수 학 강 의 록 제 51 권

LECTURE NOTES ON THE GLOBAL STABILITY OF THE MINKOWSKI SPACE

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Notes of the Series of Lectures
held at the Seoul National University

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Lecture notes on the global stability of the Minkowski space

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1 Introduction

These lecture notes originate from a series of lectures I gave during my visit at the Mathematics Department of Seoul National University. I was encouraged to submit these notes in the present, still rather crude, form for publication in their lecture notes series. Their content is the exposition of the central part of a new proof of the global stability of the Minkowski space, an important result due to D.Christodoulou and S.Klainerman at the end of the eighties, [Ch-Kl2]. This new proof which also, someway, extends the previous one, is a joint work of D.Christodoulou, S.Klainerman and the present author, [Ch-Kl-Ni].

Not all the details are given here, but I have tried to extract the general ideas of the proof from the large amount of the technical work involved. I have also used extensively, in writing these notes, the review paper by S.Klainerman and myself, "On local and global aspects of the Cauchy problem in general relativity", [Kl-Ni].

Finally I would like to express my gratitude to the Global Analysis Research Center and the Mathematics Department of Seoul National University. In particular my special thanks to Professor Dongho Chae for his warm hospitality.

2 The general statement of the result

2.1 The Einstein vacuum equations

$$R_{\mu\nu} = 0$$

where

$$R_{\mu\nu} = \frac{\partial\Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{\alpha}_{\alpha\mu}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}$$

and

$$\Gamma^{\alpha}_{\alpha\mu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$$

are the Christoffel symbols.

2.1.1 The principal part of the Ricci tensor

$$R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}\right) + ...$$

2.1.2 The costraint equations

They follow from the Bianchi identities, $G^0_\mu = 0$,

$$\begin{aligned} &\nabla^j k_{ij} - \nabla_i trk = 0 \\ &{}^{(3)}R - |k|^2 + (trk)^2 = 0 \end{aligned} \qquad (2.1)$$

2.1.3 The gauge freedom

$\Phi : \mathcal{M} \rightarrow \mathcal{M}$, a diffeomorphism (a change of coordinates).

Any pair $\{\mathcal{M}, g\}$ and $\{\mathcal{M}, \Phi^* g\}$ describes the same spacetime, that is both $g_{\mu\nu}$ and $(\Phi^* g)_{\mu\nu}$ are solutions of the Einstein vacuum equations.

2.2 The Cauchy problem for the Einstein vacuum equations

Definition 2.1 *An initial data set is a three-dimensional smooth manifold Σ with, defined on it, a Riemannian metric $\overline{g}$ and a covariant symmetric tensor field $\overline{k}$ satisfying the costraint equations.*

Definition 2.2 *To solve the Cauchy problem for the vacuum Einstein equations with a given initial data set means to find a four dimensional manifold Ω, a Lorentz metric g solution of the vacuum Einstein equations and an embedding*

$$i : \Sigma \rightarrow i(\Sigma) \equiv \Sigma_0 \subset \Omega$$

such that

$$i^*(g_0) = \overline{g} \ , \ i^*(k_0) = \overline{k}$$

where Σ_0, the "initial" hypersurface, is a Cauchy hypersurface, g_0 is the restriction of g on Σ_0 and k_0 is the second fundamental form (the extrinsic curvature) of Σ_0 relative to the metric g.

(Ω, g) is said a development of the initial data set.

2.2.1 The "hyperbolicity" of the Einstein equations

The Einstein vacuum equations are not hyperbolic in any standard sense due to their general covariance. In fact under a diffeomorphism their expression as a system of partial differential equations can change drastically. Nevertheless, they share with the hyperbolic equations the property that the Cauchy problem is well posed both for the local existence and for the uniqueness. This can be traced back to the fact that, using the general covariance, one can choose a set of coordinates, for instance the harmonic coordinates, [Br1], such that, with respect to it, the Einstein vacuum equations take the form of a quasilinear hyperbolic system. The local uniqueness theorem takes the form of a domain dependance theorem. The uniqueness is then extended to a global [1] result from the Theorem of Choquet-Bruhat and Geroch, [Br-Ge], on the existence of of a unique maximal (vacuum) Cauchy development.

[1] Here the term "global" has not the standard meaning of "global in time" used in the partial differential equations. This last one, in fact, will be true only if the causal geodesics of the maximal development are complete.

2.3 The results

The first result in solving the vacuum Einstein equations without any specific symmetry and with asymptotic flat (non compact) initial data was obtained by Choquet-Bruhat, see [Br1], who solved the local existence problem.

This result proved that, under appropriate regularity conditions and smallness conditions for the initial data, this set, $\{\Sigma_0, g_0, k_0\}$, can be integrated a finite distance into the future and the resulting solution, its (Ω, g) development, is unique [2].

More and more difficult problems can be faced, depending on how "large" is the region Ω. In particular we consider the two following problems:

2.3.1 The radiation problem

(Ω, g) is such that $\exists$ a compact set $K \subset \Sigma_0$ such that Σ_0/K is diffeomorphic to the complement of the closed ball $B_1 \subset R^3$ and, $\forall p \in \Sigma_0/K$, there is a complete null outgoing geodesic passing through p.

2.3.2 The global problem

(Ω, g) is timelike and null geodesically complete. The solution to the global problem was obtained by D.Christodolou and S.Klainerman, see [Ch-Kl2], under appropriate smallness conditions on the intial data [3]. Although this problem is more general than the radiation problem nevertheless their rather complicated proof is not completely suited [4] to prove the radiation problem.

Recently D.Christodolou, S.Klainerman and F.Nicolò, [Ch-Kl-Ni], have obtained a new proof for the radiation problem. Moreover the technique used there provides also a simpler proof of the global problem, see also [Kl-Ni].

This is the result described, in some detail, in these lecture notes. We give now, for completeness, the exact statement of the Main Theorem, but we warn the reader that, to understand it precisely, he has to go through the remaining part of this work where the structure of the proof and all the needed definitions are given.

Theorem 2.1 *Consider an initial data set $\{\Sigma_0, g, k\}$ and assume that the quantity*

$$J_0(\Sigma_0, g, k) = \sup_{\Sigma_0}\left((d_0^2+1)^3|Ric|^2\right) + \int_{\Sigma_0}\sum_{l=0}^{3}(d_0^2+1)^{l+1}|\nabla^l k|^2$$

[2]The smallness of initial data means that Σ_0 is "near" to the flat hypersurface. This condition can be significantly weakened paying the price of reducing the existence time. The regularity conditions require the control up to the fourth derivatives for $\overline{g}$ and the third derivatives for $\overline{k}$. The better regularity results are obtained by A.Fisher, J.E.Marsden, see [F-Ms1].

[3]the existence of the singularity theorems imply that this problem can be solved only for a limited class of initial data.

[4]Probably with some additional work it can also be derived from this result.

$$+\int_{\Sigma_0}\sum_{l=0}^{1}(d_0^2+1)^{l+3}|\nabla^l B|^2$$

is bounded.

There exists a sufficient large compact set $K \subset \Sigma_0$, with Σ_0/K diffeomorphic to R^3/B_1, and a unique development $(\mathcal{M},g)$ defined outside the domain of influence of K with the following properties:

i) $\mathcal{M}=\mathcal{M}^+\cup\mathcal{M}^-$ where $\mathcal{M}^+$ consists of the part of $\mathcal{M}$ which is in the future of Σ_0/K, $\mathcal{M}^-$ the one to the past.

ii) $(\mathcal{M}^+,g)$ can be foliated by a double null foliation $\{C(u)\}$, $\{\underline{C}(\underline{u})\}$ whose outgoing leaves $C(u)$ are complete [5] for all $u \leq u_1$. The boundary of K can be chosen to be the intersection of $C(u_1)\cap\Sigma_0$.

iii) The norms $\mathcal{O}$ and $\mathcal{R}$ are bounded by a constant [6].

iv) The null Riemann components have the following asymptotic behaviour:

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2}|\alpha| \leq C_0 \ , \ \sup_{\mathcal{K}} r u^{\frac{5}{2}}|\underline{\alpha}| \leq C_0 \\
&\sup_{\mathcal{K}} r^{7/2}|\beta| \leq C_0 \ , \ \sup_{\mathcal{K}} r^2 u^{\frac{3}{2}}|\underline{\beta}| \leq C_0 \\
&\sup_{\mathcal{K}} r^3|\rho| \leq C_0 \ , \ \sup_{\mathcal{K}} r^3 u^{\frac{1}{2}}|(\rho-\overline{\rho},\sigma)| \leq C_0
\end{aligned} \tag{2.2}$$

with C_0 a constant depending on the initial data.

v) $(\mathcal{M}^-,g)$ satisfies the same properties as $(\mathcal{M}^+,g)$.

vi) If $J(\Sigma_0,g,k)$ is sufficiently small we can extend $(\mathcal{M},g)$ to a smooth, complete solution compatible with the global stability of the Minkowski space.

3 The global strategy

To solve the local existence problem Choquet-Bruhat used the gauge freedom of the General Relativity. This allowed to choose a specific gauge: the harmonic gauge or "wave gauge", and trasform the Einstein equations into a "weakly coupled" system of non linear wave equations.

[5] By this we mean that the null geodesics generating $C(u)$ can be indefinitely extended toward the future.

[6] These norms are defined in the course of the proof. The norm $\mathcal{O}$ summarizes the norm of the null connection which describe the causal properties of the spacetime, the norm $\mathcal{R}$ summarizes the properties of the curvature tensor.

3.1 The wave-like coordinates

An easy algebraic computation shows that:

$$R_{\mu\nu} = R^{(h)}_{\mu\nu} + \frac{1}{2}(g_{\mu\alpha}\frac{\partial\Gamma^\alpha}{\partial x^\nu} + g_{\nu\alpha}\frac{\partial\Gamma^\alpha}{\partial x^\mu})$$

where:

$$\begin{aligned} R^{(h)}_{\mu\nu} &= -\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + H_{\mu\nu}(g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial x^\mu}) \qquad (3.3) \\ H_{\mu\nu} &= g^{\alpha\beta}g_{\rho\sigma}\Gamma^\rho_{\mu\beta}\Gamma^\sigma_{\nu\alpha} + \frac{1}{2}(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\Gamma^\alpha + g_{\nu\lambda}\Gamma^\lambda_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}\frac{\partial g_{\gamma\delta}}{\partial x^\mu} + g_{\mu\lambda}\Gamma^\lambda_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}\frac{\partial g_{\gamma\delta}}{\partial x^\nu}) \\ \Gamma^\alpha &= g^{\mu\nu}\Gamma^\alpha_{\mu\nu} \end{aligned}$$

The explicit computations to prove 3.3 are in [Fock], see also [F-Ms1]. This expression is relevant for the following reason: if the $\Gamma^\alpha = 0$ are identically zero then the vacuum Einstein equations have the form

$$R^{(h)}(g)_{\mu\nu} = 0$$

and the principal part of this system of equations,

$$-\frac{1}{2}g^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta},$$

operates in the same way on each component $g_{\mu\nu}$ so that the highest order terms are completely uncoupled. Such systems are said "weakly coupled" and are a particular case of the strict hyperbolic systems of Leray. To them we can apply general known theorems, see [Br1], [F-Ms1], to prove the local existence of the solutions. As Γ^α does not transform as a vector it is, in fact, possible to find a specific set of coordinates where this is true [7]. These coordinates are called harmonic or "wave-like" coordinates.

3.2 Local and global solutions for the non linear wave equations in Minkowski spacetime

Let us consider the nonlinear wave equation in R^{n+1}:

$$\Box\, u = F$$

with: $F = Du \cdot Du$ [8] and initial conditions: $u(0,x) = f(x)$, $\partial_t u(0,x) = g(x)$. To solve this equation the main ingredient are the energy norms, which are conserved in the linear case. In the non linear case this is not true, in general, but, nevertheless, it is possible to control them obtaining energy estimates which allow to prove the existence of the solutions at least for a finite time.

[7] More precisely the following result holds: if g is a solution of the reduced Einstein equations $R^{(h)}(g)_{\mu\nu} = 0$ with initial data $\{\overline{g}, \overline{k}\}$ such that $\Gamma^\alpha = 0$ on Σ_0, then $\Gamma^\alpha = 0$ on the whole development of the initial data set.

[8] We consider here a simple non linear term as we just want to describe some techniques to solve this problem that can be generalized to the Einstein vacuum equations. $Du \equiv (\partial_0 u, \partial_1 u, ...\partial_n u)$.

3.2.1 The local existence.

To prove the local existence one introduces the energy norm:

$$Q_0[u](t) = \left(\frac{1}{2}\int_{\Sigma_t} |Du|^2 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}||Du(t,\cdot)||_{L^2(R^n)}$$

where $|Du|^2 = |\partial_0 u|^2 + |\partial_1 u|^2 + ... + |\partial_n u|^2$. It is easy to show that in the linear case $Q_0[u](t) = Q_0[u](0)$. This is not true in the non linear case where we obtain, for a generic T, the following inequality:

$$\sup_{t\in[0,T]} ||Du(t,\cdot)||_{L^2(R^n)} \leq ||Du(0,\cdot)||_{L^2(R^n)} + 2\int_0^T ds||F||_{L^2(R^n)}$$

Recalling the previous definitions and the explicit expression of F we have

$$||F(s,\cdot)||_{L^2(R^n)} \leq ||Du(s,\cdot)||_{L^\infty}||Du(s,\cdot)||_{L^2(R^n)} \tag{3.4}$$

which, plugged in the previous inequality, gives

$$||Du(t,\cdot)||_{L^2(R^n)} \leq ||Du(0,\cdot)||_{L^2(R^n)} + 2\int_0^T ds||Du(s,\cdot)||_{L^\infty}||Du(s,\cdot)||_{L^2(R^n)}$$

Applying the Gronwall inequality we obtain

$$Q_0(t) \leq c_0 Q_0(0) \exp\int_0^t ||Du(s)||_{L^\infty} ds$$

which would allow to control $Q_0(t)$ in terms of $Q_0(0)$ if we could control, for appropriate t, $\int_0^t ||Du(s)||_{L^\infty} ds$. In particular, if we control $||Du(s)||_{L^\infty}$ in terms of $Q_0(s)$, we can control $Q_0(t)$ in terms of $Q_0(0)$, for t sufficiently small. Unfortunately this is not possible, but we can control the sup norm of a generic function f in terms of the $H^s(R^n)$ norms, where H^s is the Sobolev space and $s > \frac{n}{2}$:

$$||f||_{L^\infty(R^n)} \leq c||f||_{H^s(R^n)} \ . \tag{3.5}$$

Therefore to obtain the energy estimates we are forced to introduce a larger set of energy norms. We define

$$Q_i[u](t) = \frac{1}{2}\int_{\Sigma_t} \sum_{\alpha;|\alpha|\leq i} |\partial^\alpha Du|^2 dx$$

and as $\Box\, \partial^\alpha u = \partial^\alpha \Box\, u = \partial^\alpha F$, proceeding exactly as before, we obtain

$$Q_i[u](t) \leq Q_i[u](0) + \sqrt{2}\int_0^t ds||F(s,\cdot)||_{H^i(R^n)}$$

To repeat the previous argument we need the analogous of eq.3.4 in the case of the $H^s(R^n)$ Sobolev spaces. This is provided by the following estimate, see Proposition 3.2 in [Kl-Ni] and [Tay],

$$||f \cdot g||_{H^s} \leq c\,(||f||_{L^\infty}||g||_{H^s} + ||g||_{L^\infty}||f||_{H^s})$$

and using again the Gronwall inequality we obtain

$$Q_i(t) \leq c_i Q_i(0) \exp \int_0^t ||Du(s)||_{L^\infty} ds \tag{3.6}$$

For $n = 3$ which is the case we are interested on, the Sobolev inequality implies, for $i \geq 2$,

$$||Du(t)||_{L^\infty} \leq CQ_i(t)$$

and defining

$$\mathcal{Q}(t) \equiv \sup_{s\in[0,t]} Q_2(s)$$

we obtain

$$\mathcal{Q}(T) \leq c\mathcal{Q}(0) \exp T\mathcal{Q}(T)$$

This inequality implies a bound for $\mathcal{Q}(T)$ in the interval $[0,T]$ provided that $T\mathcal{Q}(0)$ is sufficiently small. Therefore we conclude that we control the "energy norms" $Q_i(t)$ in terms of the initial data norms for $t \in [0,T]$, for T sufficiently small, $i \leq s-1$, $s > \frac{n}{2}+1$ and initial data $(f,g) \in (H^s(R^n), H^{s-1}(R^n))$. In conclusion the ingredients for a local existence proof [9] can be summarized in the following way:

a) Generalized energy norms for $s > \frac{n}{2}+1$.
b) Estimates for the generalized energy norms for T small.
c) A fixed point mechanism [10].

3.2.2 The global existence

From the previous considerations we could expect to have a better estimate of the time T if we could control more carefully the integral $\int_0^T ||Du(t,\cdot)||_{L^\infty} dt$. This would be possible if, as in the linear case, we could take into account the decay in time of $||Du(t,\cdot)||_{L^\infty}$ [11]. This amounts, basically, to have more refined

[9] The proof, for the Einstein vacuum equations, of the local existence and uniqueness for $s > \frac{3}{2}+1$, $n = 3$, has been obtained by T.J.R.Hughes, T.Kato, J.E.Marsden, [Hu-Ka-Ms]. Recently the regularity for the non linear wave equation in R^{n+1} has been improved by D.Tataru, [Ta], up to $s > \frac{n}{2}+\frac{3}{4}$, for $n \geq 3$. To have well posedness for lower s we expect that the non linear part F has to satisfy the null condition and this fact has to be carefully exploited.

[10] The fixed point mechanism is the standard technique to complete the proof once we have the a priori estimates, see [Kl-Ni].

[11] Recall that, in the linear case, $||Du(s)||_{L^\infty} = O(s^{-\frac{n-1}{2}})$.

Sobolev estimates which could keep trace of the decay. These "Sobolev-type" estimates have been obtained by Klainerman, see [Kl3], [Kl4], and are based on the geometric (causal) properties of the Minkowski spacetime. The crucial steps of his construction are the following ones:

i) Find some generalized energy norms $E_s[u]$ such that, instead of the estimate 3.5 we have the estimate, for $t > 0$,

$$|u(t,x)| \leq \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}}E_s[u] \tag{3.7}$$

ii) Show that, due to the geometric properties of the Minkowski spacetime, the norms $E_s[Du]$ are, in the linear case, conserved in time.

Let us give a detailed description of these norms. The symmetries of the Minkowski spacetime can be described by its family of Killing and conformal Killing vector fields:

$$\begin{aligned} &T_\mu = D_\mu \\ &\Omega_{\mu\nu} = x_\mu D_\nu - x_\nu D_\mu \\ &S = t\partial_t + x^i\partial_i \\ &K_0 = (t^2+r^2)\frac{\partial}{\partial t} + 2tx^i\frac{\partial}{\partial x^i} \\ &K_\mu = -2x_\mu S + < x, x > \partial_\mu \end{aligned} \tag{3.8}$$

These generalized energy norms differ from the standard Sobolev ones as the partial derivatives are substituted by the Lie derivatives with respect to the Killing fields $T_\mu, \Omega_{\mu\nu}, S$ [12]:

$$E_0[u](t) = ||u(t,\cdot)||_{L^2(R^3)}$$

$$E_k[u](t) \quad \sum_{\{j\leq k; X_{i_1},..,X_{i_j}\}} E_0[L_{X_{i_1}}L_{X_{i_2}}...L_{X_{i_j}}u](t) \tag{3.9}$$

Using these norms Klainerman proved that, for $s > \frac{n}{2}$,

$$|Du(t,x)| \leq c\frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}}E_s[Du](t) \tag{3.10}$$

which suggests a better control of $\int_0^T ||Du(t,\cdot)||_{L^\infty(R^n)}dt$. To implement this estimate in the previous result we define $\tilde{Q}_s[u](t) \equiv E_s[Du](t)$ and repeat, in this case, the proof of the inequality 3.6. One obtains, see [Kl3], for an appropriate s_0 depending on s,

$$\tilde{Q}_s(t) \leq c_i\tilde{Q}_s(0)\exp\int_0^t E^\infty_{s_0}[Du]_{L^\infty}(t')dt'$$

[12]The reason for this choice is is that $[\square, T_\mu] = [\square, \Omega_{\mu\nu}] = 0$ and $[\square, S] = -2\square$, which imply that, in the linear wave equation these norms are conserved.

where

$$E_k^\infty[u](t) = \sum_{\{j\leq k; X_{i_1},..,X_{i_j}\}} ||L_{X_{i_1}}L_{X_{i_2}}...L_{X_{i_j}}u(t,\cdot)||_{L^\infty(R^3)}$$

and we used Lemma 3, page 330 of [Kl4] and the analogous of Proposition 3.2 of [Kl-Ni], page 83. The previous Sobolev type estimate 3.10 implies that, choosing $s > \frac{n}{2}$ sufficiently large,

$$E_{s_0}^\infty[Du]_{L^\infty}(t') \leq ct^{-\frac{n-1}{2}}\tilde{Q}_s(t')$$

and proceeding as before, one obtains the final estimate

$$\tilde{\mathcal{Q}}(T) \leq c\tilde{\mathcal{Q}}(0)\exp c\left(\int_0^T (1+t)^{-\frac{n-1}{2}}dt\right)\tilde{\mathcal{Q}}(T)$$

where $\tilde{\mathcal{Q}}(T) \equiv \sup_{t\in[0,T]}\tilde{Q}_s(t)$. Two remarks are appropriate now:

i) For $n = 3$ this result is not yet sufficient to obtain the global existence, nevertheless it gives, as proved by John and Klainerman, see [John-Kl], an almost global existence. In fact they prove that the solution exist up to a time $T = O(exp\frac{A}{\epsilon})$ where ϵ is an upper bound for the initial data.

ii) As the solution u satisfies 3.7, one could expect that the derivative Du had a better decay, but this is not true. Nevertheless a more careful investigation, see [Kl4] and [Ch-Kl1], proves that there is not any decay improvement only along the null e_3 direction. In fact we have:

$$\begin{aligned}|D_{e_4}Du(t,.)|_{L^\infty} &\leq c\frac{1}{(1+t+|x|)^{(\frac{n-1}{2}+1)}(1+|t-|x||)^{\frac{1}{2}}}E_s[Du](t)\\ |D_{e_3}Du(t,.)|_{L^\infty} &\leq c\frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{3}{2}}}E_s[Du](t)\end{aligned} \tag{3.11}$$

This suggests that, considering only some particular class of functions for the non linear term F, it would be possible to have a more specific expression in the integrand of 3.6 in which the term $D_{e_3}u$ does not appear and, therefore, the decay is sufficient to make the integral finite for any T. The appropriate choice of the non linear terms is called "The null condition", and has been introduced separately by Klainerman and Christodoulou, see [Kl2],[Ch2] [13].

The lesson we learn from the non linear wave equations in Minkowski spacetime is that, constructing a global solution the following "ingredients" seem to be crucial:

a) The symmetries of the spacetime.

b) The choice of "appropriate" null directions.

[13] For instance choosing $F(u,\partial u) = (\partial u\cdot\partial u) = \sum_a e_a(u)e_a(u) - e_4(u)e_3(u)$, we have the global existence result.

c) The choice of "appropriate" energy type norms.
d) The choice of some appropriate version of the null condition.

As, in the harmonic gauge, the vacuum Einstein equations have the form of a system of non linear wave equations with the highest order terms uncoupled, it seems reasonable to expect that the ingredients used to find a global solution for the non linear wave equations, if adapted appropriately, could be the right tools to obtain a global solution to the Einstein vacuum equations [14].
The strategy is, therefore, to implement the same kind of ideas to look for the "global" solution of the Einstein vacuum equations. This requires a careful investigation and is the subject of the following subsection.

3.3 The main ideas

3.3.1 The symmetries of the spacetime

In the search of the global solution of the nonlinear wave equation we have used in a crucial way the symmetries of the Minkowski spacetime which allowed to define some generalized energy norms, conserved in the linear case. In the case of the Einstein vacuum equations we cannot ask analogous symmetries for the spacetime as we reduce immediately to the Minkowski or the the Schwarzschild case. For instance if we require the Einstein vacuum spacetime be spherical symmetric we obtain from the Birkhoff theorem, see [Haw-El], page 369:

Theorem 3.1 *Any solution of the Einstein vacuum equations spherically symmetric is equivalent to the Schwarzschild solution.*

Nevertheless, as the symmetries play a relevant role, it will be important to control how large is the loss of symmetry of the spacetime we are trying to construct or, in other words, how far the spacetime, solution of the Einstein equations, is from the Minkowski or the Schwarzschild spacetimes.

3.3.2 The "appropriate" null directions

In the previous discussion it turns out important to select the "bad" null direction e_3 with respect to which the solution u of the non linear wave equation has the worst asymptotic behaviour. The null directions are connected to the null cones structure of the Minkowski spacetime, that is to the causal structure of the hyperbolic equations. We expect that the causal structure of the spacetime plays an analogous crucial role in solving the Einstein equations.

Here the first serious difficulty arises. In fact, differently from the non linear wave equation where the Minkowski spacetime is the "background spacetime" and therefore the causal structure, the "cones", is a priori given, solving the Einstein equations means to build the spacetime. Therefore the cone structure

[14] The previous discussion, on the other side, suggests also that to prove the global existence in the harmonic gauge can be difficult or impossible. The difficulty of extending the local solution to the global one in the harmonic gauge was already known to Choquet-Bruhat, see [Br3].

cannot be given a priori, but it is obtained together with the solution itself. This implies that we cannot prescribe the "null cones". We could, nevertheless, imagine that, provided the initial data are "small" [15] the null cones stay near and asymptotically approach the Minkowski ones.
Unfortunately this is not true for the following reason: as the initial data describe Σ_0, the space at $t = 0$, and as this hypersurface is assumed non flat, it follows that it has, globally, a certain amount of gravitational energy; moreover as the three dimensional hypersurface is asymptotically flat, from the infinity everything appears as if there is, in Σ_0, a compact region of appropriate radius, approximately spherical, which contains most of the space gravitational energy. This implies, due to the fact that the initial data have to satisfy the costraint equations, that the radial decay of the metric tensor and of the second fundamental form cannot be arbitrarily fast [16]. In fact we have the following result [17]:

i) Due to the initial costraints we cannot ask for the Riemann metric a better decay than

$$\begin{aligned} g_{ij} &= \left(1 + \frac{2m}{r}\right)\delta_{ij} + o(r^{-1}) \\ k_{ij} &= O(r^{-2}) \end{aligned}$$

ii) With these asymptotic decays the null cones will diverge logarithmically from the ones in Minkowski spacetime and, asymptotically, the points on the outgoing cones satisfy

$$r = t + 2M \log r + c$$

where M is the A.D.M. mass.

The conclusion is that the null directions cannot be assigned a priori, but they have to be built together with the spacetime. This suggests also that a covariant formulation of the Einstein equations, independent from the choice of the coordinates, is needed.

3.3.3 The covariant formulation of the Einstein vacuum equations and the energy-type norms

In the non linear wave equations the generalized energy norms are not conserved due to the presence of the non linear term F, here we expect that the energy-type norms we are going to define cannot be conserved for two reasons: first because the Einstein equations are non linear and second for the lack of symmetries of the spacetime. Anyway, before that, we have to find the energy type norms which, in the Einstein equations, play the analogous role of the $E_s[Du](t)$ norms previously introduced, eq. 3.9.

[15] This means, here, near to the ones which produce the Minkowski spacetime as solution.
[16] As this would imply the Minkowski spacetime be the only solution.
[17] We assume here that Σ_0 be maximal.

These quantities emerge naturally when we look for a covariant formulation of the Einstein equations. In the covariant approach to the Einstein equations we do not use the equations for g in their covariant form, a subset of the structure equations [18], but we use the covariant equations satisfied by the Riemann tensor which, in an Einstein vacuum spacetime, coincides with its conformal part. These equations are the second Bianchi identities [19] or "Bianchi equations" and have, in this case, the following form

$$D_{[\sigma}C_{\gamma\delta]\alpha\beta} = 0$$

where $C_{\alpha\beta\gamma\delta}$ is the conformal [20] part in the Riemann tensor decomposition,

$$\text{Riemann} = [\text{Conformal}] + [\text{part depending only on Ricci}] \ .$$

The importance of these equations for this problem is twofold: first once we have good estimates for the curvature tensor all the quantites which describe the causal structure of the spacetime, the null Ricci coefficients, or connections, see [Sp] Vol.II, which will be introduced later on, can be determined. Moreover the knowledge of the causal structure of the spacetime implies, as we will discuss in more detail later on, the knowledge of the metric tensor of the spacetime.

The second important aspect follows from the fact that the conformal part of the curvature tensor, $\mathbf{C}$, is an example of a Weyl tensor field. In fact, given a Weyl tensor field, one can define the Bel-Robinson tensor, see [Bel], and from it some integral quantities which are the analogous of the previous generalized energy norms.

The idea is, therefore, that of working at the level of the Riemann tensor instead that on the level of the metric tensor. This gives the possibility of avoiding to choose a specific coordinate system and therefore to keep the covariance of the theory.

Moreover these Bel-Robinson energy type norms are conserved in the case analogous to the linear case for the wave equation. This means to interpret the "Bianchi equations" as equations, in the Minkowski spacetime, for a generic Weyl tensor field which, although satisfying all the simmetry and traceless properties of the conformal part of the curvature tensor, has nothing to do with it. In this case it is easy to show that these integral norms are conserved and play the same role as the norms 3.9, in the linear wave equations.

Continuing the analogy we can expect to be possible to prove the boundedness of these norms in the "non linear" case, that is when the Bel-Robinson tensor is constructed from the conformal part of the curvature tensor, $\mathbf{C}$. If we control the Riemann tensor on the other side we can obtain the metric of the

[18] In which sense a subset of the structure equations can be interpreted as the Einstein equations will be described in some detail later on.

[19] The Bianchi equations are automatically satisfied by the Riemann tensor in any manifold. The link with the Einstein vacuum equations arises posing the Ricci part of the curvature tensor equal to zero.

[20] Conformal means that if instead of the metric g we consider another metric conformal to the previous one $\hat{g} = \Omega^2 g$ then $\hat{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$.

spacetime solving the structure equations, which, as we will show, amounts to solve elliptic equations and ordinary evolution equations.

To describe this approach we have, therefore, to make a thoughtful examination of the geometric structure of the Einstein spacetime.

3.3.4 Some appropriate version of the null condition

The analogous of null condition should appear at a tensor level, that is invariant under diffeomorphisms. The identification of the null condition in the Einstein vacuum equations is not obvious and we will not discuss it here.

3.4 The structure of the proof

As in the case of the non linear wave equations the proof can be divided in two parts a local one and a global one:

The local part:
We prove that, given appropriately initial conditions, there exists a finite portion of the spacetime endowed with a well defined geometric and causal structure. Moreover in this finite portion of the spacetime the energy-type norms built using the Bel-Robinson tensor are finite.

Therefore, to implement this part we need, first of all, a local existence proof for the Einstein vacuum equations. This guarantees that at least a development (in principle not maximal) of the initial data exists, what we have called "a finite portion of the spacetime". But this is not all that we need here, in fact, to solve also the "global" part of the proof, we need to prove that this development has a very detailed geometric structure and, moreover, that in this region we have a very precise control of the curvature tensor through a family of energy-type norms.

The global part:
The proof of the global part is based on a bootstrap mechanism. We consider the largest possible (development) region of the spacetime such that its geometric and causal structure is "near" to the Minkowski spacetime, in a sense which will be defined precisely later on, and that the energy-type norms, which will be defined precisely later on, are "small", that is bounded by a small constant ϵ_0.
There are now two possibilities, this region is in fact the whole spacetime and the affine parameter of all the null geodesics goes up to infinity [21] or the null geodesics affine parameters vary in a finite interval. In the first case this development would be the solution of our problem. In the second case we prove that this region can be extended. In other words we show that there is a larger region containing the previous one where the energy-type norms are still bounded by the constant ϵ_0 and where the null geodesics affine parameters vary in a larger interval. This implies that the previous region we have assumed to be the largest

[21] As it will be clear in the next discussion, we always mean the outgoing null geodesics and also the otgoing spacelike geodesics.

possible is not, in fact, the largest one, a contradiction which is avoided if and only if the largest region coincides with the whole spacetime, that is if the null outgoing geodesics are complete.

Remarks:

a) The proof of the local existence is, basically, standard in the sense that many approaches can be used to prove it, under mild conditions on the initial data.

b) The more significant part of the proof is the one relative to the global part. To perform it the geometric structure of the finite portion of the spacetime has to be examined in great detail.

4 The Bianchi equations in a fixed background spacetime

A basic tool used to solve a non linear problem is finding an appropriate linearization of it. This is true also in this case, therefore we start looking at a "linearized" version of the Bianchi equations. We recall the basic Riemann tensor properties:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}$$
$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

The Bianchi equations (Second Bianchi identities):

$$D_{[\sigma}R_{\gamma\delta]\alpha\beta} = 0$$

The Riemann tensor decomposition

$$\text{Riemann} = [\text{Conformal}] + [\text{part on Ricci}]$$

Conformal [22] curvature tensor **C**:

$$\begin{aligned} C_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}\right) \\ &\quad + \frac{1}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R \end{aligned}$$

C has all the symmetry properties of Riemann:

$$C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta} = -C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta}$$
$$C_{\alpha\beta\gamma\delta} + C_{\alpha\gamma\delta\beta} + C_{\alpha\delta\beta\gamma} = 0$$

and

$$g^{\alpha\gamma}C_{\alpha\beta\gamma\delta} = 0\ .$$

[22] Conformal means that if instead of the metric g we consider another metric conformal to the previous one $\hat{g} = \Omega^2 g$ then $\hat{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta}$.

The Bianchi equations are non linear as $\mathbf{C}$ depends on the metric g through the covariant derivatives. To "linearize" these equations we introduce the Weyl field tensor $\mathbf{W}$, which has all the simmetry properties of $\mathbf{C}$, and fix a priori the metric tensor g. $(\mathcal{M}, g)$ is, therefore, the "background spacetime" and the Bianchi equations become the linear equations for the for the Weyl field $\mathbf{W}$.

4.1 The properties of the background spacetime

Let us assume that the background spacetime be foliated by a smooth double null integrable S-foliation whose leaves are compact, spacelike, 2-surfaces diffeomorphic to S^2 [23]. Double null integrability means that this foliation implies the existence of two families of null hypersurfaces $\{C(u)\}$, $\{\underline{C}(\underline{u})\}$, which foliate the spacetime and, at their turn, are foliated by the 2-surfaces $S(u, \underline{u})$ [24].

Using this foliation one can define, at a generic point $p \in \mathcal{M}$, an "adapted" null frame $\{e_4, e_3, e_a\}$, $a \in [1,2]$ by taking an orthonormal frame $\{e_a\}$, $a \in (1,2)$ on the tangent space of the sphere $S(u, \underline{u})$ passing through p. e_4, e_3 are the vector fields associated to the directions of the null geodesics relative to $C(u)$, $\underline{C}(\underline{u})$, respectively, and normalized in such a way that $g(e_3, e_4) = -2$.

4.2 The Weyl field

$\mathbf{W}$ is a Weyl field in the background spacetime $(\mathcal{M}, g)$, if

$$W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} = W_{\gamma\delta\alpha\beta}$$
$$W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0$$

and in addition $\quad g^{\alpha\gamma} W_{\alpha\beta\gamma\delta} = 0.$

In particular $\mathbf{C}$ is a Weyl field. Let $\mathbf{W}$ satisfies the Bianchi equations

$$D_{g_{[\sigma}} W_{\gamma\delta]\alpha\beta} = 0$$

where D_g is the connection associated to the metric g. We list in the following some important properties of the Weyl field, see [Ch-Kl2].

Property 1: left and right Hodge duals are equivalent

$$^{\star}W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}$$
$$W^{\star}_{\alpha\beta\gamma\delta} = W_{\alpha\beta}{}^{\mu\nu} \frac{1}{2}\epsilon_{\mu\nu\gamma\delta}$$

$^{\star}\mathbf{W} = \mathbf{W}^{\star}$, $^{\star}(^{\star}\mathbf{W}) = -\mathbf{W}$.

Property 2: The following four sets of equations are equivalent

$$D_{[\sigma} W_{\gamma\delta]\alpha\beta} = 0 \ , \ D^{\mu} W_{\mu\nu\alpha\beta} = 0$$
$$D^{\mu\,\star} W_{\mu\nu\alpha\beta} = 0 \ , \ D_{[\sigma}{}^{\star} W_{\gamma\delta]\alpha\beta} = 0$$

[23]This is trivially true if we choose the Minkowski spacetime as background spacetime.

[24]These definitions will be discussed at length when we describe the properties of the development of the initial data.

Property 3: If W satisfy the Bianchi equations and X is a Killing (conformal Killing) vector field then $\hat{\mathcal{L}}_X W$ is also a Weyl field, solution of the Bianchi equations where $\hat{\mathcal{L}}_X W = L_X W - \frac{1}{8}\text{tr}^{(X)}\pi W$, L_X is the ordinary Lie derivative and ${}^{(X)}\pi \equiv L_X g$ is the deformation tensor relative to the vector field X. Moreover

$$\hat{\mathcal{L}}_X {}^\star W = {}^\star \hat{\mathcal{L}}_X W \ .$$

Remark: These equations have a strong analogy with the electromagnetic Maxwell equations. In fact let us define [25]

$$E = i_{(T,T)}W \ , \ H = i_{(T,T)}{}^\star W.$$

E and H, tangent to the hyperplanes $\Sigma_t \equiv \{p \in \mathcal{M}|t(p) = t\}$, determine completely the Weyl tensor field. The Bianchi equations in this decomposition are the following "Maxwell-type" equations:

$$\begin{aligned}
&\Phi^{-1}\partial_t E + \text{curl}H = \rho(E,H)\\
&\Phi^{-1}\partial_t H - \text{curl}E = \sigma(E,H)\\
&\text{div}E = k \wedge H\\
&\text{div}H = -k \wedge E
\end{aligned}$$

where ∇ is the covariant derivative with respect to Σ_t, $(\text{div}E)_i = \nabla^j E_{ij}$ and $(\text{curl}E)_{ij} = \epsilon_i^{lk}\nabla_l E_{kj}$. Analogous expressions hold for H [26].

4.3 The Bel-Robinson energy-type norms in the background spacetime

In solving the non linear wave equations the main step is to prove the boundedness of some energy-type norms which in the linear case are conserved, see the discussion in subsection 3.2.

Here we proceed in the same spirit looking for energy-type norms written in terms of the Weyl tensor, which are conserved when $\mathbf{W}$ is a solution of the "linearized" Bianchi equations. These norms are constructed from the Bel-Robinson tensor.

Bel-Robinson tensor of the Weyl field W:

$$Q_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma}W_{\beta\ \delta}^{\ \rho\ \sigma} + {}^\star W_{\alpha\rho\gamma\sigma}{}^\star W_{\beta\ \delta}^{\ \rho\ \sigma}.$$

It shows a strict analogy with the energy momentum tensor of the electromagnetic field $\mathbf{F}$:

$$T_{\alpha\beta} = F_{\alpha\rho}F_\beta^{\ \rho} + {}^\star F_{\alpha\rho}{}^\star F_\beta^{\ \rho}$$

and satisfies the following, see [Ch-Kl2],

[25] We assume the background spacetime admits a global time function. In particular later on we will choose as background spacetime, the Minkowski spacetime.

[26] The explicit expressions of $\rho(E,H)$ and $\sigma(E,H)$ are in [Ch-Kl2], page 146.

Proposition 4.1
a) Q *is symmetric and traceless relative to all pairs of indices.*
b) $Q(X_1, X_2, X_3, X_4)$ *is positive for any timelike vector fields* [27].
c) If W *is a solution of the Bianchi equations then (local conservation)*

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0$$

Definition 4.1 *Given a vector field* X *the deformation tensor of* X, ${}^{(X)}\pi = L_X g$, *and its traceless part* ${}^{(X)}\hat{\pi}$ *measure, in a precise sense, how much the diffeomorphism generated by* X *differs from an isometry or a conformal isometry, respectively.*

Proposition 4.2 *Let* $Q(W)$ *be the Bel Robinson tensor of a Weyl field* $\mathbf{W}$ *and* X, Y, Z *a triplet of vector fields. We define the covariant vector field* P *associated to the triplet:*

$$P_\alpha = Q_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta. \tag{4.12}$$

Using all the symmetry properties of Q *it follows:*

$$\begin{aligned} DivP &= DivQ_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta \\ &+ \frac{1}{2} Q_{\alpha\beta\gamma\delta} \left({}^{(X)}\hat{\pi}^{\alpha\beta} Y^\gamma Z^\delta + {}^{(Y)}\hat{\pi}^{\alpha\gamma} X^\beta Z^\delta + {}^{(Z)}\hat{\pi}^{\alpha\delta} X^\beta Y^\gamma \right) \end{aligned}$$

Thus, to any X, Y, Z Killing or conformal Killing vector fields we can associate a conserved quantity. More precisely:

Proposition 4.3 *Let* W *be a solution of Bianchi equations and* $X, Y, Z, V_1, \dots V_k$ *be Killing or conformal Killing vector fields, then*

a) $DivP = 0$ *where* P *is defined by 4.12*

b) The integral $\int_{\Sigma_t} Q[W](X, Y, Z, T_0) d\mu_{g_t}$ *is finite and constant for all* t *provided that it is finite at* $t = 0$. $d\mu_{g_t}$ *is the volume element of the induced metric* g_t *on* Σ_t, *a spacelike hypersurface, and* T_0 *is its future directed unit normal.*

c) The integrals

$$\int_{\Sigma_t} Q[\hat{\mathcal{L}}_{V_1} \hat{\mathcal{L}}_{V_2} \dots \hat{\mathcal{L}}_{V_k} W](X, Y, Z, T_0) d\mu_{g_t}$$

are finite and constant for all t *provided that they are finite at* $t = 0$.

As we have defined, in the background spacetime $(\mathcal{M}, g)$, the null hypersurfaces $C(u)$ and $\underline{C}(\underline{u})$ it is possible to introduce some different energy-type norms, integrals over these null hypersurfaces instead than over the Σ_t spacelike ones, and prove a proposition analogous to Proposition 4.3

[27] It is also possible to prove that $Q(X_1, X_2, X_3, X_4)$ is non negative for any non spacelike future directed vector fields X_1, X_2, X_3, X_4.

Proposition 4.4 *Let W be a solution of Bianchi equations and $X, Y, Z, V_1, \dots V_k$ be Killing or conformal Killing vector fields, then the integrals*

$$\int_{C(u)} Q[\hat{\mathcal{L}}_{V_1}\hat{\mathcal{L}}_{V_2}\dots\hat{\mathcal{L}}_{V_k}W](X,Y,Z,e_4)$$
$$\int_{\underline{C}(\underline{u})} Q[\hat{\mathcal{L}}_{V_1}\hat{\mathcal{L}}_{V_2}\dots\hat{\mathcal{L}}_{V_k}W](X,Y,Z,e_3) \tag{4.13}$$

are bounded uniformily in $u, \underline{u}$ provided that the corresponding integrals on Σ_0,

$$\int_{\Sigma_0} Q[\hat{\mathcal{L}}_{V_1}\hat{\mathcal{L}}_{V_2}\dots\hat{\mathcal{L}}_{V_k}W](X,Y,Z,T_0)d\mu_{g_0} \ ,$$

are finite.

Definition 4.2 *Let e_4, e_3 be the null pair of the adapted null frame. Given a Weyl field $\mathbf{W}$ we introduce the following tensor fields operating at each $p \in S$ on the subspace TS_p of the tangent space $T\mathcal{M}_p$:*

$$\alpha(\mathbf{W})(X,Y) = \mathbf{W}(X,e_4,Y,e_4) \ , \ \beta(\mathbf{W})(X) = \frac{1}{2}\mathbf{W}(X,e_4,e_3,e_4)$$
$$\rho(\mathbf{W}) = \frac{1}{4}\mathbf{W}(e_3,e_4,e_3,e_4) \ , \ \sigma(\mathbf{W}) = \frac{1}{4}\rho({}^\star\mathbf{W}) = \frac{1}{4}{}^\star\mathbf{W}(e_3,e_4,e_3,e_4)$$
$$\underline{\beta}(\mathbf{W})(X) = \frac{1}{2}\mathbf{W}(X,e_3,e_3,e_4) \ , \ \underline{\alpha}(\mathbf{W})(X,Y) = \mathbf{W}(X,e_3,Y,e_3)$$

where X, Y are arbitrary vectors tangent to S at p.

We call $\{\alpha(\mathbf{W}),\ \underline{\alpha}(\mathbf{W}),\ \beta(\mathbf{W}),\ \underline{\beta}(\mathbf{W}),\ \rho(\mathbf{W}),\ \sigma(\mathbf{W})\}$ "the null decomposition of $\mathbf{W}$ relative to e_4, e_3".
We easily check that $\alpha(\mathbf{W}), \underline{\alpha}(\mathbf{W})$ are symmetric traceless tensors, thus they have two independent components each. Together the total number of independent components of $\alpha(\mathbf{W}),\ \underline{\alpha}(\mathbf{W}),\ \beta(\mathbf{W}),\ \underline{\beta}(\mathbf{W}),\ \rho(\mathbf{W}),\ \sigma(\mathbf{W})$ account for all the ten degrees of freedom of the Weyl tensor field $\mathbf{W}$.
The null components of $\mathbf{W}$ can be expressed in terms of the null decomposition in the following way, denoting $W_{\alpha\beta\gamma\delta} \equiv \mathbf{W}(e_\alpha, e_\beta, e_\gamma, e_\delta)$ [28],

$$W_{a33b} = -\underline{\alpha}_{ab} \ , \ W_{a334} = 2\underline{\beta}_a$$

[28] Relative to a rescaling of the null pair,

$$e_4 \to e_4' = ae_4 \ , \ e_3 \to e_3' = a^{-1}e_3,$$

the null components of $\mathbf{W}$ change according to

$$\begin{aligned} &\alpha' = a^2\alpha \ \ \underline{\alpha}' = a^{-2}\underline{\alpha} \\ &\beta' = a\beta \ \ \underline{\beta}' = a^{-1}\underline{\beta} \\ &\rho' = \rho, \ \ \sigma' = \sigma \end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&W_{a44b} = -\alpha_{ab} \ , \ W_{a443} = -2\beta_a \\
&W_{a3b4} = -\rho\delta_{ab} + \sigma\epsilon_{ab} \\
&W_{a3bc} = -{}^\star({}^\star W)_{a3bc} = \epsilon^\star_{bc}\underline{\beta}_a \\
&W_{a4bc} = -{}^\star({}^\star W)_{a4bc} = -\epsilon_{bc}{}^\star\beta_a \\
&\delta_{ab}W_{a3bc} = \underline{\beta}_c \ , \ \delta_{ab}W_{a4bc} = -\beta_c \\
&W_{dcab}\delta_{da}\delta_{cb} = -2\rho \ , \ W_{3434} = -4\rho \\
&W_{ab34} = 2\epsilon_{ab}\sigma
\end{aligned}
\tag{4.15}
$$

where ${}^\star\alpha, {}^\star\underline{\alpha}, {}^\star\beta, {}^\star\underline{\beta}$ are the duals of $\alpha, \underline{\alpha}, \beta, \underline{\beta}$ relative to TS_p [29].
The Bianchi equations can be expressed in terms of the null components $\{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$ of the Weyl tensor, according to the following

Proposition 4.5 *Expressed relatively to an adapted null frame, the Bianchi equations take the form*

$$
\begin{aligned}
&\underline{\alpha}_4 \equiv \mathbf{D\!\!\!/}_4\underline{\alpha} + \frac{1}{2}tr\chi\underline{\alpha} = -\nabla\!\!\!\!/\widehat{\otimes}\underline{\beta} + \left[4\omega\underline{\alpha} - 3(\hat{\underline{\chi}}\rho - {}^\star\hat{\underline{\chi}}\sigma) + (\zeta - 4\underline{\eta})\widehat{\otimes}\underline{\beta}\right] \\
&\underline{\beta}_3 \equiv \mathbf{D\!\!\!/}_3\underline{\beta} + 2tr\underline{\chi}\ \underline{\beta} = -d\!\!\!/iv\underline{\alpha} - \left[2\underline{\omega}\underline{\beta} + (-2\zeta + \eta)\cdot\underline{\alpha}\right] \\
&\underline{\beta}_4 \equiv \mathbf{D\!\!\!/}_4\underline{\beta} + tr\chi\underline{\beta} = -\nabla\!\!\!\!/\rho + \left[2\omega\underline{\beta} + 2\hat{\underline{\chi}}\cdot\beta + {}^\star\nabla\!\!\!\!/\sigma - 3(\underline{\eta}\rho - {}^\star\underline{\eta}\sigma)\right] \\
&\rho_3 \equiv \mathbf{D}_3\rho + \frac{3}{2}tr\underline{\chi}\rho = -d\!\!\!/iv\underline{\beta} - \left[\frac{1}{2}\hat{\chi}\cdot\underline{\alpha} - \zeta\cdot\underline{\beta} + 2\eta\cdot\underline{\beta}\right] \\
&\rho_4 \equiv \mathbf{D}_4\rho + \frac{3}{2}tr\chi\rho = d\!\!\!/iv\beta - \left[\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha - \zeta\cdot\beta - 2\underline{\eta}\cdot\beta\right] \\
&\sigma_3 \equiv \mathbf{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -d\!\!\!/iv{}^\star\underline{\beta} + \left[\frac{1}{2}\hat{\chi}\cdot{}^\star\underline{\alpha} - \zeta\cdot{}^\star\underline{\beta} - 2\eta\cdot{}^\star\underline{\beta}\right] \\
&\sigma_4 \equiv \mathbf{D}_4\sigma + \frac{3}{2}tr\chi\sigma = -d\!\!\!/iv{}^\star\beta + \left[\frac{1}{2}\hat{\underline{\chi}}\cdot{}^\star\alpha - \zeta\cdot{}^\star\beta - 2\underline{\eta}\cdot{}^\star\beta\right] \\
&\beta_3 \equiv \mathbf{D\!\!\!/}_3\beta + tr\underline{\chi}\beta = \nabla\!\!\!\!/\rho + \left[2\underline{\omega}\beta + {}^\star\nabla\!\!\!\!/\sigma + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho + {}^\star\eta\sigma)\right] \\
&\beta_4 \equiv \mathbf{D\!\!\!/}_4\beta + 2tr\chi\beta = d\!\!\!/iv\alpha - \left[2\omega\beta - (2\zeta + \underline{\eta})\alpha\right] \\
&\alpha_3 \equiv \mathbf{D\!\!\!/}_3\alpha + \frac{1}{2}tr\underline{\chi}\alpha = \nabla\!\!\!\!/\widehat{\otimes}\beta + \left[4\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^\star\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta\right]
\end{aligned}
\tag{4.16}
$$

Remarks:
If the background spacetime is the Minkowski one, the terms of the Bianchi equations in square brackets are absent [30]. *They are present in a more general background spacetime and in the non linear case of the Einstein equations where* **W** *coincides with the Riemann tensor. They are products between the Weyl null components and the Ricci coefficients.*

[29] In particular $\alpha({}^\star W) = -{}^\star\alpha(W)$, $\beta({}^\star W) = -{}^\star\beta(W)$, $\underline{\alpha}({}^\star W) = {}^\star\underline{\alpha}(W)$, $\underline{\beta}({}^\star W) = -{}^\star\underline{\beta}(W)$ and $\rho({}^\star W) = \sigma(W)$, $\sigma({}^\star W) = -\rho(W)$.

[30] In the case of the Schwarzschild spacetime the only terms in parenthesis different from zero are those depending on $\omega, \underline{\omega}$.

The important result in the linearized case that, later on, we will extend to the general non linear one, consists in the asymptotic estimates which parallel the generalized Klainerman-Sobolev estimates for the linear wave equations.

4.4 The asymptotic estimates of the Weyl tensor field in the background spacetime

We control the components $\alpha(W), \underline{\alpha}(W), \beta(W), \underline{\beta}(W), \rho(W), \sigma(W)$ in terms of a set of norms of the type

$$\int_{C(u)} Q[\hat{\mathcal{L}}_{V_1}\hat{\mathcal{L}}_{V_2}\dots\hat{\mathcal{L}}_{V_k}W](X,Y,Z,e_4)$$
$$\int_{\underline{C}(\underline{u})} Q[\hat{\mathcal{L}}_{V_1}\hat{\mathcal{L}}_{V_2}\dots\hat{\mathcal{L}}_{V_k}W](X,Y,Z,e_3)$$

where for X,Y,Z we choose, if the background spacetime is the Minkowski spacetime, Killing or conformal Killing future directed vector fields, therefore $\in\{T_0,K_0\}$. We choose, in particular,

$$(X,Y,Z)=(K_0,K_0,T_0)$$
$$(X,Y,Z)=(K_0,K_0,K_0)$$

Proposition 4.6 *If we assume that the Weyl tensor W satisfy the Bianchi equations and that the following sums of integral norms are bounded,*

$$\begin{aligned}\mathcal{Q} &= \int_C Q(L_OW)(K_0,K_0,T_0,e_4)+\int_C Q(L_O^2W)(K_0,K_0,T_0,e_4)\\ &+ \int_C Q(L_{T_0}W)(K_0,K_0,K_0,e_4)+\int_C Q(L_OL_{T_0}W)(K_0,K_0,K_0,e_4)\\ \underline{\mathcal{Q}} &= \int_{\underline{C}} Q(L_OW)(K_0,K_0,T_0,e_3)+\int_{\underline{C}} Q(L_O^2W)(K_0,K_0,T_0,e_3)\\ &+ \int_{\underline{C}} Q(L_{T_0}W)(K_0,K_0,K_0,e_3)+\int_{\underline{C}} Q(L_OL_{T_0}W)(K_0,K_0,K_0,e_3)\ ,\end{aligned}$$

then the various null components of the Weyl tensor satisfy the following inequalities

$$\begin{aligned}&\sup_{\mathcal{M}}|r^{\frac{7}{2}}\alpha|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\ ,\ \sup_{\mathcal{M}}|r^{\frac{7}{2}}\beta|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\\ &\sup_{\mathcal{M}}|r^2u^{\frac{3}{2}}\underline{\beta}|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\ ,\ \sup_{\mathcal{M}}|ru^{\frac{5}{2}}\underline{\alpha}|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\\ &\sup_{\mathcal{M}}|r^3u^{\frac{1}{2}}\sigma|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\ ,\ \sup_{\mathcal{M}}|r^3u^{\frac{1}{2}}(\rho-\overline{\rho})|\leq c(\mathcal{Q}+\underline{\mathcal{Q}})\ .\end{aligned}\tag{4.17}$$

Remarks:

a) These asymptotic estimates are the equivalent of those in eq, 3.11 and the integral norms in $\mathcal{Q}$ and $\underline{\mathcal{Q}}$ are the equivalent of the norms 3.9.

b) It is clear that applying more and more Lie derivatives on W allows to obtain better regularity properties and therefore asymptotic estimates also for the derivatives of the null components of the Weyl field.

c) We know, from Proposition 4.4 that in the Minkowski spacetime $\mathcal{Q}$ and $\underline{\mathcal{Q}}$ are bounded in terms of the corresponding norms on the initial hypersurface Σ_0.

The proof of Proposition 4.6 is based on the following

Proposition 4.7 *Let F be a smooth tensor field defined on the background* [31] *spacetime $(\mathcal{M}, g)$, tangent at each point to the 2-surface $S(u, \underline{u})$ passing through that point. The following estimates hold:*

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \;\leq\; & c\Big[\Big(\int_{S(u,\underline{u}_0)} r^4|F|^4\Big)^{\frac{1}{4}} + \Big(\int_{S(u,\underline{u}_0)} r^4|r\nabla\!\!\!\!/ F|^4\Big)^{\frac{1}{4}} \\
& +\Big(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\
& + r^4|\nabla\!\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\Big]
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \;\leq\; & c\Big[\Big(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|F|^4\Big)^{\frac{1}{4}} + \Big(\int_{S(u,\underline{u}_0)} r^2\tau_-^2|r\nabla\!\!\!\!/ F|^4\Big)^{\frac{1}{4}} \\
& +\Big(\int_{C(u)\cap V(u,\underline{u})} (|F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!\!/_4 F|^2 \\
& + r^4|\nabla\!\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_4 F|^2\Big)^{\frac{1}{2}}\Big]
\end{aligned}
\tag{4.19}
$$

Similar estimates can also be obtained expressing the sup norms in terms of integrals along the incoming null hypersurfaces $\underline{C}(\underline{u})$. The results are

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r^{\frac{3}{2}}|F|) \;\leq\; & c\Big[\Big(\int_{S(u_0,\underline{u})} r^4|F|^4\Big)^{\frac{1}{4}} + \Big(\int_{S(u_0,\underline{u})} r^4|r\nabla\!\!\!\!/ F|^4\Big)^{\frac{1}{4}} \\
& +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!\!/_3 F|^2 \\
& + r^4|\nabla\!\!\!\!/^2 F|^2 + r^4|\nabla\!\!\!\!/\mathbf{D}\!\!\!\!/_3 F|^2\Big)^{\frac{1}{2}}\Big]
\end{aligned}
\tag{4.20}
$$

[31] Here the fact that we are considering a linearized problem does not matter. These Sobolev estimates are valid in a generic spacetime sufficiently regular and will be used extensively in the proof of the main Theorem.

and

$$
\begin{aligned}
sup_{S(u,\underline{u})}(r\tau_-^{\frac{1}{2}}|F|) \ \leq \ & c\Big[(\int_{S(u_0,\underline{u})} r^2\tau_-^2|F|^4)^{\frac{1}{4}} + (\int_{S(u_0,\underline{u})} r^2\tau_-^2|r\nabla\!\!\!/ F|^4)^{\frac{1}{4}} \\
& +\Big(\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!/_3 F|^2 \\
& + r^4|\nabla\!\!\!/^2 F|^2 + r^2\tau_-^2|\nabla\!\!\!/ \mathbf{D}\!\!\!/_3 F|^2\Big)^{\frac{1}{2}}\Big] \qquad (4.21)
\end{aligned}
$$

Proof: The proof of the last two propositions is based on the following lemmas.

Lemma 4.1 *Let F be a smooth tensorfield defined on the spacetime $\mathcal{M}$, tangent at every point to the sphere $S(u,\underline{u})$ containing it. Introduce the following quantities, where $V(u,\underline{u}) = J^-(S(u,\underline{u}))$,*

$$
\begin{aligned}
A(F) &\equiv \sup_{C(u)\cap V(u,\underline{u})} \left(\int_{S(u,\underline{u})} r^4|F|^4\right)^{\frac{1}{4}} \\
B(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} r^6|F|^6\right)^{\frac{1}{6}} \qquad (4.22) \\
E(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + r^2|\mathbf{D}\!\!\!/_4 F|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
A_{de.}(F) &\equiv \sup_{C(u)\cap V(u,\underline{u})} \left(\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4\right)^{\frac{1}{4}} \\
B_{de.}(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} r^4\tau_-^2|F|^6\right)^{\frac{1}{6}} \qquad (4.23) \\
E_{de.}(F) &\equiv \left(\int_{C(u)\cap V(u,\underline{u})} |F|^2 + r^2|\nabla\!\!\!/ F|^2 + \tau_-^2|\mathbf{D}\!\!\!/_4 F|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

then the following inequalities hold

$$B \ \leq \ c(I)A^{2/3}E^{1/3} \qquad (4.24)$$

$$A \ \leq \ A_0 + c(I)B^{3/4}E^{1/4} \qquad (4.25)$$

and

$$B_{de.} \ \leq \ c(I)A_{de.}^{2/3}E_{de.}^{1/3} \qquad (4.26)$$

$$A_{de.} \ \leq \ A_{de.0} + c(I)B_{de.}^{3/4}E_{de.}^{1/4} \qquad (4.27)$$

where

$$A_0 = \left(\int_{S(u,\underline{u}_0)} r^4 |F|^4 \right)^{\frac{1}{4}} \quad , \; A_{de.0} = \left(\int_{S(u,\underline{u}_0)} r^2 \tau_-^2 |F|^4 \right)^{\frac{1}{4}} \tag{4.28}$$

$\underline{u}_0 = \underline{u}|_{\Sigma_0}$ *and* $c(I)$ *is a constant depending on* $I = \sup_{C(u)} I(u, \underline{u})$*, where* $I(u,\underline{u})$ *is the isoperimetric constant of* $S(u,\underline{u})$.

The proof of Proposition 4.7 follows immediately from this lemma combined with the following form of the standard Sobolev inequalities for the sphere

Lemma 4.2 *Let* G *be a tensor field tangent to the spheres* $S(u,\underline{u})$*, then*

$$sup_{S(u,\underline{u})}|G| \leq c r^{-\frac{1}{2}} \left(\int_{S(u,\underline{u})} |G|^4 + r^4 |\nabla\!\!\!/ G|^4 \right)^{\frac{1}{4}}$$

Indeed, it suffices to apply this lemma to $G = rF$, or $G = r^{\frac{1}{2}} \tau_-^{\frac{1}{2}} F$ and then take Lemma 4.1 into account. Let us present the main steps in the proof of the nondegenerate version of Lemma 4.1.
To prove 4.24 we use the following version of the isoperimetric inequality for the sphere:

Lemma 4.3 *Let* Φ *be a scalar on a sphere* $S(u,\underline{u})$ *in* R^3*, denote by* $\bar{\Phi}$ *the average of* Φ *on* S*. Then*

$$\int_{S(u,\underline{u})} (\Phi - \bar{\Phi})^2 \leq I(u,\underline{u}) \Big(\int_{S(u,\underline{u})} |\nabla\!\!\!/ \Phi| \Big)^2 \tag{4.29}$$

Applying the lemma to the spheres $S(u,\underline{u}) \subset C(u) \cap V(u,\underline{u})$ with $\Phi = |F|^3$ and using the Holder inequality we derive

$$\int_{S(u,\underline{u})} |F|^6 \leq c \Big(r^{-2} \int_{S(u,\underline{u})} |F|^4 \Big) \Big(\int_{S(u,\underline{u})} |F|^2 + r^2 |\nabla\!\!\!/ F|^2 \Big). \tag{4.30}$$

Multiplying the equation 4.30 by r^6 and integrating with respect to $\underline{u}$ we derive eq. 4.24. To obtain 4.25 we express, with the help of the divergence theorem, the integral $\int_{S(u,\underline{u})} r^4 |F|^4$ in terms of an integral over the portion of $C(u) \cap V(u,\underline{u})$ outside the sphere $S(u,\underline{u})$. Applying also Cauchy-Schwartz this leads to

$$\int_{S(u,\underline{u})} r^4 |F|^4 \leq c \Big(\int_{C(u) \cap V(u,\underline{u})} r^6 |F|^6 \Big)^{\frac{1}{2}} \Big(\int_{C(u) \cap V(u,\underline{u})} r^2 |D\!\!\!\!/_4 F|^2 \Big)^{\frac{1}{2}}$$

which proves 4.25.

. To prove the degenerate estimates 4.26, 4.27 of the Lemma 4.1 we proceed precisely in the same way with the quantities $A_{de.}$, $B_{de.}$ and $E_{de.}$. In this case the inequality 4.26 follows easily by multiplying 4.30 by $r^4\tau_-^2$ and integrating in $\underline{u}$. The corresponding inequality 4.27 follows, as in the nondegenerate case, by applying the divergence theorem to $\int_{S(u,\underline{u})} r^2\tau_-^2|F|^4$.

The proof of Proposition 4.6 is based on Proposition 4.7. We discuss here the estimate for $\alpha(W)$, the other estimates being obtained in a similar way.

Estimate for $\alpha(W)$: Observing that

$$\begin{aligned}
&Q(W)(e_3,e_3,e_3,e_3) = 2|\underline{\alpha}|^2 \ , \ Q(W)(e_4,e_4,e_4,e_4) = 2|\alpha|^2\\
&Q(W)(e_3,e_3,e_3,e_4) = 4|\underline{\beta}|^2 \ , \ Q(W)(e_3,e_4,e_4,e_4) = 4|\beta|^2\\
&Q(W)(e_3,e_3,e_4,e_4) = 4(\rho^2+\sigma^2)
\end{aligned}$$

we obtain immediately

$$\begin{aligned}
Q(W)(K_O,K_O,T_O,e_4) &= \frac{1}{4}\underline{u}^4|\alpha|^2 + \frac{1}{2}(\underline{u}^4+2\underline{u}^2u^2)|\beta|^2\\
&\quad +\frac{1}{2}(u^4+2\underline{u}^2u^2)(\rho^2+\sigma^2) + \frac{1}{2}u^4|\underline{\beta}|^2\\
Q(W)(K_O,K_O,T_O,e_3) &= \frac{1}{4}u^4|\underline{\alpha}|^2 + \frac{1}{2}(u^4+2\underline{u}^2u^2)|\underline{\beta}|^2\\
&\quad +\frac{1}{2}(\underline{u}^4+2\underline{u}^2u^2)(\rho^2+\sigma^2) + \frac{1}{2}\underline{u}^4|\beta|^2\\
Q(W)(K_O,K_O,K_O,e_4) &= \frac{1}{4}\underline{u}^6|\alpha|^2 + \frac{3}{2}\underline{u}^4u^2|\beta|^2 + \frac{3}{2}u^4\underline{u}^2(\rho^2+\sigma^2) + \frac{1}{2}u^6|\underline{\beta}|^2\\
Q(W)(K_O,K_O,K_O,e_3) &= \frac{1}{4}u^6|\underline{\alpha}|^2 + \frac{3}{2}u^4\underline{u}^2|\underline{\beta}|^2 + \frac{3}{2}\underline{u}^4u^2(\rho^2+\sigma^2) + \frac{1}{2}\underline{u}^6|\beta|^2
\end{aligned}$$

Posing $F = r^2\alpha$ we derive

$$\begin{aligned}
\sup_{C(u)}(r^{\frac{7}{2}}|\alpha|) \le c\Bigg[&\Big(\int_{S(u,\underline{u}_0)} r^{12}|\alpha|^4\Big)^{\frac{1}{4}} + \Big(\int_{S(u,\underline{u}_0)} r^{12}|r\nabla\!\!\!/\alpha|^4\Big)^{\frac{1}{4}}\\
&+\Big(\int_{C(u)\cap V(u,\underline{u})} r^4|\alpha|^2 + r^4|r\nabla\!\!\!/\alpha|^2 + r^6|D\!\!\!\!/_4\alpha|^2 + r^4|r^2\nabla\!\!\!/^2\alpha|^2 + r^6|r\nabla\!\!\!/ D\!\!\!\!/_4\alpha|^2\Big)^{\frac{1}{2}}\Bigg]
\end{aligned}$$

The various $\int_{C(u)}$ integrals can be bounded by the Bel-Robinson norms. In fact

1) To control $\int_{C(u)\cap V(u,\underline{u})} r^4|\alpha(W)|^2$ we use the inequalities [32]

$$\int_{C(u)\cap V(u,\underline{u})} r^4|\alpha(W)|^2 \le c\int_{C(u)\cap V(u,\underline{u})} r^4|\hat{\mathcal{L}}_O\alpha(W)|^2$$

$$\int_{C(u)\cap V(u,\underline{u})} r^4|\hat{\mathcal{L}}_O\alpha(W)|^2 \le c\int_{C(u)\cap V(u,\underline{u})} r^4|\alpha(\hat{\mathcal{L}}_O W)|^2$$

[32] The proofs of this inequality and the following ones are in [Ch-Kl2].

where $|\hat{\mathcal{L}}_O f|^2 \equiv \sum_{i=1,2,3} |\hat{\mathcal{L}}_{{}^{(i)}O} f|^2$ and the r.h.s. is controlled by the integral

$$\int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T_0,e_4)\ .$$

2) To control $\int_{C(u)\cap V(u,\underline{u})} r^4|r\nabla\!\!\!\!/\,\alpha(W)|^2$ we use the inequality

$$\int_{C(u)\cap V(u,\underline{u})} r^4|r\nabla\!\!\!\!/\,\alpha(W)|^2 \le c\int_{C(u)\cap V(u,\underline{u})} r^4|\hat{\mathcal{L}}_O\alpha(W)|^2$$

and the r.h.s. is controlled by

$$\int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T_0,e_4)\ .$$

3) To control $\int_{C(u)\cap V(u,\underline{u})} r^4|r^2\nabla\!\!\!\!/^2\alpha(W)|^2$ we proceed as before obtaining

$$\int_{C(u)\cap V(u,\underline{u})} r^4 r^2\nabla\!\!\!\!/^2\alpha(W)|^2 \le c\int_{C(u)\cap V(u,\underline{u})} r^4|\hat{\mathcal{L}}_O^2\alpha(W)|^2$$

and the r.h.s. is controlled by

$$\int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T_0,e_4)\ .$$

4) To control $\int_{C(u)\cap V(u,\underline{u})} r^6|\mathbf{D}\!\!\!\!/_4\alpha(W)|^2$ we write

$$\mathbf{D}\!\!\!\!/_4\alpha(W) = -\mathbf{D}\!\!\!\!/_3\alpha(W) + \mathbf{D}\!\!\!\!/_{T_0}\alpha(W)$$

We control the term $\int_{C(u)\cap V(u,\underline{u})} r^6|\mathbf{D}\!\!\!\!/_{T_0}\alpha(W)|^2$ with the energy norm

$$\int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_{T_0} W)(\bar{K},\bar{K},\bar{K},e_4)\ .$$

For the term $\int_{C(u)\cap V(u,\underline{u})} r^6|\mathbf{D}\!\!\!\!/_3\alpha(W)|^2$ we use the Bianchi equation

$$\mathbf{D}\!\!\!\!/_3\alpha + \frac{1}{2}\mathrm{tr}\underline{\chi}\alpha = \nabla\!\!\!\!/\widehat{\otimes}\beta$$

reducing to the estimate of $\int_{C(u)\cap V(u,\underline{u})} r^4|\alpha(W)|^2$ and $\int_{C(u)\cap V(u,\underline{u})} r^4|r\nabla\!\!\!\!/\,\beta(W)|^2$. Proceeding as in 2) we conclude that the second integral is controlled again by

$$\int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T_0,e_4)\ .$$

Remarks:

a) We have not used in $\mathcal{Q}$ the integral $\int_{\Sigma_0} Q(W)(K_0, K_0, T_0, T_0)$. While this would have been possible in the Minkowski spacetime, in the general case this integral is not bounded.
b) The set of integral norms used in the linear case is not sufficient in the general case. The norms

$$\int_{C(u)} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}W)(K_0, K_0, K_0, e_4)\ ,\ \int_{C(u)} Q(\hat{\mathcal{L}}^2_{T_0}W)(K_0, K_0, K_0, e_4)$$

and the corresponding ones with respect to $\underline{C}(\underline{u})$ with e_3 instead of e_4 have to be added.
c) In the Minkowski case $\overline{\rho} = 0$.
d) Controlling the Bel-Robinson integral norms in $\mathcal{Q}$ and $\underline{\mathcal{Q}}$ allows to control the norms of the Weyl null components up to second derivatives. In particular we can control: Sup norms over $\mathcal{M}$ for the zero derivatives null components, $L^p(S)$ norms, $p \in [2,4]$, for the first derivatives null components and $L^2(C)$, $L^2(\underline{C})$ norms for the second derivatives null components. All these norms are defined with appropriate weights $r^\alpha \tau_-^\beta$.

5 The causal structure of the spacetime

In the previous section we studied the linearized Bianchi equations for the Weyl field in a given background spacetime. In particular we have introduced a set of energy-type norms constructed from the Bel-Robinson tensor and proved that they are conserved if the background spacetime is the Minkowski spacetime.

In a general spacetime these norms are not conserved, but, nevertheless, we can hope they are bounded in terms of the initial data if the spacetime has sufficiently "nice" properties. To make precise the meaning of the word "nice" we need to describe in a very detailed way the geometric structure of the spacetime. At the present level "nice" can be interpreted as "near to the Minkowski spacetime".

In this section, therefore, we describe the geometric structure of the spacetime. It is important to point out that we are not yet proving its existence, but we are assuming it given, more precisely we are assuming given a portion of it, a development of the initial data, in principle not maximal, with some defined properties. We will denote it $\mathcal{K}$ [33].

5.1 The foliation of the spacetime

We assume $\mathcal{K}$ endowed with a two dimensional foliation $\{S(u, \underline{u})\}$ with the following properties:

a) The $S(u, \underline{u})$ are closed two dimensional surfaces.

b) The foliation is double null integrable.

c) The foliation is equivariant with respect to $N = 2\Omega^2 L$, $\underline{N} = 2\Omega^2 \underline{L}$ where $g(L, \underline{L}) = -(2\Omega^2)^{-1}$.

[33] To prove that this portion exists will be a matter of the local existence result.

5.1.1 The double null integrability

Definition 5.1 *The S-foliation is said to be null-outgoing, respectively null incoming, integrable if the distribution formed by the tangent spaces of S together with the null outgoing direction, respectively null incoming, is integrable. An S-foliation which is both null outgoing and incoming integrable is called double null integrable.*

If the S-foliation is null-outgoing integrable the distribution made by the linear span formed by TS and e_4

$$p \in \mathcal{M} \longrightarrow \Delta_p \equiv \{TS \oplus e_4\}_p$$

is integrable: at each point p there is a submanifold of $\mathcal{K}$: $\mathcal{N}$, such that

$$T\mathcal{N}_p = \Delta_p \ .$$

The same holds for the null incoming integrable foliation, with the obvious substitutions:

$$p \in \mathcal{M} \longrightarrow \underline{\Delta}_p \equiv \{TS \oplus e_3\}_p$$

and $\underline{\mathcal{N}}$ instead of $\mathcal{N}$. The null hypersurfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ can be expressed, locally, as the level hypersurfaces of two functions u and $\underline{u}$.
The covariant vector n defined through $n_\mu = \partial_\mu u$ satisfies $n(e_a) = 0, a \in \{1,2\}$ and $n(e_4) = 0$. Therefore $(g^{\alpha\mu}n_\mu)\frac{\partial}{\partial x^\alpha} = (g^{\alpha\mu}\partial_\mu u)\frac{\partial}{\partial x^\alpha}$ is a null vector field proportional to e_4 and u satisfies the eikonal equation:

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \ .$$

Analogously, $\underline{u}$ satisfies

$$g^{\alpha\beta}\partial_\alpha \underline{u} \partial_\beta \underline{u} = 0 \ .$$

Defining $L^\alpha \equiv -g^{\alpha\mu}\partial_\mu u$ and $\underline{L}^\alpha \equiv -g^{\alpha\mu}\partial_\mu \underline{u}$, these vector fields satisfy

$$\mathbf{D}_L L = 0 \ , \ \mathbf{D}_{\underline{L}} \underline{L} = 0 \ .$$

Therefore, L and $\underline{L}$ are completely specified once the two optical functions u and $\underline{u}$, solutions of the eikonal equation, are given and, at each point, L and $\underline{L}$ are proportional to e_4 and e_3 respectively.

5.1.2 The equivariant property

Given a null outgoing integrable S-foliation, a null outgoing vector field N normal to each S is said to be equivariant relative to it if the leaves of the foliation are Lie transported by N. Given a null incoming integrable S-foliation, a null incoming vector field $\underline{N}$ normal to each S is said to be equivariant relative to it if the leaves of the foliation are Lie transported by $\underline{N}$.

Proposition 5.1 *Given a double null integral S-foliation the outgoing null vector field $N = 2\Omega^2 L$ and the incoming null vector field $\underline{N} = 2\Omega^2 \underline{L}$ are equivariant relative to it. Ω is defined by*

$$(2\Omega^2)^{-1} = -g(L, \underline{L}) \ .$$

If the S-foliation is double null integral each leave S belongs simultaneously to a null hypersurface $C(u)$ and to a null hypersurface $\underline{C}(\underline{u})$, therefore:

$$S(u, \underline{u}) = C(u) \cap \underline{C}(\underline{u})$$

where, locally,

$$C(u) = \{p \in \mathcal{M} | u(p) = u\}$$
$$\underline{C}(\underline{u}) = \{p \in \mathcal{M} | \underline{u}(p) = \underline{u}\}$$

$\underline{u}(p)$ and $u(p)$ being solutions of the eikonal equation.

The double null integral S-foliation implies that the spacetime is foliated by the null hypersurfaces $C(u)$ and $\underline{C}(\underline{u})$, therefore we introduce the following

Definition 5.2 *We call the foliation of the spacetime $(\mathcal{K}, g)$ made by the null hypersurfaces $C(u)$ and $\underline{C}(\underline{u})$, a "double null foliation".*

The causal structure of the spacetime is specified giving the double null foliation, a moving frame adapted to it and the null Ricci coefficients. These coefficients satisfy the "structure equations" which depend also on the Riemann tensor. The knowledge the causal structure implies the complete knowledge of the spacetime and, therefore, also of its metric tensor.

Definition 5.3 *The adapted null frame $\{e_4, e_3, e_a\}$ is*

$$e_4 = 2\Omega L \ , \ e_3 = 2\Omega \underline{L}$$
$$\{e_a\} = \textit{orthonormal frame of } TS$$

5.1.3 The null Ricci coefficients

The double null foliation and the moving frame associated to it are characterized by the null Ricci coefficients or connection coefficients, which define
a) The geometric properties of the submanifolds $S(u, \underline{u})$.
b) The way these submanifolds are situated on $C(u)$ and on $\underline{C}(\underline{u})$.
Let us describe all of them

The null second fundamental forms

$$\chi(X, Y) = g(D_X e_4, Y)$$

$$\underline{\chi}(X, Y) = g(D_X e_3, Y)$$

The torsion

$$\zeta(X) = \frac{1}{2}g(D_X e_4, e_3)$$

The null second fundamental forms can be expressed in terms of the Lie derivatives of the metric tensor g:

$$\frac{1}{2}(L_N g)(X,Y) = \chi(X,Y)\ ,\ \frac{1}{2}(L_{\underline{N}} g)(X,Y) = \underline{\chi}(X,Y)$$

$\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$ measure the change of the area of S: $|S|$ in the direction of e_4, and e_3 respectively:

$$\frac{d}{ds}|S_s|_{s=0} = \int_S \mathrm{tr}\chi$$
$$\frac{d}{d\underline{s}}|S_{\underline{s}}|_{s=0} = \int_S \mathrm{tr}\underline{\chi}$$

χ and $\underline{\chi}$ measure also the change of the length of a curve Γ on S when mapped by ϕ_s on the surface S_s,

$$\frac{dL_s}{ds}|_{s=0} = \int \frac{\chi(V,V)}{|V|}dt$$
$$\frac{dL_{\underline{s}}}{d\underline{s}}|_{s=0} = \int \frac{\underline{\chi}(V,V)}{|V|}dt$$

where V is the tangent vector to $\Gamma(t)$.

The remaining null Ricci coefficients

In the adapted null frame, they are

$$\begin{aligned}
\xi_a &= \frac{1}{2}g(\mathbf{D}_{e_4}e_4, e_a) = 0\\
\underline{\xi}_a &= \frac{1}{2}g(\mathbf{D}_{e_3}e_3, e_a) = 0\\
\eta_a &= -\frac{1}{2}g(\mathbf{D}_{e_3}e_a, e_4) = \zeta_a + \nabla\!\!\!/ \log\Omega\\
\underline{\eta}_a &= -\frac{1}{2}g(\mathbf{D}_{e_4}e_a, e_3) = -\zeta_a + \nabla\!\!\!/ \log\Omega\\
2\omega &= -\frac{1}{2}g(\mathbf{D}_{e_4}e_3, e_4) = -\frac{1}{2}D_4(\log\Omega)\\
2\underline{\omega} &= -\frac{1}{2}g(\mathbf{D}_{e_3}e_4, e_3) = -\frac{1}{2}D_3(\log\Omega)
\end{aligned}$$

The knowledge of the null Ricci coefficients specifies the properties of the foliation and, moreover, tells how the adapted null frame changes moving from point to point. The"Structure equations" they satisfy are of two types:

a) Elliptic equations on the two dimensional surfaces $S(u,\underline{u})$.

b) Evolution equation along the null geodesics generating the null cones $C(u)$ and $\underline{C}(\underline{u})$, which are ordinary differential equations, $L^2(S)$ valued.

5.1.4 The structure equations

Let

$$\{e_{(\alpha)}\} = \{e_\alpha\} = \{e_{(1)}, e_{(2)}, e_{(3)}, e_{(4)}\}$$
$$\{\theta^{(\alpha)}\} = \{\theta^\alpha\} = \{\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \theta^{(4)}\}$$
$$\theta^{(\alpha)}(e_{(\beta)}) = \delta^\alpha_\beta$$

Define

$$\mathbf{D}_{e_\alpha} e_\beta \equiv \mathbf{\Gamma}^\gamma_{\alpha\beta} e_\gamma$$
$$\mathbf{R}(e_\alpha e_\beta) e_\gamma \equiv R^\delta_{\gamma\alpha\beta} e_\delta$$

and

$$\omega^\alpha_\beta \equiv \mathbf{\Gamma}^\alpha_{\gamma\beta}\theta^\gamma$$
$$\Omega^\alpha_\beta \equiv \frac{1}{2} R^\alpha_{\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta$$

ω^α_β and $\mathbf{\Omega}^\alpha_\beta$ satisfy the following structure equations

$$d\theta^\alpha = -\omega^\alpha_\gamma \wedge \theta^\gamma$$
$$d\omega^\delta_\gamma = -\omega^\delta_\sigma \wedge \omega^\sigma_\gamma + \Omega^\delta_\gamma \ ,$$

called the first and the second structure equations respectively.

The explicit expression of the structure equations

1) The Gauss curvature K of S is connected to the spacetime curvature tensor according to the Gauss equation,

$$K = -\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho.$$

2) The null second fundamental forms χ, $\underline{\chi}$ satisfy the null Codazzi elliptic equations,

$$\mathrm{d}\!\!/\mathrm{iv}\,\hat{\chi} + \hat{\chi}\cdot\zeta = \frac{1}{2}(\nabla\!\!\!/\,\mathrm{tr}\chi + \zeta\mathrm{tr}\chi) - \beta$$
$$\mathrm{d}\!\!/\mathrm{iv}\,\underline{\hat{\chi}} - \underline{\hat{\chi}}\cdot\zeta = \frac{1}{2}(\nabla\!\!\!/\,\mathrm{tr}\underline{\chi} - \zeta\mathrm{tr}\underline{\chi}) + \underline{\beta}$$

3) The torsion ζ verifies the "torsion" equation [34],

$$\mathrm{c}\!\!/\mathrm{url}\,\zeta + \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} = \sigma\ .$$

[34] To derive the Gauss and Codazzi equations we proceed in the following way: Let Y be a vector field $\in TS$, $\nabla\!\!\!/ Y$ its covariant derivative in S,

$$\nabla\!\!\!/_\mu Y^\rho = \Pi^\gamma_\mu \Pi^\rho_\beta D_\gamma Y^\beta$$

4) The propagation equations

$$
\begin{aligned}
&\not{\mathbf{D}}_4\hat\chi + tr\chi\hat\chi - (D_4\log\Omega)\hat\chi = -\alpha \\
&D_4 tr\chi + \frac{1}{2}(tr\chi)^2 - (D_4\log\Omega)tr\chi + |\hat\chi|^2 = 0 \\
&\not{\mathbf{D}}_3\underline{\hat\chi} + tr\underline{\chi}\underline{\hat\chi} - (D_3\log\Omega)\underline{\hat\chi} = -\underline{\alpha} \\
&D_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 - (D_3\log\Omega)tr\underline{\chi} + |\underline{\hat\chi}|^2 = 0
\end{aligned}
$$

5) The equation for Ω

$$
\begin{aligned}
&\frac{1}{2}(D_4D_3\log\Omega + D_3D_4\log\Omega) + (D_3\log\Omega)(D_4\log\Omega) \\
&\quad +(|\zeta|^2 - |\not\nabla\log\Omega|^2) + 2|\zeta|^2 = -\rho
\end{aligned}
$$

6) The remaining propagation equations

$$
\begin{aligned}
&\not{\mathbf{D}}_4\underline{\hat\chi} + \frac{1}{2}\mathrm{tr}\chi\underline{\hat\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat\chi + (\mathbf{D}_4\log\Omega)\underline{\hat\chi} + \not\nabla\widehat{\otimes}\zeta - \zeta\widehat{\otimes}\zeta \\
&\quad +2\zeta\widehat{\otimes}\not\nabla\log\Omega - (\not\nabla\widehat{\otimes}\not\nabla)\log\Omega - \not\nabla\log\Omega\widehat{\otimes}\not\nabla\log\Omega = 0 \\
&\mathbf{D}_4\mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + (\mathbf{D}_4\log\Omega)\mathrm{tr}\underline{\chi} + \hat\chi\underline{\hat\chi} + 2\not{d}\mathrm{iv}\,\zeta \\
&\quad -2\not\Delta\log\Omega - 2|\zeta|^2 + 4\zeta\not\nabla\log\Omega - 2|\not\nabla\log\Omega|^2 = 2\rho \\
&\not{\mathbf{D}}_3\hat\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat\chi + \frac{1}{2}\mathrm{tr}\chi\underline{\hat\chi} + (\mathbf{D}_3\log\Omega)\hat\chi - \not\nabla\widehat{\otimes}\zeta - \zeta\widehat{\otimes}\zeta \\
&\quad -2\zeta\widehat{\otimes}\not\nabla\log\Omega - (\not\nabla\widehat{\otimes}\not\nabla)\log\Omega - \not\nabla\log\Omega\widehat{\otimes}\not\nabla\log\Omega = 0 \\
&\mathbf{D}_3\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi + (\mathbf{D}_3\log\Omega)\mathrm{tr}\chi + \underline{\hat\chi}\hat\chi - 2\not{d}\mathrm{iv}\,\zeta \\
&\quad -2\not\Delta\log\Omega - 2|\zeta|^2 - 4\zeta\not\nabla\log\Omega - 2|\not\nabla\log\Omega|^2 = 2\rho \\
&\not{\mathbf{D}}_3\zeta + 2\underline{\chi}\zeta - \mathbf{D}_3\not\nabla\log\Omega = -\underline{\beta} \\
&\not{\mathbf{D}}_4\zeta + 2\chi\zeta + \mathbf{D}_4\not\nabla\log\Omega = -\beta
\end{aligned}
$$

Among the complete set of structure equations, we identify those which do not depend on the null components of the curvature tensor. They are, precisely,

Π^γ_μ: the projection over TS. $\not{\mathbf{R}}$ is obtained computing the r.h.s. of the following equation:

$$\not{R}^\nu_{\ \mu\rho\sigma}Y^\mu = \not\nabla_\rho\not\nabla_\sigma Y^\rho - \not\nabla_\sigma\not\nabla_\rho Y^\nu$$

It is easy to obtain:

$$
\begin{aligned}
\not\nabla_\rho\not\nabla_\sigma Y^\rho &= \Pi^\lambda_\rho\Pi^\zeta_\sigma\Pi^\nu_\delta D_\lambda D_\zeta Y^\delta \\
&+ \Pi^\lambda_\rho\Pi^\nu_\delta\Pi^\tau_\sigma(D_\lambda\Pi^\tau_\zeta)D_\zeta Y^\delta \\
&+ \Pi^\lambda_\rho\Pi^\zeta_\sigma\Pi^\nu_\gamma(D_\lambda\Pi^\gamma_\delta)D_\zeta Y^\delta \\
\Pi^\lambda_\rho\Pi^\zeta_\sigma\Pi^\nu_\gamma(D_\lambda\Pi^\gamma_\delta) &= \frac{1}{2}\Pi^\zeta_\sigma(\chi^\nu_\rho e_{3\delta} + \underline{\chi}^\nu_\rho e_{4\delta})
\end{aligned}
$$

From it the result follows.

those corresponding to $\mathbf{R}(e_\alpha, e_\beta) = 0$. In other words they can be interpreted as the "Einstein vacuum equations", expressed relatively to the double null foliation [35]:

$$
\begin{aligned}
&\mathbf{D}_4 tr\chi + \frac{1}{2}(tr\chi)^2 + 2\omega tr\chi + |\hat{\chi}|^2 = 0 \\
&\mathbf{D}_4 tr\underline{\chi} + tr\chi tr\underline{\chi} - 2\omega tr\underline{\chi} = -2K + 2\mathrm{d}\!\!/\mathrm{iv}(-\zeta + \nabla\!\!\!/ log\Omega) + 2|-\zeta + \nabla\!\!\!/ log\Omega|^2 \\
&(\mathbf{D}\!\!\!/_4 \underline{\hat{\chi}}) - 2\omega\underline{\hat{\chi}} = (\nabla\!\!\!/ \widehat{\otimes}\underline{\eta}) + (\underline{\eta}\widehat{\otimes}\underline{\eta}) - \frac{1}{2}(tr\underline{\chi}\hat{\chi} + tr\chi\underline{\hat{\chi}}) \\
&(\mathbf{D}\!\!\!/_4 \zeta) + (\zeta\chi) + tr\chi\zeta = (\mathrm{d}\!\!/\mathrm{iv}\,\chi) - \nabla\!\!\!/ tr\chi - \nabla\!\!\!/ \mathbf{D}_4 + \chi\cdot\nabla\!\!\!/ \log\Omega\log\Omega \\
&\qquad\qquad -(\mathbf{D}_4\log\Omega)\nabla\!\!\!/ \log\Omega
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{D}_3 tr\underline{\chi} + \frac{1}{2}(tr\underline{\chi})^2 + 2\underline{\omega} tr\underline{\chi} + |\underline{\hat{\chi}}|^2 = 0 \\
&\mathbf{D}_3 tr\chi + tr\underline{\chi} tr\chi - 2\underline{\omega} tr\chi = -2K + 2\mathrm{d}\!\!/\mathrm{iv}(\zeta + \nabla\!\!\!/ log\Omega) + 2|\zeta + \nabla\!\!\!/ log\Omega|^2 \\
&(\mathbf{D}\!\!\!/_3 \hat{\chi}) - 2\underline{\omega}\hat{\chi} = (\nabla\!\!\!/ \widehat{\otimes}\eta) + (\eta\widehat{\otimes}\eta) - \frac{1}{2}(tr\chi\underline{\hat{\chi}} + tr\underline{\chi}\hat{\chi}) \\
&(\mathbf{D}\!\!\!/_3 \zeta) + (\zeta\underline{\chi}) + tr\underline{\chi}\zeta = -(\mathrm{d}\!\!/\mathrm{iv}\,\underline{\chi}) + \nabla\!\!\!/ \mathrm{tr}\underline{\chi} + \nabla\!\!\!/ \mathbf{D}_3\log\Omega - \underline{\chi}\cdot\nabla\!\!\!/ \log\Omega \\
&\qquad\qquad +(\mathbf{D}_3\log\Omega)\nabla\!\!\!/ \log\Omega
\end{aligned}
$$

6 The estimates of the null Ricci coefficients, given the Riemann tensor

In the previous section we have introduced the null Ricci coefficients whose knowledge specifies the causal structure of the spacetime. We need, therefore, to control them, that is to obtain estimates for their appropriate norms. Moreover these estimates should realize the "nice" properties discussed at the beginning of the previous section, that is those estimates which should allow to prove that the Bel-Robinson energy-type norms are bounded.

Let us observe that, to estimate the null Ricci coefficients, we have to use the structure equations which depend on the Riemann tensor and, on the other side, the control of the Riemann tensor is obtained through the control of the Bel-Robinson energy-type norms. Therefore a consistency of all these estimates is required and has to be proved.

The strategy we use is the following one: we assume that some appropriate norms of the Riemann tensor are bounded and examine which are the norms of the Ricci coefficients we are able to control.

The Riemann norms that we assume to be bounded are the same family of norms which we were able to control for the Weyl tensor, in the linear case, with

[35] to be considered as a closed set of equations they have to be supplemented with the equations:

$$\frac{1}{2}L_N g = \chi\ ,\ \frac{1}{2}L_{\underline{N}} g = \underline{\chi}\ .$$

the Minkowski spacetime as background spacetime. there we used a family of Bel-Robinson tensor energy-type norms, see Proposition 4.6, which in that case were proved to be bounded. Therefore the Riemann norms we assume under control are:

a) sup norms for the null components which will be bounded by a constant $\triangle_0$.

b) $L^p(S)$ norms, $p \in [2,4]$ for the first derivatives of the null components, which will be bounded by a constant $\triangle_1$.

c) $L^2(C), L^2(\underline{C})$ norms for the second derivatives, which will be bounded by a constant $\triangle_2$. The weights in r of these norms are those compatible with the Bel-Robinson integrals. We will give the whole list of these norms in the next section.

In this section we show how these null Ricci coefficients can be controlled describing the derivation of some estimates which give a good insight to the general procedure.

6.1 Some analytic tools for the evolution equations

Lemma 6.1 *Let U, V, F, $\underline{F}$ be k-covariant tensor fields tangential to S satisfying*

$$\frac{dU_{a_1\dots a_k}}{d\underline{u}} + \lambda_0 \Omega tr\chi U_{a_1\dots a_k} = F_{a_1\dots a_k}$$
$$\frac{dV_{a_1\dots a_k}}{du} + \lambda_0 \Omega tr\underline{\chi} V_{a_1\dots a_k} = \underline{F}_{a_1\dots a_k}$$

where $\lambda_0 \geq 0$ and for a generic k-tensor T: $T_{a_1\dots a_k} \equiv T(e_{a_1}, e_{a_2}, .., e_{a_k})$. Assume $|\Omega tr\chi - \overline{\Omega tr\chi}| \leq \delta_0 r^{-2}$ with $\delta_0 > 0$ sufficiently small. Posing $\lambda_1 = 2(\lambda_0 - \frac{1}{p})$, we have, along $C(u)$ or $\underline{C}(\underline{u})$:

$$|r^{\lambda_1} U|_{p,S}(u,\underline{u}) \leq c_0\left(|r^{\lambda_1} U|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{\lambda_1} F|_{p,S}(u,\underline{u}')d\underline{u}'\right)$$
$$|r^{\lambda_1} V|_{p,S}(u,\underline{u}) \leq c_0\left(|r^{\lambda_1} V|_{p,S}(u_0,\underline{u}) + \int_{u_0}^{u} |r^{\lambda_1} \underline{F}|_{p,S}(u',\underline{u})du'\right)$$

Lemma 6.2 *Consider an arbitrary compact Riemannian manifold (S,γ), verifying $k_m = \min_S r^2 K > 0$ and $k_M = \max_S r^2 K < \infty$, denoting with K the Gauss curvature of S. If the symmetric, traceless, 2-tensor ξ is a solution of*

$$\not{d}iv\xi = f$$

then there exists a constant c which depends only on ${k_m}^{-1}$, k_M and p such that, for all $2 \leq p < \infty$

$$\int_S (|\nabla\!\!\!/ \xi|^p + r^{-p}|\xi|^p) \leq c\int_S |f|^p$$

6.2 Some estimates for the non derived null Ricci coefficients

Estimate for the traceless part of the second fundamental form: $\hat{\chi}$

As $e_4 = 2\Omega L$ and $N = \Omega e_4$ is null equivariant, on scalars $\Omega \mathbf{D}_4 = \frac{d}{d\underline{u}}$. The evolution equation

$$\not{D}_4\hat{\chi} + tr\chi\hat{\chi} - (D_4 \log\Omega)\hat{\chi} = -\alpha$$

can be rewritten as

$$\frac{d}{d\underline{u}}\left(\frac{\hat{\chi}_{ab}}{\Omega}\right) + \Omega\mathrm{tr}\chi\left(\frac{\hat{\chi}_{ab}}{\Omega}\right) = -\Omega\alpha_{ab}$$

Assuming that $|r^2(\mathrm{tr}\chi - \overline{\mathrm{tr}\chi})|$ is small we can apply Lemma 6.1 and, assuming also Ω near to one, we obtain

$$\begin{aligned}|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u}) &\leq c\left(|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*}|r^{2-\frac{2}{p}}\alpha|_{p,S}\right)\\ &\leq c\left(|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u}_*) + \Delta_0\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^{\frac{3}{2}}}\right)\end{aligned}$$

Remarks:

1) Here $p \geq 2$. In fact we have assumed the sup norm of α be bounded, see eq. 4.17, therefore all the $|r^{\frac{7}{2}-\frac{2}{p}}\alpha|_{p,S}$ norms are bounded.

2) The last integral can be bounded by $c\Delta_0$ if we control $\int_{\underline{u}}^{\underline{u}_*} r^{-\frac{3}{2}}$. In a generic spacetime r is not a coordinate but a function defined by

$$4\pi r^2 = |S(u,\underline{u})|\ .$$

From it

$$\frac{\partial}{\partial\underline{u}}r(u,\underline{u}) = \frac{r(u,\underline{u})}{2}\overline{\Omega\mathrm{tr}\chi}\ ,$$

therefore if we prove that

$$\overline{\Omega\mathrm{tr}\chi} = O\left(\frac{1}{r}\right)$$

we conclude that $\frac{1}{r} = O(\frac{1}{\underline{u}})$ and control the integral.

3) The previous estimates for $|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u})$ are written in terms of an integral over $C(u)$ and its norm on $\underline{C}(\underline{u}_*)$ [36]. Therefore we need a control of $|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}$ on $\underline{C}(\underline{u}_*)$, we call it the "last slice". Denoting $\mathcal{I}_*$ a constant which

[36] In this specific estimate this could be avoided, obtaining an estimate for $\hat{\chi}$ in terms of an integral over $C(u)$ and its norm on $\Sigma_0 \cap C(u)$, but in other cases this is unavoidable.

bound all the norms of the Ricci coefficients on the last slice, we obtain [37]

$$|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_* + \Delta_0) \tag{6.31}$$

Estimate for the trace part of the second fundamental form: trχ

Rewriting its evolution equation for trχ as

$$\frac{d}{d\underline{u}}\left(\frac{\mathrm{tr}\chi}{\Omega}\right) + \frac{1}{2}\Omega\mathrm{tr}\chi\left(\frac{\mathrm{tr}\chi}{\Omega}\right) = -|\hat{\chi}|^2$$

we obtain the estimate

$$|r^{1-\frac{2}{p}}\mathrm{tr}\chi|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_* + \Delta_0) \tag{6.32}$$

We remark that this result is not enough to obtain the complete estimate of χ, as we need to control also the norms [38]

$$|r^{2-\frac{2}{p}}(\mathrm{tr}\chi - \overline{\mathrm{tr}\chi})|_{p,S}(u,\underline{u}) \text{ and } |r^{2-\frac{2}{p}}\left(\overline{\mathrm{tr}\chi} - \frac{2}{r}\right)|_{p,S}(u,\underline{u}) .$$

The control of $\underline{\hat{\chi}}$ and of tr$\underline{\chi}$ is done in a similar way, but using the evolution equation along $\underline{C}(\underline{u})$. Therefore, in this case, one does not have a dependance on the last slice norms and obtains

$$(|r^{1-\frac{2}{p}}\tau_-\underline{\hat{\chi}}|_{p,S} + |r^{1-\frac{2}{p}}\mathrm{tr}\underline{\chi}|_{p,S})(u,\underline{u}) \leq c(\mathcal{I}_0 + \Delta_0)$$

and one has, as before, to control also the norms

$$|r^{1-\frac{2}{p}}\tau_-(\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\underline{\chi}})|_{p,S}(u,\underline{u}) \ , \ |r^{2-\frac{2}{p}}\left(\overline{\mathrm{tr}\underline{\chi}} + \frac{2}{r}\right)|_{p,S}(u,\underline{u}) .$$

Remark: *To prove the estimates for the the sup norms of the non derived null Ricci coefficients requires some extra work. In fact for it we need estimates in the $L^p(S)$ norms, $p \in [2,4]$, also for the first tangential derivatives.*

6.3 Some estimates for the first derivatives of the null Ricci coefficients

Estimate of $\nabla\!\!\!\!/\,$trχ

Deriving the evolution equation for trχ

$$D_4 tr\chi + \frac{1}{2}(tr\chi)^2 - (D_4 \log\Omega)tr\chi + |\hat{\chi}|^2 = 0$$

[37]From the equation satsfied by $f = \frac{\hat{\chi}_{ab}}{\Omega}$ we could have expected a better result. In fact

$$\frac{df}{dr} + \frac{1}{r}f = r^{-\frac{7}{2}}$$

has solutions $f = O(r^{-\frac{5}{2}})$. The slower decays arises from the estimate of $\hat{\chi}$ we can prove on $\underline{C}(\underline{u}_*)$.

[38]To prove Lemma 6.1 and for subsequent estimates.

and using the commutation relations

$$\nabla\!\!\!/ \mathbf{D}_4 \mathrm{tr}\chi - \mathbf{D}\!\!\!/_4 \nabla\!\!\!/ \mathrm{tr}\chi = \chi \cdot \nabla\!\!\!/ \mathrm{tr}\chi - (\zeta + \underline{\eta})\mathbf{D}_4 \mathrm{tr}\chi$$

we obtain

$$\mathbf{D}\!\!\!/_4 \nabla\!\!\!/ \mathrm{tr}\chi + \chi \cdot \nabla\!\!\!/ \mathrm{tr}\chi = (\nabla\!\!\!/ \log\Omega)\mathbf{D}_4 \mathrm{tr}\chi + \nabla\!\!\!/ (\mathbf{D}_4 \log\Omega)\mathrm{tr}\chi - \mathrm{tr}\chi \nabla\!\!\!/ \mathrm{tr}\chi - \nabla\!\!\!/ |\hat{\chi}|^2 .$$

Defining $U_a \equiv \Omega^{-1}\nabla\!\!\!/_a \mathrm{tr}\chi$ the previous equation becomes

$$\frac{d}{d\underline{u}}U_a + \frac{3}{2}\Omega \mathrm{tr}\chi U_a = F_a$$

to which we can apply Lemma 6.1. For technical reason we define, instead,

$$\psi\!\!\!/ = \Omega^{-1}\nabla\!\!\!/ \mathrm{tr}\chi + \Omega^{-1}\mathrm{tr}\chi\zeta = U + \Omega^{-1}\mathrm{tr}\chi\zeta$$

which satisfies

$$\frac{d}{d\underline{u}}\psi\!\!\!/_a + \frac{3}{2}\Omega \mathrm{tr}\chi \psi\!\!\!/_a = F\!\!\!/_a$$

where

$$F\!\!\!/ \equiv -\Omega\hat{\chi}\cdot\psi\!\!\!/ - \nabla\!\!\!/ |\hat{\chi}|^2 - \eta|\hat{\chi}|^2 + \mathrm{tr}\chi\hat{\chi}\cdot\underline{\eta} - \mathrm{tr}\chi\beta$$

Applying Lemma 6.1 we obtain

$$|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S}(u,\underline{u}) \le c_0 \left(|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S}(u,\underline{u}_*) + \int_{\underline{u}}^{\underline{u}_*} |r^{3-\frac{2}{p}}F\!\!\!/|_{p,S} \right)$$

and

$$\begin{aligned} |r^{3-\frac{2}{p}}F\!\!\!/|_{p,S} &\le |r^{3-\frac{2}{p}}\Omega\hat{\chi}\psi\!\!\!/|_{p,S} + |r^{3-\frac{2}{p}}\nabla\!\!\!/ |\hat{\chi}|^2|_{p,S} \\ &+ \left[|r^{3-\frac{2}{p}}\eta|\hat{\chi}|^2|_{p,S} + |r^{3-\frac{2}{p}}\underline{\eta}\hat{\chi}\mathrm{tr}\chi|_{p,S} + |r^{3-\frac{2}{p}}\mathrm{tr}\chi\beta|_{p,S} \right] \end{aligned} \tag{6.33}$$

Due to the previous results for the non derived coefficients one can prove that the [....] term of eq. 6.33 satisfies

$$[...] \le (\mathcal{I}_* + \mathcal{I}_0 + \Delta_0) O\left(\frac{1}{r^2}\right) .$$

Moreover $|r^{3-\frac{2}{p}}\Omega\hat{\chi}\psi\!\!\!/|_{p,S}$ satisfies

$$|r^{3-\frac{2}{p}}\Omega\hat{\chi}\psi\!\!\!/|_{p,S} \le cr^{-2}|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S}$$

and its integral does not present problems in the Gronwall inequality.

The control of $|r^{3-\frac{2}{p}}\nabla\!\!\!/ |\hat{\chi}|^2|_{p,S}$ presents a new difficulty; we cannot estimate $\nabla\!\!\!/ \hat{\chi}$ deriving the evolution equation for $\hat{\chi}$,

$$\mathbf{D}\!\!\!/_4\hat{\chi} + tr\chi\hat{\chi} - (D_4 \log\Omega)\hat{\chi} = -\alpha ,$$

as this will produce a term $\nabla\!\!\!/ \alpha$ which cannot be estimated by a sup norm [39]. We proceed in a different way writing the Codazzi equation for $\hat{\chi}$ in terms of the tensor $\psi\!\!\!/$,

$$\text{d}\!\!\!/\text{iv}\,\hat{\chi} + \zeta\hat{\chi} = \frac{1}{2}\Omega\psi\!\!\!/ - \beta \ .$$

We apply Lemma 6.2 to this system and obtain the following $L^2(S)$ estimate:

$$|r^{3-2/p}\nabla\!\!\!/\hat{\chi}|_{p,S} \ \leq \ c\left(|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S} + r^{-\frac{1}{2}}\Delta_0\right)$$

which substituted in eq. 6.33 gives

$$|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0) + c\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}|r^{3-\frac{2}{p}}\psi\!\!\!/|_{p,S}$$

Applying, finally, the Gronwall Lemma we obtain

$$|r^{3-2/p}\psi\!\!\!/|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0)$$

This estimate, together with the previous results, implies

$$|r^{3-2/p}\nabla\!\!\!/\text{tr}\chi|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0)$$
$$|r^{3-2/p}\nabla\!\!\!/\hat{\chi}|_{p,S}(u,\underline{u}) \leq c\,(\mathcal{I}_* + \mathcal{I}_0 + \Delta_0)$$

These examples suggest how to proceed to control the norms of the remaining null Ricci coefficients and their derivatives.

6.3.1 The Last slice Lemma

It appears in the estimates of $\hat{\chi}$ and $\text{tr}\chi$ that their norms are expressed in terms of $L^2(C)$ integrals and the values of the same norms on the last incoming cone $\underline{C}_*$, the last slice. The norms on the last slice are bounded by a common constant $\mathcal{I}_*$. To use these estimates in the subsequent theorems we have to prove that also the norms on the last slice can be controlled in terms of the norms on the initial data and of the norms of the Riemann null components up to second order. This result is obtained only if we choose a well defined foliation on $\underline{C}_*$ and use it to construct the double null foliation on $\mathcal{K}$. To obtain the appropriate foliation on $\underline{C}_*$ we have to solve a non linear problem, "the last slice problem", whose main aspects are discussed in a subsequent section. Its solution implies that we can bound $\mathcal{I}_*$ in terms of $\mathcal{I}_0$, Δ_0, Δ_1, Δ_2, namely

$$\mathcal{I}_* \leq c\,(\mathcal{I}_0 + \Delta_0 + \Delta_1 + \Delta_2)$$

so that finally

$$|r^{2-\frac{2}{p}}\hat{\chi}|_{p,S}(u,\underline{u}) + |r^{1-\frac{2}{p}}\text{tr}\chi|_{p,S}(u,\underline{u}) \leq c(\mathcal{I}_0 + \Delta)$$

[39] This would be needed if we apply Lemma 6.1 to the evolution equation for $\nabla\!\!\!/\hat{\chi}$.

where $\triangle \equiv \triangle_0 + \triangle_1 + \triangle_2$.

One important conclusion to extract from the discussion about these estimates is that we need the control of the Riemann norms up to the second derivatives [40]. In the next section we will show that, on the other side, the proof of the boundedness of the Bel-Robinson energy-type norms which guarantee the control of the Riemann null components, requires good estimates for the null Ricci coefficients up to third derivatives, the "nice" properties of the spacetime, see the discussion at the beginning of Section 5. Therefore by consistency we have to control the null Ricci coefficients up to third derivatives in terms of the initial data and the estimates for the Riemann components up to second derivatives. This, technically involved result is summarized, see also [Kl-Ni], in the following

Proposition 6.1 *If the Riemann norms* [41] *are bounded by the constants $\triangle_0$, $\triangle_1$, $\triangle_2$ that we assume sufficiently small, if the initial data are bounded by $\mathcal{I}_0$ and the "Last slice" Lemma is valid, then the sum of all the appropriate norms for the null Ricci coefficients up to third derivatives, we denote globally $\mathcal{O}$, can be bounded by $\mathcal{I}_0 + \triangle_0 + \triangle_1 + \triangle_2$:*

$$\mathcal{O} \le c(\mathcal{I}_0 + \triangle_0 + \triangle_1 + \triangle_2)$$

6.4 The definition of a global Ricci norm

We collect all the Ricci coefficients norms in a quantity $\mathcal{O}$ which enters in the final theorem [42].

$$\mathcal{O} \equiv \mathcal{O}_{[3]} + \underline{\mathcal{O}}_{[3]}$$

$$\begin{aligned}
\mathcal{O}_{[3]} &\equiv \left[\mathcal{O}_3 + \tilde{\mathcal{O}}_3(\omega)\right] + \mathcal{O}_{[2]} \\
\mathcal{O}_{[2]} &\equiv \left[\mathcal{O}_2 + \tilde{\mathcal{O}}_2(\omega)\right] + \mathcal{O}_{[1]} \\
\mathcal{O}_{[1]} &\equiv \left[\mathcal{O}_1 + \sup_{p\in[2,4]}\left(\mathcal{O}_1^{p,S}(\underline{\omega}) + \mathcal{O}_0^{p,S}(\mathbf{D}_3\underline{\omega})\right)\right] + \mathcal{O}_{[0]}^{\infty} \\
\mathcal{O}_{[0]}^{\infty} &\equiv \mathcal{O}_0^{\infty} + \mathcal{O}_0^{\infty,S}(\underline{\omega}) + \sup_{\mathcal{K}}|r(\Omega - \frac{1}{2})| + \sup_{\mathcal{K}}|r^2(\overline{\mathrm{tr}\chi} - \frac{2}{r})|
\end{aligned}$$

$$\begin{aligned}
\underline{\mathcal{O}}_{[3]} &\equiv \left[\underline{\mathcal{O}}_3 + \tilde{\mathcal{O}}_3(\underline{\omega})\right] + \underline{\mathcal{O}}_{[2]} \\
\underline{\mathcal{O}}_{[2]} &\equiv \left[\underline{\mathcal{O}}_2 + \tilde{\mathcal{O}}_2(\underline{\omega})\right] + \underline{\mathcal{O}}_{[1]}
\end{aligned}$$

[40] To have the control of the sup norms of the Riemann null components we need $L^2(C, \underline{C})$ norms of the second derivatives of Riemann, see Proposition 4.7.

[41] The explicit expression of the Riemann norms with the appropriate weights are given in a subsequent section.

[42] The explicit expression of the norms $\tilde{\mathcal{O}}_2(\omega)$, $\tilde{\mathcal{O}}_3(\omega)$, $\tilde{\mathcal{O}}_2(\underline{\omega})$, $\tilde{\mathcal{O}}_3(\underline{\omega})$ will be given elsewhere.

$$\underline{\mathcal{O}}_{[1]} \equiv \left[\underline{\mathcal{O}}_1 + \sup_{p\in[2,4]}\left(\mathcal{O}_1^{p,S}(\omega) + \mathcal{O}_0^{p,S}(\mathbf{D}_4\omega)\right)\right] + \underline{\mathcal{O}}^{\infty}_{[0]}$$

$$\underline{\mathcal{O}}^{\infty}_{[0]} \equiv \underline{\mathcal{O}}_0^{\infty} + \mathcal{O}_0^{\infty,S}(\omega) + \sup_{\mathcal{K}}|r\tau_-(\overline{\mathrm{tr}\underline{\chi}} + \frac{2}{r})|$$

We define, also, the analogous quantities restricted on the initial hypersurface or on the final slice

$$\underline{\mathcal{O}}^{\infty}_{[0]}(\Sigma_0)\ ,\ \underline{\mathcal{O}}_{[1]}(\Sigma_0)\ ,\ \underline{\mathcal{O}}_{[2]}(\Sigma_0)\ ,\ \underline{\mathcal{O}}_{[3]}(\Sigma_0)$$

$$\mathcal{O}^{\infty}_{[0]}(\underline{C}_*)\ ,\ \mathcal{O}_{[1]}(\underline{C}_*)\ ,\ \mathcal{O}_{[2]}(\underline{C}_*)\ ,\ \mathcal{O}_{[3]}(\underline{C}_*).$$

The explicit expressions of the various quantities follows from the definitions:

$$\mathcal{O}_{0,1} \equiv \sup_{p\in[2,4]}\mathcal{O}^p_{0,1}\ ,\ \underline{\mathcal{O}}_{0,1} \equiv \sup_{p\in[2,4]}\underline{\mathcal{O}}^p_{0,1}$$

$$\mathcal{O}_{2,3} \equiv \sup_{p\in[2,4]}\mathcal{O}^p_{2,3}\ ,\ \underline{\mathcal{O}}_{2,3} \equiv \sup_{p\in[2,4]}\underline{\mathcal{O}}^p_{2,3}$$

and the following expressions

$q = 0,1,2$

$$\mathcal{O}^p_q = \mathcal{O}^{p,S}_q(\mathrm{tr}\chi) + \mathcal{O}^{p,S}_q(\hat{\chi}) + \mathcal{O}^{p,S}_q(\eta)$$
$$\underline{\mathcal{O}}^p_q = \mathcal{O}^{p,S}_q(\mathrm{tr}\underline{\chi}) + \mathcal{O}^{p,S}_q(\underline{\hat{\chi}}) + \mathcal{O}^{p,S}_q(\underline{\eta})$$

$q = 3$

$$\mathcal{O}^p_3 = \mathcal{O}^{p,S}_3(\mathrm{tr}\chi) + \mathcal{O}^{p,S}_3(\hat{\chi}) + \mathcal{O}^{p,S}_3(\eta)$$
$$\underline{\mathcal{O}}^p_3 = \mathcal{O}^{p,S}_3(\mathrm{tr}\underline{\chi}) + \mathcal{O}^{p,S}_3(\underline{\hat{\chi}}) + \mathcal{O}^{p,S}_3(\underline{\eta})$$

$$\mathcal{O}^{p,S}_q(X) \equiv \sup_{\mathcal{K}}\mathcal{O}^{p,S}_q(X)(u,\underline{u})$$
$$\mathcal{O}^{p,S}_q(\underline{X}) \equiv \sup_{\mathcal{K}}\mathcal{O}^{p,S}_q(\underline{X})(u,\underline{u})$$

$$\mathcal{O}^{p,S}_q(\underline{C}_*)(X) \equiv \sup_{\underline{C}_*}\mathcal{O}^{p,S}_q(X)(u,\underline{u})$$
$$\underline{\mathcal{O}}^{p,S}_q(\Sigma_0)(\underline{X}) \equiv \sup_{\Sigma_0}\mathcal{O}^{p,S}_q(\underline{X})(u,\underline{u}).$$

$$\mathcal{O}^{p,S}_q(tr\chi)(u,\underline{u}) = |r^{(2+q-\frac{2}{p})}\nabla\!\!\!\!/^q(\mathrm{tr}\chi - \overline{tr\chi})|_{p,S(u,\underline{u})}$$
$$\mathcal{O}^{p,S}_q(tr\underline{\chi})(u,\underline{u}) = |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!\!/^q(\mathrm{tr}\underline{\chi} - \overline{tr\underline{\chi}})|_{p,S(u,\underline{u})}$$

$$\mathcal{O}_q^{p,S}(\hat{\chi})(u,\underline{u}) = |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q(\hat{\chi})|_{p,S(u,\underline{u})}$$
$$\mathcal{O}_q^{p,S}(\underline{\hat{\chi}})(u,\underline{u}) = |r^{(1+q-\frac{2}{p})}\tau_-^{\frac{3}{2}}\nabla\!\!\!/^q(\underline{\hat{\chi}})|_{p,S(u,\underline{u})}$$
$$\mathcal{O}_q^{p,S}(\eta)(u,\underline{u}) = |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\eta|_{p,S(u,\underline{u})}$$
$$\mathcal{O}_q^{p,S}(\underline{\eta})(u,\underline{u}) = |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\underline{\eta}|_{p,S(u,\underline{u})}$$
$$\mathcal{O}_q^{p,S}(\omega)(u,\underline{u}) = |r^{(2+q-\frac{2}{p})}\nabla\!\!\!/^q\omega|_{p,S(u,\underline{u})}$$
$$\mathcal{O}_q^{p,S}(\underline{\omega})(u,\underline{u}) = |r^{(1+q-\frac{2}{p})}\tau_-\nabla\!\!\!/^q\underline{\omega}|_{p,S(u,\underline{u})}$$

6.5 The deformation tensors

As previously discussed, see subsection 3.3.3, the Bel-Robinson energy-type norms are not expected to be conserved in the general case due to the fact that the Bianchi equations are not linear and that the spacetime is not symmetric. To prove that, nevertheless, these norms are bounded requires to control how "far" the spacetime we are constructing, is from the completely symmetric Minkowski spacetime or from the Schwarzschild spacetime. For obtaining it we define a set of vector fields which correspond to the Killing and conformal Killing vector fields which were used in the expressions of the conserved energy-type norms for the linearized Bianchi equations, see subsection 4.4. We study how much they are not anymore Killing or conformal Killing, estimating the norms of their corresponding deformation tensors.
We consider, therefore, the following vector fields, Killing or conformal Killing when the spacetime is "flat",

$$T_0 = \frac{1}{2}(\hat{N}+\underline{\hat{N}})\ ,\ S = \frac{1}{2}(\underline{u}\hat{N}+u\underline{\hat{N}})\ ,\ K_0 = \frac{1}{2}(\underline{u}^2\hat{N}+u^2\underline{\hat{N}}) \tag{6.34}$$

We recall the explicit expression of the various components of the deformation tensor associated to a vector field X:

$$^{(X)}\pi \equiv L_X g\ ,\ ^{(X)}\hat{\pi} = {}^{(X)}\pi - \frac{1}{4}g_{\mu\nu}\mathrm{tr}\pi$$

$$\begin{aligned}
^{(X)}\pi_{ab} &= g(D_{e_a}X,e_b)+g(D_{e_b}X,e_a)\\
^{(X)}\pi_{4a} &= g(D_{\hat{N}}X,e_a)+g(D_{e_a}X,\hat{N})\\
^{(X)}\pi_{3a} &= g(D_{\underline{\hat{N}}}X,e_a)+g(D_{e_a}X,\underline{\hat{N}})\\
^{(X)}\pi_{34} &= g(D_{\underline{\hat{N}}}X,\hat{N})+g(D_{\hat{N}}X,\underline{\hat{N}})\\
^{(X)}\pi_{33} &= g(D_{\underline{\hat{N}}}X,\underline{\hat{N}})+g(D_{\underline{\hat{N}}}X,\underline{\hat{N}})\\
^{(X)}\pi_{44} &= g(D_{\hat{N}}X,\hat{N})+g(D_{\hat{N}}X,\hat{N})
\end{aligned}$$

where $e_4 \equiv \hat{N}$, $e_3 \equiv \underline{\hat{N}}$ and

$$\begin{aligned}
{}^{(X)}\hat{\pi}_{ab} &= {}^{(X)}\pi_{ab} - \frac{1}{4}\delta_{ab}tr{}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{4a} &= {}^{(X)}\pi_{4a}\ ,\ {}^{(X)}\hat{\pi}_{3a} = {}^{(X)}\pi_{3a} \\
{}^{(X)}\hat{\pi}_{34} &= {}^{(X)}\pi_{34} + \frac{1}{2}tr{}^{(X)}\pi \\
{}^{(X)}\hat{\pi}_{33} &= {}^{(X)}\pi_{33}\ ,\ {}^{(X)}\hat{\pi}_{44} = {}^{(X)}\pi_{44}
\end{aligned}$$

The deformation tensors relative to the vector fields in eq. 6.34 are

$$\begin{aligned}
&{}^{(T_0)}\hat{\pi}_{ab} \equiv{}^{(T_0)} i_{ab} = \hat{\chi}_{ab} + \hat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega + \underline{\omega})) \\
&{}^{(T_0)}\hat{\pi}_{34} \equiv{}^{(T_0)} j = \frac{1}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\omega + \underline{\omega}) \\
&{}^{(T_0)}\hat{\pi}_{4a} \equiv{}^{(T_0)} m_a = 2\underline{\eta}_a - \nabla\!\!\!\!/_a \log\Omega = \underline{\eta}_a - \zeta_a \\
&{}^{(T_0)}\hat{\pi}_{3a} \equiv{}^{(T_0)} \underline{m}_a = 2\eta_a - \nabla\!\!\!\!/_a \log\Omega = \eta_a + \zeta_a \\
&{}^{(T_0)}\hat{\pi}_{44} \equiv{}^{(T_0)} n = -4\omega = 2\mathbf{D}_4 \log\Omega \\
&{}^{(T_0)}\hat{\pi}_{33} \equiv{}^{(T_0)} \underline{n} = -4\underline{\omega} = 2\mathbf{D}_3 \log\Omega
\end{aligned}$$

$$\begin{aligned}
&{}^{(S)}\hat{\pi}_{ab} \equiv {}^{(S)}i_{ab} = \underline{u}\hat{\chi}_{ab} + u\hat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{1}{\Omega}) \\
&{}^{(S)}\hat{\pi}_{34} \equiv {}^{(S)}j = \frac{1}{2}(\underline{u}\mathrm{tr}\chi + u\mathrm{tr}\underline{\chi}) + (\underline{u}\omega + u\underline{\omega}) - \frac{1}{\Omega} \\
&{}^{(S)}\hat{\pi}_{4a} \equiv {}^{(S)}m_a = u(2\underline{\eta}_a - \nabla\!\!\!\!/_a \log\Omega) = u(\underline{\eta}_a - \zeta_a) = u{}^{(T_0)}m_a \\
&{}^{(S)}\hat{\pi}_{3a} \equiv {}^{(S)}\underline{m}_a = \underline{u}(2\eta_a - \nabla\!\!\!\!/_a \log\Omega) = \underline{u}(\eta_a + \zeta_a) = \underline{u}{}^{(T_0)}\underline{m}_a \\
&{}^{(S)}\hat{\pi}_{44} \equiv {}^{(S)}n = u(-4\omega) = 2u\mathbf{D}_4 \log\Omega = u{}^{(T_0)}n \\
&{}^{(S)}\hat{\pi}_{33} \equiv {}^{(S)}\underline{n} = \underline{u}(-4\underline{\omega}) = 2\underline{u}\mathbf{D}_3 \log\Omega = \underline{u}{}^{(T_0)}\underline{n}
\end{aligned}$$

$$\begin{aligned}
&{}^{(K_0)}\hat{\pi}_{ab} \equiv {}^{(K_0)}i_{ab} = \underline{u}^2\hat{\chi}_{ab} + u^2\hat{\underline{\chi}}_{ab} + \frac{1}{2}\delta_{ab}(\frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) \\
&\qquad\qquad + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega}) \\
&{}^{(K_0)}\hat{\pi}_{34} \equiv {}^{(K_0)}j = \frac{1}{2}(\underline{u}^2\mathrm{tr}\chi + u^2\mathrm{tr}\underline{\chi}) + (\underline{u}^2\omega + u^2\underline{\omega}) - \frac{\underline{u}+u}{\Omega} \\
&{}^{(K_0)}\hat{\pi}_{4a} \equiv {}^{(K_0)}m_a = u^2(2\underline{\eta}_a - \nabla\!\!\!\!/_a \log\Omega) = u^2(\underline{\eta}_a - \zeta_a) = u^{2(T_0)}m_a \\
&{}^{(K_0)}\hat{\pi}_{3a} \equiv {}^{(K_0)}\underline{m}_a = \underline{u}^2(2\eta_a - \nabla\!\!\!\!/_a \log\Omega) = \underline{u}^2(\eta_a + \zeta_a) = \underline{u}^{2(T_0)}\underline{m}_a \\
&{}^{(K_0)}\hat{\pi}_{44} \equiv {}^{(K_0)}n = u^2(-4\omega) = 2u^2\mathbf{D}_4 \log\Omega = u^{2(T_0)}n \\
&{}^{(K_0)}\hat{\pi}_{33} \equiv {}^{(K_0)}\underline{n} = \underline{u}^2(-4\underline{\omega}) = 2\underline{u}^2\mathbf{D}_3 \log\Omega = \underline{u}^{2(T_0)}\underline{n}
\end{aligned}$$

It follows immediately from these expressions that the control of the norms of these deformations tensors follow once that we control the global norm of the Ricci coefficients $\mathcal{O}$ [43].

6.6 The rotation vector fields

The "approximate" Killing or conformal vector fields defined in eq.6.34 do not describe all the approximate symmetries of the spacetime. We need also to define approximate "Angular momentum vector fields", $\{^{(i)}O\}$, which play the same role as the angular momentum vector fields of the Minkowski space. They are constructed as follows:

We start from the very far region of the initial surface Σ_0. In view of the strong asymptotic flatness assumptions, the manifold looks euclidean at spacelike infinity. We can, therefore, define the canonical angular momentum vector fields at infinity and then pull them back with the help of the diffeomorphism generated by the flow normal to the S surfaces along Σ_0 [44]. These vector fields can, then, be "pushed forward", in the same way, along the last slice $\underline{C}_*$ using the diffeomorphism $\underline{\phi}_\tau$ generated by the equivariant normal $\underline{N}$. Finally they are pulled back once more, along the hypersurfaces $C(u)$ with the help of the diffeomorphism generated by null-outgoing equivariant vector field N. These steps define the vector fields $^{(i)}O$ at any point of our spacetime $\mathcal{K}$.
More precisely, let $q \in S(u,\underline{u})$ be a generic point of $\mathcal{K}$. As $S(u,\underline{u})$ is diffeomorphic via $\phi_{\Delta=(\underline{u}_*-\underline{u})}$ to $S(u,\underline{u}_1) \subset \underline{C}_*$, $\exists p \in S(u,\underline{u}_1)$ such that $q = \phi_\Delta^{-1}(p)$. We define the element O of the rotation group operating over q in the following way

$$(O,q) \equiv \phi_\Delta^{-1}(O_*, p = \phi_\Delta(q))$$

where (O,q) is a point of $S(u,\underline{u})$, while $(O_*, p = \phi_\Delta(q))$ is the point of $S(u,\underline{u}_1)$ obtained applying O_* to the point p. By definition they are tangent to the S-foliation and commute with N. Moreover they satisfy the canonical commutation relations. Thus, finally, the "extended" rotation generators, or angular vector fields $\{^{(i)}O\}$, satisfy

$$\begin{aligned}
&[^{(i)}O, {}^{(j)}O] = \epsilon_{ijk}{}^{(k)}O \\
&[N, {}^{(i)}O] = 0 \\
&g({}^{(i)}O, e_4) = g({}^{(i)}O, e_3) = 0.
\end{aligned}$$

To compute the various components of the deformation tensors associated to the rotation vector fields $^{(i)}O$ we collect the expressions for the various components of the derivatives of $^{(i)}O_c = g({}^{(i)}O, e_c)$, using eq. 6.35 and, as $N = \Omega\hat{N}$, the following commutation relations

$$\begin{aligned}
[^{(i)}O, \hat{N}] &= [^{(i)}O, \Omega^{-1}N] = {}^{(i)}O(\Omega^{-1})N \\
&= -{}^{(i)}O(\log\Omega)\hat{N} = {}^{(i)}F\hat{N}
\end{aligned}$$

[43] In fact the control of the third order derivatives of the null Ricci coefficients is required when we have to control the deformation tensors associated to the rotaion vector fields.

[44] See [Kl-Ni] and the chapter on the initial data set in [Ch-Kl2].

where

$$
{}^{(i)}F \equiv -(\nabla\!\!\!/_c \log\Omega)\,{}^{(i)}O_c
$$

$$
\begin{cases}
g(\mathbf{D}_a{}^{(i)}O, e_4) = -\chi_{ab}{}^{(i)}O_b \\
g(\mathbf{D}_4{}^{(i)}O, e_a) = \chi_{ab}{}^{(i)}O_b \\
g(\mathbf{D}_4{}^{(i)}O, e_4) = 0 \\
g(\mathbf{D}_a{}^{(i)}O, e_3) = -\underline{\chi}_{ab}{}^{(i)}O_b \\
g(\mathbf{D}_3{}^{(i)}O, e_3) = 0 \\
g(\mathbf{D}_4{}^{(i)}O, e_3) = -2\underline{\eta}_b{}^{(i)}O_b \\
g(\mathbf{D}_3{}^{(i)}O, e_4) = -2\eta_b{}^{(i)}O_b
\end{cases}
\tag{6.35}
$$

Using these equations we compute most of the components of the deformation tensor relative to the rotation vector fields. Defining

$$
{}^{({}^{(i)}\Omega)}\pi \equiv {}^{(i)}\pi
$$

we obtain

$$
\begin{cases}
{}^{(i)}\pi_{ab} = g(\mathbf{D}_a{}^{(i)}O, e_b) + g(\mathbf{D}_b{}^{(i)}O, e_a) \equiv 2\,{}^{(i)}H_{ab} \\
{}^{(i)}\pi_{44} = 2g(\mathbf{D}_4{}^{(i)}O, e_4) = 0 \\
{}^{(i)}\pi_{4a} = g(\mathbf{D}_4{}^{(i)}O, e_a) + g(\mathbf{D}_a{}^{(i)}O, e_4) = 0 \\
{}^{(i)}\pi_{3a} = g(\mathbf{D}_a{}^{(i)}O, e_3) + g(\mathbf{D}_3{}^{(i)}O, e_a) \\
\qquad = -{}^{(i)}O_b\underline{\chi}_{ab} + \hat{\underline{N}}({}^{(i)}O_a) + {}^{(i)}O_b\, g(\mathbf{D}\!\!\!/_3 e_b, e_a) \equiv 4\,{}^{(i)}Z_a \\
{}^{(i)}\pi_{34} = -2(\eta_b + \underline{\eta}_b)\,{}^{(i)}O_b = -4(\nabla\!\!\!/_b \log\Omega)\,{}^{(i)}O_b = 4\,{}^{(i)}F \\
{}^{(i)}\pi_{33} = 2g(\mathbf{D}_3{}^{(i)}O, e_3) = 0
\end{cases}
\tag{6.36}
$$

We compute the commutator $[{}^{(i)}O, \hat{\underline{N}}]$:

$$
\begin{aligned}
[{}^{(i)}O, \hat{\underline{N}}] &= -4\,{}^{(i)}Z_b e_b - {}^{(i)}O_c(\nabla\!\!\!/_c \log\Omega)\hat{\underline{N}} \\
&= -4\,{}^{(i)}Z_b e_b + {}^{(i)}F\hat{\underline{N}}
\end{aligned}
$$

and, as $\underline{N} = \Omega\hat{\underline{N}}$ [45],

$$
\begin{aligned}
[{}^{(i)}O, \underline{N}] &= [{}^{(i)}O, \Omega\hat{\underline{N}}] = {}^{(i)}O(\Omega)\hat{\underline{N}} + \Omega[{}^{(i)}O, \hat{\underline{N}}] \\
&= {}^{(i)}O(\log\Omega)\underline{N} + \Omega[{}^{(i)}O, \hat{\underline{N}}] \\
&= -{}^{(i)}F\underline{N} - 4\Omega\,{}^{(i)}Z_b e_b + {}^{(i)}F\underline{N} \\
&= -4\Omega\,{}^{(i)}Z_b e_b
\end{aligned}
$$

To control the appropriate norms of the deformation tensors ${}^{(i)}\pi$ we have to control the norms of ${}^{(i)}O_a, {}^{(i)}H_{ab}, {}^{(i)}Z_b$. To do it we look for the evolution equations for these quantities along the $C(u)$ cones. They are described in the following proposition, whose proof is not reported here. From them, the norms can be estimated in terms of the norms on the $\underline{C}(\underline{u}_*)$ null hypersurface.

[45]The fact that $[{}^{(i)}O, \underline{N}] \in TS$ but is different from zero shows that one could have defined the rotation vector fields in a different way starting from the Σ_0 hypersurface and using the diffeomorphism $\underline{\phi}_t$ generated by $\underline{N}$.

Proposition 6.2 *$^{(i)}O_a, {}^{(i)}H_{ab}, {}^{(i)}Z_b$ satisfy the following evolution equations* [46]

$$
\begin{aligned}
&\frac{d}{d\underline{u}}{}^{(i)}O_b - \frac{1}{2}\Omega tr\chi^{(i)}O_b = \Omega\hat{\chi}^{(i)}O_b \\
&\frac{d}{d\underline{u}}((\nabla\!\!\!\!/_a{}^{(i)}O)_b) = \Omega\left[\hat{\chi}_{bc}(\nabla\!\!\!\!/_a{}^{(i)}O)_c - \hat{\chi}_{ac}(\nabla\!\!\!\!/_c{}^{(i)}O)_b + {}^{(i)}O(\chi_{cb}\underline{\eta}_a - \chi_{ca}\underline{\eta}_b)\right. \\
&\qquad\qquad\left. + {}^{(i)}OR_{4abc} + {}^{(i)}O_c\chi_{cb}\zeta_a + {}^{(i)}O_c(\nabla\!\!\!\!/_a\chi)_{cb}\right] \\
&\frac{d}{d\underline{u}}(\Omega\,{}^{(i)}Z_a) - \frac{1}{2}\Omega tr\chi(\Omega\,{}^{(i)}Z_a) = \left[\Omega\hat{\chi}_{ab}(\Omega\,{}^{(i)}Z_b) + 2\Omega\omega(\Omega\,{}^{(i)}Z_a)\right] \\
&\qquad +\Omega^2\left[\frac{1}{2}(L_{{}^{(i)}O} - {}^{(i)}F)(\zeta+\eta))_a - 2\zeta_b{}^{(i)}H_{ab} - \frac{1}{2}(\nabla\!\!\!\!/_a{}^{(i)}F - (\underline{\eta}+\eta)_a{}^{(i)}F)\right] \\
&\frac{d}{d\underline{u}}{}^{(i)}H_{ab} = -\Omega(\hat{\chi}_{ac}{}^{(i)}H_{cb} + \hat{\chi}_{bc}{}^{(i)}H_{ca}) + \Omega((L_{{}^{(i)}O} - {}^{(i)}F)\hat{\chi})_{ab} \\
&\qquad +\frac{1}{2}\delta_{ab}\Omega(L_{{}^{(i)}O} - {}^{(i)}F)tr\chi
\end{aligned}
\tag{6.37}
$$

How much of the null Ricci coefficients we have to control? As we said, the control of the null Ricci coefficients is necessary to control the deformation tensors. These, at their turn, have to be estimated to control the "Error" term, which is needed to bound the Bel-Robinson energy norms, see next section for more details. In the estimate of the "Error" terms the deformation tensors appear up to second derivatives and from the expressions of the evolution equations for the rotation deformation tensors, eq. 6.37, they involve the null Ricci coefficients up to third derivatives. This is the reason we need to control the null Ricci coefficients up to third order derivatives.

7 The control of the Bel-Robinson norms, assuming the null Ricci coefficients "small"

In the previous section we have estimated the null Ricci coefficients assuming some bounds on the Riemann tensor. Here we proceed in the opposite way: we assume the global null Ricci norm, $\mathcal{O}$, and the rotation deformation tensor norms, small and prove that the Bel-Robinson energy-type norms are bounded.

7.1 The Bel-Robinson energy-type norms

We recall that, as anticipated in Section 4, the family of Bel-Robinson energy-type norms we prove are bounded is sligthly larger than the one introduced in

[46]Here we assume that the null frame is such that $D\!\!\!\!/_4 e_a = 0$.

the linear case with the Minkowski spacetime as background spacetime [47]. In fact they are:

$$\mathcal{Q}(u,\underline{u}) = \mathcal{Q}_1(u,\underline{u}) + \mathcal{Q}_2(u,\underline{u})$$
$$\underline{\mathcal{Q}}(u,\underline{u}) = \underline{\mathcal{Q}}_1(u,\underline{u}) + \underline{\mathcal{Q}}_2(u,\underline{u})$$

where

$$\begin{aligned}
\mathcal{Q}_1(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \\
&+ \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_4) \\
\mathcal{Q}_2(u,\underline{u}) &\equiv \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \\
&+ \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_4) \\
&+ \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_4) \\
&+ \int_{C(u)\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T^2 W)(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned} \tag{7.38}$$

$$\begin{aligned}
\underline{\mathcal{Q}}_1(u,\underline{u}) &\equiv \left[\sup_{V(u,\underline{u})\cap\Sigma_0} |r^3\overline{\rho}|^2 + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3) + \right. \\
&\left. + \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O W)(\bar{K},\bar{K},T,e_3)\right] \\
\underline{\mathcal{Q}}_2(u,\underline{u}) &\equiv \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3) \\
&+ \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_O^2 W)(\bar{K},\bar{K},T,e_3) \\
&+ \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)(\bar{K},\bar{K},\bar{K},e_3) \\
&+ \int_{\underline{C}(\underline{u})\cap V(u,\underline{u})} Q(\hat{\mathcal{L}}_T^2 W)(\bar{K},\bar{K},\bar{K},e_3)
\end{aligned} \tag{7.39}$$

and $V(u,\underline{u}) = J^-(S(u,\underline{u}))$ is the domain of dependance of $S(u,\underline{u})$.

[47] The reason for it is in the technical aspects of the proof and will not be discussed here, see [Ch-Kl-Ni]. Choosing a smaller family of norms does not allow to obtain this result, but it could imply a weaker one

7.2 The error terms

To prove that these quantities are bounded we have to control the "Error" terms $\mathcal{E}_1(u,\underline{u})$ and $\mathcal{E}_2(u,\underline{u})$,

$$\begin{aligned}
\mathcal{E}_1(u,\underline{u}) &= \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\
&+ \frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\
&+ \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta) \\
&+ \int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta) \\
&+ \frac{1}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_2(u,\underline{u}) &= \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\
&+ \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O^2 W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma T^\delta) \\
&+ \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\
&+ \int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\
&+ \int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta) \\
&+ \frac{1}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O^2 W)_{\alpha\beta\gamma\delta}({}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\
&+ \frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\
&+ \frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\
&+ \frac{3}{2}\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_T^2 W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta)
\end{aligned}$$

The control of all the terms in which these integrals are decomposed requires a very long work. Let us give here just an example of how this is done looking at the specific term:

$$\int_{V_{(u,\underline{u})}} DivQ(\hat{\mathcal{L}}_T W)_{\beta\gamma\delta}(\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta)$$

This term, at its turn, is decomposed in many different terms. In fact $DivQ(\hat{\mathcal{L}}_X W)(\bar{K},\bar{K},\bar{K}(T_0))$ has the following structure:

$$DivQ(\hat{\mathcal{L}}_X W)(\bar{K},\bar{K},\bar{K}(T_0)) " = " (\hat{\mathcal{L}}_X W)\left[{}^{(X)}\pi DW + (D^{(X)}\pi)W\right](\bar{K},\bar{K},(\bar{K},T_0))$$

More precisely, writing this expression in terms of the null Riemann components,

$$\begin{aligned} DivQ(\hat{\mathcal{L}}_{T_0} W)(\bar{K},\bar{K},\bar{K}) &= \frac{1}{8}\tau_+^6 D(T_0,W)_{444} + \frac{3}{8}\tau_+^4\tau_-^2 D(T_0,W)_{344} \\ &+ \frac{3}{8}\tau_+^2\tau_-^4 D(T_0,W)_{334} + \frac{1}{8}\tau_-^6 D(T_0,W)_{333} . \end{aligned}$$

Let us consider the more dangerous term, the one with the highest weight,

$$\tau_+^6 D(T_0,W)_{444} = \tau_+^6 D(T_0,W)(e_4,e_4,e_4) .$$

In terms of the null Riemann components it has the structure:

$$D(T_0,W)_{444} = 4\alpha(\hat{\mathcal{L}}_{T_0} W)\cdot\Theta(T_0,W) - 8\beta(\hat{\mathcal{L}}_{T_0} W)\cdot\Xi(T_0,W)$$

where $\Theta(T_0,W)$, $\Xi(T_0,W)$ are null components of $D^\alpha(\hat{\mathcal{L}}_{T_0} W)_{\alpha\beta\gamma\delta}$ and, therefore, have the structure:

$$\left[{}^{(T_0)}\pi DW + (D^{(T_0)}\pi)W\right] .$$

Let us look at the part with the structure ${}^{(T_0)}\pi DW$. It is made by many quadratic terms,

$$\begin{aligned} \Theta(T_0,W) &= \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!\!/\,\alpha\right] + \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{n}}\,;\,\alpha_4\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{j}\,;\,\alpha_3\right] \\ &+ \mathrm{Qr}\left[{}^{(T_0)}\mathbf{i}\,;\,\nabla\!\!\!\!/\,\beta\right] + \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\beta_4\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\beta_3\right] \\ &+ \mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\nabla\!\!\!\!/\,(\rho,\sigma)\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{j}\,;\,(\rho_4,\sigma_4)\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{n}\,;\,(\rho_3,\sigma_3)\right] \\ &+ \dots\dots\dots \end{aligned}$$

We consider the first term, the other ones will be treated in a similar way,

$$DivQ(\hat{\mathcal{L}}_{T_0} W)(\bar{K},\bar{K},\bar{K}) = \tau_+^6\alpha(\hat{\mathcal{L}}_{T_0} W)\mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!\!/\,\alpha(W)\right] .$$

Therefore, we have to estimate $\int_{V_{(u,\underline{u})}}\tau_+^6\alpha(\hat{\mathcal{L}}_{T_0} W)\mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!\!/\,\alpha(W)\right]$.

$$\left|\int_{V_{(u,\underline{u})}}\tau_+^6\alpha(\hat{\mathcal{L}}_{T_0} W){}^{(T_0)}\underline{\mathbf{m}}\nabla\!\!\!\!/\,\alpha(W)\right| \le c\left|\int_{u_0}^{u} du'\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^3\alpha(\hat{\mathcal{L}}_{T_0} W)\underline{u}'^3\,{}^{(T_0)}\underline{\mathbf{m}}\nabla\!\!\!\!/\,\alpha(W)\right|$$

$$\le c\int_{u_0}^{u} du'\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|\alpha(\hat{\mathcal{L}}_{T_0} W)|^2\right)^{\frac{1}{2}}\left(\int_{C(u';[\underline{u}_0,\underline{u}])}\underline{u}'^6|{}^{(T_0)}\underline{\mathbf{m}}|^2|\nabla\!\!\!\!/\,\alpha(W)|^2\right)^{\frac{1}{2}}$$

Defining

$$\mathcal{Q}_1 \equiv \sup_{\mathcal{K}}(\mathcal{Q}_1 + \underline{\mathcal{Q}}_1)(u, \underline{u})$$

one obtains

$$\left| \int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0} W) \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}};\, \not\nabla\alpha(W)\right] \right|$$

$$\leq c \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |\alpha(\hat{\mathcal{L}}_{T_0} W)|^2 \right)^{\frac{1}{2}} \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |{}^{(T_0)}\underline{\mathbf{m}}|^2 |\not\nabla\alpha(W)|^2 \right)^{\frac{1}{2}}$$

$$\leq \mathcal{Q}_1{}^{\frac{1}{2}} \int_{u_0}^{u} du' \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |{}^{(T_0)}\underline{\mathbf{m}}|^2 |\not\nabla\alpha(W)|^2 \right)^{\frac{1}{2}}$$

To estimate the last integral we observe that, using the Ricci coefficients estimates, we are able to control $\left(\sup_{V(u,\underline{u})} |r^{2\,(T_0)}\underline{\mathbf{m}}|\right)^2$. Therefore we assume $\sup_{V(u,\underline{u})} |r^{2\,(T_0)}\underline{\mathbf{m}}| \leq c\epsilon_0$ and from it

$$\left(\int_{C(u';[\underline{u}_0,\underline{u}])} \underline{u}'^6 |{}^{(T_0)}\underline{\mathbf{m}}|^2 |\not\nabla\alpha(W)|^2 \right)^{\frac{1}{2}} = \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \frac{\underline{u}'^6}{r^4} |r^{2\,(T_0)}\underline{\mathbf{m}}|^2 |\not\nabla\alpha(W)|^2 \right)^{\frac{1}{2}}$$

$$\leq c \frac{1}{u'^2} \left(\sup_{V(u,\underline{u})} |r^{2\,(T_0)}\underline{\mathbf{m}}| \right) \left(\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_+^6 |\not\nabla\alpha(W)|^2 \right)^{\frac{1}{2}}$$

$$\leq c \frac{1}{u'^2} \left(\sup_{V(u,\underline{u})} |r^{2\,(T_0)}\underline{\mathbf{m}}| \right) \mathcal{Q}_1^{\frac{1}{2}} \leq c\epsilon_0 \frac{1}{u'^2} \mathcal{Q}_1^{\frac{1}{2}} .$$

Finally

$$\left| \int_{V_{(u,\underline{u})}} \tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0} W) \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}};\, \not\nabla\alpha(W)\right] \right| \leq c\epsilon_0 \mathcal{Q}_1{}^{\frac{1}{2}} \mathcal{Q}_1{}^{\frac{1}{2}} \int_{u_0}^{u} du' \frac{1}{u'^2} \leq c\epsilon_0 \mathcal{Q}_1$$

Conclusion: *If this kind of estimates can be obtained for all the terms composing the "Error" terms the final result is:*

$$\mathcal{E}_1(u, \underline{u}) + \mathcal{E}_2(u, \underline{u}) \leq c\epsilon_0 (\mathcal{Q}_1 + \mathcal{Q}_2)$$

This can be proved, see [Ch-Kl-Ni], and we can state the following

Proposition 7.1 *If the null Ricci coefficients and the rotation deformation tensors are bounded by a sufficiently small constant then*

$$(\mathcal{Q}_1 + \mathcal{Q}_2) \leq (\mathcal{Q}_1 + \mathcal{Q}_2)_{\Sigma_0 \cap \mathcal{M}} + c\epsilon_0 (\mathcal{Q}_1 + \mathcal{Q}_2)$$

implying

$$(\mathcal{Q}_1 + \mathcal{Q}_2) \leq \frac{1}{1 - c\epsilon_0} (\mathcal{Q}_1 + \mathcal{Q}_2)_{\Sigma_0 \cap \mathcal{M}}$$

7.3 The control of the Riemann norms in terms of the Bel-Robinson energy type norms

The introduction of the Bel-Robinson norms and the proof of their boundedness allow to control the norms of the Riemann null components. We do not prove these estimates here, which, nevertheless, are of the same type as those discussed to prove the first estimate of eq. 4.17. We collect all the norms for the Riemann tensor which can be proved to be bounded, in a unique quantity $\mathcal{R}$ which enters in the final theorem.

7.4 The definition of a global Riemann tensor norm

Although the norms we are considering are of different type: L^∞ for the Riemann components, $L^p(S)$ for the first derivatives and $L^2(C(\underline{C}))$ for the second derivatives, $\mathcal{R}$ is made only by the $L^2(C(\underline{C}))$ norms for the Riemann components up to second derivatives. In fact the other ones can be estimated in terms of them.

$$\mathcal{R} \equiv \mathcal{R}_{[2]} + \underline{\mathcal{R}}_{[2]}$$

$$\mathcal{R}_{[2]} = \mathcal{R}_{[1]} + \mathcal{R}_2\ ,\ \underline{\mathcal{R}}_{[2]} = \underline{\mathcal{R}}_{[1]} + \underline{\mathcal{R}}_2$$
$$\mathcal{R}_{[1]} = \mathcal{R}_{[0]} + \mathcal{R}_1\ ,\ \underline{\mathcal{R}}_{[1]} = \underline{\mathcal{R}}_{[0]} + \underline{\mathcal{R}}_1$$
$$\mathcal{R}_{[0]} = \mathcal{R}_0\ ,\ \underline{\mathcal{R}}_{[0]} = \underline{\mathcal{R}}_0 + \sup_{\mathcal{K}} r^3|\bar{\rho}|$$

with

$$\mathcal{R}_0 = \left(\mathcal{R}_0[\alpha]^2 + \mathcal{R}_0[\beta]^2 + \ldots + \mathcal{R}_0[\underline{\beta}]^2\right)^{1/2}$$
$$\underline{\mathcal{R}}_0 = \left(\underline{\mathcal{R}}_0[\beta]^2 + \underline{\mathcal{R}}_0[(\rho,\sigma)]^2 \ldots + \underline{\mathcal{R}}_0[\underline{\alpha}]^2\right)^{1/2}$$
$$\mathcal{R}_1 = \left(\mathcal{R}_1[\alpha]^2 + \mathcal{R}_1[\beta]^2 + \ldots + \mathcal{R}_1[\underline{\beta}]^2\right)^{1/2}$$
$$\underline{\mathcal{R}}_1 = \left(\underline{\mathcal{R}}_1[\beta]^2 + \underline{\mathcal{R}}_1[(\rho,\sigma)]^2 \ldots + \underline{\mathcal{R}}_1[\underline{\alpha}]^2\right)^{1/2}$$
$$\mathcal{R}_2 = \left(\mathcal{R}_2[\alpha]^2 + \mathcal{R}_2[\beta]^2 + \ldots + \mathcal{R}_2[\underline{\beta}]^2\right)^{1/2}$$
$$\underline{\mathcal{R}}_2 = \left(\underline{\mathcal{R}}_2[\beta]^2 + \underline{\mathcal{R}}_2[(\rho,\sigma)]^2 \ldots + \underline{\mathcal{R}}_2[\underline{\alpha}]^2\right)^{1/2}$$

and

$$\mathcal{R}_{0,1,2}[w] \equiv \sup_{\mathcal{K}} \mathcal{R}_{0,1,2}[w](u,\underline{u})$$
$$\underline{\mathcal{R}}_{0,1,2}[w] \equiv \sup_{\mathcal{K}} \underline{\mathcal{R}}_{0,1,2}[w](u,\underline{u}).$$

The terms $\mathcal{R}_{0,1,2}[w](u,\underline{u})$ denote the L^2 norms for the zero, first and second derivatives of the Riemann component w made along the portion of null hypersurface $C(u) \cap V(u,\underline{u})$, where $V(u,\underline{u}) = J^-(S(u,\underline{u}))$ is the domain of dependance of $S(u,\underline{u})$, the causal past. Writing them in detail we have:

Zero derivatives of the Riemann components:

$$\mathcal{R}_0[\alpha](u,\underline{u}) = \|r^2\alpha\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_0[\beta](u,\underline{u}) = \|r^2\beta\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_0[(\rho,\sigma)](u,\underline{u}) = \|\tau_- r(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_0[\underline{\beta}](u,\underline{u}) = \|\tau_-^2\underline{\beta}\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_0[\beta](u,\underline{u}) = \|r^2\beta\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_0[(\rho,\sigma)](u,\underline{u}) = \|r^2(\rho-\overline{\rho},\sigma-\overline{\sigma})\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_0[\underline{\beta}](u,\underline{u}) = \|\tau_- r\underline{\beta}\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_0[\underline{\alpha}](u,\underline{u}) = \|\tau_-^2\underline{\alpha}\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$

First derivatives of the Riemann components:

$$\mathcal{R}_1[\alpha](u,\underline{u}) = \|r^3\slashed{\nabla}\alpha\|_{2,C(u)\cap V(u,\underline{u})} + \|r^3\alpha_3\|_{2,C(u)\cap V(u,\underline{u})} + \|r^3\alpha_4\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_1[\beta](u,\underline{u}) = \|r^3\slashed{\nabla}\beta\|_{2,C(u)\cap V(u,\underline{u})} + \|\tau_- r^2\beta_3\|_{2,C(u)\cap V(u,\underline{u})} + \|r^3\beta_4\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_1[(\rho,\sigma)](u,\underline{u}) = \|\tau_- r^2\slashed{\nabla}(\rho,\sigma)\|_{2,C(u)\cap V(u,\underline{u})} + \|r^3(\rho,\sigma)_4\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\mathcal{R}_1[\underline{\beta}](u,\underline{u}) = \|\tau_-^2 r\slashed{\nabla}\underline{\beta}\|_{2,C(u)\cap V(u,\underline{u})} + \|\tau_- r^2\underline{\beta}_4\|_{2,C(u)\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_1[\beta](u,\underline{u}) = \|r^3\slashed{\nabla}\beta\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|r^3\beta_3\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_1[(\rho,\sigma)](u,\underline{u}) = \|r^3\slashed{\nabla}(\rho,\sigma)\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|\tau_- r^2(\rho,\sigma)_3\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_1[\underline{\beta}](u,\underline{u}) = \|\tau_- r^2\slashed{\nabla}\underline{\beta}\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|\tau_-^2 r\underline{\beta}_3\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|r^3\underline{\beta}_4\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$
$$\underline{\mathcal{R}}_1[\underline{\alpha}](u,\underline{u}) = \|\tau_-^2 r\slashed{\nabla}\underline{\alpha}\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|\tau_-^3\underline{\alpha}_3\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})} + \|\tau_- r^2\underline{\alpha}_4\|_{2,\underline{C}(\underline{u})\cap V(u,\underline{u})}$$

Second derivatives of the Riemann components:

$$\begin{aligned}
\mathcal{R}_2[\alpha](u,\underline{u}) &= \|r^4\slashed{\nabla}^2\alpha\|_{2,C\cap V} + \|r^4\slashed{\nabla}\alpha_3\|_{2,C\cap V} \\
&\quad + \|r^4\slashed{\nabla}\alpha_4\|_{2,C\cap V} + \|\tau_- r^3\alpha_{33}\|_{2,C\cap V} \\
&\quad + \|r^4\alpha_{34}\|_{2,C\cap V} + \|r^4\alpha_{44}\|_{2,C\cap V} \\
\mathcal{R}_2[\beta](u,\underline{u}) &= \|r^4\slashed{\nabla}^2\beta\|_{2,C\cap V} + \|\tau_- r^3\slashed{\nabla}\beta_3\|_{2,C\cap V} \\
&\quad + \|r^4\slashed{\nabla}\beta_4\|_{2,C\cap V} + \|r^4\beta_{34}\|_{2,C\cap V} + \|r^4\beta_{44}\|_{2,C\cap V} \\
\mathcal{R}_2[(\rho,\sigma)](u,\underline{u}) &= \|\tau_- r^3\slashed{\nabla}^2(\rho,\sigma)\|_{2,C\cap V} + \|\tau_-^2 r^2\slashed{\nabla}(\rho,\sigma)_3\|_{2,C\cap V} \\
&\quad + \|r^4\slashed{\nabla}(\rho,\sigma)_4\|_{2,C\cap V} + \|\tau_- r^3(\rho,\sigma)_{34}\|_{2,C\cap V} \\
&\quad + \|r^4(\rho,\sigma)_{44}\|_{2,C\cap V} \\
\mathcal{R}_2[\underline{\beta}](u,\underline{u}) &= \|\tau_-^2 r^2\slashed{\nabla}^2\underline{\beta}\|_{2,C\cap V} + \|\tau_- r^3\slashed{\nabla}\underline{\beta}_4\|_{2,C\cap V} \\
&\quad + \|\tau_-^2 r^2\underline{\beta}_{34}\|_{2,C\cap V} + \|r^4\underline{\beta}_{44}\|_{2,C\cap V} \\
\underline{\mathcal{R}}_2[\beta](u,\underline{u}) &= \|r^4\slashed{\nabla}^2\beta\|_{2,\underline{C}\cap V} + \|r^4\slashed{\nabla}\beta_3\|_{2,\underline{C}\cap V} \\
&\quad + \|r^6\beta_{43}\|_{2,\underline{C}\cap V} + \|\tau_- r^3\beta_{33}\|_{2,\underline{C}\cap V} \\
\underline{\mathcal{R}}_2[(\rho,\sigma)](u,\underline{u}) &= \|r^4\slashed{\nabla}^2(\rho,\sigma)\|_{2,\underline{C}\cap V} + \|\tau_- r^3\slashed{\nabla}(\rho,\sigma)_3\|_{2,\underline{C}\cap V} \\
&\quad + \|\tau_- r^3(\rho,\sigma)_{34}\|_{2,\underline{C}\cap V} + \|\tau_-^2 r^2(\rho,\sigma)_{33}\|_{2,\underline{C}\cap V}
\end{aligned}$$

$$
\begin{aligned}
\underline{\mathcal{R}}_2[\underline{\beta}](u,\underline{u}) &= \|\tau_- r^3 \nabla\!\!\!\!/^2 \underline{\beta}\|_{2,\underline{C}\cap V} + \|\tau_-^2 r^2 \nabla\!\!\!\!/ \underline{\beta}_3\|_{2,\underline{C}\cap V} \\
&+ \|r^4 \nabla\!\!\!\!/ \underline{\beta}_4\|_{2,\underline{C}\cap V} + \|\tau_- r^3 \underline{\beta}_{34}\|_{2,\underline{C}\cap V} \\
&+ \|\tau_-^3 r \underline{\beta}_{33}\|_{2,\underline{C}\cap V} \\
\underline{\mathcal{R}}_2[\underline{\alpha}](u,\underline{u}) &= \|\tau_-^2 r^2 \nabla\!\!\!\!/^2 \underline{\alpha}\|_{2,\underline{C}\cap V} + \|\tau_-^3 r \nabla\!\!\!\!/ \underline{\alpha}_3\|_{2,\underline{C}\cap V} \\
&+ \|\tau_- r^3 \nabla\!\!\!\!/ \underline{\alpha}_4\|_{2,\underline{C}\cap V} + \|\tau_-^4 \underline{\alpha}_{33}\|_{2,\underline{C}\cap V} \\
&+ \|\tau_-^2 r^2 \underline{\alpha}_{34}\|_{2,\underline{C}\cap V} + \|\tau_- r^3 \underline{\alpha}_{44}\|_{2,\underline{C}\cap V}
\end{aligned}
$$

The control of the Riemann norms can be summarized in terms of the inequality

$$\mathcal{R} \leq c(\mathcal{Q}_1 + \mathcal{Q}_2) \tag{7.40}$$

Collecting together eq. 7.40 and Proposition 7.1 we can state the following

Proposition 7.2 *If the global norm of the null Ricci coefficients $\mathcal{O}$ and the rotation deformation tensors are bounded by a sufficiently small constant ϵ_0 then*

$$\mathcal{R} \leq (\mathcal{Q}_1 + \mathcal{Q}_2)_{\Sigma_0 \cap \mathcal{K}}$$

Remarks: *Propositions 6.1 and 7.2 describe and conclude the linearization approach for this problem. The next step will be to collect all the results together, proving the required consistency and showing that the development $\mathcal{K}$ has complete null outgoing geodesics.*

8 The last slice foliation

The construction of the foliation on the last slice represents the third major idea in the proof of *C-K-N Theorem.* The easiest way to understand how such a foliation has to be constructed is to assume given a double null foliation on our original spacetime. This induces an S foliation on the last slice. It has been remarked before that one needs estimates of χ, ζ and $\underline{\omega}$ and their derivatives on $\underline{C}_*$. Let us start by considering $tr\chi$, the only propagation equation available for us on the last slice is, see subsubsection 5.1.3,

$$
\begin{aligned}
&D_3 tr\chi + \frac{1}{2} tr\underline{\chi} tr\chi + (D_3 \log\Omega) tr\chi + \hat{\underline{\chi}}\hat{\chi} - 2 d\!\!\!/iv\,\zeta - 2|\zeta|^2 \\
&-2 \Delta\!\!\!\!/ \log\Omega + 2\zeta \nabla\!\!\!\!/ \log\Omega - 2|\nabla\!\!\!\!/ \log\Omega|^2 = 2\rho
\end{aligned}
$$

According to this equation it seems that the differentiability properties of $tr\chi$, relative to the tangential directions, are the same as those for ρ. As we have discussed in the previous sections this would lead to a serious inconsistency. To avoid this difficulty we choose a special foliation on $\underline{C}_*$ which provides appropriate "final" conditions for the not underlined Ricci coefficients. The double null foliation obtained in this way will be called the "Canonical foliation". To see how to choose this foliation we write the propagation equation for $tr\chi$ in the following form

$$\frac{d}{du_*}(\Omega tr\chi) + \frac{1}{2}\Omega tr\underline{\chi}(\Omega tr\chi) + \Omega^2(\hat{\chi}\hat{\underline{\chi}} - 2\mathrm{d\!\!/iv}\,\eta - 2|\eta|^2) = 2\Omega^2\rho$$

where u_* is the value of the parameter u on the last slice. We define the scalar function

$$\mu^* = -\mathrm{d\!\!/iv}\,\zeta + \frac{1}{2}\hat{\chi}\hat{\underline{\chi}} - \rho$$

Recalling also that $\eta = \zeta + \nabla\!\!\!/ \log\Omega$, we rewrite the previous equation in the form

$$\frac{d}{du_*}(\Omega tr\chi) + \frac{1}{2}\Omega tr\underline{\chi}(\Omega tr\chi) = 2\Omega^2(\Delta\!\!\!/ \log\Omega - \mu^*) + 2\Omega^2|\eta|^2$$

u_* is here a function defined on the last slice $\underline{C}_*$ whose level surfaces determine a generic foliation. Since we have the freedom to prescribe this foliation on the last slice we ask ourselves whether there is a choice for which the dangerous term $\Delta\!\!\!/ \log\Omega - \mu^*$, the one which contains ρ, is constant on the leaves of the foliation. Therefore we require that $\log\Omega$ satisfies the equation $\Delta\!\!\!/ \log\Omega - \mu^* = -\overline{(\mu^*)}$.

To do it we look for a new foliation $u_* = u_*(v)$, expressed relative to a fixed background foliation, defined by the level surfaces of the affine parameter v, corresponding to the null geodesics generators of $\underline{C}_*$. Relative to $u_* = u_*(v)$, Ω has to satisfy the equations

$$\begin{aligned} &\Delta\!\!\!/ \log\Omega - \mu^* = -\overline{(\mu^*)} \;\;;\;\; \overline{\log 2\Omega} = 0 \\ &\frac{du_*}{dv} = (2\Omega^2)^{-1};\;\; u_*|_{S_*(0)} = u_0 \end{aligned} \tag{8.41}$$

where $S_*(0) = \underline{C}_* \cap \Sigma_0$.

Once these conditions are satisfied, the evolution equation for $tr\chi$ becomes

$$\frac{d}{du_*}(\Omega tr\chi) + \frac{1}{2}\Omega tr\underline{\chi}(\Omega tr\chi) = \Omega^2(-\overline{\hat{\chi}\hat{\underline{\chi}}} + 2\overline{\rho}) + 2\Omega^2|\eta|^2$$

which is the previous equation 8.41 with some terms averaged on $S(u, \underline{u}_*)$. Due to this, $\nabla\!\!\!/ tr\chi$ has now the right regularity properties, indeed $\nabla\!\!\!/ \overline{\rho} = 0$ [48].

To prove the existence of the "Canonical" foliation amounts to solve the system of equations 8.41, which we call the "last slice problem". This requires details which are beyond the scope of this review paper, see [Ch-Kl-Ni]. Once such a foliation is constructed locally, one proceeds, in the same way as described in the previous section, to estimate their Ricci coefficients. This is done, by using the natural propagation equations on $\underline{C}_*$, in terms of initial conditions on $\underline{C}_* \cap \Sigma_0$. As a corollary, since both the initial data and the norms associated to the background foliation are assumed small, one can extend this canonical foliation to the whole last slice, in $\mathcal{K}$. We can summarize the results in the following, see also the discussion in Section 6.3.1:

[48] The choice of the canonical foliation is needed to obtain the appropriate estimates, on the last slice, for the angular derivatives of $tr\chi$ and the other Ricci coefficents.

Theorem 8.1 *Given on $\underline{C}_*$ a background geodesic foliation whose Ricci coefficients and null curvature components satisfy appropriate smallness assumptions, it is possible to build on $\underline{C}_*$ a canonical foliation, close to the background one, whose Ricci coefficients, as well as their first three derivatives, have norms which are bounded by $\Delta = \Delta_0 + \Delta_1 + \Delta_2$ and $\mathcal{I}_0$.*

9 The Main Theorem

Let us define the following quantity

$$J_0(\Sigma_0, g, k) = \sup_{\Sigma_0}\left((d_0^2+1)^3|Ric|^2\right) + \int_{\Sigma_0}\sum_{l=0}^{3}(d_0^2+1)^{l+1}|\nabla^l k|^2 + \int_{\Sigma_0}\sum_{l=0}^{1}(d_0^2+1)^{l+3}|\nabla^l B|^2$$

where d_0 is the geodesic distance from a fixed point O on Σ,B is the Bach tensor [49]. Given a compact set K, such that Σ_0/K is diffeomorphic to the complement of the closed unit ball in R^3: B_1, d_K is the geodesic distance from the boundary ∂K, we define

$$J_K(\Sigma_0, g, k) = \sup_{\Sigma_0/K}\left((d_K^2+1)^3|Ric - {}^S Ric|^2\right) + \int_{\Sigma_0/K}\sum_{l=0}^{3}(d_K^2+1)^{l+1}|\nabla^l k|^2 + \int_{\Sigma_0/K}\sum_{l=0}^{1}(d_K^2+1)^{l+3}|\nabla^l B|^2$$

Let an initial data set $\{\Sigma_0, g, k\}$ be given and the corresponding quantity $J_0(\Sigma_0, g, k)$ bounded.

Then there exists a sufficient large compact set $K \subset \Sigma_0$, with Σ_0/K diffeomorphic to R^3/B_1, and a unique development $(\mathcal{M}, g)$ outside the domain of influence of K with the following properties:

i) $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ where $\mathcal{M}^+$ is in the future of Σ_0/K, $\mathcal{M}^-$ the one to the past.

ii) $(\mathcal{M}^+, g)$ can be foliated by a double null foliation $\{C(u)\}$, $\{\underline{C}(\underline{u})\}$ whose outgoing leaves $C(u)$ are complete for all $u \leq u_1$. The boundary of K can be chosen to be the intersection of $C(u_1) \cap \Sigma_0$.

iii) The norms $\mathcal{O}$ and $\mathcal{R}$ are bounded by a constant times J_K.

iv) The null Riemann components have the following asymptotic behaviour:

$$\sup_{\mathcal{K}} r^{7/2}|\alpha| \leq C_0\ ,\ \sup_{\mathcal{K}} r u^{\frac{5}{2}}|\underline{\alpha}| \leq C_0$$
$$\sup_{\mathcal{K}} r^{7/2}|\beta| \leq C_0\ ,\ \sup_{\mathcal{K}} r^2 u^{\frac{3}{2}}|\underline{\beta}| \leq C_0$$
$$\sup_{\mathcal{K}} r^3|\rho| \leq C_0\ ,\ \sup_{\mathcal{K}} r^3 u^{\frac{1}{2}}|(\rho - \overline{\rho}, \sigma)| \leq C_0$$

[49] It is the curl of the traceless part of the three dimensional Ricci tensor. It is relevant that this tensor does not depend on the Schwarzschild part of g.

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