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Generalized Dirichlet and Thomson Principles and Their Applications

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Introduction

This lecture note is intended to introduce the recently-developed potential theory for the non-reversible Markov processes and to explain applications of this new theory to the study of metastability of huge stochastic interacting systems.

Regarding irreducible Markov processes, it is well-known that the distribution of the process at time $t > 0$ converges to its unique invariant measure as $t \to \infty$, regardless of its starting distribution, and this asymptotic behavior is called the mixing property of Markov processes. The speed of this convergence is one of the main concerns in the study of Markov processes, as it is related to a multitude of important problems such as the performance of Markov chain Monte Carlo algorithm, equilibration of non-equilibrium physical systems, and metastability of random dynamics.

In the study of the mixing property of Markov processes, one of the most useful tools is potential theory, especially the quantity called *capacity* with respect to the Markov process under consideration. Capacity is measured for two disjoint subsets of the state space of the Markov process, and it is inversely related to how well the corresponding Markov process commutes between these two disjoint sets. Since the convergence explained above will take a long time if the Markov process cannot quickly commute between two large (with respect to the invariant measure) sets, capacity is a useful notion in the analysis of mixing properties.

Classic potential theory is developed only when the underlying Markov process is reversible with respect to its invariant measure, and has been widely used in the study of the mixing property of Markov processes (e.g., [11] or [42, Chapters 9, 10]). In the potential theory of reversible Markov processes, the so-called Dirichlet and Thomson principles provide a robust way of estimating the capacity via construction of a test function or a test flow.

Potential theory for non-reversible processes has been developed very recently. In particular, [24] and [57] established the Dirichlet and Thomson principles for non-reversible Markov processes, respectively. These formulae are far more involved than the corresponding principles for the reversible processes, and technical difficulties arise in the application of these principles. To minimize these technical issues, a more generalized version of the Dirichlet and Thomson principles were developed in [37, 56]. In the first part of the current note, we give a comprehensive review on these recent developments in the potential theory of non-reversible Markov processes based on [24, 56, 57].

In the second and third parts of this note, we explain two applications of the recently-developed potential theory to the study of metastability. The metastability is a ubiquitous phenomenon appearing when a Markov process possesses a poor mixing property because of the existence of multiple locally stable sets, or metastable sets. For example, metastability occurs for the models such as

- small random perturbations of dynamical systems (e.g., [14, 15, 23, 37, 41, 44, 45, 46, 55]),
- interacting particle systems with condensing phenomena (e.g., [5, 9, 25, 26, 35, 36, 54, 56]), and
- stochastic spin systems in the low-temperature regime (e.g., [1, 6, 10, 11, 12, 13, 16, 17, 21, 32, 33, 39, 43, 51, 52, 49]).

Readers are referred to monographs [11, 53] for more comprehensive discussions regarding the mathematical study of metastability.

The potential theory plays a crucial role in the rigorous analysis of metastability. In particular, two representative ways of quantitatively analyzing the metastable behavior are the Eyring–Kramers law [22, 29] and Markov chain model reduction [2, 3, 4, 36].

The Eyring–Kramers law describes the precise asymptotics of the mean transition time from a metastable set to other metastable sets. Since such a transition between metastable sets is the signature behavior of metastability, the Eyring–Kramers law is clearly a crucial problem. A robust methodology to prove the Eyring–Kramers law based on the potential theory (known as the potential-theoretic approach) is developed in [14]. We refer to the monograph [11] for a comprehensive review on this approach. *In Part 2, we derive the Eyring–Kramers law for a stochastic spin system known as the Ising model on a large, finite two-dimensional lattice without external field as an application of the potential theory explained in Part 1.* This part is largely based on the recent article [27]. We remark that the article [27] addresses more general situations. This article not only considers the Ising model on a two-dimensional lattice but also the Potts model (which is a generalization of the Ising model) on two- and three-dimensional lattices. In particular, the three-dimensional model is more cumbersome for carrying out rigorous analyses. Moreover, this article not only concerns the Eyring–Kramers law but also the precise analyses of the energy landscape and the typical path of transitions. In this note, we only focus on the Eyring–Kramers law for the two-dimensional model to convey the overall idea. For interested readers, we refer to the article [27] for more comprehensive results.

If there are several metastable sets and the transitions between them take place successively, it is tempting to analyze these successive transitions all at once. A natural way of carrying this out is to approximately describe, after a suitable time-rescaling, the successive transitions between metastable sets as a Markov chain whose state space consists of metastable sets of the original Markov process. This methodology for describing the metastable behavior is a special case of the Markov chain model reduction. A robust methodology for the verification of this Markov chain model reduction based on potential theory has been developed in [2, 3, 4], and this method is called the martingale approach. *In Part 3, we combine this approach and the potential theory for non-reversible processes to analyze the metastable behavior of non-reversible zero-range processes.* This part is largely based on the recent article [56]. For conciseness of the discussion, we only consider the asymmetric nearest neighbor random walk on a cycle, but the discussion given here can be applied to the general model; we refer to [56] for the interested readers.

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Part I

Potential Theory

In the first part, we review the potential theory of continuous-time Markov processes, introduce the Dirichlet and Thomson principles, and then finally explain the generalized Dirichlet and Thomson principles developed in [56]. Although we explain the whole theory in the context of continuous-time Markov processes for the convenience of the discussion, the corresponding results are also valid for discrete-time Markov chains or diffusion processes. For the discussion of diffusion processes, we refer to [37].

sectionPotential Theory of Markov Processes

0.1 Markov processes

We start by introducing several relevant notions regarding a continuous-time Markov process $(X(t))_{t\geq 0}$ on a finite set $\mathcal{H}$.

Continuous-time Markov processes

For $x\in\mathcal{H}$, we denote by $\mathbb{P}_x$ the law of the process $X(\cdot)$ starting from x, and by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$. We assume that the process $X(\cdot)$ is irreducible, in the sense that for all $x,\, y\in\mathcal{H}$[1],

$$\mathbb{P}_x[X(t)=y \text{ for some } t>0]=1\ .$$

We denote by $r:\mathcal{H}\times\mathcal{H}\to[0,\,\infty)$ the jump rate of the Markov process $X(\cdot)$. Namely, for $x,\,y\in S$, the quantity $r(x,\,y)\geq 0$ represents the rate of the jump from x to y for the Markov process $X(\cdot)$. For convenience, we set $r(x,\,x)=0$ for all $x\in\mathcal{H}$. Denote by

$$\lambda(x)=\sum_{y\in\mathcal{H}} r(x,\,y) \quad ;\ x\in\mathcal{H} \tag{0.1}$$

the holding rate of the process $X(\cdot)$ at x. Then, the dynamics $X(\cdot)$ can be described as follows: if $X(t)=x$, then the process waits for an exponential time of mean $\lambda(x)^{-1}$. Then, it jumps to $y\in\mathcal{H}$ with probability $r(x,\,y)/\lambda(x)$.

[1] In this lecture note, writing $a,\,b\in A$ always implies that a and b are *different* elements of a set A.

Embedded chain

We denote by $(\widehat{X}(n))_{n\in\mathbb{Z}^+}$ where $\mathbb{Z}^+ = \mathbb{Z}\cap[0,\,\infty)$ the discrete-time Markov chain with jump probability $p(x,\,y) = r(x,\,y)/\lambda(x)$. This chain is referred to as the embedded chain of $X(\cdot)$, and represents the jumping dynamics (irrespective of the exponential waiting time between successive jumps) of $X(\cdot)$. For $x\in\mathcal{H}$, denote by $\widehat{\mathbb{P}}_x$ the law of the embedded chain $\widehat{X}(\cdot)$ starting from x, and by $\widehat{\mathbb{E}}_x$ the expectation with respect to $\widehat{\mathbb{P}}_x$.

Invariant measure and reversibility

By irreducibility of the process $X(\cdot)$, there exists a unique probability distribution $\mu(\cdot)$ on $\mathcal{H}$ that satisfies

$$\sum_{x\in\mathcal{H}}\mu(x)r(x,\,y) = \sum_{x\in\mathcal{H}}\mu(y)r(y,\,x)\ . \tag{0.2}$$

One can readily infer from the irreducibility that

$$\mu(x) > 0 \text{ for all } x\in\mathcal{H}\ . \tag{0.3}$$

Exercise 0.1. Suppose that the Markov process $X(\cdot)$ is irreducible. Prove that there exists a unique probability distribution $\mu(\cdot)$ on $\mathcal{H}$ satisfying (0.2). Then, prove that this unique $\mu(\cdot)$ satisfies (0.3).

The distribution $\mu(\cdot)$ is called the *invariant (or stationary) distribution* since the marginal distribution of the process $X(\cdot)$ at any later time $t>0$ is μ, provided $X(0)$ is distributed according to μ. We say that the process $X(\cdot)$ is *reversible* if the following detailed balance condition holds:

$$\mu(x)r(x,\,y) = \mu(y)r(y,\,x) \quad \text{for all } x,\,y\in\mathcal{H}\ . \tag{0.4}$$

Note that (0.4) immediately implies (0.2). Such a process is called reversible since the time-reversed process has the same law with the original process. If the process $X(\cdot)$ is not reversible, it is called a non-reversible or irreversible process.

In addition, we can readily check that a measure $M(\cdot)$ on $\mathcal{H}$ given by

$$M(x) = \lambda(x)\mu(x) \quad ;\ x\in\mathcal{H} \tag{0.5}$$

is an invariant measure (not necessarily a probability measure) for the embedded chain $\widehat{X}(\cdot)$. Moreover, the chain $\widehat{X}(\cdot)$ is reversible, i.e., $M(x)p(x,\,y) = M(y)p(y,\,x)$ for all $x,\,y\in\mathcal{H}$, if and only if the original process $X(\cdot)$ is reversible.

Generator and Dirichlet form

The generator $\mathscr{L}$ associated with the process $X(\cdot)$ is an operator acting on each function $f : \mathcal{H} \to \mathbb{R}$ in a way that

$$(\mathscr{L}f)(x) = \sum_{y \in \mathcal{H}} r(x, y)(f(y) - f(x)) \quad ; \; x \in \mathcal{H} \; .$$

Namely, $\mathscr{L}f$ is another real function on $\mathcal{H}$. We denote by $L^2(\mu)$ the L^2 space of real functions on $\mathcal{H}$ with respect to the measure μ. Since $\mathcal{H}$ is a finite set, the space $L^2(\mu)$ is merely a collection of all real functions on $\mathcal{H}$. [2] Denote by $\langle \cdot, \cdot \rangle_\mu$ the inner product on $L^2(\mu)$, i.e., for $f, g : \mathcal{H} \to \mathbb{R}$,

$$\langle f, g \rangle_\mu = \sum_{x \in \mathcal{H}} f(x) g(x) \mu(x) \; .$$

The Dirichlet form associated to the process $X(\cdot)$ is defined by, for $f : \mathcal{H} \to \mathbb{R}$,

$$\mathscr{D}(f) = \langle f, -\mathscr{L}f \rangle_\mu \; . \tag{0.6}$$

This plays an important role in the potential theory. By the summation of parts and (0.2), we can write

$$\mathscr{D}(f) = \frac{1}{2} \sum_{x \in \mathcal{H}} \sum_{y \in \mathcal{H}} \mu(x) r(x, y) [f(y) - f(x)]^2 \; . \tag{0.7}$$

We note that the analyses of the reversible process are far more convenient than those of the non-reversible one, mainly because the operator $\mathscr{L}$ is self-adjoint in the space $L^2(\mu)$ in the sense that, for all $f, g : \mathcal{H} \to \mathbb{R}$,

$$\langle f, \mathscr{L}g \rangle_\mu = \langle \mathscr{L}f, g \rangle_\mu \; .$$

By the summation by parts and (0.4), we can check that both sides of the previous identity equal

$$-\frac{1}{2} \sum_{x \in \mathcal{H}} \sum_{y \in \mathcal{H}} \mu(x) r(x, y) [f(y) - f(x)][g(y) - g(x)] \; .$$

[2] Of course, this is no longer true if we consider the diffusion case.

Adjoint process

For the non-reversible case, we define the *adjoint process* $(X^\dagger(t))_{t\ge 0}$, which is another continuous-time Markov process on $\mathcal{H}$ with rate

$$r^\dagger(x,\,y) = \frac{\mu(y)r(y,\,x)}{\mu(x)} \quad ;\; x,\,y\in\mathcal{H}\;.$$

We shall denote by $\mathbb{P}_x^\dagger$ the law of the adjoint process $X^\dagger(\cdot)$ starting from x, and by $\mathbb{E}_x^\dagger$ the expectation with respect to $\mathbb{P}_x^\dagger$.

The process $X^\dagger(\cdot)$ is a time-reversed process of $X(\cdot)$, and we can notice from (0.4) that $X^\dagger(\cdot)$ is defined by the same law with $X(\cdot)$ in the reversible case; hence the time-reversing does not change the law. We define the generator for the adjoint process $X^\dagger(\cdot)$ as, for $f:\mathcal{H}\to\mathbb{R}$,

$$(\mathscr{L}^\dagger f)(x) = \sum_{y\in\mathcal{H}} r^\dagger(x,\,y)(f(y)-f(x)) \quad ;\; x\in\mathcal{H}\;.$$

The importance of the adjoint process in the context of the potential theory follows from the fact that $\mathscr{L}^\dagger$ is indeed the adjoint operator of $\mathscr{L}$ in the sense that, for all $f,\,g:\mathcal{H}\to\mathbb{R}$,

$$\langle f,\,\mathscr{L}g\rangle_\mu = \left\langle \mathscr{L}^\dagger f,\,g\right\rangle_\mu \;. \tag{0.8}$$

Exercise 0.2.
1. Verify (0.8).
2. Prove that $\langle f,\,\mathscr{L}g\rangle_\mu = 0$ if f is a constant function. In particular, for any $g:\mathcal{H}\to\mathbb{R}$, we have
$$\sum_{x\in\mathcal{H}}\mu(x)(\mathscr{L}g)(x) = 0\;.$$

Remark 0.3. Inserting $g=-f$ at (0.8), we can observe that the Dirichlet form for the adjoint process is also given as $\mathscr{D}(\cdot)$.

We can also consider the embedded chain of the adjoint process. Write $\widehat{X}^\dagger(\cdot)$ the embedded chain with respect to the process $X^\dagger(\cdot)$. One can readily verify that the jump rate $p^\dagger(\cdot,\,\cdot)$ of the chain $\widehat{X}^\dagger(\cdot)$ is given by

$$p^\dagger(x,\,y) = \frac{M(y)p(y,\,x)}{M(x)} \quad ;\; x,\,y\in\mathcal{H}\;, \tag{0.9}$$

and furthermore $M(\cdot)$ is again the invariant measure for the process $\widehat{X}^\dagger(\cdot)$. Similarly, we denote by $\widehat{\mathbb{P}}_x^\dagger$ the law of the process $\widehat{X}^\dagger(\cdot)$ starting at $x\in\mathcal{H}$, and by $\widehat{\mathbb{E}}_x^\dagger$ the expectation with respect to $\widehat{\mathbb{P}}_x^\dagger$.

0.2 Equilibrium potential and capacity

Two crucial notions in the potential theory of Markov processes are the equilibrium potential and the capacity. In this section, we define these objects and review their elementary properties.

Equilibrium potential

For $\mathcal{A} \subset \mathcal{H}$, we denote by $\tau_{\mathcal{A}}$ the hitting time of the set $\mathcal{A}$:

$$\tau_{\mathcal{A}} = \inf\{t \geq 0 : X(t) \in \mathcal{A}\} .$$

For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, we define the equilibrium potential between $\mathcal{A}$ and $\mathcal{B}$ with respect to the process $X(\cdot)$ as a function $h_{\mathcal{A},\,\mathcal{B}} : \mathcal{H} \to [0,\, 1]$ defined by

$$h_{\mathcal{A},\,\mathcal{B}}(x) = \mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] \quad ;\ x \in \mathcal{H} .$$

By definition, it is clear that

$$h_{\mathcal{B},\,\mathcal{A}} = 1 - h_{\mathcal{A},\,\mathcal{B}} . \tag{0.10}$$

The following lemma gives the basic properties of the equilibrium potential $h_{\mathcal{A},\,\mathcal{B}}$.

Lemma 0.4. *For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, the equilibrium potential $h_{\mathcal{A},\,\mathcal{B}}$ satisfies*

$$\begin{cases} h_{\mathcal{A},\,\mathcal{B}} \equiv 1 & \text{on } \mathcal{A}\ , \\ h_{\mathcal{A},\,\mathcal{B}} \equiv 0 & \text{on } \mathcal{B}\ ,\ \text{and} \\ \mathscr{L} h_{\mathcal{A},\,\mathcal{B}} \equiv 0 & \text{on } (\mathcal{A} \cup \mathcal{B})^c = \mathcal{H} \setminus (\mathcal{A} \cup \mathcal{B})\ . \end{cases} \tag{0.11}$$

Proof. The first two properties are evident from the definition of $h_{\mathcal{A},\,\mathcal{B}}$. Let us focus on the last one. Fix $x \in (\mathcal{A} \cup \mathcal{B})^c$. Then, since the process $X(\cdot)$ starting at x jumps to y with probability $r(x,\, y)/\lambda(x)$, by the Markov property we can write

$$h_{\mathcal{A},\,\mathcal{B}}(x) = \mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] = \sum_{y \in \mathcal{H}} \frac{r(x,\, y)}{\lambda(x)} \mathbb{P}_y[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] = \sum_{y \in \mathcal{H}} \frac{r(x,\, y)}{\lambda(x)} h_{\mathcal{A},\,\mathcal{B}}(y) .$$

Multiplying both sides by $\lambda(x)$ and reorganizing give us $\mathscr{L} h_{\mathcal{A},\,\mathcal{B}}(x) = 0$. □

Remark 0.5. Of course, we can define the equilibrium potential $h^{\dagger}_{\mathcal{A},\,\mathcal{B}} : \mathcal{H} \to [0,\, 1]$ with respect to the adjoint process $X^{\dagger}(\cdot)$. Then, an analogue of Lemma 0.4 holds for

$h^{\dagger}_{\mathcal{A},\,\mathcal{B}}$. It suffices to replace the last property of (0.11) with $\mathscr{L}^{\dagger} h^{\dagger}_{\mathcal{A},\,\mathcal{B}} \equiv 0$ on $(\mathcal{A} \cup \mathcal{B})^c$.

Capacity

For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, we define the capacity between $\mathcal{A}$ and $\mathcal{B}$ with respect to the process $X(\cdot)$ as

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}}) \;. \tag{0.12}$$

By the expression (0.7) of the Dirichlet form and (0.10), it holds that

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}}) = \mathscr{D}(h_{\mathcal{B},\,\mathcal{A}}) = \mathrm{cap}(\mathcal{B},\, \mathcal{A}) \;. \tag{0.13}$$

Notation 0.6. If $\mathcal{A} = \{a\}$ or $\mathcal{B} = \{b\}$ (or both), we simply write a or b instead of $\{a\}$ or $\{b\}$, respectively, in the subscript of $h_{\mathcal{A},\,\mathcal{B}}$ and $\mathrm{cap}(\mathcal{A},\, \mathcal{B})$. For instance, if $\mathcal{A} = \{a\}$ and $\mathcal{B} = \{b\}$, we write $h_{a,\,b}$ and $\mathrm{cap}(a,\, b)$, instead of $h_{\{a\},\,\{b\}}$ and $\mathrm{cap}(\{a\},\, \{b\})$, respectively.

Exercise 0.7. Let $\mathcal{H} = \mathbb{T}_N = \mathbb{Z}/(N\mathbb{Z})(= \mathbb{Z}_N)$ be a discrete torus of length N (i.e., a cycle of length N). Define a rate as

$$r(x,\, y) = \begin{cases} p & \text{if } x - y \equiv 1 \pmod N \;, \\ 1-p & \text{if } x - y \equiv -1 \pmod N \;, \\ 0 & \text{otherwise} \;, \end{cases}$$

for some $p \in [0,\, 1]$. For the Markov process $X(\cdot)$ on $\mathbb{T}_N$ with rate $r(\cdot,\, \cdot)$, answer the following questions.

1. Prove that the uniform measure $\mu(\cdot)$ on $\mathbb{T}_N$, namely,

   $$\mu(x) = \frac{1}{N} \quad \text{for all } x \in \mathbb{T}_N \;,$$

   is the unique invariant measure for the process $X(\cdot)$, and moreover that the process $X(\cdot)$ is reversible if and only if $p = 1/2$.

2. For $x,\, y \in \mathbb{T}_N$, compute $\mathrm{cap}(x,\, y)$. (cf. Notation 0.6)

3. For any non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{T}_N$, compute $\mathrm{cap}(\mathcal{A},\, \mathcal{B})$.

Next, we introduce an alternative expression for the capacity that turns out to play an important role in using the capacity in various instances. We write $\tau^{+}_{\mathcal{A}}$ for the return time to the set $\mathcal{A}$:

$$\tau_{\mathcal{A}}^{+} = \inf\{t > 0 : X(t) \in \mathcal{A} \text{ and } X(s) \neq X(0) \text{ for some } s \in [0,\, t]\}\ .$$

Namely, this time expresses the first time at which $X(t)$ arrives at $\mathcal{A}$ after leaving its initial location. In particular, if the process starts from $x \notin \mathcal{A}$, we have $\tau_{\mathcal{A}}^{+} = \tau_{\mathcal{A}}$. Recall the measure $M(\cdot)$ from (0.5).

Lemma 0.8. *For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, it holds that*

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \sum_{x\in\mathcal{A}} M(x)\mathbb{P}_x[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+}]\ .$$

Proof. By (0.6) and (0.12), we can write

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \langle h_{\mathcal{A},\,\mathcal{B}},\, -\mathscr{L}h_{\mathcal{A},\,\mathcal{B}}\rangle_{\mu} = \sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(-\mathscr{L}h_{\mathcal{A},\,\mathcal{B}})(x)\,\mu(x)\ .$$

By (0.11), we have $h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathscr{L}h_{\mathcal{A},\,\mathcal{B}})(x) = 0$ for all $x \notin \mathcal{A}$, and thus we can write

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \sum_{x\in\mathcal{A}} (-\mathscr{L}h_{\mathcal{A},\,\mathcal{B}})(x)\,\mu(x)\ .$$

Note that we used the fact that $h_{\mathcal{A},\,\mathcal{B}} \equiv 1$ on $\mathcal{A}$. By the definition of the generator and (0.10), we can further write

$$\begin{aligned}\mathrm{cap}(\mathcal{A},\, \mathcal{B}) &= \sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{H}} \mu(x)r(x,\, y)[h_{\mathcal{A},\,\mathcal{B}}(x) - h_{\mathcal{A},\,\mathcal{B}}(y)] \\ &= \sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{H}} \mu(x)r(x,\, y)[1 - h_{\mathcal{A},\,\mathcal{B}}(y)] = \sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{H}} \mu(x)r(x,\, y)h_{\mathcal{B},\,\mathcal{A}}(y)\ .\end{aligned} \tag{0.14}$$

On the other hand, by the Markov property, for $x \in \mathcal{A}$ we have

$$\mathbb{P}_x[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+}] = \sum_{y\in\mathcal{H}} p(x,\, y)\mathbb{P}_y[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}] = \sum_{y\in\mathcal{H}} p(x,\, y)h_{\mathcal{B},\,\mathcal{A}}(y)\ . \tag{0.15}$$

Since $\mu(x)r(x,\, y) = M(x)p(x,\, y)$, we can complete the proof from (0.14) and (0.15). $\square$

The capacity with respect to the adjoint process $X^{\dagger}(\cdot)$ is given by (cf. Remark 0.3)

$$\mathrm{cap}^{\dagger}(\mathcal{A},\, \mathcal{B}) = \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}}^{\dagger})\ . \tag{0.16}$$

Then, by the same reasoning as above, it holds that $\mathrm{cap}^\dagger(\mathcal{A},\,\mathcal{B}) = \mathrm{cap}^\dagger(\mathcal{B},\,\mathcal{A})$.

Now, we give two important properties of the capacity based on Lemma 0.8. The first is a somewhat unexpected property in view of the definitions (0.12) and (0.16) of capacities.

Proposition 0.9. *For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, it holds that*

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) = \mathrm{cap}^\dagger(\mathcal{A},\,\mathcal{B})\ .$$

Proof. We first claim that, for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$,

$$M(x)\mathbb{P}_x\left[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+},\, \tau_{\mathcal{B}} = \tau_y\right] = M(y)\mathbb{P}_y^\dagger\left[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}^{+},\, \tau_{\mathcal{A}} = \tau_x\right]\ . \tag{0.17}$$

To prove this, we write the left-hand side as

$$\sum_{T=1}^{\infty} \sum_{(\omega_t)_{t=0}^{T}:\omega_0=x,\,\omega_T=y} M(x) \prod_{t=0}^{T-1} p(\omega_t,\,\omega_{t+1})\ , \tag{0.18}$$

where the summation is carried out for the paths $(\omega_t)_{t=0}^{T}$ such that $p(\omega_t,\,\omega_{t+1}) > 0$ for all $t \in [\![0,\,T-1]\!]$[3] and $\omega_t \notin \mathcal{A} \cup \mathcal{B}$ for all $t \in [\![1,\,T-1]\!]$. By (0.9), we have

$$M(x) \prod_{t=0}^{T-1} p(\omega_t,\,\omega_{t+1}) = M(y) \prod_{t=0}^{T-1} p^\dagger(\omega_{t+1},\,\omega_t)\ .$$

Therefore, we can rewrite (by reversing the path) (0.18) as

$$\sum_{T=1}^{\infty} \sum_{(\omega_t)_{t=0}^{T}:\omega_0=y,\,\omega_T=x} M(y) \prod_{t=0}^{T-1} p^\dagger(\omega_t,\,\omega_{t+1})\ ,$$

where the summation is carried out for the paths $(\omega_t)_{t=0}^{T}$ such that $p^\dagger(\omega_t,\,\omega_{t+1}) > 0$ for all $t \in [\![0,\,T-1]\!]$ and $\omega_t \notin \mathcal{A} \cup \mathcal{B}$ for all $t \in [\![1,\,T-1]\!]$. By the same reasoning as above, this corresponds to the right-hand side of (0.17). Hence, we have proved (0.17).

[3] Here, for integers a and b, $[\![a,\,b]\!]$ denotes $[a,\,b] \cap \mathbb{Z}$.

Therefore, by Lemma 0.8,

$$\begin{aligned}\mathrm{cap}(\mathcal{A},\,\mathcal{B}) &= \sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{B}} M(x)\mathbb{P}_x\left[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+},\, \tau_B = \tau_y\right] \\ &= \sum_{x\in\mathcal{A}}\sum_{y\in\mathcal{B}} M(y)\mathbb{P}_y^{\dagger}\left[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}^{+},\, \tau_{\mathcal{A}} = \tau_x\right] \\ &= \sum_{y\in\mathcal{B}} M(y)\mathbb{P}_y^{\dagger}\left[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}^{+}\right] = \mathrm{cap}^{\dagger}(\mathcal{B},\,\mathcal{A})\ .\end{aligned}$$

Now, it suffices to recall (0.13). □

Proposition 0.10. *Suppose that $\mathcal{A}'$ and $\mathcal{B}'$ are non-empty disjoint subsets of $\mathcal{H}$. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of $\mathcal{A}'$ and $\mathcal{B}'$, respectively. Then, it holds that*

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) \le \mathrm{cap}(\mathcal{A}',\,\mathcal{B}')\ . \tag{0.19}$$

Proof. It suffices to prove that, the capacity is monotone in the second argument, i.e.,

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) \le \mathrm{cap}(\mathcal{A},\,\mathcal{B}')\ , \tag{0.20}$$

since by the symmetry (0.13), we can proceed as

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) \le \mathrm{cap}(\mathcal{A},\,\mathcal{B}') = \mathrm{cap}(\mathcal{B}',\,\mathcal{A}) \le \mathrm{cap}(\mathcal{B}',\,\mathcal{A}') = \mathrm{cap}(\mathcal{A}',\,\mathcal{B}')\ ,$$

provided that we have (0.20).

Now, let us prove (0.20). By Lemma 0.8, it suffices to prove

$$\sum_{x\in\mathcal{A}} M(x)\mathbb{P}_x[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+}] \le \sum_{x\in\mathcal{A}} M(x)\mathbb{P}_x[\tau_{\mathcal{B}'} < \tau_{\mathcal{A}}^{+}]\ .$$

Since $\mathcal{B}\subset\mathcal{B}'$, we trivially have $\mathbb{P}_x[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^{+}] \le \mathbb{P}_x[\tau_{\mathcal{B}'} < \tau_{\mathcal{A}}^{+}]$. □

In the investigation of the mixing property of Markov processes, use of the capacity defined above is crucial, and its (more of less accurate) estimation is required. The definition of the capacity given above is easy to understand, but it is not suitable for the estimation. Instead, the variational expression known as the Dirichlet and Thomson principles are typically used in the estimation of the capacity. The remainder of Part 1 is devoted to explain this strategy.

To explore this advanced strategy to estimate the capacity, we need to reinterpret the capacity in the context of flow structure explained below. We refer to [24, 57, 40] for more comprehensive discussions on the flow structure of Markov processes, and to [37] for the flow structure of diffusion processes.

0.3 Flow structure for reversible case

Since the flow structure is clearer when the Markov process $X(\cdot)$ is reversible, we start with this case. The general case will be treated in the next subsection.

Let us assume throughout this subsection that $X(\cdot)$ is reversible, i.e., (0.4) holds.

For $x,\, y \in \mathcal{H}$, we write $x \sim y$ if $r(x,\, y) > 0$. Since $r(x,\, y) > 0$ if and only if $r(y,\, x) > 0$, we observe that $x \sim y$ if and only if $y \sim x$. Then, we define the set of directed edges by

$$\mathfrak{E} = \{(x,\, y) \in \mathcal{H} \times \mathcal{H} : x \sim y\} \; . \tag{0.21}$$

Note that $(x,\, y) \in \mathfrak{E}$ if and only if $(y,\, x) \in \mathfrak{E}$ by the previous remark.

A function $\phi : \mathfrak{E} \to \mathbb{R}$ is called a flow if it is anti-symmetric, in the sense that

$$\phi(x,\, y) = -\phi(y,\, x) \text{ for all } x,\, y \in \mathcal{H} \; .$$

Here, $\phi(x,\, y)$ is indeed a shorthand of $\phi((x,\, y))$. This is called flow, since the quantity $\phi(x,\, y)$ represents the flux of the flow from site x to y (and hence should be $-\phi(y,\, x)$).

The *divergence* of the flow ϕ at site x is defined by

$$(\operatorname{div} \phi)(x) = \sum_{y : x \sim y} \phi(x,\, y) \; ,$$

and represents the amount of the net flow coming from x. For $\mathcal{A} \subset \mathcal{H}$, define

$$(\operatorname{div} \phi)(\mathcal{A}) = \sum_{x \in \mathcal{A}} (\operatorname{div} \phi)(x) \; .$$

A flow ϕ is called *divergence-free* at $x \in \mathcal{H}$ if $(\operatorname{div} \phi)(x) = 0$, and is called divergence-free on $\mathcal{A} \subset \mathcal{H}$ if $(\operatorname{div} \phi)(x) = 0$ for all $x \in \mathcal{A}$.

Now, we define an L^2-structure on the space of flows. Define the *conductance* between the sites as

$$c(x,\, y) = \mu(x) r(x,\, y) \quad ; \; x,\, y \in \mathcal{H} \; , \tag{0.22}$$

so that $c(x,\, y) = c(y,\, x)$ by (0.4). Denote by $\mathfrak{F}$ the space of flows. For $\phi \in \mathfrak{F}$ and $\psi \in \mathfrak{F}$, define an inner product

$$\langle \phi,\, \psi \rangle_{\mathfrak{F}} = \frac{1}{2} \sum_{(x,\, y) \in \mathfrak{E}} \frac{\phi(x,\, y) \psi(x,\, y)}{c(x,\, y)} \; . \tag{0.23}$$

The *flow norm* of a flow ϕ is naturally defined by $\|\phi\|_{\mathfrak{F}} = \langle \phi,\, \phi \rangle_{\mathfrak{F}}^{1/2}$.

Example. For $f : \mathcal{H} \to \mathbb{R}$, we define a flow Ψ_f as

$$\Psi_f(x,\, y) = c(x,\, y)[f(y) - f(x)] \quad ;\ (x,\, y) \in \mathfrak{E}\ . \tag{0.24}$$

The anti-symmetry, i.e., $\Psi_f(x,\, y) = -\Psi_f(y,\, x)$, is a consequence of (0.4). A crucial feature of this flow is the fact that

$$\|\Psi_f\|_{\mathfrak{F}}^2 = \mathscr{D}(f)\ , \tag{0.25}$$

which follows from (0.7), (0.22), and (0.23). Thus, for any two disjoint and non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, we have

$$\|\Psi_{h_{\mathcal{A},\,\mathcal{B}}}\|_{\mathfrak{F}}^2 = \mathrm{cap}(\mathcal{A},\, \mathcal{B})\ . \tag{0.26}$$

This fact will be critically used later to derive the Thomson principle.

Now, we can observe the following elementary properties.

Proposition 0.11. *With the notations as above, the followings hold.*

1. *For all $f : \mathcal{H} \to \mathbb{R}$ and $x \in \mathcal{H}$,*

$$(\mathrm{div}\, \Psi_f)(x) = \mu(x)\, (\mathscr{L} f)(x)\ .$$

   In particular, for two disjoint non-empty subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{H}$, the flow $\Psi_{h_{\mathcal{A},\,\mathcal{B}}}$ is divergence-free on $(\mathcal{A} \cup \mathcal{B})^c$.

2. *For all $f : \mathcal{H} \to \mathbb{R}$ and $\phi \in \mathfrak{F}$,*

$$\langle \Psi_f,\, \phi \rangle_{\mathfrak{F}} = -\sum_{x \in \mathcal{H}} f(x)\, (\mathrm{div}\, \phi)(x)\ .$$

Proof. The proof follows from elementary computations. For the first assertion of (1),

$$\begin{aligned}(\mathrm{div}\, \Psi_f)(x) &= \sum_{y : x \sim y} \Psi_f(x,\, y) = \sum_{y \in \mathcal{H}} c(x,\, y)[f(y) - f(x)] \\ &= \mu(x) \sum_{y \in \mathcal{H}} r(x,\, y)[f(y) - f(x)] = \mu(x)\, (\mathscr{L} f)(x)\ ,\end{aligned}$$

where the second equality holds since for y such that $x \not\sim y$, we have $c(x,\, y) = 0$. The second assertion of (1) follows directly from (0.11).

For (2), by the definition of Ψ_f,

$$\begin{aligned}\langle \Psi_f,\, \phi\rangle_{\mathfrak{F}} &= \frac{1}{2}\sum_{(x,\, y)\in\mathfrak{E}} \phi(x,\, y)[f(y)-f(x)] = -\sum_{x\in\mathcal{H}}\sum_{y:y\sim x} f(x)\phi(x,\, y)\\ &= -\sum_{x\in\mathcal{H}} f(x)\,(\operatorname{div}\phi)(x)\ .\end{aligned}$$

□

0.4 Flow structure for non-reversible case

Now, we turn to the general case that is developed in [24]. We say that $x \sim y$ if $r(x,\, y)+r(y,\, x) > 0$. Similarly as before, $x\sim y$ if and only if $y \sim x$. With this modified equivalence relationship, we define $\mathfrak{E}$ as in (0.21), and then the flow is defined as anti-symmetric functions on $\mathfrak{E}$. The divergence is also defined in an identical manner.

The difference now appears at the inner product structure. Recall (0.22) and define

$$c^s(x,\, y) = \frac{1}{2}[c(x,\, y)+c(y,\, x)] = \frac{1}{2}[\mu(x)r(x,\, y)+\mu(y)r(y,\, x)]\ ,$$

so that $c^s(x,\, y) = c^s(y,\, x)$. Then, the inner product is defined by

$$\langle \phi,\, \psi\rangle_{\mathfrak{F}} = \frac{1}{2}\sum_{(x,\, y)\in\mathfrak{E}} \frac{\phi(x,\, y)\psi(x,\, y)}{c^s(x,\, y)}\ . \tag{0.27}$$

Note that this definition is in accordance with (0.23) in the reversible case. Then, the flow norm is again defined as $\|\phi\|_{\mathfrak{F}} = \langle\phi,\, \phi\rangle_{\mathfrak{F}}^{1/2}$.

Example. For $f:\mathcal{H}\to\mathbb{R}$, define three flows as

$$\begin{aligned}\Phi_f(x,\, y) &= f(y)c(y,\, x)-f(x)c(x,\, y)\ ,\\ \Phi_f^*(x,\, y) &= f(y)c(x,\, y)-f(x)c(y,\, x)\ ,\\ \Psi_f(x,\, y) &= c^s(x,\, y)\,[f(y)-f(x)] = (1/2)(\Phi_f+\Phi_f^*)(x,\, y)\ .\end{aligned} \tag{0.28}$$

Note that the definition of Ψ_f is in accordance with (0.24), and moreover we have $\Phi_f = \Phi_f^* = \Psi_f$ in the reversible case. We remark that the relations (0.25) and (0.26) are still in force in this case. However, unlike the reversible case, the expression (0.26) for the capacity is not sufficient to derive the Dirichlet and Thomson principles, and hence the flows Φ_f and Φ_f^* have to be crucially used.

We conclude this subsection with the following proposition, which summarizes several elementary properties that will be useful later.

Proposition 0.12. *With the notations as above, the followings hold.*

1. *For all $f : \mathcal{H} \to \mathbb{R}$ and $x \in \mathcal{H}$,*

$$(\operatorname{div} \Phi_f)(x) = \mu(x)\,(\mathscr{L}^\dagger f)(x) \ \ \text{and} \ \ (\operatorname{div} \Phi_f^*)(x) = \mu(x)\,(\mathscr{L} f)(x) \ .$$

In particular, for two disjoint non-empty subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{H}$, the flows $\Phi_{h^\dagger_{\mathcal{A},\,\mathcal{B}}}$ and $\Phi^*_{h_{\mathcal{A},\,\mathcal{B}}}$ are divergence-free on $(\mathcal{A} \cup \mathcal{B})^c$.

2. *For all $f : \mathcal{H} \to \mathbb{R}$ and $\phi \in \mathfrak{F}$,*

$$\langle \Psi_f,\, \phi \rangle_{\mathfrak{F}} = -\sum_{x \in \mathcal{H}} f(x)\,(\operatorname{div} \phi)(x) \ .$$

3. *For all $f,\, g : \mathcal{H} \to \mathbb{R}$,*

$$\langle \Psi_f,\, \Phi_g \rangle_{\mathfrak{F}} = \langle -\mathscr{L} f,\, g \rangle_\mu \ \ \text{and} \ \ \langle \Psi_f,\, \Phi_g^* \rangle_{\mathfrak{F}} = \left\langle -\mathscr{L}^\dagger f,\, g \right\rangle_\mu \ .$$

Proof. Proofs of (1) and (2) are similar to those of Proposition 0.11 and hence are left to the readers. For (3), we first consider $\langle \Psi_f,\, \Phi_g \rangle_{\mathfrak{F}}$. By part (2), we can write

$$\langle \Psi_f,\, \Phi_g \rangle_{\mathfrak{F}} = -\sum_{x \in \mathcal{H}} f(x)\,(\operatorname{div} \Phi_g)(x) \ .$$

Applying part (1), we get

$$\langle \Psi_f,\, \Phi_g \rangle_{\mathfrak{F}} = -\sum_{x \in \mathcal{H}} f(x)\,[\mu(x)\,(\mathscr{L}^\dagger g)(x)] = \left\langle f,\, -\mathscr{L}^\dagger g \right\rangle_\mu \ .$$

Now the proof is completed by recalling (0.8). The proof for $\left\langle \Psi_f,\, \Phi_g^* \right\rangle_{\mathfrak{F}}$ is identical and will be omitted. □

0.5 Application of potential theory: an example

Before proceeding further regarding variational expression of the capacity, we explain an application of the potential theory in the estimate of expected hitting time or related quantities (see discussions after Proposition 0.14).

We fix two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$ throughout this subsection. We define the so-called equilibrium measure between $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{A}$ with respect to the process $X(\cdot)$ as

$$\nu_{\mathcal{A},\,\mathcal{B}}(x) = \frac{M(x)\,\mathbb{P}_x[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}^+]}{\operatorname{cap}(\mathcal{A},\,\mathcal{B})} \quad ;\ x \in \mathcal{A} \ .$$

By Lemma 0.8, $\nu_{\mathcal{A},\,\mathcal{B}}(\cdot)$ is a probability measure on $\mathcal{A}$. Similarly, we can define the equilibrium measure $\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}(\cdot)$ with respect to the adjoint process $X^{\dagger}(\cdot)$:

$$\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}(x) = \frac{M(x)\,\mathbb{P}^{\dagger}_x[\tau_{\mathcal{B}} < \tau^{+}_{\mathcal{A}}]}{\mathrm{cap}(\mathcal{A},\,\mathcal{B})} \quad ;\ x \in \mathcal{A}\ , \tag{0.29}$$

where $M(x)$ and $\mathrm{cap}(\mathcal{A},\,\mathcal{B})$ are not changed since $M(\cdot)$ is still the invariant measure for the embedded chain of the adjoint process and since Proposition 0.9, respectively.

Remark 0.13.

1. Define the boundary $\partial\mathcal{A}$ of $\mathcal{A}$ as

$$\partial\mathcal{A} = \{x \in \mathcal{A} : r(x,\,y) > 0 \text{ for some } y \notin \mathcal{A}\}\ .$$

   Note that we have $\mathbb{P}_x[\tau_{\mathcal{B}} < \tau^{+}_{\mathcal{A}}] = 0$ for $x \in \mathcal{A} \setminus \partial\mathcal{A}$. Hence, the measure $\nu_{\mathcal{A},\,\mathcal{B}}$ (as well as $\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}$) is concentrated on the boundary $\partial\mathcal{A}$.

2. If $\mathcal{A} = \{a\}$ is a singleton, the measure $\nu_{\mathcal{A},\,\mathcal{B}}$ is merely the Dirac measure on $\{a\}$.

For a probability measure π on $\mathcal{H}$, denote by $\mathbb{P}_\pi$ the law of the process $X(\cdot)$ when $X(0)$ is distributed according to π, and by $\mathbb{E}_\pi$ the associated expectation. The following proposition is the main result of the current subsection.

Proposition 0.14. *For any $f : \mathcal{H} \to \mathbb{R}$, we have that*

$$\mathbb{E}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}}\left[\int_0^{\tau_{\mathcal{B}}} f(X(t))dt\right] = \frac{\left\langle f,\, h^{\dagger}_{\mathcal{A},\,\mathcal{B}}\right\rangle_\mu}{\mathrm{cap}(\mathcal{A},\,\mathcal{B})}\ . \tag{0.30}$$

Before proving this proposition, we explain several direct applications of this proposition. First, we take $f \equiv 1$ to deduce

$$\mathbb{E}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}}[\tau_{\mathcal{B}}] = \frac{\sum_{x\in\mathcal{H}} h^{\dagger}_{\mathcal{A},\,\mathcal{B}}(x)\mu(x)}{\mathrm{cap}(\mathcal{A},\,\mathcal{B})}\ . \tag{0.31}$$

Moreover, by taking $\mathcal{A} = \{z\}$, the left-hand side becomes the mean hitting time $\mathbb{E}_z[\tau_{\mathcal{B}}]$ (cf. Remark 0.13-(2)), and thus we obtain

$$\mathbb{E}_z[\tau_{\mathcal{B}}] = \frac{\sum_{x\in\mathcal{H}} h^{\dagger}_{z,\,\mathcal{B}}(x)\mu(x)}{\mathrm{cap}(z,\,\mathcal{B})}\ . \tag{0.32}$$

Note from (0.11) that

$$\sum_{x\in\mathcal{H}} h^\dagger_{\mathcal{A},\,\mathcal{B}}(x)\mu(x) = \mu(\mathcal{A}) + \sum_{x\in(\mathcal{A}\cup\mathcal{B})^c} h^\dagger_{\mathcal{A},\,\mathcal{B}}(x)\mu(x) \le \mu(\mathcal{A}) + \mu((\mathcal{A}\cup\mathcal{B})^c) = 1-\mu(\mathcal{B})\ . \tag{0.33}$$

Hence, by deriving a lower bound on $\mathrm{cap}(\mathcal{A},\,\mathcal{B})$, we can obtain an upper bound on the expectation $\mathbb{E}_{\nu^\dagger_{\mathcal{A},\,\mathcal{B}}}[\tau_{\mathcal{B}}]$ of the hitting time from (0.31). In the next two sections, we will discuss how to get a lower and an upper bound on $\mathrm{cap}(\mathcal{A},\,\mathcal{B})$. Of course, in the real application, we may need more refined estimates than (0.33) by studying the equilibrium potential.

Next, by taking $f=\mathbf{1}_{\mathcal{C}}$ for some $\mathcal{C}\subset\mathcal{H}\setminus\mathcal{B}$, the previous proposition becomes

$$\mathbb{E}_{\nu^\dagger_{\mathcal{A},\,\mathcal{B}}}\left[\int_0^{\tau_{\mathcal{B}}}\mathbf{1}_{\mathcal{C}}(X(t))dt\right] = \frac{\sum_{x\in\mathcal{C}} h^\dagger_{\mathcal{A},\,\mathcal{B}}(x)\mu(x)}{\mathrm{cap}(\mathcal{A},\,\mathcal{B})}\ .$$

The left-hand side now measures the amount of time the process spends on $\mathcal{C}$ before arriving at $\mathcal{B}$. For this setting, the numerator of the right-hand side can be trivially bounded from above by $\mu(\mathcal{C})$ since $h^\dagger_{\mathcal{A},\,\mathcal{B}}\le 1$. Now, let us return to Proposition 0.14. The following is from the arguments given in [2, Proof of Proposition 6.10] and [3, Proof of Proposition A.2].

Proof of Proposition 0.14. It suffices to prove the proposition when $f=\mathbf{1}_{\{z\}}$ for all $z\in\mathcal{H}$. Let us fix $z\in\mathcal{H}$. If $z\in\mathcal{B}$, both sides of (0.30) are trivially 0, and hence we can assume $z\in\mathcal{H}\setminus\mathcal{B}$.

Since the embedded chain is obtained from the original Markov process via the time changing, we have

$$h^\dagger_{\mathcal{A},\,\mathcal{B}}(z) = \mathbb{P}^\dagger_z[\tau_{\mathcal{A}}<\tau_{\mathcal{B}}] = \widehat{\mathbb{P}}^\dagger_z[\widehat{\tau}_{\mathcal{A}}<\widehat{\tau}_{\mathcal{B}}]\ ,$$

where the hitting time and the return time appearing on the right-hand side are computed with respect to the process $\widehat{X}^\dagger(\cdot)$. Write

$$L_{\mathcal{A},\,\mathcal{B}} = \sup\{n\ge 0 : \widehat{X}^\dagger(n)\in\mathcal{A} \text{ and } n<\widehat{\tau}_{\mathcal{B}}\}\ ,$$

where we use the convention that $\sup\emptyset=-\infty$. With these notations, we can rewrite

$h^{\dagger}_{\mathcal{A},\,\mathcal{B}}(z)$ as

$$\begin{aligned} h^{\dagger}_{\mathcal{A},\,\mathcal{B}}(z) &= \widehat{\mathbb{P}}^{\dagger}_{z}[\widehat{\tau}_{\mathcal{A}} < \widehat{\tau}_{\mathcal{B}}] = \sum_{n=0}^{\infty} \widehat{\mathbb{P}}^{\dagger}_{z}[L_{\mathcal{A},\,\mathcal{B}} = n] \\ &= \sum_{n=0}^{\infty} \sum_{y\in\mathcal{A}} \widehat{\mathbb{P}}^{\dagger}_{z}[\widehat{X}^{\dagger}(n) = y,\, n < \widehat{\tau}_{\mathcal{B}}]\, \widehat{\mathbb{P}}^{\dagger}_{y}[\widehat{\tau}_{\mathcal{B}} < \widehat{\tau}^{+}_{\mathcal{A}}] \\ &= \sum_{y\in\mathcal{A}} \left[\widehat{\mathbb{P}}^{\dagger}_{y}[\widehat{\tau}_{\mathcal{B}} < \widehat{\tau}^{+}_{\mathcal{A}}] \sum_{n=0}^{\infty} \widehat{\mathbb{P}}^{\dagger}_{z}[\widehat{X}^{\dagger}(n) = y,\, n < \widehat{\tau}_{\mathcal{B}}] \right] , \end{aligned} \tag{0.34}$$

where the second equality follows from the Markov property.

For $u,\, v \in \mathcal{H} \setminus \mathcal{B}$ and $n \ge 0$, denote by $P(u,\, v; n,\, \mathcal{B})$ the collection of paths $(w_0,\, w_1,\, \cdots,\, w_n)$ such that $w_0 = u$, $w_n = v$, and $w_i \notin \mathcal{B}$ for all $0 \le i \le n$. Note that

$$(w_0,\, w_1,\, \cdots,\, w_n) \in P(u,\, v; n,\, \mathcal{B}) \quad \text{if and only if} \quad (w_n,\, w_{n-1},\, \cdots,\, w_0) \in P(v,\, u; n,\, \mathcal{B}) \,. \tag{0.35}$$

With this notation (noting that we assumed $z \notin \mathcal{B}$), we can write

$$M(z)\widehat{\mathbb{P}}^{\dagger}_{z}[\widehat{X}^{\dagger}(n) = y,\, n < \widehat{\tau}_{\mathcal{B}}] = \sum_{(w_0,\, w_1,\, \cdots,\, w_n)\in P(z,\, y; n,\, \mathcal{B})} \sum_{i=0}^{n-1} M(w_i) p^{\dagger}(w_i,\, w_{i+1}) \,. \tag{0.36}$$

Hence, we can deduce from (0.9), (0.35), and (0.36) that if $y \in A$, then

$$\begin{aligned} M(z)\widehat{\mathbb{P}}^{\dagger}_{z}[\widehat{X}^{\dagger}(n) = y,\, n < \widehat{\tau}_{\mathcal{B}}] &= \sum_{(w_0,\, w_1,\, \cdots,\, w_n)\in P(z,\, y; n,\, \mathcal{B})} \sum_{i=0}^{n-1} M(w_{i+1}) p(w_{i+1},\, w_i) \\ &= \sum_{(w'_0,\, w'_1,\, \cdots,\, w'_n)\in P(y,\, z; n,\, \mathcal{B})} \sum_{i=0}^{n-1} M(w'_i) p(w'_i,\, w'_{i+1}) \\ &= M(y)\widehat{\mathbb{P}}_{y}[\widehat{X}(n) = z,\, n < \widehat{\tau}_{\mathcal{B}}] \,, \end{aligned}$$

Inserting this into (0.34), we get

$$\begin{aligned} h^{\dagger}_{\mathcal{A},\,\mathcal{B}}(z) &= \sum_{y\in\mathcal{A}} \left[\frac{M(y)}{M(z)} \widehat{\mathbb{P}}^{\dagger}_{y}[\widehat{\tau}_{\mathcal{B}} < \widehat{\tau}^{+}_{\mathcal{A}}] \sum_{n=0}^{\infty} \widehat{\mathbb{P}}_{y}[\widehat{X}(n) = z,\, n < \widehat{\tau}_{\mathcal{B}}] \right] \\ &= \frac{\mathrm{cap}(\mathcal{A},\,\mathcal{B})}{M(z)} \sum_{y\in\mathcal{A}} \left[\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}(y) \sum_{n=0}^{\widehat{\tau}_{\mathcal{B}}-1} \widehat{\mathbb{P}}_{y}[\widehat{X}(n) = z] \right] \\ &= \frac{\mathrm{cap}(\mathcal{A},\,\mathcal{B})}{M(z)} \widehat{\mathbb{E}}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}} \left[\sum_{n=0}^{\widehat{\tau}_{\mathcal{B}}-1} \mathbf{1}\{\widehat{X}(n) = z\} \right] , \end{aligned} \tag{0.37}$$

where the second equality follows from the explicit formula (0.29), while the last equality follows from the Fubini theorem. Since if the original chain $X(\cdot)$ arrives at z, then it spends mean $\lambda(z)^{-1}$ exponential random time there, and hence we can conclude that

$$\mathbb{E}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}} \left[\int_0^{\tau_B} \mathbf{1}_{\{z\}}(X(t))dt \right] = \frac{1}{\lambda(z)} \widehat{\mathbb{E}}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}} \left[\sum_{n=0}^{\widehat{\tau}_{\mathcal{B}}-1} \mathbf{1}\{\widehat{X}(n) = z\} \right] .$$

Inserting this to (0.9), we get

$$\mathbb{E}_{\nu^{\dagger}_{\mathcal{A},\,\mathcal{B}}} \left[\int_0^{\tau_B} \mathbf{1}_{\{z\}}(X(t))dt \right] = \frac{\mu(z) h^{\dagger}_{\mathcal{A},\mathcal{B}}(z)}{\mathrm{cap}(\mathcal{A},\,\mathcal{B})} .$$

This completes the proof of proposition when $f = \mathbf{1}_{\{z\}}$, and we are done. □

Remark 0.15. One can expect that a quantity such as $\mathbb{E}_a[\tau_{\mathcal{B}}]$ is closely related with the mixing of the Markov process $X(\cdot)$. This relation has been explained in [42, Chapters 9 and 10]. The potential-theoretic notions are closely connected with the mixing of Markov chains.

0.6 Bound on equilibrium potential via capacities

Let us again fix two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$. We know that $h_{\mathcal{A},\,\mathcal{B}} \equiv 1$ on $\mathcal{A}$ and $h_{\mathcal{A},\,\mathcal{B}} \equiv 0$ on $\mathcal{B}$, but the value of $h_{\mathcal{A},\,\mathcal{B}}$ on $(\mathcal{A}\cup\mathcal{B})^c$ is described only in terms of the Laplace equation (cf. (0.11)), and hence the exact value is almost impossible to compute in most applications. However, in many instances, we need to bound the value of $h_{\mathcal{A},\,\mathcal{B}}$ on $(\mathcal{A}\cup\mathcal{B})^c$ to carry out an estimation. For example, with such a bound, we can carry out a much better estimate in (0.33).

In this subsection, we present the following useful upper bound on the value of $h_{\mathcal{A},\,\mathcal{B}}$ on $(\mathcal{A}\cup\mathcal{B})^c$ in terms of capacities. This bound will be frequently used in various situations. The following proof is an excerpt from [32, Section 3].

Proposition 0.16. *We have that*

$$h_{\mathcal{A},\,\mathcal{B}}(x) \le \frac{\mathrm{cap}(x,\,\mathcal{A})}{\mathrm{cap}(x,\,\mathcal{A}\cup\mathcal{B})} \quad \textit{for all } x \in (\mathcal{A}\cup\mathcal{B})^c.$$

Proof. Fix $x \in (\mathcal{A}\cup\mathcal{B})^c$. By the strong Markov property, we can write

$$\begin{aligned}\mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] &= \mathbb{P}_x[\tau_x^+ < \tau_{\mathcal{A}\cup\mathcal{B}},\, \tau_{\mathcal{A}} < \tau_{\mathcal{B}}] + \mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}\cup\mathcal{B}},\, \tau_{\mathcal{A}} < \tau_{\mathcal{B}}] \\ &= \mathbb{P}_x[\tau_x^+ < \tau_{\mathcal{A}\cup\mathcal{B}}]\mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] + \mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}} < \tau_x^+]\ .\end{aligned}$$

Therefore, we have that

$$\mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}] = \frac{\mathbb{P}_x[\tau_{\mathcal{A}} < \tau_{\mathcal{B}} < \tau_x^+]}{\mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}\cup\mathcal{B}}]} \le \frac{\mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}}]}{\mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}\cup\mathcal{B}}]}\ .$$

The proof is completed since by Lemma 0.8,

$$\begin{aligned}\mathrm{cap}(x,\,\mathcal{A}) &= M(x)\mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}}] \quad \text{and} \\ \mathrm{cap}(x,\,\mathcal{A}\cup\mathcal{B}) &= M(x)\mathbb{P}_x[\tau_x^+ > \tau_{\mathcal{A}\cup\mathcal{B}}]\ .\end{aligned}$$

□

We note that, in view of Proposition 0.9, the same result holds for $h^{\dagger}_{\mathcal{A},\,\mathcal{B}}(x)$ in place of $h_{\mathcal{A},\,\mathcal{B}}(x)$. In addition, the bound obtained in the previous proposition is particularly useful since there are numerous robust tools to estimate capacities. We discuss such robust tools in the following sections.

1 Dirichlet and Thomson Principles

In the application of the potential theory, it is important to (more or less precisely) estimate the capacity. Classic tools for this purpose are the Dirichlet and Thomson principles that we introduce in this section.

Let us fix two disjoint and non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ throughout the section. Then, we explain strategies to estimate the capacity $\mathrm{cap}(\mathcal{A},\, \mathcal{B})$.

1.1 Spaces of functions and flows

To explain the variational principles for capacities, we need to define classes of functions and flows as follows:

- For real numbers a and b, denote by $\mathfrak{C}_{a,\,b}(\mathcal{A},\, \mathcal{B})$ the set of all real-valued functions f on $\mathcal{H}$ satisfying $f|_{\mathcal{A}} \equiv a$ and $f|_{\mathcal{B}} \equiv b$, i.e.,

$$\mathfrak{C}_{a,\,b}(\mathcal{A},\, \mathcal{B}) = \{f : \mathcal{H} \to \mathbb{R} : f(x) = a,\, \forall x \in \mathcal{A} \text{ and } f(x) = b,\, \forall x \in \mathcal{B}\} \ .$$

- For $a \in \mathbb{R}$, let $\mathfrak{U}_a(\mathcal{A},\, \mathcal{B})$ be the set of all flows $\phi \in \mathfrak{F}$ which are divergence free on $(\mathcal{A} \cup \mathcal{B})^c$, i.e.,

$$(\mathrm{div}\, \phi)(x) = 0 \text{ for all } x \in (\mathcal{A} \cup \mathcal{B})^c \ ,$$

  and satisfy

$$(\mathrm{div}\, \phi)(\mathcal{A}) = -(\mathrm{div}\, \phi)(\mathcal{B}) = a \ .$$

  In particular, a flow belonging to $\mathfrak{U}_1$ is called a *unit flow*.

Example 1.1. The equilibrium potential $h_{\mathcal{A},\,\mathcal{B}}$ belongs to the class $\mathfrak{C}_{1,\,0}(\mathcal{A},\, \mathcal{B})$.

Exercise 1.2. 1. Suppose that the process $X(\cdot)$ is reversible. Prove that the flow

$$\psi_{\mathcal{A},\,\mathcal{B}} := -\frac{1}{\mathrm{cap}(\mathcal{A},\, \mathcal{B})} \Psi_{h_{\mathcal{A},\,\mathcal{B}}} \tag{1.1}$$

is a unit flow between $\mathcal{A}$ and $\mathcal{B}$. (Hint: Proposition 0.11-(1))

2. Suppose that the process $X(\cdot)$ is non-reversible. Prove that the flows

$$\phi_{\mathcal{A},\,\mathcal{B}} := -\frac{1}{\mathrm{cap}(\mathcal{A},\, \mathcal{B})} \Phi_{h^{\dagger}_{\mathcal{A},\,\mathcal{B}}} \quad \text{and} \quad \phi^*_{\mathcal{A},\,\mathcal{B}} := -\frac{1}{\mathrm{cap}(\mathcal{A},\, \mathcal{B})} \Phi^*_{h_{\mathcal{A},\,\mathcal{B}}}$$

are unit flows between $\mathcal{A}$ and $\mathcal{B}$. (Hint: Proposition 0.12-(1))

1.2 Dirichlet and Thomson principles: reversible case

We begin with the Dirichlet and Thomson principles for reversible Markov processes. Hence, in this subsection, we temporarily assume that the process $X(\cdot)$ is reversible.

The Dirichlet principle provides a minimization problem for the capacity.

Theorem 1.3 (Dirichlet principle for reversible Markov processes)**.** *We have that*

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) = \inf_{f\in\mathfrak{C}_{1,\,0}(\mathcal{A},\,\mathcal{B})} \mathscr{D}(f)\ ,$$

and the unique minimizer is given by $f = h_{\mathcal{A},\,\mathcal{B}}$.[4]

Proof. Let $f \in \mathfrak{C}_{1,\,0}(\mathcal{A},\,\mathcal{B})$. Then, write $g = f - h_{\mathcal{A},\,\mathcal{B}}$ so that $g \in \mathfrak{C}_{0,\,0}(\mathcal{A},\,\mathcal{B})$. Then,

$$\begin{aligned}\mathscr{D}(f) &= \langle h_{\mathcal{A},\,\mathcal{B}} + g,\, -\mathscr{L}(h_{\mathcal{A},\,\mathcal{B}} + g)\rangle_\mu \\ &= \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}}) + \mathscr{D}(g) - 2\,\langle g,\, \mathscr{L}h_{\mathcal{A},\,\mathcal{B}}\rangle_\mu\ ,\end{aligned}$$

where at the second equality we used the reversibility which implies the self-adjointness of $\mathscr{L}$. Since $g \equiv 0$ on $\mathcal{A}\cup\mathcal{B}$, and since $\mathscr{L}h_{\mathcal{A},\,\mathcal{B}} \equiv 0$ on $(\mathcal{A}\cup\mathcal{B})^c$, we get $\langle g,\, \mathscr{L}h_{\mathcal{A},\,\mathcal{B}}\rangle_\mu = 0$. Therefore,

$$\mathscr{D}(f) = \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}}) + \mathscr{D}(g) \ge \mathscr{D}(h_{\mathcal{A},\,\mathcal{B}})\ , \tag{1.2}$$

and the equality holds only when $\mathscr{D}(g) = 0$, i.e., when g is a constant function. Since $g \in \mathfrak{C}_{0,\,0}(\mathcal{A},\,\mathcal{B})$, g must be the zero function to obtain the equality in (1.2). This completes the proof. □

On the other hand, the Thomson principle provides a maximization problem for the capacity.

Theorem 1.4 (Thomson principle for reversible Markov processes)**.** *We have that*

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) = \sup_{\phi\in\mathfrak{U}_1(\mathcal{A},\,\mathcal{B})} \frac{1}{\|\phi\|_{\mathfrak{F}}^2}\ ,$$

and the unique maximizer is given by $\phi = \psi_{\mathcal{A},\,\mathcal{B}}$ (cf. (1.1)).

Proof. Let $\phi \in \mathfrak{U}_1(\mathcal{A},\,\mathcal{B})$. By Proposition 0.11, we have

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \phi\right\rangle_{\mathfrak{F}} = -\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x) = -\sum_{x\in\mathcal{A}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\ ,$$

[4] Note that $h_{\mathcal{A},\,\mathcal{B}} \in \mathfrak{C}_{1,\,0}(\mathcal{A},\,\mathcal{B})$ as we observed in Example 1.1.

where the second equality holds since $h_{\mathcal{A},\mathcal{B}} \equiv 0$ on $\mathcal{B}$ and $\operatorname{div}\phi \equiv 0$ on $(\mathcal{A}\cup\mathcal{B})^c$. Since $h_{\mathcal{A},\mathcal{B}} \equiv 1$ on $\mathcal{A}$ and since $(\operatorname{div}\phi)(\mathcal{A}) = 1$, we can conclude that

$$\left\langle \Psi_{h_{\mathcal{A},\mathcal{B}}},\, \phi \right\rangle_{\mathfrak{F}} = -1\ .$$

By the Cauchy–Schwarz inequality and (0.26),

$$1 = \left\langle \Psi_{h_{\mathcal{A},\mathcal{B}}},\, \phi \right\rangle_{\mathfrak{F}}^2 \le \|\Psi_{h_{\mathcal{A},\mathcal{B}}}\|_{\mathfrak{F}}^2\, \|\phi\|_{\mathfrak{F}}^2 = \operatorname{cap}(\mathcal{A},\,\mathcal{B})\, \|\phi\|_{\mathfrak{F}}^2\ .$$

This proves $\operatorname{cap}(\mathcal{A},\,\mathcal{B}) \ge \frac{1}{\|\phi\|_{\mathfrak{F}}^2}$. Since the equality of the previous Cauchy–Schwarz inequality holds only when $\phi = c\Psi_{h_{\mathcal{A},\mathcal{B}}}$ for some $c \in \mathbb{R}$, we must have $\phi = \psi_{\mathcal{A},\mathcal{B}}$ since ϕ is a unit flow. □

Remark 1.5. At this point, it is now clear how to use the Dirichlet and Thomson principles to estimate the capacity. If we take any test function $f \in \mathfrak{C}_{1,0}(\mathcal{A},\,\mathcal{B})$ and any test flow $\phi \in \mathfrak{U}_1(\mathcal{A},\,\mathcal{B})$, we can deduce from Theorems 1.3 and 1.4 that

$$\frac{1}{\|\phi\|_{\mathfrak{F}}^2} \le \operatorname{cap}(\mathcal{A},\,\mathcal{B}) \le \mathscr{D}(f)\ .$$

If one wants these lower and upper bounds to be sharp, it is necessary to take f and ϕ as objects close to the genuine optimizers, namely, as $f \approx h_{\mathcal{A},\mathcal{B}}$ and $\phi \approx \psi_{\mathcal{A},\mathcal{B}}$. For a concrete example of such a construction, we refer to [38].

We note that there is no special technical difficulty in finding such a test function. On the other hand, constructing an appropriate test flow is fundamentally more difficult, since the object that we constructed as a test flow must satisfy the divergence-free condition on $(\mathcal{A}\cup\mathcal{B})^c$, and there is no trivial way of defining such an object. This issue will be discussed in more detail in the next section.

Remark 1.6. In the reversible case, there is an alternative way, based on a Cauchy–Schwarz-type argument, of obtaining a lower bound for the capacity without relying on the Thomson principle. More precisely, if we are able to prove that $\mathscr{D}(f)$ is bounded below by a constant c for all $f \in \mathfrak{C}_{1,0}(\mathcal{A},\,\mathcal{B})$ via the Cauchy–Schwarz inequalities, then by the Dirichlet principle we have the lower bound $\mathscr{D}(f) \ge c$. This bound can be sharp if we apply the inequalities in a careful manner. We refer to [5, 9, 14] for examples of this method. This method is difficult to use when the underlying energy landscape is complicated.

1.3 Dirichlet and Thomson principles: non-reversible case

The Dirichlet and Thomson principles were known only for the reversible case, but recently the corresponding principles for the non-reversible case have been revealed.

The following theorem is a summary of these results. We no longer assume that the process $X(\cdot)$ is reversible.

Theorem 1.7. *The following variational expressions for the capacity hold.*

1. *It holds that*

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \inf_{f\in\mathfrak{C}_{1,\,0}(\mathcal{A},\mathcal{B}),\, \phi\in\mathfrak{U}_0(\mathcal{A},\mathcal{B})} \|\Phi_f - \phi\|^2 \;, \tag{1.3}$$

and the unique minimizer is given by

$$(f,\, \phi) = \left(\frac{h_{\mathcal{A},\mathcal{B}} + h^{\dagger}_{\mathcal{A},\mathcal{B}}}{2},\, -\frac{\Phi_{h^{\dagger}_{\mathcal{A},\mathcal{B}}} - \Phi^{*}_{h_{\mathcal{A},\mathcal{B}}}}{2} \right) \;, \tag{1.4}$$

2. *It holds that*

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \sup_{g\in\mathfrak{C}_{0,\,0}(\mathcal{A},\mathcal{B}),\, \psi\in\mathfrak{U}_1(\mathcal{A},\mathcal{B})} \frac{1}{\|\Phi_g - \psi\|^2} \;. \tag{1.5}$$

and the unique maximizer is given by

$$(g,\, \psi) = \left(\frac{h^{\dagger}_{\mathcal{A},\mathcal{B}} - h_{\mathcal{A},\mathcal{B}}}{2\mathrm{cap}(\mathcal{A},\, \mathcal{B})},\, -\frac{\Phi_{h^{\dagger}_{\mathcal{A},\mathcal{B}}} + \Phi^{*}_{h_{\mathcal{A},\mathcal{B}}}}{2\mathrm{cap}(\mathcal{A},\, \mathcal{B})} \right) \;. \tag{1.6}$$

In the previous theorem, the Dirichlet principle (1.3) and the Thomson principle (1.5) were established in [24] and [57], respectively. Note also that

$$-\frac{\Phi_{h^{\dagger}_{\mathcal{A},\mathcal{B}}} - \Phi^{*}_{h_{\mathcal{A},\mathcal{B}}}}{2} \in \mathfrak{U}_0(\mathcal{A},\, \mathcal{B}) \quad \text{and} \quad -\frac{\Phi_{h^{\dagger}_{\mathcal{A},\mathcal{B}}} + \Phi^{*}_{h_{\mathcal{A},\mathcal{B}}}}{2\mathrm{cap}(\mathcal{A},\, \mathcal{B})} \in \mathfrak{U}_1(\mathcal{A},\, \mathcal{B})$$

follows from Example 1.2. We now turn to the proof.

Proof. Let $f \in \mathfrak{C}_{a,\,0}(\mathcal{A},\, \mathcal{B})$. By Proposition 0.12-(3), we have that

$$\left\langle \Psi_{h_{\mathcal{A},\mathcal{B}}},\, \Phi_f \right\rangle_{\mathfrak{F}} = \langle -\mathscr{L} h_{\mathcal{A},\mathcal{B}},\, f \rangle_{\mu} = \sum_{x\in\mathcal{H}} f(x)(-\mathscr{L} h_{\mathcal{A},\mathcal{B}})(x)\mu(x) \;.$$

Since $-\mathscr{L} h_{\mathcal{A},\mathcal{B}} \equiv 0$ on $(\mathcal{A}\cup\mathcal{B})^c$ and $f = a h_{\mathcal{A},\mathcal{B}}$ on $\mathcal{A}\cup\mathcal{B}$, we can conclude from the previous identity that

$$\begin{aligned} \left\langle \Psi_{h_{\mathcal{A},\mathcal{B}}},\, \Phi_f \right\rangle_{\mathfrak{F}} &= a \sum_{x\in\mathcal{A}} h_{\mathcal{A},\mathcal{B}}(x)(-\mathscr{L} h_{\mathcal{A},\mathcal{B}})(x)\mu(x) \\ &= a\mathscr{D}(h_{\mathcal{A},\mathcal{B}}) = a\, \mathrm{cap}(\mathcal{A},\, \mathcal{B}) \;. \end{aligned} \tag{1.7}$$

Let $\phi \in \mathfrak{U}_a(\mathcal{A},\, \mathcal{B})$. Then, by Proposition 0.12-(2),

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \phi \right\rangle_{\mathfrak{F}} = -\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\operatorname{div}\phi)(x)\ .$$

Since $\operatorname{div}\phi \equiv 0$ on $(\mathcal{A}\cup\mathcal{B})^c$ and $h_{\mathcal{A},\,\mathcal{B}} = \mathbf{1}_{\mathcal{A}}$ on $\mathcal{A}\cup\mathcal{B}$,

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \phi \right\rangle_{\mathfrak{F}} = -\sum_{x\in\mathcal{A}} (\operatorname{div}\phi)(x) = -(\operatorname{div}\phi)(\mathcal{A}) = -a\ , \tag{1.8}$$

where the last equality follows from $\phi \in \mathfrak{U}_a(\mathcal{A},\, \mathcal{B})$.

Now, we first look at (1). If $f \in \mathfrak{C}_{1,\,0}(\mathcal{A},\, \mathcal{B})$ and $\phi \in \mathfrak{U}_0(\mathcal{A},\, \mathcal{B})$, then by (1.7) and (1.8),

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \Phi_f - \phi \right\rangle_{\mathfrak{F}} = \operatorname{cap}(\mathcal{A},\, \mathcal{B})\ .$$

By the Cauchy–Schwarz inequality,

$$\operatorname{cap}(\mathcal{A},\, \mathcal{B})^2 = \left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \Phi_f - \phi \right\rangle_{\mathfrak{F}}^2 \le \|\Psi_{h_{\mathcal{A},\,\mathcal{B}}}\|_{\mathfrak{F}}^2\, \|\Phi_f - \phi\|_{\mathfrak{F}}^2 = \operatorname{cap}(\mathcal{A},\, \mathcal{B})\, \|\Phi_f - \phi\|_{\mathfrak{F}}^2\ .$$

Therefore, we get $\|\Phi_f - \phi\|_{\mathfrak{F}}^2 \ge \operatorname{cap}(\mathcal{A},\, \mathcal{B})$. The equality holds only when $\Phi_f - \phi = c\Psi_{h_{\mathcal{A},\,\mathcal{B}}}$ for some $c \in \mathbb{R}$. By carefully analyzing this restriction, we can conclude that equality holds only for (1.4).

Next, we consider (2). If $g \in \mathfrak{C}_{0,\,0}(\mathcal{A},\, \mathcal{B})$ and $\psi \in \mathfrak{U}_1(\mathcal{A},\, \mathcal{B})$, then again by (1.7) and (1.8), we get

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \Phi_g - \psi \right\rangle_{\mathfrak{F}} = -1\ .$$

Thus, by the Cauchy–Schwarz inequality,

$$1 = \left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\, \Phi_g - \psi \right\rangle_{\mathfrak{F}}^2 \le \|\Psi_{h_{\mathcal{A},\,\mathcal{B}}}\|_{\mathfrak{F}}^2\, \|\Phi_g - \psi\|_{\mathfrak{F}}^2 = \operatorname{cap}(\mathcal{A},\, \mathcal{B})\, \|\Phi_g - \psi\|_{\mathfrak{F}}^2\ .$$

Hence, we get $\operatorname{cap}(\mathcal{A},\, \mathcal{B}) \ge \|\Phi_g - \psi\|_{\mathfrak{F}}^{-2}$. One can also readily check that the equality holds only for the selection (1.6). $\square$

Now, Theorem 1.7 can be used to estimate the capacity in the non-reversible case in the same manner as Remark 1.5. We note that now divergence-free test flows are needed for both upper and lower bounds, and thus we must address this technical issue directly to use these principles. Note that, in the non-reversible case, an argument such as Remark 1.6 does not exist.

Remark 1.8. In fact, Proposition 0.10 for the reversible case is a consequence of the Dirichlet principle (Theorem 1.3), since we have $\mathfrak{C}_{1,\,0}(\mathcal{A}',\, \mathcal{B}') \subset \mathfrak{C}_{1,\,0}(\mathcal{A},\, \mathcal{B})$ if $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{B} \subset \mathcal{B}'$. On the other hand, for the non-reversible case, we do not have such a simple argument since it holds that $\mathfrak{U}_0(\mathcal{A},\, \mathcal{B}) \subset \mathfrak{U}_0(\mathcal{A}',\, \mathcal{B}')$ instead of

$\mathfrak{U}_0(\mathcal{A}', \mathcal{B}') \subset \mathfrak{U}_0(\mathcal{A}, \mathcal{B})$.

1.4 Comparison result for capacity

One can observe from the Dirichlet and Thomson principles that the capacity estimates of non-reversible processes are far more complicated than those of reversible processes. Hence, if one only needs a rough capacity estimate of a non-reversible process, it would be very handy if a comparison result between the capacity of a reversible process and that of a non-reversible one exists. In this section, we provide such a result based on the Dirichlet principle. This comparison result will be used in Part 3.

Define a symmetrized rate as

$$r^s(x, y) = \frac{1}{2\mu(x)}[\mu(x)r(x, y) + \mu(y)r(y, x)] \quad ; \, x, y \in \mathcal{H} ,$$

and let $(X^s(t))_{t\geq 0}$ be a continuous-time Markov process on $\mathcal{H}$ with rate $r^s(\cdot, \cdot)$. One can observe now that the following detailed balance condition holds:

$$\mu(x)r^s(x, y) = \mu(y)r^s(y, x) .$$

Hence, $\mu(\cdot)$ is the invariant measure for the process $X^s(\cdot)$, and furthermore $X^s(\cdot)$ is a reversible process.

We write $h^s_{\mathcal{A},\mathcal{B}}$ and $\mathrm{cap}^s(\mathcal{A}, \mathcal{B})$ the equilibrium potential and the capacity, respectively, with respect to the process $X^s(\cdot)$, for two disjoint and non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$. One can easily check that the Dirichlet form of this symmetrized process is still $\mathscr{D}(\cdot)$ (cf. Remark 0.3).

Since $X^s(\cdot)$ is reversible, it could be much simpler to estimate $\mathrm{cap}^s(\mathcal{A}, \mathcal{B})$ than to estimate $\mathrm{cap}(\mathcal{A}, \mathcal{B})$. The purpose of this subsection is to compare these two capacities.

Firstly, we can show that the symmetrized capacity is always smaller.

Proposition 1.9. *For any two disjoint and non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, it holds that*

$$\mathrm{cap}^s(\mathcal{A}, \mathcal{B}) \leq \mathrm{cap}(\mathcal{A}, \mathcal{B}) .$$

Proof. Since $h_{\mathcal{A},\mathcal{B}} \in \mathfrak{C}_{1,0}(\mathcal{A}, \mathcal{B})$, by the Dirichlet principle for reversible processes (Theorem 1.3),

$$\mathrm{cap}^s(\mathcal{A}, \mathcal{B}) = \inf_{f\in\mathfrak{C}_{1,0}(\mathcal{A},\mathcal{B})} \mathscr{D}(f) \leq \mathscr{D}(h_{\mathcal{A},\mathcal{B}}) = \mathrm{cap}(\mathcal{A}, \mathcal{B}) .$$

□

We next investigate the opposite bound. To this end, we have to introduce the sector condition.

Definition 1.10. A Markov process $X(\cdot)$ is said to satisfy the sector condition with constant $C_0 > 0$ if

$$\langle f, -\mathscr{L} g\rangle_\mu^2 \le C_0 \mathscr{D}(f)\mathscr{D}(g) \tag{1.9}$$

for all $f, g : \mathcal{H} \to \mathbb{R}$.

Heuristically, this is called the sector condition since the eigenvalues of $\mathscr{L}$ satisfying (1.11) are located on a certain sector at the complex plane originating from 0. In this sense, one regards a Markov process with the sector condition as a process which is not far from reversibility. A huge class of Markov processes under consideration satisfies the sector condition. We shall check, for instance, whether the non-reversible zero-range process considered in Part III satisfies the sector condition (cf. Proposition 16.1).

Exercise 1.11. If $X(\cdot)$ is reversible, prove that one can write

$$\langle f, -\mathscr{L} g\rangle_\mu = \frac{1}{2} \sum_{x\in\mathcal{H}} \sum_{y\in\mathcal{H}} \mu(x) r(x, y)(g(y) - g(x))(f(y) - f(x)) \ , \tag{1.10}$$

and therefore $X(\cdot)$ satisfies the sector condition with constant 1.

Remark 1.12. Of course, if $X(\cdot)$ is non-reversible, the expression (1.10) does not hold, and therefore checking the inequality (1.9) is not trivial at all. To check (1.9), one usually proves inequality of the form

$$\langle f, -\mathscr{L} g\rangle_\mu \le C_1 \mathscr{D}(f) + C_2 \mathscr{D}(g) \tag{1.11}$$

for some constant $C_1, C_2 > 0$ for all $f, g : \mathcal{H} \to \mathbb{R}$. We first note that the inequality (1.9) is trivial if f or g is a constant function (cf. Exercise 0.2). Otherwise, inserting $f := \sqrt{C_2 \mathscr{D}(g)} f$ and $g := \sqrt{C_1 \mathscr{D}(f)} g$ to (1.11), we get

$$\sqrt{C_1 C_2 \mathscr{D}(f)\mathscr{D}(g)}\, \langle f, -\mathscr{L} g\rangle_\mu \le 2 C_1 C_2 \mathscr{D}(f)\mathscr{D}(g) \ .$$

Therefore, we can conclude that $X(\cdot)$ satisfies the sector condition with constant $4C_1C_2$.

Now, we are ready to establish the opposite bound of the one established in Proposition 1.9.

Proposition 1.13. *Suppose that a Markov process $X(\cdot)$ satisfies the sector condition with constant $C_0 > 0$. Then, we have that*

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) \le C_0\, \mathrm{cap}^s(\mathcal{A},\, \mathcal{B})\ .$$

Proof. We may assume that $\mathrm{cap}(\mathcal{A},\, \mathcal{B}) > 0$, as otherwise the inequality is trivial. We first note that

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B}) = \langle h_{\mathcal{A},\, \mathcal{B}},\, -\mathscr{L} h_{\mathcal{A},\, \mathcal{B}} \rangle_\mu = \left\langle h^s_{\mathcal{A},\, \mathcal{B}},\, -\mathscr{L} h_{\mathcal{A},\, \mathcal{B}} \right\rangle_\mu\ ,$$

where the second equality holds since $\mathscr{L} h_{\mathcal{A},\, \mathcal{B}} = 0$ on $(\mathcal{A} \cup \mathcal{B})^c$ and $h_{\mathcal{A},\, \mathcal{B}} = h^s_{\mathcal{A},\, \mathcal{B}}$ on $\mathcal{A} \cup \mathcal{B}$. Therefore, by the sector condition,

$$\mathrm{cap}(\mathcal{A},\, \mathcal{B})^2 \le C_0 \mathscr{D}(h^s_{\mathcal{A},\, \mathcal{B}}) \mathscr{D}(h_{\mathcal{A},\, \mathcal{B}}) = C_0 \mathrm{cap}^s(\mathcal{A},\, \mathcal{B}) \mathrm{cap}(\mathcal{A},\, \mathcal{B})\ .$$

Dividing both sides by $\mathrm{cap}(\mathcal{A},\, \mathcal{B}) > 0$ completes the proof. □

2 Generalized Dirichlet and Thomson Principles

Let us fix two disjoint and non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$. In the previous section, we explain a general strategy to estimate or bound the capacity $\mathrm{cap}(\mathcal{A},\,\mathcal{B})$ based on the Dirichlet and Thomson principles. To apply this strategy, one has to construct suitable test functions or test flows. As we have mentioned earlier, the Thomson principle for the reversible case and the Dirichlet and Thomson principles for the non-reversible case require us to construct a test flow which must be divergence-free on $(\mathcal{A}\cup\mathcal{B})^c$ which is a major technical problem in applications of these method. In this section, we introduce alternative variational principles that do not require us to construct a divergence-free flow, and hence are suitable for many applications.

2.1 Reversible case

Let us start by considering the reversible case. Hence, we assume in this subsection that the process $X(\cdot)$ is reversible. We also emphasize that we do not need to develop a generalized Dirichlet principle, since the Dirichlet principle for reversible Markov processes is not involved with the flow structure.

The generalized Thomson principle is given as follows. We write $\mathfrak{F}_0$ the collection of non-zero flows, i.e.,

$$\mathfrak{F}_0 = \{\phi \in \mathfrak{F} : \|\phi\|_{\mathfrak{F}} > 0\}\ .$$

Theorem 2.1 (Generalized Thomson principle: reversible case)**.** *It holds that*

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) = \sup_{\phi\in\mathfrak{F}_0} \frac{1}{\|\phi\|_{\mathfrak{F}}^2}\Big[\sum_{\sigma\in\mathcal{H}} h_{\mathcal{A},\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\Big]^2\ . \tag{2.1}$$

Moreover, the optimizers are given by $\phi = c\Psi_{h_{\mathcal{A},\,\mathcal{B}}}$, $c\neq 0$.

Proof. By Proposition 0.11-(2), we have that

$$\big\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\phi\big\rangle_{\mathfrak{F}} = -\sum_{x\in\mathcal{H}} h_{\mathcal{A},\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\ .$$

Thus, by the Cauchy–Schwarz inequality, it holds that

$$\begin{aligned}\Big[\sum_{x\in\mathcal{H}} h_{\mathcal{A},\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\Big]^2 &= \big\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\phi\big\rangle_{\mathfrak{F}}^2\\ &\le \|\Psi_{h_{\mathcal{A},\,\mathcal{B}}}\|_{\mathfrak{F}}^2\,\|\phi\|_{\mathfrak{F}}^2 = \mathrm{cap}(\mathcal{A},\,\mathcal{B})\,\|\phi\|_{\mathfrak{F}}^2\ .\end{aligned}$$

Hence, it holds that

$$\operatorname{cap}(\mathcal{A},\,\mathcal{B}) \geq \frac{1}{\|\phi\|_{\mathfrak{F}}^2}\left[\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\operatorname{div}\phi)(x)\right]^2 .$$

From the Cauchy–Schwarz inequality, it is clear that the equality holds only for $\phi = c\Psi_{h_{\mathcal{A},\,\mathcal{B}}}$, $c \neq 0$. $\square$

The advantage of this generalized Thomson principle is very clear. We no longer impose the divergence-free condition on test flows, and hence any flow approximating $h_{\mathcal{A},\,\mathcal{B}}$ (e.g., $\Psi_{h_{\mathcal{A},\,\mathcal{B}}}$) can be used as a test flow. For instance, if one constructed a test function f approximating $h_{\mathcal{A},\,\mathcal{B}}$ and obtained an upper bound on the capacity by injecting this test function f to the Dirichlet principle, then one can also use Ψ_f as the test flow in this generalized Thomson principle. If we encounter a technical issue in a certain region, we can modify the flow accordingly in this region to obtain a test flow. This idea was used in [27] to analyze the metastability of Ising and Potts models on large, fixed lattices without external fields. For this model, the energy landscape is extremely complex, and it is very difficult to construct a divergence-free flow. We explain a special case of this result in Part II.

Clearly, the crucial disadvantage of the generalized Thomson principle is the appearance of the equilibrium potential in the variational principle. Hence, this generalized version turns the difficulty stemming from the divergence-free restriction to the difficulty of handling the equilibrium potential. Of course, Proposition 0.16 plays an important role in controlling the equilibrium potential.

2.2 Non-reversible case

Now, we no longer assume that the process $X(\cdot)$ is reversible. Then, the variational problem becomes more complicated.

Theorem 2.2. *The followings hold.*

1. *(Generalized Dirichlet principle) We have that*

$$\mathbf{cap}(\mathcal{A},\,\mathcal{B}) = \inf_{f\in\mathfrak{C}_{1,\,0}(\mathcal{A},\,\mathcal{B}),\,\phi\in\mathfrak{F}}\left\{||\Phi_f - \phi||^2 - 2\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathbf{div}\,\phi)(x)\right\}\,, \tag{2.2}$$

and (1.4) is a minimizer.

2. (Generalized Thomson principle) We have that

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B}) = \sup_{g\in\mathfrak{C}_{0,\,0}(\mathcal{A},\,\mathcal{B}),\,\psi\in\mathfrak{F}_0} \frac{1}{||\Phi_g-\psi||^2}\left[\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\psi)(x)\right]^2\;, \tag{2.3}$$

and the constant multiples of (1.6), i.e.,

$$(g,\,\psi) = \left(c\frac{h^{\dagger}_{\mathcal{A},\,\mathcal{B}}-h_{\mathcal{A},\,\mathcal{B}}}{2\mathrm{cap}(\mathcal{A},\,\mathcal{B})},\,-c\frac{\Phi_{h^{\dagger}_{\mathcal{A},\,\mathcal{B}}}+\Phi^{*}_{h_{\mathcal{A},\,\mathcal{B}}}}{2\mathrm{cap}(\mathcal{A},\,\mathcal{B})}\right)\;,\;\; c\neq 0 \tag{2.4}$$

are maximizers.

Proof. In the proof of Theorem 1.7, we showed that for $f\in\mathfrak{C}_{a,\,0}(\mathcal{A},\,\mathcal{B})$,

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\Phi_f\right\rangle_{\mathfrak{F}} = a\,\mathrm{cap}(\mathcal{A},\,\mathcal{B})\;. \tag{2.5}$$

On the other hand, by Proposition 0.12-(2), we have

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\phi\right\rangle_{\mathfrak{F}} = -\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\;. \tag{2.6}$$

For part (1), let $f\in\mathfrak{C}_{a,\,0}(\mathcal{A},\,\mathcal{B})$. Then, by (2.5) and (2.6),

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\Phi_f-\phi\right\rangle_{\mathfrak{F}} = \mathrm{cap}(\mathcal{A},\,\mathcal{B}) + \sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\;. \tag{2.7}$$

Furthermore, by the Cauchy–Schwarz inequality and (0.26) (which still holds for non-reversible processes)

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\Phi_f-\phi\right\rangle^2_{\mathfrak{F}} \le \|\Psi_{h_{\mathcal{A},\,\mathcal{B}}}\|^2_{\mathfrak{F}}\,\|\Phi_f-\phi\|^2_{\mathfrak{F}} = \mathrm{cap}(\mathcal{A},\,\mathcal{B})\,\|\Phi_f-\phi\|^2_{\mathfrak{F}}\;. \tag{2.8}$$

By (2.7) and (2.8),

$$\mathrm{cap}(\mathcal{A},\,\mathcal{B})\,\|\Phi_f-\phi\|^2_{\mathfrak{F}} \ge \mathrm{cap}(\mathcal{A},\,\mathcal{B})^2 + 2\mathrm{cap}(\mathcal{A},\,\mathcal{B})\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\phi)(x)\;.$$

Thus, part (1) is proved if we check that the equality holds for (1.4).

The proof of part (2) is similar. For $g\in\mathfrak{C}_{0,\,0}(\mathcal{A},\,\mathcal{B})$, again by (2.5) and (2.6), we have

$$\left\langle \Psi_{h_{\mathcal{A},\,\mathcal{B}}},\,\Phi_g-\psi\right\rangle_{\mathfrak{F}} = \sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\mathrm{div}\,\psi)(x)\;.$$

Hence, by the computations as before, the proof of part (2) is completed. □

Remark 2.3. We did not attempt to characterize all the optimizers in the previous principles.

When we use these principles, it is important to control terms of the form

$$\sum_{x\in\mathcal{H}} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\operatorname{div}\phi)(x)\ .$$

For the Thomson principle, we used ψ, instead of ϕ, to denote the test flow, but in what follows we denote by ϕ the flow for the Thomson principle as well for convenience.

Indeed, this is trade-off in order to avoid the construction of a divergence-free flow. By the property of the equilibrium potential (cf. (0.11)), this summation can be decomposed into

$$(\operatorname{div}\phi)(\mathcal{A}) + \sum_{x\in(\mathcal{A}\cup\mathcal{B})^c} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\operatorname{div}\phi)(x)\ .$$

If we take the test function and flow as a good approximation of the optimizers (1.4) and (1.6), we have $(\operatorname{div}\phi)(\mathcal{A})\simeq 0$ for the Dirichlet principle and $(\operatorname{div}\phi)(\mathcal{A})\simeq 1$ for the Thomson principle. Since ϕ can be approximately divergence-free on $(\mathcal{A}\cup\mathcal{B})^c$, we also have

$$\sum_{x\in(\mathcal{A}\cup\mathcal{B})^c} h_{\mathcal{A},\,\mathcal{B}}(x)\,(\operatorname{div}\phi)(x)\simeq 0\ .$$

Since the equilibrium potential is trivially bounded by 1, we may hope

$$\sum_{x\in(\mathcal{A}\cup\mathcal{B})^c} |(\operatorname{div}\phi)(x)|\simeq 0\ ,$$

but in general it may not be true (since there are too many elements in $(\mathcal{A}\cup\mathcal{B})^c$). Instead, we need to decompose $(\mathcal{A}\cup\mathcal{B})^c$ into two regions $\mathcal{C}_1$ and $\mathcal{C}_2$ so that

$$\sum_{x\in\mathcal{C}_1} |(\operatorname{div}\phi)(x)|\simeq 0\ ,$$

but on $\mathcal{C}_2$ the summation is small because $h_{\mathcal{A},\,\mathcal{B}}$ is small. To prove that $h_{\mathcal{A},\,\mathcal{B}}$ is sufficiently small, Proposition 0.16 can be useful.

3 Collapsed Processes

In this section, we introduce the notion known as the collapsed process, which is essentially obtained by contracting a subset $\mathcal{E} \subset \mathcal{H}$ to a single point $\mathfrak{e}$. This process was introduced in [24] to study the Dirichlet principle for non-reversible processes. Moreover, in [40], it is observed that the collapsed process is a crucial notion (along with the capacity) in the precise estimate of the so-called mean jump rate, which is key to the martingale approach of metastability (cf. [2, 3, 4]).

In this section, we fix a set $\mathcal{E} \subset \mathcal{H}$. We note that the contents of the current subsection are from [40, Section 8].

3.1 Definition of collapsed process

As mentioned earlier, our aim is collapsing a set $\mathcal{E}$ into a single point $\mathfrak{e}$. To this end, let us first define the state space $\overline{\mathcal{H}} = (\mathcal{H} \setminus \mathcal{E}) \cup \{\mathfrak{e}\}$. Then, (recalling that $\mu(\cdot)$ is the invariant measure for the process $X(\cdot)$) define a rate $\overline{r} : \overline{\mathcal{H}} \times \overline{\mathcal{H}} \to [0, \infty)$ as

$$\begin{cases} \overline{r}(x,\, y) = r(x,\, y) & \text{for } x,\, y \in \mathcal{H} \setminus \mathcal{E}\ , \\ \overline{r}(x,\, \mathfrak{e}) = \sum_{z\in\mathcal{E}} r(x,\, z) & \text{for } x \in \mathcal{H} \setminus \mathcal{E}\ , \\ \overline{r}(\mathfrak{e},\, y) = \frac{1}{\mu(\mathcal{E})} \sum_{z\in\mathcal{E}} \mu(z) r(z,\, y) & \text{for } y \in \mathcal{H} \setminus \mathcal{E}\ . \end{cases} \tag{3.1}$$

The collapsed process is defined as a continuous-time Markov process $(\overline{X}(t))_{t\ge 0}$ on $\overline{\mathcal{H}}$ with rate $\overline{r}(\cdot,\, \cdot)$.

Denote by $\overline{\mathbb{P}}_x$ the law of $\overline{X}(\cdot)$ starting from x, and by $\overline{\mathscr{L}}$ and $\overline{\mathscr{D}}(\cdot)$ the generator and the Dirichlet form corresponding to the collapsed process $\overline{X}(\cdot)$, respectively. Define a probability measure $\overline{\mu}(\cdot)$ on $\overline{\mathcal{H}}$ as

$$\begin{cases} \overline{\mu}(x) \ = \ \mu(x) & \text{if } x \in \mathcal{H} \setminus \mathcal{E}\ , \\ \overline{\mu}(\mathfrak{e}) \ = \ \mu(\mathcal{E})\ . \end{cases} \tag{3.2}$$

Exercise 3.1. Answer the following questions.

1. Prove that the measure $\overline{\mu}(\cdot)$ is the invariant measure for the process $\overline{X}(\cdot)$.
2. Prove that the process $\overline{X}(\cdot)$ is reversible if the process $X(\cdot)$ is reversible. Is the converse true?

3.2 Flow space of collapsed process

Next, we investigate the flow structure with respect to the collapsed process $\overline{X}(\cdot)$. For $x,\, y \in \mathcal{H}$, we defined the conductance between x and y with respect to the original process $X(\cdot)$ as (cf. (0.22))

$$c(x,\,y) = \mu(x)r(x,\,y)\ .$$

Similarly, for $x,\,y \in \overline{\mathcal{H}}$, we define the conductance with respect to the collapsed process $\overline{X}(\cdot)$ as

$$\overline{c}(x,y) = \overline{\mu}(x)\overline{r}(x,\,y)\ .$$

Then, by (3.1) and (3.2), this conductance $\overline{c}(\cdot,\,\cdot)$ can be rewritten as

$$\begin{cases} \overline{c}(x,\,y) = c(x,\,y) & \text{for } x,\,y \in \mathcal{H}\setminus\mathcal{E}\ , \\ \overline{c}(x,\,\mathfrak{e}) = \sum_{z\in\mathcal{E}} c(x,\,z) & \text{for } x \in \mathcal{H}\setminus\mathcal{E}\ , \\ \overline{c}(\mathfrak{e},\,y) = \sum_{z\in\mathcal{E}} c(z,\,y) & \text{for } y \in \mathcal{H}\setminus\mathcal{E}\ . \end{cases} \tag{3.3}$$

Define the symmetrized conductance as

$$\overline{c}^s(x,\,y) = \frac{1}{2}[\overline{c}(x,\,y) + \overline{c}(y,\,x)] \quad ;\ x,\,y \in \overline{\mathcal{H}}\ .$$

For $x,\,y \in \overline{\mathcal{H}}$, we write $x \sim y$ if $\overline{c}^s(x,\,y) > 0$. Since $\overline{c}^s(x,\,y) = \overline{c}^s(y,\,x)$, we observe that $x \sim y$ if and only if $y \sim x$. Then, the set of directed edges are defined by

$$\overline{\mathfrak{E}} = \{(x,\,y) \in \mathcal{H}\times\mathcal{H} : x \sim y\}\ . \tag{3.4}$$

As before, we can define a flow structure on the set $\overline{\mathfrak{F}}$ of flows on $\overline{\mathfrak{E}}$ which are anti-symmetric functions on $\overline{\mathfrak{E}}$. Then, we can induce the Hilbert space structure on $\overline{\mathfrak{F}}$, as we did in Sections 0.3 and 0.4. Denote the corresponding inner product and the flow norm by $\langle\cdot,\,\cdot\rangle_{\overline{\mathfrak{F}}}$ and $\|\cdot\|_{\overline{\mathfrak{F}}}$, respectively. In particular, we can write

$$\begin{aligned} \langle\phi,\,\psi\rangle_{\overline{\mathfrak{F}}} &= \frac{1}{2}\sum_{(x,\,y)\in\overline{\mathfrak{E}}} \frac{\phi(x,\,y)\psi(x,\,y)}{\overline{c}^s(x,\,y)}\ , \text{ and} \\ \|\phi\|^2_{\overline{\mathfrak{F}}} &= \frac{1}{2}\sum_{(x,\,y)\in\overline{\mathfrak{E}}} \frac{\phi(x,\,y)^2}{\overline{c}^s(x,\,y)}\ . \end{aligned}$$

For each flow $\phi \in \mathfrak{F}$, define the collapsed flow $\overline{\phi} \in \overline{\mathfrak{F}}$ by

$$\begin{cases} \overline{\phi}(x,\,y) = \phi(x,\,y) & \text{for } x,\,y \in \mathcal{H}\setminus\mathcal{E}\ , \\ \overline{\phi}(x,\,\mathfrak{e}) = \sum_{z\in\mathcal{E}} \phi(x,\,z) & \text{for } x \in \mathcal{H}\setminus\mathcal{E}\ , \\ \overline{\phi}(\mathfrak{e},\,y) = \sum_{z\in\mathcal{E}} \phi(z,\,y) & \text{for } y \in \mathcal{H}\setminus\mathcal{E}\ . \end{cases} \tag{3.5}$$

Exercise 3.2. Prove that

$$\begin{cases} (\operatorname{div}\overline{\phi})(x) &= (\operatorname{div}\,\phi)(x) \qquad \text{for } x\in\mathcal{H}\setminus\mathcal{E}\ , \\ (\operatorname{div}\overline{\phi})(\mathfrak{e}) &= (\operatorname{div}\,\phi)(\mathcal{E})\ . \end{cases} \tag{3.6}$$

The following contraction property of the flow norm is useful later.

Lemma 3.3. *For all $\phi\in\mathfrak{F}$ and its collapsed flow $\overline{\phi}\in\overline{\mathfrak{F}}$, it holds that*

$$\|\overline{\phi}\|_{\overline{\mathfrak{F}}}\le\|\phi\|_{\mathfrak{F}}\ .$$

Moreover, the equality holds if and only if

$$\begin{cases} \phi(x,\,y)=0 & \textit{if } x,\,y\in\mathcal{E}\ ,\ \textit{and} \\ \frac{\phi(x,\,y)}{c^s(x,\,y)}=\frac{\phi(x',\,y)}{c^s(x',\,y)} & \textit{if } y\in\mathcal{E} \textit{ and } x,\,x'\in\mathcal{H}\setminus\mathcal{E} \textit{ satisfies } x\sim y \textit{ and } x'\sim y\ . \end{cases} \tag{3.7}$$

Proof. Decompose the flow norm of the flow ϕ as

$$\|\phi\|_{\mathfrak{F}}^2=\frac{A_1}{2}+A_2+\frac{A_3}{2}\ ,$$

where

$$\begin{aligned} A_1 &= \sum_{(x,\,y)\in\mathfrak{E}:x,\,y\in\mathcal{H}\setminus\mathcal{E}}\frac{\phi(x,\,y)^2}{c^s(x,\,y)}\ , \\ A_2 &= \sum_{(x,\,y)\in\mathfrak{E}:x\in\mathcal{H}\setminus\mathcal{E},\,y\in\mathcal{E}}\frac{\phi(x,\,y)^2}{c^s(x,\,y)}\ ,\ \text{and} \\ A_3 &= \sum_{(x,\,y)\in\mathfrak{E}:x,\,y\in\mathcal{E}}\frac{\phi(x,\,y)^2}{c^s(x,\,y)}\ . \end{aligned}$$

Then, decompose the flow norm of the collapsed flow $\overline{\phi}$ as

$$\|\overline{\phi}\|_{\overline{\mathfrak{F}}}^2=\frac{\overline{A}_1}{2}+\overline{A}_2\ ,$$

where

$$\begin{aligned} \overline{A}_1 &= \sum_{(x,\,y)\in\overline{\mathfrak{E}}:x,\,y\in\mathcal{H}\setminus\mathcal{E}}\frac{\overline{\phi}(x,\,y)^2}{\overline{c}^s(x,\,y)} \quad \text{and} \\ \overline{A}_2 &= \sum_{x\in\overline{\mathcal{H}}:(x,\,\mathfrak{e})\in\overline{\mathfrak{E}}}\frac{\overline{\phi}(x,\,\mathfrak{e})^2}{\overline{c}^s(x,\,\mathfrak{e})}\ . \end{aligned}$$

By (3.3) and (3.5), we immediately have that $A_1 = \overline{A}_1$.

Therefore, it suffices to prove $A_2 \ge \overline{A}_2$. For each $x \in \mathcal{H} \setminus \mathcal{E}$ adjacent to at least one point of $\mathcal{E}$, by (3.3), (3.5), and the Cauchy–Schwarz inequality, we obtain

$$\sum_{y\in\mathcal{E}:(x,\,y)\in\mathfrak{E}} \frac{\phi(x,\,y)^2}{c^s(x,\,y)} \;\ge\; \frac{\Big[\sum_{y\in\mathcal{E}:(x,\,y)\in\mathfrak{E}}\phi(x,\,y)\Big]^2}{\sum_{y\in\mathcal{E}:(x,\,y)\in\mathfrak{E}} c^s(x,\,y)} = \frac{\overline{\phi}(x,\,\mathfrak{e})^2}{\overline{c}^s(x,\,\mathfrak{e})}\;.$$

By adding this inequality over $x \in \mathcal{H} \setminus \mathcal{E}$, we obtain $A_2 \ge \overline{A}_2$, and the proof is completed. □

Exercise 3.4. For $f : \mathcal{H} \to \mathbb{R}$ which is constant over $\mathcal{E}$, prove that the flow Ψ_f satisfies the equality condition (3.7).

If a function $f : \mathcal{H} \to \mathbb{R}$ is constant over $\mathcal{E}$, we define a collapsed function $\overline{f} : \overline{\mathcal{H}} \to \mathbb{R}$ as

$$\begin{cases} \overline{f}(x) = f(x) \;\text{ if } x \in \mathcal{H}\setminus\mathcal{E}\;, \\ \overline{f}(\mathfrak{e}) = \text{the constant value of } f \text{ on } \mathcal{E}\;. \end{cases} \tag{3.8}$$

Lemma 3.5. *Suppose that the functions $f,\, g : \mathcal{H} \to \mathbb{R}$ are constant over $\mathcal{E}$, and let $\overline{f},\, \overline{g} : \overline{\mathcal{H}} \to \mathbb{R}$ be the collapsed function of $f,\, g$ (cf. (3.8)), respectively. Then, we have*

$$\langle g,\, -\mathscr{L} f\rangle_\mu = \langle \overline{g},\, -\overline{\mathscr{L} f}\rangle_{\overline{\mu}}\;. \tag{3.9}$$

In particular, we have

$$\overline{\mathscr{D}}(\overline{f}) = \mathscr{D}(f)\;. \tag{3.10}$$

Proof. Since f is constant over $\mathcal{E}$, we can write

$$\begin{aligned}&\langle g,\, -\mathscr{L} f\rangle_\mu \\ =&\frac{1}{2}\left[\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}} + \sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\mathcal{E}} + \sum_{x\in\mathcal{E}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}}\right]\mu(x)r(x,\,y)[f(y)-f(x)]g(x)\;.\end{aligned} \tag{3.11}$$

Note that the first summation is equal to

$$\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}} \overline{\mu}(x)\overline{r}(x,\,y)[\overline{f}(y)-\overline{f}(x)]\overline{g}(x)\;, \tag{3.12}$$

since $\mu = \overline{\mu}$, $r = \overline{r}$, $f = \overline{f}$, and $g = \overline{g}$ on $\mathcal{H} \setminus \mathcal{E}$. On the other hand, we have

$f(y) = \overline{f}(\mathfrak{e})$ for all $y \in \mathcal{E}$, and thus the second summation is equal to

$$\begin{aligned}&\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\mathcal{E}}\overline{\mu}(x)r(x,\,y)[\overline{f}(\mathfrak{e})-\overline{f}(x)]\overline{g}(x)\\ =&\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\overline{\mu}(x)\overline{r}(x,\,\mathfrak{e})[\overline{f}(\mathfrak{e})-\overline{f}(x)]\overline{g}(x)\;,\end{aligned} \tag{3.13}$$

where the equality follows from the second line of (3.1). Finally, a similar computation yields that the third summation is equal to

$$\begin{aligned}&\sum_{x\in\mathcal{E}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}}\mu(x)r(x,\,y)[\overline{f}(y)-\overline{f}(\mathfrak{e})]\overline{g}(\mathfrak{e})\\ =&\sum_{y\in\mathcal{H}\setminus\mathcal{E}}\overline{\mu}(\mathfrak{e})\overline{r}(\mathfrak{e},\,y)[\overline{f}(y)-\overline{f}(\mathfrak{e})]\overline{g}(\mathfrak{e})\;,\end{aligned} \tag{3.14}$$

where the equality follows from the third line of (3.1) and (3.2). By inserting (3.12), (3.13) and (3.14) into (3.11), we can conclude that

$$\begin{aligned}&\langle g,\,-\mathscr{L}f\rangle_{\mu}\\ =&\frac{1}{2}\left[\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}}+\sum_{x\in\mathcal{H}\setminus\mathcal{E}}\sum_{y\in\{\mathfrak{e}\}}+\sum_{x\in\{\mathfrak{e}\}}\sum_{y\in\mathcal{H}\setminus\mathcal{E}}\right]\overline{\mu}(x)\overline{r}(x,\,y)[\overline{f}(y)-\overline{f}(x)]\overline{g}(x)\\ =&\langle\overline{g},\,-\overline{\mathscr{L}f}\rangle_{\overline{\mu}}\;,\end{aligned}$$

and the proof of (3.9) is completed. Now, (0.6) follows from (3.9) by inserting $g = f$. □

For a function $g : \overline{\mathcal{H}} \to \mathbb{R}$, define $\overline{\Phi}_g$, $\overline{\Phi}^*_g$ and $\overline{\Psi}_g$ as, for $x,\, y \in \overline{\mathcal{H}}$,

$$\overline{\Phi}_g(x,\,y) \;=\; g(y)\overline{c}(y,\,x) - g(x)\overline{c}(x,\,y)\;, \tag{3.15}$$

$$\overline{\Phi}^*_g(x,\,y) \;=\; g(y)\overline{c}(x,\,y) - g(x)\overline{c}(y,\,x)\;, \tag{3.16}$$

$$\overline{\Psi}_g(x,\,y) \;=\; \overline{c}^s(x,\,y)(g(y) - g(x))\;. \tag{3.17}$$

Lemma 3.6. *Suppose that the function $f : \mathcal{H} \to \mathbb{R}$ is constant over $\mathcal{E}$, and let $\overline{f} : \overline{\mathcal{H}} \to \mathbb{R}$ be the collapsed function of f (cf. (3.8)). Then, the flow $\overline{\Phi_f}$, which is the collapsed flow of Φ_f defined in (0.28), coincides with the flow $\overline{\Phi}_{\overline{f}}$. Similarly, we have that*

$$\overline{\Phi^*_f} = \overline{\Phi}^*_{\overline{f}} \quad \text{and} \quad \overline{\Psi_f} = \overline{\Psi}_{\overline{f}}\;. \tag{3.18}$$

Proof. We only prove that two flows $\overline{\Phi_f}$ and $\overline{\Phi}_{\overline{f}}$ coincide, and leave the proof for the other two as exercise, since the proofs are quite similar.

Since $\overline{\Phi_f}(x,\,y) = \overline{\Phi}_{\overline{f}}(x,\,y)$ for $x,\,y \in \mathcal{H} \setminus \mathcal{E}$ holds trivially from the definitions, it suffices to prove that $\overline{\Phi_f}(x,\,\mathfrak{e}) = \overline{\Phi}_{\overline{f}}(x,\,\mathfrak{e})$ for $x \in \mathcal{H} \setminus \mathcal{E}$. This can be verified by

$$\begin{aligned}\overline{\Phi_f}(x,\,\mathfrak{e}) = \sum_{z\in\mathcal{E}} \Phi_f(x,\,z) \;&=\; \sum_{z\in\mathcal{E}} [f(z)c(z,\,x) - f(x)c(x,\,z)] \\ &=\; \overline{f}(\mathfrak{e})\overline{c}(\mathfrak{e},\,x) - \overline{f}(x)\overline{c}(x,\,\mathfrak{e}) \\ &=\; \overline{\Phi}_{\overline{f}}(x,\,\mathfrak{e})\;.\end{aligned}$$

□

Exercise 3.7. Prove (3.18).

3.3 Capacity and sector condition of collapsed process

For two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\overline{\mathcal{H}}$,we denote by $\overline{h}_{\mathcal{A},\,\mathcal{B}} : \mathcal{H} \to \mathbb{R}$ the equilibrium potential between $\mathcal{A}$ and $\mathcal{B}$, and we denote by $\overline{\mathrm{cap}}(\mathcal{A},\,\mathcal{B})$ and $\overline{\mathrm{cap}}^s(\mathcal{A},\,\mathcal{B})$ the capacity between $\mathcal{A}$ and $\mathcal{B}$ with respect to the collapsed process $\overline{X}(\cdot)$ and the symmetrized process $\overline{X}^s(\cdot)$ of $\overline{X}(\cdot)$ (which is a Markov process on $\overline{\mathcal{H}}$ associated with the generator $\frac{1}{2}(\overline{\mathscr{L}} + \overline{\mathscr{L}}^\dagger)$, where $\overline{\mathscr{L}}^\dagger$ is the adjoint generator of $\overline{\mathscr{L}}$), respectively. In general, for $\mathcal{A},\,\mathcal{B} \subset \mathcal{H} \setminus \mathcal{E}$, it is difficult to compare $\overline{\mathrm{cap}}(\mathcal{A},\,\mathcal{B})$ and $\mathrm{cap}(\mathcal{A},\,\mathcal{B})$.

Exercise 3.8. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two non-empty and disjoint subsets of $\mathcal{H}\backslash\mathcal{E}$. Then, can you prove either $\overline{\mathrm{cap}}(\mathcal{A},\,\mathcal{B}) \le \mathrm{cap}(\mathcal{A},\,\mathcal{B})$ or $\mathrm{cap}(\mathcal{A},\,\mathcal{B}) \le \overline{\mathrm{cap}}(\mathcal{A},\,\mathcal{B})$?

However, we have the following identity, which is useful in later discussions.

Lemma 3.9. *For any non-empty $\mathcal{A} \subset \mathcal{H} \setminus \mathcal{E}$, we have*

$$\overline{\mathrm{cap}}(\mathfrak{e},\,\mathcal{A}) = \mathrm{cap}(\mathcal{E},\,\mathcal{A})\;.$$

Proof. Recall that $h_{\mathcal{E},\,\mathcal{A}}(\cdot)$ denotes the equilibrium potential between $\mathcal{E}$ and $\mathcal{A}$. Since the behaviors of the processes $X(\cdot)$ and $\overline{X}(\cdot)$ are identical on $\mathcal{H} \setminus \mathcal{E}$, we immediately have that

$$h_{\mathcal{E},\,\mathcal{A}}(x) = \overline{h}_{\mathfrak{e},\,\mathcal{A}}(x) \text{ for all } x \in \mathcal{H} \setminus \mathcal{E}\;.$$

Since $h_{\mathcal{E},\,\mathcal{A}} \equiv 1$ on $\mathcal{E}$ and $\overline{h}_{\mathfrak{e},\,\mathcal{A}}(\mathfrak{e}) = 1$, we can conclude that $\overline{h}_{\mathfrak{e},\,\mathcal{A}}(\cdot)$ is the collapsed function of $h_{\mathcal{E},\,\mathcal{A}}(\cdot)$, i.e,.,

$$\overline{h}_{\mathfrak{e},\,\mathcal{A}} = \overline{h_{\mathcal{E},\,\mathcal{A}}}\;.$$

Therefore, by Lemma 3.5, we can conclude that

$$\overline{\mathrm{cap}}(\mathfrak{e},\,\mathcal{A}) = \overline{\mathscr{D}}(\overline{h}_{\mathfrak{e},\,\mathcal{A}}) = \overline{\mathscr{D}}(\overline{h_{\mathcal{E},\,\mathcal{A}}}) = \mathscr{D}(h_{\mathcal{E},\,\mathcal{A}}) = \mathrm{cap}(\mathcal{E},\,\mathcal{A})\;.$$

$\square$

Next, we assert that the sector condition of the original process is inherited by the collapsed process.

Lemma 3.10. *Suppose that the process $X(\cdot)$ satisfies the sector condition with a constant $C > 0$ (cf. Definition 1.10). Then, the process $\overline{X}(\cdot)$ also satisfies the sector condition with the same constant C. In particular, it holds for any two non-empty and disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of $\overline{\mathcal{H}}$ that*

$$\overline{\mathrm{cap}}^s(\mathcal{A},\, \mathcal{B}) \le \overline{\mathrm{cap}}(\mathcal{A},\, \mathcal{B}) \le C\, \overline{\mathrm{cap}}^s(\mathcal{A},\, \mathcal{B}) \; . \tag{3.19}$$

Proof. For two functions $f,\, g : \overline{\mathcal{H}} \to \mathbb{R}$, define their extended functions $F,\, G : \mathcal{H} \to \mathbb{R}$ as

$$F(x) \;=\; \begin{cases} f(x) & \text{if } x \in \mathcal{H} \setminus \mathcal{E}\;, \\ f(\mathfrak{e}) & \text{if } x \in \mathcal{E}\;, \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(x) & \text{if } x \in H \setminus \mathcal{E}\;, \\ g(\mathfrak{e}) & \text{if } x \in \mathcal{E}\;. \end{cases}$$

so that

$$\overline{F} = f \quad \text{and} \quad \overline{G} = g\; . \tag{3.20}$$

Hence, by (3.20), Lemma 3.5, and the sector condition of $X(\cdot)$,

$$\left\langle f, -\overline{\mathscr{L}} g \right\rangle_{\overline{\mu}} = \langle F, -\mathscr{L} G \rangle_{\mu} \le C \mathscr{D}(F) \mathscr{D}(G) = C \overline{\mathscr{D}}(f) \overline{\mathscr{D}}(g)\;,$$

and hence the process $\overline{X}(\cdot)$ also satisfies the sector condition with a constant $C > 0$. Now, (3.19) is clear from Propositions 1.9 and 1.13. $\square$

Part II

Two-dimensional Ising Model without External Field

In this second part of the lecture note, as an application of the general theory developed so far, we thoroughly analyze the metastable behavior of the Ising model on large but fixed lattice boxes. In particular, we focus on the model without an external field, which posed a longstanding mathematical challenge because of the complexity of the energy landscape. The dynamics is reversible, and the analysis is based on the Dirichlet principle (Theorem 1.3) and the generalized Thomson principle (Theorem 2.1).

The contents of the current part is based on [27] which considered more complex models, namely the Potts model and the model in three-dimensional boxes. We did not investigate these models in this note, since the two-dimensional Ising model is enough to deliver the core of our idea.

4 Ising Model on Two-dimensional Lattice

4.1 Model

In this subsection, we introduce the model and review its basic features.

Ising model

For two positive integers K, L, we write

$$\Lambda = \mathbb{T}_K \times \mathbb{T}_L \ , \tag{4.1}$$

where $\mathbb{T}_k = \mathbb{Z}/(k\mathbb{Z})$ is the discrete one-dimensional torus. For the convenience of the discussion, we assume that $K \le L$ and moreover $K \ge 5$.

We will consider the spin system on Λ; hence, we consider a spin system on the box with periodic boundary conditions. The model that we consider in this second part is defined now.

Definition 4.1 (Ising model on Λ without external field).

- Denote by $\Omega = \{+, -\}$ the set of spins and by $\mathcal{X} = \Omega^\Lambda$ the space of spin configurations on the box Λ. A configuration $\sigma \in \mathcal{X}$ is written as $\sigma = (\sigma(x))_{x\in\Lambda}$ where $\sigma(x) \in \Omega$ denotes the spin of σ at site $x \in \Lambda$.

- For x, $y \in \Lambda$, let us write $x \sim y$ if they are neighboring sites in Λ, that is, $\|x-y\| = 1$, where $\|\cdot\|$ denotes the Euclidean distance in Λ where the periodic boundary condition has to be taken into account.

- Define the Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ as

$$H(\sigma) = \sum_{x \sim y} \mathbf{1}\{\sigma(x) \neq \sigma(y)\} \quad ; \, \sigma \in \mathcal{X} \, . \tag{4.2}$$

  Note that there is no external field in this Hamiltonian; only the spin–spin interaction is considered.

- Denote by $\mu_\beta(\cdot)$ the Gibbs measure on $\mathcal{X}$ associated to the Hamiltonian H at inverse temperature $\beta > 0$, i.e.,

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} \quad ; \, \sigma \in \mathcal{X} \, , \tag{4.3}$$

  where Z_β is the partition function defined by

$$Z_\beta = \sum_{\sigma \in \mathcal{X}} e^{-\beta H(\sigma)} \, . \tag{4.4}$$

The spin system on Λ corresponding to the probability measure $\mu_\beta(\cdot)$ on $\mathcal{X}_d$ is called the Ising model.

Ground states

We denote by $\boxplus \in \mathcal{X}$ (resp. $\boxminus \in \mathcal{X}$) the configuration such that all spins are $+$ (resp. $-$), i.e., $\boxplus(x) = +$ (resp. $\boxminus(x) = -$) for all $x \in \Lambda$. We write

$$\mathcal{S} = \{\boxplus, \boxminus\} \subset \mathcal{X} \, . \tag{4.5}$$

Note that the Hamiltonian $H(\cdot)$ attains its minimum value 0 (only) at $\mathcal{S}$. Hence, $\boxplus$ and $\boxminus$ are the ground states of the model. Based on this observation, we obtain the following characterization of the partition function Z_β defined in (4.4), as well as the Gibbs measure μ_β as $\beta \to \infty$.

Proposition 4.2. *The following hold:*

1. *The partition function satisfies the asymptotics*

$$Z_\beta = 2 + O(e^{-2\beta}) \, . \tag{4.6}$$

2. We have

$$\lim_{\beta\to\infty} \mu_\beta(\boxplus) = \lim_{\beta\to\infty} \mu_\beta(\boxminus) = \frac{1}{2}\ , \quad \textit{and thus } \lim_{\beta\to\infty} \mu_\beta(\mathcal{S}) = 1\ .$$

Proof. We can readily observe that $H(\sigma) \ge 2$ for $\sigma \notin \mathcal{S}$. The estimate (4.6) comes directly from this observation along with the expression (4.4). Part (2) of the theorem follows directly from part (1) and the expression (4.3) of μ_β. □

Continuous-time Metropolis dynamics

We now define a continuous-time Metropolis-type Glauber dynamics which is a standard heat-bath dynamics in the study of the Ising model (cf. [51]). For $x \in \Lambda$, we denote by $\sigma^x \in \mathcal{X}$ the configuration obtained from σ by flipping the spin at site x.

Definition 4.3. The continuous-time Metropolis dynamics is defined as a continuous time Markov process $\{\sigma_\beta(t)\}_{t\ge 0}$ on $\mathcal{X}$ with transition rates

$$c_\beta(\sigma,\, \zeta) = \begin{cases} e^{-\beta[H(\zeta)-H(\sigma)]_+} & \text{if } \zeta = \sigma^x \neq \sigma \text{ for some } x \in \Lambda\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.7}$$

where $[a]_+ = \max\{a,\, 0\}$.

For $\sigma,\, \zeta \in \mathcal{X}$, we write $\sigma \sim \zeta$ if $c_\beta(\sigma,\, \zeta) > 0$, i.e., if ζ is obtained from σ by flipping the spin at a site (or vice versa). Note that the relationship $\sigma \sim \zeta$ does not depend on β. Moreover, the following detailed balance condition holds:

$$\mu_\beta(\sigma)\, c_\beta(\sigma,\, \zeta) = \mu_\beta(\zeta)\, c_\beta(\zeta,\, \sigma) = \begin{cases} \min\{\mu_\beta(\sigma),\, \mu_\beta(\zeta)\} & \text{if } \sigma \sim \zeta\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.8}$$

Consequently, $\mu_\beta(\cdot)$ is the unique[5] invariant measure for the Markov process $\sigma_\beta(\cdot)$, and furthermore $\sigma_\beta(\cdot)$ is reversible with respect to $\mu_\beta(\cdot)$. We denote by $\mathbb{P}_\sigma^\beta$ the law of the process $\sigma_\beta(\cdot)$ starting from σ, and by $\mathbb{E}_\sigma^\beta$ the associated expectation.

Metastability of the model

The primary concern in this second part is the metastable behavior of the process $\sigma_\beta(\cdot)$ defined above when β is large. More precisely, by the expression (4.7) of the jump rate, we can see that the dynamics $\sigma_\beta(\cdot)$ tends to lower the energy (for large β)

[5]It is clear that the Markov process $\sigma_\beta(\cdot)$ is irreducible.

since it jumps to a configuration with higher energy with exponentially small rate. Hence, in view of Proposition 4.2, the process $\sigma_\beta(\cdot)$ starting from a configuration $\boxplus$ may tend to stay in some neighborhood of $\boxplus$ for a long time. However, by the irreducibility of the process $\sigma_\beta(\cdot)$, it will eventually make a transition to $\boxminus$. Similar behavior is expected to occur when the process starts from $\boxminus$. Hence, such rare transitions between $\boxplus$ and $\boxminus$ will take place successively. This type of behavior is the metastable behavior of the process $\sigma_\beta(\cdot)$. In this part, we wish to quantitatively analyze this behavior to a precise level. For instance, we will give precise asymptotic of the mean transition time from $\boxplus$ to $\boxminus$ in the very low temperature regime, i.e., when $\beta \to \infty$.

4.2 Main results

We now explain the main results regarding the metastability of the stochastic Ising model.

Energy barrier between ground states

We first explain the energy barrier between $\boxplus$ and $\boxminus$.

- A sequence of configurations $(\omega_t)_{t=0}^T = (\omega_0, \omega_1, \dots, \omega_T) \subseteq \mathcal{X}$ for some $T \ge 0$ is called a *path* if $\omega_t \sim \omega_{t+1}$ for all $t \in [\![0, T-1]\!]$. A path $(\omega_t)_{t=0}^T$ is a path connecting two configurations σ and ζ in $\mathcal{X}$ if $\omega_0 = \sigma$ and $\omega_T = \zeta$ or vice versa.
- The *communication height* between two configurations $\sigma, \zeta \in \mathcal{X}$ is defined by

  $$\Phi(\sigma, \zeta) = \min_{(\omega_t)_{t=0}^T} \max_{t \in [\![0, T]\!]} H(\omega_t) ,$$

  where the minimum is taken over all paths connecting σ and ζ.
- The *energy barrier* between ground states is defined by

  $$\Gamma = \Gamma(K, L) := \Phi(\boxplus, \boxminus) = \Phi(\boxminus, \boxplus) ,$$

  where the last equality holds from the symmetry of the model.

The following result has been verified in [49]. We note that we have assumed $L \ge K \ge 5$.

Theorem 4.4. *The energy barrier is given by* $\Gamma = 2K + 2$.

The proof of this theorem is given in [49] based on combinatorial arguments. We do not give the proof of this in the current note in order to focus more on the role of potential theory in the analysis of the current model.

Eyring–Kramers law

Notation 4.5. In the current part, a collection $(a_\beta = a_\beta(K, L))_{\beta>0}$ of real numbers is written as $a_\beta = o_\beta(1)$ if $\lim_{\beta\to\infty} a_\beta = 0$ for all K and L.

By Theorem 4.4 and the large deviation principle, one can deduce (cf. [49]) the following estimate of the mean transition time $\mathbb{E}^\beta_{\boxplus}[\tau_{\boxminus}]$ and $\mathbb{E}^\beta_{\boxminus}[\tau_{\boxplus}]$:

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}^\beta_{\boxplus}[\tau_{\boxminus}] = \lim_{\beta\to\infty} \frac{1}{\beta} \log \mathbb{E}^\beta_{\boxminus}[\tau_{\boxplus}] = \Gamma \; . \tag{4.9}$$

Note that $\tau_{\boxplus}$ and $\tau_{\boxminus}$ represent hitting time of the set $\{\boxplus\}$ and $\{\boxminus\}$, respectively.

Along with the potential theory explained in the first part, we can derive the precise sub-exponential prefactor of the previous large-deviation estimate to get sharp asymptotics of the mean transition time.

Theorem 4.6. *There exists a constant $\kappa = \kappa(K, L) > 0$ such that*

$$\mathbb{E}^\beta_{\boxplus}[\tau_{\boxminus}] = \mathbb{E}^\beta_{\boxminus}[\tau_{\boxplus}] = (1 + o_\beta(1))\, \kappa e^{\Gamma\beta} \; . \tag{4.10}$$

Moreover, the constant κ satisfies

$$\lim_{K\to\infty} \kappa(K, L) = \begin{cases} 1/4 & \text{if } K < L \; , \\ 1/8 & \text{if } K = L \; . \end{cases} \tag{4.11}$$

Precise asymptotics such as (4.10) are called the *Eyring–Kramers law* (cf. [7] for more detail) for the Metropolis dynamics $\sigma_\beta(\cdot)$. The constant κ is explained more precisely later. Although we have not provided the formula for the constant κ at this point, there exists a complicated but explicit expression for this constant (cf. (5.8), Proposition 8.12 and Remark 8.13).

This theorem is the main result for the current part. The proof is divided into several stages. Firstly, in Section 5, we use the potential theory to reduce the proof of Theorem 4.6 to a capacity estimate. To estimate the capacity to a precise level, we need a much more accurate understanding of the energy landscape than that needed to derive (4.9). This analysis of the energy landscape is carried out in Sections 6-8. Then, the capacity estimate will be carried out in Sections 9 and 10 based on the Dirichlet principle and the generalized Thomson principle, respectively.

Remark 4.7. The followings are some comments on Theorem 4.6.

1. If $K < L$, there is only one direction for the transition between ground states, whereas if $K = L$, there are two possible directions. This is the reason for the dependency in the asymptotics of κ on the relation between K and L.

2. The constant Γ is model-independent, in the sense that it will be the same for other Glauber dynamics. However, the constant κ is model-dependent. For other Glauber dynamics, this constant may be different.

5 Application of Potential-Theoretic Approach

The proof of Theorem 4.6 is based on the potential-theoretic arguments developed in [14] and accurate analyses of the energy landscape. In this section, based on the argument developed in [14] along with the Dirichlet and the generalized Thomson principle (cf. Theorem 2.1) for reversible Markov processes, we reduce the proof of Theorem 4.6 to constructions of a test function and a test flow in Propositions 5.2 and 5.3, respectively.

5.1 Main capacity estimate

We first introduce the potential-theoretic notions. These notions are introduced in Section 0.2, but we rename these objects in the context of the Ising model.

- The *Dirichlet form* $\mathscr{D}_\beta(\cdot)$ associated with the reversible process $\sigma_\beta(\cdot)$ is given by, for $f : \mathcal{X} \to \mathbb{R}$,

$$\mathscr{D}_\beta(f) = \frac{1}{2} \sum_{\sigma,\, \zeta \in \mathcal{X}} \mu_\beta(\sigma)\, c_\beta(\sigma,\, \zeta)\, \{f(\zeta) - f(\sigma)\}^2 \ . \tag{5.1}$$

- Let $\mathcal{P}$ and $\mathcal{Q}$ be disjoint and non-empty subsets of $\mathcal{X}$. The *equilibrium potential* between $\mathcal{P}$ and $\mathcal{Q}$ is the function $h^\beta_{\mathcal{P},\,\mathcal{Q}} : \mathcal{X} \to \mathbb{R}$ defined by

$$h^\beta_{\mathcal{P},\,\mathcal{Q}}(\sigma) = \mathbb{P}^\beta_\sigma\left[\tau_{\mathcal{P}} < \tau_{\mathcal{Q}}\right] , \tag{5.2}$$

  and the *capacity* between $\mathcal{P}$ and $\mathcal{Q}$ is defined by

$$\mathrm{cap}_\beta(\mathcal{P},\, \mathcal{Q}) = \mathscr{D}_\beta(h^\beta_{\mathcal{P},\,\mathcal{Q}}) \ . \tag{5.3}$$

The following theorem is the main capacity estimate.

Theorem 5.1. *We have that*

$$\mathrm{cap}_\beta(\boxplus,\, \boxminus) = \frac{1 + o_\beta(1)}{2\kappa}\, e^{-\Gamma\beta} \ , \tag{5.4}$$

where κ is the constant appearing in Theorem 4.6.

Before proceeding to the proof of Theorem 5.1, we first explain the proof of Theorem 4.6 by assuming Theorem 5.1.

Proof of Theorem 4.6. Since $\mathbb{E}^\beta_{\boxplus}\left[\tau_{\boxminus}\right] = \mathbb{E}^\beta_{\boxminus}\left[\tau_{\boxplus}\right]$ by symmetry, we only focus on the estimate of $\mathbb{E}^{\beta}_{\boxplus}[\tau_{\boxminus}]$. By Proposition 0.14, (or more precisely, by (0.32)), we have

$$\mathbb{E}^{\beta}_{\boxplus}[\tau_{\boxminus}] = \frac{1}{\mathrm{cap}_{\beta}(\boxplus,\, \boxminus)} \sum_{\sigma\in\mathcal{X}} \mu_{\beta}(\sigma)\, h_{\boxplus,\,\boxminus}(\sigma)\ . \tag{5.5}$$

By Proposition 4.2 and the fact that $h_{\boxplus,\,\boxminus}(\boxplus) = 1$ and $h_{\boxplus,\,\boxminus}(\boxminus) \equiv 0$, we rewrite the last summation as

$$\frac{1}{2} + o_{\beta}(1) + \sum_{\sigma\in\mathcal{X}\setminus\mathcal{S}} \mu_{\beta}(\sigma)\, h_{\boxplus,\,\boxminus}(\sigma)\ .$$

Since $|h_{\boxplus,\,\boxminus}| \le 1$, again by Proposition 4.2, we have

$$\Big| \sum_{\sigma\in\mathcal{X}\setminus\mathcal{S}} \mu_{\beta}(\sigma)\, h_{\boxplus,\,\boxminus}(\sigma) \Big| \le \mu_{\beta}(\mathcal{X}\setminus\mathcal{S}) = o_{\beta}(1)\ .$$

In summary, we obtain

$$\sum_{\sigma\in\mathcal{X}} \mu_{\beta}(\sigma)\, h_{\boxplus,\,\boxminus}(\sigma) = \frac{1}{2} + o_{\beta}(1)\ .$$

Now, inserting this and Theorem 5.1 to (5.5), we can complete the proof. $\square$

5.2 The constant κ

To explain the main result for the capacity estimate, we first have to introduce the bulk constant $\mathfrak{b}$ and the edge constant $\mathfrak{e}$. The reason for the choice of the words "bulk" and "edge" will become clear as we analyze the energy landscape more deeply (cf. Remark 9.4).

Firstly, the bulk constant $\mathfrak{b}$ is defined explicitly as

$$\mathfrak{b} = \begin{cases} \frac{(K+2)(L-4)}{4KL} & \text{if } K < L\ , \\ \frac{(K+2)(L-4)}{8KL} & \text{if } K = L\ . \end{cases} \tag{5.6}$$

On the other hand, we do not provide a precise definition of the edge constant $\mathfrak{e}$ at this point. This is a complicated constant defined in (8.24) which satisfies (cf. Proposition 8.12)

$$0 < \mathfrak{e} \le \frac{1}{L}\ . \tag{5.7}$$

We stress that these constants depend on K and L even though the dependency is not highlighted in the notation.

Now, we define the constant κ as

$$\kappa = \mathfrak{b} + 2\mathfrak{e} \,. \tag{5.8}$$

We note that the bulk constant $\mathfrak{b}$ is the constant associated to the bulk part of the transition between $\boxplus$ and $\boxminus$, while the edge constant $\mathfrak{e}$ is related to the edge behavior of the transition. Since there are two edge parts (around $\boxplus$ and around $\boxminus$), the constant 2 has been multiplied in front of $\mathfrak{e}$ in (5.8). Moreover, one can readily observe that, when K (and hence L) is large, the edge constant $\mathfrak{e}$ is much smaller than $\mathfrak{b}$. Hence, the bulk effect dominates the edge effect. We also note that (4.11) follows directly from (5.6) and (5.7).

5.3 Capacity estimate

The upper bound estimate is based on the Dirichlet principle for reversible Markov processes (Theorem 1.3). To use this principle, we will prove the following proposition.

Proposition 5.2. *There exists a function $f_0 : \mathcal{X} \to \mathbb{R}$ such that $f_0 \in \mathfrak{C}_{1,0}(\{\boxplus\}, \{\boxminus\})$ and that*

$$\mathscr{D}_\beta(f_0) = \frac{1 + o_\beta(1)}{2\kappa} \, e^{-\Gamma\beta} \,. \tag{5.9}$$

Finding the test function f_0 requires a deep insight into the energy landscape, as well as the typical patterns of the Metropolis dynamics in a suitable neighborhood of saddle configurations. We construct this test function and prove Proposition 5.2 in Section 9.

To explain the lower bound of the capacity, we use the generalized Thomson principle (Theorem 2.1). For convenience, we write the flow norm associated with the process $\sigma_\beta(\cdot)$ as $\|\cdot\|_\beta$. We shall prove the following proposition later to establish the lower bound of the capacity.

Proposition 5.3. *There exists a flow ψ_0 such that*

$$\|\psi_0\|_\beta^2 = (2 + o_\beta(1))\, \kappa \, e^{\Gamma\beta} \qquad \text{and} \qquad \sum_{\sigma \in \mathcal{X}} h^\beta_{\boxplus,\boxminus}(\sigma)\, (\operatorname{div} \psi_0)(\sigma) = 1 + o_\beta(1) \,. \tag{5.10}$$

We construct the test flow ψ_0 in Section 10 (cf. Definition 10.1), and then verify in the same section that our test flow ψ_0 indeed satisfies (5.10).

We now prove Theorem 5.1 by assuming Propositions 5.2 and 5.3.

Proof of Theorem 5.1. By Theorem 1.3 and Proposition 5.2, we get

$$\mathrm{cap}_\beta(\boxplus, \boxminus) \le \mathscr{D}_\beta(f_0) = \frac{1+o_\beta(1)}{2\kappa}\, e^{-\Gamma\beta}\ . \tag{5.11}$$

On the other hand, by Theorem 2.1 and Proposition 5.3, we obtain

$$\mathrm{cap}_\beta(\boxplus, \boxminus) \ge \frac{1}{\|\psi_0\|_\beta^2} \Big[\sum_{\sigma\in\mathcal{X}} h^\beta_{\boxplus,\,\boxminus}(\sigma)\,(\mathrm{div}\,\psi_0)(\sigma) \Big]^2 = \frac{1+o_\beta(1)}{2\kappa}\, e^{-\Gamma\beta}\ . \tag{5.12}$$

The proof is completed by (5.11) and (5.12). □

Hence, to prove Theorem 4.6, it only remains to prove Propositions 5.2 and 5.3. The proof is given in the remainder of Part II.

6 Neighborhood of Configurations

For $c \in \mathbb{R}$, a path $(\omega_t)_{t=0}^T$ in $\mathcal{X}$ is called a *c-path* if we have $H(\omega_t) \le c$ for all $t \in [\![0, T]\!]$. Heuristically, if two configurations are connected by a $(\Gamma - 1)$-path, in a suitable sense, these two configurations are indistinguishable in the transition scale $e^{\beta\Gamma}$, since $\sigma_\beta(\cdot)$ commutes them in a shorter scale. Moreover, if two configurations are not connected by a Γ-path, the process $\sigma_\beta(\cdot)$ cannot commute these two configurations in the transition scale $e^{\beta\Gamma}$. The following definition of neighborhoods is inspired from these observations.

Definition 6.1 (Neighborhood of configurations)**.**

1. For $\sigma \in \mathcal{X}$, the *neighborhood* $\mathcal{N}(\sigma)$ and the *extended neighborhood* $\widehat{\mathcal{N}}(\sigma)$ are defined as

$$\mathcal{N}(\sigma) = \{\zeta \in \mathcal{X} : \exists \text{ a } (\Gamma-1)\text{-path } (\omega_t)_{t=0}^T \text{ connecting } \sigma \text{ and } \zeta\} \text{ and}$$
$$\widehat{\mathcal{N}}(\sigma) = \{\zeta \in \mathcal{X} : \exists \text{ a } \Gamma\text{-path } (\omega_t)_{t=0}^T \text{ connecting } \sigma \text{ and } \zeta\}\ .$$

   If $H(\sigma) > \Gamma - 1$ (resp. $H(\sigma) > \Gamma$), we set $\mathcal{N}(\sigma) = \emptyset$ (resp. $\widehat{\mathcal{N}}(\sigma) = \emptyset$).

2. For $\mathcal{P} \subseteq \mathcal{X}$, we define

$$\mathcal{N}(\mathcal{P}) = \bigcup_{\sigma \in \mathcal{P}} \mathcal{N}(\sigma) \quad \text{and} \quad \widehat{\mathcal{N}}(\mathcal{P}) = \bigcup_{\sigma \in \mathcal{P}} \widehat{\mathcal{N}}(\sigma)\ .$$

3. A path $(\omega_t)_{t=0}^T$ is said to be a path in $\mathcal{A} \subset \mathcal{X}$ if $\omega_t \in \mathcal{A}$ for all $t \in [\![0, T]\!]$. For $\mathcal{Q} \subset \mathcal{X}$ and $\sigma \in \mathcal{X} \setminus \mathcal{Q}$, we define

$$\widehat{\mathcal{N}}(\sigma\,;\,\mathcal{Q}) = \{\zeta \in \mathcal{X} : \exists \text{ a } \Gamma\text{-path in } \mathcal{X} \setminus \mathcal{Q} \text{ connecting } \sigma \text{ and } \zeta\}\ .$$

   If $H(\sigma) > \Gamma$, we set $\widehat{\mathcal{N}}(\sigma\,;\,\mathcal{Q}) = \emptyset$.

4. For $\mathcal{P} \subseteq \mathcal{X}$ disjoint with $\mathcal{Q}$, define

$$\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q}) = \bigcup_{\sigma \in \mathcal{P}} \widehat{\mathcal{N}}(\sigma\,;\,\mathcal{Q})\ .$$

With this notation, Theorem 4.4 is equivalent to $\mathcal{N}(\boxplus) \cap \mathcal{N}(\boxminus) = \emptyset$ and $\widehat{\mathcal{N}}(\boxplus) = \widehat{\mathcal{N}}(\boxminus)$. Since the transition must take place in the set $\widehat{\mathcal{N}}(\mathcal{S})$, analyzing the structure of this set is crucial in the energy landscape analysis. It will be carried out in Section 8.

The following lemma is useful.

Lemma 6.2. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are disjoint subsets of $\mathcal{X}$. Then, it holds that*

$$\widehat{\mathcal{N}}(\mathcal{P}\cup\mathcal{Q})=\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})\cup\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})\ .$$

Proof. Since

$$\widehat{\mathcal{N}}(\mathcal{P}\cup\mathcal{Q})=\widehat{\mathcal{N}}(\mathcal{P})\cup\widehat{\mathcal{N}}(\mathcal{Q})\ , \tag{6.1}$$

$$\widehat{\mathcal{N}}(\mathcal{Q})\supset\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})\ ,\ \text{and}\ \widehat{\mathcal{N}}(\mathcal{P})\supset\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})\ ,$$

it immediately follows that

$$\widehat{\mathcal{N}}(\mathcal{P}\cup\mathcal{Q})\supseteq\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})\cup\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})\ . \tag{6.2}$$

Let us now prove the reversed inclusion. We now assume that there exists $\sigma\in\mathcal{X}$ such that

$$\sigma\in\widehat{\mathcal{N}}(\mathcal{P}\cup\mathcal{Q})\setminus\big[\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})\cup\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})\big]\ . \tag{6.3}$$

By (6.1), we may assume without loss of generality that

$$\sigma\in\widehat{\mathcal{N}}(\mathcal{P})\setminus\big[\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})\cup\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})\big]\ .$$

Since $\sigma\in\widehat{\mathcal{N}}(\mathcal{P})\setminus\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})$, we have $\sigma\notin\mathcal{P}$. Since $\sigma\in\widehat{\mathcal{N}}(\mathcal{P})$, we can find a Γ-path connecting σ and $\mathcal{P}$. Let us assume that $(\omega_t)_{t=0}^T$ is the shortest of all such paths. We may assume that $\omega_0=\sigma$ and $\omega_T\in\mathcal{P}$.

- Suppose first that $\omega_t\notin\mathcal{Q}$ for all $t\in[\![0,\,T-1]\!]$. Then the path $(\omega_t)_{t=0}^T$ becomes a Γ-path in $\mathcal{X}\setminus\mathcal{Q}$ connecting σ and $\mathcal{P}$. This contradicts the fact that $\sigma\notin\widehat{\mathcal{N}}(\mathcal{P}\,;\,\mathcal{Q})$.

- Suppose next that $\omega_{t_0}\in\mathcal{Q}$ for some $t_0\in[\![0,\,T-1]\!]$. Then, by the minimality assumption on the length of $(\omega_t)_{t=0}^T$, we must have $\omega_t\notin\mathcal{P}$ for all $t\in[\![0,\,T-1]\!]$. Consequently, $(\omega_t)_{t=0}^{t_0}$ becomes a path in $\mathcal{X}\setminus\mathcal{P}$ connecting σ and $\mathcal{Q}$, and hence we get a contradiction to the fact $\sigma\notin\widehat{\mathcal{N}}(\mathcal{Q}\,;\,\mathcal{P})$.

Therefore, there is no σ satisfying (6.3), and we have proved the reversed inclusion relation of (6.2). □

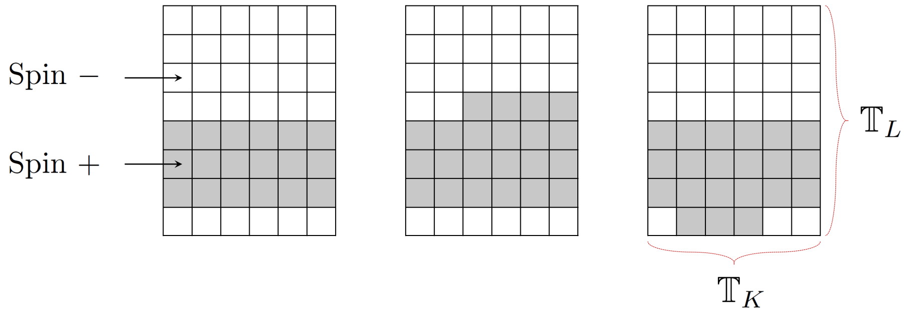

Figure 7.1: **Canonical configurations for** $(K,\, L) = (6,\, 8)$**.** White and gray boxes correspond to a box with $-$ spin and $+$ spin, respectively. Three figures represent configurations $\zeta_{2,\,3}$, $\zeta^{\mathrm{up}}_{2,\,3\,;\,3,\,4}$, and $\zeta^{\mathrm{down}}_{2,\,3\,;\,2,\,3}$, respectively.

7 Canonical Configurations and Paths

Now, we begin to analyze the energy landscape. In this section, we introduce the canonical configurations and paths, and then investigate their properties. Based on these, we study the typical configurations in the next section.

7.1 Canonical configurations

Definition 7.1 (Canonical configurations)**.** We refer to Figure 7.1 for an illustration of examples of the canonical configurations defined below. Before defining complicated notations, we note that k and ℓ are used to represent elements of $\mathbb{T}_K$ and $\mathbb{T}_L$, respectively, and v and h are used to denote vertical and horizontal lengths, respectively.

- For $\ell \in \mathbb{T}_L$ and $v \in [\![0,\, L]\!]$, denote by $\zeta_{\ell,\,v} \in \mathcal{X}$ the configuration whose spins are $+$ on
$$\mathbb{T}_K \times \{\ell + n \in \mathbb{T}_L : n \in [\![0,\, v-1]\!] \subseteq \mathbb{Z}\}\ .$$
and $-$ on the remainder. Hence, we have $\zeta_{\ell,\,0} = \boxminus$ and $\zeta_{\ell,\,L} = \boxplus$ for all $\ell \in \mathbb{T}_L$. For $v \in [\![0,\, L]\!]$, write
$$\mathcal{R}_v = \{\zeta_{\ell,\,v} : \ell \in \mathbb{T}_L\}\ . \tag{7.1}$$

- For $(\ell,\, k) \in \mathbb{T}_L \times \mathbb{T}_K$ and $(v,\, h) \in [\![0,\, L-1]\!] \times [\![0,\, K]\!]$, denote by $\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,h} \in \mathcal{X}$ the configuration whose spins are $+$ on
$$\{x \in \Lambda : \zeta_{\ell,\,v}(x) = +\} \cup \big[\,\{k + n \in \mathbb{T}_K : n \in [\![0,\, h-1]\!] \subseteq \mathbb{Z}\} \times \{\ell + v\}\,\big]$$
and $-$ on the remainder. Similarly, denote by $\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,h} \in \mathcal{X}$ whose spins are $+$

on

$$\{x \in \Lambda : \zeta_{\ell,\,v}(x) = +\} \cup \big[\,\{k+n \in \mathbb{T}_K : n \in [\![0,\, h-1]\!] \subseteq \mathbb{Z}\} \times \{\ell-1\}\,\big]$$

and $-$ on the remainder. Namely, the configuration $\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,h}$ (resp. $\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,h}$) is obtained from $\zeta_{\ell,\,v}$ by attaching a protuberance of spin $+$ of size h at the upper (resp. lower) side of the cluster of spin $+$ of $\zeta_{\ell,\,v}$.

- For $v \in [\![0,\, L-1]\!]$, define

$$\mathcal{Q}_v = \bigcup_{k\in\mathbb{T}_K} \bigcup_{h=1}^{K-1} \{\zeta^{\mathbf{up}}_{\ell,\,v\,;\,k,\,h},\, \zeta^{\mathbf{down}}_{\ell,\,v\,;\,k,\,h}\}\ . \tag{7.2}$$

  Hence, $\mathcal{Q}_v$ consists of configurations between $\mathcal{R}_v$ and $\mathcal{R}_{v+1}$.

- Finally, define

$$\mathcal{C} = \bigcup_{v=0}^{L} \mathcal{R}_v \cup \bigcup_{v=0}^{L-1} \mathcal{Q}_v\ .$$

  In the current note, the *canonical configurations* are the configurations belonging to $\mathcal{C}$.

Remark 7.2. By a direct computation, we can readily verify that $H(\sigma) \le \Gamma$ for all $\sigma \in \mathcal{C}$. In particular, we have

$$H(\sigma) = \begin{cases} \Gamma - 2 & \text{if } \sigma \in \mathcal{R}_v \text{ for some } v \in [\![1,\, L-1]\!]\ , \\ \Gamma & \text{if } \sigma \in \mathcal{Q}_v \text{ for some } v \in [\![1,\, L-2]\!]\ . \end{cases}$$

For the clarity of the discussion, we henceforth assume that $K < L$. The case $K = L$ will be discussed in Section 11. Note that the only difference for the case $K = L$ is that the configuration obtained by rotating a canonical configuration in $\mathcal{C}$ must play the same role, unlike the case $K < L$. This fact can be readily taken into account in the computations, and we refer to Section 11 or [27] for further details. Note that for $K < L$, the rows and columns play completely different roles.

7.2 Canonical paths

We now explain the crucial role of canonical configurations by describing canonical paths between $\boxminus$ and $\boxplus$ consisting of canonical configurations. The following notation is useful.

Notation 7.3. Suppose that $N \ge 2$ is a positive integer.

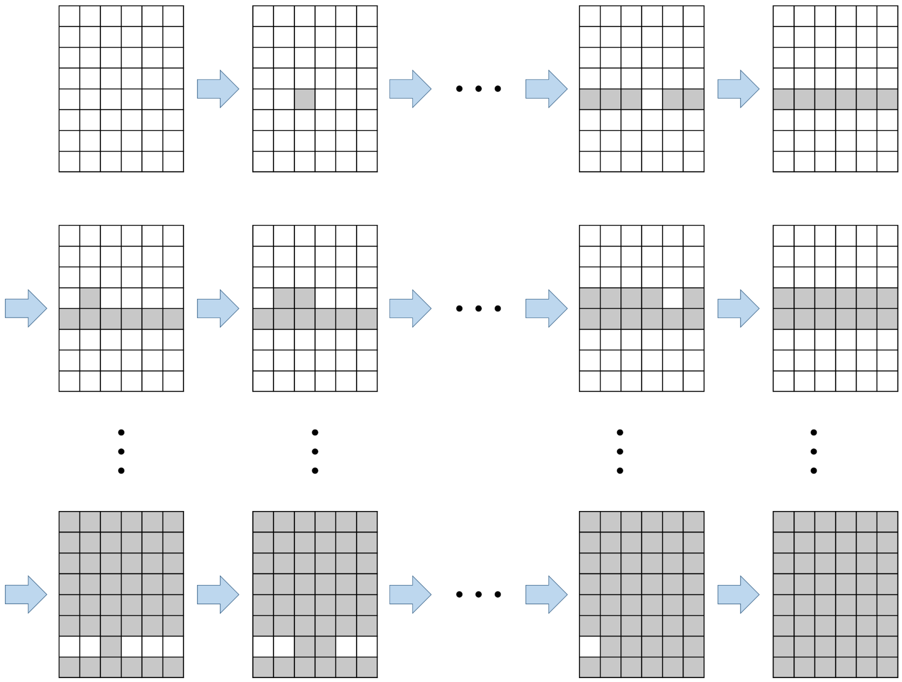

Figure 7.2: Example of a canonical path for $(K,\, L) = (6,\, 8)$.

- Denote by $\mathfrak{S}_N$ the collection of all connected subsets of $\mathbb{T}_N$, i.e.,

$$\mathfrak{S}_N = \{P \subseteq \mathbb{T}_N : P = [\![i,\, j]\!] \text{ for some } i,\, j \in \mathbb{T}_N \text{ or } P = \emptyset\} \,. \tag{7.3}$$

  Here, the set $[\![i,\, j]\!] \subseteq \mathbb{T}_N$ represents the set $\{i,\, i+1,\, \dots,\, j\}$. Note that this set can be defined even for $j < i$. For instance, for $N = 6$, the set $[\![4,\, 2]\!]$ represents $\{4,\, 5,\, 6,\, 1,\, 2\}$.

- For two sets $P,\, P' \in \mathfrak{S}_N$, we write $P \prec P'$ if $P \subseteq P'$ and $|P'| = |P| + 1$.

- A sequence $(P_n)_{n=0}^N$ of sets in $\mathfrak{S}_N$ is called an *increasing sequence* if

$$\emptyset = P_0 \prec P_1 \prec \cdots \prec P_N = \mathbb{T}_N \,.$$

  Note that, for an increasing sequence $(P_n)_{n=0}^N$ in $\mathfrak{S}_N$, we have that $|P_n| = n$ for all $n \in [\![0,\, N]\!]$.

Definition 7.4 (Canonical paths)**.** We refer to Figure 7.2 for an example of canonical path defined below.

1. We first introduce a *standard sequence* of subsets of $\Lambda = \mathbb{T}_K \times \mathbb{T}_L$ connecting the empty set and the full set Λ.

   (a) For $P,\, P' \in \mathfrak{S}_L$ with $P \prec P'$, a sequence $(A_t)_{t=0}^{K}$ of subsets of Λ is called a standard sequence connecting $\mathbb{T}_K \times P$ and $\mathbb{T}_K \times P'$ if there exists an increasing sequence $(Q_t)_{t=0}^{K}$ in $\mathfrak{S}_K$ such that

   $$A_t = (\mathbb{T}_K \times P) \cup \left[\, Q_t \times (P' \setminus P) \,\right] \quad \text{for all } t \in [\![0,\, K]\!] \;.$$

   (b) A sequence $(A_t)_{t=0}^{KL}$ of subsets of Λ is called a standard sequence connecting $\emptyset$ and Λ if there exists an increasing sequence $(P_\ell)_{\ell=0}^{L}$ in $\mathfrak{S}_L$ such that $A_{K\ell} = \mathbb{T}_K \times P_\ell$ for all $\ell \in [\![0,\, L]\!]$, and the sub-sequence $(A_t)_{t=K\ell}^{K(\ell+1)}$ is a standard sequence connecting $\mathbb{T}_K \times P_\ell$ and $\mathbb{T}_K \times P_{\ell+1}$ for all $\ell \in [\![0,\, L-1]\!]$.

2. A path $(\omega_t)_{t=0}^{KL}$ in $\mathcal{X}$ is called a *canonical path* connecting $\boxminus$ and $\boxplus$ if there exists a standard sequence $(A_t)_{t=0}^{KL}$ connecting $\emptyset$ and Λ such that

   $$\omega_t(i,\, j) = \begin{cases} - & \text{if } (i,\, j) \notin A_t \;, \\ + & \text{if } (i,\, j) \in A_t \;. \end{cases}$$

   It is easy to verify that $\omega_0 = \boxminus$ and $\omega_{KL} = \boxplus$. A canonical path connecting $\boxplus$ and $\boxminus$ is defined in a similar manner. We say that a path is a canonical path if it is a canonical path connecting either $\boxminus$ and $\boxplus$ or $\boxplus$ and $\boxminus$.

The following is an immediate consequence of the construction.

Lemma 7.5. *A canonical path consists only of canonical configurations. In particular, for any canonical path $(\omega_t)_{t=0}^{KL}$ connecting $\boxminus$ and $\boxplus$, we have that*

$$\max_{t \in [\![0,\, KL]\!]} H(\omega_t) = \Gamma \;.$$

Proof. The first assertion follows immediate from the construction. For the second assertion, it suffices to recall Remark 7.2. □

In view of the previous lemma and Theorem 4.4, a canonical path between $\boxminus$ and $\boxplus$ is an optimal path achieving the communication height between them. We emphasize here that the optimal transition may not always occur along this path. Indeed, transitions from $\boxminus$ to $\mathcal{R}_2$ and from $\mathcal{R}_{L-2}$ to $\boxplus$ may happen in a more complex manner, while transitions from $\mathcal{R}_2$ to $\mathcal{R}_{L-2}$ should happen along a canonical path. This issue is the main topic of the next section.

8 Typical Configurations

The crucial notion in the energy landscape analysis between ground states is the typical configurations defined in this section. A configuration σ is said to be a typical configuration if $\sigma \in \widehat{\mathcal{N}}(\mathcal{S})$. Therefore, the typical configurations comprise all the relevant configurations in the study of metastable transition between $\boxplus$ and $\boxminus$.

8.1 Typical configurations

Let us start by defining typical configurations.

Definition 8.1 (Typical configurations)**.** We refer to Figure 8.1 for an illustration of the typical configurations defined below.

- Define
$$\mathcal{B} = \bigcup_{v=2}^{L-2} \mathcal{R}_v \cup \bigcup_{v=2}^{L-3} \mathcal{Q}_v \; . \tag{8.1}$$
A configuration belonging to $\mathcal{B}$ is called a *bulk typical configuration.* Then, write
$$\mathcal{B}_\Gamma = \bigcup_{v=2}^{L-3} \mathcal{Q}_v = \{\sigma \in \mathcal{B} : H(\sigma) = \Gamma\} \; .$$
- Define
$$\mathcal{E}^+ = \widehat{\mathcal{N}}(\boxplus\, ;\, \mathcal{B}_\Gamma) \quad \text{and} \quad \mathcal{E}^- = \widehat{\mathcal{N}}(\boxminus\, ;\, \mathcal{B}_\Gamma) \; . \tag{8.2}$$
Then, we define $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$. A configuration belonging to $\mathcal{E}$ is called an *edge typical configuration.*

A configuration belonging to $\mathcal{B} \cup \mathcal{E}$ is called a *typical configuration.* Indeed, it holds that $\mathcal{B} \cup \mathcal{E} = \widehat{\mathcal{N}}(\mathcal{S})$, and this will be verified later.

Now, we explain the reason why we have decomposed typical configurations into bulk and edge configurations. A typical transition from $\boxminus$ to $\boxplus$ of the Metropolis dynamics can be divided into three stages. Firstly, the process passes through $\mathcal{E}^-$ to arrive at $\mathcal{B}$. Then, it goes through $\mathcal{B}$ along the canonical configurations to arrive at $\mathcal{E}^+$. Finally, the process reaches at $\boxplus$ by passing through $\mathcal{E}^+$. The behavior of the process at the second stage (i.e., in the bulk) is relatively clear, and we can understand the behavior in great detail. On the other hand, the behavior of the Metropolis dynamics on $\mathcal{E}^-$ and $\mathcal{E}^+$ is complex, and can be explained in terms of an auxiliary Markov chain defined in Definition 8.10. We are not able to write the constant appearing in the Eyring–Kramers law in a simple manner because of this complex behavior of the Metropolis dynamics in the edge typical configurations.

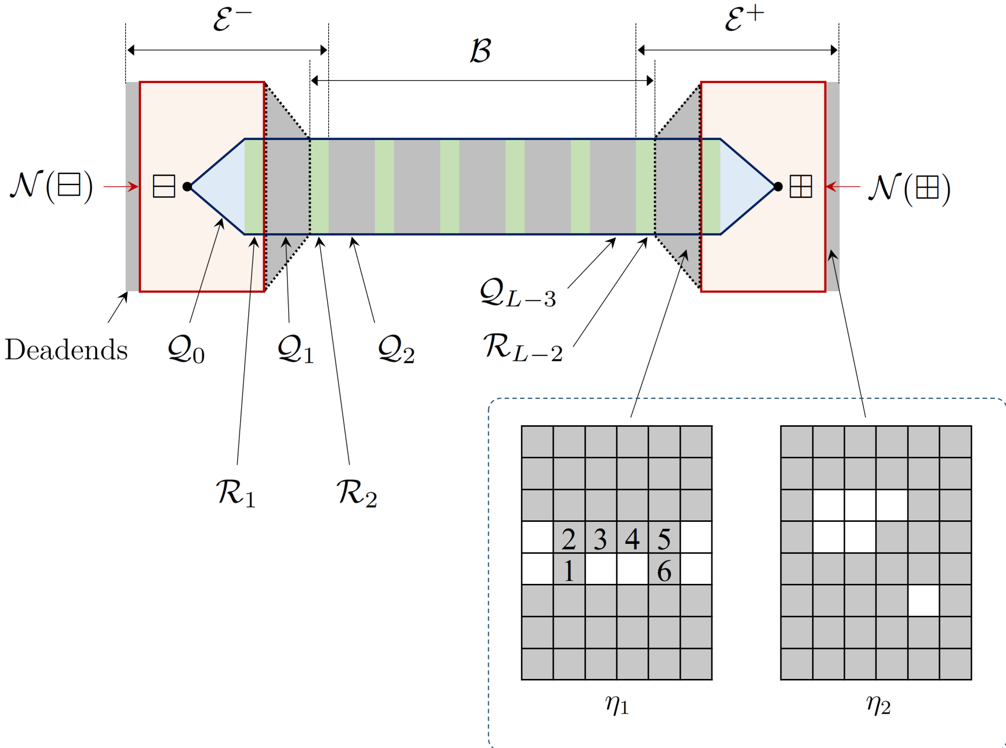

Figure 8.1: **Structure of $\widehat{\mathcal{N}}(\mathcal{S})$ and typical configurations.** Regions consisting of configurations with energy Γ are colored gray. As we will verify in Proposition 8.9, we can observe that $\widehat{\mathcal{N}}(\mathcal{S}) = \mathcal{E}^- \cup \mathcal{E}^+ \cup \mathcal{B}$, $\mathcal{E}^- \cap \mathcal{B} = \mathcal{R}_2$, and $\mathcal{E}^+ \cap \mathcal{B} = \mathcal{R}_{L-2}$. The hexagonal region enclosed by the blue line denotes the set $\mathcal{C}$ of canonical configurations between $\boxminus$ and $\boxplus$. The set $\mathcal{E}^+$ of edge typical configurations around $\boxplus$ consists of four regions. The first one is the neighborhood $\mathcal{N}(\boxplus)$ denoted by the red-enclosed box and the second one is $\mathcal{R}_{L-2}$. The third one is the region consisting of configurations with energy Γ which are connected to $\mathcal{R}_{L-2}$ via a Γ-path in $\mathcal{X} \setminus \mathcal{N}(\boxplus)$. An example of a configuration belonging to this region is η_1. In particular, the configuration η_1 is connected with a configuration in $\mathcal{R}_{L-2}$ via a Γ-path in $\mathcal{X} \setminus \mathcal{N}(\boxplus)$ which is obtained by updating six grey boxes by the order indicated in the figure. The last region is the collection of the dead-ends attached to $\mathcal{N}(\boxplus)$. This is a collection of configurations with energy Γ which are not connected to $\mathcal{R}_{L-2}$ via a Γ-path in $\mathcal{X} \setminus \mathcal{N}(\boxplus)$. An example of a dead-end configuration is η_2 which has energy $14 = 2 \cdot 6 + 2 = 2K + 2$. A similar decomposition holds for $\mathcal{E}^-$.

8.2 Characterization of configurations with low energy

To investigate the typical configurations defined above, in this subsection, we fully characterize the configurations which have energy less than Γ. Write

$$\|\sigma\|_+ = \sum_{x\in\Lambda} \mathbf{1}\{\sigma(x) = +\} \quad \text{and} \quad \|\sigma\|_- = \sum_{x\in\Lambda} \mathbf{1}\{\sigma(x) = -\} \tag{8.3}$$

which denote the number of sites with spin $+$ and $-$, respectively.

Proposition 8.2. *Suppose that $\sigma \in \mathcal{X}$ satisfies $H(\sigma) < \Gamma$. Then, either (1) or (2) below must hold.*

1. *The configuration σ belongs to $\mathcal{R}_v$ for some $v \in [\![2,\, L-2]\!]$. In particular, $\mathcal{N}(\sigma) = \{\sigma\}$.*
2. *The configuration σ belongs to $\mathcal{N}(\boxplus)$ or $\mathcal{N}(\boxminus)$.*

Remark 8.3. Two neighborhoods $\mathcal{N}(\boxplus)$ and $\mathcal{N}(\boxminus)$ are disjoint by Theorem 4.4.

Notation 8.4.
- A *horizontal bridge* (resp. *vertical bridge*) is a row (resp. column), in which all spins are identical. If a bridge consists of spin $+$ (resp. $-$), we call this bridge a $+$-bridge (resp. $-$-bridge). Then, we denote by $B_\pm(\sigma)$ the number of $\pm$-bridges in $\sigma \in \mathcal{X}$.
- A *cross* is a union of a horizontal bridge and a vertical bridge. A cross consisting of spin $+$ (resp. $-$) is called a $+$-cross (resp $-$-cross).
- We denote by $r_1, \dots, r_L$ the rows and $c_1, \dots, c_K$ the columns of $\Lambda = \mathbb{T}_K \times \mathbb{T}_L$. For $(v, h) \in [\![1,\, L]\!] \times [\![1,\, K]\!]$ and $\sigma \in \mathcal{X}$, we define

  $$\begin{aligned} H_{r_v}(\sigma) &= \sum_{x,\, y\in r_v:\, x\sim y} \mathbf{1}\{\sigma(x) \neq \sigma(y)\} \text{ and} \\ H_{c_h}(\eta) &= \sum_{x,\, y\in c_h:\, x\sim y} \mathbf{1}\{\sigma(x) \neq \sigma(y)\}\ , \end{aligned}$$

  so that we can decompose the Hamiltonian in a way that

  $$H(\sigma) = \sum_{v=1}^{L} H_{r_v}(\sigma) + \sum_{h=1}^{K} H_{c_h}(\sigma)\ . \tag{8.4}$$

  A horizontal (resp. vertical) edge denotes an edge belonging to a row (resp. column).

The following lower bound for the Hamiltonian is a consequence of notations and observations above.

Lemma 8.5. *It holds that*

$$H(\sigma) \geq 2\,[\,K + L - B_+(\sigma) - B_-(\sigma)\,]\,.$$

Proof. The lemma follows directly from (8.4) and the fact that $H_{r_v}(\sigma) \geq 2$ (resp. $H_{c_h}(\sigma) \geq 2$) if r_v (resp. c_h) is not a bridge. $\square$

We are now ready to prove Proposition 8.2.

Proof of Proposition 8.2. Fix $\sigma \in \mathcal{X}$ with $H(\sigma) < \Gamma = 2K + 2$. By Lemma 8.5, we have

$$2K + 1 \geq 2\Big[\,K + L - B_+(\sigma) - B_-(\sigma)\,\Big]\,,$$

and therefore $B_+(\sigma) + B_-(\sigma) \geq L$. Namely, there are at least L bridges. Let us take one of them and assume without loss of generality that this is a $+$-bridge. Now, we consider three cases separately.

(Case 1: σ has a $+$-horizontal bridge without a $+$-vertical one) Since $H_{c_h}(\sigma) \geq 2$ for all $h \in [\![1,\, K]\!]$, we can observe from (8.4) that $H_{r_\ell}(\sigma) = 0$ for all $\ell \in [\![1,\, L]\!]$. This implies that all rows are monochromatic, and therefore all columns are identical. Thus, again by (8.4), we get $H_{c_k}(\sigma) = 2$ for all $k \in [\![1,\, K]\!]$, and thus $\sigma \in \mathcal{R}_v$ for some $v \in [\![1,\, L]\!]$. If $v \in [\![2,\, L-2]\!]$, then it is clear that $\mathcal{N}(\sigma)$ is a singleton since any configuration obtained from σ by flipping a spin has energy greater than or equal to Γ. Thus, σ satisfies the requirements of case (1). On the other hand, if $v \notin [\![2,\, L-2]\!]$, we can readily observe that $\sigma \in \mathcal{N}(\boxminus)$ or $\mathcal{N}(\boxplus)$.

(Case 2: σ has a $+$-vertical bridge without a $+$-horizontal one) Since $\Delta H_{r_v}(\sigma) \geq 2$ for all $v \in [\![1,\, L]\!]$, we obtain from (8.4) that $2K \geq 2L$; hence, we obtain a contradiction (to the assumption that $K < L$).

(Case 3: σ has a $+$-cross) Without loss of generality, assume that $\mathbb{T}_K \times \{1\}$ and $\{1\} \times \mathbb{T}_L$ are $+$-bridges. Let us update each spin to $+$ in $[\![2,\, K]\!] \times [\![2,\, L]\!]$ in the ascending lexicographic order. The presence of spin $+$-bridges ensures that the Hamiltonian cannot increase in the course of the updates. Since we finally arrive at $\boxplus$, we can conclude that

$$\Phi(\sigma,\, \boxplus) \leq H(\sigma) < \Gamma\,.$$

Thus, we have $\sigma \in \mathcal{N}(\boxplus)$. $\square$

8.3 Properties of typical configurations

In this subsection, we investigate the structure of typical configurations introduced above. We start from two elementary lemmas.

Lemma 8.6. *Suppose that $\sigma \in \mathcal{B}$ and $\xi \in \mathcal{X}$ satisfy $\sigma \sim \xi$ and $H(\xi) \le \Gamma$. Then, the following statements hold.*

1. *We have $\xi \in \mathcal{B} \cup \mathcal{Q}_1 \cup \mathcal{Q}_{L-2} \subseteq \mathcal{C}$.*
2. *If $\sigma \in \mathcal{R}_v$ with $v \in [\![3,\, L-3]\!]$, then $\xi \in \mathcal{B}_\Gamma$.*
3. *If $\sigma \in \mathcal{B}_\Gamma$, then $\xi \in \mathcal{B}$.*

Proof. We consider two cases separately.

- **(Case 1: $\sigma \in \mathcal{R}_v$ for some $v \in [\![2,\, L-2]\!]$)** Assume that $\sigma = \zeta_{\ell,\, v}$ for some $\ell \in \mathbb{T}_L$. We can observe from the illustration given in Figure 7.1 that the only way of flipping a spin of σ in such a way that the resulting configuration has energy at most Γ is either to attach a protuberance of spin $+$ to the cluster of spin $+$ of σ or to attach a protuberance of spin $-$ to the cluster of spin $-$ of σ. This implies that

$$\xi \in \{\zeta^{\mathrm{up}}_{\ell,\, v\,;\, k,\, 1}\,,\ \ \zeta^{\mathrm{down}}_{\ell,\, v\,;\, k,\, 1}\,,\ \ \zeta^{\mathrm{up}}_{\ell,\, v-1;\, k,\, K-1}\,,\ \ \zeta^{\mathrm{down}}_{\ell+1,\, v-1;\, k,\, K-1} : k \in \mathbb{T}_K\}\ .$$

  Hence, $\xi \in \mathcal{B} \cup \mathcal{Q}_1 \cup \mathcal{Q}_{L-2}$. This observation also implies that $\xi \in \mathcal{B}_\Gamma$ if $v \in [\![3,\, L-3]\!]$, and hence part (2) is verified here as well.

- **(Case 2: $\sigma \in \mathcal{Q}_v$ for some $v \in [\![2,\, L-3]\!]$)** Suppose that $\sigma = \zeta^{\mathbf{up}}_{\ell,\, v\,;\, k,\, h}$ for some $(k,\, \ell) \in \mathbb{T}_K \times \mathbb{T}_L$ and $h \in [\![1,\, K-1]\!]$. In this case, we can observe that the only way of flipping a spin of σ without increasing the Hamiltonian is to expand or shrink the protuberance of spin $+$ attached at $\zeta_{\ell,\, v}$, and therefore

$$\xi \in \{\zeta^{\mathrm{up}}_{\ell,\, v\,;\, k,\, h-1},\, \zeta^{\mathrm{up}}_{\ell,\, v\,;\, k+1,\, h-1}\}\ .$$

  Therefore, we have $\xi \in \mathcal{B}$ and hence parts (1) and (3) are now verified. The same conclusion also holds for the case $\sigma = \zeta^{\mathrm{down}}_{\ell,\, v\,;\, k,\, h}$.

$\square$

The previous lemma implies the following result.

Lemma 8.7. *It holds that $\widehat{\mathcal{N}}(\mathcal{B}\,;\, \mathcal{C} \setminus \mathcal{B}) = \mathcal{B}$.*

Proof. Since the energy of configurations belonging to $\mathcal{B}$ do not exceed Γ, it follows immediately that

$$\widehat{\mathcal{N}}(\mathcal{B}\,;\mathcal{C}\setminus\mathcal{B})\supset\mathcal{B}\ .$$

Now, we claim the opposite inclusion, i.e.,

$$\widehat{\mathcal{N}}(\mathcal{B}\,;\mathcal{C}\setminus\mathcal{B})\subset\mathcal{B}\ . \tag{8.5}$$

Suppose the contrary that there exists $\sigma\in\widehat{\mathcal{N}}(\mathcal{B}\,;\mathcal{C}\setminus\mathcal{B})$ such that $\sigma\notin\mathcal{B}$. Since $\sigma\in\widehat{\mathcal{N}}(\mathcal{B}\,;\mathcal{C}\setminus\mathcal{B})$, there exists a Γ-path $(\omega_t)_{t=0}^{T}$ in $\mathcal{X}\setminus(\mathcal{C}\setminus\mathcal{B})=(\mathcal{X}\setminus\mathcal{C})\cup\mathcal{B}$ connecting $\mathcal{B}$ and σ. Then, as $\omega_0\in\mathcal{B}$, and $\omega_T\notin\mathcal{B}$, we can find $t_0\in[\![0,\,T-1]\!]$ such that $\omega_{t_0}\in\mathcal{B}$ and $\omega_{t_0+1}\notin\mathcal{B}$. Since $(\omega_t)_{t=0}^{T}$ is a path in $(\mathcal{X}\setminus\mathcal{C})\cup\mathcal{B}$, we get

$$\omega_{t_0+1}\in(\mathcal{X}\setminus\mathcal{B})\cap\big[\,(\mathcal{X}\setminus\mathcal{C})\cup\mathcal{B}\,\big]\subset\mathcal{X}\setminus\mathcal{C}\ .$$

On the other hand, since $\omega_{t_0}\in\mathcal{B}$ we must have $\omega_{t_0+1}\in\mathcal{C}$ by part (1) of Lemma 8.6 and thus we have a contradiction. This proves (8.5) and the proof is finished. $\square$

Next, we prove that the two sets $\mathcal{E}^+$ and $\mathcal{E}^-$ are indeed disjoint.

Proposition 8.8. *We have that $\mathcal{E}^+\cap\mathcal{E}^-=\emptyset$.*

Proof. Suppose the contrary that there exists a path $(\omega_t)_{t=0}^{T}$ is a Γ-path from $\boxminus$ to $\boxplus$ in $\mathcal{X}\setminus\mathcal{B}_\Gamma$. Define $u:[\![0,\,T]\!]\to\mathbb{R}$ as

$$u(t)=B_+(\omega_t)\quad;\ t\in[\![0,\,T]\!]\ ,$$

where $B_+(\cdot)$ is defined in Notation 8.4. Then, we have that

$$u(0)=0\ ,\ u(T)=K+L\ ,\ \text{ and }|u(t+1)-u(t)|\le 2\text{ for all }t\in[\![0,\,T-1]\!]\ . \tag{8.6}$$

Thus, the following time t^* is well defined:

$$t^*=\min\{t\in[\![0,\,T-1]\!]:u(t),\,u(t+1)\ge 2\}\ . \tag{8.7}$$

Note that, since we need to change at least $2K-1$ spins from $\boxminus$ to get $u(t)\ge 2$, we have $t^*\ge 2K-1$. Then, by (8.6), we have $B_+(\omega_{t^*})=2$ or 3. We divide the proof into three cases as in Proposition 8.2.

(Case 1: ω_{t^*} has +-horizontal bridges without a +-vertical one) For this case, if $B_+(\omega_{t^*})=3$, we have $B_+(\omega_{t^*-1})\ge 2$ and thus we get a contradiction to the minimality of t^*. Hence, we have $B_+(\omega_{t^*})=2$.

Since ω_{t^*} does have both $+$- and $-$-vertical bridges, we get $H_{c_h}(\omega_{t^*}) \geq 2$ for all $h \in [\![1, K]\!]$. By (8.4) and the fact that $H(\omega_{t^*}) \leq \Gamma = 2K+2$, we can readily observe that $\omega_{t^*} \in \mathcal{R}_2 \cup \mathcal{Q}_2$. Since $\mathcal{Q}_2 \subseteq \mathcal{B}_\Gamma$ and since $(\omega_t)_{t=0}^T$ is a path in $\mathcal{X} \setminus \mathcal{B}_\Gamma$, we can conclude that $\omega_{t^*} \in \mathcal{R}_2$. Since $H(\omega_{t^*+1}) \leq \Gamma$ and $u(t^*+1) \geq 2$, we are forced to have $\omega_{t^*+1} \in \mathcal{B}_\Gamma$ which is a contradiction.

(Case 2: ω_{t^*} has $+$-vertical bridges without a $+$-horizontal one) This case is similar to **(Case 1)**.

(Case 3: ω_{t^*} has a $+$-cross) In this case, ω_{t^*} cannot have a $-$-bridge. Thus, by (8.6), the configuration ω_{t^*} has at most three bridges. Therefore, by Lemma 8.5,

$$H(\omega_{t^*}) \geq 2(K+L-3) > \Gamma ,$$

which contradicts the fact that $(\omega_t)_{t=0}^T$ is a Γ-path.

Now, the assertion of the proposition directly follows since if $\mathcal{E}^+ \cap \mathcal{E}^- \neq \emptyset$, there must exist a Γ-path from $\boxminus$ to $\boxplus$ in $\mathcal{X} \setminus \mathcal{B}_\Gamma$. $\square$

The previous proposition implies that any Γ-path connecting $\boxminus$ and $\boxplus$ has to touch the set $\mathcal{B}_\Gamma$, i.e., has to path through bulk typical configurations.

The next proposition concerns the relationships between bulk and edge typical configurations.

Proposition 8.9. *The following properties hold:*

1. *It holds that*
$$\mathcal{E}^- \cap \mathcal{B} = \mathcal{R}_2 \quad \textit{and} \quad \mathcal{E}^+ \cap \mathcal{B} = \mathcal{R}_{L-2} \ . \tag{8.8}$$

2. *We have that* $\mathcal{E} \cup \mathcal{B} = \widehat{\mathcal{N}}(\mathcal{S})$.

Proof. (1) We only prove the first one of (8.8), as the second one follows in the same manner.

First, we have $\mathcal{B} \supset \mathcal{R}_2$ from the definition of $\mathcal{B}$. On the other hand, since the canonical path connecting $\mathcal{R}_2$ and $\boxminus$ is a Γ-path in $\mathcal{X} \setminus \mathcal{B}_\Gamma$, we also have $\mathcal{E}^- \supset \mathcal{R}_2$. Thus, we have proved that,

$$\mathcal{E}^- \cap \mathcal{B} \supset \mathcal{R}_2 \ . \tag{8.9}$$

Now, we claim that the reversed inclusion also holds. To prove this claim, we begin by observing that, since $\mathcal{B}_\Gamma$ and $\mathcal{E}$ are disjoint by definition (cf. (8.2)), we can conclude that

$$\mathcal{E}^- \cap \mathcal{B} \subset \mathcal{B} \setminus \mathcal{B}_\Gamma = \bigcup_{v \in [\![2,\, L-2]\!]} \mathcal{R}_v \ .$$

For $\sigma \in \mathcal{R}_v$ with $v \in [\![3,\, L-3]\!]$, we cannot have a path in $\mathcal{X} \setminus \mathcal{B}_\Gamma$ connecting $\boxminus$ and σ by Lemma 8.6-(2). We therefore have $\sigma \notin \mathcal{E}^-$, and thus we can conclude that

$$\mathcal{E}^- \cap \mathcal{B} \subset \mathcal{R}_2 \cup \mathcal{R}_{L-2} \ . \tag{8.10}$$

By the same reason with the inclusion $\mathcal{E}^- \supset \mathcal{R}_2$, we also have $\mathcal{E}^+ \supset \mathcal{R}_{L-2}$. Therefore, any configuration $\sigma \in \mathcal{R}_{L-2}$ cannot belong to $\mathcal{E}^-$ by Proposition 8.8; hence, from (8.10), we can deduce that

$$\mathcal{E}^- \cap \mathcal{B} \subset \mathcal{R}_2 \ . \tag{8.11}$$

This proves the claim and we are done.

(2) The inclusion $\mathcal{E} \subset \widehat{\mathcal{N}}(\mathcal{S})$ is obvious from the definition of $\mathcal{E}$, and the inclusion $\mathcal{B} \subset \widehat{\mathcal{N}}(\mathcal{S})$ also follows immediately from the fact that any bulk typical configuration is connected to $\boxminus$ (or $\boxplus$) via a part of a canonical path, which is a Γ-path (cf. Remark 7.2). Thus, we can conclude that

$$\mathcal{E} \cup \mathcal{B} \subset \widehat{\mathcal{N}}(\mathcal{S}) \ . \tag{8.12}$$

Now we prove the reversed inclusion. By Lemma 6.2 with $\mathcal{P} = \mathcal{C} \setminus \mathcal{B}$ and $\mathcal{Q} = \mathcal{B}$, we get

$$\widehat{\mathcal{N}}(\mathcal{C}) = \widehat{\mathcal{N}}(\mathcal{B}\,;\, \mathcal{C} \setminus \mathcal{B}) \cup \widehat{\mathcal{N}}(\mathcal{C} \setminus \mathcal{B}\,;\, \mathcal{B}) \ . \tag{8.13}$$

By Lemma 8.12, we have

$$\widehat{\mathcal{N}}(\mathcal{B}\,;\, \mathcal{C} \setminus \mathcal{B}) = \mathcal{B} \ . \tag{8.14}$$

Since any configuration in $\mathcal{C} \setminus \mathcal{B}$ is connected to either $\boxminus$ or $\boxplus$ via a part of a canonical path which is a Γ-path in $\mathcal{X} \setminus \mathcal{B}_\Gamma$, we obtain

$$\widehat{\mathcal{N}}(\mathcal{C} \setminus \mathcal{B}\,;\, \mathcal{B}) \subset \widehat{\mathcal{N}}(\mathcal{C} \setminus \mathcal{B}\,;\, \mathcal{B}_\Gamma) \subset \widehat{\mathcal{N}}(\mathcal{S}\,;\, \mathcal{B}_\Gamma) = \mathcal{E} \ . \tag{8.15}$$

By combining (8.13), (8.14), and (8.15), we get

$$\widehat{\mathcal{N}}(\mathcal{C}) \subset \mathcal{E} \cup \mathcal{B} \ .$$

Since $\mathcal{S} \subset \mathcal{C}$, the last inclusion implies the opposite inclusion of (8.12), and we are done. $\square$

8.4 Characterization of edge typical configurations

As mentioned before, edge typical configurations have far more complex structure than bulk ones. In this subsection, we study this complex structure in detail.

Our analysis starts with a decomposition of the form

$$\mathcal{E}^- = \mathcal{O}^- \cup \mathcal{I}^- \quad \text{and} \quad \mathcal{E}^+ = \mathcal{O}^+ \cup \mathcal{I}^+ \ ,$$

where

$$\mathcal{O}^\pm = \{\sigma \in \mathcal{E}^\pm : H(\sigma) = \Gamma\} \quad \text{and} \quad \mathcal{I}^\pm = \{\sigma \in \mathcal{E}^\pm : H(\sigma) < \Gamma\} \ .$$

Then, we analyze the structure based on this decomposition. For the concreteness of the discussion, we focus only on $\mathcal{E}^-$, as the analysis of $\mathcal{E}^+$ is essentially identical.

By Proposition 8.2, we can see that

$$\mathcal{I}^- = \mathcal{N}(\boxminus) \cup \mathcal{R}_2 \ . \tag{8.16}$$

We now construct a graph and a Markov chain which represent the asymptotic behavior of the Metropolis dynamics on $\mathcal{E}^-$. Heuristically, since the configurations belonging to $\mathcal{N}(\boxminus)$ are indistinguishable in the scale $e^{\beta\Gamma}$ (as they can be communicated by a much shorter scale), we shall identify all the configurations in $\mathcal{N}(\boxminus)$ with $\boxminus$ and define

$$\overline{\mathcal{I}}^- = \boxminus \cup \mathcal{R}_2 \ . \tag{8.17}$$

With this notation, we can write

$$\mathcal{E}^- = \mathcal{O}^- \cup \Big(\bigcup_{\sigma \in \overline{\mathcal{I}}^-} \mathcal{N}(\sigma) \Big) \ . \tag{8.18}$$

Now, we define a graph structure on the vertex set $\mathscr{V}^-$ defined by

$$\mathscr{V}^- = \mathcal{O}^- \cup \overline{\mathcal{I}}^- \ , \tag{8.19}$$

and define a continuous-time Markov chain on that graph.

Definition 8.10.
- **(Graph)** We introduce a graph structure $\mathscr{G}^- = (\mathscr{V}^-, \mathscr{E}^-)$ where for $\sigma, \sigma' \in \mathscr{V}^-$, we say that $\{\sigma, \sigma'\} \in \mathscr{E}^-$ if and only if

$$\begin{cases} \sigma, \sigma' \in \mathcal{O}^- \text{ and } \sigma \sim \sigma' \text{ or} \\ \sigma \in \mathcal{O}^-,\ \sigma' \in \overline{\mathcal{I}}^- \text{ and } \sigma \sim \xi \text{ for some } \xi \in \mathcal{N}(\sigma') \ . \end{cases}$$

- **(Markov chain)** The rate function $r^-: \mathscr{V}^- \times \mathscr{V}^- \to [0, \infty)$ is defined by, for all $\{\sigma, \sigma'\} \in \mathscr{E}$,

$$r^-(\sigma, \sigma') = \begin{cases} 1 & \text{if } \sigma, \sigma' \in \mathcal{O}^- \ , \\ |\{\xi \in \mathcal{N}(\sigma) : \xi \sim \sigma'\}| & \text{if } \sigma \in \overline{\mathcal{I}}^- \ , \ \sigma' \in \mathcal{O}^- \ , \\ |\{\xi \in \mathcal{N}(\sigma') : \xi \sim \sigma\}| & \text{if } \sigma \in \mathcal{O}^- \ , \ \sigma' \in \overline{\mathcal{I}}^- \ , \end{cases} \tag{8.20}$$

  and we finally set $r^-(\sigma, \sigma') = 0$ if $\{\sigma, \sigma'\} \notin \mathscr{E}^-$. Then, denote by $(Z^-(t))_{t\geq 0}$ a continuous-time Markov chain on $\mathscr{V}^-$ with rate $r^-(\cdot, \cdot)$. Since the rate is symmetric, the Markov chain $Z^-(\cdot)$ is reversible with respect to the uniform distribution on $\mathscr{V}^-$.

- We denote by $h^-_{\cdot,\cdot}(\cdot)$, $\mathrm{cap}^-(\cdot, \cdot)$, $D^-(\cdot)$, and $\|\cdot\|_-$ the equilibrium potential, capacity, Dirichlet form, and flow norm with respect to the Markov process $Z^-(\cdot)$, respectively. In addition, denote by L^- the generator of the process $Z^-(\cdot)$ acting on $f: \mathscr{V}^- \to \mathbb{R}$ in a way that

$$(L^- f)(\sigma) = \sum_{\sigma' \in \mathscr{V}^-: \{\sigma, \sigma'\} \in \mathscr{E}^-} r^-(\sigma, \sigma') \{f(\sigma') - f(\sigma)\} \ . \tag{8.21}$$

We first show that the Markov process $Z^-(\cdot)$ approximates in some sense the Metropolis dynamics $\sigma_\beta(\cdot)$ in $\mathcal{E}^A$.

Proposition 8.11. *Define a projection map $\Pi^-: \mathcal{E}^- \to \mathscr{V}^-$ by*

$$\Pi^-(\sigma) = \begin{cases} \xi & \text{if } \sigma \in \mathcal{N}(\xi) \text{ for some } \xi \in \overline{\mathcal{I}}^- \ , \\ \sigma & \text{if } \sigma \in \mathcal{O}^- \ . \end{cases}$$

Then, there exists a constant $C = C(K, L) > 0$ such that

1. *for $\sigma, \sigma' \in \mathcal{O}^-$, we have*

$$\Big| \frac{1}{2} e^{-\Gamma\beta} r^-(\Pi^-(\sigma), \Pi^-(\sigma')) - \mu_\beta(\sigma)\, c_\beta(\sigma, \sigma') \Big| \leq C e^{-(\Gamma+2)\beta} \ , \tag{8.22}$$

2. *for $\sigma \in \mathcal{O}^-$ and $\sigma' \in \overline{\mathcal{I}}^-$, we have*

$$\Big| \frac{1}{2} e^{-\Gamma\beta} r^-(\Pi^-(\sigma), \Pi^-(\sigma')) - \sum_{\xi \in \mathcal{N}(\sigma')} \mu_\beta(\sigma)\, c_\beta(\sigma, \xi) \Big| \leq C e^{-(\Gamma+2)\beta} \ . \tag{8.23}$$

Proof. Suppose that $\sigma, \sigma' \in \mathcal{O}^-$. If $\sigma \not\sim \sigma'$, then the left-hand side of (8.22) is clearly

0. On the other hand, if $\sigma \sim \sigma'$ so that $\{\sigma, \sigma'\} \in \mathscr{E}^-$, then by (4.8) and (8.20),

$$\Big| \frac{1}{2}e^{-\Gamma\beta} r^-(\Pi^-(\sigma), \Pi^-(\sigma')) - \mu_\beta(\sigma)\, c_\beta(\sigma, \sigma') \Big| = \Big| \frac{1}{2}e^{-\Gamma\beta} - \frac{1}{Z_\beta}e^{-\Gamma\beta} \Big|$$

since $\mu_\beta(\sigma) = \mu_\beta(\sigma') = \frac{1}{Z_\beta}e^{-\Gamma\beta}$. By (4.6), the right-hand side is $O(e^{-(\Gamma+2)\beta})$. This proves part (1).

Now, we consider part (2). Let $\sigma \in \mathcal{O}^-$ and $\sigma' \in \overline{\mathcal{I}}^-$. Similarly, we can assume $\{\sigma, \sigma'\} \in \mathscr{E}^-$ since otherwise the left-hand side of (8.23) is 0. Then, by (4.8) and (8.20), we can write

$$\begin{aligned} & \Big| \frac{1}{2}e^{-\Gamma\beta} r^-(\Pi^-(\sigma), \Pi^-(\sigma')) - \sum_{\xi \in \mathcal{N}(\sigma')} \mu_\beta(\sigma)\, c_\beta(\sigma, \xi) \Big| \\ =& \Big| \frac{1}{2}e^{-\Gamma\beta}\, |\{\xi \in \mathcal{N}(\sigma') : \xi \sim \sigma\}| - \sum_{\xi \in \mathcal{N}(\sigma') :\, \xi \sim \sigma} \min\{\mu_\beta(\sigma),\, \mu_\beta(\xi)\} \Big| \\ =& |\{\xi \in \mathcal{N}(\sigma') : \xi \sim \sigma\}| \times \Big| \frac{1}{2}e^{-\Gamma\beta} - \frac{1}{Z_\beta}e^{-\Gamma\beta} \Big| \,, \end{aligned}$$

since $\min\{\mu_\beta(\sigma),\, \mu_\beta(\xi)\} = \mu_\beta(\sigma)$ for all $\xi \in \mathcal{N}(\sigma')$. By (4.6), the last line is bounded by $KL \times O(e^{-\Gamma\beta}e^{-2\beta}) = O(e^{-(\Gamma+2)\beta})$. □

In view of this proposition, we can assert that the equilibrium potential $h^-_{\boxminus,\, \mathcal{R}_2}(\cdot)$ approximates the equilibrium potential of the Metropolis dynamics in $\mathcal{E}^-$. For this reason, the equilibrium potential $h^-_{\boxminus,\, \mathcal{R}_2}(\cdot)$ plays a significant role in the construction of the test function and flow in the next sections.

Now, we are ready to define the edge constant $\mathfrak{e}$ introduced in Section 5.2. Define

$$\mathfrak{e} = \frac{1}{|\mathscr{V}^-|\, \mathrm{cap}^-(\boxminus,\, \mathcal{R}_2)} \,. \tag{8.24}$$

The appearance of $\mathrm{cap}^-(\boxminus,\, \mathcal{R}_2)$ is quite natural in that the equilibrium potential $h^-_{\boxminus,\, \mathcal{R}_2}(\cdot)$ is the correct approximation of the equilibrium potential of the Metropolis dynamics in $\mathcal{E}^-$. We conclude this section by showing that the constant $\mathfrak{e}$ is small.

Proposition 8.12. *We have that* $\mathfrak{e} \le \frac{1}{L}$.

Proof. We use the Thomson principle (cf. Theorem 1.4) to prove the proposition. We define a test flow ψ on $\mathscr{V}^- \times \mathscr{V}^-$ (with respect to the Markov process $Z^-(\cdot)$) as

$$\begin{cases} \psi(\boxminus,\, \zeta^{\mathrm{up}}_{\ell,1\,;\,k,1}) = \frac{1}{KL} & \text{for } (k,\, \ell) \in \mathbb{T}_K \times \mathbb{T}_L \,, \\ \psi(\zeta^{\mathrm{up}}_{\ell,1\,;\,k,h},\, \zeta^{\mathrm{up}}_{\ell,1\,;\,k,h+1}) = \frac{1}{KL} & \text{for } (k,\, \ell) \in \mathbb{T}_K \times \mathbb{T}_L \text{ and } h \in [\![1,\, K-1]\!] \,. \end{cases}$$

We set $\psi(\sigma,\,\sigma')=0$ for all other cases. Notice that $\zeta_{\ell,\,1\,;\,k,\,h}^{\mathrm{up}}\in\mathcal{O}^-$ for all $h\in[\![1,\,K-1]\!]$, and that $\{\boxminus,\,\zeta_{\ell,\,1\,;\,k,\,1}^{\mathrm{up}}\}\in\mathscr{E}^-$ since $\zeta_{\ell,\,1\,;\,k,\,1}^{\mathrm{up}}\sim\zeta_{\ell,\,1}$ and $\zeta_{\ell,\,1}\in\mathcal{N}(\boxminus)$, where the latter is readily follows from the part of a canonical path connecting $\boxminus$ and $\zeta_{\ell,\,1}$ is a $(\Gamma-2)$-path. Notice that this is a unit flow from $\boxminus$ to $\mathcal{R}_2$ since

$$(\operatorname{div}\psi)(\boxminus)=\sum_{\ell\in\mathbb{T}_L}\sum_{k\in\mathbb{T}_K}\frac{1}{KL}=1\,,$$
$$(\operatorname{div}\psi)(\mathcal{R}_2)=\sum_{\ell\in\mathbb{T}_L}(\operatorname{div}\psi)(\zeta_{\ell,\,2})=\sum_{\ell\in\mathbb{T}_L}\sum_{k\in\mathbb{T}_K}\frac{-1}{KL}=-1\,,$$

and moreover we can readily check that

$$(\operatorname{div}\psi)(\sigma)=0\quad\text{for all }\sigma\in\mathscr{V}^-\setminus(\boxminus\cup\mathcal{R}_2)\,.$$

Therefore, by Theorem 1.4, we get

$$\operatorname{cap}^-(\boxminus,\,\mathcal{R}_2)\geq\frac{1}{\|\psi\|_-^2}\,. \tag{8.25}$$

It remains to evaluate the flow norm $\|\psi\|_-^2$ which is indeed equal to (since the uniform distribution is the invariant measure for the Markov process $Z^-(\cdot)$)

$$\begin{aligned}&\sum_{\ell\in\mathbb{T}_L}\sum_{k\in\mathbb{T}_K}\Big[\frac{\psi(\boxminus,\,\zeta_{\ell,\,1\,;\,k,\,1}^{\mathrm{up}})^2}{1/|\mathscr{V}^-|}+\sum_{h=1}^{K-1}\frac{\psi(\zeta_{\ell,\,1\,;\,k,\,h}^{\mathrm{up}},\,\zeta_{\ell,\,1\,;\,k,\,h+1}^{\mathrm{up}})^2}{1/|\mathscr{V}^-|}\Big]\\&=LK^2\times\frac{|\mathscr{V}^-|}{K^2L^2}=\frac{|\mathscr{V}^-|}{L}\,.\end{aligned}$$

Injecting this to (8.25) completes the proof. □

Remark 8.13. In fact, we can verify that there exist two constants $C_1,\,C_2>0$ such that
$$\frac{C_1}{KL}\leq\mathfrak{e}\leq\frac{C_2}{KL}\,.$$
We leave this as an exercise. This can be proven with a more refined test flow.

Remark 8.14. Of course, we can also establish the results corresponding to Definition 8.10, Propositions 8.11 and 8.12 for $\mathcal{E}^+$ in the completely identical manner. The constant $\mathfrak{e}$ defined for $\mathcal{E}^+$ should be in accordance with (8.24) by the symmetry of the model.

9 Upper Bound for Capacities

In this section, we construct a test function $f_0 : \mathcal{X} \to \mathbb{R}$ appearing in Proposition 5.2. For the convenience of notation, we write

$$\mathfrak{h}^{\pm}(\cdot) = h^{\pm}_{\boxminus,\mathcal{R}_2}(\cdot) \tag{9.1}$$

which is the equilibrium potential between $\{\boxminus\}$ and $\mathcal{R}_2$ with respect to the process $Z^{\pm}(\cdot)$ (cf. Definition 8.10).

9.1 Construction of test function

Now, we construct a function $f_0 : \mathcal{X} \to \mathbb{R}$. In the end, we shall verify that this function fulfills all requirements of the function f_0 appearing in Proposition 5.2. Before defining the test function explicitly, we briefly explain the gist of the idea. On edge typical configurations (i.e., on $\mathcal{E}^{\pm}$), we choose f_0 as a rescale of $\mathfrak{h}^{\pm}$. This construction mainly comes from the fact that the process $Z^{\pm}(\cdot)$ successfully characterizes the behavior of the original process on edge typical configurations by Proposition 8.11. On the other hand, on bulk typical configurations, we define f as a rescale of the equilibrium potential of a symmetric simple random walk on an one-dimensional line. This is because the Metropolis dynamics behaves as an one-dimensional random walk there thanks to the simple geometry between them.

Definition 9.1 (Test function)**.** We construct a test function $f_0 : \mathcal{X} \to \mathbb{R}$ on $\mathcal{E}$, $\mathcal{B}$, and $(\mathcal{E} \cup \mathcal{B})^c = \mathcal{X} \setminus (\mathcal{E} \cup \mathcal{B})$, separately.

1. **Construction of f_0 on edge typical configurations $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$.**

   - For $\sigma \in \mathcal{E}^-$, we recall the decomposition (8.18) of $\mathcal{E}^-$ and define

   $$f_0(\sigma) = \begin{cases} 1 - \frac{\mathfrak{e}}{\kappa}(1 - \mathfrak{h}^-(\sigma)) & \text{if } \sigma \in \mathcal{O}^- \ , \\ 1 - \frac{\mathfrak{e}}{\kappa}(1 - \mathfrak{h}^-(\xi)) & \text{if } \sigma \in \mathcal{N}(\xi) \text{ for some } \xi \in \overline{\mathcal{I}}^- \ . \end{cases} \tag{9.2}$$

   - For $\sigma \in \mathcal{E}^+$, we similarly define

   $$f_0(\sigma) = \begin{cases} \frac{\mathfrak{e}}{\kappa}(1 - \mathfrak{h}^+(\sigma)) & \text{if } \sigma \in \mathcal{O}^+ \ , \\ \frac{\mathfrak{e}}{\kappa}(1 - \mathfrak{h}^+(\xi)) & \text{if } \sigma \in \mathcal{N}(\xi) \text{ for some } \xi \in \overline{\mathcal{I}}^+ \ . \end{cases} \tag{9.3}$$

2. **Construction of f_0 on bulk typical configurations $\mathcal{B}$.** In view of (8.1), it suffices to define this object in the following two cases.

- For $\sigma \in \mathcal{R}_v$ with $v \in [\![2,\, L-2]\!]$, we set

$$f_0(\sigma) = \frac{1}{\kappa}\left[\frac{L-2-v}{L-4}\mathfrak{b} + \mathfrak{e}\right] . \tag{9.4}$$

- For $\sigma \in \mathcal{Q}_v$ with $v \in [\![2,\, L-3]\!]$, we can write $\sigma = \zeta^{\mathrm{up}}_{\ell,\, v\,;\, k,\, h}$ or $\zeta^{\mathrm{down}}_{\ell,\, v\,;\, k,\, h}$ for some $(k,\, \ell) \in \mathbb{T}_K \times \mathbb{T}_L$ and $h \in [\![1,\, K-1]\!]$. For such σ, we set

$$f_0(\sigma) = \frac{1}{\kappa}\left[\frac{(K+2)(L-2-v)-(h+1)}{(K+2)(L-4)}\mathfrak{b} + \mathfrak{e}\right] . \tag{9.5}$$

3. **Construction of f_0 on the remainder $(\mathcal{E} \cup \mathcal{B})^c$.** We define $f_0 \equiv 1$ on this set.

Remark 9.2. We note that $\mathcal{E}^-$ and $\mathcal{B}$ are not disjoint and their intersection is $\mathcal{R}_2$ by Proposition 8.9. However, we can easily check that our constructions of f_0 on $\mathcal{R}_2$ in parts (1) and (2) of the previous definition agree with the value $1 - \mathfrak{e}/\kappa$. A similar result also holds for $\mathcal{E}^+$ and $\mathcal{B}$.

9.2 Properties of test function

Now, we will confirm that the test function f_0 satisfies the requirements of f_0 appearing in Proposition 5.2.

Proposition 9.3. *The function f_0 constructed in Definition 9.1 belongs to $\mathfrak{C}_{1,0}(\{\boxplus\},\, \{\boxminus\})$ and satisfies*

$$\mathscr{D}_\beta(f_0) = \frac{1+o_\beta(1)}{2\kappa}\, e^{-\Gamma\beta} . \tag{9.6}$$

Proof. For the simplicity of notation, let us write $f = f_0$. Since we have $f(\boxminus) = 1$ and $f(\boxplus) = 0$ by part (1) of Definition 9.1, we immediately have $f \in \mathfrak{C}_{1,0}(\{\boxplus\},\, \{\boxminus\})$. Now, it remains to prove (9.6).

Let us divide the Dirichlet form $\mathscr{D}_\beta(f)$ into

$$\Big[\sum_{\{\sigma,\,\xi\}\subset(\mathcal{E}\cup\mathcal{B})^c} + \sum_{\sigma\in\mathcal{E}\cup\mathcal{B},\, \xi\in(\mathcal{E}\cup\mathcal{B})^c} + \sum_{\{\sigma,\,\xi\}\subset\mathcal{E}\cup\mathcal{B}}\Big]\, \mu_\beta(\sigma)\, c_\beta(\sigma,\, \xi)\, \{f(\xi) - f(\sigma)\}^2 , \tag{9.7}$$

where all summations are carried out for two connected configurations σ and ξ, i.e., $\sigma \sim \xi$.

The first summation is trivially 0 by part (3) of Definition 9.1. Now to consider the second summation, we recall from part (2) of Proposition 8.9 that $\mathcal{E}\cup\mathcal{B} = \widehat{\mathcal{N}}(\mathcal{S})$. This implies that $H(\sigma) \le \Gamma$ and $H(\xi) \ge \Gamma + 1$. Therefore, by (4.8), we have

$$\mu_\beta(\sigma)\, c_\beta(\sigma,\, \xi) = \mu_\beta(\xi) = \frac{1}{Z_\beta} e^{-\beta H(\xi)} = o_\beta(e^{-\Gamma\beta}) ,$$

where we implicitly used the fact that $Z_\beta \to 2$ as $\beta \to \infty$ at the last equality. Moreover, since $f(\sigma) \in [0,\, 1]$ for all $\sigma \in \mathcal{X}$ by our construction, we can assert that the second summation in (9.7) is $o_\beta(e^{-\Gamma\beta})$.

It remains to estimate the third summation of (9.7). For $\mathcal{A} \subset \mathcal{X}$, we write

$$E(\mathcal{A}) = \big\{ \{\sigma,\, \xi\} \subset \mathcal{A} : \sigma \sim \xi \big\} \; . \tag{9.8}$$

By part (1) of Proposition 8.9, we can decompose $E(\mathcal{E} \cup \mathcal{B})$ into

$$E(\mathcal{E} \cup \mathcal{B}) = E(\mathcal{B}) \cup E(\mathcal{E}^-) \cup E(\mathcal{E}^+) \; . \tag{9.9}$$

Hence, we can further decompose the third summation of (9.7) into

$$\Big[\sum_{\{\sigma,\xi\} \in E(\mathcal{B})} + \sum_{\{\sigma,\xi\} \in E(\mathcal{E}^-)} + \sum_{\{\sigma,\xi\} \in E(\mathcal{E}^+)} \Big] \, \mu_\beta(\sigma)\, c_\beta(\sigma,\, \xi)\, \{f(\xi) - f(\sigma)\}^2 \; . \tag{9.10}$$

Now, we compute the first summation of (9.10). Decompose

$$E(\mathcal{B}) = \bigcup_{v=2}^{L-3} E(\mathcal{R}_v \cup \mathcal{Q}_v \cup \mathcal{R}_{v+1}) \; ,$$

so that we can write the first summation of (9.10) as

$$\sum_{v=2}^{L-3} \sum_{\{\sigma,\xi\} \in E(\mathcal{R}_v \cup \mathcal{Q}_v \cup \mathcal{R}_{v+1})} \mu_\beta(\sigma)\, c_\beta(\eta,\, \xi)\, \{f(\xi) - f(\sigma)\}^2 \; .$$

This summation can be written as $\sum_{\ell \in \mathbb{T}_L} \sum_{k \in \mathbb{T}_K}$ of

$$\begin{aligned}
&\mu_\beta(\zeta_{\ell,\,v})\, c_\beta(\zeta_{\ell,\,v},\, \zeta^{\rm up}_{\ell,\,v\,;\,k,\,1})\, \{f(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,1}) - f(\zeta_{\ell,\,v})\}^2 \\
&\quad + \sum_{h=1}^{K-2} \mu_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h})\, c_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h},\, \zeta^{\rm up}_{\ell,\,v\,;\,k,\,h+1})\, \{f(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h+1}) - f(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h})\}^2 \\
&\quad + \sum_{h=1}^{K-2} \mu_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h})\, c_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h},\, \zeta^{\rm up}_{\ell,\,v\,;\,k-1,\,h+1})\, \{f(\zeta^{\rm up}_{\ell,\,v\,;\,k-1,\,h+1}) - f(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,h})\}^2 \\
&\quad + \mu_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,K-1})\, c_\beta(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,K-1},\, \zeta_{\ell,\,v+1})\, \{f(\zeta_{\ell,\,v+1}) - f(\zeta^{\rm up}_{\ell,\,v\,;\,k,\,K-1})\}^2 \; ,
\end{aligned}$$

and the same form of terms replacing up with down. By (4.6), (4.3), (4.8), (9.4), and (9.5), this equals $2\sum_{\ell \in \mathbb{T}_L} \sum_{k \in \mathbb{T}_K}$ (where 2 is multiplied since we have to compute up/down separately) of

$$
\begin{aligned}
&\frac{e^{-\Gamma\beta}}{Z_\beta}\cdot\frac{4\mathfrak{b}^2}{[\kappa(K+2)(L-4)]^2}+\sum_{h=1}^{K-2}\frac{e^{-\Gamma\beta}}{Z_\beta}\cdot\frac{\mathfrak{b}^2}{[\kappa(K+2)(L-4)]^2}\\
&+\sum_{h=1}^{K-2}\frac{e^{-\Gamma\beta}}{Z_\beta}\cdot\frac{\mathfrak{b}^2}{[\kappa(K+2)(L-4)]^2}+\frac{e^{-\Gamma\beta}}{Z_\beta}\cdot\frac{4\mathfrak{b}^2}{[\kappa(K+2)(L-4)]^2}\\
=&(1+o_\beta(1))\,\frac{e^{-\Gamma\beta}}{2}\,\frac{(2K+4)\mathfrak{b}^2}{(K+2)^2(L-4)^2\kappa^2}=(1+o_\beta(1))\,\frac{\mathfrak{b}^2}{(K+2)(L-4)^2\kappa^2}\,e^{-\Gamma\beta}\,.
\end{aligned}
$$

Therefore by (5.6), we can conclude that

$$
\begin{aligned}
&\sum_{\{\sigma,\,\xi\}\in E(\mathcal{B})}\mu_\beta(\sigma)\,c_\beta(\sigma,\,\xi)\,\{f(\xi)-f(\sigma)\}^2\\
&=2\,(1+o_\beta(1))\sum_{v=2}^{L-3}\sum_{\ell\in\mathbb{T}_L}\sum_{k\in\mathbb{T}_K}\frac{\mathfrak{b}^2}{(K+2)(L-4)^2\kappa^2}\,e^{-\Gamma\beta} \qquad (9.11)\\
&=(1+o_\beta(1))\,\frac{2KL(L-4)\mathfrak{b}^2}{(K+2)(L-4)^2\kappa^2}\,e^{-\Gamma\beta}=\frac{\mathfrak{b}+o_\beta(1)}{2\kappa^2}\,e^{-\Gamma\beta}\,.
\end{aligned}
$$

Next, we calculate the second summation of (9.10). By (8.18), we rewrite this summation as

$$
\begin{aligned}
&\sum_{\{\sigma_1,\,\sigma_2\}\subseteq\mathcal{O}^-}\mu_\beta(\sigma_1)\,c_\beta(\sigma_1,\,\sigma_2)\,\{f(\sigma_2)-f(\sigma_1)\}^2\\
&+\sum_{\sigma_1\in\mathcal{O}^-}\sum_{\sigma_2\in\overline{\mathcal{I}}^-}\sum_{\xi\in\mathcal{N}(\sigma_2)}\mu_\beta(\sigma_1)\,c_\beta(\sigma_1,\,\xi)\,\{f(\xi)-f(\sigma_1)\}^2\,.
\end{aligned}
$$

By Proposition 8.11, this equals $1+o_\beta(1)$ times

$$
\Big[\sum_{\{\sigma_1,\,\sigma_2\}\subseteq\mathcal{O}^-}+\sum_{\sigma_1\in\mathcal{O}^-}\sum_{\sigma_2\in\overline{\mathcal{I}}^-}\Big]\,\frac{1}{2}\,e^{-\Gamma\beta}\,r^-(\sigma_1,\,\sigma_2)\,\{f(\sigma_2)-f(\sigma_1)\}^2\,. \qquad (9.12)
$$

By (9.2), the last line becomes

$$
\begin{aligned}
&\frac{\mathfrak{e}^2}{\kappa^2}\sum_{\{\sigma_1,\,\sigma_2\}\subseteq\mathscr{V}^A}\frac{1}{2}\,e^{-\Gamma\beta}\,r^-(\sigma_1,\,\sigma_2)\,\{\mathfrak{h}^-(\sigma_2)-\mathfrak{h}^-(\sigma_1)\}^2\\
&=\frac{e^{-\Gamma\beta}\mathfrak{e}^2}{2\kappa^2}\,|\mathscr{V}^-|\,\mathrm{cap}^-(\boxminus,\,\mathcal{R}_2)=\frac{\mathfrak{e}}{2\kappa^2}e^{-\Gamma\beta}\,. \qquad (9.13)
\end{aligned}
$$

Therefore, we can conclude that

$$\sum_{\{\sigma,\xi\}\in E(\mathcal{E}^-)} \mu_\beta(\sigma)\, c_\beta(\sigma,\, \xi)\, \{f(\xi) - f(\sigma)\}^2 = \frac{\mathfrak{e} + o_\beta(1)}{2\kappa^2} e^{-\Gamma\beta}\,. \tag{9.14}$$

Similarly, we get

$$\sum_{\{\sigma,\xi\}\in E(\mathcal{E}^+)} \mu_\beta(\sigma)\, c_\beta(\sigma,\, \xi)\, \{f(\xi) - f(\sigma)\}^2 = \frac{\mathfrak{e} + o_\beta(1)}{2\kappa^2} e^{-\Gamma\beta}\,. \tag{9.15}$$

Therefore, by (9.10), (9.11), (9.14), and (9.15), we conclude that the first summation of (9.7) equals

$$\frac{\mathfrak{b} + 2\mathfrak{e} + o_\beta(1)}{2\kappa^2}\, e^{-\Gamma\beta} = \frac{1 + o_\beta(1)}{2\kappa}\, e^{-\Gamma\beta}\,,$$

as desired. $\square$

Remark 9.4. The estimates (9.11), (9.14), and (9.15) are the reason why we term $\mathfrak{b}$ and $\mathfrak{e}$ the bulk and edge constants, respectively.

We now conclude the section with a formal proof of Proposition 5.2.

Proof of Proposition 5.2 for $d = 2$. Since we have verified in the previous proposition that the function f_0 constructed in Definition 9.1 satisfies $f_0 \in \mathfrak{C}(\{\boxminus\}, \{\boxplus\})$ and the energy estimate (5.9), the proof is completed. $\square$

10 Lower Bound for Capacities

In this section, we construct the test flow ψ_0 appearing in Proposition 5.3. Construction of the test flow will be given in Section 10.1. Then, two properties of the test flow appearing in (5.10) are verified in Sections 10.2 and 10.4, respectively. Section 10.3 is devoted to providing some investigations of the equilibrium potential between $\boxminus$ and $\boxplus$, which will be used in the analyses carried out in Section 10.4.

10.1 Construction of test flow

In this subsection, we explicitly construct a test flow ψ_0.

We explain the idea before proceeding to the construction. We again use the convention (9.1) in this section. For the edge typical configurations, recall that the equilibrium potential $\mathfrak{h}^{\pm}(\cdot)$ on $\mathcal{E}^{\pm}$ is the object approximating (up to some rescaling) the equilibrium potential $h^{\beta}_{\boxminus,\,\boxplus}(\cdot)$. Hence, we define the test flow on $\mathcal{E}^{\pm}$ as a suitable modification of (a constant-multiple of) $\Psi_{\mathfrak{h}^{\pm}}$. For the bulk typical configurations $\mathcal{B}$, we know the typical behavior of the Metropolis dynamics very well, and hence we can define ψ_0 as a simple flow from $\mathcal{R}_2$ to $\mathcal{R}_{L-2}$, where the flow is constant on each edge of the transition.

Definition 10.1 (Test flow)**.** In this definition, defining $\varphi(\sigma,\,\sigma') = c$ for a flow φ implicitly implies that $\varphi(\sigma',\,\sigma) = -c$. We now construct a flow ψ_0.

1. **Construction of ψ_0 on edge typical configurations $\mathcal{E}$.** We provide an explicit construction on $\mathcal{E}^{\pm}$.

   - If $\sigma_1,\,\sigma_2 \in \mathcal{O}^{\pm}$ with $\sigma_1 \sim \sigma_2$, then we set

$$\psi_0(\sigma_1,\,\sigma_2) = \mathfrak{e}\, r^{\pm}(\sigma_1,\,\sigma_2)\,[\mathfrak{h}^{\pm}(\sigma_1) - \mathfrak{h}^{\pm}(\sigma_2)]\ . \tag{10.1}$$

   - If $\sigma_1 \in \mathcal{O}^{\pm}$ and $\sigma_2 \in \overline{\mathcal{I}}^{\pm}$, then we set, for all $\xi \in \mathcal{N}(\sigma_2)$ with $\xi \sim \sigma_1$,

$$\psi_0(\sigma_1,\,\xi) = \frac{\mathfrak{e}\, r^{\pm}(\sigma_1,\,\sigma_2)\,[\mathfrak{h}^{\pm}(\sigma_1) - \mathfrak{h}^{\pm}(\sigma_2)]}{|\{\xi' \in \mathcal{N}(\sigma_2) : \sigma_1 \sim \xi'\}|}\ . \tag{10.2}$$

2. **Construction of ψ_0 on bulk typical configurations $\mathcal{B}$.** We need to consider the following two cases:

   - For $(k,\,\ell) \in \mathbb{T}_K \times \mathbb{T}_L$ and $v \in [\![2,\,L-3]\!]$,

$$\begin{aligned}&\psi_0(\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,0},\,\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,1}) = \psi_0(\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,K-1},\,\zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,K})\\ &= \psi_0(\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,0},\,\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,1}) = \psi_0(\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,K-1},\,\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,K}) = \frac{2\mathfrak{b}}{(K+2)(L-4)}\ .\end{aligned}$$

- For $(k,\, \ell) \in \mathbb{T}_K \times \mathbb{T}_L$, $v \in [\![2,\, L-3]\!]$, and $h \in [\![1,\, K-2]\!]$,

$$\begin{aligned}&\psi_0(\zeta^{\rm up}_{\ell,\, v\,;\, k,\, h},\, \zeta^{\rm up}_{\ell,\, v\,;\, k,\, h+1}) = \psi_0(\zeta^{\rm up}_{\ell,\, v\,;\, k,\, h},\, \zeta^{\rm up}_{\ell,\, v\,;\, k-1,\, h+1}) \\ =&\psi_0(\zeta^{\rm down}_{\ell,\, v\,;\, k,\, h},\, \zeta^{\rm down}_{\ell,\, v\,;\, k,\, h+1}) = \psi_0(\zeta^{\rm down}_{\ell,\, v\,;\, k,\, h},\, \zeta^{\rm down}_{\ell,\, v\,;\, k-1,\, h+1}) = \frac{\mathfrak{b}}{(K+2)(L-4)}\;.\end{aligned}$$

3. We set $\psi_0 \equiv 0$ on all the edges which are not considered above.

10.2 Flow norm

The next proposition computes the flow norm of ψ_0 to verify the first requirement in (5.10). In the remainder of the current section, we write $\psi = \psi_0$ for the simplicity of notation.

Proposition 10.2. *For the flow $\psi = \psi_0$ constructed in Definition 10.1,*

$$\|\psi\|_\beta^2 = (1 + o_\beta(1))\, 2\kappa e^{\Gamma\beta}\;.$$

Proof. Since the support of ψ is a subset of $\mathcal{E} \cup \mathcal{B}$, by (9.9), we can write

$$\|\psi\|_\beta^2 = \Big[\sum_{\{\sigma,\,\xi\} \in E(\mathcal{E}^-)} + \sum_{\{\sigma,\,\xi\} \in E(\mathcal{E}^+)} + \sum_{\{\sigma,\,\xi\} \in E(\mathcal{B})} \Big] \frac{\psi(\sigma,\,\xi)^2}{\mu_\beta(\sigma)\, c_\beta(\sigma,\,\xi)}\;. \tag{10.3}$$

By the definition of ψ, the first summation of (10.3) can be written as

$$\sum_{\{\sigma_1,\,\sigma_2\} \in E(\mathcal{O}^-)} \frac{\psi(\sigma_1,\,\sigma_2)^2}{\mu_\beta(\sigma_1)\, c_\beta(\sigma_1,\,\sigma_2)} + \sum_{\sigma_1 \in \mathcal{O}^-} \sum_{\sigma_2 \in \mathcal{I}^-} \sum_{\xi \in \mathcal{N}(\sigma_2):\sigma_1 \sim \xi} \frac{\psi(\sigma_1,\,\xi)^2}{\mu_\beta(\sigma_1)\, c_\beta(\sigma_1,\,\xi)}\;.$$

By (10.1), (10.2), and Proposition 8.11, this equals $(1 + o_\beta(1))$ times

$$\Big[\sum_{\{\sigma_1,\,\sigma_2\} \in E(\mathcal{O}^-)} + \sum_{\sigma_1 \in \mathcal{O}^-} \sum_{\sigma_2 \in \mathcal{I}^-} \Big] \frac{2\mathfrak{e}^2 r^-(\sigma_1,\,\sigma_2)\, \{\mathfrak{h}^-(\sigma_2) - \mathfrak{h}^-(\sigma_1)\}^2}{e^{-\Gamma\beta}}\;.$$

By the definition of capacity, we can rewrite the last summation as

$$2\mathfrak{e}^2\, e^{\Gamma\beta}\, |\mathscr{V}^-|\, \mathrm{cap}^-(\boxminus,\, \mathcal{R}_2) = 2\mathfrak{e}\, e^{\Gamma\beta}\;.$$

Since we can apply a similar argument to the second summation of (10.3), we can conclude that

$$\Big[\sum_{\{\sigma,\xi\}\in E(\mathcal{E}^-)}+\sum_{\{\sigma,\xi\}\in E(\mathcal{E}^+)}\Big]\frac{\psi(\sigma,\,\xi)^2}{\mu_\beta(\sigma)\,c_\beta(\sigma,\,\xi)}=(1+o_\beta(1))\times 2\times 2\mathfrak{e}\,e^{\Gamma\beta}$$
$$=(4\mathfrak{e}+o_\beta(1))\,e^{\Gamma\beta}\,. \tag{10.4}$$

Now, we consider the third summation of (10.3). By definition, this summation is $\sum_{k\in\mathbb{T}_K,\,\ell\in\mathbb{T}_L}\sum_{v=2}^{L-3}$ of

$$\Big[\frac{\psi(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,0},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,1})^2}{\mu_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,0})\,c_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,0},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,1})}+\frac{\psi(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,K-1},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,K})^2}{\mu_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,K-1})\,c_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,K-1},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,K})}\Big]$$
$$+\sum_{h=1}^{K-2}\Big[\frac{\psi(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h+1})^2}{\mu_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h})\,c_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h+1})}+\frac{\psi(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k-1,h+1})^2}{\mu_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h})\,c_\beta(\zeta^{\mathrm{up}}_{\ell,v\,;\,k,h},\,\zeta^{\mathrm{up}}_{\ell,v\,;\,k-1,h+1})}\Big]\,,$$

and the same-form of terms can be obtained from above by replacing up with down.

By the definition of ψ, (4.8), and (4.6), this expression equals $(1+o_\beta(1))$ times

$$\Big[\frac{32\mathfrak{b}^2e^{\Gamma\beta}}{(K+2)^2(L-4)^2}+\sum_{h=1}^{K-2}\frac{8\mathfrak{b}^2e^{\Gamma\beta}}{(K+2)^2(L-4)^2}\Big]=(32+8(K-2))\,\frac{\mathfrak{b}^2e^{\Gamma\beta}}{(K+2)^2(L-4)^2}$$
$$=\frac{8\mathfrak{b}^2e^{\Gamma\beta}}{(K+2)(L-4)^2}\,.$$

Hence, by the definition of $\mathfrak{b}$, the third summation of (10.3) equals

$$(1+o_\beta(1))\times KL(L-4)\times\frac{8\mathfrak{b}^2e^{\Gamma\beta}}{(K+2)(L-4)^2}=[2\mathfrak{b}+o_\beta(1)]\,e^{\Gamma\beta}\,. \tag{10.5}$$

Therefore, by (10.3), (10.4), and (10.5), we get

$$\|\psi\|_\beta^2=2[\mathfrak{b}+2\mathfrak{e}+o_\beta(1)]\,e^{\Gamma\beta}=(1+o_\beta(1))\,2\kappa e^{\Gamma\beta}\,.$$

This finishes the proof. □

10.3 Equilibrium potential around ground states

It remains to verify the second requirement (5.10) regarding the test flow ψ_0. To this end, we first prove that the equilibrium potential is nearly constant on the neighborhood of ground states in this subsection. The main tool is Proposition 0.16 regarding the estimate of the equilibrium potential.

Lemma 10.3. *It holds that*

$$\max_{\sigma \in \mathcal{N}(\boxplus)} h^{\beta}_{\boxminus,\,\boxplus}(\sigma) = O(e^{-\beta}) \quad \textit{and} \quad \max_{\sigma \in \mathcal{N}(\boxminus)} (1 - h^{\beta}_{\boxminus,\,\boxplus}(\sigma)) = O(e^{-\beta})\ .$$

Proof. We prove the lemma only for the first estimate, because the second one follows immediately from the first since $1 - h^{\beta}_{\boxminus,\,\boxplus} = h^{\beta}_{\boxplus,\,\boxminus}$.

By Propositions 0.16 and 0.10, it holds that

$$h^{\beta}_{\boxminus,\,\boxplus}(\sigma) \le \frac{\mathrm{cap}_{\beta}(\sigma,\,\boxminus)}{\mathrm{cap}_{\beta}(\sigma,\,\{\boxminus,\,\boxplus\})} \le \frac{\mathrm{cap}_{\beta}(\sigma,\,\boxminus)}{\mathrm{cap}_{\beta}(\sigma,\,\boxplus)}\ . \tag{10.6}$$

Now, we estimate $\mathrm{cap}_{\beta}(\sigma,\,\boxplus)$ and $\mathrm{cap}_{\beta}(\sigma,\,\boxminus)$ separately.

We first give a lower bound of $\mathrm{cap}_{\beta}(\sigma,\,\boxplus)$ via the Thomson principle (Theorem 1.4). As $\sigma \in \mathcal{N}(\boxplus)$, there exists a $(\Gamma - 1)$-path $(\omega_t)_{t=0}^{T}$ connecting $\boxplus$ and σ, where T is bounded by a constant depending only on K and L. We define a test flow ϕ on $\mathcal{X}$ by

$$\phi(\omega_t,\,\omega_{t+1}) = -\phi(\omega_{t+1},\,\omega_t) = 1 \text{ for } t \in [\![0,\,T-1]\!]\ ,$$

and $\phi = 0$ otherwise. This construction implies that ϕ is a unit flow from $\{\boxplus\}$ to $\{\sigma\}$. Since $(\omega_t)_{t=0}^{T}$ is a $(\Gamma - 1)$-path, by Proposition 4.2 and (4.8),

$$\mu_{\beta}(\omega_t)\, c_{\beta}(\omega_t,\,\omega_{t+1}) = \min\{\mu_{\beta}(\omega_t),\,\mu_{\beta}(\omega_{t+1})\} \le \frac{1 + o_{\beta}(1)}{2}\, e^{-(\Gamma-1)\beta}\ .$$

Therefore, we obtain

$$\|\phi\|_{\beta}^{2} = \sum_{t=0}^{T-1} \frac{\phi(\omega_t,\,\omega_{t+1})^2}{\mu_{\beta}(\omega_t)\, c_{\beta}(\omega_t,\,\omega_{t+1})} \le \sum_{t=0}^{T-1} \frac{q + o_{\beta}(1)}{e^{-(\Gamma-1)\beta}} \le C e^{(\Gamma-1)\beta}\ .$$

Hence, by Theorem 1.4,

$$\mathrm{cap}_{\beta}(\sigma,\,\boxplus) \ge \frac{1}{\|\phi\|_{\beta}^{2}} \ge \frac{1}{C}\, e^{-(\Gamma-1)\beta}\ . \tag{10.7}$$

Next, we establish an upper bound for $\mathrm{cap}_{\beta}(\sigma,\,\boxminus)$. To this end, we first observe from our construction of f_0 (cf. Definition 9.1) that $f_0 \in \mathfrak{C}_{1,\,0}(\mathcal{N}(\boxminus),\,\mathcal{N}(\boxplus))$. Therefore, by the symmetry of capacities (cf. (0.13)), the monotonicity of capacities (cf. Proposition 0.10), and the Dirichlet principle (cf. Theorem 1.3), we have

$$\mathrm{cap}_{\beta}(\sigma,\,\boxminus) = \mathrm{cap}_{\beta}(\boxminus,\,\sigma) \le \mathrm{cap}_{\beta}(\mathcal{N}(\boxminus),\,\mathcal{N}(\boxplus)) \le \mathscr{D}_{\beta}(f_0) \le C\, e^{-\Gamma\beta} \tag{10.8}$$

for some constant $C > 0$, where the last bound follows from Proposition 9.3.

The proof is completed by (10.6), (10.7), and (10.8). $\square$

10.4 Divergence of test flow

Now, we investigate the divergence of the test flow ψ_0 constructed in Definition 10.1. For simplicity, we again write $\psi = \psi_0$ throughout the current subsection. We first check that this flow is divergence-free on bulk typical configurations.

Lemma 10.4. *We have* $(\mathrm{div}\,\psi)(\sigma) = 0$ *for all* $\sigma \in \mathcal{B} \setminus \mathcal{E}$.

Proof. Let us fix $\sigma \in \mathcal{B} \setminus \mathcal{E}$.

If $\sigma = \zeta_{\ell,\,v}$ for some $\ell \in \mathbb{T}_L$ and $v \in [\![3,\, L-3]\!]$, then we can write $(\mathrm{div}\,\psi)(\sigma)$ as

$$\sum_{k\in\mathbb{T}_K} [\psi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,1}) + \psi(\sigma,\, \zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,1}) + \psi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v-1\,;\,k,\,K-1}) + \psi(\sigma,\, \zeta^{\mathrm{down}}_{\ell+1,\,v-1\,;\,k,\,K-1})] \;.$$

By recalling Definition 10.1, this summation is equal to

$$\sum_{k\in\mathbb{T}_K} \Big[\frac{2\mathfrak{b}}{(K+2)(L-4)} + \frac{2\mathfrak{b}}{(K+2)(L-4)} - \frac{2\mathfrak{b}}{(K+2)(L-4)} - \frac{2\mathfrak{b}}{(K+2)(L-4)} \Big] = 0 \;.$$

If $\sigma = \zeta^{+}_{\ell,\,v\,;\,k,\,h}$ for some $(k,\,\ell) \in \mathbb{T}_K \times \mathbb{T}_L$, $v \in [\![2,\, L-3]\!]$, and $h \in [\![1,\, K-1]\!]$, then we can write $(\mathrm{div}\,\psi)(\sigma)$ as

$$\begin{aligned}&\phi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,h+1}) + \phi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k-1,\,h+1}) + \phi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,h-1}) + \phi(\sigma,\, \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k+1,\,h-1})\\ &= \frac{\mathfrak{b}}{(K+2)(L-4)} + \frac{\mathfrak{b}}{(K+2)(L-4)} - \frac{\mathfrak{b}}{(K+2)(L-4)} - \frac{\mathfrak{b}}{(K+2)(L-4)} = 0 \;.\end{aligned}$$

The cases $\sigma = \zeta^{\mathrm{up}}_{\ell,\,v\,;\,k,\,h}$ and $\zeta^{\mathrm{down}}_{\ell,\,v\,;\,k,\,h}$ can be handled in the same manner. This concludes the proof. $\square$

Next, we show that ψ is divergence-free on $\mathcal{R}_2$ and $\mathcal{R}_{L-2}$.

Lemma 10.5. *It holds that* $(\mathrm{div}\,\psi)(\sigma) = 0$ *for all* $\sigma \in \mathcal{R}_2 \cup \mathcal{R}_{L-2}$.

Proof. We only consider the divergence on $\mathcal{R}_2$, since the proof for $\mathcal{R}_{L-2}$ is identical. Recall the generator L^- from (8.21). Then, since the uniform measure on $\mathscr{V}^-$ is the invariant measure for the process $Z^-(\cdot)$, by the expression (0.14) of capacity,

$$\mathrm{cap}^-(\boxminus,\, \mathcal{R}_2) = -\frac{1}{|\mathscr{V}^-|} \sum_{\sigma\in\mathcal{R}_2} \sum_{\xi\in\mathscr{V}^-\setminus\mathcal{R}_2} r^-(\sigma,\, \xi)\, \{\mathfrak{h}^-(\sigma) - \mathfrak{h}^-(\xi)\} \;. \tag{10.9}$$

On the other hand, by the definition of ψ, we can write

$$\sum_{\sigma\in\mathcal{R}_2}\sum_{\xi\in\mathcal{E}^-}\psi(\sigma,\,\xi)=\mathfrak{e}\sum_{\sigma\in\mathcal{R}_2}\sum_{\xi\in\mathscr{V}^-\setminus\mathcal{R}_2} r^-(\sigma,\,\xi)\,\{\mathfrak{h}^-(\sigma)-\mathfrak{h}^-(\xi)\}\;. \tag{10.10}$$

By (10.9) and (10.10), we get

$$\sum_{\sigma\in\mathcal{R}_2}\sum_{\xi\in\mathcal{E}^-}\psi(\sigma,\,\xi)=-\mathfrak{e}\,|\mathscr{V}^-|\,\mathrm{cap}^-(\boxminus,\,\mathcal{R}_2)=-1\;, \tag{10.11}$$

where the second identity follows from the definition of $\mathfrak{e}$. On the other hand, by the definition of ψ,

$$\sum_{\sigma\in\mathcal{R}_2}\sum_{\xi\in\mathcal{B}}\psi(\sigma,\,\xi)=L\times 2K\times\frac{2\mathfrak{b}}{(K+2)(L-4)}=1\;. \tag{10.12}$$

By adding (10.11) and (10.12), we obtain

$$\sum_{\sigma\in\mathcal{R}_2}(\mathrm{div}\,\psi)(\sigma)=0\;.$$

Since $\mathrm{div}\,\psi$ is a constant function on $\mathcal{R}_2$ by symmetry, we can conclude that $(\mathrm{div}\,\psi)(\sigma)=0$ for all $\sigma\in\mathcal{R}_2$. □

Next, we show that the flow ψ is divergence-free on $\mathcal{O}^-$ and $\mathcal{O}^+$.

Lemma 10.6. *We have* $(\mathrm{div}\,\psi)(\sigma)=0$ *for all* $\sigma\in\mathcal{O}^-$ *and* $\sigma\in\mathcal{O}^+$.

Proof. We only consider the case $\sigma\in\mathcal{O}^-$ since the case $\sigma\in\mathcal{O}^+$ can be handled in the same manner. By the definition of ψ, we can write

$$(\mathrm{div}\,\psi)(\sigma)=\mathfrak{e}\,(L^-\mathfrak{h}^-)(\sigma)\;. \tag{10.13}$$

By (0.11), we can conclude from (10.13) that $(\mathrm{div}\,\psi)(\sigma)=0$ for all $\sigma\in\mathscr{V}^A\setminus(\boxminus\cup\mathcal{R}_2)$. It suffices to observe from (8.17) and (8.18) that $\mathscr{V}^-\setminus(\boxminus\cup\mathcal{R}_2)=\mathcal{O}^-$. □

We can conclude from Lemmas 10.4, 10.5, and 10.6 that the flow ψ is almost a divergence-free flow.

Proposition 10.7. *The flow* ψ *is divergence-free on* $\mathcal{X}\setminus\mathcal{N}(\mathcal{S})$.

Proof. We can decompose the set $\mathcal{X}\setminus\mathcal{N}(\mathcal{S})$ as

$$(\mathcal{B}\setminus\mathcal{E})\cup\mathcal{R}_2\cup\mathcal{R}_{L-2}\cup\mathcal{O}^-\cup\mathcal{O}^+\cup\big(\mathcal{X}\setminus(\mathcal{E}\cup\mathcal{B})\big)\;.$$

Since it follows immediately from the definition that $(\operatorname{div}\psi) \equiv 0$ on $\mathcal{X} \setminus (\mathcal{E} \cup \mathcal{B})$, we can conclude the proof from Lemmas 10.4, 10.5, and 10.6. $\square$

Now, we are ready to prove the second requirement of (5.10).

Proposition 10.8. *We have that*

$$\sum_{\sigma\in\mathcal{X}} h^{\beta}_{\boxminus,\boxplus}(\sigma)\,(\operatorname{div}\psi)(\sigma) = 1 + o_\beta(1)\ .$$

Proof. In view of Proposition 10.7, it suffices to prove that

$$\sum_{\sigma\in\mathcal{N}(\boxminus)} h^{\beta}_{\boxminus,\boxplus}(\sigma)\,(\operatorname{div}\psi)(\sigma) = 1 + o_\beta(1) \quad\text{and}\quad \sum_{\sigma\in\mathcal{N}(\boxplus)} h^{\beta}_{\boxminus,\boxplus}(\sigma)\,(\operatorname{div}\psi)(\sigma) = o_\beta(1)\ . \tag{10.14}$$

We focus only on the former, since the proof for the latter is essentially identical. By Lemma 10.3, we have

$$\begin{aligned}\sum_{\sigma\in\mathcal{N}(\boxminus)} h^{\beta}_{\boxminus,\boxplus}(\sigma)\,(\operatorname{div}\psi)(\sigma) &= (1+o_\beta(1)) \sum_{\sigma\in\mathcal{N}(\boxminus)}\ \sum_{\xi:\,\xi\sim\sigma} \psi(\sigma,\,\xi) \\ &= (1+o_\beta(1)) \sum_{\sigma\in\mathcal{N}(\boxminus)} (\operatorname{div}\psi)(\sigma)\ .\end{aligned}$$

Note that, in the previous computation, we implicitly used the fact that neither ψ nor $\mathcal{N}(\boxminus)$ depends on β.

Now, we claim that

$$\sum_{\sigma\in\mathcal{N}(\boxminus)} (\operatorname{div}\psi)(\sigma) = 1\ . \tag{10.15}$$

By (10.2), we can rewrite the left-hand side of the previous identity as

$$\sum_{\sigma\in\mathcal{N}(\boxminus)}\ \sum_{\xi\in\mathcal{O}^-:\,\xi\sim\sigma} \psi(\sigma,\,\xi) = -\sum_{\sigma\in\mathcal{N}(\boxminus)}\ \sum_{\xi\in\mathcal{O}^-:\,\xi\sim\sigma} \frac{\mathfrak{e}\, r^-(\xi,\,\boxminus)\,[\mathfrak{h}^-(\xi) - \mathfrak{h}^-(\boxminus)]}{|\{\xi' \in \mathcal{N}(\boxminus) : \xi \sim \xi'\}|}\ .$$

Since $r^-(\cdot,\,\cdot)$ is symmetric, we can further rewrite as

$$-\mathfrak{e} \sum_{\xi\in\mathcal{O}^-:\{\xi,\boxminus\}\in\mathscr{E}^A} r^-(\boxminus,\,\xi)\,[\mathfrak{h}^-(\xi) - \mathfrak{h}^-(\boxminus)]\ .$$

By (0.14) and the definition of $\mathfrak{e}$, the last display equals

$$\mathfrak{e}\,|\mathscr{V}^-|\,\operatorname{cap}^-(\boxminus,\,\mathcal{R}_2) = 1\ .$$

This completes the proof for the first estimate of (10.14) and concludes the proof. $\square$

We conclude this section with the proof of Proposition 5.3.

Proof of Proposition 5.3. Let ψ_0 be the test flow defined in Definition 10.1. Then, the two properties appearing in (5.10) for ψ_0 have been verified in Propositions 10.2 and 10.8. This completes the proof of Proposition 5.3. $\square$

11 Comments on Case $K = L$

Now, we suppose that $K = L$. We define $\theta : \mathbb{T}_K^2 \to \mathbb{T}_K^2$ as

$$\theta(k,\, \ell) = (\ell,\, k) \quad ;\ (k,\, \ell) \in \mathbb{T}_K^2\ .$$

Then, define an operator $\Theta : \mathcal{X} \to \mathcal{X}$ as, for $\sigma \in \mathcal{X}$,

$$\Theta(\sigma)(x) = \sigma(\theta(x)) \quad ;\ x \in \mathbb{T}_K^2\ .$$

Then, the collection of canonical configurations should be $\mathcal{C} \cup \Theta(\mathcal{C})$. Similarly, the definitions of bulk typical configurations and edge typical configurations should be extended to $\mathcal{B} \cup \Theta(\mathcal{B})$ and $\mathcal{E} \cup \Theta(\mathcal{E})$, respectively. With these new definitions of canonical and typical configurations, we can perform similar computations to prove the Eyring–Kramers law.

Part III

Condensing Zero-range Processes

In this third part of the lecture note, we consider a class of interacting particle systems known as the zero-range processes. The particles comprising this model are sticky and therefore tend to condensed at a site. The movements of this condensate are the metastable behavior of this model. To precisely understand the successive movements of the condensate, we use the Markov chain model reduction technique in the context of the metastability to analyze this model. According to the general methodology known as the martingale approach developed in [2, 3, 4], the proof of the Markov chain model reduction for metastable Markov processes is largely based on the potential theory.

This connection between the Markov chain model reduction and the potential theory is relatively clear if the underlying model is reversible. On the other hand, if the model is non-reversible, not only the estimates of the capacity but also deriving the Markov chain model reduction from such estimates are complicated.

In this part, we will try to explain the general method for carrying out these tasks as clearly as possible. We will use the generalized Dirichlet and Thomson principles for the non-reversible Markov processes (cf. Theorem 2.1) to derive sharp estimates of capacities between metastable sets, and then use a robust method developed in [40] to derive the Markov chain model reduction from there.

We note that the current part is largely based on the article [56].

12 Zero-range Processes

In this section, we introduce a class of zero-range processes exhibiting the condensation phenomenon.

Underlying random walk

A zero-range process is a system of interacting particles. We start by explaining the dynamics of the underlying particles comprising the zero-range process. Let $\kappa \geq 2$ be an integer and denote by

$$S = \mathbb{T}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$$

the cycle of length κ. Denote by $\mathbb{X}(\cdot)$ the continuous-time Markov process on S with rate

$$r(x,\, y) = \begin{cases} p & \text{if } y = x+1\ , \\ 1-p & \text{if } y = x-1\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

We note that the addition and subtraction in $\mathbb{T}_\kappa$ are always carried out modulo κ. We denote by $L_{\mathbb{X}}$ and $D_{\mathbb{X}}$ the generator and Dirichlet form associated with the process $\mathbb{X}(\cdot)$. We note that the potential theory of the process $\mathbb{X}(\cdot)$ has been analyzed in Exercise 0.7. We denote by $\mathbf{P}_x$, $x \in S$, the law of the underlying Markov process $\mathbb{X}(\cdot)$ starting from a site $x \in S$.

Zero-range processes

The zero-range process is defined as an interacting system of N particles, where particles basically follow the law of the process $\mathbb{X}(\cdot)$ defined above, but interact through the zero-range interaction explained below.

Let $a : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ (with the convention $\mathbb{N} = \{0,\, 1,\, 2,\, \cdots\}$) be functions defined by

$$a(n) = \begin{cases} 1 & \text{if } n = 0\ , \\ n^\alpha & \text{if } n \ge 1\ , \end{cases} \tag{12.1}$$

and

$$g(n) = \begin{cases} 0 & \text{if } n = 0\ , \\ a(n)/a(n-1) & \text{if } n \ge 1\ , \end{cases} \tag{12.2}$$

where the parameter α stands for the stickiness of constituent particles. We assume that $\alpha > 1$ in this note. We will discuss this assumption for α in Remark 14.8.

For $N \in \mathbb{N}$, define $\mathcal{H}_N \subset \mathbb{N}^S$ as the space of configurations on S with N particles:

$$\mathcal{H}_N = \Big\{\, \eta = (\eta_x)_{x \in S} \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \,\Big\}\ .$$

Here, $\eta \in \mathbb{N}^S$ represents the entire set of particle configurations on S and η_x, $x \in S$, represents the number of particles at x.

Now we are ready to define the zero-range process. For $N \in \mathbb{N}$, the zero-range process $\{\eta_N(t) : t \ge 0\}$ consisting of N particles is defined as a continuous-time Markov process on $\mathcal{H}_N$ associated with the generator

$$(\mathscr{L}_N \mathbf{f})(\eta) = \sum_{x,\, y \in S} g(\eta_x) r(x,\, y) (\mathbf{f}(\sigma^{x,\, y}\eta) - \mathbf{f}(\eta)) \quad ;\ \eta \in \mathcal{H}_N\ ,$$

for $\mathbf{f} : \mathcal{H}_N \to \mathbb{R}$, where $\sigma^{x,\,y}\eta \in \mathcal{H}_N$ represents the configuration obtained from η by sending a particle at site x to y (if possible), that is, $\sigma^{x,\,y}\eta = \eta$ if $\eta_x = 0$, and

$$(\sigma^{x,\,y}\eta)_z = \begin{cases} \eta_z - 1 & \text{if } z = x \\ \eta_z + 1 & \text{if } z = y \\ \eta_z & \text{otherwise ,} \end{cases}$$

if $\eta_x \geq 1$. Of course, we have $\sigma^{x,\,x}\eta = \eta$ for all $x \in S$ and $\eta \in \mathcal{H}_N$. For $\eta \in \mathcal{H}_N$, denote by $\mathbb{P}^N_\eta$ the law of the zero-range process $\eta_N(\cdot)$ starting from η, and denote by $\mathbb{E}^N_\eta$ the corresponding expectation.

Notation 12.1. A function on $\mathcal{H}_N$ will always be denoted by bold font such as $\mathbf{f}$ or $\mathbf{g}$ to distinguish such functions from functions on S.

Heuristically, under the zero-range dynamics defined above, one of the particles at site x jumps to site y at a rate $g(\eta_x)r(x,\,y)$. We can observe two important features of the dynamics at this point. Firstly, since the rate $g(\eta_x)r(x,\,y)$ is independent of η_z, $z \neq x$, we can observe that each particle interacts only with the particles at the same site through the function $g(\cdot)$. This is the reason that this interacting particle system is called a zero-range process.

Secondly, in view of (12.1) and (12.2), this jump rate $g(\eta_x)r(x,\,y)$ decreases as $\eta_x (\geq 2)$ becomes larger. Namely, a particle is deactivated as there are more particles grouped together with that particle. For this reason, we can observe that particles of the zero-range process are sticky. This sticky interaction eventually causes the condensation of particles as defined in the next section.

Exercise 12.2. Prove that the zero-range process defined above is irreducible.

Invariant measure

For $\eta \in \mathcal{H}_N$, let us write

$$a(\eta) = \prod_{x \in S} a(\eta_x) \ . \tag{12.3}$$

By Exercise 12.2, the zero-range process has a unique invariant measure. One can readily verify that this invariant measure $\mu_N(\cdot)$ on $\mathcal{H}_N$ is given by

$$\mu_N(\eta) = \frac{N^\alpha}{Z_N} \frac{1}{a(\eta)} \quad ; \ \eta \in \mathcal{H}_N \ , \tag{12.4}$$

where Z_N is the partition function turning μ_N into a probability measure, i.e.,

$$Z_N = N^\alpha \sum_{\eta \in \mathcal{H}_N} \frac{1}{a(\eta)} \ .$$

Exercise 12.3. 1. Prove that $\mu_N(\cdot)$ is the invariant measure for the zero-range process $\eta_N(\cdot)$.

2. Prove that the zero-range process $\eta_N(\cdot)$ is reversible if and only if $p = 1/2$.

Define

$$\Gamma_\alpha = \sum_{n=0}^{\infty} \frac{1}{a(n)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty \ ,$$

where the last inequality holds since we have assumed that $\alpha > 1$. Then, define

$$Z = \kappa \Gamma_\alpha^{\kappa-1} \ . \tag{12.5}$$

The following proposition explains the appearance of the somewhat unnecessary N^α term at (12.4).

Proposition 12.4. *We have that*

$$\lim_{N\to\infty} Z_N = Z \ .$$

Since our primary concern is the connection between the potential theory and the metastability of the zero-range processes, we shall not prove all the detailed properties of the zero-range processes. Instead, we refer to [5] for the proof. For this proposition, we refer to [5, Proposition 2.1] for the proof.

Dirichlet form

We write $\mathscr{D}_N(\mathbf{f})$, $\mathbf{f} : \mathcal{H}_N \to \mathbb{R}$, the Dirichlet form associated with the zero-range process $\eta_N(\cdot)$, i.e.,

$$\mathscr{D}_N(\mathbf{f}) = \langle \mathbf{f}, \, -\mathscr{L}_N \mathbf{f} \rangle_{\mu_N} \ .$$

By summation by parts, we can rewrite this Dirichlet form as

$$\mathscr{D}_N(\mathbf{f}) = \frac{1}{2} \sum_{x\in S} \sum_{y \in S} \mu_N(\eta)\, g(\eta_x)\, r(x,\, y)\, [\mathbf{f}(\sigma^{x,\,y}\eta) - \mathbf{f}(\eta)]^2 \ .$$

Equilibrium potentials and capacities

In the investigation of the zero-range process, both the potential theories of the underlying random walk $\mathbb{X}(\cdot)$ and of the zero-range process $\eta_N(\cdot)$ are important. Hence, in order to avoid confusion, we have to carefully define potential theoretical notions for these processes.

- Denote by τ_A and $\tau_{\mathcal{A}}$ the hitting times of the sets $A \subset S$ and $\mathcal{A} \subset \mathcal{H}_N$, respectively. In this part, the subsets of S will be denoted by plain capital letters, while the subsets of $\mathcal{H}_N$ are denoted by calligraphic capital letters.

- For two disjoint and non-empty sets A and B of S, we denote by $h_{A,\,B} : S \to [0,\,1]$ and $\mathrm{cap}_{\mathbb{X}}(A,\,B)$ the equilibrium potential and the capacity with respect to the underlying process $\mathbb{X}(\cdot)$, respectively:

$$\begin{aligned} h_{A,\,B}(x) &:= \mathbf{P}_x[\tau_A < \tau_B] \quad ; x \in S \ , \text{ and} \\ \mathrm{cap}_{\mathbb{X}}(A,\,B) &:= D_{\mathbb{X}}(h_{A,\,B}) \ . \end{aligned}$$

  For two disjoint and non-empty sets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}_N$, we denote by $\mathbf{h}_{\mathcal{A},\mathcal{B}} : \mathcal{H}_N \to [0,\,1]$ and $\mathrm{cap}_N(\mathcal{A},\,\mathcal{B})$ the equilibrium potential and the capacity with respect to the zero-range processes $\eta_N(\cdot)$, respectively:

$$\begin{aligned} \mathbf{h}_{\mathcal{A},\mathcal{B}}(\eta) = \mathbf{h}^N_{\mathcal{A},\mathcal{B}}(\eta) &:= \mathbb{P}^N_\eta\left[\tau_{\mathcal{A}} < \tau_{\mathcal{B}}\right] \ ; \ \eta \in \mathcal{H}_N \ , \text{ and} \\ \mathrm{cap}_N(\mathcal{A},\,\mathcal{B}) &:= \mathscr{D}_N(\mathbf{h}_{\mathcal{A},\mathcal{B}}) \ . \end{aligned}$$

Adjoint and symmetrized processes

Define the adjoint rate $r^\dagger(\cdot,\,\cdot)$and the symmetrized rate $r^s(\cdot,\,\cdot)$ as

$$r^\dagger(x,\,y) = \begin{cases} 1-p & \text{if } y = x+1 \ , \\ p & \text{if } y = x-1 \ , \\ 0 & \text{otherwise} \ , \end{cases} \quad \text{and} \quad r^s(x,\,y) = \begin{cases} 1/2 & \text{if } y = x+1 \ , \\ 1/2 & \text{if } y = x-1 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

so that

$$r^s(x,\,y) = \frac{1}{2}\left[r(x,\,y) + r^\dagger(x,\,y)\right] \ .$$

Denote by $(\mathbb{X}^\dagger(t))_{t\geq 0}$ and $(\mathbb{X}^s(t))_{t\geq 0}$ the Markov processes on S with rate $r^\dagger(\cdot,\,\cdot)$ and $r^s(\cdot,\,\cdot)$, respectively.

Exercise 12.5. Prove that $\mathbb{X}^\dagger(\cdot)$ and $\mathbb{X}^s(\cdot)$ are the adjoint and symmetrized processes, respectively, of the underlying process $\mathbb{X}(\cdot)$.

We write $L^\dagger_{\mathbb{X}}$ and $L^s_{\mathbb{X}}$ for the generators of the processes $\mathbb{X}^\dagger(\cdot)$ and $\mathbb{X}^s(\cdot)$, respectively. In addition, we write $h^\dagger_{A,\,B}(\cdot)$ for the equilibrium potential with respect to the process $\mathbb{X}^\dagger(\cdot)$.

Next we define two generators $\mathscr{L}_N^\dagger$ and $\mathscr{L}_N^s$ acting on $\mathbf{f} : \mathcal{H}_N \to \mathbb{R}$ as

$$(\mathscr{L}_N^\dagger \mathbf{f})(\eta) = \sum_{x,y \in S} g(\eta_x) r^\dagger(x,\, y) \left(\mathbf{f}(\sigma^{x,\, y}\eta) - \mathbf{f}(\eta)\right) \text{ and}$$
$$(\mathscr{L}_N^s \mathbf{f})(\eta) = \sum_{x,y \in S} g(\eta_x) r^s(x,\, y) \left(\mathbf{f}(\sigma^{x,\, y}\eta) - \mathbf{f}(\eta)\right) ,$$

respectively. Denote by $(\eta_N^\dagger(t))_{t\geq 0}$ and $(\eta_N^s(t))_{t\geq 0}$ the continuous-time Markov processes on $\mathcal{H}_N$ generated by $\mathscr{L}_N^\dagger$ and $\mathscr{L}_N^s$, respectively.

Exercise 12.6. Prove that $\eta_N^\dagger(\cdot)$ and $\eta_N^s(\cdot)$ are the adjoint and symmetrized processes, respectively, of the zero-range process $\eta_N(\cdot)$.

We write $\mathbf{h}_{\mathcal{A},\,\mathcal{B}}^\dagger(\cdot)$ for the equilibrium potential with respect to the adjoint process $\eta_N^\dagger(\cdot)$. We also write $\mathrm{cap}_N^s(\mathcal{A},\, \mathcal{B})$ for the capacity with respect to the symmetrized process $\eta_N^s(\cdot)$.

13 Condensation Phenomenon

In this section, we explain the condensation phenomena of the zero-range processes defined in the previous section.

Metastable valleys

We first define an auxiliary sequences to concretely define the metastable sets of the zero-range processes. For two sequences $(a_N)_{N\in\mathbb{N}}$, $(b_N)_{N\in\mathbb{N}}$ of positive real numbers, the notation $a_N \ll b_N$ implies that

$$\lim_{N\to\infty} \frac{b_N}{a_N} = \infty \ .$$

Let $(\ell_N)_{N\in\mathbb{N}}$ be sequences of positive integer such that

$$1 \ll \ell_N \ll N^{(1+\alpha)/(1+(\kappa-1)\alpha)} \ . \tag{13.1}$$

We explain later the reason for imposing this complicated upper bound for ℓ_N.

For each $x \in S$, the *metastable valley* or *metastable set* $\mathcal{E}_N^x \subset \mathcal{H}_N$ is defined as the set of configurations such that all but at most $\ell_N \ll N$ (by (13.1) since $\kappa \geq 2$) particles are condensed at site x:

$$\mathcal{E}_N^x = \{\eta \in \mathcal{H}_N : \eta_x \geq N - \ell_N\} \ .$$

Define

$$\mathcal{E}_N = \bigcup_{x\in S} \mathcal{E}_N^x \quad \text{and} \quad \Delta_N = \mathcal{H}_N \setminus \mathcal{E}_N \ . \tag{13.2}$$

Condensation of particles

The following theorem shows that the zero-range process defined above exhibit a phenomenon known as condensation of particles.

Theorem 13.1. *It holds that*

$$\lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{\kappa} \ \text{for all } x \in S \ .$$

Therefore, the invariant measure $\mu_N(\cdot)$ is concentrated on the metastable sets defined above in the sense that

$$\lim_{N\to\infty} \mu_N(\mathcal{E}_N) = 1 \quad \text{and} \quad \lim_{N\to\infty} \mu_N(\Delta_N) = 0 \ .$$

Proof. We refer to [5, display (3.2)] for a proof. □

Remark 13.2. This theorem holds for any sequence $(\ell_N)_{N\in\mathbb{N}}$ satisfying $1 \ll \ell_N \ll N$. The condition $\ell_N \ll N^{(1+\alpha)/(1+(\kappa-1)\alpha)}$ appearing in (13.1) is used only in the investigation of the metastable behavior explained in the next section (cf. conditions (H1) and (H3) introduced later in (14.5) and (14.7), respectively).

This theorem assert that, with dominating probability (as N gets larger), almost all particles are condensed at a single site. This phenomenon is called a condensation of particles. Hence, if the zero-range process starts from any configuration, it will eventually form a condensate at a certain site. Subsequently, this condensate will move around sites of S. Such movements of the condensate, which are often referred to as the inter-valley dynamics, are the metastable behavior of the zero-range processes and are our main concern that will be discussed in the next section.

14 Markov Chain Model Reduction

In this section, we introduce the main results regarding the analysis of the metastable behavior of the zero-range process, and then outline a general framework regarding the Markov chain model reduction of the metastable behavior that can be applied to the current model. This general framework is called the martingale approach, which is developed in [2, 3, 4] and then enhanced in [34].

14.1 Order process

In this section, we introduce the so-called order process which represents the inter-valley dynamics and hence plays a significant role in the Markov chain model reduction. All the definitions introduced in the current section can be made for a general class of Markov processes, but we define them only in the context of the zero-range processes for the convenience of the discussion.

Trace process

The trace process of the zero-range process $\eta_N(\cdot)$ on the set $\mathcal{E}_N$ (cf. (13.2)) is defined as

$$T^{\mathcal{E}_N}(t) = \int_0^t \mathbf{1}\{\eta_N(s) \in \mathcal{E}_N\}\, ds \quad ;\ t \ge 0\ .$$

This random time represents the total amount of time for which the zero-range process stays in $\mathcal{E}_N$ up to time t. We denote by $S^{\mathcal{E}_N} : [0,\, \infty) \to [0,\, \infty)$ the generalized inverse of the non-decreasing function $T^{\mathcal{E}_N}(\cdot)$, i.e.,

$$S^{\mathcal{E}_N}(t) = \sup\left\{s \ge 0 : T^{\mathcal{E}_N}(s) \le t\right\} \quad ;\ t \ge 0\ .$$

The trace process $(\eta_N^{\mathcal{E}_N}(t))_{t\ge 0}$ of the zero-range process $\eta_N(\cdot)$ on the set $\mathcal{E}_N$ is defined by

$$\eta_N^{\mathcal{E}_N}(t) = \eta_N(S^{\mathcal{E}_N}(t)) \quad ;\ t \ge 0\ .$$

By carefully looking at the definition, one can observe that the trajectory of $\eta_N^{\mathcal{E}_N}(\cdot)$ is obtained from that of the zero-range process $\eta_N(\cdot)$ by removing the excursion of $\eta_N(\cdot)$ on the set Δ_N (cf. (13.2)). This is the reason that the process $\eta_N^{\mathcal{E}_N}(\cdot)$ is called the trace process of $\eta_N(\cdot)$ on $\mathcal{E}_N$.

Exercise 14.1. (The answers to the following questions can be found in [2, 3])

1. Prove that $\eta_N^{\mathcal{E}_N}(\cdot)$ is indeed an irreducible continuous-time Markov process on $\mathcal{E}_N$.

2. Prove that the invariant measure of the process $\eta_N^{\mathcal{E}_N}(\cdot)$ is the conditioned measure $\mu_N^{\mathcal{E}_N}(\cdot)$ of $\mu_N(\cdot)$ on $\mathcal{E}_N$, i.e.,

$$\mu_N^{\mathcal{E}_N}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{E}_N)} \quad ; \, \eta \in \mathcal{E}_N \; .$$

3. Prove that the Markov process $\eta_N^{\mathcal{E}_N}(\cdot)$ is reversible if $\eta_N(\cdot)$ is reversible. Is the converse true?

Order process

We note that the trace process $\eta_N^{\mathcal{E}_N}(\cdot)$ includes all the information about the behavior of the process $\eta_N(\cdot)$ on $\mathcal{E}_N$. However, in view of the metastable behavior, we are only concerned with the inter-valley dynamics and are not interested in the exact location in a metastable valley within which the zero-range process is staying. Hence, the order process is defined as the process obtained from the trace process by discarding this information.

More precisely, we define a projection function $\Psi : \mathcal{E}_N \to S$ as

$$\Psi(\eta) = \sum_{x \in S} x \cdot \mathbf{1}\{x \in \mathcal{E}_N^x\}$$

and then define the *order process* as

$$Y_N(t) = \Psi(\eta_N^{\mathcal{E}_N}(N^{1+\alpha}t)) \quad ; \, t \geq 0 \; .$$

To explain the meaning of the order process, we first consider the projected trace process

$$W_N(t) = \Psi(\eta_N^{\mathcal{E}_N}(t)) \quad ; \, t \geq 0 \; .$$

This process $W_N(t)$ indicates the label of the valley at which the trace process $\eta_N^{\mathcal{E}_N}(t)$ is staying. Hence, this process captures all the relevant information regarding the inter-valley dynamics of the process $\eta_N(\cdot)$ on $\mathcal{E}_N$. We defined $Y_N(t)$ as a speeded-up version of this process, namely,

$$Y_N(t) = W_N(N^{1+\alpha}t)$$

since we observe the transitions between metastable valleys in the time scale of $N^{1+\alpha}$.

It takes a long time to move a condensate from one site to another since the particles are sticky and hence tend to keep the condensate. We can also notice that this transition time scale $N^{1+\alpha}$ is increasing in α. This is a natural result since the parameter α corresponds to the stickiness of the particles.

14.2 Markov chain model reduction via convergence of order process

Markov chain model reduction

We note here that the order process may not be a Markov process. However, one can usually prove that, in the metastable situation, the order process converges to a certain limiting Markov process $Y(\cdot)$ on S. Heuristically, this is mainly because the process entering a metastable valley will spend long enough time inside the valley to forget the entering location. This is indeed the case for the zero-range process, and the following is the main theorem regarding the Markov chain model reduction. We remark that the limiting Markov process $Y(\cdot)$ for the zero-range process is defined in the next paragraph.

Theorem 14.2. *The following hold:*

1. *Suppose that $\eta_N(0) \in \mathcal{E}_N^x$ for all $N \in \mathbb{N}$ for some $x \in S$. Then, the law of the order process $Y_N(\cdot)$ converges to the law of limiting Markov process $Y(\cdot)$ starting at x.*

2. *For all $T > 0$, it holds that*

$$\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N} \mathbb{E}_\eta^N \left[\int_0^T \mathbf{1}\{\eta_N(N^{1+\alpha}t) \in \Delta_N\} dt \right] = 0 \ .$$

If the zero-range process $\eta_N(\cdot)$ spends non-negligible amount of time at $\Delta_N = \mathcal{H}_N \setminus \mathcal{E}_N$, then the trace process $\eta_N^{\mathcal{E}_N}(\cdot)$, which is obtained by turning off the clock when the zero-range process $\eta_N(\cdot)$ stays at Δ_N, discards too much information regarding the inter-valley dynamics of the zero-range process. Part (2) of the previous theorem implies that, in the scale $N^{1+\alpha}$, the zero-range process does not spend meaningful amount of time at Δ_N and hence the trace process is indeed a good approximation of $\eta_N(\cdot)$ in view of the inter-valley dynamics. This gives authority to part (1) which asserts that the inter-valley dynamics of the trace process (and hence the zero-range process by part (2)) is approximated by the limiting Markov process $Y(\cdot)$. So far, we have explained a general way to derive the Markov chain model reduction via convergence of order process.

We discuss the strategy to prove Theorem 14.2 in Section 14.4.

Limiting Markov process

We next define the limiting Markov process $Y(\cdot)$ for the zero-range process. Define a constant by

$$I_\alpha = \int_0^1 u^\alpha (1-u)^\alpha du \; . \tag{14.1}$$

Define $a : S \times S \to [0, \infty)$ by

$$a(x,\, y) = \frac{\kappa}{\Gamma_\alpha I_\alpha} \mathrm{cap}_{\mathbb{X}}(x,\, y) \quad ;\; x,\, y \in S \; , \tag{14.2}$$

where we remind here that the notation $x,\, y \in S$ implies that x and y are different. Note that the capacity $\mathrm{cap}_{\mathbb{X}}(x,\, y)$ has been computed in Exercise 0.7. Now the limiting Markov process $(Y(t))_{t\ge0}$ is define as a continuous-time Markov process on S with rate $a(\cdot,\, \cdot)$. We denote by $\mathbf{Q}_x$ the law of process $Y(\cdot)$ starting at $x \in S$.

Since $a(x,\, y) > 0$ for all $x,\, y > 0$, the irreducibility is clear for the process $Y(\cdot)$. Denote by $\nu(\cdot)$ the uniform measure on S:

$$\nu(x) = \frac{1}{\kappa} \quad ;\; x \in S \; . \tag{14.3}$$

Exercise 14.3. Prove that that the unique invariant measure of the irreducible Markov process $Y(\cdot)$ is $\nu(\cdot)$ and furthermore, that the process $Y(\cdot)$ is reversible. (Hint: use (0.13))

A remarkable fact here is that the limiting Markov process is always reversible, while the underlying zero-range process is not, especially for $p \neq 1/2$.

14.3 Markov chain model reduction via convergence of marginal distributions

An alternative way of describing the Markov chain model reduction was developed in [34]. This methodology does not discard the excursions of the zero-range process on Δ_N (and hence does not use the trace and order processes) but proves the convergence result with a weaker notion of convergence, namely the convergence of finite dimensional distributions. This is the nature of the problem; without removing noisy excursions at Δ_N, we cannot expect the convergence in path space with the usual mode of convergence. We refer to [2] for more detail. Instead, the soft topology introduced in [30] can be used to prove the convergence.

To explain this alternative method, let us define a projection function $\widehat{\Psi} : \mathcal{H}_N \to$

$S \cup \{\mathfrak{o}\}$ as

$$\widehat{\Psi}(\eta) = \begin{cases} x & \text{if } x \in \mathcal{E}_N^x \, , \\ \mathfrak{o} & \text{if } x \in \Delta_N \, . \end{cases}$$

Then, define a process $(\widehat{Y}_N(t))_{t \ge 0}$ as

$$\widehat{Y}_N(t) = \widehat{\Psi}(\eta_N(N^{1+\alpha}t)) \quad ; \, t \ge 0$$

Then, the process $\widehat{Y}_N(\cdot)$ is a process on $\widehat{S} = S \cup \{\mathfrak{o}\}$ and may not be a Markov process. We note that the order process $Y_N(\cdot)$ is a trace process of $\widehat{Y}_N(\cdot)$ on the set S.

Define an extended limiting process $(\widehat{Y}(t))_{t \ge 0}$ on $\widehat{S}$ as a continuous-time Markov process with jump rate

$$\widehat{a}(x,\, y) = \begin{cases} a(x,\, y) & \text{if } x,\, y \in S \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

Hence, $\mathfrak{o}$ is merely a cemetery point of the Markov process $\widehat{Y}(\cdot)$. Denote by $\widehat{\mathbf{Q}}_x$, $x \in \widehat{S}$, the law of process $\widehat{Y}(\cdot)$ that starts at x.

Exercise 14.4. Prove that the measure $\widehat{\nu}(\cdot)$ on $\widehat{S}$ defined by

$$\widehat{\nu}(x) = \begin{cases} \nu(x) & \text{if } x \in S \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

is an invariant measure of the Markov process $\widehat{Y}(\cdot)$.

The following is the second way of establishing a Markov chain model reduction of the metastable behavior developed in [34].

Theorem 14.5. *For all $x \in S$ and for all $(\eta_N)_{N \in \mathbb{N}}$ such that $\eta_N \in \mathcal{E}_N^x$ for all N, the finite dimensional distributions of the process $\widehat{Y}_N(\cdot)$ under $\mathbb{P}_{\eta_N}^N$ converges to that of the law $\widehat{\mathbf{Q}}_x$, as N tends to infinity.*

The proof of this theorem is close to that of Theorem 14.2 and will be explained in the next subsection.

14.4 Martingale approach

In this section, we explain the general principle developed in [2, 3, 4, 34]. This principle, which is now called the martingale approach to the metastability reduces the proof of Theorems 14.2 and 14.5 to the verification of several sufficient conditions.

To explain the general principle in the context of zero-range process, we now explain several essential notions.

- Recall that $\eta_N^{\mathcal{E}_N}(\cdot)$ is a Markov process on $\mathcal{E}_N$. Denote by $j_N : \mathcal{E}_N \times \mathcal{E}_N \to [0, \infty)$ the jump rate of the process $\eta_N^{\mathcal{E}_N}(\cdot)$.
- For $x, y \in S$, the *mean jump rate* between two valleys $\mathcal{E}_N^x$ and $\mathcal{E}_N^y$ is defined by
$$r_N(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \sum_{\zeta \in \mathcal{E}_N^y} \mu_N(\eta)\, j_N(\eta, \zeta) \; .$$
- For each $x \in S$, let $\xi_N^x \in \mathcal{H}_N$ be the configuration such that all particles are located at site x.
- For $x \in S$, write $\breve{\mathcal{E}}_N^x = \mathcal{E}_N \setminus \mathcal{E}_N^x$.
- For $x, y \in S$, write $\breve{\mathcal{E}}_N^{x,\, y} = \mathcal{E}_N \setminus \big(\mathcal{E}_N^x \cup \mathcal{E}_N^y\big)$.

Now we introduce several sufficient conditions for the Markov chain model reduction.

- Condition **(H0)**: For all $x, y \in S$,
$$\lim_{N \to \infty} N^{1+\alpha}\, r_N(x, y) = a(x, y) \; . \tag{14.4}$$
Hence, the mean jump rate between two valleys $\mathcal{E}_N^x$ and $\mathcal{E}_N^y$ is approximately $a(x, y)/N^{1+\alpha}$. This is the reason that we accelerated the process by a factor of $N^{1+\alpha}$ in the definition of the order process. This accurate estimate of the mean jump rate is the crucial and most difficult step in the proof of Theorems 14.2 and 14.5.
- Condition **(H1)**: For all $x \in S$,
$$\lim_{N \to \infty} \sup_{\eta,\, \zeta \in \mathcal{E}_N^x} \frac{\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x)}{\mathrm{cap}_N(\eta, \zeta)} = 0 \; . \tag{14.5}$$
This condition implies that, for any $\eta, \zeta \in \mathcal{E}_N^x$, the process starting at $\eta \in \mathcal{E}_N^x$ hits the configuration $\zeta \in \mathcal{E}_N^x$ before hitting the set $\breve{\mathcal{E}}_N^x$, i.e., before arriving at one of other valleys, with dominating probability. We term this phenomenon a visiting property. We discuss this further in Remark 14.10.

  Exercise 14.6. Prove the last assertion. (Hint: Proposition 0.16)

- Condition **(H2)**: For all $x \in S$,

$$\lim_{N\to\infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} = 0 . \tag{14.6}$$

  This condition implies that the set Δ_N is negligible compared to $\mathcal{E}_N^x$ with respect to the invariant measure. *We emphasize that this condition is a direct consequence of Theorem 13.1.*

- Condition **(H3)**: For all $x \in S$,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{\eta\in\mathcal{E}_N^x} \sup_{2\delta\le s\le 3\delta} \mathbb{P}_\eta^N \left[\eta_N(N^{1+\alpha}\, s) \in \Delta_N\right] = 0 . \tag{14.7}$$

  This implies that, if the zero-range process starts from a valley it will still be in the same valley after a short time. Note that we cannot replace $\sup_{2\delta\le s\le 3\delta}$ with $\sup_{0\le s\le\delta}$, since if the process starts at the boundary of $\mathcal{E}_N^x$, then with a non-negligible probability it leaves the valley within a few steps. This condition (14.7) implies that, even after such an escape from the valley, the process returns to the valley immediately.

The next theorem is a consequence of [3, Theorem 2.1] and [34, Proposition 1.1],

Theorem 14.7. *Suppose that conditions (H0), (H1), and (H2) hold. Then, Theorem 14.2 holds. Moreover, if condition (H3) additionally holds, then Theorem 14.5 also holds.*

Therefore, to prove Theorems 14.2 and 14.5, it suffices to verify the conditions (H0), (H1), (H2), and (H3):

- Condition (H0) will be proven in Proposition 16.13. This is the most difficult part of the current problem. The main components of the proof are the estimate of capacities, the sector condition, and the argument developed in [40] based on collapsed processes.

- Conditions (H1) and (H3) are proven based on Propositions 17.3 and 17.4, respectively, based on the estimate of capacities.

- As we have mentioned above, the condition (H2) is a consequence of Theorem 13.1.

Remark 14.8. In fact, Theorem 13.1 holds only for $\alpha \ge 1$ (for the critical case $\alpha = 1$, we should be more careful about the selection of ℓ_N, see [35]) and hence the metastable behavior must be studied for all $\alpha \ge 1$. Below is the history of the research on this problem in chronological order:

1. Beltran and Landim [5] first analyzed the reversible case $p = 1/2$ with $\alpha > 1$.

2. Landim [31] analyzed the totally asymmetric case $p = 1$ with $\alpha > 3$.

3. Seo [56] analyzed the general case $p \in [0, 1]$ with $\alpha > 2$.

4. Landim, Marcondes and Seo [35, 36] analyzed the critical case $\alpha = 1$ with $p = 1/2$.

We note that the articles [5, 35, 36, 56] considered a more general case, i.e., the particle system on any finite set consisting of any underlying random walk $\mathbb{X}(\cdot)$. The articles [5, 35, 36] assumed the reversibility of the zero-range process. Moreover, [35, 36] assumed that the invariant measure for the underlying random walk $\mathbb{X}(\cdot)$ is the uniform measure on S. We also emphasize here that [31] is the first rigorous quantitative analysis of the metastable behavior of a non-reversible Markov process.

Remark 14.9. The current part of this lecture note is mainly derived from article [56]. With a more refined argument, we are able to weaken the assumption $\alpha > 2$ of [56] to $\alpha > 1$. The critical case $\alpha = 1$ for the non-reversible case is largely unknown at this moment. We discuss in the next remark the difficulty of the critical case.

Remark 14.10. If ℓ_N is too large, then there are too many configurations inside the valley and hence the visiting property explained in condition (H1) may not hold. In fact, the upper bound of ℓ_N given in (13.1) is imposed to verify condition (H1). For the critical case $\alpha = 1$, in order to ensure that (H1) is in force, we have to take ℓ_N so small that the metastable valley $\mathcal{E}_N^x$ with such ℓ_N violates Theorem 13.1 (i.e., the condition (H2)). *In conclusion, the critical zero-range process cannot satisfy two condition (H1) and (H2) simultaneously, no matter what value we give to ℓ_N.* This is the reason that the critical case cannot be handled with the martingale approach described here. Recently, [36] developed a new approach based on the analysis of the solution of certain form of resolvent equations and used this approach to investigate the metastable behavior of critical case with $p = 1/2$.

14.5 Outlook of the remainder of Part III

In the remainder of the note, we verify conditions (H0), (H1), and (H3).

- In Section 15, we explain and prove the capacity estimates between valleys. The proof is based on the generalized Dirichlet and Thomson principles and hence we need to construct the test functions and flows.

- In Section 16, we prove condition (H0).

- In Section 17, we prove conditions (H1) and (H3).

15 Estimate of Capacities

In this section, we provide, up to the construction of test objects, the estimate of the capacity between metastable valleys based on generalized Dirichlet and Thomson principles.

Main result

For $f : S \to \mathbb{R}$, the generator of the limiting Markov process $Y(\cdot)$ on S can be written as

$$(\mathfrak{L}_Y f)(x) = \sum_{y \in S \setminus \{x\}} \frac{\kappa \, \mathrm{cap}_X(x,\, y)}{\Gamma_\alpha \, I_\alpha} \, [f(y) - f(x)] \quad ; \; x \in S \, . \tag{15.1}$$

As we have mentioned before, the invariant measure for $Y(\cdot)$ is the uniform measure $\nu(\cdot)$ on S, i.e.,

$$\nu(x) = \frac{1}{\kappa} \;\text{ for all }\; x \in S \, .$$

Therefore, the Dirichlet form with respect to the process $Y(\cdot)$ acting on $f : S \to \mathbb{R}$ such a way that

$$\mathfrak{D}_Y(f) = \sum_{x \in S} \nu(x) \, f(x) \, [-(\mathfrak{L}_Y \, f)(x)] = \frac{1}{2} \sum_{x \in S} \sum_{y \in S} \frac{\mathrm{cap}_X(x,\, y)}{\Gamma_\alpha \, I_\alpha} \, [f(y) - f(x)]^2$$

Recall that $\mathbf{Q}_x$ denote the law of the process $Y(\cdot)$ starting from $x \in S$. For two disjoint non-empty sets A and B of S, the equilibrium potential and capacity between A and B with respect to the process $Y(\cdot)$ are defined by

$$\mathfrak{h}_{A,\,B}(x) = \mathbf{Q}_x(\tau_A < \tau_B) \;\text{ for } x \in S_\star \;\text{ and }\; \mathrm{cap}_Y(A,\, B) = \mathfrak{D}_Y(\mathfrak{h}_{A,\,B}) \, , \tag{15.2}$$

respectively.

For a non-empty set $A \subseteq S$, we write

$$\mathcal{E}_N(A) = \bigcup_{x \in A} \mathcal{E}_N^x \, .$$

The following theorem is the main capacity estimate for the zero-range processes

Theorem 15.1. *For disjoint, non-empty subsets A, B of S, we have that*

$$\lim_{N \to \infty} N^{1+\alpha} \, \mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) = \mathrm{cap}_Y(A,\, B) \, .$$

In addition, if $(A,\, B)$ is a partition of S, that is, $A \cup B = S$, the equilibrium potential $\mathfrak{h}_{A,\,B}(\cdot)$ becomes the indicator function on A, and hence by (15.2) we immediately obtain the following result as a corollary of the previous theorem.

Corollary 15.2. *Suppose that two disjoint, non-empty subsets A, B of S satisfy $A \cup B = S$. Then,*

$$\lim_{N\to\infty} N^{1+\alpha}\,\mathrm{cap}_N(\mathcal{E}_N(A),\,\mathcal{E}_N(B)) = \frac{1}{\Gamma_\alpha I_\alpha}\sum_{x\in A}\sum_{y\in B}\mathrm{cap}_X(x,\,y)\ .$$

Now we discuss how we can prove Theorem 15.1.

Strategy to prove Theorem 15.1

Let us now turn to the proof of Theorem 15.1, which is based on the generalized Dirichlet and Thomson principles (cf. Theorem 2.2). We explain how we can apply these principles in the context of the zero-range processes.

We start from the test functions and flows. Let us first introduce a new parameter $\epsilon > 0$ denoting small numbers. The parameter ϵ will be sent to 0 in the end (after sending N to ∞).

Remark 15.3. Henceforth, all constants are assumed to depend only on p, κ, α, and ϵ and are independent of N. Furthermore, we write $a(N,\,\epsilon) = o_N(1)$ and $b(N,\,\epsilon) = o_\epsilon(1)$ if

$$\lim_{N\to\infty} a(N,\,\epsilon) = 0 \text{ for all } \epsilon > 0 \quad \text{and}$$
$$\lim_{\epsilon\to 0}\sup_{N\in\mathbb{N}} b(N,\,\epsilon) = 0\ ,$$

respectively. The dependencies of the constant and the $o_N(1)$ term on the parameter ϵ do not incur any problem, as we always send N to infinity first before sending ϵ to 0.

Throughout the remainder of the current section, let us fix two disjoint non-empty subsets A and B of S.

In [56, Section 7], for sufficiently large $N \in \mathbb{N}$, two functions

$$\mathbf{V}_{A,\,B} = \mathbf{V}_{A,\,B}^{N,\,\epsilon} : \mathcal{H}_N \to \mathbb{R} \quad \text{and} \quad \mathbf{V}_{A,\,B}^{\dagger} = \mathbf{V}_{A,\,B}^{\dagger,N,\,\epsilon} : \mathcal{H}_N \to \mathbb{R}$$

approximating the equilibrium potentials $\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}$ and $\mathbf{h}^{\dagger}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}$, respectively, are constructed. It is also verified there that these functions enjoy the following properties.

Proposition 15.4. *For all small enough ϵ and large enough N, two functions $\mathbf{V}_{A,\,B}$ and $\mathbf{V}_{A,\,B}^{\dagger}$ satisfy the following properties:*

1. *It hold that* $\mathbf{V}_{A,\,B},\,\mathbf{V}^{\dagger}_{A,\,B} \in \mathfrak{C}_{1,\,0}(\mathcal{E}_N(A),\,\mathcal{E}_N(B))$. *Moreover, for all* $x \in S \setminus \{A,\,B\}$, *it holds that*

$$\mathbf{V}_{A,\,B}(\eta) = \mathbf{V}^{\dagger}_{A,\,B}(\eta) = \mathfrak{h}_{A,\,B}(x) \quad \textit{for all } \eta \in \mathcal{E}_N^x \;.$$

2. *It holds that*

$$N^{1+\alpha}\mathscr{D}_N(\mathbf{V}_{A,\,B}),\, N^{1+\alpha}\mathscr{D}_N(\mathbf{V}^{\dagger}_{A,\,B}) \le [1 + o_N(1) + o_\epsilon(1)]\,\mathrm{cap}_Y(A,\,B) \;.$$

We next construct test flows approximating $\Phi^*_{\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}$ and $\Phi_{\mathbf{h}^{\dagger}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}$ (cf. (0.28)). The natural candidates are $\Phi^*_{\mathbf{V}_{A,\,B}}$ and $\Phi_{\mathbf{V}^{\dagger}_{A,\,B}}$. However, the divergences of these flows are larger than required along the saddle tube between metastable sets (cf. [56, Section 7.2]) and hence we need to perform a local surgery to cancel these divergences out without impacting approximating features of the flows $\Phi^*_{\mathbf{V}_{A,\,B}}$ and $\Phi_{\mathbf{V}^{\dagger}_{A,\,B}}$. This procedure is the most complicated part in the analysis of the zero-range process. The consequences of this correction procedure can be summarized as follows.

Proposition 15.5. *For all small enough* ϵ *and large enough* N, *there exist flows*

$$\Phi_{A,\,B} = \Phi^{N,\,\epsilon}_{A,\,B} \in \mathfrak{F}_N \quad \textit{and} \quad \Phi^{\dagger}_{A,\,B} = \Phi^{\dagger,\,N,\,\epsilon}_{A,\,B} \in \mathfrak{F}_N$$

enjoying the following properties.

1. *The flows* $\Phi_{A,\,B}$ *and* $\Phi^{\dagger}_{A,\,B}$ *approximate* $\Phi^*_{\mathbf{V}_{A,\,B}}$ *and* $\Phi_{\mathbf{V}^{\dagger}_{A,\,B}}$ *in the sense that*

$$\begin{aligned} \left\| \Phi_{A,\,B} - \Phi^*_{\mathbf{V}_{A,\,B}} \right\|^2 &= [o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)} \quad \textit{and} \\ \left\| \Phi^{\dagger}_{A,\,B} - \Phi_{\mathbf{V}^{\dagger}_{A,\,B}} \right\|^2 &= [o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)} \;. \end{aligned}$$

2. *The divergence of* $\Phi_{A,\,B}$ *is negligible on* Δ_N *in the sense that*

$$\sum_{\eta \in \Delta_N} |(\mathrm{div}\ \Phi_{A,\,B})(\eta)| = o_N(1)\, N^{-(1+\alpha)} \;.$$

3. *The divergence of* $\Phi_{A,\,B}$ *is negligible on* $\mathcal{E}_N^x$, $x \in S \setminus (A \cup B)$, *in the sense that*

$$(\mathrm{div}\ \Phi_{A,\,B})(\mathcal{E}_N^x) = o_N(1)\, N^{-(1+\alpha)} \quad \textit{and} \tag{15.3}$$

$$\sum_{\eta \in \mathcal{E}_N^x} \mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\eta)\, (\mathrm{div}\ \Phi_{A,\,B})(\eta) = o_N(1)\, N^{-(1+\alpha)} \;. \tag{15.4}$$

4. The divergence of $\Phi_{A,\,B}$ satisfies

$$(\mathrm{div}\ \Phi_{A,\,B})(\mathcal{E}_N(A)) = [1+o_N(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\ \ \textit{and}$$
$$(\mathrm{div}\ \Phi_{A,\,B})(\mathcal{E}_N(B)) = -[1+o_N(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\ .$$

The flow $\Phi^{\dagger}_{A,\,B}$ also satisfies properties (2), (3), and (4).

The proof of this proposition is given in [56, Section 8].

Since the proofs of Proposition 15.4 and 15.5 are too technical and hence are not suitable as contents of a lecture note, we refer to the interested readers to the article [56]. Instead, we will now focus on how we can prove the Markov chain model reduction based on this constructions.

By (2), (3), and (4) of the previous proposition, we have the following estimate that enables the application of the generalized Dirichlet and Thomson principles.

Lemma 15.6. *We have that*

$$\sum_{\eta\in\mathcal{H}_N}\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}(\eta)\,(\mathrm{div}\ \Phi_{A,\,B})(\eta) = [1+o_N(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\ \ \textit{and} \tag{15.5}$$
$$\sum_{\eta\in\mathcal{H}_N}\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}(\eta)\,(\mathrm{div}\ \Phi^{\dagger}_{A,\,B})(\eta) = [1+o_N(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\ . \tag{15.6}$$

Proof. We only consider (15.5) since the proof of (15.6) is identical. The summation on the left-hand side of (15.5) can be divided into

$$\sum_{\eta\in\mathcal{E}_N(A)} + \sum_{\eta\in\mathcal{E}_N(B)} + \sum_{x\notin A\cup B}\sum_{\eta\in\mathcal{E}_N^x} + \sum_{\eta\in\Delta_N}\ . \tag{15.7}$$

Since $\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}\equiv 1$ on $\mathcal{E}_N(A)$, by part (4) of Proposition 15.5, the first summation is equal to

$$[1+o_N(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\ .$$

Since $\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}\equiv 0$ and $\mathcal{E}_N(B)$, the second summation in (15.7) is trivially 0. The third summation is $o_N(1)\,N^{-(1+\alpha)}$ by the second estimate of (3) of Proposition 15.5. Finally, as $|\mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}|\le 1$, the last summation is $o_N(1)\,N^{-(1+\alpha)}$ by (2) of Proposition 15.5. □

Now by accepting Proposition 15.4 and 15.5, we can complete the proof of Theorem 15.1.

Proof of Theorem 15.1. Inspired by the optimizer of Theorem 2.2-(1), let us take

$$\mathbf{f} = \frac{\mathbf{V}_{A,\,B} + \mathbf{V}^{\dagger}_{A,\,B}}{2} \quad \text{and} \quad \phi = \frac{\Phi^{\dagger}_{A,\,B} - \Phi_{A,\,B}}{2}\ . \tag{15.8}$$

Note that $\mathbf{f} \in \mathfrak{C}_{1,\,0}(\mathcal{E}_N(A),\, \mathcal{E}_N(B))$ by (1) of Proposition 15.4. Thus, by the generalized Dirichlet principle (i.e., Theorem 2.2-(1)) and Lemma 15.6, we can write

$$\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \le \|\Phi_{\mathbf{f}} - \phi\|^2 + o_N(1)\, N^{-(1+\alpha)}\ , \tag{15.9}$$

where $\|\cdot\| = \|\cdot\|_{\mathfrak{F}}$ denotes the flow norm with respect to the zero-range process $\eta_N(\cdot)$.

Let us write

$$\Phi_{A,\,B} = \Phi^{*}_{\mathbf{V}_{A,\,B}} + \Theta_N \;\text{ and }\; \Phi^{\dagger}_{A,\,B} = \Phi_{\mathbf{V}^{\dagger}_{A,\,B}} + \Theta^{\dagger}_N\ , \tag{15.10}$$

so that we have

$$\begin{aligned} \Phi_{\mathbf{f}} - \phi &= \Phi_{(\mathbf{V}_{A,\,B} + \mathbf{V}^{\dagger}_{A,\,B})/2} - \frac{\Phi_{\mathbf{V}^{\dagger}_{A,\,B}} - \Phi^{*}_{\mathbf{V}_{A,\,B}}}{2} + \frac{\Theta_N - \Theta^{\dagger}_N}{2}\ . \\ &= \Psi_{\mathbf{V}_{A,\,B}} + \frac{\Theta_N - \Theta^{\dagger}_N}{2} \end{aligned} \tag{15.11}$$

By (2) of Proposition 15.4, it holds that

$$\left\|\Psi_{\mathbf{V}_{A,\,B}}\right\|^2 = \mathscr{D}_N(\mathbf{V}_{A,B}) \le [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B)\ . \tag{15.12}$$

On the other hand, by (2) of Proposition 15.5 and definition (15.10), it holds that

$$\left\|\frac{\Theta_N - \Theta^{\dagger}_N}{2}\right\|^2 = [o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)} \tag{15.13}$$

Therefore, (15.11), (15.12), (15.13), and the triangle inequality, we can conclude that

$$\|\Phi_{\mathbf{f}} - \phi\|^2 \le [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B)\ . \tag{15.14}$$

Inserting this into (15.9), we obtain the following upper bound of the capacity.

$$\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \le [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B)\ . \tag{15.15}$$

Now we use the generalized Thomson principle to obtain the lower bound. Based on the optimizer of Theorem 2.2-(2) and our guess of the asymptotic limit of capacity

$\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B))$, we take

To this end, let

$$\mathbf{g} = \frac{\mathbf{V}^{\dagger}_{A,\,B} - \mathbf{V}_{A,\,B}}{2\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)} \quad \text{and} \quad \psi = \frac{\Phi^{\dagger}_{A,\,B} + \Phi_{A,\,B}}{2\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)}\;. \tag{15.16}$$

By (1) of Proposition 15.4, we have $\mathbf{g} \in \mathfrak{C}_{0,\,0}(\mathcal{E}_N(A),\, \mathcal{E}_N(B))$. Moreover, by Lemma 15.6, it holds that

$$\sum_{\eta\in\mathcal{H}_N} \mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}(B)}(\eta)\,(\mathrm{div}\ \psi_{A,\,B})(\eta) = 1 + o_N(1) + o_\epsilon(1)\;.$$

Therefore, by the generalized Thomson principle (i.e., Theorem 2.2-(2)), we can conclude that

$$\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \ge \frac{1 + o_N(1) + o_\epsilon(1)}{\|\Phi_{\mathbf{g}} - \psi\|^2}\;. \tag{15.17}$$

Now it remains to compute the flow norm $\|\Phi_{\mathbf{g}} - \psi\|^2$. To this end, using (15.10), let us write

$$\Phi_{\mathbf{g}} - \psi = -\frac{1}{N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)}\left[\Psi_{\mathbf{V}_{A,\,B}} + \frac{\Theta_N + \Theta_N^{\dagger}}{2}\right]\;.$$

Then, by similar computations as in the upper bound. we can conclude that

$$\|\Phi_{\mathbf{g}} - \psi\|^2 \le \frac{1 + o_N(1) + o_\epsilon(1)}{N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)}\;. \tag{15.18}$$

Combining (15.17) and (15.18), we can finally obtain the lower bound on the capacity:

$$\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \ge [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\;. \tag{15.19}$$

By the upper bound (15.15) and lower bound (15.19), we can conclude that

$$\begin{aligned}[1 + o_\epsilon(1)]\,\mathrm{cap}_Y(A,\,B) &\le \liminf_{N\to\infty} N^{1+\alpha}\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \\ &\le \limsup_{N\to\infty} N^{1+\alpha}\mathrm{cap}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \le [1 + o_\epsilon(1)]\,\mathrm{cap}_Y(A,\,B)\;,\end{aligned}$$

where the error terms $o_\epsilon(1)$ are now dependent only on ϵ. Since the two terms in the middle are independent of ϵ, by sending ϵ to 0, we can complete the proof. □

From the previous proof, the estimate obtained in Proposition (15.4) can be strengthened as follows.

Corollary 15.7. *We have that*

$$\mathscr{D}_N(\mathbf{V}_{A,\,B}) = (1 + o_N(1) + o_\epsilon(1))\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B)\ .$$

16 Estimate of Mean Jump Rates

In this section, we verify (in Proposition 16.13) the condition (H0) for the zero-range process by estimating the mean jump rate $r_N(x, y)$ for $x, y \in S$.

For the reversible case, we can readily reduce the estimate of the mean-jump rate to that of the capacity between valleys. More precisely, it has been verified in [2, Lemma 6.8] that, for the reversible case, that is, the case $p = 1/2$, the mean jump rate satisfies the following expression

$$r_N(x, y) = \frac{1}{2}\left[\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x) + \mathrm{cap}_N(\mathcal{E}_N^y, \breve{\mathcal{E}}_N^y) - \mathrm{cap}_N(\mathcal{E}_N^x \cup \mathcal{E}_N^y, \breve{\mathcal{E}}_N^{x,y})\right] \tag{16.1}$$

for all $x, y \in S$. Hence, the estimate of the mean jump rate is a direct consequence of Theorem 15.1.

Unfortunately, a the relationship (16.1) is no longer valid in the non-reversible case and the estimation of the mean jump rate $r_N(x, y)$ becomes a more challenging task. The general strategy for this task has been developed in [40, Section 8]. The following is a summary of this strategy.

1. Define the mean holding rate by

$$\lambda_N(x) = \sum_{y \in S \setminus \{x\}} r_N(x, y) \ .$$

   Then, in [3, display (A.8)], it has been verified that the holding rate $\lambda_N(x)$ satisfies

$$\lambda_N(x) = \frac{\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x)}{\mu_N(\mathcal{E}_N^x)} \ . \tag{16.2}$$

   Therefore, by estimating the capacity $\mathrm{cap}_N(\mathcal{E}_N^x, \breve{\mathcal{E}}_N^x)$ and applying Theorem 13.1, we can obtain the sharp asymptotics of $\lambda_N(x)$.

2. The second step is to compute the sharp asymptotics of $r_N(x, y)/\lambda_N(x)$ using the collapsed process introduced in Section 3. More precisely, we fix $x \in S$, and we consider a process $\overline{\eta}_N(\cdot)$ which is the collapsed process obtained by collapsing the metastable set $\mathcal{E}_N^x$ into a single point $\mathfrak{e}$. Denote by $\overline{\mathbb{P}}_{\mathfrak{e}}^N$ the law of this collapsed process starting from $\mathfrak{e}$. Then it has been proven in [3, Proposition 4.2] that

$$\frac{r_N(x, y)}{\lambda_N(x)} = \overline{\mathbb{P}}_{\mathfrak{e}}^N\left[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^{x,y}}\right] \ . \tag{16.3}$$

   Surprisingly, we can estimate the right-hand side based on the capacity estimate for the collapsed process along with the sector condition of the zero-range process which will be verified in Section 16.1.

3. Since we can obtain an estimate of $\lambda_N(x)$ and $r_N(x,\, y)/\lambda_N(x)$ by (16.2) and (16.3), we can finally obtain an estimate of the mean jump rate $r_N(x,\, y)$. This argument is rigorously explained in Section 16.

In order to focus only on the effectiveness of potential theoretic computations, we will not attempt to prove (16.2) and (16.3) in the current note; we refer to [2, 3]. Instead, we shall directly apply this strategy to verify the condition (H0) for the zero-range processes. We note that we again assume Propositions 15.4 and 15.5 (and hence all the results obtained in previous sections) throughout this section.

16.1 Sector condition

In this section, we prove the sector condition (cf. Definition 1.10) for the zero-range process. This sector condition is one of the essential ingredients of the method developed in [40] which will be applied to the zero-range process in this section.

Proposition 16.1. *There exists a constant $C_0 > 0$ such that for all $\mathbf{f},\, \mathbf{g} : \mathcal{H}_N \to \mathbb{R}$, we have*

$$\langle \mathbf{g},\, -\mathscr{L}_N \mathbf{f} \rangle_{\mu_N}^2 \le C_0\, \mathscr{D}_N(\mathbf{f})\, \mathscr{D}_N(\mathbf{g}) \;.$$

For $u \in S$, denote by $\omega^u = (\omega^u_x)_{x\in S} \in \mathcal{H}_1$ the configuration with one particle at site u, namely,

$$\omega^u_x = \begin{cases} 1 & \text{if } x = u \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $u \in S$ and $\eta \in \mathcal{H}_N$, the configuration $\eta + \omega^u \in \mathcal{H}_{N+1}$ is the one obtained from η by adding a particle from site u. Similarly, the configuration $\eta - \omega^u \in \mathcal{H}_{N-1}$ is the one obtained from η by removing a particle at site u, provided that $\eta_u \ge 1$.

With this notation, we can observe the following convenient identity: for $u \in S$ and for $\eta \in \mathcal{H}_N$ with $\eta_u \ge 1$,

$$\mu_N(\eta)\, g(\eta_u) = a_N\, \mu_{N-1}(\eta - \omega^u) \;, \tag{16.4}$$

where a_N is defined by

$$a_N = \frac{N^\alpha\, Z_{N-1}}{(N-1)^\alpha\, Z_N} \;.$$

By Proposition 12.4, it follows immediately that

$$\lim_{N\to\infty} a_N = 1 \;. \tag{16.5}$$

Proof of Proposition 16.1. Fix $\mathbf{f}, \mathbf{g} : \mathcal{H}_N \to \mathbb{R}$. By (16.4) the change of variable $\eta - \omega^x = \zeta$, we can write

$$
\begin{aligned}
&\mathscr{D}_N(\mathbf{f}) \\
=&\frac{1}{2} \sum_{\eta\in\mathcal{H}_N} \sum_{x\in S} \sum_{y\in S} \mu_N(\eta)\, g(\eta_x)\, r(x,\, y)\, [\mathbf{f}(\eta) - \mathbf{f}(\sigma^{x,\,y}\eta)]^2 \\
=&\frac{a_N}{2} \sum_{\zeta\in\mathcal{H}_{N-1}} \sum_{x\in S} \sum_{y\in S} \mu_{N-1}(\zeta)\, r(x,\, y)\, [\mathbf{f}(\zeta+\omega^x) - \mathbf{f}(\zeta+\omega^y)]^2 \ .
\end{aligned}
\tag{16.6}
$$

By a similar computation,

$$
\begin{aligned}
&\langle \mathbf{g},\, -\mathscr{L}_N \mathbf{f}\rangle_{\mu_N} \\
=&\sum_{\eta\in\mathcal{H}_N} \sum_{x\in S} \sum_{y\in S} \mu_N(\eta)\, g(\eta_x)\, r(x,\, y)\, [\mathbf{f}(\eta) - \mathbf{f}(\sigma^{x,\,y}\eta)]\, \mathbf{g}(\eta) \\
=&a_N \sum_{\zeta\in\mathcal{H}_{N-1}} \sum_{x\in S} \sum_{y\in S} \mu_{N-1}(\zeta)\, r(x,\, y)\, [\mathbf{f}(\zeta+\omega^x) - \mathbf{f}(\zeta+\omega^y)]\, \mathbf{g}(\zeta+\omega^x) \ .
\end{aligned}
\tag{16.7}
$$

For $\zeta \in \mathcal{H}_{N-1}$, write

$$
\overline{\mathbf{g}}(\zeta) = \frac{1}{\kappa} \sum_{z\in S} \mathbf{g}(\zeta+\omega^z) \ . \tag{16.8}
$$

Since we obviously have

$$
\sum_{x,\,y\in S} r(x,\, y)\, [\mathbf{f}(\zeta+\omega^x) - \mathbf{f}(\zeta+\omega^y)] = 0 \ ,
$$

we can deduce from (16.7) that

$$
\begin{aligned}
&\langle \mathbf{g},\, -\mathscr{L}_N \mathbf{f}\rangle_{\mu_N} \\
=&a_N \sum_{\zeta\in\mathcal{H}_{N-1}} \sum_{x\in S} \sum_{y\in S} \mu_{N-1}(\zeta)\, r(x,\, y)\, [\mathbf{f}(\zeta+\omega^x) - \mathbf{f}(\zeta+\omega^y)]\, [\mathbf{g}(\zeta+\omega^x) - \overline{\mathbf{g}}(\zeta)] \\
\le& \frac{a_N}{2} \sum_{\zeta\in\mathcal{H}_{N-1}} \sum_{x\in S} \sum_{y\in S} \mu_{N-1}(\zeta) r(x,\, y) \Big([\mathbf{f}(\zeta+\omega^x) - \mathbf{f}(\zeta+\omega^y)]^2 + [\mathbf{g}(\zeta+\omega^x) - \overline{\mathbf{g}}(\zeta)]^2 \Big) \\
=&\mathscr{D}_N(\mathbf{f}) + \frac{a_N}{2} \sum_{\zeta\in\mathcal{H}_{N-1}} \sum_{x\in S} \mu_{N-1}(\zeta)\, [\mathbf{g}(\zeta+\omega^x) - \overline{\mathbf{g}}(\zeta)]^2 \ ,
\end{aligned}
\tag{16.9}
$$

where the last line follows from (16.6) and the fact that $\sum_{y\in S} r(x,\, y) = 1$.

Then, by (16.8), we can write

$$\sum_{x\in S}\left[\mathbf{g}(\zeta+\omega^x)-\overline{\mathbf{g}}(\zeta)\right]^2=\frac{1}{\kappa}\sum_{u,\,v\in S}\left[\mathbf{g}(\zeta+\omega^u)-\mathbf{g}(\zeta+\omega^v)\right]^2\ . \tag{16.10}$$

by the Cauchy–Schwarz inequality,

$$\begin{aligned}[\mathbf{g}(\zeta+\omega^u)-\mathbf{g}(\zeta+\omega^v)]^2&=\left[\sum_{x=u}^{v-1}\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)\right]^2\\&\le\left[\sum_{x\in S}\left|\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)\right|\right]^2\\&\le\kappa\sum_{x\in S}[\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)]^2\ .\end{aligned}$$

Inserting this into (16.10) yields that

$$\sum_{x\in S}\left[\mathbf{g}(\zeta+\omega^x)-\overline{\mathbf{g}}(\zeta)\right]^2\le\sum_{x\in S}[\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)]^2\ .$$

Therefore, we have

$$\begin{aligned}&\frac{a_N}{2}\sum_{\zeta\in\mathcal{H}_{N-1}}\sum_{x\in S}\mu_{N-1}(\zeta)\left[\mathbf{g}(\zeta+\omega^x)-\overline{\mathbf{g}}(\zeta)\right]^2\\\le&\frac{a_N}{2}\sum_{\zeta\in\mathcal{H}_{N-1}}\sum_{x\in S}\mu_{N-1}(\zeta)[\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)]^2\\=&\frac{a_N}{2p}\sum_{\zeta\in\mathcal{H}_{N-1}}\sum_{x\in S}\mu_{N-1}(\zeta)r(x,\,x+1)[\mathbf{g}(\zeta+\omega^{x+1})-\mathbf{g}(\zeta+\omega^x)]^2\\\le&\frac{a_N}{2p}\sum_{\zeta\in\mathcal{H}_{N-1}}\sum_{x\in S}\sum_{y\in S}\mu_{N-1}(\zeta)r(x,\,y)[\mathbf{g}(\zeta+\omega^y)-\mathbf{g}(\zeta+\omega^x)]^2=\frac{1}{p}\mathscr{D}_N(\mathbf{g})\ ,\end{aligned}$$

where the last line follows from (16.6).

Finally, inserting this into (16.9), we get

$$\langle\mathbf{g},\,-\mathscr{L}_N\mathbf{f}\rangle_{\mu_N}\le\mathscr{D}_N(\mathbf{f})+\frac{1}{p}\mathscr{D}_N(\mathbf{g})\ .$$

By Remark 1.12, we are done. □

Henceforth, the constant C_0 is always used to denote the constant appearing in Proposition 16.1. The following corollary is now immediate from the above proposition and Propositions 1.9 and 1.13. We recall that $\mathrm{cap}_N^s(\cdot,\,\cdot)$ denotes the capacity with respect to the symmetrized process.

Corollary 16.2. *For any two disjoint, non-empty subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{H}_N$,*

$$\mathrm{cap}_N^s(\mathcal{A},\, \mathcal{B}) \le \mathrm{cap}_N(\mathcal{A},\, \mathcal{B}) \le C_0 \mathrm{cap}_N^s(\mathcal{A},\, \mathcal{B}) \ .$$

16.2 Capacity estimates for collapsed processes

Another essential ingredient of the method of [40] is the sharp estimate of capacity with respect to the collapsed processes. In this subsection, we explain this ingredient. In the remainder of the current section, we will fix $x_0 \in S$.

Definition of collapsed processes

We first define collapsed processes and then explain the notation regarding the collapsed process in terms of the zero-range processes.

Let $\overline{\mathcal{H}}_N = (\mathcal{H}_N \setminus \mathcal{E}_N^{x_0}) \cup \{\mathfrak{e}\}$ be the set obtained from $\mathcal{H}_N$ by collapsing the metastable set $\mathcal{E}_N^{x_0}$ into a single point $\mathfrak{e}$. Denote by $(\overline{\eta}_N(t))_{t\ge 0}$ the collapsed process on $\overline{\mathcal{H}}_N$ which is obtained from $\eta_N(\cdot)$ by collapsing the set $\mathcal{E}_N^{x_0}$ to $\mathfrak{e}$. Let $\overline{\mu}_N(\cdot)$ be a measure on $\overline{\mathcal{H}}_N$ defined by

$$\begin{cases} \overline{\mu}_N(\eta) = \mu_N(\eta) & \text{if } \eta \in \mathcal{H}_N \setminus \mathcal{E}_N^{x_0} \ , \\ \overline{\mu}_N(\mathfrak{e}) = \mu_N(\mathcal{E}_N^{x_0}) \ . \end{cases}$$

Then, by Exercise 3.1, we get the following lemma.

Lemma 16.3. *The Markov chain $\overline{\eta}_N(\cdot)$ is irreducible on $\overline{\mathcal{H}}_N$, and its unique invariant measure is $\overline{\mu}_N(\cdot)$.*

We now redefine the notation regarding the collapsed process in terms of the zero-range process

- We denote by $\overline{\mathscr{L}}_N$ the generator of the collapsed chain $\overline{\eta}_N(\cdot)$, and let $\overline{\mathscr{L}}_N^{\dagger}$ and $\overline{\mathscr{L}}_N^{s}$ denote the adjoint generator and the symmetrized generator of $\overline{\mathscr{L}}_N$, respectively (in the space $L^2(\overline{\mu}_N)$). The continuous-time Markov processes on $\overline{\mathcal{H}}_N$ generated by $\overline{\mathscr{L}}_N^{\dagger}$ and $\overline{\mathscr{L}}_N^{s}$ are denoted by $\overline{\eta}_N^{\dagger}(\cdot)$ and $\overline{\eta}_N^{s}(\cdot)$, respectively.
- Let $\overline{\mathscr{D}}_N(\cdot)$ be the Dirichlet form associated with the generator $\overline{\mathscr{L}}_N$.
- Denote by $\overline{\mathbb{P}}_\eta^N$, $\eta \in \overline{\mathcal{H}}_N$, the law of process $\overline{\eta}_N(\cdot)$ starting from η.
- We denote by $\overline{\mathfrak{F}}_N$ the space of flow associated with the collapsed process $\overline{\eta}_N(\cdot)$. The inner product and flow norm associated with this flow structure will be denoted by $\langle \cdot,\, \cdot \rangle_{\overline{\mathfrak{F}}_N}$ and $\| \cdot \|_{\overline{\mathfrak{F}}_N}$, respectively.

- For each flow $\phi \in \mathfrak{F}_N$, we denote by $\overline{\phi} \in \overline{\mathfrak{F}}_N$ the collapsed flow in the sense of (3.5).
- For each $\mathbf{f} : \mathcal{H}_N \to \mathbb{R}$ which is constant over $\mathcal{E}_N^{x_0}$, we denote by $\overline{\mathbf{f}} : \overline{\mathcal{H}}_N \to \mathbb{R}$ the collapsed function in the sense of (3.8).
- For $\mathbf{f} : \overline{\mathcal{H}}_N \to \mathbb{R}$ we define flows $\overline{\Phi}_{\mathbf{f}}$, $\overline{\Phi}_{\mathbf{f}}^*$ and $\overline{\Psi}_{\mathbf{f}}$ as in (3.15)-(3.17).
- For two disjoint non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\overline{\mathcal{H}}_N$, we denote by $\overline{\mathbf{h}}_{\mathcal{A},\mathcal{B}}$ and $\overline{\mathrm{cap}}_N(\mathcal{A}, \mathcal{B})$ the equilibrium potential and capacity between $\mathcal{A}$ and $\mathcal{B}$ with respect to the collapsed process $\overline{\eta}_N(\cdot)$. In addition, we write $\overline{\mathrm{cap}}_N^s(\mathcal{A}, \mathcal{B})$ for the capacity between $\mathcal{A}$ and $\mathcal{B}$ with respect to process $\overline{\eta}_N^s(\cdot)$.

Remark 16.4. Notice that $\overline{\mathbf{h}}_{\mathcal{A},\mathcal{B}}$ and $\overline{\mathbf{h}_{\mathcal{A},\mathcal{B}}}$ are different objects. Since the equilibrium potential $\mathbf{h}_{\mathcal{A},\mathcal{B}}$ may not be constant on $\mathcal{E}_N^{x_0}$, we may not be able to define the collapsed function $\overline{\mathbf{h}_{\mathcal{A},\mathcal{B}}}$.

By Lemma 3.10 and Proposition 16.1, we get the following proposition where C_0 is the constant appearing in Proposition 16.1

Proposition 16.5. *The collapsed process $\overline{\eta}_N(\cdot)$ satisfies a sector condition with constant C_0. Hence, for any two disjoint non-empty subsets $\mathcal{A}$, $\mathcal{B}$ of $\overline{\mathcal{H}}_N$, it holds that*

$$\overline{\mathrm{cap}}_N^s(\mathcal{A}, \mathcal{B}) \le \overline{\mathrm{cap}}_N(\mathcal{A}, \mathcal{B}) \le C_0\, \overline{\mathrm{cap}}_N^s(\mathcal{A}, \mathcal{B}) \ .$$

Capacity estimates

The following lemma, which is a direct consequence of Lemma 3.9 asserts that we are able to reduce the computation of capacity with respect to the collapsed process to that of the original zero-range process when one of the sets involved is $\{\mathfrak{e}\}$.

Lemma 16.6. *For all non-empty subsets $\mathcal{A}$ of $\mathcal{H}_N \setminus \mathcal{E}_N^{x_0}$,*

$$\overline{\mathrm{cap}}_N(\mathcal{A}, \mathfrak{e}) = \mathrm{cap}_N(\mathcal{A}, \mathcal{E}_N^{x_0}) \ .$$

In view of Exercise 3.7, the following estimate is not a simple consequence of Theorem 15.1 (or Corollary 15.2). We need an independent proof.

Proposition 16.7. *For two disjoint and non-empty subsets A and B of $S \setminus \{x_0\}$ satisfying $A \cup B = S \setminus \{x_0\}$, it holds that*

$$\overline{\mathrm{cap}}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) = [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A, B) \ .$$

The proof of this proposition will be given in next subsection.

16.3 Capacity estimates for collapsed processes

We now prove Proposition 16.7 by several steps. Throughout this subsection, we fix two disjoint and non-empty subsets A, B satisfying the condition of Proposition 16.7. Recall the test functions $\mathbf{V}_{A,\,B}$ and $\mathbf{V}^{\dagger}_{A,\,B}$ from Proposition 15.4 and the test flows $\Phi_{A,\,B}$ and $\Phi^{\dagger}_{A,\,B}$ from Proposition 15.5. Since $\mathbf{V}_{A,\,B}$ and $\mathbf{V}^{\dagger}_{A,\,B}$ are constant on $\mathcal{E}_N^{x_0}$,we can collapse them; let us write $\overline{\mathbf{V}}_{A,\,B} = \overline{\mathbf{V}_{A,\,B}}$ and $\overline{\mathbf{V}}^{\dagger}_{A,\,B} = \overline{\mathbf{V}^{\dagger}_{A,\,B}}$. Note that, by Proposition 15.4 we have that

$$\overline{\mathbf{V}}_{A,\,B}(\mathfrak{e}) = \overline{\mathbf{V}}^{\dagger}_{A,\,B}(\mathfrak{e}) = \mathfrak{h}_{A,\,B}(x_0)\ .$$

Lemma 16.8. *It holds that*

$$\left\|\overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}}\right\|^2_{\overline{\mathfrak{F}}_N} = [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\,B)\ .$$

Proof. By Exercise 3.4 and Lemma 3.6, we obtain

$$\left\|\overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}}\right\|^2_{\overline{\mathfrak{F}}_N} = \left\|\overline{\Psi_{\mathbf{V}_{A,\,B}}}\right\|^2_{\overline{\mathfrak{F}}_N} = \left\|\Psi_{\mathbf{V}_{A,\,B}}\right\|^2 .$$

It is now enough to invoke Corollary 15.7 to complete the proof.

Let $\overline{\Phi}_{A,\,B} = \overline{\Phi_{A,\,B}}$ and $\overline{\Phi}^{\dagger}_{A,\,B} = \overline{\Phi^{\dagger}_{A,\,B}}$ be the collapsed flow of $\Phi_{A,\,B}$ of $\Phi^{\dagger}_{A,\,B}$, respectively. □

Lemma 16.9. *It holds that*

$$\sum_{\eta\in\overline{\mathcal{H}}_N} \overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\eta)(\mathrm{div}\ \overline{\Phi}_{A,\,B})(\eta) = [1 + o_N(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\,B)\ \ \textit{and} \tag{16.11}$$

$$\sum_{\eta\in\overline{\mathcal{H}}_N} \overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\eta)(\mathrm{div}\ \overline{\Phi}^{\dagger}_{A,\,B})(\eta) = [1 + o_N(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\,B)\ . \tag{16.12}$$

Proof. It suffices to prove (16.11) as the proof of (16.12) is essentially the same. In view of Lemma 15.6, it suffices to check

$$\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e})(\mathrm{div}\ \overline{\Phi}_{A,\,B})(\mathfrak{e}) - \sum_{\eta\in\mathcal{E}_N^{x_0}} \mathbf{h}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\eta)(\mathrm{div}\ \Phi_{A,\,B})(\eta) = o_N(1) N^{-(1+\alpha)}\ . \tag{16.13}$$

By (3.6) and (15.3), we have

$$(\mathrm{div}\ \overline{\Phi}_{A,\,B})(\mathfrak{e}) = (\mathrm{div}\ \Phi_{A,\,B})(\mathcal{E}_N^{x_0}) = o_N(1) N^{-(1+\alpha)}\ .$$

Thus, the first term at the left-hand side of (16.13) is $o_N(1)N^{-(1+\alpha)}$. On the other hand, the second term is $o_N(1)N^{-(1+\alpha)}$ by (15.4). Hence, we have (16.13). $\square$

Now we are ready to prove Proposition 16.7 by using generalized Dirichlet and Thomson principles.

Proof of Proposition 16.7. The proof is similar to that of Theorem 15.1. We begin by recalling the functions $\mathbf{f}$, $\mathbf{g}$ and the flows ϕ, ψ from (15.8) and (15.16). Then, by the definition of the collapsing procedure, it is obvious that

$$\overline{\mathbf{f}} \in \mathfrak{C}_{1,\,0}(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \quad \text{and} \quad \overline{\mathbf{g}} \in \mathfrak{C}_{0,\,0}(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \, .$$

Since we can write

$$\overline{\Phi}_{\overline{\mathbf{f}}} - \overline{\phi} = \overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}} - \frac{\overline{\Theta}_N^{\dagger} - \overline{\Theta}_N}{2} \, , \tag{16.14}$$

where $\overline{\Theta}_N$ and $\overline{\Theta}_N^{\dagger}$ are the collapsed flows of Θ_N and $\Theta_N^{\dagger}$ defined in (15.10), respectively. By part (1) of Proposition 15.5 and Lemma 3.3, we have

$$\big\|\overline{\Theta}_N\big\|_{\overline{\mathfrak{F}}_N}^2 = (o_N(1) + o_\epsilon(1))\, N^{-(1+\alpha)} \;\; \text{and} \;\; \big\|\overline{\Theta}_N^{\dagger}\big\|_{\overline{\mathfrak{F}}_N}^2 = (o_N(1) + o_\epsilon(1))\, N^{-(1+\alpha)} \, . \tag{16.15}$$

Thus, by Theorem 2.2-(1), Lemma 16.8, and Lemma 16.9, we get the following upper bound:

$$\overline{\mathrm{cap}}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \le [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B) \, . \tag{16.16}$$

For the opposite inequality, we can repeat the same arguments with test function $\overline{\mathbf{g}}$ and test flow $\overline{\psi}$ to deduce

$$\overline{\mathrm{cap}}_N(\mathcal{E}_N(A),\, \mathcal{E}_N(B)) \ge (1 + o_N(1) + o_\epsilon(1))\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(A,\, B) \, . \tag{16.17}$$

By (16.16) and (16.17), the proof is completed. $\square$

Exercise 16.10. Prove (16.17) by using the generalized Thomson principle.

Exercise 16.11. In fact, the condition $A \cup B = S \setminus \{x_0\}$ in Proposition 16.7 is redundant. We imposed this condition only because we do not need a general result without this restriction. Prove the general result without this restriction.

16.4 Estimate of mean jump rate

Now we are ready to estimate the mean jump rate. In view of (16.3), to obtain the sharp asymptotics of the mean jump rate $r_N(x_0,\, y)$ for $x_0,\, y \in S$, the crucial object to be estimated is the probability $\overline{\mathbb{P}}_{\mathfrak{e}}^{N}[\tau_{\mathcal{E}_N^y} < \tau_{\breve{\mathcal{E}}_N^{x_0,\,y}}]$. This estimate follows from the following proposition.

Proposition 16.12. *For two disjoint and non-empty subsets A and B of $S \setminus \{x_0\}$ satisfying $A \cup B = S \setminus \{x_0\}$, we have that*

$$\lim_{N\to\infty} \overline{\mathbb{P}}_{\mathfrak{e}}^{N} \left[\tau_{\mathcal{E}_N(A)} < \tau_{\mathcal{E}_N(B)}\right] = \mathfrak{h}_{A,\,B}(x_0)\;.$$

Proof. The proof relies on Propositions 16.5, 16.7 and Lemma 16.8. Recall the equilibrium potential $\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}$ between $\mathcal{E}_N(A)$ and $\mathcal{E}_N(B)$, with respect to the collapsed chain $\overline{\eta}_N(\cdot)$. Then, by Proposition 16.7,

$$\left\|\overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\|^2_{\overline{\mathfrak{F}}_N} = \overline{\mathrm{cap}}_N(\mathcal{E}_N(A),\,\mathcal{E}_N(B)) = [1+o_N(1)+o_\epsilon(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\;. \tag{16.18}$$

By Lemma 16.8,

$$\left\|\,\overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}}\right\|^2_{\overline{\mathfrak{F}}_N} = [1+o_N(1)+o_\epsilon(1)]\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B)\;. \tag{16.19}$$

By (16.14), (16.15), (16.18), and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left\langle \overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}},\, \overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\rangle_{\overline{\mathfrak{F}}_N}\\ &= \left\langle \overline{\Phi}_{\overline{\mathbf{f}}} - \overline{\phi},\, \overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\rangle_{\overline{\mathfrak{F}}_N} + (o_N(1)+o_\epsilon(1))\,N^{-(1+\alpha)}\;,\end{aligned} \tag{16.20}$$

where $\overline{\mathbf{f}}$ and $\overline{\phi}$ are the objects defined in the proof of Proposition 16.7. By the same computation as in (2.7), we can write

$$\begin{aligned}&\left\langle \overline{\Phi}_{\overline{\mathbf{f}}} - \overline{\phi},\, \overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\rangle_{\overline{\mathfrak{F}}_N}\\ &= \overline{\mathrm{cap}}_N(\mathcal{E}_N(A),\,\mathcal{E}_N(B)) - \sum_{\eta\in\overline{\mathcal{H}}_N\setminus\mathcal{E}_N(A\cup B)} \overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\eta)\,(\mathrm{div}\,\overline{\phi})(\eta)\end{aligned} \tag{16.21}$$

Thus, by combining (16.20), (16.21) and Proposition 16.7, we get ,

$$\left\langle \overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}},\, \overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\rangle_{\overline{\mathfrak{F}}_N} = (1+o_N(1)+o_\epsilon(1))\,N^{-(1+\alpha)}\,\mathrm{cap}_Y(A,\,B) \tag{16.22}$$

Define $\mathbf{u} = \overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)} - \overline{\mathbf{V}}_{A,\,B}$. Then, by (16.18), (16.19), and (16.22) we get

$$\begin{aligned}\left\|\overline{\Psi}_{\mathbf{u}}\right\|_{\overline{\mathfrak{F}}_N}^2 &= \left\|\,\overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\|_{\overline{\mathfrak{F}}_N}^2 + \left\|\,\overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}}\right\|_{\overline{\mathfrak{F}}_N}^2 - 2\left\langle \overline{\Psi}_{\overline{\mathbf{V}}_{A,\,B}},\, \overline{\Psi}_{\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}}\right\rangle_{\overline{\mathfrak{F}}_N} \\ &= (o_N(1) + o_\epsilon(1))\, N^{-(1+\alpha)}\ .\end{aligned} \tag{16.23}$$

As $\mathbf{u}(\mathfrak{e}) = \overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e}) - \mathfrak{h}_{A,\,B}(x_0)$ and $\mathbf{u}(\eta) = 0$ for all $\eta \in \mathcal{E}_N(A \cup B)$, we can write

$$\mathbf{u} = \left(\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e}) - \mathfrak{h}_{A,\,B}(x)\right)\mathbf{u}_0$$

for some $\mathbf{u}_0 \in \mathfrak{C}_{1,\,0}(\{\mathfrak{o}\},\, \mathcal{E}_N(A \cup B))$. With this notation, we can write

$$\left\|\overline{\Psi}_{\mathbf{u}}\right\|_{\overline{\mathfrak{F}}_N}^2 = \overline{\mathscr{D}}_N(\mathbf{u}) = \left(\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e}) - \mathfrak{h}_{A,\,B}(x)\right)^2 \overline{\mathscr{D}}_N(\mathbf{u}_0)\ . \tag{16.24}$$

By the Dirichlet principle for reversible dynamics (cf. Theorem 1.3) and the sector condition for the collapsed process (cf. Proposition 16.5), we have that

$$\overline{\mathscr{D}}_N(\mathbf{u}_0) \ge \overline{\mathrm{cap}}_N^s(\mathfrak{e},\, \mathcal{E}_N(A \cup B)) \ge C_0^{-1}\, \overline{\mathrm{cap}}_N(\mathfrak{e},\, \mathcal{E}_N(A \cup B))\ . \tag{16.25}$$

By Lemma 16.6 and Theorem 15.1,

$$\begin{aligned}\overline{\mathrm{cap}}_N(\mathfrak{e},\, \mathcal{E}_N(A \cup B)) &= \mathrm{cap}_N(\mathcal{E}_N^x,\, \mathcal{E}_N(A \cup B)) \\ &= [1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}\, \mathrm{cap}_Y(x,\, A \cup B)\ .\end{aligned} \tag{16.26}$$

By (16.25) and (16.26), we can conclude that

$$\overline{\mathscr{D}}_N(\mathbf{u}_0) \ge C\,[1 + o_N(1) + o_\epsilon(1)]\, N^{-(1+\alpha)}$$

for some constant $C > 0$. Inserting this and (16.23) into (16.24), we get

$$\left[\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e}) - \mathfrak{h}_{A,\,B}(x)\right]^2 \le o_N(1) + o_\epsilon(1)\ .$$

By taking $\limsup_{N\to\infty}$ and then $\limsup_{\epsilon\to 0}$, we get

$$\limsup_{N\to\infty} \left|\overline{\mathbf{h}}_{\mathcal{E}_N(A),\,\mathcal{E}_N(B)}(\mathfrak{e}) - \mathfrak{h}_{A,\,B}(x)\right| = 0$$

and we are done. $\square$

Now we are ready to verify condition (H0) for the zero-range process.

Proposition 16.13. *The condition (H0) holds for the zero-range processes. In other* *words, for all $x, y \in S$,*

$$\lim_{N\to\infty} N^{1+\alpha}\, r_N(x,\, y) = a(x,\, y)\ .$$

Proof. By (16.2), Theorem 13.1, and Corollary 15.2, we get

$$\lambda_N(x) = \frac{\mathrm{cap}_N(\mathcal{E}_N^x,\, \breve{\mathcal{E}}_N^x)}{\mu(\mathcal{E}_N^x)} = (1+o_N(1))\, N^{-(1+\alpha)}\, \frac{\kappa}{\Gamma_\alpha I_\alpha} \sum_{y\in S\setminus\{x\}} \mathrm{cap}_X(x,\, y)\ . \tag{16.27}$$

Recall from (15.2) the definition of $\mathfrak{h}_{y,\, S\setminus\{x,y\}}$. Write

$$\tau = \inf\{t \ge 0 : Y(t) \ne Y(0)\}\ .$$

Then, one can observe that

$$\mathfrak{h}_{y,\, S\setminus\{x,y\}}(x) = \mathbf{Q}_x\left(Y(\tau) = y\right) = \frac{\mathrm{cap}_X(x,\, y)}{\sum_{y\in S\setminus\{x\}} \mathrm{cap}_X(x,\, y)}\ .$$

Thus, by (16.3) and Proposition 16.12, we get

$$\frac{r_N(x,\, y)}{\lambda_N(x)} = (1+o_N(1))\, \mathfrak{h}_{y,\, S\setminus\{x,y\}}(x) = (1+o_N(1))\, \frac{\mathrm{cap}_X(x,\, y)}{\sum_{y\in S\setminus\{x\}} \mathrm{cap}_X(x,\, y)}\ . \tag{16.28}$$

We can complete the proof by multiplying (16.27) and (16.28). □

17 Conditions (H1) and (H3)

Since we have verified conditions (H0) and (H2), it now remains to verify conditions (H1) and (H3). Verification of these conditions also use the capacity estimate obtained in Theorem 15.1 and the sector condition obtained in Proposition 16.5. We again assume the results obtained in Section 15.

We first prove the following lemma.

Lemma 17.1. *For any $x \in S$, there exists a constant C such that*

$$\inf_{\eta,\, \zeta \in \mathcal{E}_N^x} \mathrm{cap}_N(\eta,\, \zeta) \ge \frac{C}{\ell_N^{\alpha(\kappa-1)+1}}$$

Proof. We fix $x \in S$ and $\eta,\, \zeta \in \mathcal{E}_N^x$. We first find a lower bound for $\mathrm{cap}_N^s(\eta,\, \zeta)$. For $\xi,\, \xi' \in \mathcal{H}_N$, we denote by $R_N(\xi,\, \xi')$ the jump rate of the symmetrized zero-range process from ξ to ξ':

$$R_N(\xi,\, \xi') = \sum_{x \in S} \sum_{y \in S} g(\xi_x) r^s(x,\, y) \mathbf{1}\{\xi' = \sigma^{x,\, y}\xi\} \ ,$$

where $r^s(x,\, y) = \frac{1}{2}\mathbf{1}\{|x-y| = 1\}$. Take a path $(\omega_t)_{t=0}^T$ in $\mathcal{E}_N^x$ connecting η and ζ in the sense that $\omega_t \in \mathcal{E}_N^x$ for all $t \in [\![0,\, T]\!]$ and moreover satisfies

$$\omega_0 = \eta \ , \quad \omega_T = \zeta \quad \text{and} \quad R_N(\omega_t,\, \omega_{t+1}) > 0 \text{ for all } t \in [\![0,\, T-1]\!] \ .$$

The existence of such a path with $T \le C\ell_N$ where C is a constant that only depends on κ is obvious. Define a flow $\phi \in \mathfrak{F}_N$ by

$$\phi(\xi,\, \xi') = \begin{cases} 1 & \text{if } (\xi,\, \xi') = (\omega_t,\, \omega_{t+1}) \text{ for some } t \in [\![0,\, T-1]\!] \ , \\ -1 & \text{if } (\xi,\, \xi') = (\omega_{t+1},\, \omega_t) \text{ for some } t \in [\![0,\, T-1]\!] \ , \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\|\phi\|_{\mathfrak{F}_N}^2 = \sum_{t=0}^{T-1} \frac{1}{\mu_N(\omega_t) R_N(\omega_t,\, \omega_{t+1})} \ . \tag{17.1}$$

Since $g(k) \ge 1$ for all $k \ge 1$, if $\omega_{t+1} = \sigma^{x,\, y}\omega_t$ for some $x,\, y \in S$ with $|y - x| = 1$,

$$\mu_N(\omega_t) R_N(\omega_t,\, \omega_{t+1}) = \frac{N^\alpha}{Z_N} \frac{1}{a(\omega_t)} \times \frac{1}{2} g((\omega_t)_x) \ge C \frac{N^\alpha}{N^\alpha \ell_N^{\alpha(\kappa-1)}} = C \frac{1}{\ell_N^{\alpha(\kappa-1)}} \ , \tag{17.2}$$

where we use a trivial bound

$$a(\xi) = a(\xi_x) \prod_{y \in S \setminus \{x\}} a(\xi_y) \le N^\alpha \ell_N^{\alpha(\kappa-1)} \quad \text{for all } \xi \in \mathcal{E}_N^x$$

and Proposition 12.4 at the inequality of (17.2). Inserting (17.2) and the bound $T \le C\ell_N$ to (17.1), we get

$$\|\phi\|_{\mathfrak{F}_N}^2 \le C\ell_N \times \ell_N^{\alpha(\kappa-1)} = C\ell_N^{\alpha(\kappa-1)+1}$$

Since ϕ is the unit flow from $\{\eta\}$ to $\{\zeta\}$, by the Thomson principle for the reversible Markov process (cf. Theorem 1.4),

$$\mathrm{cap}_N^s(\eta,\, \zeta) \ge \frac{1}{\|\phi\|_{\mathfrak{F}_N}^2} \ge \frac{C}{\ell_N^{\alpha(\kappa-1)+1}} \, .$$

Now the proof of lemma is completed by Corollary 16.2. □

Exercise 17.2. In the previous proof, prove the existence of a path $(\omega_t)_{t=0}^T$ in $\mathcal{E}_N^x$ connecting η and ζ with $T \le C\ell_N$ for some constant C depending only on κ.

Now we verify condition (H1).

Proposition 17.3. *The condition (H1) holds for the zero-range processes.*

Proof. Fix $x \in S$. For $\eta,\, \zeta \in \mathcal{E}_N^x$, by Theorem 15.1 and Lemma 17.1, there exists $C > 0$ such that

$$\frac{\mathrm{cap}_N(\mathcal{E}_N^x,\, \breve{\mathcal{E}}_N^x)}{\mathrm{cap}_N(\eta,\, \zeta)} \le C\frac{\ell_N^{\alpha(\kappa-1)+1}}{N^{1+\alpha}} = o_N(1) \tag{17.3}$$

where the last equality follows from the condition (13.1) on ℓ_N. □

At this moment, we shall check that the condition (H3) is in force for the zero-range processes.

Proposition 17.4. *The condition (H3) holds for the zero-range processes.*

Proof. Fix $x \in S$. Recall that $\xi_N^x \in \mathcal{E}_N^x$ represent a configuration such that all the particles are located at site x. By [34, Lemma 3.4], it suffices to verify that

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbb{P}_\eta^N[\tau_{\xi_N^x} > N^{1+\alpha}\delta] = 0 \ \text{ for all } \delta > 0 \ , \ \text{ and} \tag{17.4}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{\delta < t < 3\delta} \mathbb{P}_{\xi_N^x}^N[\eta_N(N^{1+\alpha}t) \in \Delta_N] = 0 \, . \tag{17.5}$$

For (17.4), by the Markov inequality and (0.32), we have

$$\mathbb{P}^N_\eta[\tau_{\xi^x_N} > N^{1+\alpha}\delta] \le \frac{1}{N^{1+\alpha}\delta}\mathbb{E}^N_\eta\left[\tau_{\xi^x_N}\right] \le \frac{1}{N^{1+\alpha}\,\delta}\frac{1}{\mathrm{cap}_N(\eta,\,\xi^x_N)}\;, \tag{17.6}$$

where at the second inequality we use the trivial bound $h_{\eta,\,\xi^x_N} \le 1$. By Lemma 17.1,

$$\mathbb{P}^N_\eta[\tau_{\xi^x_N} > N^{1+\alpha}\delta] \le \frac{C}{\delta}\frac{\ell_N^{\alpha(\kappa-1)+1}}{N^{1+\alpha}}\;.$$

The proof of (17.4) now follows from the condition (13.1) on ℓ_N.

For (17.5), note first from the definition of μ_N that we have $\mu_N(\xi^x_N) = Z_N^{-1}$. Hence, for $t > 0$, since μ_N is the invariant measure,

$$\mathbb{P}^N_{\xi^x_N}\left[\eta_N(N^{1+\alpha}\,t)\in\Delta_N\right] \le \frac{\mathbb{P}^N_{\mu_N}\left[\eta_N(N^{1+\alpha}\,t)\in\Delta_N\right]}{\mu_N(\xi^x_N)} = \frac{\mu_N(\Delta_N)}{\mu_N(\xi^x_N)} = Z_N\,\mu_N(\Delta_N)\;.$$

Hence, (17.5) follows directly from Proposition 12.4 and Theorem 13.1. □

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