence

$$P_0(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P_0 \hookrightarrow P_{-1}(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P_{-1} \hookrightarrow \dots P_{-k}(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P_{-k} \hookrightarrow \dots.$$

It is also easy to see that $P_{k-1}(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P_{k-1}$ is generated by $P_k(\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n)P_k$ and the subset $\{e_{ij} \otimes P_k : i, j = 1, 2, \dots, n\}$ of A_{k-1}. Therefore, we have the following:

Proposition 5.3.5. *The C^*-algebra $\mathbb{Z} \ltimes_{\alpha r} \mathcal{C}_n$ is $*$-isomorphic to the C^*-algebra $\mathcal{K}(\mathcal{H}) \otimes \mathcal{O}_n$.*

By Exercise 5.1.4, we see that the Cuntz algebra $\mathcal{O}_n$ is nuclear. Note that the identity 1 of the Cuntz algebra is an infinite projection. This is a sharp distinction from simple AF algebras or irrational rotation algebras. It is known that $\mathcal{O}_n$ is not the inductive limit of postliminal C^*-algebras [**Cu77**]. On the other hand, it was recently shown [**EE**] that the irrational rotation C^*-algebra is isomorphic to the inductive limit of a sequence of direct sums of two matrix algebras over $C(\mathbb{T})$. The construction of $\mathcal{O}_n$ has been generalized to the C^*-algebras generated by partial isometries [**CK80**, **Cu81c**].

We close this section with an extension of the Cuntz algebra by the compact ideal $\mathcal{K}(\mathcal{H})$, which will be useful to compute the K-groups of the Cuntz algebras. We denote by $\mathcal{D}$ the C^*-subalgebra of $\mathcal{O}_{n+1}$ generated by $S_1, \ldots, S_n$, and by $\mathcal{J}$ the closed ideal of $\mathcal{D}$ generated by the projection $P = 1 - \sum_{i=1}^n S_i S_i^*$. As before, put $V_p = S_{p_1} S_{p_2} \ldots S_{p_k}$ for $p = (p_1, p_2, \ldots, p_k) \in \Gamma_k$. Then $\mathcal{J}$ is the closure of the linear span of elements of the form $V_p P V_q^*$ with $p, q \in \Gamma_k$, $k = 1, 2, \ldots$. From the easy relations

$$(V_p P V_q^*)(V_r P V_s)^* = \delta_{qr} V_p P V_s^*, \qquad (V_p P V_q^*)^* = V_q P V_p^*,$$

we have the following:

Proposition 5.3.6. *The ideal $\mathcal{J}$ is $*$-isomorphic to $\mathcal{K}(\mathcal{H})$, and we have the following short exact sequence:*

$$0 \to \mathcal{J} \to \mathcal{D} \to \mathcal{O}_n \to 0. \tag{5.3.5}$$

NOTE

The C^*-algebra $\mathcal{O}_n$ was studied by Cuntz [**Cu77**], in which Theorem 5.3.2, Propositions 5.3.5 and 5.3.6 have been proved. For the proof of Theorem 5.3.2, we have followed a simpler method in [**dSvD81**]. The construction of the topological dynamical system $\{X, G, \sigma\}$ was taken from [**Sp91**]. The embedding of $C^*_\lambda(\mathbb{Z}_2 * \mathbb{Z}_3)$ into the nuclear C^*-algebra $\mathcal{O}_2$ is due to Choi [**Ch79**], in which he showed that if $n = 2$ then $G \ltimes_{\alpha r} C(X) = M_2(\mathcal{O}_2)$ is actually isomorphic to $\mathcal{O}_2$ (see also [**PS79b**]). Later, Blackadar [**Bl85**] showed that every non-postliminal C^*-algebra contains a nonnuclear C^*-subalgebra. See also [**Ki!**].

5.4. K-theory for C^*-algebras

In this section, we introduce the notion of K-theory for C^*-algebras which is periodic with period two, and so enjoys the six-term exact sequence. Because the theory is now standard, we just define the K-groups and state the results. Detailed arguments are found in monographs such as [**Bl**, Chapter IV] or [**M**, Chapter 7]. Recall the definition of $K_0(A)$ for a *unital* C^*-algebra A in §3.3. We denote this group by $K_0'(A)$ for a moment. Thus, an element of $K_0'(A)$ is a formal difference $[p]-[q]$, with the identification $[p_1]-[q_1]=[p_2]-[q_2]$ if and only if there is an idempotent $r \in M_n(A)$ such that $p_1+q_2+r \sim p_2+q_1+r$. For a C^*-algebra, unital or non-unital, let A_I denote the algebra obtained from A by adjoining the identity, and $\tau : A_I \to \mathbb{C}$ the canonical quotient map. We define

$$K_0(A) = \operatorname{Ker}[\, K_0'(A_I) \xrightarrow{\tau_*} K_0'(\mathbb{C})\,]. \tag{5.4.1}$$

Therefore, every element of $K_0(A)$ is written as a formal difference $[e]-[f]$, where $e, f \in M_n(A_I)$ with $e-f \in M_n(A)$. It is easy to see that we may assume that $f = p_n$ for some large n, where p_n denotes the identity of $M_n(A_I)$. It is also easily seen that if A is unital then $K_0(A) = K_0'(A)$, and so there is no confusion. It is not so difficult to see that Theorem 3.4.1 holds for this new K_0-group. The following is one of the reason to consider the K_0-groups for non-unital C^*-algebras.

Proposition 5.4.1. *Let*

$$0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0 \tag{5.4.2}$$

be a short exact sequence of C^*-algebras. Then the sequence

$$K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J)$$

is exact.

Two $*$-homomorphisms $\phi, \psi : A \to B$ are said to be *homotopic* each other if there is a family $\{\phi_t : t \in [0,1]\}$ of $*$-homomorphisms from A to B such that $\phi_0 = \phi$, $\phi_1 = \psi$ and the map $t \mapsto \phi_t(a)$ is continuous for each $a \in A$.

Proposition 5.4.2. *If $\phi, \psi : A \to B$ are homotopic each other then we have $\phi_* = \psi_*$.*

Another important property of the K_0-group is the stability. If p is a one-dimensional projection in C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators, then the map $\phi : A \to A \otimes \mathcal{K}(\mathcal{H})$ defined by

$$\phi(a) = a \otimes p, \qquad a \in A$$

is a $*$-homomorphism, which induces a group homomorphism

$$\phi_* : K_0(A) \to K_0(A \otimes \mathcal{K}(\mathcal{H})).$$

Proposition 5.4.3. *The above homomorphism ϕ_* is independent of the choice of the projection p, and defines a group isomorphism between $K_0(A)$ and $K_0(A \otimes \mathcal{K}(\mathcal{H}))$.*

For a C^*-algebra A, we denote by

$$U_n(A) = \{u \in \mathcal{U}(M_n(A_I)) : u - 1_n \in M_n(A)\},$$

where 1_n denotes the identity of $M_n(A_I)$. This is a topological group with the norm topology. We also denote by $U_n(A)_0$ the connected component of the identity, which is a normal open subgroup of $U_n(A)$. We embed $U_n(A)$ into $U_{n+1}(A)$ by $u \mapsto \mathrm{Diag}\,(u, 1)$. Let $U_\infty(A)$ and $U_\infty(A)_0$ be the inductive limits obtained in this way. Now, we define

$$K_1(A) = U_\infty(A)/U_\infty(A)_0 = \varinjlim U_n(A)/U_n(A)_0. \tag{5.4.3}$$

Another description of $K_1(A)$ would be useful. For a C^*-algebra A, we define the *suspension* $\mathcal{S}A$ by

$$\mathcal{S}A = \{f \in C([0,1], A) : f(0) = f(1) = 0\},$$

which is an ideal of $C[0,1] \otimes A$.

Proposition 5.4.4. *For a C^*-algebra A, we have*

$$K_1(A) = K_0(\mathcal{S}A).$$

For $u \in U_n(A)$, let $(z_t)_{t\in[0,1]}$ be a path in $U_{2n}(A)$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. For example, we may put

$$z_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{2}t & \sin\frac{\pi}{2}t \\ -\sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix}.$$

If we define $e_t = z_t p_n z_t^{-1}$ for $t \in [0,1]$, then $e = (e_t)_{t\in[0,1]}$ is an idempotent in $M_{2n}((\mathcal{S}A)_I)$, where $p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the projection in $M_{2n}(A_I)$. One can show that the map

$$\theta_A : [u] \mapsto [e] - [p_n] : K_1(A) \to K_0(\mathcal{S}A) \tag{5.4.4}$$

gives a group isomorphism from $K_1(A)$ onto $K_0(\mathcal{S}A)$, where p_n is the constant function in $M_{2n}((\mathcal{S}A)_I)$ with the value $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Exercise 5.4.1. Show that the K_1-group of an AF algebra or a von Neumann algebra is $\{0\}$. Show also that $K_1(C(\mathbb{T})) = \mathbb{Z}$.

In the above description of the K_1-group, the groups $U_n(A)$ and $U_n(A)_0$ may be replaced by the group $GL_n(A)$ of all invertible elements and $GL_n(A)_0$, respectively. Now, we define the connecting map $K_1(A/J) \to K_0(J)$, to get a long exact sequence. For $u \in GL_n(A/J)$, let $w \in GL_{2n}(A)$ be a lift of $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Then the map given by

$$\partial : [u] \mapsto [wp_nw^{-1}] - [p_n] : K_1(A/J) \to K_0(J) \tag{5.4.5}$$

defines a well-defined group homomorphism from $K_1(A/J)$ to $K_0(J)$. If A is a unital C^*-algebra and a unitary u in A/J lifts to a partial isometry $v \in A$ then $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ lifts to the unitary $w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}$. Hence, we have

$$\partial([u]) = [wp_1w^{-1}] - [p_1] = \left[\begin{pmatrix} vv^* - 1 & 0 \\ 0 & 1 - v^*v \end{pmatrix}\right] = [1 - v^*v] - [1 - vv^*].$$

If $A = \mathcal{B}(\mathcal{H})$ and $J = \mathcal{K}(\mathcal{H})$ then the map ∂ sends a unitary in the *Calkin algebra* $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ to its Fredholm index. Recall that an operator $x \in \mathcal{B}(\mathcal{H})$ is said to be a *Fredholm* operator if both $\operatorname{Ker} x$ and $\operatorname{Im} x$ are finite dimensional. It is well-known that x is a Fredholm operator if and only if the image of x in the Calkin algebra is invertible. In this sense, the map ∂ in (5.4.5) is said to be the *index map*.

Proposition 5.4.5. *For the short exact sequence* (5.4.2), *we have the following long exact sequence;*

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

Exercise 5.4.2. What is the K_1-group of the Calkin algebra ?

The higher K-groups are defined inductively: $K_n(A) = K_{n-1}(\mathcal{S}A)$. We define a group homomorphism $\beta_A : K_0(A) \to K_1(\mathcal{S}A)$ as follows: Put

$$\Omega A = \{f \in C([0,1], A) : f(0) = f(1)\}.$$

Then we have the following split exact sequence

$$0 \to \mathcal{S}A \to \Omega A \xrightarrow{\eta} A \to 0, \tag{5.4.6}$$

where η is the evaluation at 1, and so we see that $K_1(\mathcal{S}A) = \operatorname{Ker}\eta_*$. For a projection $p \in M_n(A_I)$, we define $f_p(t) = e^{2\pi i t}p + (1-p)$ for $t \in [0,1]$. Then $f_p \in \Omega(GL_n(A_I)) = GL_n(\Omega(A_I))$. If $[p]-[q] \in K_0(A)$ then we see that $f_p f_q^{-1} \in GL_n(\Omega A)$ and $\eta_*([f_p f_q^{-1}]) = 0$. Now, we define

$$\beta_A : [p]-[q] \mapsto [f_p f_q^{-1}] : K_0(A) \to K_1(\mathcal{S}A) = K_2(A). \tag{5.4.7}$$

Theorem 5.4.6. *The map β_A is a group isomorphism.*

Corollary 5.4.7. *For the short exact sequence* (5.4.2), *we have the following six-term exact sequence:*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/J) \\ \uparrow{\scriptstyle\partial} & & & & \downarrow{\scriptstyle\partial} \\ K_1(A/J) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{\iota_*} & K_1(J) \end{array} \tag{5.4.8}$$

If $[e] - [p_n] \in K_0(A/J)$, where e is an idempotent in $M_n((A/J)_I)$ with $\pi(x) = e$ for $x \in M_n(A_I)$, then we see that $\exp(2\pi i x) \in M_n(J_I)$. We define

$$\exp : [e] - [p_n] \mapsto [\exp(2\pi i x)] : K_0(A/J) \to K_1(J). \tag{5.4.8}$$

Note that $z_t = \begin{pmatrix} \exp(2\pi i t x) & 0 \\ 0 & \exp(-2\pi i t x) \end{pmatrix}$ is a path in $U_{2n}(A)$ which connects $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \exp(2\pi i x) & 0 \\ 0 & \exp(-2\pi i x) \end{pmatrix}$. From this, we see that the following diagram commutes:

$$\begin{array}{ccccc} K_0(A/J) & & \xrightarrow{\exp} & & K_1(J) \\ \downarrow{\scriptstyle \beta_{A/J}} & & & & \downarrow{\scriptstyle \theta_J} \\ K_1(\mathcal{S}(A/J)) & = & K_1(\mathcal{S}A/\mathcal{S}J) & \xrightarrow{\partial} & K_0(\mathcal{S}J) \end{array}$$

In this sense, The connecting map $\partial : K_0(A/J) \to K_1(J)$ is said to be the *exponential map*.

5.5. K-theory for Crossed Products of C^*-algebras

We apply Corollary 5.4.7 to get six-term exact sequences for the crossed products of C^*-algebras. To do this, we consider the *twisted tensor products* by the C^*-algebra $\mathcal{O}_n$, for $n = 1, 2, \ldots$, where $\mathcal{O}_1 = C(\mathbb{T})$ denotes the C^*-algebra generated by a single unitary S_1 whose spectrum is the whole circle, and $\mathcal{O}_n$ with $n = 2, 3, \ldots$ is the Cuntz algebra. For a C^*-algebra A acting on a Hilbert space $\mathcal{H}$ and a family $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ of pairwise commuting unitaries in $\mathcal{B}(\mathcal{H})$, we define the twisted tensor product $A \times_{\mathcal{U}} \mathcal{O}_n$ by the C^*-subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{O}_n$ generated by $A \otimes 1$ and $U_1 \otimes S_1, \ldots, U_n \otimes S_n$. If $n = 1$ then this reduces to the usual crossed product $\mathbb{Z} \ltimes_{\alpha r} A$ with a single $*$-automorphism $\alpha = \operatorname{Ad} U_1$.

We denote by $\mathcal{E}$ the C^*-subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{O}_{n+1}$ generated by $A \otimes 1$ and $U_1 \otimes S_1, \ldots U_n \otimes S_n$. We define a canonical endomorphism Φ of $\mathcal{O}_{n+1}$ by

$$\Phi(x) = \sum_{i=1}^{n+1} S_i x S_i^*, \qquad x \in \mathcal{O}_{n+1}.$$

Then $\{\Phi(S_1), \ldots, \Phi(S_n)\}$ is a family of isometries with pairwise orthogonal ranges such that $\sum_{i=1}^{n} \Phi(S_i)\Phi(S_i)^* < 1$. Applying Theorem 5.3.2, we see that the map

$$\phi : aU_i \otimes S_i \mapsto aU_i \otimes \Phi(S_i), \qquad a \in A,\ i = 1, 2, \ldots, n \tag{5.5.1}$$

extends to an isomorphism from $\mathcal{E}$ into $\mathcal{B}(\mathcal{H}) \otimes \mathcal{O}_{n+1}$. We denote by $\widehat{\mathcal{E}}$ the C^*-algebra generated by $\mathcal{E}$ and $\phi(\mathcal{E})$. We also define the map β on $\mathcal{E}$ by

$$\beta(x) = (1 \otimes S_{n+1})x(1 \otimes S_{n+1})^*, \qquad x \in \mathcal{E}, \tag{5.5.2}$$

then from the relation $S_{n+1}S_{n+1}^* = 1 - \sum_{i=1}^n S_i S_i^*$, we see that

$$\beta(x) = \phi(x) - \sum_{i=1}^{n} (1 \otimes S_i)x(1 \otimes S_i)^* \in \widehat{\mathcal{E}}. \tag{5.5.3}$$

Therefore, the C^*-algebra $\widehat{\mathcal{E}}$ is nothing but the C^*-subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{O}_{n+1}$ generated by $\mathcal{E}$ and $\beta(\mathcal{E})$. We denote by $\mathcal{I}$ (respectively $\widehat{\mathcal{I}}$) the closed ideal of $\mathcal{E}$ (respectively $\widehat{\mathcal{I}}$) generated by $\beta(A \otimes 1)$ (respectively $\beta(\mathcal{E})$). We also denote by $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the compact ideal of $\mathcal{B}(\mathcal{H})$.

Proposition 5.5.1. *We have the following short exact sequences:*

$$0 \to \mathcal{I} \xrightarrow{i} \mathcal{E} \xrightarrow{q} A \times_{\mathcal{U}} \mathcal{O}_n \to 0, \tag{5.5.4}$$

$$0 \to \widehat{\mathcal{I}} \xrightarrow{\widehat{i}} \widehat{\mathcal{E}} \xrightarrow{\widehat{q}} A \times_{\mathcal{U}} \mathcal{O}_n \to 0, \tag{5.5.5}$$

where $\mathcal{I}$ (respectively $\widehat{\mathcal{I}}$) is $*$-isomorphic to $\mathcal{K} \otimes A$ (respectively $\mathcal{K} \otimes \mathcal{E}$). These two sequences are related by the following commuting diagram;

$$\begin{array}{ccccccc} \mathcal{K} \otimes A & \simeq & \mathcal{I} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{q} & A \times_{\mathcal{U}} \mathcal{O}_n \\ \downarrow{\scriptstyle \mathrm{id}\otimes k} & & & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{K} \otimes \mathcal{E} & \simeq & \widehat{\mathcal{I}} & \xrightarrow{\widehat{i}} & \widehat{\mathcal{E}} & \xrightarrow{\widehat{q}} & A \times_{\mathcal{U}} \mathcal{O}_n \end{array} \tag{5.5.6}$$

where $k : A \to \mathcal{E}$ is the inclusion map given by $a \mapsto a \otimes 1$, and $j : \mathcal{E} \to \widehat{\mathcal{E}}$ is also the inclusion map.

Proof. First, we show that $\mathcal{I} \simeq \mathcal{K} \otimes A$. This follows from the similar argument as in Proposition 5.3.6 if $n \geq 2$. When $n = 1$, it is clear that $\mathcal{I}$ is the closure of the linear span of elements of the form $(U_1 \otimes S_1)^i \beta(x)(U_1 \otimes S_1)^{*j}$ for $x \in A$ and $i, j = 1, 2, \ldots$. Therefore, the map

$$(U_1 \otimes S_1)^i \beta(x)(U_1 \otimes S_1)^{*j} \mapsto e_{ij} \otimes \beta(x)$$

extends to an isomorphism from $\mathcal{I}$ onto $\mathcal{K} \otimes \beta(A \otimes 1) \simeq \mathcal{K} \otimes A$, where $\{e_{ij}\}$ is the canonical basis for the compact ideal $\mathcal{K}$.

Consider the the short exact sequence (5.3.5), here we also take account into the case $n = 1$ as well as $n \geq 2$ by the above argument. By Proposition 4.4.8, the C^*-algebra $\mathcal{D}$ is nuclear, and so we have the following short exact sequence:

$$0 \longrightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{J} \longrightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{D} \longrightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{O}_n \longrightarrow 0$$

by Proposition 4.1.8. From the relation $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{J}) \cap \mathcal{E} = \mathcal{I}$, we get the short exact sequence (5.5.4). The isomorphism $\widehat{\mathcal{I}} \simeq \mathcal{K} \otimes \mathcal{E}$ and the exactness of (5.5.5) are similar as above. The commutativity of the diagram (5.5.6) also follows from the uniqueness of the C^*-algebra $A \times_{\mathcal{U}} \mathcal{O}_n$. $\square$

On the K-groups levels, it is easy to see that the following diagrams

(5.5.7)
$$\begin{array}{ccc} K_\#(\mathcal{I}) & \xrightarrow{i_*} & K_\#(\mathcal{E}) \\ \simeq\downarrow & & \downarrow \mathrm{id} \\ K_\#(A) & \xrightarrow{\beta_*} & K_\#(\mathcal{E}) \end{array} \qquad \begin{array}{ccc} K_\#(\widehat{\mathcal{I}}) & \xrightarrow{\widehat{i}_*} & K_\#(\widehat{\mathcal{E}}) \\ \simeq\downarrow & & \downarrow \mathrm{id} \\ K_\#(\widehat{\mathcal{E}}) & \xrightarrow{\beta_*} & K_\#(\widehat{\mathcal{E}}) \end{array}$$

commute. We apply the six term exact sequence (5.4.8) to the short exact sequences (5.5.4) and (5.5.5). By (5.5.6) and (5.5.7), we have the following commuting diagram;

(5.5.8)
$$\begin{array}{ccccccccccccc} K_1(A) & \xrightarrow{\beta_*} & K_1(\mathcal{E}) & \xrightarrow{q_*} & K_1(A \times_{\mathcal{U}} \mathcal{O}_n) & \xrightarrow{\partial} & K_0(A) & \xrightarrow{\beta_*} & K_0(\mathcal{E}) & \xrightarrow{q_*} & K_0(A \times_{\mathcal{U}} \mathcal{O}_n) & \xrightarrow{\partial} & \\ \downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & & \downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & & \\ K_1(\mathcal{E}) & \xrightarrow{\beta_*} & K_1(\widehat{\mathcal{E}}) & \xrightarrow{\widehat{q}_*} & K_1(A \times_{\mathcal{U}} \mathcal{O}_n) & \xrightarrow{\widehat{\partial}} & K_0(\mathcal{E}) & \xrightarrow{\beta_*} & K_0(\widehat{\mathcal{E}}) & \xrightarrow{\widehat{q}_*} & K_0(A \times_{\mathcal{U}} \mathcal{O}_n) & \xrightarrow{\widehat{\partial}} & \end{array}$$

with exact rows. In order to get the six term exact sequence for $A \times_{\mathcal{U}} \mathcal{O}_n$ from the diagram (5.5.8), we need the following:

Lemma 5.5.2.

(i) *The homomorphisms* $k_* : K_\#(A) \to K_\#(\mathcal{E})$ *are injective.*

(ii) *The homomorphisms* $\beta_* : K_\#(\mathcal{E}) \to K_\#(\widehat{\mathcal{E}})$ *are equal to the homomorphisms* $(\mathrm{id} - \sum_{i=1}^n (\widehat{\alpha}_i^{-1})_*) \circ j_*$, *where* $\widehat{\alpha}_i$ *denotes the automorphism* $\mathrm{Ad}\,(U_i \otimes 1)$ *of* $\widehat{\mathcal{E}}$.

Before the proof of Lemma 5.5.2, we state and prove the following fundamental theorem.

Theorem 5.5.3. *Let* A *be a* C^**-algebra. Then we have the following commuting diagram;*

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\sigma} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \times_{\mathcal{U}} \mathcal{O}_n) \\
\uparrow{\scriptstyle \partial} & & & & \downarrow{\scriptstyle \partial} \\
K_1(A \times_{\mathcal{U}} \mathcal{O}_n) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\sigma} & K_1(A)
\end{array}$$

where $\sigma = \mathrm{id} - \sum_{i=1}^n (\alpha_i^{-1})_*$ *and* $\iota : A \to A \times_{\mathcal{U}} \mathcal{O}_n$ *is the inclusion* $a \mapsto a \otimes 1$.

Proof. First, we compare two homomorphisms $\beta_*, j_* : K_\#(\mathcal{E}) \to K_\#(\widehat{\mathcal{E}})$ in the diagram (5.5.8). Because the map $\widehat{\alpha}_i : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ sends $\mathcal{E}$ into itself, we see that $\mathrm{Im}\,\beta_* \subseteq \mathrm{Im}\,j_*$ by Lemma 5.5.2 (ii). Now, we show that j_* is an epimorphism. For $x \in K_\#(\widehat{\mathcal{E}})$, we have $\widehat{q}_*(x) = q_*(y) = \widehat{q}_* \circ j_*(y)$ for some $y \in K_\#(\mathcal{E})$. Because

$$x - j_*(y) \in \mathrm{Ker}\,\widehat{q}_* = \mathrm{Im}\,\beta_* \subseteq \mathrm{Im}\,j_*,$$

we have $x - j_*(y) = j_*(z)$ for some $z \in K_\#(\mathcal{E})$, and so $x = j_*(y + z)$ with $y + z \in K_\#(\mathcal{E})$.

Because k_* is a monomorphism by Lemma 5.5.2 (i), we apply the Five lemma of homological algebra to see that all vertical arrows in (5.5.8) are isomorphisms. We identify $K_\#(\mathcal{E})$ and $K_\#(\widehat{\mathcal{E}})$ in the bottom row with $K_\#(A)$ by the isomorphisms k_* and $j_* \circ k_*$, respectively. Then it is easy to see that β_* and $\widehat{q}_*$ become $\mathrm{id} - \sum_{i=1}^n (\alpha_i^{-1})_*$ and ι_* by Lemma 5.5.2 (ii), respectively. □

Proof of Lemma 5.5.2. We first show that $k_* : K_1(A) \to K_1(\mathcal{E})$ is a monomorphism. It is easy to see that we may work with A and $\mathcal{E}$ instead with the matrix algebras. Let v_0 and v_1 be unitaries in A such that $k_*[v_0] = k_*[v_1]$.

This says that there is a continuous function $t \mapsto w_t \in \mathcal{U}(\mathcal{E})$ such that $w_0 = v_0 \otimes 1$ and $w_1 = v_1 \otimes 1$. Put

$$w'_t = w_t(\theta(w_t)^* + 1 \otimes S_{n+1}S^*_{n+1}), \qquad t \in [0,1],$$

where θ is an endomorphism of $\mathcal{E}$ defined by

$$\begin{aligned} a \otimes 1 &\mapsto a \otimes (1 - S_{n+1}S^*_{n+1}), && a \in A, \\ U_i \otimes S_i &\mapsto U_i \otimes S_i(1 - S_{n+1}S^*_{n+1}), && i = 1, 2, \dots, n. \end{aligned}$$

Then we have $w'_j = v_j \otimes S_{n+1}S^*_{n+1} + 1 \otimes (1 - S_{n+1}S^*_{n+1})$ for $j = 0, 1$. It is also easy to see that w'_t lies in the C^*-algebra $\mathcal{I}_I$ generated by $\mathcal{I}$ and the identity of $\mathcal{E}$. This shows that $[v_0] = [v_1]$ in $K_1(\mathcal{I}) = K_1(A)$.

We replace A and $U_1, \dots U_n$ by $\Omega A = C(\mathbb{T}) \otimes A$ and $1 \otimes U_1, \dots 1 \otimes U_n$, respectively, to get an another monomorphism

$$(\mathrm{id} \otimes k)_* : K_1(\Omega A) \to K_1(\Omega \mathcal{E}).$$

Note that the split short exact sequence (5.4.6) gives the isomorphism

$$K_1(\Omega A) \simeq K_1(\mathcal{S}A) \oplus K_1(A) \simeq K_0(A) \oplus K_1(A),$$

by Theorem 5.4.6. From this we see that $k_* : K_0(A) \to K_0(\mathcal{E})$ is also a monomorphism.

For the proof of the second statement, we may also work on $\mathcal{E}$ and $\widehat{\mathcal{E}}$. Let p be a projection in $\mathcal{E}$. Then by the relation (5.5.3) we see that $\phi(p)$ is the orthogonal sum of the projections $\beta(p)$ and $(1 \otimes S_i)p(1 \otimes S_i)^*$, $i = 1, 2, \dots, n$. As elements of $K_0(\widehat{\mathcal{E}})$, we have

$$\begin{aligned} [(1 \otimes S_i)p(1 \otimes S_i)^*] &= [\widehat{\alpha}_i^{-1}((U_i \otimes S_i)p(U_i \otimes S_i)^*)] \\ &= (\widehat{\alpha}_i^{-1})_*[(U_i \otimes S_i)p(U_i \otimes S_i)^*] = (\widehat{\alpha}_i^{-1})_*[p], \end{aligned}$$

and so it follows that

$$\phi_* = \beta_* + \sum_{i=1}^{n} (\widehat{\alpha}_i^{-1})_* \circ j_*, \tag{5.5.9}$$

on the K_0-group level. This is also the case on the K_1-group level, because $\phi(u)$ is the product of the unitaries

$$\beta(u) + (1 - \beta(1)), \quad (1 \otimes S_i)u(1 \otimes S_i)^* - (1 - 1 \otimes S_i S_i^*)), \quad i = 1, 2, \ldots, n,$$

for a unitary u in $\mathcal{E}$.

It remains to show that $\phi_* = j_*$ by (5.5.9). To do this, we show that $\phi, j : \mathcal{E} \to \widehat{\mathcal{E}}$ are homotopic each other. If we denote by $W = \sum_{i,j=1}^{n+1} S_i S_j S_i^* S_j^*$ then we have $\phi(U_i \otimes S_i) = (1 \otimes W)(U_i \otimes S_i)$ for $i = 1, 2, \ldots, n$. It is not so difficult to see that $1 \otimes W$ is a self-adjoint unitary in $\widehat{\mathcal{E}}$. Let $W = E - F$ be the spectral decomposition of W and put $W_t = E + e^{\pi i t} F$ for $t \in [0, 1]$. We define $\phi_t : \mathcal{E} \to \widehat{\mathcal{E}}$ by

$$\begin{aligned} a \otimes 1 &\mapsto a \otimes 1, \qquad a \in A, \\ U_i \otimes S_i &\mapsto (1 \otimes W_t)(U_i \otimes S_i), \qquad i = 1, 2, \ldots, n, \end{aligned}$$

for $t \in [0, 1]$. Noting that ϕ and j coincide on $A \otimes 1 \subseteq \mathcal{E}$, we see that $\phi_t : \mathcal{E} \to \widehat{\mathcal{E}}$ is a continuous path of homomorphisms with $\phi_0 = j$ and $\phi_1 = \phi$. □

Now, we apply Theorem 5.5.3 to the case when A is the trivial C^*-algebra $\mathbb{C}1$ and $U_1 = \cdots = U_n = 1$. Then the twisted tensor product is nothing but the Cuntz algebra itself. Therefore, we have the following commuting diagram:

$$\begin{array}{ccccc} K_0(\mathbb{C}) & \xrightarrow{1-n} & K_0(\mathbb{C}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}_n) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(\mathcal{O}_n) & \xleftarrow{\iota_*} & K_1(\mathbb{C}) & \xleftarrow{1-n} & K_1(\mathbb{C}) \end{array}$$

Noting that $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, we have the following:

Corollary 5.5.4. *We have $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n) = 0$.*

This classify the Cuntz algebras. Note that the K_0-group of the Cuntz algebra is not an ordered group. See Proposition 3.3.5. Next, we consider the case of $n = 1$, in which the twisted tensor product reduces to the usual crossed product $\mathbb{Z} \ltimes_{\alpha r} A$ as was mentioned before. Then Theorem 5.5.3 reduces to the following:

Theorem 5.5.5. *For a C^*-algebra A with a $\mathbb{Z}$-action α, we have the following six-term exact sequence;*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\sigma} & K_0(A) & \xrightarrow{\iota_*} & K_0(\mathbb{Z} \ltimes_\alpha A) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(\mathbb{Z} \ltimes_\alpha A) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\sigma} & K_1(A) \end{array}$$

where $\sigma = (1_A)_* - (\alpha_{-1})_*$.

Now, we apply Theorem 5.5.5 to the irrational rotation C^*-algebra $A_\theta = \mathbb{Z} \ltimes_\theta C(\mathbb{T})$. It is easy to see that both of two σ's in Theorem 5.5.5 become the zero maps, and so we have the following exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & K_0(A_\theta) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(A_\theta) & \longleftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \end{array}$$

Hence, we see that $K_0(A_\theta) \simeq K_1(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Especially, A_θ is not an AF algebra by Exercise 5.4.1. It can be shown that $K_1(A_\theta)$ is generated by $[U]$ and $[V]$, where U and V are unitaries with the relation (5.2.4). By the discussion in §5.2, we also see that the unique trace of A_θ induces an isomorphism of $K_0(A_\theta)$ onto the group $\mathbb{Z} + \mathbb{Z}\theta$.

We close this section to state a generalization of Theorem 5.5.5 to the free group F_n on n generators.

Theorem 5.5.6. *Let F_n be the free group with generators $a_1, \ldots, a_n$, and α an action of F_n on a C^*-algebra A. Then we have the following six-term exact sequence;*

$$\begin{array}{ccccc} (K_0(A))^n & \xrightarrow{\beta} & K_0(A) & \xrightarrow{\iota_*} & K_0(F_n \ltimes_{\alpha r} A) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(F_n \ltimes_{\alpha r} A) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\beta} & (K_1(A))^n \end{array}$$

where $\beta(\gamma_1 \oplus \cdots \oplus \gamma_n) = \sum_{j=1}^{n} (\gamma_j - (\alpha_{a_j^{-1}})_* \gamma_j)$.

If $A = \mathbb{C}1$ and α is the trivial action then $F_n \ltimes_{\alpha r} \mathbb{C}$ is nothing but the reduced group C^*-algebra $C^*_\lambda(F_n)$. Hence, we see that $K_1(C^*_\lambda(F_n)) = \mathbb{Z}^n$.

Because the inclusion $\mathbb{C} \xrightarrow{\iota} F_n \ltimes_{\alpha r} \mathbb{C}$ induces an isomorphism ι_* on the K_0-level, we also see that $K_0(C^*_\lambda(F_n)) = \mathbb{Z}$ with the generator $[1]_0$. Therefore, we have the following.

Corollary 5.5.7. *There is no nontrivial projection in* $C^*_\lambda(F_n)$.

NOTE

Theorem 5.5.3 is due to Cuntz [**Cu81b**], who combined and simplified his former argument [**Cu81a**] for Corollary 5.5.4 and Pimsner and Voiculescu's argument [**PV80b**] for Theorem 5.5.5. The sequence (5.5.4) is said to be the *Toeplitz extension*. For an another proof of Theorem 5.5.5, we refer to Blackadar's book [**Bl**, §10] together with further results on the K-theory of the crossed products by finite groups or continuous groups such as $\mathbb{T}$ and $\mathbb{R}$. See also the survey article [**Rf85**]. Theorem 5.5.6 is also due to Pimsner and Voiculescu [**PV82**]. Corollary 5.5.7 gives us an example of a simple C^*-algebra without nontrivial projections. See also the paragraph just after Theorem 3.5.5. For a history and survey surrounding the question of nontrivial projections, we refer to the article [**Va89**].

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