Generalized surface quasi-geostrophic equations: wellposedness and dynamical properties

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ABSTRACT

In these notes, we shall study the following family of partial differential equations

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp P(\Lambda)\theta, \end{cases}$$

with $\theta(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ and $u(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$, which is commonly referred to as the generalized surface quasi-geostrophic (gSQG) equation. This system models the advection of a scalar θ by an incompressible velocity field u which is determined by θ at each moment of time. Here, $P(\Lambda)$ is a radial Fourier multiplier with $\Lambda = (-\Delta)^{\frac{1}{2}}$, which determines the regularity of the velocity; the most widely studied cases are $P(|\xi|) = |\xi|^{-2}$ and $P(|\xi|) = |\xi|^{-1}$, which correspond to the two-dimensional Euler and surface quasi-geostrophic equations, respectively. Rather recently, there has been increasing interest in the study of the systems (gSQG), as the models exhibit various distinct features depending on the constitutive relation $P(\Lambda)$. Of special interest was the class of so-called α-SQG equations given by $u = \nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta$. It turns out that, there exist *three critical* cases:

- $P(|\xi|) = |\xi|^{-2}$ (2D Euler): critical for global wellposedness
- $P(|\xi|) = |\xi|^{-1}$ (SQG): critical for scaling
- $P(|\xi|) = 1$ (trivial model): critical for local wellposedness.

While there is an extensive body of literature dedicated to the wellposedness issue for the system (gSQG), the results are scattered in many papers mostly written in the past decade. In these notes, we plan to give a rough overview of the local solution theory for the family of equations (gSQG). We also discuss some recent progress in long time dynamics and singularity formation.

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1 Introduction

In these notes, we shall consider the *inviscid generalized surface quasi-geostrophic* models, which are given by

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} P(\Lambda)\theta. \end{cases} \tag{gSQG}$$

Here $\theta(t,\cdot) : \Omega \to \mathbb{R}$ denotes the *active scalar* and $u(t,\cdot) : \Omega \to \mathbb{R}^2$ the velocity, for a given two-dimensional domain Ω which will be usually taken to be either the whole space $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. The operator $\nabla^{\perp}$ is defined by $(-\partial_{x_2}, \partial_{x_1})$ and $P(\Lambda)$ is a multiplier. This is a particular example of an *active scalar equation*, where a transported quantity determines the velocity at each instant of time by means of an operator, which can be considered as a generalization of the "Biot–Savart law" from electromagnetism. Note that the velocity in (gSQG) is always divergence-free, so that the flow is area-preserving. Moreover, the velocity satisfies the slip boundary condition, which means that the fluid particles can slide on the boundary (when it is nonempty). The radial Fourier multiplier $P(\Lambda)$ determines the regularity of the system. The primary goal of these notes is to understand the basic problem of wellposedness and dynamical phenomena for different choices of the operator $P(\Lambda)$.

1.1 Motivation

We shall discuss a few concrete examples belonging to the general class of equations described by (gSQG) and some motivations for the study of the generalizations.

1.1.1 Two-dimensional vorticity equation

The primary example of the gSQG model is the two-dimensional incompressible Euler equations, which read in the *velocity formulation* as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

where $u(t,\cdot) : \Omega \to \mathbb{R}^2$ is the velocity field of an incompressible fluid defined in a two-dimensional region Ω. Here, $\nabla \cdot u = 0$ enforces that the velocity at each moment time is area-preserving. Unlike the compressible fluid equations, the pressure $p(t,\cdot) : \Omega \to \mathbb{R}$ is determined solely by the incompressibility constraint; indeed, taking the

Leray projector $\mathbf{P}$ one can rewrite (1) purely in terms of the velocity:

$$\partial_t u + \mathbf{P}(u \cdot \nabla u) = 0.$$

While this expression is simple, it is rather difficult to deeply understand the operator $u \mapsto \mathbf{P}(u \cdot \nabla u)$. It turns out that much more information about the solution can be obtained from the *vorticity formulation*: introducing $\omega = \nabla \times u$ and simply taking the curl of (1), we obtain

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \quad u = \nabla^\perp \Delta^{-1} \omega. \end{cases} \tag{2}$$

The second equation comes from solving the system of equations $\nabla \cdot u = 0, \nabla \times u = \omega$. The precise form of the operator Δ^{-1} depends on the spatial domain Ω. In the case of $\mathbb{R}^2$, it is given by the convolution against the kernel

$$\mathbf{K}(x) = \frac{1}{2\pi} \ln \frac{1}{|x|}.$$

In the torus case, the kernel is obtained by the periodic version of this kernel. In this case, it is simpler to define Δ^{-1} using Fourier series directly. Now, note that (2) is nothing but (gSQG) for $\omega = \theta$ and $P(\xi) = -|\xi|^{-2}$.

1.1.2 Surface quasi-geostrophic equation

The surface quasi-geostrophic (SQG) equation has been introduced in the mathematical community by Constantin, Majda, and Tabak [38, 37] who emphasized two main purposes of the model. First, it is an approximate two-dimensional model of geostrophic flow, with ability to capture the behavior of *sharp front formation* between boundaries of different air masses. Second, it is an interesting mathematical model for the three-dimensional vorticity equation of incompressible fluid flow, which shares many characteristic features. To begin with, the equation of motion is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ \qquad u = R^\perp \theta \end{cases} \tag{3}$$

where $R^\perp = (-R_2, R_1)$ is the perpendicular of the vector Riesz transform R, which is defined with the multiplier $-|\xi|^{-1} i\xi$ (up to a real constant). That is, $R^\perp = -\nabla^\perp \Lambda^{-1}$, and it simply corresponds to (gSQG) with $P(\xi) = -|\xi|^{-1}$.

To begin with, to explain the analogy with the three-dimensional vorticity dynamics, we recall the vorticity form of the 3D incompressible and inviscid fluid motion:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nabla u \omega, \\ \quad u = \nabla \times (-\Delta)^{-1} \omega. \end{cases} \tag{4}$$

In this equation, $\omega(t,\cdot), u(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ denote the vorticity and velocity of the fluid, respectively. Note that the vorticity (which is now a vector field in 3D) is being advected and stretched by the velocity field, which is self-generated by the relation $u = \nabla \times (-\Delta)^{-1}\omega$. Now, taking the gradient perp of (3) (we write $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$), we see that for the variable $\nabla^\perp\theta$, we have the evolution equation

$$\begin{aligned} \partial_t(\nabla^\perp\theta) + u\cdot\nabla(\nabla^\perp\theta) &= \nabla u(\nabla^\perp\theta), \\ u &= \Lambda^{-1}(\nabla^\perp\theta). \end{aligned}$$

In this form, $\nabla^\perp\theta$ is being advected and stretched by the velocity field in the same way with ω from (4), so that we enjoy the Cauchy formula

$$\nabla^\perp\theta(t,\Phi(t,x)) = \nabla\Phi(t,x)\nabla^\perp\theta_0(x)$$

analogously to the original one for solutions of (4):

$$\omega(t,\Phi(t,x)) = \nabla\Phi(t,x)\omega_0(x)$$

where Φ is the flow map generated by u. Furthermore, if you compare the rules for determining the velocity in these two equations, we see that u is regular exactly by order 1 in both cases; the scaling of the equations coincides. This suggests that (3) could share some dynamical features of the three-dimensional fluid flow, although it is a two-dimensional model. Indeed, further studies began to reveal that there are several fundamental differences between the two. Still, the authors in [37] present the following blow-up criteria which is motivated by the corresponding ones in the 3D Euler case ([6, 36]): if either

$$\int_0^T \|\nabla^\perp\theta(t,\cdot)\|_{L^\infty}dt \quad \text{or} \quad \int_0^T \|\alpha(t,\cdot)\|_{L^\infty}dt$$

is finite, then the solution is regular up to T. Here, α is defined by

$$\frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}\cdot\nabla u\cdot\frac{\nabla^\perp\theta}{|\nabla^\perp\theta|}$$

which measures the stretching rate.

Now, returning to the physical motivation for the study of (3), it should be mentioned that $\theta(t,\cdot)$ denotes the temperature of the fluid, and the evolution equation has been derived as the approximate boundary dynamics of the Boussinesq system defined in the upper half-space with small Rossby and Ekman numbers and constant potential vorticity ([121, pp. 345–368 and 653–670]). In this connection, the most interesting question is whether sharp gradients could develop in time for the level sets of the temperature. Preliminary numerical simulations reported in [37, Fig 1] shows formation of large gradients, but it is still a very challenging open problem to determine whether there is finite-time singularity formation, yet alone the problem of showing large gradient growth. Some rather recent progress on this issue will be given in §3.

At this point, it is worthwhile to mention the scaling criticality of the SQG equations and the mathematical challenges arising from it. One can see the detailed discussion in [70]. As one can see from the relation $u = R^{\perp}\theta$, the velocity field scales in the same way with the active scalar, and it is barely insufficient to control the L^{∞} norm of u using the strongest conservation law available in the equation, which is the L^{∞} norm of θ. Only if the velocity were slightly more regular, one could get hands on the pointwise estimates on the velocity at least near certain steady states, by leveraging the strongest conservation law with a stability result in a weaker norm, say in the L^1 of the active scalar. The pointwise estimate, in turn, gives control on the particle trajectories and the flow map. This is the starting point of the so-called Lagrangian bootstrap scheme.

1.1.3 Alpha-SQG models and assumptions on P

Given the two-dimensional Euler and SQG equations, it is natural to consider the interpolating models

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = -\nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \end{cases} \tag{5}$$

where $\alpha \in \mathbb{R}$. The 2D Euler and SQG equations corresponds to the case $\alpha = 0$ and $\alpha = 1$, respectively. These models correspond to the case $P(\xi) = -|\xi|^{-2+\alpha}$. We shall often write instead

$$u = -\nabla^{\perp}\Lambda^{-2+\alpha},$$

(recall that $\Lambda = (-\Delta)^{\frac{1}{2}}$) or even change the multiplicative constant. One can consider more general P which are in between $-|\xi|^{-2}$ and $-|\xi|^{-1}$, for all sufficiently large ξ. We shall refer to this regime of (gSQG) as **regular**; we can attach a value of $\alpha > 0$ for any regular P as the minimum value satisfying the inequality

$$0 \le -P(\xi) \le |\xi|^{-2+\alpha}$$

for all $|\xi| \ge \Xi_0$ for some Ξ_0. By definition, for the regular gSQG equations, the velocity is not more singular than the advected scalar. In the SQG case (3), the velocity has exactly the same regularity with the scalar, which makes it critical for many arguments. Next, one may *extrapolate* from the cases of the two-dimensional Euler and SQG equations, and consider any real value of α. When it is even more regular than 2D Euler, then the model is wellposed globally in time, using the conservation of the L^{∞} norm of θ.

A more interesting case is when $\alpha > 1$, in which case the velocity is actually more singular than the scalar field. We shall call the regime $1 < \alpha \le 2$ as **intermediate**;

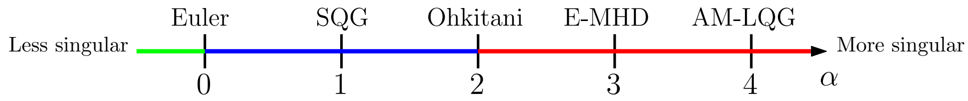

Figure 1: gSQG models

although the velocity is rather singular, it turns out that the equations in this regime share many properties as in the regular case; in particular, local wellposedness for smooth solutions and smooth patches for the α-SQG models in this regime has been proved in [20]. The proofs can be extended to more general multipliers. See Figure 1.

Now, note that when $\alpha = 2$, namely if $u = -\nabla^{\perp}\theta$, then we have

$$u \cdot \nabla\theta \equiv 0,$$

so that the equation becomes trivial. While this cancellation may seem to suggest local wellposedness even in the region when $\alpha > 2$, it turns out that (gSQG) is strongly illposed in this **singular** regime ([23, 22]). Therefore, our studies of the gSQG equations will be mainly focused on the regular and intermediate regimes.

At this point, let us specify some reasonable assumptions that we are going to put on $P(\xi)$:

- P is radial, namely, we can abuse notation and write $P(\xi) = P(|\xi|)$,
- P is infinitely differentiable possibly except at the origin,
- P satisfies
$$|\partial^k P(\xi)| \lesssim_k \langle|\xi|\rangle^{-k}|P(\xi)|$$
for all $k \geq 1$. Here $\langle|\xi|\rangle = \sqrt{1+|\xi|^2}$.

As we shall see, although the gSQG solutions for different velocities have a lot of properties in common, there are interesting analytical and geometrical differences that occur depending on P. In most parts of these notes, we shall focus on the α-SQG case, though.

1.1.4 Logarithmically corrected models

Lastly, there has been considerable interested in the *slightly* regularized/singularized models. Such modified models bring interesting mathematical challenges, for instance in terms of proving local/global wellposedness and propagation of singular structures. To illustrate such models, let us first consider the case of the two-dimensional Euler equations, where $P(|\xi|) = -|\xi|^{-2}$. Here, there are two popular singularizations that have been studied: first, we have loglog Euler

$$u = \nabla^{\perp}\Delta^{-1}\log\log(10-\Delta)\omega, \tag{6}$$

and log Euler

$$u = \nabla^{\perp}\Delta^{-1}\log(10-\Delta)\omega, \tag{7}$$

defined by their respective multipliers. In the log Euler case, there have been interest in the case of the power law

$$u = \nabla^{\perp}\Delta^{-1}\log^{\gamma}(10-\Delta)\omega,$$

with some $\gamma > 0$ as well. For these slightly singularized models, the key question is whether you can extend the global regularity result for the two-dimensional Euler, since this global result is "critical." Next, slightly singularized versions of the SQG equations have been considered mainly in view of the criticality of global regularity for the dissipative system ([88, 89]).

Finally, since the case $\alpha = 2$ is critical for local regularity, one may ask the question of local wellposedness for both slightly regularized and singularized systems. In the log singular case, namely when

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \\ u = \nabla^{\perp}\log(10-\Delta)\theta, \end{cases} \tag{8}$$

It was proved in [20] that this system is locally wellposed with any small power of fractional dissipation. On the other hand in [22] it was established that the system is locally wellposed in a scale of Sobolev spaces with index decreasing in time, without any dissipation. Ohkitani [117, 118] has obtained this model as a rescaled limit of the α-SQG systems with $\alpha \to 2^-$. To see this, consider the velocity laws

$$u^{\varepsilon} = \nabla^{\perp}(-\Delta)^{-\varepsilon}\theta, \qquad \tilde{u}^{\varepsilon} = \nabla^{\perp}((-\Delta)^{-\varepsilon}-1)\theta.$$

Note that

$$\partial_t\theta + u^{\varepsilon}\cdot\nabla\theta = \partial_t\theta + \bar{u}^{\varepsilon}\cdot\nabla\theta$$

simply because $\nabla^{\perp}\theta\cdot\nabla^{\theta} \equiv 0$. Hence, we may rewrite the α-SQG equation with α slightly smaller than 2 as follows:

$$\partial_t\theta + \nabla^{\perp}((-\Delta)^{-\varepsilon}-1)\theta\cdot\nabla\theta = 0.$$

Introducing the rescaled time variable $\tau = \varepsilon t$,

$$\partial_\tau\theta + \nabla^{\perp}\left(\frac{(-\Delta)^{-\varepsilon}-1}{\varepsilon}\right)\theta\cdot\nabla\theta = 0.$$

The multiplier in the large brackets is $(|\xi|^{-2\varepsilon}-1)/\varepsilon$. Fixing some $\xi \neq 0$ and taking the pointwise limit as $\varepsilon \to 0^+$, the multiplier becomes $-c\ln|\xi|$, with some absolute constant c. Therefore, it may be claimed that the "limiting equation" (although in a rescaled time variable) is given by

$$\partial_\tau \theta - \nabla^\perp \ln(-\Delta)\theta \cdot \nabla\theta = 0.$$

Ohkitani even conjectured global regularity (for some special initial data) for this model, based on numerical simulations.

Since Ohkitani's model is illposed in standard function spaces, we claim that it is more natural to consider the *logarithmically regularized* equation as the critical limit of α-SQG equations as $\alpha \to 2$:

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta = 0, \\ u = \nabla^\perp \log^{-1}(10 - \Delta)\theta. \end{cases} \tag{9}$$

This can be seen as the borderline equation which barely belongs to the intermediate regime, and we expect many interesting phenomena to occur for the solutions.

1.1.5 Examples of singular SQG equations

There are a few places in physics where equations of the form (gSQG) with more singular multiplier P show up.

Asymptotic model in the LQG equation. A most notable example is the so-called large-scale quasi-geostrophic (LQG) equation, which can be written in the form

$$\partial_t \psi + \nabla^\perp \Delta\psi \cdot \nabla\psi = 0. \tag{10}$$

This corresponds to the case $P(|\xi|) = -|\xi|^2$, and a very interesting feature of this model is that it is purely local, namely the equation does not consist of any non-local operators. To explain the physical relevance of this model, we consider the dynamics of shallow water QG (quasi-geostrophic) flow, which accounts for the ocean front dynamics. Some concrete physical situations include Jupiter's atmospheric pattern and Great Red Spot ([125, 121]). The governing equation is the so-called Charney–Hasegawa–Mima (CHM) equation

$$\partial_t q + \nabla^\perp \psi \cdot \nabla q = 0.$$

Here, the advected variable q denotes the *potential vorticity* (PV) and is related by the stream function ψ by $q = (\Delta - L_D^{-2})\psi$. Note that this is nothing but the gSQG equation with the velocity defined by $u = \nabla^\perp(\Delta - L_D^{-2})^{-1}q$. In this formulation, the length scale L_D denotes the *Rossby deformation length*, which roughly speaking captures the ratio between the action of gravity and large-scale planetary rotation. Relevance of the CHM equation in plasma dynamics is discussed in [69]. Letting L be the characteristic length of the flow, one may consider two limiting situations where (i) $L \ll L_D$ and (ii) $L \gg L_D$. In the first scenario, the flow is essentially governed by the 2D Euler dynamics. On the other hand, in the second scenario, formally we obtain the model (10) in the rescaled timescale tL_D^2. This is sometimes referred to as the asymptotic model (AM), see [12].

Hall–MHD system. For yet another singular example, we may consider the Hall–magneto-hydrodynamics (Hall–MHD) equations (without dissipation):

$$\begin{cases} \partial_t B + \nabla \times ((\nabla \times B) \times B) = 0, \\ \nabla \cdot B = 0, \end{cases} \tag{11}$$

where $B(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the magnetic field. Strictly speaking, this is the "electron part" of the full Hall–MHD system, see [120]. To see the relation with the family of gSQG systems, one may take the so-called "$2 + \frac{1}{2}$-dimensional ansatz": assuming magnetic field B of the form

$$B = \nabla\psi \times e_3 + \Lambda[\phi]\, e_3$$

for some scalars ψ, ϕ independent of the last coordinate x_3, the above system reduces to

$$\begin{cases} \partial_t \psi + \nabla^\perp \Lambda[\phi] \cdot \nabla\psi = 0, \\ \partial_t \phi + \Lambda^{-1}(\nabla^\perp \Lambda(\Lambda[\psi]) \cdot \nabla\psi) = 0. \end{cases} \tag{12}$$

While this system can be viewed as a "vector version" of gSQG, one can simplify it even further: taking the ansatz $\psi \simeq \phi$ (which roughly propagates in time, assuming existence of a smooth solution), we see that the limiting equation is

$$\partial_t \psi + \nabla^\perp \Lambda[\psi] \cdot \nabla\psi = 0,$$

which simply corresponds to (gSQG) with $P(|\xi|) = |\xi|$.

1.1.6 Active scalars

In this section we shall introduce various model equations which have structural similarities with the gSQG equations. For each model, we shall focus on the key features that distinguishes it from the others. Given some statement regarding the gSQG equations, it will be an interesting problem to investigate to which generality such a statement extends. Before we proceed, we shall refer to the following class of transport equations as *active scalar* systems:

$$\begin{cases} \partial_t \theta + u \cdot \nabla\theta = 0, \\ \qquad\quad u = \mathbf{K}[\theta]. \end{cases} \tag{13}$$

It is assumed that at each moment of time, u is well-defined in terms of θ by some relation θ. Of course one can consider more general classes of equations than (13);

one can replace θ by some vector/matrix/tensor (system of active scalars), have multiple velocities, and so on. As in the case of gSQG equations, we shall be interested in incompressible models, as the behavior of the solutions are very different in the compressible case. Then, the velocity has the form

$$u = \nabla^{\perp}\psi,$$

where ψ is determined from θ by some functional relation $\psi = \mathbf{G}[\theta]$. A recent textbook [35] discusses wellposedness theory of several active scalar models.

Incompressible porous media. The incompressible porous media (IPM) equation and its variants have been widely studied. In two spatial dimensions, the equations read

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ u = \partial_{x_1} \nabla^{\perp} (-\Delta)^{-1} \rho. \end{cases} \tag{14}$$

Here, ρ denotes the density of some medium, which is being affected by gravity acting in the negative x_2 direction. This makes the equation anisotropic, unlike the gSQG equations which enjoys rotational invariance. Furthermore, a peculiar property of this model is that, while it is inviscid, it has a monotone decreasing quantity. To see this, compute

$$\begin{aligned} \frac{d}{dt} \int x_2 \rho dx &= - \int x_2 \nabla \cdot (u\rho) dx \\ &= \int u_2 \rho dx = \int \partial^2_{x_1 x_1} (-\Delta)^{-1} \rho \, \rho dx \\ &= - \int |\partial_{x_1} (-\Delta)^{-\frac{1}{2}} \rho|^2 dx \le 0. \end{aligned}$$

In these computations, it is assumed that ρ is sufficiently smooth and decaying at infinity. Note that the integral

$$I = \int x_2 \rho dx$$

is nothing but the gravitational potential energy of the medium. Decrease of I shows intrinsic stabilizing nature of the system: heavier regions of the fluid tends to sink while lighter tends to rise. This system is very interesting as there is coexistence of asymptotic stability ([54, 18]) and infinite-time infinite gradient growth ([96]).

One can note that IPM is naturally associated with the following 2D Boussinesq type system, where ρ satisfies the transport equation as in the above and u satisfies

$$\partial_t u + u \cdot \nabla u = -\nabla p + \kappa u + (0, -g\rho)^T. \tag{15}$$

Here $\kappa, g > 0$ are constants. Taking these constants to be 1 and setting the left hand side to be zero results in the velocity formula in (14).

Vlasov–Poisson equation. The Vlasov–Poisson (VP) system is a popular model in both astrophysics and plasma dynamics. We consider the simplest case of 1×1 dimensional VP. The dependent variable is the distribution function $f(t,x,v) : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which we assume to be non-negative. The system reads

$$\begin{cases} \partial_t f + v\partial_x f + u\partial_v f = 0, \\ \rho(t,x) = \int f(t,x,v)dv, \\ \quad u(t,x) = \pm\partial_x \Delta^{-1}\rho. \end{cases} \tag{16}$$

In the above, the choice of sign for the velocity u is determined by whether the particle interaction is repulsive or attractive. This model is *incompressible* in the sense that in the phase space $\mathbb{R} \times \mathbb{R}$, the phase velocity (v,u) is incompressible as

$$\partial_x(v) + \partial_v(u) \equiv 0.$$

In particular, all the $L^p_{x,v}$ norms of f are conserved in time, assuming some regularity and decay of f:

$$\|f(t,\cdot)\|_{L^p(\mathbb{R}\times\mathbb{R})} = \|f_0\|_{L^p(\mathbb{R}\times\mathbb{R})}.$$

Unlike the other models that we have seen, it is easy to prove global existence and uniqueness of smooth solutions for (16). A peculiar property of this model is the *velocity smoothing effect*: the density function ρ is smoother (in an averaged sense) than it is naively expected. Note that the density satisfies the equation

$$\partial_t \rho + \partial_x \left(\int v f dv \right) = 0.$$

See [68] for the precise statement and proof for the smoothing effect. A simple way to see the smoothing effect is to solve the linear transport equation

$$\partial_t f + v\partial_x f = 0,$$

with the initial data $f_0(x,v) = \mathbf{1}_{[0,1]^2}(x,v)$. One very interesting topic is to understand what happens for "very singular" data; see the last chapter of [108]. Many variations of the VP systems have been studied, which can be obtained either by modifying the relation between u and ρ or by coupling VP with other systems.

Oldroyd-B system. The Oldroyd-B system in two spatial dimensions describes the evolution of a symmetric matrix of added stress:

$$\begin{cases} \partial_t \sigma + u \cdot \nabla\sigma = (\nabla u)\sigma + \sigma(\nabla u)^T, \\ \qquad -\Delta u + \nabla p = \nabla \cdot \sigma, \\ \qquad \nabla \cdot u = 0. \end{cases} \tag{17}$$

Here $\sigma(t,\cdot)$ is a symmetric 2×2 matrix-valued function, which introduces a significant complication compared with the scalar-valued PDEs. As in the case of the IPM equation, the major open problem is the global regularity of smooth solutions. This system can be obtained by a closure of a kinetic model ([102, 35]).

Transport-Stokes system. The transport-Stokes system appears in sedimentation theory. The main variable ρ denotes the density of suspensions, and satisfies

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \Delta u + \nabla p = \rho\, e_3, \\ \qquad \nabla \cdot u = 0. \end{cases} \tag{18}$$

Here, $\rho(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ and $u(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$. We use e_3 to denote the vector $(0,0,1)^T$. The name of the equation is self-explanatory; the density is being transported by the (incompressible) velocity which satisfies the steady Stokes equation. Justification of the model and wellposedness is given in [112, 103, 73, 5, 113].

Axisymmetric Euler and lake equations. Given a two-dimensional domain Ω and a given "depth" function $b : \Omega \to \mathbb{R}_+$, the lake equation is given by

$$\partial_t \left(\frac{\omega}{b}\right) + u \cdot \nabla \left(\frac{\omega}{b}\right) = 0, \tag{19}$$

where u is the solution of

$$\nabla \cdot (bu) = 0, \quad \nabla \times u = \omega. \tag{20}$$

When $\partial\Omega$ is nonempty, one needs the boundary condition $u \cdot n = 0$ on $\partial\Omega$ where n is the outerward unit normal vector ([104, 105]). When the depth function takes the form $b(x) = |x|$, (19) can be identified with the three-dimensional axisymmetric Euler equations without swirl. In the case $b(x_1, x_2) = x_1 x_2$, this is related with the so-called birotational Euler equations ([31]).

Magnetic relaxation equation. The two-dimensional magnetic relaxation equations (MRE) read

$$\begin{cases} \partial_t B + u \cdot \nabla B = B \cdot \nabla u, \\ (-\Delta)^\gamma u = B \cdot \nabla B + \nabla p. \end{cases} \tag{21}$$

Here, B, u are time-dependent vector fields on a two-dimensional domain Ω and $\gamma \in \mathbb{R}$ is a parameter. What is very interesting in (21) is that the velocity is *quadratic* in terms of the magnetic field B. In the special case $\gamma = 0$, the kinematic relation is equivalent with $\omega = B \cdot \nabla j$, where $\omega = \nabla \times u$ is the vorticity of the fluid and $j = \nabla \times B$ is the current density of the magnetic field. These equations were used to study the process of magnetic relaxation in the ideal magnetohydrodynamics (MHD) model ([7, 114, 9, 86]). This makes the study of long-time dynamics of (21) particularly important, although already the problem of global existence of solutions is a challenging problem, especially when $\gamma = 0$.

1.2 Overview

We shall begin with a general discussion about the first questions that arise, given a partial differential equation. The rest of these notes is mostly about various attempts in answering those basic questions in the very specific context of generalized surface quasi-geostrophic equations.

Notion of wellposedness. Given a partial differential equation which describes the time evolution of a quantity, the very first question that one needs to answer is that of local wellposedness (or local regularity); namely, if we are given an initial datum, then whether there is a corresponding solution to the PDE, at least for a small interval of time. This is a delicate question: even when the PDE is "nice", the answer usually depends on the precise choice of the function space. To be more precise, assume that we are given a PDE

$$\partial_t \theta = \mathbf{T}[\theta] \tag{22}$$

where $\mathbf{T}$ is a partial differential operator, possibly nonlinear. Let X be a Banach space of functions and Y be a time-dependent Banach function space. That is, Y consists of functions which are dependent on both time and space. Then, we say that (22) is locally wellposed from X to Y, in the sense of Hadamard, if

- Existence: for any $\theta_0 \in X$, there exist a solution $\theta \in Y$ such that $\theta(t=0) = \theta_0$.
- Uniqueness: there is at most one solution that belongs to Y with the prescribed initial data θ_0.
- Continuous dependence: the solution map (which is well-defined when the previous two conditions are satisfied) is continuous.

The first two items are reasonable: we would like to be guaranteed the unique existence of a solution in the future, if we know its initial state exactly. Note that existence does not imply uniqueness and vice versa. Indeed, the definition of uniqueness does not require existence. The third requirement, which is somewhat more delicate to prove, is natural especially from the practical point of view. If the PDE indeed gives a good description of a physical process, then we would like to be able to know what happens to the physical system at a later time, given the current measurements of the system. This is the primary motivation for the study of the initial value problem. Assume that we have existence, uniqueness, and some way of computing the solution at a later time, for some PDE. Still, if the solution does not depend continuously on the initial data, then it is very unlikely that the computed solution gives a good approximation of the future state, especially since it is impossible to exactly measure the current state. In practice, one would like to have much stronger notion of dependence on the initial data.

For transport equations including the gSQG equations, it is usually expected that the regularity of the solution does not change with time. (The equations are "time reversible.") Indeed, no matter how smooth the advecting velocity is, "singular features" in θ will remain essentially unchanged in time. Therefore, once the choice of the initial data space X is made, it is customary for transport type equations to take Y to be either $L^\infty([0,T];X)$ or $C([0,T];X)$. Then, we can be more specific about the above requirements of Hadamard:

- Existence: for any $\theta_0 \in X$, there exist some $T > 0$ and a solution θ to (22) belonging to $C([0,T);X)$ satisfying $\theta(t=0) = \theta_0$.
- Uniqueness: let θ_1 and θ_2 be two solutions to (22) belonging to $C([0,T_1];X)$ and $C([0,T_2];X)$, respectively. Assume further that $\theta_1(t=0) = \theta_2(t=0)$. Then, we actually have that $\theta_1 \equiv \theta_2$ on $[0, \min\{T_1, T_2\}]$.
- Continuous dependence: Let θ be a solution to (22) belonging to $C([0,T];X)$ for some $T > 0$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\tilde{\theta}_0$ satisfies $\|\tilde{\theta}_0 - \theta(t=0)\|_X < \delta$, then there is a solution $\tilde{\theta} \in C([0,T];X)$ to (22) satisfying $\tilde{\theta}(t=0) = \tilde{\theta}_0$ and we have $\|\tilde{\theta} - \theta\|_{C([0,T];X)} < \varepsilon$.

While the precise notion of wellposedness highly depends on the choice of the function space, when someone says that a PDE is wellposed without explicitly specifying the space, then usually that person has in mind "sufficiently nice" function spaces, most traditional examples being H^∞ and H^ω. Here H^∞ is by definition the space of infinitely differentiable functions for which any derivative is square integrable in space. The space H^ω consists of real analytic functions with square integrable derivatives. When the domain is bounded, these are simply identified with C^∞ and C^ω, respectively.

Next, we mention that sometimes we are forced to use the space $L^\infty([0,T];X)$ rather than $C([0,T];X)$. Even in that case, we can require the solution to belong to $C_w([0,T];X)$ where C_w means continuity in some "weak" topology of X. This ensures that the time restriction $\theta(t=0)$ is well-defined. For an example, we consider the linear transport equation in $\mathbb{R}$:

$$\partial_t \theta + \partial_x \theta = 0.$$

If the initial data θ_0 is smooth and decays at infinity, it can be proved that $\theta(t,x) = \theta_0(x-t)$ is the unique smooth solution. Note that even when θ_0 is merely a bounded function (not necessarily differentiable or even continuous), the formula $\theta_0(x-t)$ defines a bounded function for all time, which solves the linear transport equation in the sense of distributions; namely, if $\varphi \in C_c^\infty(\mathbb{R}^2)$ is a *test function*, then

$$\iint \theta \partial_t \varphi + \theta \partial_x \varphi dx dt = 0.$$

That is, in this case the derivatives on θ are interpreted in the distributional sense. Furthermore, it can be proved that $\theta_0(x-t)$ is the only bounded function with this property. Therefore we would like to say that this transport equation is wellposed in $X = L^\infty$, but we see that in general the solution is not continuous in time in the (norm) topology of L^∞. In this case, we are forced to use the weak topology in L^∞.

Now assume that we are given a PDE which is wellposed in the above sense. Then, this gives rise to the notion of *maximal lifespan* of a solution. Given an initial data θ_0, we can always uniquely associate $T^* \in (0,\infty]$ such that the corresponding unique solution exists in $C([0,T^*);X)$ but not in any larger time interval (unless $T^* = \infty$). This is simply because if we are given a solution θ in $C([0,T);X)$ and the limit $\lim_{t\to T}\theta =: \theta(T)$ exists, then we can we can solve the PDE starting from the new initial data $\theta(T)$, thereby obtaining a solution in $[0, T+\epsilon)$ for some $\epsilon > 0$. Of course this argument requires that the operator defining the PDE is time-independent.

Let us discuss some consequences of continuous dependence. Usually, the *trivial solution* (or zero solution) $\theta \equiv 0$ is a solution to (22). Since it is defined globally in time, using continuous dependence with respect to the trivial solution, we obtain that for any $T, \varepsilon > 0$, there exists $\delta > 0$ such that if θ_0 satisfies $\|\theta_0\|_X < \delta$ then the corresponding solution θ exists for $[0,T]$, and satisfies

$$\|\theta\|_{C([0,T];X)} < \varepsilon.$$

That is, small initial data are guaranteed to live for a long time interval. If the PDE at hand has a scaling symmetry, this can be often applied to obtain a lower bound on the maximal lifespan of large initial data. We shall discuss this in the concrete examples of α-SQG equations later. Next, in most cases, the way to obtain wellposedness in a given function space X is to derive an *a priori bound* of the form

$$\frac{d}{dt}\|\theta\|_X \le F(\|\theta\|_X). \tag{23}$$

Here $F : [0,\infty) \to [0,\infty)$ is a non-decreasing continuous function satisfying $F(0) = 0$. We use the term "a priori" since at the stage of deriving the inequality, we actually do not have existence of a solution; we *assume* that a solution exists in $C([0,T];X)$ (or even in C^∞) and then prove that the bound holds. Still, it is usually the first step towards establishing the existence. Assuming that we are given (23), we see that $\|\theta(t)\|_X \le Y(t)$, where Y is the solution to the ODE

$$\begin{cases} \dot{Y}(t) = F(Y(t)), \\ \ Y(0) = \|\theta_0\|_X. \end{cases} \tag{24}$$

In particular, we see that if the solution to this ODE exists in $[0,T)$ then the PDE solution must exist as well. Furthermore, this shows that if $T^* < \infty$ is the maximal lifespan of a solution, then we must have

$$\liminf_{t\to T^*} \|\theta(t)\|_X = \infty. \tag{25}$$

Indeed, assume towards a contradiction that we can find an increasing sequence $\{t_k\}$ converging to T^* such that $\|\theta(t_k)\|_X \leq C$ for some C independent of k. Then, applying the comparison principle with (24) with initial time t_k shows that the solution must exist for a time interval of the form $[t_k, t_k + \epsilon)$ with ϵ depending only on C, so that by taking k sufficiently large, we obtain a contradiction to the statement that T^* is the maximal lifespan. A statement like (25) is called a *blow-up criterion* since it is equivalent to having the solution in X becoming nonexistent exactly at time T^*. There is significant interest in finding the optimal blow-up criterion; clearly, a criterion which involves a weaker quantity is stronger. The task of finding the sharp criterion is essentially equivalent with understanding precisely what happens at a potential blow-up moment. A blow-up criterion can be called as a *continuation criteria* or a sufficient condition for regularity propagation, simply because if we are given a solution θ in $C([0, T^*); X)$ and it satisfies

$$\liminf_{t \to T^*} \|\theta(t)\|_X < \infty \tag{26}$$

then we are guaranteed to extend the solution past T^*. Ideally, one would like to have that, in a nonlinear PDE, there is a (family of) *critical* norm Z, such that it is a strictly weaker quantity than the norm X but provides a sharp continuation criterion for the solution. One may naively think that the quantity Z should correspond to the strongest coercive conservation law in a PDE but usually it turns out to be not the case.

Dynamical properties. Given a partial differentiable equation which is locally wellposed in some space X and an initial datum $\theta_0 \in X$, we see that there are two cases, either

- (i) the solution is global in time;
- (ii) there is finite time blow-up.

To begin with, when there is a non-empty set of initial data in X for which there is blow-up, then a natural question is to understand what are some simple and natural characterizations of the set of initial data. Next, it is an important question to ask what is the *asymptotics of blow-up*, namely what happens to the solutions very close to the blow-up time. This is interesting since in many equations, there is some (rather explicit) universal profile appearing on solutions near singularity, after a proper rescaling in space. Furthermore, a more ambitious problem is to figure out how to extend the solution after the time of singularity (is there life after death?). On the other hand, when the set of initial data leading to finite time blow-up is empty, we can say that the PDE is globally wellposed in the space X. In this case, the most interesting problem is to figure out what happens to the solution in the long time limit $t \to \infty$. Sometimes it is hoped (or believed) that there are a few special solutions to the PDE upon consideration towards which other solutions converge as $t \to \infty$. This type of questions on *long time dynamics* are notoriously difficult to answer for many PDEs featuring non-local interactions. Therefore, rather than trying to answer long time behavior directly, one can start by looking for the candidates which could be responsible for behavior of other solutions as $t \to \infty$. This is usually the motivation for the study (classification, characterization, etc) of *steady states*, which are simply solutions to (22) for which $\partial_t \theta \equiv 0$; in other words, they are given by solutions to the differential equation $\mathbf{T}[\theta] \equiv 0$. While this equation may look very simple, when $\mathbf{T}$ is nonlinear and non-local, it is surprisingly difficult to understand the set of steady solutions; it could be huge and has a complicated structure. Furthermore, one can try to look for the next simplest solutions, which are time-periodic solutions.

A case study: inviscid Burgers' equation. We would like to provide a non-trivial yet simple example and investigate how the discussion above applies. To this end we introduce the one-dimensional inviscid Burgers' equation:

$$\partial_t \theta + \theta \partial_x \theta = 0. \tag{27}$$

Here, $\theta(t, \cdot)$ is a real-valued function defined either on $\mathbb{R}$ or $\mathbb{T}$. This is one of the simplest PDEs where blow-up (singularity formation) of smooth solutions occurs in finite time. To begin with, let us discuss local wellposedness. For now, let us take $X = C^1(\mathbb{T})$ although there are many other natural options. Then, taking the derivative of (27) we obtain

$$\partial_t(\partial_x \theta) + \theta \partial_x(\partial_x \theta) = -(\partial_x \theta)^2. \tag{28}$$

Let us now proceed by assuming that we are given a solution $\theta \in C([0,T); C^1(\mathbb{T}))$. In the following argument, the time variable will be restricted to $[0,T)$. Then, we can introduce the flow map Φ on $[0,T)$ satisfying

$$\frac{d}{dt}\Phi(t,x) = \theta(t, \Phi(t,x)), \qquad \Phi(0,x) = x.$$

It can be shown that on $[0,T)$, $\Phi(t,\cdot) : \mathbb{T} \to \mathbb{T}$ is well-defined and a C^1 diffeomorphism of $\mathbb{T}$. (This simply means that it is bijective and the inverse map is also C^1.) Then, writing (28) *along the flow*, we obtain

$$\frac{d}{dt}(\partial_x \theta)(t, \Phi(t,x)) = -(\partial_x \theta)^2(t, \Phi(t,x)).$$

In this equation, we consider $x \in \mathbb{T}$ as fixed and describe the time evolution of the quantity $(\partial_x \theta)(t, \Phi(t,x))$, which is a function of t only. We are assuming that $\partial_x \theta \in C^0$, so that $\partial_x \theta(t, \Phi(t,x)) \in C^0$ for each $t \in [0,T)$. For simplicity, let us define the C^1 norm of $\theta(t,\cdot)$ by $\max_{x \in \mathbb{T}} |\partial_x \theta(t,x)|$. Then, we arrive at

$$\frac{d}{dt}\|\theta\|_{C^1} \le \|\theta\|_{C^1}^2. \tag{29}$$

This a priori estimate can be used to prove local-in-time existence of a solution, on the time interval within which the unique solution to the ODE

$$\dot{Y} = Y^2, \qquad Y(0) = \|\theta_0\|_{C^1}$$

does not become infinite. This ODE is explicitly solvable, with

$$Y(t) = \frac{1}{\|\theta_0\|_{C^1}^{-1} - t},$$

which gives, by the comparison principle for ODE solutions,

$$\|\theta(t,\cdot)\|_{C^1} \leq Y(t) = \frac{1}{\|\theta_0\|_{C^1}^{-1} - t}.$$

Therefore, given an initial data θ_0 belonging to C^1, a unique C^1 solution is guaranteed in the time interval $[0, \|\theta_0\|_{C^1}^{-1})$. Note that the length of this interval is inversely proportional to the initial data size $\|\theta_0\|_{C^1}$, which is natural since intuitively large solutions are more dynamic. Before we proceed further, let us discuss briefly what happens if we change the function space, say to $C^\infty(\mathbb{T})$. In principle, it is possible that we can be guaranteed with a longer time of existence by changing the function space. In the case of the inviscid Burgers' equation, it is not the case and indeed, the quantity $\|\theta\|_{C^1}$ actually provides a blow-up criterion. To illustrate this point we return to (29) and rewrite it as follows:

$$\frac{d}{dt}\ln(\|\theta\|_{C^1}) \leq \|\theta\|_{C^1}.$$

Integrating in time from 0 to some t satisfying $t < T$, we obtain that

$$\ln\left(\frac{\|\theta(t)\|_{C^1}}{\|\theta_0\|_{C^1}}\right) \leq \int_0^t \|\theta(\tau,\cdot)\|_{C^1} d\tau$$

or in other words,

$$\|\theta(t,\cdot)\|_{C^1} \leq \|\theta_0\|_{C^1} \exp\left(\int_0^t \|\theta(\tau,\cdot)\|_{C^1} d\tau\right).$$

Therefore, we see that

$$\int_0^T \|\theta(\tau,\cdot)\|_{C^1} d\tau < \infty$$

is a continuation criterion; if it holds, then the unique solution is guaranteed to be continue-able past time T. Let us take initial data $\theta_0 \in C^\infty(\mathbb{T})$. By definition, $C^\infty(\mathbb{T}) = \cap_{k\geq 0} C^k(\mathbb{T})$ and to show that the solution belongs to C^∞, we need to estimate the C^k norm for all $k \geq 0$. For simplicity, let us denote

$$\|f\|_{C^k} := \sup_{x\in\mathbb{T}} |\partial_x^{(k)} f(x)|.$$

Assuming that there exists a C^∞ solution of (27) in a time interval $[0,T)$, to obtain a C^2 estimate we differentiate (28) once more:

$$\partial_t(\partial_x^2\theta) + \theta\partial_x(\partial_x^2\theta) = -3(\partial_x\theta)(\partial_x^2\theta). \tag{30}$$

Along the flow, the equation is simply

$$\frac{d}{dt}(\partial_x^2\theta)(t,\Phi(t,x)) = -3((\partial_x\theta)(\partial_x^2\theta))(t,\Phi(t,x)).$$

Taking absolute values, we obtain that

$$\frac{d}{dt}\|\theta(t,\cdot)\|_{C^2} \leq 3\|\theta(t,\cdot)\|_{C^1}\|\theta(t,\cdot)\|_{C^2},$$

which shows that

$$\|\theta(t,\cdot)\|_{C^2} \leq \|\theta_0\|_{C^2}\exp\left(3\int_0^t \|\theta(\tau,\cdot)\|_{C^1}d\tau\right).$$

We claim that any higher C^k norm obeys a similar inequality. To demonstrate this in the case $k=3$, we differentiate (30) to obtain

$$\partial_t(\partial_x^3\theta) + \theta\partial_x(\partial_x^3\theta) = -4(\partial_x\theta)(\partial_x^3\theta) - 3|\partial_x^2\theta|^2. \tag{31}$$

Writing along the flow and using the previous bound for the C^2 norm gives

$$\frac{d}{dt}\|\theta(t,\cdot)\|_{C^3} \leq 4\|\theta(t,\cdot)\|_{C^1}\|\theta(t,\cdot)\|_{C^3} + \|\theta_0\|_{C^2}^2\exp\left(6\int_0^t \|\theta(\tau,\cdot)\|_{C^1}d\tau\right).$$

Then one can integrate this inequality in time, to obtain a bound for $\|\theta(t,\cdot)\|_{C^3}$. A slightly more elegant way is to return to (31) and apply the Landau–Kolmogorov inequality

$$\|f\|_{C^2}^2 \leq C\|f\|_{C^1}\|f\|_{C^3}.$$

This gives, again after composing with the flow map,

$$\frac{d}{dt}\|\theta(t,\cdot)\|_{C^3} \leq C\|\theta(t,\cdot)\|_{C^1}\|\theta(t,\cdot)\|_{C^3}$$

which then immediately gives

$$\|\theta(t,\cdot)\|_{C^3} \leq \|\theta_0\|_{C^3}\exp\left(C\int_0^t \|\theta(\tau,\cdot)\|_{C^1}d\tau\right).$$

Here is the conclusion: if the initial data θ_0 belongs to C^k for some $k\in\mathbb{N}$, the local-in-time solution in $C([0,T);C^k(\mathbb{T}))$ can be continued past T *if and only if*

$\int_0^T \|\theta(\tau,\cdot)\|_{C^1} d\tau < \infty$. The important observation here is that this continuation criterion is completely independent of k. In particular, we obtain local unique solution in the space $X = C^\infty(\mathbb{T})$.

Next, in the case of Burgers' equation, there is a simple way to tell that the estimate (29) is *sharp*, namely that the inequality is actually achieved for some solutions. To this end, given some non-constant initial data (any constant function defines a steady state) $\theta_0 \in C^\infty(\mathbb{T})$, let us assume that we can pick a point $x^* \in \mathbb{T}$ such that

$$-\partial_x \theta_0(x^*) = \max_{x\in\mathbb{T}} |\partial_x \theta_0| = \|\theta_0\|_{C^1}.$$

The unique local-in-time smooth solution is guaranteed in the time interval $[0, T^*)$ where $T^* := 1/\|\theta_0\|_{C^1}$. Now, on this time interval, we consider the flow map and set $x(t) := \Phi(t, x^*)$ for simplicity. Then, (28) gives

$$\frac{d}{dt}(-\partial_x \theta(t, x(t))) = (-\partial_x \theta(t, x(t)))^2.$$

This *equality* not only guarantees that $-\theta(t, x(t)) > 0$ for $t \in [0, T^*)$ but also

$$\|\theta(t,\cdot)\|_{C^1} \geq -\partial_x \theta(t, x(t)) = \frac{1}{\|\theta_0\|_{C^1}^{-1} - t}.$$

Comparing this with the upper bound for $\|\theta(t,\cdot)\|_{C^1}$, we actually conclude that

$$\|\theta(t,\cdot)\|_{C^1} = \frac{1}{\|\theta_0\|_{C^1}^{-1} - t}$$

exactly holds for any $t \in [0, T^*)$. In particular, the solution blows up exactly at time T^*! For the Burgers' equation, we can see more or less exactly what is happening at the blow-up time. To simplify the analysis, we consider $x^* = 0$: namely $-\partial_x \theta_0(0)$ is the maximum value of $\partial_x \theta_0$. From this assumption, we have that the second derivative satisfies $\partial_{xx}\theta_0(0) = 0$. Next, generically, the third derivative of θ_0 at 0 will not vanish, and the previous assumption gives that $\partial_{xxx}\theta_0(0) > 0$. This gives

$$\theta_0(x) = -ax + bx^3 + o(x^3),$$

for some positive constants $a, b > 0$. We shall assume for simplicity that $a = 1$; this gives that the blow up time is exactly $t^* = 1$. Now that the flow map is simply

$$\Phi(t, x) = x + t\theta_0(x),$$

we see that the solution is

$$\theta(t, x + t\theta_0(x)) = \theta_0(x).$$

At $t^* = 1$,

$$-x + bx^3 + o(x^3) = \theta(1, x + (-x + bx^3 + o(x^3))).$$

In other words,

$$-x + o(x) = \theta(1, bx^3 + o(x^3)).$$

Change of variables $z = bx^3 + o(x^3)$ gives that (this is invertible for x sufficiently small)

$$\theta(1, z) = -b^{-\frac{1}{3}} z^{\frac{1}{3}}.$$

That is, the Burgers' equation *generically* develops a $C^{1/3}$-cusp at the time of singularity. While the L^∞ norm of θ is conserved from the transport nature of the equation, somehow some C^α norms remain bounded uniformly up to the time of blow-up. Furthermore, one may prove that the solution develops a *self-similar* cusp near the blow-up time: in the concrete case of the data

$$\theta_0(x) = -x + x^3$$

defined in $\mathbb{R}$, for each $0 \le t < 1$, we can explicitly calculate the flow map inverse $A = A(t, x) = \Phi^{-1}(t, x)$ via

$$A - t(A - A^3) = A + t\theta_0(A) = x$$

which gives

$$tA^3 + (1 - t)A - x = 0.$$

The unique real solution is then given by

$$A(t, x) = \sqrt[3]{-\frac{x}{2t} + \sqrt{\frac{x^2}{4t^2} + \frac{1}{27}\left(\frac{1-t}{t}\right)^3}} - \sqrt[3]{-\frac{x}{2t} + \sqrt{\frac{x^2}{4t^2} + \frac{1}{27}\left(\frac{1-t}{t}\right)^3}}.$$

We can then explicitly obtain

$$\theta(t, x) = \frac{\theta_0(A(t, x))}{1 + t(-1 + 3A^2(t, x))} \simeq \frac{1}{1-t} F\left(\frac{x}{(1-t)^{\frac{3}{2}}}\right),$$

where

$$F(z) = \frac{1}{3 + z^{\frac{2}{3}}}.$$

Then, we can ask whether there is a way to continue the solution after this blow-up time. It turns out that for this equation it is possible: while there are infinitely many distributional solutions after the blow-up time, there is one (and only one) which is "physically relevant." One can refer to some standard texts in PDE [62] for the proof.

Problem 1.2.1. *A solution to* (27) *is called self-similar if it has the form*

$$\theta(t,x) = g(t)F(h(t)x)$$

for some functions g, h, F *of one variable.*

- *Find all* C^∞*-smooth self similar solutions.*
- *Find all (locally integrable) self-similar solutions.*
- *Discuss their relevance for the initial value problem of general initial data.*

One can first consider the above questions for the simpler case of Riccati equation (considered as a PDE rather than an ODE)

$$\partial_t f = f^2 \tag{32}$$

instead of (27).

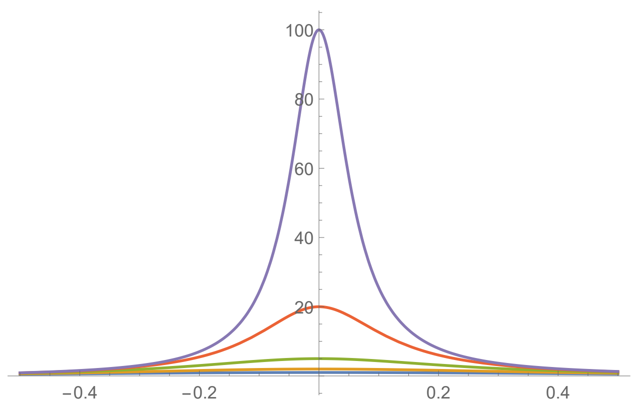

Figure 2: Solutions to (32) at $t = 0, 0.5, 0.8, 0.95, 0.99$

For smooth initial data $f_0(x) : \mathbb{R} \to \mathbb{R}_+$, the explicit solution formula is given by $f(t,x) = \frac{f_0(x)}{1-tf_0(x)}$. We may assume, essentially without loss of generality, that the maximum point of the initial data is $x = 0$, with the maximum value equal to 1. Furthermore, assume that the second derivative of f_0 at $x = 0$ is nonzero, which is generically true. One can see that the solution blows up at $t = 1$ (Figure 1.2).

Then, one can perform height normalization: $f \mapsto \frac{f(t,x)}{\max_x(f(t,x))}$. Then this forces the solution to have the maximum value equal to 1 for all times (Figure 1.2, left).

One sees that the functions become "sharper" as one approaches the blow-up time. This motivates us to consider fatness normalization: that is, we can always rescale the independent variable in a way that the Hessian (which is assumed to be nonzero) is normalized: $g \mapsto g(t, (-(\partial_{xx}g)(t,0))^{-\frac{1}{2}}x)$.

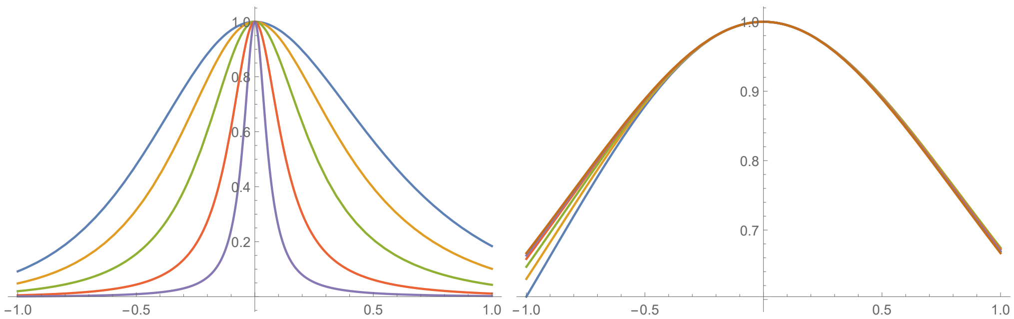

Figure 3: Solutions to (32) at $t = 0, 0.5, 0.8, 0.95, 0.99$, after renormalization

Then one sees that the solutions are converging to a curve (Figure 1.2, right) as $t \to 1$. Using the solution formula, one can check that the limiting curve is given by the function $1/(1+x^2)$, which is interesting for many reasons.

One can analyze various one-dimensional transport equations similarly as in the Burgers case. One may consider transport equations of the form

$$\partial_t \rho + u \partial_x \rho = 0 \tag{33}$$

where u is determined by ρ according to a given constitutive law, or more generally

$$\partial_t \rho + u \partial_x \rho = f \rho \tag{34}$$

where again f is determined by ρ at each moment of time. It is useful to have various strategies for the proof of singularity formation, and indeed many are known for the case of the inviscid Burgers' equation. A common approach is to consider the time evolution of a quantity

$$\int \rho(t,x) F(t,x) dx$$

for some F to be chosen cleverly depending on the form of the equation.

Problem 1.2.2. *Consider the problem of singularity formation in the following concrete cases. That is, investigate conditions on the initial data which guarantees finite time blow up or global regularity.*

- (33) *with* $u(t,x) = \int_0^x \rho(y)dy$*. Take the domain to be the half-line* $\{x \geq 0\}$*.*
- (33) *with* $u(t,x) = \frac{1}{x}\int_0^x \rho(y)dy$*.*

Wellposedness of gSQG. Let us now focus on the specific case of generalized surface quasi-geostrophic equations (gSQG). As we have discussed in the above, the very first question to be asked is whether given a sufficiently nice initial data θ_0

(e.g. in $C^\infty(\mathbb{T}^2)$), there exists a unique corresponding solution θ to (gSQG). As we have discussed earlier, one can give a rather precise answer to this question: *there is local wellposedness if and only if the multiplier P is not singular.* For this reason, in this text we are mainly concerned with the non-singular regime. Now that we are given local wellposedness, the next step is to determine whether locally unique C^∞ solutions are global in time or not. Unfortunately, we do not have a definite answer to this basic problem for (gSQG), except for the case of the two-dimensional Euler equations and slightly singularized equations. (Another notable exception is the finite time singularity formation result for smooth patches in the upper half-plane [94].) The gSQG equations are both *non-linear* and *non-local*, which makes it difficult to understand long-time dynamics. Here, by "long time", we are referring to any timescales which are strictly longer than the one in which local smooth solution is guaranteed by the a priori estimate. There are no apparent (dynamical) mechanisms for regularity and the known conservation laws are too weak to control the solution, except for the case of 2D Euler.

The specific topics covered in these notes can be viewed as several attempts to attack the problem of long time dynamics. Let us informally describe some of these attempts.

- **Blow-up criteria**. To begin with, one can try to obtain non-trivial blow-up criteria, which are *necessary* conditions for the finite time singularity formation to occur. Usually, they take the form of the time integral of a *critical* norm of the solution. Apart from the intrinsic interest, they often provide tests as to whether a singular behavior observed in numerical computations is a candidate for blow-up. For instance, the Beale–Kato–Majda criterion [6] allows us, roughly speaking, to distinguish double exponential growth of the gradient from finite-time blow up. Now, given the transport nature of the gSQG systems, it makes a lot of sense to consider blow-up criteria which are *geometric*; that is, what should happen to the level sets of θ at the time of blow-up? For instance, two initially disjoint level sets can "collide" at the singular time. It might happen that some initially smooth level curve becomes non-smooth. In the case of gSQG equations, one can sometimes rule out some of these possibilities. Closely related to this, one may investigate whether a specific *type* of singularity formation is allowed by the PDE. The most widely tested blow-up scenario is locally self-similar singularity.

- **Weak solutions**. In general, one can construct globally defined *weak solutions*, primarily using the fact that the quantity $\|\theta(t,\cdot)\|_{L^p}$ is conserved in time for smooth solutions. For the proof, see the works [122, 109, 5]. The weak solutions provide natural candidates for solutions after a potential singularity formation for smooth solutions.

  While global existence of a weak solution, given an initial datum, is not very difficult to get, the difficult part for them is to prove uniqueness. (However, see [39] for uniqueness of patch solutions.) Indeed, using convex integration techniques, it is possible to show non-uniqueness of "very" weak solutions ([10]). Recent groundbreaking work of Vishik [128, 129] gives non-uniqueness of forced 2D Euler solutions with $\omega \in L^p$. Apart from the uniqueness issue, another very interesting set of problems is regarding the dynamics of weak solutions.

- **Patch solutions**. Given global existence of weak solutions, one can consider certain specific classes of weak solutions, towards the goal of achieving uniqueness and finite-time singularity formation within the class. The most widely studied such class of solutions is *patches*, which are characteristic functions of a moving domain. The study of patch solutions in the Euler case can be motivated from atmosphere dynamics. Even within the class of patch solutions, one may consider the ones with either smooth or irregular boundaries. In the former, it can be shown that the smoothness of the patch boundary propagates at least locally in time. More precisely, given a smooth patch initial data, there exists a local solution of the same form, which is unique in the class of smooth patch solutions. Then, one can ask the problem of finite time singularity formation within this restricted class of solutions. It is interesting to note that while there are a few numerical simulations suggesting finite time singularity formation for smooth gSQG patches, no convincing such simulations does not seem available in the class of smooth gSQG solutions. In the case of patches with non-smooth boundaries, having a wellposedness theory is a challenging problem. In the 2D Euler case, a unique patch solution is guaranteed by the Yudovich theorem, but not much is known as to how the unique solution looks like, given an initial patch with singular boundary.

  More generally, it is possible to consider solutions with different types of singularities, and study how they evolve in time.

- **Gradient growth**. While it is not clear whether smooth solutions to (gSQG) could blow up in finite time, there are results showing *infinite growth* of smooth solutions in certain norms in infinite time. For instance, one can try to prove a *lower bound* on the C^1 norm of the solution in time, assuming that it remains smooth for all time. Roughly speaking, this question asks whether one can rigorously prove that different level sets of θ become very close with each other, which is easily observed in numerical computations and experiments. This is already a very difficult task, and existing results require rather special constructions of the initial data. An important outstanding question is whether such growth is "generic" in a sense.

- **Simpler model equations**. Recalling that one of the principal motivation for studying SQG was its formal analogy with the three-dimensional vorticity dynamics, it is reasonable to consider even simpler toy models for (gSQG) which share some characteristic features. A few of those models can be obtained directly from the gSQG equations under some ansatz on the solutions. For many such models, finite time singularity could be established.

- **Special solutions**. Lastly, one may construct and study special solutions, most notably steady and time-periodic ones. Then one can try to understand, by linearization, what happens to perturbations of such solutions. It could be already a challenging problem to study linearized behavior, in particular to understand the behavior of the trajectories associated with the initial velocity field. Another large class of solutions come from point vortex approximation, although it requires some work to regard them as approximate solutions.

1.3 Preliminaries

1.3.1 Fourier transform

Given an integrable function f defined in $\mathbb{R}$, we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The inverse transform of g is defined by

$$\check{g}(x) = \int_{\mathbb{R}} g(\xi) e^{2\pi i x \xi} d\xi.$$

A Fourier multiplier $\mathbf{M}$ is simply a multiplication operator in the Fourier side. Given a function $M(\xi) : \mathbb{R} \to \mathbb{R}$, we define $\mathbf{M}f$ as the function whose Fourier transform is given by $M(\xi)\hat{f}(\xi)$. We say that M is the multiplier of $\mathbf{M}$. This operator can be first defined on the class of Schwartz functions and then extended to an appropriate function class depending on the nature of the multiplier.

In the definition $u = \nabla^{\perp} P(\Lambda)\theta$, we have that

$$\hat{u}(\xi) = (i\xi)^{\perp} P(|\xi|)\hat{\theta}(\xi).$$

Therefore, to understand this operator in the physical space, we need to obtain the inverse Fourier transform of $P(|\xi|)$. This will be computed in a few important cases below.

1.3.2 Basic properties of gSQG

We shall assume that we are given a sufficiently smooth (and decaying, when the domain is unbounded) solution to (gSQG).

Symmetries and steady states. The gSQG equations enjoy many symmetries; to begin with, it has translation symmetry; if $v \in \mathbb{R}^2$ and θ is a solution, then

$$\theta^v(t,x) := \theta(t, x+v)$$

is again a solution. Next, if $\mathbf{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, then

$$\theta^{\mathbf{R}}(t,x) := \theta(t, \mathbf{R}x)$$

is a solution. This is because we are assuming that the multiplier $P(\Lambda)$ is isotropic (there is no preferred direction). Moreover, if $\mathbf{F}$ is a reflection, then

$$\theta^{\mathbf{F}}(t,x) := -\theta(t, \mathbf{F}x)$$

is a solution. Lastly, one may consider scaling invariance, when $P(\Lambda)$ is of power type, namely when $P(\xi) = |\xi|^\gamma$ for some $\gamma \in \mathbb{R}$. In this case, for any $\lambda > 0$,

$$\theta^\lambda(t,x) := \lambda^a \theta(t, \lambda x)$$

is again a solution. To see this, we simply compute that

$$u^\lambda(t,x) = \nabla^\perp \Lambda^\gamma \theta^\lambda(t,x) = \nabla^\perp \Lambda^\gamma(\lambda^a \theta(t,\lambda x)) = \lambda^{a+1+\gamma} u(t, \lambda x)$$

so that

$$u^\lambda \cdot \nabla \theta^\lambda = \lambda^{2a+2+\gamma}(u \cdot \nabla \theta)(t, \lambda x).$$

This equals

$$\partial_t \theta^\lambda(t,x) = \lambda^a \partial_t \theta(t, \lambda x)$$

so that we need to select $a = 2a + 2 + \gamma$, or $a = -2 - \gamma$. In particular, when $\gamma = -2$, we have that $a = 0$. This is interesting since this symmetry does not change the time variable. Of course, there is a simpler scaling symmetry which changes the time;

$$\theta_\mu(t,x) := \mu \theta(\mu t, x)$$

is a solution for any $\mu \in \mathbb{R}$. Note that this does not change the spatial variable and therefore this symmetry holds for any multiplier P. This simply states that if we magnify our dependent variable, the same dynamics occurs but just in a faster timescale. It is interesting to note the special case $\mu = -1$; this shows that reversing the sign of the advected scalar corresponds to moving backwards in time.

Assuming that we have the uniqueness of the solution, a consequence of having symmetry is the propagation of symmetry of the solution in time; if the initial data

θ_0 is invariant under some group $\mathcal{G}$ of symmetries of the equation (which does not change the time variable), then the solution is forced to be invariant under $\mathcal{G}$ for each t. As an example, one can impose the time-scaling invariance: assume that a solution θ to α-SQG satisfies

$$\theta(t,x) = \mu\theta(\mu t, x)$$

for *all* $\mu > 0$ and t, x. Then, the solution is determined by the data at $t = 1$: simply taking $\mu = 1/t$ gives (for $t > 0$) $\theta(t,x) = t^{-1}\theta_1(x) := t^{-1}\theta(1,x)$. This seems impossible since the maximum of $|\theta|$ is supposed to be preserved in time. Of course, θ_1 can be simply unbounded and indeed one can construct solutions which satisfy this property. On the other hand, one can combine scaling symmetries in t and x and consider invariant solutions; see [106, 107].

A particularly simple case, which gives rise to a set of steady states, is the 1-dimensional group of rotations around the origin. If θ_0 is invariant under this group, then it means that $\theta_0(x)$ is a radial function (function depending only on $|x|$) and this defines a steady state, since the stream function $\psi := P(\Lambda)\theta$ is also radial, which means that $u = \nabla^\perp\psi$ is a radial function multiplied by the vector $x^\perp = (-x_2, x_1)$. On the other hand, $\nabla\theta$ is a radial function multiplied by the vector x, so that $u \cdot \nabla\theta \equiv 0$.

Another simple 1-dimensional group of symmetries is the set of all translations in a fixed direction. For instance, one can consider θ_0 which is a function of the x_1 variable only. This is a steady state as well, for a similar reason.

Slightly more interesting examples of steady states come from eigenfunctions of the Laplacian on $\mathbb{T}^2$. Let us say

$$-\Delta f = \lambda f,$$

for some $\lambda > 0$. Then, $\Lambda f = \sqrt{\lambda} f$ and we have $P(\Lambda)f = P(\sqrt{\lambda})f$. Then we see that f is a steady state.

Conserved quantities. To begin with, any L^p norm of θ is formally conserved in time, since u is divergence free. Actually, for any $\mu \in \mathbb{R}$, the measure of the set

$$\{x : \theta(t,x) > \mu\}$$

is constant in time, from which L^p conservation for any $1 \le p \le \infty$ follows. There is a slightly more non-trivial conservation law, which has fundamental consequences. In the equation

$$\partial_t\theta + \nabla^\perp P(\Lambda)\theta \cdot \nabla\theta = 0,$$

we may rewrite the nonlinearity as

$$\nabla^\perp\theta \cdot \nabla P(\Lambda)\theta$$

and then we compute

$$\frac{d}{dt}\frac{1}{2}\int \theta P(\Lambda)\theta = \int \partial_t\theta P(\Lambda)\theta$$

(this is because P is a symmetric operator)

$$= \int (\nabla^\perp\theta\cdot\nabla)P(\Lambda)\theta\, P(\Lambda)\theta = \frac{1}{2}\int(\nabla^\perp\theta\cdot\nabla)(P(\Lambda)\theta)^2 = 0.$$

We have integrated by parts in the last step. This shows that the quantity $\|P^{\frac{1}{2}}(\Lambda)\theta\|_{L^2}^2$ is conserved in time. This is nothing but (a constant multiple) of the kinetic energy of the fluid in the case of 2D Euler.

As an immediate application of this additional conservation law, one can prove *nonlinear* stability of steady states which are supported on the lowest modes. This observation has appeared in [110] to prove "absence of turbulence". Later Denisov [46] applied it to obtain infinite gradient growth. Assume that the domain is $\mathbb{T}^2$ and we index the Fourier modes by $k \in \mathbb{Z}$. Without loss of generality, we may assume that $\int_{\mathbb{T}^2}\theta = 0 = \hat{\theta}(0)$. Recall that $\bar{\theta} = \sin(x_1)$ is a steady state for any gSQG. Consider θ_0 which is close in L^2 to $\bar{\theta}$, and let $\theta(t,\cdot)$ be the solution of gSQG with some P. Then, from conservation laws, we have

$$\sum_{k\in\mathbb{Z}^2}|\hat{\theta}(t,k)|^2 = \sum_{k\in\mathbb{Z}^2}|\hat{\theta}_0(k)|^2 =: A_0$$

and

$$\sum_{k\in\mathbb{Z}^2}|P(k)||\hat{\theta}(t,k)|^2 = \sum_{k\in\mathbb{Z}^2}|P(k)||\hat{\theta}_0(k)|^2 = B_0.$$

Let $P_1 = P((\pm1,0)) = P((0,\pm1)) > 0$. Then, we control

$$\sum_{|k|>1}(P_1 - |P(k)|)|\hat{\theta}(t,k)|^2 = P_1A_0 - B_0.$$

The right hand side is positive and small since θ_0 is a sufficiently small perturbation of $\bar{\theta}$. Now, if $P_1 - |P(k)| \geq c_0$ uniformly for $|k| > 1$ (which is the case for many situations), then we conclude that for all t as long as the solution exists,

$$\sum_{|k|>1}|\hat{\theta}(t,k)|^2 \leq \frac{P_1A_0 - B_0}{c_0}.$$

This is intuitively clear: if most of the energy is contained in the lowest mode at the initial time, then the energy cannot cascade forward much since then it needs to be balanced with some backwards transfer, to satisfy two coercive conservation laws.

We now observe some other conservation laws, which may not be coercive. On $\Omega = \mathbb{R}^2$, we have the angular momentum

$$I = \int_{\mathbb{R}^2} |x|^2 \theta(t,x) dx.$$

This quantity is meaningful when the domain is rotationally symmetric. To see that this is conserved, we compute

$$\begin{aligned}
\frac{dI}{dt} &= -\int_{\mathbb{R}^2} |x|^2 \nabla(\theta u) dx = \int_{\mathbb{R}^2} 2x \cdot u\theta dx \\
&= \int_{\mathbb{R}^2} 2(x_1(-\partial_{x_2} P\theta) + x_2 \partial_{x_1} P\theta)\theta dx \\
&= \int_{\mathbb{R}^2} 2(x_1 P\theta \, \partial_{x_2}\theta - x_2 P\theta \, \partial_{x_1}\theta) dx \\
&= \int_{\mathbb{R}^2} 2(x_1 \theta \, \partial_{x_2} P\theta - x_2 \theta \, \partial_{x_1} P\theta) dx = 0.
\end{aligned}$$

In the last step, we have used that P is a symmetric operator and that it commutes with the multiplication operator x_i for $i = 1, 2$.

This conservation law is particularly useful when θ has a *sign*; namely, if $\theta_0 \geq 0$, then we have that $I \geq 0$ for all t as long as the solution exists. Similarly, we have the conservation of mean

$$\int_{\mathbb{R}^2} \theta(t,x) dx.$$

Next, there are some *monotone* quantities, under special symmetry assumptions on the solution. If θ is odd symmetric with respect to x_1, namely if $\theta(x_1, x_2) = -\theta(x_1, -x_2)$ and $\theta \geq 0$ on $\{x_2 \geq 0\}$, then

$$\frac{d}{dt} \int_{\mathbb{R}^2} x_1 \theta(t,x) dx \geq 0.$$

For the proof in the case of 2D Euler, see [76]. Similarly, if θ is odd with respect to both axes and non-negative in the first quadrant, one can obtain the same monotone quantity.

Anti-derivative formulation. It is often a good idea to reformulate a PDE using more regular variables, especially when one is concerned about low regularity issues. It also allows us to easily access low frequencies of the solution. In the case of Euler, the natural variable is u, which is order one smoother than ω and satisfies the equation

$$\partial_t u + u \cdot \nabla u + \nabla p = 0.$$

Note that here p is determined purely by the incompressibility constraint; taking the divergence of both sides and using $\nabla \cdot u = 0$ gives

$$\Delta p = -\nabla(u \cdot \nabla u).$$

One sees that while the variable u is more regular, one needs to then solve the above non-linear and non-local equation, which is quite difficult. For the SQG case, one can introduce the variable $v = \Lambda^{-1} u$, and see that v satisfies (Buckmaster–Shkoller–Vicol [10])

$$\partial_t v + u \cdot \nabla v + (\nabla u)^{\perp} \cdot v = -\nabla \pi. \tag{35}$$

Similar formulation can be derived in general for the case of generalized SQG equations; indeed, v can be defined precisely as the "incompressible velocity" whose curl is given by θ. In other words, v is the unique solution to the system $\nabla \times v = \theta$, $\nabla \cdot v = 0$. Then, by taking the curl of (35), the pressure term vanishes, and we are left with

$$\partial_t \theta + \nabla \times (u \cdot \nabla v) + \nabla \times ((\nabla u)^{\perp} \cdot v) = 0.$$

Then,

$$\nabla \times (u \cdot \nabla v) = u \cdot \nabla(\nabla \times v) + \partial_1 u_1 \partial_1 v_2 + \partial_1 u_2 \partial_2 v_2 - \partial_2 u_1 \partial_1 v_1 - \partial_2 u_2 \partial_2 v_1$$

and

$$\begin{aligned} \nabla \times ((\nabla u)^{\perp} \cdot v) &= \partial_1(\partial_2 u_1 v_1 + \partial_2 u_2 v_2) - \partial_2(\partial_1 u_1 v_1 + \partial_1 u_2 v_2) \\ &= \partial_{12} u_1 v_1 + \partial_{12} u_2 v_2 - \partial_{21} u_1 v_1 - \partial_{21} u_2 v_2 \\ &\quad + \partial_2 u_1 \partial_1 v_1 + \partial_2 u_2 \partial_1 v_2 - \partial_1 u_1 \partial_2 v_1 - \partial_1 u_2 \partial_2 v_2. \end{aligned}$$

We see that

$$\partial_{12} u_1 v_1 + \partial_{12} u_2 v_2 - \partial_{21} u_1 v_1 - \partial_{21} u_2 v_2 = 0$$

and

$$\partial_2 u_1 \partial_1 v_1 + \partial_2 u_2 \partial_1 v_2 - \partial_1 u_1 \partial_2 v_1 - \partial_1 u_2 \partial_2 v_2$$

cancels with $\nabla \times (u \cdot \nabla v) - u \cdot \nabla(\nabla \times v)$. Therefore we get that $\partial_t \theta + u \cdot \nabla \theta = 0$.

Kernel representation. We recall some Fourier transform formulas, first in the one-dimensional setup. Here we use the convention that

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

The Fourier transform of the constant function $\mathbf{1}$ is $\delta(\xi)$ and vice versa (i.e. the Fourier transform of $\delta(x)$ is $\mathbf{1}$). Next, in the range $0 > \alpha > -1$, we have that

$$\mathcal{F}[|x|^\alpha] = -C_\alpha |\xi|^{-\alpha-1}.$$

Here,

$$C_\alpha = \frac{2\sin(\frac{\pi\alpha}{2})\Gamma(\alpha+1)}{(2\pi)^{\alpha+1}}.$$

We now move on to the two-dimensional case, and recall that the Fourier transform of $|x|^{-\alpha}$ for $0 < \alpha < 2$ is given by

$$C_\alpha |\xi|^{-(2-\alpha)}, \qquad C_\alpha = \frac{(2\pi)^\alpha}{\pi 2^\alpha} \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)}.$$

It is interesting to note that the Fourier transform of $|x|^{-1/2}$ is simply $|\xi|^{-1/2}$. Using this formula, we see that in the case of α-SQG, we explicitly have $u = \nabla^\perp \psi$ with the stream function

$$\psi(x) = C_\alpha \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} \theta(y) dy. \tag{36}$$

In the upper half-plane case $y_2 \geq 0$, we can obtain the kernel by the method of reflection;

$$\psi(x) = C_\alpha \int_{\mathbb{R}^2} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x-\bar{y}|^\alpha} \right] \theta(y) dy$$

where $\bar{y} = (y_1, -y_2)$. Taking $\nabla^\perp$, we obtain

$$u(x) = C'_\alpha \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}} \theta(y) dy. \tag{37}$$

This is integrable for $0 < \alpha < 1$, and when $\alpha = 1$ (SQG case) it should be interpreted in the principal value sense;

$$u(x) = C P.V. \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^3} \theta(y) dy = C \lim_{\epsilon \to 0} \int_{\{|x-y| > \epsilon\}} \frac{(x-y)^\perp}{|x-y|^3} \theta(y) dy.$$

In the following, we shall usually normalize the coefficient of the kernel in a way that the constant C'_α in (37) is equal to 1. This only rescales the time variable in the gSQG evolution. Now, when the kernel becomes more singular than the SQG case $1 < \alpha < 2$, then we have that

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\alpha}} (\theta(y) - \theta(x)) dy.$$

Here, it is required that θ has some differentiability, so that we gain some (possibly fractional) factor of $|y-x|$ from $\theta(y)-\theta(x)$.

We now return to the regular case $0<\alpha<1$ and obtain expressions for the gradient of u, which is the key quantity controlling the flow. This computation needs to be done carefully, since the kernel becomes singular (non-integrable) after a direct differentiation. We first rewrite (37) as

$$u(x)=\int \frac{y^\perp}{|y|^{2+\alpha}}\theta(x-y)dy$$

with a change of variables, and then

$$\partial_{x_1}u(x)=\int \frac{y^\perp}{|y|^{2+\alpha}}\partial_{x_1}(\theta(x-y))dy=-\int \frac{y^\perp}{|y|^{2+\alpha}}\partial_{y_1}(\theta(x-y))dy$$

assuming that $\theta\in C^1$, say. Then, we integrate by parts carefully

$$\begin{aligned}-\int \frac{y^\perp}{|y|^{2+\alpha}}\partial_{y_1}(\theta(x-y))dy&=-\lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{y^\perp}{|y|^{2+\alpha}}\partial_{y_1}(\theta(x-y))dy\\ &=-\lim_{\epsilon\to 0}\left[-\int_{|y|>\epsilon}\partial_{y_1}\left(\frac{y^\perp}{|y|^{2+\alpha}}\right)\theta(x-y)dy+\int_{|y|=\epsilon}\frac{y^\perp}{|y|^{2+\alpha}}\theta(x-y)dy\right]\end{aligned}$$

In the first expression, if we look at the first component of u, we obtain

$$\lim_{\epsilon\to 0}\int_{|y|>\epsilon}(2+\alpha)\frac{y_1y_2}{|y|^{4+\alpha}}\theta(x-y)dy$$

and the key trick we use here is that for any $\epsilon>0$, we have exact cancellation of

$$\int_{|y|>\epsilon}(2+\alpha)\frac{y_1y_2}{|y|^{4+\alpha}}\theta(x)dy=\theta(x)\int_{|y|>\epsilon}(2+\alpha)\frac{y_1y_2}{|y|^{4+\alpha}}dy=0$$

so that

$$\lim_{\epsilon\to 0}\int_{|y|>\epsilon}(2+\alpha)\frac{y_1y_2}{|y|^{4+\alpha}}\theta(x-y)dy=\lim_{\epsilon\to 0}\int_{|y|>\epsilon}(2+\alpha)\frac{y_1y_2}{|y|^{4+\alpha}}\left[\theta(x-y)-\theta(x)\right]dy$$

and then the last limit is absolutely convergent in the range $0<\alpha<1$ assuming that θ is C^1. On the other hand, the second term

$$\int_{|y|=\epsilon}\frac{y^\perp}{|y|^{2+\alpha}}\theta(x-y)dy=\int_{|y|=\epsilon}\frac{y^\perp}{|y|^{2+\alpha}}\left[\theta(x-y)-\theta(x)\right]dy$$

and thanks to the C^1 regularity of θ, we obtain that this limit vanishes as $\epsilon\to 0$. One may change variables again in the limit, and obtain finally that

$$\partial_{x_1}u_1=(2+\alpha)\int \frac{(x_2-y_2)(x_1-y_1)}{|x-y|^{4+\alpha}}(\theta(y)-\theta(x))dy. \tag{38}$$

For the other component, we have

$$\partial_{x_1} u_2 = \int \frac{(2+\alpha)(x_1-y_1)^2 - |x-y|^2}{|x-y|^{4+\alpha}} (\theta(y) - \theta(x)) dy. \tag{39}$$

The formula for $\partial_{x_2} u_1$ is obtained by taking the negative sign of the above with roles of x_1 and x_2 switched. Lastly, we can find $\partial_{x_2} u_2 = -\partial_{x_1} u_1$.

Logarithmic kernel representation. We now consider the case of $P(|\xi|)$ whose inverse Fourier transform is not explicitly known. We shall denote the kernel by $K(|x|)$. The relevant pieces of information are the asymptotics of K and its derivatives in the small scale limit $|x| \to 0$. We shall follow the procedure of Kwon [101, Lemma 3.1]. To begin with, we borrow the inequality

$$\frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-at} t^{\gamma-1} dt = a^{-\gamma}, \tag{40}$$

where $a > 0$. Following [101], we may obtain asymptotics of the kernel corresponding to $P(|\xi|) = \ln^{-\gamma}(10 + |\xi|^2)$. Applying (40) to $a = \ln(10 + |\xi|^2)$,

$$\begin{aligned} \ln^{-\gamma}(10+|\xi|^2) &= \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-t \ln^{-\gamma}(10+|\xi|^2)} t^{\gamma-1} dt \\ &= \frac{1}{\Gamma(\gamma)} \int_0^\infty (10+|\xi|^2)^{-t} t^{\gamma-1} dt. \end{aligned}$$

Now we apply (40) again, to obtain that

$$\ln^{-\gamma}(10+|\xi|^2) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-10\beta} e^{-\beta|\xi|^2}\, \beta^{t-1} d\beta\, t^{\gamma-1} dt.$$

To handle the singularity at $\beta = 0$, we use

$$\int_0^\infty e^{-|\xi|^2 s} ds = |\xi|^{-2},$$

which holds for any $\xi \neq 0$. Then

$$\ln^{-\gamma}(10+|\xi|^2) = \frac{|\xi|^2}{\Gamma(\gamma)} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-10\beta} \int_0^\infty e^{-(\beta+s)|\xi|^2}\, ds\, \beta^{t-1} d\beta\, t^{\gamma-1} dt.$$

Now we are ready to take the inverse Fourier transform:

$$\begin{aligned} K(|x|) &\sim \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty \int_0^\infty e^{-10\beta} \frac{1}{\beta+s} \Delta_x (e^{-|x|^2/4(\beta+s)}) ds\, \beta^{t-1} d\beta\, t^{\gamma-1} dt \\ &\sim \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty \int_0^\infty e^{-10\beta} \frac{1}{\beta+s} (\frac{|x|^2}{(s+\beta)^2} + \frac{C}{(s+\beta)}) (e^{-|x|^2/4(\beta+s)}) ds\, \beta^{t-1} d\beta\, t^{\gamma-1} dt. \end{aligned}$$

We now make a change of variables $\beta = |x|^2\alpha$, $s = |x|^2\tau$:

$$K(|x|) \sim |x|^{2(t-1)} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty \int_0^\infty e^{-10\alpha|x|^2} (\frac{1}{(\tau+\alpha)^3} + \frac{C}{(\tau+\alpha)^2}) e^{-1/4(\alpha+\tau)}\, d\tau\, \alpha^{t-1} d\alpha\, t^{\gamma-1} dt.$$

We may compute

$$\begin{aligned} &\int_0^\infty (\frac{1}{(\tau+\alpha)^3} + \frac{C}{(\tau+\alpha)^2}) e^{-1/4(\alpha+\tau)}\, d\tau \\ &= 16 - \frac{4(1+4\alpha)e^{-1/4\alpha}}{\alpha} + 4C(1-e^{-1/4\alpha}) =: H(\alpha). \end{aligned}$$

Then

$$|x|^2 K(|x|) \sim \int_0^\infty \frac{|x|^{2t}}{\Gamma(t)} \int_0^\infty e^{-10\alpha|x|^2} H(\alpha)\, \alpha^{t-1} d\alpha\, t^{\gamma-1} dt.$$

We are now ready to obtain lower and upper bounds. For a lower bound, we can restrict the α integral to the region $0 < \alpha < 1$ and $0 < t < 1$: this gives, using a lower bound on H,

$$|x|^2 K(|x|) \gtrsim e^{-10|x|^2} \int_0^1 \frac{|x|^{2t}}{t\Gamma(t)} t^{\gamma-1} dt \gtrsim e^{-10|x|^2} \ln^{-\gamma}(10 + \frac{1}{|x|}),$$

where we have used that $t\Gamma(t) \sim 1$ for $0 < t < 1$. On the other hand, we see that the upper bound

$$|x|^2 K(|x|) \lesssim \ln^{-\gamma}(10 + \frac{1}{|x|})$$

holds. With an integration by parts, one may obtain a polynomial decay in $\langle x \rangle$.

1.3.3 Flow map

Since (gSQG) is a transport equation, it is natural to solve θ along the *flow map* defined by the velocity u. We shall assume that we are given a solution to (gSQG), in a time interval, such that $u \in L^\infty([0,T]; Lip(\Omega))$. Then, we can uniquely solve for the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}\Phi(t,x) = u(t,\Phi(t,x)), \\ \Phi(0,x) = x, \end{cases} \tag{41}$$

and then we have

$$\theta(t,\Phi(t,x)) = \theta_0(x),$$

where θ_0 is the initial data. For each $0 \leq t \leq T$, the map $\Phi(t, \cdot)$ is bi-Lipschitz from Ω to itself (it can be shown using some topological arguments and flow estimates given in below), and we may denote the inverse by Φ_t^{-1}, so that

$$\theta(t, x) = \theta_0(\Phi_t^{-1}(x)).$$

From these expressions of the solution, we have that if $\theta_0 \geq c$ in a region $A \subset \Omega$, then $\theta(t, \cdot) \geq c$ in the region $\Phi(t, A)$. In particular, if $\theta_0 \geq 0$ everywhere in Ω, the same holds for the solution as well. This is a basis for the *Lagrangian viewpoint* of (gSQG).

We shall observe some elementary estimates on the flow map. Given two points $x \neq x'$, one often needs to understand how the distance between the trajectories change in time. To this end we compute

$$\frac{d}{dt}|\Phi(t,x) - \Phi(t,x')|^2 = 2(\Phi(t,x) - \Phi(t,x')) \cdot (u(t, \Phi(t,x)) - u(t, \Phi(t,x'))).$$

Using the mean value theorem, we obtain that

$$\left| \frac{d}{dt}|\Phi(t,x) - \Phi(t,x')|^2 \right| \leq 2\|\nabla u(t,\cdot)\|_{L^\infty} |\Phi(t,x) - \Phi(t,x')|^2.$$

Integrating the inequality in time, one obtains

$$\exp\left(\int_0^t -\|\nabla u(s,\cdot)\|_{L^\infty} ds\right) \leq \frac{|\Phi(t,x) - \Phi(t,x')|}{|x - x'|} \leq \exp\left(\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} ds\right).$$

It is important that one not only has an upper bound on the distance but also a lower bound; the latter tells us that *as long as the velocity is Lipschitz continuous*, two "fluid particles" cannot collide in finite time. To be more precise, for

$$\lim_{t \to T} |\Phi(t,x) - \Phi(t,x')| = 0$$

to occur, it is *necessary* (but not sufficient) to have that

$$\lim_{t \to T} \int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} ds = \infty.$$

For incompressible fluid models, it is usually very difficult to prove the existence of fluid particles which collide in finite time. Even proving collision in infinite time is an interesting problem.

1.3.4 Linear transport in L^p

We consider regularity and uniqueness issues for the *linear* transport equation in L^p, which is a basic building block for nonlinear results that comes later. This material is taken from DiPerna–Lions [49]. For simplicity, consider in $\mathbb{T}^n$ the following

$$\partial_t \theta + v \cdot \nabla \theta + d\theta = 0. \tag{42}$$

Here, $v : [0,T] \times \mathbb{T}^n \to \mathbb{R}^n$ and $d : [0,T] \times \mathbb{T}^n \to \mathbb{R}$ are *given* vector and scalar functions of time and space, respectively. Assuming that θ is a smooth function of time and space, we can obtain a pointwise equality

$$\partial_t|\theta|^p + v \cdot \nabla|\theta|^p + pd|\theta|^p = 0$$

and then this gives, after integrating in $\mathbb{T}^n$,

$$\frac{d}{dt}\int |\theta|^p dx \le \|pd + \nabla \cdot v\|_{L^\infty} \int |\theta|^p dx$$

and then integrating in time,

$$\|\theta(t)\|_{L^p} \le \exp\left(\int_0^t \|d(\tau) - \frac{1}{p}\nabla \cdot v(\tau)\|_{L^\infty} d\tau\right) \|\theta_0\|_{L^p}.$$

This was for $1 \le p < \infty$, but one may take the limit $p \to \infty$ in the last inequality, (recalling that $\|f\|_{L^p}$ converges to $\|f\|_{L^\infty}$ in $\mathbb{T}^n$, assuming that $f \in L^\infty$) which gives

$$\|\theta(t)\|_{L^\infty} \le \exp\left(\|d\|_{L^1_t L^\infty}\right) \|\theta_0\|_{L^\infty}.$$

We shall use the simplifying notation $L^p_t X := L^p(I; X)$ with $\|f\|_{L^p_t X} = (\int_I \|f\|_X^p dt)^{1/p}$, where I is an interval of time. It is clear that the term $d\theta$ in (42) could change the extreme values of θ.

Now, following DiPerna–Lions, we shall verify the above L^p estimates under minimal regularity assumptions on the coefficients and the solution. To begin with, for $\theta \in L^\infty_t L^p$, (42) should be interpreted in the weak sense. That is, by a solution we mean that for any smooth scalar function ϕ of time and space, we have

$$\int \theta_0(x)\phi(0,x)dx + \int_0^T \int \theta(\partial_t\phi + \nabla \cdot (\phi v) - d\phi)dxdt = 0.$$

Note that for this formulation to make sense, under the hypothesis $\theta \in L^\infty_t L^p$, it is *necessary* to have that

$$d, \quad \nabla \cdot v, \quad v \in L^1([0,T]; L^q), \tag{43}$$

where q will always mean the conjugate exponent of p; $1/p + 1/q = 1$. These assumptions are actually sufficient to guarantee the *existence* of a solution to the initial value problem for (42); to be more precise,

Proposition 1.3.1 ([49, Proposition II.1])**.** *For any $\theta_0 \in L^p(\mathbb{T}^n)$ for $1 \le p \le \infty$, there exists a weak solution θ belonging to $L^\infty([0,T]; L^p)$, assuming* (43)*.*

To achieve quantitative estimates for the distance between weak solutions and approximate smooth solutions, we shall need

Proposition 1.3.2 ([49, Theorem II.1])**.** *Given $\theta \in L^\infty([0,T];L^p)$ a weak solution, assume further that*

$$v, \quad d \in L^1([0,T];L^\alpha)$$

where $\alpha \geq q$. Then, the mollified function $\theta_\epsilon := \theta * \rho_\epsilon$ satisfies (42) *with an error:*

$$\partial_t \theta_\epsilon + v \cdot \nabla \theta_\epsilon + d\theta_\epsilon = R_\epsilon, \tag{44}$$

where

$$\|R_\epsilon\|_{L^1([0,T];L^\beta)} \longrightarrow 0, \qquad \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}.$$

In the above proposition, $\{\rho_\epsilon\}_{\epsilon>0}$ is the "standard" mollifiers, defined by

$$\rho_\epsilon(x) := \frac{1}{\epsilon^n}\rho(\frac{x}{\epsilon})$$

where $\rho \geq 0$ is some radial compactly supported smooth bump function.

Proof. For simplicity we shall fix a time moment and estimate the remainder in L^β. Directly applying the mollifier to (42), we note that

$$R_\epsilon = ((v \cdot \nabla\theta)_\epsilon - v \cdot \nabla\theta_\epsilon) + ((d\theta)_\epsilon - d\theta_\epsilon) = I + II.$$

Therefore it suffices to estimate I and II. For I, we can write $I = I_1 - I_2$ where

$$I_2 = (\theta\nabla \cdot v)_\epsilon,$$

$$I_1 = \int \theta(y)(v(y) - v(x)) \cdot \nabla\rho_\epsilon(x-y)dy.$$

Since it is easy to see that

$$\|I_2 - \theta\nabla \cdot v\|_{L^\beta} \to 0$$

as $\epsilon \to 0$, it suffices to show that I_1 has the same limit in L^β. This is clear if θ, v were smooth. We would like to show that the same holds by an approximation, and for this we need uniform bound of I_1 in L^β.

To this end, using Hölder's inequality, we bound

$$\|\int \theta(y)(v(y) - v(x)) \cdot \nabla\rho_\epsilon(x-y)dy\|_{L^\beta} \lesssim \|\theta\|_{L^p}\left(\int dx \int_{|x-y|\lesssim\epsilon} \epsilon^{-\alpha}|v(y) - v(x)|^\alpha dy\right)^{1/\alpha}.$$

This is possible since the x-domain of integration is bounded. Then, after a change of variables, the latter integral is

$$\left(\int\int_{|z|\lesssim 1}\left(\int_0^1 |\nabla v(x + t\epsilon z)|dt\right)^\alpha dzdx\right)^{1/\alpha}.$$

Using Hölder's inequality and Fubini's theorem, we see that this is bounded by $\lesssim \|\nabla v\|_{L^\alpha}$. ☐

We are ready to prove the uniqueness result:

Theorem 1.3.1 (Uniqueness for linear transport)**.** *Let* $1 \le p \le \infty$ *and* $\theta_0 \in L^p$. *Assume that* (43) *and* $\|\nabla \cdot v\|_{L^1 L^\infty} < \infty$ *hold. Then, there is a unique solution in* $L^\infty([0,T];L^p)$ *corresponding to* θ_0.

Proof. For simplicity we take $d \equiv 0$. We have

$$\partial_t \theta_\epsilon + v \cdot \nabla \theta_\epsilon = R_\epsilon,$$

and with another mollification in time, we can make θ_ϵ to be a function smooth in time and space, at the cost of adding another term to R_ϵ. Then we can write pointwise

$$\partial_t |\theta_\epsilon|^p + v \cdot \nabla |\theta_\epsilon|^p = p R_\epsilon |\theta_\epsilon|^{p-1} \operatorname{sgn}(\theta_\epsilon).$$

We can then integrate in space, then in time, and take the limit $\epsilon \to 0$ to deduce the result. □

1.3.5 Elementary Inequalities

In this section, we shall collect a few well-known inequalities that will be used throughout the notes. We start with Hardy's inequality.

Lemma 1.3.3 (Hardy)**.** *Let* f *be a smooth function on* $[0,1]$ *which vanishes near* 0. *Then, for any* $\ell \in [0,1]$ *we have*

$$\|x^{-1} f(x)\|_{L^2(0,\ell)} \le 2\|f'(x)\|_{L^2(0,\ell)}, \qquad \|x^{-2} f(x)\|^2_{L^2(0,\ell)} \le 2\|f''(x)\|^2_{L^2(0,\ell)}.$$

Lemma 1.3.4 (Calculus inequalities)**.** *The following inequalities are used to distribute the derivatives in the "Fourier sense." The point is that* s *does not need to be an integer.*

- *For* $s \ge 1$,

$$\left||\xi|^s - |\eta|^s\right| \le C_s |\xi - \eta| \left(|\xi - \eta|^{s-1} + |\eta|^{s-1}\right).$$

- *For* $s \ge 2$,

$$\left||\xi|^s - |\eta|^s - |\xi - \eta|^s\right| \le C_s |\eta||\xi - \eta| \left(|\xi - \eta|^{s-2} + |\eta|^{s-2}\right).$$

- *For* $s \ge 3$,

$$\left||\xi|^s - |\eta|^s - |\xi - \eta|^s - s(\xi - \eta) \cdot \eta |\xi - \eta|^{s-2}\right| \le C_s (|\eta|^2 |\xi - \eta|^{s-2} + |\xi - \eta||\eta|^{s-1}).$$

The above inequalities hold uniformly for any vectors $\xi, \eta \in \mathbb{R}^2$; $C_s > 0$ is a constant depending only on s.

Proof. When s is an even integer, the inequalities are clear by expansion. To establish the inequalities in the general case, one may consider the functions

$$A(\rho) = (1-\rho)\xi + \rho\eta, \quad B(\rho) = (1-\rho)(\xi - \eta)$$

and use the mean value theorem to expand the differences: for instance, we have

$$|\xi|^s - |\eta|^s = \int_0^1 \partial_\rho |A(\rho)|^s \,\mathrm{d}\rho = -s(\eta - \xi) \cdot \int_0^1 |A(\rho)|^{s-2} A(\rho)\,\mathrm{d}\rho,$$

and

$$|\xi|^s - |\eta|^s - |\xi - \eta|^s = -s(\eta - \xi) \int_0^1 \left(|A(\rho)|^{s-2} A(\rho) - |B(\rho)|^{s-2} B(\rho) \right) \mathrm{d}\rho.$$

We leave the details as an exercise. □

1.3.6 Singular integrals

We review some properties of singular integral operators (SIOs). We consider the classical ones, defined by the principal value integration against a translation-invariant kernel:

$$\begin{aligned} \mathbf{P}f(x) &= P.V. \int_{\mathbb{R}^n} P(x-y) f(y) dy \\ &= \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} P(x-y) f(y) dy. \end{aligned}$$

If $f \in C^1 \cap L^1(\mathbb{R}^n)$ and P satisfies the mean zero condition

$$\int_{\mathbb{R}^n} P(y) dy = \lim_{\epsilon \to 0} \int_{\epsilon < |y| < 1/\epsilon} P(y) dy = 0$$

then we may rewrite

$$\mathbf{P}f(x) = \int_{|x-y|<1} P(x-y)(f(y) - f(x)) dy + \int_{|x-y|\geq 1} P(x-y) f(y) dy \tag{45}$$

which can be used to show that $\mathbf{P}f$ is well-defined pointwise. In many applications, it suffices to consider kernels of the *classical type*,

- P is of $-n$ homogeneous; that is, $P(y) = |y|^{-n}\sigma(y/|y|)$ for some σ defined on $\mathbb{S}^{n-1}$,
- σ is sufficiently smooth,

- $\int_{\mathbb{S}^{n-1}} \sigma = 0$.

The primary example is given by the operator $\nabla^2(-\Delta)^{-1}$ in $\mathbb{R}^n$. Another fundamental examples are the Hilbert and Riesz transforms.

The following Hölder estimate is well-known:

Proposition 1.3.5. *If $f \in C^{k,\alpha} \cap L^1(\mathbb{R}^n)$ then $\mathbf{P}f \in C^{k,\alpha}(\mathbb{R}^n)$ for any $k \geq 0$ and $0 < \alpha < 1$, with*

$$\|\mathbf{P}f\|_{\dot{C}^\alpha} \lesssim \frac{1}{\alpha(1-\alpha)} \|f\|_{\dot{C}^\alpha}.$$

We give a sketch of the proof, and all the details can be found in [108].

Proof. We take the case $k = 0$, but the general case is not more difficult. To begin with, from the representation (45) we see that $\mathbf{P}f$ is pointwise well defined when f is just C^α with any $\alpha > 0$. This is because for $|x - y| \ll 1$,

$$|P(x-y)||f(x) - f(y)| \lesssim |x-y|^{\alpha - n},$$

which is integrable in y.

Next, we move on to the $\dot{C}^\alpha$ bound: we write

$$\mathbf{P}f(x) - \mathbf{P}f(x') = \int P(x-y)(f(y) - f(x))dy - \int P(x'-y)(f(y) - f(x'))dy.$$

Here, we have used the mean zero property of σ to insert $f(x)$ and $f(x')$. We need to show that this difference is bounded by a constant times $|x - x'|^\alpha$ in absolute value. For this purpose we split the integral into near and far-field regions; when y is close to either x or x', we can bound each term separately. Say we consider the values of y with $|x - y| < R$ for some R to be determined. Then,

$$\left| \int_{|x-y|<R} P(x-y)(f(x) - f(y))dy \right| \lesssim \|f\|_{\dot{C}^\alpha} \int_{|x-y|<R} \frac{1}{|x-y|^n} |x-y|^\alpha dy \lesssim \|f\|_{\dot{C}^\alpha} R^\alpha.$$

This shows that we can take $R \sim |x - x'|$. Furthermore, the y domain of integration over which we can obtain this bound does not need to be really circular. Note that $|x - y| \lesssim |x - x'|$ implies that $|x' - y| \lesssim |x - x'|$. Therefore, we may combine the differences in the far field region defined by $\{|x - y| > 100|x - x'|\}$ (say) and treat the remainder as in the above. In this far field region, we may rewrite the difference as

$$\int_{|x-y|>100|x-x'|} (P(x-y) - P(x'-y))(f(y) - f(x'))dy$$

again using that

$$\int_{|x-y|>100|x-x'|} P(x-y)(f(x) - f(x'))dy = 0.$$

Then,

$$
\begin{aligned}
&\left| \int_{|x-y|>100|x-x'|} (P(x-y) - P(x'-y))(f(y) - f(x'))dy \right| \\
&\lesssim \|f\|_{\dot{C}^\alpha} \int_{|x-y|>100|x-x'|} |\nabla P(x^* - y)||x-x'||y-x'|^\alpha dy \\
&\lesssim \|f\|_{\dot{C}^\alpha} \int_{|x-y|>100|x-x'|} |y-x'|^{-n-1}|x-x'||y-x'|^\alpha dy \\
&\lesssim \|f\|_{\dot{C}^\alpha} |x-x'|^\alpha.
\end{aligned}
$$

Here, we have used the mean value theorem to P and used that $|x^* - y| \gtrsim |x - x'|$ holds for any point x^* between x and x'. This finishes the proof. $\square$

1.3.7 Estimate on Riesz kernels

Following the computations in the case of SIOs, one can obtain estimates on the Riesz kernels $(-\Delta)^\beta$ and their generalizations. In the context of α-SQG equations, this immediately gives regularity of the stream function. To this end we consider for $0 < \alpha < 2$

$$
\psi(x) = \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} \theta(y) dy,
$$

assuming that $\theta \in C^\beta \cap L^1(\mathbb{R}^2)$. From scaling, we expect ψ to be more regular by order $2 - \alpha$.

We mainly consider the case $1 < \alpha < 2$. When $0 < \alpha < 1$, we can differentiate both sides once in x to reduce to the former case. Furthermore, the case $\alpha = 1$ leads to a SIO after a differentiation, which we have treated in the above.

To begin with, it is not difficult to see that ψ is pointwise well-defined:

$$
\begin{aligned}
|\psi(x)| &\le \int_{|x-y|\le R} + \int_{|x-y|>R} \frac{1}{|x-y|^\alpha} |\theta(y)| dy \\
&= I + II,
\end{aligned}
$$

and

$$
I \le \|\theta\|_{L^\infty} \int_{|x-y|\le R} \frac{1}{|x-y|^\alpha} dy \le C R^{2-\alpha} \|\theta\|_{L^\infty},
$$

$$
II \le \frac{1}{R^\alpha} \|\theta\|_{L^1}.
$$

Now, we expect ψ to be differentiable with order $2 - \alpha + \beta$. Assume for a moment that this does not exceed 1. Consider

$$
\psi(x) - \psi(x') = \int \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] \theta(y) dy
$$

We consider the region

$$I = \int_{y:|(x+x')/2-y|<2|x-x'|} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] \theta(y) dy.$$

Denote the domain by A. A change of variables give

$$I = \int_A \frac{1}{|x-y|^\alpha} (\theta(y) - \theta(y - x + x')) dy.$$

Then,

$$\begin{aligned} |I| &\leq \int_A \frac{1}{|x-y|^\alpha} dy \|\theta\|_{\dot{C}^\beta} |x-x'|^\beta \\ &\leq C \|\theta\|_{\dot{C}^\beta} |x-x'|^{2-\alpha+\beta}. \end{aligned}$$

Next,

$$II = \int_{A^c} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] \theta(y) dy.$$

We may rewrite

$$\begin{aligned} 2II = &\int_{A^c} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] (\theta(y) - \theta(x)) dy \\ &+ \int_{A^c} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] (\theta(y) - \theta(x')) dy. \end{aligned}$$

Here, the point is that

$$\int_{A^c} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] dy = 0$$

by antisymmetry in x and x'. In the region A^c, we may estimate

$$\begin{aligned} \left| \int_{A^c} \left[\frac{1}{|x-y|^\alpha} - \frac{1}{|x'-y|^\alpha} \right] (\theta(y) - \theta(x)) dy \right| &\leq C \|\theta\|_{\dot{C}^\beta} |x-x'| \int_{A^c} \frac{1}{|x-y|^{\alpha+1}} |x-y|^\beta \\ &\leq C \|\theta\|_{\dot{C}^\beta} |x-x'|^{2-\alpha+\beta} \end{aligned}$$

using the mean value theorem. The other term can be estimated in the same way. We have arrived at the following:

Proposition 1.3.6. *We have that* $\psi \in C^{2-\alpha+\beta}$*, assuming that* $0 < 2 - \alpha + \beta < 1$.

1.4 Notation

We shall use the letters c, C to denote various constants which may change from a line to another. We write $A \lesssim B$ (or equivalently $B \gtrsim A$) if $A \leq CB$ holds for some absolute constant $C > 0$. Next, we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Given some large parameter, we shall write $A \simeq B$ or $A \approx B$ if $A/B \to 1$ as the large parameter goes to infinity.

1.4.1 Standard function spaces

We introduce various norms that will be used.

- Lebesgue spaces: given a function f and $1 \leq p < \infty$, we define

$$\|f\|_{L^p}^p := \int |f|^p \, dx.$$

  In the case $p = \infty$,

$$\|f\|_{L^\infty} := \operatorname{esssup}|f(x)|.$$

  We shall use the same notation when f is a vector or matrix-valued function.

- L^2-based Sobolev spaces: Given a scalar valued function f, and an integer $m \geq 0$, we shall use the notation $\nabla^m f$ to denote the 2^m-dimensional vector consisting of all possible partial derivatives of f of order m. Then, we define

$$\|f\|_{\dot{H}^m} := \|\nabla^m f\|_{L^2}, \quad \|f\|_{H^m}^2 := \sum_{j=0}^{m} \|f\|_{\dot{H}^j}^2.$$

- Sobolev spaces: similarly, we consider for $1 \leq p < \infty$

$$\|f\|_{\dot{W}^{m,p}} := \|\nabla^m f\|_{L^p}, \quad \|f\|_{W^{m,p}}^2 := \sum_{j=0}^{m} \|f\|_{\dot{W}^{j,p}}^2.$$

- Hölder spaces: $C^{k,\alpha}$, where $k \in \mathbb{N} \cup \{0\}$ and $0 \leq \alpha \leq 1$. It is important to distinguish the endpoint cases $C^{k,1}$ with $C^{k+1,0}$. For $k = 0$ and $0 < \alpha \leq 1$, we define

$$\|f\|_{\dot{C}^{0,\alpha}} = \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}.$$

  Then,

$$\|f\|_{C^{0,\alpha}} = \|f\|_{\dot{C}^{0,\alpha}} + \|f\|_{L^\infty}.$$

On the other hand, $C^{0,0}$ denotes simply the space of continuous and bounded functions C^0. Then, $\dot{C}^{k+1,\alpha}$ can be defined as the space of functions (with a corresponding norm) whose first order partial derivatives exist and belong to $\dot{C}^{k,\alpha}$.

- Bounded mean oscillation (BMO): we say $f \in BMO$ if

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| dx$$

is uniformly bounded for all cubes Q, where

$$f_Q = \frac{1}{|Q|}\int_Q f(x) dx$$

is the mean of f in Q. We can use the supremum of the above quantity to define the BMO seminorm. Note that bounded functions belong to BMO but not vice versa.

1.4.2 Analytic function space

Definition 1.4.1 (Analytic function)**.** *We say $f \in H^\infty(\mathbb{T})$ is **analytic** if there exist constants $\rho > 0, M > 0$ such that for all $k \in \mathbb{N}$*

$$\|\partial^k f\|_{L^\infty(\mathbb{T})} \le M\frac{k!}{\rho^k}.$$

We denote the space of analytic functions by $C^\omega(\mathbb{T})$.

The definition can be generalized to higher dimensional domains in a similar way, by requiring the above bound to any multi-index. One can also treat $\mathbb{R}^d$ or $\mathbb{R}^k \times \mathbb{T}^{d-k}$ in a similar way. We observe that if f is analytic in the above sense, then the Taylor series converges locally; indeed using the Taylor theorem we have that for any N

$$\left| f(x) - \sum_{k \le N} \frac{\partial^k f(a)}{k!}(x-a)^k \right| \le C\|\partial^{N+1} f\|_{L^\infty} \frac{|x-a|^{N+1}}{(N+1)!}.$$

Using the definition of analyticity, we see that the right hand side goes to zero as $N \to \infty$, as long as $|x-a| < \rho$. That is, we see that the constant ρ in the definition corresponds to the **radius of analyticity**.

A fundamental observation which is the basic tool in studying PDEs in analytic class is given:

Proposition 1.4.2 (Sobolev characterization of analytic class)**.** *Fix some $r \ge 0$. Then $f \in C^\omega(\mathbb{T})$ if and only if there exist $\rho > 0, M > 0$ such that*

$$\|\partial^n f\|_{H^r(\mathbb{T})} \le M\frac{n!}{\rho^n}. \tag{46}$$

Proof. Exercise using the Sobolev embedding. ☐

Proposition 1.4.3 (An exhaustion structure of C^ω)**.** *Fix some $r \geq 0$. Then we have*

$$C^\omega(\mathbb{T}) = \bigcup_{\tau > 0} e^{-\tau\Lambda} H^r(\mathbb{T}).$$

Here, the Banach space $e^{-\tau\Lambda} H^r(\mathbb{T})$ is defined by

$$e^{-\tau\Lambda} H^r(\mathbb{T}) = \left\{ f \in H^r(\mathbb{T}) : \| e^{\tau\Lambda} f \|_{H^r} < \infty \right\}.$$

Remark 1.4.4. *Note that by definition, $e^{-\tau\Lambda} H^r(\mathbb{T})$ consists of functions which are given by applying the operator $e^{-\tau\Lambda}$ to H^r–functions. The negative sign is not a typo.*

Proof. Assume that $f \in C^\omega(\mathbb{T})$. Then we compute

$$\begin{aligned} \| e^{\tau\Lambda} f \|_{H^r}^2 &= \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} \sum_k (1+k^2)^r k^m |\hat{f}_k|^2 \\ &\leq \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} M^2 \frac{((m/2)!)^2}{\rho^m}. \end{aligned}$$

We have used (46) with

$$\| f \|_{H^{m/2}}^2 \leq \| f \|_{L^2} \| f \|_{H^m}.$$

Using Stirling's formula and Hadamard root test, we see that the summation in m is convergent whenever $\tau < \rho$. The other direction is easier and left as an exercise. Inspecting the proof shows that $\tau > 0$ exactly corresponds to the radius of analyticity. ☐

Proposition 1.4.5 (Algebra property)**.** *Let $\tau > 0$ and $r > 1/2$. Then $e^{-\tau\Lambda} H^r(\mathbb{T})$ is an algebra. That is, it is closed under multiplication with estimate*

$$\| fg \|_{e^{-\tau\Lambda} H^r(\mathbb{T})} \leq C_r \| f \|_{e^{-\tau\Lambda} H^r(\mathbb{T})} \| g \|_{e^{-\tau\Lambda} H^r(\mathbb{T})}.$$

Proof. Exercise. This is precisely the reason why we inserted r in the Sobolev characterization of C^ω in the first place. ☐

2 Wellposedness

Let us give a brief overview of this section. To begin with, in §2.1, we present several explicit computations, which are rather elementary but turn out to be very helpful in understanding the dynamics of gSQG equations. In §2.2, we consider the problem of local regularity of smooth solutions. This will guarantee the unique existence of a smooth solution associated with a smooth initial datum, at least locally in time. We consider local regularity in both Sobolev and Hölder spaces. Several blow up criteria are given. Local regularity in analytic function spaces is treated in §2.3. Then in §2.4, we study local regularity of smooth patches, which constitute a class of weak solutions but can be treated analogously to smooth solutions. Next, §2.5 gives the global in time existence of weak solutions. In particular, even after potential singularity formation for smooth solutions, there is always at least one weak solution.

2.1 Explicit computations

We shall perform a few explicit computations.

2.1.1 Radial solutions and the circular patch

Recall that any radial profile $\theta(x) = f(|x|)$ defines a steady state. However, we can see how the form of the velocity changes depending on the regularity of the kernel. As a particular case, one may take $\bar{\theta} = \mathbf{1}_B$, where $B = \{|x| \leq 1\}$ is the unit disc. This steady state is sometimes referred to as the Rankine vortex in the 2D Euler case. We start with computing the associated stream function:

$$\bar{\psi}(x) = \int_{|y|\leq 1} \frac{1}{|x-y|^\alpha} dy.$$

Here, a trick is to differentiate the relation

$$\Lambda^{-2+\alpha}\bar{\theta} = \bar{\psi}$$

in r: on the right hand side, we obtain the angular part of the velocity $\bar{u}^\theta$, which equals

$$\begin{aligned} -\Lambda^{-2+\alpha}\delta_{\partial B} &= \int_{|y|=1} \frac{1}{|x-y|^\alpha} d\sigma(y) \\ &= \int_0^{2\pi} \frac{d\theta}{(r^2+1-2r\cos(\theta))^{\alpha/2}}. \end{aligned}$$

This formula can be used to compute the rotation speed of ∂B, and the regularity of $\bar{u}$. We see that if $r > 0$ is away from $r = 1$, this defines a smooth function of r. When $r = 1$, for $0 < \alpha < 1$ we may compute that

$$\bar{u}^\theta(1) = \int_0^{2\pi} \frac{d\theta}{(2 - 2\cos(\theta))^{\alpha/2}} = \frac{1}{2^{\alpha/2}} 2^{1-\alpha/2} \sqrt{\pi} \frac{\Gamma((1-\alpha)/2)}{\Gamma(1-\alpha/2)}.$$

This quantity becomes infinite as $\alpha \to 1^-$. That is: *the tangential velocity for the SQG circular patch is infinite.* Now we consider the regularity of $\bar{u}^\theta$. When $\alpha = 1$, we have already seen that $\bar{u}^\theta \notin L^\infty$; this is not surprising since $\bar{u}^\theta$ is a Riesz transform applied to $\bar{\theta}$, which is merely L^∞, and Riesz transforms do not respect L^∞. The BMO bound of Riesz transforms show that in this case the tangential velocity blows up like a log at $r = 1$. Moving on to the case of $0 < \alpha < 1$, note that the kernel is uniformly smooth away from the region $\theta = 0$. Hence, we may replace $\sin\theta \approx \tau$ and $\cos\theta \approx 1$. Writing $\xi = r - 1$, we are led to

$$\bar{u}^\theta(r) \sim \int_0^1 \frac{1}{(\xi^2 + \tau^2)^{\alpha/2}} d\tau.$$

While this is uniformly bounded in ξ, differentiating gives

$$\partial_r \bar{u}^\theta(r) \sim \int_0^1 \frac{\xi}{(\xi^2 + \tau^2)^{\alpha/2+1}} d\tau \sim \xi^{-\alpha}.$$

This immediately shows that $\bar{u}^\theta$ belongs to $C^{1-\alpha}$ but not better, which makes sense; in the 2D Euler case $\alpha = 0$, $\bar{u}^\theta \in C^{0,1}$.

Lastly, in the intermediate regime, we can see that $\bar{u}^\theta$ is not bounded at $r = 1$ and blows up with the rate dictated by the $C^{1-\alpha}$ regularity.

Similarly, one can consider shear steady states: $\bar{\theta} = f(x_2)$, where f is a smooth and decaying function of one variable. Formally, it defines a steady state.

Problem 2.1.1. *Compute the corresponding velocity vector field.*

2.1.2 Effects of symmetry

We consider the case of 2D Euler for simplicity, and understand the effect of having rotational symmetries of the vorticity on the vorticity. It is helpful to first recall the case of radial vortex, namely the vorticity given by

$$\omega(r, \theta) = h(r)$$

for some function $h : [0, \infty) \to \mathbb{R}$. Here (r, θ) denotes the usual polar coordinates on $\mathbb{R}^2$. Using the polar coordinates, the velocity can be written in general

$$u(r, \theta) = u^r(r, \theta)\mathbf{e}^r + u^\theta(r, \theta)\mathbf{e}^\theta$$

and we shall refer to u^r and u^θ as radial and angular components of the velocity, respectively. Then, under the assumption $\omega(r,\theta) = h(r)$, we recall that

$$u(r,\theta) = \left(\frac{1}{r}\int_0^r sh(s)ds\right)\mathbf{e}^\theta.$$

It should be emphasized that this simple formula encapsulates many cancellations. To begin with, the radial part of the velocity completely vanishes, and u^θ is only dependent on r. More interestingly, the formula for $u^\theta(r)$ involves only the values of the vorticity in the ball $B(0,r)$. In other words, the contributions to the Biot–Savart law coming from the vorticity in $\mathbb{R}^2\backslash B(0,r)$ completely cancel with each other. There is one more cancellation that can be seen this formula, which is that if

$$\int_{B(0,r_0)} \omega dx = 0,$$

then the above formula can be replaced with

$$u^\theta(r) = \frac{1}{r}\int_{r_0}^r sh(s)ds$$

for any r. In particular, in the region $r \geq r_0$, this formula shows that the distribution of the vorticity in the region $B(0,r_0)$ does not matter at all! This fact has many interesting consequences.

2.1.3 Solutions with m-fold rotational symmetry

In general, any vorticity can be expanded in the form

$$\omega(r,\theta) = \sum_m \left(h_{m,s}(r)\sin(m\theta) + h_{m,c}(r)\cos(m\theta)\right)$$

where $m \geq 1$ for the case of sines and $m \geq 0$ for cosines. This is nothing but the Fourier series expansion repeated for any $r > 0$. We say that ω is m-fold symmetric if

$$\omega(r,\theta) = \omega(r,\theta + 2\pi/m), \qquad \forall r,\theta.$$

In terms of the previous expansion, m-fold symmetry holds if and only if $h_{k,s}$ and $h_{k,c}$ identically vanish for all k which are not integer multiples of m. For simplicity, we take

$$\omega(r,\theta) = h(r)\sin(m\theta)$$

for some scalar valued function h and study its implications on the velocity. It turns out that the stream function is given by the formula

$$\Delta^{-1}\omega = H(r)\sin(m\theta), \qquad H(r) = \frac{1}{r^m}\int_0^r s^{2m-1}\int_s^\infty \frac{h(\tau)}{\tau^{m-1}}$$

so that

$$u(r,\theta) = H'(r)\sin(m\theta)\,\mathbf{e}^\theta - \frac{mH(r)}{r}\cos(m\theta)\,\mathbf{e}^r.$$

In this explicit and simple computation, we immediately see that the case $m = 2$ is distinguished: this is because the last integral

$$\int_s^\infty \frac{h(\tau)}{\tau^{m-1}}$$

has the logarithmic divergence for $h \sim 1$. To see this effect more clearly, we take

$$\omega(r,\theta) = \mathbf{1}_{[0,R]}(r)\sin(m\theta), \quad \text{i.e.} \quad h(r) = \mathbf{1}_{[0,R]}(r).$$

Then, we can explicitly compute H: in the region $r \le R$,

$$H(r) = \begin{cases} \frac{r^2}{m^2-4} + \frac{R^2}{2m(2-m)}(\frac{r}{R})^m, & m \neq 2, \\ \frac{r^2}{4}\ln\frac{R}{r} + \frac{r^2}{16} & m = 2. \end{cases}$$

Focusing on the case $m = 2$, from the formula

$$H(r) = \frac{r^2}{4}\ln\frac{R}{r} + \frac{r^2}{16},$$

we see that

$$\begin{cases} u^\theta = (\frac{r}{2}\ln\frac{R}{r} + O(r))\sin(2\theta), \\ u^r = (-\frac{r}{2}\ln\frac{R}{r} + O(r))\cos(2\theta). \end{cases}$$

In particular, even though the vorticity is bounded, $|\nabla u| \gtrsim \ln\frac{R}{r}$. This is indeed the "standard" counterexample to the L^∞ bound for the singular integral operator $\nabla^2\Delta^{-1}$. A systematic treatment is given in Elgindi [53].

Coming back to the formula for $m \neq 2$, it is interesting to see what happens for the case $h(r) = r^a$, where a is allowed to be any real number. (We do not tackle the question of the uniqueness of the operator Δ^{-1} here.) Explicitly, if $a + 2 \neq \pm m$ then

$$\omega = r^a\sin(m\theta), \quad \Delta^{-1}\omega = \frac{1}{(a+2)^2 - m^2}r^{a+2}\sin(m\theta).$$

Then, the convective derivative becomes

$$\mathbf{u}\cdot\nabla\omega = u^\theta\frac{1}{r}\partial_\theta\omega + u^r\partial_r\omega = -\frac{\sin(2m\theta)}{m}r^{2a} + O(\frac{r^{2a}}{m^2}).$$

The flow becomes a specific perturbation of the radial vortex in the limit $m \to \infty$.

Proof of the above explicit formulae is simple. We write the ansatz $\Delta^{-1}\omega = H(r)\sin(m\theta)$. Then from

$$h(r)\sin(m\theta) = \Delta(H(r)\sin(m\theta)),$$

we obtain the ODE

$$\partial_{rr}H + \frac{1}{r}\partial_r H - \frac{m^2}{r^2}H = h.$$

This type of ODEs are referred to as *Euler's equations*. It can be integrated: to begin with,

$$\partial_r\left(\frac{m}{r^m}H + \frac{1}{r^{m-1}}\partial_r H\right) = \frac{h}{r^{m-1}}$$

and

$$\frac{1}{r^{2m-1}}\partial_r(r^m H) = \int_r^\infty \frac{h(s)}{s^{m-1}}$$

which gives

$$H(r) = \frac{1}{r^m}\int_0^r s^{2m-1}\int_s^\infty \frac{h(\tau)}{\tau^{m-1}}.$$

Lastly, from the above form of the stream function, we can recover velocity by $u^r = -\frac{1}{r}\partial_\theta\Delta^{-1}\omega$ and $u^\theta = \partial_r\Delta^{-1}\omega$.

2.1.4 Bahouri–Chemin state

On $\mathbb{T}^2$, which we represent by $[-1,1]^2$, consider the function

$$\omega(x_1, x_2) = \operatorname{sgn}(x_1 x_2).$$

This is sometimes referred to the Bahouri–Chemin steady state. (Check that it is indeed a steady state to the gSQG equations.) Based on the discussion above, one can consider the following problems.

Problem 2.1.2. *For each $0 < \alpha \le 2$, calculate the asymptotic of $\Lambda^{-\alpha}\omega$ and $\nabla\Lambda^{-\alpha}\omega$ near $x = 0$. Why is this steady state interesting?*

2.1.5 Singular steady states

Elgindi–Huang constructed singular steady states to the 2D Euler equations, which have the radially homogeneous form. One motivation comes from the study of dynamics of singular (weak) solutions to Euler. More specifically, Theorem 3.1 of [52] gives, for each $0 < s < 1$, a positive solution to

$$\Delta\psi = -\frac{1}{\psi^s}, \qquad \text{in} \qquad (\mathbb{R}_+)^2. \tag{47}$$

One may consider this equation for $s < 0$, which gives rise to more regular steady states ([1]). The stream function ψ is strictly positive on the quadrant $(\mathbb{R}_+)^2$ and then one can extend it as an odd function of both variables in $\mathbb{R}^2$. The proof is given in the Appendix of their paper. Following their notation, this solution satisfies radial homogeneous with degree $-\alpha$

$$\psi(r, \theta) = r^{-\alpha+1} K(\theta) \tag{48}$$

where $K(0) = K(\pi/2) = 0$ and $K > 0$ on $\theta \in (0, \pi/2)$. Then, comparing this with (47), one can check that

$$s = \frac{1+\alpha}{1-\alpha}.$$

The range $s \in [0, 1]$ corresponds to $\alpha \in [-1, 0]$. Abe excluded the existence of homogeneous solutions in this range assuming $u \in C^2(\mathbb{R}^2 \backslash \{0\})$. This is not inconsistent with their result because the solutions obtained does not satisfy $u \in C^2(\mathbb{R}^2 \backslash \{0\})$. Actually, u is only Hölder continuous. Both limits $s \to 0$ ($\alpha = -1$) and $s \to 1$ ($\alpha = 0$) are interesting. In the limit $s \to 0$, the sequence of steady states converge to the Bahouri–Chemin state where the vorticity is identically 1 on the positive quadrant. In the limit $s \to 1$, the solutions converge to a vortex sheet.

To see how one can get existence of solutions, plugging in the ansatz (48) to (47), one obtains the equation

$$\frac{4}{(1+s)^2} K + K'' = -\frac{1}{K^s}, \qquad \theta \in (0, \pi/2). \tag{49}$$

One can impose the additional condition $K'(\pi/4) = 0$. This means that one seeks for solutions which are even symmetric across the diagonal line. The key trick is to multiply both sides of (49) by K' and observe that

$$\left(\frac{(K')^2}{2} + \frac{2}{(1+s)^2} K^2 + \frac{2}{1-s} K^{1-s} \right)' = 0.$$

When $s \leq 1$, one can integrate both sides and use that the value of $K(0) = 0$ by the boundary condition. In the case $s > 1$, one should be more careful. Directly integrating gives

$$\frac{(K')^2}{2} + \frac{2}{(1+s)^2} K^2 + \frac{2}{1-s} K^{1-s} = C_s$$

and then we obtain that

$$K' = \pm 2\left(C_s - \frac{1}{(1+s)^2}K^2 - \frac{1}{1-s}K^{1-s}\right)^{1/2}.$$

We choose the + sign, since we are interested in solutions K increasing from 0 at $\theta = 0$ to the maximal value at $\theta = \pi/4$ where the derivative is zero for the first time. This gives the relation

$$C_s = \frac{1}{(1+s)^2}K^2(\pi/4) - \frac{1}{s-1}K^{1-s}(\pi/4) \tag{50}$$

The function $X \mapsto \frac{1}{(1+s)^2}X^2 - \frac{1}{s-1}X^{1-s}$ is strictly increasing on $X > 0$ and its range cover entire real line. Therefore, for any $C_s \in \mathbb{R}$, there is a unique positive real solution to (50), which we denote by $k(C_s)$.

We need to prove that there exists a choice of C_s such that $K(0) = 0$. For this we write

$$\frac{dK}{2\left(C_s - \frac{1}{(1+s)^2}K^2 + \frac{1}{s-1}K^{1-s}\right)^{1/2}} = d\theta$$

and integrate from $\theta = 0$ to $\pi/4$. This gives

$$f(C_s) := \int_0^{k(C_s)} \frac{dk}{2\left(C_s - \frac{1}{(1+s)^2}k^2 + \frac{1}{s-1}k^{1-s}\right)^{1/2}} = \frac{\pi}{4}.$$

The function on the LHS is integrable near $k = 0$. As $C_s \to -\infty$, we have that $k(C_s) \to 0^+$. Therefore, $f(-\infty) = 0$. When $s = 3$, one can check that $\lim_{C_s \to \infty} f(C_s) = \pi$. In general, one can require $2m$-fold symmetry instead of 2-fold, which gives existence by taking C_s satisfying $f(C_s) = \pi/(2m)$.

Problem 2.1.3. *In the special case $s = 3$, this integral can be calculated explicitly. Find K explicitly.*

2.1.6 Singular self-similar states

In the case of α-SQG equations, it is possible to combine scaling symmetries in t and x to obtain equations for the singular and self-similar solutions. However, it should be mentioned that it is highly non-trivial to prove *existence* to such reduced equations (this was achieved for instance in [61, 66]). To see how it is done, consider the ansatz

$$\omega(t,x) = t^{-1}\Omega(t^{-\mu}r, \theta)$$

in the Euler case, where (r, θ) is the polar coordinates. Here, $\mu > 0$ is some positive constant, and in terms of the rescaled radial variable $z = t^{-\mu} r$, it is further assumed that Ω satisfies the "boundary condition:"

$$\lim_{z \to \infty} \Omega(z, \theta) = G(\theta)$$

for some given function G on the unit circle. (Note that for each fixed t, $\omega(t, \cdot)$ has a singularity of order $r^{-1/\mu}$ as $r \to 0$.) Then 2D Euler reduces to a steady transport system defined on $\mathbb{R}^2$:

$$-\Omega - \mu z \partial_z \Omega + \nabla^\perp \Psi \cdot \nabla \Omega = 0,$$

where Ψ is the stream function corresponding to Ω, and $\nabla, \nabla^\perp$ are taken with respect to the new coordinates system (z, θ). The exponents $\mu = 0$ and 1 are special and correspond to 0-homogeneous data and sheets of uniform density. If G is N-periodic, namely $G(\theta) = \mathring{g}(N\theta)$ for some 2π-periodic function $\mathring{g}$, then naturally one can look for N-periodic solutions. The above computations show that as $N \to \infty$, the nonlinearity becomes $O(N^{-1})$ and therefore it makes sense to consider it perturbatively. See the seminar work of Elling [61].

Problem 2.1.4. *Calculate the flow map (more or less explicitly, or at least asymptotically) for all of the explicit solutions mentioned in this section.*

2.2 Local wellposedness for smooth solutions

In this section, we shall treat the most basic question, which is the local existence and uniqueness of sufficiently smooth solutions. It will be convenient to separate the questions of existence and uniqueness.* We first consider the uniqueness problem, which is simpler. For simplicity, most of the time we shall restrict ourselves to the case of α-SQG, where the Biot–Savart law is simply $u = -\nabla^\perp \Lambda^{-2+\alpha}$. Recall that $\alpha = 0$ corresponds to the 2D Euler case.

2.2.1 Uniqueness

A uniqueness statement is usually framed in terms of a stability result.

Lemma 2.2.1. *We have uniqueness of the solution to α-SQG in the regular regime, under the assumption that $\theta \in L^\infty([0, T]; H^s)$ with $s > 1 + \alpha$. This follows from the*

*In general, given a function space, existence does not imply uniqueness and vice versa.

following stability estimate. Let θ and $\bar{\theta}$ be two solutions belonging to $L^\infty([0,T];H^s)$ to gSQG with initial data θ_0 and $\bar{\theta}_0$, respectively. Then we have

$$\|(\theta-\bar{\theta})(t)\|_{L^2} \leq C\|\theta_0-\bar{\theta}_0\|_{L^2}\exp\left(\int_0^t (\|\theta(\tau)\|_{H^s}+\|\bar{\theta}(\tau)\|_{H^s})d\tau\right).$$

Remark 2.2.2. *Note that the uniqueness statement is "conditional" in the sense that the uniqueness holds upon assuming some specific regularity of the solution. In the case of the above, the precise statement is that:* ***assuming*** *that there is a solution belonging to the space $L^\infty([0,T];H^s)$, then there cannot be two different solutions in $L^\infty([0,T];H^s)$ satisfying the same initial condition. Here, one might be worried that continuity in time of the solution is not assumed, but actually $L^\infty([0,T];H^s)$ automatically implies $C_*([0,T];H^s)$ (weakly continuous in time), using the fact that θ is a solution to the gSQG equation. Of course, the uniqueness statement will be slightly weaker if one simply assumes $C([0,T];H^s)$ instead.*

Proof. Let us assume that there are two solutions $\theta, \tilde{\theta}$ to (gSQG), and denote the corresponding velocities by u, $\tilde{u}$, respectively. Then, by taking the difference we have

$$\partial_t(\theta-\tilde{\theta}) = -(u-\tilde{u})\cdot\nabla\theta - \tilde{u}\cdot\nabla(\theta-\tilde{\theta}).$$

Then,

$$\frac{1}{2}\frac{d}{dt}\|\theta-\tilde{\theta}\|_{L^2}^2 \leq C\|\theta-\tilde{\theta}\|_{L^2}\|(u-\tilde{u})\cdot\nabla\theta\|_{L^2}.$$

For simplicity, we shall assume that there exists some $1\leq q\leq\infty$ such that

$$\|u-\tilde{u}\|_{L^q} \leq C\|\theta-\tilde{\theta}\|_{L^2}$$

holds, which would give

$$\frac{1}{2}\frac{d}{dt}\|\theta-\tilde{\theta}\|_{L^2}^2 \leq C\|\nabla\theta\|_{L^{q^*}}\|\theta-\tilde{\theta}\|_{L^2}^2$$

where q^* is the dual exponent of q. Using

$$\|\nabla\theta\|_{L^{q^*}} \leq C\|\theta\|_{H^s},$$

(this is the place where the assumption $\theta\in H^s$ with $s>1+\alpha$ is used) we obtain

$$\|\theta-\tilde{\theta}\|_{L^2}^2(t) \leq \|\theta-\tilde{\theta}\|_{L^2}^2(t=0)\exp\left(\int_0^t \|\theta(\tau,\cdot)\|_{H^s}d\tau\right).$$

This is a stability estimate which works for any two solutions. In particular, when the initial data coincide, namely if $\theta_0=\tilde{\theta}_0$, then we have $\theta=\tilde{\theta}$ for any t as long as the solution remains in H^s. □

The proof shows that it suffices to have $\nabla\theta \in L^\infty_t L^{q^*}$ for uniqueness. However, when the solution belongs to only $L^\infty([0,T]; C^\beta)$, this proof does not work since the gradient in general does not have some integrability. One can still obtain uniqueness with a different proof.

Lemma 2.2.3. *We have uniqueness of the solution to α-SQG in the regular regime, under the assumption that $\theta \in L^\infty([0,T]; C^\beta \cap H^{-1})$ with $\beta > \alpha$. This follows from the following stability estimate. Let θ and $\bar\theta$ be two solutions belonging to $L^\infty([0,T]; C^\beta \cap H^{-1})$ to gSQG with initial data θ_0 and $\bar\theta_0$, respectively. Then we have*

$$\|(\theta - \bar\theta)(t)\|_{\dot H^{-1}} \leq C\|\theta_0 - \bar\theta_0\|_{\dot H^{-1}} \exp\left(\int_0^t (\|\theta(\tau)\|_{C^\beta} + \|\bar\theta(\tau)\|_{C^\beta}) d\tau\right).$$

Proof. We just sketch the argument. Assume that $\theta, \bar\theta$ are two solutions. We write $u, \bar u$ be the corresponding velocities. Then we compute with

$$\zeta := \nabla(-\Delta)^{-1}(\theta - \bar\theta)$$

that

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|_{L^2}^2 = -\int (u - \bar u)\cdot \nabla\theta(-\Delta)^{-1}(\theta - \bar\theta) - \int \bar u \cdot \nabla(\theta - \bar\theta)(-\Delta)^{-1}(\theta - \bar\theta)$$

The first term equals, using divergence-free property,

$$-\int (u - \bar u)\cdot \nabla\theta(-\Delta)^{-1}(\theta - \bar\theta) = \int \theta(u - \bar u)\cdot \nabla(-\Delta)^{-1}(\theta - \bar\theta) = \int \theta\Lambda^\alpha(\zeta^\perp)\cdot\zeta$$

One can expand

$$\begin{aligned}\int \theta\Lambda^\alpha(\zeta^\perp)\cdot\zeta &= C_\alpha \iint \theta(x)\frac{\zeta^\perp(x) - \zeta^\perp(y)}{|x-y|^{2+\alpha}}\cdot\zeta(x)dydx \\ &= -\frac{C_\alpha}{2}\iint \frac{(\theta(x) - \theta(y))}{|x-y|^{2+\alpha}}\zeta^\perp(y)\zeta(x)dydx\end{aligned}$$

Using that $\theta \in C^\beta$ for $\beta > \alpha$, we can bound the integral by

$$\left|\int \theta\Lambda^\alpha(\zeta^\perp)\cdot\zeta\right| \leq C\|\theta\|_{C^\beta}\|\zeta\|_{L^2}^2$$

The second term equals

$$\int (\nabla\cdot\zeta)\,(u\cdot\zeta) = -\sum_{i,j}\int \zeta_i\partial_i u_j\zeta_j$$

after integration by parts and using that ζ is a gradient. Then we can bound

$$\left|\int (\nabla\cdot\zeta)\,(u\cdot\zeta)\right| \leq C\|\nabla u\|_{L^\infty}\|\zeta\|_{L^2}^2 \leq C\|\theta\|_{C^\alpha}\|\zeta\|_{L^2}^2.$$

This finishes the proof. $\square$

Let us now describe Yudovich's uniqueness theorem, which handles the "borderline" case of two stability results in the above.

Lemma 2.2.4 (Yudovich [135]). *Consider the two-dimensional Euler equations, with initial data $\omega_0 \in L^\infty(\mathbb{T}^2)$. There is a unique solution in the class $L^\infty([0,T]; L^\infty(\mathbb{T}^2))$ for any $T > 0$.*

Proof. This highly non-trivial result of Yudovich requires a clever trick of optimizing in the Hölder inequality. (By the way, there is an alternative proof based on the Lagrangian formulation, see [111].) We are going to assume that $u_0 = 0$ for simplicity. Since $u = 0$ is a solution, the goal is to prove that $u = 0$. To begin with, we need to consider the velocity formulation:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0,$$

since the velocity is a more regular variable. We would like to understand

$$\frac{d}{dt} \int |u|^2 = -2 \int (u \cdot \nabla u) \cdot u.$$

Naively, we would like to bound the RHS by

$$\left| -2 \int (u \cdot \nabla u) \cdot u \right| \leq C \|\nabla u\|_{L^\infty} \int |u|^2$$

but unfortunately $\omega \in L^\infty$ does not guarantee that $\|\nabla u\|_{L^\infty} < \infty$. Instead, we first use Hölder's inequality for some $p > 2$:

$$\left| -2 \int (u \cdot \nabla u) \cdot u \right| \leq C \|u\|_{L^2} \|\nabla u\|_{L^p} \|u\|_{L^{p'}}$$

where p' is defined by

$$\frac{1}{2} + \frac{1}{p} + \frac{1}{p'} = 1.$$

In turn, we bound

$$\|u\|_{L^{p'}} \leq C \|u\|_{L^2}^{\alpha} \|\nabla u\|_{L^p}^{1-\alpha},$$

where the value of $0 < \alpha < 1$ can be found by comparing the scaling exponents in both sides of the inequality:

$$-\frac{2}{p'} = \alpha(-1) + (1-\alpha)(1 - \frac{2}{p}),$$

or we obtain

$$\alpha = \frac{p-2}{p-1}.$$

Summarizing,

$$\left|-2\int(u\cdot\nabla u)\cdot u\right|\leq C\|u\|_{L^2}^{1+\alpha}\|\nabla u\|_{L^p}^{2-\alpha}.$$

Note that as $p\to\infty$, we have that $\alpha\to 1$, which makes sense. Now, we recall the singular integral bound

$$\|\nabla u\|_{L^p}\leq Cp\|\omega\|_{L^p}\leq Cp\|\omega_0\|_{L^p}$$

for $p\geq 2$, say. Applying this inequality,

$$\frac{d}{dt}\|u\|_{L^2}^2\leq Cp^{2-\alpha}\|u\|_{L^2}^{1+\alpha}.$$

The punchline is that one can optimize the RHS in p. Towards a contradiction, assume that $\|u\|_{L^2}>0$ (and is small) and we can differentiate $p^{2-\alpha}\|u\|_{L^2}^{1+\alpha}$ in p to find a value p^* for which the derivative vanishes. Indeed, we can take

$$p^*\sim\ln(1/\|u\|_{L^2}^2).$$

Again, recall that we are assuming $\|u\|_{L^2}$ is positive and small, so that $p^*\geq 2$ is large. Therefore, denoting for simplicity $X=\|u\|_{L^2}^2$, we arrive at the differential inequality

$$\left|\frac{dX}{dt}\right|\lesssim X\ln\frac{1}{X}.$$

In turn, this implies

$$\left|\frac{d}{dt}\frac{1}{X}\right|\lesssim\frac{1}{X}\ln\frac{1}{X}\qquad\text{and}\qquad\left|\frac{d}{dt}\ln\frac{1}{X}\right|\lesssim\ln\frac{1}{X}.$$

Integrating the last estimate in time, we arrive at the inequality

$$\ln\frac{1}{X}\sim\ln\frac{1}{X_0}.$$

This is a contradiction, since this inequality implies that if X is finite at some time moment, then it should be finite for all t. However, at $t=0$ we are assuming that $\|u_0\|_{L^2}=0$, which implies $\ln\frac{1}{X}=+\infty$. This shows that if $\|u_0\|_{L^2}=0$ and $\omega\in L^\infty$, then $\|u\|_{L^2}=0$ for all times. The case of general initial data is left as a simple exercise. $\square$

Problem 2.2.5. *Formulate Yudovich's uniqueness statement in terms of a stability estimate.*

Remark 2.2.6. *The above proof requires $u \in L^2$, which is often not satisfied for bounded vorticities. Even when the vorticity is compactly supported in $\mathbb{R}^2$, unless the total circulation is zero (namely, $\int_{\mathbb{R}^2} \omega = 0$), the velocity in $\mathbb{R}^2$ decays exactly with rate $|x|^{-1}$ and not better. This can be explicitly seen in the case of the radial vortex, see [108]. When the vorticity decays slowly in space, the corresponding velocity can decay more slowly and even grow in space. A simple example is given by the vorticity which is identically a nonzero constant. Then it can be argued that the velocity should grow linearly in space. In these cases, the Yudovich argument should be modified. A natural way is to use appropriately weighted L^2 spaces (cf. [56]), which should work but it could be quite difficult to obtain estimates of the pressure in such weighted spaces. A completely different way to obtain uniqueness is to use the Lagrangian approach; see below and [111].*

Problem 2.2.7. *Extend the Yudovich uniqueness theorem appropriately to the case of α-SQG equations. (See [4, 87] for instance.)*

2.2.2 More on Yudovich theorem

We now discuss some interesting extensions of Yudovich's theorem. Before we do that, let us clarify the notion of *existence class* and *uniqueness class*. One (who is reasonably familiar with PDE theory) might have the impression that uniqueness is simply harder to obtain than existence, but this is in general false; uniqueness does not imply existence and vice versa. Concretely, we can take a pair of function spaces (X, Y) and say that it is an existence class (for some PDE) if for any data in X, one can guarantee the existence of a solution in Y. (For us, $Y = L^\infty_t X$ is the usual choice, and in this case we can simply say that X is an existence class.) Similarly, a pair (X, Y) (or just X if we fix $Y = L^\infty_t X$) is an uniqueness class if for any given data X, there is at most one solution belonging to Y and achieving that initial data. To be concrete, we consider now the 2D Euler equations in vorticity form. We have that H^s for $s > 1$ is both an existence and uniqueness class. As a consequence, if $X \subset H^s$ for some $s > 1$, then X is automatically an uniqueness class. However, one can imagine a space like $B^s_{p,\infty}$ (the Besov space with the summability index equal to ∞) which is potentially not an existence class. This is simply because the proof of a priori estimate not only requires sufficient regularity but also some structural property of the function space. On the other hand, one can also give examples of existence classes which are (presumably) not uniqueness classes, again in the case of 2D Euler. If $\omega_0 \in L^p$ then one can establish the existence of a solution in $L^\infty_t L^p$. With forcing L^p was already shown to be not an uniqueness class for $p < \infty$ (see below). Even at low (but critical) regularity, there are spaces which are presumably *only* uniqueness classes. The most notable example for 2D Euler is the BMO space that we have defined in the above. Several different proofs of uniqueness in $L^1 \cap BMO$ are known ([130, 4]); the difficulty in proving existence in this space is that the BMO norm cannot be characterized by the sequence of L^p norms; such spaces are sometimes referred to as *Yudovich classes* [45].

Yudovich spaces. An Yudovich space Y^Θ is defined in terms of a function $\Theta : [1,\infty) \to [1,\infty)$: we say that $f \in Y^\Theta$ if $f \in L^p$ for all $1 \le p < \infty$ and

$$\|f\|_{Y^\Theta} := \sup_{p\ge 1} \frac{\|f\|_{L^p}}{\Theta(p)} < \infty.$$

The space $L^1 \cap L^\infty$ simply corresponds to the case when Θ is a uniformly bounded function. By changing the growth rate of $\Theta(p)$ as $p \to \infty$, one can allow various singular functions in the space Y^Θ. Yudovich himself introduced this spaces and extended the uniqueness theorem to the "slightly" unbounded case, which roughly speaking contains vorticities blowing up like $\ln\ln$ at a few points ([136]). Concrete examples are $\Theta(p) \sim \ln p$, $\Theta(p) \sim \ln p \ln\ln p$, and so on (asymptotics for p large).

Problem 2.2.8. *Find the asymptotic for p large of the following:*

- $\|\log|x|\|_{L^p}$,
- $\|(\log|x|)^a\|_{L^p}$, *for some* $a > 0$,
- $\|\log\log|x|\|_{L^p}$.

In the above assume that the functions are defined on $\mathbb{R}^d$ *with some* $d \ge 1$ *and the* L^p *norm is taken in the ball* $0 < |x| \le 1$.

This is closely related to the Osgood's lemma for uniqueness from ODE theory. As we have mentioned earlier, non-uniqueness of Euler for L^p vorticity was obtained by Vishik ([128, 129]) with forcing, see also the monograph [2] and [3]. The idea is very roughly connected with the issue of (non-)uniqueness for ODE. To illustrate the point we consider the following simple ODE

$$\begin{cases} \dfrac{df}{dt} = u(f), \\ \quad f(0) = 0. \end{cases} \tag{51}$$

We assume here that $f : [0,T] \to \mathbb{R}_{\ge 0}$ for some $T > 0$ and $u : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is non-decreasing, $u(0) = 0$, and smooth except at 0; $\lim_{z\to 0^+} u'(z) = \infty$. While $f \equiv 0$ is a solution, there could be other solutions, e.g. $f(t) \sim t^\beta$ when u is a power type function. In any case, for initial data strictly positive, there is a unique solution simply because we assumed $u \ge 0$, so the solution is also non-decreasing in time. Assume further that u is strictly positive except at 0. Then we can define for any $\varepsilon > 0$ the time it takes for the unique solution to (51) starting at ε to reach some number, say 1. (It can be replaced with any positive number.) This time is given by simply integrating (51):

$$\int_\varepsilon^1 \frac{1}{u(f)} df = T_\varepsilon.$$

Then we see that

$$\lim_{\varepsilon\to 0} T_\varepsilon = \int_0^1 \frac{1}{u(z)} dz.$$

That is, local integrability of the function $z \mapsto 1/u(z)$ is equivalent with finiteness of the **escape time from the equilibrium** ($T := \lim_{\varepsilon\to 0} T_\varepsilon$). For (51), non-uniqueness occurs if and only if the escape time from 0 is finite. As an exercise, one can work out the case of $u(f) = f(\ln f)^\beta$ for $\beta > 0$. The point is that, non-uniqueness can be seen as an extreme form of instability for the equilibrium solution.

BMO case. A typical example of a BMO function is given by $f(x) = \ln|x|$ for $|x| \le 1$. While it can be computed that $\|f\|_{L^p} \sim p$ for p large, much information is lost when one regards BMO as a type of Yudovich space. Indeed, a non-trivial statement is that the singular integral operators of the "classical type" are bounded in BMO, and in the case of 2D Euler, it gives that $\nabla u \in BMO$ for $\omega \in L^1 \cap BMO$. On the other hand, the L^p estimates for the operator $\mathbf{T} = \nabla^2(-\Delta)^{-1}$ says that $\|\mathbf{T}f\|_{L^p} \lesssim p\|f\|_{L^p}$, for p large. Therefore, while $\|\mathbf{T}f\|_{BMO} \lesssim \|f\|_{L^\infty}$ follows naturally from the L^p estimates, the stronger estimate $\|\mathbf{T}f\|_{BMO} \lesssim \|f\|_{BMO}$ does not. Moreover, invariance of the Besov space $B^0_{\infty,\infty}$ under $\mathbf{T}$ is clear from the definition of the space, but BMO is not exactly $B^0_{\infty,\infty}$.

We have the following uniqueness result for BMO vorticity:

Lemma 2.2.9. *For any $T > 0$, there can be at most one solution ω to the 2D Euler equations belonging to $L^\infty([0,T]; BMO)$ for the same initial data.*

The proof is actually parallel to the argument of Yudovich (see [4] for instance). While this is essentially the best uniqueness statement for 2D Euler, it is important to note that *existence* is not clear in the BMO space, for BMO initial data. Moreover, it is important to understand exactly what type of functions belongs to BMO. For instance, one can compute that for functions locally near $r = 0$ of the form $f = g(\theta)\ln r$ (in polar coordinates), $f \in BMO$ if and only if g is a constant. For instance, the most singular part of $\nabla^2(-\Delta)^{-1}(\ln r \sin(2\theta))$ is given by a constant multiple of $(\ln r)^2 \sin(2\theta)$.

2.2.3 Local regularity in H^s spaces

We now consider local existence in H^s.

Theorem 2.2.1 (Local wellposedness in H^s)**.** *The equation* (gSQG) *in the regular and intermediate regimes is locally wellposed in $H^s(\mathbb{R}^2, \mathbb{T}^2)$ for any s sufficiently large. That is, given initial data $\theta_0 \in H^s$, there are $T > 0$ and a unique solution in $\theta \in C([0,T]; H^s)$ to* (gSQG) *with $\theta(0) = \theta_0$. To be precise, we require*

- *$s > 1 + \alpha$ when $0 \le \alpha \le 1$.*

- *$s > 3$ when $1 < \alpha < 2$, namely in the singular regime.*

*Furthermore, the **solution map** $\theta_0 \to \theta$ is continuous from H^s to $C([0,T];H^s)$.*

Remark 2.2.10. *What is contained in the continuity statement is that the (guaranteed) existence time interval length T also depends continuously in θ_0.*

Remark 2.2.11. *In the log singular case, we have local wellposedness in a scale of time-decaying Sobolev spaces ([23]): for any $s_0 > 4$ so that $\theta_0 \in H^{s_0}$, there exist $T > 0$, a continuous function of time $s(t) > 4$ with $s(0) = s_0$ defined in $t \in [0,T]$, and a solution $\theta \in C([0,T];H^{s(t)})$ to (8) with initial data θ_0.*

While we take the case of $\mathbb{R}^2$ for simplicity, the proof in the $\mathbb{T}^2$ case is similar, just using the Fourier series instead of the Fourier transform. We shall first start with a rather general uniqueness lemma, which is applicable not only in the Sobolev case but also in the Hölder case, which will be treated below.

Proof of Theorem 2.2.1. We shall proceed in a few steps. As it is usual, the first step is to obtain an estimate of the solution, assuming that a smooth solution exists in a time interval.

A priori estimate. We assume that there exists a solution to (gSQG) belonging to the class $\theta \in C([0,T];H^s(\mathbb{R}^2))$ for some $T > 0$. The following argument will be restricted to $0 \le t < T$.

Regular regime. We begin with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^s}^2 = -\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)}\int |\xi|^s\left(\widehat{u}(\xi-\eta)\cdot i\eta\widehat{\theta}(\eta)\right)\mathrm{d}\eta\,\mathrm{d}\xi.$$

We then "distribute the derivative" as follows:

$$|\xi|^s = |\eta|^s + (|\xi|^s - |\eta|^s)\,.$$

The point is that

$$-\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)}\int \left(\widehat{u}(\xi-\eta)\cdot i\eta|\eta|^s\widehat{\theta}(\eta)\right)\mathrm{d}\eta\,\mathrm{d}\xi = 0$$

by anti-symmetry (this corresponds to the term where all the derivatives have fallen on $\nabla\theta$), so that we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^s}^2 = -\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)}\int (|\xi|^s - |\eta|^s)\left(\widehat{u}(\xi-\eta)\cdot i\eta\widehat{\theta}(\eta)\right)\mathrm{d}\eta\,\mathrm{d}\xi.$$

Now, taking absolute values in the right hand side and using

$$||\xi|^s - |\eta|^s| \le C_s|\xi-\eta|(|\xi-\eta|^{s-1} + |\eta|^{s-1}),$$

we have that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^s}^2\right| &\le C_s \int |\xi|^s|\widehat{\theta}(\xi)| \int \Big(|\xi-\eta||\widehat{u}(\xi-\eta)||\eta|^s|\widehat{\theta}(\eta)| + |\xi-\eta|^s|\hat{u}(\xi-\eta)||\eta||\widehat{\theta}(\eta)|\Big) \\ &= I + II.\end{aligned}$$

Therefore, to treat the term I, we see that it is necessary to have $|\xi|\widehat{u}(\xi) \in L^1_\xi$ with $\||\xi|\widehat{u}(\xi)\|_{L^1_\xi} \lesssim \|\theta\|_{H^s}$, which is guaranteed if $s > 1+\alpha$. The term II can be bounded under the same condition. This gives the desired estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^s}^2\right| \le C_s\|\theta\|_{H^s}^3.$$

Singular regime. Apparently, the above argument in the regular regime does not close when u is more singular than θ. The point is that the nonlinearity can be viewed in two ways,

$$u\cdot\nabla\theta = \nabla^\perp\psi\cdot\nabla\theta = -\nabla^\perp\theta\cdot\nabla\psi$$

and we use the last expression when all the derivatives hit ψ. Then we see that there is a cancellation since ψ is a symmetric operator applied to θ. For concreteness, let us assume that $\widehat{\psi}(\xi) = |\xi|^{-2+\alpha}\widehat{\theta}(\xi)$ with $1<\alpha<2$. Of course one can treat more general case as well. Now, we return to the expression

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^s}^2 = -\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)} \int (|\xi|^s - |\eta|^s)\Big(\widehat{u}(\xi-\eta)\cdot i\eta\widehat{\theta}(\eta)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

and rewrite the right hand side as

$$\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)} \int (|\xi|^s-|\eta|^s)\Big(i(\xi-\eta)|\xi-\eta|^{-2+\alpha}\widehat{\theta}(\xi-\eta)\cdot i\eta^\perp\widehat{\theta}(\eta)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi =: I + II.$$

Then we see that we can do some symmetrization in the expression

$$I := \mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)} \int |\xi-\eta|^s\Big(i(\xi-\eta)|\xi-\eta|^{-2+\alpha}\widehat{\theta}(\xi-\eta)\cdot i\eta^\perp\widehat{\theta}(\eta)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi;$$

since

$$I' := \mathrm{Re}\int |\xi|^{s-1+\frac{\alpha}{2}}\overline{\widehat{\theta}(\xi)} \int |\xi-\eta|^s\Big(i(\xi-\eta)|\xi-\eta|^{-1+\frac{\alpha}{2}}\widehat{\theta}(\xi-\eta)\cdot i\eta^\perp\widehat{\theta}(\eta)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

vanishes. Hence it suffices to estimate the difference $I - I'$ and

$$II := -\mathrm{Re}\int |\xi|^s\overline{\widehat{\theta}(\xi)} \int (|\xi|^s - |\eta|^s - |\xi-\eta|^s)\Big(\widehat{u}(\xi-\eta)\cdot i\eta\widehat{\theta}(\eta)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

In II, the point is that now

$$||\xi|^s - |\eta|^s - |\xi-\eta|^s| \lesssim |\eta||\xi-\eta|^{s-1} + |\xi-\eta||\eta|^{s-1}.$$

On the other hand, in $I - I'$, we can factor as

$$|\xi - \eta|^s(\xi - \eta)|\xi - \eta|^{-1+\frac{\alpha}{2}}|\xi|^{s-1+\frac{\alpha}{2}}\left(|\xi - \eta|^{-1+\frac{\alpha}{2}} - |\xi|^{-1+\frac{\alpha}{2}}\right)$$

which gives a factor of $|\eta|$ as well, using that $2 > \alpha > 1$. Therefore, this allows us to close an H^s estimate, assuming that s is large enough.

Problem 2.2.12. *Check that the sharp threshold is $s > 1 + \alpha$ again in this case.*

Existence. To prove the existence of a smooth solution, one can consider the *regularized system*, e.g.

$$\partial_t \theta^\kappa + u^\kappa \cdot \nabla \theta^\kappa = \kappa \Delta \theta^\kappa,$$
$$\theta_0^\kappa = \theta_0 * \varphi_\kappa.$$

Not only the equation but also the initial data has been regularized. For the above problem, a unique local in time solution $\theta^{(\kappa)}$ can be obtained by Duhamel's principle; the idea is to view the nonlinear term as a perturbation. To be more precise, we rewrite the equation as

$$\theta^\kappa(t) = e^{t\kappa\Delta}\theta_0^\kappa + \int_0^t e^{(t-\tau)\kappa\Delta}(-u^\kappa \cdot \nabla\theta^\kappa)(\tau)d\tau$$

and solve the equation using a fixed point theorem on $L^\infty([0,T];H^s)$ with T sufficiently small. A priori, the local existence time may depend on κ but the key point is that the above H^s a priori estimate is valid, *uniformly in* κ. This can be used to show that for any $\kappa > 0$, there is a uniform time of existence $T > 0$, on which we have

$$\|\theta^\kappa\|_{L^\infty([0,T];H^s)} \le 2\|\theta_0^\kappa\|_{H^s} \le 4\|\theta_0\|_{H^s}.$$

Some details of this argument can be found in [20]. Therefore, we obtain a sequence of time-dependent functions $\{\theta^\kappa\}$ which are bounded uniformly in the space

$$C([0,T];H^s)$$

as well as in

$$Lip([0,T];H^{s-2}).$$

To see the latter statement, one may just estimate

$$\|\partial_t\theta^\kappa\|_{H^{s-2}} \le \|u^\kappa \cdot \nabla\theta^\kappa\|_{H^{s-2}} + \|\kappa\Delta\theta^\kappa\|_{H^{s-2}} \le C\kappa\|\theta^\kappa\|_{H^s} + C\|\theta^\kappa\|_{H^s}^2.$$

Applying the Aubin–Lions lemma, we have a subsequence θ^{κ_j} which is strongly convergent in $L^\infty([0,T];H^{s-1})$. It is not difficult to show that the limit belongs to $L^\infty([0,T];H^s)$ and satisfies the gSQG equation with initial data θ_0. This gives the existence of a solution, in the class $L^\infty([0,T];H^s)$, but there is some additional argument necessary to upgrade the solution to $C([0,T];H^s)$. The standard way is to use the so-called Bona–Smith approximation introduced in [8]. Here we present a slick argument which is also well known.

To begin with, we have that the solution θ belongs to the class $C_w([0,T];H^s)$, simply because it is obtained as a sub-sequential limit of θ^κ which belongs to $C([0,T];H^s)$ and the weak continuity in time is preserved in the weak limit. We now try to upgrade weak continuity to strong continuity, say at time $t=0$. (The argument at other time moments is the same.) Recall that given weak continuity, it suffices to prove norm convergence to get strong continuity; that is, if $t_k \to 0$ then $\|\theta(t_k)\|_{H^s} \to \|\theta_0\|_{H^s}$. However, weak continuity already implies that in the time limit, the H^s norm can only drop. Assume towards a contradiction that we can find a sequence $t_k > 0$ such that $\|\theta_0\|_{H^s} < \limsup_{k\to\infty} \|\theta(t_k)\|_{H^s}$. But then, we recall the a priori estimate in H^s, which asserts that for $t>0$ sufficiently small,

$$\|\theta(t)\|_{H^s} < (1+\varepsilon)\|\theta_0\|_{H^s}.$$

This is a contradiction, by taking $\varepsilon > 0$ sufficiently small.

Continuity of the solution map. Here we give a proof of the continuity of the solution map in H^s communicated to us by T. Elgindi. The usual proof is again based on the Bona–Smith ([8]) approximation, but this one is essentially the same in spirit, just a quantitative version of Bona–Smith. Recall that if $f:\mathbb{R}\to\mathbb{R}$ is an integrable function, then it has a modulus of integrability: there exists a function δ such that

$$\int_{[a-r,a+r]} |f| \le \delta(r)$$

for any $a \in \mathbb{R}$. A function in H^s has a similar property; $f \in H^s(\mathbb{R}^2)$ simply means that its Fourier transform $\hat{f}$ is such that

$$\int_{\mathbb{R}^2} (1+|\xi|^2)^s |\hat{f}|^2 d\xi < \infty.$$

Then we can actually find a function $a(|\xi|) > 0$ such that

$$\int_{\mathbb{R}^2} (1+a(|\xi|))(1+|\xi|^2)^s |\hat{f}|^2 d\xi < \infty.$$

We can require a to satisfy some nice properties, e.g. it is infinitely differentiable with derivatives decaying sufficiently fast. Now, an important point is that if we have a convergent sequence of initial data $\theta_n \to \theta$ in H^s, then there is a **uniform** function $a(|\xi|) > 0$ such that

$$\|\theta_n\|^2_{H^{s;a}} := \int_{\mathbb{R}^2} (1+a(|\xi|))(1+|\xi|^2)^s |\hat{\theta}_n|^2 d\xi \le 2\|\theta\|^2_{H^s}$$

for all sufficiently large n. We now have the following

Claim. If a is a "nice" multiplier, then we have propagation of the $H^{s;a}$ norm in time for the solution of (gSQG).

Problem 2.2.13. *Prove the **Claim**.*

We can finish the proof that $\theta_n(t) \to \theta(t)$ in H^s. The point is that there is a uniform bound

$$\|\theta_n(t)\|_{H^{s;a}} \lesssim \|\theta_n(t=0)\|_{H^{s;a}} \lesssim \|\theta\|_{H^s},$$

and we can leverage it with a very weak convergence, say

$$\theta_n(t) \longrightarrow \theta(t)$$

strongly in L^2. This is because the H^s space is sitting strictly between L^2 and $H^{s;a}$, no matter how small a is. This latter L^2 convergence simply follows from the stability lemma 2.2.1. This finishes the proof, modulo the **Claim**. See the remark below. □

Remark 2.2.14. *In the regular and intermediate regimes, we have **propagation of regularity**: given a gSQG equation, we can simply pick some s_0 such that the equation is locally wellposed in H^{s_0} and then for any $s \geq s_0$, we can prove the propagation estimate*

$$\frac{d}{dt}\|\theta\|_{H^s} \leq C_s\|\theta\|_{H^{s_0}}\|\theta\|_{H^s},$$

which immediately shows that the higher H^s norm cannot blow up unless $\|\theta\|_{H^{s_0}}$ blows up. This is one of the motivations to obtain local regularity in low Sobolev spaces. In particular, we obtain that any H^∞ initial data remains in H^∞ for a nonzero interval of time.

However, it is important to notice that in the log singular case, the above proof does not give local propagation of H^∞. It only says that for any k, there is a time interval $[0, T_k]$ with $T_k > 0$ depending on k such that the solution belongs to $L^\infty([0, T_k]; H^k)$.

Problem 2.2.15. *Obtain local wellposedness in appropriate Sobolev spaces for the models introduced in §1.1.6.*

2.2.4 Local regularity in $C^{k,\alpha}$ spaces

We now consider the problem of local regularity in Hölder spaces, which is more elementary in some sense. See [132, 21, 44, 43] where various local wellposedness statements in Hölder spaces are introduced. In particular [44] gives local wellposedness without a decay condition at infinity for solutions. Note that in the statement below, the kernel is assumed to be in the regular regime, unlike the Sobolev theorem which allows for the intermediate regime (unfortunately, we have *illposedness* of the intermediate regime in Hölder spaces: [43]). We take the domain to be either $\mathbb{T}^2$ or $\mathbb{R}^2$.

Theorem 2.2.2 (Local wellposedness in Hölder spaces)**.** *In the regular regime ($0 \leq \alpha < 1$), the α-SQG equation is locally wellposed in the space $C^{k,\beta} \cap H^{-1}$ with any $k + \beta > \alpha$.*

That is, given an initial data $\theta_0 \in C^{k,\beta} \cap H^{-1}$, there exist $T > 0$ and a unique solution θ to α-SQG belonging to $C_*([0,T); C^{k,\beta} \cap H^{-1})$.

Proof. We consider the Lagrangian framework. The first step is to obtain an a priori estimate of the solution in the Hölder space.

A priori estimate. For simplicity, assume that $0 < \beta < 1$. We shall perform a C^β estimate: begin with the flow formula

$$\theta(t, \Phi(t,x)) = \theta_0(x)$$

and then from bijectivity of $\Phi(t, \cdot)$, we have that

$$\|\theta(t,\cdot)\|_{C^\beta} = \sup_{x \neq x'} \frac{|\theta(t,\Phi(t,x)) - \theta(t,\Phi(t,x'))|}{|\Phi(t,x) - \Phi(t,x')|^\beta}$$

It will be useful to introduce the shorthand (after fixing some points x, x' with $x' \neq x$)

$$\begin{aligned} \theta &= \theta(t,x), & \theta' &= \theta(t,x'), \\ \theta \circ \Phi &= \theta(t,\Phi(t,x)), & \theta \circ \Phi' &= \theta(t,\Phi(t,x')), \\ \Phi &= \Phi(t,x), & \Phi' &= \Phi(t,x'). \end{aligned}$$

We can then compute that

$$\begin{aligned} &\frac{d}{dt} \frac{\theta(t,\Phi(t,x)) - \theta(t,\Phi(t,x'))}{|\Phi(t,x) - \Phi(t,x')|^\beta} \\ &\quad = -\beta \frac{\theta(t,\Phi(t,x)) - \theta(t,\Phi(t,x'))}{|\Phi(t,x) - \Phi(t,x')|^{2+\beta}} (u(t,\Phi(t,x)) - u(t,\Phi(t,x'))) \cdot (\Phi(t,x) - \Phi(t,x')). \end{aligned}$$

Taking absolute values, for

$$D(x,x') := \frac{|\theta(t,\Phi(t,x)) - \theta(t,\Phi(t,x'))|}{|\Phi(t,x) - \Phi(t,x')|^\beta},$$

we obtain the estimate

$$\left| \frac{dD}{dt} \right| \leq C_\beta \|\nabla u\|_{L^\infty} D.$$

Therefore, the crucial point is that we need to choose β so that u is Lipschitz continuous: from the Hölder estimate,

$$\|\nabla u\|_{L^\infty} \leq C_\beta \|\theta\|_{C^\beta}$$

$\beta > \alpha$, when $\alpha < 1$. Then taking the supremum in $x \neq x'$ gives

$$\frac{d}{dt}\|\theta\|_{C^\beta} \leq C_\beta \|\theta\|_{C^\beta}^2.$$

Integrating this in time gives $T > 0$ such that we have

$$\sup_{t\in[0,T]} \|\theta(t,\cdot)\|_{C^\beta} \leq 2\|\theta_0\|_{C^\beta}.$$

It is not difficult to extend this type of a priori estimate for $\beta > 1$, although there are some critical values of β for which this does not work.

We need to separately consider the case $\alpha = 1$, where we shall again assume that $0 < \beta < 1$ but estimate θ in the $C^{1,\beta}$ norm. Indeed in this case, the equation has been shown to be *strongly illposed* in C^1 (or in $C^{0,1}$), [60, 87]. In this case, before moving on to the Lagrangian coordinates, we first take a derivative:

$$\partial_t \nabla^\perp \theta + u \cdot \nabla \nabla^\perp \theta = \nabla u \nabla^\perp \theta$$

and then we take the flow: for $V = \nabla^\perp \theta$,

$$\frac{d}{dt} V(t, \Phi(t,x)) = \nabla u(t, \Phi(t,x)) V(t, \Phi(t,x)).$$

Then arguing similarly as in the above,

$$\frac{d}{dt}\|V\|_{C^\beta} \leq C_\beta \left(\|\nabla u\|_{L^\infty} \|V\|_{C^\beta} + \|V\|_{L^\infty} \|\nabla u\|_{C^\beta} \right).$$

Now, the point is that for any $0 < \beta < 1$,

$$\|\nabla u\|_{C^\beta} \leq C_\beta \|V\|_{C^\beta}$$

so we can close an C^β estimate for V:

$$\frac{d}{dt}\|V\|_{C^\beta} \leq C_\beta \|V\|_{C^\beta}^2.$$

Integrating in time, we obtain $T > 0$ such that

$$\sup_{t\in[0,T]} \|V(t,\cdot)\|_{C^\beta} \leq 2\|V_0\|_{C^\beta}. \tag{52}$$

Existence of a solution. Given the a priori estimate of the form (52), one may construct the solution in a variety of ways. One way is to mollify the initial data, namely for any $\varepsilon > 0$, take the initial data

$$\theta_0^\varepsilon := \varphi_\varepsilon * \theta_0 \in C^\infty$$

and we can smoothly truncate it outside of a ball, say $B(0, 1/\varepsilon)$. Then, we consider the initial value problem

$$\begin{aligned} \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon &= 0, \\ u^\varepsilon &= \nabla^\perp P(\Lambda)\theta^\varepsilon, \\ \theta^\varepsilon(t=0) &= \theta_0^\varepsilon. \end{aligned}$$

Here, the point is that we already have existence of a solution from the Sobolev well-posedness. Furthermore, since the solution is smooth, the above argument regarding the Hölder a priori estimate is valid for the solution, which gives

$$\sup_{t\in[0,T]} \|\theta^\varepsilon\|_{C^\beta} \le 2\|\theta_0^\varepsilon\|_{C^\beta} \le 4\|\theta_0\|_{C^\beta}$$

uniformly for all sufficiently small $\varepsilon > 0$. We have that

$$\{\theta^\varepsilon\}_{n\ge 0} \text{ uniformly bounded in } L^\infty([0,T]; C^\beta) \cap Lip([0,T]; C^{\beta-1}).$$

Therefore, in the limit $\varepsilon \to 0$, there is a subsequence which is strongly convergent in $C([0,T]; C^{\beta'})$ for $0 < \beta' < \beta$. Then, since the velocity is not worse in regularity compared to θ, we have the same convergence for the corresponding sequence of velocities. If we denote the limits by θ and u, respectively; that is, if

$$\theta^{\varepsilon_k} \longrightarrow \theta, \qquad u^{\varepsilon_k} \longrightarrow u$$

in $C([0,T]; C^{\beta'})$, then we first have that

$$u = \nabla^\perp P(\Lambda)\theta,$$

and

$$\partial_t \theta^{\varepsilon_k} \longrightarrow \partial_t \theta, \qquad u^\varepsilon \cdot \nabla \theta^\varepsilon \longrightarrow u \cdot \nabla \theta$$

in the sense of distributions, so that

$$\partial_t \theta + u \cdot \nabla \theta = 0.$$

Finally, it is clear that $\theta(t = 0) = \theta_0$ and that θ inherits the uniform bound of $L^\infty([0,T]; C^\beta)$. Therefore, we have the existence of a solution. $\square$

Eulerian–Lagrangian approach. The above existence proof may be not satisfactory since it involves using the Sobolev existence theory. Indeed, in the Lagrangian scheme, it is customary to obtain a solution with an iterative scheme. We shall take $\alpha < 1$, $T > 0$ small so that the above a priori estimate does not blow up, and define the following sequence of functions

$$\begin{aligned} \partial_t \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} &= 0, \\ \theta^{n+1}(t=0) &= \theta_0, \end{aligned}$$

$$u^{n+1} = \nabla^\perp P(\Lambda)\theta^{n+1}.$$

The initial step is simply defined by

$$\theta^0 := \theta_0$$

for all $t \in [0,T]$. This trivially verifies the hypothesis

$$\|\theta^n\|_{L^\infty([0,T];C^\beta)} \leq 2\|\theta_0\|_{C^\beta} \tag{53}$$

in the case $n = 0$. We shall verify this bound inductively. Assuming that we have the bound for some $n \geq 0$, we can first obtain the Lipschitz bound on u^n:

$$\|u^n\|_{L^\infty([0,T];Lip)} \leq C_\beta \|\theta^n\|_{L^\infty([0,T];C^\beta)} \leq 2C_\beta \|\theta_0\|_{C^\beta},$$

so that on $[0,T]$, we have a unique solution of the flow map

$$\frac{d}{dt}\Phi^n(t,x) = u^n(t, \Phi^n(t,x)), \qquad \Phi^n(0,x) = x.$$

In turn, the flow map Φ^n defines the solution to the transport equation for θ^{n+1} via

$$\theta^{n+1}(t, \Phi^n(t,x)) = \theta_0(x).$$

This formula allows us to conclude

$$\|\theta^{n+1}\|_{L^\infty([0,T];C^\beta)} \leq 2\|\theta_0\|_{C^\beta}$$

by taking $T > 0$ smaller if necessary, but in a way independent of n. To see this, we simply take two points $x \neq x'$ and compute

$$\frac{\theta^{n+1}(t, \Phi^n(t,x)) - \theta^{n+1}(t, \Phi^n(t,x'))}{|\Phi^n(t,x) - \Phi^n(t,x')|^\beta} = \frac{\theta_0(x) - \theta_0(x')}{|\Phi^n(t,x) - \Phi^n(t,x')|^\beta}.$$

After taking absolute values, we rewrite the right hand side as

$$\frac{|\theta_0(x) - \theta_0(x')|}{|\Phi^n(t,x) - \Phi^n(t,x')|^\beta} = \frac{|\theta_0(x) - \theta_0(x')|}{|x - x'|^\beta} \frac{|x - x'|^\beta}{|\Phi^n(t,x) - \Phi^n(t,x')|^\beta}.$$

On the other hand, we can estimate

$$D = \frac{|\Phi^n(t,x) - \Phi^n(t,x')|}{|x-x'|}$$

from above and below by taking the time derivative of the quantity

$$\frac{\Phi^n(t,x) - \Phi^n(t,x')}{|x-x'|}.$$

This gives

$$\exp\left(-C\|\nabla u^n\|_{L^1([0,t];L^\infty)}\right) \le D \le \exp\left(C\|\nabla u^n\|_{L^1([0,t];L^\infty)}\right).$$

Combining the estimates,

$$\frac{|\theta^{n+1}(t,\Phi^n(t,x)) - \theta^{n+1}(t,\Phi^n(t,x'))|}{|\Phi^n(t,x) - \Phi^n(t,x')|^\beta} \le C\|\theta_0\|_{C^\beta} D^{-\beta} \le C\|\theta_0\|_{C^\beta} \exp\left(C\beta\|\nabla u^n\|_{L^1([0,t];L^\infty)}\right).$$

First taking the supremum for all $x \neq x'$ and then taking for t in $[0,T]$, we obtain that

$$\|\theta^{n+1}\|_{L^\infty([0,T];C^\beta)} \le C\|\theta_0\|_{C^\beta} \exp\left(CT\beta\|\nabla u^n\|_{L^\infty([0,T];L^\infty)}\right).$$

This verifies (53) for $n+1$. Therefore, we obtain a sequence

$$\{\theta^n\}_{n\ge 0} \text{ uniformly bounded in } L^\infty([0,T];C^\beta).$$

Similarly

$$\{u^n\}_{n\ge 0} \text{ uniformly bounded in } L^\infty([0,T];C^{1,\beta-\alpha}).$$

Next,

$$\{\Phi^n\}_{n\ge 0} \text{ uniformly bounded in } L^\infty([0,T];C^{1,\beta-\alpha}).$$

Then we can find a convergent subsequence and take the limit to obtain a solution. Here, naively taking a convergent subsequence θ^{n_k} does not work, since we do not know a priori whether (even after taking a further subsequence) u^{n_k-1} is converging to $\nabla^\perp P(\Lambda)\theta$, where θ is the sub-sequential limit of θ^{n_k}. One way to overcome this is to show that the full sequence θ^n is actually convergent in some weak norm. This is similar to the proof of uniqueness.

We fix some $T > 0$, and set $d^n := \sup_{t\in[0,T]} \|\theta^{n+1}(t) - \theta^n(t)\|_{L^2}$. From the equation

$$\partial_t(\theta^{n+1} - \theta^n) + u^n \cdot \nabla(\theta^{n+1} - \theta^n) + (u^n - u^{n-1}) \cdot \nabla\theta^n = 0,$$

we obtain

$$\frac{d}{dt}\|\theta^{n+1}(t)-\theta^n(t)\|_{L^2} \lesssim \|\nabla\theta^n\|_{L^{q^*}}\|\theta^n(t)-\theta^{n-1}(t)\|_{L^2} \lesssim d^n$$

for some q^*. To proceed, it is important that we are assuming **uniform** bound in n of the quantity $\|\nabla\theta^n\|_{L^{q^*}}$. Strictly speaking this does not follows from the uniform C^β bound if β is small but we shall assume that β is sufficiently large for this to hold. Then, uniformly in n we have the bound

$$\|\theta^{n+1}(t)-\theta^n(t)\|_{L^2} \le Cd^n t,$$

since we are assuming that $\|\theta^{n+1}(t)-\theta^n(t)\|_{L^2}=0$ at $t=0$. In other words,

$$d^{n+1} \le CTd^n,$$

so that $d^{n+1} < d^n/2$ if T is chosen small. This shows that the sequence θ^n is Cauchy in $L^\infty_t L^2$. We denote the (unique) limit in this topology by θ. This removes ambiguity of the sub-sequential limit. That is, any subsequence strongly convergent in $L^\infty([0,T];C^\gamma)$ with $\gamma<\beta$ should converge to θ, simply because convergence in C^γ implies convergence in L^2.

Lagrangian approach. A yet another way to get existence is to use the purely Lagrangian framework, which simply rewrites the equation *purely* in terms of the flow map $\Phi(t,\cdot)$. Let us explain how this can be done. We write $K(\cdot)$ for the kernel of the multiplier $P(\Lambda)$. Then,

$$\frac{d}{dt}\Phi(t,x) = u(t,\Phi(t,x)) = \int_{\mathbb{R}^2}(\nabla^\perp K)(\Phi(t,x)-y)\theta(t,y)dy.$$

Then, we make a change of variables defined by $y=\Phi(t,x')$: since Φ is area preserving, we have that

$$\frac{d}{dt}\Phi(t,x) = \int_{\mathbb{R}^2}(\nabla^\perp K)(\Phi(t,x)-\Phi(t,x'))\theta_0(x')dx'.$$

Integrating in time, we have that

$$\Phi(s,x) = x + \int_0^s\int_{\mathbb{R}^2}(\nabla^\perp K)(\Phi(t,x)-\Phi(t,x'))\theta_0(x')dx'dt. \tag{54}$$

That is, if we have a (reasonably) smooth solution to gSQG, then the resulting flow map Φ should solve the integrodifferential equation (54). The key idea in the purely Lagrangian framework is to try to find a solution to (54) directly; if we are given with some family of area-preserving diffeos of $\mathbb{R}^2$ parameterized by t, say $\{\Psi(t,\cdot)\}_{t\in[0,T]}$, then we can obtain a new family by applying the operator $\mathbf{T}$, defined by

$$\mathbf{T}[\Psi](s) := x + \int_0^s\int_{\mathbb{R}^2}(\nabla^\perp K)(\Psi(t,x)-\Psi(t,x'))\theta_0(x')dx'dt.$$

Then, solutions of (54) are simply fixed points of $\mathbf{T}$. We can take the time interval to be sufficiently small to guarantee the existence of a fixed point for $\mathbf{T}$.

Problem 2.2.16. *Prove propagation of regularity statements for the local in time Hölder solutions.*

2.2.5 Continuation criteria

Recall that once a *continuation criterion* is satisfied (a hypothesis on the solution), then the classical solution extends to a larger time interval. Therefore, a criterion is stronger if the hypothetical assumption is weaker. We shall prove an analogue of the well known Beale–Kato–Majda criterion ([6]) for the d-dimensional Euler equations, which states that

$$\text{If } \|\omega\|_{L^1(0,T;L^\infty)} \leq M \text{ then } \|u\|_{L^\infty(0,T;H^m)} \leq C(M, m, \|u_0\|_{H^m}).$$

In particular, we are able to extend the solution beyond the time interval $[0, T)$, given $m > d/2 + 1$. In $2D$, this implies global regularity since we have a trivial bound $M \leq T\|\omega_0\|_{L^\infty}$. The proof is simple, and it relies on the standard energy estimate, one *key* time-independent harmonic analysis estimate, and then an application of the Gronwall inequality (or, a comparison principle of ODEs). The first ingredient is simply

$$\left|\frac{d}{dt}\|u\|_{H^m}\right| \lesssim_m \|\nabla u\|_{L^\infty}\|u\|_{H^m}$$

which is the basic H^m-estimate on classical solutions of the Euler equations. The harmonic analysis bound takes the form

$$\|\nabla u\|_{L^\infty} \lesssim_m (1 + \log(10 + \|u\|_{H^3})\|\omega\|_{L^\infty} + \|\omega\|_{L^2}). \tag{55}$$

The fact that we have a log of the higher Sobolev norm in u is crucial. (Note that we cannot close an a priori estimate if we have any power of log strictly larger than 1.) The fact that the L^2-norm of vorticity appears is slightly annoying, so we first absorb it in the $O(1)$-term (once we fix T). The standard energy estimate at the level of the vorticity gives

$$\|\omega(t)\|_{L^2} \lesssim \|\omega_0\|_{L^2} \exp(\int_0^t \|\omega(s)\|_{L^\infty} ds) < +\infty$$

since we are assuming $L^1_t L^\infty_x$-control on the vorticity.

A heuristic discussion of the key estimate (55)**.** To gain some insight into the inequality, we imagine a 1D situation where we attempt to bound $\mathbf{H}v$ in L^∞ ($\mathbf{H}$ being the Hilbert transform) using a log of some high Sobolev norm, together with v in L^∞. First, some dependence on the high Sobolev norm is essential, simply because if we assume that v is the Heaviside step function, then $\mathbf{H}v$ will logarithmically diverge precisely at the origin. Now, we smooth out v in a way that it makes a sharp transition from 0 to 1 in a neighborhood of the origin of width ϵ. Then, evaluating the Hilbert transform at the origin cost approximately $\int_0^\epsilon x^{-1}dx$ which is $-\log \epsilon$. Note that $\log \|v\|_{H^m}$ also scales like this number.

We now prove (55). It suffices to show

Lemma 2.2.17. *For a smooth function f we have*

$$\|\mathbf{R}_a\mathbf{R}_b f\|_{L^\infty} \lesssim \|f\|_{L^2} + \|f\|_{L^\infty} \log(e + \frac{\|f\|_{H^4}}{\|f\|_{H^2}}).$$

Here, $\mathbf{R}_i$ is the Riesz transform with respect to the coordinate x_i.

Remark 2.2.18. *This is a general statement, not very specific to the Riesz transforms. For example, the lemma works with $\mathbf{R}_a\mathbf{R}_b$ replaced by any Calderon–Zygmund operators of the classical type. Indeed we only need homogeneity and the existence of a finite number of derivatives; it will be clear from the proof.*

Remark 2.2.19. *The BKM criterion was extended to allow for $\|\omega\|_{BMO}$ in [100, 116]. This is strictly related to the improved results on uniqueness (e.g. BMO vorticity) that we have seen in the above.*

Proof. We fix $x \in \mathbb{R}^d$ $(d = 2, 3)$ and the task is to bound

$$Mf(x) := C \int_{\mathbb{R}^d} m(\xi)\hat{f}(\xi)e^{ix\xi}d\xi,$$

where $m(\cdot)$ is the multiplier of the operator $\mathbf{R}_a\mathbf{R}_b$ (the only fact relevant is that it is smooth and homogeneous of degree 0). As always, we decompose dyadically. With the sequence of Schwartz class functions Ψ_k supported in the Fourier annulus of radii $\sim 2^k$, we consider

$$|Mf(x)| \le |(Mf)_0(x)| + |(Mf)_{[0,N]}(x)| + |(Mf)_{\ge N}(x)|.$$

The first term is simply bounded by the L^2-norm of f, (this is simply the Fourier side manifestation of the fact that the global contribution of the integral against the Riesz kernel is bounded by f in L^2 in the physical side) and the right hand side is bounded by an appropriate high Sobolev norm of f, and it is important to extract the decay in N:

$$|(Mf)_{\ge N}(x)| \lesssim 2^{-N}\|f\|_{H^4}.$$

Then we define N in a way that $2^N \approx \|f\|_{H^4}/\|f\|_{L^2}$. It only remains to control the middle part, and this is actually where the form of the multiplier (e.g. differentiability) is important.

We write

$$(Mf)_{[0,N]}(x) = (f * K_N)(x),$$

where

$$K_N(x) = c\int e^{ix\xi}m(\xi)\Psi_{[0,N]}(\xi)d\xi.$$

It only remains to establish the estimate

$$\|K_N\|_{L^1} \lesssim 1 + N,$$

or equivalently,

$$\|\int_{\mathbb{R}^d} m(\xi)\Phi_k(\xi)e^{ix\xi}d\xi\|_{L^1} \lesssim 1.$$

We define the function in the above formula by $K_k(x)$ and we state

Claim. The function K_k is supported on $x \lesssim 2^{-k}$.[†], and $\|K_k\|_{L^\infty} \lesssim 2^{dk}$.

The second statement is obvious, and to show the first, let us assume that $|x| \gg 2^{-k}$. Then assume without loss of generality that $|x_1| \gg 2^{-k}$. We will integrate in parts with respect to this variable, where we gain by powers of x_1 (which is *huge*). We obtain

$$|K_k(x)| \lesssim \frac{1}{|x_1|}\int_{\mathbb{R}^d} \xi\Psi_k^{(1)}(\xi)d\xi.$$

Integration by parts can be repeated. This finishes the proof. □

We now turn to the case of the SQG equation, namely

$$\partial_t\theta + \mathcal{R}[\theta]\cdot\nabla\theta = 0. \tag{56}$$

Taking the gradient gives

$$\partial_t\nabla^\perp\theta + \mathcal{R}[\theta]\cdot\nabla(\nabla^\perp\theta) = \nabla\mathcal{R}[\theta]\nabla^\perp\theta.$$

We claim the bound

$$\frac{d}{dt}\|\nabla^\perp\theta\|_{H^{m-1}} \leq C(\|\nabla\mathcal{R}[\theta]\|_{L^\infty} + \|\nabla^\perp\theta\|_{L^\infty})\|\nabla^\perp\theta\|_{H^{m-1}}.$$

Here we take m sufficiently large. Then, we use the above harmonic analysis bound

$$\|\nabla\mathcal{R}[\theta]\|_{L^\infty} \leq C\|\nabla^\perp\theta\|_{L^\infty}(1 + \log(10 + \|\theta\|_{H^m})),$$

to obtain

$$\frac{d}{dt}(1 + \log(10 + \|\theta\|_{H^m})) \leq C\|\nabla^\perp\theta\|_{L^\infty}(1 + \log(10 + \|\theta\|_{H^m})).$$

This gives that

[†]What it *really* means is that $|K_k(x)| \lesssim (1 + x/2^{-k})^{-100}$ (of course, the power is arbitrary).

Proposition 2.2.20. *For the SQG equation, there is no blow-up of an H^m solution at T unless the quantity*

$$\int_0^T \|\nabla^\perp \theta(t)\|_{L^\infty} dt$$

becomes infinite.

Now we consider some *geometric* blow-up criteria, which provide some intuition as to how the solutions look like near the blow-up time. They could be applied to check whether sharp gradients appearing in numerical simulations are true candidates for finite-time singularity. We begin with an elementary but elegant result of Cordoba–Fefferman [41].

Condition of Cordoba–Fefferman. In [41], the authors consider the so-called hyperbolic saddle scenario (or X-point configuration) for sharp front development of gSQG solutions. This scenario corresponds to the situation where certain level sets of $\theta(t,\cdot)$ evolve as hyperbolas becoming closer with each other, thereby developing a large gradient. Such a behavior was observed already in the original work of Constantin–Majda–Tabak [38, 37]. Therefore, it is natural to ask whether a finite time singularity can be obtained by *touching* (or collapsing) of two hyperbolic level sets. Such a collapse is generically observed in singularity formation for two and higher dimensional compressible fluid systems.

For simplicity, assume that there exist two time-evolving level sets $\Gamma_\pm(t)$ in $[0,T)$ for $\theta(t,\cdot)$ which are graph-type, that is

$$\Gamma_\pm(t) = \{x \in [a,b] \times \mathbb{R} : x_2 = f_\pm(t,x_1)\}$$

for some scalar valued C^1 functions $f_\pm : [a,b] \times [0,T) \to \mathbb{R}$. It is assumed that

$$f_-(t,x_1) < f_+(t,x_1),$$

for all $x_1 \in [a,b]$ and $0 \le t < T$. Most importantly, assume that at the critical time T they *completely collapse* into a common curve Γ. That is,

$$\lim_{t\to T}(f_+(t,x_1) - f_-(t,x_1)) = 0 \tag{57}$$

for all $x_1 \in [a,b]$. With these assumptions, the main result of [41] is the following:

Theorem 2.2.3. *Consider the transport equation*

$$\partial_t\theta + u \cdot \nabla\theta = 0,$$

where $u = \nabla^\perp\psi$ for some ψ. Assume that $u(t,\cdot)$ is Lipschitz continuous for all $t < T$. Then, for the above scenario to occur, it is necessary to have

$$\int_0^T \|u(t,\cdot)\|_{L^\infty([a,b]\times[f_-(t,x_1),f_+(t,x_1)])} dt = +\infty.$$

In particular, it is necessary to have

$$\limsup_{t\to T} \|u(t,\cdot)\|_{L^\infty} = +\infty.$$

This criterion is interesting since one does not need to specify at all the relation between the transported scalar θ and the velocity u. In particular the regularity of u relative to θ does not matter. Not surprisingly, the key ingredient in the proof of the above theorem is the incompressibility of the velocity field. For compressible equations, such a behavior can indeed occur with velocity fields which are uniformly bounded up to the time of shock formation.

The proof is simple enough to be described here.

Proof. To begin with, we consider the evolution of the area between the levels Γ_+ and Γ_-; namely

$$A(t) := [a,b] \times [f_-(t,x_1), f_+(t,x_1)].$$

Then,

$$\begin{aligned} \frac{d}{dt}|A(t)| = \psi(t,(a,f_+(t,a))) - \psi(t,(a,f_-(t,a))) \\ - (\psi(t,(b,f_+(t,b))) - \psi(t,(b,f_-(t,b)))). \end{aligned} \tag{58}$$

The is just a consequence of incompressibility. The change in time of $|A(t)|$ only comes from the flux through the lateral boundaries of $A(t)$, which is nothing but $\pm \int u_1$. Using that $u_1 = -\partial_{x_2}\psi$ together with the fundamental theorem of calculus gives the above identity.

Now, assume that the total collapse along $[a,b]$ occurs at T and define

$$\widetilde{A}(t) := [\tilde{a},\tilde{b}] \times [f_-(t,x_1), f_+(t,x_1)].$$

Here,

$$\tilde{a}(t) = a + \int_t^T \|u(s,\cdot)\|_{L^\infty(A(s))} ds$$

and

$$\tilde{b}(t) = b - \int_t^T \|u(s,\cdot)\|_{L^\infty(A(s))} ds.$$

Towards a contradiction, assume that

$$\int_0^T \|u(t,\cdot)\|_{L^\infty(A(t))} dt < \infty.$$

Then, we can pick $t^* < T$ such that $\tilde{a}(t), \tilde{b}(t) \in [a,b]$ for all $[t^*, T)$. For $t > t^*$, we compute

$$\begin{aligned}\frac{d}{dt}\widetilde{A}(t) = \|u(t,\cdot)\|_{L^\infty(A(t))}&(f_+(t,\tilde{b}) - f_-(t,\tilde{b}) + f_+(t,\tilde{a}) - f_-(t,\tilde{a})) \\ &+ \int_{\tilde{a}(t)}^{\tilde{b}(t)} \partial_t[f_+(t,x_1) - f_-(t,x_1)]dx_1.\end{aligned}$$

The last term equals

$$\psi(t,(\tilde{a}, f_+(t,\tilde{a}))) - \psi(t,(\tilde{a}, f_-(t,\tilde{a}))) - (\psi(t,(\tilde{b}, f_+(t,\tilde{b}))) - \psi(t,(\tilde{b}, f_-(t,\tilde{b})))).$$

Taking absolute values and using the mean value theorem, we have the bound

$$\int_{\tilde{a}(t)}^{\tilde{b}(t)} \partial_t[f_+(t,x_1) - f_-(t,x_1)]dx_1 \leq \|u_1(t)\|_{L^\infty(A(t))} \left(f_+(t,\tilde{a}) - f_-(t,\tilde{a}) + f_+(t,\tilde{b}) - f_-(t,\tilde{b})\right).$$

Therefore, we have that

$$\frac{d}{dt}\widetilde{A}(t) \geq 0,$$

whenever $t > t^*$. This is a contradiction to (57). □

In the specific case of SQG equations, one can go a step further and obtain a lower bound on the distance between two level sets, under the assumption that the collapse is *semi-uniform* ([40]): it simply means that for all $0 \leq t < T$ and $x_1 \in [a,b]$, there exists a uniform constant $c_0 > 0$ such that

$$\min_{x_1 \in [a,b]} (f_+(t,x_1) - f_-(t,x_1)) \geq c_0 \max_{x_1 \in [a,b]} (f_+(t,x_1) - f_-(t,x_1)).$$

We can simply define

$$\delta(t) = \frac{1}{b-a}|A(t)|$$

so that

$$\delta(t) \sim f_+(t,x_1) - f_-(t,x_1)$$

for any x_1 and t. The main result of [40] states the following.

Theorem 2.2.4. *Under the assumption of semi-uniform collapse in the SQG case,*

$$\delta(t) > \exp(-e^{At+B})$$

for some constants A, B.

Proof. We start with

$$\frac{d}{dt}|A(t)| \le 2 \sup_{x_1 \in [a,b]} |\psi(t,x_1,f_+(t,x_1)) - \psi(t,x_1,f_-(t,x_1))|,$$

which is immediate from (58). Now, the standard potential estimate for $\psi := (-\Delta)^{-\frac{1}{2}}\theta$ gives

$$|\psi(t,x) - \psi(t,x')| \le C\|\theta\|_{L^1 \cap L^\infty} |x-x'| \ln \frac{1}{|x-x'|}$$

where we assume that $|x-x'| \ll 1$ and $\theta \in L^1 \cap L^\infty(\mathbb{R}^2)$. We could have assumed from the beginning that $\delta(t) \ll 1$, and under the semi uniform hypothesis, $\delta(t) \sim |A(t)|$. Therefore,

$$\frac{d}{dt}|A(t)| \lesssim |A(t)| \ln \frac{1}{|A(t)|}.$$

This finishes the proof. □

Remark 2.2.21. *Repeating the above argument in the Euler and gSQG in the regular regime, one obtains at most exponential convergence of level sets:*

$$\delta(t) > \exp(-(At+B)).$$

Constantin–Majda–Tabak criterion. We describe the geometric criterion of Constantin–Majda–Tabak ([38, 37]). To state this criterion we recall the evolution equation for $\nabla^\perp\theta$:

$$D_t \nabla^\perp \theta = \nabla u \nabla^\perp \theta. \tag{59}$$

Here $D_t = \partial_t + u \cdot \nabla$. We define ξ to be the directional field of $\nabla^\perp\theta$, namely

$$\xi(t,x) := \frac{\nabla^\perp\theta(t,x)}{|\nabla^\perp\theta(t,x)|}.$$

We simply set $\xi = 0$ whenever $\nabla^\perp\theta(t,x) = 0$. Then, taking the dot product of (59) with $\nabla^\perp\theta(t,x)$, it follows that

$$D_t|\nabla^\perp\theta| = \alpha|\nabla^\perp\theta| \tag{60}$$

where

$$\alpha(t,x) := (S(t,x)\xi(t,x)) \cdot \xi(t,x) \tag{61}$$

with $S(t,x)$ being the symmetric part of $\nabla u(t,x)$. Then, it is not difficult to see that blow-up occurs at T if and only if

$$\int_0^T \|\alpha(t,\cdot)\|_{L^\infty} dt = \infty.$$

Next, noting that

$$\kappa = (\xi \cdot \nabla)\xi \cdot \xi^\perp$$

is the curvature of a level curve of θ, [34] derived the interesting equation

$$D_t(\kappa|\nabla^\perp\theta|) = (\nabla^\perp\theta \cdot \nabla)q,$$

with

$$q = (\nabla u \xi) \cdot \xi^\perp.$$

Integrating this equation over a bunch of smooth level sets,

$$\frac{d}{dt}\int_{x:C_1 \le \theta(t,x) \le C_2} \kappa|\nabla^\perp\theta| = 0.$$

Now, singularity formation requires a region of $|\nabla^\perp\theta| \gg 1$. Therefore, based on the above conservation law it is forced that either

- the curvature oscillates very rapidly, or
- the curvature goes to 0, namely the corresponding level set becomes straight.

Strictly related results have been obtained in the work of Chae [19].

Constantin–Fefferman–Majda criterion. We now describe the criterion of Constantin–Fefferman–Majda ([36]), adapted to the SQG equation. Recall that ξ is the direction field of $\nabla^\perp\theta$. By definition, $\xi = 0$ whenever $\nabla^\perp\theta$ vanishes. We define that a set $\Omega_0 \subset \mathbb{R}^2$ is *smoothly directed* if

- ξ is Lipschitz continuous in Ω_0,
- there exists some $\rho > 0$ such that

$$\sup_{q_0 \in \Omega_0} \int_0^T \|\nabla\xi(t,\cdot)\|^2_{L^\infty(B_\rho(\Phi(t,q_0)))} dt < \infty.$$

Under the above key assumption, one can show that there is no singularity formation from particles starting in Ω_0:

Theorem 2.2.5. *Assume that Ω_0 is smoothly directed and a solution θ to SQG remains smooth in $[0,T)$. Furthermore, assume that the velocity is bounded L^1 in time:*

$$\sup_{q_0\in\Omega_0}\int_0^T \|u(t,\cdot)\|_{L^\infty(B_\rho(\Phi(t,q_0)))}dt < \infty.$$

Then,

$$\sup_{q_0\in\Omega_0} |\nabla^\perp\theta(t,\Phi(t,q_0))| < \infty.$$

Proof. The starting point is the principal value formula for the velocity gradient in the SQG case:

$$\nabla u(x) = -P.V.\int \frac{y^\perp}{|y|}\otimes(\nabla\theta)(x+y)\,\frac{dy}{|y|^2}. \tag{62}$$

Recalling the definition of α in (61), we see that

$$\alpha(x) = P.V.\int \hat{y}\cdot\xi^\perp(x)\,\xi(x+y)\cdot\xi^\perp(x)|\nabla^\perp\theta(x+y)|\,\frac{dy}{|y|^2}.$$

Usually this is handled using the regularity of $\nabla^\perp\theta$. This time, we use regularity of ξ instead. We divide

$$\alpha = \alpha_1 + \alpha_2$$

where α_2 corresponds to the integral in the region $|y| \gtrsim \rho$. It is easy to estimate α_2; here, we can use the alternative expression

$$(\nabla u)(x) = -\int \frac{1}{|y|}(\nabla\nabla^\perp\theta)(x+y)dy.$$

Then, one can write down a similar expression for α_2 to deduce

$$|\alpha_2(x)| \le C\rho^{-2}\|\theta\|_{L^2} \le C\rho^{-2}\|\theta_0\|_{L^2}.$$

It only remains to handle

$$\alpha_1(x) := P.V.\int \chi(\frac{|y|}{\rho})\hat{y}\cdot\xi^\perp(x)\,\xi(x+y)\cdot\xi^\perp(x)|\nabla^\perp\theta(x+y)|\,\frac{dy}{|y|^2}.$$

Rewriting

$$\xi(x+y)\cdot\xi^\perp(x) = (\xi(x+y)-\xi(x))\cdot\xi^\perp(x)$$

and using the mean value theorem, we immediately see that

$$|\alpha_1(x)| \le C\|\nabla\xi\|_{L^\infty(B(\rho,x))} \int \chi(\frac{|y|}{\rho})|\nabla^\perp\theta(x+y)|\,\frac{dy}{|y|}.$$

The trick now is to write $|\nabla^\perp\theta(x+y)| = (\xi\cdot\nabla^\perp\theta)(x+y)$ and integrate by parts in y to obtain that

$$\begin{aligned}\int \chi(\frac{|y|}{\rho})|\nabla^\perp\theta(x+y)|\,\frac{dy}{|y|} = &-\int \chi(\frac{|y|}{\rho})\nabla^\perp\cdot\xi(x+y)\theta(x+y)\,\frac{dy}{|y|}\\ &-\int \nabla^\perp\chi(\frac{|y|}{\rho})\cdot\xi(x+y)\theta(x+y)\,\frac{dy}{|y|}\\ &+P.V.\int \chi(\frac{|y|}{\rho})\theta(x+y)\xi(x+y)\cdot\hat{y}^\perp\,\frac{dy}{|y|^2}.\end{aligned}$$

The first two terms are easy to bound, just using $\|\theta\|_{L^\infty}$. To resolve the singularity in the last integral, we rewrite

$$\xi(x+y) = (\xi(x+y)-\xi(x)) + \xi(x)$$

to obtain

$$\begin{aligned}P.V.\int \chi(\frac{|y|}{\rho})\theta(x+y)\xi(x+y)\cdot\hat{y}^\perp\,\frac{dy}{|y|^2} &= -\xi(x)\cdot u(x)\\ +\xi(x)\cdot\int\left(1-\chi(\frac{|y|}{\rho})\right)\theta(x+y)\hat{y}^\perp\,\frac{dy}{|y|^2}&\\ +P.V.\int \chi(\frac{|y|}{\rho})\theta(x+y)(\xi(x+y)-\xi(x))\cdot\hat{y}^\perp\,\frac{dy}{|y|^2}.&\end{aligned}$$

The second term in the right hand side can be easily bounded by $\|\theta\|_{L^2}$ using Hölder's inequality. The last integral can be bounded by

$$C(\rho\|\nabla\xi\|_{L^\infty(B(\rho,x))}\|\theta\|_{L^\infty} + \rho^{-1}\|\theta\|_{L^2}).$$

So far we have obtained the bound

$$\begin{aligned}|\alpha(x)| \lesssim\ &\|\nabla\xi\|_{L^\infty(B(\rho,x))}\|u\|_{L^\infty(B(\rho,x))}\\ &+(1+\rho\|\nabla\xi\|_{L^\infty(B(\rho,x))})(\|\nabla\xi\|_{L^\infty(B(\rho,x))}\|\theta_0\|_{L^\infty} + \rho^{-2}\|\theta_0\|_{L^2}).\end{aligned}$$

We are ready to complete the proof. Take a point $q_0\in\Omega_0$ and recall the formula

$$D_t|\nabla^\perp\theta| = \alpha|\nabla^\perp\theta|.$$

We have shown that

$$|\alpha(t,x)| \lesssim 1 + \|u\|_{L^\infty(B(\rho,x))} + \|\nabla\xi\|^2_{L^\infty(B(\rho,x))},$$

where the implicit constant depends only on ρ and $\|\theta_0\|_{L^p}$. Integrating along particle trajectories finishes the proof. □

Remark 2.2.22. *In the SQG case we can simply bound*

$$\|u\|_{L^\infty} \lesssim \|\theta_0\|_{L^2} + \|\theta_0\|_{L^\infty} \ln(10 + \|\nabla^\perp \theta\|_{L^\infty}).$$

Therefore, the uniform boundedness assumption on the velocity can be dropped if we are going to replace the assumption on $\nabla\xi$ to the global one.

2.3 Cauchy–Kowalevskaya type theorem

In this section, we provide a proof of the Cauchy–Kowalevskaya type theorem for the gSQG equations, which provides local regularity in the class of analytic functions. The following arguments require only that $P \lesssim 1$, that is, P is a bounded multiplier.

Theorem 2.3.1 (wellposedness in the analytic class)**.** *In the regular and intermediate regimes,* (gSQG) *is locally wellposed in the analytic class.*

Proof. We obtain a priori estimates in the analytic class. Let us start by writing out (gSQG) in Fourier:

$$\partial_t \hat{\theta}(\xi) = \int_{\mathbb{R}^2} (\xi - \eta)^\perp \cdot P(|\xi - \eta|)\hat{\theta}(\xi - \eta)\eta\hat{\theta}(\eta)d\eta$$

For $\tau = \tau(t) > 0$, we define

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\||\xi|^s e^{\tau|\xi|}\hat{\theta}(\xi)\|_{L^2}^2 &= \dot{\tau}\int |\xi|^{2s+1}e^{2\tau|\xi|}|\hat{\theta}(\xi)|^2 d\xi \\ +\mathrm{Re}\int |\xi|^{2s}e^{2\tau|\xi|}\overline{\hat{\theta}(\xi)} &\int_{\mathbb{R}^2}(\xi-\eta)^\perp \cdot P(|\xi-\eta|)\hat{\theta}(\xi-\eta)\eta\hat{\theta}(\eta)d\eta d\xi \\ &= I + II. \end{aligned}$$

Strictly speaking, we also need to estimate

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau|\xi|}\hat{\theta}(\xi)\|_{L^2}^2$$

as well, but we shall omit the details. For the variable

$$Z(\xi) = e^{\tau|\xi|}\hat{\theta}(\xi),$$

we have

$$II = +\mathrm{Re}\iint |\xi|^{s+\frac{1}{2}}\overline{Z(\xi)}|\xi|^{s-\frac{1}{2}}(\xi-\eta)^\perp \cdot P(|\xi-\eta|)e^{\tau(|\xi|-|\eta|)}\hat{\theta}(\xi-\eta)\eta Z(\eta)d\eta d\xi.$$

We write

$$|\xi|^{s-\frac{1}{2}} = |\xi-\eta|^{s-\frac{1}{2}} + |\eta|^{s-\frac{1}{2}} + R(\xi,\eta)$$

and decompose accordingly

$$II = II_1 + II_2 + II_3.$$

Then, we can estimate with

$$W(\xi) = |\xi|^{s+\frac{1}{2}}|Z(\xi)|$$

that

$$\begin{aligned}|II_2| &\leq C\iint W(\xi)|\xi-\eta|P(|\xi-\eta|)Z(|\xi-\eta|)W(\eta)d\eta d\xi \\ &\leq C\||\xi|P(|\xi|)Z(|\xi|)\|_{L^1_\xi}\|W(\xi)\|^2_{L^2_\xi}.\end{aligned}$$

Next, we can handle II_1 similarly:

$$\begin{aligned}|II_1| &\leq C\iint W(\xi)W(\xi-\eta)|\eta||Z(\eta)|d\xi d\eta \\ &\leq C\||\xi|Z(|\xi|)\|_{L^1_\xi}\|W(\xi)\|^2_{L^2_\xi},\end{aligned}$$

where we have used simply that $P\lesssim 1$. Lastly, to estimate II_3, we take absolute values and use the calculus inequality

$$|R| \lesssim |\eta||\xi-\eta|(|\xi-\eta|^{s-\frac{5}{2}} + |\eta|^{s-\frac{5}{2}})$$

to bound

$$|III_3| \leq III_{31} + III_{32},$$

where

$$\begin{aligned}III_{32} &\leq C\iint W(\xi)|\xi-\eta|^2P(|\xi-\eta|)Z(|\xi-\eta|)|\eta|^{s-\frac{3}{2}}|Z(\eta)|d\xi d\eta \\ &\leq C\||\xi|^2P(|\xi|)Z(|\xi|)\|_{L^1_\xi}\|\tilde{W}(\xi)\|^2_{L^2_\xi}.\end{aligned}$$

Here,

$$\tilde{W}(\xi) = (|\xi|^{s+\frac{1}{2}}+1)|Z(\xi)|.$$

A similar bound can be obtained for III_{31}. Collecting the bounds and using again that $P\lesssim 1$, we obtain that

$$|II| \leq C\||\xi|(1+|\xi|)P(|\xi|)Z(|\xi|)\|_{L^1_\xi}\|\tilde{W}(\xi)\|^2_{L^2_\xi}.$$

Now, the point is that we can choose $\dot{\tau}$ to be a sufficiently large negative so that

$$\dot{\tau} + C\||\xi|(1+|\xi|)P(|\xi|)Z(|\xi|)\|_{L^1_\xi} \leq 0.$$

The estimate can be closed since the quantity $C\||\xi|(1+|\xi|)P(|\xi|)Z(|\xi|)\|_{L^1_\xi}$ is dominated by $\|\tilde{W}(\xi)\|_{L^2_\xi}$. This gives

$$\frac{1}{2}\frac{d}{dt}\||\xi|^s e^{\tau|\xi|}\hat{\theta}(\xi)\|_{L^2}^2 \leq 0,$$

or

$$\||\xi|^s e^{\tau(t)|\xi|}\hat{\theta(t)}(\xi)\|_{L^2}^2 \leq \||\xi|^s e^{\tau_0|\xi|}\hat{\theta}_0(\xi)\|_{L^2}^2.$$

This finishes the proof of the analytic a priori estimate. Note furthermore propagation of analytic regularity by Sobolev regularity; this is because the quantity

$$C\||\xi|(1+|\xi|)P(|\xi|)Z(|\xi|)\|_{L^1_\xi}$$

is bounded simply by a Sobolev norm of θ. That is, initially analytic data do not leave the analytic class unless it blows up in a finite regularity class.

For the existence, one can again regularize the equation using dissipation. Uniqueness is clear because we are assuming very strong regularity for the solutions. $\square$

Problem 2.3.1. *Try to extend CK type theorem to the case of "slightly singular" multipliers.*

2.4 Local regularity for smooth patches

In this section we discuss the evolution of patch solutions, which are given by the characteristic functions of a time-dependent domain. They model interesting physical situations, e.g. region of strong vortex in the case of 2D Euler. We achieve significant theoretical and computational simplification by assuming that the solutions are constant inside and outside the moving set: essentially, it becomes a one-dimensional PDE, although the price one needs to pay is that the equation takes an integro-differential form. Since the accompanying velocity is not smooth anymore, one would expect that the boundary of the patch would become irregular in time. This is not the case at least for locally in time and this shows the ability to propagate an-isotropic regularity for the family of gSQG equations. Understanding the possibility of small scale creation within this class of solutions, including finite-time singularity formation, is a very active research topic.

2.4.1 Contour dynamics equation

In this section, we follow the presentation in [63, 20]. We shall consider initial data of the form

$$\theta_0(x) = \mathbf{1}_\Omega$$

where $\Omega \in \mathbb{R}^2$ is a bounded open set, with a sufficiently smooth boundary. This means that near any point on the boundary, we can rotate the set in a way that locally, the boundary can be described as the graph of a smooth function. We shall parameterize the boundary by $\gamma \in \mathbb{T} = [-\pi, \pi)$, in a way that it is of constant speed:

$$z_0 : \mathbb{T} \to \partial\Omega, \qquad |\partial_\gamma z_0(\gamma)|^2 = C.$$

Even at later times, we shall measure the regularity of the patch boundary using a constant speed parametrization (CSP). We need to impose the chord-arc condition

$$\frac{|z(t,\gamma) - z(t,\gamma-\eta)|}{|\eta|} > 0, \tag{63}$$

for all $\gamma, \eta \in \mathbb{T}$, which in particular ensures that the curve describing the patch boundary is one-to-one. The notation z is appropriate, since it is sometimes convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider z as a complex number. We claim that the evolution of the α-SQG equation with the above patch initial data can be described by the integro-differential equation

$$\partial_t z(t,\gamma) = C_\alpha \int_{\mathbb{T}} \frac{\partial_\gamma z(t,\gamma) - \partial_\gamma z(t,\gamma-\eta)}{|z(t,\gamma) - z(t,\gamma-\eta)|^\alpha} d\eta. \tag{64}$$

By a simple change of variables, the right hand side is equal to

$$C_\alpha \int_{\mathbb{T}} \frac{\partial_\gamma z(t,\gamma) - \partial_\gamma z(t,\eta)}{|z(t,\gamma) - z(t,\eta)|^\alpha} d\eta,$$

and we shall use both formulas. This claim consists of two parts: first, it means that if we have a solution to the curve evolution equation (64), this gives a well-defined weak solution of (gSQG), and second, this is the *unique* weak solution under certain assumptions. From now on, we shall neglect the constant C_α. The equation of the form (64) is commonly referred to as *contour dynamics equation* (CDE), simply because it shows how the contour changes with time.

To begin with, let us check that if Ω is the unit disc, then it defines a steady state. We choose the initial parametrization $z_0(\gamma) = e^{i\gamma}$, and compute that the right hand side of (64) is given by

$$\int_{\mathbb{T}} \frac{i(e^{i\gamma} - e^{i(\gamma-\eta)})}{|e^{i\gamma} - e^{i(\gamma-\eta)}|^\alpha} d\eta = ie^{i\gamma} \int_{\mathbb{T}} \frac{1-e^{i\eta}}{|1-e^{i\eta}|^\alpha} d\eta = ic_\alpha e^{i\gamma}.$$

This is tangent to the circle (which is $\partial\Omega$), so it confirms that the unit disc patch is a steady solution.

Derivation of CDE. Now let us formally derive the CDE. For this purpose, we shall fix a patch Ω and recall that the velocity vector field associated to the patch is given by

$$u(z) = \int_\Omega \nabla^\perp K|z - z'|dz'.$$

We just recall the Green's formula

$$\int_\Omega \partial_{x_j} f = \int_{\partial\Omega} f n_j ds.$$

That is,

$$u(z) = \int_{\partial\Omega} n^\perp K(z - z')dz'.$$

Using the boundary parametrization and evaluating at $z(\gamma)$, we obtain that

$$u(z(\gamma)) = \int_{\mathbb{T}} z'(\eta)K(z(\gamma) - z(\eta))d\eta.$$

In the 2D Euler case, this is simply

$$\frac{1}{2\pi}\int_{\mathbb{T}} \ln\frac{1}{|z(\gamma) - z(\eta)|} z'(\eta)d\eta.$$

Moving on to the α-SQG case, we obtain (with a constant that we neglect)

$$\int_{\mathbb{T}} \frac{1}{|z(\gamma) - z(\eta)|^\alpha} z'(\eta)d\eta.$$

This is apparently different with the one given in (64). This additional term can be inserted since

$$z'(\gamma)\int_{\mathbb{T}} \frac{1}{|z(\gamma) - z(\eta)|^\alpha} d\eta$$

is clearly *tangent* to the contour at the point $z(\gamma)$. This is simply because for patch solutions, we are only interested in the motion of the boundary as a whole and not really tracking the individual "fluid particles." This inclusion clearly makes the kernel more regular.

Weak solutions. We now define the notion of weak solutions for (gSQG): θ is a weak solution (a distribution defined in space-time) if for any $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^2)$, we have

$$\int_0^T \int_{\mathbb{R}^2} \theta(t,x)\left(\partial_t\varphi(t,x) + u(t,x)\cdot\nabla\varphi(t,x)\right)dxdt = 0.$$

For this definition to make sense, we need the product (θu) to be a well-defined distribution. When $\theta \in L^\infty$ (and decaying at infinity), for gSQG equations with kernel singular up to the usual SQG case, we have that $u \in BMO$ (or better), so in particular we have that $\theta u \in L^p_{loc}$ for any p. In the intermediate regime, this is not true anymore, and an observation is necessary to even define properly the notion of weak solutions. To this end, the authors in [20] observed the following. We begin with

$$\int \theta u \cdot \nabla \varphi = \int \Lambda^{2-\alpha}[\psi]\, \nabla^\perp \psi \cdot \nabla \varphi.$$

On the other hand, we compute that

$$\begin{aligned}\int \psi[\Lambda^{2-\alpha}\nabla^\perp \cdot, \nabla\varphi]\psi &= \int \psi \Lambda^{2-\alpha}[\nabla^\perp \cdot (\nabla\varphi\psi)] - \int \psi \nabla\varphi \Lambda^{2-\alpha} \cdot \nabla^\perp \psi \\ &= \int \Lambda^{2-\alpha}[\psi]\nabla\varphi \cdot \nabla^\perp\psi + \int \nabla^\perp \cdot (\psi\nabla\varphi)\Lambda^{2-\alpha}[\psi] \\ &= 2\int \Lambda^{2-\alpha}[\psi]\, \nabla^\perp\psi \cdot \nabla\varphi.\end{aligned}$$

Combining the above identities, we arrive at

$$\int \theta u \cdot \nabla\varphi = \frac{1}{2}\int \psi[\Lambda^{2-\alpha}\nabla^\perp \cdot, \nabla\varphi]\psi,$$

where the point is that $[\Lambda^{2-\alpha}\nabla^\perp \cdot, \nabla\varphi]\psi \in L^2_l oc$ if $\theta \in L^\infty$ and decaying at infinity. This shows that the nonlinearity

$$\int \theta u \cdot \nabla\varphi$$

is well-defined even in the intermediate regime.

Weak solutions and CDE. The following proposition shows the relation with the CDE and the notion of weak solutions defined in the above.

Proposition 2.4.1 ([63, Proposition 3.2])**.** *Assume that the curve $z(t,\gamma)$ satisfies* (63) *and* (64)*. Then, $\theta(t,x)$ defines a weak solution. Conversely, if we have a patch-type weak solution whose boundary curve is smooth and satisfies* (63) *for each moment of time, then* (64) *is satisfied.*

Proof. We assume that θ is a patch weak solution, whose boundary curve is given by $z(t,\gamma)$ at time t. We also set $\theta(t,\cdot) = \mathbf{1}_{\Omega(t)}$. For φ smooth, we need to have

$$0 = \int_0^T \int_{\mathbb{R}^2} \theta(t,x)\left(\partial_t\varphi(t,x) + u(t,x)\cdot\nabla\varphi(t,x)\right) dxdt =: I + II.$$

First, we can compute that

$$
\begin{aligned}
I &= \int_0^T \int_{\Omega(t)} \partial_t \varphi(t,x) dx dt \\
&= \int_0^T \int_{\mathbb{R}^2} \mathbf{1}_{\Omega(t)} \partial_t \varphi(t,x) dx dt.
\end{aligned}
$$

Using integration by parts and

$$
\partial_t \mathbf{1}_{\Omega(t)} = \partial_t z \cdot \partial_\gamma^\perp z \delta(x-z)
$$

(here, only normal component of $\partial_t z$ contributes which gives $\partial_t z \cdot \partial_\gamma^\perp z / |\partial_\gamma z|$ but the tangential integration factor is $|\partial_\gamma z|\delta$), we obtain that

$$
I = -\int_0^T \int_{\mathbb{T}} \partial_t z(t,\gamma) \cdot \partial_\gamma^\perp z(t,\gamma) \varphi(t, z(t,\gamma)) d\gamma dt.
$$

On the other hand,

$$
\begin{aligned}
II &= \int_0^T \int_{\Omega(t)} u \cdot \nabla \varphi dx dt \\
&= \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega(t) \cap \{d(x, \partial\Omega(t)) \geq \varepsilon\}} u \cdot \nabla \varphi dx dt.
\end{aligned}
$$

This is valid since $u \in BMO(\mathbb{R}^2)$ and in particular belongs to L^1. For each $\varepsilon > 0$, we are in a region where u is C^∞ smooth. Therefore, we may integrate by parts with

$$
\Omega_\epsilon(t) = \Omega(t) \cap \{d(x, \partial\Omega(t)) \geq \varepsilon\}
$$

to obtain

$$
\begin{aligned}
II &= \lim_{\varepsilon \to 0} \int_0^T \int_{\partial\Omega_\varepsilon(t)} (u\varphi) d\sigma(x) dt \\
&= \int_0^T \int_{\mathbb{T}} u(t, z(t,\gamma)) \cdot \partial_\gamma^\perp z(t,\gamma) \varphi(t, z(t,\gamma)) d\gamma dt
\end{aligned}
$$

where we have used that u is divergence free. However, we may compute explicitly $u(t, z(t,\gamma))$. Next, from $I + II = 0$ for any φ, we conclude that

$$
z_t \cdot \partial_\gamma^\perp z + u \cdot \partial_\gamma^\perp z \equiv 0
$$

on $\partial\Omega(t)$. This derives the CDE. The other direction can be obtained in a similar way. $\square$

Local regularity for smooth patches. To prepare for local wellposedness, we recall the CDE

$$\partial_t z = \int_{\mathbb{T}} \frac{z_\gamma(t,\gamma) - z_\gamma(t,\eta)}{|z(t,\gamma) - z(t,\eta)|^\alpha} d\eta.$$

We observe the following symmetric structure:

$$\begin{aligned}\frac{1}{2}\|z\|_{L^2}^2 = \int_{\mathbb{T}} z(\gamma)\cdot z_t(\gamma) d\gamma &= \iint z(\gamma) \frac{z_\gamma(t,\gamma) - z_\gamma(t,\eta)}{|z(t,\gamma) - z(t,\eta)|^\alpha} d\eta d\gamma \\ &= \frac{1}{2}\iint (z(\gamma) - z(\eta)) \frac{z_\gamma(t,\gamma) - z_\gamma(t,\eta)}{|z(t,\gamma) - z(t,\eta)|^\alpha} d\eta d\gamma\end{aligned}$$

by antisymmetry. Then, we may rewrite it as

$$= \frac{1}{2(2-\alpha)} \iint (\partial_\gamma + \partial_\eta)|z(\gamma) - z(\eta)|^{2-\alpha} d\gamma d\eta = 0.$$

This shows that in the H^k estimate, the top derivative term should cancel. In the following, we shall keep using this trick;

$$\iint A(\gamma)B(\gamma,\eta) d\eta d\gamma = \frac{1}{2}\iint (A(\gamma) - A(\eta))B(\gamma,\eta) d\eta d\gamma$$

if B is anti-symmetric; $B(\gamma,\eta) = -B(\eta,\gamma)$.

Next, to enforce the arc-chord condition, we shall propagate an upper bound on the quantity

$$F(t,\gamma,\eta) := \frac{|\eta|}{|z(t,\gamma) - z(t,\gamma-\eta)|}.$$

When $\eta = 0$, we define

$$F(t,\gamma,0) := \frac{1}{|z_\gamma(t,\gamma)|}.$$

We are ready to state the main theorem for gSQG patches.

Theorem 2.4.1. *Let $z_0 \in H^k(\mathbb{T})$ with $k \geq 3$. Assume further that $F_0 < \infty$. Then, there is a corresponding local unique solution to CDE belonging to $C^1([0,T]; H^k(\mathbb{T}))$ for some $T > 0$.*

Remark 2.4.2. *In general, it is known that bounded weak solutions are not unique for the SQG equations ([10, 24]). However, it is not known whether bounded weak solutions can be non-unique even in the (highly restricted class of) patch solutions. The above result definitely shows that the answer is no for smooth boundary patches.*

Proof in the regular case. We divide the proof into a few steps.

1. A priori estimates. First, we obtain an a priori estimate for the H^3 norm. In the above, we have seen conservation of the L^2 norm. Then, we need to compute

$$\frac{1}{2}\frac{d}{dt}\|\partial_\gamma^3 z\|_{L^2}^2 = \int_{\mathbb{T}} \partial_\gamma^3 z \cdot \partial_\gamma^3 z_t d\gamma.$$

The principal term is when all the derivatives fall on the numerator (all other terms have three derivatives falling on z or less); this gives

$$\begin{aligned}
I &= \iint \partial_\gamma^3 z \cdot \frac{\partial_\gamma^4 z(\gamma) - \partial_\gamma^4 z(\gamma-\eta)}{|z(\gamma)-z(\gamma-\eta)|^\alpha} d\eta d\gamma \\
&= \frac{1}{4}\iint \frac{\partial_\gamma |\partial_\gamma^3 z(\gamma) - \partial_\gamma^3(\gamma-\eta)|^2}{|z(\gamma)-z(\gamma-\eta)|^\alpha} d\eta d\gamma \\
&= \frac{\alpha}{4}\iint \frac{(z(\gamma)-z(\gamma-\eta))\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta))|\partial_\gamma^3 z(\gamma)-\partial_\gamma^3(\gamma-\eta)|^2}{|z(\gamma)-z(\gamma-\eta)|^{\alpha+2}} d\eta d\gamma.
\end{aligned}$$

This is nothing but the (anti)symmetric structure that we have seen in the above. Then, we can take absolute values. The integrand to begin with can be bounded by

$$\lesssim \frac{|\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta)||\partial_\gamma^3 z(\gamma)-\partial_\gamma^3(\gamma-\eta)|^2}{|z(\gamma)-z(\gamma-\eta)|^{\alpha+1}}$$

and we can then bound the denominator using the quantity F;

$$\frac{1}{|z(\gamma)-z(\gamma-\eta)|^{\alpha+1}} \lesssim \frac{|F|^{1+\alpha}}{|\eta|^{1+\alpha}}.$$

Moreover,

$$\frac{|\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta)|}{|\eta|} \lesssim \|z\|_{C^2}.$$

We are using the fact that $|\eta|^{-\alpha}$ is integrable on $\mathbb{T}$. This gives

$$|I| \lesssim \|F\|_{L^\infty}^{1+\alpha}\|z\|_{C^2}\|z\|_{H^3}^2 \lesssim \|F\|_{L^\infty}^{1+\alpha}\|z\|_{H^3}^3.$$

All the other terms can be bounded by a similar quantity: we obtain

$$\frac{d}{dt}\|z\|_{H^3} \lesssim \|F\|_{L^\infty}^{3+\alpha}\|z\|_{H^3}^4.$$

It remains to obtain a bound on $\|F\|_{L^\infty}$. For this we compute

$$\frac{d}{dt}\|F\|_{L^p}^p \lesssim p\iint |F|^{p+1}\frac{|z_t(\gamma)-z_t(\gamma-\eta)|}{|\eta|}d\eta d\gamma.$$

Using the equation directly, it is not very difficult to obtain the pointwise bound

$$\frac{|z_t(\gamma) - z_t(\gamma - \eta)|}{|\eta|} \lesssim \|z\|_{C^2}^2 \|F\|_{L^\infty}^{1+\alpha} \lesssim \|z\|_{H^3}^2 \|F\|_{L^\infty}^{1+\alpha}.$$

This gives

$$\frac{d}{dt}\|F\|_{L^p} \lesssim \|z\|_{H^3}^2 \|F\|_{L^\infty}^{2+\alpha} \|F\|_{L^p}.$$

But then we can take the limit $p \to \infty$. This allows to close the estimate

$$\frac{d}{dt}(\|F\|_{L^\infty} + \|z\|_{H^3}) \lesssim (\|F\|_{L^\infty} + \|z\|_{H^3})^{7+\alpha}.$$

That is, if the quantity $M(t) = \|F(t)\|_{L^\infty} + \|z(t)\|_{H^3}$ is finite at the initial time, then there exists a time $T > 0$ depending only on $M(0)$ such that $M(t)$ remains finite on $[0, T)$.

2. Existence. The existence of a solution to (64) can be proved by introducing a sequence of regularized systems preserving the antisymmetric structure. In [63], the following regularization was used:

$$\partial_t z^\epsilon(t,\gamma) = \phi_\epsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\epsilon * z^\epsilon(t,\gamma) - \phi_\epsilon * z^\epsilon(t,\gamma-\eta))}{|z^\epsilon(t,\gamma) - z^\epsilon(t,\gamma-\eta)} d\eta.$$

The additional mollification $\phi\epsilon*$ in front of the right hand side is to ensure the antisymmetric structure. The point is that, for any $\epsilon > 0$, this system no longer loses derivatives and therefore local existence and uniqueness can be proved directly by the contraction mapping principle.

3. Uniqueness. To obtain uniqueness, assume that there are two solutions z_1, z_2 corresponding to the same initial data z_0 and define the difference by $z = z_1 - z_2$. Then, we can perform an L^2 estimate for z:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2 &= \iint z(\gamma) \cdot \left(\frac{\partial_\gamma z_1(\gamma) - \partial_\gamma z_1(\gamma-\eta)}{|z_1(\gamma) - z_1(\gamma-\eta)|^\alpha} - \frac{\partial_\gamma z_1(\gamma) - \partial_\gamma z_1(\gamma-\eta)}{|z_2(\gamma) - z_2(\gamma-\eta)|^\alpha} \right) \\
&\quad + \iint z(\gamma) \cdot \frac{\partial_\gamma z(\gamma) - \partial_\gamma z(\gamma-\eta)}{|z_2(\gamma) - z_2(\gamma-\eta)|^\alpha} d\eta d\gamma \\
&= I + II.
\end{aligned}$$

The term II can be handled using the antisymmetric structure. For the other term, we can estimate after taking absolute values

$$\begin{aligned}
|I| &\lesssim \|F_1\|_{L^\infty}^\alpha \|F_2\|_{L^\infty}^\alpha \|z_1\|_{C^2} \iint |\eta|^{1-\alpha} |z(\gamma)| \left| \left| \frac{z_1(\gamma) - z_1(\gamma-\eta)|}{\eta} \right|^\alpha - \left| \frac{z_2(\gamma) - z_2(\gamma-\eta)|}{\eta} \right|^\alpha \right| \\
&\lesssim \|F_1\|_{L^\infty} \|F_2\|_{L^\infty} \|z_1\|_{C^2} \iint |\eta|^{-\alpha} |z(\gamma)|\, |z(\gamma) - z(\gamma-\eta)| \\
&\lesssim \|F_1\|_{L^\infty} \|F_2\|_{L^\infty} \|z_1\|_{C^2} \|z\|_{L^2}^2.
\end{aligned}$$

This gives, with a constant depending on the solutions z_1 and z_2,

$$\frac{d}{dt}\|z\|_{L^2}^2 \lesssim \|z\|_{L^2}^2.$$

This finishes the proof of uniqueness. $\square$

To move on to the SQG and more singular case where $\alpha \geq 1$, we need to exploit an additional cancellation. An important idea of [63] is to modify the CDE as

$$\partial_t z = \int_{\mathbb{T}} \frac{z_\gamma(t,\gamma) - z_\gamma(t,\eta)}{|z(t,\gamma) - z(t,\eta)|^\alpha} d\eta + \Lambda(t,\gamma) z_\gamma(t,\gamma). \tag{65}$$

To begin with, for any $\Lambda(t,\gamma)$, (65) does not alter the shape of the boundary, simply because z_γ is *tangential* to the curve. However, the key point is that Λ can be chosen in a unique way that

$$z_\gamma \cdot z_\gamma^2 \equiv 0,$$

which gives a crucial cancellation. Note that this condition is equivalent with the one that

$$|\partial_\gamma z|^2 \equiv A(t). \tag{66}$$

is constant in γ (but not in t). Namely, ensuring that the parametrization has a normalized speed gives an additional cancellation. We pursue generality and consider

$$\partial_t z(t,\gamma) = \int_{\mathbb{T}} \frac{z_\gamma(t,\gamma) - z_\gamma(t,\eta)}{D(|z(t,\gamma) - z(t,\eta)|)} d\eta + \Lambda(t,\gamma) z_\gamma(t,\gamma). \tag{67}$$

Now, by differentiating (66) in t, plugging in (67) for $\partial_t z$, and integrating in space (using that $\int \partial_\gamma \Lambda d\gamma = 0$), we obtain that

$$\begin{aligned} \Lambda(t,\gamma) = \frac{\gamma+\pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma z(\gamma)}{|\partial_\gamma z(\gamma)|^2} \cdot \partial_\gamma \left(\int_{\mathbb{T}} \frac{z_\gamma(\gamma) - z_\gamma(\gamma-\eta)}{D(|z(\gamma) - z(\gamma-\eta)|)} d\eta \right) d\gamma \\ - \int_{-\pi}^{\gamma} \frac{\partial_\gamma z(\eta)}{|\partial_\gamma z(\eta)|^2} \cdot \partial_\eta \left(\int_{\mathbb{T}} \frac{z_\gamma(\eta) - z_\gamma(\eta-\xi)}{D(|z(\eta) - z(\eta-\xi)|)} d\xi \right) d\eta. \end{aligned} \tag{68}$$

Note the normalization $\Lambda(t,-\pi) = \Lambda(t,\pi) = 0$.

We now perform a priori estimates with the modified equations (67)–(68). As an exercise, we compute

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|z\|_{L^2}^2 &= \int_{\mathbb{T}} \Lambda(\gamma) z(\gamma) \cdot \partial_\gamma z(\gamma) d\gamma \\ &= -\frac{1}{2}\int_{\mathbb{T}} \partial_\gamma \Lambda(\gamma) |z|^2(\gamma) d\gamma \\ &\lesssim \|\partial_\gamma \Lambda\|_{L^\infty} \|z\|_{L^2}^2. \end{aligned}$$

This motivates us to study

$$\partial_\gamma \Lambda = \frac{1}{A}\left(\frac{1}{2\pi}\int_{\mathbb{T}} I(\gamma)d\gamma - I(\gamma)\right)$$

where $A = |\partial_\gamma z(\gamma)|^2$ and

$$I(\gamma) = \partial_\gamma z(\gamma)\cdot\partial_\gamma\left(\int_{\mathbb{T}}\frac{z_\gamma(\gamma)-z_\gamma(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)}d\eta\right).$$

We compute

$$\partial_\gamma\left(\int_{\mathbb{T}}\frac{z_\gamma(\gamma)-z_\gamma(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)}d\eta\right) = \int_{\mathbb{T}}\frac{z_\gamma^2(\gamma)-z_\gamma^2(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)}d\eta$$
$$-\int_{\mathbb{T}}\frac{(z_\gamma(\gamma)-z_\gamma(\gamma-\eta))D'(|z(\gamma)-z(\gamma-\eta)|)}{D^2(|z(\gamma)-z(\gamma-\eta)|)}\frac{(z(\gamma)-z(\gamma-\eta))\cdot\partial_\gamma(z(\gamma)-z(\gamma-\eta))}{|z(\gamma)-z(\gamma-\eta)|^2}d\eta.$$

To see how the first term can be bounded, we take a cutoff length L_0 and note that we can find $D_0 > 0$ such that

$$D(|z(\gamma)-z(\gamma-\eta)|) \geq D_0(L_0, \|F\|_{L^\infty})$$

whenever $|\eta| > L_0$. The precise form of D_0 will depend on D. Therefore, the integral in this regime can be bounded by

$$\int_{|\eta|>L_0}\frac{z_\gamma^2(\gamma)-z_\gamma^2(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)}d\eta \lesssim \frac{1}{D_0}\|\partial_\gamma^2 z\|_{L^\infty}.$$

Next, when $|\eta| \leq L_0$, we can write

$$\frac{z_\gamma^2(\gamma)-z_\gamma^2(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)} = \frac{z_\gamma^2(\gamma)-z_\gamma^2(\gamma-\eta)}{|z(\gamma)-z(\gamma-\eta)|}\cdot\frac{|z(\gamma)-z(\gamma-\eta)|}{D(|z(\gamma)-z(\gamma-\eta)|)}$$

and since the map $\eta \to |z(\gamma)-z(\gamma-\eta)|$ is one-to-one, (this gives the choice of L_0)

$$\int_{|\eta|\leq L_0}\frac{z_\gamma^2(\gamma)-z_\gamma^2(\gamma-\eta)}{D(|z(\gamma)-z(\gamma-\eta)|)}d\eta \lesssim \|z_\gamma^3\|_{L^\infty}\|F\|_{L^\infty}^2\int_0^\delta\frac{h}{D(|h|)}dh.$$

Therefore, we see an integral condition

$$\int_0^\delta \frac{h}{D(|h|)}dh < \infty \tag{69}$$

on D for the boundedness of this integral. Here δ is a small positive constant depending on L_0 and z. On the other hand, for the second term, following similar computations we arrive at the condition

$$\int_0^\delta \frac{-hD'(|h|)}{D^2(|h|)}dh < \infty. \tag{70}$$

Here for simplicity we assume that $D' < 0$. Let us now proceed to verify the conditions (69)–(70) for some specific choices for D.

- $D(|h|) = |h|^\alpha$, $\alpha < 2$. This is considered in [20]. Then, we have

$$\int_0^\delta \frac{h}{D} dh = \int_0^\delta \frac{1}{|h|^{\alpha-1}} dh < \infty, \qquad \int_0^\delta \frac{-hD'(|h|)}{D^2(|h|)} dh \sim \int_0^\delta \frac{1}{|h|^{\alpha-1}} dh < \infty.$$

- Logarithmically regularized models: $D(|h|) = |h|^2 \log^\beta(10 + |h|^{-1})$. Then

$$\int_0^\delta \frac{h}{D} dh = \int_0^\delta \frac{1}{|h| \log^\beta(10 + |h|)} dh < \infty$$

requires $\beta > 1$. Next,

$$D'(|h|) \sim |h| \log^\beta(10 + |h|^{-1}) = |h|^{-1} D(|h|),$$

which gives

$$\int_0^\delta \frac{-hD'(|h|)}{D(|h|)} dh \sim \int_0^\delta \frac{h}{D} dh < \infty$$

under the same assumption on β.

Under the above conditions on D, we can bound $|I(\gamma)|$ and in turn

$$\|\partial_\gamma \Lambda\|_{L^\infty} \lesssim \|z\|_{C^1} \|z\|_{C^3} (1 + \|F\|_{L^\infty}^4).$$

To control $\|z\|_{C^3}$, we need at least H^4 on z. Now, we see how the addition of a new term helps us estimate the H^4. In the expression

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_\gamma^4 z|^2 d\gamma = \int_{\mathbb{T}} \partial_\gamma^4 z(\gamma) \cdot \partial_\gamma^4 \int_{\mathbb{T}} \frac{\partial_\gamma z(\gamma) - \partial_\gamma z(\gamma - \eta)}{|D(z(\gamma) - z(\gamma - \eta))|} d\eta d\gamma + \cdots,$$

where $\cdots$ represent terms involving Λ. There are several potentially dangerous terms. In the top order term,

$$\begin{aligned}
&\int_{\mathbb{T}} \partial_\gamma^4 z(\gamma) \cdot \int_{\mathbb{T}} \frac{\partial_\gamma^5 z(\gamma) - \partial_\gamma^5 z(\gamma - \eta)}{|D(z(\gamma) - z(\gamma - \eta))|} d\eta d\gamma \\
&\quad = \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} (\partial_\gamma^4 z(\gamma) - \partial_\gamma^4 z(\gamma - \eta)) \cdot \frac{\partial_\gamma^5 z(\gamma) - \partial_\gamma^5 z(\gamma - \eta)}{|D(z(\gamma) - z(\gamma - \eta))|} d\eta d\gamma
\end{aligned}$$

using symmetry. Then, integrating by parts, we obtain

$$= \frac{1}{4} \iint |\partial_\gamma^4 z(\gamma) - \partial_\gamma^4 z(\gamma - \eta)|^2 \frac{D'}{D^2} \frac{(z(\gamma) - z(\gamma - \eta)) \cdot (z_\gamma(\gamma) - z_\gamma(\gamma - \eta))}{|z(\gamma) - z(\gamma - \eta)|} d\eta d\gamma$$

Here, the problem is that while we need to cancel the singularity in D'/D^2 when $|\eta| \ll 1$, we cannot use the regularity of the difference $\partial_\gamma^4 z(\gamma) - \partial_\gamma^4 z(\gamma - \eta)$ since we have only H^4 of z. Regarding this point, we note that while

$$(z(\gamma) - z(\gamma - \eta)) \cdot (z_\gamma(\gamma) - z_\gamma(\gamma - \eta)) = O(\eta^2),$$

we actually have

$$\lim_{\eta\to 0}\frac{(z(\gamma)-z(\gamma-\eta))\cdot(z_\gamma(\gamma)-z_\gamma(\gamma-\eta))}{|\eta|^2}=z_\gamma(\gamma)\cdot z_\gamma^2(\gamma)=0.$$

Therefore, this crucial cancellation comes in to give that actually

$$|(z(\gamma)-z(\gamma-\eta))\cdot(z_\gamma(\gamma)-z_\gamma(\gamma-\eta))|\lesssim\|z\|_{C^3}^2|\eta|^3.$$

This allows us to estimate

$$\begin{aligned}&\left|\iint|\partial_\gamma^4 z(\gamma)-\partial_\gamma^4 z(\gamma-\eta)|^2\frac{D'}{D^2}\frac{(z(\gamma)-z(\gamma-\eta))\cdot(z_\gamma(\gamma)-z_\gamma(\gamma-\eta))}{|z(\gamma)-z(\gamma-\eta)|}d\eta d\gamma\right|\\&\quad\lesssim\|z\|_{C^3}^2\|z\|_{H^4}^2\lesssim\|z\|_{H^4}^4\end{aligned}$$

with some powers of $\|F\|_{L^\infty}$ on the right hand side. In the other extreme term, we need to control

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^4 z(\gamma)\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta))\partial_\gamma^4\frac{1}{|D(z(\gamma)-z(\gamma-\eta))|}d\eta d\gamma.$$

Here, we shall require new conditions on higher derivatives on D, but the examples that we have discussed in the above satisfy these. To handle the singularity in $|\eta|\ll 1$, we still need a cancellation. This time, we need to use the algebraic identity

$$\partial_\gamma^4 z\cdot\partial_\gamma z=-3\partial_\gamma^2 z\cdot\partial_\gamma^3 z.$$

This follows from

$$0=\partial_\gamma(z_\gamma(\gamma)\cdot z_\gamma^2(\gamma))=|z_\gamma^2(\gamma)|^2+z_\gamma(\gamma)\cdot z_\gamma^3(\gamma)$$

and

$$0=\partial_\gamma(|z_\gamma^2(\gamma)|^2+z_\gamma(\gamma)\cdot z_\gamma^3(\gamma))=3z_\gamma^2\cdot z_\gamma^3+z\cdot z_\gamma^4.$$

The above identity allows us to write

$$\partial_\gamma^4 z(\gamma)\cdot(z(\gamma)-z(\gamma-\eta))=O(\eta)+C\partial_\gamma^2 z\cdot\partial_\gamma^3 z,$$

and the last term does not contain top order terms so that it can receive a derivative from the other top order term, which is $\partial_\gamma^4(z(\gamma)-z(\gamma-\eta))$. These are the main ideas in the proof of local regularity for patches in the intermediate regime.

2.4.2 On singularity formation for patches

A patch-type solution can become singular in two ways:

- breakdown of the boundary regularity,
- touching of different pieces.

The second type of singularity is commonly referred to as a *splash*. Of course, at the singular time, both can happen simultaneously. Gancedo–Strain [65] proved absence of splash singularities in the SQG case $\alpha = 1$ (unlike water waves), assuming uniform $C^{1,1}$ bound on the boundary up to the potential blow-up time. However, there are important assumptions in this work: (i) singularity is occurring at a *single point* and (ii) only two segments of the patch boundary become involved at the splash singularity ("the splash is simple"). Removing the second assumption in particular brings fundamental difficulties, as it leads to a "linear term" in the front dynamics. Here we present a recent result of Jeon–Zlatos [80]:

Theorem 2.4.2. *Assume $0 < \alpha \le 1/2$. Further assume that we are given a patch solution to the α-SQG with boundary regularity $C^{1,\alpha/(1-\alpha)}$ uniformly bounded on $[0,T)$ for some $T > 0$. Then, there is* no *splash singularities at $t = T$.*

Various numerical simulations suggest that blow-up of gSQG patches happens with two different pieces of boundary touching each other while forming a corner-like structure. This theorem confirms such a behavior, since it shows that at the time of splash, the boundary is required to form a cusp which is strictly worse than $C^{1,\alpha/(1-\alpha)}$. It is interesting to note that this becomes C^1 as $\alpha \to 0$. A closely related result can be found in Kiselev–Luo [92].

There are several high resolution numerical computations which suggest singularity formation for the gSQG patch dynamics [124]. They involve a self-similar type of small scale creation.

2.4.3 Global regularity in the graph case

In this section, we briefly describe a rather recent breakthrough of Cordoba–Gomez-Serrano–Ionescu where global regularity of gSQG patches were obtained in the *small graph* case. See more recent developments in [75]. For simplicity, consider the patch defined in the whole plane $\mathbb{R}^2$ and assume that the patch interface is a *graph*; namely, for each t, there is a function h of one variable $h(t,x)$ such that

$$\partial\Omega(t) = \{(x, h(t,x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Then, the CDE can be written in terms of the profile h; for some p depending on α in the α-SQG equation, we have

$$\partial_t h(t,x) = \int_{\mathbb{R}} \frac{h'(t,x) - h'(t,x-y)}{(|y|^2 + |h(t,x) - h(t,x-y)|^2)^{p/2}} dy. \tag{71}$$

If h is sufficiently regular, namely when $h_0 \in H^3(\mathbb{R})$, it can be shown that (by repeating the proof above) there is a unique local solution to (71) belonging to $L^\infty([0,T]; H^3)$.

Theorem 2.4.3 (Cordoba–Gomez-Serrano–Ionescu [42])**.** *In the case $p > 1$, there is a global solution for the case when the initial data is a graph with sufficiently small slope.*

In the following we give a sketch of the proof of *global wellposedness* for small data.

Observation. We look at local and far-field contributions to $\partial_t h$.

- Case $|y| \ll 1$: we have $|y|/|y|^p$, which is integrable for $p < 2$ (assuming smoothness of h).
- Case $|y| \gg 1$: we have $|h'(x)|/|y|^p$, which requires $p > 1$ for integrability (assuming decay of h).

A power series expansion: We may write the kernel as

$$\int_{\mathbb{R}} \frac{h'(x) - h'(x-y)}{|y|^p} \left(1 + \sum_{n \geq 1} c_n \cdot \left(\frac{h(x) - h(x-y)}{y} \right)^{2n} \right) dy$$

with some combinatorial coefficients c_n and we do not expect cancellation between terms with different powers.

The Linear Part: We define

$$\mathbf{L}h(x) = \int_{\mathbb{R}} \frac{h'(x) - h'(x-y)}{|y|^p} dy,$$

and we would like to see it as a multiplier. With the inverse Fourier transform, (up to a multiplicative constant which we ignore)

$$\mathbf{L}h(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1 - e^{iy\cdot\xi}}{|y|^p} dy \right) i\xi \hat{h}(\xi) e^{-ix\cdot\xi} d\xi$$

and a direct computation (scaling observation suffices) shows that the integral inside the round brackets evaluate to $C|\xi|^{p-1}$. Hence,

$$\widehat{\mathbf{L}h}(\xi) = C i\xi |\xi|^{p-1} \hat{h}(\xi).$$

See also recent [92] where the dispersion relation (as a pseudo-differential operator) is obtained in the general case.

The Cubic Part: We have

$$N_3h(x) = \int_{\mathbb{R}} \frac{h'(x) - h'(x-y)}{|y|^p} \cdot \frac{h(x) - h(x-y)}{y} \cdot \frac{h(x) - h(x-y)}{y} dy.$$

Taking the Fourier transform,

$$\begin{aligned}\widehat{N_3h}(\xi) &= \int_{\mathbb{R}^5} \hat{h}(\eta_1) i\eta_1 e^{ix\eta_1} \frac{1 - e^{-iy\eta_1}}{|y|^p} \\ &\quad \cdot \hat{h}(\eta_2) e^{ix\eta_2} \frac{1 - e^{-iy\eta_2}}{y} \cdot \hat{h}(\eta_3) e^{ix\eta_3} \frac{1 - e^{-iy\eta_3}}{y} e^{-ix\xi}\, dx d\eta_1 d\eta_2 d\eta_3 dy \\ &= \int i\eta_1 \hat{h}(\eta_1)\hat{h}(\eta_2)\hat{h}(\eta_3) \frac{1 - e^{-iy\eta_1}}{|y|^p} \frac{1 - e^{-iy\eta_2}}{y} \frac{1 - e^{-iy\eta_3}}{y} e^{-ix(\xi - \eta_1 - \eta_2 - \eta_3)} dx dy d\eta,\end{aligned}$$

and the dx-integral produces $\delta_0(\xi - \eta_1 - \eta_2 - \eta_3)$. Hitting this against $d\eta_1$,

$$\widehat{N_3h}(\xi) = \int_{\mathbb{R}^2} i(\xi - \eta_2 - \eta_3)\hat{h}(\xi - \eta_2 - \eta_3)\hat{h}(\eta_2)\hat{h}(\eta_3) m(\xi - \eta_2 - \eta_3, \eta_2, \eta_3) d\eta_2 d\eta_3.$$

with the multiplier

$$m(\xi_1, \xi_2, \xi_3) = \int_{\mathbb{R}} \frac{1 - e^{-iy\xi_1}}{|y|^p} \cdot \frac{1 - e^{-iy\xi_2}}{y} \cdot \frac{1 - e^{-iy\xi_3}}{y} dy.$$

Observation. In the case $m \equiv 1$, this operator is simply $\partial_x(h^3)$. In our case, $m \sim |\eta_1|^{p-1}$, even assuming $\eta_2, \eta_3 \ll |\eta_1|$.

A model problem: Consider the simplified system

$$\partial_t h + i\Lambda h = N_3 h,$$

where $\Lambda(\xi) = \xi|\xi|^{p-1}$ and $N_3 h \sim |\nabla|^p h \cdot (h')^2$. The whole argument is a bootstrap on two kind of bounds:

- Propagation of Sobolev Regularity: we have

  $$\frac{d}{dt}\|D^\alpha h\|_{L^2}^2 \lesssim \|D^\alpha h\|_{L^2}^2 \cdot \|h'\|_{L^\infty}^2$$

  (ignoring a loss of ∇ on the RHS), and hence we need to ensure

  $$\int_0^\infty \|h'\|_{L^\infty}^2 dt < +\infty.$$

- Propagation of *pointwise* decay: in view of the above, we need (at least)

  $$\|h'(t)\|_{L^\infty} \lesssim \frac{1}{\sqrt{t}},$$

  and regarding this point, the *hope* is that, at a "lower" level of the derivative, we have

  $$\|h\|_{L^\infty} \lesssim \|e^{-it\Lambda} h_0\|_{L^\infty}$$

  based on the analysis of the linear propagator.

2.4.4 Curvature dynamics

We consider the evolution equation for the curvature of a smooth patch, which has been derived recently in Kiselev–Luo [91]. We parameterize the patch boundary by γ, and denote $\mathbf{T}$ and $\mathbf{N} = -\mathbf{T}^{\perp}$ to be the unit tangent and normal vectors. We follow the convention that the boundary curve is oriented counterclockwise.

Our starting point is simply

$$\partial_t \gamma = u(t, \gamma) = u_\tau \mathbf{T} + u_n \mathbf{N}$$

where

$$u_\tau = u \cdot \mathbf{T}, \qquad u_n = u \cdot \mathbf{N}.$$

Next, if we consider the arc-length function $g = |\dot{\gamma}|$, then

$$\partial_t (g^2) = 2\dot{\gamma} \cdot \partial_t \dot{\gamma} = 2g\mathbf{T} \cdot \partial_\xi (u_\tau \mathbf{T} + u_n \mathbf{N}).$$

Here, $\dot{a} := \partial_\xi a$, where ξ is the parametrization of γ. We also define $\partial_s := g^{-1}\partial_\xi$ to be the derivative with respect to the arc-length parametrization.

The curvature is defined by

$$\kappa = -\partial_s \mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \partial_s \mathbf{N}.$$

Then, we can see that

$$\partial_t (g^2) = 2g(\dot{u}_\tau + u_n \kappa g).$$

Furthermore, using that

$$u \cdot \partial_\xi \mathbf{T} = u \cdot (-\kappa g \mathbf{N}) = -u_n \kappa g,$$

we can simplify

$$\partial_t g = g \partial_s u \cdot \mathbf{T}. \tag{72}$$

Next, one can obtain evolution equations for $\mathbf{T}, \mathbf{N}$, and κ. For instance,

$$\begin{aligned} \partial_t \mathbf{T} = \partial_t \partial_s \gamma &= -(\partial_s u \cdot \mathbf{T})\partial_s \gamma + \partial_s(\partial_t \gamma) \\ &= (\partial_s u \cdot \mathbf{N})\mathbf{N}. \end{aligned} \tag{73}$$

Then from $0 = \partial_t (\mathbf{N} \cdot \mathbf{T})$,

$$\partial_t \mathbf{N} = -(\partial_s u \cdot \mathbf{N})\mathbf{T}. \tag{74}$$

To obtain equation for the curvature, it is actually easier to introduce the angle $\theta = \theta(\xi)$ such that $\mathbf{T} = (\cos\theta, \sin\theta)$ and $\mathbf{N} = (\sin\theta, -\cos\theta)$. Then we have simply

$$\kappa = \partial_s \theta$$

and

$$\partial_t \kappa = -2\kappa \partial_s u \cdot \mathbf{T} - \partial_s^2 u \cdot \mathbf{N}. \tag{75}$$

The above equations are completely general evolution laws for curves transported by a smooth velocity field. We now specialize to the case of the Euler vortex patches and observe surprising cancellations. Similar computations can be done for the gSQG patches, although one needs to remove the tangential component.

Theorem 2.4.4 ([91]). *Assume that the Euler patch boundary is at least $W^{2,p}$ regular for some $p > 2$. Then, the curvature equation can be written as*

$$\partial_t \kappa = a(\xi)\kappa + \pi \mathbf{H}[\kappa](\xi) + F(\xi) \tag{76}$$

where $a = -\partial_s u \cdot \mathbf{T}(\xi)$, $\mathbf{H}$ is the Hilbert transform defined on $\mathbb{T}$, and F is "smooth" in a precise sense defined in Propositions 5.1 and 5.2 of [91].

The right hand side of (76) is surprisingly regular and the main terms are very simple. It can be applied, for instance, to prove *strong Illposedness* of the patch evolution in integer Hölder spaces C^k for $k \geq 2$. Namely, there exists a patch whose boundary is initially C^k but not so for $t \neq 0$ small.

2.4.5 Evolution of a corner

We consider the evolution of a corner-type patch by Euler flow ([16, 79]). To this end, we consider the domain Ω_0 satisfying the following:

- In the region $[-M, M]^2$, we have

  $$\Omega_0 \cap [-M, M]^2 = \{x : 0 < x_2 < x_1/\beta\} \cap [-M, M]^2,$$

  for some $M \geq 2$ and $\beta > 0$.

- Ω_0 is convex and $\partial\Omega_0$ is C^∞ smooth except at $(0, 0)$.

We compute the velocity u_0 associated with Ω_0 along the horizontal line

$$\{(x_1, 0), 0 < x_1 < 1\}.$$

We take $(a, 0)$ for some $0 < a < 1$ and compute

$$2\pi u_{0,2}(a, 0) = \int_{\Omega_0} \frac{a - y_1}{(a - y_1)^2 + y_2^2} dy_1 dy_2 = A(a) + B(a),$$

where

$$A(a) := \int_{\Omega_0 \cap [-M,M]^2} \frac{a - y_1}{(a - y_1)^2 + y_2^2} dy_1 dy_2$$

and

$$B(a) := \int_{\Omega_0 \setminus [-M,M]^2} \frac{a - y_1}{(a - y_1)^2 + y_2^2} dy_1 dy_2.$$

The far-field integral $B(a)$ and its derivatives can be estimated as follows:

$$\begin{aligned} |B(a)| &\leq \int_{\Omega_0 \setminus [-M,M]^2} \frac{1}{|a - y_1|} dy_1 dy_2 \\ &\leq \frac{1}{M-1} |\Omega_0| \leq \frac{2}{M} |\Omega_0| \end{aligned}$$

since $M \geq 2$. Similarly from

$$B'(a) = \int_{\Omega_0 \setminus [-M,M]^2} \frac{1}{(a - y_1)^2 + y_2^2} - \frac{2(a - y_1)^2}{((a - y_1)^2 + y_2^2)^2} dy_1 dy_2,$$

we derive

$$|B'(a)| \leq \int_{\Omega_0 \setminus [-M,M]^2} \frac{C}{(a - y_1)^2} dy_1 dy_2 \leq \frac{C}{M^2} |\Omega_0|$$

and

$$|B''(a)| \leq \frac{C}{M^3} |\Omega_0|.$$

We now rewrite $A(a)$ as

$$\begin{aligned} A(a) &= \int_0^{M/\beta} \int_{\beta y_2}^{M} \frac{a - y_1}{(a - y_1)^2 + y_2^2} dy_1 dy_2 \\ &= -\frac{1}{2} \int_0^{M/\beta} \ln((a - M)^2 + y_2^2) - \ln((a - \beta y_2)^2 + y_2^2) dy_2 \\ &=: A_1(a) + A_2(a). \end{aligned}$$

We differentiate once in a:

$$A_2'(a) = \int_0^{M/\beta} \frac{a - \beta y_2}{y_2^2 + (a - \beta y_2)^2} dy_2.$$

Integrating,

$$\begin{aligned} A_2'(a) &= -\frac{1}{2(1+\beta^2)} \left(2\arctan\left(\frac{a}{-a\beta + (1+\beta^2) y_2}\right) + \beta \ln(a^2 - 2a\beta y_2 + (1+\beta^2) y_2^2) \right) \Bigg|_0^{M/\beta} \\ &= -\frac{1}{2(1+\beta^2)} \left(2\arctan\left(\frac{a}{-a\beta + (1+\beta^2) M\beta}\right) + \beta \ln(a^2 - 2aM + (1+\beta^2) M^2/\beta^2) \right) \\ &\quad + \frac{1}{2(1+\beta^2)} \left(2\arctan\left(-\frac{1}{\beta}\right) + 2\beta \ln(a) \right). \end{aligned}$$

We can explicitly evaluate $A_1'(a)$ similarly. Then we see that, after differentiating in a once more,

$$A''(a) = \frac{\beta}{1+\beta^2}\frac{1}{a} + D(a),$$

where $D(a)$ is a function which is smooth and uniformly bounded for $0 \leq a \leq 1$. Combining this with the above estimate for $B''(a)$, we conclude that

$$\partial_a^2 u_{0,2}(a,0) > \frac{\beta}{1+\beta^2}\frac{1}{2a}$$

for all $0 < a$ sufficiently small depending on $M, |\Omega_0|$ and a few absolute constants. Now, from (75), we have

$$\frac{\partial}{\partial t}\left(e^{2\int_0^t \partial_s u\cdot \mathbf{T}}\kappa\right) = -e^{2\int_0^t \partial_s u \cdot \mathbf{T}}\partial_s^2 u \cdot \mathbf{N}.$$

At $t = 0$ and $x = (a, 0)$ for all a sufficiently small, we see that $\partial_s^2 u \cdot \mathbf{N} = -\partial_s^2 u_{0,2} = -\partial_a^2 u_{0,2} < 0$. In particular, it follows that $\kappa(t,(a,0)) < 0$ for all sufficiently small negative t, using Taylor expansion of κ in time. For the case of small positive t, we can compute the velocity along the other part of the boundary $\partial\Omega_0$. Combining this computation with the above curvature evolution (76) gives the following result.

Proposition 2.4.3. *There exists a vortex patch which is convex at $t = 0$ but not at $t \neq 0$ small.*

We can fix some small $a > 0$ and smooth out the corner in a way depending on a such that the resulting initial patch still loses convexity instantaneously at the point $(a, 0)$ for $t \neq 0$. Furthermore, it is possible to modify this regularized patch boundary $\partial\Omega_0$ in a way that its curvature is *strictly positive* except for the point $(a, 0)$, at which the curvature becomes positive for $t < 0$ and negative for $t > 0$. After a shift of time, this gives the following:

Proposition 2.4.4. *There exists a strictly convex smooth patch such that it loses convexity after some finite time.*

We can repeat similar computations in the gSQG case. In the regular α-SQG case, we have this time

$$u_{0,2}(a,0) = \int_{\Omega_0} \frac{a-y_1}{((a-y_1)^2+y_2^2)^{1+\alpha/2}} dy_1 dy_2 = A(a) + B(a),$$

where

$$A(a) := \int_{\Omega_0\cap[-M,M]^2} \frac{a-y_1}{((a-y_1)^2+y_2^2)^{1+\alpha/2}} dy_1 dy_2.$$

Then

$$
\begin{aligned}
A(a) &= \int_0^{M/\beta}\int_{\beta y_2}^{M} \frac{a-y_1}{((a-y_1)^2+y_2^2)^{1+\alpha/2}}dy_1dy_2 \\
&= -\frac{1}{\alpha}\int_0^{M/\beta} \frac{1}{((a-M)^2+y_2^2)^{\alpha/2}} - \frac{1}{((a-\beta y_2)^2+y_2^2)^{\alpha/2}}dy_2 \\
&=: A_1(a)+A_2(a).
\end{aligned}
$$

We differentiate once in a:

$$
A_2'(a) = -\int_0^{M/\beta} \frac{a-\beta y_2}{(y_2^2+(a-\beta y_2)^2)^{1+\alpha/2}}dy_2.
$$

We rewrite the above as

$$
-\frac{1}{(1+\beta^2)^{1+\alpha/2}}\int_0^{M/\beta} \frac{-(y_2-\beta a/(1+\beta^2))+C_1}{((y_2-\beta a/(1+\beta^2))^2+C_2)^{1+\alpha/2}}dy_2 =: D_1(a)+D_2(a)
$$

where

$$
D_2(a) = -\frac{1}{(1+\beta^2)^{1+\alpha/2}}\int_0^{M/\beta} \frac{C_1}{((y_2-\beta a/(1+\beta^2))^2+C_2)^{1+\alpha/2}}dy_2.
$$

Here

$$
C_1 = a - \frac{\beta}{1+\beta^2}a, \qquad C_2 = \frac{a^2}{1+\beta^2} - \frac{\beta^2}{(1+\beta^2)^2}a^2.
$$

The main term in $D_2(a)$ is given by

$$
-\frac{2}{(1+\beta^2)^{1+\alpha/2}}\int_0^{\beta a/(1+\beta^2)} \frac{C_1}{(y_2^2+C_2)^{1+\alpha/2}}dy_2.
$$

After a change of variables,

$$
\begin{aligned}
&= -\frac{2}{(1+\beta^2)^{1+\alpha/2}}\frac{C_1}{C_2^{(\alpha+1)/2}}\int_0^{\beta a/(\sqrt{C_2}(1+\beta^2))} \frac{1}{(1+z^2)^{1+\alpha/2}}dz \\
&= -\frac{2}{(1+\beta^2)^{1+\alpha/2}}\frac{C_1}{C_2^{(\alpha+1)/2}}F_\alpha(\beta a/(\sqrt{C_2}(1+\beta^2))).
\end{aligned}
$$

Here F_α is the anti-derivative of $z \mapsto \frac{1}{(1+z^2)^{1+\alpha/2}}$ satisfying $F_\alpha(0)=0$. Note that F_α is uniformly bounded in $\mathbb{R}$ and the argument $\beta a/(\sqrt{C_2}(1+\beta^2))$ is independent of a.

Differentiating in a, we obtain the main term:

$$
A''(a) = C_\beta \frac{1}{a^{1+\alpha}} + \textit{bounded}.
$$

This can then be used to establish an analogue of Proposition 2.4.4 in the gSQG case.

Remark 2.4.5. *We give some references regarding gSQG patch dynamics: ref*

2.4.6 Propagation of singular structures

Wellposedness of smooth vortex patches can be seen as a special case of "propagation of singular structures", simply because the patches are not smooth (not even continuous) across the boundary. Another well-known example of singular structure propagation is given by the analytic vortex sheets, which are locally wellposed in the analytic class. It is also known that analyticity breaks down in finite time. In the next section, we shall introduce the dynamics of point vortices, which is even more singular than vortex sheets in some sense. On the other hand, one may consider more mild singularities. An approach initiated by Elgindi is to prepare ansatz of the form

$$\omega(t,x) = \mathring{\omega}(|x - \Phi(t,x_0)|) + \widetilde{\omega}(t,x),$$

where $\mathring{\omega}(\cdot)$ is some fixed and explicit singular (unbounded at the origin) radial profile and $\widetilde{\omega}$ is considered as a perturbation. The case of point vortex can be considered as a special (limiting) case. The radial assumption on the singular profile is convenient since it gives rise to the cancellation in the most singular part of the nonlinearity. The singularity of the vorticity is allowed to move in time; its trajectory is denoted by $\Phi(t,x_0)$. In this setup, the goal is to write down the evolution equation for the remainder and prove that certain regularity of it can be propagated in time. One needs to have a good understanding of the linearized dynamics against the singular radial profile and then build appropriate function spaces to control the perturbation dynamics. Proving existence and uniqueness are non-trivial. At least when the singularity of $\mathring{\omega}$ is very mild so that certain extension of Yudovich's theorem applies, then uniqueness is guaranteed and it only remains to close certain a priori estimates. This was done in [45] for loglog vortices and for some related ones.

2.5 Global weak solutions

In this section, we sketch the proof of global-in-time existence of a bounded weak solutions to the gSQG equations, based on [20]. It is an interesting problem to construct global weak solutions in the singular regime. A very interesting difficult problem is to obtain more dynamical information for weak solutions. We shall fix the physical domain to be $\mathbb{T}^2$ for simplicity.

Theorem 2.5.1. *For $P(\Lambda) \lesssim 1$ and $\theta_0 \in L^\infty(\mathbb{T}^2)$, there exists a corresponding global in time weak solution to* (gSQG)*.*

Proof. We consider the Galerkin approximation of gSQG: define $\mathbb{P}_n$ be the Fourier projector onto Fourier modes with frequencies not exceeding n. Given θ_0, take

$$\begin{cases} \partial_t \theta_n + \mathbb{P}_n(u_n \cdot \nabla \theta_n) = 0, \\ \qquad\qquad u_n = \nabla^{\perp} P\theta_n, \\ \qquad\qquad \theta_{n,0} = \mathbb{P}_n \theta_0. \end{cases} \tag{77}$$

Thanks to the projection $\mathbb{P}_n$ in the evolution equation, there is a unique global in time solution θ_n to (77). We observe the following *uniform* bounds for the solution sequence:

$$\|\theta_n(t,\cdot)\|_{L^2} \leq \|\theta_0\|_{L^2},$$

$$\|P^{-\frac{1}{2}}\psi_n(t,\cdot)\|_{L^2} \leq \|P^{-\frac{1}{2}}\psi_0\|_{L^2}.$$

Here, $\psi_n = P\theta_n$ and $\psi_0 = \theta_0$.

Now, using that θ_n is a solution to (77), we have that

$$\int \partial_t \theta_n(t,x)\phi(x)dx = \int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi dx$$

for any ϕ smooth. The key trick is to use the anti-symmetric structure in the nonlinearity to rewrite

$$\int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi dx = \frac{1}{2} \int \psi_n [P^{-1}\nabla^{\perp}\cdot, \nabla \mathbb{P}_n \phi] \psi_n \, dx.$$

Then, we see that

$$\begin{aligned} \left| \int \theta_n u_n \cdot \nabla \mathbb{P}_n \phi dx \right| &\lesssim \|\psi_n\|_{L^2} \|P^{-1}\psi_n\|_{L^2} \|\phi\|_{H^4} \\ &\lesssim \|\theta_0\|_{L^2}^2 \|\phi\|_{H^4}. \end{aligned}$$

This shows that we can bound

$$\|\partial_t \theta_n\|_{H^{-4}} \lesssim \|\theta_0\|_{L^2}^2.$$

Therefore, Aubin–Lions lemma gives us a subsequence, still denoted by θ_n, which verifies

$$\theta_n \longrightarrow \theta$$

in $C([0,T];L^2)$ for any $T > 0$. In particular, $\theta(t=0) = \theta_0$. More importantly, for a given text function ϕ, the nonlinearity

$$\int \theta u \cdot \nabla \phi dx$$

is well-defined, with $u = \nabla^\perp P\theta$. We can rewrite

$$\int \theta u \cdot \nabla\phi dx = \frac{1}{2}\int \psi[P^{-1}\nabla^\perp\cdot, \nabla\phi]\psi\, dx.$$

To prove that θ is indeed a weak solution, we need to verify

$$\int_0^T \int \theta(\partial_t\phi + u\cdot\nabla\phi)dxdt = \int \theta_0\phi(0)dx$$

for any smooth test function ϕ depending on space-time. It is relatively easy to prove that

$$\int \theta_{n,0}\phi(0)dx \to \int \theta_0\phi(0)dx$$

and

$$\int_0^T \int \theta_n\partial_t\phi dxdt \to \int_0^T \int \theta\partial_t\phi dxdt.$$

Therefore, it only remains to show that

$$\int_0^T \int \theta_n(u_n\cdot\nabla\mathbb{P}_n\phi)dxdt \to \int_0^T \int \theta(u\cdot\nabla\phi)dxdt.$$

With the above rewriting, this is equivalent with showing

$$\iint \psi_n[P^{-1}\nabla^\perp\cdot, \nabla\mathbb{P}_n\phi]\psi_n\, dxdt - \iint \psi[P^{-1}\nabla^\perp\cdot, \nabla\phi]\psi\, dxdt \to 0.$$

This difference can be written as the sum of three terms

$$\begin{aligned}
&= \iint \psi_n[P^{-1}\nabla^\perp\cdot, \nabla(\mathbb{P}_n\phi - \phi)]\psi_n\, dxdt \\
&\quad + \iint (\psi_n - \psi)[P^{-1}\nabla^\perp\cdot, \nabla\phi]\psi_n\, dxdt \\
&\quad + \iint \psi[P^{-1}\nabla^\perp\cdot, \nabla\phi](\psi_n - \psi)\, dxdt.
\end{aligned}$$

It is not difficult to show that the first two integrals converge to 0 as $n\to\infty$. For the last term, we can further rewrite

$$\iint \psi[P^{-1}\nabla^\perp\cdot, \nabla\phi](\psi_n - \psi)\, dxdt = \iint P^{-1}\psi P[P^{-1}\nabla^\perp\cdot, \nabla\phi](\psi_n - \psi)\, dxdt$$

and then we estimate

$$\left|\iint \psi[P^{-1}\nabla^\perp\cdot, \nabla\phi](\psi_n - \psi)\, dxdt\right| \le \|P^{-1}\psi\|_{L^2}\|P[P^{-1}\nabla^\perp\cdot, \nabla\phi](\psi_n - \psi)\|_{L^2}.$$

It is not difficult to check that the last term goes to 0. This finishes the proof. $\square$

Problem 2.5.1. *Construct a weak solution to gSQG equations satisfying that* θ *has unbounded support for some* $t \neq 0$ *although* θ_0 *has compact support in* $\mathbb{R}^2$*.*

3 Dynamical behavior

Local wellposedness theory essentially says that for a short interval of time there is a unique solution which looks like the initial data in regular function spaces. In this section, we discuss some concrete examples of *long-time* dynamics of solutions to (gSQG). In other words, we are interested in the behavior of solutions *after* the time interval on which the solutions behave similarly with the initial data. Unfortunately, we are very far from having a general picture of long time dynamics for the solutions to (gSQG). Instead, we shall look at a few specific situations in which we can say at least something non-trivial about the long time dynamics.

3.1 Point vortex

In this section, we discuss the point vortex dynamics, the primary motivation being understanding of the dynamics of smooth solutions to gSQG equations. Intuitively, if θ is highly concentrated on a few localized regions in space, then it may be reasonable to replace it by a number of weighted Dirac deltas. This is the basis of several numerical schemes. At this point, it should be mentioned that it is a highly nontrivial task to justify point vortex solutions as actual solutions (in some sense) to the PDE.

Let us derive the governing equations. We hypothesize that

$$\theta(t,x) = \sum_{i=1}^{N} \Gamma_i \delta_{X_i(t)} \tag{78}$$

where Γ_i is constant in time, called the *intensity* of the ith vortex, and $X_i \neq X_j$ whenever $i \neq j$. That is, at each moment of time, $\theta(t,\cdot)$ is a measure defined on $\mathbb{R}^2$, acting on a continuous function $h \in C(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2} h(x) d\theta(x) = \sum_{i=1}^{N} \Gamma_i h(X_i(t)).$$

For now, we can be completely general about the Biot–Savart kernel and assume that the corresponding stream function is given by

$$\psi(t,x) = \sum_{i=1}^{N} \Gamma_i G(|x - X_i(t)|)$$

for some radial function G. (In the case of 2D Euler, $G(z) = \frac{1}{2\pi}\ln\frac{1}{|z|}$.) Then, we may define the corresponding velocity by

$$u(t,x) = \sum_{i=1}^{N} \Gamma_i \frac{(x - X_i(t))^{\perp}}{|x - X_i(t)|} \cdot \nabla G(|x - X_i(t)|).$$

Already in the case of 2D Euler, we immediately observe that

- $|u(x)| \sim \frac{1}{|x - X_i(t)|}$ as $x \to X_i(t)$, so that u does not even belong to L^2_{loc}.

To derive the ODE for the points $X_i(t)$, we have no choice but to neglect the self-interaction velocity. A heuristic justification comes from that each point vortex generates a purely radial flow, so that it should not move itself. (This is why justifying the point vortex as a solution to the PDE is difficult; see [111].) This gives

$$\frac{d}{dt} X_j(t) = \sum_{i \neq j} \Gamma_i \frac{(X_j(t) - X_i(t))^{\perp}}{|X_j(t) - X_i(t)|} \cdot \nabla G(|X_j(t) - X_i(t)|). \tag{79}$$

We see that for $j = 1, \cdots, N$, the ODE system (79) is closed. This is commonly referred to as the point vortex system. The initial value problem for the ODE is locally wellposed. Let us look into the system in more detail now.

Conserved quantities. It is easy to observe that the ODE system (79) enjoys the same conservation laws with smooth solutions of gSQG. To begin with, we consider the "energy", which is defined by

$$H[X_1, \cdots, X_N] := \sum_{i \neq j} \Gamma_i \Gamma_j G(|X_i - X_j|).$$

Indeed, we see that (79) can be written in the form

$$\begin{cases} \Gamma_j \dfrac{dX_{j,1}}{dt} = \dfrac{\partial H}{\partial X_{j,2}}, \\ \Gamma_j \dfrac{dX_{j,2}}{dt} = -\dfrac{\partial H}{\partial X_{j,1}}. \end{cases} \tag{80}$$

This shows not only that H is conserved in time, but actually the point vortex system is *Hamiltonian* (to be precise, one needs to consider the rescaled variable $\sqrt{\Gamma_j} X_j$). An immediate consequence is that in the phase space, the Lebesgue measure

$$\prod_{i=1}^{N} dX_i$$

is preserved by the dynamics.

Next, we can check the conservation laws which correspond to the center of mass and angular impulse, respectively. Note that

$$M[X_1, \cdots, X_N] := \sum_{i=1}^{N} \Gamma_i X_i \tag{81}$$

and

$$I[X_1, \cdots, X_N] := \sum_{i=1}^{N} \Gamma_i |X_i|^2 \tag{82}$$

are constant in time. Note that (82) defines a coercive quantity only when the signs of Γ_i are all the same. In particular, when I is coercive, it is prohibited that $|X_i| \to \infty$ for any i. This already suggests that the dynamics of point vortices could be much simpler when they are all of the same sign.

Case of one, two, or three vortices. Given the conservation laws, it can be seen that the motion is completely integrable if $N \leq 3$. When $N = 1$, the point vortex is steady, although in domains with boundary there is some motion already in this case. We now move on to the case $N = 2$. From the Hamiltonian conservation, which is simply

$$H \sim G(|X_1 - X_2|)$$

in the case of two vortices, it follows at once that $|X_1 - X_2|$ is a constant of motion. (It is assumed that $|\nabla G|$ is nowhere zero.) To proceed further, one needs to distinguish the case $\Gamma_1 + \Gamma_2 = 0$. When this condition is not satisfied, two vortices rotate around each other in circles, with a common angular speed. In the special case $\Gamma_1 + \Gamma_2 = 0$, the vortices move in parallel with each other. The dynamics in this case is identical to that of one point vortex defined in the upper half-plane $\mathbb{R}^2_+$. Next, in the case of three point vortices, one can write down a formula for the solution but it is not very simple (see the references in [111]). Already in the Euler case, there is a "finite time singularity" for the case $N = 3$. To be more precise, three point vortices can collide in a finite time. An example of such initial configuration is given by

$$\Gamma_1 = 2, \quad \Gamma_2 = 2, \quad \Gamma_3 = -1$$

with

$$X_1(0) = (-1, 0), \quad X_2(0) = (1, 0), \quad X_3(0) = (1, \sqrt{2}).$$

Then, it turns out that for all $i \neq j$, $D_{ij}(t) := |X_i(t) - X_j(t)|$ satisfies the ODE

$$\frac{d}{dt} D_{ij} = -\frac{1}{3\sqrt{2}\pi} D_{ij}(0),$$

which means that $D_{ij}(t) = 0$ exactly at $t = 3\sqrt{2}\pi$. However, it can be shown that the set of initial configurations leading to such a finite time collapse is of measure zero ([111]).

Quasi-periodic motion for 4 vortices. Assuming that G decays sufficiently fast at infinity, the interactions of points far away from each other will be weak, and one can imagine the situation that the set of point vortices is split into a few groups which are very far from each other group, but the vortices are very close within each group. If this can be justified, it leads to several very interesting dynamics. Already when there are four point vortices of equal strength (say $\Gamma = 1$ for all of them), one can group them into two groups of two, so that the distance between the groups is very large, while the distance between the elements of each group is very small. Then all the vortices will rotate around a very large circle slowly, with each group elements rotating in small circles at the same time. This (and much more) was established in an important work of Khanin [85], with a very nice application of the celebrated Kolmogorov–Arnol'd–Moser theory.

Thomson polygon. For any $N \geq 2$, one can consider the N-point configuration which rotates uniformly around a point. For simplicity, simply define

$$X_j(0) = \left(\sin(\frac{2\pi j}{N}), \cos(\frac{2\pi j}{N})\right)$$

for $0 \leq j < N$ with $\Gamma_j = \Gamma$ for some $\Gamma \neq 0$ for all j. That is, initially the vortices are located on the vertices of a regular N-gon centered at the origin. One can study stability of the Thompson polygon; it turns out that this configuration becomes unstable if N is large.

Von Karman street. The point vortex system can be used to give a mathematical description of the famous Von Karman street, which arise in flows past obstacles. The description requires two infinite arrays of positive and negative point vortices. To be more precise, let $0 < a < b$ and $0 < h$ be three parameters. Then, we define

$$X_{+,j}(0) = \left(jb, \frac{h}{2}\right), \quad j \in \mathbb{Z},$$

and

$$X_{-,j}(0) = \left(jb + a, -\frac{h}{2}\right), \quad j \in \mathbb{Z}$$

with $\Gamma_{+,j} = 1$ and $\Gamma_{-,j} = -1$ for all $j \in \mathbb{Z}$. Then, it can be shown that the entire vortex configuration moves by a rigid motion with velocity $(V, 0)$ for some $V > 0$ depending only on a, b, h. One can study the stability properties of this infinite array of vortices.

Problem 3.1.1. *Consider the point vortex system of one or two vortices in the case of* $\mathbb{R}^2_+$ *and* $(\mathbb{R}_+)^2$*.*

3.2 Steady, traveling, and rotating states

In this section, we study some examples and properties of steady solutions and their variants. We shall mostly focus on explicit examples for the 2D Euler case but it can be shown that there are analogous examples in the gSQG case.

Steady states as critical points of energy. We recall the principle of Arnol'd that steady states are critical points of the energy functional. It takes some work to be made precise. For simplicity we focus on the Euler case. Recall that in this case the energy is simply

$$E[\omega] = \frac{1}{2}\int \omega\psi dx, \quad \psi = (-\Delta)^{-1}\omega.$$

If ψ has some decay at infinity, using integration by parts this is equal to the usual kinetic energy of the fluid. However, we shall use this expression (sometimes referred to as the "pseudo-energy") since it is more general and well-defined just under the assumption $\omega \in L^1 \cap L^\infty$.

To really prove the principle of Arnol'd, we first fix some vorticity $\bar{\omega} \in L^1 \cap L^\infty$ and consider the associated class $\mathcal{A}$ which consists of functions $\omega \in L^1 \cap L^\infty$ satisfying $\int \omega = \int \bar{\omega}$. (To avoid some technical issues, one needs to work with a bounded domain in $\mathbb{R}^2$ or with $\mathbb{T}^2$, but the case of $\mathbb{R}^2$ can be treated with some additional preparations.) Recall that $\bar{\omega}$ defines a steady solution (in the weak sense) if

$$\int \bar{\omega}\nabla^\perp\bar{\psi} \cdot \nabla\phi = 0$$

holds for all test functions ϕ, where $-\Delta\bar{\psi} = \bar{\omega}$. This requirement can be alternatively viewed as follows: given a text function ϕ, define the vector field $v = \nabla^\perp\phi$, which is incompressible. Then, consider the advection equation

$$\partial_t f + v \cdot \nabla f = 0, \qquad f(t = 0) = \bar{\omega}.$$

There is a unique solution which verifies

$$\|f(t, \cdot)\|_{L^p} = \|\bar{\omega}\|_{L^p}, \qquad \int f(t, \cdot) = \int \bar{\omega}.$$

Note that

$$\frac{d}{dt}|_{t=0}E(t) := \frac{d}{dt}|_{t=0}E[f(t, \cdot)] = \int -v \cdot \nabla\bar{\omega}\bar{\psi} = \int \bar{\omega}\nabla^\perp\bar{\psi} \cdot \nabla\phi.$$

That is, if $\bar{\omega}$ is a steady weak solution, then $\frac{dE}{dt} = 0$ for *all variations* coming from smooth incompressible vector fields. The converse statement holds as well.

In this connection, *stable* steady states are naturally associated with non-degenerate extremal points of the energy functional. This is not difficult to explain, and some concrete examples will be given below. Let $\bar\omega = \mathrm{argmax}_{\omega\in\mathcal{A}} E[\omega]$. Then, we see from the above that $\bar\omega$ needs to be a steady state. (Otherwise, there is a variation which strictly increases the energy within $\mathcal{A}$.) We now need to make precise the notion of an extreme point and stability.

- Assume that $\bar\omega$ is the *strict local* maximizer of E in $\mathcal{A}$ with the L^1 topology. In formulas, we assume that there exists some $\delta_0 > 0$ and a continuous increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $\omega \in \mathcal{A}$ satisfies $\|\omega - \bar\omega\|_{L^1} < \delta_0$, then
$$\mu(\|\omega - \bar\omega\|_{L^1}) \le E[\bar\omega] - E[\omega].$$
- We say $\bar\omega$ is L^1-stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\omega_0 - \bar\omega\|_{L^1} < \delta$ then $\sup_{t\in\mathbb{R}} \|\bar\omega - \omega(t,\cdot)\|_{L^1} < \varepsilon$, where $\omega(t,\cdot)$ is the solution corresponding to ω_0. This is sometimes referred to as the Lyapunov stability, especially to distinguish it from stronger notions of stability (cf. asymptotic stability).

Then, one can proceed to the proof that a strict local L^1 maximizer of the energy is L^1 stable. To be precise it needs to be assumed that $\int \bar\omega \neq 0$, unless one restricts the notion of stability to include only $\omega_0 \in \mathcal{A}$. Under this additional assumption, given some initial data ω_0 close to $\bar\omega$ in L^1, one can first perturb ω_0 to fix the mass condition, by simply multiplying it with a constant which is close to 1. Let us still denote the resulting data by ω_0, and the corresponding solution by $\omega(t,\cdot)$. (Two solutions are simply related by a rescaling of the time variable.) Then, using that E is conserved in time and $\omega(t,\cdot) \in \mathcal{A}$, we have a chain of inequalities

$$\mu(\|\omega(t,\cdot) - \bar\omega\|_{L^1}) \le E[\bar\omega] - E[\omega(t,\cdot)] = E[\bar\omega] - E[\omega_0] < C(\bar\omega)\|\bar\omega - \omega_0\|_{L^1} < \delta.$$

Therefore, we conclude that

$$\|\omega(t,\cdot) - \bar\omega\|_{L^1} < \mu^{-1}(\delta) =: \varepsilon.$$

While this statement is nice and clean, the problematic part is that in reality, there are only a few steady states of 2D Euler which satisfy all the necessary assumptions. (One example is given by the Rankine vortex.) Furthermore, we are not using all the conserved quantities of 2D Euler. Indeed, the important idea (which again goes back to Arnol'd) is to *combine* various conserved quantities together and consider the maximization (or minimization) problem. Alternatively (although it is a delicate issue to really determine if these two approaches are equivalent), one can consider a narrower admissible class by imposing certain values of additional conserved quantities. It turns out that by one way or other, one can obtain existence and stability of rotating and traveling states, which we are now going to explain.

Rotating and traveling states. A solution to 2D Euler is (uniformly) rotating around the origin if there exists some Ω such that

$$\omega(t,x) = \omega_0(R_{\Omega t}x).$$

Here, $R_{\Omega t}$ is the counter-clockwise rotation matrix by angle Ωt defined on $\mathbb{R}^2$, which is simply $(r, \theta) \mapsto (r, \theta + \Omega t)$ in polar coordinates. Similarly, a solution is (uniformly) traveling if there exists a vector V such that

$$\omega(t, x) = \omega_0(x - Vt).$$

These solutions are sometimes called as *relative* equilibrium points. A steady solution is trivially both rotating and traveling, with $\Omega = 0$ and $V = 0$, respectively. Because translation and rotation are symmetries of the equation, it is natural to attempt to give a variational characterization of uniformly traveling and rotating states, similar to the one above for steady states. For the case of rotation, one may restrict the class of variations to the ones which do not change the angular momentum, namely

$$\frac{d}{dt} I[f(t, \cdot)] = 0, \qquad I[f] := \int |x|^2 f.$$

Similarly, in the case of traveling states, one should restrict the variations to those which fix the center of mass.

Some non-trivial rotating and traveling states. The above variational principles can be applied to construct some non-trivial (namely, not steady) rotating and traveling states. This procedure is conceptually very simple, but requires hard work to rigorously carry out. We provide an example.

Consider the following admissible class of vorticities on $\mathbb{R}^2$:

$$\mathcal{A} = \left\{ 0 \le \omega \le 1 : \omega \text{ is compactly supported}, \int x\omega = 0, \int \omega = \pi \right\}.$$

Then, consider the maximization

$$\max_{\omega \in \mathcal{A}} E[\omega].$$

It is known that the maximum is attained uniquely when $\omega = \mathbf{1}_D$, namely the characteristic function on the unit disc ([131, 126, 13]). It is not surprising that there is a maximum, since the energy becomes larger when the vorticity is concentrated. The requirement $\omega \le 1$ gives a restriction on the degree of concentration. The uniqueness statement follows from the moving plane method. While this is classical, we can tweak the admissible class, in a way that no discs can belong to it:

$$\widetilde{\mathcal{A}}_\epsilon = \left\{ 0 \le \omega \le 1 : \omega \text{ is compactly supported}, \int x\omega = 0, \int \omega = \pi, \int |x|^2\omega = \epsilon + \int |x|^2 \mathbf{1}_D \right\}.$$

Here $\epsilon > 0$ is a small constant. When ϵ is very small, it is conceivable that the maximum of the kinetic energy is attained for a shape which is slightly perturbed from the disc. This turns out to be *essentially* (see [126] for details) correct and gives rise to non-trivial uniform rotating solutions to the 2D Euler, which are nothing but rotating ellipses that are named after Kirchhoff. One can tweak the admissible class further by putting

$$\widetilde{\mathcal{A}}_\epsilon^m = \widetilde{\mathcal{A}}_\epsilon \cap \{\omega \text{ is } m - \text{fold symmetric}\}$$

for some $m \geq 2$. This gives the so-called m-waves of Kelvin [11, 119, 71, 29]. One may adapt this principle to the case of gSQG equations.

Special solutions to the point vortex system (79) that we have studied earlier also provide non-trivial solutions. For instance, it can be done for the case of Thompson polygons and Von Karman streets ([15, 67, 123]). In the case of Thompson polygon, one can fix some $N \geq 2$ and set up the following maximization problem: maximize the kinetic energy in the class

$$\mathcal{A}_\varepsilon^N = \left\{0 \leq \omega \leq \frac{1}{\varepsilon} : \omega \text{ is } N \text{ fold symmetric}, \int \omega = N\pi, \int |x|^2\omega = N\pi \right\}.$$

If one can establish the existence of a energy maximum, it can be shown that the vorticity defines a uniformly rotating solution, which behaves similarly with the Thomson polygon. This was done in [14], when $\varepsilon > 0$ is sufficiently small. See this reference for the history of this problem.

3.3 Gradient growth I: energy pump

Starting this section, we shall obtain a few results on the growth of solutions to the gSQG equations in various norms. Such results can be considered as quantifying "small scale creation," which is a characteristic feature of incompressible fluid models. For this purpose we need to really go beyond the a priori estimates and understand some details of the PDE. In this section, we describe the "energy pump" by Kiselev–Nazarov [93] for the gSQG equations. While it is in general an extremely difficult task to prove small scale creation for incompressible fluid models based on Fourier analysis, this work is a notable exception. The argument beautifully combines the conservation laws, symmetries, and most importantly the special nonlinear structure of the gSQG equations. Their main theorem states the following.

Theorem 3.3.1. *Given a gSQG equation in the regular and intermediate regime, take s sufficiently large depending on P. Then, for any $A > 0$, there exists a C^∞ smooth initial data θ_0 in $\mathbb{T}^2$ such that $\|\theta_0\|_{H^s} \leq 1$ but the corresponding solution to the gSQG equation satisfies*

$$\limsup_{t \to T^*} \|\theta(t, \cdot)\|_{H^s} \geq A.$$

Here, $0 < T^* \leq \infty$ is the lifespan of the smooth solution corresponding to θ_0.

To prove the above result, we proceed in a few steps.

Writing the equation in Fourier. We take the domain to be $\mathbb{T}^2$ and take the case $P(\lambda) = \lambda^{-\alpha}$. The equation in terms of the Fourier series takes the form

$$\frac{d}{dt}\hat{\theta}_k = \frac{1}{2}\sum_{\ell+m=k} m\ell^{\perp}(|\ell|^{-\alpha} - |m|^{-\alpha})\hat{\theta}_\ell\hat{\theta}_m. \tag{83}$$

Choice of the initial data. We take $\theta_0 := p$ which is characterized by its Fourier coefficients:

$$\hat{p}_e = \hat{p}_{-e} = 1, \quad \hat{p}_g = \hat{p}_{g+e} = \hat{p}_{-g} = \hat{p}_{-g-e} = \tau,$$

where $e = (1,0)$ and $g = (0,2)$. Later τ will be taken to be a small positive constant. One can pick $g = (0,n)$ for any integer $n \geq 2$. We denote the unique local in time smooth solution to be $\theta(t,\cdot)$.

Conservation laws. We have

$$\sum_k |\hat{\theta}_k|^2(t) = 2 + 4\tau^2$$

and

$$\sum_k |k|^{-\alpha}|\hat{\theta}_k|^2(t) = 2 + 2\tau^2(2^{-\alpha} + 5^{-\frac{\alpha}{2}}).$$

Stability. Subtracting the above two conserved quantity gives

$$(1-2^{-\alpha})\sum_{|k|>1}|k|^{\alpha}|\hat{\theta}_k|^2(t) \leq \sum_k (1-|k|^{-\alpha})|\hat{\theta}_k|^2(t) = C_\alpha\tau^2,$$

for some constant $C_\alpha > 0$. That is,

$$\sum_{|k|>1}|k|^{\alpha}|\hat{\theta}_k|^2(t) \lesssim_\alpha \tau^2.$$

This in turn gives a *global lower bound* on the first mode: for $\tau > 0$ sufficiently small (depending only on $\alpha > 0$), using that

$$2 + 4\tau^2 = \sum_k |\hat{\theta}_k|^2(t) = 2|\hat{\theta}_e|^2(t) + \sum_{|k|>1}|k|^{\alpha}|\hat{\theta}_k|^2(t),$$

we obtain in particular that

$$\hat{\theta}_e(t) > \frac{1}{2}.$$

We have used the continuity in time of the Fourier coefficients, which holds as long as the solution is well-defined. Moreover, we used that for any t, the nonzero Fourier coefficients with $|k| = 1$ are only $k = \pm e$, which follows from the *symmetry* of the initial data.

The pump. The following quadratic form is indeed a *pump*, as it will become clear shortly. We define

$$Q[\hat{\theta}] = \sum_{k\in\mathbb{Z}^2_+} \Phi(k)\hat{\theta}_k\hat{\theta}_{k+e} := \sum_{k\in\mathbb{Z}^2_+} (k_1 + \frac{1}{2})\hat{\theta}_k\hat{\theta}_{k+e}.$$

Note carefully that in the above definition, the summation is only over the Frequencies with $k \in \mathbb{Z}^2_+ := \mathbb{Z} \times \mathbb{Z}_+$. We write

$$\frac{d}{dt}Q[\hat{\theta}] = I + II,$$

where

$$I := \frac{1}{2} \sum_{k\in\mathbb{Z}^2_+} \Phi(k) \Bigg[\hat{\theta}_k \sum_{\ell+m=k+e,\ell,m\neq\pm e} \ell m^\perp(|\ell|^{-\alpha} - |m|^{-\alpha})\hat{\theta}_\ell\hat{\theta}_m$$
$$+ \hat{\theta}_{k+e} \sum_{\ell+m=k,\ell,m\neq\pm e} \ell m^\perp(|\ell|^{-\alpha} - |m|^{-\alpha})\hat{\theta}_\ell\hat{\theta}_m\Bigg]$$

and

$$II := \hat{\theta}_e \sum_{k\in\mathbb{Z}^2_+} ke^\perp\Phi(k) \Big[(1 - |k|^{-\alpha})\hat{\theta}^2_k - (1 - |k + 2e|^{-\alpha})\hat{\theta}_k\hat{\theta}_{k+2e}$$
$$- (1 - |k + e|^{-\alpha})\hat{\theta}^2_{k+e} + (1 - |k - e|^{-\alpha})\hat{\theta}^2_{k-e}\Big].$$

Key estimates. We first bound the term I from above. Using a very naive estimate

$$\Big|\ell m^\perp(|\ell|^{-\alpha} - |m|^{-\alpha})\Big| \lesssim_\alpha |k|^2$$

for $\ell + m = k$, we obtain

$$|I| \le C_\alpha \sum_k |\Phi(k)||k|^2|\hat{\theta}_k| \sum_{\ell\neq\pm e} |\hat{\theta}_\ell|^2 \le C_\alpha\tau^2 \sum_k |k|^3|\hat{\theta}_k|.$$

On the other hand, we rewrite II to extract a lower bound:

$$II = \hat{\theta}_e \sum_{k_2>0} k_2 \sum_{k_1\in\mathbb{Z}} \frac{1}{2} \Big[(1 - |k - e|^{-\alpha})\hat{\theta}^2_{k-e} + (1 - |k + e|^{-\alpha})\hat{\theta}^2_{k+e}$$
$$+ 2 \left[(k_1 + \frac{1}{2})(1 - |k - e|^{-\alpha}) - (k_1 - \frac{1}{2})(1 - |k + e|^{-\alpha})\right] \hat{\theta}_{k-e}\hat{\theta}_{k+e}\Bigg].$$

The right hand side is an infinite sum of the expressions of the form

$$A\hat{\theta}_{k-e}^2 + B\hat{\theta}_{k+e}^2 + C\hat{\theta}_{k-e}\hat{\theta}_{k+e}.$$

Inspecting the coefficients, we arrive at the lower bound

$$II \geq c_\alpha \sum_{k\in\mathbb{Z}_+^2} k_2 \frac{\hat{\theta}_k^2}{|k|^{2+\alpha}} \geq c_\alpha \sum_{k\in\mathbb{Z}_+^2} \frac{\hat{\theta}_k^2}{|k|^{2+\alpha}}.$$

The crucial point is that II assumes a strictly positive lower bound. Naively, we have

$$\frac{d}{dt}Q \gtrsim_\alpha O(\tau^2) + \sum_{k\in\mathbb{Z}_+^2} \frac{\hat{\theta}_k^2}{|k|^{2+\alpha}}.$$

Conclusion. To begin with, we may assume that the solution is global in time, since otherwise there is nothing to prove. Now, there are three scenarios, at least one of which must occur. In each case, one can simply take $\tau > 0$ sufficiently small in a way depending on A to finish the proof. Before we inspect the scenarios, note that

$$Q \leq \sum_{k\in\mathbb{Z}_+^2} |k|\hat{\theta}_k^2.$$

- Case 1: At some T, we have $\sum_k |k|^3|\hat{\theta}_k|(T) \geq \tau^{1/2}$. By interpolation, this implies a large growth of the H^s norm at time T.
- Case 2: We may assume that Case 1 never occurs. However, instead suppose that at some T, we have for the first time

  $$\sum_{k\in\mathbb{Z}_+^2} \frac{\hat{\theta}_k^2(T)}{|k|^{2+\alpha}} = 2\tau^{5/2}.$$

  Then, for $0 < t < T$, we have that

  $$\frac{d}{dt}Q > 0$$

  and in particular $Q(T) > Q_0 = O(\tau^2)$. The important point is that we can now interpolate

  $$Q \leq \sum_{k\in\mathbb{Z}_+^2} |k|\hat{\theta}_k^2 \leq \left(\sum_{k\in\mathbb{Z}_+^2} \frac{\hat{\theta}_k^2}{|k|^{2+\alpha}}\right)^{5/6} \left(\sum_{k\in\mathbb{Z}_+^2} |k|^{16+5\alpha}\hat{\theta}_k^2\right)^{1/6}.$$

This at $t = T$ enforces a large growth of

$$\left(\sum_{k \in \mathbb{Z}^2_+} |k|^{16+5\alpha} \hat{\theta}_k^2 \right)^{1/6} .$$

- Case 3: We may now assume that neither Case 1 nor 2 occur. In this case, we see that II grows without any bound in time.

3.4 Gradient growth II: Lagrangian approach

We have seen a result showing large growth of Sobolev norms for gSQG solutions, which was based on Fourier analysis. Here, we provide a gentle introduction to the features of the Lagrangian approach, towards the goal of proving growth of norms.

The Lagrangian over the Eulerian approach. There are two complementary approaches to the study of hydrodynamic PDEs, namely Lagrangian (based on the flow map and particle trajectory) and Eulerian (Fourier and harmonic analysis). In the Eulerian approach, one takes the Fourier transform of the system and studies the evolution of Fourier modes. Although such Fourier-based methods are more traditional, it is becoming clear that to extract detailed information about the Euler solution, one has to work with the flow maps and the Lagrangian trajectories. In the following, we shall demonstrate how the Lagrangian approach works to give detailed information about the solution.

Bootstrap arguments through the Lagrangian approach. Let us briefly explain how to perform bootstrap arguments based on particle trajectories to obtain detailed information about the flow. Assume that on some interval of time $[0, T]$, a Lipschitz velocity field $u(t, x)$ is given. Then, recall that the flow map $\Phi(t, x) : [0, T] \times \Omega \to \Omega$ is defined by solving the ODE

$$\frac{d}{dt}\Phi(t, x) = u(t, \Phi(t, x)), \qquad \Phi(0, x) = x \tag{84}$$

for each fixed $x \in \Omega$. Next, for each fixed t, the map $\Phi(t, \cdot)$ is invertible, and let us denote the inverse by Φ_t^{-1}. In the context of gSQG equations, the solution is simply expressed by

$$\theta(t, x) = \theta_0(\Phi_t^{-1}(x)). \tag{85}$$

Furthermore, using the evolution equation for $\nabla^\perp \theta$ one can see that

$$\nabla^\perp \theta(t, x) = \nabla \Phi(t, \Phi_t^{-1}(x)) \nabla^\perp \theta_0(\Phi_t^{-1}(x)). \tag{86}$$

Note that knowing the flow map (or its inverse) precisely recovers the solution in terms of simple formula (85). On the other hand, recall that the velocity $u(t,\cdot)$ at any instant of time is determined by $\theta(t,\cdot)$ by the fixed-in-time operator $\nabla^{\perp}P$. That is, we have a circle of relations $u \mapsto \Phi \mapsto \theta \mapsto u$, which is a "decomposition" of the nonlinear PDE (gSQG) into three simpler problems, only one of them involving differentiation in time.

In this context, bootstrap-type arguments have the following general form: first, using some "simple" a priori information on the solution (symmetries, positivity, topology of level sets, etc) which is guaranteed for all time, one obtains crude bounds on the flow maps, which in turn gives more information on the solution θ via (85), (86). Then, using the relation $\theta \mapsto u$ and (84), one may obtain a "more refined" information on the flow, with a careful analysis of the ODE system (84). In principle, the information so obtained can be feed back into (85), (86) to extract even more control on the solution.

Recently such arguments were used in a few different contexts: (i) to show existence of Euler solutions with norm growth ([95, 46, 47, 78, 133, 139, 28, 30, 26, 27, 96]), (ii) to obtain blow-up/regularity for $1D$ model equations for Euler ([32, 50, 25, 72]), and (iii) to study vortex patch dynamics ([48, 59, 57, 74, 94, 97]). See [99, 89] for a survey of recent results.

This type of argument is most effective when used in conjunction with a specific *scenario*, which we now discuss, as it not only enables the very first step of the bootstrap argument, but also provides sharp and conditional estimates throughout the whole argument.

Stability of the instability. We show how to use the bootstrap argument to turn stability into instability. For simplicity, we shall assume that the domain is $\mathbb{T}^2$ and consider the case of 2D Euler. Furthermore, assume that we are given a vorticity $\bar{\omega}$ which is an L^1-stable steady state with velocity $\bar{u}$. By L^1-stability of a steady state, we simply mean the following: there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists $\delta > 0$ such that once

$$\|\bar{\omega} - \omega_0\|_{L^1} < \delta,$$

then if we denote the solution to 2D Euler corresponding to ω_0 by $\omega(t)$, we have

$$\|\bar{\omega} - \omega(t)\|_{L^1} < \varepsilon$$

for all $t \geq 0$. Several non-trivial examples of L^1 stable vorticities are known. Then, writing

$$\omega_0 = \bar{\omega} + \widetilde{\omega}_0$$

for $\|\widetilde{\omega}_0\|_{L^1} \ll 1$, the perturbation equation is given by

$$\partial_t \widetilde{\omega} + (\widetilde{u} + \bar{u}) \cdot \nabla \widetilde{\omega} = -\widetilde{u} \cdot \nabla \bar{\omega}.$$

Note that the right hand side is a lower order term, assuming that $\bar{\omega}$ is sufficiently regular. Then we see that mainly, the perturbation is simply being advected by the total velocity $u = \widetilde{u} + \bar{u}$. This equation is to be compared with the linear evolution, which is

$$\partial_t \widetilde{\omega}^{lin} + \bar{u} \cdot \nabla \widetilde{\omega}^{lin} = -\widetilde{u}^{lin} \cdot \nabla \bar{\omega}.$$

If $\bar{\omega}$ is explicitly given, then one can more or less solve the above linear transport equation. The key point is that generically, $\bar{u}$ will have hyperbolic points, i.e. stagnation points at which the matrix $\nabla \bar{u}$ is hyperbolic. Therefore, while this linear equation is stable in the L^p sense, the gradient of the perturbation will grow exponentially in general.

Note that the only difference in the evolution equations for the nonlinear and linear is the term $\widetilde{u} \cdot \nabla \widetilde{\omega}$, which is supposed to be small, being quadratic in the perturbation. However, this term involves the gradient of the perturbation, which we cannot control by L^p stability for any p. This issue shows clearly why we need to take the Lagrangian approach: we can compare the linear flow map Φ^{lin} and the actual (nonlinear) flow map Φ, since by definition, they are generated by u^{lin} and u which are very close with each other. To see this, L^1 stability with L^∞ conservation implies that

$$\|\widetilde{\omega}\|_{L^2} \le \|\widetilde{\omega}\|_{L^1}^{\frac{1}{2}} \|\widetilde{\omega}\|_{L^\infty}^{\frac{1}{2}} \ll 1$$

for all $t \ge 0$. This gives, in turn

$$\|\widetilde{u}\|_{L^\infty} \le C \|\widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \ll 1.$$

This then can be used to show that for each x,

$$|\Phi(t,x) - \Phi^{lin}(t,x)| \ll 1$$

for an interval of time which scales like $1/\|\widetilde{u}\|_{L^\infty} \gg 1$. On the same time interval, it can be shown that

$$|\Phi_t^{-1}(x) - (\Phi_t^{lin})^{-1}(x)| \ll 1.$$

This can be used to translate large gradient growth in the linear equation to the nonlinear one. However, it should be mentioned that this arguments works on a time interval which is determined by the smallness of the initial perturbation, and to go beyond this time, a special argument (clever bootstrapping) is required in general.

Example 3.4.1 (Shear flow in a channel)**.** *We consider one of the simplest examples of gradient growth, with the domain* $\Omega = \mathbb{T} \times [-1, 1]$ *and stable steady state* $\bar{u}(x_1, x_2) = (x_2, 0)$. *An essentially equivalent example is given on the annulus [115].*

Problem 3.4.2. *Prove gradient growth for the vorticity which is given by a perturbation of the above steady state. First, consider the case when the vorticity perturbation is supported on the boundary of Ω. The growth occurs even when the perturbation does not touch the boundary [127, 51].*

Hyperbolic flow scenario. By a *scenario*, one is usually referring to a simple qualitative character of the fluid configurations, including but not limited to, symmetries, sign of the vorticity, flows in certain domains, and hyperbolic/rotational flow-lines. As an example, we may consider an initial vorticity invariant under a rotation $\mathcal{O}$ of $\mathbb{R}^2$; i.e., $\omega_0(\mathcal{O}x) = \omega_0(x)$. Then the solution automatically satisfies $\omega(t, \mathcal{O}x) = \omega(t, x)$. The *hyperbolic flow scenario* is obtained by arranging the vorticity to have symmetries in a way to ensure that the associated velocity is *hyperbolic* at a fixed point, say the origin, with preferably fixed separatrices for all time. This idea was very effective in proving infinite-time growth of the vorticity gradient and other geometric quantities of the gSQG equations ([95, 133, 139, 90, 94, 55, 46, 47]). Such a hyperbolic scenario can be used together with the maximum principle: if one has a sign (either non-negative or non-positive) on the initial vorticity, the solution to the Euler equations keeps the same sign for all times. In the context of the gSQG equations, one can just take θ which is non-negative on the first quadrant, and extend it as a function on the entire plane as an odd function in both coordinates to guarantee hyperbolic behavior at the origin.

Problem 3.4.3. *Consider the gSQG equation in $\mathbb{T}^2$ where the Biot–Savart law is more regular than the 2D Euler case. Prove infinite time exponential growth of the gradient, and show that it is the sharp rate.*

Shape of the boundary. Recent investigations have shown that the shape of the physical domain may have a significant impact on the dynamics; to just provide a few examples, (i) fluid in domains with boundary are more prone to generate small scales ([137, 95, 94]), (ii) cusp-type singularity of the boundary can cause finite time blow-up even in $2D$ ([98]), and on the other hand (iii) in certain domains the flow is "more stable" in the long time limit ([78, 77, 58]).

3.5 Gradient growth III: Key Lemma

3.5.1 Key Lemma for Euler

To motivate the "Key Lemma" of Kiselev–Sverak, we make a few observations on the velocity $u = \nabla^{\perp} P(\Lambda)\theta$ simply using the Taylor expansion. Clearly, one can write

$$u(x) = u(x_0) + \nabla u(x_0)(x - x_0) + O(x - x_0)^2,$$

when $u \in C^2$. For simplicity, we assume that $x_0 = 0$. Recalling that u is incompressible, we have that $\partial_1 u_1 + \partial_2 u_2 = 0$. More importantly, the derivatives at the origin can be expressed using the kernel; in the α-SQG case, we simply have

$$u(0) = -\int_{\mathbb{R}^2} \frac{y^\perp}{|y|^{2+\alpha}} \theta(y) dy.$$

Next, from (38), (39), we have

$$\partial_{x_1} u_1(0) = (2+\alpha) \int \frac{y_1 y_2}{|y|^{4+\alpha}} (\theta(y) - \theta(0)) dy$$

and

$$\partial_{x_1} u_2(0) = \int \frac{(2+\alpha)(y_1)^2 - |y|^2}{|y|^{4+\alpha}} (\theta(y) - \theta(0)) dy.$$

We see what is the form of the velocity gradient under the *odd-odd scenario*, namely when θ is odd with respect to both axes. We shall see that this induces a hyperbolic flow at the origin with both axes being the separatrices. Assuming that $\theta \in C^1$, the symmetry forces $\theta(0) = 0$ and $u(0) = 0$. The latter follows from the fact that

$$\int_{\mathbb{R}^2} y_i |y|^{-a} \theta(y) dy = 0$$

for any a and $i = 1, 2$ as long as the kernel is integrable. Similarly, we have that

$$\int_{\mathbb{R}^2} (y_i)^2 |y|^{-a} \theta(y) dy = 0$$

This gives $\partial_{x_1} u_2(0) = 0 = \partial_{x_2} u_1(0)$. Therefore, we explicitly have

$$u(x) = \mathbf{I} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} + O(|x|^2)$$

where

$$\mathbf{I} := (2+\alpha) \int_{\mathbb{R}^2} \frac{y_1 y_2}{|y|^{4+\alpha}} \theta(y) dy = 4(2+\alpha) \int_{y \geq 0} \frac{y_1 y_2}{|y|^{4+\alpha}} \theta(y) dy.$$

While this formula holds for points close to the origin, for practical purposes we need a result which holds for all points, with a quantitative estimate on the remainder term. To see whether such a result is achievable, we can first work on the case of the Hilbert transform which is arguably the simplest nontrivial singular integral transform.

Case of the Hilbert transform. On the real line, the Hilbert transform is defined by

$$\mathbf{H} f(x) = P.V. \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} f(y) dy.$$

If f is *odd*, then we explicitly have that

$$\mathbf{H}f(0) = -P.V.\frac{2}{\pi}\int_{y\geq 0}\frac{f(y)}{y}dy.$$

This kernel is integrable, since if f is odd and C^1 then it is forced that $f(0) = 0$. Now, for $x > 0$, we can write (neglecting P.V. for simplicity)

$$\mathbf{H}f(x) = \frac{1}{\pi}\int_{y\geq 0}\left[\frac{1}{x-y} - \frac{1}{x+y}\right]f(y)dy.$$

Then, we divide the integral into the regions $y \geq 2x$ and $0 < y < 2x$. In the latter, we consider the inverse of the Zygmund transform

$$\Lambda^{-1}f(x) := \int_0^x \mathbf{H}f(y)dy.$$

Note that if we integrate the kernel in x,

$$\int_{0<y<2x} \ln\frac{x-y}{x+y}f(y)dy$$

and note that by a rescaling change of variables $y = xz$ (for x fixed) we have that

$$\int_{0<y<2x} \ln\frac{x-y}{x+y}f(y)dy = x\int_{0<z<2} \ln\frac{1-z}{1+z}f(xz)dz$$

and taking the absolute value gives that

$$\left| x\int_{0<z<2} \ln\frac{1-z}{1+z}f(xz)dz\right| \leq C\|f\|_{L^\infty}x.$$

An important point is that the function

$$\frac{\Lambda^{-1}f}{x}$$

scales like the Hilbert transform of f, so that in L^∞ it should not be bounded by $\|f\|_{L^\infty}$. This shows that the "inner part" actually satisfies this property, under the odd symmetry. That is, there is a cancellation of singularities due to symmetry. Now, in the region $y \geq 2x$, we would like to replace the kernel by $-2/y$. Let us look at the difference:

$$\int_{y\geq 2x}\left[\frac{1}{x-y} - \frac{1}{x+y} + \frac{2}{y}\right]f(y)dy = \int_{y\geq 2x} x\left[\frac{1}{(x-y)y} - \frac{1}{(x+y)y}\right]f(y)dy.$$

Therefore, we see that this difference satisfies, after integrating in x,

$$\left|\int_x\int_{y\geq 2x} x\left[\frac{1}{(x-y)y} - \frac{1}{(x+y)y}\right]f(y)dy\right| \leq C\|f\|_{L^\infty}x.$$

We have arrived at a version of the Key Lemma for the Hilbert transform.

Lemma 3.5.1. *Let f be a bounded odd function satisfying $f(0) = 0$. Then, the inverse Zygmund transform applied to f satisfies*

$$\left| \frac{\Lambda^{-1} f(x)}{x} + \frac{2}{\pi} \int_{y \geq 0} \frac{f(y)}{y} dy \right| \leq C \|f\|_{L^\infty}.$$

One can consider the following one-dimensional toy model for 2D Euler: in $\mathbb{R}_+$, consider the equation

$$\partial_t \theta + u \cdot \nabla \theta = 0,$$

where u on $\mathbb{R}_+$ is defined by

$$u(t,x) = x \int_{y \geq 2x} y^{-1} \theta(t,y) dy.$$

One can modify the weight y^{-1} to more singular powers to get a model for gSQG. A slightly more "realistic" model would be to take $\partial_x u = H[\theta]$ ([17]).

Key Lemma for Euler. We are in a position to describe the Key Lemma. To begin with, we consider the case of Euler equations. The following general version is from [59].

Lemma 3.5.2. *Assume that $\omega \in L^\infty(\mathbb{R}^2)$ is compactly supported. Then, the corresponding velocity $u = \nabla^\perp \Delta^{-1} \omega$ satisfies*

$$\left| u(r,\theta) - u(0) - \frac{1}{2\pi} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} r I^s(r) + \frac{1}{2\pi} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} r I^c(r) \right| \leq Cr \|\omega\|_{L^\infty}, \tag{87}$$

where $C > 0$ is an absolute constant. (In particular, it is independent on the size of the support of ω.) Here,

$$u(0) = \left(-\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \sin(\theta)\omega(r,\theta) d\theta dr, \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \cos(\theta)\omega(r,\theta) d\theta dr \right)^T,$$

$$I^s(r) := \int_r^\infty \int_0^{2\pi} \sin(2\theta) \frac{\omega(s,\theta)}{s} d\theta ds,$$

and

$$I^c(r) := \int_r^\infty \int_0^{2\pi} \cos(2\theta) \frac{\omega(s,\theta)}{s} d\theta ds.$$

Proof. One way of proving the lemma is to consider a decomposition of the vorticity in polar coordinates. Namely, one may write

$$\omega = \sum_m f_m(r) \sin(m\theta) + \sum_m g_m(r) \cos(m\theta)$$

and estimate the velocity coming from each term. □

Remark 3.5.3. *This corresponds simply to the Taylor expansion of the velocity at the origin. The non-local Taylor coefficients are precisely $I^s(0)$ and $I^c(0)$, which can be computed by the explicit integrals given in the above.*

Remark 3.5.4. *Similar results can be obtained for the Riesz transforms in higher dimensions.*

Problem 3.5.5. *Consider the logarithmic singular vorticity*

$$\omega = \log \frac{1}{|x|} g(\theta) \mathbf{1}_{|x| \leq 1}$$

where g is a bounded function of the angle. Calculate precisely the unbounded part of ∇u.

We can now specialize the above lemma to the *hyperbolic scenario*: we shall mean that ω (θ in the case of gSQG) satisfies, for each time,

- odd with respect to both axes ("odd-odd"),
- non-negative in the first quadrant.

Then, we immediately see that $u(0) = 0$ and

$$I^c(r) \equiv 0$$

for all r. Then, (87) simplifies into

$$\left| u(r,\theta) - \frac{1}{2\pi} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} r I^s(r) \right| \leq Cr \|\omega\|_{L^\infty}. \tag{88}$$

For further applications it will be actually more useful to use Cartesian coordinates. In the case of the x_1-component, we have that

$$u_1(x) = \frac{1}{2\pi} x_1 I^s(|x|) + O(|x|).$$

Similarly,

$$u_2(x) = -\frac{1}{2\pi} x_2 I^s(|x|) + O(|x|).$$

In situations where $I^s \gg 1$ near the origin, it will be very useful to replace the $O(|x|)$ term with $O(x_1)$ in the case of u_1 (and $O(x_2)$ in the case of u_2), since otherwise the $O(|x|)$ term could be actually dominate the other in the region $0 < x_1 \ll x_2 \sim |x|$. This was actually done in the original Key Lemma of Kiselev–Sverak [95], although it is impossible avoid a logarithmic error.

Lemma 3.5.6. *Assume the hyperbolic flow scenario together with $\omega \in L^\infty$ compactly supported. Then, we have that*

$$\frac{u_1(x)}{x_1} = \frac{1}{2\pi} I^s(|x|) + B_1(x)$$

and

$$\frac{u_2(x)}{x_2} = -\frac{1}{2\pi} I^s(|x|) + B_2(x)$$

with

$$|B_i(x)| \le C\|\omega\|_{L^\infty} \min\left\{1, \ln \frac{x_{3-i}}{x_i}\right\} \tag{89}$$

for $i = 1, 2$.

Problem 3.5.7. *Prove sharpness of the upper bound in $|B_i|$.*

Proof. The proof is rather straightforward, and we borrow the argument from [139]. To begin with, we note that the bound for u_2 follows from that of u_1, using that the velocity transforms as

$$(u_1(x_1, x_2), u_2(x_1, x_2)) \mapsto -(u_2(x_2, x_1), u_1(x_2, x_1))$$

upon the transformation $\omega(x_1, x_2) \mapsto \omega(x_2, x_1)$. The starting point is simply to rewrite the kernel using the odd symmetry of the vorticity. That is, first using the odd symmetry in x_1,

$$\begin{aligned} 2\pi\, u_1(x) &= \int_{\mathbb{R}^2} \frac{-(x_2 - y_2)}{|x - y|^2} \omega(y) dy \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}} \left[-\frac{(x_2 - y_2)}{|x - y|^2} + \frac{(x_2 - y_2)}{|\bar{x} - y|^2} \right] \omega(y) dy, \end{aligned}$$

where $\bar{x} := (-x_1, x_2)$. Using that

$$-\frac{(x_2 - y_2)}{|x - y|^2} + \frac{(x_2 - y_2)}{|\bar{x} - y|^2} = \frac{(x_2 - y_2)(-4x_1 y_1)}{|x - y|^2 |\bar{x} - y|^2},$$

we obtain

$$2\pi\, u_1(x) = -4x_1 \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{(x_2 - y_2) y_1}{|x - y|^2 |\bar{x} - y|^2} \omega(y) dy.$$

This shows that u_1 is “divisible” by x_1, which is intuitively clear. Then, using the odd symmetry in x_2, we get

$$2\pi\, u_1(x) = -4x_1 \int_{(\mathbb{R}_+)^2} \left[\frac{(x_2 - y_2) y_1}{|x - y|^2 |\bar{x} - y|^2} + \frac{(-x_2 - y_2) y_1}{|x + y|^2 |\tilde{x} - y|^2} \right] \omega(y) dy.$$

Here, $\tilde{x} := (x_1, -x_2)$. We shall set

$$K(x,y) := -\frac{(x_2 - y_2)y_1}{|x-y|^2|\bar{x}-y|^2} - \frac{(-x_2 - y_2)y_1}{|x+y|^2|\tilde{x}-y|^2}$$

for $x, y \in (\mathbb{R}_+)^2$. The behavior of K is different for different points. Given $x = (x_1, x_2)$, it turns out to be natural to divide the positive quadrant into four regions,

$$\begin{aligned} A &:= [2x_1, \infty) \times [2x_2, \infty), \quad B := [2x_1, \infty) \times [0, 2x_2), \\ C &:= (0, 2x_1) \times (0, 2x_2), \quad D := (0, 2x_1) \times [2x_2, \infty). \end{aligned}$$

(See Figure 4.) As in the case of the Hilbert transform, the main term comes from the region A. The point is that, in this region, $|x-y|, |\bar{x}-y|, |x+y|, |\tilde{x}-y| \sim |y|$. More precisely, it is easy to see that

$$\left| \int_A \left[\frac{(x_2-y_2)y_1}{|x-y|^2|\bar{x}-y|^2} - \frac{(x_2-y_2)y_1}{|y|^4} \right] \omega(y)dy \right| \le C\|\omega\|_{L^\infty}.$$

That is, up to $O(1)$ error, $\int_A K\omega dy$ can be replaced with

$$\int_A \left[\frac{-(x_2-y_2)y_1}{|y|^4} + \frac{(x_2+y_2)y_1}{|y|^4} \right] \omega(y)dy = \int_A \left[\frac{2y_1y_2}{|y|^4} \right] \omega(y)dy.$$

Next, it is easy to deal with the "local" region C. In this case, there was no reason to symmetrize in x_2. We shall estimate

$$\left| \int_C -\frac{(x_2-y_2)y_1}{|x-y|^2|\bar{x}-y|^2} \omega(y)dy \right| \le C\|\omega\|_{L^\infty} \int_C \frac{y_1|x_2-y_2|}{|x-y|^2|\bar{x}-y|^2} dy,$$

the other term in K being easier to estimate. We make change of variables $z_i = x_i - y_i$ for $i = 1, 2$ and bound the above by

$$C\|\omega\|_{L^\infty} \int_0^{2x_1} \int_0^{2x_2} \frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)} dz_2 dz_1.$$

We can consider two cases; (i) $x_1 \gtrsim x_2$, (ii) $x_1 \lesssim x_2$. In the first case, the bound is simpler thanks to the x_1^2 term in the denominator;

$$\int_0^{2x_1} \int_0^{2x_2} \frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)} dz \lesssim \int_{B(0,x_1)} \frac{1}{x_1} \frac{1}{|z|} dz \lesssim 1.$$

When $x_1 \lesssim x_2$, we *need to* make use of the other term z_2^2. In this case, we consider separately the regions $z_2 \lesssim x_1$ and $z_2 \gtrsim x_1$. Bounding the first region is parallel to the case (i). Next, when $z_2 \gtrsim x_1$, we bound

$$\begin{aligned} \int_0^{2x_1} \int_{4x_1}^{2x_2} \frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)} dz &\lesssim x_1 \int_0^{2x_1} \int_{4x_1}^{2x_2} \frac{1}{z_2^3} dz \\ &\lesssim x_1^2 \int_{4x_1}^{2x_2} \frac{1}{z_2^3} dz_2 \lesssim 1. \end{aligned}$$

This concludes that

$$\left|\int_C K(x,y)\omega(y)dy\right| \leq C\|\omega\|_{L^\infty}.$$

The estimate in the region D is similar. In this case, we actually always have $z_2 \gtrsim x_1$ (under the same change of variables as before), but we have a fast decay of K in z_2, as we have seen. This gives that

$$\left|\int_D K(x,y)\omega(y)dy\right| \leq C\|\omega\|_{L^\infty}.$$

The interesting (and dangerous) region is B. Under the same change of variables, we can bound the integral by

$$C\|\omega\|_{L^\infty}\int_{2x_1}^{\infty}\int_0^{2x_2}\frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)}dz_2dz_1.$$

Still, if we have $x_1 \gtrsim x_2$ (say $x_1 \geq x_2$ for concreteness), we can bound using the change of inequalities $z_1 \gtrsim x_1 \gtrsim x_2 \gtrsim z_2$ as follows:

$$\int_{2x_1}^{\infty}\int_0^{2x_2}\frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)}dz_2dz_1 \lesssim \int_{2x_1}^{\infty}\int_0^{2x_2}\frac{z_2}{z_1^3}dz_2dz_1 \lesssim 1.$$

The difficulty arises when $x_2 \geq x_1$. Even in this case, in the sub-region where $z_1 \geq 2x_2$, we can proceed as in the case of $x_1 \gtrsim x_2$, using that $z_1 \gtrsim x_2 \gtrsim z_2$. The only remaining integral is

$$\int_{2x_1}^{2x_2}\int_0^{2x_2}\frac{(z_1+x_1)z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)}dz_2dz_1.$$

To clarify the situation, we *explicitly evaluate* this integral. We replace z_1+x_1 by z_1 and note that it is equal to

$$\int_0^{2x_2}\frac{1}{2}\ln\left(\frac{(2x_2)^2+z_2^2}{(2x_1)^2+z_2^2}\right)\frac{z_2}{(x_1^2+z_2^2)}dz_2$$

When $z_2 \lesssim x_1$,

$$\ln\left(\frac{(2x_2)^2+z_2^2}{(2x_1)^2+z_2^2}\right) \sim \ln\frac{x_2}{x_1},$$

and then

$$\int_0^{2x_1}\frac{z_2}{x_1^2+z_2^2}dz_2 \lesssim 1.$$

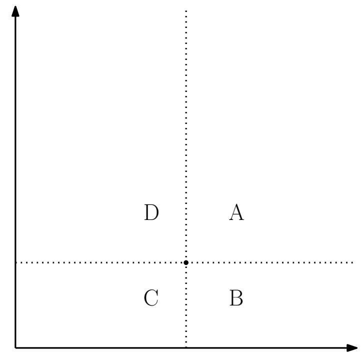

Figure 4: A decomposition of the positive quadrant.

Lastly, when $z_2 \gtrsim x_1$, estimating the above gives

$$\left(\ln \frac{x_2}{x_1}\right)^2.$$

This is not good. Actually, we note that we could have estimated from below

$$|\bar{x} - y|^2 \gtrsim z_1^2 + z_2^2$$

instead. Then,

$$\int_{2x_1}^{2x_2} \int_0^{2x_2} \frac{(z_1 + x_1)z_2}{(z_1^2 + z_2^2)(z_1^2 + z_2^2)} dz_2 dz_1 \sim \int_0^{2x_2} \frac{z_2}{(2x_1)^2 + z_2^2} dz_2 \sim \ln(1 + \frac{x_2}{x_1}),$$

as desired. This finishes the proof. $\square$

Remark 3.5.8. *When $\omega \in L^\infty$ and not better, the logarithmic error term is actually unavoidable near the axis. However, if we assume further that $\nabla\omega \in L^\infty$, clearly $\nabla u \in L^\infty$ and such a logarithmic term must disappear. It is an interesting question how to see this quantitatively. Zlatos [139] proved that*

$$|B_1(x)| \lesssim x_2 \|\nabla\omega\|_{L^\infty([0,x_2]^2)}.$$

3.5.2 Key Lemma and double exponential growth

The Key Lemma was introduced in the groundbreaking work [95] to construct solutions to 2D Euler showing double exponential in time growth of the gradient, when the fluid domain is given by a disc. We provide a sketch of the proof, with a simplified setup.

Proposition 3.5.9. *Consider 2D Euler in the upper half plane $\mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$ with an external strain field: namely, ω satisfies*

$$\begin{aligned} \partial_t \omega + (u + \bar{u}) \cdot \nabla \omega = 0, \\ u = \nabla^\perp \Delta^{-1} \omega, \end{aligned} \tag{90}$$

where $\bar{u} := 10C(-x_1, x_2)^T$ with C from (89). Furthermore, assume that $\omega_0 \in C_c^\infty(\mathbb{R}^2_+)$ satisfies $\|\omega_0\|_{L^\infty} < 2$, odd in x_1, and non-positive in the first quadrant. Lastly, assume that $\omega_0 = -1$ in the region

$$\{(x_1, x_2) : \delta < x_1 < M, x_2 \leq x_1\},$$

for some $0 < \delta < M$; see Figure 5. Then, the unique global in time solution to (90) satisfies

$$\|\nabla \omega(t, \cdot)\|_{L^\infty} \geq c \exp(c \exp(ct))$$

for all t for some $c > 0$.

Remark 3.5.10. *The point is that, if one considers the linear equation*

$$\partial_t \omega + \bar{u} \cdot \nabla \omega = 0,$$

then only exponential in time growth of the gradient can occur. Therefore, in the above statement, the role of nonlinearity is essential in the double exponential growth.

Proof. The first step is to observe the velocity on the diagonal, namely when $x_1 = x_2$. Applying Lemma 3.5.6 on the diagonal, we see that $u + \bar{u}$ is signed: $I^s < 0$ always and the constant $10C$ in front of $\bar{u}$ is arranged in a way that (using $\|\omega(t, \cdot)\|_{L^\infty} < 2$ for all t) we have $(u + \bar{u})_1 < 0$ and $(u + \bar{u})_2 > 0$ for all t.

More generally, we note that the logarithmic error in the Key Lemma does not enter for the first component of the velocity below the diagonal. Defining $V = (u + \bar{u})_1$ for simplicity, we see that

$$-\frac{1}{2\pi} I^s(t, |x|) + 8C \leq -\frac{V(t, x)}{x_1} \leq -\frac{1}{2\pi} I^s(t, |x|) + 12C$$

for $x = (x_1, x_2)$ with $x_2 \leq x_1$. Here, a useful further simplification is achieved by comparing $I^s(t, |x|)$ with $I^s(t, x_1)$; under the assumption $x_2 \leq x_1$, it can be shown that

$$|I^s(t, |x|) - I^s(t, x_1)| \leq C \|\omega\|_{L^\infty},$$

by increasing C if necessary. Therefore, we have

$$-\frac{1}{2\pi} I^s(t, x_1) + 6C \leq -\frac{V(t, x)}{x_1} \leq -\frac{1}{2\pi} I^s(t, x_1) + 14C. \tag{91}$$

The estimate (91) is then used to construct a *barrier*: we shall take $\phi(t)$, $\overline{\phi}(t)$ in a way that the region

$$R(t) := \big\{(x_1, x_2) : \phi(t) < x_1 < \overline{\phi}(t), x_2 \le x_1\big\}$$

satisfies $\omega(t, R(t)) = -1$ for all t. Furthermore, we would like to have that the ratio $\overline{\phi}(t)/\phi(t)$ grows exponentially in time.

To this end, we can simply take

$$\frac{1}{\phi(t)}\frac{d}{dt}\phi(t) = -\frac{1}{2\pi}I^s(t, \phi(t)) - 6C, \qquad \phi(0) = \delta.$$

Similarly, we consider the trajectory

$$\frac{1}{\overline{\phi}(t)}\frac{d}{dt}\overline{\phi}(t) = -\frac{1}{2\pi}I^s(t, \overline{\phi}(t)) - 14C, \qquad \overline{\phi}(0) = M.$$

Then, to show that ω equals -1 on $R(t)$, it suffices to observe that the "fluid particles" starting outside of $R(0)$ cannot enter the region $R(t)$ for all t. The particles cannot enter through the diagonal since the velocity on the diagonal is always pointing northwest. Next, they cannot enter the lateral boundaries due to (91).

The punchline is that

$$-\frac{1}{2\pi}I^s(t, \phi(t)) = -\frac{1}{2\pi}I^s(t, \overline{\phi}(t)) - c_0 \ln\frac{\bar{\phi}(t)}{\phi(t)} =: -a(t) - c_0 \ln\frac{\bar{\phi}(t)}{\phi(t)}$$

for some absolute constant $c_0 > 0$ (possibly with some bounded error which we may incorporate in C). In other words,

$$\frac{d}{dt}\ln\phi(t) = -a(t) - c_0 \ln\frac{\bar{\phi}(t)}{\phi(t)} - 6C,$$

$$\frac{d}{dt}\ln\overline{\phi}(t) = -a(t) - 14C.$$

Subtracting the two equations,

$$\frac{d}{dt}\ln\frac{\bar{\phi}(t)}{\phi(t)} = -8C + c_0 \ln\frac{\bar{\phi}(t)}{\phi(t)}.$$

At this point, recall that C is just some absolute constant and we were free to choose δ and M. We take them in a way that $M/\delta \gg 16C/c_0$. Then, using a continuity argument, it can be shown that

$$\frac{d}{dt}\ln\frac{\bar{\phi}(t)}{\phi(t)} \ge \frac{c_0}{2}\ln\frac{\bar{\phi}(t)}{\phi(t)},$$

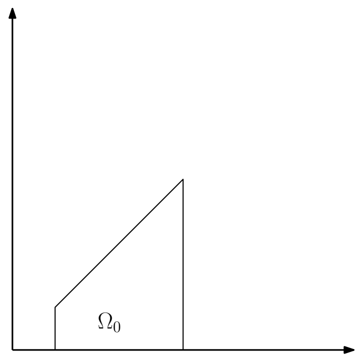

Figure 5: Kiselev–Sverak "train"

or in other words,

$$\ln \frac{\bar{\phi}(t)}{\phi(t)} \geq \exp(\frac{c_0}{2}t) \ln \frac{M}{\delta}$$

This already shows that the velocity gradient in L^∞ grows exponentially in time. Using a bound on $\overline{\phi}(t)$, we can show that $\phi(t)$ converges to 0 double exponentially in time. Since $\omega(t,(0,0)) = 0$ and $\omega(t,(\phi(t),0)) = -1$, the proof is complete by the mean value theorem. (All the details can be found in [95].) □

3.5.3 Key Lemma for gSQG

Not surprisingly, various versions of the Key Lemma have been obtained for the gSQG equations and they proved to be very useful for many purposes:

- Singularity formation for carefully designed data
- Infinite growth of certain Sobolev norms in time
- Illposedness in critical and supercritical function spaces.

The key point is to utilize the trivial stability (by area-preserving feature) of the Bahouri–Chemin solution, which exhibits an explicit singular behavior on the axes. One wants to approximate the velocity field with the explicit one coming from the Bahouri–Chemin solution, and the difficult part is to decide how to control the error.

We state a version of the Key Lemma for generalized SQG equations (gSQG) with $1 < \alpha < 2$. We state the lemma in $\mathbb{R}^2$ for simplicity, but it holds for also in $\mathbb{T}^2$ with minor modifications. The point in the statement below is that the error is controlled with the critical Sobolev norm, thereby making it useful for the study of critical problems.

Lemma 3.5.11. *Let $\theta \in$ be odd-odd symmetric and signed on the first quadrant. Let $x = (x_1, x_2)$ satisfy $x_1 > x_2 > 0$. Then, $u = \nabla^\perp \Lambda^{-2+\alpha}\theta$ satisfies*

$$\left| \frac{u_1(x)}{x_1} - C_\alpha \int_{Q(2x)} \frac{y_1 y_2}{|y|^{4+\alpha}} \theta(y)\,\mathrm{d}y \right| \le B_1(x) \tag{92}$$

and

$$\left| \frac{u_2(x)}{x_2} + C_\alpha \int_{Q(2x)} \frac{y_1 y_2}{|y|^{4+\alpha}} \theta(y)\,\mathrm{d}y \right| \le \left(1 + \log \frac{x_1}{x_2}\right) B_2(x), \tag{93}$$

where

$$Q(2x) = [2x_1, \infty) \times [2x_2, \infty)$$

and B_1, B_2 satisfy

$$|B_1(x)| + |B_2(x)| \le C(\|\theta\|_{H^{1+\alpha}} + \|\theta\|_{L^\infty}).$$

3.5.4 Singularity formation

Using a form of the key lemma, one can try to obtain singularity formation for solutions to the gSQG equations. Assume the following two things:

- One has local wellposedness of α-SQG for a class of functions which allow $\theta_0 = 1$ in the "train" depicted in Figure 5.
- Simply replace the velocity field by the main term of the Key Lemma.

Then, it is possible to show that the train must hit the origin in finite time, by proceeding similarly to the proof of Proposition 3.5.9. This is based on a contradiction argument: it is assumed that there is a solution in the wellposedness class in a sufficiently long period of time. It is actually quite difficult to justify each of the above two requirements. This was done in [97, 94, 64, 138] using several very clever ideas but the precise setup for local wellposedness is quite complicated to be described here.

We now give some details of the proof of singularity formation result of [94]. Before describing the statement, let us investigate the kernel for α-SQG, following the notation of [94]. We begin with

$$u_1(x) = -\int_{\mathbb{R}^2} \frac{y_2 - x_2}{|x-y|^{2+\alpha}} (-\theta(y)) dy$$

To clarify, we are going to take θ to be non-positive in the first quadrant, and that is why we put $-\theta$ in the Biot–Savart law. As in [94], given $y = (y_1, y_2)$, write $\bar{y} = (y_1, -y_2)$ and $\tilde{y} = (-y_1, y_2)$. Then, after symmetrization, we have

$$u_1(x) = -\int_{(\mathbb{R}_+)^2} \left[\frac{y_2 - x_2}{|x-y|^{2+\alpha}} - \frac{y_2 - x_2}{|x-\tilde{y}|^{2+\alpha}} - \frac{y_2 + x_2}{|x+y|^{2+\alpha}} + \frac{y_2 + x_2}{|x-\bar{y}|^{2+\alpha}} \right] (-\theta(y)) dy.$$

We want $u_1 < 0$ and therefore we seek a lower bound on the kernel, which will be denoted by $K_1(x, y)$ from now on. To this end, one can note that

$$-\frac{y_2 + x_2}{|x + y|^{2+\alpha}} + \frac{y_2 + x_2}{|x - \bar{y}|^{2+\alpha}} \geq 0$$

for all $x, y \in (\mathbb{R}_+)^2$, so that we may neglect these two terms altogether. On the other hand, unfortunately the sum of the other two terms are not signed:

$$\frac{y_2 - x_2}{|x - y|^{2+\alpha}} - \frac{y_2 - x_2}{|x - \tilde{y}|^{2+\alpha}} \geq 0 \qquad \text{if and only if} \qquad y_2 \geq x_2.$$

That is, in Figure 4, the vorticity in the regions A, D pushes the point towards the x_2-axis while the vorticity in the other regions do the opposite. (The fact that this must be the case can actually be immediately seen from the point vortex idea.) This led the authors of [94] to decompose $u_1 = u_1^{bad} + u_1^{good}$: $u_1^{good}(x)$ is by definition the integral coming from the region $y_2 \geq x_2$:

$$u_1^{good}(x) = -\int_{y_2 \geq x_2, y_1 \geq 0} K_1(x, y)(-\theta(y))dy.$$

Similar considerations give that the regions A, B in Figure 4 are "good" for u_2. Note that the region A, which is good for both components of the velocity, is precisely the one featured in the Key Lemma.

Again, for the initial data θ_0, we are going to take it to be odd-odd symmetric and non-positive on the first quadrant. Furthermore, we may take $0 \leq |\theta_0| \leq 1$. This gives global L^∞ and actually also $C^{1-\alpha}$ bound for u for $0 < \alpha < 1$, which is convenient.

Now, one introduces the time dependent "train"

$$R(t) = \{x : X(t) < x_1 < 1, 0 < x_2 < x_1\}$$

for some $1 > X(t) > 0$, and assume that $\theta_0 \equiv -1$ on $R(0)$. The goal is then to prove that for some explicitly chosen $X(t)$ which reaches 0 at some critical time $T^* > 0$, the solution $\theta(t)$ (which is assumed to uniquely exist and belongs to some function space on $[0, T^*]$) contains $R(t)$ in its support. For such a choice of $X(t)$ to be possible, two main ingredients are naturally

- an upper bound on the bad part of the velocity
- a lower bound on the good part of the velocity.

To begin with, the bad part is controlled by considering the worst case scenario where the bad region is fully occupied by the solution. This gives (Lemma 4.3 of [94])

$$u_1^{bad}(x) \leq C x_1^{1-\alpha}, \qquad 0 \leq x_2 \leq x_1$$

and similarly

$$u_2^{bad}(x) \geq -Cx_2^{1-\alpha}, \qquad 0 \leq x_1 \leq x_2.$$

The constant C is uniformly bounded near $\alpha = 0$, which is consistent with the fact that 2D Euler does not feature a unbounded bad term in the good region. In the following, we shall only focus on the case $0 < \alpha \ll 1$.

On the other hand, to control the good part, one assumes that the support of θ contains a triangle; given $x \in (\mathbb{R}_+)^2$, define

$$A(x) = \{y : x_1 < y_1 < x_1 + 1, x_2 < y_2 < x_2 + y_1 - x_1\}.$$

Then, by calculating the key integral over the region $A(x)$, one obtains for all $|x|$ sufficiently small (independent of α) the lower bound of the form

$$u_1^{good}(x) \leq -\frac{c}{\alpha} x_1^{1-\alpha}$$

and

$$u_2^{good}(x) \geq \frac{c}{\alpha} x_2^{1-\alpha}$$

where $c > 0$ is a universal constant. We have arrived at Proposition 4.5 of [94], which is the main tool in the proof of singularity formation.

Proposition 3.5.12. *For $0 < \alpha \ll 1$, there are universal constants $c_0, c_1 > 0$ such that for all $x \in (\mathbb{R}_+)^2, |x| \leq c_0$ satisfying $(-\theta) = 1$ on $A(x)$, we have*

$$u_1(x) \leq -\frac{c_1}{\alpha} x_1^{1-\alpha} \quad \textit{for} \quad x_2 \leq x_1$$

and

$$u_2(x) \geq \frac{c_1}{\alpha} x_2^{1-\alpha} \quad \textit{for} \quad x_1 \leq x_2.$$

3.6 Long time existence

3.6.1 Convergence between equations

In this section, we consider the relation between gSQG solutions with different velocity kernels and obtain long time existence in some special limiting cases as a consequence. This type of idea was used in Yu–Zheng–Jiu [134] for the first time, and it is reminiscent of the so-called shadowing theorem for the Euler and Navier–Stokes equations [33]. We present a recent result on the long time existence of gSQG equations from [22].

Difference equation. Assume that θ^j is a solution to

$$\partial_t \theta^j + \nabla^\perp P_j(\Lambda)\theta^j \cdot \nabla \theta^j = 0$$

for $j = 1, 2$. Let us assume that θ^2 is relatively more regular. Then, we may write the equation for the difference:

$$\begin{aligned} &\partial_t(\theta^1 - \theta^2) + \nabla^\perp P_1(\Lambda)\theta^1 \cdot \nabla(\theta^1 - \theta^2) \\ &\quad + \nabla^\perp P_1(\Lambda)(\theta^1 - \theta^2) \cdot \nabla \theta_2 + \nabla^\perp (P_2(\Lambda) - P_1(\Lambda))\theta_2 \cdot \nabla \theta_2 = 0. \end{aligned} \tag{94}$$

Taking the L^2 inner product with $(\theta^1 - \theta^2)$, we obtain that

$$\frac{d}{dt}\|\theta^1 - \theta^2\|_{L^2} \lesssim_{\theta_2} \|\theta^1 - \theta^2\|_{L^2} + \|\nabla^\perp (P_2(\Lambda) - P_1(\Lambda))\theta_2\|_{L^1}.$$

When we derive this estimate, we assume that P_1 is an intermediate operator; namely $P_1(\Lambda) \lesssim 1$. For the last term, we use Fourier series

$$\begin{aligned} \|\nabla^\perp (P_2(\Lambda) - P_1(\Lambda))\theta_2\|_{L^1} &\lesssim \sum_{k \in \mathbb{Z}^2} |P_2(k) - P_1(k)||k||\hat{\theta}_{2,k}| \\ &= \sum_{k \in \mathbb{Z}^2} |k|^{1-s}|P_2(k) - P_1(k)|\, |k|^s |\hat{\theta}_{2,k}| \\ &\lesssim \||k|^{1-s}|P_2(k) - P_1(k)|\|_{\ell_2} \|\theta_2\|_{H^s(\mathbb{T}^2)}. \end{aligned}$$

Then, we obtain with $\epsilon_s := \||k|^{1-s}|P_2(k) - P_1(k)|\|_{\ell_2}$ that

$$\frac{d}{dt}\|\theta^1 - \theta^2\|_{L^2} \lesssim_{\theta_2} \|\theta^1 - \theta^2\|_{L^2} + \epsilon_s.$$

Integrating in time

$$\begin{aligned} \|\theta^1 - \theta^2\|_{L^2}(t) &\le e^{Ct}\left(\|\theta^1_0 - \theta^2_0\|_{L^2} + \int_0^t Ce^{-C\tau}\epsilon_s d\tau\right) \\ &\le e^{Ct}\|\theta^1_0 - \theta^2_0\|_{L^2} + (e^{Ct} - 1)\epsilon_s. \end{aligned}$$

In particular, if the initial data coincide,

$$\|\theta^1 - \theta^2\|_{L^2}(t) \le (e^{Ct} - 1)\epsilon_s.$$

In these estimates, the constant C depend on some Sobolev norm of θ_2, taken supremum in the interval $[0, t]$.

Limit to the trivial model. We consider the sequence of solutions

$$\begin{aligned} \partial_t \theta^\alpha + \nabla^\perp \Lambda^{-\alpha}\theta^\alpha \cdot \nabla \theta^\alpha &= 0, \\ \theta^\alpha(t = 0) &= \theta_0 \end{aligned} \tag{95}$$

where $\theta_0 \in C^\infty(\mathbb{T}^2)$ is *fixed*. We assume, without loss of generality, that $\int_{\mathbb{T}^2} \theta_0 = 0$. We shall compare this to the solution in the case $\alpha = 0$, which is simply $\theta^0(t) \equiv \theta_0$. We now compute the error:

$$\begin{aligned}
\epsilon_s^2 &= \sum_{k\neq 0, k\in\mathbb{Z}^2} |k|^{2(1-s)}|1-|k|^{-\alpha}|^2 \\
&\lesssim \int_{|\xi|\geq 1, \xi\in\mathbb{R}^2} |\xi|^{2(1-s)}|1-|\xi|^{-\alpha}|^2 d\xi \sim_s \alpha^2,
\end{aligned}$$

whenever $s > 2$. This gives that

$$\|\theta^\alpha - \theta_0\|_{L^2}(t) \leq \alpha(e^{Ct}-1).$$

Here, $C > 0$ only depends on the initial data θ_0. In particular, we see that for any fixed $T > 0$, $\|\theta^\alpha - \theta_0\|_{L^2}(T) \to 0$ as $\alpha \to 0$, with some *rate*.

Time rescaling and the Ohkitani model. We may rewrite the equation (95) as

$$\frac{\partial_t}{\alpha}\theta^\alpha + \nabla^\perp\left(\frac{\Lambda^{-\alpha}-I}{\alpha}\right)\theta^\alpha\cdot\nabla\theta^\alpha = 0.$$

Therefore, if we introduce the time rescaling

$$\tilde{\theta}^\alpha(\tau, x) := \theta^\alpha(\tau/\alpha, x),$$

then

$$\partial_\tau\tilde{\theta}^\alpha + \nabla^\perp\Gamma_\alpha\tilde{\theta}^\alpha\cdot\nabla\tilde{\theta}^\alpha = 0, \tag{96}$$

where $\Gamma_\alpha := \frac{\Lambda^{-\alpha}-I}{\alpha}$. We may compare this with the solution to the Ohkitani model,

$$\partial_\tau\tilde{\theta} + \nabla^\perp\Gamma\tilde{\theta}\cdot\nabla\tilde{\theta} = 0$$

with the same initial data and

$$\Gamma(\Lambda) = -\ln\Lambda.$$

Applying the previous L^2 estimate for the difference,

$$\|\tilde{\theta}^\alpha - \tilde{\theta}\|_{L^2}(\tau) \leq \epsilon_s(e^{C\tau}-1).$$

In this case,

$$\begin{aligned}
\epsilon_s^2 &= \sum_{k\neq 0, k\in\mathbb{Z}^2} |k|^{2(1-s)}\left|\frac{|k|^{-\alpha}-1}{\alpha} + \ln|k|\right|^2 \\
&\lesssim \int_{|\xi|\geq 1, \xi\in\mathbb{R}^2} |\xi|^{3-2s}\left|\frac{|\xi|^{-\alpha}-1}{\alpha} + \ln|\xi|\right|^2 d\xi \sim_s \alpha^2,
\end{aligned}$$

again when $s > 2$. That is, $\|\tilde{\theta}^\alpha - \tilde{\theta}\|_{L^2}(\tau) \lesssim_s \alpha(e^{C\tau} - 1)$. Therefore, we can take $\tau^* > 0$ depending only on C and s and obtain that

$$\|\tilde{\theta}^\alpha - \tilde{\theta}\|_{L^2}(\tau) \leq C\tau\alpha, \qquad 0 \leq \tau \leq \tau^*.$$

Higher-order difference estimate. We may rewrite (94) as

$$\begin{aligned} &\partial_t(\theta^1 - \theta^2) + \nabla^\perp P_1(\Lambda)(\theta^1 - \theta^2) \cdot \nabla(\theta^1 - \theta^2) \\ &\quad + \nabla^\perp P_1(\Lambda)\theta^2 \cdot \nabla(\theta^1 - \theta^2) \\ &\quad - \nabla^\perp \theta^2 \cdot \nabla P_1(\Lambda)(\theta^1 - \theta^2) + \nabla^\perp (P_2(\Lambda) - P_1(\Lambda))\theta^2 \cdot \nabla\theta^2 = 0. \end{aligned} \tag{97}$$

For small time, we have for $\varphi := \theta^1 - \theta^2$ the L^2 estimate

$$\|\varphi\|_{L^2}(t) \lesssim_s \epsilon_s t.$$

For a sufficiently large s. For $m \geq 3$, we can perform H^m estimate for (97):

$$\frac{d}{dt}\|\varphi\|_{H^m} \lesssim \|\varphi\|_{H^m}^2 + \|\theta_2\|_{H^s}\|\varphi\|_{H^m} + \|\theta_2\|_{H^s}^2\epsilon_s$$

with a larger $s > 0$ depending on m. Therefore, with a bootstrap argument, we can conclude a similar estimate for H^m

$$\|\varphi\|_{H^m}(t) \lesssim_s \epsilon_s t.$$

One just needs to keep in mind that s depends on m, $s \geq s_0 + m$. For $m \geq 3$, H^m is a blow-up criterion for intermediate gSQG.

Based on the above considerations, one can obtain long-time existence of smooth solutions for the gSQG equations in the intermediate regime ([22]).

4 Illposedness results

In this section, we collect several situations where the generalized SQG equations are *illposed* in the strongest sense of Hadamard. It is worth spending some time to clarify the definition of illposedness. For this, let us recall the Hadamard's notion of wellposedness for a given PDE, from a space X to another Y;

- Existence: for any $\theta_0 \in X$, there exist a solution $\theta \in Y$ such that $\theta(t = 0) = \theta_0$.
- Uniqueness: there is at most one solution that belongs to Y with the prescribed initial data θ_0.
- Continuous dependence: the solution map (which is well-defined when the previous two conditions are satisfied) is continuous.

Then, we simply say that a PDE is illposed if at least one of the above three requirements do not hold:

- Non-continuous dependence: *even if* the solution map is defined from X to Y, it is not continuous.
- Non-uniqueness: for some initial data $\theta_0 \in X$, there are *at least two* solutions belonging to Y corresponding to θ_0.
- Non-existence: for some $\theta_0 \in X$, there does not exist a solution $\theta \in Y$ such that $\theta(t = 0) = \theta_0$.

Note that these items are written in reverse order compared to the wellposedness case. This is intentional as one may argue that the non-continuity of the solution is the weakest sense of illposedness and so on. Next, it is worth emphasizing again that the notion of illposedness highly depends not only on the data space X but also on the solution space Y. It is natural to take of course $Y = L^\infty([0,T]; X)$ for the equations we consider. In this concrete case, we shall say that a PDE has nonexistence in X, if there is some $\theta_0 \in X$ such that for any $T > 0$, there are no solutions $\theta \in L^\infty([0,T]; X)$ for the PDE satisfying $\theta(t = 0) = \theta_0$. (It is assumed that the space X is not too rough so that $\theta(t = 0)$ is well-defined at least in the weak sense.) We also introduce the notion of norm-inflation, which is stronger than non-continuous dependence. For simplicity, assume that the 0 function is a solution to the given PDE. We say that there is a norm inflation from X to Y if for any $\epsilon > 0$ there exist $c > 0$ and a data $\theta_0 \in X$ with $\|\theta_0\|_X < \epsilon$ and $\|\theta\|_Y \geq c$ where θ is an associated solution.

This section is organized as follows. In Section 4.1, we consider the question of illposedness when the multiplier is singular. Then, Section 4.2 gives illposedness in the presence of the physical boundary. This shows that when the boundaries are present, the active scalar is required to vanish on the boundary, unless we work with a very well-prepared class of solutions.

4.1 Illposedness in the very singular regime

The goal of this section is to sketch the proof of the fact that gSQG equations in the singular regime are illposed in the sense of Hadamard. More precisely, no matter how regular the initial data is, in the scale of Sobolev spaces, the same regularity will not propagate in time for gSQG dynamics. While the gSQG equations are nonlinear, under the contradiction hypothesis that the equations are wellposed, one may argue that the linearized dynamics gives a good approximation to the nonlinear solution, at least for a small time interval. This requires that we are looking at solutions which are close to a specific known solution at the initial time. For the specific solution, we take steady states which are *degenerate* in a precise sense that will be described below. Then, the main work goes into proving that the linearized dynamics near degenerate steady states are illposed in Sobolev spaces.

4.1.1 Steady states and Linearization

In what follows, we use the notation (x,y) for a point in $\mathbb{R}^2$. Recall that functions depending only on one spatial coordinate defines steady states to (gSQG), which are *shear flows*: setting $\bar{\theta}(x,y) = f(y)$ forces that $(-\Delta)^{-1+\frac{\alpha}{2}}\bar{\theta} =: F$ is again a pure function of y and hence $\bar{u} = (-F', 0)$, giving $\bar{u}\cdot\nabla\bar{\theta} = 0$. Since $F' = (-\partial_{yy})^{-1+\frac{\alpha}{2}} f$, we see that the linearization takes the form

$$\partial_t\theta + (-\partial_y)^{\alpha-1} f \partial_x\theta = -\partial_y f \partial_x (-\Delta)^{-1+\frac{\alpha}{2}}\theta. \tag{98}$$

For the simplicity of presentation, we shall focus on the *local* case; that is, when $\alpha \in 2\mathbb{N}$. (The analysis in this case is much simpler and rather elementary; the general case is handled in [23]. See [84, 83, 82] for further developments.) However, the case $\alpha = 2$ is not interesting at all, and it turns out that the analysis for $\alpha \geq 6$ can be done in a parallel way with the case $\alpha = 4$. Hence we will focus on the $\alpha = 4$ case. For the linearized equation, the $H^{\frac{\alpha}{2}-1}$-norm of the solution is still under control; assuming that θ is a sufficiently smooth solution to (98), we have that

$$\frac{d}{dt}\|\theta\|_{H^{\frac{\alpha}{2}-1}} \lesssim \|\theta\|_{H^{\frac{\alpha}{2}-1}}.$$

Based on the above estimate, one can obtain without much difficulty the *existence* of a solution to (98). However, for higher Sobolev norms, one cannot obtain such an estimate, and at least for very specific profiles f, illposedness can be seen just based on the H^m-estimates.

4.1.2 Main result for the linear dynamics

The main result for the linearized equation is stated as follows. We take the y-domain to be $\mathbb{T}$ for simplicity.

Theorem 4.1.1 (Linear illposedness: sharp norm growth)**.** *Consider stationary solution of the form $\bar{\theta} = f(y)$, where f is any smooth function on $\mathbb{T}$ satisfying*

$$f(y_0) = f'(y_0) = f''(y_0) = 0, f'''(y_0) \neq 0$$

for some $y_0 \in \mathbb{T}$. Then, there exist $y_1 > 0$ and profiles

$$G(y) \in C^\infty(y_0, y_0 + y_1),$$

and

$$g(y) \in C_c^\infty(y_0 + \frac{1}{3}y_1, y_0 + \frac{2}{3}y_1)$$

satisfying the following: for any $\lambda \geq 1$, if we consider the initial data

$$\theta_0 := \mathrm{Re}\left[e^{i\lambda(x+G(y))}\right] g(y)$$

then any *H^1-solution to* (98) *with initial data θ_0 defined on some* $[0,T)$ *satisfies*

$$\|\nabla\theta(t)\|_{L^2_x L^p_y} \gtrsim (1 - C_0\lambda t)e^{c_0(1-\frac{2}{p})\lambda^2 t}\frac{\|\nabla\theta_0\|^2_{L^2}}{\|\nabla\theta_0\|_{L^2_x L^{p'}_y}}, \quad t \in [0,T) \tag{99}$$

for $2 \leq p \leq \infty$ with some absolute constants $c_0, C_0 > 0$ depending only on f, where $1/p' + 1/p = 1$.

Remark 4.1.1. *The statement of the above theorem is not vacuous as it can be shown that there exists at least one global H^1-solution for* (98) *with any H^1 initial data. One can achieve by mollifying the equation and passing to a weak limit; see for instance [84, Appendix A]. Of course, the lower bound in* (99) *is meaningful only for $0 < t \lesssim \lambda^{-1}$.*

As a corollary, we obtain the linear illposedness in high Sobolev spaces. Conceptually, this is simple thanks to the superposition principle.

Corollary 4.1.2 (Linear illposedness: nonexistence)**.** *Under the same assumptions on f as in Theorem 4.1.1, there exists an initial data $\theta_0 \in C^\infty(\mathbb{T})$ such that* any *H^1-solution to* (98) *with initial data θ_0 defined on* $[0,T)$ *escapes $H^{1+\epsilon}$ for any $\epsilon > 0$; that is, $\|\theta(t)\|_{H^{1+\epsilon}} = +\infty$ for any $t > 0$.*

4.1.3 Main results for nonlinear dynamics

As we have mentioned earlier, by combining linear illposedness with a contradiction argument, one can actually obtain nonlinear illposedness for gSQG equations in the singular regime. We shall focus on the case $P(\Lambda) = \Lambda^2 = -\Delta$.

Theorem 4.1.2 (Nonlinear illposedness: unboundedness)**.** *Assume that for some $\epsilon, \delta, r, s > 0$, the solution map for* (gSQG) *exists as a map*

$$\mathcal{B}(0; H^r) \to L^\infty([0,\delta]; H^s).$$

Then the solution map is unbounded for $s \geq 4$.

Remark 4.1.3. *The above statement in particular implies norm inflation: indeed, we can show that for any $\epsilon > 0$, there exists $C^\infty(\mathbb{T}^2)$ initial data satisfying*

$$\|\theta_0\|_{H^r} < \epsilon, \qquad \sup_{t\in(0,\delta]} \|\theta(t)\|_{H^s} > \epsilon^{-1},$$

assuming that a solution exists in $L^\infty_t H^s$.

Theorem 4.1.3 (Nonlinear illposedness: nonexistence)**.** *Assume that $s > 4 + \frac{1}{2}$. For any $\epsilon, \delta > 0$, there exists initial data $\theta_0 \in H^s$ satisfying $\|\theta_0\|_{H^s} < \epsilon$ such that* no *corresponding solution can exist in $L^\infty([0,\delta]; H^s)$.*

The proof of nonlinear illposedness results are firmly based on the linear illposedness mechanism that we are going to discuss in detail below.

4.1.4 Linear illposedness

The goal in the remainder of this section is to prove the linear illposedness statement; the proof of nonlinear statements can be found in [84, 23]. We now proceed to the detailed study of linear dynamics. The linearized equation is given by

$$\partial_t\theta + f'''(y)\partial_x\theta = f'(y)\partial_x\Delta\theta. \tag{100}$$

Previous discussions show that the H^1-norm of θ should be (formally) under control. Indeed, a straightforward computation shows the following

Proposition 4.1.4. *Let θ be a sufficiently smooth solution to* (100) *in the time interval $[0,T)$ with initial data $\theta_0 \in H^1(\mathbb{T}^2)$. Then, for $0 < t < T$, we have*

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 = -\int f''''(y)\partial_x\theta\partial_y\theta.$$

In particular, we have

$$\|\nabla\theta(t)\|_{L^2} \leq \|\nabla\theta_0\|_{L^2}\exp(Ct)$$

for $0 \leq t < T$, where $C > 0$ depends only on f.

Proposition 4.1.5 (Generalized energy identity)**.** *Let θ be an H^1-solution to* (100) *and $\tilde{\theta}$ be a sufficiently smooth solution to*

$$\partial_t \tilde{\theta} - f'(y)\partial_x \Delta \tilde{\theta} = \epsilon.$$

Then, we have that

$$\frac{d}{dt}\langle \nabla\theta, \nabla\tilde{\theta}\rangle = -\langle \partial_x\theta, f'''\partial_{xx}\tilde{\theta}\rangle + \langle \partial_y\theta, f'''\partial_x\partial_y\tilde{\theta}\rangle + \langle \nabla\theta, \nabla\epsilon\rangle. \tag{101}$$

Note that when $\theta = \tilde{\theta}$ and $\epsilon = -f'''\partial_x\theta$, we recover Proposition 4.1.4 from (101).

Proof. We simply compute

$$\begin{aligned}\frac{d}{dt}\langle \nabla\theta, \nabla\tilde{\theta}\rangle = &\langle f'\partial_x\Delta\nabla\theta, \nabla\tilde{\theta}\rangle + \langle \nabla\theta, f'\partial_x\Delta\nabla\tilde{\theta}\rangle + \langle f''\partial_x\Delta\theta, \partial_y\tilde{\theta}\rangle + \langle \partial_y\theta, f''\partial_x\Delta\tilde{\theta}\rangle \\ &+ \langle -f'''\partial_x\nabla\theta, \nabla\tilde{\theta}\rangle + \langle \nabla\theta, \nabla\epsilon\rangle.\end{aligned}$$

Combining the first, second, and fifth terms on the right hand side, we get

$$\langle f'\partial_x\Delta\nabla\theta, \nabla\tilde{\theta}\rangle + \langle \nabla\theta, f'\partial_x\Delta\nabla\tilde{\theta}\rangle + \langle -f'''\partial_x\nabla\theta, \nabla\tilde{\theta}\rangle = -\langle \nabla\theta, 2f''\partial_y\partial_x\nabla\tilde{\theta}\rangle.$$

Then, we compute that the third and fourth terms add up to

$$\langle f''\partial_x\Delta\theta, \partial_y\tilde{\theta}\rangle + \langle \partial_y\theta, f''\partial_x\Delta\tilde{\theta}\rangle = \langle \nabla\theta, 2f''\partial_y\partial_x\nabla\tilde{\theta}\rangle + \langle \partial_y\theta, f'''\partial_x\partial_y\tilde{\theta}\rangle - \langle \partial_x\theta, f'''\partial_{xx}\tilde{\theta}\rangle.$$

This finishes the proof. □

4.1.5 Degenerating wavepackets

Now we shall write down an approximate equation for (100) with a specific choice of f near $y = 0$ and solve it approximately. Namely, we pick

$$f(y) = \frac{y^3}{3}$$

in the region $|y| \leq \frac{1}{2}$. It turns out that this particular function simplifies the computations below significantly. (However, what one just needs is that the profile f is vanishing at least of second order at some point.) The approximate solutions we construct will be denoted by degenerating wavepackets, as their $W^{1,p}$-norms degenerate with an exponential rate in frequency for $1 \leq p < 2$. To this end, we propose

$$\partial_t\tilde{\theta} = f'(y)\partial_x(\partial_{xx}\tilde{\theta} + \partial_{yy}\tilde{\theta}). \tag{102}$$

The goal of the following sections is to establish the key statement in this paper; namely

Proposition 4.1.6 (Construction of degenerating wavepackets)**.** *There exists a profile $g(y) \in C^\infty(0,\infty)$ such that for any $\lambda \in \mathbb{N}$, there is a linear map*

$$h_0 \mapsto \tilde{\theta}^{(\lambda)}(t,x,y;h_0) := e^{i\lambda(x+\lambda^2 t+g(y))}h(t,y)$$

defined on functions $h_0 \in C_c^\infty(0,\frac{1}{10})$ satisfying the following properties:

1. *(initial data) $\tilde{\theta}^{(\lambda)}(0,x,y;h_0) = e^{i\lambda(x+g(y))}h_0(y)$;*
2. *(regularity) for any $k,\ell \geq 0$ and $t \geq 0$,*
$$\|(\lambda^{-3}\partial_t)^k(\lambda^{-1}\partial_x)^\ell \nabla\tilde{\theta}^{(\lambda)}(t)\|_{L^2} \lesssim \|h_0\|_{H^{1+k}};$$
3. *(degeneration) for any $1 \leq p \leq 2$ and $t \geq 0$,*
$$\|\nabla\tilde{\theta}^{(\lambda)}(t)\|_{L_x^2 L_y^p} \lesssim e^{-(\frac{2}{p}-1)\lambda^2 t}\|h_0\|_{W^{1,p}};$$
4. *(error bounds) for $t \geq 0$,*
$$\|\epsilon(t)\|_{H^1} \lesssim \|h_0\|_{H^3}.$$

In the following, $\tilde{\theta}^{(\lambda)}$ will be denoted as the degenerating wavepacket solution associated with h_0 at frequency λ. The proof of this proposition is carried out in 4.1.8 with ingredients from 4.1.6–4.1.7; we first derive the form of $\tilde{\theta}^{(\lambda)}$ after a suitable renormalization in 4.1.6 and obtain estimates for this solution in the renormalized coordinates in 4.1.7.

Remark 4.1.7. *In view of Proposition 4.1.4, it is natural to rewrite the equation in terms of $\nabla\theta$:*

$$\partial_t \nabla\theta + f'''\partial_x\nabla\theta + \begin{pmatrix} 0 \\ f''''\partial_x\theta \end{pmatrix} = f'\partial_x\Delta\nabla\theta + \begin{pmatrix} 0 \\ f''\partial_x\Delta\theta \end{pmatrix}$$

and reformulate the equation in terms of the second component; setting $v := \partial_y\theta$ gives that

$$\partial_t v + f'''\partial_x v + f''''\partial_x\partial_y^{-1}v = f'\partial_x\Delta v + f''\partial_{xxx}\partial_y^{-1}v + f''\partial_x\partial_y v. \tag{103}$$

One can then proceed with the associated approximate equation

$$\partial_t\tilde{v} + \partial_x(f'\partial_{yy} + f''\partial_y)\tilde{v} - f'\partial_{xxx}\tilde{v} = 0$$

instead of (102) *and arrive at the key proposition 4.1.6 as well. While this form has the advantage that the L^2 norm is conserved in time (rather than the $\dot{H}^1$-norm), computations are somewhat more complicated and one needs to work with the operator ∂_y^{-1}.*

4.1.6 Renormalization

The goal in this section is to derive the form of an approximate solution of (102). Given $\lambda \in \mathbb{N}$, we make the change of variables

$$\eta = \ln y, \quad \tau = \lambda^2 t.$$

In the following we are only concerned with the region $\{0 < y < 1\}$ and hence η takes values in $(-\infty, 0)$. Furthermore, we may separate x-dependence completely by writing

$$\tilde{\theta}(t, x, y) = e^{i\lambda x}\phi(\lambda^2 t, \eta(y)).$$

Then

$$\partial_t \tilde{\theta} = \lambda^2 e^{i\lambda x}\partial_\tau \phi,$$

$$y^2\partial_x(\partial_{xx}\tilde{\theta} + \partial_{yy}\tilde{\theta}) = i\lambda e^{i\lambda x}(-y^2\lambda^2\phi + y\partial_y(y\partial_y\phi) - y\partial_y\phi)$$

so that (102) turns into

$$\partial_\tau \phi = i\lambda^{-1}(-\lambda^2 e^{2\eta}\phi + \partial_{\eta\eta}\phi - \partial_\eta\phi) =: \tilde{\mathcal{L}}[\phi].$$

We look for an approximate solution $\tilde{\phi}$ to $[\partial_\tau - \tilde{\mathcal{L}}]\phi = 0$ of the form

$$\tilde{\phi} = \lambda^{-1}e^{i\lambda\Phi(\tau,\eta)}h(\tau, \eta)$$

where Φ and h are smooth functions independent of λ to be determined. (The prefactor λ^{-1} was inserted to normalize the corresponding $\tilde{\theta}$ in H^1 rather than in L^2 as $\lambda \to +\infty$.) We now compute

$$\partial_\tau \tilde{\phi} = \lambda^{-1}\left[i\lambda\partial_\tau\Phi h + \partial_\tau h\right]e^{i\lambda\Phi},$$

$$\partial_\eta \tilde{\phi} = \lambda^{-1}\left[i\lambda\partial_\eta\Phi h + \partial_\eta h\right]e^{i\lambda\Phi},$$

$$\partial_{\eta\eta}\tilde{\phi} = \lambda^{-1}\left[-\lambda^2(\partial_\eta\Phi)^2 h + 2i\lambda\partial_\eta\Phi\partial_\eta h + i\lambda\partial_{\eta\eta}\Phi h + \partial_{\eta\eta}h\right]e^{i\lambda\Phi},$$

so that

$$\begin{aligned}[\partial_\tau - \tilde{\mathcal{L}}]\tilde{\phi} = e^{i\lambda\Phi}\big[& i\left(\partial_\tau\Phi + (\partial_\eta\Phi)^2 + e^{2\eta}\right)h \\ &+ \lambda^{-1}\left(\partial_\tau h + 2\partial_\eta\Phi\partial_\eta h + (\partial_{\eta\eta}\Phi - \partial_\eta\Phi)h\right) - i\lambda^{-2}(\partial_{\eta\eta}h - \partial_\eta h)\big].\end{aligned}$$

Matching the terms of order 1 and λ^{-1}, we obtain the equations

$$\partial_\tau\Phi + (\partial_\eta\Phi)^2 = -e^{2\eta} \tag{104}$$

and

$$\partial_\tau h + 2\partial_\eta \Phi \partial_\eta h + (\partial_{\eta\eta}\Phi - \partial_\eta \Phi)h = 0 \tag{105}$$

respectively. Given Φ and h solving (104)–(105), we have that

$$[\partial_t - \tilde{\mathcal{L}}]\tilde{\phi} = -i\lambda^{-1} e^{i\lambda\Phi}(\partial_{\eta\eta}h - \partial_\eta h).$$

Taking the ansatz $\Phi(\tau, \eta) = -\tau - G(\eta)$ for some $G \geq 0$, we have that G must satisfy

$$(G')^2 = 1 - e^{2\eta}.$$

We then simply take the solution

$$\Phi(\tau, \eta) = -\tau - \eta - \int_{-\infty}^{\eta} \left[\sqrt{1 - e^{2\eta'}} - 1\right] d\eta' \tag{106}$$

for (104); then (105) reduces to simply

$$\partial_\tau h - 2\sqrt{1 - e^{2\eta}}\partial_\eta h + \left(\frac{e^{2\eta}}{\sqrt{1 - e^{2\eta}}} + \sqrt{1 - e^{2\eta}}\right) h = 0. \tag{107}$$

In the regime $\eta \ll -1$, we have approximately

$$\partial_\tau h - 2\partial_\eta h + h = 0$$

whose solution is given explicitly by

$$h(\tau, \eta) = e^{-\tau} h_0(\eta + 2\tau). \tag{108}$$

The goal in the following is to show that the actual solution h behaves essentially like the "ideal" one given in (108). Indeed, the form of Φ in (106) is chosen in a way that the characteristics for h proceeds to the left, so that (107) reduces to the simple one above in the limit $\tau \to +\infty$.

4.1.7 Estimates on the degenerating wavepackets

We define the η-characteristics by

$$\frac{d}{d\tau} Y(\tau, \eta) = -2\sqrt{1 - e^{2Y(\tau,\eta)}}, \quad Y(0, \eta) = \eta \tag{109}$$

for $\eta \leq -1$. Interestingly, this ODE can be integrated explicitly, with the solution

$$Y(\tau, \eta) = \frac{1}{2} \ln\left(1 - \tanh^2(2\tau + a(\eta))\right), \tag{110}$$

where

$$a(\eta) = -\eta + \ln(1 + \sqrt{1 - e^{2\eta}}) = \tanh^{-1}\left(\sqrt{1 - e^{2\eta}}\right), \tag{111}$$

which is valid for any $\eta \leq 0$. While it is not necessary at all to have the explicit solution to carry out the necessary analysis, let us take advantage of it for simplicity. Note that from (110), we have

$$2\tau + a(\eta) = a(Y).$$

For later use, we compute

$$(\partial_\tau Y)(\tau, \eta) = -2\tanh(2\tau + a(\eta)) = -2\sqrt{1 - e^{2Y}},$$

and

$$(\partial_\eta Y)(\tau, \eta) = \frac{\tanh(2\tau + a(\eta))}{\tanh(a(\eta))} = \frac{\sqrt{1 - e^{2Y}}}{\sqrt{1 - e^{2\eta}}}.$$

Now rewriting (107) along the characteristics, we have

$$\frac{d}{d\tau} h(\tau, Y) = -(\sqrt{1 - e^{2Y}} + \frac{e^{2Y}}{\sqrt{1 - e^{2Y}}}) h(\tau, Y)$$

so that

$$h(\tau, Y(\tau, \eta)) = h_0(\eta) \exp\left(-\int_0^\tau \sqrt{1 - e^{2Y(\tau', \eta)}} + \frac{e^{2Y(\tau', \eta)}}{\sqrt{1 - e^{2Y(\tau', \eta)}}}\, d\tau'\right).$$

Note that by integrating (109) in τ,

$$Y(\tau, \eta) - \eta = -2\int_0^\tau \sqrt{1 - e^{2Y(\tau', \eta)}} d\tau'.$$

Moreover, differentiating (109) in η gives

$$\frac{d}{d\tau}(\partial_\eta Y) = \frac{2e^{2Y}}{\sqrt{1 - e^{2Y}}}(\partial_\eta Y)$$

so that

$$\frac{d}{d\tau}\ln(\partial_\eta Y) = \frac{2e^{2Y}}{\sqrt{1 - e^{2Y}}}$$

and integrating in τ,

$$\ln((\partial_\eta Y)(\tau, \eta)) = \int_0^\tau \frac{2e^{2Y}}{\sqrt{1 - e^{2Y}}} d\tau'$$

since $\ln((\partial_\eta Y)(0,\eta)) = 0$. From the previous computations, we conclude that

$$\exp\left(-\int_0^\tau \sqrt{1-e^{2Y(\tau',\eta)}} + \frac{e^{2Y(\tau',\eta)}}{\sqrt{1-e^{2Y(\tau',\eta)}}}\, d\tau'\right) = (\partial_\eta Y)^{-\frac{1}{2}} e^{\frac{Y(\tau,\eta)-\eta}{2}}.$$

Denoting the inverse of $Y(\tau,\cdot)$ by Y_τ^{-1},

$$h(\tau,\eta) = (\partial_\eta Y)^{-\frac{1}{2}}(\tau, Y_\tau^{-1}) e^{\frac{\eta - Y_\tau^{-1}}{2}} h_0 \circ Y_\tau^{-1}(\eta).$$

Now to simplify the above expression, recall

$$2\tau + a(\eta) = a(Y),$$

which gives

$$a(Y_\tau^{-1}) = a(\eta) - 2\tau.$$

Using the formula for $a(\cdot)$ in (111),

$$\frac{\eta - Y_\tau^{-1}}{2} - \frac{1}{2}\ln\frac{1+\sqrt{1-e^{2\eta}}}{1+\sqrt{1-e^{2Y_\tau^{-1}}}} = -\tau$$

and

$$\partial_\eta Y \circ Y_\tau^{-1} = \frac{\tanh(a(\eta))}{\tanh(a(Y_\tau^{-1}))} = \frac{\sqrt{1-e^{2\eta}}}{\sqrt{1-e^{2Y_\tau^{-1}}}}$$

we obtain

$$h(\tau,\eta) = e^{-\tau} h_0 \circ Y_\tau^{-1}(\eta) \frac{A(\eta)}{A(Y_\tau^{-1}(\eta))},$$

where

$$A(\eta) = \sqrt{1 + \frac{1}{\sqrt{1-e^{2\eta}}}}.$$

Note that $A(\eta) \sim 1$ for all $\eta \leq -1$. In particular we have $A \circ Y_\tau^{-1} \sim 1$ as well.

4.1.8 Proof of Proposition 4.1.6

We are now ready to proceed to the proof of key Proposition.

Regularity. Note that the case $k = \ell = 0$ is covered by the statement for degeneration with $p = 2$, which will be proved below. Moreover, the case $\ell > 0$ is straightforward from the explicit form of $\tilde{\theta}(t)$. The prove the statement for $k > 0$, it suffices to note that $(\partial_t^k h)$ satisfies the same equation with h since the coefficients are independent of time. We omit the details.

Degeneration property. We now establish the degeneration property. Beginning with

$$\partial_y \tilde{\theta} = i\partial_y \Phi e^{i\lambda(x+\Phi)} h + \lambda^{-1} e^{i\lambda(x+\Phi)} \partial_y h,$$

we have

$$\|\partial_y \tilde{\theta}(t)\|_{L^2_x L^p_y} \lesssim \|(\partial_y \Phi) h\|_{L^p_y} + \lambda^{-1} \|\partial_y h\|_{L^p_y}.$$

Switching to the η-coordinates,

$$\begin{aligned}
\|(\partial_y \Phi) h\|^p_{L^p_y} &= \int (\partial_\eta \Phi)^p |h|^p e^{(1-p)\eta} d\eta \\
&\lesssim e^{-\tau p} \int |h_0 \circ Y_\tau^{-1}|^p e^{\eta(1-p)} d\eta \\
&\lesssim e^{-\tau(2-p)} \int |h_0(\eta)|^p (\partial_\eta Y_\tau) d\eta \\
&\lesssim e^{-\tau(2-p)}.
\end{aligned}$$

We have used the simple bounds

$$|\partial_\eta Y(\tau)| \lesssim 1, \quad |\partial_\eta \Phi| \lesssim 1$$

and the fact that the support of $h_0 \circ Y_\tau^{-1}$ is contained in the interval $[-\tau - c, -\tau - c']$ for some absolute constants $c > c' > 0$ to replace $e^{\eta(1-p)}$ inside the integral with $e^{-\tau(1-p)}$. Similarly, we first write

$$\|\partial_y h\|^p_{L^p_y} = \int |\partial_\eta h|^p e^{(1-p)\eta} d\eta$$

and

$$\partial_\eta h = e^{-\tau} \left[(\partial_\eta h_0 \circ Y_\tau^{-1}) \partial_\eta Y_\tau^{-1} \frac{A}{A \circ Y_\tau^{-1}} + h_0 \circ Y_\tau^{-1} \partial_\eta \left(\frac{A}{A \circ Y_\tau^{-1}}\right) \right].$$

From the simple bounds

$$\partial_\eta Y_\tau^{-1} \lesssim 1, \quad \left| \frac{A}{A \circ Y_\tau^{-1}} \right| \lesssim 1, \quad \left| \partial_\eta \left(\frac{A}{A \circ Y_\tau^{-1}}\right) \right| \lesssim 1,$$

we deduce that

$$\|\partial_y h\|^p_{L^p_y} \lesssim e^{-\tau(2-p)} \int |\partial_\eta h_0|^p + |h_0|^p d\eta \lesssim e^{-\tau(2-p)}$$

similarly as in the above. From these we conclude that

$$\|\partial_y \tilde{\theta}(t)\|_{L_x^2 L_y^p} \lesssim e^{-\frac{2-p}{p}\lambda^2 t} \|h_0\|_{W_y^{1,p}}.$$

Error estimate. Recall that

$$[\partial_\tau - \mathcal{L}]\,\tilde{\phi} = -i\lambda^{-2}(\partial_{\eta\eta} h - \partial_\eta h)e^{i\lambda\Phi}.$$

Switching back to the (t,x,y)-coordinates,

$$\epsilon(t) := [\partial_t - \mathcal{L}]\,\tilde{\theta}(t) = -ie^{i\lambda x}\left[(\partial_{\eta\eta} h - \partial_\eta h)e^{i\lambda\Phi}\right](t,y).$$

It suffices to prove the following global-in-time bounds for h:

Lemma 4.1.8. *For all $\tau \geq 0$, we have the estimates*

$$\|\partial_\eta^{(m)} h\|_{L^2(e^{\eta/2}d\eta)} \lesssim e^{-c_m\tau}\|h_0\|_{H_y^m}$$

for some $c_m > 0$ and

$$\|\partial_\eta^{(m)} h\|_{L^2(e^{-\eta/2}d\eta)} \lesssim \|h_0\|_{H_y^m}$$

with implicit constants independent of $\tau \geq 0$. Here $\partial_\eta^{(m)}$ denotes the m-th order derivative in η and we have defined

$$\|f\|^2_{L^2(e^{\pm\eta/2}d\eta)} := \int_{-\infty}^{0} (f(\eta))^2 e^{\pm\eta}\, d\eta.$$

Assuming the bounds stated in the above lemma for the moment, it is easy to estimate $\epsilon(t)$: to begin with,

$$\partial_x\epsilon(t) = \lambda e^{i\lambda x}\left[(\partial_{\eta\eta} h - \partial_\eta h)e^{i\lambda\Phi}\right](t,y)$$

and

$$\begin{aligned}\|\partial_x\epsilon(t)\|_{L^2} &\lesssim \lambda(\|\partial_{\eta\eta}h\|_{L_y^2} + \|\partial_\eta h\|_{L_y^2}) \\ &\lesssim \lambda(\|\partial_{\eta\eta}h\|_{L^2(e^{-\eta/2}d\eta)} + \|\partial_\eta h\|_{L^2(e^{-\eta/2}d\eta)}) \lesssim \lambda\|h_0\|_{H_y^2}.\end{aligned}$$

Next,

$$\partial_y\epsilon(t) = \lambda e^{i\lambda x}\left[(\partial_{\eta\eta} h - \partial_\eta h)\partial_y\Phi e^{i\lambda\Phi}\right](t,y) - ie^{i\lambda x}\left[(\partial_y\partial_{\eta\eta} h - \partial_y\partial_\eta h)e^{i\lambda\Phi}\right](t,y)$$

and note that

$$\begin{aligned}\|\partial_y\Phi\partial_\eta h\|_{L_y^2} + \|\partial_y\Phi\partial_{\eta\eta} h\|_{L_y^2} &= \|\partial_y\Phi\partial_\eta h\|_{L_y^2} + \|\partial_{\eta\eta}\Phi e^{-\eta}\partial_\eta h\|_{L_y^2} \\ &\lesssim \|\partial_{\eta\eta}h\|_{L^2(e^{-\eta/2}d\eta)} + \|\partial_\eta h\|_{L^2(e^{-\eta/2}d\eta)}\end{aligned}$$

where we have used $|\partial_\eta \Phi| \lesssim 1$. Similarly,

$$\|\partial_y \partial_{\eta\eta} h\|_{L^2_y} + \|\partial_y \partial_\eta h\|_{L^2_y} \lesssim \|\partial_{\eta\eta\eta} h\|_{L^2(e^{-\eta/2} d\eta)} + \|\partial_{\eta\eta} h\|_{L^2(e^{-\eta/2} d\eta)}$$

so that

$$\begin{aligned}\|\partial_y \epsilon(t)\|_{L^2} &\lesssim (1+\lambda)(\|\partial_{\eta\eta\eta} h\|_{L^2(e^{-\eta/2} d\eta)} + \|\partial_{\eta\eta} h\|_{L^2(e^{-\eta/2} d\eta)} + \|\partial_\eta h\|_{L^2(e^{-\eta/2} d\eta)}) \\ &\lesssim (1+\lambda)\|h_0\|_{H^3_y}.\end{aligned}$$

This gives the desired error estimate.

Proof of Lemma 4.1.8. To estimate high order norms of h, it is better to return to the equation:

$$\partial_\tau h + 2\partial_\eta \Phi \partial_\eta h + (\partial_{\eta\eta} \Phi - \partial_\eta \Phi) h = 0.$$

Indeed, multiplying both sides of the above by he^η and integrating, we see immediately that

$$\frac{1}{2} \frac{d}{d\tau} \|h\|^2_{L^2(e^{\eta/2} d\eta)} = 2 \int \partial_\eta \Phi h^2 e^\eta \leq -c_0 \|h\|^2_{L^2(e^{\eta/2} d\eta)}$$

so

$$\|h\|_{L^2_y} = \|h\|_{L^2(e^{\eta/2} d\eta)} \lesssim e^{-c_0 \tau} \|h_0\|_{L^2_y}$$

for some $c_0 > 0$. On the other hand, using the weight $e^{-\eta/2}$, we see easily that

$$\|h\|_{L^2(e^{-\eta/2} d\eta)} = \|h_0\|_{L^2(e^{-\eta/2} d\eta)} \lesssim \|h_0\|_{L^2_y}.$$

Differentiating the equation in η,

$$\partial_\tau(\partial_\eta h) + 2\partial_\eta \Phi \partial_\eta(\partial_\eta h) + 3\partial_{\eta\eta} \Phi \partial_\eta h - \partial_\eta \Phi \partial_\eta h + (\partial_{\eta\eta\eta} \Phi - \partial_{\eta\eta} \Phi) h = 0$$

and proceeding similarly as above,

$$\frac{1}{2} \frac{d}{d\tau} \|\partial_\eta h\|^2_{L^2(e^{\eta/2} d\eta)} + 2 \int \partial_{\eta\eta} \Phi (\partial_\eta h)^2 e^\eta - 2 \int \partial_\eta \Phi (\partial_\eta h)^2 e^\eta = -\int (\partial_{\eta\eta\eta} \Phi - \partial_{\eta\eta} \Phi) h \partial_\eta h e^\eta.$$

Using

$$\partial_{\eta\eta} \Phi \geq 0, \quad \partial_\eta \Phi \leq -c_0, \quad |\partial_{\eta\eta\eta} \Phi - \partial_{\eta\eta} \Phi| \lesssim 1,$$

we obtain

$$\frac{1}{2} \frac{d}{d\tau} \|\partial_\eta h\|_{L^2(e^{\eta/2} d\eta)} \lesssim -\|\partial_\eta h\|_{L^2(e^{\eta/2} d\eta)} + \|h\|_{L^2(e^{\eta/2} d\eta)}$$

after canceling a factor of $\|\partial_\eta h\|_{L^2(e^{\eta/2}d\eta)}$. From the decay of $\|h\|_{L^2(e^{\eta/2}d\eta)}$ in τ, we obtain this time

$$\|\partial_\eta h\|_{L^2_y} = \|\partial_\eta h\|_{L^2(e^{\eta/2}d\eta)} \lesssim e^{-c_1\tau}\|h_0\|_{H^1_y}$$

for some $c_1 > 0$. Using the weight $e^{-\eta/2}$, we have this time

$$\frac{1}{2}\frac{d}{d\tau}\|\partial_\eta h\|^2_{L^2(e^{-\eta/2}d\eta)} + 2\int \partial_{\eta\eta}\Phi(\partial_\eta h)^2 e^{-\eta} = -\int(\partial_{\eta\eta\eta}\Phi - \partial_{\eta\eta}\Phi)h\partial_\eta h e^{-\eta}.$$

This time, we need to observe the pointwise decay of $\partial_{\eta\eta}\Phi$ and $\partial_{\eta\eta\eta}\Phi$ on the support of $h(\tau)$; for the former, we have

$$\partial_{\eta\eta}\Phi = \frac{e^{2\eta}}{\sqrt{1-e^{2\eta}}}$$

but the denominator is bounded uniformly away from 0 while $\eta \le -2\tau + C_0$ for η belonging to the support of $h(\tau)$. In particular we have $|\partial_{\eta\eta}\Phi| \lesssim e^{-4\tau}$. Indeed it is easy to observe that the same bound holds for any higher order derivatives of Φ in η. Therefore, we conclude that

$$\frac{d}{d\tau}\|\partial_\eta h\|_{L^2(e^{-\eta/2}d\eta)} \lesssim e^{-4\tau}\|h_0\|_{L^2(e^{-\eta/2}d\eta)}$$

which gives

$$\|\partial_y h\|_{L^2_y} = \|\partial_\eta h\|_{L^2(e^{-\eta/2}d\eta)} \lesssim \|h_0\|_{H^1_y}$$

with the implicit constant *independent* of $\tau \ge 0$.

So far we have established the statements of the lemma for $m = 0, 1$. The case $m \ge 2$ can be proved inductively on m, and let us just sketch the proof for $m = 2$: differentiating the equation for h twice in η,

$$\begin{aligned}
&\partial_\tau(\partial_{\eta\eta}h) + 2\partial_\eta\Phi\partial_\eta(\partial_{\eta\eta}h) - \partial_\eta\Phi(\partial_{\eta\eta}h)\\
&\quad = -4\partial_{\eta\eta}\Phi\partial_{\eta\eta}h - 2\partial_{\eta\eta\eta}\Phi\partial_\eta h - 2\partial_\eta(\partial_{\eta\eta}\Phi - \partial_\eta\Phi)\partial_\eta h - \partial_{\eta\eta}(\partial_{\eta\eta}\Phi - \partial_\eta\Phi)h - \partial_{\eta\eta}\Phi\partial_{\eta\eta}h.
\end{aligned}$$

We have reorganized the terms in a way that each term in the right hand side of the equation has Φ with at least two η derivatives, which decay exponentially in τ pointwise on the support of $h(\tau)$. Therefore,

$$\begin{aligned}
&\left|\frac{1}{2}\frac{d}{d\tau}\|\partial_{\eta\eta}h\|_{L^2(e^{\pm\eta/2}d\eta)} - (1\pm 1)\int \partial_\eta\Phi(\partial_\eta h)^2 e^{\pm\eta}\right|\\
&\quad \lesssim e^{-4\tau}\left(\|\partial_{\eta\eta}h\|_{L^2(e^{\pm\eta/2}d\eta)} + \|\partial_\eta h\|_{L^2(e^{\pm\eta/2}d\eta)} + \|h\|_{L^2(e^{\pm\eta/2}d\eta)}\right).
\end{aligned}$$

From this it is straightforward to conclude the statements of the lemma for $m = 2$. We omit the details. $\square$

4.1.9 Proof of Theorem 4.1.1

We are in a position to complete the proof of Theorem 4.1.1. Recall that $f(y) = \frac{y^3}{3}$ on the support of $h(\tau(t))$ for all $t \geq 0$. Hence, taking some $\lambda \geq 1$ and $h_0 \in C_c^\infty(0, \frac{1}{10})$ and applying (101) gives

$$\frac{d}{dt}\langle \nabla\theta, \nabla\tilde{\theta}\rangle = -2\langle \partial_x \theta, \partial_{xx}\tilde{\theta}\rangle + 2\langle \partial_y \theta, \partial_x\partial_y\tilde{\theta}\rangle + \langle \nabla\theta, \nabla\epsilon\rangle.$$

Using the error estimate from Proposition 4.1.6, we bound the right hand side as

$$\lesssim \left(\lambda\|\nabla\tilde{\theta}\|_{L^2} + \|\nabla\epsilon\|_{L^2}\right)\|\nabla\theta\|_{L^2} \lesssim \lambda\|\nabla\theta_0\|_{L^2}^2.$$

Therefore,

$$\langle \nabla\theta(t), \nabla\tilde{\theta}(t)\rangle \geq (1 - C_0\lambda t)\|\nabla\theta_0\|_{L^2}^2$$

for some absolute constant $C_0 > 0$ and then using the degeneration estimate,

$$\langle \nabla\theta(t), \nabla\tilde{\theta}(t)\rangle \leq \|\nabla\theta(t)\|_{L_x^2 L_y^p}\|\nabla\tilde{\theta}(t)\|_{L_x^2 L_y^{p'}} \lesssim e^{-\frac{2-p'}{p'}\lambda^2 t}\|\nabla\theta_0\|_{L_x^2 L_y^{p'}}\|\nabla\theta(t)\|_{L_x^2 L_y^p}.$$

Combining the bounds, we conclude that

$$\|\nabla\theta(t)\|_{L_x^2 L_y^p} \gtrsim (1 - C_0\lambda t)e^{(1-\frac{2}{p})\lambda^2 t}\frac{\|\nabla\theta_0\|_{L^2}^2}{\|\nabla\theta_0\|_{L_x^2 L_y^{p'}}}.$$

This completes the proof. $\square$

4.2 Illposedness with physical boundaries

It is a rather well-known fact that the two-dimensional Euler equations is wellposed in domains with boundary, *even when the vorticity is not required to vanish on the boundary.* In this section, we shall demonstrate that this property is a rather special feature of the Euler, and does not carry over to the case of gSQG. Namely, in the case of (gSQG), there is strong illposedness when the advected scalar does not vanish on the boundary. For simplicity, we shall take the case of the upper half-plane, $\mathbb{R}_+^2 = \{(x_1, x_2) : x_2 \geq 0\}$.

4.2.1 Euler case

In the case of the two-dimensional Euler equations, we have the following classical result. The domain $\mathbb{R}_+^2$ can be replaced by any sufficiently nice domains with a boundary.

Theorem 4.2.1. *The two-dimensional Euler equation is wellposed in the space* $C_c^\infty(\mathbb{R}^2_+)$.

Proof. The key technical lemma is as follows:

Lemma 4.2.1. *If* $\omega \in C_c^\alpha(\mathbb{R}^2_+)$, *then* $\nabla u \in C^\alpha(\mathbb{R}^2_+)$, *where* $u = \nabla^\perp \Delta^{-1}_{\mathbb{R}^2_+}\omega$.

It should be emphasized that this lemma is not trivial. The way the operator $\Delta^{-1}_{\mathbb{R}^2_+}$ is defined is that we first extend ω as a function $\tilde{\omega}$ defined on $\mathbb{R}^2$ by the odd extension in x_2, namely

$$\tilde{\omega}(x) = -\omega(x_1, -x_2)$$

if $x_2 < 0$. In particular, this extension is not really defined on the boundary $\partial\mathbb{R}^2_+ = \{x_2 = 0\}$ unless ω vanishes there. That is, $\tilde{\omega}$ is not a C^α function on $\mathbb{R}^2$ and therefore the standard Schauder estimate $\nabla^2\Delta^{-1} : C^\alpha \to C^\alpha$ from $\mathbb{R}^2$ is not applicable.

The key point in the above lemma is the *half-moon* computation, which explicitly shows that if we take the transform $\nabla^2\Delta^{-1}$ to the function $\mathbf{1}_{|x|<1}\mathrm{sgn}(x_2)$, the result is C^∞ smooth in the ball $\{|x| < 1/2\}$. This computation boils down to the special property of the kernel $\sigma(\cdot)$ for $\nabla^2\Delta^{-1}$ which is that its angular integral in any interval of length π vanishes.

Equipped with the lemma, propagation of C^∞ regularity is straightforward: from

$$\partial_t\omega + u \cdot \nabla\omega = 0,$$

we obtain

$$\frac{d}{dt}\|\omega\|_{C^\alpha} \le C_\alpha\|\nabla u\|_{L^\infty}\|\omega\|_{C^\alpha} \le C_\alpha\|\omega\|^2_{C^\alpha}.$$

For higher derivatives, we can start with one:

$$\partial_t\partial\omega + u \cdot \nabla\partial\omega = -\partial u \cdot \nabla\omega,$$

and we again obtain

$$\frac{d}{dt}\|\partial\omega\|_{C^\alpha} \le C_\alpha\|\nabla u\|_{C^\alpha}\|\partial\omega\|_{C^\alpha} \le C_\alpha\|\partial\omega\|^2_{C^\alpha}.$$

Next, to estimate $\omega \in C^{2,\alpha}$, we need to obtain uniform Hölder regularity of $\nabla^2 u$ up to the boundary. Here is a cute argument. We already know that x_1-derivatives are bounded, since the odd extension $\tilde{\omega}$ is a C^∞ smooth function of x_1. We can also exchange the order of the derivatives. Therefore, it only remains to consider $\partial^2_{x_2}u$. But

$$\partial_{x_2}(\partial_{x_2}u_2) = -\partial_{x_1}(\partial_{x_2}u_1)$$

by the incompressibility condition, and

$$\partial_{x_2}(\partial_{x_2}u_1) = \partial_{x_1}(\partial_{x_2}u_2) - \omega$$

which is again smooth. Continuing this way, we obtain local in time propagation of C^∞ regularity. For the global regularity, we need to carry over the log estimate

$$\|\nabla u\|_{L^\infty} \lesssim \|\omega\|_{L^\infty} \log(10 + \|\omega\|_{C^\alpha})$$

to the case of $\mathbb{R}^2_+$, which can be done similarly as in the $\mathbb{R}^2$ case, once we have the half moon lemma. □

Problem 4.2.2. *Let $\omega = \mathbf{1}_O$ be a smooth vortex patch, namely $O \subset \mathbb{R}^2$ is a bounded open set with C^∞ smooth boundary. Prove that the corresponding velocity $u = \nabla^\perp\Delta^{-1}\omega$ belongs to both $C^\infty(\overline{O})$ and $C^\infty(\mathbb{R}^2\backslash O)$. Show that the same holds if $\omega = f\,\mathbf{1}_O$ where $f \in C^\infty(\mathbb{R}^2)$.*

Problem 4.2.3. *Prove the following "quarter moon lemma," if ω is C^α_c in the quadrant $(\mathbb{R}_+)^2$ and vanishes at the origin $\omega(0,0) = 0$, then we have*

$$\nabla u \in C^\alpha((\mathbb{R}_+)^2).$$

4.2.2 Alpha-SQG case

Let us now consider more singular case than Euler, and describe the work of [81, 138]. Assume that the kernel is given by

$$\psi(x) = \int_{\mathbb{R}^2} h(|x-y|)\ln(|x-y|)\tilde{\omega}(y)dy.$$

Here, $\tilde{\omega}$ is the vorticity defined on $\mathbb{R}^2$ by the odd extension in x_2. We are assuming that the modified kernel is slightly more singular than the Euler one, so we require that $h(r) \to \infty$ as $r \to 0$.

We first perform some explicit computations in the α-SQG case, to get a hint of what happens to the velocity. Namely, we take

$$\psi(x) = \int |x-y|^{-\alpha}\tilde{\omega}(y)dy$$

with $\alpha > 0$ small. Since we are assuming that ω is smooth on the upper half-plane, taking x_1 derivatives are allowed as many time as we want. Now we compute

$$u_1(x) = -\partial_{x_2}\psi(x) = C\int |x-y|^{-\alpha-2}(x_2-y_2)\tilde{\omega}(y)dy.$$

We are not going to precisely track the multiplicative constants which are irrelevant for our purpose here. Here, a trick is to perform an integration by parts after splitting the integral into two regions,

$$\partial_{x_2}\psi = \int \partial_{x_2}(|x-y|^{-\alpha})\tilde{\omega} = \int_{y_2>0} + \int_{y_2<0} =: I + II,$$

where

$$I = \int_{y_2>0} -|x-y|^{-\alpha}\partial_{x_2}\tilde{\omega} + \int_{y_2=0} |x-y|^{-\alpha}\omega(y)dy.$$

A similar computation can be done for the other term II. The first term on the right hand side of the above is smooth, and we shall neglect it. Therefore, we obtain that

$$u_1(x) = C\int_{\mathbb{R}} |x-(y_1,0)|^{-\alpha}\omega(y_1,0)dy_1.$$

Now, instead of taking another x_2 derivative, we first specialize to points $x = (0, x_2)$: when $x_2 = 0$ we have

$$u_1(0,0) = C\int_{\mathbb{R}} \frac{1}{|y_1|^{\alpha}}\omega(y_1,0)dy_1.$$

This is to be compared with

$$u_1(0,z) = C\int_{\mathbb{R}} \frac{1}{(y_1^2+z^2)^{\frac{\alpha}{2}}}\omega(y_1,0)dy_1.$$

We now look at the scaling of the difference $D(z) := u_1(0,z) - u_1(0,0)$:

$$D(z) = C\int_{\mathbb{R}} \left[\frac{1}{(y_1^2+z^2)^{\frac{\alpha}{2}}} - \frac{1}{(y_1^2)^{\frac{\alpha}{2}}}\right]\omega(y_1,0)dy_1.$$

For a fixed z, we make a change of variables $y = z\xi$, so that

$$D(z) = C\int_{\mathbb{R}} z^{1-\alpha}\left[\frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} - \frac{1}{(\xi^2)^{\frac{\alpha}{2}}}\right]\omega(z\xi,0)d\xi.$$

Here, the point is that

$$\begin{aligned}\lim_{z\to 0^+}\frac{D(z)}{z^{1-\alpha}} &= C\lim_{z\to 0^+}\int_{\mathbb{R}} \left[\frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} - \frac{1}{(\xi^2)^{\frac{\alpha}{2}}}\right]\omega(z\xi,0)d\xi \\ &= C\lim_{z\to 0^+}\int_{\mathbb{R}} \left[\frac{1}{(1+\xi^2)^{\frac{\alpha}{2}}} - \frac{1}{(\xi^2)^{\frac{\alpha}{2}}}\right]\omega(0,0)d\xi\end{aligned}$$

by the dominated convergence theorem. Therefore, we obtain the conclusion that in the α SQG case, the velocity is at best $C^{1-\alpha}$ and not better. Advection by such a velocity field will immediately destroy C^1 regularity of ω, unless it is locally a constant. This leads to the following result.

Theorem 4.2.2. *The initial value problem for the α-SQG with any $0 < \alpha \leq 1$ on the upper half-plane is illposed for smooth data. To be more precise, there exist initial data $\theta_0 \in C_c^\infty(\mathbb{R}^2_+)$ such that there is no solution in $L^\infty([0,T]; C^1(\mathbb{R}^2_+))$ for any $T > 0$ corresponding to the initial data θ_0.*

Even when the kernel is only "slightly" more singular than the Euler case, it is expected that the dynamics is still illposed in the upper half-plane in the sense that C^∞ initial data will instantaneously lose C^∞ regularity with time.

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