수 학 강 의 록 제 53 권

OVERDETERMINED PDE SYSTEMS OF GENERIC TYPE

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Preface. Given an overdetermined PDE system we are concerned , first of all, with the local existence of solutions without any side conditions. This is the solovability question. As a necessary condition for the existence of solutions all the compatibility conditions must be fulfilled. The purpose of these lectures is to present a theory together with an algorithm of finding *all* the compatibility conditions and constructing an equivalent *involutive* system in order to obtain the general solutions for the overdetermined PDE systems of *generic* type (Definition 4.1). We make use of various generalized versions of the Frobenius theorem on the involutivity. The simplest versions of the generalized Frobenius theorem seem to be known to R. Bryant [Bry], S. Wang [Wang], and perhaps to other experts in the area of exterior differential systems. The generalized Frobenius theorems presented in §6 are obtained by the author independently and seem to comprise most of those known previously.

This is the note of some ten lectures that the author gave at Seoul National University in the fall, 2007. He wants to thank the audience for their interest and patience. The author also expresses deep gratitude to Chung-Ki Cho, Jae-Nyun Yoo, Sungyeon Kim, Jongwon Oh and Jaesung Cho, who shared their ideas in many valuable discussions, and to Masatake Kuranishi who taught me the essential ideas of prolongation, and to Robert Bryant who kindly showed the coordinate free

computation §12. The author thank also Robert Foote, Gerd Schmalz and Dmitri Zaitsev for their interest and collaboration. Finally, the author thanks Daniel M. Burns who introduced the equivalence problem, prolongation and the complete system to the author. Around 1980 he observed and suggested to the author that the complete system is of central importance in understanding overdetermined systems, in particular, proving the rigidity and the regularity of mappings that are defined by overdetermined PDE systems.

The author wishes this little monograph to be useful as a field manual in exploiting the wide open unexplored world of overdetermined systems.

Chong-Kyu Han

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§§7,8,11,12 are examples or applications. The readers may skip these sections. §2 is a brief review of the notions of prolongation and involutivity in the theory of exterior differential system.

§1. PDE systems and EDS.

Any partial differential equation (PDE) or system of PDE's can be expressed as an exterior differential system (EDS) in a jet space. An EDS is a system of equations on a manifold M defined by equating to zero a number of exterior differential forms $\theta = (\theta^1, \theta^2, \cdots)$. When all the forms are 1-forms, it is called a Pfaffian system. An integral manifold of an EDS θ is a submanifold $i : N \to M$ such that $i^*\theta = 0$. Given a PDE system, we shall discuss in this section how to set it up as an EDS, in particular, as a Pfaffian system with independence condition. Let $u = (u^1, \ldots, u^q)$ be a system of functions of p independent variables $x = (x^1, \ldots, x^p)$. Let $X \subset \mathbb{R}^p$ and $U \subset \mathbb{R}^q$ be open. We define the m-th jet space by

$$J^m(X, U) := \{(x, u^{(m)}), x \in X, u \in U\},$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. For a smooth function $u = f(x)$ the p-dimensional submanifold of $J^m(X, U)$

$$x \mapsto (x, f^{(m)}), \quad x \in X$$

is called the m-th jet graph of $f(x)$ and denoted by $(j^m f)(x)$. For a finite sequence $J = \{j_1, j_2, \ldots\}$ of integers $\{1, \ldots, p\}$, let

$$u_J = (\frac{\partial}{\partial x^{j_1}} \frac{\partial}{\partial x^{j_2}} \cdots)u.$$

Since

$$du = \sum_{j=1}^{p} u_j dx^j,$$

the m-th jet graph is an integral manifold of the the following

EDS in $J^m(X,U)$: For each $\alpha = 1, \dots, q$

$$
\begin{aligned}
\theta^\alpha &:= du^\alpha - \sum_{j=1}^{p} u_j^\alpha dx^j \\
\theta_j^\alpha &:= du_j^\alpha - \sum_{k=1}^{p} u_k^\alpha dx^k \\
&\vdots \\
\theta_J^\alpha &:= \sum_{k=1}^{p} u_{Jk}^\alpha dx^k, \quad |J| = m-1,
\end{aligned}
\tag{1.1}
$$

where $|J|$ is the length of the sequence J. (1,1) is called the contact system of $J^m(X,U)$. Then $j^m f(x)$ is an integral manifold of (1.1), on which

$$dx^1 \wedge \cdots \wedge dx^p \neq 0 \tag{1.2}$$

(1.2) is called an independence condition. As examples, let us consider the cases of first order and the second order PDE's:

1.1 First order PDE for one unknown function in two independent variables.

Consider a PDE for unknown function $u(x,y)$

$$\triangle(x, y, u, u_x, u_y) = 0 \tag{1.3}$$

with $\frac{\partial \triangle}{\partial u_y} \neq 0$. Let $\mathcal{S}_\triangle$ be the submanifold of $J^1(X,U) = \{(x, y, u, u_x, u_y)\} = \mathbb{R}^5$ defined by (1.3). The contact form of $J^1(X,U)$ is

$$\theta := du - u_x dx - u_y dy. \tag{1.4}$$

$(J^1(X,U),\theta)$ is indeed a "contact manifold" in the sense that

$$(d\theta)^2 \wedge \theta = -2dx \wedge dy \wedge du \wedge du_x \wedge du_y \neq 0.$$

Then the Pfaffian system

$$(\mathcal{S}_\triangle, \theta), \tag{1.5}$$

defines a distribution $\mathcal{D}$ of 3-dimensional planes consisting of tangent vectors to $\mathcal{S}_\triangle$ that are annihilated by θ. Then for a C^∞function $u = f(x,y)$ the following are equivalent:

a) $u = f(x,y)$ is a solution of (1.3).
b) The first jet graph $(j^1 f)(x,y)$ is contained in $\mathcal{S}_\triangle$.
c) The first jet graph $(j^1 f)(x,y)$ is an integral manifold of the EDS in $J^1(X,U)$ given by a 0-form (1,3) and a 1-form (1.4) with the independence condition

$$dx \wedge dy \neq 0. \tag{1.6}$$

d) The first jet graph $(j^1 f)(x,y)$ is an integral manifold of the Pfaffian system (1.5) with the independence condition (1.6).

Our approach is based on d): the observation that a solution of (1.3) corresponds in a one-to-one manner to an integral manifold of (1.5) satisfying (1.6).

1.2. *Cauchy problems for the first order PDE's*

Consider the following Cauchy problem:

$$\begin{aligned} &\triangle(x,y,u,u_x,u_y) = 0 \\ &u(x,0) = \phi(x), \quad a < x < b, \end{aligned} \tag{1.7}$$

where we assume $\frac{\partial \Delta}{\partial u_y} \neq 0$. From the Cauchy data it follows that $u_x(x,0) = \phi'(x)$. Substituting this in (1.7) we have $\triangle(x, 0, \phi(x), \phi'(x), u_y) = 0$. This determines a curve

$$\gamma(x) := (x, 0, \phi(x), \phi'(x), u_y(x))$$

such that $\gamma^*\theta = 0$. A solution is a mapping

$$\Gamma : (a,b) \times (-\epsilon, \epsilon) \to \mathcal{S}_\Delta$$

such that $\Gamma^*\theta = 0$ and $\Gamma(x,0) = \gamma(x)$.

1.3. *Boundary value problems for the first order PDE's*

Let D be a domain in $\mathbb{R}^2$. Consider the following boundary value problem: Find $u \in C^\infty(\bar{D})$ satisfying

$$\begin{aligned} &\triangle(x,y,u,u_x,u_y) = 0 \quad \text{in } D \\ &u(x,y) = \psi(x,y), \qquad \text{for } (x,y) \in bD \end{aligned} \tag{1.8}$$

The boundary data cannot be given arbitrarily. There can be local or global constraints on the boundary data, which are the compatibility conditions to the differential equations in the interior. In order to see the local conditions, suppose that bD is locally given by curve $x(t), y(t)$. Let $\psi(t) := \psi(x(t), y(t))$. Then the values of u_x and u_y at the boundary points satisfy

$$\begin{aligned} \Delta(x(t), y(t), \psi(y), u_x, u_y) &= 0 \\ \frac{du}{dt} = u_x x'(t) + u_y y'(t) &= \psi'(t) \end{aligned}$$

1.4 Problem. *Find the global conditions that the boundary data must satisfy.*

1.5 Second order PDE for $u(x,y)$.

Given a PDE of second order

$$\triangle(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0 \tag{1.9}$$

We assume $\frac{\partial\triangle}{\partial u_{yy}} \neq 0$. On the second jet space

$$J^2(X,U) = \{(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})\} = \mathbb{R}^8$$

we consider the contact system $\theta = (\theta^0, \theta^1, \theta^2)$, where

$$\begin{aligned} \theta^0 &:= du - u_x dx - u_y dy \\ \theta^1 &:= du_x - u_{xx} dx - u_{xy} dy \\ \theta^2 &:= du_y - u_{xy} dx - u_{yy} dy \end{aligned} \tag{1.10}$$

Let $\mathcal{S}_\triangle \subset J^2(X,U)$ be the submanifold defined by (1.9). Then (1.10) defines a distribution $\mathcal{D}$ of 4-dimensional tangent planes of the 7-dimensional manifold $\mathcal{S}_\triangle$. Then for the same reason as in the cases of the first order, we see that solutions of (1.9) are in one-to-one correspondence to the integral manifolds of the Pfaffian system $(\mathcal{S}_\triangle, \theta)$ given by (1.10) with the independence condition (1.6).

§2. Prolongation of Pfaffian systems.

In this section, we briefly review basic notions of the prolongation and the involutivity. We refer the readers to [BCGGG], [GJ] and [Kura]. Let M be a smooth (C^∞) manifold of dimension n and let $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p$, $s + p \leq n$, be a set of linearly independent smooth 1-forms on M . We are concerned with the problem of finding a smooth submanifold $N \subset M$ of dimension p which satisfies

(2.1)

$\theta^\alpha|_N = 0, \quad \alpha = 1, \cdots, s$ (Pfaffian system)

$\Omega|_N \neq 0, \quad$ where $\Omega = \omega^1 \wedge \ldots \wedge \omega^p$ (independence condition).

Such a submanifold N is called an integral manifold of dimension p satisfying the independence condition, or simply a 'solution' of (2.1). To find a solution of (2.1) we consider subbundles $I \subset J \subset T^*M$. Here $I = \langle \theta^1, \cdots, \theta^s \rangle$ and $J = \langle \theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p \rangle$, where $\langle \cdots \rangle$ denotes the linear span of what are inside. Let $\mathcal{D}$ be the $(n-s)$-dimensional plane field annihilated by $\theta^1, \cdots, \theta^s$. For $k = 1, \cdots, p$, an integral manifold of (2.1) of dimension k is a submanifold of M of dimension k whose tangent spaces belong to $\mathcal{D}$. An integral manifold N of dimension p such that $\Omega|_N \neq 0$ is a solution of (2.1). If $N$is an integral manifold of (2.1) then $\theta^\alpha|_N = 0$, and therefore, $d\theta^\alpha|_N = 0$, for each $\alpha = 1, \cdots, s$. A k-dimensional integral element is a k-dimensional subspace (x, E) of T_xM, for some $x \in M$, on which $\theta^\alpha = 0$ and $d\theta^\alpha = 0$, for all $\alpha = 1, \cdots, s$. By $V(I, J)$ we denote the set of all p-dimensional integral elements (x, E) satisfying $\Omega|_E \neq 0$. Basic idea of the theory is that we can find a solution by constructing k-dimensional integral manifold N^k with N^{k-1} as initial data, inductively for $k = 1, \cdots, p$, so that we have a nested sequence of integral manifolds

$$N^0 \subset N^1 \subset \cdots \subset N^p.$$

Let

$$\{x\} = E^0 \subset E^1 \subset \cdots \subset E^p = E$$

be the corresponding flag of integral elements. The notion of involutivity is the existence of such a flag for each element of $V(I, J)$ so that the Cauchy problem is well-posed in each step and the solutions to the $(k+1)^{st}$ Cauchy problem remain solutions to the family of k^{th} Cauchy problem with data smoothly changing in $(k+1)^{st}$ direction . If the system is analytic (C^ω) one can construct such a nested sequence of integral manifolds by using the Cauchy-Kowalewski theorem. This is the idea of the Cartan-Kähler theorem which asserts that an involutive analytic Pfaffian system has analytic solutions. If (I, J) is not involutive we construct an involutive system which is equivalent to the original system by repeating the process of the following two steps:

Step 1. *Reduce (2.1) to a submanifold $M' \subset M$ so that $V'(I, J) \to M'$ is surjective:*

Let M_1 be the image of $V(I, J) \to M$. If $M = M_1$ then we do nothing. If $M_1 \neq M$ then we note that any integral manifold of (I, J) must lie in M_1, and so we set

$$V_1(I, J) = \{(x, E) \in V(I, J) : E \subset T_x M_1\}.$$

Now consider the projection

$$V_1(I, J) \to M_1$$

with image M_2. If $M_2 = M_1$ we stop; otherwise we continue as before. Eventually we arrive either at the empty set, in which case (I, J) has no integral manifolds, or else at M' with $V'(I, J) \to M'$ being surjective and with all $(x, E) \in V'(I, J)$ satisfying $E \subset T_x M'$.

Step 2. *Assuming $V(I, J) \to M$ is surjective we do prolongation.*

To recall the definitions, let $G_p(M)$ be the Grassmann bundle of p-planes in TM. Let $\pi^1, \cdots, \pi^r$ be a set of 1-forms so that

$$\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p, \pi^1, \cdots, \pi^r$$

form a basis of T^*M. Let $(x,E) \in V(I,J)$. Since $\Omega|_E \neq 0$, on a neighborhood of $(x,E) \in G_p(M)$ we have $\theta^\alpha = m^\alpha_\rho \omega^\rho$, $\pi^\epsilon = \ell^\epsilon_\rho \omega^\rho$, (summation convention for $\rho = 1,\cdots,p$) and $\Omega \neq 0$. Thus $\{m^\alpha_\rho, \ell^\epsilon_\rho\}$ are local fibre coordinates in $G_p(M)$. The canonical system on $G_p(M)$ is the set of the tautological 1-forms

$$\begin{aligned} &\theta^\alpha - m^\alpha_\rho \omega^\rho, \quad \alpha = 1,\cdots,s \\ &\pi^\epsilon - \ell^\epsilon_\rho \omega^\rho, \quad \epsilon = 1,\cdots,r, \end{aligned} \tag{2.2}$$

where the summation convention is used for $\rho = 1,\cdots,p$. The first prolongation $(I^{(1)},J^{(1)})$ is the restriction to $M^{(1)} := V(I,J) \subset G_p(M)$ of the canonical system. Since $m^\alpha_\rho = 0$ on $V(I,J)$ the problem of finding a solution of (2.1) is equivalent to finding a submanifold $N^{(1)} \subset M^{(1)}$ of dimension p satisfying

$$\begin{aligned} &\theta^\alpha|_{N^{(1)}} = 0, \quad (\pi^\epsilon - \ell^\epsilon_\rho \omega^\rho)|_{N^{(1)}} = 0 \\ &\Omega|_{N^{(1)}} \neq 0. \end{aligned} \tag{2.3}$$

Integral manifolds of (I,J) and those of $(I^{(1)},J^{(1)})$ are in a one-to-one correspondence. The k-th prolongation $(I^{(k)},J^{(k)})$ on $M^{(k)} = V(I^{(k-1)},J^{(k-1)})$ is defined inductively to be the first prolongation of $(I^{(k-1)},J^{(k-1)})$ on $M^{(k-1)}$. We have a version of the Cartan-Kuranishi theorem [Kura] :

Theorem 2.1. *Let $(I^{(k)},J^{(k)})$, $k = 1,2,\cdots$, be the sequence of prolongations of (1.1). Suppose that , for each k, $M^{(k)}$ is a smooth submanifold of $G_p(M^{(k-1)})$ and that the projection $M^{(k)} \to M^{(k-1)}$ is a surjective submersion. Then there is k_0 such that prolongations $(I^{(k)},J^{(k)})$ are involutive for $k \geq k_0$.*

§3 Pfaffian systems of Frobenius type.

Let $\Lambda = \bigoplus_{j=0}^{n} \Lambda^j(M)$ be the exterior algebra of differential forms of M. A subalgebra $\mathcal{I} \subset \Lambda$ is called an algebraic ideal if $\mathcal{I} \wedge \Lambda \subset \mathcal{I}$. Consider the algebraic ideals $\mathcal{I}$ and $\mathcal{J}$ generated by $\{\theta^1, \cdots, \theta^s\}$ and $\{\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p\}$, respectively. (2.1) is quasi-linear if $d\mathcal{I} \subset \mathcal{J}$, namely,

$$d\theta^\alpha = \sum_{\beta=1}^{s} \phi_\beta^\alpha \wedge \theta^\beta + \sum_{\rho=1}^{p} \psi_\rho^\alpha \wedge \omega^\rho,$$

for some 1-forms $\phi_\beta^\alpha, \psi_\rho^\alpha$, for each $\alpha = 1, \cdots, s$. Existence of solutions has been studied mainly for the quasi-linear systems. (2.1) is said to be of Frobenius type if $s + p = n$, that is, if $\{\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p\}$ is a coframe of M. It is easy to see that Frobenius types are quasi-linear. In this section we focus our interest to the systems of Frobenius type. In this case no further equations are obtained by prolongation and the existence of general integral manifolds is determined only by Step 1 of §2. The notion of involutivity is very subtle as we see in [BCGGG]. However, for $V(I, J)$ of Frobenius type the following are equivalent (see [GJ] Chapter 3):

i) $V(I, J) \to M$ is surjective.
ii) (I, J) is integrable in the sense of the Frobenius theorem.
iii) (I, J) is invoultive.

It is easy to prove the following

Lemma 3.1. *Let M be a smooth manifold of dimension n. Let $\theta := (\theta^1, \cdots, \theta^s)$ be a set of independent 1-forms on M and $\mathcal{D} :=< \theta >^\perp$ be the $(n-s)$-dimensional plane field annihilated by θ. Suppose that N is a submanifold of M of dimension $n - r$, for some $r \leq s$, defined by $T_1 = \cdots = T_r = 0$, where T_i are smooth real-valued functions of M such that $dT_1 \wedge \cdots \wedge dT_r \neq 0$. Then the following are equivalent :*

(i) $\mathcal{D}$ *is tangent to* N.
(ii) $dT_j \equiv 0$, mod θ, *on* N, *for each* $j = 1, \ldots, r$.

In (ii) mod θ means that modulo the algebraic ideal $\mathcal{I}$. Thus (ii) is equivalent to saying that for each $j = 1, \cdots, r$ we have $dT_j \wedge \theta^1 \wedge \cdots \wedge \theta^s = 0$, on N. Our basic observation is the following algorithm for Step 1: For each $\alpha = 1, \cdots, s$, set

$$d\theta^\alpha = T^\alpha_{ij}\omega^i \wedge \omega^j, \quad \text{mod } \theta, \quad (\text{summation convention for} i, j = 1, \cdots, p)$$

where T^α_{ij} are skew symmetric in (ij). Let $\mathcal{T}_1$ be the set of functions $\{T^\alpha_{ij}\}$. If $\mathcal{T}_1$ are identically zero then $V(I, J) \to M$ is surjective, which is the Frobenius integrability condition for θ, and by Frobenius theorem we have $(n-p)$-parameter family of integral manifolds. Otherwise, let M_1 be the common zero set of $\mathcal{T}_1$ and set

$$dT^\alpha_{ij} = T^\alpha_{ij,k}\omega^k, \text{ mod } \theta.$$

Let $\mathcal{T}_2$ be the set of functions $T^\alpha_{ij,k}$. If $\mathcal{T}_2$ are identically zero on M_1 then $V_1(I, J) \to M_1$ is surjective, and by Frobenius theorem there exist $(\dim M_1 - p)$-parameter family of solutions. If $\mathcal{T}_2$ are not identically zero, let M_2 be the submanifold of M_1 defined by $\mathcal{T}_2 = 0$ and continue as before. Eventually we arrive either at an empty set, in which case there is no integral manifolds, or at an integrable Pfaffian system on a submanifold $M' \subset M$, in which case there exist $(\dim M' - p)$-parameter family of integral manifolds.

§4. Overdetermined PDE systems of generic type.

Let $u = (u^1, \cdots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \cdots, x^p)$. Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u^{(m)}) = 0, \quad \lambda = 1, \cdots \ell, \tag{4.1}$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. We assume that (4.1) is over-determined, that is, $\ell > q$.

As we differentialte (4.1) μ times we have partial derivatives of u up to order $m + \mu$. Since

$$\frac{\text{number of equations}}{\text{number of partial derivatives}} \to \frac{\ell}{q}, \qquad \text{as } \mu \to \infty,$$

generically it is possible to solve for all the partial derivatives of u of a sufficiently high order, say k, as functions of derivatives of lower order, by the implicit function theorem, namely,

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}), \tag{4.2}$$

for all multi-index K with $|K| = k$, and for all $\alpha = 1, \cdots, q$. (4.2) is called a complete system of order k and we say (4.1) admits prolongation to a complete system of order k.

4.1 Definition (overdetermined PDE system of generic type). *System (4.1) with $\ell > q$ (overdetermined) is said to be generic type if it admits prolongation to a complete system (4.2) of finite order k.*

(4.2) can be regarded as a Pfaffian system of Frobenius type on a manifold M: Let M be the submanifold of the $J^{(k-1)}(X, U)$ defined by (4.1) and its derivatives. Then the set of solutions of (4.1) is in natural one-to-one correspondence with the set of integral manifolds of the Pfaffian system of Frobenius type (M, θ), where the Pfaffian system θ consists of the contact forms (1.1) with $|J| \leq k-2$ and

$$\theta^\alpha_J := du^\alpha_J - \sum_{j=1}^{p} H^\alpha_{Jj}(x, u^{(k-1)})dx^j, \quad |J| = k-1. \tag{4.3}$$

K. Yamaguchi and T. Yatsui studied in [YY1] and [YY2] the equivalence problem by point transformations and formulated the geometry of $J^{(k-1)}$ with the distribution of p planes $\mathcal{D}$ given by (4.2). Solving (4.1) is finding an integral manifold of $\mathcal{D}$ on the submanifold M that is defined by (4.1) and its prolongation. We do this by Step 1 as in §2. For examples, we consider a PDE systems of first order for one unknown function $u(x, y)$

$$\begin{cases} u_x = A(x, y, u) \\ u_y = B(x, y, u). \end{cases} \tag{4.4}$$

This is a complete system of first order. In this case M is the whole 0-th order jet space $\mathbb{R}^3 = \{(x, y, u)\}$ we consider the Pfaffian system

$$\theta := du - Adx - Bdy = 0 \tag{4.5}$$

with the independence condition

$$dx \wedge dy \neq 0. \tag{4.6}$$

Then $d\theta = T dx \wedge dy, \quad \bmod \theta, \quad$ where

$$T = A_y + A_u B - B_x - B_u A. \tag{4.7}$$

If $T = 0$ on $\mathbb{R}^3$, then for any initial condition $u(x_0, y_0) = u_0$ there exists a unique solution satisfying the initial condition and thus there exists a 1-parameter family of solutions.

Example 4.2.

$$(4.8)\qquad \begin{cases} u_x = a(x,y) + u \\ u_y = b(x,y). \end{cases}$$

By (4.7) $T = a_y + b - b_x$. If the functions a and b satisfy $a_y + b - b_x = 0$, then there exists 1-parameter family of solutions. If T does not vanish identically then $T = 0$ gives a relation between x and y, which violates the condition $dx \wedge dy \neq 0$, hence no function $u(x,y)$ can be a solution of (4.8).

Example 4.3.

$$(4.9)\qquad \begin{cases} u_x = a(x,y) + u^2 \\ u_y = b(x,y), \quad b \neq 0. \end{cases}$$

In this case $\theta = du - (a+u^2)dx - bdy$ and $T = a_y + 2ub - b_x$. T cannot be identically zero for all (x,y,u). Thus $T = 0$ gives

$$(4.10)\qquad u = \frac{1}{2b}(-a_y + b_x).$$

Thus, if there is a solution then (4.10) is the solution and (4.10) is indeed a solution if it satisfies (4.9), namely,

$$(4.11)\qquad \begin{cases} \{\frac{1}{2b}(-a_y + b_x)\}_x = a + \{\frac{1}{2b}(-a_y + b_x)\}^2 \\ \{\frac{1}{2b}(-a_y + b_x)\}_y = b. \end{cases}$$

However, we derive (4.11) as follows: dT modulo θ on the submanifold $\{T = 0\}$ is

$$\{a_{xy} + \frac{b_x}{b}(-a_y + b_x) - b_{xx} + 2b[a + (\frac{-a_y + b_x}{2b})^2]\}dx$$
$$+ \{a_{yy} + \frac{b_y}{b}(-a_y + b_x) - b_{xy} + 2b^2\}dy.$$

By setting the coefficient of dx and the coefficient of dy to be zero, we obtain (4.11).

Now we consider systems of second order

$$\left\{\begin{array}{l} u_x + u_y = b(x,y) \\ u_{yy} = a(x,y,u,u_x,u_y). \end{array}\right. \tag{4.12}$$

Differentiate the first equation of (4.12) with respect to x and y, respectively, and solving for all the second order derivatives of u , we obtain

$$\left\{\begin{array}{l} u_{xx} = b_x - b_y + a \\ u_{xy} = b_y - a \\ u_{yy} = a. \end{array}\right. \tag{4.13}$$

Thus (4.12) admits prolongation to a complete system of second order. Let $J^1(\mathbb{R}^2,\mathbb{R}^1) = \{(x,y,u,p,q)\} = \mathbb{R}^5$ be the space of the first jets, where the first jet-graph of a function $u(x,y)$ is given by $p = u_x$ and $q = u_y$. Let M be a real submanifold of dimension 4 defined by the first equation of (4.12), that is, $p+q = b(x,y)$. We consider 1-forms

$$\left\{\begin{array}{l} \theta^0 = du - pdx - (b-p)dy \\ \theta^1 = dp - (b_x - b_y + a)dx - (b_y - a)dy \\ \theta^2 = dq - (b_y - a)dx - ady. \end{array}\right.$$

Observe that on M we have $\theta^2 = -\theta^1$ and that $\{\theta^0, \theta^1\}$ defines a 2-dimensional plane field $\mathcal{D}$ on M, whose integral manifolds are the first jet graph of solutions. To check the Frobenius integrability conditions we see that on M

$$\left\{\begin{array}{l} d\theta^0 = 0, \quad \mod \quad \{\theta^0, \theta^1\} \\ d\theta^1 = Tdx \wedge dy, \quad \mod \quad \{\theta^0, \theta^1\}, \end{array}\right.$$

where

$$T = -b_{yy} + a_y + a_u b + a_x + a_p b_x + a_q b_y. \tag{4.14}$$

If $T = 0$ holds identically on M, then there exists 2-parameter family of solutions. Otherwise, we restrict $\{\theta^0, \theta^1\}$ to the submanifold M_1 of M given by $T = 0$.

Example 4.4 (Linear Case).

$$\begin{cases} u_x + u_y = b(x, y) \\ u_{yy} = \alpha(x, y) + c_1 u + c_2 u_x + c_3 u_y. \end{cases}$$

Then

$$T(b^{(2)}, \alpha^{(1)}, c_1, c_2, c_3) = -b_{yy} + \alpha_x + \alpha_y + c_1 b + c_2 b_x + c_3 b_y,$$

which depends only on (x, y). If the functions α, b and the constants c_1, c_2, c_3 satisfy $T = 0$ then there are 2-parameter family of solutions. If T is not identically zero, $T = 0$ gives a relation between x and y, which violates $dx \wedge dy \neq 0$, therefore, no function $u(x, y)$ can be a solution.

Example 4.5 (Nonlinear case).

$$\begin{cases} u_x + u_y = b(x), \quad (b' \neq 0) \\ u_{yy} = \alpha(x, y) + u_x^2. \end{cases}$$

In this example we assume b depends only on x, for simplicity. Then on $M := \{p + q = b(x)\}$ we have independent 1-forms

$$\begin{cases} \theta^0 = du - p dx - (b - p) dy \\ \theta^1 = dp - (b' + \alpha + p^2) dx + (\alpha + p^2) dy. \end{cases}$$

Then on M

$$d\theta^1 \equiv T dx \wedge dy, \quad \text{mod} \quad \{\theta^0, \theta^1\}$$

where $T = \alpha_y + \alpha_x + 2pb'$. Hence $T = 0$ implies $p = -\frac{1}{2b'}(\alpha_x + \alpha_y)$, which defines a 3-dimensional submanifold M_1 of M. If $dT \equiv 0, \text{mod}\{\theta^0, \theta^1\}$ on M_1 then there is a 1-parameter family of solutions. In fact, if $u(x, y)$ is a solution, so is $u +$constant. On M_1

$$dT = (\alpha_x + \alpha_y)_x dx + (\alpha_x + \alpha_y)_y dy + 2b' dp + 2pb'' dx,$$

substituting $p = -\frac{\alpha_x + \alpha_y}{2b'}$ and $dp = (b' + \alpha + p^2)dx + (\alpha + p^2)dy$ and setting each of the coefficients to dx and to dy to be zero we obtain

(4.15)

$$\begin{cases} (\alpha_x + \alpha_y)_x + 2(b')^2 + 2b'\alpha + \frac{(\alpha_x + \alpha_y)^2}{2b'} - \frac{b''}{b'}(\alpha_x + \alpha_y) = 0 \\ (\alpha_x + \alpha_y)_y + 2b'\alpha + \frac{(\alpha_x + \alpha_y)^2}{2b'} = 0. \end{cases}$$

If the functions α and b satisfy (4.15) there is a 1-parameter family of solutions. Otherwise, no solutions exist.

Now we discuss the regularity and finiteness of solutions of overdetermined PDE systems of generic type.

Let M be the submanifold of $J^{(k-1)}(X, U)$ defined by (4.1) and its derivatives. Then the set of solutions of (4.1) is in natural one-to-one correspondence with the set of integral manifolds of the Pfaffian system of Frobenius type (M, θ), where the Pfaffian system θ consists of the contact form (1.1) with $|J| \leq k - 2$ and (4.3).

Then by the fundamental theorem of ODE we have

4.6 Theorem. *Suppose that (4.1) is generic type that admits complete prolognation (4.2) of order k. Suppose further that (4.1) is C^∞(C^ω, respectively) in its arguments. If f is a* *solution of* (4.1) *of class* C^k, *then* f *is* C^∞ (C^ω, *respectively). A solution is uniquely determined by its* $(k-1)$*-jet at a point.*

Regularity of mappings of various geometric structures, various local rigidity phenomena can be proved by using Theorem 10.2. Some of the results along these lines are found in [CH1],[CH2],[H1],[H2] [H4] and [HY].

§5. Frobenius Theorem.

We recall the Frobenius theorem on the integrability of vector fields (cf. [War] Chapter 1). Let M^m be a smooth manifold of dimension m. Let $X_1, \dots, X_p$ be smooth vector fields that are linearly independent at every point. Let $\mathcal{D}$ be the distribution of p-dimensional tangent planes spanned by $X_1, \dots, X_p$. A submanifold N is called an integral manifold of $\mathcal{D}$ if at every point of N, each X_j , $j = 1, \dots, p$, is tangent to N . The distribution $\mathcal{D}$ is said to be integrable if

$$[X_i, X_j](x) \in \mathcal{D}, \quad \forall x \in M \tag{5.1}$$

The Frobenius theorem states

5.1. Theorem. *Suppose that $\mathcal{D}$ is a smooth distribution spanned by a set of smooth vector fields $X_1, \dots, X_p$ that satisfies the integrability condition (5.1). Then at any point $x \in M$ there exists a unique smooth integral manifold N^p through x.*

Given a smooth distribution $\mathcal{D}$ of p-dimensional tangent planes let

$$\theta = (\theta^1, \dots, \theta^s), \quad s + p = m \tag{5.2}$$

be a system of linearly independent 1-forms that defines $\mathcal{D}$, that is, for (x, V) a tangent vector , $V \in \mathcal{D}$ if and only if $\theta^\alpha(V) = 0$, for each $\alpha = 1, \dots, s$. Typically, θ is found by taking smooth vector fields $Y_1, \dots, Y_s$, $p + s = m$, so that

$$X_1, \dots, X_p, Y_1, \dots, Y_s \tag{5.3}$$

span the whole tangent space at every point of M and then taking the dual 1-forms

$$\omega^1, \dots, \omega^p, \theta^1, \dots, \theta^s. \tag{5.4}$$

Let $\mathcal{I}$ be the algebraic ideal of the exterior algebra $\Omega := \bigoplus_{k=0}^{m} \Omega^k$, where Ω^k is the set of smooth k-forms and $\Omega^0 := C^\infty(M)$ is the ring of smooth functions on M. Each Ω^k is a module over $C^\infty(M)$. $\mathcal{I}$ is the set of all elements of Ω of the form $\sum_{\alpha=1}^{s} \theta^\alpha \wedge \phi^\alpha, \quad \phi^\alpha \in \Omega$. The ideal $\mathcal{I}$ is said to be closed if

$$d\mathcal{I} \subset \mathcal{I} \tag{5.5}$$

5.2. Exercise.

Let $X_1, \ldots, X_p$ be smooth vector fields that are linearly independent at every point of M^m and let $\mathcal{D}$ be the distribution spanned by those vector fields. Let $\theta^1, \ldots, \theta^s$, $s + p = m$, be smooth 1-forms that defines $\mathcal{D}$. Then the following are equivalent:

a) $\mathcal{D}$ is integrable in the sense (5.1).
b) $\mathcal{I}$ is closed.
c) For each $\alpha = 1, \ldots, s$,

$$d\theta^\alpha = 0, \qquad \mathrm{mod}\ (\theta^1, \ldots, \theta^s). \tag{5.6}$$

Hint: Use the identity: For any 1-form θ

$$d\theta(X_i, X_j) = X_i\theta(X_j) - X_j\theta(X_i) - \theta([X_i, X_j]) \tag{5.7}$$

Then we may state the Frobenius theorem as follows:

5.3. Theorem. *Let M be a smooth manifold of dimension m and let $\theta = (\theta^1, \ldots, \theta^s)$ be a system of smooth 1-forms that are linearly independent at every point of M. If θ satisfies the integrability condition (5.6) then for any point $x \in M$ there* *exists a unique integral manifold N of dimension $p := m - s$ through x. Therefore, M is foliated by s-parameter family of integral manifolds.*

Now given a point of M we are interested in the existence of a single integral manifold at the point rather than s-parameter family of integral manifolds. We set for each $\alpha = 1, \dots, s$

$$d\theta^\alpha = \sum_{i,j=1}^{p} T^\alpha_{ij} \omega^i \wedge \omega^j, \qquad \text{mod } \theta, \tag{5.8}$$

where $T^\alpha_{ji} = -T^\alpha_{ij}$. The set of functions $T = (T^\alpha_{ij})$ is called the torsion of the Pfaffian system θ. T is the obstruction to the existence of integral manifolds: The Frobenius theorem is that if T vanishes identically then M is foliated by integral submanifolds. If a submanifold $i : N^p \hookrightarrow M^m$ is an integral manifold of (5.2) then $i^*\theta = 0$, which implies $i^* d\theta = 0$, and therefore, if N^p is an integral manifold of (5.2) then $T = 0$ on N. Thus N is contained in the variety $T = 0$. This fact seems to have been well observed by experts (cf. [Bry], [Wang]). Independently from the predated observations we obtained several generalizations of Theorem 5.3, which are based on the the following

5.4. Theorem. *Let M^m be a smooth manifold and let $\theta := (\theta^1, \cdots, \theta^s)$ be a system of smooth 1-forms that are linearly independent at every point of M. Let n be an integer such that $2 \le n \le p := m - s$. Suppose that $i : N^n \hookrightarrow M^m$ is a submanifold of dimension n, defined by $\rho^1 = \cdots = \rho^{m-n} = 0$, where ρ^j are smooth real-valued functions of M such that $d\rho^1 \wedge \cdots \wedge d\rho^{m-n} \neq 0$. Then the following are equivalent:*
(i) $i^*\theta^\alpha = 0, \quad \alpha = 1, \cdots, s$.
(ii) $\forall \alpha = 1, \cdots, s, \theta^\alpha \equiv 0,$ mod $(\rho^1, \cdots, \rho^{m-n}, d\rho^1, \cdots, d\rho^{m-n})$.

Lemma 5.5. *Let (t, x), where $t = (t_1, \cdots, t_d)$, $x = (x_1, \cdots, x_n)$, be the standard coordinates of $\mathbb{R}^{d+n}$. Suppose that f is a C^∞ function defined on a neighborhood of the origin such that $f(0, x) = 0$. Then $f(t, x) = \sum_{j=1}^{d} t_j g^j(t, x)$, for some C^∞ functions $g^1, \cdots, g^d$ defined on a smaller neighborhood of the origin.*

Proof of Lemma 5.5.

$$
\begin{aligned}
f(t, x) &= \int_0^1 \frac{\partial}{\partial \tau} f(\tau t, x) d\tau \\
&= \int_0^1 \sum_{j=0}^{d} t_j f_j(\tau t, x) d\tau, \qquad \text{where } f_j = \frac{\partial f}{\partial t_j} \\
&= \sum_{j=1}^{d} t_j \int_0^1 f_j(\tau t, x) d\tau.
\end{aligned}
$$

Let $g^j(t, x) = \int_0^1 f_j(\tau t, x) d\tau$, for each $j = 1, \cdots, d$. Then it is standard to show that g^j are C^∞.

Proof of Theorem 5.4:

i) $\Rightarrow$ *ii)*: Choose independent 1-forms $\omega^1, \ldots, \omega^n$ so that

$$
d\rho^1, \ldots, d\rho^{m-n}, \omega^1, \ldots, \omega^n
$$

span T^*M. Then

$$
i^*(\omega^1 \wedge \cdots \wedge \omega^n) \neq 0
$$

Set

$$
\theta^\alpha = \sum_{j=1}^{m-n} a_j^\alpha d\rho^j + \sum_{j=1}^{n} b_j^\alpha \omega^j \tag{5.9}
$$

Since $i^*\theta^\alpha = 0$ and $i^*(d\rho^j) = 0$, pulling back (5.9) by i we have

$$0 = \sum_{j=1}^{n} b_j^\alpha (i^*\omega^j).$$

Therefore, for each α, j, we have $b_j^\alpha = 0$ on N, which implies by Lemma 5.5

$$b_j^\alpha = \sum_{k=1}^{m-n} h_{jk}^\alpha \rho^k, \tag{5.10}$$

for some smooth function h_{jk}^α. Substituting (5.10) for b_j^α in (5.9) we have

$$\theta^\alpha = \sum_{j=1}^{m-n} a_j^\alpha d\rho^j + \sum_{j=1}^{n}\sum_{k=1}^{m-n} \rho^k h_{jk}^\alpha \omega^j \tag{5.11}$$

$ii) \Rightarrow i)$: Suppose that

$$\theta^\alpha = \sum_{j=1}^{m-n} \rho^j \psi_j^\alpha + \sum_{j=1}^{m-n} h_j^\alpha d\rho^j \tag{5.12}$$

for some 1-forms ψ_j^α and smooth functions h_j^α. Apply any tangent vector $(x, V) \in TN$ to (5.12). Since $\rho^j(x) = 0$ and $d\rho^j(V) = 0$, we have $\theta^\alpha(V) = 0$, which implies that $i^*\theta^\alpha = 0$. $\square$

§6 Generalization of the Frobenius theorem.

Let M^m and $\theta = (\theta^1, \dots, \theta^s)$ be the same as in the Frobenius theorem (Theorem 5.3), and let $\omega^1, \dots, \omega^p$, $p + s = m$, be a complementary set of 1-forms as in (5.4). There are two directions in generalizing the Frobenius theorem: One is reducing the Pfaffian system $\theta = 0$ to a submanifold $M' \subset M^m$ of dimension m', $p \le m' \le m - 1$. Reducing Pfaffian system to a lower dimensional manifold is important in its own right regardless of the existence of integral manifolds. The other is the existence of integral manifolds of lower dimensions and foliation by lower dimensional integral manifolds: We want to find conditions on T^α_{ij} that imply there exists an integral manifold of dimension less than p.

6.1 Reduction of $\theta = 0$ to a submanifold.

We want to find real valued functions $\rho_1, \dots, \rho_d$ such that

i) $d\rho_1 \wedge \cdots \wedge d\rho_d \neq 0$
ii) $d\rho_j \equiv 0, \mod (\rho_1, \dots, \rho_d, \theta^1, \dots, \theta^s)$, for $j = 1, \dots, d$.

Then the problem is reduced to $M' := \{x \in M : \rho_j(x) = 0, j = 1, \dots, d\}$. Let $m' = m - d$ be the dimension of M'. Let $i : M' \hookrightarrow M$ be the inclusion map. Then ii) implies that the rank of $i^*\theta$ is constantly $s' = s - d$. If the torsions vanish on M', then by the Frobenius theorem there exists a s'-parameter family of integral manifolds of dimension

$$m' - s' = (m - d) - (s - d) = m - s = p.$$

6.2 Example. Reduction to a submanifold, to a system without solutions.

In $\mathbb{R}^4 = \{(x,y,z,w)\}$ consider the following two independent 1-forms

$$\theta = (\theta^1, \theta^2), \quad \text{where } \theta^1 = dz + xdy, \quad \theta^2 = dw + wdx.$$

Let $\rho(x,y,z,w) = w$ and let $M' = \{w = 0\}$. Since $d\rho = dw \neq 0$ and $d\rho = dw \equiv 0, \quad \text{mod } (w, dw, \theta^1, \theta^2)$, the Pfaffian system $\theta = 0$ reduces to M'.

But there is no integral manifolds in $i : M' \hookrightarrow \mathbb{R}^4$ for the following reason: $i^*\theta^2 = 0$, $i^*\theta^1 = dz + xdy$ and

$$d(i^*\theta^1) = i^*(d\theta^1) = dx \wedge dy \neq 0 \mod (i^*\theta).$$

6.3 Example. Reduction to a submanifold, to an involutive system.

In $\mathbb{R}^4 = \{(x,y,z,w)\}$ given 1-forms

$$\theta = (\theta^1, \theta^2), \quad \text{where } \theta^1 = dz + zdy, \quad \theta^2 = dw + w(1+y)dx.$$

As in Example 6.2 we can easily check that the Pfaffian system $(\mathbb{R}^4; \theta^1, \theta^2)0$ reduces the submanifold $M' = \{w = 0\}$. In the submanifold $i : M' \hookrightarrow \mathbb{R}^4$ we have $i^*\theta^2 = 0$, $i^*\theta^1 = dz + zdy$ and

$$\begin{aligned} i^*(d\theta^1) &= dz \wedge dy \\ &= -zdy \wedge dy \quad \text{mod } \theta \\ &= 0. \end{aligned}$$

Hence, the reduced system (M', θ^1) is involutive so that there exists a 1-parameter family of integral manifolds of dimension 2. Similarly, $M'' = \{z = 0\}$ gives another reduction. Then for the inclusion map $i : M'' \hookrightarrow \mathbb{R}^4$, we have

$$d(i^*\theta^2) = i^*(d\theta^2) = wdy \wedge dx \neq 0.$$

Therefore, torsion is w. In fact, the plane $w = 0, z = 0$ is the only integral manifold that is contained in M''.

6.4 Example. Reduction to a pair of submanifolds, to a pair of involutive systems.

In $\mathbb{R}^4 = \{(x, y, z, w)\}$ we consider 1-forms $\theta = (\theta^1, \theta^2)$, where

$$\theta^1 = dz + wf(x, y)dw, \quad \theta^2 = dw + zg(x, y)dz.$$

As in Example 6.2 we can easily check that the Pfaffian system $(\mathbb{R}^4; \theta^1, \theta^2)$ reduces to the submanifold $M' = \{w = 0\}$. In the submanifold $i : M' \hookrightarrow \mathbb{R}^4$ we have $i^*\theta^1 = dz$, $i^*\theta^2 = zg(x, y)dz$, so that the original system reduces to (M', θ^1). Since $d(i^*\theta^1) = 0$, the reduced system is involutive. Similarly, $M'' = \{z = 0\}$ gives another reduction. Then for the inclusion map $i : M'' \hookrightarrow \mathbb{R}^4$, we have $i^*\theta^1 = wf(x, y)dw$, $i^*\theta^2 = dw$, so that the original system reduces to (M'', θ^2). Since $d(i^*\theta^2) = 0$, the reduced system (M'', θ^2) is involutive.

Next we discuss the cases where there exists exactly one integral manifold of dimension p. These cases may be regarded as reduction of the Pfaffian system to p-dimensional submanifold. For possible applications we discuss the cases with degenerate torsions:

On $\mathbb{R}^3 = \{(x, y, z)\}$ consider a 1-form

$$\theta = dz + f(x, y, z)dy, \tag{6.1}$$

where $f(x, y, z)$ is a smooth (C^∞) real valued function defined on an open neighborhood of the origin. We are concerned with the existence of integral manifolds of (6.1). Suppose that M is an integral manifold of (6.1). Since $\theta|_M = 0$ we have $(d\theta)|_M = 0$. Now

$$\begin{aligned} d\theta &= (f_x dx + f_y dy + f_z dz) \wedge dy \\ &= f_x dx \wedge dy, \qquad \mod \theta. \end{aligned}$$

The obstruction to the existence of integral manifolds is the torsion

$$T = f_x.$$

If T is identically zero then by the Frobenius theorem there exists a 1-parameter family of integral manifolds.

In order construct examples with singular torsion sets we set

$$T = f_x = z(z - g(x,y)) = z^2 - zg(x,y). \tag{6.2}$$

I want $z = 0$ is the only integral manifold, so that we require

$$\begin{cases} f(x,y,0) = 0 \\ f_x = z^2 - zg(x,y). \end{cases}$$

Second condition implies that

$$f(x,y,z) = z^2 x - zG(x,y),$$

where $G_x = g$. Now any pair (G, g) with $G_x = g$ yields the torsion (6.2).

6.5 Example. Degenerate torsion with an isolated integral manifold. $G(x,yz) = x^2, \quad g(x,y) = 2x$: *Let* $\theta = dz + (z^2x - zx^2)dy$. *Then* $d\theta \equiv (z^2 - 2zx)dx \wedge dy$, *mod* θ. *Therefore,* $T = z(z - 2x)$. *The zero set of* T *is two planes intersecting along* y*-axis, among which the plane* $z = 0$ *is an integral manifold.*

6.6 Example. Degenerate torsion with an isolated integral manifold. *Let* $f_x = z(z^2 - x^2 - y^2)$, *so that* $f(x,y,z) = z^3x - zx^3/3 - zy^2x$. *Then the zero set of the torsion is given by* $z(z^2 - x^2 - y^2) = 0$. *This variety is the union of the plane* $z = 0$ *and the cone* $z^2 - x^2 - y^2 = 0$. $z = 0$ *is an integral manifold.*

Now we study by using Theorem 5.4, the existence of integral manifold $i : N^n \hookrightarrow M^m$, $2 \leq n \leq p$, of the Pfaffian system

$$\theta^\alpha = 0, \quad \alpha = 1, \ldots, s, \quad s + p = m \tag{6.3}$$

Suppose that N is an integral manifold of (6.3). Then $i^*\theta^\alpha = 0$ implies that $d(i^*\theta^\alpha) = i^*(d\theta^\alpha) = 0$. Let $\omega^1, \ldots, \omega^p$ be the complementary set of 1-forms. We set as usual

$$d\theta^\alpha = \sum_{i,j=1}^{p} T^\alpha_{ij} \omega^i \wedge \omega^j, \quad \text{mod } \theta, \quad \alpha = 1, \cdots s, \tag{6.4}$$

where $T^\alpha_{ji} = -T^\alpha_{ij}$. Consider $\binom{p}{2} := p(p-1)/2$ linearly independent differential 2-forms $\omega^i \wedge \omega^j$ arranged in lexico-graphical order. Let

$$\mathcal{T} = (T^\alpha_{ij}) \tag{6.5}$$

be the matrix of size $s \times \binom{p}{2}$. We shall call $\mathcal{T}$ torsion of the Pfaffian system (6.3).

Proposition 6.7. *Let M be a smooth manifold of dimension m and let $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p$ be a system of smooth 1-forms as in (6.3)-(6.4). Suppose that N is an integral manifold of (6.3) of dimension n $(2 \leq n \leq p)$. Then there exists $\binom{p}{2} \times \binom{n}{2}$ matrix valued smooth function A of rank $\binom{n}{2}$ defined on N such that*

$$\mathcal{T} A = 0. \tag{6.6}$$

In particular, if N^p is an integral manifold of maximal dimension then $\mathcal{T} = 0$ on N^p.

Proof. After re-ordering if necessary, we may assume that $\omega^1 \wedge \cdots \wedge \omega^n|_N \neq 0$. Set

$$\omega^\lambda|_N = \sum_{i=1}^{n} a^\lambda_i \omega^i|_N, \quad \lambda = n+1, \cdots, p. \tag{6.7}$$

Then the restriction to N of (6.4) reads

(6.8)

$$0 = \sum_{\substack{i<j \\ i,j=1,\cdots,n}} \tau^\alpha_{ij} \omega^i \wedge \omega^j, \quad \text{where}$$

$$\tau^\alpha_{ij} = T^\alpha_{ij} + \sum_{\mu=n+1}^{p} T^\alpha_{i\mu} a^\mu_j - \sum_{\lambda=n+1}^{p} T^\alpha_{j\lambda} a^\lambda_i + \sum_{\substack{\lambda<\mu \\ \lambda,\mu=n+1,\cdots,p}} T^\alpha_{\lambda\mu}(a^\lambda_i a^\mu_j - a^\lambda_j a^\mu_i)$$

$\alpha = 1, \cdots, s$. Since $\omega^i \wedge \omega^j$, $i < j$, are independent on N (6.8) implies

(6.9)

$$T^\alpha_{ij} + \sum_{\mu=n+1}^{p} T^\alpha_{i\mu} a^\mu_j - \sum_{\lambda=n+1}^{p} T^\alpha_{j\lambda} a^\lambda_i + \sum_{\substack{\lambda<\mu \\ \lambda,\mu=n+1,\cdots,p}} T^\alpha_{\lambda\mu}(a^\lambda_i a^\mu_j - a^\lambda_j a^\mu_i) = 0,$$

for each $\alpha = 1, \cdots, s$ and each pair (ij) with $i < j$, $i, j = 1, \cdots, n$. In matrices we write (6.9) as

$$\mathcal{T} A = 0, \tag{6.10}$$

where A is a matrix of size $\binom{p}{2} \times \binom{n}{2}$ given as follows: for a pair $I = (ij)$ with $i < j$, $i, j = 1, \cdots, n$, I-th column of A is

$$(\underset{\substack{\uparrow \\ (ij)^{th}}}{0 \cdots \quad 1} \cdots \quad \underset{\substack{\uparrow \\ (i\mu)^{th}}}{a^\mu_j} \cdots \quad \underset{\substack{\uparrow \\ (j\lambda)^{th}}}{-a^\lambda_i} \cdots \quad \underset{\substack{\uparrow \\ (\lambda\mu)^{th}}}{\underbrace{a^\lambda_i a^\mu_j - a^\lambda_j a^\mu_i}} \cdots)^t$$

for $n < \lambda < \mu$. Observe that the first $\binom{n}{2}$ rows or A is the identity matrix, therefore A is of maximal rank. In particular, if $n = p$ then A is the identity matrix of size $\binom{p}{2}$, therefore, $\mathcal{T}$ is identically zero on an integral manifold of maximal dimension p. □

Observe that (6.10) is a system of $\binom{n}{2}$ independent linear equations on the $\binom{p}{2}$ columns of $\mathcal{T}$. Hence we have

Theorem 6.8. *If N is an integral manifold of (6.3) of dimension n, $(2 \leq n \leq p)$, then the number of linearly independent columns of $\mathcal{T}$ is at most $\binom{p}{2} - \binom{n}{2}$.*

Definition 6.9. *Given a set of smooth functions $T^\alpha, \alpha = 1, \cdots, k$ on M a smooth function ρ is said to be a common factor of T^α's if $T^\alpha = \rho\phi^\alpha$, for some smooth function ϕ^α for each $\alpha = 1, \cdots, k$. A set of smooth real-valued functions $\rho_1, \cdots, \rho_d$, $(d \leq m)$, is said to be non-degenerate if $d\rho_1 \wedge \cdots \wedge d\rho_d \neq 0$.*

Theorem 6.10. *Let $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p$ be 1-forms of M^m, $s + p = m$, as in (6.3)-(6.4). Let n, $2 \leq n \leq p$, be an integer. Then there exists an integral manifold N of (6.3) of dimension n if and only if there exists a non-degenerate set of functions $\rho = (\rho_1, \cdots, \rho_{m-n})$ having the following properties: on the common zero set of ρ the first $\binom{n}{2}$ columns $\mathcal{T}_1, \cdots, \mathcal{T}_{\binom{n}{2}}$ belong to the linear span of $\mathcal{T}_\lambda, \lambda = \binom{n}{2} + 1, \cdots, \binom{p}{2}$, on N, where $\mathcal{T}_\lambda$ is the λ-th column of $\mathcal{T}$, and*

$$\theta^\alpha = 0, \quad mod\ (\rho, d\rho). \tag{6.11}$$

Corollary 6.11. *Under the same hypotheses as in Theorem 6.10 suppose that $s \geq \binom{p}{2} - \binom{n}{2} + 1$. Then there exists an integral manifold of (6.3) of dimension n if and only if there exists a non-degenerate set of common factors $\rho_1, \cdots, \rho_{m-n}$ of determinants of square submatrices of $\mathcal{T}$ of size $\binom{p}{2} - \binom{n}{2} + 1$ that satisfies (6.11).*

Corollary 6.12. *Under the same hypotheses as in Theorem 6.10, if $s = 1$, then there exists an integral manifold of dimension n if and only if there exists a non-degenerate set of smooth functions $\rho = (\rho_1, \cdots, \rho_{m-n})$ such that*

$$T_{ij} = 0, \ mod\ (\rho, d\rho, T_{\lambda\mu} : \ either\ \lambda > n\ or\ \mu > n) \tag{6.12}$$

that satisfies (6.11).

Proposition 6.13. *Suppose that a submanifold $N^n \subset M^m$ given as the common zero set of smooth real valued functions $\rho_1, \cdots, \rho_{m-n}$ is an integral manifold of (6.3) of dimension $n < p$. Then N is contained in a (unique) integral manifold $\tilde{N}$ of maximal dimension p if and only if there exist smooth functions $\tau_1, \cdots, \tau_s$ with the following properties:*

i) $d\tau_1 \wedge \cdots \wedge d\tau_s \neq 0, \quad mod\ (\tau_1, \cdots, \tau_s)$

ii) $\rho_j \equiv 0, \quad mod\ (\tau_1, \cdots, \tau_s)$, *for each* $j = 1, \cdots, m-n$

iii) $\theta^\alpha \equiv 0, \quad mod\ (\tau_1, \cdots, \tau_s, d\tau_1, \cdots, d\tau_s)$.

$\rho_1, \cdots, \rho_{m-n}$ and $\tau_1, \cdots, \tau_s$ in the above propositions can be obtained from the factorization of the coefficients of $d\theta^\alpha$: Let τ^α_{ij} be the LHS of (6.9) for each $\alpha = 1, \ldots, s$, $i, j = 1, \ldots, n$, and let T^α_{ij}, $\alpha = 1, \ldots, s$ and $i, j = 1, \ldots, p$ be as in (6.4). Then $\rho_1, \ldots, \rho_{m-n}$ nondegenerate are common factors of τ^α_{ij} and $\tau_1, \ldots, \tau_s$ are nondegenerate common factors of T^α_{ij}.

Remark. *If each 1-form of (6.3)-(6.4) is real-analytic (C^ω), then T^α_{ij} are (C^ω) and therefore, factorizable into a product of finitely many complex valued functions f with $df(P) \neq 0$. The factorization is unique modulo unit.*

Now we are concerned with the problem of deciding whether M is foliated by integral manifolds of dimension $n < p$. In the case $s = 1$, i.e., the Pfaffian system (6.3) consists of a single 1-form θ, is the classical Pfaff problem (see [BCGGG Chapter II). Let θ be a smooth 1-form on a smooth manifold M^m. The rank r is defined by the conditions

$$\theta \wedge (d\theta)^r \neq 0, \quad \theta \wedge (d\theta)^{r+1} = 0.$$

There is a second integer t defined by

$$(d\theta)^t \neq 0, \quad (d\theta)^{t+1} = 0.$$

Elementary arguments show that there are two cases:

$$(i) t = r;$$
$$(ii) t = r + 1.$$

The first is the case $\theta \wedge (d\theta)^r \neq 0$ and $(d\theta)^{r+1} = 0$ and the second is the case $(d\theta)^{r+1} \neq 0$ and $\theta \wedge (d\theta)^{r+1} = 0$. We have

6.14 Theorem. *Let θ be a 1-form. In a neighborhood suppose r and t are constant. Then θ has the normal form*

$$\begin{aligned} \theta &= y^0 dy^1 + \cdots + y^{2r} dy^{2r+1}, \quad && \text{if } r + 1 = t \\ \theta &= dy^1 + y^2 dy^3 + \cdots y^{2r} dy^{2r+1}, \quad && \text{if } r = t. \end{aligned} \tag{6.13}$$

In these expressions, the y's are independent functions and are therefore parts of a local coordinate system.

Proof: See [BCGGG] page 40 .

6.15 Corollary. *Let θ be a smooth 1-form of rank r on M^m, $2r + 1 \leq m$. Then M is foliated by integral manifolds of dimension $m-(r+1)$. Integral manifolds are given by $y_{2k-1} =$ const, $k = 1, \ldots, r + 1$. In particular, if θ is of rank 0, which is the case of the Frobenius integrability, then M is foliated by integral manifolds of dimension $m - 1$.*

6.16 Problem. *Let M and θ be the same as in Theorem 6.14. How many different foliations do there exist for the Pfaffian system (M, θ)?*

6.17 Problem. *Generalize Corollary 6.15 and Problem 6.16 to the system $\theta = (\theta^1, \ldots, \theta^s)$, $s > 1$.*

§7. Complex submanifolds in real hypersurfaces.

We present in this section an application of the generalization of the Frobenius theorem that we discussed in §6. This section is part of [HT]. We shall discuss the existence of complex submanifolds in terms of derivatives of the Levi-form.

Let M be a smooth (C^∞) real hypersurface in $\mathbb{C}^{n+1}$, with coordinates (z, w), where $z = (z_1, \cdots, z_n)$, defined on a neighborhood U of our reference point P. Let M be defined by $r(z, \bar{z}, w, \bar{w}) = 0$, where r is a C^∞ real-valued function defined on U such that $dr \neq 0$ on M. We assume $r_w \neq 0$. In this section we discuss conditions for M to admit complex submanifolds through P. We first present a necessary and sufficient condition for a complex hypersurface to exist through P and then extend our arguments to the cases of complex submanifolds of higher codimensions.

Let

$$\theta = \sqrt{-1}\partial r. \tag{7.1}$$

Since $dr = \partial r + \bar{\partial} r = 0$ on M we have $\bar{\theta} = -\sqrt{-1}\bar{\partial} r = \sqrt{-1}\partial r = \theta$, therefore, θ is a real 1-form on M. Then

$$H(M) := \{v \in T(M) : \theta(v) = 0\}$$

is the bundle of maximal complex subspaces of $T(M)$. A real submanifold $N \subset M$ of dimension $2n$ is a complex submanifold if and only if N is an integral manifold of $H(M)$. In order to find conditions on r for an integral manifold to exist we use a generalization of the Frobenius theorem to non-involutive cases with degenerate torsions. We adopt the definitions and notations from [BCGGG] and [GJ]. Generalized Frobenius theorem with non-degenerate torsions has been studied in [H].

Then our problem is finding an integral manifold of dimension $2n$ of the exterior differential system

$$(r, \quad \theta). \tag{7.2}$$

If N is an integral manifold of (7.2) then $r|_N = 0$ and $\theta|_N = 0$, therefore, $dr|_N = 0$ and $d\theta|_N = 0$. Since $\theta = \bar{\theta}$, mod (dr), and

$$\frac{1}{\sqrt{-1}}\theta = \sum_{i=1}^{n} r_i dz_i + r_w dw$$

we have

$$\begin{aligned} dw &= -\frac{1}{r_w}\sum r_j dz_j, \qquad \text{mod } (dr, \theta) \\ d\bar{w} &= -\frac{1}{r_{\bar{w}}}\sum r_{\bar{j}} d\bar{z}_j, \qquad \text{mod } (dr, \theta). \end{aligned} \tag{7.3}$$

Since

$$\frac{1}{\sqrt{-1}}d\theta = \bar{\partial}\partial r$$
$$= \sum_{i,j=1}^{n} \{r_{i\bar{j}} d\bar{z}_j \wedge dz_i + r_{i\bar{w}} d\bar{w} \wedge dz_i + r_{w\bar{j}} d\bar{z}_j \wedge dw + r_{w\bar{w}} d\bar{w} \wedge dw\}$$

by substituting (7.3) for dw and for $d\bar{w}$ we have

$$\frac{1}{\sqrt{-1}}d\theta \equiv \sum_{i,j=1}^{n} T_{i\bar{j}} d\bar{z}_j \wedge dz_i, \qquad \text{mod } (dr, \theta) \tag{7.4}$$

where

$$T_{i\bar{j}} = r_{i\bar{j}} - r_{i\bar{w}}\frac{r_{\bar{j}}}{r_{\bar{w}}} - r_{w\bar{j}}\frac{r_i}{r_w} + r_{w\bar{w}}\frac{r_{\bar{j}}}{r_{\bar{w}}}\frac{r_i}{r_w}. \tag{7.5}$$

$(T_{i\bar{j}})$ is a hermitian matrix, which is coefficients of the Levi form of M. If M is Levi flat, that is, if

$$T_{i\bar{j}} \equiv 0, \text{ mod } (r), \quad \forall i, j = 1, \cdots, n$$

then by the Frobenius theorem M is foliated by complex hypersurfaces. The functions $T_{i\bar{j}}$, mod (r), are the obstruction to the existence of integral manifolds, which is generally called *torsion* for the exterior differential system (7.2).

Definition 7.1. *A real valued function ρ defined on U is a factor of the Levi-form $(T_{i\bar{j}})$ if $T_{i\bar{j}} \equiv 0$, mod (r, ρ), for each i, j.*

Our main observation is that if a complex hypersurface exits it is given as the zero set of a factor ρ of the Levi-form. A necessary and sufficient condition for the existence of a complex hypersurface is that $\theta(v) = 0$ for all vectors $v \in T_x\mathbb{C}^{n+1}$ with $r(x) = \rho(x) = 0$, $dr(v) = 0$ and $d\rho(v) = 0$, which is a condition on the derivatives of r up to third order. We have

Theorem 7.2. *Let M be a real hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$, given as a zero set of a smooth real-valued function r with $r_w \neq 0$ defined on a small neighborhood $U \subset \mathbb{C}^{n+1}$ of a point $P \in M$. Let θ and $T_{i\bar{j}}$ be the same as defined by (7.1) and (7.4). Then there exists a complex hypersuface N in M through P if and only if there is a factor ρ of the Levi-form such that*
i) $\rho(P) = 0$, $\quad (dr \wedge d\rho)(P) \neq 0$
ii) $\theta \equiv 0$, $\quad$ mod $(r, \rho, dr, d\rho)$.

Proof. First we recall a well known fact: Let (t, x), where $t = (t_1, \cdots, t_d)$, $x = (x_1, \cdots, x_m)$, be the standard coordinates of $\mathbb{R}^{d+m}$. Suppose that f is a C^∞ function defined on a neighborhood of the origin such that $f(0, x) = 0$. Then

$$f(t, x) = \sum_{j=1}^{d} t_j g^j(t, x)$$

for some C^∞ functions $g^1, \cdots, g^d$ defined on a smaller neighborhood of the origin.

Now suppose that N is a complex hypersurface through P.

Then

$$0 = \frac{1}{\sqrt{-1}} d\theta|_N = \sum_{i,j=1}^{n} (T_{i\bar{j}}|_N) d\bar{z}_j \wedge dz_i.$$

Since $d\bar{z}_j \wedge dz_i$ are independent on N, we have $T_{i\bar{j}}|_N = 0$, for each $i, j = 1, \cdots, n$. Now choose any smooth real-valued function ρ on U such that N is the common zero set of r and ρ and such that $d\rho \wedge dr \neq 0$ on N. We take a local coordinate system $(r, \rho, x_1, \cdots, x_{2n})$ of $\mathbb{C}^{n+1}$. Then $T_{i\bar{j}} \equiv 0$, mod (r, ρ). Now *ii*) follows from observing that the following are equivalent:

a) $T_x N = H_x(M), \quad \forall x \in N$.
b) For $x \in N$ and for $v \in T_x N$ implies $\theta(v) = 0$.
c) For $v \in T_x(\mathbb{C}^{n+1})$ with $r(x) = \rho(x) = 0, dr(v) = 0$, $d\rho(v) = 0$ we have $\theta(v) = 0$.
d) $\theta \equiv 0$, mod $(r, \rho, dr, d\rho)$.

Conversely, suppose that ρ is a factor of the Levi-form with the properties i) and ii). Let N be the common zero set of r and ρ. Then the property i) implies that N, near P, is a smooth $(2n)$-dimensional submanifold of M containing P and *ii*) implies that any tangent vector to N belongs to $H(M)$, hence, N is a complex hypersurface. □

Example 7.3. *Quadric real hypersurfaces in $\mathbb{C}^2$: Let Q be the zero set of*

$$r = w + \bar{w} + az\bar{z} + \lambda z\bar{w} + \bar{\lambda} w\bar{z} + bw\bar{w},$$

where $a, b \in \mathbb{R}$, and $\lambda \in \mathbb{C}$ are constants.

We shall show that if Q contains a complex hypersurface through the origin then Q is Levi flat. We have

$$\theta = \sqrt{-1}\{(a\bar{z} + \lambda\bar{w})dz + (1 + \bar{\lambda}\bar{z} + b\bar{w})dw\},$$

and

$$\begin{aligned} \frac{1}{\sqrt{-1}} d\theta &= \bar{\partial}\partial r \\ &\equiv T d\bar{z} \wedge dz, \qquad \text{mod } (\theta, dr), \end{aligned}$$

where

$$T = a - \lambda \frac{\bar{a}z + \bar{\lambda}w}{1 + \lambda z + \bar{b}w} - \bar{\lambda}\frac{a\bar{z} + \lambda\bar{w}}{1 + \bar{\lambda}\bar{z} + b\bar{w}} + b\frac{\bar{a}z + \bar{\lambda}w}{1 + \lambda z + \bar{b}w}\frac{a\bar{z} + \lambda\bar{w}}{1 + \bar{\lambda}\bar{z} + b\bar{w}}.$$

Let $\mathcal{T}$ be T multiplied by the common denominator:

$$\begin{aligned} \mathcal{T} = a &+ (ab - \lambda\bar{\lambda})w + (ab - \lambda\bar{\lambda})\bar{w} + (-\lambda\bar{\lambda}a + ba^2)z\bar{z} \\ &+ (-\lambda\bar{\lambda}^2 + b\bar{\lambda}a)\bar{z}w + (-\bar{\lambda}\lambda^2 + ba\lambda)z\bar{w} + (ab^2 - \lambda\bar{\lambda}b)w\bar{w}. \end{aligned}$$

Therefore, in order for the origin to be a zero of $\mathcal{T}$ the coefficient a must be zero and in that case Q contains the complex line $w = 0$. We have

$$r = w + \bar{w} + \lambda z\bar{w} + \bar{\lambda}w\bar{z} + bw\bar{w}$$

and

$$\mathcal{T} = -\lambda\bar{\lambda}(w + \bar{w}) - \lambda\bar{\lambda}^2\bar{z}w - \bar{\lambda}\lambda^2 z\bar{w} - \lambda\bar{\lambda}bw\bar{w}.$$

Observe that

$$\mathcal{T} = -\lambda\bar{\lambda}r \equiv 0, \quad \text{mod } (r),$$

therefore Q is Levi flat.

Example 7.4. *Cubic real hypersurfaces in* $\mathbb{C}^2 = \{(z,w)\}$*: Let* $z = x+iy$ *and* $w = u+iv$*. Consider the zero set* M *of*

$$\begin{aligned} r &= 2u(1+2y) + 8vx^2 \\ &= (w+\bar{w})(1+\frac{z-\bar{z}}{i}) + \frac{w-\bar{w}}{i}(z+\bar{z})^2. \end{aligned}$$

We shall show that M is not Levi flat and a complex line $w=0$ is contained in M. We have $dr = 16xvdx + 4udy + 2(1+2y)du + 8x^2dv$ and

$$\begin{aligned} \theta &= i\partial r \\ &= [w+\bar{w}+2(w-\bar{w})(z+\bar{z})]dz + [i+z-\bar{z}+(z+\bar{z})^2]dw, \end{aligned}$$

therefore,

$$\begin{aligned} dw &= -\frac{w+\bar{w}+2(w-\bar{w})(z+\bar{z})}{i+z-\bar{z}+(z+\bar{z})^2}dz, \quad \bmod \theta \\ d\bar{w} &= -\frac{w+\bar{w}-2(w-\bar{w})(z+\bar{z})}{-i+\bar{z}-z+(z+\bar{z})^2}d\bar{z}, \quad \bmod \theta. \end{aligned}$$

Then

$$\begin{aligned} d\theta &= i\bar{\partial}\partial r \\ &= [2(w-\bar{w})d\bar{z} + (1-2(z+\bar{z}))d\bar{w}] \wedge dz + [-1+2(z+\bar{z})]d\bar{z}\wedge dw \\ &\quad \text{substituting the above for } dw \text{ and } d\bar{w} \\ &= Td\bar{z}\wedge dz, \end{aligned}$$

where

$$\begin{aligned} T &= 2(w-\bar{w}) - (1-2(z+\bar{z}))\frac{w+\bar{w}-2(w-\bar{w})(z+\bar{z})}{-i+\bar{z}-z+(z+\bar{z})^2} \\ &\qquad - (-1+2(z+\bar{z}))\frac{w+\bar{w}+2(w-\bar{w})(z+\bar{z})}{i+z-\bar{z}+(z+\bar{z})^2} \\ &= 4iv - (1-4x)\frac{2u-4vi\cdot 2x}{-i-2yi+4x^2} - (-1+4x)\frac{2u+4vi\cdot 2x}{i+2yi+4x^2}. \end{aligned}$$

To see that M is not Levi flat consider a curve $\sigma(x) = (x, 0, -4x^3, x)$, which lies on M and passes through the origin. Observe that $T(\sigma(x))$, after multiplying by the product of the denominators, is a polynomial in x of degree 6 without constant term, therefore, does not vanish identically. We also have

$$dT = \zeta_1(x, y, u, v)dx + \zeta_2(x, y, u, v)dy + adu + (4i + \zeta)dv,$$

where $\zeta_j(x, y, 0, 0) = 0, \quad$ for $j = 1, 2$ and $\zeta(0) = 0$. The submanifold $r = T = 0$ is the complex line $w = 0$, along which we have

$$dT = adu + (4i + \zeta)dv$$
$$dr = 2(1 + 2y)du + 8x^2 dv$$

and

$$\theta = (i + z - \bar{z} + (z + \bar{z})^2)dw$$
$$= (i + 2iy + 4x^2)(du + idv).$$

Thus we have

$$\theta \equiv 0, \quad \text{mod } (r, T, dr, dT).$$

Now we discuss the cases of complex submanifolds of higher codimensions. Suppose that N^{2k} is a complex submanifold of complex dimension $k, 1 \leq k \leq n$, through P. Then the Levi-form restricted to N is zero, that is,

$$d\theta(L, \bar{L}) := \sqrt{-1} \sum_{i,j=1}^{n} T_{i\bar{j}} a_i \bar{a}_j = 0,$$

for any complex vector $L = (a_1, \cdots, a_n, b) \in \mathbb{C}^{n+1}$, which is tangent to N. Therefore, at P the null space of the Levi form is of complex dimension $\geq k$, which implies that

$$\text{rank } [T_{i\bar{j}}] \leq n - k. \tag{7.6}$$

Let $\tau_1, \cdots, \tau_m$ be the determinant of the square submatrices of $[T_{i\bar{j}}]$ of size $n-k+1$. Then each τ_j is a polynomial in $T_{i\bar{j}}$ of degree $n-k+1$. Then (7.6) is equivalent to

$$\tau_j|_N = 0. \tag{7.7}$$

Thus a complex submanifold N is contained in the common zero set of $\tau_j, j = 1, \cdots, m$. If N is defined as a common zero set of real-valued functions $r, \rho_1, \cdots, \rho_d$ with $dr \wedge d\rho_1 \wedge \cdots \wedge d\rho_d \neq 0$, where $d = 2n+1-2k$, then each τ_j must be zero on the common zero set of $r, \rho_1, \cdots, \rho_d$. This implies $\tau_j \equiv 0$, mod $(r, \rho_1, \cdots, \rho_d)$. For each $\rho_\nu, \nu = 1, \cdots, d$ let $\theta^\nu = \sqrt{-1}\partial\rho_\nu$, then the common zero set of $r, \rho_1, \cdots, \rho_d$ is a complex manifold if and only if $\theta_\nu \equiv 0$, mod $(r, \rho_1, \cdots, \rho_d, dr, d\rho_1, \cdots, d\rho_d)$. Thus we have the following

Theorem 7.5. *Let M be a real hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$, given as a zero set of a smooth real-valued function r with $dr \neq 0$ defined on a small neighborhood $U \subset \mathbb{C}^{n+1}$ of a point $P \in M$. Let θ and $T_{i\bar{j}}$ be the same as defined by (7.1) and (7.4). Then there exists a complex submanifold N of complex dimension k through P if and only if there is a smooth real-valued functions $\rho_1, \cdots, \rho_d$, where $d = 2n+1-2k$, defined on U such that*

i) For each $\nu = 1, \cdots, d$, $\rho_\nu(P) = 0$, and $dr \wedge d\rho_1 \wedge \cdots \wedge d\rho_d(P) \neq 0$

ii) Each $\tau_j, j = 1, \cdots, m$, of (2.7) is zero modulo $(r, \rho_1, \cdots, \rho_d)$.

iii) For each ν, $\theta^\nu \equiv 0$, mod $(r, \rho_1, \cdots, \rho_d, dr, d\rho_1, \cdots, d\rho_d)$.

Example 7.6. *Complex curve through the origin in $M^5 \subset \mathbb{C}^3 = \{(z_1, z_2, w)\}$: Let*

$$r = w + \bar{w} + az_1\bar{z}_1 + \lambda(z_1)^2\bar{z}_2 + \bar{\lambda}z_2(\bar{z}_1)^2,$$

where a is a nonzero real constant and λ is a nonzero complex constant.

In this example we have

$$T := [T_{i\bar{j}}] = \begin{bmatrix} a & 2\lambda z_1 \\ 2\bar{\lambda}\bar{z}_1 & 0 \end{bmatrix}$$

so that $\det T = -4\lambda\bar{\lambda} z_1 \bar{z}_1 = 0$ implies that $z_1 = 0$. Thus we take $\rho_1 = \Im w$, $\rho_2 = \Re z_1$, and $\rho_3 = \Im z_1$. Let N be the set of common zeros of $(r, \rho_\nu, \nu = 1, 2, 3)$, which is a complex line $(0, \zeta, 0)$. Then modulo $(r, \rho_1, \rho_2, \rho_3)$ we have

$$\begin{aligned} dr &= dw + d\bar{w}, & \theta &= \sqrt{-1}dw \\ d\rho_1 &= \tfrac{1}{2\sqrt{-1}}(dw - d\bar{w}), & \theta^1 &= \tfrac{1}{2}dw \\ d\rho_2 &= \tfrac{1}{2}(dz_1 + d\bar{z}_1), & \theta^2 &= \tfrac{\sqrt{-1}}{2}dz_1 \\ d\rho_3 &= \tfrac{1}{2\sqrt{-1}}(dz_1 - d\bar{z}_1), & \theta^3 &= \tfrac{1}{2}(dz_1). \end{aligned}$$

We see that $\theta \equiv 0$, and $\theta^\nu \equiv 0$ for $\nu = 1, 2, 3$, modulo $(r, \rho_1, \rho_2, \rho_3, dr, d\rho_1, d\rho_2, d\rho_3)$.

§8. Integrable submanifolds in almost complex manifolds.

In this section we discuss the existence of integrable submanifolds in almost complex manifolds [HL]. This is another application of the generalized Frobenius theorem that we discussed in §6.

Let (M^{2m}, J) be a smooth almost complex manifold. For a real tangent vector $X \in TM$ let $X' = 1/2(X - \sqrt{-1}JX)$ and $X'' = 1/2(X + \sqrt{-1}JX)$. The complex vectors X' and X'', which we shall call $(1,0)$ part of X and $(0,1)$ part of X, respectively, are eigenvectors of J associated with the eigenvalues $+i$, and $-i$, respectively. Then we have $X = X' + X''$

and the decomposition of the complexified tangent bundle:

$$T_{\mathbb{C}}M = T'M + T''M \ ,$$

where $T'M$ and $T''M$ are the set of all $(1,0)$ vectors and $(0,1)$ vectors, respectively. Then we see that $\overline{T'M} = T''M$. On a neighborhood of the reference point $P \in M$ let $L_1, \dots, L_m$ and $\bar{L}_1, \cdots, \bar{L}_m$ be smooth sections of $T'M$ and $T''M$, respectively, that are linearly independent at every point. Let $\theta^1, \dots, \theta^m, \bar{\theta}^1, \dots, \bar{\theta}^m$ be the dual 1-forms. For a smooth function ρ we define $\partial\rho = \sum_{j=1}^{m}(L_j\rho)\theta^j$ and $\bar{\partial}\rho = \sum_{j=1}^{m}(\bar{L}_j\rho)\bar{\theta}^j$. Then we see that $d\rho = \partial\rho + \bar{\partial}\rho$. Now we consider a submanifold N^{2n} defined as the common zero set of real-valued functions $\rho_1, \dots, \rho_{2d}$ on a neighborhood of $P \in M$ such that

$$d\rho_1 \wedge \cdots \wedge d\rho_{2d}(P) \neq 0, \tag{8.1}$$

where $d = m - n$.

Definition 8.1. *The* rank $(\partial\rho_1, \cdots, \partial\rho_{2d})$ *at P is the maximal number k with*

$$\partial\rho_{\alpha_1} \wedge \cdots \wedge \partial\rho_{\alpha_k}(P) \neq 0 \ .$$

Proposition 8.2. *Suppose that $\rho_1, \cdots, \rho_{2d}$ be real valued functions on a neighborhood of P of (M^{2m}, J), $d \leq m$, with $d\rho_1 \wedge \cdots \wedge d\rho_{2d}(P) \neq 0$. Then*

$$d \leq \text{rank}\,(\partial\rho_1, \cdots, \partial\rho_{2d}) \leq 2d \ .$$

Proof.
(8.2)

$$\begin{aligned} d\rho_1 \wedge \cdots \wedge d\rho_{2d} &= (\partial\rho_1 + \bar{\partial}\rho_1) \wedge \cdots \wedge (\partial\rho_{2d} + \bar{\partial}\rho_{2d}) \\ &= (\partial\rho_1 \wedge \cdots \wedge \partial\rho_{2d}) + \text{mixed terms} + (\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_{2d}) \ , \end{aligned}$$

where mixed term means those terms that contain both $\partial\rho_\alpha$'s and $\bar{\partial}\rho_\alpha$'s. If rank $(\partial\rho_1, \cdots, \partial\rho_{2d}) \leq d-1$ then each term in the last line of (1.2) contains either $\partial\rho_\alpha$'s more than d times or $\bar{\partial}\rho_\alpha$'s more than d times. Hence, each term of the last line of (8.2) is zero at P, which contradicts to $d\rho_1 \wedge \cdots \wedge d\rho_{2d}(P) \neq 0$.

Definition 8.3. *A submanifold N is said to be J-invariant if for all $X \in TN$ we have $JX \in TN$.*

Proposition 8.4. *Let N^{2n} be a submanifold of (M^{2m}, J) passing through $P \in M$ given as a common zero set of real valued functions $\rho_1, \cdots, \rho_{2d}$ that satisfy (8.1). Let $T'N = \{X - \sqrt{-1}JX : X \in TN, JX \in TN\}$ and $T''N = \{X + \sqrt{-1}JX : X \in TN, JX \in TN\}$. Then the following are equivalent:*

i) N is J-invariant.

ii) For each $x \in N$, T'_xN and T''_xN are of complex dimension n.

iii) $\operatorname{rank}(\partial\rho_1, \cdots, \partial\rho_{2d})(x) = d$, $\forall x \in N$.

Proof.
i)⇒ ii): Suppose that N is J-invariant. Then it is easy to see that there exist linearly independent real vector fields $X_1, JX_1, \cdots, X_n, JX_n$ tangent to N. Thus $2n$ complex vectors $X'_k := 1/2(X_k - \sqrt{-1}JX_k)$, and $X''_k := 1/2(X_k + \sqrt{-1}JX_k)$, $k = 1, \cdots, n$, are linearly independent, which implies ii).

ii) ⇒ iii): Suppose that for each $x \in N$, T'_xN is of complex dimension n. Since

$$\begin{aligned} T'N &= \{L \in T'M : \partial\rho_\alpha(L) = 0, \quad \alpha = 1, \cdots, 2d\} \\ &= \bigcap_{\alpha=1}^{2d} (\operatorname{Ker}\partial\rho_\alpha \cap T'M) \\ &= \left(\bigcap_{\alpha=1}^{2d} (\operatorname{Ker}\partial\rho_\alpha)\right) \cap T'M, \end{aligned}$$

has fibre of complex dimension n at each point $x \in N$, we have that $(\partial\rho_1, \cdots, \rho_{2d})$ has rank $m - n = d$ at x.

iii) ⇒ i): Since T'_xM is of complex dimension $m = n + d$, the intersection of the null spaces of $\partial\rho_\alpha : T'_xM \to \mathbb{C}$, $\alpha = 1, \cdots, 2d$, is of complex dimension $m - d = n$, and therefore, contains linearly independent vectors $X'_1, \cdots, X'_n$. Then for each $\alpha = 1, \cdots, 2d$ and each $k = 1, \cdots, n$ we have

$$
\begin{aligned}
0 &= \partial t_\alpha(X'_k) = d\rho_\alpha(X'_k) \\
&= d\rho_\alpha(1/2(X_k - \sqrt{-1}JX_k)) \\
&= 1/2(d\rho_\alpha(X_k) - \sqrt{-1}d\rho_\alpha(JX_k)) .
\end{aligned}
\tag{8.3}
$$

By (8.3) we have $d\rho_\alpha(X_k) = 0$ and $d\rho_\alpha(JX_k) = 0$. This implies that $\{X_k, JX_k : k = 1, \cdots, n\}$ are tangent to N. Since $\{X'_k, k = 1, \cdots, n\}$ are independent the set of vectors $\{X_1, JX_1, \cdots, X_n, JX_n\}$ is a basis for T_xN which is J-invariant . Therefore, N is J-invariant. □

Now let $\theta = (\theta^1, \ldots, \theta^m)$ and $\bar{\theta} = (\bar{\theta}^1, \ldots, \bar{\theta}^m)$ be defined on a neighborhood U of $P \in M$. Let Ω^0 be the ring of smooth complex-valued functions defined on U. For each pair of integers (p, q) where $p, q = 1, \ldots, m$, let $\Omega^{p,q}$ be the module over Ω^0 generated by differential $(p+q)$-forms of type

$$\theta^{\lambda_1} \wedge \cdots \wedge \theta^{\lambda_p} \wedge \bar{\theta}^{\mu_1} \wedge \cdots \wedge \bar{\theta}^{\mu_q} .$$

Then for each integer $k = 1, \ldots, 2m$ the module of smooth k-forms on U has decomposition

$$\Omega^k = \bigoplus_{p+q=k} \Omega^{p,q} .$$

We consider the exterior algebra

$$\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^{2m} .$$

Definition 8.5. *For any elements ω and ϕ_λ, $(\lambda = 1, \ldots)$ of Ω^* we write*

$$\omega \equiv 0, \quad \mathrm{mod}\ (\phi_\lambda, \lambda = 1, \ldots) ,$$

if ω belongs to the algebraic ideal generated by ϕ_λ's. By mod $\bar{\theta}$, *we shall mean* mod $(\bar{\theta}^1, \dots, \bar{\theta}^m)$.

We refer to [BCGGG] and [GJ] for basics on exterior differential systems. Now we set

$$d\bar{\theta}^\lambda \equiv \sum_{\mu<\nu} T^\lambda_{\mu\nu} \theta^\mu \wedge \theta^\nu, \quad \operatorname{mod} \bar{\theta}, \tag{8.4}$$

where $\lambda, \mu, \nu = 1, \dots, m$. The system of complex-valued functions $T^\lambda_{\mu\nu}$ shall be called the torsion of the almost complex structure J. If the torsion is identically zero then J is said to be integrable. The Newlander-Nirenberg theorem states that if J is integrable then there exist smooth complex-valued functions $z_1, \dots, z_m$ such that each θ^i is spanned by usually $(1,0)$-forms $dz_1, \dots, dz_m$, which implies that (M, J) is a complex manifold with coordinates $z_1, \dots, z_m$. In the case that the torsion is not identically zero we shall show that various rank conditions imply the existence of integrable submanifolds. To show the ideas we shall discuss the simplest case of $m = 3$ and $n = 2$ first. (8.4) with $m = 3$ is

$$\begin{aligned} d\bar{\theta}^1 &\equiv T^1_{12}\theta^1 \wedge \theta^2 + T^1_{13}\theta^1 \wedge \theta^3 + T^1_{23}\theta^2 \wedge \theta^3, \quad \operatorname{mod} \bar{\theta} \\ d\bar{\theta}^2 &\equiv T^2_{12}\theta^1 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3 + T^2_{23}\theta^2 \wedge \theta^3, \quad \operatorname{mod} \bar{\theta} \\ d\bar{\theta}^3 &\equiv T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + T^3_{23}\theta^2 \wedge \theta^3, \quad \operatorname{mod} \bar{\theta}. \end{aligned} \tag{8.5}$$

We shall call the 3×3 matrix of $T^\lambda_{\mu\nu}$'s the torsion matrix. Let N be a smooth submanifold of real dimension 4 passing through a point $P \in (M^6, J)$. Assume that N is J-invariant and that (N, J) is integrable. Let Z_1 and Z_2 be independent smooth sections of $T'N$. We apply (Z_1, Z_2) to (8.5). By assumption that (N, J) is integrable, the bracket $[Z_1, Z_2]$ is a section of $T'N$ and therefore, the left side of (8.5) is zero and we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T^1_{12} \\ T^2_{12} \\ T^3_{12} \end{bmatrix} \alpha + \begin{bmatrix} T^1_{13} \\ T^2_{13} \\ T^3_{13} \end{bmatrix} \beta + \begin{bmatrix} T^1_{23} \\ T^2_{23} \\ T^3_{23} \end{bmatrix} \gamma, \tag{8.6}$$

where $\alpha = \theta^1 \wedge \theta^2(Z_1, Z_2)$, $\beta = \theta^1 \wedge \theta^3(Z_1, Z_2)$ and $\gamma = \theta^2 \wedge \theta^3(Z_1, Z_2)$. (8.6) implies that the three columns of the torsion matrix is linearly dependent at every point of N. Thus on N the determinant of the torsion matrix is zero. Let

$$\det \begin{bmatrix} T^1_{12} & T^1_{13} & T^1_{23} \\ T^2_{12} & T^2_{13} & T^2_{23} \\ T^3_{12} & T^3_{13} & T^3_{23} \end{bmatrix} := \tau := s + \sqrt{-1}t. \tag{8.7}$$

Definition 8.6. *Let ϕ be a smooth real-valued function on a neighborhood of $P \in M$. A real-valued smooth function t is called a non-degenerate factor of ϕ if $dt \neq 0$ and $\phi = t\phi'$ for some smooth function ϕ'.*

We assume that s and t in (8.7) satisfy the non-degeneracy condition

$$ds \wedge dt \neq 0\ . \tag{8.8}$$

Otherwise, we use a non-degenerate factor s of the real part and a non-degenerate factor t of the imaginary part of (8.7) that satisfy (8.8).

Theorem 8.7. *Let (M^6, J) be a smooth almost complex manifold. Let $\theta = (\theta^1, \theta^2, \theta^3)$ and $\bar\theta = (\bar\theta^1, \bar\theta^2, \bar\theta^3)$ be independent sections of $(T'M)^*$ and $(T''M)^*$, respectively, on a neighborhood of a point $P \in M$. Suppose that $s + \sqrt{-1}t$ is the determinant of the torsion matrix (8.7) that satisfy the non-degeneracy condition (8.8). Let N be the common zero set of s and t. Then N is a complex submanifold if and only if*

$$\partial s \wedge \partial t \equiv 0, \quad \text{mod } (s, t). \tag{8.9}$$

Proof. Suppose that N is a complex submanifold. Then N is J-invariant . Therefore, by Proposition 8.4

$$\text{rank}\,(\partial s, \partial t)(x) = 1, \quad \forall x \in N,$$

so that (8.9) holds.

Conversely, if (8.9) holds, then by Proposition 8.4 N is J-invariant. Then for the almost complex manifold (N, J) the torsion is zero. Therefore, by the Newlander-Nirenberg theorem (N, J) is a complex manifold. □

As for the general dimensions, suppose that a submanifold N of real dimension $2n$ is J-invariant and that (N, J) is integrable. Let $Z_1, \cdots, Z_n$ be independent smooth sections of $T'N$. For each pair (ij) of $i, j = 1, \cdots, n$ with $i < j$, apply (Z_i, Z_j) to (8.4). By assumption that (N, J) is integrable the bracket $[Z_i, Z_j]$ is a section of $T'N$, therefore, the left side of (8.4) becomes zero. Thus we have

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathcal{T} A, \tag{8.10}$$

where

$$\mathcal{T} = \begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{m-1,m} \\ \vdots & \vdots & \ddots & \vdots \\ T^m_{12} & T^m_{13} & \cdots & T^m_{m-1,m} \end{bmatrix} \tag{8.11}$$

and

(8.12)

$$A = \left[\begin{matrix} \theta^1 \wedge \theta^2(Z_1, Z_2) & \theta^1 \wedge \theta^2(Z_1, Z_3) & \cdots & \theta^1 \wedge \theta^2(Z_{n-1}, Z_n \\ \theta^1 \wedge \theta^3(Z_1, Z_2) & \theta^1 \wedge \theta^3(Z_1, Z_3) & \cdots & \theta^1 \wedge \theta^3(Z_{n-1}, Z_n \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{m-1} \wedge \theta^m(Z - 1, Z_2) & \theta^{m-1}\theta^m(Z_1, Z_3) & \cdots & \theta^{m-1} \wedge \theta^m(Z_{n-1}, \end{matrix} \right.$$

For any positive integer k let $\binom{k}{2} = \frac{1}{2}k(k-1)$. Then $\mathcal{T}A$, where $\mathcal{T}$ is the torsion matrix of size $m \times \binom{m}{2}$ and A is a matrix of size $\binom{m}{2} \times \binom{n}{2}$ whose entries are smooth complex-valued functions that depend on N and choice of $(Z_1, \cdots, Z_n)$. We observe that A is of maximal rank $\binom{n}{2}$, in fact, by a complex linear change of frame $Z_1, \cdots, Z_n$ the first $\binom{n}{2}$ rows of $A(P)$ can be made the identity matrix and the other rows all to be zeros. Therefore, (8.10) gives $\binom{n}{2}$ independent linear equations on m columns of $\mathcal{T}$, so that the number of linearly independent columns in $\mathcal{T}$ is less than or equal to $\binom{m}{2} - \binom{n}{2}$, which is equivalent to that all the square matrix of $\mathcal{T}$ of size $\binom{m}{2} - \binom{n}{2} + 1$ is of determinant zero. Thus for the general dimensions we have

Theorem 8.8. *Let (M^{2m}, J), $m \geq 2$, be a smooth almost complex manifold. Let $\theta = (\theta^1, \cdots, \theta^m)$ and $\bar{\theta} = (\bar{\theta}^1, \cdots, \bar{\theta}^m)$ be independent sections of $T'M$ and $T''M$, respectively. Let $\mathcal{T}$ be the associated torsion matrix (8.11). Let n be an integer with $2 \leq n \leq m - 1$. Let $\tau_1, \cdots, \tau_\ell$ be the determinant of all the square matrices of size $\binom{m}{2} - \binom{n}{2} + 1$. Let N^{2n} be a submanifold of M defined as common zero set of real-valued functions $\rho_1, \cdots, \rho_{2d}$, $d = m - n$, that satisfy $d\rho_1 \wedge \cdots \wedge d\rho_{2d} \neq 0$. Then N is a complex submanifold if and only if*

$$\tau_k \equiv 0, \quad \text{mod } (\rho_1, \cdots, \rho_{2d}) \tag{8.13}$$

and

$$\text{rank}\, (\partial\rho_1, \cdots, \partial\rho_{2d})(x) = d, \quad \forall x \in N. \tag{8.14}$$

Proof. Following the same argument as in the proof of Theorem 8.7, we see that (8.14) implies that N is J-invariant and that (8.13) implies that on N the torsion vanishes, so that the conclusion follows from the Newlander-Nirenberg theorem. □

As in Theorem 8.7 $\rho_1, \cdots, \rho_{2d}$ may be chosen from the real part and imaginary part of $\tau_1, \cdots, \tau_\ell$ or their non-degenerate factors. The condition (8.14) or (8.9) in the case $m = 3$ is a system of second order PDE's on the almost complex structure tensor J. To see this, consider an almost complex manifold (M^6, J) that is a small perturbation of the standard complex structure J_{st} of $\mathbb{C}^3$ by setting

$$\bar{\theta}^{\alpha} = d\bar{z}_{\alpha} + \sum_{\beta=1}^{3} A_{\alpha}^{\beta}(z, \bar{z})dz_{\beta}, \quad \alpha = 1, 2, 3, \tag{8.15}$$

where $A_{\alpha}^{\beta}(0) = 0$. Then the entries of the torsion matrix are rational functions in A_{α}^{β} and their derivatives. Since (8.9) involves the derivatives of the elements of the torsion matrix, the condition (8.9) is a system of second order PDE's on A_{α}^{β}'s.

§9 Equivalence problem of G-structures.

We shall discuss in this section some historical background, or the author's motivation, of studying the complete prolongation

Let M be a C^∞ manifold of dimension n and G be a linear subgroup of $GL(n;\mathbb{R})$. A G-structure on M is reduction of coframe bundle of M to a subbundle with the structure group G. For instance, a Riemannian structure on M is a $SO(n)$-structure and the subbundle in this case is the orthonormal coframe bundle of M.

Now let M and $\tilde{M}$ be manifolds of dimension n with G-structure. The equivalence problem is deciding whether there exists a structure preserving mapping $f : M \to \tilde{M}$. Locally, this is a question of existence of solutions for an overdetermined system of first order partial differential equations in cases where G is a sufficiently small group.

E.Cartan's method to this problem is as follows: We fix coframes $\theta = (\theta^1, \cdots, \theta^n)^t$ of M and $\tilde{\theta} = (\tilde{\theta}^1, \cdots, \tilde{\theta}^n)^t$ of $\tilde{M}$ adapted to the G-structure, where θ and $\tilde{\theta}$ are defined over an open set U of M and an open set $\tilde{U}$ of $\tilde{M}$, respectively. Then the question is whether there exists a mapping $f : M \to \tilde{M}$ that satisfies

$$f^*\tilde{\theta}^\alpha = a^\alpha_\beta\, \theta^\beta, \tag{9.1}$$

where $a := [a^\alpha_\beta(x)]_{n\times n}$ is a G-valued function of M. In terms of local coordinates, (9.1) is a system of first order partial differential equations for $f = (f^1, \cdots, f^n)$ and system of algebraic equations for $a^\alpha_\beta(x)$. Thus we consider the product $U \times G$ and the tautological 1-form Θ, which is a vector valued 1-form defined by $\Theta = g\theta$ on $U \times G$, namely

$$\Theta_{(x,g)} = g\theta_x, \ \forall x \in U, \ \ \forall g \in G, \tag{9.2}$$

where θ_x is a column vector $(\theta_x^1, \cdots, \theta_x^n)^t$. G acts on $U \times G$ on the left by the action defined by

$$h(x, g) = (x, hg), \quad \forall x \in U, \quad \forall g, h \in G.$$

Proposition 9.1. *A diffeomorphism $f : U \to \tilde{U}$ satisfies (9.1) if and only if there exists a diffeomorphism $F : U \times G \to \tilde{U} \times G$ satisfying*

i) $F^*\tilde{\Theta} = \Theta$

ii) the following diagram commutes:

$$\begin{array}{ccc} U \times G & \xrightarrow{F} & \tilde{U} \times G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ U & \xrightarrow{f} & \tilde{U} \end{array}$$

iii) $F(x, gh) = gF(x, h)$, *for each* $x \in U$, *and* $g, h \in G$.

Proof. Suppose that f satisfies $f^*\tilde{\theta} = g_0\theta$, where g_0 is a G-valued function on M. Define $F : U \times G \to \tilde{U} \times G$ by $F(x, g) = (f(x), gg_0^{-1}(x))$. Then F satisfies ii) and iii). Moreover,

$$F^*\tilde{\Theta} = F^*(\tilde{g}\tilde{\theta}) = (g{g_0}^{-1})f^*\tilde{\theta} = (g{g_0}^{-1})g_0\theta = g\theta = \Theta.$$

Conversely, suppose that $F : U \times G \to \tilde{U} \times G$ satisfies i) - iii). Define $f : U \to \tilde{U}$ and $g_0 : U \to G$ by $F(x, e) = (f(x), g_0(x)^{-1})$, where e is the identity of G. Then $F(x, g) = gF(x, e) = (f(x), gg_0^{-1})$, and i) implies that

$$g\theta = F^*(\tilde{g}\tilde{\theta}) = (g{g_0}^{-1})f^*\tilde{\theta}$$

therefore, $f^*\tilde{\theta} = g_0\theta$. □

Now apply d to (9.2). We get

$$d\Theta = dg \wedge \theta + gd\theta;$$

substituting $\theta = g^{-1}\Theta$, we obtain

$$d\Theta = dgg^{-1} \wedge \Theta + gd\theta. \tag{9.3}$$

Now let $d\theta^i = \sum_{j,k=1}^n b^i_{jk}\theta^j \wedge \theta^k$, $(b^i_{jk} = -b^i_{kj})$. We want to find 1-forms ω^i_j, $i, j = 1, \cdots, n$, such that

$$d\theta^i = -\omega^i_j \wedge \theta^j \tag{9.4}$$

and

$$[\omega^i_j(x)] \in \mathcal{G}, \quad \text{for each } x \in U,$$

where $\mathcal{G}$ is the Lie algebra of G. This Lie algebra valued 1-form $\omega = [\omega^i_j]$ is called a torsion-free connection (see [Chern]). If G is too big, such $[\omega^i_j]$ is not unique. If G is too small such $[\omega^i_j]$ does not exist. For example, if $G = O(n, \mathbb{R})$, then there exists a unique torsion free connection $[\omega^i_j]$. We assume the unique existence of torsion free connection ω for the G-structure.

Substitute $d\theta = -\omega \wedge \theta$ and $\theta = g^{-1}\Theta$ in (9.3), to get

$$d\Theta = dgg^{-1} \wedge \Theta - g\omega \wedge g^{-1}\Theta = (dgg^{-1} - g\omega g^{-1}) \wedge \Theta.$$

Let

$$\Omega = -(dgg^{-1} - g\omega g^{-1}), \tag{9.5}$$

then Ω is a $\mathcal{G}$-valued 1-form on $U \times G$ and we have

$$d\Theta = -\Omega \wedge \Theta. \tag{9.6}$$

Now it is easy to show

PROPOSITION 9.2. *Let Θ^i and Ω^i_j, $i, j = 1, \cdots, n$, be the 1-forms defined by (9.2) and (9.5) on $U \times G$. Then Θ^i, Ω^i_j spans the cotangent space at each point of $U \times G$. Furthermore, if $\tilde{\Theta}^i, \tilde{\Omega}^i_j$ are the corresponding 1-forms on $\tilde{U} \times G$ and*

$$F : U \times G \to \tilde{U} \times G$$

is the mapping as in Proposition 1.1, then

$$F^* \tilde{\Omega}^i_j = \Omega^i_j. \tag{9.7}$$

The set $\{\Theta^i, \Omega^i_j\}$ is called a complete set of invariants for the equivalence problem. Ω is called a torsion-free connection form on $U \times G$. Note that ω is a 1-form on the base manifold U and that the restriction of Ω on each fibre is the Maurer-Cartan form of G.

Now let G be a Lie-subgroup of $GL(n; \mathbb{R})$. Suppose that a manifold E of dimension n has a G-structure and $\pi : Y \to E$ is the associated principal bundle. The equivalence problem is finding canonically a system of differential 1-forms

$$\omega^1, \cdots, \omega^N, \quad \text{where} \quad N = n + \text{dim}G, \tag{9.8}$$

so that a mapping $f : E \to \tilde{E}$ preserves the G-structure if and only if there exists a mapping $F; Y \to \tilde{Y}$, which is a lift of f,that is, $\tilde{\pi} \circ F = f \circ \pi$, and such that

$$F^* \tilde{\omega}^i = \omega^i, \quad i = 1, \cdots N, \tag{9.9}$$

where $\tilde{E}$ is a manifold of dimension n with a G-structure and $\tilde{\pi} : \tilde{Y} \to \tilde{E}$ is the associated principal bundle and $\tilde{\omega}^i$ are the corresponding 1-forms on $\tilde{Y}$. (9.8) is called a complete system of invariants of the G-structure and (9.9) is a complete system of order 1 for F. It turns out that (9.9) is equivalent to a complete system of order 2 for f . In the following we present a direct construction of a complete system for Riemannian isometries.

Example 9.3. Let (M, g) and $(\tilde{M}, \tilde{g})$ be smooth Riemannian manifolds with Riemannian metric g and $\tilde{g}$, respectively. A C^1 map $u : M \to \tilde{M}$ is an isometry if

$$u^*\tilde{g} = g. \tag{9.10}$$

In term of local coordinates (9.10) can be written as

$$u_i^\alpha u_j^\beta \tilde{g}_{\alpha\beta}(u) = g_{ij}(x), \quad \text{(summation convention)}, \tag{9.11}$$

for each $i, j = 1, \ldots, n$. By applying ∂_k to (9.11) we have

$$(u_{ik}^\alpha u_j^\beta + u_i^\alpha u_{jk}^\beta)\tilde{g}_{\alpha\beta}(u) + u_i^\alpha u_j^\beta \frac{\partial \tilde{g}_{\alpha\beta}}{\partial u_\gamma}(u) u_k^\gamma = \frac{\partial g_{ij}}{\partial x_k}(x), \tag{9.12}$$

for each $i, j, k = 1, \ldots, n$. We may assume that $u(0) = 0, g_{ij}(0) = \delta_{ij}, \tilde{g}_{\alpha\beta}(0) = \delta_{\alpha\beta}$, and $u_j^\alpha(0) = \delta_j^\alpha$. Then at the reference point 0 (9.12) is

$$u_{jk}^i + u_{ik}^j = -\tilde{g}_{ij,k}(0) + g_{ij,k}(0). \tag{9.13}$$

By permuting the indices $\{i, j, k\}$ in (2.12) we get

$$u_{ki}^j + u_{ji}^k = -\tilde{g}_{jk,i}(0) + g_{jk,i}(0) \tag{9.14}$$

and

$$u_{ij}^k + u_{kj}^i = -\tilde{g}_{ki,j}(0) + g_{ki,j}(0). \tag{9.15}$$

Then (9.13) + (9.15) – (9.14) yields

$$2u_{jk}^i = -\tilde{g}_{ij,k}(0) + \tilde{g}_{jk,i}(0) - \tilde{g}_{ki,j}(0) + g_{ij,k}(0) - g_{jk,i}(0) + g_{ki,j}(0).$$

Therefore, on a neighborhood of $(0, u(0), u_i(0))$ in the space of first jets of u we have

$$u_{jk}^i = H_{jk}^i(x, u^{(1)}), \tag{9.16}$$

which is a complete system of order 2.

§10. Infinitesimal automorphisms and mappings of G-structures.

Let $U \subset \mathbb{R}^n$ be an open set. By a frame over U we shall mean a set of C^∞1-forms $e = (e_1, \dots, e_n)$ which are linearly independent at every point of U. In this section we work with vector fields rather than 1-forms. We adopt definitions and notations of [Kob] and use the summation convention. Arguments in this section , especially Lemma 10.3, can be better presented in higher order jet or infinite jet, however, we present as it is in [H2]. Let Let G be a Lie subgroup of $GL(n, \mathbb{R})$ and $F(U)$ be the frame bundle. A G structure is a subbundle P of $F(U)$ with the structure group G. Let $\tilde{P}$ be a G-structure on $\tilde{U} \subset \mathbb{R}^n$. A C^1 map $f : U \to \tilde{U}$ is called a G-mapping if for any frame e belonging to P f_*e is a frame belonging to $\tilde{P}$. If we fix frames e and $\tilde{e}$ belonging to P and $\tilde{P}$, respectively, this condition can be written as

$$f_* e_j = \tilde{a}^i_j \tilde{e}_i, \tag{10.1}$$

where $[\tilde{a}^i_j]$ is a G-valued function on $\tilde{U}$. (10.1) is a dual expression of (9.1). A vector field $X = u^i e_i$ is called an infinitesimal automorphism of the G-structure P if its local flow map $\exp(tX)$ is a 1-parameter group of G-self mappings. Set

$$L_X e_j = a^i_j e_i,$$

where L is the Lie derivative. Then X is an infinitesimal G-automorphism if and only if the $n \times n$ matrix valued function $[a^i_j]$ is $\mathcal{G}$-valued, where $\mathcal{G}$ is the Lie algebra of G (cf. [Kob]). Let

$$[e_i, e_j] = b^k_{ij} e_k.$$

Then

$$\begin{aligned} L_X e_j &= [u^i e_i, e_j] \\ &= (-e_j u^i + u^k b^i_{kj}) e_i, \end{aligned}$$

so we have

$$a_j^i = -e_j u^i + u^k b_{kj}^i.$$

Since $\mathcal{G}$ is a linear subspace of $gl(n;\mathbb{R})$, it is defined by linear equations, namely,

$$\mathcal{G} = \{y_j^i \in \mathbb{R}^{n^2} : c_{i\lambda}^j y_j^i = 0, \quad \lambda = a, \dots , N\},$$

where N is the codimension of $\mathcal{G}$ in $gl(n;\mathbb{R})$ and c are constants. Thus we get

$$c_{i\lambda}^j(-e_j u^i + u^k b_{kj}^i) = 0, \quad \lambda = 1, \dots , N.$$

To express the above in terms of local coordinates, let $X = \xi^i(\partial/\partial x_i)$ and let $(\partial/\partial x_j) = b_j^i e_i$, then $u^i = b_j^i \xi^j$ and the above equation becomes

$$c_{i\lambda}^j[-e_j(b_k^i \xi^k) + b_t^k b_{kj}^i \xi^t] = 0, \quad \lambda = 1, \dots , N. \tag{10.2}$$

(10.2) is a system of linear PDE of first order for $\xi = (\xi^1, \dots , \xi^n)$ with C^∞coefficients. We have

10.1 Theorem. *Let G be a Lie subgroup of $GL(n;\mathbb{R})$. Suppose that U and $\tilde{U}$ are open neighborhoods of the origin of $\mathbb{R}^n$ with G-sturctures P and $\tilde{P}$, respectively. Let $f : U \to \tilde{U}$ be a G-mapping of class C^k for some sufficiently large k. Suppose that the equation (10.2) for the infinitesimal automorphisms of P admits prolongation to a complete system of order m and that $\tilde{P}$ has the same property. Then f satisfies a complete system of order $m + 1$.*

10.2 Definition (*G*-structure of Frobenius type). *A G-structure P on a C^∞manifold M is of Frobenius type of order m if the equation (10.2) for the infinitesimal automorphisms of P admits prolongation to a complete system of order m.*

Now let P be a G-structure of Frobenius type of order m on an open set $U \subset \mathbb{R}^n$. For all α with $|\alpha| = m$, and for all $i = 1, \dots, n$, let

$$\partial^\alpha \xi^i = H^i_\alpha(x, \partial^\beta \xi : |\beta| < m) \tag{10.3}$$

be prolongation of (10.2) to a complete system. Note that each H^i_α is linear in $\partial^\beta \xi$. We introduce new variables

$$p_\beta = (p^1_\beta, \dots, p^n_\beta) \text{ for } \partial^\beta \xi = (\partial^\beta \xi^1, \dots, \partial^\beta \xi^n).$$

For each positive integer k, the kth order jet space is

$$J^k := U \times \mathbb{R}^{(k)} = \{(x, \xi, p)\},$$

where $p = (p_\beta : |\beta| \leq k)$ and (k) is the number of the variables (ξ, p). In J^k consider the submanifold $\triangle^k$ which is defined by (10.2) and all the equations obtained by differentiating (10.2) in all possible ways up to order $k-1$. Let $X = \xi^i(\partial/\partial x_i)$ be a C^k vector field. Then the k-th jet graph of X is the submanifold of J^k

$$j^k_X(x) := (x, \xi(x), \partial^\beta \xi(x) : |\beta| \leq k)$$

and kth order contact system is

$$\Omega^k := \{\omega \in T^*(J^k) : (j^k_X)^* \omega = 0, \quad \forall \text{vector fields } X \text{ on } U\}.$$

Then Ω^k is spanned by

$$\begin{aligned} \omega^k &:= d\xi^k - p^i_j dx^j \quad \text{and} \\ \omega^i_\beta &:= dp^i_\beta - p^i_{(\beta,j)} dx^j, \end{aligned}$$

where (β, j) denotes the multi-index $(\beta_1, \dots, \beta_j + 1, \dots, \beta_n)$ if $\beta = (\beta_1, \dots, \beta_n)$. We have

10.3 Lemma. *Let P be a G-structure on U and let J^k, $\triangle^k$ and Ω^k be as above. Then the following are equivalent:*
(i) P is of Frobenius type of order m;
(ii) the $(m-1)$th contact system Ω^{m-1} defines an n-dimensional distribution $\mathcal{D}$ on $\triangle^{m-1}$ such that $dx^1 \wedge \cdots \wedge dx^n \neq 0$ on each integral element of $\mathcal{D}$.

Proof of Lemma 10.3. $(i) \Longrightarrow (ii)$: Let (10.3) be prolongation of (10.2) to a complete system. (10.3) is equivalent to the total differential equation

$$d(\partial^\beta \xi^i) = H^i_{(\beta,j)}(x, \partial^\gamma \xi : |\gamma| \leq m-1)dx^j, \quad \forall \beta \text{ with } |\beta| = m-1,$$

for all $i = 1, \ldots, n$. This implies that on $\triangle^{m-1}$

$$\Omega^i_\beta := dp^i_\beta - H^i_{(\beta,j)}(x, \xi, p_\gamma)dx^j = 0,$$

for all β with $|\beta| = m-1$, and for all $i = 1, \ldots, n$. Let $\mathcal{D}$ be the distribution $\triangle^{m-1}$ given by $\Omega^{(m-1)} = 0$. Then on each integral element of $\mathcal{D}$ we have

$$\begin{aligned} d\xi^i &= p^i_j dx^j, \\ dp^i_\beta &= p^i_{(\beta,j)}dx^j, \quad \forall \beta \text{ with } |\beta| < m-1, \\ dp^i_\beta &= H^i_{(\beta,j)}(x, \xi, p)dx^j, \quad \forall \beta \text{ with } |\beta| = m-1, \\ dx^1 &\wedge \cdots \wedge dx^n \neq 0. \end{aligned}$$

Therefore, $\mathcal{D}$ is an n-dimensional distribution.

$(ii) \Longrightarrow (i)$: Let $\mathcal{D}$ be the distribution as in (ii). Let $\phi^1, \ldots, \phi^\nu$ be differential 1-form on $\triangle^{m-1}$ which generate the differential ideal associated with $\mathcal{D}$, where

$$\nu = (\text{dimension of } \triangle^{m-1}) - n.$$

Set

$$\phi^j = a^j dx + b^j d\xi + c^j dp, \quad j = 1, \ldots, \nu,$$

where a^j, b^j and c^j are row vectors and dx, $d\xi$ and dp are column vectors so that $a^j dx = a^j_1 dx^1 + \cdots + a^j_n dx^n$, and so forth. Since each integral element of $\mathcal{D}$ is n-dimensional subspace of $T(\triangle^{m-1})$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$, we can solve $\phi^j = 0$, $j = 1, \dots, \nu$, for $d\xi$ and dp we get

$$\begin{cases} d\xi^i = h^i_j dx^j \\ dp^i_\beta = h^i_{(\beta,j)} dx^j, \quad \forall \beta \text{ with } |\beta| \leq m-1, \forall i = 1, \dots, n, \end{cases}$$

where h are C^∞functions on $\triangle^{m-1}$. This implies that if $\xi = (\xi^1, \dots, \xi^n)$ satisfies (10.2) then

$$d(\partial^\beta \xi^i) - h^i_{(\beta,j)}(x, \partial^\gamma \xi : |\gamma| \leq m-1) dx^j = 0,$$

for all β with $|\beta| \leq m-1$. In particular, if $|\beta| = m-1$, the above equation is equivalent to

$$\partial^{(\beta,j)} \xi^i = h^i_{(\beta,j)}(x, \partial^\gamma \xi : |\gamma| \leq m-1), \quad |\beta| = m-1,$$

which is a complete system of order m. □

Proof of Theorem 10.1. Let (10.3) be the complete system for the infinitesimal automorphism of P and let Ω^k be the kth contact system on $\triangle^k \subset J^k$ for each $k = 1, \dots, m-1$. For each multi-index β with $|\beta| = m-1$ and each $i = 1, \dots, n$, let $\Omega^i_\beta := dp^i_\beta - H^i_{(\beta,j)} dx^j$, where H are the same as in the complete system (10.3). Let $\mathcal{D}$ be the distribution as in Lemma 10.3. We will put tilde on the corresponding notions on $\tilde{U} : \tilde{J}^{m-1} := \tilde{U} \times \mathbb{R}^{(m-1)} = (\tilde{x}, \tilde{\xi}, \tilde{p})$, and so forth. A C^{m+1} diffeomorphism $f : U \to \tilde{U}$ naturally defines a C^1 diffeomorphism $F^k : J^k \to \tilde{J}^k$ for each $k = 1, \dots, m-1$ as follows: Let $F^k(x, \xi, p) = (\tilde{x}, \tilde{\xi}, \tilde{p})$. Then

$$\tilde{x}^i(x, \xi, p) = f^i(x) \tag{10.4}$$

$$\tilde{\xi}^i(x,\xi,p) = \xi^\lambda \frac{\partial f^i}{\partial x^\lambda}, \quad i = 1,\dots,n \tag{10.5}$$

and define $\tilde{p}(x,\xi,p)$ by chain rule, namely

$$\begin{aligned}\tilde{p}^i_j(x,\xi,p) &= \frac{\partial \tilde{\xi}^i}{\partial \tilde{x}^k} \\ &= \frac{\partial \tilde{\xi}^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tilde{x}^j}\end{aligned} \tag{10.6}$$

substitute (10.5) for ξ^i and p^λ_μ for $\frac{\partial \xi^\lambda}{\partial x^\mu}$ the right hand side of (10.6) becomes

$$= \left(p^\lambda_\mu \frac{\partial f^i}{\partial x^\lambda} + \xi^\lambda \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda}\right) \frac{\partial x^\mu}{\partial \tilde{x}^i}$$

each $\frac{\partial x^\mu}{\partial \tilde{x}^j}$ is an entry of $\left[\frac{\partial f^i}{\partial x^j}\right]^{-1}_{i,j=1,\dots,n}$, therefore a C^∞function of $\frac{\partial f^i}{\partial x^j}$, $i,j = 1,\dots,n$, so

$$= \xi^\lambda \frac{\partial^2 f^i}{\partial x^\mu \partial x^\lambda} \frac{\partial x^\mu}{\partial \tilde{x}^j} + a^\mu_\lambda p^\lambda_\mu,$$

where a^μ_λ are C^∞functions in $(\partial^\gamma f : |\gamma| \le 1)$. Now let $\beta = (\beta_1,\dots,\beta_n)$ be a multi-index and $(j_1,\dots,j_{|\beta|})$ denotes the sequence $\underbrace{1,\dots,1}_{\beta_1 times},\underbrace{2,\dots,2}_{\beta_2 times},\underbrace{n,\dots,n}_{\beta_3 times})$. Then by induction on $|\beta|$ we obtain

$$\begin{aligned}&\tilde{p}^i_\beta(x,\xi,p) \\ &= \xi^\lambda \left[\frac{\partial^{|\beta|+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \cdots \partial x^{\lambda_{|\beta|}}} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda_{|\beta|}}}{\partial \tilde{x}^{j_{|\beta|}}} + a_{\beta,\lambda}\right] \\ &+ a^\gamma_{\beta,\lambda} p^\lambda_\gamma, \quad |\gamma| \le |\beta|,\end{aligned} \tag{10.7}$$

where a are C^∞functions in $(\partial^\gamma f : |\gamma| \le |\beta|)$. Then we claim

(1) $F^{m-1}(\triangle^{m-1}) = \tilde{\triangle}^{m-1}$ and

(2) $F_*^{m-1}(\mathcal{D}) = \tilde{\mathcal{D}}$.

Proof of the claim

(1) A C^1 vector field $X = \xi^i(\partial/\partial x^i)$ is an infinitesimal automorphism of P if and only if f_*X is an infinitesimal automorphism of $\tilde{P}$. This implies that $F^1(\triangle^1) = \tilde{\triangle}^1$. Then it is clear that $F^k(\triangle^k) = \tilde{\triangle}^k$ for $k = 2, \dots, m-1$.

(2) For each $k = 1, \dots, m-1$, we have $(F^k)^*(\tilde{\Omega}^k) = \Omega^k$ which is immediate from the definition of the contact system. In particular $(F^{m-1})^*(\tilde{\Omega}^{m-1}) = \Omega^{m-1}$. Thus we have

$$\begin{aligned} v \in \mathcal{D} &\iff v \in T(\triangle^{m-1}) \text{ and } v \text{ annihilates } \Omega^{m-1} \\ &\iff F_*v \in T(\tilde{\triangle}^{m-1}) \text{ and } F_*v \text{ annihilates } \tilde{\Omega}^{m-1} \\ &\iff F_*^{m-1}v \in \tilde{\mathcal{D}}. \end{aligned}$$

□

Now we compute $F^*\tilde{\Omega}^i_\beta$, $|\beta| = m-1$:

(10.8)

$$(F^{m-1})^*\tilde{\Omega}^i_\beta = (F^{m-1})^*(d\tilde{p}^i - \tilde{H}^i_{(\beta,j)}(\tilde{x}, \tilde{\xi}, \tilde{p})d\tilde{x}^j)$$

substitute (10.4)-(10.7) for $\tilde{x}, \tilde{\xi}$, and $\tilde{p}$, respectively,

$$= \left[\frac{\partial^m f^i}{\partial x^\lambda \partial x^{\lambda_1} \cdots \partial x^{\lambda_{m-1}}} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda_{m-1}}}{\partial \tilde{x}^{j_{m-1}}} + a_{\beta,\lambda}\right] d\xi^\lambda$$

$$+ a^\gamma_{\beta,\lambda} dp^\lambda_\gamma, \quad |\gamma| \leq m-1$$

$$+ \left[\xi^\lambda \frac{\partial^{m+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \cdots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda-1}}{\partial \tilde{x}^{j_{m-1}}} + b^i_{\beta,k}\right] dx^k$$

where a are C^∞functions of $(\partial^\gamma f : |\gamma| \leq m-1)$ and b are C^∞functions of $(x, \xi, p, \partial^\gamma f : |\gamma| \leq m)$.

By the proof of Lemma 10.3, $\tilde{\mathcal{D}}$ on $\tilde{\triangle}^{m-1}$ is given by

$$\begin{aligned} \tilde{\Omega}^{m-1} &= 0 \\ \tilde{\Omega}^i_\beta &= 0, \forall i = 1, \dots, n, \forall \beta \text{ with } |\beta| = m-1. \end{aligned}$$

Recall that $\tilde{\Omega}^{m-1}$ is the contact system and $\tilde{\Omega}^i_\beta$ are 1-forms defined by the complete system. Since $F^{m-1}_*(\mathcal{D}) = \tilde{\mathcal{D}}$, $(F^{m-1})^*\tilde{\Omega}^i_\beta$ is a linear combination of $\{\omega^i, \omega^i_\gamma, \Omega^i_\delta : i = 1, \dots, n, |\gamma| < m-1, |\delta| = m-1\}$ where ω are contact forms. So we set

(10.9)

$$(F^{m-1})^*\tilde{\Omega}^i_\beta = c^i_{\beta,\lambda}\omega^\lambda + c^{i,\gamma}_{\beta,\lambda}\omega^\lambda_\gamma + c^{i,\delta}_{\beta,\lambda}\Omega^\lambda_\delta$$
$$= c^i_{\beta,\lambda}(d\xi^\lambda - p^\lambda_k dx^k) + c^{i,\gamma}_{\beta,\lambda}(dp^\lambda_\gamma - p^\lambda_{(\gamma,k)}dx^k) + c^{i,\delta}_{\beta,\lambda}(dp^\lambda_\delta - H^\lambda_{(\delta,k)}dx^k),$$

where c are C^1 functions on Δ^{m-1}, $|\gamma| \le m-2$ and $|\delta| = m-1$.

By equating the components of $d\xi$ and dp in (10.8) and (10.9) we obtain c's as C^∞functions in $(x, \xi, p, \partial^\gamma f : |\gamma| \le m)$ for $(x, \xi, p) \in \Delta^{m-1}$. Substitute this in (10.9) and equate the components of dx^k in (10.8) and (10.9) to obtain

$$\xi^\lambda \frac{\partial^{m+1} f^i}{\partial x^\lambda \partial x^{\lambda_1} \cdots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda-1}}{\partial \tilde{x}^{j_{m-1}}} = C^\infty \text{ function in } (x, \xi, p, \partial^\gamma f : |\gamma| \le m), \tag{10.10}$$

where $(x, \xi, p) \in \Delta^{m-1}$. Since (10.2) is a system of linear partial differential equations of first order obtained from the structure equations of the Lie algebra $\mathcal{G}$, we see that $dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \wedge \cdots \wedge d\xi^n \neq 0$ on $\Delta^1 \subset J^1$ and therefore on $\Delta^{m-1} \subset J^{m-1}$. Thus there exits a C^∞function $p(x, \xi)$ such that $(x, \xi, p(x, \xi)) \in \Delta^{m-1}$, $\forall (x, \xi)$. For each $j = 1, \dots, n$, the restriction of (10.10) to the submanifold $\{(x, \xi, p(x, \xi)) : \xi = (0, \dots, 0, \underbrace{1}_{j^{th}} 0. \dots, 0)\}$ is

$$\frac{\partial^{m+1} f^i}{\partial x^j \partial x^{\lambda_1} \cdots \partial x^{\lambda_{m-1}} \partial x^k} \frac{\partial x^{\lambda_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{\lambda-1}}{\partial \tilde{x}^{j_{m-1}}} = C^\infty \text{ function of } (x, \partial^\gamma f : |\gamma| \le m). \tag{10.11}$$

Here $i, j, j_1, \dots, j_{m-1}$ and k are arbitrary. Since the matrix

$$\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right)_{i,j=1,\dots,n}$$

is nonsingular and each $\partial x^i/\partial \tilde{x}^j$ is a C^∞function in (∂f), from (10.11) we have

$$\frac{\partial^{m+1} f^i}{\partial x^j \partial x^{\lambda_1} \cdots \partial x^{\lambda_{m-1}} \partial x^k} = C^\infty \text{ function in } (x, \partial^\gamma f : |\gamma| \leq m).$$

□

Now we will show that every G-structure of finite order is of Frobenius type. We recall the definitions first. Let G be a Lie subgroup of $GL(n, \mathbb{R})$ and $\mathcal{G}$ be the associated Lie algebra. The kth prolongation $\mathcal{G}^{(k)}$ of $\mathcal{G}$ is the space of symmetric multi-linear mappings

$$t : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{(k+1) times} \to \mathbb{R}^n$$

such that, for each fixed $v_1, \ldots, v_k \in \mathbb{R}^n$, the linear transformation

$$v \in \mathbb{R}^n \mapsto t(v, v_1, \ldots, v_k) \in \mathbb{R}^n \text{ belongs to } \mathcal{G}.$$

G is said to be of finite order k if $\mathcal{G}^{(k)} = 0$ and $\mathcal{G}^{(k-1)} \neq 0$. Riemannian structures and conformal structures (when dimension ≥ 3) are of finite order 1 and 2, respectively (cf. [Kob],[Stern]).

10.4 Theorem. *Let P be a G-structure on $U \subset \mathbb{R}^n$. If G is of finite order $m-1$ $(m \geq 1)$, then P is of Frobenius type of order m.*

Proof. Since $\mathcal{G}$ is a linear subspace of $gl(n, \mathbb{R})$, it is defined by $\mathcal{G} = \{(y^i_j) \in gl(n, \mathbb{R}) : \sum_{i,j=1}^n c^j_{i\lambda} y^i_j = 0, \lambda = 1, \ldots, N\}$, where the $c^j_{i\lambda}$ are constants and N is the codimension of $\mathcal{G}$ in $gl(n,\mathbb{R})$. Therefore, as a linear space, $\mathcal{G}^{(m-1)}$ is isomorphic to the subspace of

$$\mathbb{R}^{n^{m+1}} = \left(y^i_{j_1\cdots j_m}\right), \quad \text{each } i,j \in \{1,\dots,n\},$$

which is defined by the following system of linear equations:

$$(10.12) \qquad \begin{aligned} &\sum_{i,j=1}^{n} c^{j_1}_{i\lambda} y^i_{j_1\cdots j_m} = 0, \quad \lambda = 1,\dots,N \\ &\text{and the symmetry conditions on the subscripts} \\ &y^i_{j_1j_2\cdots j_m} - y^i_{j_2j_1\cdots j_m} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &y^i_{j_1\cdots j_{m-1}j_m} - y^i_{j_1\cdots j_mj_{m-1}} = 0 \end{aligned}$$

Since the only solution of (10.12) is $y = 0$, there exists n^{m+1} independent equations in (10.12). Let

$$(10.13) \qquad g^1(y) = 0, \dots, g^{n^{m+1}}(y) = 0.$$

Now we fix a frame $(e_1,\dots,e_n)$ belonging to P. Let $X = \sum_{i=1}^n x^i e_i$ be an infinitesimal automorphism of P. Define ξ^i_j by $[e_j, X] = \sum_{i=1}^n \xi^i_j e_i$. Then the matrix $[\xi^i_j]$ belong to $\mathcal{G}$. For any sequence $(j_2,\dots,j_k)$, each $j \in \{1.\dots,n\}$, we denote by $\xi^i_{jj_2\cdots j_k}$ the derivative $(e_{j_k}\cdots e_{j_2})(\xi^i_j)$. Then in the Jacobi identity

$$[e_k,[e_j,X]] - [e_j,[e_k,X]] = [[e_k,e_j],X]$$

substitute $\sum_{i=1}^n \xi^i_j e_i$ and $\sum_{i=1}^n \xi^i_k e_i$ for $[e_j,X]$ and $[e_k,X]$, respectively, we obtain $\xi^i_{jk} - \xi^i_{kj} = \langle \xi^\lambda, \xi^\lambda_\mu \rangle$, where $\langle , \rangle$ denotes a linear combination of the variables inside with C^∞coefficients. By induction on the number of the subscripts we see that a transposition for any two subscripts in $\xi^i_{j_1\cdots j_k}$ makes a difference by a linear combination of $\{\xi^\lambda_J : |J| < k, \lambda = 1, \ldots, n\}$, where $J = (j_1 j_2 \cdots)$ is a seqence of subscripts and $|J|$ is the size of J. Moreover, since $\mathcal{G}$ is a linear space, for each fixed $j_2 \cdots j_k$, the matrix of the derivatives

$$\left[\xi^i_{j_1 j_2 \cdots j_k}\right]_{i,j_1=1,\ldots,n} \qquad \text{belongs to } \mathcal{G}.$$

Now at each point $x \in M$, consider

$$\sum_{i,j_1=1}^{n} c^{j_1}_{i\lambda} \xi^i_{j_1 j_2 \cdots j_m} = 0, \quad \lambda = 1, \ldots, N$$

(10.14) and the symmetries in the subscripts

$$\xi^i_{\underline{j_1 j_2} \cdots j_m} - \xi^i_{\underline{j_2 j_1} \cdots j_m} + \delta^i_{j_1 j_2} = 0$$

....

$$\xi^i_{j_1 \cdots \underline{j_{m-1} j_m}} - \xi^i_{j_1 \cdots \underline{j_m j_{m-1}}} + \delta^i_{j_{m-1} j_m} = 0,$$

where each δ is a linear combination with C^∞coefficients of

$$\{\xi^t_J : |J| \le m-1, t = 1, \ldots, n\}. \tag{10.15}$$

Let

$$g^1(x, \xi) = 0, \ldots, g^{n^{m+1}}(x, \xi) = 0 \tag{10.16}$$

be the equations corresponding to (10.13). Since the last n^{m+1} columns of the Jacobian matrix $(\partial g(x,\xi)/\partial \xi)$ is equal to $(\partial g/\partial y)$, which is nonsingular, we can solve(10.16) to obtain $\xi^i_{j_1 \cdots j_m}$ as a linear combination of (10.15) with C^∞coefficients, for each $i, j_1, \ldots, j_m$. This completes the proof.

§11. Multi-contact structures given by two vector fields.

This section is an application of Theorem 4.6. Finite dimensionality (or rigidity) of the solution space of (4.1) can be shown by constructing a complete prolongation. We present the finiteness of infinitesimal automorphisms of multi-contact structures given by two vector fields in $\mathbb{R}^3$, which is part of [HOS].

Let M be a smooth (C^∞) manifold. A multi-contact structure on M is a set of C^∞ subbundles $H_1, \dots, H_k$, $(k \geq 2)$, of the tangent bundle TM that satisfies the following non-degeneracy conditions:
$H^{(1)} = H = H_1 \oplus \cdots \oplus H_k$ and $H^{(j)} = [H^{(j-1)}, H] + H^{(j-1)}$, $j = 2, 3, \cdots$, are subbundles of TM. Moreover,

$$(11.1) \qquad H^{(N)} = TM, \quad \text{for some positive integer } N.$$

A local diffeomorphism f of M is called a generalized contact map if f preserves the multi-contact structure, that is,

$$(11.2) \quad f_*(H_i, P) \subset (H_i, f(P)), \quad \forall i = 1, \dots, k, \quad \forall P \in M.$$

A smooth vector field V is an infinitesimal automorphism of the multi-contact structure iff its flow maps $\exp tV$, $-\epsilon < t < \epsilon$, are generalized contact maps of this structure. The set of infinitesimal automorphisms is closed under the bracket, thus forms a Lie algebra. A. Koranyi raised the question whether the local group of generalized contact maps is finite dimensional, or equivalently, whether the Lie algebra of infinitesimal automorphisms is finite dimensional. We have

Theorem 11.1. *Let X and Y be C^∞ vector fields on $\mathbb{R}^3$ such that $X, Y, [X, Y]$ span the whole tangent bundle of $\mathbb{R}^3$.*

Then the set of infinitesimal automorphisms of the multi-contact structure given by X and Y is finite dimensional, with dimension at most 8.

Proof. Let $Z = [X, Y]$ and set

$$V = rX + sY + pZ. \tag{11.3}$$

Then V is an infinitesimal automorphism of the multi-contact structure given by X and Y if and only if

$$[V, X] = \lambda X, \quad [V, Y] = \mu Y, \tag{11.4}$$

for some functions λ and μ. We set

$$[Z, X] = a_1X + b_1Y + c_1Z, \quad [Z, Y] = a_2X + b_2Y + c_2Z. \tag{11.5}$$

We shall work in C^∞ category, that is, all the coefficients a_i, b_i and c_i, $i = 1, 2$, are assumed to be C^∞. By substituting (11.3) in (11.4) and substituting (11.5) for $[Z, X]$ and for $[Z, Y]$, respectively, and then equating the corresponding components, we have the following system of linear partial differential equations of first order:

$$(11.6) \qquad \begin{aligned} -Xr + a_1p &= \lambda & (11.6a)\\ -Xs + b_1p &= 0 & (11.6b)\\ -Xp - s + c_1p &= 0 & (11.6c)\\ -Yr + a_2p &= 0 & (11.6d)\\ -Ys + b_2p &= \mu & (11.6e)\\ -Yp + r + c_2p &= 0. & (11.6f) \end{aligned}$$

(11.6) is a system of six equations for five unknowns r, s, p, λ, μ :. However, if we know p we determine s, r, λ, and μ by (11.6c), (11.6f), (11.6a), and (11.6e), respectively. We shall express each of the third order partial derivatives of p as a linear combination with C^∞ coefficients of partial derivatives of lower order of p. This is a complete system of order 3 for p and the conjecture follows from Theorem 11.1.

Now we construct a complete system of order 3 for p: By (11.6c) $s = -Xp + c_1p$ and by (11.6f) $r = Yp - c_2p$. Substituting these in (11.6b) and (11.6d), respectively, we have

$$\begin{aligned} X^2p - c_1Xp + (-Xc_1 + b_1)p &= 0 \\ Y^2p - c_2Yp + (-Yc_2 - a_2)p &= 0. \end{aligned} \tag{11.7}$$

Rewrite (11.7) as

$$\begin{aligned} X^2p &\in < Xp, p > \\ Y^2p &\in < Yp, p >, \end{aligned}$$

where $< >$ denotes the set of linear combinations of the variables inside with C^∞ coefficients. The coefficients are polynomials in $a_i, b_i, c_i, \quad i = 1, 2,$ and their derivatives in X and Y. For algebraic calculation of linear differential operators that are applied to p we write the above as

$$X^2 \in < X, 1 > \tag{11.8}$$

and

$$Y^2 \in < Y, 1 > . \tag{11.9}$$

Applying X repeatedly to (11.8) and applying Y repeatedly to (11.9), we obtain by induction on n

$$X^n \in < X, 1 >, \qquad \text{for } n = 2, 3, \cdots , \tag{11.10}$$

and

$$Y^n \in < Y, 1 >, \qquad \text{for } n = 2, 3, \cdots . \tag{11.11}$$

All the third order linear operators composed of X and Y only are straight-forward from (11.8)-(11.9): We have

$$YX^2 \in < YX, Y, X, 1 > \tag{11.12}$$

and

$$XY^2 \in < XY, Y, X, 1 > . \tag{11.13}$$

If we switch the order of X and Y in a third order operator in (11.12) - (11.13) then the difference is in $< ZY, ZX, Y^2, YX, X^2, Z, Y, X, 1 >$ Analogously to the CR geometry, we obtain $Z = [X, Y]$ directional derivatives by commuting X and Y, e.g., to get Z^n we start with Y^nX^n and by commuting them we change it into X^nY^n, where Z^n and other terms are obtained in the process of commutation. To do this we introduce the following notations: For each pair of non-negative integers m, n with $n \geq m$ let $\Gamma_{n,m}$ denote the set of all linear combinations of $\{Z^tY^jX^k : \quad t \leq m, \quad t + j + k \leq n\}$ with coefficients that are polynomials in $a_i, b_i, c_i \quad (i = 1, 2)$ and their derivatives in X and Y of any order. Let $\Gamma_n = \bigcup_{m=0}^{n} \Gamma_{n,m}$.

To obtain ZX^2 we apply Y to (11.10) with $n = 3$:

$$YX^3 \in Y < X, 1 > \subset < YX, Y, X, 1 > .$$

On the other hand

$$YX^3 = (YX)XX = (XY - [XY])XX = (XY - Z)XX = XYXX - ZXX.$$

The second last term reduces to order 2 as follows: By making repeated use of (11.12)-(11.13)

$$\begin{aligned} X(YXX) &\in X < YX, Y, X, 1 > \subset < XYX, XY, XX, Y, X, 1 > \\ &\subset < YXX, ZY, ZX, YY, YX, XX, Z, Y, X, 1 > \subset \Gamma_{2,1}. \end{aligned}$$

Thus we have

$$ZX^2 \in \Gamma_{2,1}. \tag{11.14}$$

Similarly, by applying X to (11.11) with $n = 3$ we have

$$ZY^2 \in \Gamma_{2,1}. \tag{11.15}$$

To obtain ZYX we apply Y^2 to (11.8): we have

$$ZYX \in \Gamma_2, \tag{11.16}$$

and by applying Y^2 to (11.10) with $n = 3$ we have

$$Z^2X \in \Gamma_2. \tag{11.17}$$

To obtain Z^2Y we apply X^2 to (11.11) with $n = 3$: We have

$$Z^2Y \in \Gamma_2. \tag{11.18}$$

Finally, apply Y^3 to (11.10) with $n = 3$, to obtain

$$Z^3 \in \Gamma_2. \tag{2.17}$$

(11.10) - (11.19) is a polongation of (11.7) to a complete system of order 3. A solution is determined by the second jet of p at a point, which is given by ten numbers that satisfy two equations of (11.7). Therefore, the solution space is at most eight dimensional. □

§12 Infinitesimal isometry of (M^2, g).

This section is due to R. Bryant. He kindly showed this coordinate-free calculation when the author visited him to MSRI, Berkeley, in January 2002.

Let (M, g) be a smooth Riemannian manifold of dimension n. A smooth vector field ξ on M is an infinitesimal isometry (or a Killing field) if and only if ξ satisfies

$$L_\xi g = 0, \tag{12.1}$$

where L is the Lie derivative. In terms of local coordinates $x = (x^1, \cdots, x^n)$ (12.1) becomes

$$\xi_i^\lambda g_{\lambda j} + \xi_j^\lambda g_{\lambda i} - \xi^\lambda g_{ij,\lambda} = 0, \quad i, j = 1, \cdots, n, \tag{12.2}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda}$ (summation convention for $\lambda = 1, \cdots, n$). Since (12.2) is symmetric in (i, j) the number of equations in (12.2) is $\frac{n(n+1)}{2}$ whereas the number of unknowns is n so that (12.2) is overdetermined if $n \geq 2$. In this section we shall present a coordinate-free computation of prolongation of (12.1) with $n = 2$ to a complete system of order 2 and discuss the existence of solutions. Let $\{e_1, e_2\}$ be an orthonormal frame over a 2-dimensional Riemannian manifold M and let ω^1, ω^2 be the dual coframe. Then

$$g = \omega^1 \circ \omega^1 + \omega^2 \circ \omega^2,$$

where $\phi \circ \eta := \frac{1}{2}(\phi \otimes \eta + \eta \otimes \phi)$ is the symmetric product of 1-forms. Recall also that there exist a uniquely determined 1-form ω_2^1 (Levi-Civita connection) and a function K (Gaussian curvature) satisfying

$$\begin{aligned} d\omega^1 &= -\omega_2^1 \wedge \omega^2 \\ d\omega^2 &= \quad \omega_2^1 \wedge \omega^1 \end{aligned} \tag{12.3}$$

and

$$d\omega_2^1 = K\omega^1 \wedge \omega^2. \tag{12.4}$$

Furthermore, Lie derivatives of ω^i, $i = 1, 2$ with respect to a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ are

$$\begin{aligned} L_\xi \omega^1 &= d(\xi \lrcorner \omega^1) + \xi \lrcorner d\omega^1 \\ &= d\xi^1 - \omega_2^1(\xi)\omega^2 + \xi^2 \omega_2^1 \quad \text{by (12.3)} \end{aligned} \tag{12.5}$$

and similarly

$$\begin{aligned} L_\xi \omega^2 &= d(\xi \lrcorner \omega^2) + \xi \lrcorner d\omega^2 \\ &= d\xi^2 + \omega_2^1(\xi)\omega^1 - \xi^1 \omega_2^1. \end{aligned} \tag{12.6}$$

By (12.5) and (12.6), we have

$$\begin{aligned} \frac{1}{2} L_\xi g &= (L_\xi \omega^1) \circ \omega^1 + (L_\xi \omega^2) \circ \omega^2 \\ &= (d\xi^1 + \xi^2 \omega_2^1) \circ \omega^1 + (d\xi^2 - \xi^1 \omega_2^1) \circ \omega^2. \end{aligned}$$

On the other hand, the covariant derivative of ξ is a $(1, 1)$ tensor field given by

$$\begin{aligned} \nabla \xi &= \nabla(\xi^1 e_1 + \xi^2 e_2) \\ &= (d\xi^1 + \xi^2 \omega_2^1) \oplus e_1 + (d\xi^2 - \xi^1 \omega_2^1) \oplus e_2. \end{aligned}$$

By setting

$$\begin{aligned} d\xi^1 + \xi^2 \omega_2^1 &= \xi_1^1 \omega^1 + \xi_2^1 \omega^2, \\ d\xi^2 - \xi^1 \omega_2^1 &= \xi_1^2 \omega^1 + \xi_2^2 \omega^2 \end{aligned} \tag{12.7}$$

and substituting in the above we have

$$\frac{1}{2} L_\xi g = \xi_1^1 \omega^1 \circ \omega^1 + (\xi_2^1 + \xi_1^2)\omega^1 \circ \omega^2 + \xi_2^2 \omega^2 \otimes \omega^2.$$

By (12.1), ξ is an infinitesimal isometry if and only if

$$\xi_1^1 = \xi_2^2 = 0, \ \xi_2^1 + \xi_1^2 = 0. \tag{12.8}$$

Substituting (12.8) in (12.7) we see that a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ is an infinitesimal isometry if and only if

$$
\begin{aligned}
d\xi^1 &= -\xi^2\omega_2^1 + \xi_2^1\omega^2, \\
d\xi^2 &= \xi^1\omega_2^1 + \xi_1^2\omega^1,
\end{aligned}
\tag{12.9}
$$

which is a coordinate-free version of (12.2) with $n = 2$ expressed as an exterior differential system. Prolongation of (12.9) to a complete system is differentiating (12.9) and expressing $(d\xi^1, d\xi^2, d\xi_2^1)$ in terms of (ξ^1, ξ^2, ξ_2^1) : We apply d to (12.9) and substitute (12.9), (12.3) and (12.4) for $d\xi^i$, $d\omega^i$ and $d\omega_2^1$, respectively, to obtain

$$
\begin{aligned}
(d\xi_2^1 - K\xi^2\omega^1) \wedge \omega^2 = 0, \\
(d\xi_2^1 + K\xi^1\omega^2) \wedge \omega^1 = 0.
\end{aligned}
$$

Hence we have

$$
d\xi_2^1 = K(\xi^2\omega^1 - \xi^1\omega^2). \tag{12. 10}
$$

The system (12.9) and (12.10) is a prolongation of (12.1) to a complete system.
Now consider the Euclidean space $\mathbb{R}^3$ of variables (ξ^1, ξ^2, ξ_2^1). Then the submanifold of the first jet space of ξ defined by (12.8) may be identified with $\mathcal{S} := M \times \mathbb{R}^3$.
On $M \times \mathbb{R}^3$ consider the Pfaffian system $\theta = (\theta^1, \theta^2, \theta^3)$ given by

$$
\begin{aligned}
\theta^1 &= d\xi^1 + \xi^2\omega_2^1 - \xi_2^1\omega^2, \\
\theta^2 &= d\xi^2 - \xi^1\omega_2^1 + \xi_2^1\omega^1, \\
\theta^3 &= d\xi_2^1 - K\xi^2\omega^1 + K\xi^1\omega^2.
\end{aligned}
\tag{12.11}
$$

We check the Frobenius integrability conditions for (12.11): By (12.3) and (12.4) we have

$$
d\theta^1, d\theta^2 \equiv 0 \bmod \theta
$$

and

$$d\theta^3 \equiv (K_1\xi^1 + K_2\xi^2)\omega^1 \wedge \omega^2 \mod \theta$$

where $K_i = dK(e_i)$, $i = 1, 2$ so that $dK = K_1\omega^1 + K_2\omega^2$.
Thus (12.11) is integrable if and only if $T := K_1\xi^1 + K_2\xi^2$ is identically zero on $M \times \mathbb{R}^3$, which is equivalent to $K_1 = K_2 = 0$ i.e. K is constant. In this case, there exist 3 parameter family of solutions by the Frobenius theorem. Otherwise, assuming $dT \neq 0$ on $T = 0$, we consider a submanifold $\mathcal{S}'$ of dimension 4 defined by $T = 0$.
Differentiating $dK = K_1\omega^1 + K_2\omega^2$, we see by (12.3) that

$$\begin{aligned} 0 &= d^2K \\ &= (dK_1 + K_2\omega_2^1)\omega^1 + (dK_2 - K_1\omega_2^1)\omega^2. \end{aligned} \tag{12.12}$$

Thus we put

$$dK_1 = -K_2\omega_2^1 + K_{11}\omega^1 + K_{12}\omega^2, \tag{12.13}$$

$$dK_2 = K_1\omega_2^1 + K_{21}\omega^1 + K_{22}\omega^2. \tag{12.14}$$

By substituting (12.13), (12.14) in (12.12) we have $K_{12} = K_{21}$.
On $\mathcal{S}'$, we have by (12.11), (12.13) and (12.14)

$$\begin{aligned} dT &= \xi^1 dK_1 + K_1 d\xi^1 + \xi^2 dK_2 + K_2 d\xi^2 \\ &\equiv (K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1)\omega^1 + (K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1)\omega^2 \mod \theta. \end{aligned}$$

We set

$$\begin{aligned} T_1 &= K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1, \\ T_2 &= K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1. \end{aligned} \tag{12.15}$$

If $T_1, T_2 \equiv 0$ on $\mathcal{S}'$, $i^*\theta^1, i^*\theta^3, i^*\theta^3$ have rank 2. Then $\mathcal{S}'$ is foliated by two dimensional integral manifolds and therefore there are 2 parameter family of solutions. But this implies that $K_1 = K_2 = 0$ which is impossible. Let

$$A = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \end{pmatrix}.$$

If $\det A = 0$, A has rank 2 and $\mathcal{S}'' = \{T = T_1 = T_2 = 0\}$ is a 3-dimensional submanifold of $\mathcal{S}$. If we have $dT_1, dT_2 \equiv 0 \bmod \theta^1, \theta^2, \theta^3$ on $\mathcal{S}''$, the Frobenius theorem imply that $\mathcal{S}''$ is foliated by two dimensional integral manifolds and therefore there exists 1 parameter family of solutions. To calculate dT_1, dT_2 we differentiate (12.13). Then we have

(12.16)

$$\begin{aligned} 0 &= d^2K_1 \\ &= (dK_{11} + 2K_{12}\omega_2^1 + K_2K\omega^2)\omega^1 + (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^2. \end{aligned}$$

Thus we put

$$dK_{11} = -2K_{12}\omega_2^1 + K_{111}\omega^1 + K_{112}\omega^2, \tag{12.17}$$

$$dK_{12} = (K_{11} - K_{22})\omega_2^1 + K_{121}\omega^1 + K_{122}\omega^2. \tag{12.18}$$

By substituting (12.17), (12.18) in (12.16) we have $K_{112} = K_{121} - K_2K$.

Differentiating (12.14), we have

(12.19)

$$\begin{aligned} 0 &= d^2K_2 \\ &= (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^1 + (dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1)\omega^2. \end{aligned}$$

By substituting (12.17), (12.18) in (12.19) we have

$$(dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1 - K_{122}\omega^1)\omega^2 = 0.$$

Thus we put

$$dK_{22} = 2K_{12}\omega_2^1 + (K_{122} - K_1K)\omega^1 + K_{222}\omega^2. \tag{12.20}$$

On $\mathcal{S}''$, we have by (12.11), (12.17), (12.18) and (12.20)

$$dT_1 \equiv (K_{111}\xi^1 + (K_{121} - K_2K)\xi^2 - 2K_{12}\xi^1_2)\omega^1 + (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^2 \mod \theta$$

and

$$dT_2 \equiv (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^1 + ((K_{122} - K_1K)\xi^1 + K_{222}\xi^2 + 2K_{12}\xi^1_2)\omega^2 \mod \theta.$$

We summarize the discussions of this section in the following

12.1 Theorem. *Let M be a Riemannian manifold of dimension 2. Let*

$$K = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \\ K_{111} & K_{121} - K_2K & -2K_{12} \\ K_{121} & K_{122} & K_{11} - K_{22} \\ K_{122} - K_1K & K_{222} & 2K_{12} \end{pmatrix}.$$

(i) If K has rank 0, there exist 3 parameter family of infinitesimal isometries,
(ii) If K has rank 2 and $(K_1, K_2) \neq 0$, there exist 1 parameter family of infinitesimal isometries,
(iii) If K has rank 3, there exists only trivial infinitesimal isometry.

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