수 학 강 의 록 제 52 권

COVERING STRUCTURE OF HOLOMORPHIC FUNCTIONS

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Covering structure of holomorphic functions

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Chapter 1

Preliminaries

These are extended notes of my lectures given at Seoul National University in September, 2003. I would like to express hearty thanks to Professor Jong-Kyu Han and Professor Moonja Jeong (at the university of Suwon) for inviting me and giving me the chance to make these lecture notes.

In the sequel, we could discuss the case of general Riemann surfaces. But for the sake of simplicity, we will restrict ourselves to the case of $\mathbb{C}$ and its subdomains D with smooth boundary, which we call *non-degenerate planar domain.* We say that a non-degenerate planar domain D is *n-ply connected* if the boundary of D in $\mathbb{C}$ has just n connected components.

1.1 Quasiconformal maps

Definition We call an orientation-preserving homeomorphism ϕ of a domain D onto another D' a *quasiconformal map* if ϕ is ACL on every rectangle

$$R = \{z = x + iy \mid a \leq x \leq b, c \leq y \leq d\}$$

in D (i.e., $\phi(x+iy)$ is absolutely continuous on $[a, b]$ and on $[c, d]$, respectively, with respect to x and y for almost every fixed y and x), and there is a constant $k < 1$ such that

$$|\phi_{\bar{z}}| \leq k|\phi_z|$$

almost everywhere on D.

Here, we set $K = (1+k)/(1-k)$ and we also call ϕ a *K-qc* map. Further, the infimum of such K is called the *maximal dilatation* of ϕ, and denoted by K_ϕ.

In general, ϕ_z is non-vanishing almost everywhere on D, and hence we can consider the quotient

$$\mu(\phi) = \frac{\phi_{\bar{z}}}{\phi_z},$$

which is called the *complex dilatation* of ϕ. Note that, if ϕ is K-qc, then the essential supremum of $\mu(\phi)$ satisfies

$$\|\mu(\phi)\|_\infty = \text{ess.sup}_D|\mu(\phi)| = k < 1,$$

and the maximal dilatation of ϕ equals $(1+k)/(1-k)$.

Remark When we can take 0 as k, namely when ϕ is 1-qc, then a classical lemma of Weyl means that ϕ is actually a conformal map.

Definition Fix a domain D. Then we say that another domain $\tilde{D}$ is *quasiconformally equivalent* to D if there is a quasiconformal map $\phi : D \to \tilde{D}$.

Consider a pair $(\tilde{D}, \phi)$ of such a domain $\tilde{D}$ and a quasiconformal map $\phi : D \to \tilde{D}$. We say that such a pair (D_1, ϕ_1) is *Teichmüller equivalent* to another pair (D_2, ϕ_2) if there is a conformal map $g : D_1 \to D_2$ such that $\phi_2 \circ \phi_1^{-1}$ is homotopic to g relative boundary (i.e. by the homotopy

$$H(z,t) : \overline{D_1} \times [0,1] \to \overline{D_2}$$

such that $H(z,t) = g(z)$ on the boundary of D_1, where $\overline{D_j}$ is the closure of D_j).

We denote by $T(D)$ the set of all Teichmüller equivalence classes of pairs $(\tilde{D}, \phi)$ as above, and call it the *Teichmüller space* of D.

The Teichmüller space $T(D)$ has a natural distance d_T: For every two points $[D_j, \phi_j]$ in $T(D)$, we set

$$d_T([D_1,\phi_1],[D_2,\phi_2]) = \inf_{\phi_2\circ\phi_1^{-1}:D_1\to D_2} K_{\phi_2\circ\phi_1^{-1}},$$

where (D_j, ϕ_j) moves in the Teichmüller equivalence class $[D_j, \phi_j]$ for each j. Then it is well-known that d_T is a distance and complete. We call d_T the *Teichmüller distance.* The Teichmüller space $T(D)$ is always equipped with this distance and the topology induced from it.

Example 1 *The Teichmüller space $T(\mathbb{C})$ of the complex place consists of the single point, which means that every quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ is homotopic to the identity on $\mathbb{C}$.*

Example 2 *If D is the unit disc $U = \{|z| < 1\}$, then the Teichmüller space $T(U)$ is called the* universal Teichmüller space, *and especially denoted by $T(1)$.*

Here, we can forget the boundary correspondence and consider the complex structures of D only. Then we have the following deformation spaces.

Definition We say that a pair (D_1, ϕ_1) as above is *conformally equivalent* to another pair (D_2, ϕ_2) if there is a conformal map $g : D_1 \to D_2$ such that $\phi_2 \circ \phi_1^{-1}$ is homotopic to g.

We denote by $T^{\#}(D)$ the set of all conformal equivalence classes of pairs $(\tilde{D}, \phi)$ as above, and call it the *reduced Teichmüller space* of D.

The reduced Teichmüller space $T^{\#}(D)$ has a natural distance $d_{T^{\#}}$: For every two points $[D_j, \phi_j]^{\#}$ in $T^{\#}(D)$, we set

$$d_{T^{\#}}([D_1, \phi_1]^{\#}, [D_2, \phi_2]^{\#}) = \inf_{\phi_2 \circ \phi_1^{-1} : D_1 \to D_2} K_{\phi_2 \circ \phi_1^{-1}},$$

where (D_j, ϕ_j) moves in the conformal equivalence class $[D_j, \phi_j]^{\#}$ for each j. The reduced Teichmüller space is a quotient space of the Teichmüller space and again it is well-known that $d_{T^{\#}}$ is a distance and complete. We call $d_{T^{\#}}$ the *reduced Teichmüller distance.* The reduced Teichmüller space $T^{\#}(D)$ is always equipped with this distance and the topology induced from it.

Example 3 *The reduced Teichmüller space $T^{\#}(U)$ of the unit disc consists of a single point.*

Example 4 *Let D be a non-degenerate doubly connected domain. Then the reduced Teichmüller space $T^{\#}(D)$ is homeomorphic to $\mathbb{R}$.*

More precisely, every point can be represented by a pair of a domain $D_r = \{1 < |z| < r\}$ with $r \in (1, +\infty)$ and a suitable quasiconformal map $\phi : D \to D_r$.

Example 5 (Cf. [43]) *Let D be a non-degenerate triply connected domain. Then the reduced Teichmüller space $T^{\#}(D)$ is homeomorphic to $\mathbb{R}^3$.*

Example 6 *In general, if D is a Riemann surface of genus g with n smooth boundary components and if $2g + n > 2$, then the reduced Teichmüller space $T^{\#}(D)$ is homeomorphic to $\mathbb{R}^{6g+3n-6}$.*

Actually, it is well-known that $T^{\#}(D)$ can be identified with the Fricke space of a Fuchsian model G of D (cf. [27]). Since G is a free group with $2g+n-1$ real Möbius (hyperbolic) transformations as generators, $T(D)$ is real $(6g+3n-6)$-dimensional if $2g+n>2$.

In particular, if D is a non-degenerate n-ply connected planar domain with $n \geq 3$, then $T^{\#}(D)$ is homeomorphic to $\mathbb{R}^{3n-6}$.

Remark The exceptional cases where $2g+n \leq 2$ are essentially the ones that D is one of $\hat{\mathbb{C}}$, $\mathbb{C}$, U, D_r, and tori. Also we may consider $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $U^* = U - \{0\}$ as exceptional domains.

Here we see in Example 3 that $T^{\#}(U)$ is trivial, and similarly we can see that $T^{\#}(\hat{\mathbb{C}})$, $T^{\#}(\mathbb{C})$, $T^{\#}(\mathbb{C}^*)$, and $T^{\#}(U^*)$ are trivial. Also as in Example 4, we can prove that if D is a torus, then $T^{\#}(D)$ is homeomorphic to $\mathbb{R}^2$.

Now, we can consider several extra-structures over the complex structure on D. The holomorphic covering structure is such one, which is the main topic of this note.

1.2 Hurwitz spaces

Let D be a domain as above, and $f : D \to \tilde{D}$ be a possibly incomplete and branched holomorphic covering of $\tilde{D}$ by D. We say that a point α in D is a *singular value* of the covering projection f if, for every neighborhood U of α in $\tilde{D}$, there exists a connected component V of $f^{-1}(U)$ such that $f : V \to U$ is not a biholomorphic surjection. In other words, a point α is not a singular value of f if and only if α is *evenly covered* by f, i.e. we can find a neighborhood U of α in $\tilde{D}$ such that f maps every connected component of $f^{-1}(U)$ biholomorphically onto U.

We denote by S_f the set of all singular values of f, and call it the *singular value set* of f.

In the sequel, for the sake of simplicity, we assume that the singular value set is *countable.*

Definition Let f_1 and f_2 be possibly incomplete and branched holomorphic covering projections of $\tilde{D}$ by D. We say that f_1 and f_2 determine the same *covering structure* if there are biholomorphic self-maps g and h of D and $\tilde{D}$, respectively, such that

$$f_2 = h \circ f_1 \circ g.$$

We denote by $\mathcal{C}_f$ the covering structure determined by $f : D \to \tilde{D}$. We call the set of all covering structures determined by $g : D \to \tilde{D}$ which are quasiconformally equivalent to f is the *prime Hurwitz space* of f, and denoted by $H^{\#}(f)$. Here we say that $f : D \to \tilde{D}$ and $g : D \to \tilde{D}$ are *quasiconformally equivalent* if there are quasiconformal self-maps ϕ and ψ of D and $\tilde{D}$, respectively, such that

$$g = \psi \circ f \circ \phi.$$

Here, we can define the *Hurwitz distance* $d_{H^{\#}}$ by

$$d_{H^{\#}}(\mathcal{C}_{f_1}, \mathcal{C}_{f_2}) = \inf_{\phi} K(\phi),$$

where the infimum is taken over all quasiconformal self-maps ψ or ϕ of D and $\tilde{D}$ satisfying

$$f_1 = \psi \circ f_2 \circ \phi.$$

Theorem 1.2.1 *Let $f : D \to \tilde{D}$ be as above. Then $d_{H^{\#}}$ is actually a distance and complete on $H^{\#}(f)$.*

Proof. Every quasiconformal self-map ψ of $\tilde{D} - S_f$ can be extended uniquely to a quasiconformal self-map $\tilde{\psi}$ of $\tilde{D}$ without changing the maximal dilatation, and hence $K_\phi = K_\psi$ if they satisfy $f_1 = \psi \circ f_2 \circ \phi$. Hence the standard arguments show the assertions. ∎

Example 7 *The set $\mathcal{A}_n$ of all covering structures determined by polynomials with degree n in general position (considered as holomorphic self-maps of $\mathbb{C}$, i.e. the derivatives have mutually different simple critical values in $\mathbb{C}$) is a prime Hurwitz space.*

Here a *critical value* of f is the image of a critical point, i.e. a zero of the derivative, of f.

Example 8 *The set $\mathcal{L}_{m,n}$ of all covering structures determined by Laurent series*

$$a_{-m}z^{-m} + + \cdots + a_n z^n \qquad (a_{-m}a_n \neq 0)$$

in general position (considered as holomorphic self-maps of $\mathbb{C}^*$) is a prime Hurwitz space.

Example 9 *The set $\mathcal{B}_n$ of all covering structures determined by rational functions with simple poles in general position (considered as holomorphic self-maps of $\hat{\mathbb{C}}$) is a prime Hurwitz space.*

Example 10 *Set*

$$f(z) = \int^{z} P(t)e^{Q(t)}dt$$

with polynomials $P(z)$ and $Q(z)$ of degree p and q, respectively. Assume that $f(z)$ is in general position, *i.e. has $(p+q)$ singular values. If Q is not a constant, then f is transcendental, and the prime Hurwitz space $H^{\#}(f)$ of f is called a* transcendental prime Hurwitz space.

Prime Hurwitz spaces are conceptually natural, but not so easy to deal with. Hence we consider some finer equivalence relation as follows.

Definition We say that projections f_1 and f_2 as above determine the same *isomorphism class* if there is a biholomorphic self-map g of D such that

$$f_2 = f_1 \circ g.$$

We call the set of all isomorphism classes of $g : D \to \tilde{D}$ which are quasiconformally equivalent to f the *Hurwitz space* of f, and denote it by $H(f)$.

If the group $\mathrm{Aut}(\tilde{D})$ of biholomorphic self-homeomorphisms of $\tilde{D}$ is trivial, then $H^{\#}(f) = H(f)$.

If $\tilde{D}$ is essentially one of

$$\mathbb{C}, \quad \mathbb{C}^*, \quad U, \quad U^*, \quad D_r$$

with $r \in (1, +\infty)$, then we consider only *normalized* quasiconformal self-maps, i.e. those which fix $\{0,1\} \cap \overline{\tilde{D}}$ pointwise. And we can define the *normalized Hurwitz distance* d_H by

$$d_H(\mathcal{C}_{f_1}, \mathcal{C}_{f_2}) = \inf_{\phi} K(\phi),$$

where the infimum is taken over all normalized quasiconformal self-maps ψ of $\tilde{D}$ and quasiconformal ones ϕ of D satisfying

$$f_1 = \psi \circ f_2 \circ \phi.$$

Theorem 1.2.2 *Let $f : D \to \tilde{D}$ be as above. Then d_H is a distance and complete on $H(f)$.*

Remark The advantage of this definition consists in the fact that the singular value set is the same for every functions in the same isomorphism class.

Historically, the Hurwitz space of a rational function f is defined algebraically. Natanzon showed in [39] that such Hurwitz spaces are the same as $\mathrm{Top}(f)$, which is the set of all isomorphism classes of g *topologically equivalent* to f, i.e. there are self-homeomorphisms ϕ and ψ of D and $\tilde{D}$, respectively, such that $g = \psi \circ f \circ \phi$. More generally, $\mathrm{Top}(f) = H(f)$ for a Speiser functions defined in Chapter 3. Also see [51].

Example 11 *The* Hurwitz space *$H_{0,n}[n]$ of genus 0 and degree n with type $[n]$ is the space of all isomorphism classes of polynomials of degree n in general position, and corresponds to $\mathcal{A}_n$.*

The Hurwitz space *$H_{0,n}[1^n]$ of genus 0 and degree n with type $[1^n] = (1, \cdots, 1)$ is the space of all isomorphism classes of rational functions of degree n in general position with n simple poles, and corresponds to $\mathcal{B}_n$.*

In this note we will discuss about the case of $H_{0,n}[1^n]$ more closely in Chapter 2. Next, a Hurwitz space of a transcendental entire function is called a *transcendental Hurwitz space.* The Hurwitz spaces corresponding to the prime Hurwitz spaces in Example 10 are such examples, and will be discussed more closely in Chapter 3.

Remark The Hurwitz spaces of a rational function can be compactified naturally. See [16], [17], and [41]. We will explain in §2 of Chapter 3 another geometric compactification of the transcendental Hurwitz space of finite type.

Chapter 2

Covering structure of Bell representations

In this chapter, we consider Bell representations. The definition and the importance of such representations are explained in §1. In §2, we introduce the natural deformation space of Bell representation, and explain the connection with the Hurwitz space $H_{0,n}[1^n]$. (These results are obtained in the reseach with Professor M. Jeong.) Here we need a precise value of so-called Hurwitz numbers, which is derived in §3. §4 consists of some open problems.

2.1 Backgrounds

Let D be a domain in $\mathbb{C}$. Consider the subspace $A^2(D)$ of the Hilbert space $L^2(D)$ (of all square integrable functions on D with respect to the Lebesque meaure on $\mathbb{C}$) consisting of all elements in $L^2(D)$ holomorphic on D. Then there is the natural projection

$$P : L^2(D) \to A^2(D),$$

which is called the *Bergman projection.* The coresponding kernel $K(z, w)$ is called the *Bergman kernel.*

Example 12 *When D is the unit disc U,*

$$K(z,w) = \frac{1}{\pi(1-z\overline{w})^2}.$$

Hence the Bergman kernel function $K(z, w)$ associated to a simply connected domain D can be written by using the Riemann map $f_a(z)$ (determined uniquely by the conditions $f_a(a) = 0$ and $f_a'(a) > 0$) and its derivative:

$$K(z, w) = \frac{f_a'(z)\overline{f_a'(w)}}{\pi(1 - f_a(z)\overline{f_a(w)})^2}.$$

Now, fix a point a in D, and let f_a be the *Ahlfors map* associated with the pair (D, a). Among all holomorphic functions h which map D into the unit disc and satisfy $h(a) = 0$, the Ahlfors map f_a is the unique function which maximizes $h'(a)$ under the condition $h'(a) > 0$. Such proper holomorphic maps can recover the Bergman projections and kernels in general.

Theorem 2.1.1 *Let $f : D_1 \to D_2$ be a proper holomorphic map between planar (proper) domains. Let P_j be the Bergman projection for D_j. Then*

$$P_1(f' \cdot (\phi \circ f)) = f' \cdot ((P_2\phi) \circ f)$$

for all $\phi \in L^2(D_2)$.

But the translation formula for the Bergman kernels is not so simple in general. For instance, it is hard to write down the following formula explicitly.

Proposition 2.1.2 *Let $f : D_1 \to D_2$ be a proper holomorphic map between planar (proper) domains. Then the Bergman kernels $K_j(z, w)$ associated to D_j transform according to*

$$f'(z)K_2(f(z), w) = \sum_{k=1}^{m} K_1(z, F_k(w))\overline{F_k'(w)}$$

for $z \in D_1$ and $w \in D_2 - V$ where the multiplicity of the map f is m and the functions F_k with $k = 1, \cdots, m$ denote the local inverses to f and V is the set of critical values.

S. Bell obtained several kinds of simpler representations of Bergman kernel functions.

Theorem 2.1.3 ([6]) *For a non-degenarate multiply connected planar domain D, we can find two points a, b in D such that*

$$K(z, w) = f_a'(z)\overline{f_b'(w)}R(z, w)$$

with a rational combination $R(z, w)$ of f_a and f_b.

Here we say that a function $R(z,w)$ is a *rational combination of* f_a *and* f_b if it is a rational function of

$$f_a(z), \quad f_b(z), \quad \overline{f_a(w)}, \quad \overline{f_b(w)}.$$

Such a representation as above has the following variant.

Theorem 2.1.4 ([10]) *For a non-degenarate multiply connected planar domain D, we can find two points a and b in D such that*

$$K(z,w) = \frac{f_a'(z)\overline{f_a'(w)}}{(1-f_a(z)\overline{f_a(w)})^2}\left(\sum_{j,k} H_j(z)\overline{K_k(w)}\right)$$

where f_a *and* f_b *are the Ahlfors functions, H and K are rational functions of them, and the sum is a finite sum.*

Actually, we can use any proper holomorphic maps.

Theorem 2.1.5 ([7]) *Let D be a non-degenarate multiply connected planar domain, and f a proper holomorphic map of D onto the unit disc U. Then* $K(z,w)$ *is an algebraic function of*

$$f(z), \quad f'(z), \quad \overline{f(w)}, \quad \overline{f'(w)}.$$

Moreover, we have the following

Theorem 2.1.6 ([7]) *Let D be a non-degenerate multiply connected planar domain. The following conditions are equivalent.*

(1) The Bergman kernel $K(z,w)$ *associated to D is algebraic, i.e. an algebraic function of z and* $\overline{w}$.

(2) The Ahlfors map $f_a(z)$ *is an algebraic function of z.*

(3) There is a proper holomorphic mapping $f : D \to U$ *which is an algebraic function.*

(4) Every proper holomorphic mapping from D onto the unit disc U is an algebraic function.

Theorem 2.1.7 ([9]) *Let D be a non-degenerate multiply connected planar domain. There are two holomorphic functions* F_1 *and* F_2 *on D such that the Bergman kernel on D is a rational combination of* F_1 *and* F_2 *if and only if there is a proper holomorphic map f of D onto U such that f and f' are algebraically dependent: i.e. there is a polynomial Q such that* $Q(f,f') = 0$.

Then, for every proper holomorphic map f of D to U, f and f' are algebraically dependent.

Proposition 2.1.8 **([9])** *Let D be a* simply connected *planar (proper) domain. The Bergman kernel on D is a rational combination of a function of a complex variable if and only if the Riemann map f of D and f' are algebraically dependent.*

Finally, we note the following facts.

Proposition 2.1.9 **([7])** *Suppose that $K(z,w)$ is algebraic. Let f be a proper holomorphic map to U. Then $K(z,w)$ is an algebraic function of $f(z)$ and $\overline{f(w)}$.*

Corollary 1 **([7])** *Let D_1 and D_2 have algebraic Bergman kernels. Then every biholomorphic map of D_1 onto D_2 is algebraic.*

Now the issue is to find a family of canonical domains which admit a *simple* proper holomorphic map to U. Bell propose such a family, and actually, they are enough.

Theorem 2.1.10 **([28])** *Every non-degenerate n-ply connected planar domain with $n > 1$ is mapped biholomorphically onto a domain $W_{\mathbf{a},\mathbf{b}}$, defined by*

$$W_{\mathbf{a},\mathbf{b}} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex vectors

$$\mathbf{a} = (a_1, a_2, \cdots, a_{n-1}), \quad \mathbf{b} = (b_1, b_2, \cdots, b_{n-1}).$$

To prove this theorem, suppose that a non-degenerate n-ply connected planar domain D is given arbitrarily. Here we may assume that the boundary of D consists of exactly n smooth simple closed curves $\{\gamma_j\}$. Fix a point a in D, and let f_a be the Ahlfors function associated to the pair (D, a). Then f_a maps D properly and holomorphically onto the unit disc U. Moreover, f_a can be extended to a continuous map of the closure $\overline{D}$ of D onto the closed unit disc so that every component γ_j is mapped homeomorphically onto the unit circle.

Lemma 2.1.11 *There is a compact Riemann surface R (without boundary) of genus 0 and a holomorphic injection ι of D into R such that*

$$f_a \circ \iota^{-1}$$

can be extended to a meromorphic function, say F, on R.

Proof. Since there are only a finite number of zeros of f'_a, there is a positive constant ρ such that $\rho < 1$ and that

$$W = \{\rho < |\zeta| < 1\},$$

where ζ is the complex coordinate on the target plane of the map f_a, contains no critical values. Hence every component W_j of $f_a^{-1}(D)$ is mapped biholomorphically onto D by the restriction $f_a|_{W_j}$, of f_a to W_j.

Now we consider the disjoint union $\mathbf{R}$ of D and n copies V_j $(j = 1, \cdots, n)$ of

$$V = \{\rho < |\zeta|\} \cup \{\infty\}.$$

Identify every subdomain W_j of D with the subdomain W'_j of V_j corresponding to W by the biholmorphic map corresponding to $f_a|_{W_j}$. Then the resulting set, which we denote by $R = \mathbf{R}/f_a$, has a natural complex structure induced from those on D and on every V_j, and hence is a Riemann surface. Here the natural inclusion map ι of D into R is a holomorphic injection, and using the complex coordinate ζ_j on the copy V_j corresponding to ζ on V, we have

$$f_a \circ \iota^{-1}(\zeta_j) = \zeta$$

on W'_j by the definition.

Now, since topologically R is obtained from D by attaching a disc along each boundary curves of D, R is a simply connected compact Riemann surface without boundary, and hence in particular, is of genus 0. Also we can extend $F = f_a \circ \iota^{-1}$ to a meromorphic function on the whole R by setting $F(\zeta_j) = \zeta$ and $F(\infty) = \infty$ on the whole V_j for every j. ∎

Next, the following uniformization theorem (which is also called the generalized Riemann mapping theorem) is classical and well-known. As references, we cite for instance [20] and [27].

Proposition 2.1.12 *Every simply connected Riemann surface is mapped biholomorphically onto one of U, $\mathbb{C}$, and $\hat{\mathbb{C}}$.*

Corollary 2 *There is a biholomorphic map h of the above Riemann surface R onto $\hat{\mathbb{C}}$, and hence $F \circ h^{-1}$ is a rational function.*

Here, we may assume that

$$f(\infty) = \infty$$

by applying to f the pre-composition of a Möbius transformation S which sends ∞ to a pole of f, i.e. by replacing h to $S^{-1} \circ h$, if necessary.

Lemma 2.1.13 *Let w be the complex variable of the above rational function f. Then f has the following partial fraction decomposition:*

$$f(w) = Cw + C' + \sum_{k=1}^{n-1} \frac{A_k}{w - B_k}.$$

Here A_k, B_k, C and C' are complex constants, every A_k and C are non-zero, and $\{B_k\}$ are mutually distinct.

Proof. Since f has exactly n simple poles, as is seen from the construction, and one of them is ∞ by the above assumption, f is of degree exactly n and has $n-1$ finite, mutually distinct, simple poles, say $B_1, \cdots, B_{n-1}$. Hence we can write $f(w)$ as

$$f(w) = \frac{P(w)}{Q(w)}$$

with polynomials $P(w)$ of degree exactly n and

$$Q(w) = (w - B_1) \cdots (w - B_{n-1}).$$

Thus it is easily to see that the partial fraction decomposition of f is such as desired. ∎

Proof of Theorem 2.1.10 We replace the complex variable w of f to

$$z = T(w) = Cw + C'$$

by applying to f the precomposition by the affine transformation T. Further set

$$a_k = CA_k, \quad b_k = CB_k + C'$$

for every k. Then we conclude that

$$f \circ T^{-1}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}.$$

Thus the domain

$$\left\{ \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

can be expressed also as

$$\{|F \circ h^{-1} \circ T^{-1}(z)| < 1\} = (T \circ h \circ \iota)(D),$$

which is mapped biholomorphically onto D by the holomorphic injection $(T \circ h \circ \iota)^{-1}$. ■

Theorem 2.1.10 can be considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains.

We call a domain $W_{\mathbf{a},\mathbf{b}}$ as in Theorem 2.1.10 a *Bell representation*of W. The function $f_{\mathbf{a},\mathbf{b}}$ defined by

$$f_{\mathbf{a},\mathbf{b}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic map from $W_{\mathbf{a},\mathbf{b}}$ onto U which is rational. Hence a Theorem 2.1.6 implies the following corollary.

Corollary 3 *Every non-degenerate n-ply connected planar domain with $n > 1$ is biholomorphic to a domain with algebraic Bergman kernel.*

Actually, it is a very classical fact that, for such a function $f = f_{\mathbf{a},\mathbf{b}}$ as above, f and f' are algebraically dependent, i.e. there is a polynomial $P(z_1, z_2)$ such that

$$P(f, f') = 0.$$

Hence Theorem 2.1.7 implies the following results, which gives more close information about the simplicity/complexity of the Bergamn kernel.

Proposition 2.1.14 *There are two holomorhic functions F_1 and F_2 on $W = W_{\mathbf{a},\mathbf{b}}$, which are algebraic, such that the Bergman kernel on it is a rational combination of F_1 and F_2.*

2.2 The coefficeint body

Now we define a deformation space of Bell representations. See [30] for the details.

Definition The locus $\mathbf{B}_n$ in $\mathbb{C}^{2n-2}$ consists of $(\mathbf{a}, \mathbf{b})$ such that the corresponding domain $W_{\mathbf{a},\mathbf{b}}$ is a non-degenerate n-ply connected planar domain.

We call this locus $\mathbf{B}_n$ the *coefficient body for non-degenerate n-ply connected canonical domains.*

It is obvious that $\mathbf{B}_n$ is contained in the product space

$$(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C},$$

which has the same homotopy type as that of

$$(S^1)^{n-1} \times F_{0,n-1}\mathbb{C},$$

where

$$F_{0,n-1}\mathbb{C} = \{(z_1, \cdots, z_{n-1} \in \mathbb{C}^{n-1} \mid z_j \neq z_k \quad \text{if} \quad j \neq k\}$$

is the configuration space of $n-1$ points in $\mathbb{C}$.

To clarify the structure of the coefficient body, it is more convenient to consider the following modification.

Definition We set

$$\mathbf{B}_n^* = \{(a_1, \cdots, a_{n-1}, \mathbf{b}) \mid (a_1^2, \cdots, a_{n-1}^2, \mathbf{b}) \in \mathbf{B}_n\},$$

and call it the *modified coefficient body* (of degree n).

Clearly, $\mathbf{B}_n^*$ is contained in

$$(\mathbb{C}^*)^{n-1} \times F_{0,n-1}\mathbb{C}.$$

Also it is invariant under the reflection

$$S_k : (a_1, \cdots, a_k, \cdots, a_{n-1}, \mathbf{b}) \mapsto (a_1, \cdots, -a_k, \cdots, a_{n-1}, \mathbf{b})$$

of $\mathbb{C}^{2n-2}$ for every k. And $\mathbf{B}_n$ can be identified with the quotient space of $\mathbf{B}_n^*$ by the action of the group $G =< S_1, \cdots, S_{n-1} >$ generated by these reflections. Thus $\mathbf{B}_n^*$ is 2^{n-1}-sheeted smooth holomorphic covering of $\mathbf{B}_n$ with the covering transformation group G.

In the sequel, we assume that $n > 2$, since $\mathbf{B}_2$ and $\mathbf{B}_2^*$ are explicitly known. See Example 12 below.

First, note that $\mathbf{B}_n^*$ is circular in the following sense.

Proposition 2.2.1 *For every* $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ *and every* $\theta \in \mathbb{R}$, $e^{i\theta}(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$.

Another important property is "star-shapedness" of $\mathbf{B}_n^*$.

Proposition 2.2.2 *For every* $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$ *and every* $0 < r \leq 1$, $r(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n^*$.

Also, we can know the homotopy type of $\mathbf{B}_n^*$ and $\mathbf{B}_n$.

Theorem 2.2.3 $\mathbf{B}_n^*$ *and hence* $\mathbf{B}_n$ *are domains and have the same homotopy type as that of*

$$(S^1)^{n-1} \times F_{0,n-1}\mathbb{C}.$$

Corollary 4 *The modified coefficient body* $\mathbf{B}_n^*$ *is a circular domain homeomorphic to* $\mathbf{B}_n$.

Remark The fundamental group of $F_{0,n-1}\mathbb{C}$ is called the *pure braid group*, and its structure is well-known. See for instance [12].

Now, Theorem 2.2.3 follows from the following two lemmas.

Lemma 2.2.4 *The coefficient body* $\mathbf{B}_n$ *is the set of all* $(\mathbf{a}, \mathbf{b})$ *such that*

$$f'_{\mathbf{a},\mathbf{b}}(z) = 0$$

has $2n-2$ *solutions* $c_1, \cdots, c_{2n-2}$ *counted with multiplicities such that*

$$|f_{\mathbf{a},\mathbf{b}}(c_j)| < 1$$

for every j. *The set* $\mathbf{B}_n^*$ *is characterized in the same way.*
In particular, $\mathbf{B}_n$ *and* $\mathbf{B}_n^*$ *are open subsets of* $\mathbb{C}^{2n-2}$.

Next set

$$\rho(\mathbf{b}) = \min_{j \neq k} |b_j - b_k|.$$

And for a sufficiently small $\epsilon > 0$ with $\epsilon \leq 1/(6n)$, we set

$$\begin{aligned} \mathbf{B}_n^\epsilon &= \{(\mathbf{a}, \mathbf{b}) \in \mathbb{C}^{2n-2} \mid \rho(\mathbf{b}) > 0, |b_k| \leq 1/2, \\ &\quad 0 < |a_k| \leq \epsilon\sqrt{\rho(\mathbf{b})}, 1 \leq k \leq n-1\}. \end{aligned}$$

Note that $\rho(\mathbf{b}) \leq 1$.

Lemma 2.2.5 *$\mathbf{B}_n^*$ has the same homotopy type as that of $\mathbf{B}_n^\epsilon$.*

Proof. First we show that

$$\mathbf{B}_n^\epsilon \subset \mathbf{B}_n^*$$

Suppose that $(\mathbf{a},\mathbf{b}) \in \mathbf{B}_n^\epsilon$. If we set

$$C_k = \{z \in \mathbb{C} \mid |b_k - z| = \epsilon\rho(\mathbf{b})\}$$

then $z \in C_k$ implies that $|z| \leq \frac{2}{3}$, and

$$|b_j - z| \geq (1-\epsilon)\rho(\mathbf{b}) > \rho(\mathbf{b})/2$$

for every $j \neq k$, and hence

$$\begin{aligned} |g_{\mathbf{a},\mathbf{b}}(z)| &\leq |z| + \sum_{j=1}^{n-1} \left| \frac{a_j^2}{z - b_j} \right| \\ &\leq \frac{2}{3} + \frac{\epsilon^2\rho(\mathbf{b})}{\epsilon\rho(\mathbf{b})} + (n-2)\frac{\epsilon^2\rho(\mathbf{b})}{\rho(\mathbf{b})/2} \\ &= \frac{2}{3} + (1 + (2n-4))\epsilon < 1. \end{aligned}$$

On the other hand, if we set

$$\tilde{C}_k = \{|b_k - z| = |a_k^2|/2\}$$

then $|a_k^2|/2 < \epsilon^2\rho(\mathbf{b})$, and $z \in \tilde{C}_k$ implies that

$$\begin{aligned} |g_{\mathbf{a},\mathbf{b}}(z)| &\geq \frac{|a_k^2|}{|z - b_k|} - |z| - \sum_{j \neq k} \left| \frac{a_j^2}{z - b_j} \right| \\ &\geq 2 - \frac{2}{3} - (n-2)\frac{\epsilon^2\rho(\mathbf{b})}{\rho(\mathbf{b})/2} \\ &= 2 - \frac{2}{3} - (2n-4)\epsilon^2 > 1. \end{aligned}$$

Thus

$$\{z \in \mathbb{C} \mid |g_{\mathbf{a},\mathbf{b}}(z)| = 1\}$$

has a component in

$$\{z \in \mathbb{C} \mid |a_k^2|/2 < |z - b_k| < \epsilon\rho(\mathbf{b})\},$$

and $W^*_{\mathbf{a},\mathbf{b}}$ is disjoint from $\{|z - b_k| \leq |a_k^2|/2\}$, for every k, which implies that $W^*_{\mathbf{a},\mathbf{b}}$ is non-degenerate and n-ply connected.

Next for every $(\mathbf{a}_0, \mathbf{b}_0) = (a_{1,0}, \cdots, a_{n,0}, b_{1,0}, \cdots, b_{n,0}) \in \mathbf{B}^*_n$, let $\ell_{\mathbf{a}_0,\mathbf{b}_0}$ be the ray

$$\{(r\mathbf{a}_0, r\mathbf{b}_0) \mid 0 < r \leq 1\}.$$

Then by Proposition 2.2.2, $\ell_{\mathbf{a}_0,\mathbf{b}_0} \subset \mathbf{B}^*_n$. Also since $\rho(r\mathbf{b}_0) = r\rho(\mathbf{b}_0)$, we conclude that

$$|ra_{k,0}| = r|a_{k,0}| = \epsilon' \sqrt{\rho(r\mathbf{b}_0)},$$

where

$$\epsilon' = \sqrt{r}|a_{k,0}|/\sqrt{\rho(\mathbf{b}_0)},$$

which in turn tends to 0 as r does.

Now, fix an $\epsilon > 0$ with $\epsilon \leq 1/(6n)$. Then, $(r\mathbf{a}_0, r\mathbf{b}_0) \in \mathbf{B}^\epsilon_n$ for every sufficiently small r. Hence we can construct a deformation retraction

$$r_\epsilon : \mathbf{B}^*_n \to \mathbf{B}^\epsilon_n,$$

by mapping the point $(\mathbf{a}_0, \mathbf{b}_0)$ to the nearest point in $\mathbf{B}^\epsilon_n$ along $\ell_{\mathbf{a}_0,\mathbf{b}_0}$. This retraction is clearly the identity on $\mathbf{B}^\epsilon_n$, and we conclude the assertion. ∎

Example 13

$$\mathbf{B}^*_2 = \{(a, b) \in \mathbb{C}^2 : a \neq 0, |b + 2a| < 1, |b - 2a| < 1\},$$

which is biholomorphic to the polydisc deleted the diagonal.

Next, for every point in the set

$$\left\{(a, b) \in \mathbf{B}^*_2 : \left|\frac{4a^2}{1 - \overline{(b + 2a)}(b - 2a)}\right| = \frac{4r}{4 + r^2}\right\}$$

corresponds to the same domain for every given $r > 2$.

Next, we give typical examples of points in $\mathbf{B}_3$. Consider the case that

$$f(z) = f_{4a^2,4a^2,b,-b}(z) = z + \frac{4a^2}{z-b} + \frac{4a^2}{z+b}$$

with $a, b \in \mathbb{C} - \{0\}$.

Theorem 2.2.6 *The complex vector* $(4a^2, 4a^2, b, -b)$ *belongs to* $\mathbf{B}_3$ *if and only if*

$$|b^2 + 4a^2 + 4a(a^2+b^2)^{1/2}| \cdot |b^2 - 2a^2 + 2a(a^2+b^2)^{1/2}|^2 < |b|^4$$

where the same value of $(a^2+b^2)^{1/2}$ *is taken in each term.*

Now, holomorphic functions can be parametrized by the set of critical values. In fact, such a parametrization can be considered for any functions in general position, and was used to give local parameters of the classical Hurwitz spaces.

Definition Let Γ be the set of all points $(\mathbf{a}, \mathbf{b})$ of $\mathbf{B}_n$ such that the corresponding rational function $f_{\mathbf{a},\mathbf{b}}$ has a non-simple critical point or has a pair of critical points whose images are the same. We call Γ the *collision locus.*

Then for every point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$, the rational function $f_{\mathbf{a},\mathbf{b}}$ has $2n-2$ simple critical values. We denote the set of simple critical values of $f_{\mathbf{a},\mathbf{b}}$ by

$$S_{\mathbf{a},\mathbf{b}} = \{\alpha_1, \cdots, \alpha_{2n-2}\},$$

where, letting $\{c_j\}_{j=1}^{2n-2}$ be the set of the simple critical points of $f_{\mathbf{a},\mathbf{b}}$, $\alpha_j = f_{\mathbf{a},\mathbf{b}}(c_j)$ for every j. This set can be considered as a point in the unordered configuration space $B_{0,2n-2}\mathbb{C}$ of $2n-2$ points on $\mathbb{C}$, i.e. the quotient space of $F_{0,2n-2}\mathbb{C}$ by the symmetric group $\mathfrak{S}_{2n-2}$:

$$B_{0,2n-2}\mathbb{C} = F_{0,2n-2}\mathbb{C}/\mathfrak{S}_{2n-2}.$$

Moreover by Lemma 2.2.4, we see that $S_{\mathbf{a},\mathbf{b}}$ is actually a point of the unordered configuration space $B_{0,2n-2}U$ of $2n-2$ points on the unit disc U.

Thus we can define the projection

$$\pi_S : \mathbf{B}_n - \Gamma \to B_{0,2n-2}U$$

by setting

$$\pi_S(\mathbf{a}, \mathbf{b}) = S_{\mathbf{a},\mathbf{b}}.$$

We have the following theorem about the projection π_S.

Theorem 2.2.7 *The projection* π_S *is a*

$$(2n-2)!\,n^{n-3}$$

-sheeted proper holomorphic covering of $B_{0,2n-2}U$ *for every* $n > 2$.

To show this theorem, first recall that, for every point $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n - \Gamma$, the critical points $c_1, \cdots, c_{2n-2}$ of $f_{\mathbf{a},\mathbf{b}}$ are the solutions of the algebraic equation

$$\prod_{j=1}^{n-1}(z-b_j)^2\left(1-\sum_{k=1}^{n-1}\frac{a_k}{(z-b_k)^2}\right)=0.$$

Hence c_j moves holomorphically with respect to $(\mathbf{a}, \mathbf{b})$. Since so does the image α_j of c_j for each $j = 1, \cdots, 2n-2$, the map π_S is holomorphic.

Definition The *marked Hurwitz space* $MH_{0,n}[1^n]$ of genus 0 and degree n with type $[1^n] = (1, \cdots, 1)$ and with the ordered poles is the set of all isomorphism classes of rational functions in general position of degree n such that poles are simple and ordered.

And we show by the following two lemmas that, for every point S in $B_{0,2n-2}U$, $\pi_S^{-1}(\mathrm{S})$ consists of $(2n-2)!\,n^{n-3}$ points.

Lemma 2.2.8 $\mathbf{B}_n - \Gamma$ *can be identified with the subset* MH_nU *of marked Hurwitz space* $MH_{0,n}[1^n]$, *consisting of all isomorphism classes of rational functions whose critical values are in* U, *by the mapping* ι *which maps* $(\mathbf{a}, \mathbf{b})$ *to the isomorphism class of* $f_{\mathbf{a},\mathbf{b}}$.

Proof. By Lemma 2.2.4, every $f = f_{\mathbf{a},\mathbf{b}}$ with $(\mathbf{a}, \mathbf{b}) \in \mathbf{B}_n - \Gamma$ determines a point in MH_nU. Here we always assume that the order of poles is $b_1, \cdots, b_{n-1}, \infty$.

Next suppose that $(\mathbf{a}', \mathbf{b}')$ is also in $\mathbf{B}_n - \Gamma$. If $g = f_{\mathbf{a}',\mathbf{b}'}$ is in the isomorphism class of f, then there is a Möbius transformation A such that

$$f = g \circ A$$

and since A maps poles of f to those of g keeping the order, A fixes ∞ and hence is affine, which we write as $A(z) = pz + q$. Then

$$z+\sum_{k=1}^{n-1}\frac{a_k}{z-b_k}=pz+q+\sum_{k=1}^{n-1}\frac{a'_k}{A(z)-b'_k}.$$

Hence A should be the identity map. This implies that

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}'),$$

and hence ι is injective.

Finally, for every point in MH_nU, take a representative (a rational function) f in the class. Then the poles of f are simple and ordered. By applying precomposition of a suitable Möbius transformation which sends ∞ to a pole if necessary, we may assume that f has the form

$$f(z) = az + b + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}.$$

Again by another precomposition of an affine transformation, we may assume that $a = 1, b = 0$, i.e. $f = f_{\mathbf{a},\mathbf{b}}$ with some $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$. Thus $\iota : \mathbf{B}_n - \Gamma \to MH_nU$ is surjective. ∎

Now, fix a point $\mathrm{S} = \{\alpha_j\}_{j=1}^{2n-2}$ in $B_{0,2n-2}U$. And fix a set of mutually disjoint cuts (simple smooth arcs) ℓ_j from α_j to a mutually distinct boundary point ω_j of U for every j. Here we assume that $\omega_1, \cdots, \omega_{2n-2}$ are located with this order (with respect to the counter-clockwise direction) on the boundary ∂U of U.

Lemma 2.2.9 *The number of points in the preimage $\pi_S^{-1}(\mathrm{S})$ of S by π_S is always*

$$(2n-2)!\, n^{n-3}.$$

Proof. For every point $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$, $f_{\mathbf{a},\mathbf{b}}$ gives a representative of the point $\iota((\mathbf{a}, \mathbf{b}))$ in MH_nU over S. In other words, $f_{\mathbf{a},\mathbf{b}}$ gives an n-sheeted branched holomorphic covering of $\widehat{\mathbb{C}}$ by $\widehat{\mathbb{C}}$ with critical values S and ordered simple poles $b_1, \cdots, b_{n-1}, \infty$.

Recall that $f = f_{\mathbf{a},\mathbf{b}}$ also gives the branched covering of U by $W_{\mathbf{a},\mathbf{b}}$. This covering can be reconstructed as follows: Set $\Omega = U - \cup_{j=1}^{2n-2} \ell_j$. Then the preimage $f^{-1}(\Omega)$ consists of n domains Ω_k, the order of which is naturally defined as follows: Let γ_k be the component of $f_{\mathbf{a},\mathbf{b}}^{-1}(\partial U)$ surrounding the k-th pole. Then Ω_k is the component whose boundary contains the part of γ_k which is projected by $f_{\mathbf{a},\mathbf{b}}$ onto the subarc of ∂U from ω_{2n-2} to ω_1 (which contains no ω_j).

Let ℓ_j^k be the "slit" on Ω_k over ℓ_j (i.e. the part of the boundary corresponding to the preimage $f^{-1}(\ell_j)$ on Ω_k) for every k and j. Then each ℓ_j^k is divided by some critical point into two arcs, which can be considered as two sides of the "slit" ℓ_j^k. And for every j, there is a pair, say $\{\Omega_{k(j)}, \Omega_{k'(j)}\}$ such that sides of these "slits" are glued "crosswise" along $\ell_j^{k(j)}$ and $\ell_j^{k'(j)}$. (Here two sides of every other "slit" ℓ_j^k is glued trivially.) Hence we have a transposition $\sigma_j = (k(j)\, k'(j))$ of ordered n sheets at ℓ_j when we move counter-clockwise along ∂U for each j. Since $W_{\mathbf{a},\mathbf{b}}$ has exactly n boundary components,

$$\sigma_{2n-2} \circ \cdots \circ \sigma_1$$

should be the identical permutation. And apply all such gluings as above, we can reconstruct the branched covering $f : W_{\mathbf{a},\mathbf{b}} \to U$.

Thus for every $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ and with fixed cuts $\{\ell_j\}$, we have an ordered factorization of the identical permutation into $2n-2$ transpositions. And since $W_{\mathbf{a},\mathbf{b}}$ is connected, such transpositions generate the full symmetric group $\mathfrak{S}_n$.

Conversely, for every such an ordered factorization of the identical permutation, we can construct an n-sheeted branched covering of $\widehat{\mathbb{C}}$ by itself, and hence also of U by an n-connected domain W, having the set S as simple critical values. Then W has n boundary components, and hence by the argument as in the proof of Theorem 2.1.10, we can find a point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n - \Gamma$ such that $W_{\mathbf{a},\mathbf{b}}$ is biholomorphic to W and $f_{\mathbf{a},\mathbf{b}}$ belongs to the isomorphism class of the covering projection of the above covering. In other words, $(\mathbf{a}, \mathbf{b}) \in \pi_S^{-1}(\mathrm{S})$. Also it is clear that different such factorizations give different covering structures, and hence different $(\mathbf{a}, \mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ by Lemma 2.2.8.

On the other hand, it is known, and will be proved in the next section, that the number of such (transitive minimal) ordered factorizations of the identical permutation on $\{1, \cdots, n\}$ into transpositions is

$$(2n-2)!\, n^{n-3},$$

which shows the assertion. ∎

Finally, we have

Lemma 2.2.10 *π_S is locally biholomorphic, and evenly covered.*

Proof. Fix a point S in $B_{0,2n-2}U$ and a point $(\mathbf{a},\mathbf{b})$ in $\pi_S^{-1}(\mathrm{S})$ arbitrarily. Then it is classically well-known (or can be shown by a standard arguments in the quasiconformal deformation theory) that we can find a neighborhood V of S and a holomorphic function ϕ of V into $\mathbf{B}_n$ such that

$$\phi(\mathrm{S}) = (\mathbf{a},\mathbf{b})$$

and for every $(\mathbf{a}',\mathbf{b}')$ in $\phi(V)$, $f_{\mathbf{a}',\mathbf{b}'}$ gives the same factorization of the identical permutation as $f_{\mathbf{a},\mathbf{b}}$ does. Here if V is sufficiently small, we can consider the natural bijection between S and the set S$'$ of critical values of $f_{\mathbf{a}',\mathbf{b}'}$ for every $(\mathbf{a}',\mathbf{b}') \in \phi(V)$. And we take as the "slits" ℓ'_j for S$'$ the image of ℓ_j by a self-diffeomorphism of $U \cup \partial U$ which is the identity outside mutually disjoint simply connected, relatively compact, neighborhoods of each α_j in U and induces the above bijection between S and S$'$.

Then from the construction, $\pi_S \circ \phi$ is the identity. And since the number of points in the preimage $\pi_S^{-1}(\mathrm{S})$ is a finite constant by the above lemma, we conclude that π_S is locally biholomorphic, and also evenly covered. ■

Thus π_S gives an unbranched $(2n-2)!\, n^{n-3}$-sheeted, holomorphic covering of $B_{0,2n-2}U$ by $\mathbf{B}_n - \Gamma$. In particular, it is proper, which completes the proof of Theorem 2.2.7.

Example 14 *In the case $n = 3$, such ordered factorizations are*

$$\{(p\,q),(p\,q),(p\,r),(p\,r)\}, \qquad (p\,q),(p\,r),(p\,r),(p\,q)\},$$

$$\{(p\,q),(p\,r),(r\,q),(p\,r)\}, \qquad \{(p\,q),(p\,r),(q\,p),(q\,r)\},$$

where we can take any bijection of $\{p,q,r\}$ to $\{1,2,3\}$. Hence we have 4! different ordered (transitive minimal) factorizations of the identical permutations on $\{1,2,3\}$.

2.3 Hurwitz numbers

The *Hurwitz number* H^g_α is the number of branched holomorphic coverings of $\hat{\mathbb{C}}$ by a closed Rieman surfaces of genus g, in general position, with prescribed branching represented by a partition α over ∞. More precesely, H^g_α is the number of the isomorphism classes of holomorphic maps of a closed Riemann surface of genus g to $\hat{\mathbb{C}}$ with the type of the poles given by α and with $r = n + J + 2(g-1)$ finite specified simple critical values. Here

$$\alpha = (\alpha_1, \cdots, \alpha_J)$$

is a partition of $\{1, \cdots, n\}$ with J parts, and this relation is denoted by $\alpha \vdash n$. The number J of parts of α is denoted also by $\ell(\alpha)$.

Recall the following formula in [19] (also cf. [21]), which gives an important representation of the Hurwitz numbers by Hodge integrals.

Theorem 2.3.1

$$H^g_\alpha = \frac{r!}{\#\mathrm{Aut}(\alpha)} \prod_{j=1}^{J} \frac{\alpha_j^{\alpha_j}}{\alpha_j!} \int_{\overline{\mathcal{M}_{g,J}}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{\prod(1 - \alpha_j \psi_j)}$$

Here λ_i and ψ_j are Chow classes of codimension i and 1, respectively, on the Deligne-Mumford compactification $\overline{\mathcal{M}_{g,J}}$ of the moduli space of type (g, J) (i.e. of genus g and with J punctures), where $1 \leq j \leq J$ and $0 \leq i \leq g$ with $\lambda_0 = 1$. More precisely, $\mathcal{M}_{g,J}$ has J naural line bundles L_j (the cotangent space to the j-th marked point), and a natural rank g vector bundle E (the Hodge bundle whose fibers correspond to the abel differentials on the curve). And

$$\psi_j = c_1(L_j), \qquad \lambda_i = c_i(E),$$

where c_i is the i-th Chern class.

In the sequel of this section, we restrict ourselves to the case of *genus* $g = 0$.

Definition Let $\mathcal{C}_\alpha$ be the conjugacy class in the symmetric group $\mathfrak{S}_n$ on $\{1, \cdots, n\}$ corresponding the partition α.

Example 15 *Transpositions are in* $\mathcal{C}_{[2\,1^{n-2}]}$, *and* $\ell([2\,1^{n-2}]) = n - 1$.

Definition Fix a permutation $\pi \in \mathcal{C}_\alpha$ arbitrarity, let c_α be the number of of *minimal, transitive, ordered factorization* of π into transpositions, i.e. number of ordered sets of j_0 transpositions $(\sigma_1, \cdots, \sigma_{j_0})$ such that

1. $\pi = \sigma_{j_0} \cdots \sigma_1$,

2. every $\sigma_k \in \mathcal{C}_{[2\,1^{n-2}]}$,
3. $\{\sigma_k\}$ generate $\mathfrak{S}_n$, and
4. j_0 is minimal subject to (1), (2), and (3).

Now, for $\pi \in \mathfrak{S}_n$, let $\alpha(\pi)$ be the corresponding partition and set

$$\kappa(\pi) = \ell(\alpha(\pi)).$$

Let $\sigma = (a\,b)$ be a transposition. Then there are two cases for the action of the product $\sigma\pi$ (from the right).

Join: If a, b are in different cycles of π, then

$$\kappa(\sigma\pi) = \kappa(\pi) - 1.$$

Cut: If not,

$$\kappa(\sigma) = \kappa(\pi) + 1.$$

Proposition 2.3.2

$$j_0 = n + \ell(\alpha) - 2,$$

which we denote by $\mu(\alpha)$.

Proof. Suppose that $(\sigma_1, \cdots, \sigma_j)$ is a transitive ordered factorization of $\pi \in \mathcal{C}_\alpha$ (i.e. satisfying (a), (b), and (c)). Let G be the graph on vertices labelled $\{1, \cdots, n\}$, and edges labelled $\{1, \cdots, j\}$, in which the edge labelled k connects the vertices interchanged by σ_k for every k. (By (c), G is connected.)

Let T be the *spanning tree* of G, i.e. consisting of those edges e_k such that $G - e_k$ is disconnected. And suppose that $G - T$ contains edges corresponding to a joins and b cuts. Then since T contains $n-1$ edges corresponding to joins,

$$\ell(\alpha) = n - (n - 1 + a) + b.$$

Since G has $n - 1 + a + b$ edges,

$$j = n - 1 + a + b = n + \ell(\alpha) - 2 + 2a \geq n + \ell(\alpha) - 2.$$

On the other hand, it is easy to see that the equality can occur if $a = 0$.

■

Theorem 2.3.3 *Let $\alpha = [\alpha_1 \cdots \alpha_J] \vdash n$. Then*

$$c_\alpha = n^{J-3}(n+J-2)! \prod_{j=1}^{J} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!}.$$

Corollary 5 (cf. [14])

$$c_{[n]} = n^{n-2}, \qquad c_{[1^n]} = (2n-2)!\, n^{n-3}.$$

Further, (since the Hurwitz numbers H^0_α are just c_α up to suitable multiplicative constants) we conclude that

$$H^0_{[n]} = n^{n-3}, \qquad H^0_{[1^n]} = \frac{(2n-2)!\, n^{n-3}}{n!}.$$

To prove Theorem 2.3.3, we recall that joins and cuts can be represented by a differential operator.

Definition Let $h(\alpha)$ be the size of $\mathcal{C}_\alpha$ and $\mathbf{K}_\alpha$ be the sum of all elements of $\mathcal{C}_\alpha$ in the group algebra $\mathbb{C}\mathfrak{S}_n$.

Let $(p_1, \cdots, p_j, \cdots)$ be indeterminates (variables). If $\alpha = [\alpha_1, \cdots, \alpha_J]$, then set

$$p_\alpha = p_{\alpha_1} \cdots p_{\alpha_J}.$$

And for every $\pi \in \mathfrak{S}_n$, set

$$\Phi(\pi) = p_{\alpha(\pi)}.$$

Then we can extend Φ linearly to the whole $\mathbb{C}\mathfrak{S}_n$.

Proposition 2.3.4

$$\Phi(\mathbf{K}_{[2\,1^{n-2}]}\pi) = \Delta\Phi(\pi),$$

where

$$\Delta = \frac{1}{2} \sum_{i,j \geq 1} \left(p_{i+j}\, ij \frac{\partial^2}{\partial p_i \partial p_j} + p_i p_j (i+j) \frac{\partial}{\partial p_{i+j}} \right).$$

Proof. Let $\sigma = (a, b)$ be an arbitrary transposition. If σ is join for π, let i, j be the lengths of cycles containing a, b respectively. Then a $p_i p_j$ in $\Phi(\pi)$ is replaced by a p_{i+j} in $\Phi(\sigma\pi)$. If σ is cut for π, then similarly, a p_{i+j} in $\Phi(\pi)$ is replaced by a $p_i p_j$ in $\Phi(\sigma\pi)$.

Counting the number of cases, we have the assertion. ∎

Lemma 2.3.5 *The generating series*

$$F = \sum_{n\geq 1} \left(\sum_{\alpha \vdash n} \frac{h(\alpha) c_\alpha}{\mu(\alpha)!} p_\alpha \right) \frac{z^n}{n!}$$

satisfies the differential equation

$$L(F) = 0,$$

where

$$L = \Delta - z\frac{\partial}{\partial z} - \sum_{j\geq 1} p_j \frac{\partial}{\partial p_j} + 2.$$

Proof. Extend F to the generating function

$$\tilde{F} = \sum_{n\geq 1} \left(\sum_{\alpha \vdash n} \frac{h(\alpha) c_\alpha}{\mu(\alpha)!} u^{\mu(\alpha)} p_\alpha \right) \frac{z^n}{n!}.$$

Now, suppose that $(\sigma_1, \cdots, \sigma_{j_0})$ is a minimal, transitive, ordered factorization of π, and remove the edge corresponding to σ_{j_0} from the graph G corresponding to π. This is equivalent to modify the generating series $\tilde{F}$ to

$$\frac{\partial \tilde{F}}{\partial u}.$$

On the other hand, if σ_{j_0} is a cut, then $(\sigma_1, \cdots, \sigma_{j_0 - 1})$ is a minimal transitive ordered factorization of an element in $\mathfrak{S}_n$, and a p_{i+k} in $\Phi(\pi)$ is replaced by a $p_i p_k$ in $\Phi(\sigma_{j_0}\pi)$. And if $\sigma_{j_0} = (a, b)$ is a join, then let i, k be the length of cycles containing a, b, respectively. Then $(\sigma_1, \cdots, \sigma_{j_0 - 1})$ represents a pair of minimal transitive ordered factorizations of elements in $\mathfrak{S}_i$ and in $\mathfrak{S}_k$. Hence a $p_i p_k$ in $\Phi(\pi)$ is replaced a p_{i+k} in $\Phi(\sigma_{j_0}\pi)$. Hence we conclude that

$$\frac{\partial \tilde{F}}{\partial u} = \Delta \tilde{F}.$$

On the other hand, by setting $u = 1$, we have

$$\Delta F = \left.\frac{\partial \tilde{F}}{\partial u}\right|_{u=1} = z\frac{\partial F}{\partial z} + \sum_{i\geq 1} \frac{\partial F}{\partial p_j} - 2F,$$

which implies the assertion. ■

Thus if we show the following lemma, we see that F $\hat{F}$ defined below, for the constant terms and the coefficient of z (as the series of z) are the same.

Lemma 2.3.6 *The series*

$$\hat{F} = \sum_{n\geq 1} \left(\sum_{\alpha\vdash n} n^{J-3} \prod_{j=1}^{J} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!} h(\alpha) p_\alpha \right) \frac{z^n}{n!}$$

satisfies

$$z\frac{\partial}{\partial z} L(\hat{F}) = 0.$$

Proof. First, let $s(z, p_j)$ be the unique solution of

$$s = z \exp\left(\sum_{j=1}^{\infty} \frac{j^j}{j!} p_j s^j \right).$$

Then since the coefficeint

$$\sum_{\alpha\vdash n} \frac{n^J}{n!} \left(\prod_{j=1}^{J} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!} h(\alpha) p_\alpha \right)$$

of z^n/n^3 in $\hat{F}$ is the coefficient of s^n in

$$\exp\left(n \sum_{j=1}^{\infty} \frac{j^j}{j!} p_j s^j \right),$$

which is also the coefficient of z^n/n^3 in

$$n \log\left(\frac{s}{z}\right),$$

we can show that

$$\left(z\frac{\partial}{\partial z}\right)^2 \hat{F} = \sum_{j=1}^{\infty} \frac{j^j}{j!} p_j s^j.$$

Here, since

$$z\frac{\partial s}{\partial z} = s\left(1 - \sum_{j=1}^{\infty} \frac{j^{j+1}}{j!} p_j s^j\right)^{-1},$$

we have

$$\begin{aligned} z\frac{\partial \hat{F}}{\partial z} &= \int_0^s \left(\sum_{j=1}^{\infty} \frac{j^j}{j!} p_j s^{j-1}\right)\left(1 - \sum_{j=1}^{\infty} \frac{j^{j+1}}{j!} p_j s^j\right) ds \\ &= \sum_{j=1}^{\infty} \frac{j^{j-1}}{j!} p_j s^j - \frac{1}{2}\left(\sum_{j=1}^{\infty} \frac{j^j}{j!} p_j s^j\right)^2. \end{aligned}$$

Also since

$$\frac{\partial s}{\partial p_i} = \frac{i^i}{i!} s^i z \frac{\partial s}{\partial z},$$

we can show that

$$z\frac{\partial^2 \hat{F}}{\partial z \partial p_i} = \frac{i^{i-1}}{i!} s^i$$

for every $i \geq 1$, which implies that

$$\frac{\partial \hat{F}}{\partial p_i} = \frac{i^{i-2}}{i!} s^i - \frac{i^{i-1}}{i!} \sum_{j=1}^{\infty} \frac{j^{j+1}}{j!} p_j \frac{s^{i+j}}{i+j}.$$

Thus setting

$$S_\ell = \sum_{i,j \geq 1,\, i+j=\ell} \frac{i^i}{i!} \frac{j^{j-1}}{j!},$$

$$T_{k,\ell} = \frac{k^{k+1}}{k!} \sum_{i \geq 1,\, j \geq 0,\, i+j=\ell} \frac{i^i}{i!} \frac{j^j}{j!} \frac{1}{k+j},$$

we see that

$$\begin{aligned}&\frac{\partial}{\partial z}L(\hat{F})\\ &= \sum_{j=1}^{\infty}\left(\left(S_j-\frac{j^j}{j!}+\frac{j^{j-1}}{j!}\right)p_js^j+\left(\frac{1}{2}\frac{(2j)^{2j}}{(2j)!}-T_{j,j}\right)p_j^2s^{2j}\right)\\ &+\sum_{i,j\geq 1, i\neq j}\left(\frac{(i+j)^{i+j}}{(i+j)!}-T_{i,j}-T_{j,i}\right)p_ip_js^{i+j}.\end{aligned}$$

On the other hand, let $w(z)$ be the unique solution of $w = ze^w$. Then S_j is just the coefficient of z^j of

$$\frac{w}{1-w}w = \frac{w}{1-w} - w.$$

Hence we conclude that

$$S_j = \frac{j^j}{j!} - \frac{j^{j-1}}{j!}.$$

Next let $u(z)$ and $v(z)$ be the unique solutions of $u = xe^u$ and $v = ye^v$, respectively, and set

$$T(x,y) = \frac{v}{1-v}\left(\frac{x}{y-x} - \frac{u}{v-u}\frac{1}{1-u}\right).$$

Then we can see that

$$T(x,y) + T(y,x) = \sum_{j=1}^{\infty}\frac{j^j}{j!}\frac{xy^j - yx^j}{y-x},$$

and that $T_{i,j} + T_{j,i}$ is the coefficient of x^iy^j of $T(x,y)+T(y,x)$, which implies that

$$T_{i,j} + T_{j,i} = \frac{(i+j)^{i+j}}{(i+j)!}.$$

Thus we have proved the assertion. ■

2.4 Open problems

Problem Determine the *Ahlfors locus* of $\mathbf{B}_n$ which consists of all $(\mathbf{a}, \mathbf{b})$ such that $f_{\mathbf{a},\mathbf{b}}$ gives an Ahlfors map (, or more precisely, $e^{i\theta} f_{\mathbf{a},\mathbf{b}}$ with a suitable $\theta \in \mathbb{R}$ is an Ahlfors map).

Problem Fix a point $(\mathbf{a}, \mathbf{b})$ in $\mathbf{B}_n$, and let $W = W_{\mathbf{a},\mathbf{b}}$ be the corresponding n-ply conenncted damain. Determine the *leaf* $E(W)$ of $\mathbf{B}_n$ for W, consisting of all points in $\mathbf{B}_n$ which correspond to n-ply connected domains biholomorphically equivalent to W.

Problem Determine the *collision locus* Γ of $\mathbf{B}_n$. (Recall that the collision locus of $\mathbf{B}_2$ is empty.)

Chapter 3

Transcendental Hurwitz spaces of finite type

In this chapter, we consider the Hurwitz spaces of so-called *structurally finite* transcendental entire functions, which can be considered as the simplest transcendental Hurwitz spaces. Structurally finite functions are introduced in §1 using configuration trees. We explain in §2 the geometric compacification of the Hurwitz spaces of finite type, and in §3 the relation between the covering structures and the dynamical structures of such functions. We give some open problems in §4.

3.1 Configuration trees

We have assumed that the singular value set is *closed and countable.* But this condition is still too weak. Here, we consider subclasses as follows.

Definition The projection $f : D \to \tilde{D}$ as before is called a *function with a finite number of singularity clusters* if the singular value set S_f of f has only a finite number of accumulation points in $\tilde{D}$.

Further, we call a function f with a finite number of singularity clusters an *approximate Speiser function* if the set S_f is bounded in $\tilde{D}$. The projection f is called a *Speiser function* if f has only a finite number of singular values.

Also, we have various kinds of singlarities. And we can consider the folloing kind of singularity.

Definition For a function $f : D \to \tilde{D}$ with a finite number of singularity clusters, we say that a point α is *virtually evenly covered by* f *with respect to a neighborhood* U if α is not a *critical value* of f and if there is a simple path L from the boundary of U to α such that every connected component D of $f^{-1}(U-L)$ is relatively compact and f is a biholomorphic map of D onto $U-L$.

We call a point which is not virtually evenly covered by π with respect to any neighborhood of α a *singular value of the covering* by f.

By definition, a point which is evenly covered by f is virtually evenly covered by f. Hence a singular value of the covering by f is a singular value of f. Also note that we include accumulation points of asymptotic values (even if they are not asymptotic values) in the set of all singular values of the covering by f. Here we say that α is an *asymptotic value* of the projection f if there is a path exiting D, i.e., the image of a proper continuous map of $[0,+\infty)$ into D, along which f tends to α.

Proposition 3.1.1 *Suppose that a singular value* α *of the covering by* $f: D \to \tilde{D}$ *is an isolated point in the set of all singular values of* f*. Then* α *is a singular value of* f*. In particular, if* f *is a Speiser function, then every singular value of the covering by* f *is a singular value of* f*.*

Proof. If α is isolated in the set of all singular values of the covering by f and not a critical value, then there is a disk U with center α such that $U-\{\alpha\}$ contains no singular values. Since α is a singular value of the covering by f, for any path from the boundary of U to α, there is a relatively non-compact connected component V of $f^{-1}(U-L)$, since f is always a biholomorphic map of V onto $U-L$ in this case. Thus α is an asymptotic value, which implies the assertion. ∎

Now we will give several examples. For the sake of simplicity, we consider the case that $D=\mathbb{C}$ only.

Example 16 *The entire function*

$$g(z)=\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(a+\frac{z}{n}\right)e^{-z/n}$$

is a function with a finite number of singularity clusters, but not an approximate Speiser function.

As another example, we give

$$g(z)=z\sin z.$$

Example 17 *The entire function*

$$g(z) = \frac{z-c}{z} \sin z$$

is an approximate Speiser function, but not a Speiser function.
As another example, we give

$$g(z) = \frac{\sin z}{z}.$$

Example 18 *The entire function*

$$g(z) = \sin z$$

is a Speiser function.
As another example, we give

$$g(z) = \exp z.$$

Next, to describe the covering structure of entire functions, we use the following kind of configuration graph.

Definition A *plain configuration tree* T is a planar tree with countably many vertices, one of which is marked as *the initial vertex* v_T (and hence every edge has an orientation towards v_T).

The tree T is colored as follows:

1. There are white vertices and black ones.

2. There are white edges, black ones, and red ones.

3. Every connected component of the set of all white vertices and white edges can be identified with the tree $\mathbb{R}$ with vertices $\mathbb{Z}$, and hence is called a $\mathbb{Z}$*-unit.*

4. Every edge not in any $\mathbb{Z}$-unit is colored black or red, according as it starts from a black vertex or from a white vertex.

The triple $(T, (S, \pi), s_T)$ of a plain configuration tree T, the configuration data $(S, \pi) = (\alpha_1, \cdots, \alpha_n)$, and the data map s_T is called a *decorated ideal configuration tree (DICT).*

Here a *configuration data* (S, π) is the cyclically ordered finite set of the *(singularity) data* $S = \{\alpha_j \in \hat{\mathbb{C}}\}_{j=1}^n \in B_{0,n}(\hat{\mathbb{C}})$, which is called the *plain data set.* The cyclic order π of S is called the *marking* of S. Any realization of this marking as a disjoint union of arcs from each point in S to ∞ is called a *spider at* ∞ for this marking.

Finally, the *data map* $s_T(v)$ is a surjection of the set $V_0(T)$ of all non-inital vertices of the plain configuration tree T to S such that s_T is a constant on every $\mathbb{Z}$-unit.

On the set $V(T) = V_0(T) \bigcup \{v_T\}$, we can consider three integer-valued functions; the *norm* N_T, the *age* A_T, and the *height* H_T (from v_T). See [51], §7.

Example 19 *For a Speiser entire function f, let S_f be the singular value set of f. Also fix a spider at ∞ for the marking of S_f, and let D be the simply connected domain obtained from $\mathbb{C}$ by deleting all legs of this spider.*

Then every component of $f^{-1}(D)$ is bibohomorphic to D and is called a plate. *we consider each plate as a vertex, and if two plates are connected (along a slit over a leg of the spider), then the corresponding two vertices are conencted by an edge.*

Here if the slit deleted the singular value is proper in $\mathbb{C}$, then the edge is to be white, and if not (and hence one end of the slit ends at a critical point), then the edge is to be black. Also if a vertex where a white edge ends is to be white, and the others are black. Take a vertex as the intial vertex.

Those edges such that the corresponding slits which determines the same logarithmic singularity form a $\mathbb{Z}$-unit with connecting white vertices. Here if a vertex should belongs to several, say n, $\mathbb{Z}$-units, then dupulicate the vertex to n vertices, and connect them by red edges so that the $\mathbb{Z}$-units are disjoint. Next if there is a black edge starts from a white vertex, then we add a reduction pair, *which is a pair of a red edge and its ending black vertex.*

This graph is actually a tree. We call it a plain configuration tree of *f, and denote it by T_f.*

Theorem 3.1.2 *Let f be a Speiser entire function. Then the Hurwitz space $H(f)$ of f coincides with* Top(f) *and has a finite dimension.*

Proof. Consider the data map $\tilde{H}(f) \to B_{0,n}(\mathbb{C})$ which maps an isomorphism class to the deta set, where n is the number of points in S_f. Then we can see that this is a local homeomorphism. ∎

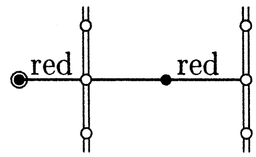

Figure 3.1: A configuration tree of $a \exp z^2 + b$

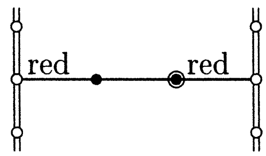

Figure 3.2: Another tree of $a \exp z^2 + b$

Now, when we change the initial vertex to another vertex, we delete all reduction pairs whose red edges have the opposite orientation in the new tree, and attach a new pair to every white vertex such that a black edge now starts from it. Here when we delete a reduction pair, we should regard that every edge having ended at the black vertex in the reduction pair of the old tree ends now at the white vertex from which the deleted red edge started.

We say that such a new configuration tree is obtained from the old one by a *change of the initial vertex.* Further, if a white vertex is the initial one, then we may attach a reduction pair and regard that the new black vertex is the initial one. Thus we can always assume that *the initial vertex is black.*

We say that two configuration trees are *equivalent* if, after suitable changes of the initial vertices of both, they are identical including colors.

Example 20 *Both of Figures 1 and 2 are configuration trees of the same entire function* $a \exp z^2 + b$ $(a \neq 0)$. *Here the concentric bigger circle indicates the initial vertex. Figure 2 is a configuration tree equivalent to that in Figure 1.*

Definition We say that a configuration tree T is *realizable by an entire function* (with respect to some configuration data) if there is an entire function f whose tree is equivalent to T.

Remark In general, the realizability of a configuration tree depends on the configuration data. For instance, if the singularity data consists of a single value, then the tree in Figure 1 is not realizable with respect to this configuration data.

Definition We say that a tree T is *eventually monotone* if there is a compact subtree E of T such that every connected component of $T - E$ is either purely white or purely black.

Proposition 3.1.3 *Let T be a locally compact and eventually monotone configuration tree realizable by an approximate Speiser entire function of order ρ. Then the number of non-compact connected components of $T - E$ is not greater than* $\max\{2\rho, 1\}$ *for every compact subset E of T. In particular, the number of $\mathbb{Z}$-units is not greater than ρ*

Example 21 *Another typical example of a configuration tree is a dual of a colored* tree dessin *of a Belyi function (cf. [48]). For the case of*

$$f(z) = A \int_0^z t^3(t-1)^2(t-c)(t-\overline{c})dt$$

satisfying $f(1) = 1$ and $f(c) = 0$, see Figure 3.

Here, a Belyi function *is a polynomial with only two critical values* $0, 1$. *The tree dessin of f is colored so that every point in $f^{-1}(0)$ is represented by a green (white in Figure 3) vertex and every point in $f^{-1}(1)$ is represented by a red (black in Figure 3) vertex.*

Now, edges in the tree dessin TD of f correspond to black vertices in a configuration tree T_f of f. A green or red vertex of TD correspond to black edges with the singularity datum 0 or 1, respectively.

In the converse process, we attach a red vertex to every free end point of an edge in TD. Also, if several neighboring edges in T_f are attached the same singularity datum, then the corresponding non-free end points of edges in TD should reduce to a single vertex in TD.

Finally, we define important subclasses of Speiser functions.

Definition We say that a Speiser entire finction f is *structurally tame* if a plain configuration tree of f associated with some marking has the bounded age.

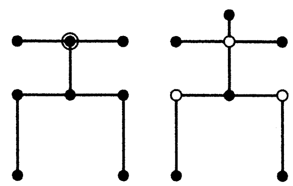

Figure 3.3: A configuration tree and the tree dessin of $f(z)$

Proposition 3.1.4 *Let f, g are structurally tame, then so is $f \circ g$.*

A proof will appear in [55].

Corollary 6 *For every structurally tame entire function f, every itertion f^n is structurally tame.*

Further, we can define a smaller subclass, again by using configuration trees.

Definition If the tree T_f of an entire function f consists of p black edges and q $\mathbb{Z}$-units, then we say that f is *structurally finite* and *of type* (p, q).

Recall that almost evenly covered covering structures are just those induced from structurally finite entire functions. We also characterize structural finiteness as follows.

Definition The *core* of a configuration tree is the smallest connected closed subtree containing all black vertices and non-white edges. And we call the tree *virtually compact* if the core is compact.

Theorem 3.1.5 (Virtual compactness) *The configuration tree of every structurally finite entire function is virtually compact. Conversely, every virtually compact configuration tree is realizable (with respect to a suitable data) by a structurally finite entire function.*

Remark A virtually compact tree is locally finite, eventually monotone. Every structurally finite entire function is structurally tame.

A typical example of an entire function with virtually non-compact but monotone tree is the sine function.

Now, to consider the deformation space of structurally finite entire function, we consider the following topology.

Definition Let f be a non-affine entire function. Then the *full deformation set* $FD(f)$ of f is the set of all entire functions g such that there is a quasiconformal self-map ϕ of $\mathbb{C}$ satisfying the *qc-L^∞ condition*:

$$D_f(g;\phi) = \|f - g \circ \phi\|_\infty \left(= \sup_{\mathbb{C}} |f - g \circ \phi| \right) < \infty.$$

It is clear that, if $g \in FD(f)$, then $f \in FD(g)$. And, for every pair of functions f_1 and f_2 in $FD(f)$, we set

$$d(f_1, f_2) = \inf \left(\log K(\phi_1 \circ \phi_2^{-1}) + \|f_1 \circ \phi_1 - f_2 \circ \phi_2\|_\infty \right),$$

where the infimum is taken over all normalized quasiconformal self-maps ϕ_1 and ϕ_2 of $\mathbb{C}$ satisfying the qc-L^∞ conditions $D_f(f_j;\phi_j) < \infty$.

This d is actually a distance, and $FD(f)$ equipped with this distance is a complete metric space. We call the distance d defined above the *synthetic Teichmüller distance* on $FD(f)$. The space $FD(f)$ equipped with this synthetic Teichmüller distance is called the *full synthetic deformation space* of f and is denoted as $FSD(f)$.

In the case of stracturally finite functions, we can show the following theorem.

Theorem 3.1.6 (Inclusion Theorem) *For a structurally finite entire function f, the full deformation set $FD(f)$ contains all structurally finite entire functions of the same type as that of f.*

Definition We define the set $SF_{p,q}$, where $p + q \geq 1$, by setting

$$SF_{p,q} = \left\{ \int_0^z (c_p t^p + \cdots + c_0) e^{a_q t^q + \cdots + a_1 t} dt + b \right\}$$

with $c_p a_q \neq 0$ if $q > 0$, and we note that $SF_{p,0} = \mathrm{Poly}_{p+1}$; the set of all polynomials of degree exactly $p + 1$.

Such primitive functions have already appeared as typical examples in various contexts. See for instance, [1], [3], [15], and [42].

Now the topological characterization of structurally finiteness in [51] shows the following

Corollary 7 *Every element of $SF_{p,q}$ is structurally finite and of type (p, q).*

Thus, the Inclusion Theorem implies that $FD(f)$ contains $SF_{p,q}$ for every $f \in SF_{p,q}$. In particular, the synthetic Teichmüller distance is finite on $SF_{p,q} \times SF_{p,q}$.

Actually, we can show the following

Theorem 3.1.7 (Representation Theorem [50]) *An entire function is structurally finite and of type (p,q) if and only if it belongs to $SF_{p,q}$.*

Corollary 8 *Let f be an element of $SF_{p,q}$ in general position. Then the Hurwitz space $H(f)$ is coincident with the sublocus $SS\mathcal{F}_{p,q}$ consisting of the isomorphism classes of all functions in $SF_{p,q}$ in general position.*

Definition For every $f \in SF_{p,q}$, we set $SD(f) = SF_{p,q}$ and equip it with the synthetic Teichmüller topology, which we call the *synthetic deformation space* of f.

Proposition 3.1.8 *For every $P \in SF_{p,0} = \mathrm{Poly}_{p+1}$, $SD(P) = FSD(P)$.*

On the other hand, we have the following

Example 22 (Melting of $\mathbb{C}$-decorations) *The functions*

$$f_j(z) = \left(1 + \frac{z}{j}\right) e^z \in SF_{1,1}$$

converge to $g(z) = e^z \in SF_{0,1}$ with respect to the synthetic Teichmüller topology as j tend to $+\infty$.

Indeed, fix an $\epsilon > 0$, and set $U = \{|w| < \epsilon\}$. Then for every sufficiently large j, the unique critical value of f_j is contained in U, and hence there is a conformal map ϕ_j of $W = \{\mathrm{Re}\, z > \log \epsilon\}$ into $\mathbb{C}$ such that $f = f_j \circ \phi_j$ on W, which we can extend to a quasiconformal self-map $\tilde{\phi}_j$ of $\mathbb{C}$ so that the maximal dilatation of $\tilde{\phi}_j$ tends to 1. Moreover, we have

$$D_f(f_j; \tilde{\phi}_j) \le \sup_{\partial W} (|f| + |f_j \circ \tilde{\phi}_j|) = 2\epsilon,$$

which implies the assertion.

Similarly,

$$g_j(z) = e^{2z} + \frac{2}{j} e^z$$

are structurally infinite, but converge to $g(2z)$ with respect to the synthetic Teichmüller topology as j tend to $\pm\infty$.

Also we can see that for every structurally finite entire function f in $SF_{p,q}$ with $q > 0$, $FD(f)$ is always so large that it contains structurally infinite entire functions, and in general $SD(f)$ is neither closed nor open in $FSD(f)$. Hence we need to consider slightly larger complete subspaces. The following theorem provides such spaces and also shows that $SD(f)$ can be considered as a stratum of $FSD(f)$.

Theorem 3.1.9 (Completeness Theorem) *Suppose that f belongs to the set $SF_{p,q}$ with $q > 0$, and let $\{f_j\}$ be a sequence in $SD(f)$ converging to some g in $FSD(f)$. Then g is structurally finite and of type (p', q) with $p' \leq p$.*

In particular,

$$SF_{\leq p,q} = \cup_{p' \leq p} SF_{p',q}$$

with the synthetic Teichmüller topology is a complete metric space, and hence a completion of $SD(f)$.

Corollary 9 *For every $q > 0$, $SF_{0,q}$ equipped with the synthetic Teichmüller topology is complete.*

Now $SF_{\leq p,q}$ has another natural topology induced from the coefficients of representatives. For instance, we define the line element ds by

$$ds = \frac{\sum_{m=0}^{p} |dc_m|}{\sum_{m=0}^{p} |c_m|} + \frac{|da_q|}{|a_q|} + \sum_{n=1}^{q-1} |da_n| + |db|$$

at every

$$f(z) = \int_0^z (c_p t^p + \cdots + c_0) e^{a_q t^q + \cdots + a_1 t} dt + b$$

in $SF_{\leq p,q}$. This distance is complete, and we call the induced topology the *coefficient topology* on $SF_{\leq p,q}$. Thus the Completeness Theorem shows the following

Corollary 10 (Equivalence Theorem) *The synthetic Teichmüller topology is equivalent to the coefficient topology on $SF_{p,q}$ for every p and q.*

Finally, we state the following theorem about the size of the Julia set. Proofs will be given in [52].

Theorem 3.1.10 *For every transcendental structurally finite entire function f, the Hausdorff dimension of its Julia set $J(f)$ is two.*

Remark Compare with a theorem of Stallard ([49] II): For every transcendental entire function such that the set of all singular values is bounded, the Hausdorff dimension of $J(f)$ is greater than 1.

As for the area of the Julia set, we have the following

Theorem 3.1.11 (Hyperbolic implies area zero) *Let f be a (not necessarily transcendental) structurally finite entire function. If f is hyperbolic, then $J(f)$ has vanishing area.*

Remark Devaney-Keen proved in [15] that, if the Schwarzian derivative of a meromorphic f is polynomial (such an f is structurally finite if f is entire) and f is hyperbolic, then the Julia set has vanishing area.

3.2 Geometric compactification

We can compactify the transcendental Hurwitz space of finite type as in the case of algebraic Hurwitz spaces. See [54] for the details.

Definition We say that a DICT $(T', (S', \pi'), s_{T'})$ is obtained from a DICT $(T, (S, \pi), s_T)$ by a *fundamental move* for the pair (α_k, α_{k+1}) of neighboring elements in (S, π) if the following three conditions are satisfied for this k:

i) The marked configuration data (S', π') of T' is obtained from (S, π) by interchanging two elements α_k and α_{k+1}. The induced bijection σ_k of S onto S' satisfies that $\pi' = \sigma_k \circ \pi$, and is called the *$k$-th exchanging map* of S.

ii) The tree T' is obtained from T by applying simple moves as follows, which we call the *canonical move* of T for the pair (α_k, α_{k+1}).

If a vertex v_j with $s_T(v_j) = \alpha_j$ is black, let $\hat{L}_j = L_j$ be the black edge starting from v_j. If a vertex v_j with $s_T(v_j) = \alpha_j$ is white, then let $\hat{L}_j$ be the $\mathbb{Z}$-unit containing v_j, L_j^* be the red edge starting from $\hat{L}_j$, and L_j be the white edge, *forward* with respect to the direction of $\hat{L}_j$, and having a common vertex with L_j^*.

Fix a pair (v_k, v_{k+1}) of vertices such that $s_T(v_k) = \alpha_k$ and $s_T(v_{k+1}) = \alpha_{k+1}$. Suppose that $\hat{L}_k$ and $\hat{L}_{k+1}$ are not disjoint or connected by only red edges. (Otherwise, we do nothing.) Then after reducing the tree completely (cf. [51]), we may assume that $\hat{L}_k \bigcup L_k^*$ and $\hat{L}_{k+1} \bigcup L_{k+1}^*$ have a unique common vertex v. Change the initial vertex to v, reduce the tree completely if necessary, apply the following move, and then change the initial vertex again to the original one. Applying these procedure to every pair of vertices such as above, we have the tree T'.

1. If both of L_k and L_{k+1} are black, then apply the inverse simple move of L_k along L_{k+1}.

2. If L_k^* exists and L_{k+1} is black, then apply the inverse simple move of L_k^* along L_{k+1}.

3. If L_{k+1}^* exists and L_k is black, then apply the inverse simple move of L_k along L_{k+1} in the tree obtained by the inverse simple move of L_k along L_{k+1}^*.

4. If both of L_k^* and L_{k+1}^* exist, then apply the inverse simple move of L_k^* along L_{k+1} in the tree obtained by the inverse simple move of L_k^* along L_{k+1}^*.

iii) Finally, identifying $V_0(T)$ with $V_0(T')$,

$$s_{T'} = \sigma_k \circ s_T,$$

possibly after an allowed change of the cyclic order indices.

Now let $(T, (S, \pi), s_T)$ be a DICT, and fix a data α in S. Let E_α be the union of all black and red edges, starting from vertices v with $s_T(v) = \alpha$, which we call the *colliding-locus* of T for α. Then every connected component E of E_α, which is called a *colliding component* (for α), consists of

1. either a single black edge only ,

2. or a single red edge only,

3. or more than one black edges and no red edges,

4. or a single red edge and at least one black edge,

5. or more than two red edges.

If a colliding component E satisfies the thrid, the forth, or the fifth condition, then we say that E is *of realizable type*, *of melting type*, or *of rearranging type*, respectively. In these cases, we can *unify* T at E, i.e. all edges contained in E are ended at the same vertex v of T which is the youngest, i.e.

the vertex with the minimal age among $V_0(T) \bigcup E$. The unified edges in E have a *linear order* induced from the cyclic order of S. We call this order of edges in E the *ribbon structure* of T at E.

Definition We say that two DICTs $(T_1, (S_1, \pi_1), s_{T_1})$ and $(T_2, (S_2, \pi_2), s_{T_2})$ are *equivalent* if, $S_1 = S_2$, and by applying a finite number of suitable fundamental moves, reducing trees completely, unifying the trees at every colliding component, and applying suitable changes of the intial vertices, we have the same configuration tree (including colors) with the same ribbon structures and the data maps.

We denote by $\mathcal{DICT}_{p,q}$ the set of equivalence classes of all DICTs of type (p, q). We equip $\mathcal{DICT}_{p,q}$ with the natural topology.

Further, if the data S contains ∞, then we say that such a DICT is *of divergent type.* We denote by $\mathcal{D}i\mathcal{ICT}_{p,q}$ the subset of $\mathcal{DICT}_{p,q}$ consisting of equivalence classes of DICTs of divergence type. And set

$$\mathcal{D}e\mathcal{ICT}_{p,q} = \mathcal{DICT}_{p,q} - \mathcal{D}i\mathcal{ICT}_{p,q}.$$

We denote by $\mathcal{D}m\mathcal{ICT}_{p,q}$ and $\mathcal{D}r\mathcal{ICT}_{p,q}$ the sets of equivalence classes of all DICT having colliding components of melting type and of rearranging type, respectively.

Finally we set

$$\mathcal{DCT}_{p,q} = \mathcal{D}e\mathcal{ICT}_{p,q} - \left(\mathcal{D}m\mathcal{ICT}_{p,q} \bigcup \mathcal{D}r\mathcal{ICT}_{p,q}\right),$$

and representatives of the elements in $\mathcal{DCT}_{p,q}$ is simply called a *decorated configuration tree (DCT)* of type (p, q).

Definition We say that a DICT $\tau = (T, S, s_T)$ is *realizable by an entire function f* if f is represented by a DICT $(T', S', s_{T'})$, with T' completely reduced and unified at every colliding component, equivalent to τ under the following specifications;

1. every black edge and its starting black vertex v of T' represent a Maskit surgery attaching a quadratic block, and
2. every red edge and its starting $\mathbb{Z}$-unit $\hat{L}$ of T' represent a Maskit surgery attaching an exp-block,

where the corresponding data value $s_{T'}(v)$ or $s_{T'}(V_0(T')\bigcap \hat{L})$ is the critical value of the quadratic block or the asymptotic value of the exp-block, respectively. Also Maskit surgeries are to be done inductively with respect to the age and the marking.

Furthermore, the direction of every $\mathbb{Z}$-unit corresponds to the clockwise rotation around the associated data value, and the ribbon structure at every data value corresponds to the counter-clockwise order around the data value. We call such a DICT (T, S, s_T) as above a *DICT of f*.

We say that a configuration data S of type $p+q$ is *simple* if S consists of $p+q$ mutually distinct values. We call a DCT with a simple data a *simple DCT* (i.e. a *SCT*). We denote by

$$\mathcal{SCT}_{p,q}$$

the set of all equivalence classes of SCTs of type (p, q).

Proposition 3.2.1 *A DICT is realizable by an entire function if and only if it is a DCT.*

Also, we have

$$\mathcal{SSF}_{p,q} = \mathcal{SCT}_{p,q}.$$

When $q = 0$, then $\mathcal{DICT}_{p,0}$ is enough to compactify $\mathcal{SSF}_{p,0}$. Note that in this case,

$$\mathcal{D}m\mathcal{ICT}_{p,0} = \emptyset, \quad \mathcal{D}r\mathcal{ICT}_{p,0} = \emptyset$$

and hence

$$\mathcal{DICT}_{p,0} = \mathcal{D}i\mathcal{ICT}_{p,0} \bigcup \mathcal{DCT}_{p,0}.$$

Theorem 3.2.2 *$\mathcal{SSF}_{p,0}$ is dense in $\mathcal{DICT}_{p,0}$, and $\mathcal{DICT}_{p,0}$ is compact.*

But, when $q > 0$, $\mathcal{DICT}_{p,q}$ is not compact.

Definition A *dupulicated ideal configuration tree (DuICT)* $\tau = (\mathbf{T}, (\mathbf{S}, \Pi), \mathbf{s_T})$ is an ordered set of DICTs, which are called *components* of τ,

$$\{(T_j, (S_j, \pi_j), s_{T_j})\}_{j=0}^{J}$$

with the pairings

$$\{(e_k, e_{k'})\}_{k=1}^{K}$$

of forward and backward white infinite rays, respectively, in a $\mathbb{Z}$-unit of T_k and of $T_{k'}$, which are called *white ends*, satisfying that

1. T_j is of type (p_j, q_j) with $p_j \leq p$ and $q_j \leq q$,

2. the initial vertx of T_j with $j \geq 1$ is the vertex of a reduction pair, which we call the *initial reduction pair* of T_j, from a vertex not contained in $\bigcup_k (e_k \bigcup e_{k'})$,

3. $K \leq p + q$,

4. $$\sum_j p_j = p, \quad \sum_j q_j = q + K,$$

5. the indices (k, k') are mutually distinct and $k \neq k'$ for every pair (k, k'),

6. for every subset E_0 of data values in $\mathbf{S} = \bigcup_j S_j$, the cyclic order of $E_0 \bigcap S_j$ induced from the marking of S_j are the same for every S_j,

7. the data value s_k on $V_0(T_k) \bigcap e_k$ equals to the value $s_{k'}$ on $V_0(T_{k'}) \bigcap e_{k'}$, and

8. by connecting $T_k - e_k$ and $T_{k'} - e_{k'}$ at the relative boundary points for every k, and deleting all initial reduction pairs of T_j with $j > 0$, we have a plain configuration tree T_* of type (p, q) (where the initial vertex is that of T_0).

Example 23 *Identify elements s_k with $s_{k'}$ in S for every pairing indices (k, k'), and set S_* be the resulting set. Let ι_j be the natural inclusion of S_j into S_*. Then we can give the set S_* a marking so that the restriction to $\iota_j(S_j)$ is the same as the marking of S_j for every j. We denote S_* with this marking by (S_*, π_*). Also we can define the data map s_{T_*} so that s_{T_*} restricted to $T_j - \bigcup_k (e_k \bigcup e_{k'})$ equals to s_{T_j} under the canonical inclusion of (S_j, π_j) to (S_*, π_*) induced from ι_j.*

We say that the DuICT τ is obtained from a DICT $(T_*, (S_*, \pi_*), s_{T_*})$ by applying the duplication *of a $\mathbb{Z}$-unit K times.*

Here, every pairing $(e_k, e_{k'})$ of forward and backward white ends determines a single ideal point in a standard manner, which we call the hyper-vertex $*_k$ *corresponding to the pairing.*

Definition We say that two DuICTs

$$\{(T_j^1, (S_j^1, \pi_j^1), s_{T_j^1})\}_{j=0}^{J^1}, \quad \{(T_j^2, (S_j^2, \pi_j^2), s_{T_j^2})\}_{j=0}^{J^2}$$

with the pairing

$$\{(e_k^1, e_{k'}^1)\}_{k=1}^{K^1}, \quad \{(e_k^2, e_{k'}^2)\}_{k=1}^{K^2}$$

are *equivalent* if,

1. $J^1 = J^2$ and $K^1 = K^2$,
2. every DICT $(T_j^1, (S_j^1, \pi_j^1), s_{T_j^1})$ is equivalent to the DICT $(T_j^2, (S_j^2, \pi_j^2), s_{T_j^2})$, and hence we can assume that they are the same,
3. every pairing $(e_k^1, e_{k'}^1)$ is *equivalent* to the pairing $(e_k^2, e_{k'}^2)$ for every k, i.e. $T_k^1 = T_k^2$, $T_{k'}^1 = T_{k'}^2$, and $e_k^1 \bigcap e_k^2$ and $e_{k'}^1 \bigcap e_{k'}^2$ are non-compact.

We denote by

$$\mathcal{DuICT}_{p,q}$$

the set of all equivalence classes of proper DuICTs of type (p, q).

We set

$$\widehat{\mathcal{SSF}_{p,q}} = \mathcal{DuICT}_{p,q} \bigcup \mathcal{DICT}_{p,q}.$$

Here the topology can be defined naturally so that $\widehat{\mathcal{SSF}}_{p,q}$ is a compactification of $\mathcal{SSF}_{p,q}$.

Theorem 3.2.3 *The space $\widehat{\mathcal{SSF}}_{p,q}$ is compact and $\mathcal{SSF}_{p,q}$ is a dense subset of it.*

3.3 Covering structures VS Dynamical structures

Back to the covering structure of polynomials, we note the following

Theorem 3.3.1 *Suppose that f is a polynomial of degree $N \geq 2$ such that f' is not a Ritt polynomial*

$$(z-d)^m P((z-d)^\ell),$$

where m and ℓ are non-negative integers, P is a polynomial, $d \in \mathbb{C}$, and $\ell > 1$. If another polynomial g satisfies that $g \circ g \in \mathcal{C}_{f \circ f}$, then $g \in \mathcal{D}_f$.

This theorem follows from a result by Ritt in [46], or directly from the following simple lemma.

Lemma 3.3.2 (Lenstra-Schneps lemma [48]) *Suppose that $P^{(*)}$ and $Q^{(*)}$ are polynomials with $P \circ Q = P^* \circ Q^*$ and the degrees of Q and Q^* are the same. Then there exists a similarity A such that $Q^* = A \circ Q$.*

Remark See [45], where Pilgrim shows that the dynamical structure of an extra-clean Balyi polynomial P is determined by the covering structure of $P \circ P$.

In general, a covering structure $\mathcal{C}_f$ corresponds to a complex two-dimensional family consisting of dynamical structures. An exception is the case of a non-linear polynomial f with a single critical point. When $f(z) = z^N$, then $\mathcal{C}_f$ contains all

$$g(z) = c_1(z-d)^N + c_2 \qquad (c_1 \neq 0).$$

And for every such g, $\mathcal{D}_g = \mathcal{D}_{P_c}$ with a suitable $P_c(z) = z^N + c$. Hence $\mathcal{C}_f$ corresponds to a complex one-dimensional family of dynamical structures, i.e.

$$\{\mathcal{D}_{P_c} \mid c \in \mathbb{C}\}.$$

And we can show a similar theorem as the above theorem also for the case of structurally finite transcendental entire functions, by using the following characterization of such functions.

Proposition 3.3.3 (Cf. [51]) *An entire function $f(z)$ is structurally finite if and only if f is a Speiser function and, applying the resolutions of a finite number of singularities of f^{-1} (with respect to a given spider at ∞) to the covering $f : \mathbb{C} \to \mathbb{C}$, we have the trivial covering of $\mathbb{C}$ by a countable number of $\mathbb{C}$.*

Here in general, the *resolution of a singularity* σ of π^{-1} (which is either a critical point of π or a logarithmic singurality of π^{-1}) for a Speiser covering $\pi : R \to \mathbb{C}$ of $\mathbb{C}$ by a, not necessarily connected, Riemann surface R with respect to a given spider at ∞, is the operation defined as follows;

1. cut R along all components of $\pi^{-1}(\ell)$ tending to σ, where ℓ is the leg of the spider ending at the singular value corresponding to σ, and

2. paste each component of the surface obtained in the first operation along the newly appearing borders over ℓ, if exist, so that $\pi : R \to \mathbb{C}$ induces a holomorphic covering $\pi' : R' \to \mathbb{C}$ of $\mathbb{C}$ by the resulting, not necessarily connected, Riemann surface R'.

Theorem 3.3.4 *Suppose that f is a structurally finite transcendental entire functions such that f' is neither a Ritt function*

$$(z-d)^m P((z-d)^\ell) e^{Q((z-d)^\ell)}$$

nor an exponential function

$$e^{cz+d},$$

where P and Q are polynomials, m and ℓ are non-negative integers, $d \in \mathbb{C}$, $c \in \mathbb{C} - \{0\}$, and $\ell > 1$. If another entire function g satisfies that $g \circ g \in \mathcal{C}_{f\circ f}$, then $g \in \mathcal{D}_f$.

This theorem is a generalization of Theorem 2 in [57] (cf. [58]). The proof below is different from, and simpler than, that of Theorem 2 in [57]. Also see [1], [13] and [32].

Example 24 *Let $f(z) = ae^{bz} + c$ with $ab \neq 0$. Then $\mathcal{C}_{f\circ f}$ contains $g \circ g$ for every g with the same form as that of f. Recall that every such $g \in \mathcal{D}_{e_\lambda}$, where $e_\lambda(z) = e^{\lambda z}$ with a suitable $\lambda \in \mathbb{C} - \{0\}$.*

To prove Theorem 3.3.4, first we note the following fact, which is an easy consequence of Proposition 3.3.3.

Lemma 3.3.5 *Such a function g as in Theorem 3.3.4 is structurally finite.*

Thus as in the case of polynomials, this theorem follows from the lemma below.

Lemma 3.3.6 (Transcendental Lenstra-Schneps Lemma) *Let f and g be structurally finite transcendental entire functions. Suppose that other structurally finite transcendental entire funcitons f^* and g^* satisfy the equation $f \circ g = f^* \circ g^*$. Then there exists a similarity A such that $g = A \circ g^*$ (and hence $f = f^* \circ A^{-1}$).*

Example 25 *If one of f, f^*, g, and g^* is structurally infinite, then the assertion of the above lemma does not necessarily hold. A typical example is a logarithmic lift:*

$$f(z) = e^z, \quad f^*(z) = ze^z, \quad g(z) = z + e^z, \quad g^*(z) = e^z.$$

Another typical example is

$$f(z) = e^{z^2}, \quad f^*(z) = e^{1-z^2}, \quad g(z) = \sin z, \quad g^*(z) = \cos z.$$

Here g and g^* determine the same covering structure, but the assertion of the lemma does not hold.

On the other hand, we can show the following proposition by the same argument as in the proof of Lemma 3.3.5.

Proposition 3.3.7 *Suppose that f and g are structurally finite, that g^* is transcendental, and that $f \circ g = f^* \circ g^*$ with another entire function f^*. Then f^* is structurally finite.*

Finally, we note the following corollaries of the transcendental Lenstra-Schneps Lemma.

Proposition 3.3.8 *Let f and g be structurally finite transcendental entire functions. Suppose that $f \circ g = g \circ f$. Then $g = A \circ f$ and also $f = g \circ A^{-1}$ with a suitable similarity A.*

Moreover suppose that neither f nor g has the form

$$\int_d^z P((t-d)^\ell) e^{Q((t-d)^\ell)} dt + d$$

with a suitable integer $\ell > 1$, polynomials P and Q, and $d \in \mathbb{C}$. Then $f = g$.

Corollary 11 *Let f and g be structurally finite transcendental entire functions. Suppose that $f \circ g = g \circ f$. Then the Julia sets of f and g coincide with each other.*

3.4 Open problems

Problem Determine the conditions for a given geometrically tame configuration tree to be realizable by entire functions.

Conjecture Let f be a Speiser entire function. Then the Hausdorff dimension of the Julia set of f is two.

Problem Determine the conditions for two given Speiser functions to commute.

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