수 학 강 의 록 제 6 권

Lecture Notes on Geometric Analysis

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LECTURE NOTES ON GEOMETRIC ANALYSIS

PETER LI

§0 INTRODUCTION

This set of lecture notes originated from a series of lectures given by the author at a Geometry Summer Program in 1990 at the Mathematical Sciences Research Institute in Berkeley. During the Fall quarter of 1990, the author also taught a course in Geometric Analysis at the University of Arizona. For that purpose, the lecture notes were revised and expanded. During the author's visit with the Global Analysis Research Institute at Seoul National University, he was encouraged to submit these notes in the present, but still rather crude, form for publication in their lecture notes series. The readers should be aware that these notes are meant to address the entry level geometric analysts by introducing the basic techniques in geometric analysis in the most economical way. The theorems discussed are choosen sometimes for their fundamental usefulness and sometimes for purpose of demonstrating various techniques. In many cases, they do not represent the best possible results which are available. Moreover, little time was spent on historical accounts and chronological ordering.

The author would like to express his gratitute to the Global Analysis Research Institute and the Mathematics Department of Seoul National University for their hospitality. In particular, special thanks to Professor Dong Pyo Chi, Professor Hyeong In Choi, Professor Hyuk Kim, and the geometry graduate students for making his visit a memorable one.

§1. FIRST AND SECOND VARIATIONAL FORMULAS FOR AREA

Let M be a Riemmannian manifold of dimension m with metric denoted by ds^2. In terms of local coordinates $\{x_1, \dots, x_m\}$ the metric is written in the form

$$ds^2 = g_{ij}\, dx_i\, dx_j,$$

where we are adopting the convention that repeated indices are being taken the summation over. If X and Y are tangent vectors at a point $p \in M$, we will also denote their inner product by

$$ds^2(X, Y) = \langle X, Y \rangle.$$

If we denote $\mathcal{S}(TM)$ to be the set of smooth vector fields on M, then the Riemannian connection $\nabla : \mathcal{S}(TM) \times \mathcal{S}(TM) \to \mathcal{S}(TM)$ satisfies the following properties:

(1) $\nabla_{(f_1 X_1 + f_2 X_2)} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$, for all $X_1, X_2, Y \in \mathcal{S}(TM)$ and for all $f_1, f_2 \in C^\infty(M)$;

(2) $\nabla_X(g_1Y_1 + g_2Y_2) = X(g_1)Y_1 + g_1\nabla_X Y_1 + X(g_2)Y_2 + g_2\nabla_X Y_2$, for all $X, Y_1, Y_2 \in \mathcal{S}(TM)$ and for all $g_1, g_2 \in C^\infty(M)$;
(3) $X\langle Y, Z\rangle = \langle \nabla_X Y, X\rangle + \langle Y, \nabla_X Z\rangle$, for all $X, Y, Z \in \mathcal{S}(TM)$; and
(4) $\nabla_X Y - \nabla_Y X = [X, Y]$, for all $X, Y \in \mathcal{S}(TM)$.

Property (3) says that ∇ is compatible with the Riemannian metric, while property (4) means that ∇ is torsion free. The curvature tensor of the Riemannian metric is then given by

$$\mathcal{R}_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for $X, Y, Z \in \mathcal{S}(TM)$. The curvature tensor satisfies the properties:
(1) $\mathcal{R}_{XY}Z = -\mathcal{R}_{YX}Z$, for all $X, Y, Z \in \mathcal{S}(TM)$;
(2) $\mathcal{R}_{XY}Z + \mathcal{R}_{YZ}X + \mathcal{R}_{ZX}Y = 0$, for all $X, Y, Z \in \mathcal{S}(TM)$; and
(3) $\langle \mathcal{R}_{XY}Z, W\rangle = -\langle \mathcal{R}_{ZW}X, Y\rangle$, for all $X, Y, Z, W \in \mathcal{S}(TM)$.

The sectional curvature of the 2-plane section spanned by a pair of orthonormal vectors X and Y are defined by

$$K(X, Y) = \langle \mathcal{R}_{XY}Y, X\rangle.$$

If we take $\{e_1, \dots, e_m\}$ to be an orthonormal basis of the tangent space of M, then the Ricci curvature is defined to be the symmetric 2-tensor given by

$$\mathcal{R}_{ij} = \sum_{k=1}^{m} \langle \mathcal{R}_{e_i,e_k}e_k, e_j\rangle.$$

Observe that

$$\mathcal{R}_{ii} = \sum_{k\neq i} K(e_i, e_k).$$

Let N be an n-dimensional submanifold in M with $n < m$. The Riemannian metric ds_M^2 defined on M when restricted to N induces a Riemannian metric ds_N^2 on N. One checks easily that for vertor fields $X, Y \in \mathcal{S}(TM)$, if we define

$$\nabla_X^t Y = (\nabla_X Y)^t$$

to be the tangential component of $\nabla_X Y$ to N, then ∇^t is the Riemannian connection of N with respect to ds_N^2. The normal component of ∇ yields the second fundamental form of N. In particular, one defines

$$-\vec{II}(X, Y) = (\nabla_X Y)^n$$

and checks that it is tensorial with respect to $X, Y \in \mathcal{S}(TM)$. Taking the trace of the bilinear form $\vec{II}$ over the tangent space of N yields that mean curvature vector, given by

$$\mathrm{tr}\vec{II} = \vec{H}.$$

In the remaining of this section we will derive the first and second variational formulas for the area functional of a submanifold. Let $N^n \subset M^m$ be a n-dimensional submanifold of a m-dimensional manifold M with $m > n$. Consider a 1-parameter family of deformations of N given by $N_t = \phi(N,t)$ for $t \in (-\epsilon, \epsilon)$ with $N_0 = N$. Let $\{x_1, \ldots, x_n\}$ be a coordinate system around a point $p \in N$. We can consider $\{x_1, \ldots, x_n, t\}$ to be a coordinate system of $N \times (-\epsilon, \epsilon)$ near the point $(p, 0)$. Let us denote $e_i = d\phi\left(\frac{\partial}{\partial x_i}\right)$ for $i = 1, \ldots, n$ and $T = d\phi\left(\frac{\partial}{\partial t}\right)$. The induced metric on N_t from M is then given by $g_{ij} = \langle e_i, e_j \rangle$. We may futher assume that $\{x_1, \ldots, x_n\}$ form a normal coordinate system at $p \in N$. Hence $g_{ij}(p,0) = \delta_{ij}$ and $\nabla_{e_i} e_j(p,0) = 0$. Let us define dA_t to be the area element of N_t with respect to the induced metric. For t sufficiently close to 0, we can write $dA_t = J(x,t)\, dA_0$. With respect to the normal coordinate system $\{x_1, \ldots, x_n\}$, the function $J(x,t)$ is given by

$$J(x,t) = \frac{\sqrt{g(x,t)}}{\sqrt{g(x,0)}}$$

with $g(x,t) = \det(g_{ij}(x,t))$. To compute the first variation for the area of N, we compute $J'(p,t) = \frac{\partial J}{\partial t}(p,t)$. By the assumption that $g_{ij}(p,0) = \delta_{ij}$, we have $J'(p,0) = \frac{1}{2} g'(p,0)$. However,

$$\begin{aligned} g &= \det(g_{ij}) \\ &= \sum_{j=1}^{n} g_{1j}\, c_{1j} \end{aligned}$$

where c_{ij} are the cofactors of g_{ij}. Therefore

$$\begin{aligned} g'(p,0) &= \sum_{j=1}^{n} g'_{1j}(p,0)\, c_{1j}(p,0) + \sum_{j=1}^{n} g_{1j}(p,0)\, c'_{1j}(p,0) \\ &= g'_{11}(p,0) + c'_{11}(p,0). \end{aligned}$$

By induction on the dimension, we conclude that $g'(p,0) = \sum_{i=1}^{n} g'_{ii}$. On the other hand,

$$\begin{aligned} g'_{ii} &= T\langle e_i, e_i \rangle \\ &= 2\langle \nabla_T e_i, e_i \rangle \\ &= 2\langle \nabla_{e_i} T, e_i \rangle \end{aligned}$$

because $\{x_1, \ldots, x_n, t\}$ form a coordinate system for $N \times (-\epsilon, \epsilon)$. Let us point out that the quantity $\sum_{i=1}^{n} \langle \nabla_{e_i} T, e_i \rangle$ is now well-defined under orthonormal change of basis and hence is globally defined. If we write $T = T^t + T^n$ where T^t is the tangential component of T on N and T^n its normal component, then

$$\begin{aligned} \sum_{i=1}^{n} \langle \nabla_{e_i} T, e_i \rangle &= \sum_{i=1}^{n} \langle \nabla_{e_i} T^t, e_i \rangle + \sum_{i=1}^{n} \langle \nabla_{e_i} T^n, e_i \rangle \\ &= \operatorname{div}(T^t) + \sum_{i=1}^{n} e_i \langle T^n, e_i \rangle - \sum_{i=1}^{n} \langle T^n, \nabla_{e_i} e_i \rangle \\ &= \operatorname{div}(T^t) + \langle T^n, \vec{H} \rangle \end{aligned}$$

where $\overrightarrow{H}$ is the mean curvature vector of N. Hence the first variation for the volume form at the point $(p,0)$ is given by

$$\frac{d}{dt}dA_t|_{(p,0)} = (\mathrm{div}T^t + \langle T^n, \overrightarrow{H}\rangle)dA_0|_{(p,0)}.$$

However, the right hand side is intrinsically defined independent on the choice of coordinates and this formula is valid at any arbitrary point.

If T is a compactly supported variational vector field on N, then using the divergence theorem the first variation of the area of N is given by

$$\frac{d}{dt}A(N_t)\bigg|_{t=0} = \int_N \langle T^n, \overrightarrow{H}\rangle.$$

This shows that the mean curvature of N is identically 0 if and only if N is a critical point of the area functional. Such a manifold is said to be *minimal.*

When N is a curve in M that is parametrized by arc-length with unit tangent vector e, then the first variational formula for length can be written as

$$\begin{aligned}\frac{d}{dt}L\bigg|_{t=0} &= \langle T^t, e\rangle|_0^l - \int_0^l \langle T^n, \nabla_e e\rangle \\ &= \langle T, e\rangle|_0^l - \int_0^l \langle T, \nabla_e e\rangle.\end{aligned}$$

We will now proceed to derive the second variational formula for area. Let $\phi : N \times (-\epsilon,\epsilon)\times(-\epsilon,\epsilon) \longrightarrow M$ be a 2-parameter families of variations of N. Following the similar notation, we denote $d\phi(\frac{\partial}{\partial x_i}) = e_i$ for $i = 1,\dots,n$, and we denote the variational vector fields by $d\phi(\frac{\partial}{\partial t}) = T$ and $d\phi(\frac{\partial}{\partial s}) = S$. In terms of a general coordinate system, the first partial derivative of J can be written as

$$\frac{\partial J}{\partial t}(x,t,s) = \sum_{i,j=1}^n g^{ij}\langle \nabla_{e_i}T, e_j\rangle\, J(x,t,s),$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}). Differentiating this with respect to s and evaluating at $(p,0,0)$ we have

$$\begin{aligned}\frac{\partial^2 J}{\partial s \partial t} &= \sum_{i,j=1}^n S\left(g^{ij}\langle\nabla_{e_i}T, e_j\rangle J\right) \\ &= \sum_{i,j=1}^n (Sg^{ij})\langle\nabla_{e_i}T, e_j\rangle J + \sum_{i,j=1}^n g^{ij}\left(S\langle\nabla_{e_i}T, e_j\rangle\right) J \\ &\quad + \sum_{i,j=1}^n g^{ij}\langle\nabla_{e_i}T, e_j\rangle S(J) \\ &= \sum_{i,j=1}^n (Sg^{ij})\langle\nabla_{e_i}T, e_j\rangle + \sum_{i=1}^n S\langle\nabla_{e_i}T, e_i\rangle \\ &\quad + \left(\sum_{i=1}^n\langle\nabla_{e_i}T, e_i\rangle\right)\left(\sum_{j=1}^n\langle\nabla_{e_j}S, e_j\rangle\right).\end{aligned} \tag{1.1}$$

Differentiating the formula $\sum_{k=1}^n g^{ik}g_{kj} = \delta_{ij}$, we obtain

$$\sum_{k=1}^n (Sg^{ik})g_{kj} = -\sum_{k=1}^n g^{ik}(Sg_{kj}).$$

Hence

$$\begin{aligned} Sg^{ij} &= -\sum_{k,l=1}^n g^{ik}(Sg_{kl})g^{lj} \\ &= -Sg_{ij} \\ &= -S\langle e_i, e_j\rangle \\ &= -\langle \nabla_S e_i, e_j\rangle - \langle \nabla_S e_j, e_i\rangle \\ &= -\langle \nabla_{e_i} S, e_j\rangle - \langle \nabla_{e_j} S, e_i\rangle. \end{aligned}$$

The first term on the right hand side of (1.1) becomes

$$\sum_{i,j=1}^n (Sg^{ij})\langle \nabla_{e_i}T, e_j\rangle = -\sum_{i,j=1}^n \langle \nabla_{e_i}S, e_j\rangle\langle \nabla_{e_i}T, e_j\rangle - \sum_{i,j=1}^n \langle \nabla_{e_j}S, e_i\rangle\langle \nabla_{e_i}T, e_j\rangle.$$

The second term on the right hand side of (1.1) can be written as

$$\begin{aligned} \sum_{i=1}^n S\langle \nabla_{e_i}T, e_i\rangle &= \sum_{i=1}^n \langle \nabla_S \nabla_{e_i}T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}T, \nabla_S e_i\rangle \\ &= \sum_{i=1}^n \langle \mathcal{R}_{Se_i}T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}\nabla_S T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}T, \nabla_{e_i}S\rangle \end{aligned}$$

where the term $\langle \mathcal{R}_{Se_i}T, e_i\rangle$ on the right hand side denotes the curvature tensor of M. Therefore, we have

$$\begin{aligned} \frac{\partial^2 J}{\partial s\partial t} &= -\sum_{i,j=1}^n \langle \nabla_{e_i}S, e_j\rangle\langle \nabla_{e_i}T, e_j\rangle - \sum_{i,j=1}^n \langle \nabla_{e_j}S, e_i\rangle\langle \nabla_{e_i}T, e_j\rangle \\ &\quad + \sum_{i=1}^n \langle \mathcal{R}_{Se_i}T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}\nabla_S T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}T, \nabla_{e_i}S\rangle \\ &\quad + \left(\sum_{i=1}^n \langle \nabla_{e_i}T, e_i\rangle\right)\left(\sum_{j=1}^n \langle \nabla_{e_j}S, e_j\rangle\right). \end{aligned} \tag{1.2}$$

When N is a geodesic parametrized by arc-length in M with unit tangent vector given by e, the second variational formula for the length is given by

$$\begin{aligned} \left.\frac{\partial^2 L}{\partial s\partial t}\right|_{(s,t)=(0,0)} &= \int_0^l \{-\langle \nabla_e S, e\rangle\langle \nabla_e T, e\rangle + \langle \mathcal{R}_{Se}T, e\rangle\} \\ &\quad + \int_0^l \{\langle \nabla_e \nabla_S T, e\rangle + \langle \nabla_e T, \nabla_e S\rangle\}. \end{aligned}$$

If we futher assumed that N is a geodesic, ie. $\nabla_e e \equiv 0$, then we have

$$\begin{aligned}\left.\frac{\partial^2 L}{\partial s \partial t}\right|_{(s,t)=(0,0)} &= \int_0^l \{-(e\langle S,e\rangle)(e\langle T,e\rangle) + \langle \mathcal{R}_{Se}T,e\rangle\} \\ &\quad + \int_0^l \{e\langle \nabla_S T,e\rangle + \langle \nabla_e T, \nabla_e S\rangle\} \\ &= \int_0^l \{\langle \nabla_e T, \nabla_e S\rangle + \langle \mathcal{R}_{Se}T,e\rangle - (e\langle S,e\rangle)(e\langle T,e\rangle\} \\ &\quad + \langle \nabla_S T, e\rangle|_0^l .\end{aligned}$$

When N is a general n-dimensional manifold and if the two variational vector fields are the same and are normal to N, then (1.2) becomes

$$\begin{aligned}\left.\frac{\partial^2 J}{\partial t^2}\right|_{t=0} &= -\sum_{i,j=1}^n \langle \nabla_{e_i}T, e_j\rangle^2 - \sum_{i,j=1}^n \langle \nabla_{e_j}T, e_i\rangle\langle \nabla_{e_i}T, e_j\rangle \\ &\quad + \sum_{i=1}^n \langle \mathcal{R}_{Te_i}T, e_i\rangle + \sum_{i=1}^n \langle \nabla_{e_i}\nabla_T T, e_i\rangle \\ &\quad + \sum_{i=1}^n |\nabla_{e_i}T|^2 + \left(\sum_{i=1}^n \langle \nabla_{e_i}T, e_i\rangle\right)^2 \\ &= -\sum_{i,j=1}^n \langle \nabla_{e_i}T, e_j\rangle^2 - \sum_{i,j=1}^n \langle \nabla_{e_j}T, e_i\rangle\langle \nabla_{e_i}T, e_j\rangle \\ &\quad - \sum_{i=1}^n \langle \mathcal{R}_{e_iT}T, e_i\rangle + \operatorname{div}(\nabla_T T)^t + \langle (\nabla_T T)^n, \vec{H}\rangle \\ &\quad + \sum_{i=1}^n |\nabla_{e_i}T|^2 + \langle T, \vec{H}\rangle^2.\end{aligned} \tag{1.3}$$

On the other hand, if $\{e_{n+1}, \dots, e_m\}$ denotes an orthonormal set of vectors normal to N in M, then

$$\sum_{i=1}^n \langle \nabla_{e_i}T, \nabla_{e_i}T\rangle = \sum_{i,j=1}^n \langle \nabla_{e_i}T, e_j\rangle^2 + \sum_{i=1}^n \sum_{\nu=n+1}^m \langle \nabla_{e_i}T, e_\nu\rangle.$$

Also

$$\begin{aligned}\langle \nabla_{e_i}T, e_j\rangle &= \langle T, \vec{II}_{ij}\rangle \\ &= \langle \nabla_{e_j}T, e_i\rangle\end{aligned}$$

where $\vec{II}_{ij}$ denotes the second fundamental form with value in the normal bundle of N.

Hence, (1.3) becomes

$$\frac{\partial^2 J}{\partial t^2}\bigg|_{t=0} = -\langle T, \vec{II}_{ij}\rangle^2 - \sum_{i=1}^{n}\langle \mathcal{R}_{e_i T}T, e_i\rangle + \mathrm{div}(\nabla_T T)^t$$
$$+ \langle (\nabla_T T)^n, \vec{H}\rangle + \sum_{i=1}^{n}\sum_{\nu=n+1}^{m}\langle \nabla_{e_i}T, e_\nu\rangle^2 + \langle T, \vec{H}\rangle^2.$$

Therefore, the second variational formula for area in terms of compactly supported normal variations is given by

$$\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} = \int_N \{-\langle T, \vec{II}_{ij}\rangle^2 - \sum_{i=1}^{n}\langle \mathcal{R}_{e_i T}T, e_i\rangle + \langle (\nabla_T T)^n, \vec{H}\rangle\}$$
$$+ \int_N \{\sum_{i=1}^{n}\sum_{\nu=n+1}^{m}\langle \nabla_{e_i}T, e_\nu\rangle^2 + \langle T, \vec{H}\rangle^2\}.$$

For the special case when N is an orientable codimension-1 minimal submanifold of an orientable manifold M, we can write any normal variation in the form $T = \psi e_m$ where e_m is a unit normal vector field to N. Then the second variational formula can be written as

$$\frac{d^2}{dt^2}A(N_t)\bigg|_{t=0} = \int_N \{-\langle T, \vec{II}_{ij}\rangle^2 - \mathcal{R}(T,T) + \sum_{i=1}^{n}\langle \nabla_{e_i}T, e_m\rangle^2\}$$
$$= \int_N \{-\psi^2 h_{ij}^2 - \psi^2\mathcal{R}(e_m, e_m) + |\nabla\psi|^2\}$$

where $\vec{II}_{ij} = h_{ij}\, e_m$, $\mathcal{R}(T,T)$ denotes the Ricci curvature of M in the direction of T, and we have used the fact that

$$\langle \nabla_{e_i}T, e_m\rangle = \psi\langle \nabla_{e_i}e_m, e_m\rangle + e_i(\psi)\langle e_m, e_m\rangle$$
$$= e_i(\psi).$$

If we further assume restrict the variation to be given by hypersurfaces which are constant distant from N, the variational vector field is then given by e_m with $\nabla_{e_m}e_m \equiv 0$. This situation is particularly useful for the purpose of controling the growth of the volume of geodesic balls of radius r. In this case, if we write $\vec{H} = H\, e_m$, the first variational formula for the area element becomes

$$\frac{\partial J}{\partial t}(x,0) = H(x)\, J(x,0), \tag{1.4}$$

and the second variational formula can be written as

$$\frac{\partial^2 J}{\partial t^2}(x,0) = -\sum_{i,j=1}^{m-1} h_{ij}^2(x)\, J(x,0) - \mathcal{R}(e_m, e_m)(x)\, J(x,0) + H^2(x)\, J(x,0). \tag{1.5}$$

§2 Bishop Comparison Theorem

In this section, we will develop a volume comparison theorem originally proved by Bishop (see [B-C]). Let $p \in M$ be a point in a complete Riemannian manifold of dimension m. In terms of polar normal coordinates at p, we can write the volume element as

$$J(\theta, r) dr \wedge d\theta$$

where $d\theta$ is the area element of the unit $(m-1)$-sphere. By the Gauss lemma, the area element of submanifold $\partial B_p(r)$ which is the boundary of the geodesic ball of radius r is given by $J(\theta, r)d\theta$. By the first and second variational formulas (1.4) and (1.5), if $x = (\theta, r)$ is not in the cut-locus of p, we have

$$\begin{aligned} J'(\theta, r) &= \frac{\partial J}{\partial r}(\theta, r) \\ &= H(\theta, r)\, J(\theta, r) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} J''(\theta, r) &= \frac{\partial^2 J}{\partial r^2}(\theta, r) \\ &= -\sum_{i,j=1}^{m-1} h_{ij}^2(\theta, r)\, J(\theta, r) - \mathcal{R}_{rr}(\theta, r)\, J(\theta, r) + H^2(\theta, r)\, J(\theta, r) \end{aligned} \tag{2.2}$$

where $\mathcal{R}_{rr} = \mathcal{R}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$, $H(\theta, r)$, and $(h_{ij}(\theta, r))$ denote the Ricci curvature in the radius direction, the mean curvature and the second fundamental form of $\partial B_p(r)$ at the point $x = (\theta, r)$ with respect to the unit normal vector $\frac{\partial}{\partial r}$, respectively.

Using the inequalities

$$\begin{aligned} \sum_{i,j=1}^{m-1} h_{ij}^2 &\geq \sum_{i=1}^{m-1} h_{ii}^2 \\ &\geq \frac{\left(\sum_{i=1}^{m-1} h_{ii}\right)^2}{m-1} \\ &= \frac{H^2}{m-1}, \end{aligned} \tag{2.3}$$

we can estimate (2.2) by

$$\begin{aligned} J'' &\leq \frac{m-2}{m-1} H^2\, J - \mathcal{R}_{rr}\, J \\ &= \frac{m-2}{m-1} (J')^2\, J^{-1} - \mathcal{R}_{rr}\, J. \end{aligned} \tag{2.4}$$

By the fact that any metric is locally euclidean, we have the initial conditions

$$J(\theta, r) \sim r^{m-1}$$

and

$$J'(\theta, r) \sim (m-1) r^{m-2},$$

as $r \to 0$. Let us point out that if M is a simply connected constant curvature space form with constant sectional curvature K, then all the above inequalities become equalities. In particular (2.4) becomes

$$J'' = \frac{m-2}{m-1} (J')^2 \, J^{-1} - (m-1) K \, J.$$

Theroem 2.1. *(Bishop [B-C]) Let M be a complete Riemannian manifold of dimension m, and p a fixed point of M. Let us assume that the Ricci curvature tensor of M at any point x is bounded below by $(m-1)K(r(p,x))$ for some function K depending only on the distance from p. If $J(\theta, r)\, d\theta$ is the area element of $\partial B_p(r)$ as defined above and $\overline{J}(r)\, d\theta$ is the solution of the ordinary differential equation*

$$\overline{J}'' = \frac{m-2}{m-1} (\overline{J}')^2 \, \overline{J}^{-1} - (m-1) K \, J$$

with initial conditions

$$\overline{J}(r) \sim r^{m-1}$$

and

$$\overline{J}'(r) \sim (m-1) r^{m-2},$$

as $r \to 0$, then within the cut-locus of p, the function $\frac{J(\theta,r)}{\overline{J}(r)}$ is a non-increasing function of r. Also, if $\overline{H}(r) = \frac{\overline{J}'}{\overline{J}}$, then $H(\theta, r) \leq \overline{H}(r)$ whenever (θ, r) is within cut-locus of p. In particular, if K is a constant, then $\overline{J} d\theta$ corresponds to the area element on the simply connected space form of constant curvature K.

Proof. By setting $f = J^{\frac{1}{m-1}}$, (2.1) and (2.4) can be written as

$$f' = \frac{1}{m-1} H \, f$$

and

$$\begin{aligned} f'' &\leq \frac{-1}{m-1} \mathcal{R}_{rr} \, f \\ &\leq -K \, f. \end{aligned}$$

The initial conditions become

$$f(\theta, 0) = 0$$

and

$$f'(\theta, 0) = 1.$$

Let $\overline{f} = \overline{J}^{\frac{1}{m-1}}$ be the corresponding function defined using $\overline{J}$. The function $\overline{f}$ satisfies

$$\overline{f}'' = -K\overline{f},$$

$$\overline{f}(0) = 0,$$

and

$$\overline{f}'(0) = 1.$$

Observe that when K = constant the function $\overline{f} > 0$ for all values of $r \in (0, \infty)$ when $K \leq 0$, and for $r \in (0, \frac{\pi}{\sqrt{K}})$ when $K > 0$. In general, $\overline{f} > 0$ on an interval $(0, a)$ for some $a > 0$. At those values of r we can define

$$F(\theta, r) = \frac{f(\theta, r)}{\overline{f}(r)}.$$

We have

$$F' = \overline{f}^{-2}(f'\overline{f} - f\overline{f}')$$

and

$$\begin{aligned} F'' &= \overline{f}^{-1} f'' - 2\overline{f}^{-2} f'\overline{f}' - \overline{f}^{-2} f\overline{f}'' + 2\overline{f}^{-3} f(\overline{f}')^2 \\ &\leq -2\overline{f}^{-1}\overline{f}' F'. \end{aligned}$$

Hence

$$\begin{aligned} (\overline{f}^2 F')' &= \overline{f}^2(F'' + 2\overline{f}^{-1}\overline{f}' F') \\ &\leq 0. \end{aligned}$$

Integrating from ϵ to r yields

$$\begin{aligned} F'(r) &\leq F'(\epsilon)\overline{f}^2(\epsilon)\overline{f}^{-2}(r) \\ &= (\overline{f}(\epsilon) f'(\epsilon) - f(\epsilon)\overline{f}'(\epsilon))\overline{f}^{-2}(r). \end{aligned}$$

Letting $\epsilon \to 0$, the initial conditions of f and $\overline{f}$ implies that

$$F'(r) \leq 0.$$

In particular, $\overline{f} f' - \overline{f}' f \leq 0$, which implies

$$H(\theta, r) \leq \overline{H}(r).$$

Moreover, F is a non-increasing function of r implies that $\frac{J(\theta, r)}{\overline{J}(r)}$ is also a non-increasing function of r.

By computing the area element and the mean curvature of the constant curvature space form explicitly, we have the follow corollary.

Corollary 2.1. *Under the same assumption of the theroem, if K is a constant, then*

$$H \leq \begin{cases} (m-1)\sqrt{K}\cot(\sqrt{K}r), & \text{for } K>0 \\ (m-1)r^{-1}, & \text{for } K=0 \\ (m-1)\sqrt{-K}\coth(\sqrt{-K}r), & \text{for } K<0, \end{cases}$$

and

$$\frac{J(\theta,r)}{\overline{J}(r)}$$

is a non-increasing function of r, where

$$\overline{J}(r) = \begin{cases} \sin^{m-1}(\sqrt{K}r), & \text{for } K>0 \\ r^{m-1}, & \text{for } K=0 \\ \sinh^{m-1}(\sqrt{-K}r), & \text{for } K<0. \end{cases}$$

Let us take this opportunity to point out that this estimate implies that when $K>0$, one must encounter a cut point along any geodesic which has length $\frac{\pi}{\sqrt{K}}$. In particular, this proves Myers' theorem.

Corollary 2.2. *(Myers) Let M be an m-dimensional complete Riemannian manifold with Ricci curvature bounded from below by*

$$\mathcal{R}_{ij} \geq (m-1)K$$

for some constant $K>0$. Then M must be compact with diameter d bounded from above by

$$d \leq \frac{\pi}{\sqrt{K}}.$$

Corollary 2.3. *Let M be an m-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a constant $(m-1)K$. Suppose $\overline{M}$ is an m-dimensional simply connected space form with constant sectional curvature K. Let us denote $A_p(r)$ to be the area of the boundary of the geodesic ball $\partial B_p(r)$ centered at $p \in M$ of radius r and $\overline{A}(r)$ to be the area of the boundary of a geodesic ball $\overline{B}(r)$ of radius r in $\overline{M}$. Then for $0 \leq r_1 \leq r_2 < \infty$, we have*

$$A_p(r_1)\overline{A}(r_2) \geq A_p(r_2)\overline{A}(r_1). \tag{2.5}$$

If we denote $V_p(r)$ and $\overline{V}(r)$ to be the volume of $B_p(r)$ and $\overline{B}(r)$ respectively, then for $0 \leq r_1 \leq r_2, r_3 \leq r_4 < \infty$ we have

$$\left(V_p(r_2) - V_p(r_1)\right)\left(\overline{V}(r_4) - \overline{V}(r_3)\right) \geq \left(V_p(r_4) - V_p(r_3)\right)\left(\overline{V}(r_2) - \overline{V}(r_1)\right). \tag{2.6}$$

Proof. Let us define $C(r)$ to be a subset of the unit tangent sphere $S_p(M)$ at p such that for all $\theta \in C(r)$ the geodesic given by $\gamma(s) = \exp_p(s\theta)$ is minimizing up to $s = r$. Clearly for $r_1 \leq r_2$ we have $C(r_2) \subset C(r_1)$. By Theroem 2.1, we have

$$J(\theta, r_1)\,\overline{J}(r_2) \geq J(\theta, r_2)\,\overline{J}(r_1)$$

for $\theta \in C(r_2)$. Integrating over $C(r_2)$ yields

$$\begin{aligned}\int_{C(r_2)} J(\theta, r_1)\, d\theta\, \overline{J}(r_2) &\geq \int_{C(r_2)} J(\theta, r_2)\, d\theta\, \overline{J}(r_1)\\ &= A_p(r_2)\, J(r_1).\end{aligned}$$

On the other hand

$$\begin{aligned}A_p(r_1) &= \int_{C(r_1)} J(\theta, r_1)\, d\theta\\ &\geq \int_{C(r_2)} J(\theta, r_1)\, d\theta.\end{aligned}$$

Hence, together with the fact that

$$\overline{A}(r) = \alpha_{m-1}\, J(r)$$

with α_{m-1} being the area of the unit $(m-1)$-sphere, we conclude (2.5).

To see (2.6), we first assume that $r_1 \leq r_2 \leq r_3 \leq r_4$, in which case we simply integrate the inequality

$$A_p(t_1)\,\overline{A}(t_2) \geq A_p(t_2)\,\overline{A}(t_1)$$

over $r_1 \leq t_1 \leq r_2$ and $r_3 \leq t_2 \leq r_4$. For the case when $r_1 \leq r_3 \leq r_2 \leq r_4$, we write

$$\begin{aligned}&(V_p(r_2) - V_p(r_1))\,(\overline{V}(r_4) - \overline{V}(r_3))\\ &\quad= (V_p(r_3) - V_p(r_1))\,(\overline{V}(r_2) - \overline{V}(r_3)) + (V_p(r_3) - V_p(r_1))\,(\overline{V}(r_4) - \overline{V}(r_2))\\ &\qquad+ (V_p(r_2) - V_p(r_3))\,(\overline{V}(r_2) - \overline{V}(r_3)) + (V_p(r_2) - V_p(r_3))\,(\overline{V}(r_4) - \overline{V}(r_2))\\ &\quad\geq (V_p(r_2) - V_p(r_3))\,(\overline{V}(r_3) - \overline{V}(r_1)) + (V_p(r_4) - V_p(r_2))\,(\overline{V}(r_3) - \overline{V}(r_1))\\ &\qquad+ (V_p(r_2) - V_p(r_3))\,(\overline{V}(r_2) - \overline{V}(r_3)) + (V_p(r_4) - V_p(r_2))\,(\overline{V}(r_2) - \overline{V}(r_3))\\ &\quad= (V_p(r_4) - V_p(r_3))\,(\overline{V}(r_2) - \overline{V}(r_1)).\end{aligned}$$

Let us point out that equality in (2.6) holds if and only if $C(r_1) = C(r_4)$ and $J(\theta, r) = \overline{J}(r)$ for all $0 \leq r \leq r_4$ and $\theta \in C(r_1)$. In particular, if $r_1 = 0$ then $J(\theta, r) = \overline{J}(r)$ for all $r \leq r_4$ and $\theta \in S_p(M)$. This implies that $B_p(r_4)$ is isometric to $\overline{B}(r_4)$.

Theorem 2.2. *Let M be an m-dimensional complete Riemannian manifold with non-negative Ricci curvature. Then the volume growth of M must satisfy the following estimates:*

(1) *(Bishop) If α_{m-1} is the area of the unit $(m-1)$-sphere, then*

$$V_p(r) \le \frac{\alpha_{m-1}}{m} r^m$$

for all $p \in M$ and $r \ge 0$.

(2) *(Yau [Y 2]) For all $p \in M$, there exists a constant $C(V_p(1)) > 0$ depending only on the volume of the geodesic ball centered at p of radius 1, such that*

$$V_p(r) \ge C\, r$$

for all $r > 2$.

Proof. Applying (2.6) to $r_1 = 0 = r_3$ and $r_4 = r$, we have

$$V_p(r_2)\,\overline{V}(r) \ge V_p(r)\,\overline{V}(r_2).$$

Observing that

$$\lim_{r_2 \to 0} \frac{V_p(r_2)}{\overline{V}(r_2)} = 1,$$

and the upper bound follows.

To prove the lower bound, let $x \in \partial B_p(1+\rho)$. Then (2.6) and the curvature assumption implies that

$$V_x(2+\rho) - V_x(\rho) \le V_x(\rho)\,\frac{(2+\rho)^m - \rho^m}{\rho^m}.$$

However, since the distance between p and x is $r(p,x) = 1+\rho$, we have $B_p(1) \subset (B_x(2+\rho) \setminus B_x(\rho))$. Hence

$$V_p(1) \le V_x(2+\rho) - V_x(\rho).$$

Also $B_x(\rho) \subset B_p(1+2\rho)$, we have

$$V_x(\rho) \le V_p(1+2\rho).$$

Therefore, we conclude that

$$V_p(1) \le V_p(1+2\rho)\,\frac{(2+\rho)^m - \rho^m}{\rho^m}.$$

The lower bounded follows by setting $r = 1+2\rho$ and observing that

$$\frac{(2+\rho)^m - \rho^m}{\rho^m} = O(\rho{-1})$$

as $\rho \to \infty$.

We would like to remark that if we assume that for a sufficiently small $\epsilon > 0$ the Ricci curvature has a lower bound of the form

$$\mathcal{R}_{ij}(x) \geq -\epsilon(1 + r(p,x))^{-2},$$

then one can show that M must have infinite volume. On the other hand, if the Ricci curvature is bounded from below by

$$\mathcal{R}_{ij}(x) \geq -C_0(1 + r(p,x))^{-2-\delta}$$

for some constants $C_0, \delta > 0$, then and the upper bound is also valid and is of the form

$$V_p(r) \leq Cr^m$$

where $C(C_0, \delta, m) > 0$ is a constant depending on C_0, δ and m.

It is also a good exercise to show that if a complete manifold has Ricci curvature bounded from below by

$$\mathcal{R}_{ij} \geq \epsilon r(p,x)^{-2}$$

for some constant $\epsilon > \frac{1}{4}$ and for all $r > 1$, then M must be compact.

The next theorem shows that when the upper bound of the diameter given by Myers' theorem is achieved, then the manifold must be isometrically a sphere.

Theorem 2.3. *(Cheng [Cg]) Let M be a complete m- dimensional Riemannian manifold with Ricci curvature bounded from below by*

$$\mathcal{R}_{ij} \geq (m-1)K$$

for some constant $K > 0$. If the diameter d of M satisfies

$$d = \frac{\pi}{\sqrt{K}},$$

then M is isometric to the standard sphere of radius $\frac{1}{\sqrt{K}}$.

Proof. By scaling, we may assume that $K = 1$. Let p and q be a pair of points in M which realize the diameter. The volume comparison theorem implies that

$$V_p(d) \leq V_p(\frac{d}{2}) \frac{\overline{V}(d)}{\overline{V}(\frac{d}{2})}.$$

The assumption that $d = \frac{\pi}{\sqrt{K}}$ implies that

$$\frac{\overline{V}(d)}{\overline{V}(\frac{d}{2})} = 2,$$

hence

$$V_p(d) \leq 2V_p(\frac{d}{2}).$$

Similarly, we have

$$V_q(d) \leq 2V_q(\frac{d}{2}).$$

However, by the triangle inequality and the fact that $d = r(p,q)$, we have $B_p(\frac{d}{2}) \cap B_q(\frac{d}{2}) = \emptyset$. Therefore,

$$\begin{aligned} 2V(M) &= V_p(d) + V_q(d) \\ &\leq 2(V_p(\frac{d}{2}) + V_q(\frac{d}{2})) \\ &\leq 2V(M) \end{aligned}$$

where $V(M)$ denotes the volume of M. This implies that the inequalities in volume comparison theorem are in fact equalities. Hence by the remark following Corollary 2.3 M must be the standard sphere.

§3 Bochner-Weitzenböck Formulas

The Bochner-Weitzenböck formulas, sometimes referred to as the Bochner technique, is one of the most important techniques in the theory of geometric analysis. There are many formulas which can be derived for various situations. In this section, we will only derive the formula for differential forms so as to illustrate the flavor of this technique.

For convenience sake, we will also introduce the moving frame notation. Let $\{e_1, \ldots, e_m\}$ be locally defined orthonormal frame fields of the tangent bundle. Let us denote the dual coframe fields by $\{\omega_1, \ldots, \omega_m\}$. They have the property that $\omega_i(e_j) = \delta_{ij}$. The connection 1-forms w_{ij} are given by exterior differentiating the ω_i's, and are given by the Cartan's 1st structural equations

$$d\omega_i = \omega_{ij} \wedge \omega_j$$

where

$$\omega_{ij} + \omega_{ji} = 0.$$

Cartan's 2nd structural equations yield the curvature tensor

$$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},$$

with

$$\Omega_{ij} = \frac{1}{2}\mathcal{R}_{ijkl}\,\omega_l \wedge \omega_k.$$

Now let us consider the case that N is an n-dimensional submanifold of M. Let $\{e_1, \ldots, e_m\}$ be an adopted orthonormal frame field of M such that $\{e_1, \ldots, e_n\}$ are orthonormal to N. We will now adopt the indexing convention that $1 \leq i,j,k \leq n$ and $n+1 \leq \nu, \mu \leq m$. The second fundamental form of N is given by

$$\omega_{\nu i} = h^{\nu}_{ij}\,\omega_j.$$

Relating the two notations, we have the formulas

$$\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle,$$

$$\mathcal{R}_{ijkl} = \langle \mathcal{R}_{e_i e_j} e_l, e_k \rangle,$$

and

$$h^\nu_{ij} = \langle \vec{II}(e_i, e_j), e_\nu \rangle.$$

The sectional curvature of the 2-plane section spanned by e_i and e_j is given by $\mathcal{R}_{ijij}$ and the Ricci curvature is given by

$$\mathcal{R}_{ij} = \sum_{k=1}^{m} \mathcal{R}_{ikjk}.$$

Let $f \in C^\infty(M)$ be a smooth function defined on M. Its exterior derivative is given by

$$df = f_i\, \omega_i. \tag{3.1}$$

The second covariant derivative of f can be defined by

$$f_{ij}\, \omega_j = df_i + f_j\, \omega_{ji}. \tag{3.2}$$

Exterior differentiating (3.1), and applying the 1st structural equations, we have

$$\begin{aligned} 0 &= df_i \wedge \omega_i + f_i\, d\omega_i \\ &= df_i \wedge \omega_i + f_i\, \omega_{ij} \wedge \omega_j \\ &= (df_i + f_j \omega_{ji}) \wedge \omega_i \\ &= f_{ij}\, \omega_j \wedge \omega_i. \end{aligned}$$

This implies that $f_{ij} - f_{ji} = 0$ for all i and j. The symmetric 2-tensor given by $(f_{ij}\omega_j \wedge \omega_i)$ is called the *hessian* of f. Taking the trace of the hessian, we define the *Laplacian* of f by

$$\Delta f = f_{ii}.$$

The third covariant derivative of f is defined by

$$f_{ijk}\, \omega_k = df_{ij} + f_{kj} \wedge \omega_{ki} + f_{ik} \wedge \omega_{kj}.$$

Exterior differentiating (3.2) gives

$$df_{ij} \wedge \omega_j + f_{ij}\, d\omega_j = df_j \wedge \omega_{ji} + f_j\, d\omega_{ji}.$$

However, the 1st and 2nd structural equations imply that

$$\begin{aligned} 0 &= -df_{ij} \wedge \omega_j - f_{ij}\, d\omega_j + df_j \wedge \omega_{ji} + f_j\, d\omega_{ji} \\ &= -df_{ij} \wedge \omega_j - f_{ij}\, \omega_{jk} \wedge \omega_k + df_j \wedge \omega_{ji} + f_j\, \omega_{jk} \wedge \omega_{ki} + f_j\, \Omega_{ji} \\ &= -(df_{ij} + f_{ik}\, \omega_{kj}) \wedge \omega_j + (df_j + f_k\, \omega_{kj}) \wedge \omega_{ji} + f_j\, \Omega_{ji} \\ &= -(df_{ij} + f_{ik}\, \omega_{kj}) \wedge \omega_j + f_{jk}\, \omega_k \wedge \omega_{ji} + f_j\, \Omega_{ji} \\ &= -(df_{ij} + f_{ik}\, \omega_{kj} + f_{kj}\, \omega_{ki}) \wedge \omega_j + f_j\, \Omega_{ji} \\ &= -f_{ijk}\, \omega_k \wedge \omega_j + \frac{1}{2} f_j\, \mathcal{R}_{jikl}\, \omega_l \wedge \omega_k. \end{aligned}$$

This yields the Ricci identity

$$\begin{aligned} f_{ijk} - f_{ikj} &= \frac{1}{2} f_l \, (\mathcal{R}_{lijk} - \mathcal{R}_{likj}) \\ &= f_l \, \mathcal{R}_{lijk}. \end{aligned}$$

Contracting the indices k and i by setting $k = i$ and summing over $1 \leq i \leq m$, we have

$$f_{iji} - f_{iij} = f_l \, R_{lk}.$$

For $p \leq m$, we will now take the convention on the indices so that $1 \leq i, j, k, l \leq m$, $1 \leq \alpha, \beta, \gamma \leq p$, and $1 \leq a, b, c, d \leq p-1$. Let $\omega \in \Lambda^p(M)$ be an exterior p-form defined on M. Then in terms of the basis, we can write

$$\omega = a_{i_1, \ldots, i_p} \, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}$$

where the summation is being performed over the multi-index $I = (i_1, \ldots, i_p)$. With this understanding, we can write

$$\omega = a_I \, \omega_I.$$

Exterior differentiating yields

$$\begin{aligned} d\omega &= da_I \wedge \omega_I + a_I \, d\omega_I \\ &= da_I \wedge \omega_I + a_I \, (-1)^{p-\alpha} \omega_{i_p} \wedge \cdots \wedge d\omega_{i_\alpha} \wedge \cdots \wedge \omega_{i_1} \\ &= da_I \wedge \omega_I + a_I \, \omega_{i_\alpha j_\alpha} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{j_\alpha} \wedge \cdots \wedge \omega_{i_1} \\ &= (da_I + a_{i_1 \ldots j_\alpha \ldots i_p} \, \omega_{j_\alpha i_\alpha}) \wedge \omega_I. \end{aligned}$$

One defines the covariant derivatives $a_{i_1, \ldots i_p, j}$ by

$$\sum_{j=1}^{m} a_{i_1 \ldots i_p, j} \, \omega_j = da_{i_1, \ldots i_p} + \sum_{\substack{1 \leq \alpha \leq p \\ j_\alpha}} a_{i_1 \ldots j_\alpha \ldots i_p} \, \omega_{j_\alpha i_\alpha}$$

for each multi-index $I = (i_1, \ldots, i_p)$. Similarly, for $(p-1)$-forms, we have

$$a_{i_1 \ldots i_{p-1}, j} \, \omega_j = da_{i_1 \ldots i_{p-1}} + a_{i_1 \ldots j_a \ldots i_{p-1}} \, \omega_{j_a i_a}.$$

Exterior differentiating this, we have

$$\begin{aligned} da_{i_1 \ldots i_{p-1}, j} \wedge \omega_j + a_{i_1 \ldots i_{p-1}, k} \, \omega_{kj} \wedge \omega_j = {} & da_{i_1 \ldots j_a \ldots i_{p-1}} \wedge \omega_{j_a i_a} + a_{i_1 \ldots j_a \ldots i_{p-1}} \, \omega_{j_a k} \wedge \omega_{k i_a} \\ & + \frac{1}{2} a_{i_1 \ldots j_a \ldots i_{p-1}} \, \mathcal{R}_{j_a i_a k l} \, \omega_l \wedge \omega_k. \end{aligned}$$

The left hand side becomes

$$\begin{aligned} da_{i_1\ldots i_{p-1},j} \wedge \omega_j + a_{i_1\ldots i_{p-1},k}\, \omega_{kj} \wedge \omega_j \\ &= da_{i_1\ldots i_{p-1},j} \wedge \omega_j + a_{i_1\ldots i_{p-1},k}\, \omega_{kj} \wedge \omega_j \\ &= a_{i_1\ldots i_{p-1},jk}\, \omega_k \wedge \omega_j - a_{i_1\ldots k_a \ldots i_{p-1},j}\, \omega_{k_a i_a} \wedge \omega_j \\ &= a_{i_1\ldots i_{p-1},jk}\, \omega_k \wedge \omega_j + da_{i_1\ldots k_a\ldots i_{p-1}} \wedge \omega_{k_a i_a} \\ &\quad + \sum_{b\neq a} a_{i_1\ldots j_b\ldots k_a\ldots i_{p-1}}\, \omega_{j_b i_b} \wedge \omega_{k_a i_a} \\ &\quad + a_{i_1\ldots j_a\ldots i_{p-1}}\, \omega_{j_a k_a} \wedge \omega_{k_a i_a}. \end{aligned}$$

Equating to the right hand side gives

$$a_{i_1\ldots i_{p-1},jk}\, \omega_k \wedge \omega_j + \sum_{b\neq a} a_{i_1\ldots j_b\ldots k_a\ldots i_{p-1}}\, \omega_{j_b i_b} \wedge \omega_{k_a i_a} = \frac{1}{2}\mathcal{R}_{j_a i_a lk}\, a_{i_1\ldots j_a\ldots i_{p-1}}\, \omega_k \wedge \omega_l.$$

We now claim that the second term on the left hand side is identically 0. Indeed,

$$\begin{aligned} \sum_{b\neq a} a_{i_1\ldots j_b\ldots k_a\ldots i_{p-1}}\, \omega_{j_b i_b} \wedge \omega_{k_a i_a} &= \sum_{b<a} a_{i_1\ldots j_b\ldots k_a\ldots i_{p-1}}\, \omega_{j_b i_b} \wedge \omega_{k_a i_a} \\ &\quad + \sum_{b>a} a_{i_1\ldots j_b\ldots k_a\ldots i_{p-1}}\, \omega_{j_b i_b} \wedge \omega_{k_a i_a}, \end{aligned}$$

and the claim follows by interchanging the role of k_a and j_b in the second term. Hence

$$a_{i_1\ldots i_{p-1},jk}\, \omega_k \wedge \omega_j = \frac{1}{2}\mathcal{R}_{j_a i_a lk}\, a_{i_1\ldots j_a\ldots i_{p-1}}\, \omega_k \wedge \omega_l$$

which implies that

$$a_{i_1\ldots i_{p-1},lk} - a_{i_1\ldots i_{p-1},kl} = \mathcal{R}_{j_a i_a lk}\, a_{i_1\ldots j_a\ldots i_{p-1}}.$$

Similarly, for p-forms, we also have

$$a_{i_1\ldots i_p,jk} - a_{i_1\ldots i_p,kj} = \mathcal{R}_{l_\alpha i_\alpha jk}\, a_{i_1\ldots l_\alpha\ldots i_p}. \tag{3.3}$$

Let us now compute the Laplacian $\Delta\omega = -d\delta\omega - \delta d\omega$ for p-forms. First we have

$$\begin{aligned} d\omega &= a_{I,j}\, \omega_j \wedge \omega_I \\ &= \sum_{i_1<i_2<\cdots<i_{p+1}} \sum_{\sigma(i_1,\ldots,i_{p+1})} \operatorname{sgn}(\sigma)\, a_\sigma \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_1}, \end{aligned}$$

where $\sigma(i_1, \ldots, i_{p+1})$ denotes a permutation of the set $(i_1, \ldots, i_{p+1})$ and $\operatorname{sgn}(\sigma)$ is the sign of σ. Recall that if ω is a p-form then

$$\delta\omega = (-1)^{m(p+1)+1} * d * \omega.$$

The linear operator $* : \Lambda^p(M) \to \Lambda^{m-p}(M)$ is determined by

$$*(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}) = \mathrm{sgn}(\sigma(I, I^c))\, \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}},$$

where $\sigma(I, I^c)$ denotes a permutation by sending

$$(i_p, \ldots i_1, i_m, \ldots, i_{p+1}) \mapsto (1, \ldots, m).$$

Let us now define

$$\begin{aligned} \beta &= *\omega \\ &= a_{i_1 \ldots i_p} \,\mathrm{sgn}(\sigma(I, I^c))\, \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}}. \end{aligned}$$

By setting $b_{k_1 \ldots k_{m-p}} = \mathrm{sgn}(\sigma(I, I^c))\, a_{i_1 \ldots i_p}$ with $K = (k_1, \ldots, k_{m-p}) = (i_{p+1}, \ldots, i_m) = I^c$, we can write

$$\beta = b_K\, \omega_K$$

and

$$\begin{aligned} d\beta &= b_{K,j}\, \omega_j \wedge \omega_K \\ &= (db_{k_1 \ldots k_{m-p}} + b_{k_1 \ldots j_\theta \ldots k_{m-p}}\, \omega_{j_\theta k_\theta}) \wedge \omega_K \end{aligned}$$

for $1 \le \theta \le m - p$. On the other hand, we also have

$$\begin{aligned} d\beta &= \mathrm{sgn}(\sigma(I, I^c))\, da_{i_1 \ldots i_p} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &\quad + b_{k_1 \ldots j_\theta \ldots k_{m-p}}\, \omega_{j_\theta k_\theta} \wedge \omega_K \\ &= \mathrm{sgn}(\sigma(I, I^c))\, a_{i_1 \ldots i_p, j}\, \omega_j \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &\quad - \mathrm{sgn}(\sigma(I, I^c))\, a_{i_1 \ldots j_\alpha \ldots i_p}\, \omega_{j_\alpha i_\alpha} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &\quad + b_{k_1 \ldots j_\theta \ldots k_{m_p}}\, \omega_{j_\theta k_\theta} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &= \mathrm{sgn}(\sigma(I, I^c))\, a_{i_1 \ldots i_\alpha \ldots i_p, i_\alpha}\, \omega_{i_\alpha} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &\quad - \mathrm{sgn}(\sigma(I, K))\, a_{i_1 \ldots j_\alpha \ldots i_p}\, \omega_{j_\alpha i_\alpha} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &\quad + b_{k_1 \ldots j_\theta \ldots k_{m-p}}\, \omega_{j_\theta k_\theta} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}}. \end{aligned}$$

However

$$b_{k_1 \ldots j_\theta \ldots k_{m-p}} = 0,$$

unless $j_\theta = i_\alpha$ for some α, and

$$\begin{aligned} b_{k_1 \ldots i_\alpha \ldots k_{m-p}} &= \mathrm{sgn}(\sigma(i_p, \ldots, k_\theta, \ldots, i_1, k_{m-p}, \ldots, i_\alpha, \ldots, k_1))\, a_{i_1 \ldots \underset{\alpha^{\text{th}}\text{slot}}{k_\theta} \ldots i_p} \\ &= -\mathrm{sgn}(\sigma(i_p, \ldots, i_1, i_m, \ldots, i_{m-p}))\, a_{i_1 \ldots j_\alpha \ldots i_p}. \end{aligned}$$

Using the skew symmetry $\omega_{i_\alpha k_\theta} = -\omega_{k_\theta i_\alpha}$, we have

$$d\beta = \mathrm{sgn}(\sigma(I, I^c))\, a_{i_1 \ldots i_\alpha \ldots i_p, i_\alpha}\, \omega_{i_\alpha} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}}.$$

Hence

$$\begin{aligned}*d\beta &= \operatorname{sgn}(\sigma(I, I^c)) \operatorname{sgn}(\sigma(i_\alpha, i_m, \dots, i_{p+1}, i_p, \dots, \hat{i}_\alpha, \dots, i_1)) \\ &\quad \times a_{i_1 \dots i_\alpha \dots i_p, i_\alpha} \, \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1} \\ &= (-1)^{m-\alpha+p(m-p)} a_{i_1 \dots i_p, i_\alpha} \, \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1}.\end{aligned}$$

Therefore

$$\delta\omega = \sum_{1 \le \alpha \le p} (-1)^{\alpha+p^2+1} a_{i_1 \dots i_\alpha \dots i_p, i_\alpha} \, \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1}.$$

Computing directly gives,

$$\begin{aligned}-\Delta\omega &= \delta(a_{i_1 \dots i_p, j} \, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}) + d[(-1)^{\alpha+p^2+1} a_{i_1 \dots i_p, i_\alpha} \, \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1}] \\ &= (-1)^{\alpha+(p+1)^2+1} a_{i_1 \dots i_p, j i_\alpha} \, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1} \\ &\quad + (-1)^{(p+1)+(p+1)^2+1} a_{i_1 \dots i_p, jj} \, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\ &\quad + (-1)^{\alpha+p^2+1} a_{i_1 \dots i_p, i_\alpha j} \, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1}.\end{aligned}$$

However, by (3.3),

$$a_{i_1 \dots i_p, i_\alpha j} = a_{i_1 \dots i_p, j i_\alpha} + \mathcal{R}_{k_\beta i_\beta i_\alpha j} \, a_{i_1 \dots k_\beta \dots i_p}.$$

Hence

$$\begin{aligned}\Delta\omega &= (-1)^{\alpha+p^2} \mathcal{R}_{k_\beta i_\beta i_\alpha j} \, a_{i_1 \dots k_\beta \dots i_p} \, \omega_j \wedge \cdots \wedge \hat{\omega}_{i_\alpha} \wedge \cdots \wedge \omega_{i_1} + a_{i_1 \dots i_p, jj} \, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\ &= -\mathcal{R}_{k_\beta i_\beta j_\alpha i_\alpha} \, a_{i_1 \dots k_\beta \dots i_p} \, \omega_{i_p} \wedge \cdots \wedge \omega_{j_\alpha} \wedge \cdots \wedge \omega_{i_1} + a_{I, jj} \, \omega_I.\end{aligned}$$

If we define

$$E(\omega) = \mathcal{R}_{k_\beta i_\beta j_\alpha i_\alpha} \, a_{i_1 \dots k_\beta \dots i_p} \, \omega_{i_p} \wedge \cdots \wedge \omega_{j_\alpha} \wedge \cdots \wedge \omega_{i_1}$$

and the Bochner Laplacian by

$$\nabla^* \nabla \omega = a_{I, jj} \, \omega_I,$$

then

$$\Delta\omega = \nabla^* \nabla \omega - E(\omega).$$

Remark 3.1. *If $\omega = f$ is a smooth function, then*

$$\Delta f = \sum_{i=1}^{m} f_{ii}.$$

Remark 3.2. *If $\omega = a_i \, \omega_i$ is a 1-form, then*

$$\begin{aligned}E(\omega) &= \mathcal{R}_{kiji} \, a_k \, \omega_j \\ &= \mathcal{R}_{jk} \, a_k \, \omega_j.\end{aligned}$$

Lemma 3.1. *If M is compact, the operator δ is the adjoint operator to d, ie.*

$$\int_M \langle d\omega, \eta\rangle = \int_M \langle \omega, \delta\eta\rangle$$

for all $\omega, \eta \in \Lambda^p(M)$.

Proof. Note that by the definition of $*$, we have

$$\begin{aligned} **(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}) &= *(\operatorname{sgn}(\sigma(I, I^c))\, \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &= \operatorname{sgn}(\sigma(I^c, I)) \operatorname{sgn}(\sigma(I, I^c))\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}. \end{aligned}$$

On the other hand, it is clear that

$$\operatorname{sgn}(\sigma(I, I^c)) = (-1)^{(m-p)p} \operatorname{sgn}(\sigma(I^c, I)).$$

Hence, we have the formula that $** = (-1)^{(m-p)p}$ on p-forms. We also observe that

$$\begin{aligned} (\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}) \wedge *(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}) &= \operatorname{sgn}(\sigma(I, I^c))\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \wedge \omega_{i_m} \wedge \cdots \wedge \omega_{i_{p+1}} \\ &= \omega_1 \wedge \cdots \wedge \omega_m. \end{aligned}$$

Also by linearity, if $\omega = a_I\, \omega_I$ and $\theta = b_I\, \omega_I$ then

$$\begin{aligned} \omega \wedge *\theta &= a_I b_I\, \omega_1 \wedge \cdots \wedge \omega_m \\ &= \langle \omega, \theta\rangle dV. \end{aligned}$$

Let us now consider $\omega \in \Lambda^{p-1}(M)$ and $\eta \in \Lambda^p(M)$, then

$$\begin{aligned} \langle d\omega, \eta\rangle &= d\omega \wedge *\eta \\ &= d(\omega \wedge *\eta) + (-1)^p \omega \wedge d * \eta \\ &= d(\omega \wedge *\eta) + (-1)^p (-1)^{(m-p+1)(p-1)} \omega \wedge ** d * \eta \\ &= d(\omega \wedge *\eta) + \omega \wedge *\delta\eta \\ &= d(\omega \wedge *\eta) + \langle \omega, \delta\eta\rangle. \end{aligned}$$

Integrating both sides over M and the lemma follows from Stoke's theorem.

Lemma 3.2. *(Bochner [B]) Let $\omega = \sum_I a_I\, \omega_I$ be a p-form on M. Then*

$$\Delta|\omega|^2 = 2\langle \Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2\langle E(\omega), \omega\rangle.$$

Proof. The norm of ω is given by

$$|\omega|^2 = \sum_I a_I^2.$$

Let us choose an orthonormal[2] coframe in a neighborhood of $x \in M$ by parallel translating an orthonormal frame $\{e_1, \ldots, e_m\}$ at the point x. Hence at x, we have $\omega_{ij} = 0$. Moreover, $\nabla_{e_i} e_j = 0$ along the geodesic tangent to e_i which implies that $\nabla_{e_i} \nabla_{e_i} e_j = 0$ at x. Computing

$$\begin{aligned}\Delta|\omega|^2 &= (a_I^2)_{jj} \\ &= 2a_I\,(a_I)_{jj} + 2((a_I)_j)^2.\end{aligned}$$

Note that in general, $(a_I)_j \neq a_{I,j}$ and $(a_I)_{jj} \neq a_{I,jj}$ since the terms on the left denote differentiations of the function a_I while the terms on the right denote covariant differentiations of the p-form. However, by the choice of our frame,

$$\begin{aligned}(a_I)_j &= da_I(e_j) \\ &= a_{I,j} - a_{i_1 \ldots j_\alpha \ldots i_p}\, \omega_{j_\alpha i_\alpha}(e_j) \\ &= a_{I,j}\end{aligned}$$

at the point x. Similarly

$$\begin{aligned}a_{I,jj} &= da_{i_1 \ldots i_p, j}(e_j) \\ &= d[da_I(e_j) + a_{i_1 \ldots j_\alpha \ldots i_p}\, \omega_{j_\alpha i_\alpha}(e_j)](e_j) \\ &= (a_I)_{jj} + a_{i_1 \ldots j_\alpha \ldots i_p}\, e_j(\omega_{j_\alpha i_\alpha}(e_j))\end{aligned}$$

at x. On the other hand,

$$\begin{aligned}e_j(\omega_{j_\alpha i_\alpha}(e_j)) &= e_j \langle \nabla_{e_j} e_{j_\alpha}, e_{i_\alpha} \rangle \\ &= \langle \nabla_{e_j} \nabla_{e_j} e_{j_\alpha}, e_{i_\alpha} \rangle + \langle \nabla_{e_j} e_{j_\alpha}, \nabla_{e_j} e_{i_\alpha} \rangle \\ &= 0\end{aligned}$$

at x. Therefore,

$$\begin{aligned}\Delta|\omega|^2 &= 2\langle \nabla^* \nabla \omega, \omega \rangle + 2|\nabla \omega|^2 \\ &= 2\langle \Delta \omega, \omega \rangle + 2|\nabla \omega|^2 + 2\langle E(\omega), \omega \rangle\end{aligned}$$

at the point x. Now we observe that both sides of the equation is globally defined, hence this formula is valid on M.

Theorem 3.1. *Let M be a compact m-dimensional Riemannian manifold without boundary. Suppose $\mathcal{R}_{ij} \geq 0$, then any harmonic 1-form ω must be parallel and $\mathcal{R}(\omega, \omega) \equiv 0$. In particular, this implies that the first betti number $b_1(M)$ of M must be at most m. If in addition there exists a point $x \in M$ such that the Ricci curvature is positive, then $b_1 = 0$.*

Proof. Let ω be a harmonic 1-form. By the Bochner formula and Remark 3.2, we have

$$\begin{aligned}\Delta|\omega|^2 &= 2|\nabla \omega|^2 + 2\langle E(\omega), \omega \rangle \\ &= 2|\nabla \omega|^2 + 2\langle \mathcal{R}_{jk}\, a_j\, \omega_k, a_i\, \omega_i \rangle \\ &= 2|\nabla \omega|^2 + 2\langle \mathcal{R}_{ji}\, a_j\, a_i \rangle \\ &= 2|\nabla \omega|^2 + 2\mathcal{R}(\omega, \omega),\end{aligned} \tag{3.4}$$

which is nonnegative. Hence, by the maximum principle and the fact that M is compact, $|\omega|^2$ must be identically constant. Moreover, (3.4) implies that

$$|\nabla\omega|^2 = 0$$

and

$$\mathcal{R}(\omega,\omega) = 0$$

on M. Since the dimension of parallel 1-forms are at most m, we conclude that the dimension of harmonic 1-forms are at most m. The betti number bound follows from Hodge decomposition theorem. If we further assume that $\mathcal{R}(x) > 0$ for some point $x \in M$, then the fact that $\mathcal{R}(\omega,\omega) \equiv 0$ implies that $\omega(x) = 0$. On the other hand, since $|\omega|^2$ is constant, this show that $\omega \equiv 0$. Hence $b_1(M) = 0$.

We would like to remark that if ω is a parallel 1-form on M, then by the deRham decomposition theorem, the universal covering $\tilde{M}$ of M must be isometrically a product of $\mathbf{R} \times N$, where $\mathbf{R}$ is given by ω. Hence, if M is a compact manifold with nonnegative Ricci curvature, then its universal covering $\tilde{M}$ must be a product of $\mathbf{R}^k \times N$ for some manifold N with nonnegative Ricci curvature and $k = b_1(M)$.

Definition. *The curvature operator of a Riemmanian manifold is a linear map* $\mathcal{S} : \Lambda^2(M) \to \Lambda^2(M)$ *given by*

$$\mathcal{S}(\omega_i \wedge \omega_j) = \mathcal{R}_{jikl}\, \omega_l \wedge \omega_k.$$

Theorem 3.2. *Let M be a compact manifold. If the curvature operator is nonnegative on M, then any harmonic p-form must be parallel. Hence the p-th betti number of M is at most $\binom{m}{p}$. Moreover, if there exists a point $x \in M$ such that $\mathcal{S}(x) > 0$, then $b_p(M) = 0$.*

Proof. Similar to the case of 1-forms, we have the identity

$$\Delta|\omega|^2 = 2|\nabla\omega|^2 + 2\langle E(\omega), \omega\rangle,$$

where

$$\langle E(\omega), \omega\rangle = \mathcal{R}_{k_\beta i_\beta j_\alpha i_\alpha}\, a_{i_1 \dots k_\beta \dots i_p}\, a_{i_1 \dots j_\alpha \dots i_p}.$$

Let us now define the 2-form $\overline{\omega} = a_{i_1 \dots j_\alpha \dots i_p}\, \omega_{j_\alpha} \wedge \omega_{i_\alpha}$. By the definition of $\mathcal{S}$, we have

$$\mathcal{S}(\overline{\omega}) = \mathcal{R}_{i_\alpha j_\alpha kl}\, a_{i_1 \dots j_\alpha \dots i_p}\, \omega_l \wedge \omega_k.$$

Hence,

$$\begin{aligned}
\langle \mathcal{S}(\overline{\omega}), \overline{\omega}\rangle &= \langle \mathcal{R}_{i_\alpha j_\alpha kl}\, a_{i_1 \dots j_\alpha \dots i_p}\, \omega_l \wedge \omega_k, a_{i_1 \dots j_\beta \dots i_p}\, \omega_{j_\beta} \wedge \omega_{i_\beta}\rangle \\
&= \mathcal{R}_{i_\alpha j_\alpha i_\beta j_\beta}\, a_{i_1 \dots j_\alpha \dots i_p}\, a_{i_1 \dots j_\beta \dots i_p} \\
&= \langle E(\omega), \omega\rangle.
\end{aligned}$$

The theorem now follows from the argument of Theorem 3.1.

Definition. *A vector field* $X = a_i e_i$ *is a Killing vector field if* $a_{i,j} + a_{j,i} = 0$ *for all* $1 \le i, j \le m$.

Lemma 3.3. *The infinitesimal generator of a 1-parameter family of isometries of* M *is a Killing vector field on* M.

Proof. Let $\phi_t : M \to M$ be a 1-parameter family of isometries parametrized by $t \in (-\epsilon, \epsilon)$. If $\{e_1, \ldots, e_m\}$ is an orthonormal frame at a point $x \in M$ given by the normal coordinates centered at x. Then the fact that ϕ_t are isometries implies that

$$\begin{aligned} \langle d\phi_t(e_i), d\phi_t(e_j)\rangle &= \langle e_i, e_j\rangle \\ &= \delta_{ij}. \end{aligned}$$

Differentiating with respect to t and evaluate at $t = 0$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\langle d\phi_t(e_i), d\phi_t(e_j)\rangle|_{t=0} \\ &= \langle \nabla_T d\phi_t(e_i), d\phi_t(e_j)\rangle + \langle \nabla_T d\phi_t(e_j), d\phi_t(e_i)\rangle, \end{aligned} \tag{3.5}$$

where T is the infinitesimal vector field given by ϕ_t.

However, since one can view $\{\frac{\partial}{\partial t}, e_1, \ldots, e_m\}$ as tangent vectors given by a coordinate system of $(-\epsilon, \epsilon) \times M$, we have the property that

$$[T, d\phi_t(e_i)] = 0$$

for all $1 \le i \le m$. Hence we can rewrite (3.5) as

$$0 = \langle \nabla_{e_i} T, e_j\rangle + \langle \nabla_{e_j} T, e_i\rangle$$

which is exactly the condition that T is a Killing vector field.

Theorem 3.3. *Let* M *be a compact manifold with nonpositive Ricci curvature. Then any Killing vector field on* M *must be parallel. Moreover, if there exists a point* $x \in M$ *such that the Ricci curvature satisfies* $\mathcal{R}(x) < 0$, *then there are no non-trivial Killing vector fields. In particular, this implies that* M *does not have any 1-parameter family of isometries.*

Proof. Let $X = a_i e_i$ be a Killing vector field. Its dual 1-form is given by $\omega = a_i \omega_i$. The commutation formula yields

$$\begin{aligned} \Delta|\omega|^2 &= 2a_{i,j}^2 + 2a_{i,jj}\, a_i \\ &= 2|\nabla\omega|^2 - 2a_{j,ij}\, a_i \\ &= 2|\nabla\omega|^2 - 2\mathcal{R}(X, X). \end{aligned}$$

We now apply the same argument as in the proof of Theorem 3.1.

§4 Laplacian Comparison Theorem

Let M be a complete m-dimensional manifold. Suppose p is a fixed point in M, let us consider the distance function $r_p(x) = r(p,x)$ to p. When there is no ambiguity, the subscript will be deleted and we will simply write $r(x)$. The distance function in general is not smooth due to the presence of cut-points. However, it can be seen that it is a Lipschitz function with Lipschitz constant 1. In particular, we have

$$|\nabla r|^2 = 1$$

almost everywhere on M. Though r might not be a C^2 function, one can still estimate its Laplacian in the sense of distribution.

Theorem 4.1. *Let M be a complete m-dimensional Riemannian manifold with Ricci curvature bounded from below by*

$$\mathcal{R}_{ij} \geq (m-1)K$$

for some constant K. Then the Laplacian of the distance function satisfies

$$\Delta r(x) \leq \begin{cases} (m-1)\sqrt{K}\cot(\sqrt{K}r), & \text{for } K > 0 \\ (m-1)r^{-1}, & \text{for } K = 0 \\ (m-1)\sqrt{-K}\coth(\sqrt{-K}r), & \text{for } K < 0 \end{cases}$$

in the sense of distribution.

Proof. For a smooth function f, let $x \in M$ be a point such that $\nabla f(x) \neq 0$. Then locally the level surface N of f through the point x is a smooth hypersurface. Let $\{e_1, \ldots, e_{m-1}\}$ be an orthonormal vector field tangent to N. Let us denote e_m to be the unit normal vector to N. The Laplacian of f at x is defined to be

$$\begin{aligned}
\Delta f(x) &= \sum_{i=1}^{m-1}(e_ie_i - \nabla_{e_i}e_i)f(x) + (e_me_m - \nabla_{e_m}e_m)f(x) \\
&= \sum_{i=1}^{m-1}(e_ie_i - (\nabla_{e_i}e_i)^t)f(x) - \sum_{i=1}^{m-1}(\nabla_{e_i}e_i)^n f(x) + (e_me_m - \nabla_{e_m}e_m)f(x) \\
&= \Delta_N f(x) + \vec{H} f(x) + (e_me_m - \nabla_{e_m}e_m)f(x) \\
&= \vec{H} f(x) + (e_me_m - \nabla_{e_m}e_m)f(x)
\end{aligned}$$

where Δ_N denotes the Laplacian of N with respect to the induced metric and $\vec{H}$ denotes the mean curvature vector of N. If we take $f = r$, then for the point x which is not in the cut-locus of p, we have $N = \partial B_p(r)$. Moreover, the unit normal vector $e_m = \frac{\partial}{\partial r}$. Hence,

$$\Delta r(x) = H(x)$$

where $H(x)$ is the mean curvature of $\partial B_p(r)$ with respect to $\frac{\partial}{\partial r}$. Therefore according to Corollary 2.1 the theorem is true for those point which is not in the cut- locus of p.

Using the same notation as in Theorem 2.1, let us denote

$$\overline{H}(r) = \begin{cases} (m-1)\sqrt{K}\cot(\sqrt{K}r), & \text{for } K > 0 \\ (m-1)r^{-1}, & \text{for } K = 0 \\ (m-1)\sqrt{-K}\coth(\sqrt{-K}r), & \text{for } K < 0. \end{cases}$$

To show that Δr has the desired estimate in the sense of distribution, it suffices to show that for any nonnegative compactly supported smooth function ϕ, we have

$$\int_M (\Delta\phi)\, r \leq \int_M \phi\, \overline{H}(r).$$

In terms of polar normal coordinates at p, we can write

$$\int_M \phi\, \overline{H}(r) = \int_0^\infty \int_{C(r)} \phi\, \overline{H}(r)\, J(\theta, r)\, d\theta\, dr.$$

On the other hand, for each $\theta \in S_p(M)$ if we denote $R(\theta)$ to be the maximum value of $r > 0$ such that the geodesic $\gamma(s) = \exp_p(s\theta)$ minimizes up to $s = r$, then by Fubini's theorem we can write

$$\int_0^\infty \int_{C(r)} \phi\, \overline{H}(r)\, J(\theta, r)\, d\theta\, dr = \int_{S_p(M)} \int_0^{R(\theta)} \phi\, \overline{H}(r)\, J(\theta, r)\, dr\, d\theta.$$

However, for $r < R(\theta)$, we have

$$\begin{aligned} \overline{H}(r)\, J(\theta, r) &\geq H(\theta, r)\, J(\theta, r) \\ &= \frac{\partial}{\partial r} J(\theta, r). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_M \phi\, \overline{H}(r) &\geq \int_{S_p(M)} \int_0^{R(\theta)} \phi\, \frac{\partial J}{\partial r}\, dr\, d\theta \\ &= -\int_{S_p(M)} \int_0^{R(\theta)} \frac{\partial \phi}{\partial r}\, J\, dr\, d\theta + \int_{S_p(M)} [\phi\, J]_0^{R(\theta)}\, d\theta \\ &= -\int_M \frac{\partial \phi}{\partial r} + \int_{S_p(M)} \phi(\theta, R(\theta))\, J(\theta, R(\theta))\, d\theta \\ &\geq -\int_M \langle \nabla\phi, \nabla r \rangle, \end{aligned}$$

where we have used the fact that both ϕ and J are nonnegative and $J(\theta, 0) = 0$. On the other hand, since r is Lipschitz, we have

$$-\int_M \langle \nabla\phi, \nabla r \rangle = \int_M (\Delta\phi) r,$$

which proves the theorem.

We are now ready to prove a structural theorem for manifolds with nonnegative Ricci curvature. Let us first define the notions of a line and a ray in a Riemannian manifold.

Definition. *A line is a normal geodesic $\gamma : (-\infty, \infty) \longrightarrow M$ such that any of its finite segment $\gamma|_a^b$ is a minimizing geodesic.*

Definition. *A ray is half-line, which is a normal minimizing geodesic $\gamma_+ : [0, \infty) \longrightarrow M$.*

Theorem 4.2. *(Cheeger-Gromoll [C-G]) Let M be a complete manifold with nonnegative Ricci curvature of dimension m. If there exists a line in M, then M is isometric to $\mathbf{R} \times N$, the product of a real line and an $(m-1)$-dimensional manifold N with nonnegative Ricci curvature.*

Proof. Let $\gamma_+ : [0, \infty) \longrightarrow M$ be a ray in M. One defines the Buseman function β_+ with respect to γ_+ by

$$\beta_+(x) = \lim_{t \to \infty} (t - r(\gamma_+(t), x)).$$

We observe that β_+ is a Lipschitz function with Lipschitz constant 1. Moreover, by the Laplacian comparison theorem,

$$\begin{aligned}\Delta \beta_+(x) &\geq -\lim_{t \to \infty} (m-1) r(\gamma_+(t), x)^{-1} \\ &= 0\end{aligned}$$

in the sense of distribution. If γ is a line, then $\gamma_+(t) = \gamma(t)$ and $\gamma_-(t) = \gamma(-t)$ for $t \geq 0$ are rays. The corresponding Buseman functions β_+ and β_- are subharmonic in the sense of distribution. In particular, $\beta_- + \beta_+$ is also subharmonic on M. On the other hand, since γ is a line, the triangle inequality implies that

$$\begin{aligned}2t &= r(\gamma(-t), \gamma(t)) \\ &\leq r(\gamma(-t), x) + r(\gamma(t), x).\end{aligned}$$

Hence

$$t - r(\gamma_-(t), x) + t - r(\gamma_+(t), x) \leq 0,$$

and by taking the limit as $t \to \infty$ we have

$$\beta_-(x) + \beta_+(x) \leq 0.$$

Moreover, it is also clear that

$$\beta_-(x) + \beta_+(x) = 0$$

for all x on γ. However, by the strong maximum principle, since the subharmonic function $\beta_- + \beta_+$ has an interrior maximum, it must be identically constant. In particular, both β_- and β_+ are harmonic and $\beta_- \equiv -\beta_+$. By regularity theory, β_+ is a smooth harmonic function with $|\nabla \beta_+| \leq 1$, and $|\nabla \beta_+| \equiv 1$ on the geodesic γ. For simplicity, let us write $\beta = \beta_+$. The Bochner formula gives

$$\begin{aligned}\Delta |\nabla \beta|^2 &= 2\beta_{ij}^2 + 2\mathcal{R}_{ij} \beta_i \beta_j + 2\langle \nabla \beta, \nabla \Delta \beta \rangle \\ &\geq 0.\end{aligned}$$

Hence by the fact that β achieves its maximum in the interrior of M and the maximum principle for subharmonic functions, $\beta \equiv 1$ on M, $\beta_{ij} \equiv 0$, and $\nabla \beta$ is a parallel vector field on M. This implies that M must split, which proves the theorem.

Corollary 4.1. *Let M be a complete m-dimensional Riemannian manifold with nonnegative Ricci curvature. If M has at least 2 ends, then there exists a compact $(m-1)$-dimensional manifold N of nonnegative Ricci curvature such that*

$$M = \mathbf{R} \times N.$$

Proof. The assumption that M has at least 2 ends implies that there exsits a compact set $D \subset M$ such that $M \setminus D$ has at least 2 unbounded components. Hence there are 2 unbounded sequences of points $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$, such that the minimal geodesics γ_i joining x_i to y_i must intersect D. By compactness of D, if $p_i \in \gamma_i \cap D$ and $v_i = \gamma_i'(p_i)$, then by passing through a subsequence we have $p_i \to p$ and $v_i \to v$ for some $p \in D$. We now claim that the geodesic $\gamma : (-\infty, \infty) \longrightarrow M$ given by the initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$ is a line.

To see this, let us consider an arbitrary segment $\gamma|_{[a,b]}$ of γ. By continuity of initial conditions of second order ordinary differential equation, we know that $\gamma_i|_{[a,b]} \to \gamma|_{[a,b]}$ because $(p_i, v_i) \to (p, v)$. However, by the assumption, $\gamma_i|_{[a,b]}$ are minimizing geodesics, hence $\gamma|_{[a,b]}$ is also minimizing. Therefore,γ is a line.

Theorem 4.2 now implies that $M = \mathbf{R} \times N$. The compactness of N follows from the assumption that M has at least 2 ends.

Another application of the Laplacian comparison theorem is the eigenvalue comparison theorem of Cheng.

Theorem 4.3. *(Cheng [Cg]) Let M be a compact Riemannian manifold of dimension m. Assume that the Ricci curvature of M is bounded by*

$$\mathcal{R}_{ij} \geq (m-1)K$$

for some constant K. Let us consider $\overline{M}$ to be the simply connected space form with constant curvature K. Suppose M has boundary ∂M. Let us denote $\mu_1(M)$ to be the first nonzero eigenvalue of the Dirichlet Laplacian on M and i to be the inscribe radius of M. If $\overline{B}(i)$ is a geodesic ball in $\overline{M}$ with radius i and $\mu_1(\overline{B}(i))$ its first Dirichlet eigenvalue, then

$$\mu_1(M) \leq \mu_1(\overline{B}(i)).$$

When M has no boundary, let us denote $\lambda_1(M)$ to be the first nonzero eigenvalue of the Laplacian and d to be the diameter of M. If $\overline{B}(\frac{d}{2})$ is a geodesic ball in $\overline{M}$ with radius $\frac{d}{2}$ and $\mu_1(\overline{B}(\frac{d}{2}))$ its first Dirichlet eigenvalue, then

$$\lambda_1(M) \leq \mu_1(\overline{B}(\frac{d}{2})).$$

Proof. Let us first consider the case when M has boundary. Let $B_p(i)$ be an inscribed ball in M. By the monotonicity of eigenvalues, it suffices to show that

$$\mu_1(B_p(i)) \leq \mu_1(\overline{B}(i)).$$

Let $\overline{u}$ be the first Dirichlet eigenfunction on $\overline{B}(i)$. By the uniqueness of $\overline{u}$, we may assume that $\overline{u} \geq 0$. If we denote $\overline{p} \in \overline{B}(i)$ to be its center and $\overline{r}$ to be the distance function to $\overline{p}$, then $\overline{u}(\overline{r})$ must be a function of $\overline{r}$ alone. By the fact that $\overline{u} \geq 0$ and $\overline{u}(i) = 0$, the strong maximum principle implies that $\frac{\partial \overline{u}}{\partial \overline{r}}(i) < 0$. If there is some value of $\overline{r} < i$ such that $\frac{\partial \overline{u}}{\partial \overline{r}} > 0$ then this would imply that $\overline{u}$ has a interrior local minimum. However, this violates the strong maximum principle. Hence, $\overline{u}' = \frac{\partial \overline{u}}{\partial \overline{r}} \leq 0$.

Let us define a Lipschitz function on $B_p(i)$ by $u(r) = \overline{u}(r)$, where r denotes the distance function to p. Clearly, u satisfies the Dirichlet boundary condition. Computing the Laplacian of u gives

$$\begin{aligned}\Delta u &= \overline{u}'\Delta r + \overline{u}''|\nabla r|^2\\ &\geq \overline{u}'\Delta r + \overline{u}''.\end{aligned}$$

On the other hand, if we denote $\overline{\Delta}$ to be the Laplacian on $\overline{M}$, then

$$\begin{aligned}-\mu_1(\overline{B}(i))\,\overline{u} &= \overline{\Delta}\overline{u}\\ &= \overline{u}'\overline{\Delta}\overline{r} + \overline{u}''.\end{aligned}$$

By Theorem 4.1 and the fact that $\overline{u}' \leq 0$, we conclude that

$$\Delta u \geq -\mu_1(\overline{B}(i))\,u$$

in the sense of distribution. Hence, by the Rayleigh principle for eigenvalues, we conclude that

$$\begin{aligned}\mu_1(B_p(i)) &\leq \frac{\int_{B_p(i)} |\nabla u|^2}{\int_{B_p(i)} u^2}\\ &= \frac{-\int_{B_p(i)} u\Delta u}{\int_{B_p(i)} u^2}\\ &\leq \mu_1(\overline{B}(i)).\end{aligned}$$

To prove the upper bound for the case when M has no boundary, we consider the disjoint balls $B_p(\frac{d}{2})$ and $B_q(\frac{d}{2})$ centered at a pair of points p and q which realize the diameter. By the above estimate,

$$\mu_1(B_p(\frac{d}{2})) \leq \mu_1(\overline{B}(\frac{d}{2}))$$

and

$$\mu_1(B_q(\frac{d}{2})) \leq \mu_1(\overline{B}(\frac{d}{2})).$$

We now claim that $\lambda_1(M) \leq \max\{\mu_1(M_1), \mu_1(M_2)\}$ for any disjoint pair of open subsets $M_1, M_2 \subset M$. This will establish the theorem by setting $M_1 = B_p(\frac{d}{2})$ and $M_2 = B_q(\frac{d}{2})$.

To see the claim, let u_1 and u_2 be nonnegative first Dirichlet eigenfunctions on M_1 and M_2 respectively. By multiplying a constant, we may assume that $\int_{M_i} u_i = 1$ for $i = 1, 2$. Let us define a Lipschitz function on M by

$$u = \begin{cases} u_1, & \text{on } M_1 \\ -u_2, & \text{on } M_2 \\ 0, & \text{on } M \setminus (M_1 \cup M_2). \end{cases}$$

Clearly, $\int_M u = 0$. Hence by the Rayleigh principle,

$$\begin{aligned} \lambda_1(M)\left(\int_{M_1} u_1^2 + \int_{M_2} u_2^2\right) &= \lambda_1(M)\int_M u^2 \\ &\leq \int_M |\nabla u|^2 \\ &= \int_{M_1} |\nabla u_1|^2 + \int_{M_2} |\nabla u_2|^2 \\ &= \mu_1(M_1)\int_{M_1} u_1^2 + \mu_1(M_2)\int_{M_2} u_2^2 \\ &\leq \max\{\mu_1(M_1), \mu_1(M_2)\}\left(\int_{M_1} u_1^2 + \int_{M_2} u_2^2\right). \end{aligned}$$

This establishes the claim.

§5 Poincaré Inequality and the First Eigenvalue

In this section, we will obtain lower estimates for the first eigenvalue for the Laplcian on a compact manifold. For the moment, we will primarily concern with the case when M has no boundary. The following lower bound was proved by Lichnerowicz [Lz] while Obata [O] considered the case when the estimate is achieved.

Theorem 5.1. *(Lichnerowicz and Obata) Let M be an m-dimensional compact manifold without boundary. Suppose that the Ricci curvature of M is bounded from below by*

$$\mathcal{R}_{ij} \geq (m-1)K$$

for some constant $K > 0$, then the first nonzero eigenvalue of the Laplacian on M must satisfy

$$\lambda_1 \geq mK.$$

Moreover, equality holds if and only if M is isometric to a standard sphere of radius $\frac{1}{\sqrt{K}}$.

Proof. Let u be a nonconstant eigenfunction satisfiying

$$\Delta u = -\lambda u,$$

with $\lambda > 0$. Consider the smooth function

$$Q = |\nabla u|^2 + \frac{\lambda}{m} u^2$$

defined on M. Computing its Laplacian

$$\begin{aligned}
\Delta Q &= (2u_j u_{ji} + \frac{2\lambda}{m} u u_{ii})_i \\
&= 2u_{ji}^2 + 2u_j u_{jii} + \frac{2\lambda}{m} u_i^2 + \frac{2\lambda}{m} u u_{ii} \\
&= 2u_{ji}^2 + 2\mathcal{R}_{ij} u_i u_j + 2u_j(\Delta u)_j + \frac{2\lambda}{m}|\nabla u|^2 + \frac{2\lambda}{m} u(\Delta u)
\end{aligned}$$

where we have used the Ricci identity and the convention that summation is being performed on repeated indices. On the other hand

$$\begin{aligned}
\sum_{i,j=1}^{m} u_{ji}^2 &\geq \sum_{i=1}^{m} u_{ii}^2 \\
&\geq \frac{\left(\sum_{i=1}^{m} u_{ii}\right)^2}{m} \\
&= \frac{(\Delta u)^2}{m} \\
&= \frac{\lambda^2 u^2}{m}.
\end{aligned}$$

Hence, by the assumption on the Ricci curvature, we have

$$\begin{aligned}
(5.1) \qquad \Delta Q &\geq \frac{2\lambda^2 u^2}{m} + 2(m-1)K|\nabla u|^2 - 2\lambda|\nabla u|^2 + \frac{2\lambda}{m}|\nabla u|^2 - \frac{2\lambda^2 u^2}{m} \\
&= 2(m-1)\left(K - \frac{\lambda}{m}\right)|\nabla u|^2.
\end{aligned}$$

If $\lambda \leq mK$, then Q is a subharmonic function. By the compactness of M and the maximum principle, Q must be identically constant and all the above inequalities are equalities. In particular, the right hand side of (5.1) must be identically 0. Hence $\lambda = mK$ because u is nonconstant. Moreover,

$$|\nabla u|^2 + \frac{\lambda}{m} u^2 = \frac{\lambda}{m}|u|_\infty^2$$

where $|u|_\infty = \sup_M |u|$. If we normalize u such that $|u|_\infty = 1$, and observe that at the maximum and the minimum points of u its gradient must vanish, then we conclude that $\max u = 1 = -\min u$ and

$$\frac{|\nabla u|}{\sqrt{1-u^2}} = \sqrt{K}.$$

Integrating this along a minimal geodesic γ joining the points where $u = 1$ and $u = -1$, we have

$$\begin{aligned} d\sqrt{K} &\geq \int_\gamma \frac{|\nabla u|}{\sqrt{1-u^2}} \\ &\geq \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} \\ &= \pi \end{aligned}$$

where d denotes the diameter of M. However, by Cheng's theorem (Theorem 2.3), M must be the standard sphere.

We will now give a sharp lower bound for the first eigenvalue on manifolds with nonnegative Ricci curvature. The estimate of Lichnerowicz becomes trivial in this case, since the Ricci curvature does not have a positive lower bound. However, one could still estimate the first eigenvalue in terms of the diameter of M alone.

Let λ_1 be the least nontrivial eigenvalue of a compact manifold and let ϕ be the corresponding eigenfunction. By multiplying with a constant it is possible to arrange that

$$a - 1 = \inf_M \phi, \qquad a + 1 = \sup_M \phi$$

where $0 \leq a(\phi) < 1$ is the median of ϕ.

Lemma 5.1. *(Li-Yau [L-Y]) Suppose M^n is a compact manifold without boundary whose Ricci curvature is nonnegative. Then the first nontrivial eigenvalue satisfies*

$$\lambda_1 \geq \frac{\pi^2}{(1+a)d^2}$$

where d is the diameter of M.

Proof. Setting $\lambda = \lambda_1$ and $u = \phi - a$ the equation becomes

$$\Delta u = -\lambda(u + a).$$

Let $P = |\nabla u|^2 + cu^2$ where $c = \lambda(1 + a)$. Let $x_0 \in M$ be the point where P is maximum. If $|\nabla u(x_0)| \neq 0$ we may rotate the frame so that $u_1(x_0) = |\nabla u(x_0)|$. Differentiating in the e_i direction yields

$$\frac{1}{2} P_i = u_m u_{mi} + c u u_i$$

so at x_0

$$0 = u_1 (u_{11} + cu)$$

and

$$u_{ij} u_{ij} \geq u_{11}^2 = c^2 u^2. \tag{5.2}$$

Covariant differentiating with respect to e_i again, using the commutation formula, (5.2), the definition of P, and evaluating at x_0, we have

$$\begin{aligned}
0 &\geq \frac{1}{2}\Delta P \\
&= u_{mi}u_{mi} + u_m u_{mii} + cu_1^2 + cu\Delta u \\
&\geq c^2u^2 + u_m(\Delta u)_m + R_{mp}u_m u_p + cu_1^2 - c\lambda u(u+a) \\
&\geq c^2u^2 - \lambda u_1^2 + cu_1^2 - c\lambda u(u+a) \\
&= (c-\lambda)\left(u_1^2 + cu^2\right) - ac\lambda u \\
&\geq a\lambda P(x_0) - ac\lambda.
\end{aligned}$$

Hence, for all $x \in M$

$$|\nabla u(x)|^2 \leq \lambda(1+a)\left(1-u(x)^2\right). \tag{5.3}$$

Also (5.3) is trivially satisfied if $\nabla u(x_0) = 0$.

Let γ be the shortest geodesic from the minimizing point of u to the maximizing point. The length of γ is at most d. Integrating the gradient estimate (5.3) along this segment with respect to arclength, we obtain

$$d\sqrt{\lambda(1+a)} \geq \sqrt{\lambda(1+a)}\int_\gamma ds \geq \int_\gamma \frac{|\nabla u| ds}{\sqrt{1-u^2}} \geq \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \pi.$$

In view of Lemma 5.1 and known examples, Li and Yau conjectured that the sharp estimate

$$\lambda_1 \geq \frac{\pi^2}{d^2}$$

should hold for compact manifolds with nonnegative Ricci curvature. In fact, if the first eigenspace has multiplicity greater than 1, this was verified in [L 2]. This conjecture was finally proved by Zhong and Yang by applying the maximum principle to a judicious choice of test function.

Lemma 5.2. *The function*

$$z(u) = \frac{2}{\pi}\left(\arcsin(u) + u\sqrt{1-u^2}\right) - u$$

defined on $[-1,1]$ *satisfies*

$$\dot{z}u + \ddot{z}(1-u^2) + u = 0; \tag{5.4}$$

$$\dot{z}^2 - 2z\ddot{z} + \dot{z} \geq 0; \tag{5.5}$$

$$2z - \dot{z}u + 1 \geq 0; \tag{5.6}$$

and

$$(1-u^2) \geq 2|z|. \tag{5.7}$$

Proof. Differentiating yields

$$\dot{z} = \frac{4}{\pi}\sqrt{1-u^2} - 1$$

and

$$\ddot{z} = \frac{-4u}{\pi\sqrt{1-u^2}}.$$

Clearly (5.4) is satisfied.

To see (5.5), we have

$$\dot{z}^2 - 2z\ddot{z} + \dot{z} = \frac{4}{\pi\sqrt{1-u^2}}\left\{\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin(u)\right) - (1+u^2)\right\}.$$

Since the right hand side is an even function, it suffices to check that

$$\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin(u)\right) - (1+u^2) \geq 0$$

on $[0,1]$. Computing its derivative

$$\frac{d}{du}\left\{\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin(u)\right) - (1+u^2)\right\} = \frac{4}{\pi}\arcsin(u) - 2u$$

which is nonpositive on $[0,1]$. Hence

$$\begin{aligned}\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin(u)\right) - (1+u^2) &\geq \left[\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin(u)\right) - (1+u^2)\right]_{u=1} \\ &= 0.\end{aligned}$$

Inequality (5.6) follows easily because

$$2z - \dot{z}u + 1 = \frac{4}{\pi}\arcsin(u) + 1 - u$$

which is obviously nonnegative.

To see (5.7), we will consider the cases $-1 \leq u \leq 0$ and $0 \leq u \leq 1$ seperately. It is clearly that the inequality is valid at -1, 0, and 1. Let us set

$$f(u) = 1 - u^2 - \frac{4}{\pi}\left(\arcsin(u) + u\sqrt{1-u^2}\right) + 2u.$$

Then

$$\dot{f} = -2u - \frac{4}{\pi}(2\sqrt{1-u^2}) + 2,$$

$$\ddot{f} = -2 + \frac{8u}{\pi\sqrt{1-u^2}},$$

and

$$\dddot{f} = \frac{8}{\pi(1-u^2)^{\frac{3}{2}}}.$$

When $-1 \leq u \leq 0$, $\ddot{f} \leq 0$. Hence $f(u) \geq \min\{f(-1), f(0)\} = 0$. For the case $0 \leq u \leq 1$, $\dddot{f} \geq 0$. Hence

$$\begin{aligned}\dot{f} &\leq \max\{\dot{f}(0), \dot{f}(1)\} \\ &= \max\{2 - \frac{8}{\pi}, 0\} \\ &= 0.\end{aligned}$$

Therefore $f(u) \geq f(1)$ which proves (5.7).

Lemma 5.3. *Suppose M is a compact manifold without boundary whose Ricci curvature is nonnegative. Assume that a nontrivial eigenfunction ϕ corresponding to the eigenvalue λ is normalized so that for $0 \leq a < 1$, $a+1 = \sup \phi$ and $a - 1 = \inf_M \phi$. By setting $u = \phi - a$, its gradient must satisfy the estimate*

$$|\nabla u|^2 \leq \lambda(1-u^2) + 2a\lambda z(u) \tag{5.8}$$

where

$$z(u) = \frac{2}{\pi}\left(\arcsin(u) + u\sqrt{1-u^2}\right) - u. \tag{5.9}$$

Proof. We will first prove an estimate similar to (5.8) for $u = \epsilon(\phi - a)$ where $0 < \epsilon < 1$. The lemma will follow by letting $\epsilon \to 1$. By the definition of u, we have

$$\Delta u = -\lambda(u + \epsilon a)$$

with $-\epsilon \leq u \leq \epsilon$. By (5.3) we may assume $a > 0$. Consider the function

$$Q = |\nabla u|^2 - c(1-u^2) - 2a\lambda z(u),$$

where by (5.3) and (5.7) we can choose c large enough so that $\sup_M Q = 0$. The lemma follows if $c \leq \lambda$, for a sequence of $\epsilon \to 1$, hence we may assume that $c > \lambda$.

Let the maximizing point of Q be x_0. We claim that $|\nabla u(x_0)| > 0$ since otherwise $\nabla u(x_0) = 0$ and

$$\begin{aligned}0 &= Q(x_0) \\ &= -c(1-u^2)(x_0) - 2a\lambda z(x_0) \\ &\leq -(c - a\lambda)(1-\epsilon^2)\end{aligned}$$

by (5.7), which is a contradiction.

Differentiating in the e_i direction gives

$$\frac{1}{2}Q_i = u_m u_{mi} + cuu_i - a\lambda \dot{z} u_i. \tag{5.10}$$

At x_0, we can rotate the frame so $u_1(x_0) = |\nabla u(x_0)|$ and using $Q_i = 0$, we have

$$u_{mi}u_{mi} \geq u_{11}^2 = (cu - a\lambda\dot{z})^2. \tag{5.11}$$

Differentiating again, using the commutation formula, $Q(x_0) = 0$, (5.7), (5.10), and (5.11), we get

$$\begin{aligned}
0 \geq& \frac{1}{2}\Delta Q(x_0) \\
=& u_{mi}u_{mi} + u_m u_{mii} + cu_1^2 + cu\Delta u - a\lambda\ddot{z}u_1^2 - a\lambda\dot{z}\Delta u \\
=& u_{mi}u_{mi} + u_m(\Delta u)_m + R_{mp}u_m u_p + (c - a\lambda\ddot{z})\, u_1^2 + (cu - a\lambda\dot{z})\,\Delta u \\
\geq& (cu - a\lambda\dot{z})^2 + (c - \lambda - a\lambda\ddot{z})\left[c\left(1-u^2\right) + 2a\lambda z\right] \\
& - \lambda\,(cu - a\lambda\dot{z})\,(u + \epsilon a) \\
=& - ac\lambda\left\{(1-u^2)\ddot{z} + u\dot{z} + \epsilon u\right\} + a^2\lambda^2\left\{-2z\ddot{z} + \dot{z}^2 + \epsilon\dot{z}\right\} \\
& + a\lambda(c-\lambda)\left\{-u\dot{z} + 2z + 1\right\} + (c-\lambda)(c - a\lambda).
\end{aligned}$$

However by (5.4), (5.5), and (5.6),we conclude that

$$\begin{aligned}
0 &\geq ac\lambda(1-\epsilon)u - a^2\lambda^2(1-\epsilon)\dot{z} + (c-\lambda)(c-a\lambda) \\
&\geq -ac\lambda(1-\epsilon) - a^2\lambda^2(1-\epsilon)(\frac{4}{\pi} - 1) + (c - la)(c - a\lambda) \\
&\geq -(c+\lambda)\lambda(1-\epsilon) + (c-\lambda)^2.
\end{aligned}$$

This implies that

$$c \leq \lambda\left\{\frac{2 + (1-\epsilon) + \sqrt{(1-\epsilon)(7-\epsilon)}}{2}\right\}.$$

Clearly, when $\epsilon \to 1$ this yields the desired estimate.

Theorem 5.2. *(Zhong-Yang [Z-Y]) Suppose M is a compact manifold without boundary whose Ricci curvature is nonnegative. Let $a \geq 0$ be the median of a normalized first eigenfunction with $a + 1 = \sup\phi$ and $a - 1 = \inf\phi$, and let d be the diameter. Then the first nontrivial eigenvalue satisfies*

$$d^2\lambda_1 \geq \pi^2 + \frac{6}{\pi}\left(\frac{\pi}{2} - 1\right)^4 a^2 \geq \pi^2(1 + .02a^2).$$

Proof. Arguing with $u = \phi - a$ as before, let γ be the shortest geodesic from the minimizing point of u to the maximizing point with length at most d. Integrating the gradient estimate (5.8) along this segment with respect to arclength and using oddness, we have

$$\begin{aligned} d\lambda^{\frac{1}{2}} &\geq \lambda^{\frac{1}{2}} \int_\gamma ds \\ &\geq \int_\gamma \frac{|\nabla u| ds}{\sqrt{1-u^2+2az(u)}} \\ &\geq \int_0^1 \left\{ \frac{1}{\sqrt{1-u^2+2az}} + \frac{1}{\sqrt{1-u^2-2az}} \right\} du \\ &\geq \int_0^1 \frac{1}{\sqrt{1-u^2}} \left\{ 2 + \frac{3a^2z^2}{1-u^2} \right\} du \\ &\geq \pi + 3a^2 \left(\int_0^1 \frac{z}{\sqrt{1-u^2}} \right)^2 \\ &= \pi + \frac{3a^2}{\pi^2} \left(\frac{\pi}{2} - 1 \right)^4 . \end{aligned}$$

This technique applies to manifolds with boundary. Let M^n be a compact manifold with smooth boundary whose Ricci curvature is nonnegative. Suppose that the second fundamental form of ∂M is nonnegative. Then the first nontrivial eigenvalue of the Laplacian with Neumann boundary conditions also satisfies the inequality (5.8). The proof runs the same as Lemma 5.3 except that the possibility of the maximum of the test function Q at the boundary must be handled. In fact, the boundary convexity assumption implies that the maximum of Q cannot occur on the boundary.

The next theorem gives an estimate of the first eigenvalue for general compact Riemannian manifold without boundary. The estimate depends on the lower bound of the Ricci curvature, the upper bound of the diameter, and the dimension of M alone.

Theorem 5.3. *(Li-Yau [L-Y]) Let M be a compact m-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature of M is bounded from below by*

$$\mathcal{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$, and d denotes the diameter of M. Then there exists constants $C_1(m), C_2(m) > 0$ depending on m alone, such that, the first nonzero eigenvalue of M satisfies

$$\lambda_1 \geq \frac{C_1}{d^2} \exp(-C_2 d\sqrt{R}).$$

Proof. Let u be a nonconstant eigenfunction satisfying

$$\Delta u = -\lambda u.$$

By the fact that

$$-\lambda \int_M u = \int_M \Delta u = 0,$$

u must change sign. Hence we may normalize u to satisfy $\min u = -1$ and $\max u \leq 1$. Let us consider the function

$$v = \log(a+u)$$

for some constant $a > 1$. The function v satisfies

$$\begin{aligned}\Delta v &= \frac{\Delta u}{a+u} - \frac{|\nabla u|^2}{(a+u)^2} \\ &= \frac{-\lambda u}{a+u} - |\nabla v|^2.\end{aligned}$$

Let us define $Q(x) = |\nabla v|^2(x)$. Differentiating Q in the e_i direction gives

$$Q_i = 2v_j v_{ji}.$$

Its Laplacian is given by

$$\begin{aligned}\Delta Q &= 2v_{ji}^2 + 2v_i v_{jii} \\ &\geq 2v_{ji}^2 + 2\langle \nabla v, \nabla \Delta v\rangle - 2(m-1)R|\nabla v|^2.\end{aligned} \tag{5.12}$$

However, similar to the proof of Theorem 5.1, we have

$$\begin{aligned}v_{ji}^2 &\geq \frac{(\Delta v)^2}{m} \\ &= \frac{1}{m}\left(\frac{\lambda u}{a+u} + Q\right)^2 \\ &\geq \frac{1}{m}\left(Q^2 + \frac{2\lambda u}{a+u}Q\right).\end{aligned}$$

Also,

$$\begin{aligned}\langle \nabla v, \nabla \Delta v\rangle &= -\lambda\left\langle \nabla v, \nabla\left(\frac{u}{a+u} + Q\right)\right\rangle \\ &= -\frac{a\lambda}{a+u}Q - \langle \nabla v, \nabla Q\rangle.\end{aligned}$$

Hence (5.12) becomes

$$\begin{aligned}\Delta Q + 2\langle \nabla v, \nabla Q\rangle &\geq \frac{2}{m}Q^2 + \left(\frac{4\lambda}{m} - 2(m-1)R - \frac{2(m+2)a\lambda}{m(a+u)}\right)Q \\ &\geq \frac{2}{m}Q^2 + \left(\frac{4\lambda}{m} - 2(m-1)R - \frac{2(m+2)a\lambda}{m(a-1)}\right)Q.\end{aligned} \tag{5.13}$$

If $x_0 \in M$ is a point where Q achieved its maximum, then

$$0 \geq \Delta Q(x_0) + 2\langle \nabla v, \nabla Q\rangle(x_0).$$

Hence, we have

$$0 \geq \frac{2}{m}Q^2(x_0) + \left(\frac{4\lambda}{m} - 2(m-1)R - \frac{2(m+2)a\lambda}{m(a-1)}\right) Q(x_0),$$

which implies that

$$\begin{aligned} Q(x) &\leq Q(x_0) \\ &\leq -\left(2\lambda - m(m-1)R - \frac{(m+2)a\lambda}{(a-1)}\right) \\ &= \frac{(m+2)a\lambda}{a-1} - m(m-1)R \end{aligned}$$

for all $x \in M$. Integrating $Q^{\frac{1}{2}} = |\nabla \log(a+u)|$ along a minimal geodesic γ joining the points at which $u = -1$ and $u = \max u$, we have

$$\begin{aligned} \log\left(\frac{a}{a-1}\right) &\leq \log\left(\frac{a+\max u}{a-1}\right) \\ &\leq \int_\gamma |\nabla \log(a+u)| \\ &\leq d\sqrt{\frac{(m+2)a\lambda}{a-1} - m(m-1)R} \end{aligned}$$

for all $a > 1$. Setting $t = \frac{a-1}{a}$, we have

$$(m+2)\lambda \geq t\left(\frac{1}{d^2}\left(\log\frac{1}{t}\right)^2 - m(m-1)R\right)$$

for all $0 < t < 1$. Maximizing the right hand side as a function of t by setting $t = \exp(-1 - \sqrt{1 + m(m-1)Rd^2})$, we obtain the estimate

$$\begin{aligned} \lambda &\geq \frac{2}{(m+2)d^2}(1 + \sqrt{1+m(m-1)Rd^2}) \exp(-1 - \sqrt{1+m(m-1)Rd^2}) \\ &\geq \frac{C_1}{d^2}\exp(-C_2 d\sqrt{R}) \end{aligned}$$

as claimed.

We would like to point out that when M is a compact manifold with boundary, there are corresponding estimates for the first Dirichlet eigenvalue and the first nonzero Neumann eigenvalue using the maximum principle. In fact, Reilly [R] proved the Lichnerowicz-Obata result for the Dirichlet eigenvalue on manifolds whose boundary has nonnegative mean curvature with respect to the outward normal vector. Recently, Escobar [E] established the Lichnerowicz-Obata result for the first nonzero Neumann eigenvalue on manifolds whose boundary is convex with respect to the second fundamental form. There are analogous estimates to that of Theorem 5.3 for both the Dirichlet and Neumann eigenvalues on manifolds with boundary. In general, the estimate for the Dirichlet eigenvalue [L-Y] will depend also on the lower bound of the mean curvature of the boundary with respect to the outward normal, and the estimate for the Neumann eigenvalue will depend also on the lower bound of the second fundamental form of the boundary and the ϵ-ball condition (see [Cn]). However, when the boundary is convex, the Neumann eigenvalue has an estimate similar to manifolds without boundary.

Corollary 5.1. *(Li-Yau [L-Y]) Let M be a compact m-dimensional Riemannian manifold whose boundary is convex in the sense that the second fundamental form is nonnegative with respect to the outward pointing normal vector. Suppose that the Ricci curvature of M is bounded from below by*

$$\mathcal{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$, and d denotes the diameter of M. Then there exists constants $C_1(m), C_2(m) > 0$ depending on m alone, such that, the first nonzero Neumann eigenvalue of M satisfies

$$\lambda_1 \geq \frac{C_1}{d^2} \exp(-C_2 d\sqrt{R}).$$

Proof. In view of the proof of Theorem 5.3, it suffices to show that the maximum value for the functional Q does not occur on the boundary of M. Supposing the contrary that the maximum point for Q is $x_0 \in \partial M$. Let us denote the outward pointing unit normal vector by e_m, and assume that $\{e_1, \ldots, e_{m-1}\}$ are orthonormal tangent vectors to ∂M. Since Q satisfies the differential inequality (5.13), the strong maximum principle implies that

$$e_m(Q)(x_0) > 0.$$

However,

$$e_m(Q) = 2\sum_{i=1}^{m}(e_i v)(e_m e_i v).$$

Using the Neumann boundary condition on u, we conclude that $e_m v = 0$. Moreover, since the second covariant derivative of v is defined by

$$v_{ij} = (e_i e_j - \nabla_{e_i} e_j)\, v,$$

we have

$$\begin{aligned} e_m(Q) &= 2\sum_{\alpha=1}^{m-1}(e_\alpha v)(v_{m\alpha} + \nabla_{e_m} e_\alpha v) \\ &= 2\sum_{\alpha=1}^{m-1}(e_\alpha v)\,(v_{\alpha m} + \nabla_{e_m} e_\alpha v) \\ &= 2\sum_{\alpha=1}^{m-1}(e_\alpha v)\,(e_\alpha e_m v - \nabla_{e_\alpha} e_m v + \nabla_{e_m} e_\alpha v)\,. \end{aligned}$$

Using $e_m v = 0$ again, the fact that the second fundamental form is defined by

$$h_{\alpha\beta} = \langle \nabla_{e_\alpha} e_m, e_\beta \rangle,$$

we conclude that

$$\begin{aligned} e_m(Q) &= -2 \sum_{\alpha,\beta=1}^{m-1} (e_\alpha v) h_{\alpha\beta} (e_\beta v) + 2 \sum_{\alpha,\beta=1}^{m-1} (e_\alpha v) \langle \nabla_{e_m} e_\alpha, e_\beta \rangle (e_\beta v) \\ &\leq 2 \sum_{\alpha,\beta=1}^{m-1} (e_\alpha v) \langle \nabla_{e_m} e_\alpha, e_\beta \rangle (e_\beta v). \end{aligned}$$

On the other hand, since

$$\langle \nabla_{e_m} e_\alpha, e_\beta \rangle = -\langle \nabla_{e_m} e_\beta, e_\alpha \rangle,$$

we conclude that

$$2 \sum_{\alpha,\beta=1}^{m-1} (e_\alpha v) \langle \nabla_{e_m} e_\alpha, e_\beta \rangle (e_\beta v) = -2 \sum_{\alpha,\beta=1}^{m-1} (e_\alpha v) \langle \nabla_{e_m} e_\beta, e_\alpha \rangle (e_\beta v).$$

Hence

$$e_m(Q) \leq 0,$$

which is a contradiction.

When M is a complete manifold, it is often useful to have a lower bound of the first eigenvalue for the Dirichlet Laplacian on a geodesic ball. This is provided by the next theorem.

Theorem 5.4. *(Li-Schoen [L-S]) Let M be a complete manifold of dimension m. Let $p \in M$ be a fixed point such that $B_p(2\rho) \cap \partial M = \emptyset$ for $2\rho \leq d$. Assume that the Ricci curvature on $B_p(2\rho)$ satisfies*

$$\mathcal{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$. For any $\alpha \geq 1$, there exists constants $C_1(\alpha), C_2(m, \alpha) > 0$, such that for any compactly supported function f on $B_p(\rho)$

$$\int_{B_p(\rho)} |\nabla f|^\alpha \geq C_1 \rho^{-\alpha} \exp(-C_2(1 + \rho\sqrt{R})) \int_{B_p(\rho)} |f|^\alpha.$$

In particular, the first Dirichlet eigenvalue of $B_p(\rho)$ satisfies

$$\mu_1 \geq C_1 \rho^{-2} \exp(-C_2(1 + \rho\sqrt{R})).$$

Proof. Let $q \in \partial B_p(2\rho)$. By the triangle inequality $B_p(\rho) \subset (B_q(3\rho) \setminus B_q(\rho))$. Theorem 4.1 implies that

$$\begin{aligned} \Delta r &\leq (m-1)\sqrt{R} \coth(r\sqrt{R}) \\ &\leq (m-1)(r^{-1} + \sqrt{R}) \end{aligned}$$

for $r(x) = r(q, x)$. For $k > m - 2$, we have

$$\begin{aligned}\Delta r^{-k} &= -kr^{-k-1}\Delta r + k(k+1)r^{-k-2} \\ &\geq -k(m-1)r^{-k-1}(r^{-1} + \sqrt{R}) + k(k+1)r^{-k-2} \\ &= kr^{-k-1}((k+2-m)r^{-1} - (m-1)\sqrt{R}) \\ &\geq k(3\rho)^{-k-1}((k+2-m)(3\rho)^{-1} - (m-1)\sqrt{R})\end{aligned}$$

on $B_p(\rho)$. Choosing $k = m - 1 + 3(m-1)\rho\sqrt{R}$ this becomes

$$\Delta r^{-k} \geq k(3\rho)^{-k-2} \tag{5.14}$$

on $B_p(\rho)$.

Let f be a nonnegative function supported on $B_p(\rho)$. Multiplying (5.14) with f and integrating over $B_p(\rho)$ yields

$$\begin{aligned}k(3\rho)^{-k-2}\int_{B_p(\rho)} f &\leq \int_{B_p(\rho)} f\Delta r^{-k} \\ &= -\int_{B_p(\rho)} \langle \nabla f, \nabla r^{-k}\rangle \\ &\leq k\int_{B_p(\rho)} |\nabla f| r^{-k-1} \\ &\leq k\rho^{-k-1}\int_{B_p(\rho)} |\nabla f|.\end{aligned}$$

Hence

$$\int_{B_p(\rho)} |\nabla f| \geq C_1\rho^{-1}\exp(-C_2(1+\rho\sqrt{R}))\int_{B_p(\rho)} f.$$

This shows the case when $\alpha = 1$ by simply applying the above inequality to $|f|$.

For $\alpha > 1$, we set $f = |g|^\alpha$. Then we have

$$\begin{aligned}\alpha\left(\int_{B_p(\rho)} |\nabla g|\right)^{\frac{1}{\alpha}}\left(\int_{B_p(\rho)} |g|^\alpha\right)^{\frac{\alpha-1}{\alpha}} &\geq \alpha\int_{B_p(\rho)} |g|^{\alpha-1}|\nabla g| \\ &= \int_{B_p(\rho)} |\nabla g^\alpha| \\ &\geq C_1\rho^{-1}\exp(-C_2(1+\rho\sqrt{R}))\int_{B_p(\rho)} |g|^\alpha,\end{aligned}$$

which implies the desired inequality.

§6 Gradient Estimate and Harnack Inequality

In this section we will derive a gradient estimate and Harnack inequality of Yau [Y 1] for positive harmonic functions on a complete Riemannian manifold. As consequences, Liouville type theorems can be proved for complete manifolds with nonnegative Ricci curvature.

Theorem 6.1. (Yau [Y 1]) *Let M be a complete Riemannian manifold of dimension m. Assume that the geodesic ball $B_p(2\rho) \cap \partial M = \emptyset$. Suppose the Ricci curvature on $B_p(2\rho)$ is bounded from below by*

$$\mathcal{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$. If u is a positive harmonic function on $B_p(2\rho)$, then there exists a constant $C(m) > 0$ depending only on m such that

$$\sup_{B_p(\rho)} |\nabla \log u|^2 \leq C(R + \rho^{-2}).$$

In particular,

$$\sup_{B_p(\rho)} u \leq \left(\inf_{B_p(\rho)} u\right) \exp(C(\rho^{-1} + \rho R)).$$

Proof. If we set the function $v = \log u$, then

$$\Delta v = -|\nabla v|^2. \tag{6.1}$$

Let us define a cut-off function $\phi(r(x))$ given by a function of $r(x) = r(p,x)$ alone, such that

$$\phi(r) = \begin{cases} 1, & \text{for } r \leq \rho \\ 0, & \text{for } r \geq 2\rho \end{cases}$$

with $\phi \geq 0, 0 \geq (\phi')^2\phi^{-1} \geq C\rho^{-2}$, and $|\phi''| \leq C\rho^{-2}$ for some constant $C > 0$.

Consider the function $Q = |\nabla v|^2$ which is supported on $B_p(2\rho)$. Let $X_0 \in B_p(2\rho)$ be a point at which Q achieves its maximum. Since the distant function is not smooth in general, we observe that Q is only smooth when x is not in the cut-locus of p. Let us first assume this is the case. Computing the Laplacian using the Ricci identity and the lower bound of the Ricci curvature, we have

$$\begin{aligned}
\Delta Q &= (\Delta\phi)|\nabla v|^2 + 2\langle \nabla\phi, \nabla|\nabla v|^2\rangle + \phi(\Delta|\nabla v|^2) \\
&= \phi^{-1}(\Delta\phi)Q + 2\phi^{-1}\langle \nabla\phi, (\nabla Q - \nabla\phi)|\nabla v|^2)\rangle + \phi(2v_{ji}^2 + 2v_j v_{jii}) \\
&\geq (\phi^{-1}(\Delta\phi)) - 2\phi^{-2}|\nabla\phi|^2)Q + 2\phi^{-1}\langle\nabla\phi, \nabla Q\rangle + 2\phi v_{ji}^2 \\
&\quad - 2(m-1)RQ + 2\phi\langle\nabla v, \nabla\Delta v\rangle.
\end{aligned}$$

However (6.1) and the inequality

$$\begin{aligned}
v_{ji}^2 &= \frac{(\Delta v)^2}{m} \\
&= \frac{Q^2}{m}
\end{aligned}$$

implies that

$$\Delta Q \le (\phi^{-1}(\Delta\phi) - 2\phi^{-2}|\nabla\phi|^2 - 2(m-1)R)Q + 2\phi^{-1}\langle\nabla\phi, \nabla Q\rangle$$
$$+ \frac{2}{m}\phi^{-1}Q^2 - 2\langle\nabla v, \nabla Q\rangle + 2\phi^{-1}Q\langle\nabla v, \nabla\phi\rangle.$$

Hence, evaluating at the maximum point x_0 and using $\nabla Q(x_0) = 0$ and $\Delta Q(x_0) \le 0$, we conclude that

$$0 \ge Q + m\left(\frac{1}{2}\Delta\phi - \phi^{-1}|\nabla\phi|^2 - (m-1)R\right) + m\langle\nabla\phi, \nabla v\rangle,$$

unless $Q(x_0) = 0$ and the theorem follows. However,

$$|m\langle\nabla\phi, \nabla v\rangle| \le \frac{1}{2}Q + \frac{m^2}{2}\phi^{-1}|\nabla\phi|^2$$

implies that

$$Q \le m(2(m-1)R + (2+m)\phi^{-1}|\nabla\phi|^2 - \Delta\phi).$$

On the other hand, the assumption on ϕ and the Laplacian comparison theorem assert that

$$\begin{aligned}\phi^{-1}|\nabla\phi|^2 &= \phi^{-1}(\phi')^2 \\ &\le C\rho^{-2}\end{aligned}$$

and

$$\begin{aligned}\Delta\phi &= \phi'(\Delta r) + \phi'' \\ &\ge \phi'(m-1)(r^{-1} + \sqrt{R}) + \phi'' \\ &\ge -C\rho^{-2}(1 + \rho\sqrt{R}).\end{aligned}$$

Hence

$$\begin{aligned}C(\rho^{-2} + R) &\ge Q(x_0) \\ &\ge \sup_{B_p(2\rho)} \phi|\nabla v|^2 \\ &\ge \sup_{B_p(\rho)} |\nabla \log u|^2.\end{aligned}$$

When x_0 is a cut-point to p, we consider a minimizing geodesic γ joining p to x_0 with $\gamma(0) = p$ and $\gamma(r(x_0)) = x_0$. Let $q = \gamma(\epsilon)$ for sufficiently small $\epsilon > 0$. Clearly, x_0 is not a cut-point to q. Let us define the function

$$\psi(x) = \phi(r_q(x) + \epsilon) >$$

Since ψ is a non-increasing function, and because

$$r_q(x_0) + \epsilon = r_p(x_0)$$

and

$$\begin{aligned} r_q(x) + \epsilon &= r_q(x) + r_p(q) \\ &\le r_p(x), \end{aligned}$$

we have

$$\phi(x) \ge \psi(x)$$

with

$$\psi(x_0) = \psi(x_0).$$

Therefore x_0 is also a maximum point for the function $\psi|\nabla v|^2$ which is now smooth. The theorem now follows by performing the above computation on $\psi|\nabla v|^2$ and letting $\epsilon \to 0$.

To prove the Harnack inequality, one simply consider the minimal geodesic γ joining any 2 points in $B_p(\rho)$. We observe that by the triangle inequality, the length of γ must be at most 2ρ. The Harnack inequality now follows by integrating the upper bound of $|\nabla \log u|$ along γ.

Corollary 6.1. *Let M be an m-dimensional complete noncompact Riemannian manifold without boundary. Suppose the Ricci curvature of M is nonnegative. Then any positive harmonic function on M must be constant.*

Proof. By Theorem (6.1) and the curvature assumption, we have

$$\sup_{B_p(\rho)} |\nabla \log u|^2 \le C\rho^{-2}$$

for any $\rho > 0$. Taking the limit as $\rho \to \infty$ yields

$$\sup_M |\nabla \log u|^2 \le 0$$

which implies $\log u \equiv$ constant.

Corollary 6.2. *(Cheng) Let M be a m-dimensional complete noncompact Riemannian manifold without boundary. Suppose the Ricci curvature of M is nonnegative. Let u be a sublinear growth harmonic function on M, ie.*

$$|u(x)| = o(r_p(x))$$

with respect to some fixed point $p \in M$. Then u must be identically constant.

Proof. For any $\rho > 0$, the function $u + \sup_{B_p(2\rho)} |u|$ is a positive harmonic function on $B_p(2\rho)$. Hence Theorem (6.1) implies that

$$\sup_{B_p(\rho)} |\nabla \log(u + \sup_{B_p(2\rho)} |u|)|^2 \le C\rho^{-2}.$$

In particular,

$$\begin{aligned} \sup_{B_p(\rho)} |\nabla u|^2 &\le C\rho^{-2} \sup_{B_p(\rho)} (u + \sup_{B_p(2\rho)} |u|)^2 \\ &\le 2C\rho^{-2} \sup_{B_p(2\rho)} |u|^2 \end{aligned}$$

which tends to 0 as $\rho \to \infty$ by the assumption on u. Hence u must be identically constant.

§7 Mean Value Inequality

We will prove a version of mean value inequality which is adopted to the theory of subharmonic functions on a Riemannian manifold. Let us first prove a theorem of Yau.

Lemma 7.1. *(Yau [2]) Let M be a complete Riemannian manifold. Suppose $p \in M$ is a point such that the geodesic ball $B_p(2\rho)$ centered at p of radius 2ρ satisfies $B_p(2\rho) \cap \partial M \neq \emptyset$. Let f be a nonnegative subhamonic function defined on $B_p(2\rho)$. Then for any constant $\alpha > 1$, there exists a constant $C(\alpha) > 0$ depending only on α such that*

$$\int_{B_p(\rho)} f^{\alpha-2}|\nabla f|^2 \leq C\rho^{-2} \int_{B_p(2\rho)} f^{\alpha}.$$

In particular, if M has no boundary, then there does not exist any nonconstant, nonnegative, L^α subharmonic function.

Proof. Let $\phi(r(x))$ be a cut-off function given by a function of $r(x) = r(p,x)$ alone. We may take ϕ to satisfy

$$\phi(r) = \begin{cases} 1, & \text{for } r \leq \rho \\ 0, & \text{for } r \geq 2\rho \end{cases}$$

with $\phi \geq 0$, and $0 \geq (\phi')^2\phi^{-1} \geq C\rho^{-2}$ for some constant $C > 0$. Let us now consider the integral

$$\begin{aligned} 0 &\geq \int_{B_p(2\rho)} \phi^2 f^{\alpha-1}\Delta f \\ &= -(\alpha-1)\int_{B_p(2\rho)} \phi^2 f^{\alpha-2}|\nabla f|^2 - 2\int_{B_p(2\rho)} \phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle. \end{aligned}$$

However, an algebraic inequality implies that

$$2\left|\int_{B_p(2\rho)} \phi f^{\alpha-1}\langle\nabla\phi,\nabla f\rangle\right| \leq \frac{\alpha-1}{2}\int_{B_p(2\rho)} \phi^2 f^{\alpha-2}|\nabla f|^2 + \frac{2}{\alpha-1}\int_{B_p(2\rho)} |\nabla\phi|^2 f^\alpha.$$

Hence, we have

$$\begin{aligned} \frac{\alpha-1}{2}\int_{B_p(\rho)} f^{\alpha-2}|\nabla f|^2 &\leq \frac{\alpha-1}{2}\int_{B_p(2\rho)} \phi^2 f^{\alpha-2}|\nabla f|^2 \\ &\leq \frac{2}{\alpha-1}\int_{B_p(2\rho)} |\nabla\phi|^2 f^\alpha \\ &\leq \frac{C}{\rho^2(\alpha-1)}\int_{B_p(2\rho)} f^\alpha. \end{aligned}$$

If M has no boundary, we simply take $\rho \to \infty$. The fact that f is L^α implies that the integral

$$\int_M f^{\alpha-2}|\nabla f|^2 = 0,$$

which implies that $|\nabla f| \equiv 0$. Hence f must be a constant function.

Theorem 7.1. *(Li-Tam [L-T]) Let M be a complete Riemannian manifold of dimension m. Let $p \in M$ be a fixed point such that the geodesic ball $B_p(4\rho)$ centered at p of radius 4ρ satisfy $B_p(2\rho) \cap \partial M = \emptyset$. Suppose f is a nonnegative subharmonic function defined on $B_p(4\rho)$. Assume that the Ricci curvature on $B_p(4\rho)$ is bounded by $\mathcal{R}_{ij} \geq -(m-1)R$ for some constant $R \geq 0$. Then there exists constants $C_3, C_4(m) > 0$ with C_4 depending only on m such that*

$$\sup_{x \in B_p(\rho)} f^2 \leq C_3(1 + \exp(C_4 \rho \sqrt{R})) V_p(4\rho)^{-1} \int_{B_p(4\rho)} f^2.$$

Proof. Let h be a harmonic function on $B_p(4\rho)$ obtained by the solving the Dirichlet boundary problem

$$\Delta h = 0 \quad \text{on } B_p(2\rho),$$

and

$$h = f \quad \text{on } \partial B_p(2\rho).$$

Since f is nonnegative, by the maximum principle h is positive on the ball $B_p(\rho)$. Moreover,

$$f \leq h \quad \text{on } B_p(\rho).$$

The Harnack inequality (Theorem 6.1) implies that

$$\sup_{B_p(\rho)} h \geq \left(\inf_{B_p(\rho)} h \right) \exp(C(\rho^{-1} + \rho R)).$$

Hence, in particular, we have.

$$\sup_{B_p(\rho)} f^2 \leq \sup_{B_P(\rho)} h^2 \leq \exp(2C(\rho^{-1} + \rho R)) V_p(\rho)^{-1} \int_{B_p(\rho)} h^2. \tag{7.1}$$

We will now estimate the L^2-norm of h in terms of the L^2-norm of f. By triangle inequality, we observe that

$$\int_{B_p(\rho)} h^2 \leq 2 \int_{B_p(\rho)} (h-f)^2 + \int_{B_p(\rho)} f^2 \leq 2 \int_{B_p(2\rho)} (h-f)^2 + \int_{B_p(4\rho)} f^2. \tag{7.2}$$

However, since the functions $h - f$ vanishes on $\partial B_p(2\rho)$, the Poincaré inequality (Theorem 5.3) implies that

$$\int_{B_p(2\rho)} (h-f)^2 \leq C_1 \rho^2 \exp(C_2(1 + \rho\sqrt{R})) \int_{B_p(2\rho)} |\nabla(h-f)|^2 \tag{7.3}$$

for some constants $C_1 > 0$ and $C_2(m) > 0$. By triangle inequality again, we have

$$\int_{B_p(2\rho)} |\nabla(h-f)|^2 \leq 2\int_{B_p(2\rho)} |\nabla h|^2 + 2\int_{B_p(2\rho)} |\nabla f|^2.$$

The fact that a harmonic function minimizes Dirichlet integral among functions with the same boundary data asserts that

$$\int_{B_p(2\rho)} |\nabla(h-f)|^2 \leq 4\int_{B_p(2\rho)} |\nabla f|^2.$$

Now the argument in Lemma (7.1) implies that

$$\int_{B_p(2\rho)} |\nabla f|^2 \leq C\rho^{-2}\int_{B_p(4\rho)} f^2$$

for some constant $C > 0$. Hence together with (7.1), (7.2), (7.3), and the volume comparison (Corollary 2.3), the theorem follows.

Let us point out that the fact that the constant in the mean value inequality depends only on the lower bound of the Ricci curvature and the radius of the ball is essential in some of the geometric application. In fact, it is well known that one can prove another version of the mean value inequality by using an iteration method of Moser. However, the constant in this case will depends on the Sobolev constant which, unlike the first eigenvalue, cannot be estimated by the Ricci curvature and the radius alone.

We will now give an application of this mean value inequality to study the space of harmonic functions on a certain class of manifolds. This result can be viewed as a generalization of Yau's Liouville theorem. Let us first prove a lemma.

Lemma 7.2. *(Li [L 1]) Let $\mathcal{H}$ be a finite dimensional space of L^2 functions defined over a set D. If $V(D)$ denotes the volume of the set D, then there exists a function f_0 in $\mathcal{H}$ such that*

$$\dim \mathcal{H} \int_D f_0^2 \leq V(D) \sup_D f_0^2.$$

Proof. Let $f_1, \ldots, f_k$ be an orthonormal basis for $\mathcal{H}$ with respect to the L^2 inner product. Let us consider the function

$$F(x) = \sum_{i=1}^k f_i(x)$$

which is well-defined under orthonormal change of basis. Clearly

$$\dim \mathcal{H} = \int_D F(x).$$

Now let us consider the subspace $\mathcal{H}_p$ of $\mathcal{H}$ which consists of functions f vanishing at $p \in D$. The space is clearly of at most codimension 1. Otherwise, there are f_1 and f_2 in the compliment of $\mathcal{H}_p$ which are linearly independent. This implies that both $f_1(p) \neq 0$ and $f_2(p) \neq 0$. However, clearly the linearly combination

$$f_1(p)f_2 - f_2(p)f_1$$

is a function in $\mathcal{H}_p$, which is a contradiction. This implies that by a change of orthonormal basic, there exist f_0 in the orthorgonal compliment of $\mathcal{H}_p$ and has unit L^2-norm, such that

$$F(p) = f_0^2(p).$$

Hence, in particular, if we choose $p \in D$ such that F achieves its maximum then

$$\begin{aligned}\dim \mathcal{H} &= \int_D F \\ &\leq V(D)F(p) \\ &= V(D)f_0^2(p) \\ &= V(D)\sup_D f_0^2.\end{aligned}$$

This proves that lemma.

Theorem 7.2. *(Li-Tam [L-T]) Let M be an m-dimensional complete noncompact Riemannian manifold without boundary. Suppose that the Ricci curvature of M is nonnegative on $M \setminus B_p(1)$ for some unit geodesic ball centered at $p \in M$. Let us assume that the lower bound of the Ricci curvature on $B_p(1)$ is given by*

$$\mathcal{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$. If we denote $\mathcal{H}'(M)$ to be the space of functions spanned by the set of harmonic functions f which has the property that when restricted to each unbounded component of $M \setminus D$ is either bounded from above or from below for some compact subset $D \subset M$, then $\mathcal{H}'(M)$ is of finite dimensional. Moreover, there exists a constant $C(m,R) > 0$ depending only on m and R, such that, the dimension of $\mathcal{H}'(M)$ is bounded from above by $C(m,R)$.

Proof. By the definition of $\mathcal{H}'(M)$, there exists $R_0 > 1$ such that

$$f = \sum_{i=1}^{m} v_i,$$

where each v_i is bounded on one side of each end of $M \setminus B_p(R_0)$. Let E be an end of $M \setminus B_p(R_0)$. If v is a harmonic function defined on M which is positive on E and if x is a point in E with $r(p,x) \geq 2R_0$, then by applying Theorem 6.1 to the ball $B_x(\frac{r(p,)}{2})$ and using the curvature assumption, there is a constant $C > 0$ independent of v, such that

$$|\nabla v|(x) \leq Cr^{-1}(p,x)v(x). \tag{7.4}$$

Since all the $v_i's$ are bounded on one side on E, there are constants $a_1, \cdots, a_m$ and $\epsilon_i = \pm 1$ such that the harmonic functions $u_i = a_i + \epsilon_i v_i$ are positive on E. Hence, by applying (7.4) to $u_1, \cdots, u_m$, we can estimate the gradient of f by

$$\begin{aligned} |\nabla f| &\leq \sum_{i=1}^{m} |\nabla v_i| \\ &= \sum_{i=1}^{m} |\nabla u_i| \\ &\leq Cr^{-1}(p,x) \sum_{i=1}^{m} u_i(x). \end{aligned} \tag{7.5}$$

This implies that for any given $\delta > 0$, using the fact that $|\nabla f|$ is a subharmonic function on $M \setminus B_p(1)$, the maximum principle implies that

$$|\nabla f|(x) - \left(\sup_{\partial E}(|\nabla f|)\right) \leq C\delta \left(\sum_{i=1}^{m} u_i(x) + 1\right)$$

for all $x \in E$. Letting $\delta \to 0$, we conclude that

$$\sup_{E}(|\nabla f) \leq \sup_{\partial E}(|\nabla f|).$$

Since E is an arbitrary end of $M \setminus B_p(R_0)$, we have

$$\sup_{M \setminus B_p(R_0)} (|\nabla f|) \leq \sup_{\partial B_p(R_0)} (|\nabla f|) \tag{7.6}$$

In fact, we claim that

$$\sup_{M \setminus B_p(1)} |\nabla f| \leq \sup_{\partial B_p(1)} |\nabla f|.$$

This follows from applying the maximum principle to the subharmonic function $|\nabla f|$ on the set $M \setminus B_p(1)$ and (7.6). In particular, this implies that

$$\sup_{M} |\nabla f| \leq \sup_{B_p(1)} |\nabla f|. \tag{7.7}$$

Let us now consider the codimension-1 subspace $\mathcal{H}'_p(M)$ of $\mathcal{H}'(M)$ defined by

$$\mathcal{H}'_p(M) = \{f \in \mathcal{H}'(M) | f(p) = 0\}.$$

Clearly $\mathcal{H}'_p(M) = \mathcal{H}'(M) \setminus$ constant. For any $f \in \mathcal{H}'_p(M)$, the fundamental theorem of calculus implies that

$$\sup_{B_p(4)} f^2 \leq 4 \sup_{B_p(4)} |\nabla f|^2.$$

Together with (7.7), we have

$$\sum_{B_p(4)} f^2 \le 4 \sup_{B_p(1)} |\nabla f|^2.$$

Applying the gradient estimate (Theorem 6.1) to the function $f + \sup_{B_p(2)} |f|$ yields

$$\begin{aligned}\sup_{B_p(4)} f^2 &\le 4 \sup_{B_p(1)} |\nabla f| 2 \\ &\le 4C \sup_{B_p(1)} \left(f + \sup_{B_p(2)} |f| \right)^2 \\ &\le 16C \sup_{B_p(2)} f^2.\end{aligned}$$

However, this together with the mean value inequality (Theorem 7.1) when applied to the nonnegative subharmonic function $|f|$ asserts that there exists constants $C_3, C_4(n) > 0$ such that

$$V_p(4) \sup_{B_p(4)} f^2 \le C_3 \exp(C_4\sqrt{k}) \int_{B_p(4)} f^2. \tag{7.8}$$

On the other hand Lemma 7.2 implies that for any finite dimensional subspace $\mathcal{H}$ of $\mathcal{H}_p^+(M)$, there exists a function f_0 such that

$$\dim \mathcal{H} \int_{B_p(4)} f_0^2 \le V_p(4) \sup_{B_p(4)} f_0^2.$$

Hence applying (7.8) to f_0 yields the estimate

$$\dim \mathcal{H} \le C_3 \exp(C_4\sqrt{k}).$$

Since this estimate holds for any finite dimensional subspace $\mathcal{H}$, this implies that

$$\dim \mathcal{H}_p^+(M) \le C_3 \exp(C_4\sqrt{k}).$$

Therefore,

$$\dim \mathcal{H}^+(M) \le C_3 \exp(C_4\sqrt{k}) + 1$$

as to be proven.

Let us remark that if M has nonnegative Ricci curvature, then (7.7) can be written as

$$\sup_M |\nabla f| \le |\nabla f|(p).$$

However, since $|\nabla f|$ is a subharmonic function on M, the maximum principle implies that $|\nabla f|$ must be identically constant. If f is not a constant function, we can apply the Bochner formula to $|\nabla f|$ again, and conclude that ∇f is a parallel vector field. This implies that M must split and that f is a linear growth harmonic function. In particular, f cannot be a positive harmonic function. Hence f must be identically constant, which recovers Yau's theorem.

§8 Reilly's Formula and Applications

We will discuss a few applications of the integral version of Bochner's formula derived by Reilly [R]. In particular, this formula is useful in the studying of embedded minimal surfaces and surfaces with constant mean curvature. Let us first point out some standard formulas about submanifolds in $\mathbf{R}^{m+1}$ and $\mathbf{S}^{m+1}$.

Lemma 8.1. *Let $\{x_1, \dots, x_{m+1}\}$ be rectangular coordinates of $\mathbf{R}^{m+1}$, and let us denote the position vector by $X = (x_1, \dots, x_{m+1})$. If M is a submanifold of $\mathbf{R}^{m+1}$ with the induced metric and if $\overrightarrow{II}$ and $\overrightarrow{H}$ denotes the second fundamental form and the mean curvature vector of M, then*

$$H_M(X) = -\overrightarrow{II}$$

and

$$\Delta_M X = -\overrightarrow{H},$$

where $H_M(X)$ and $\Delta_M(X)$ are the Hessian of X and the Laplacian of X computed on M.

Proof. Let $H(X)$ be the Euclidean Hessian of X, then we have

$$H(X) = 0$$

since the x_i's are coordinate functions. On the other hand, if e_i and e_j are tangential to M, then

$$\begin{aligned} H(X)_{ij} &= (e_i e_j - \nabla_{ei} e_j)X \\ &= H_M(X)_{ij} - \langle \nabla_{ei} e_j, e_\nu \rangle X_\nu \\ &= H_M(X)_{ij} + \langle \overrightarrow{II}_{ij}, \nu \rangle \nu \\ &= H_M(X)_{ij} + \overrightarrow{II}_{ij}. \end{aligned}$$

Hence, the lemma follows.

Corollary 8.1. *A submanifold M of $\mathbf{R}^{m+1}$ is minimal if and only if the coordinate functions are harmonic. In particular, there are no compact minimal submanifolds in $\mathbf{R}^{m+1}$ other than points.*

Lemma 8.2. *Let N be an n-dimesnional submanifold of the standard unit sphere $\mathbf{S}^m$, then N is minimal if and only if all the coordinate functions of $\mathbf{S}^m \subset \mathbf{R}^{m+1}$ are eigenfunctions of N satisfying*

$$\Delta_N X = -nX.$$

Proof. By Lemma 8.1, and using the fact that the position vector X is also the unit normal vector on $\mathbf{S}^m$, we have

$$\begin{aligned} H_{\mathbf{S}^m}(X) &= -\overrightarrow{II} \\ &= -\delta_{ij} X. \end{aligned}$$

Appying the formula that

$$H_{\mathbb{S}^m}(X)_{\alpha\beta} = H_N(X)_{\alpha\beta} + \overrightarrow{II}_N(X)$$

for tangent vectors e_α and e_β which are tangential to N, and by taking the trace, we have

$$\begin{aligned} -nX &= \Delta_N X + \overrightarrow{H}_N(X) \\ &= \Delta_N X + \overrightarrow{H}_N. \end{aligned}$$

This proves the lemma.

The following integral formula was proved by Reilly in his work of repoving Aleksandrov's theorem.

Theorem 8.1. *(Reilly)Let D be a manifold of dimension $m+1$ with boundary given by a smooth m-dimensional manifold M. Suppose f is function defined on D satisfying*

$$\Delta f = g \qquad \text{on } D$$

and

$$f = u \qquad \text{on } M,$$

then

$$\frac{m}{m+1}\int_D g^2 \geq \int_M H(f_\nu)^2 + 2\int_M f_\nu \Delta_M u + \int_M \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}u_\alpha u_\beta + \int_D \mathcal{R}_{ij} f_i f_j$$

where H and $h_{\alpha\beta}$ denote the mean curvature and the second fundamental form of M with respect to the outward unit normal ν, Δ_M is the Laplacian on M, and $\mathcal{R}_{ij}$ is the Ricci curvature of D. Moreover, equality holds if and only if

$$f_{ij} = \frac{g\delta_{ij}}{m+1}$$

on D.

Proof. Let us consider the Bochner formula

$$\begin{aligned} \frac{1}{2}\Delta|\nabla f|^2 &= f_{ij}^2 + f_i f_{ijj} \\ &= f_{ij}^2 + f_i(\Delta f)_i + \mathcal{R}_{ij} f_i f_j \\ &= f_{ij}^2 + \langle \nabla f, \nabla g\rangle + \mathcal{R}_{ij} f_i f_j. \end{aligned}$$

Using the inequality

$$\begin{aligned} \sum_{i,j=1}^{m+1} f_{ij}^2 &\geq \frac{\left(\sum_{i=1}^{m+1} f_{ii}\right)^2}{m+1} \\ &= \frac{g^2}{m+1}, \end{aligned}$$

we have

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{g^2}{m+1} + \langle\nabla f, \nabla g\rangle + \mathcal{R}_{ij}f_if_j.$$

Integrating this over D yields

$$\frac{1}{2}\int_D \Delta|\nabla f|^2 \geq \frac{1}{m+1}\int_D g^2 + \int_D \langle\nabla f, \nabla g\rangle + \int_D \mathcal{R}_{ij}f_if_j. \tag{8.1}$$

Integrating by parts, the last term on the right hand side becomes

$$\int_D \langle\nabla f, \nabla g\rangle = -\int_D g^2 + \int_M g f_\nu$$

where ν is the outward unit normal to M. Hence (8.1) becomes

$$\frac{1}{2}\int_D \Delta|\nabla f|^2 \geq -\frac{m}{m+1}\int_D g^2 + \int_M g f_\nu + \int_D \mathcal{R}_{ij}f_if_j. \tag{8.2}$$

On the other hand, if we pick orthonormal frame $\{e_1, \dots, e_{m+1}\}$ near the boundary of D such that $\{e_1, \dots, e_m\}$ are tangential to M, and $\nu = e_{m+1}$ is the outward unit normal vector, then divergence theorem implies that

$$\frac{1}{2}\int_D \Delta|\nabla f|^2 = \int_M \sum_{i=1}^{m+1}(e_if)(e_{m+1}e_if).$$

Using the boundary data of f, and choosing $\nabla_{e_{m+1}}e_{m+1} = 0$ at a point, we conclude that

$$\begin{aligned}\sum_{i=1}^{m+1}(e_if)(e_{m+1}e_if) &= (e_{m+1}f)(e_{m+1}e_{m+1}f) + \sum_{\alpha=1}^{m}(e_\alpha f)(e_{m+1}e_\alpha f) \\ &= (e_{m+1}f)\left(\Delta f - \sum_{\alpha=1}^{m} f_{\alpha\alpha}\right) + \sum_{\alpha=1}^{m}(e_\alpha f)(e_{m+1}e_\alpha f) \\ &= f_\nu(g - Hf_\nu - \Delta_M u) + \sum_{\alpha=1}^{m}(e_\alpha f)(e_{m+1}e_\alpha f)\end{aligned} \tag{8.3}$$

where Δ_M is the Laplacian on M and H is the mean curvature of M with respect to the unit normal ν. However,

$$\begin{aligned}e_{m+1}e_\alpha f &= e_\alpha e_{m+1}f + \nabla_{e_{m+1}}e_\alpha f - \nabla_{e_\alpha}e_{m+1}f \\ &= e_\alpha e_{m+1}f + \sum_{\beta=1}^{m}\langle\nabla_{e_{m+1}}e_\alpha, e_\beta\rangle f_\beta - \sum_{\beta=1}^{m}\langle\nabla_{e_\alpha}e_{m+1}, e_\beta\rangle f_\beta,\end{aligned} \tag{8.4}$$

because

$$\begin{aligned}\langle\nabla_{e_{m+1}}e_\alpha, e_{m+1}\rangle &= -\langle e_\alpha, \nabla_{e_{m+1}}e_{m+1}\rangle \\ &= 0\end{aligned}$$

and

$$\langle \nabla_{e_\alpha} e_{m+1}, e_{m+1}\rangle = \frac{1}{2} e_\alpha |e_{m+1}|^2$$
$$= 0.$$

Using (8.3), (8.4), and the fact that

$$\langle \nabla_{e_\alpha} e_{m+1}, e_\beta\rangle = -\langle e_{m+1}, \nabla_{e_\alpha} e_\beta\rangle$$
$$= h_{\alpha\beta},$$

we can write

(8.5)
$$\frac{1}{2}\int_D \Delta|\nabla f|^2$$
$$= \int_M g f_\nu - \int_M H(f_\nu)^2 - \int_M f_\nu \Delta_M u + \int_M \sum_{\alpha=1}^m (e_\alpha f)(e_\alpha e_{m+1} f)$$
$$+ \sum_{\alpha,\beta=1}^m \int_M \langle \nabla_{e_{m+1}} e_\alpha, e_\beta\rangle f_\alpha f_\beta - \int_M \sum_{\alpha,\beta=1}^m h_{\alpha\beta} u_\alpha u_\beta.$$

On the other hand,

$$\int_M \sum_{\alpha,\beta=1}^m \langle \nabla_{e_{m+1}} e_\alpha, e_\beta\rangle f_\alpha f_\beta = -\int_M \sum_{\alpha,\beta=1}^m \langle e_\alpha, \nabla_{e_{m+1}} e_\beta\rangle f_\alpha f_\beta$$
$$= -\int_M \sum_{\alpha,\beta=1}^m \langle \nabla_{e_{m+1}} e_\alpha, e_\beta\rangle f_\alpha f_\beta$$

implies that it must be identically 0. Also integrating by parts yields

$$\int_M \sum_{\alpha=1}^m (e_\alpha f)(e_\alpha e_{m+1} f) = -\int_M f_\nu \Delta_M u.$$

Combining this with (8.2) and (8.5), we have

$$-\int_M H(f_\nu)^2 - 2\int_M f_\nu \Delta_M u - \int_M \sum_{\alpha,\beta=1}^m h_{\alpha\beta} u_\alpha u_\beta \geq -\frac{m}{m+1}\int_D g^2 + \int_D \mathcal{R}_{ij} f_i f_j$$

which was to be proved. Equality case is clear from the above argument.

Theorem 8.2. *(Aleksandrov-Reilly) Any compact embedded hypersuface of constant mean curvature in* $\mathbf{R}^{m+1}$ *is a standard sphere.*

Proof. Let M^m be compact embedded hypersurface in $\mathbf{R}^{m+1}$ with constant mean curvature H. By compactness, it is clear that $H > 0$. After scaling, we may assume that $H = m$.

The assumption that M is embedded implies that M must enclose a bounded domain D in $\mathbf{R}^{m+1}$. Let us now consider the solution f of the boundary value problem

$$\Delta f = -1 \quad \text{on } D$$

and

$$f = 0 \quad \text{on } M = \partial D.$$

Applying Theorem 8.1 to f, we have

$$\frac{V(D)}{m+1} \geq \int_M f_\nu^2. \tag{8.6}$$

Schwarz inequality now implies that

$$\begin{aligned} A(M)\int_M f_\nu^2 &\geq \left(\int_M f_\nu\right)^2 \\ &= \left(\int_D \Delta f\right)^2 \\ &= V^2(D), \end{aligned}$$

where $A(M)$ is the area of M. Therefore, together with (8.6), we obtain the inequality

$$A(M) \geq (m+1)V(D). \tag{8.7}$$

On the other hand, if $X = (x_1, \ldots, x_{m+1})$ are the coordinate functions of $\mathbf{R}^{m+1}$, then one checks that their Euclidean Hessian is identically zero. Hence, in particular,

$$\begin{aligned} 0 &= \sum_{\alpha=1}^{m} X_{\alpha\alpha} \\ &= \Delta_M X + H\, X_{m-1} \\ &= \Delta_M X + m\, X_{m+1}. \end{aligned}$$

Now, let us consider

$$\begin{aligned} 0 &= \int_D \langle X, \Delta X\rangle \\ &= -\int_D |\nabla X|^2 + \int_M \langle X, X_{m+1}\rangle \\ &= -(m+1)V(D) - \frac{1}{m}\int_M \langle X, \Delta_M X\rangle \\ &= -(m+1)V(D) + \frac{1}{m}\int_M |\nabla_M X|^2 \\ &= -(m+1)V(D) + A(M), \end{aligned}$$

where we have used the fact that $|\nabla X|^2 = m+1$ and $|\nabla_M X|^2 = m$.

Hence, the inequalities which were used to derive (8.7) are all equalities. In particular,

$$f_{ij} = -\frac{\delta_{ij}}{m+1} \quad \text{on } D. \tag{8.8}$$

and

$$f_{m+1} = \text{constant} \quad \text{on } M.$$

Computing the difference between the Hessian of f on M and the Hessian of f on D, and using the fact that $f = 0$ on M, we have

$$f_{\alpha\beta} = h_{\alpha\beta} f_{m+1}$$

for all $1 \leq \alpha, \beta \leq m$, where $h_{\alpha\beta}$ is the second fundamental form of M. Applying (8.8), we conclude that

$$-\frac{\delta_{\alpha\beta}}{m+1} = h_{\alpha,\beta} f_{m+1}.$$

Using that the mean curvature of M is m, we obtain

$$f_{m+1} = -\frac{1}{m+1}$$

and

$$h_{\alpha\beta} = \delta_{\alpha\beta}.$$

The Gauss curvature equations implies that M has constant sectional curvature 1. In particular, by Lichnerowicz theorem, the first non-zero eigenvalue of M satisfies

$$\lambda_1(M) \geq m. \tag{8.9}$$

Now, let us consider the embedding function X. We compute that

$$\begin{aligned} \Delta_M X &= -mX_{m+1} \\ &= -me_{m+1}, \end{aligned} \tag{8.10}$$

where we have used the fact that the Euclidean Hessian of X is 0 and the mean curvature of M is m. On the other hand, we have

$$\begin{aligned} mA(M) &= \int_M |\nabla_M X|^2 \\ &= -\int_M \langle X, \Delta_M X \rangle \\ &= m \int_M \langle X, e_{m+1} \rangle \\ &\leq m \left(\int_M |X|^2 \right)^{\frac{1}{2}} \left(\int_M |e_{m+1}|^2 \right)^{\frac{1}{2}} \\ &= m \left(\int_M |X|^2 \right)^{\frac{1}{2}} \sqrt{A(M)}. \end{aligned} \tag{8.11}$$

Without loss of generality, we may choose the origin to be the center of gravity of M, so that $\int_M X = 0$. Under this assumption, the Poincaré inequality assets that

$$\int_M |\nabla_M X|^2 \geq \lambda_1(M) \int_M |X|^2.$$

Applying this and (8.9) to (8.11), we have

$$\begin{aligned} A(M) &\leq \frac{1}{m} \int_M |\nabla_M X|^2 \\ &\leq A(M). \end{aligned}$$

Hence, all the inequalities becomes equalitites. In particular,

$$\lambda_1(M) = m,$$

which implies that M is isometric to the unit m-sphere. Moreover, X satisfies the equation $\Delta_M X = -mX$. Hence together with (8.10), we conclude that $X = e_{m+1}$. This implies that $|X| = 1$ on M, and M is the unit sphere.

Theorem 8.3. *(Choi-Wang* [C-W]*) Let M^m be a compact embedded oriented minimal hypersuface in a compact oriented Riemannian manifold N^{m+1}. Suppose that the Ricci curvature of N is bounded from below by*

$$R_{ij} \geq mR$$

for some positive constant R. Then the first non-zero eigenvalue of M has a lower bound given by

$$\lambda_1(M) \geq \frac{mR}{2}.$$

Proof. The assumption that N has positive Ricci curvature implies that its first homology group $H^1(N, \mathbf{R})$ is trivial. By an exact sequence argument, we conclude that M divides N into 2 connected components N_1 and N_2 with $\partial N_1 = M = \partial N_2$. Let us denote D to be one of the component to be choosen later. If u is the first nonconstant eigenfunction on M, satifying

$$\Delta_M u = -\lambda_1(M) u,$$

then let f be the solution of

$$\Delta f = 0, \quad \text{on } D,$$

with boundary condition

$$f = u, \quad \text{on } M.$$

Applying Theorem 8.1, we have

$$0 \geq -2\lambda_1(M) \int_M u f_\nu + \int_M h_{\alpha\beta} u_\alpha u_\beta + mR \int_D |\nabla f|^2.$$

On the other hand

$$\begin{aligned} 2\int_M uf_\nu &= 2\int_M ff_\nu \\ &= \int_D \Delta(f^2) \\ &= 2\int_D |\nabla f|^2. \end{aligned}$$

Hence, we have

$$(2\lambda_1(M) - mR)\int_D |\nabla f|^2 \geq \int_M h_{\alpha\beta}u_\alpha u_\beta. \tag{8.12}$$

Let us observe that the right hand side is independent of the extended function f. If we choose a different component of $N \setminus M$ to perform this computation, the second fundamental form will differ by a sign, hence we may choose a component, say N_1, so that

$$\int_M h_{\alpha\beta}u_\alpha u_\beta \geq 0.$$

Hence together with (8).12, we conclude that either $\lambda_1(M) \geq \frac{mR}{2}$, or $\nabla f = 0$ on N_1. However, the latter is impossible because f has boundary value u which is nonconstant. This proves the estimate.

§9 Isoperimetric Inequalitites and Sobolev Inequalities

In this section, we will show that a class of isoperimetric inequalities which occur in geometry are in fact equivalent to a class of Sobolev type inequalities. The relationship between these inequalities were exploited in the study of eigenvalues of the Laplacian as early as the 1920's by Faber [F] and Krahn [K]. The equivelence was first formally established by Federer-Fleming [F-F] (also see [Bm]) in 1960. In 1970, Cheeger [C] observed that the same argument can apply to estimating the first eigenvalue of the Laplacian.

We will first define the isoperimetric and Sobolev constants on a manifold. Let us assume that M is a compact Riemannian manifold with or without boundary ∂M.

Definition 9.1. *If $\partial M \neq \phi$, we define the Dirichlet α-isoperimetric constant of M by*

$$ID_\alpha(M) = \inf_{\substack{\Omega\subset M \\ \partial\Omega\cap\partial M=\emptyset}} \frac{A(\partial\Omega)}{V(\Omega)^{\frac{1}{\alpha}}}$$

where the infimum is taken over all subdomains $\Omega \subset M$ with the properties that $\partial\Omega$ is a hypersurface not intersecting ∂M.

Simillarly, we define the Neumann α-isoperimetric constant of M.

Definition 9.2. *The Neumann α-isoperimetric constant of M is defined by*

$$IN_\alpha(M) = \inf_{\substack{\partial\Omega_1 = S = \partial\Omega_2 \\ M = \Omega_1 \cup S \cup \Omega_2}} \frac{A(S)}{\min\{V(\Omega_1), V(\Omega_2)\}^{\frac{1}{\alpha}}}$$

where infimum is taken over all hypersurfaces S dividing M into 2 parts denoted by Ω_1 and Ω_2. Note that in this case, there is no assumption on whether M has boundary or not.

Definition 9.3. *If $\partial M \neq \emptyset$, we define the Dirichlet α-Sobolev constant of M by*

$$SD_\alpha(M) = \inf_{\substack{f \in H_{1,1}(M) \\ f|_{\partial M} = 0}} \frac{\int_M |\nabla f|}{(\int_M |f|^\alpha)^{\frac{1}{\alpha}}}$$

where infimum is taken over all functions f in the first Sobolev space with Dirichlet boundary condition.

We also define the Neumann α-Sobolev constant of M.

Definition 9.4. *The Neumann α-Sobolev constant of M is defined by*

$$SN_\alpha(M) = \inf_{f \in H_{1,1}(M)} \frac{\int_M |\nabla f|}{(\inf_{k \in \mathbb{R}} \int_M |f - k|^\alpha)^{\frac{1}{\alpha}}}$$

where the first infimum is taken over all functions f in the first Sobolev space, and the second infimum is taken over all real numbers k. Again, there is no assumption on whether M has boundary or not.

Theorem 9.1. *$ID_\alpha(M) = SD_\alpha(M)$.*

Proof. To see that $ID_\alpha(M) \leq SD_\alpha(M)$, it suffices to show that for any Lipschitz function f defined on M with boundary condition $f|_{\partial M} \equiv 0$, we have

$$\int_M |\nabla f| \geq ID_\alpha(M) \left(\int_M |f|^\alpha \right)^{\frac{1}{\alpha}}.$$

Without loss of generality, we may assume $f \geq 0$. Let us define $M_t = \{x \in M | f(x) > t\}$. By the co-area formula,

$$\begin{aligned} \int_M |\nabla f| &= \int_0^\infty A(\partial M_t) dt \\ &\geq ID_\alpha(M) \int_0^\infty V(M_t)^{\frac{1}{\alpha}} dt. \end{aligned} \tag{9.1}$$

We now claim that for any $s \geq 0$, we have the inequality

$$\left(\int_0^s V(M_t)^{\frac{1}{\alpha}} dt \right)^\alpha \geq \alpha \int_0^s t^{\alpha - 1} V(M_t) dt.$$

This is obvious for the case $s = 0$. Differentiating both sides as functions of s, we have

$$\frac{d}{ds}\left(\int_0^s V(M_t)^{\frac{1}{\alpha}}dt\right)^{\alpha} = \alpha\left(\int_0^s V(M_t)^{\frac{1}{\alpha}}dt\right)^{\alpha-1} V(M_s)^{\frac{1}{\alpha}} \tag{9.2}$$

and

$$\frac{d}{ds}\left(\alpha\int_0^s t^{\alpha-1}V(M_t)dt\right) = \alpha s^{\alpha-1}V(M_s). \tag{9.3}$$

Observing that $\int_0^s V(M_t)^{\frac{1}{\alpha}}dt \le sV(M_s)^{\frac{1}{\alpha}}$, because $M_s \subset M_t$ for $t \le s$, we conclude that (9.2) is greater than or equal to (9.3). Integrating from 0 to ∞ yields the inequality as claimed.

Applying this inequality to (9.1) yields

$$\int_M |\nabla f| \ge ID_\alpha(M)\left(\alpha\int_0^\infty t^{\alpha-1}V(M_t)dt\right)^{\frac{1}{\alpha}}.$$

However, the co-area formula implies that

$$\begin{aligned}\alpha\int_0^\infty t^{\alpha-1}V(M_t)dt &= \int_0^\infty \frac{d(t^\alpha)}{dt}\int_t^\infty \int_{\partial V(M_s)} \frac{dA_s}{|\nabla f|}ds dt\\ &= \int_0^\infty t^\alpha \int_{\partial V(M_t)} \frac{dA_t}{|\nabla f|}|dt\\ &= \int_M f^\alpha.\end{aligned}$$

This proves $ID_\alpha(M) \le SD_\alpha(M)$.

We will now prove that $ID_\alpha(M) \ge SD_\alpha(M)$. Let Ω be a subdomain of M with smooth boundary $\partial\Omega$ such that $\partial\Omega \cap \partial M = \phi$. We define

$$N_\epsilon = \{x \in \Omega | d(x, \partial\Omega) < \epsilon\}.$$

Note that for $\epsilon > 0$ sufficiently small, $d(x, \partial\Omega)$ is a smooth function. Define

$$f_\epsilon(X) = \begin{cases} 0, & \text{on} \quad M \setminus \Omega \\ \frac{1}{\epsilon}d(x, \partial\Omega), & \text{on} \quad N_\epsilon \\ 1, & \text{on} \quad \Omega \setminus N_\epsilon, \end{cases}$$

Clearly f_ϵ is a Lipschitz function defined on M with Dirichlet boundary condition. Moreover,

$$\int_M |\nabla f| = \int_0^\epsilon \frac{1}{\epsilon}A(\partial N_t \setminus \partial\Omega)dt.$$

On the other hand, we have

$$\int_M |\nabla f| \geq SD_\alpha(M)\left(\int_M |f|^\alpha\right)^{\frac{1}{\alpha}}$$
$$\geq SD_\alpha(M)V(\Omega \setminus N_\epsilon)^{\frac{1}{\alpha}}.$$

Hence

$$\frac{1}{\epsilon}\int_0^\epsilon A(\partial N_t \setminus \partial\Omega)dt \geq SD_\alpha(M)V(\Omega \setminus N_\epsilon)^{\frac{1}{\alpha}}.$$

Letting $\epsilon \to 0$ yields

$$A(\partial\Omega) \geq SD_\alpha(M)V(\Omega)^{\frac{1}{\alpha}}.$$

Since Ω is arbitrary, this proves $ID_\alpha(M) \geq SD_\alpha(M)$.

Theorem 9.2.. *$IN_\alpha(M) \leq SN_\alpha(M)$ and $SN_\alpha(M) \leq \max\{1, 2^{1-\frac{1}{\alpha}}\}IN_\alpha(M)$ for all $\alpha >$ 0.*

Proof. Let f be a Lipschitz function defined on M. Let $k \in \mathbf{R}$ be chosen such that

$$M_+ = \{x \in M | f(x) - k > 0\}$$

and

$$M_- = \{x \in M | f(x) - k < 0\}$$

satisfy the conditions that $V(M_+) \leq \frac{1}{2}V(M)$ and $V(M_-) \leq \frac{1}{2}(M)$. To show that $SN_\alpha(M) \geq IN_\alpha(M)$, it suffices to show that

$$\int_M |\nabla u| \geq IN_\alpha(M)\left(\int_M |u|^\alpha\right)^{\frac{1}{\alpha}}$$

for $u = f - k$. Note that if

$$M_t = \{x \in M | u(x) > t\}$$

then for $t > 0$, we have

$$V(M_t) \leq V(M_+) \leq \frac{1}{2}V(M).$$

This implies that

$$\min\{V(M_t), V(M \setminus M_t))\} = V(M_t).$$

Hence, $A(\partial M_t) \geq IN_\alpha(M)V(M_t)$. Therefore by the same proof of Theorem 9.1, we have

$$\int_{M_+} |\nabla u| \geq IN_\alpha(M)\left(\int_{M_+} |u|^\alpha\right)^{\frac{1}{\alpha}}.$$

The same argument also gives

$$\int_{M_-} |\nabla u| \geq IN_\alpha(M)\left(\int_{M_-} |u|^\alpha\right)^{\frac{1}{\alpha}}.$$

Hence

$$\int_M |\nabla u| \geq IN_\alpha(M)\left[\left(\int_{M_+}|u|^\alpha\right)^{\frac{1}{\alpha}}+\left(\int_{M_-}|u|^\alpha\right)^{\frac{1}{\alpha}}\right]$$
$$\geq IN_\alpha(M)\left(\int_M |u|^\alpha\right)^{\frac{1}{\alpha}}.$$

This proves $SN_\alpha(M) \geq IN_\alpha(M)$.

To prove that $\max\{1, 2^{1-\frac{1}{\alpha}}\}IN_\alpha(M) \geq SN_\alpha(M)$, we consider any hyppersurface S dividing M into two parts denoted by Ω_1 and Ω_2. Let us assume that $V(\Omega_2) \leq V(\Omega_1)$. For $\epsilon > 0$ sufficiently small, let us define

$$N_\epsilon = \{x \in \Omega_2 \,|\, d(x, S) < \epsilon\}$$

and the function

$$f_\epsilon(x) = \begin{cases} 1, & \text{on } \Omega_1 \\ 1 - \frac{1}{\epsilon}d(x,S), & \text{on } N_\epsilon \\ 0, & \text{on } \Omega_2 - N_\epsilon, \end{cases}$$

Let k_ϵ be choosen such that

$$\int_M |f_\epsilon - k_\epsilon|^\alpha = \inf_{k\in\mathbf{R}} \int_M |f_\epsilon - k|^\alpha.$$

Clear, $0 \leq k_\epsilon \leq 1$. By using a similar argument to the proof of Theorem 9.1, we have

$$\int_M |\nabla f_\epsilon| = \int_{N_\epsilon} |\nabla f_\epsilon|$$
$$\geq SN_\alpha(M)\left(\int_M |f_\epsilon - k_\epsilon|^\alpha\right)^{\frac{1}{\alpha}}$$
$$\geq SN_\alpha(M)\left(\int_{\Omega_1} |f_\epsilon - k_\epsilon|^\alpha + \int_{\Omega_2\setminus N_\epsilon} |ff_\epsilon - k_\epsilon|^\alpha\right)^{\frac{1}{\alpha}}$$
$$\geq SN_\alpha(M)((1-k_\epsilon)^\alpha V(\Omega_1) + k_\epsilon^\alpha V(\Omega_2 \setminus N_\epsilon))^{\frac{1}{\alpha}}$$
$$\geq SN_\alpha(M)((1-k_\epsilon)^\alpha + k_\epsilon^\alpha)^{\frac{1}{\alpha}} V(\Omega_2 \setminus N_\epsilon)^{\frac{1}{\alpha}}.$$

We now observe that, $(1-k)^\alpha + k^\alpha \geq 2^{1-\alpha}$ for all $0 \leq k \leq 1$ and $\alpha > 1$, also $(1-k)^\alpha + k^\alpha \geq 1$ for all $0 \leq k \leq 1$ and $\alpha \leq 1$. Hence by taking $\epsilon \to 0$, the left hand side of (9.4) tends to $A(S)$ while the right hand side of (9.4) is bounded from below by $SN_\alpha(M)\min\{1, 2^{\frac{1-\alpha}{\alpha}}\}V(\Omega_2)^{\frac{1}{\alpha}}$. This establishes the inequality $\max\{1, 2^{1-\frac{1}{\alpha}}\}IN_\alpha(M) \geq SN_\alpha(M)$.

Let us point out that when the dimension of M is m and $\alpha > \frac{m}{m-1}$, then by the fact that the volume of geodesic balls of radius r behaves like

$$V(r) \sim \frac{\alpha_{m-1}}{m} r^m$$

and the area of the their boundary is asymptotic to

$$A(r) \sim \alpha_{m-1} r^{m-1},$$

it is clear that both $ID_\alpha(M) = 0 = IN_\alpha(M)$. Hence it is only interesting to consider those $\alpha \leq \frac{m}{m-1}$.

Corollary 9.1. *(Cheeger [C]) Let M be a compact Riemannian manifold. If $\partial M \neq \emptyset$, let us denote $\mu_1(M)$ to beo the first Dirichlet eigenvalue on M and $\lambda_1(M)$ to be its first nonzero Neumann eigenvalue for the Laplacian. When $\partial M = \emptyset$, we will denote the first nonzero eigenvalue of M by $\lambda_1(M)$ also. Then*

$$\mu_1(M) \geq \frac{ID_1(M)^2}{4}$$

and

$$\lambda_1(M) \geq \frac{IN_1(M)^2}{4}.$$

Proof. By Theorem 9.1, to see that

$$\mu_1(M) \geq \frac{ID_1(M)^2}{4},$$

it suffices to show that for any Lipschitz function f with Dirichlet boundary condition, it must satisfy

$$\int_M |\nabla f|^2 \geq \frac{SD_1(M)^2}{4} \int_M f^2.$$

Applying the definition of $SD_1(M)$ to the function f^2, we have

$$\int_M |\nabla f^2| \geq SD_1(M) \int_M f^2. \tag{9.4}$$

On the other hand,

$$\begin{aligned}\int_M |\nabla f^2| &= 2\int_M |f||\nabla f| \\ &\leq 2\left(\int_M f^2\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, the desired inequality follows from this and (9.4).

For the Neumann eigenvalue, we simply observe that if u is the first eigenfunction satisfying

$$\Delta u = \lambda_1(M)u,$$

then u must change sign. If we denote $M_+ = \{x \in M | u(x) > 0\}$ and $M_- = \{x \in M | u(x) < 0\}$, then

$$\mu_1(M_+) = \lambda_1(M) = \mu_1(M_-).$$

Let us assume that $V(M_+) \leq V(M_-)$. In particular, this implies that $ID_1(M_+) \geq IN_1(M)$. Hence by our previous argument,

$$\begin{aligned}\lambda_1(M) &= \mu_1(M_+)\\ &\geq \frac{ID_1(M_+)^2}{4}\\ &\geq \frac{IN_1(M)^2}{4}.\end{aligned}$$

This proves the corollary.

Corollary 9.2. *Let M be a compact Riemannian manifold with boundary. For any function $f \in H_{1,2}(M)$ and $f|_{\partial M} \equiv 0$, we have*

$$\int_M |\nabla f|^2 \geq \left(\frac{2-\alpha}{2} ID_\alpha(M)\right)^2 \left(\int_M |f|^{\frac{2\alpha}{2-\alpha}}\right)^{\frac{2-\alpha}{\alpha}}.$$

Proof. By applying Theorem (9.1) and the definition of $SD_\alpha(M)$ to the function $|f|^{\frac{2}{2-\alpha}}$, we obtain

$$\int_M |\nabla |f|^{\frac{2}{2-\alpha}}| \geq ID_\alpha(M) \left(\int_M |f|^{\frac{2\alpha}{2-\alpha}}\right)^{\frac{1}{\alpha}}.$$

On the other hand, Schwarz's inequality implies that

$$\begin{aligned}\int_M |\nabla |f|^{\frac{2}{2-\alpha}}| &= \frac{2}{2-\alpha}\int_M |f|^{\frac{\alpha}{2-\alpha}} |\nabla f|\\ &\leq \frac{2}{2-\alpha}\left(\int_M |f|^{\frac{2\alpha}{2-\alpha}}\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^2\right)^{\frac{1}{2}}.\end{aligned}$$

This proves the corollary.

§10 Lower Bounds of Isoperimetric Inequalities

The purpose of this section is to give lower bounds of the isoperimetric inequalities in terms of the diameter, the volume, and the lower bound of the Ricci curvature of the manifold. The estimate was proved by Croke [Cr] with the aid of Berger-Kazdan's lemma [Bs], Santalo's formula [S], and the notion of visibility angle first considered by Yau [Y 3]. Let us first explain the lemma of Berger-Kazdan.

Let $\gamma : [0, \pi] \to M$ be a normal geodesic on M with length π. Assume that there are no conjugate points on γ. Let $\{e_1, e_2, \dots, e_m\}$ be a parallel orthonormal frame field defined on γ with $e_1 = \gamma'$. The Jacobi equation along $\gamma(s)$ is given by

$$\nabla_{e_1} \nabla_{e_1} V - \mathcal{R}_{e_1 V} e_1 = 0. \tag{10.1}$$

In particular, if V_i is solution of (10.1) with initial conditions

$$V_i(0) = 0$$

and

$$V_i'(0) = e_i$$

for all $2 \le i \le m$, then one deduces that $\langle e_1, V_i \rangle = 0$ for all s. Hence, we can express

$$V_i = \sum_{j=2}^{m} b_{ij} e_j$$

and (10.1) becomes

$$\sum_{k=2}^{m} b_{ik}'' e_k - \sum_{k=2}^{m} b_{ik} \mathcal{R}_{e_1 e_k} e_1 = 0.$$

Taking the dot product with e_j, this implies that the matrix-valued function $B = (b_{ij})$ satisfies

$$B'' + B\mathcal{R} = 0$$

with

$$\mathcal{R} = (\mathcal{R}_{ki}) = (\langle \mathcal{R}_{e_1 e_k} e_j, e_1 \rangle).$$

Moreover, B satisfies the intitial conditions

$$B(0) = 0, \quad \text{and} \quad B'(0) = I.$$

Note that $\mathcal{R}$ is symmetric and B is invertible by the assumption that γ has no conjugate points. Let A be the transpose of B. Clearly, by the symmetry of $\mathcal{R}, A$ satisfies

$$A'' + \mathcal{R}A = 0 \tag{10.2}$$

with initial conditions

$$A(0) = 0 \text{ and } A'(0) = I. \tag{10.3}$$

Moreover, the solutions V_i is given by $V_i = Ae_i$.

Lemma 10.1. *(Berger-Kazdan [Bs]) Let $A_t(s)$ be a $(n \times n)$-matrix valued function defined on $[0,\pi]$. Suppose A_t satisfies (10.2) with initial conditions $A_t(t) = 0$ and $A_t'(t) = I$. Assume that $\mathcal{R}$ is symmetric on $[0,\pi]$, and $A_0(s)$ is invertible for all $s \in [0,\pi]$. If $\rho(s) > 0$ is a continuous function defined on $[0,\pi]$ satisfying $\rho(\pi - s) = \rho(s)$, then*

$$\int_0^\pi \int_t^\pi \rho(s-t) \det A_t(s) ds\, dt \geq \int_0^\pi \int_t^\pi \rho(s-t) \sin^n(s-t) ds\, dt.$$

Equality holds if and only if $\mathcal{R}(s) = I$ on $[0,\pi]$ and $A_t(s) = \sin(s-t)I$.

Proof. Let us denote $A = A_0$, and A^* to be its adjoint. Taking the adjoint of (10.2), we have $(A^*)'' + \mathcal{R}^* A^* = 0$ Since $\mathcal{R} = \mathcal{R}^*$, we have $(A^*)'' + \mathcal{R}A^* = 0$. In particular, this implies that

$$\begin{aligned} ((A^*)'A - A^*A')' &= (A^*)''A - A^*A'' \\ &= 0. \end{aligned}$$

Hence, together with the initial conditions of A we conclude that

$$(A^*)'A = A = A^*A'. \tag{10,4}$$

We now claim that $A_t(s)$ is given by

$$A_t(s) = A(s)\left(\int_t^s (A^*A)^{-1}(\tau)d\tau\right) A^*(t). \tag{10.5}$$

Indeed, $A_t(t) = 0$ and

$$\begin{aligned} A_t'(s) &= A'(s)\left(\int_t^s (A^*A)^{-1}(\tau)d\tau\right) A^*(t) + A(s)(A^*A)^{-1}(s)A^*(t) \\ &= A'(s)\left(\int_t^s (A^*A)^{-1}(\tau)d\tau\right) A^*(t) + (A^*)^{-1}(s)A^*(t), \end{aligned}$$

which implies that $A_t'(t) = 0$. Also

$$\begin{aligned} A_t''(s) &= A''(s)\left(\int_t^s (A^*A)^{-1}(\tau)d\tau\right) A^*(t) \\ &\quad + A'(s)(A^*A)^{-1}(s)A^*(t) + ((A^*)^{-1})'(s)A^*(t) \\ &= -\mathcal{R}(s)A_t(s) + \left(A'(s)A^{-1}(s)(A^*)^{-1}(s) + ((A^*)^{-1})'(s)\right) A^*(t). \end{aligned}$$

On the other hand, differenitating the identity $A^*(A^*)^{-1} = I$ yields

$$(A^*)'(A^*)^{-1} + A^*((A^*)^{-1})' = 0.$$

Hence applying (10.4), we conclude that

$$\begin{aligned}((A^*)^{-1})' &= -(A^*)^{-1}(A^*)'(A^*)^{-1}\\ &= -(A^*)^{-1}A^*A'A^{-1}(A^*)^{-1}\\ &= -A'A^{-1}(A^*)^{-1}.\end{aligned}$$

This established (10.5).

Let us denote $\phi = (\det A)^{\frac{1}{n}}$. Applying Jensen's inequality, we have

(10.6)
$$\begin{aligned}&\left(\int_t^s \phi^{-2}(\tau)d\tau\right)^{-n} \det\left(\int_t^s (A^*A)^{-1}(\tau)\phi^2(\tau)d\tau\right)\\ &= \det\left(\left(\int_t^s \phi^{-2}(\tau)d\tau\right)^{-1}\left(\int_t^s (A^*A)^{-1}(\tau)\phi^2(\tau)\phi^{-2}(\tau)d\tau\right)\right)\\ &\geq \left(\int_t^s \phi^{-2}(\tau)d\tau\right)^{-1}\left(\int_t^s \det\left((A^*A)^{-1}\phi^2\right)\phi^{-2}d\tau\right)\\ &= \left(\int_t^s \phi^{-2}(\tau)d\tau\right)^{-1}\left(\int_t^s \phi^{2n-2}(\tau)\det((A^*A)^{-1}(\tau))d\tau\right)\\ &= 1.\end{aligned}$$

Hence

$$\begin{aligned}\det A_t(s) &= \det A(s)\det\left(\int_t^s (A^*A)^{-1}(\tau)d\tau\right)\det A(t)\\ &\leq \phi^n(s)\phi^n(t)\left(\int_t^s \phi^{-2}(\tau)d\tau\right)^n,\end{aligned}$$

and

$$\left(\det A_t(s)\right)^{\frac{1}{n}} \geq \phi(s)\phi(t)\left(\int_t^s \phi^{-2}(\tau)d\tau\right).$$

On the other hand, Hölder inequality implies that

(10.7)
$$\begin{aligned}&\left(\int_0^\pi\int_t^\pi \rho(s-t)\det A_t(s)dsdt\right)^{\frac{1}{n}}\left(\int_0^\pi\int_t^\pi \rho(s-t)\sin^n(s-t)dsdt\right)^{\frac{n-1}{n}}\\ &\geq \int_0^\pi\int_t^\pi \rho(s-t)\left(\det A_t(x)\right)^{\frac{1}{n}}\sin^{n-1}(s-t)dsdt.\end{aligned}$$

Therefore,

(10.8)
$$\begin{aligned}\int_0^\pi \int_t^\pi \rho(s-t)\det A_t(s)ds\,dt \\ \geq \left(\int_0^\pi \int_t^\pi \rho(s-t)\phi(s)\phi(t)\sin^{n-1}(s-t)\int_t^s \phi^{-2}(\tau)\,d\tau\,ds\,dt\right)^n \\ \times \left(\int_0^\pi \int_t^\pi \rho(s-t)\sin^n(s-t)ds\,dt\right)^{1-n}.\end{aligned}$$

Clearly, equality holds if and only if equality holds on both (10.6) and (10.7). Equality holds on (10.6) if and only if $A_0 = (A^*A)^{-1}\phi^2$ is a constant matrix on $[0,\pi]$. Differentiating A_0^{-1} and using (10.4), we have

$$\begin{aligned}0 &= (A^*)'A\phi^{-2} + A^*A'\phi^{-2} - 2A^*A\phi^{-3}\phi' \\ &= 2A^*A'\phi^{-2} - 2A^*A\phi^{-3}\phi' \\ &= 2(A^*\phi^{-1})(A\phi^{-1})'.\end{aligned}$$

This implies that $A\phi^{-1} = A_1$ is a constant matrix. Taking the determinant of $A = \phi A_1$, we conclude that $\det A_1 = 1$. Using the initial condition (10.3), we conclude that

$$I = A'(0) = \phi'(0)A_1,$$

hence $A_1 = I$ and $A = \phi I$. On the other hand, equality on (10.7) implies that $\det A_t(s) = \sin^n(s-t)$. In particular, this implies that $A(s) = \sin(s)I$.

In view of (10.8), to prove the lemma, it suffices to show that the functional defined by

$$G(\phi) = \int_0^\pi \int_t^\pi \rho(s-t)\phi(s)\phi(t)\sin^{n-1}(s-t)\int_t^s \phi^{-2}(\tau)\,d\tau\,ds\,dt$$

satisfies the property that

$$G(\phi) \geq G(\sin),$$

because of the identity

$$\int_t^s \sin^{-2}(\tau)d\tau = \frac{\sin(s-t)}{\sin s \sin t}.$$

Observe that the definition of ϕ, the assumption that A is invertible on $(0,\pi)$, and the fact that A satisfies (10.2) implies that ϕ has at most zeros of order 1 at 0 and π. Hence, we may write $\phi(s) = (\sin s)(\exp u(s))$, where $u(s)$ is bounded from below and blows up at most at the order of log at 0 and π. Hence we may apply Jensen's inequality and conclude that

$$\begin{aligned}
G(\phi) &= \int_0^\pi \int_t^\pi \rho(s-t)\phi(s)\phi(t)\sin^{n-1}(s-t)\int_t^s \phi^{-2}(\tau)\,d\tau\,ds\,dt \\
&= \int_0^\pi \int_t^\pi \int_t^s \rho(s-t)\exp(u(s)+u(t)-2u(\tau))\sin^{n-1}(s-t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt \\
&\geq \mu(\Omega)\exp\left\{\mu(\Omega)^{-1}\int_0^\pi \int_t^\pi \int_t^s \rho(s-t)(u(s)+u(t)-2u(\tau))\right. \\
&\qquad \left.\times\sin^{n-1}(s-t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt\right\}
\end{aligned}$$

with $\Omega = \{(\tau,s,t)|t\leq\tau\leq s, t\leq s\leq\pi, 0\leq t\leq\pi\}$ and

$$\mu(\Omega) = \int_0^\pi \int_t^\pi \int_t^s \rho(s-t)\sin^{n-1}(s-t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt.$$

Hence, we have reduced to showing that

$$\int_0^\pi \int_t^\pi \int_t^s \rho(s-t)(u(s)+u(t)-2u(\tau))\sin^{n-1}(s-t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt = 0.$$

Let us define $\eta(s) = \rho(s)\sin^{n-1}(s)$. Clearly, $\eta(s) = \eta(\pi - s)$. Let us rewrite the integral

$$\begin{aligned}
\int_0^\pi \int_t^\pi \int_t^s & \eta(s-t)(u(s)+u(t)-2u(\tau))\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt \\
&= \int_0^\pi \int_t^\pi \int_t^s \eta(s-t)u(s)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt \\
&\quad + \int_0^\pi \int_t^\pi \int_t^s \eta(s-t)u(t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt \\
&\quad - \int_0^\pi \int_t^\pi \int_t^s \eta(s-t)u(\tau)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt.
\end{aligned}$$

The first term on the right hand side can be written as

$$\int_0^\pi \int_t^\pi \int_t^s \eta(s-t)u(s)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt = \int_0^\pi u(s)\int_t^\pi \eta(s-t)\sin(s-t)ds\,dt.$$

By changing the order of integration, the second term on the right can be written as

$$\begin{aligned}
\int_0^\pi \int_t^\pi \int_t^s \eta(s-t)u(t)\frac{\sin s\sin t}{\sin^2\tau}\,d\tau\,ds\,dt &= \int_0^\pi \int_t^\pi \eta(s-t)u(t)\sin(s-t)ds\,dt \\
&= \int_0^\pi u(t)\int_0^t \eta(s-t)\sin(s-t)dt\,ds.
\end{aligned}$$

Also the third term can be written as

$$-2\int_0^\pi\int_t^\pi\int_t^s \eta(s-t)u(\tau)\frac{\sin s\sin t}{\sin^2\tau}d\tau\, ds\, dt$$
$$=2\int_0^\pi u(\tau)\int_\tau^\pi\int_0^\tau \eta(s-t)\frac{\sin s\sin t}{\sin^2\tau}dt\, ds\, d\tau.$$

Hence

$$\int_0^\pi\int_t^\pi\int_t^s \eta(s-t)(u(s)+u(t)-2u(\tau))\frac{\sin s\sin t}{\sin^2\tau}d\tau\, ds\, dt$$
$$=\int_0^\pi u(t)\sin^{-2}(t)f(t)dt$$

where

$$f(s)=\sin^2(t)\int_t^\pi \eta(s-t)\sin(s-t)ds+\sin^2(t)\int_0^t \eta(t-s)\sin(t-s)ds$$
$$-2\int_t^\pi\int_0^t \eta(s-\tau)\sin s\sin\tau\, d\tau\, ds.$$

We now claim that f is identically 0 on $[0,\pi]$. Clearly, $f(0)=0$. Computing its derivative,

$$f'(t)=2\sin t\cos t\int_t^\pi \eta(s-t)\sin(s-t)ds+\sin^2 t\int_t^\pi \frac{\partial}{\partial t}\big(\eta(s-t)\sin(s-t)\big)ds$$
$$+2\sin t\cos t\int_0^t \eta(t-s)\sin(t-s)ds+\sin^2 t\int_0^t \frac{\partial}{\partial t}\big(\eta(t-s)\sin(t-s)\big)ds$$
$$+2\int_0^t \eta(t-\tau)\sin t\sin\tau\, d\tau-2\int_t^\pi \eta(s-t)\sin s\sin t\, ds$$
$$=2\sin t\int_t^\pi \eta(s-t)\big(\cos t\ \sin(s-t)-\sin s\big)ds$$
$$-\sin^2 t\int_t^\pi \frac{\partial}{\partial s}\big(\eta(s-t)\sin(s-t)\big)ds$$
$$+2\sin t\int_0^t \eta(t-s)\big(\cos t\ \sin(t-s)+\sin s\big)ds$$
$$-\sin^2 t\int_0^t \frac{\partial}{\partial s}\big(\eta(t-s)\sin(t-s)\big)ds$$
$$=-2\sin^2 t\int_t^\pi \eta(s-t)\cos(s-t)ds-\sin^2 t\big(\eta(\pi-t)\sin(\pi-t)\big)$$
$$+2\sin^2 t\int_0^t \eta(t-s)\cos(t-s)ds+\sin^2 t\big(\eta(t)\sin(t)\big).$$

Using the fact that $\eta(t)=\eta(\pi-t)$ and $\sin(t)=\sin(\pi-1)$, we conclude that

$$\sin^{-2}t\, f'(t)=-2\int_t^\pi \eta(s-t)\cos(s-t)ds+2\int_0^t \eta(t-s)\cos(t-s)ds.$$

If we set $F(t) = \sin^{-2} t\, f'(t)$, we observe that $F(0) = 2\int_0^\pi \eta(s)\cos s\, ds$. Using the symmetry of η and the fact that $\cos t = -\cos(\pi - t)$, we have $F(0) = 0$. Differentiating F with respect to t, we obtain

$$\begin{aligned}
F'(t) &= -2\int_t^\pi \frac{\partial}{\partial t}(\eta(s-t)\cos(s-t))\, ds + 2\int_0^t \frac{\partial}{\partial t}(\eta(t-s)\cos(t-s))\, ds + 4\eta(0) \\
&= -2\int_t^\pi \frac{\partial}{\partial s}(\eta(s-t)\cos(s-t))\, ds - 2\int_0^t \frac{\partial}{\partial s}(\eta(t-s)\cos(t-s))\, ds + 4\eta(0) \\
&= 2\eta(\pi - t)\,\cos(\pi - t) + 2\eta(t)\,\cos t \\
&= 0.
\end{aligned}$$

This implies that $F(t) = 0$ for all $t \in [0, \pi]$, and hence $f(t) = 0$ on $[0, \pi]$. This proves the lemma.

We are now ready to give an estimate on the isoperimetric inequality $IN_{\frac{m}{m-1}}(M)$ for compact manifolds without boundary. Let us first set up the following notation. Let M be a manifold with boundary ∂M. The unit tangent bundle of M is denoted by SM, and $\pi : SM \to M$ is the projection map. Given any unit vector $v \in SM$, we denote $\gamma_v(s)$ to be the normal geodesic with initial conditions $\gamma_v(0) = \pi(v)$ and $\gamma_v'(0) = v$. The geodesic flow $\zeta^t : SM \to SM$ on SM is given by

$$\zeta^t(v) = \gamma_v'(t).$$

Let us define $\tilde{\ell}(v)$ to be the smallest value of t such that $\gamma_v(t) \in \partial M$. Clearly, if the geodesic γ_v is confined in the interior of M, then $\tilde{\ell}(v) = \infty$. The map $\zeta^t(v)$ is obviously defined for all $t \le \tilde{\ell}(v)$. We also define

$$\ell(v) = \sup\{t | \gamma_v \text{ minimizes up to } t \text{ and } t \le \tilde{\ell}(v)\}.$$

Observe that $\ell(v) < \infty$ because M is compact, and $\ell(v) \le \tilde{\ell}(v)$. The set of unit tangent vectors $v \in SM$ such that the geodesic γ_{-v} minimizes up to the boundary is given by

$$UM = \{v \in SM \,|\, \ell(-v) = \tilde{\ell}(-v)\}.$$

Let us denote $U_p = \pi^{-1}|_{UM}(p)$ to be the preimage set of π when restricted to UM. If S_p is the unit tangent sphere at the point p, then the relative measure of U_p is denoted by

$$\omega_p(M) = \frac{m(U_p)}{m(S)p)}.$$

Definition 10.1. *The visibility angle of M with respect to its boundary ∂M is defined by $\omega(M) = \inf_{p \in M} \omega_p(M)$.*

Let $p \in \partial M$ be a point on the boundary, we denote v_p to be the inward pointing unit normal vector to ∂M at p. Define

$$S^+\partial M = \{v \in SM | \pi(v) \in \partial M \text{ and } \langle v, \nu_{\pi(v)}\rangle \ge 0\}$$

to be the set of inward pointing tangent bundle over ∂M. The volume of the standard unit m-sphere is denoted by α_m. It is clear that, the volume of the set $S^+\partial M$ is given by

$$V(S^+\partial M) = \frac{\alpha_{m-1}}{2} V(\partial M).$$

The following integral formula was proved by Santaló [S].

Proposition 10.1. *(Santaló) Let f be an integrable function defined on U.M. Then*

$$\int_{UM} f(v)dv = \int_{S+\partial M}\int_0^{\ell(u)} f(\zeta^r(u))\langle u, \nu_{\pi(u)}\rangle dr\, du.$$

In particular, by setting $f = 1$, we have

$$V(UM) = \int_{S+\partial M} \ell(u)\langle u, \nu_{\pi(u)}\rangle du.$$

Theorem 10.1. *Let M^m be a complete manifold with boundary, ∂M. Then*

$$A^m(\partial M) \geq C_1\omega^{m+1}(M)V^{m-1}(M),$$

where $C_1 = 2^{m-1}\alpha_{m-1}^m\alpha_m^{1-m}$. Equality holds if and only if $\omega(M) = 1$ and M is isometric to a hemisphere of the standard sphere.

Proof. Let $J(v,t)$ be the area element of $\partial B_{\pi(v)}(t)$, the boundary of the goedesic ball centered at $\pi(v)$ with radius t, at the point (u,t) in terms of normal polar coordinate at $\pi(v)$. For any $p \in M$, we have

$$V(M) = \int_{S_p}\int_0^{\ell(v)} J(v,t)dt\, dv.$$

Integrating over all points $p \in M$, this implies that

$$\begin{aligned} V^2(M) &= \int_M\int_{S_p}\int_0^{\ell(v)} J(v,t)dt\, dv\, dp \\ &= \int_{SM}\int_0^{\ell(0)} J(v,t)dt\, dv \\ &\geq \int_{UM}\int_0^{\ell(0)} J(v,t)dt\, dv \\ &= \int_{S+\partial M}\int_0^{\ell(u)}\int_0^{\ell(\zeta^r(u))} J(\zeta^r(u),t)\, dt\, \langle u, \mu_{\pi(u)}\rangle dr\, du \end{aligned} \tag{10.9}$$

by Proposition (10.1). We now observe that $\ell(\zeta^r(u)) \geq \ell(u) - r$, hence by (10.9), we have

$$V^2M \geq \int_{S+\partial M}\int_0^{\ell(u)}\int_0^{\ell(u)-r} J(\zeta^r(u),t)\, dt\, \langle u, \mu_{\pi(u)}\rangle dr\, du. \tag{10.10}$$

Let us now observe that by rescaling the metric by $\frac{\pi}{\ell(u)}$, and setting

$$\det A_r(t) = J(\zeta^r(u), t + r)$$

along the geodesic γ_u, Lemma 10.1 implies that

$$\int_0^{\ell(u)} \int_0^{\ell(u)-r} J(\zeta^r(u), t)\, dt\, dr = C_3\, \ell^{m+1}(u),$$

where

$$\begin{aligned} C_3 &= \pi^{-(m+1)} \int_0^{\pi} \int_t^{s} \sin^{m-1}(t-s)\, dt\, ds \\ &= 2^{-1} \alpha_m \pi^{-m} \alpha_{m-1}^{-1}. \end{aligned}$$

Hence (10.10) becomes

$$V^2(M) \geq C_3 \int_{S^+\partial M} \ell^{m+1}(u)\, \langle u, \nu_\pi(u) \rangle\rangle\, du. \tag{10.11}$$

On the other hand, Hölder inequality implies that

(10.12)

$$\begin{aligned} \left(\int_{S^+\partial M} \ell^{m+1}(u) \langle u, \nu_{\pi(u)} \rangle du\right) \left(\int_{S^+\partial M} \langle u, \nu_\pi(u) \rangle du\right)^m &\geq \left(\int_{S^+\partial M} \ell(u) \langle u, \nu_{\pi(u)} \rangle du\right)^{m+1} \\ &= V^{m+1}(UM) \\ &\geq (\alpha_{m-1}\omega(M)V(m))^{m+1}. \end{aligned}$$

Evaluating the integral

$$\int_{S^+\partial M} \langle u, \nu_{\pi(u)} \rangle du = \frac{\alpha_m}{2\pi} A(\partial M)$$

and applying (10.12) to (10.11), the desired estimate follows. It is clear that equality holds if and only if $\omega(M) = 1, \ell(u)$ is identically constant for $u \in S^+\partial M$, and equality holds for Lemma 10.1. This is equivalent to M being a hemisphere of the standard sphere.

With the aid of Theorem 10.1, we are ready to estimate the isoperimetric inequality for some cases in terms of the lower of the Ricci curvature, the upper bound of the diameter, and the lower bound of the volume. The following argument for estimating the visibility angle was first proved by Yau in [Y 3].

Corollary 10.1. *Let M^m be a compact Riemannian manifold without boundary. Suppose the Ricci curvature of M is bounded from below by $\mathcal{R}_{ij} \geq (m-1)K$, for some constant K. Let $d = d(M)$ and $V(M)$ be the diameter and the volume of M, respectively. Then*

$$ID_{\frac{m}{m-1}}(M) \geq C_4 \left(\frac{V(M)}{\bar{V}(d)}\right)^{\frac{m+1}{m}}$$

where $\bar{V}(d)$ denotes the volume of a geodesic ball of radius d in the simply connected space form of constant K sectional curvature, and $C_4 = 2^{-\frac{2}{m}}\alpha_{m-1}\alpha_m^{\frac{1-m}{m}}$.

Proof. Let S be a hypersurface dividing M into two components M_1 and M_2. Let us assume that $V(M_1) \geq V(M_2)$, hence $V(M_1) \geq 2V(M)$. By Theorem 10.1, we have

$$\frac{A^m(S)}{\min\{V(M_1), V(M_2)\}^{m-1}} = \frac{A^m(S)}{V^{m-1}(M_2)} \geq C_1\omega^{m+1}(M_2).$$

On the other hand, for any point $p \in M_2$, if γ is a minimizing geodesic joining p to a point $x \in M_1$ then $\gamma'(0)$ must be in U_p. Hence if we write the metric in terms of normal polar coordinates at p, we can estimate the volume of M_1 by

$$\begin{aligned} V(M_1) &\leq \int_0^d \int_{U_p} J(v,r)\,dv\,dr \\ &\leq m(U_p)\int_0^d \bar{J}(r)\,dr \\ &= \omega_p(M_2)\,\bar{V}(d). \end{aligned}$$

The corollary follows by using the assumption that $V(M_1) \geq 2V(M)$ and the definition of $SN_{\frac{m}{m-1}}(M)$.

Corollary 10.2. *Let M^m be a complete Riemannian manifold. Let us assume that the geodesic ball of radius R centered at a point $p \in M$ satisfies that $B_p(R) \cap \partial M \neq \emptyset$. Suppose the Ricci curvature is bounded below by $\mathcal{R}_{ij} \geq (m-1)K$ on $B_p(R)$ for some constant K. Then for any $0 < r < R$, we have*

$$ID_{\frac{m}{m-1}}(B_p(r)) \geq C_1^{\frac{1}{m}}\left(\frac{V_p(R) - V_p(r)}{\bar{V}(r+R)}\right)^{\frac{m+1}{m}}$$

where $\bar{V}(r+R)$ denotes the volume of a geodesic ball of radius $r+R$ in the simply connected space form of constant K sectional curvature.

Proof. By the definition of $ID_{\frac{m}{m-1}}(B_p(r))$ and Theorem 10.1, it suffices to estimate $\omega(D)$ for any proper subdomain of $B_p(r)$. However, it is clear that $\omega(D) \geq \omega(B_p(r))$ because $D \subset B_p(r)$. Hence

$$ID_{\frac{m}{n-1}}(B_p(r)) \geq C_1^{\frac{1}{m}}\omega^{\frac{m+1}{m}}(B_p(r)).$$

Following the same argument as in the proof of Corollary 10.1, for any $x \in B_p(r)$, we have

$$\begin{aligned} V_p(R) - V_p(r) &\leq \int_0^{r+R}\int_{U_x} J(v,r)dv\,dr \\ &\leq m(U_x)\int_0^{r+R}\bar{J}(r)dr \\ &\leq \omega_x(B_p(r))\bar{V}(r+R). \end{aligned}$$

Hence the corollary follows.

§11 Harnack Inequality and Regularity Theory of De Giorgi-Nash-Moser

In the section, we will present Moser's version of the De Giorgi-Nash-Moser's regularity theory, which was first discovered independently by De Giorgi and Nash. The iteration procedure of Moser was particularly useful in the theory of geometric analysis. We will attempt to cover this in most generality and keep explicit account on the dependency of various geometric and analytic constants. In applying this type of argument in the study of geometric PDE, often the explicit geometric dependency is crucial. As a result of these estimates, one derives a mean value inequality for nonnegative subsolutions and a Harnack inequality for positive solutions of a fairly general class of elliptic operators. In particular, it gives a C^α estimate for solutions of any second order elliptic operators of divergence form with only measurable coefficients. This regularity result was the original motivation for the development of this theory. We shall point out that the mean value inequality and the Harnack inequality derived from this argument applies to a slightly more general class of equation, while the ones given in earlier sections has less geometric dependency but requiring more smoothness on the operator. Both approaches are important in the theory of geometric analysis, while each is more suitable for different type of situation. The following account of Moser's argument which has been adopted to a more geometrically setting is a modification of the lecture notes of Schoen in [Sc].

In terms of notations, let us define the average value of a function f defined on a geodesic ball $B_p(R)$ by

$$\fint_{B_p(R)} f\, dV = V_p(R)^{-1} \int_{B_p(R)} f\, dV.$$

When the point p is fixed, the average L^q-norm of f over $B_p(R)$ is defined by

$$\Vert\!\!| f \Vert_{q,R} = \left(\fint_{B_p(R)} f\, dV \right)^{\frac{1}{q}}$$

and the regular L^q-norm is defined by

$$\Vert f \Vert_{q,R} = \left(\int_{B_p(R)} f\, dV \right)^{\frac{1}{q}}.$$

Lemma 11.1. *Let M be a complete manifold of dimension m. Let us assume that the geodesic ball $B_p(R)$ centered at p with radius R satisfies $B_p(R) \cap \partial M = \emptyset$. Suppose that $u \in H_{1,2}(B_p(R))$ is a nonnegative function defined on $B_p(R)$ such that*

$$\Delta u \geq -fu.$$

Let us defined the value $\nu = \frac{m}{2}$ for $m > 2$, adn $1 < \nu < \infty$ be arbitrary for $m = 2$. Assume that the function f is nonnegative on $B_p(R)$ and its L^q norm is finite for some $\nu < q \leq \infty$, with

$$A = \Vert\!\!| f \Vert_{q,R} = \left(\int_{B_p(R)} f^q \right)^{\frac{1}{q}}$$

for $\nu < q < \infty$ and

$$A = \|f\|_{\infty,R} = \sup_{B_p(R)} f$$

for $q = \infty$. If μ is the conjugate of ν such that $\frac{1}{\nu} + \frac{1}{\mu} = 1$, then let $C_s > 0$ be a constant such that the Sobolev inequality takes the form

$$\fint_{B_p(R)} |\nabla\phi|^2 \geq \frac{C_s}{R^2}\left(\fint_{B_p(R)} \phi^{2\mu}\right)^{\frac{1}{\mu}},$$

for all compactly supported function defined on $B_p(R)$ which is is $H_{1,2}(B_p(R))$. Given a $\theta \leq 1$, let us assume that

$$\frac{V_p(R)}{V_p(\theta R)} \leq C_v,$$

then for any $k > 0$, there exists constant $C_5 > 0$ depending only on k, ν, q, C_s, and C_v such that

$$\|u\|_{\infty,\theta R} \leq C_5\left((A\,R^2)^{\frac{q}{q-\nu}} + (1-\theta)^{-2}\right)^{\frac{\nu}{k}} \fint\|u\|_{k,R}.$$

Proof. By rescaling, without loss of generality, we may assume that $V_p(R) = 1$. For any arbitrary constant$a \geq 1$, the assumption of u implies that

$$\int \phi^2 f u^{2a} \geq -\int \phi^2 u^{2a-1}\Delta u,$$

for any compactly supported Lipschitz function on $B_p(R)$. Integrating by parts, the right hand side yields

$$\begin{aligned}-\int \phi^2 u^{2a-1}\Delta u &= 2\int \phi u^{2a-1}\langle\nabla\phi, \nabla u\rangle + (2a-1)\int \phi^2 u^{2a-2}|\nabla u|^2\\ &\geq 2\int \phi u^{2a-1}\langle\nabla\phi,\nabla u\rangle + a\int \phi^2 u^{2a-2}|\nabla u|^2.\end{aligned}$$

However, using the identity

$$\int |\nabla(\phi u^a)|^2 = \int |\nabla\phi|^2 u^{2a} + 2a\int \phi u^{2a-1}\langle\nabla\phi,\nabla u\rangle + a^2\int \phi^2 u^{2a-2}|\nabla u|^2,$$

we have

$$\begin{aligned}a\int \phi^2 f u^{2a} + \int |\nabla\phi|^2 u^{2a} &\geq \int |\nabla(\phi u^a)|^2\\ &\geq \frac{C_s}{R^2}\left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{1}{\mu}}.\end{aligned} \tag{11.1}$$

Let us choose $\phi(r)$ to be the Lipschitz cut-off function depending only on the distance to p, given by

$$\phi = \begin{cases} 0 & \text{on} \quad B_p(R) \setminus B_p(\rho+\sigma) \\ \dfrac{\rho+\sigma-r}{\sigma} & \text{on} \quad B_p(\rho+\sigma) \setminus B_p(\rho) \\ 1 & \text{on} \quad B_p(\rho). \end{cases}$$

When $q = \infty$, (11.1) implies that there is a constant $C_1 > 0$ depending only on C_s and C_v such that

$$\begin{aligned}\left(\int_{B_p(\rho)} u^{2a\mu}\right)^{\frac{1}{\mu}} &\leq V_p(\rho)^{\frac{1}{\nu}} \left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{1}{\mu}} \\ &\leq C_1 \left(aA\,R^2 + \frac{R^2}{\sigma^2}\right) \int_{B_p(\rho+\sigma)} u^{2a}.\end{aligned}$$

Hence,

$$\left(C_1 \left(aA\,R^2 + \frac{R^2}{\sigma^2}\right)\right)^{\frac{1}{2a}} \|u\|_{2a,\rho+\sigma} \geq \|u\|_{2a\mu,\rho}. \tag{11.2}$$

When $\frac{m}{2} < q < \infty$, by the Hölder inequality, we have

$$\begin{aligned} a\int \phi^2 f u^{2a} &\leq aA \left(\int (\phi^2 u^{2a})^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\ &\leq aA \left(\int \phi^2 u^{2a}\right)^{\frac{\mu(q-1)-q}{q(\mu-1)}} \left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{1}{q(\mu-1)}}. \end{aligned} \tag{11.3}$$

However, applying the inequality

$$x^{\epsilon} \leq \delta^{\frac{\epsilon-1}{\epsilon}} x + \delta \epsilon^{\frac{1}{1-\epsilon}} \left(\frac{1}{\epsilon} - 1\right)$$

by setting $\epsilon = \frac{\mu(q-1)-q}{q(\mu-1)}$ and

$$x = (aA\,V_p(2R)^{\frac{1}{q}})^{\frac{q(\mu-1)}{\mu(q-1)-q}} \left(\int \phi^2 u^{2a}\right) \left(\int (\phi^2 u^{2a})^{\mu}\right)^{-\frac{1}{\mu}},$$

we have

$$\begin{aligned} &aA \left(\int \phi^2 u^{2a}\right)^{\frac{\mu(q-1)-q}{q(\mu-1)}} \left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{q-\mu(q-1)}{q\mu(\mu-1)}} \\ &\leq \delta^{\frac{\epsilon-1}{\epsilon}} (aA)^{\frac{q(\mu-1)}{\mu(q-1)-q}} \left(\int \phi^2 u^{2a}\right) \left(\int (\phi^2 u^{2a})^{\mu}\right)^{-\frac{1}{\mu}} + \delta \epsilon^{\frac{1}{1-\epsilon}} \left(\frac{1}{\epsilon} - 1\right). \end{aligned} \tag{11.4}$$

Multiplying through by

$$\left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{1}{\mu}}$$

and choosing δ so that

$$\delta\epsilon^{\frac{1}{1-\epsilon}}\left(\frac{1}{\epsilon}-1\right)=\frac{C_s}{2R^2},$$

(11.3) and (11.4) becomes

$$a\int \phi^2 f u^{2a} \le C_2\left(\frac{C_s}{R^2}\right)^{\frac{-\mu}{\mu(q-1)-q}}(aA)^{\frac{q(\mu-1)}{\mu(q-1)-q}}\left(\int \phi^2 u^{2a}\right)+\frac{C_s}{2R^2}\left(\int (\phi^2 u^{2a})^{\mu}\right)^{\frac{1}{\mu}}$$

for some constant $C_2>0$ depending only on μ, q and C_v. Hence together with (11.1), we have

$$\left(C_3(aAR^2)^{\frac{q(\mu-1)}{\mu(q-1)-q}}+\frac{R^2}{\sigma^2}\right)^{\frac{1}{2a}}\|u\|_{2a,\rho+\sigma}\ge \|u\|_{2a\mu,\rho}. \tag{11.5}$$

In any event, (11.2) and (11.5) imply that we have the inequality

$$\left(C_2(aAR^2)^{\alpha}+\frac{R^2}{\sigma^2}\right)^{\frac{1}{2a}}\|u\|_{2a,\rho+\sigma}\ge \|u\|_{2a\mu,\rho} \tag{11.6}$$

with $\alpha=\frac{q(\mu-1)}{\nu(q-1)-q}$. Let us now choose the sequences of a_i, ρ_i, and σ_i, such that

$$a_0=\frac{k}{2},\quad a_1=\frac{k\mu}{2},\quad \cdots,\quad a_i=\frac{k\mu^i}{2},\quad \cdots,$$

$$\sigma_0=2^{-1}(1-\theta)R,\quad \sigma_1=2^{-2}(1-\theta)R,\quad \cdots,\quad \sigma_i=2^{-(1+i)}(1-\theta)R,\quad \cdots,$$

and

$$\rho_{-1}=R,\quad \rho_0=R-\sigma_0,\quad \rho_1=R-\sigma_0-\sigma_1,\quad \cdots,\quad \rho_i=R-\sum_{j=0}^{i}\sigma_j,\quad \cdots.$$

Observe that $\lim_{i\to\infty}\rho_i=\theta R$. Applying (11.6) to $a=a_i, \rho=\rho_i$, and $\sigma=\sigma_i$, and iterating the inequality, we conclude that

$$\|u\|_{2a_{i+1},\rho_i}\le \prod_{j=0}^{i}\left(C_2(a_iAR^2)^{\alpha}+\frac{R^2}{\sigma_i^2}\right)^{\frac{1}{2a_i}}\|u\|_{k,R}.$$

On the other hand, we have the inequality

$$\begin{aligned}\lim_{i\to\infty}V(\theta R)^{\frac{-1}{2a_{i+1}}}\|u\|_{2a_{i+1},\rho_i} &\ge \lim_{i\to\infty}V(\theta R)^{\frac{-1}{2a_{i+1}}}\|u\|_{2a_{i+1},\theta R}\\ &=\|u\|_{\infty,\theta R}.\end{aligned}$$

Therefore, letting $i \to \infty$, we conclude that

$$\|u\|_{\infty,\theta R} \le \prod_{j=0}^{\infty}\left(C_2\left(\frac{kAR^2\mu^i}{2}\right)^{\alpha} + \frac{84^i}{(1-\theta)^2}\right)^{\frac{1}{k\mu^i}} \|u\|_{k,R}.$$

The product can be estimated by using the fact that

$$\prod_{i=0}^{\infty} B^{\mu^{-i}} = B^{\frac{\mu}{\mu-1}}$$

and the fact that $\sum_{i=0}^{\infty} i\mu^{-i}$ is finite. Hence we have

$$\begin{aligned}
&\prod_{j=0}^{\infty}\left(C_2\left(\frac{kAR^2\mu^i}{2}\right)^{\alpha} + \frac{84^i}{(1-\theta)^2}\right)^{\frac{1}{k\mu^i}} \\
&\le \prod_{j=0}^{\infty}\left(C_2\left(\frac{kAR^2}{2}\right)^{\alpha} + \frac{8}{(1-\theta)^2}\right)^{\frac{1}{k\mu^i}} \max\{\mu^{\alpha}, 4\}^{\frac{i}{k\mu^i}} \\
&\le C_3^{\frac{1}{k}}\left((AR^2)^{\alpha} + \frac{1}{(1-\theta)^2}\right)^{\frac{\mu}{k(\mu-1)}},
\end{aligned}$$

where $C_3 \ge 0$ depends only on k, μ, q and C_s alone. This proves the desired inequality for $k \ge 2$.

For those values of $k < 2$, we begin with the case $k = 2$. In that case, the inequality takes the form

$$\begin{aligned}
\|u\|_{\infty,\eta\rho} &\le C_4\left((AR^2)^{\alpha} + \frac{1}{(1-\eta)^2\rho^2}\right)^{\frac{\mu}{2(\mu-1)}} \|u\|_{2,\rho} V_p(\theta R)^{-\frac{1}{2}} \\
&\le C_4\left((AR^2)^{\alpha} + \frac{1}{(1-\eta)^2\rho^2}\right)^{\frac{\mu}{2(\mu-1)}} \|u\|_{k,\rho}^{\frac{k}{2}} \|u\|_{\infty,\rho}^{1-\frac{k}{2}} V_p(\theta R)^{-\frac{1}{2}},
\end{aligned}$$

for any $\theta R \le \rho \le R$, and $\eta < 1$. Let us choose the sequences of ρ_i and η_i to be

$$\rho_0 = \theta R, \quad \rho_1 = \theta R + 2^{-1}(1-\theta)R, \quad \cdots, \quad \rho_i = \theta R + (1-\theta)R\sum_{j=1}^{i} 2^{-j}, \quad \cdots,$$

and

$$\eta_i \rho_i = \rho_{i-1}.$$

Applying (11.7) to the pair ρ_i and η_i and iterating the inequality yields

(11.8)

$$\|u\|_{\infty,\theta R} \le \|u\|_{\infty,R}^{(1-\frac{k}{2})^i} \prod_{j=1}^{i}\left(C_4\left((AR^2)^{\alpha} + \frac{2^j}{(1-\theta)^2R^2}\right)^{\frac{\mu}{2(\mu-1)}} \|u\|_{k,R}^{\frac{k}{2}} V_p(\theta R)^{-\frac{1}{2}}\right)^{(1-\frac{k}{2})^{j-1}}.$$

Letting $i \to \infty$, the term

$$\|u\|_{\infty,R}^{(1-\frac{k}{2})^i} \to 1,$$

and

$$\prod_{j=1}^{\infty} \|u\|_{k,R}^{\frac{k}{2}(1-\frac{k}{2})^{j-1}} = \|u\|_{k,R}.$$

Hence, (11.8) implies that

$$\|u\|_{\infty,\theta R} \leq C_5 \left((A\,R^2)^\alpha + \frac{1}{(1-\theta)^2 R^2}\right)^{\frac{\mu}{k(\mu-1)}} \|u\|_{k,R} V_p(\theta R)^{-\frac{1}{k}}.$$

Substituting the values of α and the fact that $\mu = \frac{\nu}{\nu-1}$, the desired inequality follows.

Lemma 11.2. *Let M be a complete manifold. Suppose that the geodesic ball $B_p(R)$ centered at p with radius R satisfies $B_p(R) \cap \partial M = \emptyset$. Let $u \geq 0$ be a function in $H_{1,2}(B_p(R))$, satisfying the inequality*

$$\Delta u \leq Au$$

in the weak sense for some constant $A \geq 0$ on $B_p(R)$. Let us denote $\nu = \frac{m}{2}$ for $m > 2$, and $1 < \nu < \infty$ be arbitrary when $m = 2$. If μ is the conjugate of ν such that $\frac{1}{\nu} + \frac{1}{\mu} = 1$, then let C_s be the Sobolev constant such that

$$\fint_{B_p(R)} |\nabla\psi|^2 \geq \frac{C_s}{R^2} \left(\fint_{B_p(R)} \psi^{2\mu}\right)^{\frac{1}{\mu}},$$

for all compactly supported $H_{1,2}(B_p(R))$ functions. Let us assume that the first non-zero Neumann eigenvalues $\lambda_1\left(\frac{R}{4}\right)$ and $\lambda_1(\frac{R}{2})$ of the balls $B_p(\frac{R}{4})$ and $B_p(\frac{R}{2})$ satisfy the estimate

$$\min\left\{\frac{R^2}{16}\lambda_1\left(\frac{R}{4}\right), \frac{R^2}{4}\lambda_1\left(\frac{R}{2}\right)\right\} \geq C_p$$

for some constant $C_p > 0$. Also, let us denote the upper of the ratio of the volumes of balls by

$$\frac{V_p(R)}{V_p(\frac{R}{16})} \leq C_v.$$

Then for $k > 0$ sufficiently small, there exists constant $C > 0$ depending only on the quantities k, ν, C_v, C_p, C_s, and $(AR^2 + 1)$ such that

$$\|u\|_{k,\frac{R}{8}} \leq C_{20} \inf_{B_p(\frac{R}{16})} u.$$

Proof. The function u^{-1} satisfies

$$\begin{aligned}\Delta u^{-1} &= -u^{-2}\Delta u + 2u^{-3}|\nabla u|^2\\ &\geq -Au^{-1}.\end{aligned}$$

By applying Lemma 11.1 to u^{-1}, we have

$$\text{(11.9)} \qquad \begin{aligned}\left(\inf_{B_p(\frac{R}{16})} u\right)^{-1} &= \sup_{B_p(\frac{R}{16})} u^{-1} \\ &\leq C_5(AR^2+1)^{\frac{\nu}{k}} ⨍\!\|u^{-1}\|_{k,\frac{R}{8}}.\end{aligned}$$

Clearly, the lemma follows if we can estimate the product

$$⨍\!\|u^{-1}\|_{k,\frac{R}{4}} \cdot ⨍\!\|u\|_{k,\frac{R}{4}}$$

from above for some value of $k > 0$.

To achieve this, let us consider the function

$$w = \beta + \log u$$

where $\beta = -⨍_{B_p(R)} \log u$. The function w satisfies

$$\begin{aligned}\Delta w &= \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \\ &\leq A - |\nabla w|^2,\end{aligned}$$

hence

$$\text{(11.10)} \qquad |\nabla w|^2 \leq A - \Delta w.$$

Let $\psi(r)$ be a cut-off function defined by

$$\psi = \begin{cases} 0 & \text{on} \quad M \setminus B_p(R) \\ \dfrac{2(R-r)}{R} & \text{on} \quad B_p(R) \setminus B_p(\frac{R}{2}) \\ 1 & \text{on} \quad B_p(\frac{R}{2}). \end{cases}$$

Multiplying (11.10) by ψ^2 and integrating, we have

$$\begin{aligned}\int \psi^2 |\nabla w|^2 &\leq \int \psi^2 A - \int \psi^2 \Delta w \\ &= \int \psi^2 A + 2\int \psi \langle \nabla \psi, \nabla w\rangle \\ &\leq \int \psi^2 A + 2\int |\nabla \psi|^2 + \frac{1}{2}\int \psi^2 |\nabla w|^2.\end{aligned}$$

We deduce that

$$\int_{B_p(\frac{R}{2})} |\nabla w|^2 \le \int \psi^2 |\nabla w|^2 \le 2A \int \psi^2 + 4 \int |\nabla \psi|^2 \le 2A\, V_p(R) + \frac{14 V_p(R)}{R^2}. \tag{11.11}$$

However, the Poincaré inequality and the choice of β implies that

$$\frac{4C_p}{R^2} \int_{B_p(\frac{R}{2})} w^2 \le \int_{B_p(\frac{R}{2})} |\nabla w|^2.$$

Hence, we have

$$\int_{B_p(\frac{R}{2})} w^2 \le C_6 V_p(R) \tag{11.12}$$

where $C_6 > 0$ is a constant depending only on C_p and $(AR^2 + 1)$. Applying the Schwarz inequality, we also have

$$\int_{B_p(\frac{R}{2})} |w| \le C_6^{-\frac{1}{2}} V_p(R). \tag{11.13}$$

On the other hand, let $\phi(r)$ to be the Lipschitz cut-off function depending only on the distance to p, given by

$$\phi = \begin{cases} 0 & \text{on} \quad B_p(\frac{R}{2}) \setminus B_p(\rho + \sigma) \\ \dfrac{\rho + \sigma - r}{\sigma} & \text{on} \quad B_p(\rho + \sigma) \setminus B_p(\rho) \\ 1 & \text{on} \quad B_p(\rho). \end{cases}$$

Then multiplying $\phi^2 |w|^{2a-2}$ to (11.10) for $a \ge 2$, and integrating by parts yields

$$\begin{aligned} \int \phi^2 |w|^{2a-2} |\nabla w|^2 &\le A \int \phi^2 |w|^{2a-2} - \int \phi^2 |w|^{2a-2} \Delta w \\ &\le A \int \phi^2 |w|^{2a-2} + 2 \int \phi |w|^{2a-2} \langle \nabla \phi, \nabla w \rangle \\ &\quad + (2a-2) \int \phi^2 |w|^{2a-3} |\nabla w|^2. \end{aligned} \tag{11.14}$$

Using (11.11) and the inequalities,

$$\int \phi^2 |w|^{2a-2} \le \int_{B_p(\rho+\sigma)} |w|^{2a-2},$$

$$2\int \phi|w|^{2a-2}\langle\nabla\phi,\nabla w\rangle \le \frac{1}{4}\int \phi^2|\nabla w|^2|w|^{2a-2} + 4\int |\nabla\phi|^2|w|^{2a-2}$$
$$\le \frac{1}{4}\int \phi^2|\nabla w|^2|w|^{2a-2} + \frac{4}{\sigma^2}\int_{B_p(\rho+\sigma)} |w|^{2a-2},$$

and

$$(2a-2)\int \phi^2|w|^{2a-3}|\nabla w|^2 \le \frac{1}{4}\int \phi^2|w|^{2a-2}|\nabla w|^2 + (8a-12)^{2a-3}\int \phi^2|\nabla w|^2$$
$$\le \frac{1}{4}\int \phi^2|w|^{2a-2}|\nabla w|^2 + (8a-12)^{2a-3}\int_{B_p(\rho+\sigma)} |\nabla w|^2,$$

(11.14) becomes

(11.15)
$$\int \phi^2|w|^{2a-2}|\nabla w|^2 \le \left(2A+\frac{8}{\sigma^2}\right)\int_{B_p(\rho+\sigma)} |w|^{2a-2} + 2(8a-12)^{2a-3}\int_{B_p(\rho+\sigma)} |\nabla w|^2$$
$$\le \left(2A+\frac{8}{\sigma^2}\right)\int_{B_p(\rho+\sigma)} |w|^{2a-2}$$
$$+ 4(8a-12)^{2a-3}\left(A+\frac{8}{R^2}\right)V_p(R).$$

By setting $a=2$, $\rho=\frac{R}{4}$, and $\sigma=\frac{R}{4}$, and combining with (11.12) we have

$$\int_{B_p(\frac{R}{4})} w^2\,|\nabla w|^2 \le C_7 R^{-2}\,V_p(R)$$

for some constant $C_7>0$ depending only on C_p and (AR^2+1). On the other hand,

$$\int_{B_p(\frac{R}{4})} w^2\,|\nabla w|^2 = \frac{1}{4}\int_{B_p(\frac{R}{4})} |\nabla\left(\operatorname{sgn}(w)\,w^2\right)|^2$$
$$\ge \frac{1}{4}\lambda_1\left(\frac{R}{4}\right)\int_{B_p(\frac{R}{4})}\left(\operatorname{sgn}(w)\,w^2 - V_p\left(\frac{R}{4}\right)^{-1}\int_{B_p(\frac{R}{4})}\operatorname{sgn}(w)\,w^2\right)^2$$
$$= \frac{4C_p}{R^2}\left(\int_{B_p(\frac{R}{4})}|w|^4 - V_p\left(\frac{R}{4}\right)^{-1}\left(\int_{B_p(\frac{R}{4})}\operatorname{sgn}(w)\,w^2\right)^2\right)$$
$$\ge \frac{4C_p}{R^2}\left(\int_{B_p(\frac{R}{4})}|w|^4 - V_p\left(\frac{R}{4}\right)^{-1}\left(\int_{B_p(\frac{R}{4})} w^2\right)^2\right).$$

Hence combining with (11.12), we have

(11.16)
$$\int_{B_p(\frac{R}{4})} |w|^4 \le C_8 V_p(R)$$

for some constant $C_8 > 0$ depending on C_p, (AR^2+1) and C_v. Using Schwarz inequality, we also conclude that

$$\int_{B_p(\frac{R}{4})} |w|^3 \leq C_8^{\frac{3}{4}} V_p(R). \tag{11.17}$$

For general $a \geq 2$, the Schwarz inequality implies that

$$|\nabla(\phi\,|w|^a)|^2 \leq 2|\nabla\phi|^2|w|^{2a} + 2a^2\phi^2|w|^{2a-2}|\nabla w|^2.$$

Combining this with (11.15), we conclude that

$$\int |\nabla(\phi\,|w|^a)|^2 \leq \frac{2}{\sigma^2}\int_{B_p(\rho+\sigma)} |w|^{2a} + 4a^2\left(A+\frac{4}{\sigma^2}\right)\int_{B_p(\rho+\sigma)} |w|^{2a-2} + 8(8a-12)^{2a-1}\left(A+\frac{8}{R^2}\right)V_p(R).$$

Using the inequality

$$|w|^{2a-2} \leq |w|^{2a} + 1,$$

we have

$$\int |\nabla(\phi\,|w|^a)|^2 \leq C_9 a^2\left(A+\frac{1}{\sigma^2}\right)\int_{B_p(\rho+\sigma)} |w|^{2a} + C_{10}(8a-12)^{2a-1}\left(A+\frac{1}{R^2}+\frac{1}{\sigma}\right)V_p(R)$$

for some universal constants $C_9, C_{10} > 0$. Hence, applying the Sobolev inequality

$$\frac{C_s}{R^2}V_p(R)^{\frac{1}{\nu}}\left(\int_{B_p(\rho)} |w|^{2a\mu}\right)^{\frac{1}{\mu}} \leq \frac{C_s}{R^2}V_p(R)^{\frac{1}{\nu}}\left(\int (\phi^2|w|^{2a})^{\mu}\right)^{\frac{1}{\mu}} \leq \int |\nabla(\phi\,|w|^a)|^2,$$

we conclude that

$$\| w\|_{2a\mu,\rho} \leq C_{11}^{\frac{1}{2a}} a^{\frac{1}{a}}\left(AR^2+\frac{R^2}{\sigma^2}\right)^{\frac{1}{2a}} \| w\|_{2a,\rho+\sigma} + C_{12}^{\frac{1}{2a}}(8a)\left(AR^2+1+\frac{R^2}{\sigma^2}\right)^{\frac{1}{2a}}$$

where $C_{11}, C_{12} > 0$ are constants depending only on C_s and C_v. Consider the sequences of a_i, ρ_i and σ_i given by

$$a_0 = 2, \quad a_1 = 2\mu, \quad \cdots, \quad a_i = 2\mu^i, \quad \cdots,$$

$$\sigma_0 = 2^{-4}R, \quad \sigma_1 = 2^{-5}R, \quad \cdots, \quad \sigma_i = 2^{-(4+i)}R, \quad \cdots,$$

and

$$\rho_0 = \frac{R}{4} - \sigma_0, \quad \rho_1 = \frac{R}{4} - \sigma_0 - \sigma_1, \quad \cdots, \quad \rho_i = \frac{R}{4} - \sum_{j=0}^{i} \sigma_j, \quad \cdots.$$

If we adopt the convention that $\rho_{-1} = \frac{R}{4}$, then applying the inequality to a_i, ρ_i and σ_i, we have

$$\Vert w\Vert_{4\mu^{i+1},\rho_i} \leq (2\mu^i)^{\frac{1}{2\mu^i}}\, 2^{\frac{4+i}{2\mu^i}}\, C_{13}^{\frac{1}{4\mu^i}} \Vert w\Vert_{4\mu^i,\rho_{i-1}} + 16\mu^i\, 2^{\frac{4+i}{2\mu^i}}\, C_{14}^{\frac{1}{4\mu^i}}.$$

where $C_{13}, C_{14} > 0$ are constants depending only on C_s, C_v, and (AR^2+1). Iterating this by running $i = 0, \cdots, \ell$ gives

$$\begin{aligned} \Vert w\Vert_{4\mu^{\ell+1},\rho_\ell} \leq & \prod_{i=0}^{\ell} (2\mu^i)^{\frac{1}{2\mu^i}}\, 2^{\frac{4+i}{2\mu^i}}\, C_{13}^{\frac{1}{4\mu^i}} \Vert w\Vert_{4,\frac{R}{4}} \\ & + \sum_{i=0}^{\ell-1} 16\mu^i\, 2^{\frac{4+i}{2\mu^i}}\, C_{14}^{\frac{1}{4\mu^i}} \prod_{j=i+1}^{\ell} \left((2\mu^j)^{\frac{1}{2\mu^j}}\, 2^{\frac{4+j}{2\mu^j}}\, C_{13}^{\frac{1}{4\mu^j}} \right) \\ & + 16\mu^\ell\, 2^{\frac{4+\ell}{2\mu^\ell}}\, C_{14}^{\frac{1}{4\mu^\ell}}. \end{aligned}$$

Using the equality $\sum_{i=0}^{\infty} \mu^{-i} = \frac{m}{2}$, and the fact that $\sum_{i=0}^{\infty}(3+i)\mu^{-i}$ is finite, we conclude that

$$\begin{aligned} \Vert w\Vert_{4\mu^{\ell+1},\rho_\ell} & \leq C_{15} \left(\Vert w\Vert_{4,\frac{R}{4}} + \sum_{i=0}^{\ell} \left(\mu^i\, 2^{\frac{4+i}{2\mu^i}} \right) \right) \\ & \leq C_{16} \left(\Vert w\Vert_{4,\frac{R}{4}} + 4\mu^\ell \right) \end{aligned} \tag{11.18}$$

where the constants $C_{15}, C_{16} > 0$ depend only on m, C_s, C_v, and (AR^2+1).

For each integer $j \geq 4$, let ℓ be such that $4\mu^\ell < j \leq 4\kappa^{\ell+1}$. Using the fact that $\rho_\ell \geq \frac{R}{8}$, Hölder inequality and the estimate (11.18) implies that

$$\begin{aligned} \fint_{B_p(\frac{R}{8})} |w|^j & \leq \left(\fint_{B_p(\frac{R}{8})} |w|^{4\mu^{\ell+1}} \right)^{\frac{j}{4\mu^{\ell+1}}} \\ & \leq C_{16}^j (\Vert w\Vert_{4,\frac{R}{4}} + j)^j. \end{aligned}$$

Hence together with (11.12), (11.13), (11.16), and (11.17), we have

$$\begin{aligned} \fint_{B_p(\frac{R}{8})} e^{k|w|} & = \sum_{j=0}^{\infty} (j!)^{-1} k^j \fint_{B_p(\frac{R}{8})} |w|^j \\ & \leq C_{17} + \sum_{j=5}^{\infty} (j!)^{-1} (C_{18} k j)^j, \end{aligned}$$

where $C_{17}, C_{18} > 0$ are constants depending only on C_p, C_v, and $(AR^2 + 1)$. However, using Stirling's inequality $j^j < j!e^j$, we conclude that

$$⨍_{B_p(\frac{R}{8})} e^{k|w|} \le C_{17} + \sum_{j=5}^{\infty}(C_{18}ke)^j.$$

Therefore, by choosing $k < (C_{18}e)^{-1}$, the infinite series converges and we obtain the estimate

$$\int_{B_p(\frac{R}{8})} e^{k|w|} \le C_{19} \tag{11.19}$$

where $C_{19} > 0$ is a constant depending on m, C_p, C_v C_s, and $(AR^2 + 1)$.

Let us now observe that

$$\begin{aligned} e^{k\beta}u^k &= e^{kw} \\ &\le e^{k|w|} \end{aligned}$$

and

$$\begin{aligned} e^{-k\beta}u^{-k} &= e^{-kw} \\ &\le e^{k|w|} \end{aligned}$$

imply that

$$\|u^{-1}\|_{k,\frac{R}{8}} \cdot \|u\|_{k,\frac{R}{8}} \le \left(⨍_{B_p(\frac{R}{8})} e^{k|w|}\right)^2.$$

The lemma now follows by applying (11.9) and (11.19).

By combining Lemma 11.1 and Lemma 11.2, we obtain the following Harnack inequality.

Theorem 11.1. *Let M be a complete manifold of dimension m. Let us assume that the geodesic ball $B_p(R)$ centered at p with radius R satisfies $B_p(R) \cap \partial M = \emptyset$. Suppose that $u \in H_{1,2}(B_p(R))$ is a nonnegative function defined on $B_p(R)$ such that*

$$|\Delta u| \le Au.$$

Then there exists a constant $C_{21} > 0$ depending on the quantities $(AR^2 + 1), m, C_p, C_s$, and C_v such that

$$\sup_{B_p(16R)} u \le C_{21} \inf_{B_p(16R)} u.$$

Corollary 11.1. *Let M be a manifold of dimension m. Suppose that ds_0^2 is a complete metric on M such that there is a point $p \in M$ and for all R, the quantities C_p, C_s, and C_v are all bounded independent of R. Then for any ds^2 metric on M which is equivalent to ds_0^2, there does not exist any nonconstant positive harmonic functions for the Laplacian with respect to ds^2. In particular, any manifold which is quasi-isometric to Euclidean space endowed with the standard flat metric has no non-constant positive harmonic functions.*

Proof. To see this, we first observe the properties that C_p, C_s, and C_v are uniformly bounded is a quasi-isometric invariant. Hence one Theorem 11.1 implies that any positive harmonic function u defined on M must satisfies the Harnack inequality

$$\sup_{B_p(16R)} u \leq C_{21} \inf_{B_p(16R)} u.$$

On the other hand, since u is positive, by translation, we may assume that $\inf_M u = 0$. Hence, by taking $R \to \infty$, we conclude that

$$\sup_M u \leq C_{21} \inf_M u = 0.$$

Therefore, u must be identically 0.

Corollary 11.2. *Let M, ds_0^2, and ds^2 satisfy the hypothesis of Corollary 11.1. Suppose $u \in H_{1,2}(B_p(1))$ satisfies the differential inequality*

$$|\Delta u| \leq A$$

in the weak sense for some constant $A > 0$. Then u must be Hölder continuous at the point p.

Proof. Let us denote $s(R) = \sup_{B_p(R)} u$ and $i(R) = \inf_{B_p(R)} u$. Applying Theorem 11.1 to the functions $s(R) - u$ and $u - i(R)$, we have

$$s(R) - i(\frac{R}{16}) \leq C_{21} \left(s(R) - s(\frac{R}{16}) \right)$$

and

$$s(\frac{R}{16}) - i(R) \leq C_{21} \left(i(\frac{R}{16}) - i(R) \right).$$

Adding the two inequalities yield

$$\omega(R) + \omega(\frac{R}{16}) \leq C_{21} \left(\omega(R) - \omega(\frac{R}{16}) \right)$$

where $\omega(R) = s(R) - i(R)$ denotes the oscillation of u on $B_p(R)$. This implies that

$$\omega(\frac{R}{16}) \leq \gamma \omega(R)$$

for $\gamma = \frac{C_{21}-1}{C_{21}+1} < 1$. Iterating this inequality gives

$$\omega(16^{-k}) \leq \gamma^k \omega(1).$$

Setting $r = 16^{-k}$, we see that u is Hölder continuous with Hölder exponent $-\frac{\log \gamma}{\log 16}$.

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