수 학 강 의 록

제 50 권

INTRODUCTION TO THE CLASSIFICATION OF MANIFOLDS

STEVEN HUTT

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
held at the Seoul National University

Steven Hutt
UBK Asset Management
7 Baker Street
London W1M 1AB
England

펴낸날 : 2000년 2월 15일
지은이 : Steven Hutt
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터 〔TEL : 82-2-880-6562〕

Introduction to the Classification of Manifolds

Steven Hutt

Preface

These notes are based on seminars given at Seoul National University in the Spring of 1998.

CONTENTS

1 Surgery Theory

1.1 Introduction

We begin with two fundamental questions:

• Existence: Given a topological space X can we find a closed manifold M and a homotopy equivalence $f : M \to X$?

• Uniqueness: Given a homotopy equivalence $f : M \to N$ of closed manifolds, can we find a homotopy $f \simeq g : M \to N$ such that g is a diffeomorphism?

The motivation for answering these questions comes from the following program:

- We are given a manifold problem, eg. construct a manifold admitting a certain group action.
- We 'solve' the problem in the homotopy category using the techniques of algebraic topology to obtain a topological space (homotopy type).
- We try to 'improve' our topological space to a manifold.
- We count the number of distinct ways if doing this.

Basic necessary homotopy conditions for a space X to be homotopy equivalent to a closed manifold are:

1. The space X must be of the homotopy type of a *finite* CW-complex. Often it is possible to construct a CW-complex with the required properties, but with cells in infinitely many dimensions. The problem of finding a homotopy equivalent finite CW-complex was solved by Wall using algebraic K-theory [34].
2. The space X must have the homological properties of a manifold. I particular, it must satisfy Poincaré duality between its cohomology and homology.
3. The Poincaré space X must have a 'normal bundle', which is compatible with the Poincaré duality structure (cf. Wu's formula relating Steenrod squares in a manifold to characteristic classes of the normal bundle [29]).

Basic necessary conditions for a homotopy equivalence to be homotopic to a diffeomorphism:

1. A diffeomorphism is a *simple* homotopy equivalence: roughly speaking the homotopy equivalence does not twist the cells around too much.

2. A homotopy is a map on a cylinder. A weaker version is to ask for a map on a homotopy cylinder, ie. an h-cobordism. Surgery theory constructs h-cobordisms.

3. The h-cobordism should be trivial, ie. a cylinder. This is the h-cobordism theorem, and uses algebraic K-theory.

1.2 Poincaré Duality

Let M be a closed, orientable n-dimensional manifold (topological, PL or smooth). An orientation/funadamental class for M is a choice of generator $[M] \in H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

Proposition 1.1 *The fundamental class determines a Poincaré duality isomorphism between cohomology and homology*

$$[M]\cap : H^{n-*}(M) \to H_*(M)$$

There are many proofs. For example:

- Simplicial: Choose a triangulation $\{\sigma\}$ for the manifold and take the dual cell decomposition $\{\sigma'\}$. Then we have face relations

$$\sigma_1 \leq \sigma_2 \Longleftrightarrow \sigma_2' \leq \sigma_1'.$$

  So that there is an isomorphism of chain complexes $\{\sigma\}^* \to \{\sigma'\}$. See [26].

- Geometric: Choose a Morse function and, using the flow lines, consider the induced cellular decomposition. Poincaré duality is obtained by reversing the direction of the flow. See [16].

- Sheaf Theoretic: Let $\mathcal{C}_M$ be the canonical Godement resolution on M and D_M its Verdier dual. Then since M is locally Euclidean there is defined a quasi-isomorphism of sheaves $\mathcal{C}_M \to \Sigma^n D_M$. The assembly/global sections of this quasi-isomorphism determine the usual Poincaré duality isomorphism. See [9]

- Homotopy: The Poincaré isomorphism may also be realised on the level of spectra, ie. spaces which are stabilised with respect to suspension. Embed the manifold M^n in the sphere S^{n+k}, k large, and take a tubular neighbourhood $M \subset (U, \partial U)$. Then the composition

$$S^{n+k} \longrightarrow U/\partial U \xrightarrow{\Delta} U/\partial U \wedge U_+$$

  where the first map is 'collapse', induces the slant product

$$D(U/\partial U) \longrightarrow \Sigma^{k+l} U_+ \cong \Sigma^{k+l} M_+.$$

This map is a stable homotopy equivalence and is equivalent to the Poincaré duality isomorphism on noting the Atiyah-Thom equivalence

$$D(U/\partial U) = D(T\nu_M) \cong \Sigma^{k+l} M_+.$$

See [23]

In all cases there is a passage from *local* to *global*:

$$\text{locally Euclidean} \Longrightarrow \text{local PD} \Longrightarrow \text{global PD}.$$

Corollary 1.2 *Any CW-comlex homotopy equivalent to a manifold satisfies Poincaé duality.*

The full statement of Poincaré duality must take into account non-orientability and the role of the fundamental group. Since this is often a source of confusion, a proper definition is in order. (This account is from Ranicki [23].)

Let X be a finite CW-complex and $\tilde{X}$ the universal cover with group of covering transformations $\pi = \pi_1(X)$. Then π acts on the left on $\tilde{X}$ and hence detemines an action of the group ring $\mathbb{Z}[\pi]$ on $H_*(\tilde{X})$

$$\mathbb{Z}[\pi] \times H_*(\tilde{X}) \longrightarrow H_*(\tilde{X}) \ \ (g, x) \mapsto g_* x.$$

Thus the homology groups $H_*(\tilde{X})$ are left $\mathbb{Z}[\pi]$-modules. Similarly for the cohomology groups $H^*(\tilde{X})$

$$\mathbb{Z}[\pi] \times H^*(\tilde{X}) \longrightarrow H^*(\tilde{X}) \ \ (g, x) \mapsto (g^{-1})^* x.$$

In the case X is a manifold:

Goal: Construct an isomorphism of $\mathbb{Z}[\pi]$-modules

$$[X]\cap : H^{n-*}(\tilde{X}) \to H_*(\tilde{X})$$

However, suppose π is infinite, so that $\tilde{X}$ is a non-compact (infinite) CW-complex. Then:

- The groups $H_*(\tilde{X})$ are built from *finite* linear combinations of cells.
- The groups $H^{n-*}(\tilde{X})$ are built from *arbitrary* homomorphisms $\phi : C(\tilde{X}) \to \mathbb{Z}$. In particular, they may have *infinite* support.

Thus we cannot hope to construct an isomorphism from cohomology to homology. There are at least two solutions:

- Use the compactly supported cohomology groups $H^*_{\text{cpt}}(\tilde{X})$. This is appropriate for a sheaf theoretic approach, but is not directly amenable to algebra.

- Use the $\mathbb{Z}[\pi]$-dual of $C(\tilde{X})$ instead of the $\mathbb{Z}$-dual. The homology groups will be finitely generated as $\mathbb{Z}[\pi]$-modules. Unfortunately, they will become *right* $\mathbb{Z}[\pi]$-modules.

We get round this problem using an involution:

Definition 1.3 *An involution on a ring R is a function*

$$A \to A \quad a \mapsto \bar{a}$$

satisfying

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{(ab)} = \bar{b} \cdot \bar{a}, \quad \bar{1} = 1$$

for all $a, b \in R$.

Example 1.4 Let $\mathbb{Z}[\pi]$ be the group ring of the group π with elements

$$\Sigma_{g \in \pi} n_g g$$

where $n_g \in \mathbb{Z}$ and $\{g \in \pi \,|\, n_g \neq 0\}$ is finite.

An orientation character on a group π is a group homomorphism

$$\omega : \pi \to \mathbb{Z}_2 = \{\pm 1\}.$$

The ω-twisted involution on the group ring $\mathbb{Z}[\pi]$ is given by

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \quad a = \Sigma_{g \in \pi} n_g g \mapsto \bar{a} = \Sigma_{g \in \pi} n_g \omega(g) g^{-1}.$$

The special case $\omega = 1$ is the oriented involution on $\mathbb{Z}[\pi]$,

Definition 1.5 *Let R be a ring with involution. The dual of a left R-module M is defined to be the left R-module*

$$M^* = Hom_R(M, R)$$

with R acting by

$$R \times M^* \to M^* \quad (a, f) \mapsto (x \mapsto f(x) \cdot \bar{a}).$$

Similarly we may define the dual of an R-module map $f" M \to N$ by

$$f^* : N^* \to M^* \quad g \mapsto (x \mapsto g(f(x))).$$

Similarly for the dual of a chain complex of R-modules.

Definition 1.6 *An oriented cover $(\tilde{X}, \pi, \omega)$ of a CW-complex X is a regular covering of X with group of covering translations π, together with an orientation character $\omega : \pi \to \mathbb{Z}_2$.*

Example 1.7 Let M be a closed manifold, $\pi = \pi_1(M)$ and universal cover $\tilde{M}$. Then there is defined a map

$$\omega_M : \pi \to \mathbb{Z}_2$$

depending on whether parallel transport along a loop preserves or swaps the local orientation. Thus there is defined an oriented cover $(\tilde{M}, \pi, \omega_M)$.

Let X be a space with orientation cover $(\tilde{X}, \pi, \omega)$:

- Homology groups X are $H_*(C(\tilde{X}))$.
- The (π, ω)-cohomology $\mathbb{Z}[\pi]$-modules of $\tilde{X}$ are defined to be the homology groups of the dual complex $C(\tilde{X})^*$, where $\mathbb{Z}[\pi]$ has the ω-*twisted* involution.

Example 1.8 For a finite group π, the $(\pi, +1)$-cohomology modules may be identified with the usual cohomology groups via the map

$$\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\tilde{X}), \mathbb{Z}[\pi]) \to \mathrm{Hom}_{\mathbb{Z}}(C(\tilde{X}), \mathbb{Z}) \quad f \mapsto (\sigma \mapsto f(\sigma)_1),$$

where $f(\sigma)_1$ is the coefficient of 1 in $f(\sigma) \in \mathbb{Z}[\pi]$.

Similarly, for infinite π, there is induced an isomorphism

$$H^*_{(\pi,\omega)}(\tilde{X}) \cong H^*_{\mathrm{cpt}}(\tilde{X}).$$

Example 1.9 Let $X = \mathbb{RP}^2$ be the projective plane, and for $\epsilon = \pm 1$ define the oriented cover

$$(\tilde{X}, \pi, \omega) = (S^2, \mathbb{Z}_2, \epsilon).$$

Let

$$A = \mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z}[T]/(T^2 - 1)$$

with the ω-twisted involution $\bar{T} = \epsilon T$. Then the cellular A-module chain complex of S^2 is

$$C(S^2) : \ldots \to 0 \to A \overset{1+T}{\to} A \overset{1-T}{\to} A,$$

and the dual A-module chain complex is

$$C(S^2)^* : A \overset{1-\bar{T}}{\to} A \overset{1+\bar{T}}{\to} A \to \ldots,$$

so that $H^0_{(\mathbb{Z}_2,\epsilon)}(S^2) = \mathbb{Z}^\epsilon$ is the $\mathbb{Z}[\mathbb{Z}_2]$-module defined by $\mathbb{Z}$ with the generator $T \in \mathbb{Z}_2$ acting by ϵ.

Consider a CW complex with oriented cover $(\tilde{X}, \pi, \omega)$. Then by acyclic models [29] there is a π-equivariant diagonal chain approximation

$$\tilde{\Delta} : C(\tilde{X}) \to C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})$$

Apply the functor $\mathbb{Z}^\omega \otimes_{\mathbb{Z}[\pi]} -$ to obtain a $\mathbb{Z}$-module map

$$\begin{aligned} \Delta = 1 \otimes \tilde{\Delta} : \mathbb{Z}^\omega \otimes_{\mathbb{Z}[\pi]} C(\tilde{X}) = C(X; \mathbb{Z}^\omega) &\to \mathbb{Z}^\omega \otimes_{\mathbb{Z}[\pi]} (C(\tilde{X}) \otimes_{\mathbb{Z}} C(\tilde{X})) \\ &= C(\tilde{X})^t \otimes_{\mathbb{Z}[\pi]} C(\tilde{X})) \end{aligned}$$

where $C(\tilde{X})^t$ denotes the right $\mathbb{Z}[\pi]$-module cellular chain complex with action

$$C(\tilde{X})^t \times \mathbb{Z}[\pi] \to C(\tilde{X})^t \quad (x, a) \mapsto \bar{a}x.$$

Combining this with the slant map, for each class $x \in H_m(X; \mathbb{Z}^\omega)$ we obtain a $\mathbb{Z}[\pi]$-module cap product map

$$x \cap - : H^n_{(\pi,\omega)}(\tilde{X}) \to H_{n-m}(\tilde{X}).$$

From now on we shall simply write $H^n_{(\pi,\omega)}(\tilde{X}) = H^n(\tilde{X})$.

Finally we have

Definition 1.10 *An m-dimensional Poincaré complex X is a finite CW complex with an orientation character $\omega(X) : \pi_1(X) \to \mathbb{Z}_2$ and a fundamental homology class $[X] \in H_m(X; \mathbb{Z}^{\omega(X)})$. such that the cap products*

$$[X] \cap - : H^*(\tilde{X}) \to H_{m-*}(\tilde{X})$$

are isomorphisms, where $\tilde{X}$ is the universal cover of X.

Example 1.11 An m-dimensional manifold M, with orientation cover $(\tilde{M}, \pi_1(M), \omega_M)$ given as above, is a Poincaré complex.

Example 1.12 (Browder) Every finite H-space is a Poincaré complex.

Example 1.13 (Bieri) Every 2-dimensional Poincaré complex is a surface.

1.3 Normal Data for Manifolds

Recall that isomorphism classes of k-plane bundles ξ over a space X are classified by homotopy classes of maps

$$X \to BO(k) = G_k(\mathbb{R}^\infty)$$

for some classifying space $BO(k)$.
Whitney sum with trivial bundles determines maps

$$\cdots \to BO(k) \to BO(k+1) \to \cdots$$

and taking the direct limit we obtain the classifying space BO for stable vector bundles. See [20] or [15] for example.

Example 1.14 A k-plane bundle $\xi : S^1 \to BO(k)$ over S^1 is classified by an element $\xi \in O(k)$ such that

$$E(\xi) = \mathbb{R}^k \times I / \{(x, 0) = (\xi(x), 1)\}.$$

We say ξ is orientable iff $\xi \in SO(k) \subset O(k)$.

Definition 1.15 *Given a k-plane bundle $\xi : X \to BO(k)$ over a connected space, we define the orientation character*

$$\omega_1(\xi) \in H^1(X'\mathbb{Z}_2) = Hom(\pi_1(X), \mathbb{Z}_2)$$

to be the group homomorphism

$$\omega = \omega_1(\xi) : \pi_1(X) \to \mathbb{Z}_2 = \{\pm 1\}$$

taking value $\omega(g) = +1$ (resp. -1) on a loop $g : S^1 \to X$ such that $g^*\xi$ is orientable (resp. non-orientable). We say ξ is orientable if $\omega = +1$.

Definition 1.16 *The Thom space of a k-plane bundle*

$$\xi : X \to BO(k)$$

is the pointed space

$$T(\xi) = D(\xi)/S(\xi)$$

where, having chosen a metric on $E(\xi)$, we take $D(\xi)$ to be the unit disk bundle, and $S(\xi)$ to be the unit sphere bundle.

The Thom space has the structure of a CW-complex with one 0-cell at the base point, and one $(n+k)$-cell for each n-cell of X. It should be regarded as a twisted suspension of X. Stabilisation of the bundle corresponds to suspension of the Thom space.

Theorem 1.17 *An oriented k-plane bundle ξ admits a Thom class $U_\xi \in \tilde{H}^k(T\xi)$ inducing isomorphisms*

$$U_\xi \cap : \tilde{H}_*(T\xi) \to H_{*-k}(X)$$

and

$$U_\xi \cup : H^*(X) \to \tilde{H}^{*+k}(T\xi).$$

A similar twisted version holds for a non-orientable bundle. See [29] for a proof.

Let M^n be an oriented manifold and

$$e : M^n \hookrightarrow \mathbb{R}^{n+k}$$

an embedding. Then by the tubular neighbourhood theorem, the embedding e extends to an embedding

$$E(\nu_e) \hookrightarrow \mathbb{R}^{n+k}$$

where $E(\nu_i)$ is the total space of a k-plane bundle $\nu_M \to BO(k)$ over M. Then

$$\tau_M \oplus \nu_e = e^*\tau_{\mathbb{R}^{n+k}} = M \times \mathbb{R}^k$$

so that ν_M is a stable inverse to the tangent bundle τ_M.

The stable class of ν_M

$$\nu_M : M \to BO = \lim_k BO(k)$$

is thus well-defined.

Notice that we get more than just a stable bundle. There is a collapse map

$$S^{n+k} = \mathbb{R}^{n+k}/\infty \to T\nu_M = U/\partial U,$$

defining a stable homotopy class

$$\alpha_M \in \pi_k^S(T\nu_M)$$

called the normal invariant of M. If $h : \pi_*^S(M) \to H_*(M)$ is the Hurewitz map then

$$h(\alpha_X) \cap U_{\nu_M} = [M] \in H_n(M; \mathbb{Z}).$$

Thus α_M is a stable version of $[M]$. In fact, the stable cap product

$$\alpha_M \cap : DT\nu \to \Sigma^{k+l} M_+$$

is a stable version of the Poincaré duality isomorphism

$$[M] \cap : H^*(M) \to H_{n-*}(M).$$

Notice also that we may recover M up to cobordism from its normal invariant by transversality. Simply make $\alpha_M : S^{n+k} \to T\nu_M$ transverse to the zero-section $M \subset T\nu_M$.

1.4 Normal Data for Poincaré Spaces

We want to recover as much as the normal data of a manifold in the weaker situation of a Poincaré complex. Since a Poincaré complex is essentially a *homotopy* object, we can only hope to recover the manifold normal data up to homotopy. The following theorem is due to Spivak [31]:

Theorem 1.18 *Let X be an n-dimensional Poincaré complex. Then there is a spherical fibration over X, written ξ_X, with fibre a homotopy $(k-1)$-sphere, and a 'homotopy' normal invariant $\alpha_X : S^{n+k} \to T\xi_X$ such that*

$$h(\alpha_X) \cap U_{\xi_X} = [X] \in H_n(X).$$

Such data is essentially unique up to homotopy.

In the case of a manifold M, the manifold could be recovered up to cobordism from the normal invariant $\alpha_M : S^{n+k} \to T\nu_M$ by an application of transversality. However, in the case of the Thom space of a spherical fibration we do not appear to have a transversality result. (Actually, such a result does exist but it is rather involved. See [8] for example.)

The theorem is proved as in the manifold case, by embedding X in $\mathbb{R}^{n+k}$. However, we do not have a tubular neighbourhood theorem for Poincaré complexes so we cannot conclude that a regular neighbourhood admits a bundle structure. Instead, a spectral sequence argument involving the Poincaré duality isomorphism must be used to identify the homotopy fibre as a $(k-1)$-sphere. The spherical fibration ξ_X (stabilized) is called the Spivak normal fibration.
There is an interesting intermediate version where X is assumed to satisfy a local form of Poincaré duality, ie. it has the local homology of Euclidean space. In this case, it can be shown that the projection $p : \partial U \to X$ may be chosen so that the fibres have the Cech cohomology of a $(k-1)$-sphere, but in general the projection will not be locally trivial (cf. manifold approximate fibrations: non-trivial germs!).
Just as normal bundles are classified by homotopy classes of maps into BO(k), so there is a spherical fibration classification theorem due to Stasheff:

Theorem 1.19 *Fibre homotopy classes of $(k-1)$-spherical fibrations over a finite CW-complex are in one-to-one correspondence with the homotopy classes of maps $X \to BG(k)$ to the classifying space of the monoid $G(k)$ of homotopy equivalences $S^{k-1} \to S^{k-1}$.*

In particular, the Spivak normal fibration of a Poincaré complex X may be regarded as a homotopy class of maps $\xi_X : X \to BG(k)$. Similarly, there is a stable version $\xi_X : X \to BG$ obtained by taking Whitney sums with trivial fibrations.

Because the homotopy normal invariant α_X is a stable homotopy version of the fundamental class, the composition

$$S^{n+k} \overset{\alpha_X}{\to} T\xi_X \overset{\Delta}{\to} T\xi_X \wedge X_+$$

determines a stable homotopy equivalence

$$DT\xi_X \to \Sigma^{k+l} X_+$$

as in the case of a manifold. Thus the Thom space $T\xi_X$ is the S-dual of X_+. This map is a stable homotopy version of the Poincaré duality isomorphism for X.
It is easily shown (from the definition of a classifying space of a monoid) that the homotopy groups

$$\pi_i(BG) \cong \lim_k \pi_{i+k-2}(S^{k-1}) = \pi_{i-1}^S$$

where π_{i-1}^S is the $(i-1)$-stable stem, ie. the stable homotopy of a sphere.
Every orthogonal map $\mathbb{R}^k \to \mathbb{R}^k$ determines a homotopy equivalence $S^{k-1} \to S^{k-1}$ by restriction. Thus there are defined forgetful maps of classifying spaces

$$J_k : BO(k) \to BG(k) \quad \text{and} \quad J : BO \to BG.$$

Define the space G/O to fit into the fibration

$$G/O \to BO \xrightarrow{J} BG.$$

Lemma 1.20 *Homtopy classes of maps $X \to G/O$ are in one-to-one correspondence with equivalence classes of pairs*

$$(\textit{stable VB } \eta \textit{ over } X, \textit{ a fibre homotopy } J\eta \simeq *).$$

A stable spherical fibration $\xi : X \to BG$ admits a bundle reduction iff the map ξ lifts up to homotopy to a map $\xi' : X \to BO$.
The collection of different reductions are classified by maps $X \to G/O$.

Example 1.21 (Madsen-Milgram) We have

$$\pi_3(BG(3)) = \pi_4(S^2) = \mathbb{Z}_2$$

. Choose $\omega = 1 \in \pi_3(BG(3))$ and consider the spherical fibration

$$S^2 \to S(\omega) \to S^3.$$

Since S^2 and S^3 are both Poincaré complexes, by a spectral sequence argument $S(\omega)$ is a 5-dimensional Poincaré complex. Let

$$\theta(\omega) : S^4 \to S^2$$

be the adjoint of ω. Then it can be shown that $S(\omega)$ has the cell structure

$$S(\omega) = (S^3 \wedge S^2) \cup_{[\iota_3,\iota_2]+\theta(\omega)} D^5$$

where $[\iota_3, \iota_2] : S^4 \to S^3 \vee S^2$ is the Whitehead product on $\iota_j : S_j \hookrightarrow S^3 \vee S^2$, $j = 3, 2$. Recall that the standard product $S^3 \times S^2$ can be written

$$S^3 \times S^2 = (S^3 \vee S^2) \cup_{[\iota_3,\iota_2]} D^5$$

and that the suspension

$$\Sigma(S^3 \times S^2) \cong S^4 \vee S^3 \vee S^6$$

since $\Sigma[\iota_3, \iota_2]$ is null homotopic. It follows that

$$T(\omega) = S^3 \cup_{\Sigma\theta(\omega)} D^6.$$

Furthermore, the Spivak normal fibration of $S(\omega)$ is classified by

$$\nu_{S(\omega)} : S(\omega) \to S^3 \xrightarrow{-\omega} BG(k)$$

so that

$$T(\nu_{S(\omega)}) = (S^k \cup_{\Sigma\theta(-\omega)} D^{3+k}) \vee S^{2+k} \vee S^{5+k}.$$

Here $\Sigma\theta(-\omega) \in \pi_{k+2}(S^k) = \mathbb{Z}_2$ is non-zero.

In fact, $T(\nu_{S(\omega)})$ is not of the homotopy type of the Thom space of a vector bundle. This is proved as follows:

Consider the Steenrod square operations Sq^i applied to the space

$$S(\omega) = (S^3 \vee S^2) \cup_{[\iota_3, \iota_2]+\theta(\omega)} D^5.$$

This space has cells e^0, e^2, e^3, e^5. The possibilities for non-trivial Steenrod squares are:

- $Sq^2 : e^3 \mapsto e^5$ – but Steenrod squares commute with suspensions and e^5 is attatched to e^2 by a Whitehead product which suspends to zero. Hence $Sq^2 = 0$
- $Sq^3 : e^2 \mapsto e^5$ – here e^5 is attatched to e^2 via a Whitehead product, so this makes no contribution, but it is also attatched via $\theta(\omega)$ which may pick up a Steenrod square. However, the Adem relation $Sq^3 = Sq^2 \cdot Sq^1$ implies that Sq^3 is zero here, since there is no 3-cell attatched to e^2.

Thus on $S(\omega)$ the total Steenrod square $Sq = 1$. However, the Steenrod relation

$$Sq^2 \cdot Sq^2 + Sq^3 \cdot Sq^1 = 0$$

determines a secondary operation Φ on $S(\omega)$ which does detect $\theta(\omega)$.

Since $Sq = 1$ on $S(\omega)$, it follows from a theorem of Wu that the total Stiefel-Whitney class of the Spivak normal fibration of $S(\omega)$ is 1. In particular $\omega_2(\nu_{S(\omega)}) = 0$. Thus if $\nu_{S(\omega)}$ admits a bundle reduction ξ, then ξ may be clasified by a map

$$\xi : S(\omega) \to BSpin(k)$$

by definition of $BSpin$. Thus there is an induced map of Thom spaces

$$T(\xi) \to T(\gamma^k_{BSpin})$$

where γ^k_{BSpin} is the universal bundle over $BSpin(k)$. Now Φ is non zero on $T(\xi)$ but it can be shown to be zero on $T(\gamma^k_{BSpin})$. Hence, by naturality of secondary operations, we have a contradiction, and the Spival normal fibration $\nu_{S(\omega)}$ cannot admit a bundle reduction.

We conclude that the Poincaré complex $S(\omega)$ is not of the homotopy type of a manifold.

1.5 Normal Invariants

Suppose that X is an m-dimensional Poincaré complex with Spivak normal fibration $\xi : X \to BG(k)$, k large, and homotopy normal invariant $\alpha_X : S^{n+k} \to T\xi$. Our goal is to manufacture a homotopy equivalence

$$f : M^m \to X^m$$

for some closed manifold M. As a humble beginning, let us try to construct a map (not necessarily a homotopy equivalence) which preserves fundamental classes, ie. induces an isomorphism

$$f_* : H_m(M) \to H_m(X).$$

Think of this as the first step to building a homotopy equivalence. The key tool for manufacturing manifolds is of course transversality. We would like to apply transversality to the homotopy normal invariant $\alpha_X : S^{n+k} \to T\xi_X$, but of course ξ_X is only a fibration, so this does not work. We need to assume the existence of bundle data, (which is certainly necessary for a homotopy equivalence to exist).

Definition 1.22 *A normal invariant (η, ρ) on an m-dimensional Poincaré complex X consists of a vector bundle $\eta : X \to BO(k)$, k large, together with a map $\rho : S^{n+k} \to T\eta$ such that*

$$h(\rho) \cap U_\eta = [X] \in H_m(X).$$

Two normal invariants (η_1, ρ_1) and (η_2, ρ_2) are said to be equivalent if there exists a bundle isomorphism $\eta_1 \cong \eta_2$ taking ρ_1 to ρ_2. We write $\mathcal{T}(X)$ for the set of equivalence classes of normal invariants of X. Of course, $\mathcal{T}(X)$ may be empty.

Note that by the homotopy uniqueness of the Spivak normal fibration, the bundle η has the fibre homotopy type of the Spivake normal fibration.
Apply transversality to the map $\rho : S^{n+k} \to T\eta$ to obtain a degree 1 map (ie. preserves fundamental classes)

$$f : M^m \to X.$$

Notice that transversality also gives us a bundle map $b : \nu_M \to \eta$ over f. The pair

$$(f, b) : (M, \nu_M) \to (X, \xi)$$

is called a normal map. See [3] for details.

More accurately, transversality gives us a bordism class of normal maps:

Definition 1.23 *Two normal maps*

$$(f_1, b_1) : M_1 \to X, \quad (f_2, b_2) : M_2 \to X$$

are said to be normally bordant if there exists:

1. *a cobordism $(W; M_1, M_2)$,*
2. *a map $F : W \to X$ extending f_1 and f_2,*
3. *a bundle map $B : \nu_W \to \xi$ extending b_1 and b_2.*

We write $N(X)$ for the set of equivalence classes of normal maps. Again $N(X)$ may be empty.

Proposition 1.24 *Let X be an m-dimensional Poincaré complex admitting a normal invariant. Then the following are in one-to-one correspondence:*

1. *equivalence classes of normal invariants,*

2. *normal bordism classes of normal maps, for varying reductions of ξ_X,*

3. *homotopy classes of maps $X \to G/O$.*

Boardman and Vogt [2] have shown that the space G/O admits a delooping $B(G/O)$ so that there is a fibration sequence

$$G/O \to BO \to BG \to B(G/O).$$

In particular, a Poincaré complex X with Spivak normal fibration $\xi : X \to BG(k)$ admits a normal invariant iff the map

$$[X, BG] \to [X, B(G/O)]$$

takes ξ to zero. In particular, the element in $[X, B(G/O)]$ is an obstruction to the existence of a manifold structure on X.

Clearly then G/O plays a critical role in surgery theory, along with its topological version G/TOP and G/PL for topological and PL manifolds respectively. They are the surgery classifying spaces. The study of their homotopy type is equivalent to a linearization of the classification problem. We give a brief review of their homotopy properties. Further details can be found in [15]

- The study of G/O as the fibre of the J homomorpism was properly initiated by Adams, and later completely solved by Sullivan. After p-adically completing, there is for each prime p a homotopy equivalence

  $$G/O[p] \cong BSO[p] \times \mathrm{Cok}(J[p])$$

  where $\mathrm{Cok}(J[p])$ is the mapping cone on the p-adic J-map. The first factor is well understood, but the second factor involves the homotopy groups of spheres, and so is currently intractable.

- The map $G/PL \to G/TOP$ is an isomorphism on all homotopy groups, except dimension 4, where both groups are $\mathbb{Z}$ but the map is multiplication by 2, ie. the fibre is $K(\mathbb{Z}_2, 3)$. It follows there is an obstruction (the Kirby-Siebenmann invariant) in $H^4(M; \mathbb{Z}_2)$ to triangulating a topological manifold M^n, $n \geq 5$.

- Before the 1970s there were no techniques for extending the classical surgery theory on smooth and PL manifolds to topological manifolds. This was due mainly to a lack of understanding about homeomorphisms on

Euclidean space. Following Kirby's 'torus trick', which reduced the problem to classifying fake PL-tori, Kirby and Siebenmann developed surgery techniques in the topological category in the early 70s [14]. It follows immediately from the surgery exact sequence and the high dimensional Poincaré conjecture that

$$\pi_i(G/TOP) = 0, \mathbb{Z}_2, 0, \mathbb{Z} \quad \text{for } i \equiv 1, 2, 3, 0 \text{ mod } 4 \text{ respectively.}$$

This may be expressed topologically as a homotopy equivalence

$$\Omega^4 G/TOP \cong \mathbb{Z} \times G/TOP$$

Notice the extra $\mathbb{Z}$ appearing. This turns out to count the number of local types for exotic homology manifolds.

1.6 Surgery on Manifolds

In the classification of manifolds we wish to address two questions:

- When is a space homotopy equivalent to a manifold?
- When is a homotopy equivalence of manifolds homotopic to a diffeomorphism?

In the previous sections we have studied the homotopy properties of manifolds, and arrived at the idea of a normal map as a first step in the process. The construction of normal maps was closely related to the homotopy theory of the associated surgery classifying spaces.

We now assume we are given a normal map and move on to the problem of determining whether its normal bordism class contains a homotopy equivalence. The key tool for constructing cobordisms and bordisms in a managed way is surgery theory [35]. Surgery theory produces normal bordisms, and, in particular, can be used to reduce the connectivity of a given normal map within a normal bordism class. We shall see that

Lemma 1.25 *For $2n+1 \leq m$, every normal map $(f, b) : M^m \to X^m$ is normal bordant to an n-connected normal map $(f', b') : M' \to X$.*

Thus f is 'almost' a homotopy equivalence, except for possible non-zero homotopy groups in the middle dimensions $\pi_{n+1}(f)$. There are obstructions to killing these homotopy groups which lie in the L-groups of $\mathbb{Z}[\pi_1(X)]$. Their triviality is sufficient to be able to do surgery to a homotopy equivalence. We leave surgery on maps till later, and begin with surgery on manifolds. Further details can be found in [35] [3].

Consider the space $D^{m+1} = D^{n+1} \times D^{m-n}$ with boundary

$$\partial(D^{n+1} \times D^{m-n}) = (S^n \times D^{m-n}) \cup_{S^n \times S^{m-n-1}} (D^{n+1} \times S^{m-n-1}).$$

In other words the manifolds

$$S^n \times D^{m-n} \quad \text{and} \quad D^{n+1} \times S^{m-n-1}$$

share the same boundary $S^n \times S^{m-n-1}$ and are cobordant via D^{m+1}.

Let M be an m-dimensional manifold. A framed n-embedding in M consists of the following data:

- An n-embedding $e : S^n \to M^m$.
- A framing of the normal bundle ν_e of e, ie. an extension of e to an embedding
$$\bar{e} : S^n \times D^{m-n} \to M^m.$$

We say e is the core of $\bar{e}$. Of course, it is a non-trivial condition to require that ν_e is framed. In addition, it is important to remember that there may be many distinct framings. Thus the input data for a surgery consists of *both* the embedding e and the particular choice of framing $\bar{e}$. (We shall see that different framings have different effects on the surgery.)

Definition 1.26 *Let $(e, \bar{e})$ be a framed n-embedding. The effect of n-surgery on $\bar{e}$ is the m-dimensional manifold*

$$M' = closure(M - \bar{e}(S^n \times D^{m-n})) \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}.$$

The trace of the surgery is the cobordism $(W; M; M')$ defined by

$$W^{m+1} = M \times I \cup_{S^n \times D^{m-n} \times \{1\}} D^{n+1} \times D^{m-n}.$$

To construct the effect of surgery M' from M we first cut out the interior of $S^n \times D^{m-n}$ from M and then sew back in $D^{n+1} \times S^{m-n-1}$ along the boundary $S^n \times S^{m-n-1}$. Hence the terminology 'surgery'.

Notice that the trace of the surgery is simply obtained by adding a handle $D^{n+1} \times D^{m-n}$ to $M \times I$ via the attatching map $\bar{e}$. Thus it is a particularly simple cobordism: an elementary cobordism.

There is a kind of symmetry in the surgery operation. If M' is the effect of an n-surgery $\bar{e} : S^n \times D^{m-n} \to M$ on M, then there is an obvious dual embedding $\bar{e}^* : D^{n+1} \times S^{m-n-1} \to M'$ such that M is the effect of the $(m-n-1)$-surgery $\bar{e}^*$ on M', with the same trace but turned upside-down. A symmetric description of W looks something like this:

$$(M \times I) \cup_{S^n \times D^{m-n}} D^{m+1} \cup_{D^{n+1} \times S^{m-n-1}} (M' \times I).$$

Remark Let $(W; M, M')$ be an arbitrary cobordism, and

$$(f; \{0\}, \{1\}) : (W; M, M') \to (I; \{0\}, \{1\})$$

a Morse function on W [16]. Let $M_t = f^{-1}(t)$ and $W_t = f^{-1}([0,t])$. Then for t a regular value, $(W_t; M, M_t)$ is a cobordism. Further, at a critical value c of

f, $W_{c+\delta}$ is obtained from $W_{t-\delta}$ by adding a handle. Thus every cobordism is built from a sequence of elementary cobordisms or surgeries.

Here are some simple examples:

Example 1.27 Consider the standard framed embedding above

$$e : S^n \times D^{m-n} \subset S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1} = \partial(D^{n+1} \times D^{m-n}) = S^m.$$

The effect of surgery on this embedding is the m-dimensional manifold

$$D^{n+1} \times S^{m-n-1} \cup D^{n+1} \times S^{m-n-1} = S^{n+1} \times S^{m-n-1}.$$

Example 1.28 Consider the standard embedding

$$e : S^0 \times D^1 \to S^1$$

and modify it by the twist

$$\omega : D^1 \to D^1 \quad t \mapsto -t,$$

to give the twisted embedding

$$e_\omega : S^0 \times D^1 \xrightarrow{1 \sqcup \omega} S^0 \times D^1 \xrightarrow{e} S^1$$

Notice this has the same core as e, but the framing is non-standard. The result of surgery on e_ω is the same as e, namely S^1, but the trace of the surgery $(W; S^1, S^1)$ is a non-orientable cobordism. In particular, it is not a cylinder. In fact, it is a punctured Moebius band.

Example 1.29 The connected sum of two manifolds $M_1 \sharp M_2$ is obtained the disjoint union $M_1 \sqcup M_2$ by surgery on the embedding

$$S^0 \times D^m = D^m \sqcup D^m \hookrightarrow M_1 \sqcup M_2.$$

Since our ultimate goal is to use surgery to construct homotopy equivalences, several questions need to be answered:

- What is the homotopy theoretic effect of a surgery?
- When is surgery possible and what are the possible obstructions?
- Surgery constructs cobordant manifolds. How can we use this to construct homotopy equivalences or even diffeomorphisms?

1.7 The Homotopy Effect of Surgery

The operation of surgery on a framed n-embedding

$$\bar{e} : S^n \times D^{m-n} \to M^m$$

can be thought of homotopically as a pair of operations. First we attatch an $(n+1)$-cell to M via the core $e : S^n \to M$ forming the cofibration sequence

$$S^n \xrightarrow{e} M \to M \cup e^{n+1}.$$

Notice that this kills the homotopy class $e \in \pi_n(M)$ - this is the main point of the surgery operation: to modify the homotopy type. The resulting space $M \cup e^{n+1}$ has the homotopy type of the trace of the surgery

$$W^{m+1} = M \times I \cup_{S^n \times D^{m-n} \times \{1\}} D^{n+1} \times D^{m-n} \cong M \cup e^{n+1},$$

(just collapse the D^{m-n} coordinate to 0.) To recover the homotopy type of the effect of the surgery (and restore Poincaré duality) we must remove a dual $(m-n-1)$-cell, ie. form the cofibration sequence

$$M' \to M \cup e^{n+1} \xrightarrow{f} S^{m-n-1}.$$

Thus homotopically, surgery consists of adding a cell and removing a dual cell. In particular, homotopically

$$W \cong M \cup_x e^{n+1} \cong M' \cup_{x^*} e^{m-n}.$$

Remark Since we have a purely homotopy theoretic description of surgery on a manifold, we have a recipe for doing surgery on Poincaré complexes: add a cell and remove its Poincaré dual. Everything works fine except for removing the dual cell: we can construct a map $f : X \cup e^{n+1} \to S^{m-n-1}$ but then we need to build a space X', the effect of Poincaré surgery, removing the $(m-n-1)$-cell, ie. construct a cofibration

$$X' \to X \cup e^{n+1} \xrightarrow{f} S^{m-n-1}.$$

In other words, we must extend the map f to the left as a cofibration. In the manifold case this is possible because of the strong local topology. For a general Poincaré complex we must proceed by other means. In general there are higher Massey product obstructions to removing cells [10]. But for the case of a Poincaré complex, these turn out to be zero, and Poincaré surgery is possible [8].

Let $\pi_n(X)$ denote the nth-homotopy group of a space X. The fundamental group action

$$\pi_1(X) \times \pi_n(X) \to \pi_n(X)$$

is obtained by changing the reference path, making $\pi_n(X)$ into a $\mathbb{Z}[\pi_1(X)]$-module.

- For $x \in \pi_1(X)$ let $\langle x \rangle \subset \pi_1(X)$ denote the normal subgroup generated by x.
- For $x \in \pi_n(X)$ let $\langle x \rangle \subset \pi_n(X)$ denote the $\mathbb{Z}[\pi_1(X)]$-submodule generated by x.

Given a map $f : S^n \to X$ representing $x \in \pi_n(X)$, the homotopy groups of the mapping cone

$$Y = X \cup_f e^{n+1}$$

are given by

$$\pi_i(Y) = \begin{cases} \pi_i(X), & i < n, \\ \pi_i(X)/\langle x \rangle, & i = n \end{cases}$$

Recall, the trace W^{m+1} of an n-surgery on a framed n-embedding $\bar{e} : S^n \times D^{m-n} \to M$ with core e has homotopy type

$$W \cong M \cup_x e^{n+1} \cong M' \cup_{x^*} e^{m-n}.$$

In calculating the effect of surgery on homotopy groups we may distinguish several cases:

1. Case $2n+2 \leq m$: in this case $n+1 < m-n$ so that

   $$\pi_n(M)/< x > = \pi_n(W) = \pi_n(M').$$

   and x is killed in $\pi_n(M')$.

2. Case $2n+1 = m$: in this case $n+1 = m-n$ so that

   $$\pi_n(M)/< x > = \pi_n(W) = \pi_n(M')/< x^* >.$$

   so that π_n is neither increased nor decreased by an n-surgery.

3. Case $2n = m$: in this case π_{n-1} may change. For example, if M' is the result of a trivial n surgery on M^{2n} then

   $$M' = M\sharp(S^{n+1} \times S^{n-1})$$

   so that

   $$\pi_{n-1}(M') = \pi_{n-1}(M) \oplus \mathbb{Z}.$$

Thus, in the context of a normal map $f : M \to X$, we may hope to construct a homotopy equivalence by successively killing the homotopy classes in the kernel of

$$f_* : \pi_i(M) \to \pi_i(X)$$

by surgeries. We shall see that indeed this is possible up to the middle dimension, at which point some non trivial obstructions are encountered.

In the meantime, here are some examples of the effects of surgery.

Example 1.30 Let $0 \in \pi_n(M)$ be the zero map $0 : S^n \to * \subset M^m$. Then 0 may be killed by a surgery on the embedding

$$S^n \times D^{m-n} \subset S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1} = D^m \subset M^n.$$

The effect of the surgery is the connected sum

$$M' = M\sharp(S^{n+1} \times S^{m-n-1}).$$

Example 1.31 For $m \geq 2$, consider the stable normal bundle map $\nu_{\mathbb{RP}^m} \to BO$ where

$$(\nu_{\mathbb{RP}^m})_* = 1 : \pi_1(\mathbb{RP}^m) = \mathbb{Z}_2 \to \pi_1(BO) = \mathbb{Z}_2.$$

The homotopy class $1 \in \pi_1(\mathbb{RP}^m)$ is represented by the standard embedding

$$e : S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^m.$$

But the normal bundle $\nu_e : S^1 \to BO(m-1)$ has non-zero first Stiefel-Whitney class

$$\omega_1(\nu_e) = 1 \in H^1(S^1; \mathbb{Z}_2).$$

Hence $1 \in \pi_1(\mathbb{RP}^m)$ cannot be killed by a surgery.

It is important to realise that the effect of surgery is determined not just be the core of the surgery, which certainly affects the resulting homotopy type, but also by the embedding itself $\bar{e} : S^n \times D^{m-n} \to M$. In particular, given an embedding $e : S^n \to M^m$ with trivial normal bundle $\nu_e : S^n \to BO(m-n)$, the number of extensions to a map $\bar{e} : S^n \times D^{m-n} \to M$ is in one-to-one correspondence with maps $\omega : S^n \to O(m-n)$. Simply let ω act on a fixed embedding in the obvious way. Isotopic embeddings correspond to homotopic maps.

Example 1.32 Begin with the standard embedding

$$e : S^n \times D^{m-n} \to S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1} = S^m.$$

We know that the effect of surgery on e is the product space $S^{n+1} \times S^{m-n-1}$. Now pick any element

$$\omega \in \pi_n(O(m-n)) = \pi_{n+1}(BO(m-n)).$$

We can think of ω in two ways. Firstly, $\omega \in \pi_n(O(m-n))$ determines a twist on the embedding e:

$$e_\omega : S^n \times D^{m-n} \to S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1} = S^m\,;\ (x,y) \to (x, \omega(x)(y)).$$

In particular, the result of surgery on e_ω may be written

$$S^n \times D^{m-n} \cup_\omega D^{n+1} \times S^{m-n-1}.$$

Secondly, $\omega \in \pi_{n+1}(BO(m-n))$ determines a disk bundle

$$(D^{m-n}, S^{m-n-1}) \to (E(\omega), S(\omega)) \to S^{n+1}.$$

The link between the two is the 'clasping function' description of bundles over spheres. It is easily seen that

$$S(\omega) = S^n \times D^{m-n} \cup_\omega D^{n+1} \times S^{m-n-1}.$$

The trace W^{m+1} of the surgery is the manifold $E(\omega)\backslash D^{m+1}$. Clearly the result of surgery $S(\omega)$ depends on the embedding e_ω, and, in particular, the choice of framing of e.

Example 1.33 Let $\omega_0 \in \pi_1(BO(2))$ classify the trivial bundle over S^1 and $\omega_1 \in \pi_1(BO(2))$ the non-orientable bundle. Then

$$S(\omega_0) = S^1 \times S^1, \quad \text{and} \quad S(\omega_1) = K^2 \text{ the Klein bottle.}$$

Example 1.34 We have $\pi_3(SO(4)) = \pi_4(BSO(4))$ is generated by maps

$$\omega : S^3 \to SO(4) = S^3 \times \mathbb{R}P^3.$$

There is defined an isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_3(SO(4))$$

taking the pair $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$ to the map $\omega(h, j) : S^3 \to SO(4)$ given by

$$\omega(h, j) : S^3 \to SO(4) \quad x \mapsto \{y \mapsto x^h y x^j\},$$

where the latter products are quaternionic multiplication. For k odd, write

$$\omega_k = \omega((1+k)/2, (1-k)/2) \in \pi_4(BSO(4))$$

and let $S(\omega_k)$ be the space as defined above. Then each $S(\omega_k)$ is a smooth 7-dimensional closed manifold homeomorphic to S^7. But for $k^2 \not\equiv 1 (\text{mod } 7)$, $S(\omega_k)$ is not diffeomorphic to S^7. These are the original Milnor exotic spheres.

1.8 Surgery on Normal Maps

Having described surgery on manifolds, we now extend the technique to surgery on normal maps $(f, b) : (M, \nu_M) \to (X, \eta)$, the goal being to construct a normal bordism to a homotopy equivalence.

Let M be an m-dimensional manifold and $f : M \to X$ a map. An n-embedding ϕ in f consists of the following data:

- An n-embedding $e : S^n \to M$.
- An extension $d : D^{n+1} \to X$ of $f \cdot e$, ie. a null homotopy $f \cdot e \simeq *$.

A framed n-embedding Φ in f consists of the following data:

- A framed n-embedding $\bar{e} : S^n \times D^{m-n} \to M$.
- An extension $\bar{d} : D^{n+1} \times D^{m-n} \to X$ of $f \cdot \bar{e}$.

Definition 1.35 *Let Φ be a framed n-embedding in the map $f : M^m \to X$. The effect of n-surgery on Φ is the effect of n-surgery on $\bar{e}$*

$$M' = cl(M \backslash \bar{e}(S^n \times D^{m-n})) \cup D^{n+1} \times S^{m-n-1}$$

together with the extension of f to the trace cobordism

$$(g; f, f') : (W; M, M') \to X \times (I, \{0\}, \{1\})$$

given by

$$g = (f \times I) \cup \bar{d} : W = (M \times I) \cup_{\bar{e}} D^{n+1} \times D^{m-n} \to X \times I.$$

Again, not every n-embedding $e : S^n \to M^m$ extends to a framed n-embedding $\bar{e} : S^n \to M^m$. The normal bundle $\nu_e : S^n \to BO(m-n)$ determines the framing obstruction

$$\nu_e \in \pi_n(BO(m-n))$$

such that $\nu_e = 0$ if and only if e extends to a framed embedding $\bar{e}$.

Suppose that $\phi = (e, d)$ is an n-embedding in $f : M \to X$. Then

$$\nu_e = 0 \in \pi_n(BO(m-n)) \iff \phi \text{ extends to a framed } n\text{-embedding } \Phi.$$

Again we may calculate the effect on homotopy groups of an n-surgery Φ on f:

$$\pi_i(g) = \begin{cases} \pi_i(f) & \text{if } i \leq n \\ \pi_{n+1}(f)/\langle\phi\rangle & \text{if } i = n+1. \end{cases}$$

Similarly, if $\phi' \in \pi_{m-n}(f')$ is the dual $(m-n-1)$-embedding in $f' : M' \to X$ then

$$\pi_j(g) = \begin{cases} \pi_j(f') & \text{if } j \leq m-n-1 \\ \pi_{m-n}(f')/\langle\phi'\rangle & \text{if } j = m-n. \end{cases}$$

Thus for $2n+2 \leq m$

$$\pi_i(f') = \pi_i(g) = \begin{cases} \pi_f(f) & \text{if } i \leq n \\ \pi_{n+1}(f)/\langle\phi\rangle & \text{if } i = n+1. \end{cases}$$

The element $x \in \pi_{n+1}(f)$ represented by ϕ is therefore killed by the n-surgery.

The next step is to incorporate the normal data. Let $(f, b) : M^m \to X$ be a normal map. Let $x \in \pi_{n+1}(f)$ be a homotopy element which we wish to kill by normal surgery. We must construct the following data:

- An n-embedding ϕ in f consisting of an embedding $e : S^n \to M$ and an extension $d : D^n \to X$ of $f \cdot e$, representing the class $x \in \pi_{n+1}(f)$.

- A framed n-embedding Φ in f, extending ϕ, consisting of a framing $\bar{e} : S^n \times D^{m-n} \to M$ of e together with an extension $\bar{d} : D^{n+1} \times D^{m-n} \to X$ of $f \cdot \bar{e}$.
- A commutative diagram β of normal bundles over Φ.

We call such a collection (Φ, β) a framed n-embedding in the normal map (f, b). As before, we may define an n-surgery on (f, b) using (Φ, β). The result is the same as an n-surgery on Φ, but now everything is covered by compatible normal maps.

What are the obstructions to killing a class $x \in \pi_{n+1}(f)$ by a normal n-surgery? Clearly, we must represent x by a framed n-embedding (Φ, β) in (f, b). We have:

$$\phi = (e, d) \mapsto \Phi = (\bar{e}, \bar{d}) \iff \nu_e = 0 \in \pi_n(BO(m-n)).$$

This is the obstruction to framing the normal bundle ν_e.

$$\Phi = (\bar{e}, \bar{d}) \mapsto (\Phi, \beta) \iff \nu_b(\Phi) = 0 \in \pi_{n+1}(BO).$$

The class $\nu_b(\Phi) : S^n \to O$ measures the difference

$$(\text{framing } \bar{e} \oplus \epsilon^\infty \text{ of } \nu_e \oplus \epsilon^\infty) - (\text{framing of } \bar{e} \oplus \epsilon^\infty \text{ from } d : f \cdot e \simeq *.)$$

And finally

$$\phi = (e, d) \mapsto (\Phi, \beta) \iff \nu_b(\phi) = 0 \in \pi_{n+1}(BO, BO(m-n))$$

given by

$$\nu_b(\phi) = (\nu_e, \text{ stable framing of } \bar{e} \text{ from } d : f \cdot e \simeq *.)$$

Thus we can in principal determine when an element $x \in \pi_{n+1}(f)$ can be killed by an n-surgery on (f, b).

1.9 Constructing Homotopy Equivalences

We now apply the surgery techniques above to the construction of homotopy equivalences from normal maps, modulo the surgery obstructions to be defined. We shall largely supress the normal data since, although it plays a crucial role, a full account is rather involved. Instead we focus on the main ideas.

Let X be an n-dimensional Poincaré complex with (η, ρ) a normal invariant and $(f, b) : M^n \to X$ a normal map. Since f is a degree one map (ie. it preserves fundamental classes) it follows easily from Poincaré duality that the induced map on homology

$$f_* : H_*(\tilde{M}) \to H_*(\tilde{X})$$

is always a split epimorphism. So we define the kernel $\mathbb{Z}[\pi]$-modules

$$K_*(f) = \mathrm{Ker}(f_*) = H_{*+1}(f).$$

These kernel groups also satisfy a form of Poincaré duality

$$[M]\cap : K^{n-i}(f) \cong K_i(f).$$

In addition, f is a homotopy equivalence, if and only if it induces an isomorphism on π_1 and $K_i(f) = 0$ for all $i \geq 0$. It follows from Poincaré duality and the universal coefficient theorem for cohomology, that f is a homotopy equivalence if and only if f induces an isomorphism on π_1 and $K_i(f) = 0$, $2i \leq n$.

Let us assume, by induction, that

$$K_j(f) = \pi_{j+1}(f) = 0, \quad j < i$$

and we are in the Hurewitz dimension i, so that $x \in K_i(f) = \pi_{i+1}(f)$ is represented by a map $\phi : S^i \to M$ together with a given null-homotopy $f \cdot \phi \simeq *$. Then clearly

$$\phi^* \nu_M = \phi^* f^*(\eta)$$

is a trivial bundle. Since the tangent bundle of a sphere is stably trivial, we conclude that for $i < [n/2]$, ϕ may be chosen to be an embedding (Whitney) with trivial normal bundle (stable range).

The normal map may therefore be modified as follows:

- Do surgery on M via the framed embedding ϕ to obtain a trace cobordism $(W; M, M')$.
- Extend the map $f : M \to X$ to a map

  $$F : (W; M, M') \to (X \times I; X \times \{0\}, \{1\}),$$

  using the null-homotopy $f \cdot \phi \simeq *$.
- Extend the normal map $b : \nu_M \to \eta$ to a normal map

  $$B : \nu_W \to \eta \times I.$$

Thus we have a normal cobordism

$$(F, B) : (W; M, M') \to (X \times I; X \times \{0\}, \{1\})$$

restricting to a new normal map

$$(f', b') : M' \to X$$

such that x is killed in $K_i(f')$.

Iterating gives

Proposition 1.36 *A normal map $(f, b) : M^n \to X$ is normally bordant to an $[n/2]$-connected normal map.*

As in the case of manifold surgery, when we do surgery on a class $x \in K_i(f)$, this class is killed. But a new dual class is introduced in dimension $(n-i-1)$. So long as $i < [n/2]$, the new class is in dimension $\geq [n/2]$ so we are ok. However, we encounter problems in the middle dimension.

Example 1.37 Suppose $m = 2k$ and we do surgery on a trivial $S^{k-1} \times D^{k+1}$. Then the result of surgery is the connected sum $M \sharp S^k \times S^k$, so that

$$K_k(f) = K_k(f') \oplus \mathbb{Z} \oplus \mathbb{Z}$$

1.10 Problems in the Middle Dimensions

Suppose M is an m-dimensional manifold with $m = 2k$. Let $f : M \to X$ be a normal map, which we may assume is k-connected. It is no longer the case that every class in $K_k(f) = \pi_{k+1}(f)$ can be represented by an embedded sphere with trivial normal bundle, so we cannot automatically do surgery to kill $K_k(f)$. By the Poincaré duality property of the kernels, $K_k(f)$ is the only non zero kernel group, and so by a chain complex argument (together with possible stabilization by trivial surgeries) we may assume $K_k(f)$ is a free $\mathbb{Z}[\pi]$-module.

Then Poincaré duality gives a pairing

$$\lambda : K_k(f) \times K_k(f) \to \mathbb{Z}[\pi].$$

This pairing will play a crucial role in the surgery obstructions. It may be described geomtrically as follows. Recall the following theorem of Haefliger [7]

Theorem 1.38 *Regular homotopy classes of immersions $\phi : S^k \to M^{2k}$ are in one-to-one correspondence, via the tangent map, with stable homotopy classes of stable bundle monomorphisms $\tau_{S^k} \to \phi^* \tau_M$.*

It follows that each element of $K_k(f)$ can be represented by an immersion together with a reference path to a fixed point $p_0 \in M$. The immersions may be chosen with trivial normal bundle, and this uniquely determines the regular homotopy class.

Thus let S_1 and S_2 be two such immersed k-spheres meeting transversely at a finite set of points P. Using the reference paths, each intersection point $p \in P$ determines a fundamental group element g_p and an orientation $\epsilon_p = \pm 1$. We may therefore define the above intersection form by the formula

$$\lambda(S_1, S_2) = \Sigma_P \epsilon_p g_p \in \mathbb{Z}[\pi].$$

Similarly we may count self-intersection points of an immersed sphere S_1 intersecting transversally with itself. This is not quite well defined, since there is no natural order for the local intersections. However, if the order is exchanged then $\epsilon_p g_p$ becomes $(-1)^k \epsilon_p \bar{g}_p$. Thus self-intersection defines a map

$$\mu : K_k(f) \to \mathbb{Z}[\pi]/\{r - (-1)^k \bar{r}\}.$$

The triple $(K_k(f), \lambda, \mu)$ defines a $\mathbb{Z}[\pi]$-valued quadratic form on the free $\mathbb{Z}[\pi]$-module $K_k(f)$. It satisfies the properties [35]

1. λ is a bilinear map
2. $\lambda(x, y) = (-1)^k \bar{\lambda(y, x)}$
3. $\lambda(x, x) = \mu(x) + (-1)^k \bar{\mu(x)}$
4. $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y)$
5. $\mu(xa) = \bar{a}\mu(x)a$
6. for $k \geq 3$, $x \in K_k(f)$ is represented by an embedding with trivial normal bundle if and only if $\mu(x) = 0$

How does this relate to a normal map $(f, b) : M \to X$? Suppose $K_x(f)$ contains a hyperbolic plane, ie. a $\mathbb{Z}[\pi]$-submodule

$$H = \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi]$$

with generators $\{x, y\}$ such that $\mu(x) = \mu(y) = 0$ and $\lambda(x, y) = 1$. Then we may represent x and y geometrically as a pair of embedded k-spheres which intersect transversely at a point. A neighbourhood of this pair looks like a handle $S^k \times S^k - D^{2k}$ with boundary S^{2k-1} in M. In particular, surgery on one of them replaces the interior of the neighbourhood with a disk D^{2k} and so kills both x and y.

Definition 1.39 *The even dimensional surgery obstruction group $L_{2k}(\mathbb{Z}[\pi])$ is the group of stable isomorphism classes of $(-1)^k$-quadratic forms (L, λ, μ) on free $\mathbb{Z}[\pi]$-modules L, modulo hyperbolic forms.*

The case $m = 2k + 1$ is rather more delicate, and will be not be dealt with here. The reader is referred to [23]. For now we simply state that the central result of surgery

Theorem 1.40 *(Wall) Let $(f, b) : M^m \to X$ be a normal map, with $m \geq 5$. Then there is defined a surgery obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi])$ such that (f, b) is normally bordant to a homotopy equivalence if and only if $\sigma(f, b) = 0$*

By definition, the L-groups are 4-periodic. In general the groups $L_n([\pi])$ are difficult to calculate.

Example 1.41 Here are some L-groups.

Let R be the ring $\mathbb{R}$ with identity involution. Then

$$L_n(\mathbb{R}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

In fact, for k even, an element of $L_{2k}(\mathbb{R})$ is simply a symmetric form on a finite dimensional vector space over $\mathbb{R}$. The isomorphism $L_{2k}(\mathbb{R}) \cong \mathbb{Z}$ is the signature.

For the ring $\mathbb{Z}_2$ we have $+1 = -1$ so from the definition $L_0(\mathbb{Z}_2) = L_2(\mathbb{Z}_2)$. Classically, any quadratic form (L, λ, μ) over $\mathbb{Z}_2$ admits a symlectic basis $\{x_1, \ldots, x_{2m})$ for L such that

$$\lambda(x_i, x_j) = \begin{cases} 1 & \text{if } |i-j| = m \\ 0 & \text{otherwise} \end{cases}$$

Then the Arf invariant defines an isomorphism [3]

$$L_{2k}(\mathbb{Z}_2) \to \mathbb{Z}_2 \quad (L, \lambda, \mu) \mapsto c(L, \lambda, \mu)$$

where

$$c(L, \lambda, \mu) = \Sigma_{i=1}^{m} \mu(x_i)\mu x_i + m.$$

We have

$$L_{2k}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ \mathbb{Z}_2 & \text{if } k \text{ is odd.} \end{cases}$$

where the first isomorphism is signature/8 and the second is the Arf invariant.

There is a neat description of the L-groups as *algebraic* cobordism classes of chain complexes of $\mathbb{Z}[\pi]$-modules with a (quadratic) Poincaré duality self-isomorphism [24]. These should be thought of as the chain homotopy version of quadratic forms. With this description, functors from manifolds and normal maps to surgery obstructions become extremely natural, and the surgery exact sequence takes on a particularly pleasant form.

In addition, this general categorical viewpoint allows us to consider cobordism classes of more general Poincaré objects. For example, Poincaré sheaves. It turns out that these cobordism groups recover the normal invariant set $N(M)$, and allow a local analysis of surgery obstructions [9]. This is important for spaces with bad local topology such as exotic homology manifolds or stratified spaces. The surgery obstruction becomes an assembly map.

1.11 Manifold Structure Set

We have considered the obstructions to improving a Poincaré complex to a manifold. Let us now consider the problem of counting the number of distinct manifolds within a given homotopy type. Essentially, we have proved

Theorem 1.42 *Let M be a closed n-dimensional manifold. Then there is an exact sequence of pointed sets*

$$S(M) \to N(M) \xrightarrow{\theta} L_n(\mathbb{Z}[\pi]).$$

where $S(M)$ is the structure set of M, and θ is the surgery obstruction map.

Recall $N(M) = [M, G/O] = \mathcal{T}(M)$.

The structure set $S(M)$ is the central object of interest, and essentially counts the number of manifolds homotopy equivalent to M. Elements of $S(M)$ consist of homotopy equivalences $f : N \to M$ modulo the following relation. Two elements (N, f), $N', f')$ are equivalent if there exist an h-cobordism $(W; N, N')$ and an extension

$$(F; f, f') : (W; N, N') \to X.$$

Recall a cobordism $(W; N, N')$ is an h-cobordism if the inclusions

$$N \hookrightarrow W \hookleftarrow N'$$

are homotopy equivalences.

Remark In general, an h-cobordism is not a cylinder (which would give the homotopy relation). There is a torsion obstruction in algebraic K-theory. It is important to allow the more general h-cobordism because we have only constructed homotopy equivalences rather than simple homotopy equivalences. For simply connected spaces h-cobordisms are cylinders. More later.

Example 1.43 Consider the special case S^m, $m \geq 1$. Then the smooth structure set $S(S^m)$ is a group. In fact, there is a one-to-one correspondence

$$S(M) \to \Theta^m; (f : \Sigma^m \to S^m) \mapsto [\Sigma^m],$$

where Θ^m is the group diffeomorphism classes of m-dimensional homotopy spheres. Addition is by connected sum. For example, $\Theta^7 = \mathbb{Z}_{28}$ [13].

Example 1.44 There is also a topological version of the structure set consisting of topological manifolds. In this case G/O must be replaced by G/TOP, but otherwise the sequence has a similar form. In the special case S^m, $m \geq 5$ we have $S^{\mathrm{TOP}}(S^m) = 1$ by the high dimensional Poincaré conjecture .

Example 1.45 Every surface admits a unique smooth structure, and every homotopy equivalence of surfaces is homotopic to a diffeomorphism. Hence the structure set is trivial.

The surgery exact sequence actually extends to the left to give an exact sequence

$$\cdots \to [M\times I, \partial(M\times I); G/O, *] \xrightarrow{\theta} L_{n+1}(\mathbb{Z}[\pi]) \to S(M) \to [M; G/O] \to L_n(\mathbb{Z}[\pi]).$$

Let us describe the maps:

An element $x \in S(M)$ represented by a homotopy equivalence $f : N \to M$ determines a fibre homotopy equivalence $\nu_M \simeq (f^{-1})^*\nu_N$ and hence an element in $t \in [M; G/O]$ such that $\nu_M + t = (f^{-1})^*\nu_N$.

An element $t \in [M; G/O]$ classifies a bundle reduction $\tilde{\nu}$ of the Spivak normal fibration $J\nu_M : M \to BG(k)$. Apply transversality to

$$S^{n+k} \to T\tilde{\nu}$$

to obtain a normal map $(f, b) : N \to M$ so that

$$\theta(t) = \sigma(f, b) \in L_n(\mathbb{Z}[\pi]).$$

The group $L_{n+1}(\mathbb{Z}[\pi])$ acts on $S(M)$ as follows. By the Wall Realization Theorem [35] each element $\sigma \in L_{n+1}(\mathbb{Z}[\pi])$ may be realized as a rel ∂ surgery obstruction

$$\sigma = \sigma(g, c)$$

where (g, c) is a normal bordism

$$(g, c) : (W; N_0, N_1) \to M \times (I; \{0\}, \{1\})$$

between homotopy equivalences $f_0 : N_0 \to M$ and $f_1 : N_1 \to M$. The action is then

$$L_{n+1}(\mathbb{Z}[\pi]) \times S(M) \to S(M)\,;\ (\sigma, (N_0, f_1)) \mapsto (N_1, f_1).$$

Two elements $(N_0, f_0), (N_1, f_1) \in S(M)$ have the same image in $[M; G/O]$ if and only if there exists $x \in L_{n+1}(\mathbb{Z}[\pi])$ such that

$$(N_1, f_1) = x(N_0, f_0) \in S(M).$$

1.12 Simply Connected Spaces

Let M be a manifold of dimension $n = 4k$. Then the cup-product determines a symmetric bilinear form

$$\lambda : H^{2k}(M; \mathbb{Q}) \times H^{2k}(M; \mathbb{Q})\ ;\ (x, y) \mapsto \langle x \cup y, [M] \rangle$$

which by Poincaré duality is non-singular. Thus we may define the signature of M

$$\sigma(M) = \text{signature}(H^{2k}(M; \mathbb{Q}), \lambda).$$

This is a homotopy invariant and so may also be defined for $4k$-dimensional Poincaré complexes.

In particular, suppose X is a $n = 4k$-dimensional Poincaré complex, (ρ, η) a normal invariant determining a normal map $(f, b) : M \to X$ for some M. Suppose in addition X is simply connected so that

$$L_{4k}(\mathbb{Z}[\pi]) = L_{4k}(\mathbb{Z}) = \mathbb{Z}$$

where the latter identification is signature/8. Then we may write the surgery obstruction as

$$\sigma(f, b) = (\sigma(M) - \sigma(X))/8 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}.$$

For a general $4k$-dimensional manifold M we also have the Hirzebruch signature theorem [11]. Let τ_M denote the tangent bundle of M and $p_*(M) = p_*(\tau_M) \in H^{4*}(M)$ the total Pontryagin class of M. Thus

$$p_*(M) = \Sigma_{i \geq 0} p_i(M)$$

where

$$p_i(M) \in H^{4i}(M), \quad \text{for } i \geq 0.$$

The L-genus of M is the rational cohomology class

$$\mathcal{L}(M) \in H^{4*}(M; \mathbb{Z})$$

with components $\mathcal{L}_{4k}(M) \in H^{4k}(M; \mathbb{Q})$ given by certain rational polynomials in the Pontryagin classes $p_i(M)$. For example

$$\mathcal{L}_1 = \frac{1}{3} p_1, \quad \mathcal{L}_2 = \frac{1}{45}(7p_2 - (p_1)^2).$$

The L-genus is determined by and determines the rational Pontryagin classes. It arises naturally from the consideration of multiplicative rational invariants on cobordism rings. It is a local invariant of M. For example, it contains the signatures of submanifolds $N^{4k} \subset M$ with trivial normal bundle.

In general, the rational Pontryagin classes are not homotopy invariants, being constructed from the characteristic classes of a bundle. (Although they ARE topological invariants!). However, we can ask if certain linear combinations are homotopy invariants.

Theorem 1.46 *(Hirzebruch Signature Theorem) Let M be a closed manifold of dimension $n = 4k$. Then*

$$\sigma(M) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}.$$

So that $\langle \mathcal{L}(M), [M] \rangle$ is in fact a homotopy invariant. It turns out that for simply connected manifolds, this is essentially the only homotopy invariant linear combination of the rational Pontryagin classes. (For non-simply connected manifolds, the higher signatures $\mathcal{L}_\pi$ are appropriate, and then homotopy invariance is equivalent to the Novikov conjecture.)

What about Poincaré complexes? Pontryagin classes are not defined for spherical fibrations since rationally $H^*(BSG; \mathbb{Q}) = 0$, $* > 0$.

Let X be a $4k$-dimensional simply connected Poincaré complex with η a bundle reduction of the Spivak normal fibration ξ_X. As usual this defines a normal map

$$(f, b) : M \to X$$

for some manifold M. Then (f, b) is normal bordant to a homotopy equivalence $f' : M' \to$ if and only if $\sigma(M) = \sigma(X)$ or equivalenty

$$\sigma(X) = \langle \mathcal{L}(-\eta), [X]_{\mathbb{Q}} \rangle \in L_{4k}(\mathbb{Z}) = \mathbb{Z}.$$

Hence, X is homotopy equivalent to a manifold, if and only if the Spivak normal fibration of X admits a bundle reduction compatible with the Hirzebruch signature formula. We can regard the map θ in the surgery exact sequence

$$S(X) \to [X, G/O] \xrightarrow{\theta} L_{4k}(\mathbb{Z})$$

as measuring the deviation of the normal invariant from the Hirzebruch formula. Further details on the Hirzebruch signature theorem and its generalization, the Novikov conjecture, can be found in [6].

1.13 Some Applications

We shall allow topological manifolds in this section, in order to avoid the groups $\pi_*(G/O)$. Thus we shall consider the topological surgery exact sequence

$$\cdots \to S^{\mathrm{TOP}}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi])$$

of an n-dimensional, closed topological manifold.

In addition, if we work with simply connected manifolds, then every h-cobordism is a cylinder. In this case we may write $S^{\mathrm{TOP}}(M)$ as the set of equivalence classes of homotopy equivalences $f : N \to M$ with $(N_1, f_1) \sim (M_2, f_2)$ iff there exists a homeomorphism $\phi : N_1 \to N_2$ such that $f_1 \simeq f_2 \cdot \phi$.

Fake Projective Spaces

We follow [35]. Suppose we are given a free action of S^1 on S^{2n-1}. Let M^{2n-2} be the quotient manifold. Then there is a principal S^1 bundle

$$S^1 \to S^{2n-1} \to M.$$

This is classified by a map $f : M \to B(S^1) = \mathbb{CP}^\infty$. By dimension we may assume $f : M \to \mathbb{CP}^{n-1}$. Let $\tilde{f} : S^{2n-1} \to S^{2n-1}$ be the covering map with fibre $1 : S^1 \to S^1$ (since it is a pullback).

Now

$$H_{2n-1}(\tilde{f}) \cong \pi_{2n-1}(\tilde{f}) \cong \pi_{2n-1}(f) \cong H_{2n-1}(f),$$

and there is an exact sequence

$$0 = H_{2n-1}(\mathbb{CP}^{n-1}) \to H_{2n-1}(f) \to H_{2n-2}(M) \to H_{2n-2}(\mathbb{CP}^{n-1}) \to H_{2n-2}(f) = 0$$

Hence $H_{2n-1}(\tilde{f}) = 0$ and $\tilde{f}$ is a degree one map, ie. a homotopy equivalence. It follows from the homotopy properties of fibrations that $f : M \to \mathbb{CP}^{n-1}$ is a homotopy equivalence aswell.

Conversely a homotopy equivalence $f : M \to \mathbb{CP}^{n-1}$ determines a principal S^1 bundle via the map

$$M \to \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^\infty,$$

with total space S^{2n-1}.

Lemma 1.47 *There is a one-to-one correspondence between $S(\mathbb{CP}^{n-1})$ and free actions of S^1 on S^{2n-1}.*

To calculate $S(\mathbb{CP}^{n-1})$ we first calculate the set $[\mathbb{CP}^{n-1}, G/TOP]$. The cofibration

$$\mathbb{CP}^{k-1} \to \mathbb{CP}^k \to S^{2k}$$

determines an exact sequence

$$\pi_{2k}(G/TOP) \xrightarrow{f^*} [\mathbb{CP}^k, G/TOP] \to [\mathbb{CP}^{k-1}, G/TOP] \to \pi_{2k-1}(G/TOP) = 0.$$

By naturality of the surgery obstruction, the composition

$$\pi_{2k}(G/TOP) \xrightarrow{f^*} [\mathbb{CP}^k, G/TOP] \xrightarrow{\theta(\mathbb{CP}^{2k})} L_{2k}(\mathbb{Z})$$

is just the surgery obstruction map $\theta(S^{2k})$ fitting into the topological surgery exact sequence

$$\cdots \to S(S^{2k}) \to \pi_{2k}(G/TOP) \xrightarrow{\theta(S^{2k})} L_{2k}(\mathbb{Z}) \to \ldots.$$

But by the higher dimensional Poincaré conjecture $S(S^{2k}) = 0$, so that $\theta(S^{2k})$ is an isomorphism.

We conclude that $\theta(\mathbb{CP}^k)$ splits $L_{2k}(\mathbb{Z}) = \pi_{2k}(G/TOP) \xrightarrow{f^*} [\mathbb{CP}^k, G/TOP]$. In other words we have a split short exact sequence

$$0 \to L_{2k}(\mathbb{Z}) \to [\mathbb{CP}^k, G/TOP] \to [\mathbb{CP}^{k-1}, G/TOP] \to 0.$$

Thus we may inductively construct a one-to-one correspondence

$$[\mathbb{CP}^{n-1}, G/Top] \to \Sigma_{k=1}^{n-1} L_{2k}(\mathbb{Z}).$$

The map can be described geometrically as follows. An element in $g \in [\mathbb{CP}^{n-1}, G/Top]$ determines a normal map

$$(f, b) : M \to \mathbb{CP}^{n-1}$$

. For each $\mathbb{CP}^k \subset \mathbb{CP}^{n-1}$, make f transverse to $\mathbb{CP}^k$ to obtain a normal map

$$(f_k, b_k) : N^{2k} \to \mathbb{CP}^k$$

with surgery obstruction $\sigma(f_k, b_k) \in L_{2k}(\mathbb{Z})$. Then the above one-to-one correspondence takes

$$g \mapsto \Sigma_{k=1}^{n-1} \sigma(f_k, b_k)$$

Thus we have calculated the normal invariants of $\mathbb{CP}^{n-1}$. Return now to the surgery exact sequence

$$0 = L_{2n-1}(\mathbb{Z}) \to S(\mathbb{CP}^{n-1}) \to [\mathbb{CP}^{n-1}, G/TOP] \xrightarrow{\theta} L_{2n-2}(\mathbb{Z}).$$

According to our description above, the map θ is the projection

$$\Sigma_{k=1}^{n-1} L_{2k}(\mathbb{Z}) \to L_{2n-2}(\mathbb{Z}).$$

In conclusion

$$S(\mathbb{CP}^{n-1}) \cong \Sigma_{k=1}^{n-2} L_{2k}(\mathbb{Z}).$$

Remark Let $f : M \to \mathbb{CP}^{2k-1}$ be a homotopy equivalence. We now have a simple way to determine if f is homotopic to a homeomorphism. Simply check the surgery obstructions $\sigma(f_k, b_k)$ for each normal map $(f_k, b_k) : M_k \to \mathbb{CP}^k \subset \mathbb{CP}^{n-1}$. If they are all zero, then f is homtopic to a homeomorphism! Thus:

homeomorphisms are detected by 'local' homotopy equivalences!

This is a special case of Sullivan's characteristic variety theorem [32].

Exotic Spheres

Milnor's construction in 1956 of exotic 7-spheres was one of the first steps in the development of surgery theory [17]. Milnor showed that the smooth structure set $S(S^7)$ is non-trivial.

Theorem 1.48 *There exists a 7-dimensional manifold Σ^7 which is homeomorphic but not diffeomeorphic to the standard smooth S^7.*

Recall the isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_3(SO(4)) \cong \pi_4(BSO(4)),$$

taking a pair $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$ to the 4-plane bundle $\omega = \omega(h, j)$ over S^4 with

$$\chi(\omega) = h + j, \quad p_1(\omega) = 2(h - j) \in H^4(S^4) = \mathbb{Z}.$$

Let

$$(D^4, S^3) \to (E(\omega), S(\omega)) \to S^4$$

be the (D^4, S^3)-bundle over S^4 associated to ω. Then by the Gysin sequence or the Serre spectral sequence

$$H_i(S(\omega)) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 7 \\ \operatorname{coker}(h + j : \mathbb{Z} \to \mathbb{Z}) & \text{if } i = 3 \\ \ker(h + j : \mathbb{Z} \to \mathbb{Z}) & \text{if } i = 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $S(\omega)$ is simply connected, choosing $h + j = 1$ we obtain a 7-dimensional homotopy sphere $S(\omega)$ which is the bondary of an 8-dimensional manifold manifold $E(\omega)$. In addition, if $k \in \mathbb{Z}$ is odd, set

$$h = (1 + k)/2, \quad \text{and} \quad j = (1 - k)/2.$$

Thus for $\omega_k = \omega(h, j)$ we have

$$\chi(\omega_k) = 1, \quad p_1(\omega_k) = 2k.$$

Write

$$(W_k, \Sigma_k) = (E(\omega_k), S(\omega_k)).$$

Now, by the homology of Σ_k, there exists a Morse function $f : \Sigma_k \to \mathbb{R}$ with only two critical points: a minimum and a maximum. The Morse flow therefore determines a diffeomorphism $\Sigma_k - p\{pt.\}$. It follows that Σ_k is *homeomorphic* to S^7. (The smooth structure may not extend to a neighbourhood of the puncture point.)

Consider the tangent bundle of W_k classified by

$$\tau_{W_k} = \tau_{S^4} \oplus \omega_k : W_k \simeq S^4 \to BSO(8),$$

so that

$$p_1(W_k) = 2k \in H^4(W_k) = \mathbb{Z}.$$

Suppose now that there exists a diffeomorphism

$$f : \Sigma_k \to S^7$$

and use f to build the smooth 8-dimensional manifold

$$M = W_k \cup_f D^8.$$

Then $H^4(M) = H^4(T\omega_k) \cong H_0(S^4) = \mathbb{Z}$ generated by the Thom class $U_{\omega_k} \in H^4(T\omega_k)$. But

$$\langle U_{\omega_k} \cup U_{\omega_k}, [M] \rangle = \chi(U_{\omega_k}) = 1 \in \mathbb{Z},$$

so that the symmetric form

$$(H^4(M), \lambda) = (\mathbb{Z}, 1)$$

has signature $\sigma(M) = 1$. But, by the Hirzebruch signature theorem

$$1 = \sigma(M) = \langle \mathcal{L}_2(M), [M] \rangle \in \mathbb{Z}$$

where $\mathcal{L}_2(M) = (7p_2(M) - p_1(M)^2)/45 \in H^8(M) = \mathbb{Z}$. We conclude that

$$\frac{1}{45}(7p_2(M) - 4k^2) = 1$$

or $k^2 \equiv 1$, (mod 7). Thus for $k^2 \not\equiv 1$ (mod 7), Σ^7 is a topological 7-sphere, with a non-standard smooth structure.

1.14 From Homotopy Equivalences to Diffeomorphisms

Let N, M be n-dimensional manifolds. Suppose $f : N \to M$ is a homotopy equivalence. We seek methods to determine when f is homotopic to a diffeomorphism. A full acount of the obstructions encountered involves simple homotopy, which we will deal with later. For now we will content ourselves with an h-cobordism $(W; N, N')$ together with an extension of f to W

$$F : (W; N, N') \to M \times (I; \{0\}, \{1\})$$

such that $f' : N' \to M$ is a diffeomorphism. (There is then a single K-theoretic obstruction to a homotopy.)

By definition of the structure set $S(M)$, the above data exists if and only if

$$[h] = [1_M] \in S(M),$$

where $1_M : M \to M$ is the identity diffeomorphism.

Firstly, let us consider bundle data. If f is a diffeomorphism then, of course, the normal bundles ν_M and $(f^{-1})^*\nu_N$ are isomorphic. If f is merely a homotopy equivalence, then we only have a fibre homotopy equivalence between the corresponding spherical fibrations, or equivalently a fibre homotopy trivialisation

$$t(f) : J(\nu_M - (f^{-1})^*\nu_N) \simeq * : M \to BG.$$

Equivalently, there is defined a fibre homotopy equivalence of spherical fibrations

$$a : J\nu_N \to J\nu_M$$

over the homotopy equivalence f. By the uniqueness of the Spivak normal fibration

$$T(a)_*(\rho_N) = \rho_M \in \pi^S_{m+k}(T\nu_M).$$

Both $t(f)$ and a are uniquely determined determined by the homotopy equivalence f and gives rise to a unique element, also written $t(f)$,

$$t(f) : M \to G/O$$

with image in BO given by the stable difference

$$\nu_M - (f^{-1})^*\nu_N : M \to BO.$$

This determines the map

$$t : S(M) \to [M, G/O]\, ;\, f \mapsto t(f),$$

in the surgery exact sequence.

Suppose now the classifying map $t(g) : M \to G/O$ is null homotopic, ie. $t(f) = 0 \in [M, G/O]$. This can be interpreted in the following way. There exists an isomorphism of normal bundles over f

$$c : \nu_N \to \nu_M$$

such that $Jc \simeq a : J\nu_N \to J\nu_M$, and so in addition

$$T(c)_*(\rho_N) = \rho_M \in \pi^S_{m+k}(T\nu_M).$$

In other words, the homotopy equivalence f takes the normal invariant (ν_N, ρ_N) to the normal invariant (ν_M, ρ_M). But there is one-to-one correspondence between normal invariants $\mathcal{T}(M)$ and bordism classes of normal maps $N(M)$. It follows that

$$(f, c) = (1_M, 1_{\nu_M}) \in N(M).$$

In summary:

Proposition 1.49 *A homotopy equivalence $f : N^m \to M^m$ determines a classifying map $t(f) \in [M, G/O]$. The class $t(f) = 0$ if and only if there is an extension of f to a normal bordism*

$$((F; f, 1_M), (B; c, 1_{\nu_N})) : (W^{m+1}; N, M) \to M \times (I; \{0\}, \{1\}).$$

So, modulo the above obstuction, we have succeeded in constructing a normal bordism from the homotopy equivalence to a diffeomorphism. The next step is to try to do surgery rel ∂ on the normal map $(F, B) : (W; N, M) \to M \times I$ construct a rel ∂ homotopy equivalence

$$H : (W'; N, M) \to M \times (I; \{0\}, \{1\}).$$

Notice that $(W; N, M)$ is an h-cobordism and we are just a K-theory obstruction from a diffeomorphism $g : N \to M$.

There is a relative version of surgery, where we do surgery away from the boundary, on the interior of the manifold. Thus the homotopy type of the manifold is modified but the boundary stays unchanged. As in the case of closed manifold, we have

Theorem 1.50 *Let $(M, \partial M)$ be an m-dimensional manifold, $m \geq 6$, with boundary ∂M. Let*

$$(f, b) : (M, \partial M) \to (X, \partial X)$$

be a normal map, which is a homotopy equivalence on the boundary. Then there is defined a rel ∂ surgery obstruction

$$\sigma(f, b) \in L_{m+1}(\mathbb{Z}[\pi_1(X)])$$

such that $\sigma(f, b) = 0$ if and only if (f, b) is normal bordant, rel ∂, to a homotopy equivalence.

Refering back to the normal map (F, B) above, there is defined a surgery obstruction

$$\sigma(F, B) \in L_{m+1}(\mathbb{Z}[\pi_1(M)])$$

such that $\sigma(F, B)$ is zero, if and only if we can replace $(W; N, M)$ with an h-cobordism $(W'; N, M)$, ie. the sequence

$$L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S(M) \to [M, G/O]$$

is exact.

Summarising, we have a two stage process for determining if a homotopy equivalence $f : N \to M$ is h-cobordant to a diffeomorphism

- Are the bundles ν_N, ν_M compatible with f, ie. is $t(f) = 0 \in [M, G/O]$?
- If so, can we do rel ∂ surgery on the resulting normal bordism $(F, B) : (W; N, M) \to M \times (I; \{0\}, \{1\})$ to obtain an h-cobordism $(W'; N, M)$

The final step is to know when an h-cobordism is trivial.

1.15 The $\pi - \pi$ Theorem

Suppose we have a normal map of pairs $(f, b) : (M, N) \to (X, Y)$. Then we can ask whether we can do surgery on (f, b) to replace it with a homotopy equivalence of pairs. There are two possible interpretations for what 'surgery' should mean here:

- We assume that on the boundary $f_| : N \to Y$ is already a homotopy equivalence. In which case we need only do surgery on the interior of M away from the boundary N. The setup is close to surgery on closed manifolds, except that we carry a boundary around with us, but it is unchanged by the operation. It turns out that there is a rel ∂ surgery obstruction
$$\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$$
such that $\sigma(f, b) = 0$ if and only if we may surgery rel ∂ (f, b) to a homotopy equivalence of pairs.
- Alternatively we make no assumption about $f_| : N \to Y$, and we try to do surgery on both M and N simultaneously to get a homotopy equivalence of pairs.

The latter is more complicated and involves embedding disks $(D^n, S^{n-1}) \subset (M, N)$ and defining a surgery operation on such framed embeddings. In addition, the surgery obstructions take values in a relative L-group

$$\sigma(f, b) \in L_n(\pi_1(X), \pi_1(Y))$$

which is complicated to define [35].

Nevertheless, the notation at least suggests that if $\pi_1(Y) \cong \pi_1(X)$ then the relative L-group should be zero and so we should be able to do surgeries on (f, b) to get a homotopy equivalence of pairs. This is the content of Wall's $\pi - \pi$ theorem [35].

Theorem 1.51 (Wall) *Let (X, Y) be a Poincaré pair of dimension $m \geq 6$. Suppose the inclusion $Y \subset X$ gives an isomorphism of fundamental groups $\pi_1(Y) \cong \pi_1(X)$. Then given any normal map of pairs*

$$(f, b) : (M, N) \to (X, Y)$$

we can perform surgery to make (f, b) a homotopy equivalence of pairs.

For an m-dimensional manifold pair (M, N) with $N = \partial M$, we perform surgery as follows. Consider an n-embedding

$$e : (D^n, S^{n-1}) \to (M, N)$$

with framing

$$\bar{e} : (D^n \times D^{m-n}, S^{n-1} \times D^{m-n}) \to (M, N).$$

Then the result of surgery on the framed n-embedding $\bar{e}$ is the manifold with boundary (M', N') given by

$$(\mathrm{cl}(M \backslash \bar{e}(D^n \times D^{m-n})) , \mathrm{cl}(N \backslash \bar{e}(S^{n-1} \times D^{m-n})) \cup (D^n \times S^{m-n-1})).$$

In particular, M' is the result of surgery on $\bar{e}$ restricted to $S^{n-1} \times D^{m-n}$.

We will sketch a proof of the $\pi - \pi$ theorem in the even dimensional case $m = 2k$, $k \geq 3$.

As in the case of surgery on closed manifolds, we may assume we have already done surgery so that

- The map $f : M \to X$ is k-connected, and the map $f : N \to Y$ is $(k-1)$-connected.
- Since $k \geq 3$, the fundamental groups are isomorphic:

$$\pi = \pi_1(M) = \pi_1(N) = \pi_1(X) = \pi_1(Y).$$

- $K_k(f)$ is the only non-trivial kernel. It is a finitely generated, free $\mathbb{Z}[\pi]$-module with basis $\{e_i\}$

Again, since the fundamental groups are isomorphic, the Hurewicz map determines isomorphisms

$$\pi_{k+1}(f) \cong \pi_{k+1}(\tilde{f}) \cong H_{k+1}(\tilde{f}) = K_k(f).$$

Each element $e_i \in K_k(f)$ is therefore represented by a map

$$g_i : (D^k, S^{k-1}) \to (M, N)$$

together with a pairwise null homotopy of $f \cdot g_i$ in (X, Y). Since $2k = m$ we have

- The maps $\{e_i\}$ may be chosen to be mutually transverse immersions.
- The normal bundles of these immersions are framed.

So we have framed immersions

$$\bar{g}_i : (D^k \times D^k, S^{k-1} \times D^k) \to (M, N).$$

We now show that these immersions are in fact regularly homotopic to disjoint embeddings. The result will then follow by doing surgery on each g_i to kill the generator $e_i \in K_k(f)$.

It is sufficient to prove that the core embeddings

$$g_i : (D^k, S^{k-1}) \to (M, N)$$

are regular homotopic to disjoint embeddings. We employ the Whitney trick. Since the $\{g_i\}$ are mutually transverse, they intersect in a finite collection of points P in the interior of M. Suppose the disks D_1^k, D_2^k have an intersection point $p \in D_1^k \cap D_2^k \subset \text{int}(M)$. Choose arcs α_1, α_2 in D_1^k, D_2^k, respectively, from p to N. Since $\pi_1(M) = \pi_1(N)$ we can find a regular 2-simplex D^2 in M with sides α_1, α_2, and $\alpha \subset N$. As in the proof of the Whitney trick, slide the disk D_2^k across D^2 to remove the intersection point p.

Now do surgery on the disjoint embeddings to kill $K_k(f)$.

1.16 Application: Browder's Embedding Theorem

We have shown how surgery theory can be used to determine when a Poincaré space is homotopy equivalent to a manifold. Here we use surgery theory to detect when a 'Poincaré embedding' is homotopy equivalent to a manifold embedding. This extends the surgery classification of manifolds to submanifolds.

To some extent, this material is more of historical interest than practical use. It was once hoped that one could analyse Poincaré embeddings using the methods of homotopy theory. Then this could be used to study submanifolds. It turns out that the homotopy theory of Poincaré embeddings is extremely complicated. There are cohomological obstructions of all orders to the existence of Poincaré embeddings. In fact, it is a difficult problem in unstable homotopy theory. Although the existence of Poincaré embeddings follows from previous results concerning surgery on Poincaré spaces (eg. Hausmann and Vogel), they either employ manifold techniques (and therefore undermine the original reason for study) or are sufficiently abstruse to defy practical application. Nevertheless, the Browder embedding theorem which follows was one of the first applications of surgery, and nicely illustrates its use.

We arrived at the definition of a Poincaré space by extracting all of the homotopy properties of manifolds, such as Poincaré duality and normal data. The same principle is used in defining a Poincaré embedding. There are several possible definitions. We follow Levitt, as described in Wall [35].

Definition 1.52 *Let M^m, V^{m+q} be Poincaré spaces. An embedding of M in V consists of the following data:*

- *A $(q-1)$-spherical fibration ξ over M with projection $p : E \to M$.*
- *A Poinaré pair (C, E).*
- *A homotopy equivalence $h : C \cup M(p) \to V$, where $M(p)$ is the mapping cylinder of p and $C \cap M(p) = E$.*

Write $i : M \to V$ for the restriction of h to M. Of course, i is in general not an embedding.

It follows from the uniqueness of the Spivak normal fibration that there is a stable equivalence of spherical fibrations

$$J\nu_M = i^* J\nu_V \oplus \xi.$$

If ξ admits a bundle reduction η then we may form the bundle

$$i^*\nu_V \oplus \eta.$$

We say that η is compatible if there is a bundle isomorphism

$$\nu_M = i^*\nu_V \oplus \eta.$$

It is easily seen that every smoothly embedded submanifold $N \subset M$ determines a Poincaré embedding. We take $\xi = J\nu$ where ν is the normal bundle of N in M. Browder's embedding theorem states that for codimension ≥ 3, the converse is also true:

Theorem 1.53 *Let V^{m+q}, M^m be closed manifolds with $m + q \geq 5$ and $q \geq 3$. Let $(\xi, (C, E), h)$ be a Poincaré embedding of M in V. Suppose ξ admits a compatible bundle reduction η. Then there is an embedding $j : M \to V$ inducing the given Poincaré embedding up to homotopy.*

Proof Identify ξ with η and consider the map

$$h^{-1} : V \to C \cup M(p).$$

Making h^{-1} transverse to $M \subset M(p)$ we obtain a map $f : M' \to M$ covered by a bundle map $c : \xi' \to \xi$, where ξ' is the normal bundle of the embedding $i' : M' \to V$. Since η is compatible, we have

$$b = c \oplus 1 : \nu_{M'} = \xi' \oplus i'^*\nu_V \to \xi \oplus i^*\nu_V = \nu_M,$$

so that f is covered by a normal map $b : \nu_{M'} \to \nu_M$. It follows that (f, b) is trivial in $N(M)$, ie. is normal bordant to the identity map $1_M : M \to M$

$$(F; f, 1_M) : (L; M', M) \to M.$$

Let A be the total space of the bundle $F^*\xi$ with zero section L. Then the restriction of A to M' is a tubular neighbourhood $E(\xi')$ of M' in V. Form the manifold W by gluing $V \times I$ to A via $E(\xi')$:

$$W = (V \times I) \cup_{E(\xi')} A.$$

Write $A' = A \cap V \times \{1\}$. We build a normal map of quadruples

$$(W; V \times \{0\} \cup M(p), V \times \{1\} - A' \cup S(F^*\xi), E) \to ((C \cup M(p)) \cup_{h^{-1}} (V \times I), V \cup M(p), C, E)$$

The map is a homotopy equivalence on $V \times \{0\} \cup M(p)$. In addition, since $q \geq 3$, $\pi_1(C) = \pi_1(V)$ so that, by the $\pi - \pi$ theorem, we may surger the normal map to a homotopy equivalence of quadruples. Then the resulting cobordism W' is homotopy equivalent to $V \times I$, and so is an h-cobordism. Assuming the K-theory obstructions are zero (which we can achieve by requiring h to be a simple homotopy equivalence), we obtain a cylinder and hence the required manifold embedding.

2 Algebraic K-Theory and Manifolds

Algebraic K-theory arises naturally in many areas of mathematics from the geometry of varieties to number theory. We shall study its applications to the classification of manifolds. These include Wall's finiteness obstruction, the Siebenmann End Theorem and the s-cobordism theorem. The first two are concerned with finiteness. Wall's finiteness obstruction detects whether a given space is homotopy equivalent to a finite CW-complex, while the Siebenmann End Theorem detects when an open manifold admits a boundary. In the former, the input is a finitely dominated space, ie. a homotopy direct summand of a finite CW-complex. In the latter, an open manifold with tame end. The s-cobordism theorem detects when a cobordism with the homotopy type of a cylinder is in fact diffeomorphic to a cylinder. The applications of these results, and algebraic K-theory itself, are pervasive in the study of manifolds.

More recent work on stratified manifolds (cf. Weinberger [37]), extending the classical manifold theory, depends essentially on developing local versions of algebraic K-theory via sheaf theoretic constructions. Again, the classification of stratified manifolds depends essentially on algebraic K-theory. Furthermore, there are interesting differences between the properties of PL (or smooth) stratified manifolds and weaker topological versions, due to the existence or not of regular neighbourhoods. A regular neighbourhood acts as a kind of barrier to K-theory, limiting its influence to within each stratum. In the topological setting, where regular neighbourhoods may not exist, K-theory can leak to neighbouring strata, leading to more subtle behaviour.

We begin with an introduction to projective modules and the definition of the K-groups. The reader is referred to Rosenberg [25], Silvester [28] and Milnor [19] for further details.

2.1 Projective Modules

Let R be a ring with unit $1 \in R$. Typical examples for us will be $\mathbb{Z}$, $\mathbb{Q}$, group rings $\mathbb{Z}[\pi]$ of fundamental groups π, and matrix rings.

Let M be a left R-module. (In these notes we shall simply say R-module to denote a left R-module.) Thus M is an abelian group with operation $+$ and there is a left action

$$R \times M \to M\,;\ (r, m) \mapsto rm,$$

such that for all $r, s \in R$, $m, n \in M$

1. $r(m + n) = rm + rn$,
2. $(r + s)m = rm + sm$,
3. $(rs)m = r(sm)$,
4. $1m = m$.

A map of R-modules $f : M \to N$ is a homomorphism if for all $r \in R$, $m, n \in M$

1. $f(m + n) = f(m) + f(n)$,
2. $f(rm) = rf(m)$.

Definition 2.1 *The R-module M is finitely generated if there exists a finite collection $\{m_1, \ldots, m_q\} \in M$ such that every $m \in M$ can be written $m = \Sigma r_i m_i$ for some $r_i \in R$.*

Example 2.2

1. If R is a field, then a finitely generated R-module is a finite dimensional vector space over R.
2. Let X be a CW-complex, $\pi = \pi_1(X)$ the fundamental group of X and $\tilde{X}$ the universal cover of X. Then the cellular chain complex $C(\tilde{X})$ is a chain complex of $\mathbb{Z}[\pi]$-modules. If X is a finite CW-complex the $C(\tilde{X})$ is finitely generated as a $\mathbb{Z}[\pi]$-module.
3. Let $E \to X$ be a vector bundle. Then the space of continuous sections $\Gamma(E)$ is a $C(X)$-bundle, where $C(X)$ is the ring of continuous functions on X (*not* the cellular chain complex).

Definition 2.3 *An R-module M is said to be free if there is a collection $\{m_\alpha\}$ of elements of M such that each element $m \in M$ has a unique expression as a finite sum*

$$m = \Sigma r_\alpha m_\alpha, \quad r_\alpha \in R.$$

The collection $\{m_\alpha\}$ is called a basis for M.

Clearly, M is free if and only if $M \cong \oplus_\alpha R_\alpha$, where R_α is an isomorphic copy of R.

Warning Cardinality of basis is not necessarily an invariant of M. We say M satisfies the Invariant Basis Property (IBP) if $n \neq m \Rightarrow R^n \not\cong R^m$. For example, any commutative ring or group ring satisfies the IBP. But, in general, for a field F, the endomorphism ring $\mathrm{End}_F(F^\infty)$ does not satisfy the IBP.

Example 2.4

1. Every finite n-dimensional vector space V admits a basis $\{v_1, \dots v_n\}$.

2. The complex $C(\tilde{X})$ is a chain complex of free $\mathbb{Z}\pi$-modules. If X is a finite CW-complex, each $C(\tilde{X})_k$ has a finite basis, obtained by choosing a lift $\tilde{e}_i \subset \tilde{X}$ of each k-cell $e_i \subset X$.

3. For a vector bundle $E \to X$ and space of functions $C(X)$, the set of continuous sections $\Gamma(E)$ is a free $C(X)$-module $\Leftrightarrow$ E is a trivial vector bundle.

Lemma 2.5 *Let F be a free R-module, $f : M \to N$ an epimorphism and $g : F \to N$ a homomorphism. There exists a homomorphism $h : F \to M$ such that $f \cdot h = g$.*

Proof For each basis element x_α of F choose $h(x_\alpha) = m_\alpha$, where $f(m_\alpha) = g(x_\alpha)$, using the fact that f is onto. Since F is free, we may define a homomorphism

$$h : F \to M \, ; \, x = \Sigma r_\alpha x_\alpha \mapsto \Sigma r_\alpha m_\alpha, \quad r_\alpha \in R.$$

Then h is well defined, but not necessarily unique. In fact, any two such maps h_1, h_2 differ by a map into $\ker(f)$.

This lifting property is abstracted in the following definition.

Definition 2.6 *An R-module P is said to be projective if for any epimorphism $f \; M \to N$ and morphism $g : P \to N$ there exists a morphism $h : P \to M$ such that $f \cdot h = g$.*

Clearly, every free module is projective, but the converse is not true.

Lemma 2.7 *The following are equivalent for an R-module P*

1. *P is projective.*

2. *Every epimorphism $f : M \to P$ splits, ie. there is a morphism $g : P \to M$ such that $f \cdot g = 1_P$.*

3. *There exists an R-module Q such that $P \oplus Q$ is free.*

Proof

$1 \Rightarrow 2$ Immediate.

$2 \Rightarrow 3$ Let $\{x_\alpha\}$ be a generating set for P and F a free R-module with basis $\{y_\alpha\}$ (same cardinality). Then there is an epimorphism

$$\pi : F \to P \, ; \, \Sigma r_\alpha y_\alpha \mapsto \Sigma r_\alpha x_\alpha, \quad r_\alpha \in R.$$

Let $Q = \ker(\pi)$ with inclusion $j : Q \to F$. By (2) π splits via a morphism $i : P \to F$. Define a morphism

$$\omega : P \oplus Q \to F\,;\ (x, y) \mapsto i(x) + j(y).$$

Then ω is easily seen to be an isomorphism.

$3 \Rightarrow 1$ Consider the natural splitting

$$P \xrightarrow{i_1} P \oplus Q \xrightarrow{\pi_1} P.$$

Let $f : M \to N$ be an epimorphism and $g : P \to N$. To show that P is projective we must construct a morphism $h : P \to M$ such that $f \cdot h = g$. Instead, consider the map

$$P \oplus Q \xrightarrow{\pi_1} P \xrightarrow{g} N.$$

Since $P \oplus Q$ is free, there exists a lift h' of $g \cdot \pi_1$ across f, so that $f \cdot h' = g \cdot \pi_1$. Setting $h = h' \cdot i_1$ gives the required morphism.

Remark M is finitely generated and projective iff there exists a Q such that $P \oplus Q \cong R^n$.

2.2 The Grothendieck group $K_0(R)$

Let R be a ring and let $\operatorname{Proj} R$ denote the set of isomorphism classes of finitely generated projective R-modules. (Since finitely generated, projective R-modules are direct summands of R^n, $\operatorname{Proj} R$ is a set.) Then we have the direct sum map

$$\operatorname{Proj} R \times \operatorname{Proj} R \to \operatorname{Proj} R\,;\ (M, N) \mapsto M \oplus N,$$

so that $\operatorname{Proj} R$ is an abelian semigroup ($\oplus$ is associative and commutative) with identity the zero R-module.

In general $\operatorname{Proj} R$ is not a group, and neither does it satisfy the cancellation property.

Lemma 2.8 *Let S be an abelian semigroup. Then there exists an abelian group G and a semigroup map $\phi : S \to G$ such that for any abelian group H and semigroup map $\chi : S \to H$ there exists a unique homomorphism of groups $\theta : G \to H$ such that $\theta \cdot \phi = \chi$.*

In other words, if F : abelian groups $\Rightarrow$ abelian semigroups is the forgetful functor then

$$\mathrm{Hom}_{\text{ab-semigrp}}(S, FH) = \mathrm{Hom}_{\text{ab-grp}}(G, H),$$

so that the functor $S \Rightarrow G$ is a left adjoint to the forgetful functor F. It is easily seen that any such G and ϕ are unique up to isomorphism.

The proof of the lemma follows.

Proof We give two alternative constructions of G. Firstly, let F be the free abelian group on elements of S. For $s \in S$ write $\langle s \rangle$ for the corresponding generator of F. Let R be the subgroup of F generated by the expressions

$$\langle s_1 \rangle + \langle s_2 \rangle - \langle s_1 + s_2 \rangle$$

so that the first two operands are in F and the last in S. Let G be the abelian group $G = F/R$ and ϕ the map

$$\phi : S \to G\,;\; s \mapsto \langle s \rangle + R \in F/R.$$

taking $s \in S$ to the coset $\langle s \rangle + R \in F/R$. Then

$$\phi(s_1+s_2) = \langle s_1 + s_2 \rangle + R = (\langle s_1 \rangle + \langle s_2 \rangle) + R = (\langle s_1 \rangle + R) + (\langle s_2 \rangle + R) = \phi(s_1) + \phi(s_2).$$

Write $[s] = \langle s \rangle + R \in F/R$. By collecting terms we see that every element of G can be written in the form $[s_1] - [s_2]$ for some $s_1, s_2 \in S$.

For a semigroup map $\chi : S \to H$ into an abelian group H, define a group homomorphism $\theta : G \to H$ such that $\theta \cdot \phi = \chi$, by

$$\theta : G \to H\,;\; [s_1] - [s_2] \mapsto \chi(s_1) - \chi(s_2).$$

In the second approach we define G to be the set of equivalence pairs (x, y) of elements $x, y \in S$, subject to the relation $(x, y) \sim (u, v)$ iff there exists $t \in S$ with

$$x + v + t = u + y + t.$$

If $[x, y]$ denotes the equivalence class of (x, y) then define

$$[x, y] + [x', y'] = [x + x', y + y'].$$

Then G is a group:

- $+$ is abelian and associative
- for any $x, y \in S$, $[x, x] = [y, y]$ in G. This is the zero element in G.
- -[x,y]=[y,x]

Define

$$\phi : S \to G\,;\; x \mapsto [x + x, x],$$

so that $[x, y] = \phi(x) - \phi(y)$ and the image of ϕ generates G.

Exercise Verify that these two constructions give the universal group as required and find a natural isomorphism between them. In either case, we call the resulting group G the Grothendieck group of S.

Definition 2.9 *For a ring R define the abelian group $K_0(R)$ to be the Grothendieck group of the abelian semigroup $Proj\, R$.*

Thus, according to the first construction, every element of $K_0(R)$ can be written as a difference $[P]-[Q]$ where P and Q are finitely generated projective R-modules. (Exercise: Show that every element can in fact be written as a difference $[P]-[R^n]$.)

Example 2.10 Let F be a field so that a finitely generated projective F-module is simply a finite dimensional vector space over F. Define

$$d : K_0(F) \to \mathbb{Z}\,;\ [P]-[Q] \mapsto \dim P - \dim Q.$$

Then d is well-defined since dimension is additive:

$$\dim (P \oplus Q) = \dim P + \dim Q.$$

Define

$$\eta : \mathbb{Z} \to K_0(F)\,;\ n \mapsto [F^n]$$

where for $n < 0$ we write $[F^n] = -[F^{-n}]$. Then

1. $d \cdot \eta(n) = d([F^n]) = n$
2. $\eta \cdot d([P]-[Q]) = \eta(\dim P - \dim Q) = [F^{\dim P - \dim Q}]$.

But assuming $\dim P \geq \dim Q$ we have

$$Q \oplus F^{\dim P - \dim Q} = P$$

so that

$$[F^{\dim P - \dim Q}] = [P]-[Q].$$

Hence we have

Lemma 2.11 *For a field F, dimension defines an isomorphism*

$$d : K_0(F) \to \mathbb{Z}.$$

More generally, the dimension of a finitely generated projective R-module should take values in $K_0(R)$.

It is important to restrict ourselves to finitely generated projective R-modules. Suppose in the construction of $K_0(R)$ we allow countably generated projective R-modules. For such a module P, the countable sum

$$P \oplus P \oplus P \oplus \cdots$$

is also countably generated, projective. Thus

$$P \oplus (P \oplus P \oplus \cdots) \cong P \oplus P \oplus P \oplus \cdots$$

so that

$$[P] = [P \oplus P \oplus \cdots] - [P \oplus P \oplus \cdots] = 0.$$

Hence the associated Grothendieck group would be trivial since every element is a difference of terms $[P]$.

Exercise 2.12 R-modules P, Q are said to be stably isomorphic if for some integer n we have

$$P \oplus R^n \cong Q \oplus R^n.$$

Show that if P and Q are finitely generated, projective then

$$[P] = [Q] \in K_0(R) \Leftrightarrow P \text{ and } Q \text{ are stably isomorphic.}$$

Show that finitely generated, projective $\mathbb{Z}$-modules are stably isomorphic if and only if they are isomorphic. (Use the structure theorem for finitely generated abelian groups.) Conclude that

$$K_0(\mathbb{Z}) \cong \mathbb{Z}$$

by rank.

2.3 K_0 from Idempotents

Projective modules may also be described in terms of idempotent matrices and this gives a useful alternative description of $K_0(R)$.

Suppose P is a finitely generated, projective module, and $P \oplus Q \cong R^n$. Consider the projection map

$$p : R^n \xrightarrow{\pi} P \xhookrightarrow{i} R^n.$$

Then clearly $p^2 = p$. Since R^n has a canonical basis, we may represent the map p by an $n \times n$ matrix also written p. (The matrix p acts on the *right* of R^n since R^n is a left R-module.) Thus p is an idempotent matrix, $p^2 = p$. We see then that each finitely generated, projective R-module P gives rise to an idempotent matrix p. Of course, there are many different splittings $P \oplus Q \cong R^n$, so that p is not uniquely defined.

On the other hand, if p is an idempotent $n \times n$ matrix then Rp (the image of p in R^n) is a finitely generated, projective R-module. In fact, it is easily shown that

$$Rp \oplus R(1-p) \cong R^n.$$

However, different idempotent matrices can give rise to isomorphic projective modules. The exact relationship is given by the following lemma.

Lemma 2.13 *Let p and q be idempotent matrices of size $n \times n$ and $m \times m$ respectively. Then p and q determine isomorphic projective modules if and only if there exist conjugate $N \times N$ matrices*

$$\begin{pmatrix} p & 0 \\ 0 & 0_{N-n} \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 \\ 0 & 0_{N-m} \end{pmatrix}.$$

Proof Set $N = m + n$. First we show p and q determine isomorphic modules. Clearly

$$R^n p \cong R^N \begin{pmatrix} p & 0 \\ 0 & 0_m \end{pmatrix}$$

so we may assume $n = m$, and p, q are conjugate, ie. $upu^{-1} = q$ for some $u \in GL(n, R)$. But then

$$Rq = Rupu^{-1} \cong Rpu^{-1} \cong Rp.$$

Next, suppose $\alpha : R^n p \to R^m q$ is an isomorphism. Define maps a and b by

$$a : R^n \xrightarrow{p} R^n p \xrightarrow{\alpha} R^m p \hookrightarrow R^m$$

and

$$b : R^m \xrightarrow{q} R^m q \xrightarrow{\alpha^{-1}} R^n p \hookrightarrow R^n.$$

Then

$$ab = p, \quad ba = q, \quad a = pa = aq, \quad b = qb = bp.$$

Thus

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \in GL(N, R).$$

Furthermore

$$\begin{aligned} \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} & \\ = \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} & \\ = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \sim \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} & \end{aligned}$$

We shall use the following notation

- $M(n, R) = n \times n$ matrices over R
- $M(n, R) \hookrightarrow M(n+1, R)\,;\, a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$
- $GL(n, R) =$ invertible $n \times n$ matrices over R
- $GL(n, R) \hookrightarrow GL(n+1, R)\,;\, a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.
- $M(R) = \bigcup_n M(n, R), \quad GL(R) = \bigcup_n GL(n, R)$
- $\mathrm{Idem}(R) =$ idempotents in $M(R)$
- $\mathrm{Idem}(R)_{GL(R)} =$ orbit set under conjugation action of $GL(R)$

The operation

$$\mathrm{Idem}(R) \times \mathrm{Idem}(R) \to \mathrm{Idem}(R)\,;\ (p,q) \mapsto \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

passes to the orbit set $\mathrm{Idem}(R)_{GL(R)}$. Then $\mathrm{Idem}(R)_{GL(R)}$ is an abelian semigroup. We have shown

Theorem 2.14 *The map*

$$\mathrm{Proj}\, R \to \mathrm{Idem}(R)_{GL(R)}\,;\ P \mapsto [p],$$

is an isomorphism of abelian semigroups.

Hence $K_0(R)$ is the Grothendieck group of $\mathrm{Idem}(R)_{GL(R)}$. Using this description we may obtain 'Morita equivalence' for K_0.

Theorem 2.15 *There is a natural isomorphism*

$$K_0(R) \to K_0(M_n(R)).$$

Proof

$$\mathrm{Idem}(M_n(R)) = \mathrm{Idem}(R), \quad GL(M_n(R)) = GL(R).$$

2.4 Change of Rings

Let $f : R \to S$ be a unital homomorphism of rings. Given a (left) R-module M, we may define a (left) S-module $S \otimes_f M$ where R acts on the right of S via f, ie.

$$S \times R \to S\,;\ (s,r) \mapsto sf(r).$$

Note that $S \otimes_f R \cong S$. Also, if P is a finitely generated, projective R-module then $P \oplus Q \cong R^n$ for some Q, so that

$$(S \otimes_f P) \oplus (S \otimes_f Q) \cong S \otimes_f (P \oplus Q) \cong S \otimes_f R^n \cong S^n.$$

In other words, $S \otimes_f P$ is a finitely generated, projective S-module. Thus there is defined a homomorphism

$$K_0(f) : K_0(R) \to K_0(S)\,;\ [P] \mapsto [S \otimes_f P],$$

and K_0 becomes a covariant functor from rings to abelian groups.

Example 2.16

1. For R a ring, let $R[x]$ be the ring of polynomials in the indeterminate x. Then $f : R \to R[x]\,;\ r \mapsto r$ and $g : R[x] \to R\,;\ p \mapsto p(0)$ satisfy $g \cdot f = 1_R$. Hence
$$K_0(R[x]) \cong K_0(R) \oplus \ker K_0(g).$$

2. For R a ring, π a multiplicative group, we write $R[\pi]$ for the group ring of finite formal sums $\Sigma_i r_i g_i$, where $r_i \in R$, $g_i \in \pi$. Then $f : R \to R[\pi]\,;\, r \mapsto r \cdot 1$ and $g : R[\pi] \to R\,;\, \Sigma r_i g_i \mapsto \Sigma r_i$ satisfy $g \cdot f = 1_R$. Hence

$$K_0(R[\pi]) \cong K_0(R) \oplus \ker K_0(g).$$

3. For a ring R with unit 1, let $\iota : \mathbb{Z} \to R\,;\, 1 \mapsto 1$. Thus there is defined a map

$$\iota_* = K_0(\iota) : \mathbb{Z} = K_0(\mathbb{Z}) \to K_0(R).$$

Definition 2.17 *The reduced K_0-group of the ring R is defined by*

$$\tilde{K}_0(R) = K_0(R)/\iota_*(\mathbb{Z}).$$

Example 2.18

1. For F a field, $\tilde{K}_0(F) = 0$.
2. $\tilde{K}_0(\mathbb{Z}) = 0$.

In the case $f : R \to S$ is surjective there is an alternative description of the map $f_* : K_0(R) \to K_0(S)$. Let $J \lhd R$ be the kernel of f, so that J is a two-sided ideal of R.

Suppose P is a finitely generated, projective R-module. Then

$$JP = \{\Sigma r_i x_i \,|\, r_i \in J, x_i \in P\}$$

is a submodule of P. Thus the quotient P/JP is an R-module. Define

$$S \times P/PJ \to P/PJ\,;\, (s, [x]) \mapsto [rx]$$

where $f(r) = s$. This is easily seen to be well-defined and makes P/PJ into an S-module, which we write $\bar{P}$. Since the map is bilinear over R, there is an induced map

$$S \otimes_f P \to \bar{P}\,;\, s \otimes p \mapsto \overline{rp}, \quad f(r) = s.$$

This map is an isomorphism with inverse $\bar{p} \mapsto 1 \otimes p$.

For rings S, T the cartesian product $R = S \times T$ is the ring obtained from componentwise addition and multiplication.

Exercise 2.19 Show that

$$K_0(S \times T) \cong K_0(S) \oplus K_0(T).$$

(Hint: In general, there is no splitting map $i : S \to S \times T$. However, show that such a splitting map does exist on the level of K_0.)

The group of units $R^\times$ of a ring R is given by

$$R^\times = \{r \in R \,|\, rs = sr = 1, \text{ for some } s \in R\}.$$

Proposition 2.20 *(Nakayama's Lemma) Let R be a ring and $J \lhd R$ a two-sided ideal such that $1 + J \subset R^\times$. If M is a non-trivial, finitely generated R-module, then $JM \neq M$.*

Proof Let $x_1, \ldots, x_n$ be a set of generators of M with $x_n \notin Rx_1 + \cdots + Rx_{n-1}$. If $JM = M$ then $x_n \in JM$ so that

$$x_n = j_1 x_1 + \cdots + j_n x_n$$

and so

$$(1 - j_n)x_n \in Rx_1 + \cdots Rx_{n-1}.$$

But $1 - j_n \in R$ so $x_n \in Rx_1 + \cdots + Rx_{n-1}$, a contradiction.

Corollary 2.21 *With J as above, if N is a submodule of M with $N + JM = M$, then $N = M$.*

Proof If $N + JM = M$ then $J(M/N) = M/N$ and so $M/N = 0$.

Theorem 2.22 *Let $f : R \to S$ be a surjective ring homomorphism with $1 + \ker f \subset R^\times$. Then $f_* : K_0(R) \to K_0(S)$ is injective.*

Proof Let $J = \ker f$. Suppose P, Q are finitely generated, projective R-modules such that $\bar{P} = P/JP$ and $\bar{Q} = Q/JQ$ are isomorphic as S-modules. Let $\theta : \bar{P} \to \bar{Q}$ be such an isomorphism. Then θ is also an R-module isomorphism. Since P is projective, there exists a map $\theta' : P \to Q$ over the isomorphism θ. We show that θ' is also an isomorphism. We have

$$Q = \theta' P + \ker(\pi_Q) = \theta' P + JQ,$$

where $\pi_Q : Q \to \bar{Q}$ is the natural map. Hence, by the corollary to Nakayama's lemma, $Q = \theta' P$, and θ' is onto.

Since Q is projective, there exists a splitting map $i : Q \to P$ such that $\theta' \cdot i = 1_Q$. Hence

$$P \cong iQ \oplus \ker\theta'.$$

But $\ker\theta' \subset \ker(\pi_Q \cdot \theta') = \ker(\theta \cdot \pi_P) = \ker(\pi_P) = JP$. Hence again, by the corollary to Nakayama's lemma, $P = iQ$ and so $\ker\theta' = 0$.

We have shown that $\theta' : P \to Q$ is an isomorphism. Suppose then $[P] - [Q] \in \ker f_*$ so that $[\bar{P}] - [\bar{Q}] = 0 \in K_0(S)$, ie. $\bar{P}$, $\bar{Q}$ are stably isomorphic over S

$$\bar{P} \oplus S^m \cong \bar{Q} \oplus S^n$$

for some $m, n \in \mathbb{Z}$. Then

$$\overline{P \oplus R^m} \cong \overline{Q \oplus R^n}.$$

From the previous argument $P \oplus R^m \cong Q \oplus R^n$ and $[P] = [Q] \in K_0(R)$. Thus $f_* : K_0(R) \to K_0(S)$ is injective, as required.

Definition 2.23 *A ring R is said to be local if the non-units of R form a two-sided ideal of R, ie. $R - R^\times \lhd R$.*

In fact, it is sufficient for $R - R^\times$ to be an additive group.

Example 2.24

1. $\mathbb{Z}/p^n$, for p prime
2. $k[[x]]$, the ring of formal power series over a field k
3. $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid b \neq 0, p \nmid b\}$, the ring $\mathbb{Z}$ localised at the prime p

For a local ring R, set $J := R - R^\times$, the ideal of non-units.

Lemma 2.25 *The ideal J is the unique maximal ideal in R and $1 + J \subset R^\times$.*

Proof Clearly J is maximal. For if $J \subsetneq I \subset R$ then there exists $x \in I$ such that $x \notin J$. Thus x is a unit, and hence $I = R$. Also J is unique, for if I is maximal and there exists $x \in I\backslash J$, then $I = R$ so that we may assume $I \subset J$ and hence $I = J$. Lastly, if $1 + x \notin R^\times$ for $x \in J$, then $1 + x \in J$ and so $1 \in J$, a contradiction.

Lemma 2.26 *If R is local then $K_0(R) \cong \mathbb{Z}$ generated by $[R]$.*

Proof Since J is maximal, R/J is a skew field so $K_0(R/J) \cong \mathbb{Z}$. But $1 + J \subset R^\times$ so the natural map $K_0(R) \to K_0(R/J)$ is injective. This map takes $[R]$ to $[R/J] \neq 0$. Hence $K_0(R) \cong \mathbb{Z}$.

Exercise 2.27 Show that $K_0(\mathbb{Z}/m) \cong \mathbb{Z}^n$, where n is the number of distinct primes dividing m.

For a local ring R, we may regard the isomorphism $K_0(R) \cong \mathbb{Z}$ as a rank function on finitely generated, projective R-modules. In particular, it shows that finitely generated, projective R-modules are stably free, ie. $P \oplus R^n \cong R^N$ for some $n, N \in \mathbb{Z}$. In this case, we say rank $P = N - n$.

The following stronger result is true.

Theorem 2.28 *Let R be a local ring. Then finitely generated, projective R-modules are free.*

Proof Let R be a local ring with $J = R - R^\times$, the unique maximal ideal. Suppose P is a finitely generated, projective R-module with $P \oplus Q \cong R^n$. Set $\bar{R} = R/J$, a skew field, and $\bar{P} = P/JP$, $\bar{Q} = Q/JQ$ so that $\bar{P} \oplus \bar{Q} \cong \bar{R}^n$.

Since $\bar{R}$ is a field, we may choose a basis $\bar{x}_1, \ldots, \bar{x}_n$ of $\bar{R}^n$ such that $\bar{x}_1, \ldots, \bar{x}_k$ is a basis for $\bar{P}$ and $\bar{x}_{k+1}, \ldots, \bar{x}_n$ is a basis for $\bar{Q}$. Choose elements $x_1, \ldots, x_k$

in P and $x_{k+1}, \dots, x_n$ in Q lifting the $\bar{x}_i$. We claim that $x_1, \dots, x_n$ is a basis for R^n, from which the result will follow.

Suppose then $x_i = (a_{i1}, \dots, a_{in}) \in R^n$. We must show that the matrix $A = (a_{ij})$ has a two-sided inverse. Since $\bar{x}_1, \dots, \bar{x}_n$ is a basis for $\bar{R}^n$, $\bar{A} = (\bar{a}_{ij})$ has an inverse $\bar{B}$ so that $\bar{B}\bar{A} = I$, ie. $BA \equiv I \bmod J$. Consider the $(1,1)$ entry in BA. It is congruent to 1 mod J and hence is invertible. Thus by elementary row operations we may introduce zeros below this entry. Similarly for the $(2,2)$ entry. Continuing in this way, there is an invertible matrix C such that CBA is diagonal with entries congruent to 1 mod J, and hence is invertible. In particular, A admits a left inverse. Similarly, A admits a right inverse, and these two inverses are necessarily equal.

We conclude from the theorem that for local rings R, there is a rank function

$$\text{rank} : \text{Proj}\, R \to \mathbb{Z};\ P \mapsto \text{rank}\, P,$$

where $\text{rank}\, R^n = n$.

How can one construct a rank function in more general circumstances? Suppose R is a ring and we are given a ring homomorphism $f : R \to S$, where S is a ring such that $K_0(S) \cong \mathbb{Z}$ generated by $[S]$, eg. a local ring.

Consider the composition

$$\mathbb{Z} \xrightarrow{\iota} R \xrightarrow{f} S$$

inducing

$$K_0(\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\iota_*} K_0(R) \xrightarrow{f_*} K_0(S) \cong \mathbb{Z}.$$

Then $f_* \cdot \iota_* = 1_{\mathbb{Z}}$. Thus

$$K_0(R) \cong \text{image}\iota_* \oplus \ker f_*,$$

or equivalently

$$K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R).$$

We may think of $f_* : K_0(R) \to \mathbb{Z}$ as defining a rank function (the rank at f) of finitely generated, projective R-modules. Of course, this function depends on the choice of f and S. Similarly, the splitting of $K_0(R)$ is not canonical.

Example 2.29 Let $P \lhd R$ be a prime ideal, so that R/P is an integral domain. If F is the field of fractions of R/P then we have a natural map $\omega : R \to F$ and we obtain $\omega_* : K_0(R) \to K_0(F) \cong Z$. For M a finitely generated, projective R-module, define

$$\text{rank}_P(M) = \omega_*([M]) = \dim(F \otimes_\omega M).$$

2.5 Topological K-theory

All topological spaces are assumed to be Hausdorff and compact, unless otherwise specified.

For a topological space X let $C(X)$ denote the ring of continuous functions from X into either $\mathbb{R}$ or $\mathbb{C}$, with pointwise addition and multiplication.

Recall a (locally trivial) n-dimensional vector bundle (over $\mathbb{R}$ or $\mathbb{C}$) consists of a map $p : E \to X$ with structure

1. For all $x \in X$, $p^{-1}(x)$ is an n-dimensional vector space.
2. There are structure maps
$$+ : E \times_p E \to E, \quad \times : \mathbb{R} \times E \to E$$
restricting on each fibre to the given vector space structure.
3. For each $x \in X$ there is an open neighbourhood $U \ni x$ and a fibrewise linear isomorphism $p^{-1}(U) \to U \times \mathbb{R}^n$ over U.

All the standard vector space operations can be performed fibrewise and extended to vector bundles.

Let $\mathrm{Vect}(X)$ be the set of isomorphism classes of finite dimensional vector bundles on X. Then $\mathrm{Vect}(X)$ is a commutative ring with addition Whitney sum

$$\mathrm{Vect}(X) \times \mathrm{Vect}(X) \to \mathrm{Vect}(X)\,;\ (E_1, E_2) \mapsto E_1 \oplus E_2.$$

If $p : E \to X$ is a vector bundle, we write $\Gamma(E)$ for the set of continuous sections of E

$$\Gamma(E) = \{s : X \to E \,|\, p \cdot s = 1_X\}.$$

Then $\Gamma(E)$ is a $C(X)$-module in the obvious way. The following theorem is due to Swan [33].

Theorem 2.30 *The functor Γ induces a map*

$$\Gamma : \mathit{Vect}(X) \to \mathit{Proj}\, C(X).$$

The map is an isomorphism of semigroups.

Proof Let $p : E \to X$ be a vector bundle. First we must show that $\Gamma(E)$ is finitely generated and projective over $C(X)$. Choose a finite open cover $\{U_i\}$ of X such that $E_{|U_i} \cong U_i \times R^n$. The canonical sections of $U_i \times R^n$ determine sections $\{e_1^i, \ldots, e_n^i\}$ of $E_{|U_i}$. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{U_i\}$.

Suppose $v \in \Gamma(E)$. Then $v_i := v_{|U_i} = \Sigma_j \alpha_j^i e_j^i$, for some $\alpha_j^i \in \mathbb{R}$. So

$$v = \Sigma_i \lambda_i v_i = \Sigma_i \lambda_i (\Sigma_j \alpha_j^i e_j^i) = \Sigma_{ij} \alpha_j^i (\lambda_i e_j^i).$$

Thus $\{\lambda_i e_j^i\}$ generates $\Gamma(E)$ as a $C(X)$-module, ie. $\Gamma(E)$ is finitely generated.

To show that $\Gamma(E)$ is projective, choose generators $s_1, \ldots, s_k$ of $\Gamma(E)$. This determines an epimorphic bundle map

$$\phi : X \times \mathbb{R}^k \to E\,;\; (x; v_1, \ldots, v_k) \mapsto \Sigma v_i s_i(x).$$

Then we have a short exact sequence

$$0 \to E' \to X \times R^k \to E \to 0.$$

Choose the standard metric for $X \times R^k$ and any metric for E (using partitions of unity). Let ϕ^* be the adjoint of ϕ so that

$$\langle \phi v, w \rangle = \langle v, \phi^* w \rangle \text{ for all } v \in X \times \mathbb{R}^k, w \in E.$$

Then ϕ^* is injective and $\phi^* : E \to E^\perp$ so

$$E \oplus E^\perp \cong X \times R^k$$

and

$$\Gamma(E) \oplus \Gamma(E^\perp) \cong \Gamma(X \times \mathbb{R}^k) = C(X)^k.$$

Suppose now P is a finitely generated, projective R-module and

$$P \oplus Q \cong C(X)^n = \Gamma(X \times R^r).$$

Consider then

$$E = \{(x, v) \in X \times \mathbb{R}^k \mid v = f(x) \text{ for some } f \in P\}.$$

Let $p : E \to X$ be the projection onto X. Then E clearly satisfies the first two vector bundle properties. It remains to show E is locally trivial.

Let $x \in X$ and choose $e^1, \ldots, e^r \in P$ such that $\{e^1(x), \ldots, e^r(x)\}$ is a basis for $E_x = p^{-1}(x)$. Since linear independence is an open condition, there exists an open neighbourhood U of x such that $\{e^1(y), \ldots, e^r(y)\}$ is linearly independent for all $y \in U$. Similarly, for $f^1, \ldots, f^{n-r} \in Q$ over some open subset V of x. Thus $e^1, \ldots, e^r, f^1, \ldots, f^{n-r}$ are linearly independent over $U \cap V$. Dimension counting (in $P \oplus Q$) we conclude that they form a basis over $U \cap V$. Thus $e^1, \ldots, e^r$ trivialise E over $U \cap V$.

For a compact, Hausdorff space X we write $K^0(X)$ for the Grothendieck group of $Vect(X)$. Then we have

Corollary 2.31 *There is a natural isomorphism*

$$K^0(X) \to K_0(C(X)).$$

The abelian group $K^0(X)$ is the natural target for additive invariants of vector bundles over X.

Theorem 2.32 *Let $f, g : Y \to X$ with $f \simeq g$ (homotopic) and Y paracompact. Then for all vector bundles E on X, we have $f^*E \cong g^*E$.*

Corollary 2.33 *If $X \cong Y$ are homotopy equivalent then Vect(X) $\cong$ Vect(Y). If X is contractible, paracompact then every vector bundle on X is trivial.*

We conclude from the homotopy theorem that K^0 is a contravariant functor from (paracompact) spaces and homotopy classes of maps to abelian groups.

Assume X is connected.

Example 2.34 The point map $X \to *$ induces a homomorphism

$$K^0(*) \cong \mathbb{Z} \to K^0(X).$$

If X is pointed, then the inclusion $* \to X$ determines a homomorphism

$$K^0(X) \to K^0(*) \cong \mathbb{Z}.$$

Thus for a connected space, $\mathbb{Z}$ is canonically a direct summand in $K^0(X)$ and we write

$$K^0(X) \cong \mathbb{Z} \oplus \tilde{K}^0(X).$$

Write $\mathrm{Vect}_n(X)$ for the set of isomorphism classes of n-dimensional vector bundles on X. Then if $L \in \mathrm{Vect}_1(X)$ denotes the 1-dimensional trivial line bundle over X we have stabilisation maps

$$\cdots \mathrm{Vect}_{n-1}(X) \stackrel{\oplus L}{\to} \mathrm{Vect}_n(X) \stackrel{\oplus L}{\to} \mathrm{Vect}_{n+1}(X) \to \cdots.$$

The maps

$$\mathrm{Vect}_n(X) \to \tilde{K}^0(X)\,;\ E \mapsto [E] - n$$

are compatible with stablisation, so that there is defined a map

$$\lim_n \mathrm{Vect}_n(X) \to \tilde{K}^0(X).$$

Theorem 2.35 *The map*

$$\lim_n \mathit{Vect}_n(X) \to \tilde{K}^0(X).$$

is a 1-1 correspondence.

Proof Every element $\tilde{K}^0(X)$ can be written as a difference $[E^n] - [F^n]$. Choose E_1^m, F_1^m such that

$$E^n \oplus E_1^m \cong n + m \cong F^n \oplus F_1^m.$$

Then

$$[E^n] - [F^n] = [E^n] - [E^n \oplus E_1^m] - [F^n] + [F^n \oplus F_1^m] = [E^n \oplus F_1^m] - [n+m].$$

Hence the map is onto. A similar argument shows that it is into.

Corollary 2.36

$$[X, BGL(\mathbb{R})] \cong \tilde{K}^0(X).$$

For example, if $X = S^n$, then by Bott periodicity, the reduced groups $\tilde{K}^0(S^n)$ are 8- periodic in n.

2.6 Wall's Finiteness Obstruction

We say a topological space is homotopically finite if it is homotopy equivalent to a finite CW-complex.

Example 2.37 A compact smooth manifold is homotopically finite. To see this, choose a Morse function and consider the associated finite CW-decomposition.

A compact topological manifold is also homotopically finite, but this is much harder to prove.

Definition 2.38 *A space X is homotopy dominated by a space Y if there are maps $d : Y \to X$ and $u : X \to Y$ such that $d \cdot u \simeq 1_X$ The domination is pointed if $d \cdot u \simeq 1_X$ is pointed.*

A space X is said to be finitely dominated if there is a finite CW-complex which dominates X.

Clearly a homotopically finite space is finitely dominated, but in general the converse is not true. We shall prove [34]:

Theorem 2.39 *(Wall) Let X be a finitely dominated space. Then there is an obstruction*

$$\sigma(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

such that X is homotopically finite if and only if $\sigma(X) = 0$.

Roughly speaking, a homotopically finite space corresponds to a free, finitely generated module, while a finitely dominated space, which is a homotopy split summand of a finite space, corresponds to a finitely generated, projective module.

Lemma 2.40 *Every compact topological manifold is finitely dominated.*

Proof Let M^n be a compact topological manifold. Cover M by open balls B_i, $i = 1, \ldots, s$ and for each i let $\phi_i : M \to S^n$ be a map that sends B_i homeomorphically onto S^n − north pole, and $M - B_i$ onto the north pole. Then

$$\prod \phi_i : M \hookrightarrow \prod S^n \hookrightarrow \mathbb{R}^{(n+1)s}.$$

We conclude that M embeds in Euclidean space. Suppose then $M^n \subset \mathbb{R}^k$. We construct a retraction from a neighbourhood U of M onto M. Assume by induction we have a neighbourhood U_l of M and a retraction

$$r_l : (U_l - M)^{(l)} \cup M \to M,$$

where $(U_l - M)^{(l)}$ is an l-skeleton of $U_l - M$ which gets finer and finer as we apporach M. Let $\{\Delta_i\}$ be the $(l+1)$-simplices of $U_l - M$, so that the diameters of $r_l(\partial\Delta_i)$ get smaller and smaller as i tends to infinity. Since M is locally Euclidean, $r_{l|\partial\Delta_i}$ extends to a map $r_{l+1} : \Delta_{l+1} \to M$ for i large. Choose U_{l+1}

small enough so that r_{l+1} is defined on $U_{l+1}^{(l+1)}$. This completes the induction step.

Now suppose $r : U \to M$ is such a retraction, and take any finite polyhedron K with $M \subset K \subset U$. Clearly, $d = r_{|K}$ is a domination of M with right inverse $u = i : M \hookrightarrow K$.

Theorem 2.41 *(Mather) If X is dominated by an n-dimensional complex, then X is homotopy equivalent to an $(n+1)$-dimensional complex.*

We first quote without proof:

Lemma 2.42 *For a map $f : X \to Y$, let $M(f) = Y \cup_f (X \times I)$ be the mapping cylinder of f. Then*

1. *If $f_1, f_2 : X \to Y$ are homotopic, then $M(f_1)$ and $M(f_2)$ are homotopy equivalent rel $X \cup Y$.*
2. *If $f : X \to Y$ and $g : Y \to Z$ then*

$$M(g \cdot f) \cong M(f) \cup_Y M(g).$$

We now prove Mather's theorem:

Proof Let $d : K \to X$ be a domination of X with right inverse $u : X \to K$. Suppose K is n-dimensional. Since $d \cdot u \simeq 1_X$ we have

$$X \times \mathbb{R}^1 = \cup_{i=-\infty}^{\infty} M(1_X) \cong \cup_{i=-\infty}^{\infty} M(u) \cup_K M(d) \cong \cup_{i=-\infty}^{\infty} M(d) \cup_X M(u) \cong \cup_{i=-\infty}^{\infty} M(u \cdot d).$$

But $M(u \cdot d)$ is $(n+1)$-dimensional. Hence result.

Consider the above construction, but on the torus $X \times S^1$ instead of $X \times \mathbb{R}$. By the same kind of argument we can show that $X \times S^1$ is homotopy equivalent to the mapping torus of $u \cdot d : K \to K$. We have

Lemma 2.43 *If X is finitely dominated, then $X \times S^1$ is homotopy finite.*

We now introduce the main construction [34] for building homotopy finite spaces out of finitely dominated spaces (modulo an obstruction) . The method of attack is to try to increase the connectivity of the domination map $d : K \to X$ by adding cells to K to kill the homotopy groups of d until we get a homotopy equivalence. Notice that since K and X are finite dimensional (but X may have infinitely many cells in each dimension) this process (if possible) will take a finite number of steps.

We begin by modifying the fundamental groups. Consider the maps

$$d_* : \pi_1(K) \to \pi_1(X), \quad u_* : \pi_1(X) \to \pi_1(K),$$

such that $d_* \cdot u_* = 1$. Note that $\pi_1(K)$ is finitely generated, since K is finite. Also d_* is onto. To make d_* an isomorphism we add 2-cells to K to kill the kernel of d_*. We must add only a *finite* number of 2-cells. Recall that the effect on $\pi_1(K)$ of adding a 2-cell via an attaching map $g \in \pi_1(K)$ is to kill the normal subgroup generated by g.

Lemma 2.44 *$\ker d_*$ is normally generated in $\pi_1(K)$ by finitely many elements.*

Proof Let $\{g_i\}$ be a finite generating set for $\pi_1(K)$. (For example, choose the 1-cells.) Set $\alpha = u_* \cdot d_* : \pi_1(K) \to \pi_1(K)$, and

$$P = \{g_i \alpha(g_i^{-1})\} \subset \ker(d_*),$$

noting that

$$d_*(g_i \alpha(g_i^{-1})) = d_*(g_i) d_* \alpha(g_i^{-1}) = d_*(g_i) d_*(g_i^{-1}) = 1.$$

Consider the length 2 word $g_i g_j$. Then

$$g_i g_j \alpha((g_i g_j)^{-1}) = g_i g_j \alpha(g_j^{-1}) \alpha(g_i^{-1}) = \{g_i [g_j \alpha(g_j^{-1})] g_i^{-1}\} g_i \alpha(g_i) \in N(P)$$

where $N(P) \subset \ker d_*$ is the normal closure of P.

Hence, by induction on word length

$$g\alpha(g^{-1}) \in N(P), \quad \text{for all } g \in G.$$

But if $g \in \ker d_*$ then $\alpha(g^{-1}) = u_* d_*(g^{-1}) = u_*(1) = 1$. Hence

$$g = g\alpha(g^{-1}), \quad \text{for all } g \in \ker d_*$$

and so $\ker d_* \subset N(P)$.

We conclude that $\ker d_* = N(P)$ and $\ker d_*$ is finitely normally generated.

Proposition 2.45 *If $d : K \to X$ is a finite domination of CW-complexes, we can attach finitely many 2-cells to K to form a complex $\bar{K}$ and extend d to a map $\bar{d} : \bar{K} \to X$ such that $\bar{d} : \pi_1(\bar{K}) \cong \pi_1(X)$.*

Proof Choose finitely many attaching maps $\alpha_i : S^1 \to K$ so that $\{[\alpha_i]\}$ normally generates $\ker d_*$. Set

$$\bar{d} : \bar{K} = K \cup \cup_{\alpha_i} e_i^2 \to X.$$

Then $\bar{d}_*$ is onto, since d_* is onto. Claim $\bar{d}_*$ is into. Suppose then $[\omega] \in \ker \bar{d}_*$. Then by the cellular approximation theorem, we may assume ω maps into the 1-skeleton of $\bar{K}$. This means that ω lifts up to homotopy to a map $[\omega'] \in \ker d_*$. Hence $[\omega'] \in N(\{\alpha_i\})$. But

$$j_*(g^{-1}\alpha_i g) = j_*(g^{-1}) j_*(\alpha_i) j_*(g_i) = j_*(g^{-1}) j_*(g) = 1,$$

since j_* kills α_i. (Here j is the natural map $j : K \to \bar{K}$.)

Hence $j_*([\omega']) = 0$, ie. $[\omega] = 0$.

By the previous lemma we may assume $d : K \to X$ induces an isomorphism on fundamental groups. We want to continue adding cells to K to make d more and more highly connected. The first step is to show finite generation of the higher homotopy groups.

Assume X is a CW-complex with fundamental group $\pi = \pi_1(X)$. Write $\mathbb{Z}\pi$ for the group ring. Let $\tilde{X}$ be the universal cover of X so that passing to homology $H_i(\tilde{X})$ is a $\mathbb{Z}\pi$-module. If X is a finite complex then $H_i(\tilde{X})$ is a finitely generated $\mathbb{Z}\pi$-module.

Note that $\pi_i(\tilde{X})$ is also a $\mathbb{Z}\pi$-module - we don't have to worry about basepoints since $\tilde{X}$ is simply connected.

Let $d : K \to X$ be a finite domination inducing an isomorphism on fundamental groups π. Write $\tilde{K}$ and $\tilde{X}$ for the universal covers, and $\tilde{d} : \tilde{K} \to \tilde{X}$ for any lift of d. We may choose a lift $\tilde{u} : \tilde{X} \to \tilde{K}$ such that $\tilde{d} \cdot \tilde{u} \simeq 1$.

Consider then the homology exact sequence of $\tilde{d}$

$$\cdots \to H_2(\tilde{K}) \xrightarrow{d_*} H_2(\tilde{X}) \to H_2(\tilde{X}, \tilde{K}) \to 0.$$

But d_* is onto, since it has a right inverse. Hence $H_2(\tilde{X}, \tilde{K}) = 0$, and there are short exact sequences

$$0 \to H_{k+1}(\tilde{X}, \tilde{K}) \to H_k(\tilde{K}) \to H_k(\tilde{X}) \to 0$$

for $k \geq 2$.

Theorem 2.46 *Let $d : K \to X$ be a finite domination between CW-complexes. Suppose*

$$\pi_k(d) = 0, \quad 0 \leq k \leq n-1, \ n \geq 2.$$

Then we can attach finitely many n-cells to K to form $\bar{K}$ and extend d to a map $\bar{d} : \bar{K} \to X$ so that

$$\pi_k(\bar{d}) = 0, \quad 0 \leq k \leq n.$$

Proof We must show that $\pi_n(d)$ is finitely generated as a $\mathbb{Z}\pi$-module. The result will then follow as above by adding finitely many n-cells to K.

By the relative Hurewicz theorem

$$\pi_n(d) \cong \pi_n(\tilde{d}) \cong H_n(\tilde{X}, \tilde{K}).$$

Thus we must show $H_n(\tilde{X}, \tilde{K})$ is finitely generated as a $\mathbb{Z}\pi$-module, where

$$0 \to H_n(\tilde{X}, \tilde{K}) \to H_{n-1}(\tilde{K}) \xrightarrow{d_*} H_{n-1}(\tilde{X}) \to 0$$

is split by u_*.

In general, if C is a chain complex of free, finitely generated R-modules, it does *not* follow that the homology groups $H_i(C)$ are finitely generated. However:

Lemma 2.47 *If C is a chain complex of free, finitely generated R-modules and $H_i(C) = 0$, $i < n$, then $H_n(C)$ is finitely generated.*

Proof Since $H_n(C)$ is a quotient of the kernel of $\partial_n : C_n \to C_{n-1}$ it is sufficient to show that $\ker(\partial_n)$ is finitely generated. Clearly $\ker(\partial_0)$ is free and finitely generated. By

$$0 \to \ker(\partial_1) \to C_1 \to \ker(\partial_0) \to H_0(C) = 0$$

we see that $\ker(\partial_1)$ is a split summand of C_1 so that it is finitely generated, projective. The result then follows by induction.

Note that we could equally have shown that $\ker(\partial_n)$ is stably free.

We must show $H_n(\tilde{X}, \tilde{K}) \cong H_n(C)$ for some chain complex C satisfying the conditions of the above lemma. Let $\tilde{\alpha} = \tilde{u} \cdot \tilde{d} : \tilde{K} \to \tilde{K}$, and set C to be the cellular chain complex of the mapping cone of $\tilde{\alpha}$. Thus C is free and finitely generated in each dimension. By assumption

$$H_i(C) = 0, \quad i < n - 1$$

and

$$\cdots \to H_{n-1}(\tilde{K}) \xrightarrow{\tilde{\alpha}_*} H_{n-1}(\tilde{K}) \to H_{n-1}(C) \to 0.$$

In particular, C satisfies the conditions of the above lemma, so that $H_{n-1}(C)$ is finitely generated. We must show $H_n(\tilde{X}, \tilde{K}) \cong H_{n-1}(C)$.

By the above, $H_{n-1}(C) \cong \operatorname{coker}(\tilde{\alpha}_*)$. But $\tilde{\alpha} \cdot \tilde{\alpha} \simeq \tilde{\alpha}$ so that

$$0 \to \ker(\tilde{\alpha}_*) \to H_{n-1}(C) \to \operatorname{im}(\tilde{\alpha}_*) \to 0$$

splits via $\tilde{\alpha}_*$. Thus

$$\operatorname{coker}(\tilde{\alpha}_*) \cong \ker(\tilde{\alpha}_*) \cong \ker(\tilde{d}_*) = H_n(\tilde{X}, \tilde{K}),$$

and we are done.

We have shown that any finite domination of X by an n-dimensional complex may be improved to a finite domination $d : K \to X$ where

1. $\dim K = n$,
2. $d_* : \pi_i(K) \to \pi_i(X)$ is an isomorphism, $i < n$.

Note also that since $\dim K = n$, we have $H_i(K) \cong H_i(X) = 0$, $i > n$.

We wish to add $(n+1)$-cells to K to kill the kernel of $d_* : \pi_n(K) \to \pi_n(X)$, but *without* introducing new non-trivial homology in H_{n+1}. This is certainly possible if $\ker(\pi_n(K) \to \pi_n(X))$ is free and finitely generated. In fact, we only require $\ker(d_*)$ to be stably free, since we can always replace K by $K \vee \vee_i S^n$.

Lemma 2.48 *$\ker(d_*)$ is finitely generated, projective as a $\mathbb{Z}\pi$-module.*

Proof As above, $\ker(d_*) \cong H_{n+1}(\tilde{X}, \tilde{K})$, and $H_i(\tilde{X}, \tilde{K}) = 0$, $i < n+1$. By Mather's theorem, X is homotopy equivalent to an $(n+1)$-dimensional complex, so that $\ker(d_*)$ is the $(n+1)$st-homology of an $(n+1)$-dimensional complex with trivial homology, $i < n+1$.

Suppose then C is an $(n+1)$-dimensional chain complex of free $\mathbb{Z}\pi$-modules (not necessarily finitely generated) and $H_i(C) = 0$, $i < n+1$. Then as above we can show $H_{n+1}(C)$ is projective. Since $C_{n+2} = 0$, $H_{n+1}(C) = \ker(\partial_{n+1} : C_{n+1} \to C_n)$. Clearly $\ker(\partial_0) = C_0$ is free. By

$$0 \to \ker(\partial_i) \to C_i \to \ker(\partial_{i-1}) \to H_{i-1}(C) = 0,$$

$i \leq n+1$, we see that $\ker(\partial_{n+1})$ is projective. As before, it is also finitely generated.

We wanted to show that $\ker(d_*)$ is stably free, but we have only been able to show that it is finitely generated, projective.

Recall that $\tilde{K}_0(\mathbb{Z}\pi)$ is isomorphic to the set of equivalence classes of finitely generated, projective $\mathbb{Z}\pi$-modules, where $P \sim Q$ if and only if there exist finitely generated, free F_1, F_2 such that

$$P \oplus F_1 \cong Q \oplus F_2.$$

In other words, $P \sim Q$ if and only if they are stably isomorphic.

It follows that the finitely generated, projective $\mathbb{Z}\pi$-module $\ker(d_*) = H_{n+1}(\tilde{d}_*)$ determines an element in $\tilde{K}_0(\mathbb{Z}\pi)$. It can be shown that this class is independent of the particular way in which the cells were added to K in the above process.

Definition 2.49 *Let X be a finitely dominated space. The Wall finiteness obstruction*

$$\sigma(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$$

is defined to be the class $\sigma(X) = [H_{n+1}(\tilde{d}_*)]$, where $d : K \to X$ is any finite domination such that $\dim K = n$ and $\pi_i(d) = 0$, $i < n$.

We have shown

Theorem 2.50 *(Wall) If X is a finitely dominated space, the element $\sigma(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$ vanishes if and only if X has the homotopy type of a finite complex.*

Corollary 2.51 *Every simply connected, finitely dominated space is homotopy equivalent to a finite complex.*

Corollary 2.52 *Every simply connected topological manifold is homotopy equivalent to a finite complex.*

Following the construction carefully, we see that if X is finitely dominated by an n-dimensional complex, then X is homotopy equivalent to an $(n+1)$-dimensional finite complex. In fact, this can be improved to an n-dimensional finite complex.

2.7 Application to Embedding Theory

Let X and M be compact manifolds without boundary, and suppose $X \subset M$ as a closed subset. We are interested in constructing 'nice' manifold neighbourhoods for X in M. The strongest situation is

Example 2.53 (Tubular Nighbourhood Theorem) Let X be smoothly embedded in M. Then there exists a smooth vector bundle E over X diffeomorphic to a neighbourhood of X in M.

Choose a metric on E and let D be the associated disk bundle, with boundary ∂D. Let N be the manifold $M - \mathrm{int} D$ with boundary ∂D, so that $M = N \cup_{\partial D} D$. Let $N^\circ = N - \partial N$ so that N° is an open manifold. Using a collar of ∂N in N we see that N° is homotopy equivalent to N. In particular, even though N° is an open manifold, it is homotopy finite.

Using the tubular neighbourhood, there is also defined a homeomorphism $M - X \cong N^\circ$. (Isotope $M - X$ away from X, along the fibres of the tubular neighbourhood, into N°.) Thus, for a smooth embedding, the open manifold $M - X$ is homotopy finite.

For general, non-smooth embeddings we do not have such an analysis of $M - X$. In fact, the structure of $M - X$ can be very wild. In the topological setting, even locally flat submanifolds do not in general admit topological bundle neighbourhoods [27]. Instead, we must be satisfied with something rather weaker.

Definition 2.54 *Let X,M be compact, topological manifolds with empty boundary. Suppose $X \subset M$ as a closed subset. A mapping cylinder neighbourhood (MCN) of X in M consists of a closed manifold V together with a map $p : V \to X$ such that the mapping cylinder of p, $M(p)$, is homeomorphic to a neighbourhood of X in M.*

Clearly, the mapping cylinder $M(p)$ is then a topological manifold with boundary V. This definition does not address how nice the map p is, or more precisely, whether it has some kind of bundle structure. However, the existence of a mapping cylinder neighbourhood provides much of the functionality of a tubular neighbourhood. (In practice, p is almost never a topological bundle map, but instead can often be shown to be a *manifold approximate fibration*, a kind of local homotopy bundle. This is the case, for example, when X is tame in M, modulo K-theory obstructions citeweinberger.)

Example 2.55 A tubular neighbourhood of a smooth embedding is a mapping cylinder neighbourhood.

It is non-trivial to prove

Theorem 2.56 *Suppose for all $x \in X$ there exists arbitrarily small neighbourhoods $V \ni x$ in M such that the local fundamental groups $\pi_1(V - X) = 0$, ie. X is 1-LC. Then X admits a mapping cylinder neighbourhood.*

The existence of the MCN follows from Quinn's Tame End Theorem [22] for non-trivial local π_1, where a K-theoretic obstruction to an MCN is given. For trivial local π_1, as above, the K-groups are all trivial, and hence so is the obstruction. The proof of Quinn's theorem procedes by a careful accounting of the Wall finiteness obstruction of $(M-X) \cap V$ for arbitrarily small neighbourhoods V in M of points $x \in X$. These local obstructions are patched together to form an element in the homology of X with coefficents in K-theory. This element is then the sole obstruction to an MCN.

The proof of Quinn's theorem is beyond the scope of these notes. However, we will consider a (much cruder) global K-theoretic obstruction to a mapping cylinder neighbourhood, which is similar in flavour to Quinn's obstruction.

Example 2.57 Consider the Fox-Artin wild arc I_1. This is a wildly embedded interval in $\mathbb{R}^3$. Choose a second smoothly embedded interval I_2 so that the union $I_1 \cup I_2$ is an embedded circle. Choose points x_1 and x_2 on I_1 and I_2 respectively. Let γ_1 and *gamma*$_2$ be small loops with centers x_1 and x_2 perpendicular to I_1 and I_2 respectively. Extend γ_1 and γ_2 so as to be based loops.

These loops cannot be homotopic in $\mathbb{R}^3 - S^1$. For if $H : S^1 \times I \to \mathbb{R}^3 - S^1$ is such a homotopy then since $S^1 \times I$ is compact there exists an $\epsilon > 0$ such that for all $t \in I$, $x \in S^1$

$$d(H(x,t), X) > \epsilon,$$

ie. throughout the homotopy S^1 is always more than ϵ away from X. Let B be a ball of radius ϵ around b. Then H must avoid B. Clearly this is impossible.

Suppose, however, X admits a mapping cylinder neighbourhood. Thus there is a closed manifold of X in $\mathbb{R}^3$ homeomorphic to the mapping cylinder on some map $p : V \to X$, where V is a manifold. Let I be the portion of X between x_1 and x_2 containing b.Let N be the mapping cylinder of $p_| : P^{-1}(I) \to I$. Then

$$\partial N = D_1^2 \cup_{\gamma_1} Z \cup_{\gamma_2} D_2^2$$

where D_1 and D_2 are 2-disks and $N \cong I$, ie. N is contractible. Hence

$$\tilde{H}_i(\partial N) \cong H_{i+1}(N, \partial N) \cong H^{3-i-1}(N) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Thus ∂N is a 2-dimensional homology sphere, ie. a topological 2-sphere. Hence Z is a cylinder and $\gamma_1 \simeq \gamma_2$, a contradiction.

We conclude that X does not admit a mapping cylinder neighbourhood in $\mathbb{R}^3$.

Lemma 2.58 *If $X \subset M$ admits a MCN then $M - X$ is homotopy finite.*

Proof The proof is similar to the tubular neighbourhood case. Let $N \subset M$ be the mapping cylinder neighbourhood. The mapping cylinder structure in the neighbourhood of X in M allows us to pull back the open manifold $M - X$ into the closed manifold with boundary $\mathrm{cl}(M - N)$, so that they are homotopy equivalent. The latter is a compact, topological manifold, and hence is homotopy finite.

Homotopy finiteness of $M - X$ is therefore an obstruction to the existence of a mapping cylinder neighbourhood.

We now define the property of tameness for $X \subset M$. Tameness should be thought of as a kind finite domination. It basically ensures that $M - X$ can be pulled away from X in a way that is *controlled* over X. This is a weaker notion than the existence of a MCN, but it turns out the two properties differ only be K-theory!

Definition 2.59 *Let $X \subset M$ be an embedding, and $r : N \to X$ a retraction from a closed neighbourhood of X in M. We say X is tame in M if for every neighbourhood U of X in M and every $\epsilon > 0$ there exists a neighbourhood V of X in M and a homotopy $H : (M - X) \times I \to M - X$ such that*

1. $h = 1$ *on* $(M - X) \times \{0\} \cup (M - U) \times I$
2. $h((u - X) \times I) \subset U - X$
3. $h((M - X) \times \{1\}) \subset M - V$
4. *the tracks of* $r \cdot H$ *have diameter* $< \epsilon$

We have the following heirarchy of embeddings

smooth embeddings $\subset$ MCN embeddings $\subset$ tame embeddings

Lemma 2.60 *Suppose $X \subset M$ is tame. Then $M - X$ is finitely dominated.*

Proof We may choose V above so that $M - V$ is a compact manifold with boundary, ie. has finite homotopy type. Then

$$M - X \xrightarrow{H_1} M - V \hookrightarrow M - X \simeq 1_{M-X},$$

so that $M - X$ is finitely dominated by $M - V$.

We conclude that for a tamely embedded $X \subset M$, there is defined a finiteness obstruction

$$\sigma(M - X) \in \tilde{K}_0(\mathbb{Z}\pi)$$

where $\pi = \pi_1(M - X)$.

Corollary 2.61 *If X admits an MCN then $\sigma(M - X) = 0$.*

Of course, the obstruction $\sigma(M - X) = 0$ is only part of the story, and on its own is not a sufficient obstruction to an MCN.

Let us consider the special case of embeddings $S^1 \subset S^n$. Then $\pi = \pi_1(S^n - S^1)$ is a finitely presented, perfect group, $H_1(\pi) = H_2(\pi) = 0$. We have (Ferry and Pederson):

Theorem 2.62 *For each $n \geq 7$, and each finitely presented perfect group π with $H_1(\pi) = H_2(\pi) = 0$, and each $\sigma \in \tilde{K}_0(\mathbb{Z}\pi)$, there is a tame embedding $S^1 \subset S^n$ such that*

$$\pi_1(S^n - S^1) = \pi, \quad \sigma(S^n - S^1) = \sigma.$$

Thus all the finiteness obstructions are realised by tame embeddings. In the case of an embedded S^1, the homology groups with coeficients in K-theory mentioned above take on a particularly simple form

$$\tilde{H}_0(S^1; \tilde{K}_\times(\mathbb{Z}\pi)) \cong \tilde{K}_0(\mathbb{Z}\pi) \oplus \tilde{K}_{-1}(\mathbb{Z}\pi).$$

The obstruction to a mapping cylinder neighbourhood for an emnbedding $S^1 \subset S^n$ therefore has two components. The first, in $\tilde{K}_0(\mathbb{Z}\pi)$ is the finiteness obstruction $\sigma(S^n - S^1)$ discussed above, which obstructs the existence of a finite complex homotopy equivalent to $S^n - S^1$. Even if this obstruction is zero, we only have an abstract finite complex $K \cong S^n - S^1$, but no way of relating this to X itself to build a MCN. It is necessary therefore to consider K as a space *over* X or, more precisely, as a space with *control* over X. This gives rise to the notion of a controlled finiteness obstruction, and in particular, to our second component in the negative K-group $\tilde{K}_{-1}(\mathbb{Z}\pi)$

3 $K_1(R)$, Whitehead torsion and applications

3.1 The Group $K_1(R)$

Let R be a ring with unit. Recall the infinite general linear group $GL(R)$ is the union of the sequence

$$GL(1, R) \subset GL(2, R) \subset \cdots \subset GL(n, R) \subset \cdots$$

with inclusions

$$GL(n, R) \to GL(n+1, R)\,;\; A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The elementary $n \times n$ matrix $e_{ij}(a)$, $a \in R$, $i \neq j$, is given by

$$e_{ij}(a) = \begin{pmatrix} 1 & & & \\ & 1 & a & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

with a in the ith row and jth column. We write $E(n, R)$ for the subgroup of $GL(n, R)$ generated by all $n \times n$ elementary matrices. Then

$$E(1, R) \subset E(2, R) \subset \cdots \subset E(n, R) \subset \cdots$$

with union $E(R)$, the group of elementary matrices. Note that multiplication on the left (right) by an element of $E(R)$ corresponds to an elementary row (column) operation. We have [36]:

Lemma 3.1 (Whitehead) *The subgroup $E(R) \subset GL(R)$ is the commutator subgroup of $GL(R)$. Thus the quotient $GL(R)/E(R)$ is abelian.*

Proof The identity

$$e_{ij}(a)e_{jk}(1)e_{ij}(a)^{-1}e_{jk}(1)^{-1} = e_{ik}(a), \quad i \neq j \neq k \neq i,$$

shows that each elementary matrix is a commutator.

For matrices $A, B \in GL(n, R)$, the identities

1.
$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

2.
$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 - A & 1 \end{pmatrix}$$

3.
$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=n+1}^{2n} e_{ij}(A_{ij})$$

show that each commutator is a product of elementary matrices.

Definition 3.2 *For a ring R with unit, define $K_1(R)$ to be the abelian group $GL(R)/E(R) = GL(R)_{ab}$.*

Then K_1 is a functor from rings to abelian groups. The group structure on $K_1(R)$ is induced from matrix multiplication in $GL(R)$, ie.

$$[A] \cdot [B] = [AB] \in K_1(R).$$

However, it is common to think of $K_1(R)$ as an additive group. In fact, suppose $A, B \in GL(R)$. Then we write

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(R).$$

Using the identity

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

and the fact that $\begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \in E(R)$, we see

$$[A \oplus B] = [AB \oplus 1] = [AB] \in K_1(R).$$

Hence the group structure is also induced by the block sum $\oplus$.

Let $R^\times$ be the group of units in R so that $R^\times = GL(1, R)$. Then there is induced a map

$$\iota : R^\times \to K_1(R).$$

Furthermore, if R is commutative then

$$\det : GL(n, R) \to R^\times$$

determines a homomorphism

$$\det : K_1(R) \to R^\times$$

which is split by ι.

Definition 3.3 *For a commutative ring R, we write $SK_1(R)$ for the kernel of $det : K_1(R) \to R^\times$.*

Clearly then

$$K_1(R) \cong SK_1(R) \oplus R^\times.$$

3.2 Fields and Local Rings

Lemma 3.4 *For a field F*

$$det : K_1(F) \to F^\times$$

is an isomorphism, ie. $SK_1(F) = 0$.

Proof Let A be an $n \times n$ invertible matrix over F. Since F is a field, a standard row reduction argument allows us to replace A by a diagonal matrix $(a, 1, \ldots, 1)$ so that $\det A = a$. Thus $[A] = [a] \in K_1(F)$.

There are two essential steps in proving the above lemma

1. Use the existence of the determinant function to construct a splitting

$$R^\times = GL(1, R) \hookrightarrow GL(R) \stackrel{det}{\to} R^\times$$

and hence a splitting

$$R^\times_{\mathrm{ab}} \to K_1(R) \to R^\times_{\mathrm{ab}}.$$

2. By row operations, show that any element of $K_1(R)$ can be represented by a diagonal matrix $(a, 1, \ldots, 1)$.

In general, there is no determinant function for non-commutative rings, so that (1) may fail. In addition, (2) can fail when a column of an invertible matrix contains no unit, so that we cannot do row reduction.

In the case of local rings these problems can be avoided. Recall a ring is local if the set of non-units forms an ideal.

Lemma 3.5 *Let R be a local ring (not necessarily commutative). Then the inclusion*

$$R^\times = GL(1,R) \hookrightarrow GL(R)$$

induces an onto map

$$R^\times_{ab} \to K_1(R).$$

Proof Let $A \in GL(R)$. If the first column of A contains a unit, then we can proceed as in the previous proof. This is the case: since R is local the non-units form a proper ideal. But if the first column of A consists only of non-units, since A is invertible, some linear combination of them is 1. This is a contradiction.

We now show that local rings (not necessarily commutative) admit a determinant function.

Theorem 3.6 *Let R be a local ring. Then there exists a unique 'determinant' map*

$$det : GL(R) \to R^\times_{ab}$$

satisfying the following three properties

a) $det(EA) = det(A)$ for $A \in GL(R)$, $E \in E(R)$

b) $det(I) = 1$

c) $det(diag(1, \cdots, a, \cdots; 1)A) = \bar{a}det(A)$.

In addition, the determinant map satisfies

d) $det(AB) = det(A)det(B)$

e) $det(A) = -det(A')$ where A' is obtained from A by swapping rows

f) $det(A^T) = det(A)$

We begin by proving the uniqueness of the determinant function, and then proceed to inductively construct it.

Suppose then we have a map det satisfying the properties a), b), c). Since R is local, any matrix A in $GL(R)$ can be row reduced to a diagonal matrix $D = \text{diag}(a, 1, \cdots, 1)$. Hence

$$\det(A) \stackrel{a}{=} \det(D) \stackrel{c}{=} \bar{a}\det(I) \stackrel{b}{=} \bar{a},$$

and det is uniquely determined.

We now construct $\det_n : GL(n,R) \to R^\times_{ab}$ by induction on n. First define $\det_1 : GL(1,R) \to R^\times_{ab}$ to be the abelianization map. Then $\det_1$ satisfies the properties a),b),c).

Next, suppose by induction we have defined maps $\det_k : GL(k,R) \to R^\times_{ab}$, $k < n$, satisfying a),b),c), together with the compatibility relation

$$\det_{k+l}(A \oplus I_l) = \det_k(A), \quad A \in GL(k,R), \ k, k+l < n.$$

We then define $\det_n$ as follows. Let $A \in GL(n, R)$ with rows $A_1, \ldots, A_n$. Let $B \in GL(n, R)$ be a left inverse, $BA = I$. Let $b_1, \ldots, b_n$ be the entries in the first row of B. Then

$$b_1 A_1 + \cdots + b_n A_n = (1, 0, \ldots, 0).$$

Set $A_j = (a_{j1} B_j)$ with $B_j \in R^{n-1}$ so that $\Sigma b_j B_j = 0$. Since R is local, at least one b_j is a unit, say $b_i \in R$. Thus

$$b_i^{-1} b_1 B_1 + \cdots + b_i^{-1} b_{i-1} B_{i-1} + B_i + \cdots + b_i^{-1} b_n B_n = 0.$$

ie. B_i is a linear combination of $B_1, \ldots, \hat{B}_i, \ldots, B_n$. Hence we may row reduce A to the matrix

$$\begin{pmatrix} a_{11} & B_1 \\ \vdots & \vdots \\ a_{i-1\,i} & B_{i-1} \\ b_i^{-1} & 0 \\ a_{i+1\,1} & B_{i+1} \\ \vdots & \vdots \\ a_{nn} & B_n \end{pmatrix}$$

Define

$$\det_n(A) = (-1)^i b_i^{-1} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \hat{B}_i \\ \vdots \\ B_n \end{pmatrix}$$

Continuing in this way, det is well defined.

Example 3.7 We give an example of a commutative ring with $SK_1(R) \neq 0$. Let R be the ring

$$R = \mathbb{R}[x, y]/(x^2 + y^2 - 1).$$

Then R is the ring of polynomial functions on

$$S^1 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

In fact, any polynomial $f(x, y) \in \mathbb{R}[x, y]$ determines a function $f : \mathbb{R}^2 \to \mathbb{R}$, and two such functions agree on S^1 iff they differ by a multiple of $x^2 + y^2 = 1$.

An element of $SL(n, R)$ is thus a matrix of polynomial functions on S^1, or equivalently, a map $S^1 \to SL(n, \mathbb{R})$. Taking homotopy classes, we have a map

$$SL(n, R) \to \pi_1(SL(n, \mathbb{R})).$$

This map is easily seen to be trivial on elementary matrices. Consider the inclusion

$$i_n : SO(n) \to SL(n, \mathbb{R}).$$

The Gram-Schmidt process determines a retraction

$$r_n : SL(n,\mathbb{R}) \to SO(n),$$

and so we have a map

$$SL(n,R) \to \pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

There is then a commutative diagram

$$\begin{array}{ccccccccc} SL(2,R) & \subset & SL(3,R) & \subset & \cdots & \subset & SL(n,R) & \subset & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \mathbb{Z} & \to & \mathbb{Z}/2 & = & \cdots & = & \mathbb{Z}/2 & = & \end{array}$$

so that in the limit we obtain a map $SL(R) \to \mathbb{Z}/2$ and hence a map $SK_1(R) \to \mathbb{Z}/2$.

To show $SK_1(R) \neq 0$, consider the element

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL(2,R).$$

This determines a map

$$S^1 \to SL(2,\mathbb{R})\,;\,(x,y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

or equivalently

$$S^1 \to SO(2)\,;\, e^{i\theta t} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which is a generator for $\pi_1(SO(2)) = \mathbb{Z}$. It follows that the class

$$\left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\right] \in SK_1(R)$$

is non-zero.

3.3 Whitehead Groups

Let X be a finite CW-complex with fundamental group π. Write $C(X)$ for the cellular chain complex of X so that $C(X)_i$ is the free abelian group with one generator for each cell of X and the boundary map $\partial_i : C(X)_i \to C(X)_{i-1}$ is induced by incidence numbers. Clearly then, $C(X)$ is a finite chain complex of free, finitely generated $\mathbb{Z}$-modules with a *preferred basis*, namely, the cells of X.

Now let $\tilde{X}$ be the universal cover of X. Then $\tilde{X}$ has the structure of a CW-complex and the group π acts cellularly on $\tilde{X}$, so that $C(\tilde{X})$ is a finite chain complex of free, finitely generated $\mathbb{Z}\pi$-modules. However, there is now no preferred choice of basis for $C(\tilde{X})$, since we must make a choice of lift for each cell. To obtain a $\mathbb{Z}\pi$-basis for $C(\tilde{X})$, for each cell of X, pick a covering cell in $\tilde{X}$. Of course, this choice is not unique - different choices differ by an element in π.

For topological applications, this non-uniqueness motivates the following definition

Definition 3.8 *Let π be a (multiplicative) group with integral group ring $\mathbb{Z}\pi$. The Whitehead group of π is given by*

$$Wh(\pi) = K_1(\mathbb{Z}\pi)/\{\pm g \mid g \in \pi\},$$

ie. the quotient of $K_1(\mathbb{Z}\pi)$ by the units $\pm g$.

Thus, $\mathrm{Wh}(\pi)$ is the obstruction group to row reducing invertible matrices over $\mathbb{Z}\pi$ to elements $\pm g, g \in \pi$.

The main application to topology is Whitehead torsion, hence the name. The calculation of the Whitehead group is usually rather complicated. See [21] for a comprehensive account. See [18] for an excellent introduction.

Let C be a chain complex over $\mathbb{Z}\pi$. Recall a chain contraction $s : C_* \to C_{*+1}$ is a map such that $\partial s + s\partial = 1$. A chain complex which admits a chain contraction is said to be contractible.

Example 3.9 If X is a contractible CW-complex, then, by the cellular approximation theorem, $C(\tilde{X})$ admits a chain contraction.

Recall a chain complex C is said to be acyclic if $H_*(C) = 0$.

Lemma 3.10 *Let C be a chain complex of finitely generated, projective $\mathbb{Z}\pi$-modules. Then C is acyclic iff C is contractible.*

Example 3.11 Let $\phi : F_1 \to F_2$ be an isomorphism of finitely generated, free $\mathbb{Z}\pi$-modules. Then the chain complex C given by

$$0 \to F_1 \xrightarrow{\phi} F_2 \to 0$$

is acyclic with chain contraction ϕ^{-1}. If F_1, F_2 have preferred basis then ϕ is represented by a matrix A. Thus formally we obtain an element

$$\tau(A) \in \mathrm{Wh}(\pi).$$

More generally, let C be a finite chain complex of finitely generated, free $\mathbb{Z}\pi$-modules, with preferred bases. Suppose C is acyclic and let $s : C_* \to C_{*+1}$ be a chain contraction. Consider

$$\partial + s : \oplus_{\text{even}} C_i \to \oplus_{\text{odd}} C_i,$$

then

$$(\partial + s)(\partial + s) = \partial^2 + s\partial + \partial s + s^2 = 1 + s^2.$$

But $1+s^2$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ s^2 & 1 & 0 & \cdots \\ 0 & s^2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

where the (i,j)th entry maps $C_j \to C_i$. Hence $1+s^2$ is a chain isomorphism, as are $(\partial+s)^2$ and $\partial+s$.

Now let $i : \oplus_{\text{even}} C_i \to \oplus_{\text{odd}} C_i$ be any isomorphism mapping bases to bases and define

$$\tau(C) = \tau(i \cdot (\partial + s)) \in \mathrm{Wh}(\pi)$$

where $\tau(i \cdot (\partial+s))$ is the image in $\mathrm{Wh}(\pi)$ of the matrix representing $i \cdot (\partial + s)$. Since i sends bases to bases, any other i' determines a permutation matrix $i \cdot i' \in E(\mathbb{Z}\pi)$ so that $\tau(C)$ is independent of the choice of i.

Notice also that $1+s^2$ is a product of elementary matrices, so that $\tau(1+s^2) = 0$ and

$$\tau((\partial+s)\cdot i^{-1}) = -\tau(i \cdot (\partial+s)) \in \mathrm{Wh}(\pi).$$

Finally, we must show the definition of $\tau(C)$ is independent of the choice of chain contraction s. That is, for some other chain contraction $\bar{s}$

$$-\tau(i\cdot(\partial+s)) + \tau(i\cdot(\partial+\bar{s})) = \tau((\partial+s)\cdot i^{-1}\cdot i\cdot(\partial+\bar{s})) = \tau((\partial+s)(\partial+\bar{s})) = 0.$$

Lemma 3.12 *Given chain contractions s, $\bar{s}$, there exist maps $\{F_k : C_k \to C_{k+2}\}$ such that $\partial F - F\partial = s - \bar{s}$, ie. s, $\bar{s}$ are chain homotopic contractions.*

Proof Define $\{F_k\}$ inductively, starting with $F_{-1} = 0 : C_{-1} = 0 \to C_1$. Assume F_{k-1} is defined with the desired property, then

$$0 = \partial(F_{k-1}\partial + s - \bar{s}) = \partial Fk - 1\partial + \partial s - \partial\bar{s} = (F_{k-2}\partial + s - \bar{s})\partial + \partial s - \partial \bar{s}.$$

Thus, setting $F_k = s(F_{k-1}\partial + s - \bar{s})$ we have

$$\partial F_k - F_{k-1}\partial = \partial s(F_{k-1}\partial + s - \bar{s}) - F_{k-1}\partial = (1 - s\partial)(F_{k-1}\partial + s - \bar{s}) - F_{k-1}\partial = s - \bar{s}.$$

Hence result.

It follows that

$$\begin{aligned}(\partial+s)(\partial+\bar{s}) = (\partial + \partial F - F\partial + \bar{s})(\partial+\bar{s}) &= \partial F \partial \bar{s} \partial + \partial \bar{s} + \partial F \bar{s} - F\partial\bar{s} + \bar{s}^2 \\ &= 1 + \partial F \partial + (\partial F \bar{s} - F\partial\bar{s} + \bar{s}^2).\end{aligned}$$

But $1 + \partial F \partial$ has inverse $1 - \partial F \partial$ so that

$$(\partial+s)(\partial+\bar{s})(1 - \partial F\partial) = 1 + \text{terms of degree } 2.$$

Hence $(\partial+s)(\partial+\bar{s})(1-\partial F\partial)$ has a matrix blocked like $1+s^2$. It suffices to show that $\tau(1+\partial F\partial) = 0$.

Lemma 3.13 *The following relations hold*

1. $\partial s \partial = s$
2. $s\partial s\partial = s\partial$
3. $\partial s\partial s = \partial s$
4. $(\partial s)(s\partial) = (s\partial)(\partial s) = 0$

ie. ∂s *and* $s\partial$ *are complementary projections.*

Consider then the matrix

$$A = \begin{pmatrix} s\partial & \partial s \\ \partial s & s\partial \end{pmatrix}$$

then $A = A^{-1}$. Thus

$$\begin{aligned} \tau(1 + \partial F\partial) &= \tau\left(\begin{pmatrix} s\partial & \partial s \\ \partial s & s\partial \end{pmatrix}\begin{pmatrix} 1 + \partial F\partial & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s\partial & \partial s \\ \partial s & s\partial \end{pmatrix}\right) \\ &= \tau\begin{pmatrix} 1 & 0 \\ \partial F\partial 1 & \end{pmatrix} = 0. \end{aligned}$$

We conclude that $\tau(C) \in \mathrm{Wh}(\pi)$ is well-defined.

Example 3.14 Suppose (X, Y) is a finite CW-pair, with the inclusion map $Y \hookrightarrow X$ a homotopy equivalence. Then $C(\tilde{X}, \tilde{Y})$ is an acyclic finite chain complex of free $\mathbb{Z}\pi_1(Y)$-modules. Define the Whitehead torsion of the pair (X, Y) by

$$\tau(X, Y) = \tau(C(\tilde{X}, \tilde{Y})) \in \mathrm{Wh}(\mathbb{Z}\pi_1(Y)).$$

Note that $C(\tilde{X}, \tilde{Y})$ as a preferred basis only up to the action of $\pi_1(Y)$, but this is sufficient to give a well-defined class in $\mathrm{Wh}(\mathbb{Z}\pi_1(Y))$.

We now introduce the subject of simple homotopy, which is a refined version of homotopy, taking into account the way the homotopy equivalence 'twists' the cells. The definitive account may be found in [5].

Definition 3.15 *Let* (X, Y) *be a finite CW-pair. We say* X *is an elementary expansion of* Y *if* $X = Y \cup_f B^n$, *where* B^n *is a standard n-disk,* $F \subset \partial B^n$ *is a standard* $(n-1)$*-disk, and* $f : F \to Y^{(n-1)}$ *satisfies* $f(\partial F) \subset Y^{(n-2)}$. *We write* $Y \overset{e}{\nearrow} X$ *or* $X \overset{e}{\searrow} Y$.

If

$$Y = Y_0 \overset{e}{\nearrow} Y_1 \overset{e}{\nearrow} Y_2 \overset{e}{\nearrow} \cdots \overset{e}{\nearrow} Y_n$$

we write $Y \nearrow X$. If

$$X = X_n \overset{e}{\searrow} X_{n-1} \overset{e}{\searrow} X_{n-1} \overset{e}{\searrow} \cdots \overset{e}{\searrow} X_0 = Y$$

we write $X \searrow Y$.

Definition 3.16 *Two spaces, X and Y, are said to be simple homotopy equivalent if there is a chain*

$$Y = Y_0 \nearrow Y_1 \searrow Y_2 \cdots \nearrow Y_{n-1} \nearrow Y_n = X.$$

Such a chain is called a formal deformation from Y to X and we write $Y \rightsquigarrow X$.

Note that every $Y \rightsquigarrow X$ can be rewritten $Y \nearrow Z \searrow X$.

How does an elementary expansion affect the torsion of a CW-pair? Let (X,Y) be a finite CW-pair with $i : Y \hookrightarrow X$ a homotopy equivalence. Let $X \overset{e}{\nearrow} X'$ so that (X',Y) is again a finite CW-pair, with $i' : Y \hookrightarrow X'$ a homotopy equivalence. We wish to compare $\tau(X,Y)$ and $\tau(X',Y)$. Since $X \overset{e}{\nearrow} X'$ we have

$$X' = X \cup e^{n-1} \cup e^n$$

where $e^{n-1} = \partial B^n - F$ and $e^n = B^n$. Choose orientations of $[e^{n-1}]$ and $[e^n]$ in $C(X',Y)$ so that

$$\partial[e^n] = [e^{n-1}] + c$$

for $c \in C(X,Y)_{n-1}$. Thus the chain complex $C(X',Y)$ looks like

$$\cdots \to C(X,Y)_{n+1} \to C(X,Y)_n \oplus [e^n] \xrightarrow{\begin{pmatrix} \partial & c \\ 0 & 1 \end{pmatrix}} C(X,Y)_{n-1} \oplus [e^{n-1}] \to C(X,Y)_{n-2} \to \cdots$$

We can then extend a chain contraction s of $C(X,Y)$ to a chain contraction s' of $C(X',Y)$ simply by setting $s'([e^n]) = 0$ and $s'([e^{n+1}]) = [e^n] - s(c)$. Then for n even, we have

$$\partial' + s' : \oplus_{\text{even}} C(X',Y)_i \to \oplus_{\text{odd}} C(X',Y)_j$$

has matrix or block form

$$\begin{pmatrix} \partial + s & * \\ 0 & 1 \end{pmatrix} : \oplus_{\text{even}} C(X,Y)_i \oplus [e^n] \to \oplus_{\text{odd}} C(X,Y)_j \oplus [e^{n-1}].$$

We have proved

Lemma 3.17 *If $X \rightsquigarrow X'$ rel Y, then $\tau(X,Y) = \tau(X',Y)$.*

Thus elementary expansions preserve torsion. We shall in fact show

Theorem 3.18 *Let (X,Y) be a finite CW-pair with $Y \hookrightarrow X$ a homotopy equivalence. Then*

$$\tau(X,Y) = 0 \iff X \rightsquigarrow Y \text{ rel } Y.$$

First we show

Lemma 3.19 *(Cell Trading Lemma) Let (X,Y) be a finite CW-pair with $\pi_k(X,Y) = 0$, $0 \le k \le n$. Then $X \rightsquigarrow X'$ rel Y, with $\dim(X' - Y) \ge n+1$.*

Proof Assume by induction $X = Y \cup \{e_i^n\} \cup \cdots$. Let $\phi : (B^n, S^{n-1}) \to (X, Y)$ be a characteristic map for e_1^n, ie. ϕ maps $\circ B^n$ homeomorphically onto the interior of e_1^n. Since $\pi_n(X, Y) = 0$ there is a relative homotopy

$$\Phi : (B^n, S^{n-1}) \times I \to (X, Y)$$

with $\Phi_0 = \phi$ and $\Phi(B^n \times 1) \subset Y$. Consider Φ as a map from $B^{n+1} \subset \partial B^{n+2}$ to X and form

$$X' = X \cup_\Phi B^{n+2}.$$

Let $C = \partial B^{n+2} - \mathrm{int} B^{n+1}$ so that C is an $(n+1)$-disk. By 'pulling over the top', C collapses into Y and we have an elementary collapse $F : C \cup Y \to Y$.

Consider the topological pushout defining the space $\bar{X}$

$$\begin{array}{ccccc} C \cup Y & \hookrightarrow & X' & \to & X'/(C \cup Y) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \hookrightarrow & \bar{X} & \to & \bar{X}/Y \end{array}$$

so that the righthand vertical map is a homeomorphism. We see that $\bar{X}$ has one less n-cell than X but one more $(n+2)$-cell (rel Y). We claim that $X \rightsquigarrow \bar{X}$ rel Y.

Consider the *homotopy* pushout $X' \cup (C \cup Y) \times I \cup_F Y$ of the above diagram, together with the natural homotopy equivalence

$$X' \cup (C \cup Y) \times I \cup_F Y \to \bar{X}.$$

The homotopy pushout admits 3 collapses:

1. Extend the collapse of C into Y linearly along I to collapse $C \times I$ into $Y \times I$, and at the same time collapsing $C \subset X'$ into Y in the process.

2. Collapse the cylinder $Y \times I$ along I into X'.

3. Collapse the whole of the mapping cylinder on F into X'.

Thus we have

$$X' \cup (C \cup Y) \times I \cup_F Y \overset{1,2}{\rightsquigarrow} \bar{X} \text{ rel } Y,$$

and

$$X' \cup (C \cup Y) \times I \cup_F Y \overset{3}{\searrow} X' \searrow X,$$

where the latter collapses B^{n+2}. We conclude that $X \rightsquigarrow \bar{X}$ rel Y.

We can now use the cell trading lemma to prove the main theorem. Suppose (X, Y) is a finite CW-pair with $Y \hookrightarrow X$ a homotopy equivalence. Assume $\tau(X, Y) = 0$. We must show $X \rightsquigarrow Y$ rel Y. Since $\pi_k(X, Y) = 0$ for all k, we can trade cells between any dimensions k, $k+2$. This means that $X \rightsquigarrow X'$ rel Y for some $X' = Y \cup \{e_i^n\} \cup \{e_j^{n+1}\}$. Then the chain complex $C(\tilde{X}', \tilde{Y})$ is

$$0 \to C(\tilde{X}', \tilde{X})_{n+1} \xrightarrow{\partial} C(\tilde{X}', \tilde{Y})_n \to 0$$

for some isomorphism ∂. Let us distinguish two cases:
Case 1: Suppose with respect to the preferred bases $\{e_i^n\}$, $\{e_j^{n+1}\}$ that $\partial = 1$. Recall the construction of ∂ geometrically. Let $\phi_j : S^n \to Y \cup \{e_i^n\}$ be the attaching map for e_j^{n+1}. Then

$$\partial[e_j^{n+1}] = \Sigma_i \alpha_{ij}[e_i^n]$$

where α_{ij} is the degree of the map

$$S^n \xrightarrow{\phi_j} Y \cup \{e_i^n\} \xrightarrow{/Y} \vee_i S_i^n \xrightarrow{\pi_i} S^n.$$

Since $\partial = 1$ we see that $\alpha_{ij} = \delta_{ij}$ (Kronecker delta). In particular, we may arrange for each ϕ_j to take the upper open hemisphere of S^n homeomorphically onto e_j^n, and the lower closed hemisphere into Y. Hence $e_j^n \cup_{\phi_j} e_j^{n+1}$ is an $(n+1)$-ball, which collapses into Y. This holds for all j, so that $X \rightsquigarrow Y$ rel Y.
Case 2: In general, since $\tau(X, Y) = 0$, the matrix of ∂ with respect to the bases $\{[e_i^n]\}$, $\{[e^{n+1}]\}$ is stably a product of elementary matrices and diagonal matrices with elements π group elements on the diagonal. Each of these algebraic properties of ∂ has a geometric counterpart:

1. Stabilization of the matrix ∂ corresponds geometrically to adding trivial pairs of cells e^n, e^{n+1} with incidence number 1
2. Multiplication by an elementary matrix, say $e_{jk}(a)$, corresponds to pulling the attaching map of e_j^{n+1} over e_k^{n+1} (which is equivalent to replacing ϕ_j with $\phi_j + a\phi_k$, where $a\phi_k$ is trivialized by e_k^{n+1}.)
3. Multiplication by a diagonal matrix with entries $\pm g$ corresponds to choosing different lifts for e_j^{n+1} in $(\tilde{X}, \tilde{Y})$

We conclude that by a sequence of geometric moves, we may ensure $\partial = 1$ and so reduce to case 1. This concludes the proof of the main theorem.

One consequence of the above theorem is a geometric description of the Whitehead group. Suppose A is a space with $\pi = \pi_1(A)$. We consider the collection of pairs (X, A) with X a finite CW-complex rel A. Define an equivalence relation $(X, A) \sim (X', A)$ iff $X \rightsquigarrow X'$ rel A, and set Wh(A) to be the set of equivalence classes. Then $\tau : Wh(A) \to \text{Wh}(\pi)$ is a 1-1 correspondence. This kind of geometric approach to Whitehead torsion is useful in that it allows the concept to be extended to different circumstances. For example, we can talk about equivariant torsion, topological torsion or controlled torsion, simply by specifying additional geometric conditions on the construction of Wh(A).

Definition 3.20 *Let $f : X \to Y$ be a homotopy equivalence of finite CW-complexes. The torsion $\tau(f)$ of f is defined to by*

$$\tau(f) = \tau(M(f), X) \in Wh(\pi_1(Y)),$$

where $M(f)$ is the mapping cylinder of f. In the case $\tau(f) = 0$ we say f is a simple homotopy equivalence.

Here are some properties of torsion. All maps are homotopy equivalences of finite CW-complexes.

1. If $f_0 \simeq f_1 : X \to Y$ are homotopic, then $\tau(f_0) = \tau(f_1)$.
2. If $f : X \to Y$ and $g : Y \to Z$ then
$$\tau(g \cdot f) = \tau(g) + g_* \tau(f).$$
3. $\tau(f \times g) = \tau(f)\chi(Y) + \tau(g)\chi(X)$.
4. If $f : X \to Y$ is in addition a homeomorphism, then $\tau(f) = 0$, ie. torsion is a homeomorphism invariant.

Each of these properties is formal, except the last, which was an outstanding problem for some time, before it was finally shown by Chapman [4]. (Of course, if f is a *cellular* homeomorphism then the result is obvious. But this is not the case in general. An application of the cellular approximation theorem will result in a map which is in general not a homeomorphism.)

As you may expect, there are connections between finiteness obstructions and torsion. Suppose $d : K \to X$ is a finite domination with right inverse $u : X \to K$, so that $d \cdot u \simeq 1_X$. Write $T(u \cdot d)$ for the mapping torus of $u \cdot d : K \to K$
$$T(u \cdot d) = K \times I / \{(z, 0) \sim (u(d(z)), 1)\}.$$
Then $T(u \cdot d)$ is homotopy finite, since K is, and, as in the proof of Mather's theorem $T(u \cdot d)$ is homotopy equivalent to $X \times S^1$. Hence $X \times S^1$ is homotopy finite. Consider the composition
$$\phi : T(u \cdot d) \to X \times S^1 \xrightarrow{1 \times -1} X \times S^1 \to T(u \cdot d),$$
where $-1 : S^1 \to S^1$ is reflection about the origin. This determines an element
$$\tau(\phi) \in \mathrm{Wh}(\pi \times \mathbb{Z}),$$
noting that $\pi_1(X \times S^1) = \pi \times \mathbb{Z}$.

Using the properties of τ listed above, it is straightforward to show that if X is homotopy finite then $\tau(\phi) = 0$.

So we see that a finite domination determines two invariants
$$\sigma(X) \in \tilde{K}_0(\mathbb{Z}\pi), \quad \tau(\phi) \in \mathrm{Wh}(\pi \times \mathbb{Z}).$$
They are related by the Bass-Heller-Swan formula [1]
$$\mathrm{Wh}(\pi \times \mathbb{Z}) \cong \mathrm{Wh}(\pi) \times \tilde{K}_0(\mathbb{Z}\pi) \times \mathrm{Nils}$$
which identifies $\sigma(X)$ and $\tau(\phi)$.

3.4 Lens Spaces and $\mathrm{Wh}(\mathbb{Z}/m)$

Lens spaces are 3-dimensional manifolds with cyclic fundamental group. They provided the first examples of non-diffeomorphic, homotopy equivalent manifolds. This application was one of the original motivations for the development of Whitehead torsion and the calculation of $\mathrm{Wh}(\mathbb{Z}/m)$. See [18].

Given an invertible 2×2 matrix over $\mathbb{Z}$,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

use complex multiplication in $S^1 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ to define a diffeomorphism

$$f_M : S^1 \times S^1 \to S^1 \times S^1 \; ; \; (x, y) \mapsto (x^a y^b, x^c y^d)$$

inducing

$$(f_M)_* = M : H_1(S^1 \times S^1) = \mathbb{Z}\oplus\mathbb{Z} \to H_1(S^1 \times S^1) = \mathbb{Z}\oplus\mathbb{Z} \; ; \; (x, y) \mapsto (ax+by, cx+dy).$$

The identification space

$$L = S^1 \times D^2 \cup_M S^1 \times D^2$$

is obtained by gluing together two copies of the solid torus $S^1 \times D^2$ along their boundary $S^1 \times S^1$ using the diffeomorphism f_M.

Definition 3.21 *Let $m, n \geq 0$ be coprime integers. The lens space $L(m, n)$ is the closed, oriented 3-dimensional manifold defined by*

$$L(m, n) = S^1 \times D^2 \cup_M S^1 \times D^2$$

where $M = \begin{pmatrix} p & m \\ q & n \end{pmatrix}$ for any $p, q \in \mathbb{Z}$ such that $np - mq = 1$.

Lemma 3.22 *The oriented diffeomorphism class of $L(m, n)$ depends only on the class of M under the relation*

$$M \sim ANB, \quad A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

In particular, there is an orientation preserving diffeomorphism

$$L(m, n) \to L(m, am + n).$$

Also, $L(-m, n)$ is $L(m, n)$ with the opposite orientation.

Example 3.23

1. $L(0, 1) = S^2 \times S^1$
2. $L(1, n) = S^3$

3. $L(2,1) = SO(3) = \mathbb{RP}^3$

Proposition 3.24 *For* $m \geq 2$

$$L(m,n) = S^3/(\mathbb{Z}/m)$$

the quotient of the free $\mathbb{Z}/m$*-action on the 3-sphere*

$$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$$

given by

$$t : S^3 \to S^3 \,;\, (z_1, z_2) \mapsto (z_1\omega, z_2\omega^n)$$

with $\omega = e^{2\pi i/m}$ *and* $\mathbb{Z}/m = \langle t \mid t^m \rangle$.

In particular, $\pi_1(L(m,n)) \cong \mathbb{Z}/m$, generated by the canonical class $t \in \pi_1(L(m,n))$.

It can be shown that $L(m,n)$ admits a CW-structure with one cell in each dimension ≤ 3. Let $\widetilde{L(m,n)}$ be the universal cover of $L(m,n)$ so that the cellular chain complex $C(\widetilde{L(m,n)})$ is a bounded chain complex of free, finitely generated $\mathbb{Z}[\mathbb{Z}/m]$-modules:

$$\cdots \to \mathbb{Z}[\mathbb{Z}/m] \stackrel{t^n-1}{\to} \mathbb{Z}[\mathbb{Z}/m] \stackrel{\Sigma}{\to} \mathbb{Z}[\mathbb{Z}/m] \to t-1 \to \mathbb{Z}[\mathbb{Z}/m],$$

where each map is multiplication, and Σ is the norm element

$$\Sigma = 1 + t + \cdots + t^{m-1} \in \mathbb{Z}[\mathbb{Z}/m].$$

To obtain torsion invariants we work over the rationals. Thus consider $C(\widetilde{L(m,n)};\mathbb{Q})$

$$\cdots \to \mathbb{Q}[\mathbb{Z}/m] \stackrel{t^n-1}{\to} \mathbb{Q}[\mathbb{Z}/m] \stackrel{\Sigma}{\to} \mathbb{Q}[\mathbb{Z}/m] \stackrel{t-1}{\to} \mathbb{Q}[\mathbb{Z}/m].$$

Let $\epsilon : \mathbb{Q}[\mathbb{Z}/m] \to \mathbb{Q};\, \Sigma a_i g_i \mapsto \Sigma a_i$, be the augmentation map with kernel $N \triangleleft \mathbb{Q}[\mathbb{Z}/m]$. Then as rings (or $\mathbb{Q}$-algebras)

$$\mathbb{Q}[\mathbb{Z}/m] \cong N \oplus (\Sigma)$$

where $(\Sigma) = \mathbb{Q}\Sigma$. Note that N and (Σ) are mutually annihilating. It follows that $C(\widetilde{L(m,n)};\mathbb{Q})$ also splits as a direct sum of

$$N \stackrel{t^n-1}{\to} N \stackrel{0}{\to} N \stackrel{t-1}{\to} N$$

and

$$(\Sigma) \stackrel{0}{\to} (\Sigma) \stackrel{\Sigma}{\to} (\Sigma) \stackrel{0}{\to} (\Sigma).$$

So the first chain complex is a finite chain complex of finitely generated, free N-modules with trivial homology. (It also has a preferred basis under the projection.) Hence it has torsion

$$\tau \in K_1(N)/\{\pm g\}.$$

Since N is a semisimple algebra it is isomorphic to a product of fields so that

$$K_1(N) = B^\times.$$

In fact, we can calculate τ by hand. Define a chain contraction s of

$$N_3 \stackrel{t^n-1}{\to} N_2 \stackrel{0}{\to} N_1 \stackrel{t-1}{\to} N_0$$

such that

$$\partial + s = \begin{pmatrix} t-1 & 0 \\ 0 & t^n - 1 \end{pmatrix} : N_1 \oplus N_3 \to N_0 \oplus N_2.$$

In other words,

$$\tau = (t-1)(t^n - 1) \in N^\times.$$

Note that a different choice of lift (and orientation) for the cells to $\widetilde{L(m,n)}$ will change τ by an element $\pm t^r$ so τ is well-defined in $N^\times/\{\pm g\}$.

To distinguish lens spaces by torsion we must understand the relations between elements of the form $(t^n - 1)$. Firstly,

$$(t^n - 1) = -t^n(t^{-n} - 1)$$

so that

$$t^n - 1 = t^{-n} - 1, \quad \text{mod } \pm g.$$

We quote

Lemma 3.25 *(Franz Independence Lemma) The units*

$$t^n - 1, \quad 1 \leq n < \frac{m}{2}, \ (m, n) = 1,$$

in $N^\times$ do not satisfy any multiplicative relations.

Regarding conditions for $L(m,n)$ and $L(m,n')$ to be diffeomorphic:

1. $L(m,n) = -L(m,-n)$ so assume $1 \leq n < \frac{m}{2}$, $(m,n) = 1$.
2. $L(m,n)$ together with a choice of lift of cells to the universal cover, determines an element $\tau = (t^n - 1)(t-1) \in N^\times$. A new choice of lift determines a new element $\tau = \pm t^r(t^n - 1)(t-1) \in N^\times$.
3. $(t^n - 1)(t-1) = (t^{n'} - 1)(t-1)$, $1 \leq n, n' < \frac{m}{2}$, $(m,n) = (m,n') = 1$ if and only if $n = n'$.

We have then

Theorem 3.26 *(Classification) Two lens spaces are diffeomorphic if and only if they have the same torsion in $N^\times/\{\pm g\}$.*

The following theorem is also true, but more difficult to prove. See [5] [12].

Theorem 3.27 *Let $L(m,n)$, $L(m,n')$ be lens spaces. Then the following are equivalent:*

1. *There is a homotopy equivalence $f: L(m,n) \to L(m,n')$.*

2. *There exist units $u \in \mathbb{Z}[\mathbb{Z}/m]^\times$, $r \in (\mathbb{Z}/m)^\times$ such that*

$$u(t^{n'r} - 1)(t^r - 1) = (t^n - 1)(t - 1) \in \mathbb{Q}[\mathbb{Z}/m].$$

3. *$n \equiv \pm n' r^2$, (mod m) for some $r \in (\mathbb{Z}/m)^\times$.*

In addition

$$\tau(f) = \tau(L(m,n))\tau(L(m,n'))^{-1} \in \text{im}(\text{Wh}(\mathbb{Z}/m) \hookrightarrow N^\times/\{\pm g\}).$$

Corollary 3.28 *Lens spaces are diffeomorphic if and only if they are simple homotopy equivalent.*

Example 3.29 *Consider the lens spaces $L(5,1)$, $L(5,2)$. Then*

$$1 \equiv \pm 2r^2, \quad (mod\ 5)$$

has no solution. Hence $L(5,1)$ and $L(5,2)$ are not homotopy equivalent.

Example 3.30 *The lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent since $1 = 2 \cdot 2^2$ (mod 7). But*

$$\tau(L(7,1)) = (t-1)^2, \quad \tau(L(7,2)) = (t^2 - 1)(t - 1)$$

so they are not diffeomorphic.

3.5 The h-cobordism theorem

Recall a cobordism $(W; M, M')$ is called an h-cobordism if the inclusion maps $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences. In other words, W is a homotopy cylinder.

The following is due to Smale.

Theorem 3.31 *(h-Cobordism) Let $(W; M, M')$ be an h-cobordism with $\dim W \geq 6$. Suppose M is simply-connected. Then W is diffeomorphic to $M \times [0,1]$.*

Note that for simpiy-connected W, M, M', we have that $(W; M, M')$ is an h-cobordism if and only if $H_*(W, M) = 0$.

Example 3.32 Suppose W is a compact manifold with boundary. Suppose in addition

1. W and ∂W are simply connected.

2. dim W $= n \geq 6$.

3. $\tilde{H}_*(W) = 0$.

Then W is diffeomorphic to a standard n-disk. In fact, if we pick a point $x \in W$ in the interior of W and take a small ball B around x, then the complement $V = W - \text{int}B$ is an h-cobordism. Everything is simply connected so that V is a cylinder on ∂B. Hence $W = B \cup \partial B \times I$ is a standard disk.

Example 3.33 (Poincaré Conjecture, $n \geq 6$) Let M^n be a manifold of the homotopy type of an n-sphere, $n \geq 6$. Then the High Dimensional Poincaré Conjecture states that M is homeomorphic to an n-sphere.

To see this, let B be a standard disk in M and set $V = M - \text{int}B$. Then a simple argument shows $\tilde{H}_*(V) = 0$, so that V is diffeomorphic to a standard n-disk. Thus M is obtained by gluing to standard n-disks together along their boundary by some diffeomorphism $g : S^{n-1} \to S^{n-1}$. Now, topologically isotoping one of the disks to a point we obtain $S^n = D^n \cup *$. (The isotopy is not smooth at the final stage.) Hence M is homeomorphic to a standard n-sphere.

In this section we outline the proof of the h-cobordism theorem. The key input is a calculus of handles, which allows us to translate homology or homotopy data about a cobordism into topological data. The methods are very similar to the techniques used to show a homotopy equivalence with trivial torsion is simple.

We already know that every cobordism is obtained by a sequence of surgeries, and that the trace of a surgery adds a handle to the original manifold. Thus every cobordism (and hence every manifold) can be decomposed into a collection of handles. Let us recall first some notation concerning handles.

Let W be a manifold of dimension n. Let $H = D^p \times D^q$, $p + q = n$, be an n-disk such that $W \cap H = S^{p-1} \times D^q \subset \partial W$. Then we say H is a handle of index p on W. There are various subsets of H of note:

1. $D^p \times 0$ is the core of H and $0 \times D^q$ the cocore.

2. $S^{p-1} \times 0$ is the attaching sphere (a-sphere) and $0 \times S^{q-1}$ the belt sphere (b-sphere).

3. $S^{p-1} \times D^q$ is the a-tube and $D^p \times S^{q-1}$ the b-tube.

We can think of H as being attached to ∂W via an attaching embedding $f : S^{p-1} \times D^q \to \partial W$, so that adding a handle to W via f determines a new manifold $W \cup_f H$. It is easily shown that isotopic embeddings determine diffeomorphic manifolds.

Suppose now we added two handles to W

$$W \cup H^{(r)} \cup H^{(s)}.$$

That is to say, we first add an r-handle $H^{(r)}$ to W and then an s-handle $H^{(s)}$ to $W \cup H^{(r)}$. Suppose that $s \leq r$. The handle $H^{(s)}$ is attatched to $W \cup H^{(r)}$ via its a-tube $S^{s-1} \times D^{n-s}$ or equivalently, via a thickened a-sphere S^{s-1}. The cocore of $H^{(r)}$ is $0 \times D^{n-r}$. Thus, S^{s-1} and D^{n-r} intersect in general position in an $m = (s-1) + (n-r) - n = s - r - 1$ dimensional manifold. But $s \leq r$ so that $m \leq -1$. Thus we may assume that the a-tube of $H^{(s)}$ misses the cocore of $H^{(r)}$. In particular, we may slide the handle $H^{(s)}$ off of $H^{(r)}$.

We have then

Lemma 3.34 *(Reordering) Let $W \cup H_1 \cup H_2 \cup H_3 \cup \cdots$. Then we may assume $indexH_i \leq indexH_{i+1}$, ie. each handle is attached to handles of lower dimension.*

The next step is to consider relations between handles of consecutive dimensions, say $W \cup H^{(r)} \cup H^{(r+1)}$. Putting everything into general position, we see that the attaching map of $H^{(r+1)}$ intersects the b-sphere (boundary of the cocore) of $H^{(r)}$ transversly in a finite number of algebraic points $\{\pm x_i\}$, with sign of x_i depending on whether orientation is preserved (+) at x_i or not (−).

Definition 3.35 *The incidence number $\epsilon(H^{(r+1)}, H^{(r)})$ of handles is the algebraic sum $\Sigma \pm x_i$.*

The incidence number $\epsilon(H^{(r+1)}, H^{(r)})$ is closely related to the boundary relations in the cellular chain complex C of $W \cup H^{(r)} \cup H^{(r+1)}$. We know that $H^{(r)}$ contributes an r-cell e^r, and hence a generator $[e^r] \in C_r$. Similarly $H^{(r+1)}$ contributes a generator $[e^{r+1}] \in C_{r+1}$. By definition of the cellular chain complex, the boundary may $d : C_{r+1} \to C_r$ satisfies

$$d[e^{r+1}] = \epsilon(e^{r+1}, e^r)[e^r] + \cdots$$

where the incidence number $\epsilon(e^{r+1}, e^r)$ is given by the degree of the composition

$$S^r \to W \cup e^r \to S^r.$$

The first map is the attaching map for e^{r+1} and the second map collapses W to a point. But degree can be measured geometrically by making the map transverse to a point (eg. the cone point of e^r) and algebraically counting the number of points in the inverse image. These points are in fact the $\{x_i\}$ above.

We see then that the geometric and chain theoretic incidence numbers are equal

$$\epsilon(H^{(r+1)}, H^{(r)}) = \epsilon(e^{r+1}, e^r).$$

This observation is critical. It allows one to trade homological information with geometric information. In practise, one must have a method for converting algebraic methods to geometric methods. This is where the calculus of handle moves enters. The reordering lemma is one such move. We require three more: cancellation, creation and addition.

Two handles $H^{(r)}$ and $H^{(r+1)}$ are said to be complementary if their a-sphere and b-sphere (respectively) meet transversely in a single point.

Lemma 3.36 *(Cancellation) Consider $W \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary. Then there is a diffeomorphism $W \cup H^{(r)} \cup H^{(r+1)} \to W$ which is the identity outside of a neighbourhood of $H^{(r)} \cup H^{(r+1)}$.*

In fact, if W has simply connected boundary, $n-r \geq 4$, $r \geq 2$ and $n \geq 6$ (to use the Whitney trick), the conclusion of the lemma holds true only assuming $\epsilon(H^{(r+1)}, H^{(r)}) = \pm 1$.

The lemma is proved by showing that such a pair of handles are in a standard form so that $H^{(r)} \cup H^{(r+1)}$ is diffeomorphic to an n-disk attached by an $(n-1)$-face.

Conversely, we work the argument backwards. Start with W and attach an n-disk B^n to ∂W via an $(n-1)$-face B^{n-1}. Clearly this does not change the diffeomorphism type of W. But we can choose to regard B^n as a pair of complementary handles $H^{(r)} \cup H^{(r+1)}$. In other words,

Lemma 3.37 *(Creation) Complementary handles may be freely introduced into W without changing the diffeomorphism type.*

Finally we consider the addition move. Consider $W \cup H_1^{(r)}$, $r \geq 2$, with attaching map $f_1 : S^{r-1} \to M'$. (Here $(W; M, M')$ is a cobordism.) Let $[f_1] \in \pi_{r-1}(M')$ denote the homotopy class of f.

Now consider another r-handle $H_2^{(r)}$ attached to W but disjoint from $H_1^{(r)}$, so that we have $W \cup H_1^{(r)} \cup H_2^{(r)}$. Write $f_2 : S^{r-1} \to M'$ for the attaching map of $H_2^{(r)}$.

Lemma 3.38 *(Addition) Let M' be simply connected and $n - r \geq 2$, $r \geq 2$. Then f_2 may be isotoped to a map f_3 such that $[f_3] = [f_2] \pm [f_1]$ in $\pi_{r-1}(M')$ and $im f_3 \cap im f_1 = \emptyset$.*

The proof is straightforward. Imagine the two spheres $S_1 = f_1(S^{r-1})$ and $S_2 = f_2(S^{r-1})$ embedded disjointly in M'. Take hold of a point of S_2 and, keeping everything except a neighbourhood of this point in S_2 fixed, pull the point and the neighbourhood to S_1 and then over the top of the handle $H_1^{(r)}$. (Imagine a snail $H_2^{(r)}$ slowly smothering a pebble $H_2^{(r)}$!).

Armed with these handle moves we can now sketch a proof of the h-cobordism theorem. Suppose then $(W; M, M')$ is an h-cobordism with $\pi_1(M) = 0$. We may assume that W has a handle decomposition

$$W = (M \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_k \cup (M' \times I)$$

where, by the reordering lemma, the handles are attached in order of increasing index. Let us write $W^{(r)}$ for $M \times I$ union all the handles of index $\leq r$.

We proceed by induction, successively eliminating handles of each index by various moves, while maintaining the diffeomorphism type. Once all the handles have been eliminated we will have a cylinder.

We begin with the 0-handles in W. By construction $W^{(0)}$ is the disjoint union of $M \times I$ with a finite number of n-balls. Since $M \subset W$ is a homotopy equivalence, each 0-handle must be connected by a 1-handle to either another 0-handle or to $M \times I$. Thus for each 0-handle there is a complementary 1-handle. By the cancellation lemma, the 0-handles can be eliminated.

Now consider the 1-handles. Let $H^{(1)}$ be a 1-handle and α an arc in the b-tube parallel to the core. By general position, we may assume α misses any 2-handles so that it lies in the boundary of $W^{(2)}$. Since we have an h-cobordism, $\pi_1(W^{(2)}, M) = 0$ so that there is a 2-disk in $W^{(2)}$ with boundary $\alpha \cup \beta$, where β lies in M. Now thicken D^2 to an n-ball B^n. By the Creation lemma we may regard B^n as a pair of complementary handles $H^{(2)} \cup H^{(3)}$ so that $H^{(2)}$ has a-sphere D^2. By construction the pair $(H^{(2)}, H^{(1)})$ is complementary so by the Cancellation lemma they may be eliminated, leaving only the 3-handle.

Now suppose by induction that W has a handlebody decomposition with no handles of index $< s$. Suppose $2 \leq s \leq n-4$ (so that we can use the addition lemma). Let $H^{(s)}$ be an s-handle to be eliminated. Let $H_i^{(s+1)}$ be the collection of $(s+1)$-handles, and consider the incidence numbers $\epsilon_i = \epsilon(H_i^{(s+1)}, H^{(s)})$. By repeated application of addition lemma, we can modify the ϵ_i by moving the $(s+1)$-handles so that $\epsilon_1 = \pm 1$ and $\epsilon_i =$, $i > 1$. Thus $H^{(s)}$ and $H_1^{(s+1)}$ are complementary handles, so that $H^{(s)}$ can be eliminated.

Let us summarise our situation. We have an h-cobordism $(W; M, M')$, and by a sequence of handle moves we have eliminated all s-handles for $2 \leq s \leq n-4$. Also, we may assume there are no n or $(n-1)$-handles, by regarding them as dual 0 and 1-handles and applying the above argument. Similarly for the $(n-2)$-handles. Thus we are only potentially left with some $(n-3)$-handles. However, since W is an h-cobordism, there cannot in fact be any $(n-3)$-handles, otherwise $H_{n-3}(W, M)$ would be non-zero. So we have eliminated all the handles, and W is a cylinder, ie. W is diffeomorphic to $M \times I$.

What happens in the non simply connected case? As usual we have to either work in the universal cover or add reference paths to our handles. In any case, the there is a handle calculus as before consisting of reordering, cancellation, creation and addition moves. The proof procedes in a different manner however, Recall in the simply connected case we eliminated the 1-handles by replacing them with 3-handles. Similarly we may replace s-handles by $(s+2)$-handles. In particular, suppose there are no handles of index $< s$. Let $H^{(s)}$ be an r-handle. Let $H_i^{(s+1)}$ be the collection of $(s+1)$-handles and

$$\epsilon_i = \epsilon(H_i^{(s+1)}, H^s).$$

Since $H_s(W, M) = 0$ there must exist integers n_i such that $\Sigma n_i \epsilon_i = 1$. Add a complementary pair $(H^{(s+2)}, H^{(s+1)}$ and apply the addition lemma moving $H^{(s+1)}$ over the $H_i^{(s+1)}$ to arrange that $\epsilon(H^{(s+1)}, H^s) = \Sigma n_i \epsilon_i = 1$. We cancel $H^{(s+1)}$ and H^s leaving the $(s+2)$-handle $H^{(s+2)}$.

Together with some low dimensional arguments, we can ensure that W has handles only in consecutive dimensions $(n-3)$, $(n-2)$. Let A be the matrix over $\mathbb{Z}\pi$ of incidence numbers. Then since $H_*(W, M) = 0$, A is a nonsingular matrix. Define the torsion of $(W; M, M')$ to be

$$\tau(W, M) = \tau(A) \in \mathrm{Wh}(\pi).$$

By construction of the Whitehead group the element $\tau(W, M)$ is zero if and only if A can be reduced to the identity matrix by the moves

1. Replace A by $A \oplus I$
2. Add a multiple of one row to another
3. Reorder the rows or columns
4. multiply a row by an element of π or -1

But each of these algebraic moves can be realised by handle moves

1. introduce a complementary pair of handles
2. apply the addition lemma
3. reindex the handles
4. change the reference path or orientation of a handle

We conclude that if $\tau(W, M) = 0$ we may cancel the $(n-3)$ and $(n-2)$-handles to obtain a cylinder. Thus we have the s-cobordism theorem of Barden-Mazur-Stallings:

Theorem 3.39 *(s-cobordism)*

> *Let $(W^n; M, M')$, $n \geq 6$, be an h-cobordism. Then W is diffeomorphic to $M \times I$ if and only if*
>
> $$\tau(W, M) = 0 \in Wh(\pi_1(M)).$$

Actually, it can be shown that there is a 1-1 correspondence

$$\tau : \{\text{h-cobordisms on } M\} \leftrightarrow \mathrm{Wh}(\pi_1(M)).$$

Thus, h-cobordisms on M are classified by the Whitehead group of $\pi_1(M)$.

3.6 Siebenmann's End Theorem

We conclude with an application of both the Wall finiteness obstruction in $\tilde{K}_0(\mathbb{Z}\pi)$ and Whitehead torsion in $\mathrm{Wh}(\pi)$. Siebenmann's end theorem [30] is concerned with the problem of putting a boundary on a non-compact manifold. At first sight this appears to be of little concern to compact manifolds. However this is not the case. There are numerous circumstances where the topology of compact manifolds is closely related to the properties of non-compact manifolds. The most obvious is in embedding theory, when the existence of a nice neighbourhood of an embedded manifold depends intimately on the topology of the open complement. But the influence is much deeper and extends to splitting problems, existence of bundle structures, topological surgery and more.

Let M^n be an open manifold. We shall assume M has empty boundary, although this is not at all necessary. Since we are not concerned with smooth structure here we will work with topological manifolds, but the methods are the same in either category (including PL). We wish to enquire as to whether M admits a boundary. By this we mean that there exists a manifold with boundary $(M', \partial M')$ together with a homeomorphism $M \cong M' - \partial M'$.

Clearly the existence of a boundary has something to do with the properties of M 'at infinity', or in other words, away from compact subsets. Any property which has 'support' in a compact subset of M will ultimately be of no interest in deciding the existence of a boundary. Only those properties that persist at infinity are relevant. This is formalised algebraically in the concept of a pro-group. A pro-group (G_i, h_i) is a sequence of groups and homomorphisms

$$G_1 \stackrel{h_2}{\leftarrow} G_2 \stackrel{h_3}{\leftarrow} G_3 \stackrel{h_4}{\leftarrow} \cdots.$$

For example, we can think of our open manifold M as being decomposed into a collection of cocompact (compact complement) subsets $\{C_i\}$ such that $\cap_i C_i = \emptyset$. A choice of proper ray $p : [0, \infty) \to N$ allows us to construct a (preliminary version) fundamental pro-group

$$\pi_1(C_0) \leftarrow \pi_1(C_1) \leftarrow \pi_1(C_2) \leftarrow \cdots.$$

We can in a sense pass to infinity immediately by taking the inverse limit. But this is too dramatic, since our geometric constructions will not take place 'at infinity' (which doesn't exist yet!), but in each successive C_i. Another way of focusing our attention at infinity, but which is more relevant to us, is to use pro-equivalence. We say two pro-groups (G_i, h_i) and (G'_i, h'_i) are pro-equivalent if there is a commuting diagram

$$\begin{array}{ccccccccccc}
G_{i_1} & & \leftarrow & & G_{i_2} & & \leftarrow & & G_{i_3} & & \leftarrow & & \cdots \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & & \\
& & G'_{j_1} & & \leftarrow & & G'_{j_2} & & \leftarrow & & G'_{j_3} & \cdots &
\end{array}$$

We say that (G_i, h_i) is pro-stable if it is pro-equivalent to a constant pro-group $(G, 1)$. Notice that pro-equivalent pro-groups have isomorphic inverse limits. The converse is not necessarily true.

An important class of pro-groups are those satisfying the Mittag-Leffler condition. We say (G_i, h_i) is Mittag-Leffler is it is equivalent to a pro-group (G'_i, h'_i) where each h'_i is an epimorphism.

A clean neighbourhood of infinity in M is a codimension 0 closed submanifold $V \subset M$ such that $M - \text{int}V$ is compact. A system of clean neighbourhoods of infinity consists of a collection of neighbourhoods of infinity $\{V_i\}$ such that $V_i \subset V_{i-1}$ and $\cap_i V_i = \emptyset$. We say M is connected at infinity if there exists a system $\{V_i\}$ of clean neighbourhoods of infinity such that the pro-set

$$\pi_0(V_1) \leftarrow \pi_0(V_2) \leftarrow \cdots$$

is pro-equivalent to the trivial pro-set $(*, *)$.

Suppose M is connected at infinity. Let $\{V_i\}$ be a system of neighbourhoods of infinity and $p : [0, \infty) \to M$ be a proper ray. The π_1-system of M at infinity with respect to the ray p is the pro-equivalence class of the pro-group

$$\pi_1(V_1) \leftarrow \pi_1(V_2) \leftarrow \pi_1(V_3) \leftarrow \cdots$$

where we take a base point b_i in V_i on the ray p, and the homomorphisms are obtained by using the arc from b_i to b_{i-1}. Note that the pro-equivalence class is independent of the choice of basepoints so constructed. In fact, the pro-equivalence class only depends on the proper homotopy class of the ray, and not even on the system of neighbourhoods of infinity. However, there may be different proper homotopy classes of ray, even if M is connected at infinity.

Lemma 3.40 *If the fundamental group system of M at infinity, with respect to a ray p, is stable then any two rays p, p' are proper homotopic.*

Definition 3.41 *A manifold M is said to be tame at infinity if each clean neqihbourhood of infinity is finitely dominated.*

We may now state Siebenmann's theorem

Theorem 3.42 *Let M^n, $n \geq 6$, be a connected at infinity open manifold. Then M admits a boundary if and only if*

1. *the π_1-system of M is stable*
2. *M is tame at infinity*
3. *An invariant $\sigma(\epsilon) \in \tilde{K}_0(\pi_1 \epsilon)$ is zero.*

Here $\pi_1(\epsilon)$ is defined to be the inverse limit $\overset{\lim}{\leftarrow} \{\pi_1(V_i)\}$ where $\{V_i\}$ is a system of clean neighbourhoods of infinity in M.

The proof is fairly involved, and we shall only indicate the highlights. The basic ideas are as follows

1. We may easily construct a sequence of clean neighbourhoods of infinity $\{V_i\}$.
2. Suppose the inclusions $\partial V_i \to V_i$ are homotopy equivalences. Then $C_i = V_i - \mathrm{int} V_{i+1}$ is an h-cobordism, for each i. By an Eilenberg swindle we may push the torsion τ_i of each C_i to infinity. Hence M is homemorphic to $\partial V_1 \times [0, \infty)$ and admits a boundary.
3. It remains to show that we may choose the inclusions $\partial V_i \to V_i$ to be homotopy equivalences. The trick is to exchange handles between V_i and V_{i+1} to increase the connectivity of $\partial V_i \to V_i$. Everything is ok except in a critical dimension where a certain projective module needs to be free - hence the finiteness obstruction.

Assume then we are given M^n, $n \geq 6$, a manifold which is connected at infinity, tame at infinity and such that the π_1-system of M at infinity is stable. Thus, we may assume there is a system of clean neighbourhoods of infinity, together with a commutative diagram

$$\begin{array}{ccccccccccc}
\pi_1(V_1) & & \longleftarrow & & \pi_1(V_2) & & \longleftarrow & & \pi_1(V_3) & & \longleftarrow & \cdots \\
& \nwarrow & & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & & \\
& & G & & \longleftarrow & & G & & \longleftarrow & & G \cdots &
\end{array}$$

for some group G. Since V_i is finitely dominated and the composition $G \to \pi_1(V_i) \to G$ is the identity, we may conclude that the kernel of $\pi_1(V_i) \to \pi_1(V_{i-1})$ is normally generated by finitely many elements. Consider then a loop in V_i which bounds a 2-disk in V_{i-1}. First make the disk transverse to ∂V_i, so that it intersects in a finite number of circles in the interior of the disk. Starting with the innermost circles and working out, each such circle bounds a small 2- disk. If this small disk lies in $V_{i-1} - V_i$ then add a regular to the disk. If instead the small disk lies in V_i excise out a regular neighbourhood of the disk from V_i. Continue in this way until the original 2-disk is disjoint from ∂V_i. Let V_i' be the result of this exercise, for each i. Then $\pi_1(V_i') \cong G$ for all i.

A similar argument allows us to arrange that $\pi_1(\partial V_i') \cong \pi_1(V_i')$. We have shown

Lemma 3.43 *With M as above, we may assume that there is a system of clean neighbourhoods of infinity $\{V_i\}$ such that*

$$\pi_1(\partial V_i) \cong \pi_1(V_i) \cong \pi_1(\epsilon), \quad \textit{for all } i$$

for some group $\pi_1(\epsilon)$.

So much for the fundamental group. How can we arrange for $C_i = V_i - \mathrm{int}V_{i+1}$ to be an h-cobordism? Clearly, it is necessary and sufficient for the homology groups of the universal cover $H_k(\tilde{V}_i, \partial\tilde{V}_i)$, $k \geq 0$, to be zero. Suppose an element $[\alpha]$ in $H_k(\tilde{V}_i, \partial\tilde{V}_i)$ may be represented by an embedded k-disk $(D^k, S^{k-1}) \subset (V, \partial)$ (with a reference path to the base point). Then we excise this disk, by removing the interior of a regular neighbourhood, to obtain a new manifold $(V_i', \partial V_i')$ fitting into the long exact sequence

$$0 \to H_{k+1}(\tilde{V}_i, \partial\tilde{V}_i) \to H_{k+1}(\tilde{V}_i', \partial\tilde{V}_i') \to \mathbb{Z}\pi(\epsilon) \to H_k(\tilde{V}_i, \partial\tilde{V}_i) \to H_k(\tilde{V}_i', \partial\tilde{V}_i') \to 0.$$

The effect of excising this (D^k, S^{k-1}) is therefore to kill $[\alpha]$ in $H_k(\tilde{V}_i, \partial\tilde{V}_i)$ and to potentially introduce new classes in dimension $k+1$.

So we could procede by induction and successively kill $H_k(\tilde{V}_i, \partial\tilde{V}_i)$. There is one problem however: once we reach the middle dimension we have trouble embedding the disks D^k using standard embedding methods. However, it turns out that the embedded disks are already there, inside the handle decomposition.

Proposition 3.44 *Let M be as above. There exists an arbitrarily small clean neighbourhood of infinity V such that $\pi_1(\partial V) \cong \pi_1(V) \cong \pi_1(\epsilon)$ and $H_i(\tilde{V}, \partial\tilde{V}) = 0$, $i \leq n-3$.*

Proof Let us assume that we have already constructed a clean neighbourhood of infinity V such that $H_i(\tilde{V}, \partial\tilde{V}) = 0$, $i \leq k$, $k < n-3$. We can show, in the usual way, that $H_{k+1}(\tilde{V}, \partial\tilde{V})$ is finitely generated as a $\mathbb{Z}\pi(\epsilon)$-module. So a set of generators of $H_{k+1}(\tilde{V}, \partial\tilde{V})$ has support in a compact subset of V. Choose then a new clean neighbourhood of infinity V' such that $H_i(\tilde{V}, \partial\tilde{V}) = 0$, $i \leq k$ and

$$H_{k+1}(\tilde{C}, \partial\tilde{V}) \to H_{k+1}(\tilde{V}, \partial\tilde{V})$$

is onto. Here $C = V - \text{int}V'$ is a compact subset (containing the support of $H_{k+1}(\tilde{V}, \partial\tilde{V})$). Since the map is onto, we conclude that all the handles of index $k+1$ in V may be assumed to be in C. Also a straightforward argument shows

$$H_i(\tilde{C}, \partial\tilde{V}) = 0, \quad i \leq k-1,$$

so that C contains handles of index k and $k+1$. Reorder these handles that they are attached in order of increasing index and let U be the new boundary after ading all the k-handles.

Consider now an element $[\alpha] \in H_{k+1}(\tilde{C}, \partial\tilde{V})$ written $[\alpha] = \Sigma n_i g_i [e_i]$ where the e_i are $(k+1)$-handles with a reference path. Introduce a cancelling pair of handles H_1, H_2 of index $(k+1)$ and $(k+2)$ and use the addition lemma on H_1 pushing over the $g_i e_i$ so that H_1 represents the class $[\alpha]$. Since α is a cycle in $H_{k+1}(\tilde{C}, \partial\tilde{V})$ we may assume that the cell H_1 is attatched to ∂V. (This requires the Whitney trick to move the attaching map, hence the dimension restrictions in the proposition.) Finally then , we have a $(k+1)$-handle H_1 attached to ∂V and representing the class $[\alpha] \in H_{k+1}(\tilde{C}, \partial\tilde{V})$. Excising such handles allows us to kill $H_{k+1}(\tilde{C}, \partial\tilde{V})$.

So we have constructed a system of clean neighbouroods of infinity $\{V_i\}$ such that

1. $\pi_1(\partial V_i) \cong \pi_1(V_i) \cong \pi_1(\epsilon)$ with inclusion maps inducing isomorphisms.
2. $H_i(\tilde{V}_i, \partial\tilde{V}_i) = 0, \quad i \leq n-3.$

Regarding n and $(n-1)$-handles as dual 0 and 1-handles we can apply the above techniques to kill $H_n(\tilde{V}_i, \partial\tilde{V}_i)$ and $H_{n-1}(\tilde{V}_i, \partial\tilde{V}_i)$. Thus we are only left with $H_{n-2}(\tilde{V}_i, \partial\tilde{V}_i)$. The situation is similar to the argument in the construction of the Wall finiteness obstruction. Indeed, in that case, as in this, we needed the homology group to be (stably) free. Again, using the same kind of argument, $H_{n-2}(\tilde{V}_i, \partial\tilde{V}_i)$ need not be free, but it is at least a projective $\mathbb{Z}\pi_1(\epsilon)$-module. We obtain an element

$$\sigma(\epsilon) = \sigma(H_{n-2}(\tilde{V}_i, \partial\tilde{V}_i) \in \tilde{K}_0(\mathbb{Z}\pi_1(\epsilon)).$$

If this element is zero, after adding suitably many cancelling pairs, we may assume $H_{n-2}(\tilde{V}_i, \partial\tilde{V}_i)$ is free, in which case we can excise $(n-2)$-handles from V_i to kill $H_{n-2}(\tilde{V}_i, \partial\tilde{V}_i)$ but leave $H_{n-1}(\tilde{V}_i, \partial\tilde{V}_i)$ unchanged.

References

[1] H. Bass, A. Heller and R. Swan, *The Whitehead group of a polynomial extension*, Math. Publ. d'IHES **22** (1964), 61-80.

[2] J.M. Boardman and R. Vogt, *Homotopy invariant algebraic structures*, Lecture Notes in Mathematics **347** Springer-Verlag, New York, 1973.

[3] W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65, Springer-Verlag, New York, 1972.

[4] T. Chapman, *Topological invariance of Whitehead torsion*, Amer. J. of Math. **96** (1974), 488-497.

[5] M. Cohen, *A course in simple homotopy theory*, Springer-Verlag, New York, 1973.

[6] S.C. Ferry, A.A. Ranicki and J. Rosenberg, *Novikov conjectures, index theorems and rigidity I,II*, L.M.S. lecture note series **227**, Cambridge University Press, 1995.

[7] A. Haefliger, *Differentiable embeddings of S^n in S^{n+q} for $q > 2$*, Annals of Math. **83** (1966), 402-436.

[8] J.C. Hausmann and P. Vogel, *Geometry on Poincar'e spaces*, Princeton University Press, 1993.

[9] S. Hutt, *Poincaré Sheaves on Topological Spaces*, preprint.

[10] S. Hutt, *A counterexample to the extension of cofibrations*, Topology and Applications, **50** (1993), 87-97.

[11] F. Hirzebruch, *Topological methods in algebraic geometry*, Classics in Mathematics, Springer-Verlag, New York, 1995. (Originally printed as Vol 131 of Grundlehren der mathematische Wissenschaften.)

[12] M. Kervaire, S. Maumary and G. deRham, *Torsion et type simple d'homotopy*, LNM **48** (1967).

[13] M. Kervaire and J. Milnor, *Groups of homotopy spheres*, Annals of Math. **77** (1963), 504-537.

[14] R.C. Kirby and L.C. Siebenmann, *Foundational essays on topological manifolds, smoothings and triangulations*, Annals of Mathematics Studies, **88** Princeton University Press, 1977.

[15] I. Madsen and R.J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, **92**, Princeton University Press, 1979.

[16] J. Milnor, *Morse theory*, Princeton University Press, 1963.

[17] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Annals of Math. **64** (1956), 399-405.

[18] J. Milnor, *Whitehead torsion*, BAMS **72** (1966), 358-426.

[19] J. Milnor, *Introduction to algebraic K-theory*, Annals of Math. Studies **72** (1971), Princeton University Press.

[20] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton University Press, 1974.

[21] R. Oliver, *Whitehead groups of finite groups*, Cambridge University Press, 1988.

[22] F. Quinn, *Ends of Maps I,II,III,IV*, Annals of Math. **110** (1979), 275-331; Inven. Math. **68** (1982), 353-424; J. Diff. Geo. **17** (1982), 503-521; Amer. J. of Math. **108** (1986), 1139-1162.

[23] A.A. Ranicki, *The algebraic theory of surgery I,II*, Proc. LMS **40** (1980), 87-192, 193-283.

[24] A.A. Ranicki, *Algebraic L-theory and topological manifolds*, Cambridge Tracts in Mathematics **102**, Cambridge University Press, 1992.

[25] J. Rosenberg, *Algebraic K-theory and Its Applications*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1994.

[26] C.P. Rourke, B.J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69, Springer-Verlag, New York, 1972.

[27] C. Rourke and B. Sanderson, *On topological neighbourhoods*, Comp. Math. **22** (1970), 387-424.

[28] J.R. Silvester, *Introduction to Algebraic K-theory*, Chapman and Hall, London, 1981.

[29] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.

[30] L. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, 1965, Princeton Ph.D. thesis.

[31] M. Spivak, *Spaces satisfying Poincaré duality*, Topology **6** (1969), 77-102.

[32] D. Sullivan, *Geometric topology seminar notes*, unpublished notes, Princeton, 1968.

[33] R. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264-277.

[34] C.T.C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math. **81** (1965), 56-59; *Finiteness conditions for CW-complexes, II*, Proc. Royal Soc. Ser. A **295** (1966), 129-139.

[35] C.T.C. Wall, *Surgery on compact manifolds*, L.M.S. Monographs, Academic Press, London, 1970.

[36] J.H.C. Whitehead, *Simple homotopy types*, Amer. J. Math. **72** (1952), 163-165.

[37] S. Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, The University of Chicago Press, 1994.

Lecture Notes Series

1. M.-H. Kim (ed.), Topics in algebra, algebraic geometry and number theory, 1992
2. J. Tomiyama, The interplay between topological dynamics and theory of C*-algebras, 1992 ; 2nd Printing, 1994
3. S. K. Kim, S. G. Lee and D. P. Chi (ed.), Proceedings of the 1st GARC Symposium on pure and applied mathematics, Part I, 1993

   H. Kim, C. Kang and C. S. Bae (ed.), Proceedings of the 1st GARC Symposium on pure and applied mathematics, Part II, 1993
4. T. P. Branson, The functional determinant, 1993
5. S. S.-T. Yau, Complex hyperface singularites with application in complex geometry, algebraic geometry and Lie algebra, 1993
6. P. Li, Lecture notes on geometric analysis, 1993
7. S.-H. Kye, Notes on operator algebras, 1993
8. K. Shiohama, An introduction to the geometry of Alexandrov spaces, 1993
9. J. M. Kim (ed.), Topics in algebra, algebraic geometry and number theory II, 1993
10. O. K. Yoon and H.-J. Kim, Introduction to differentiable manifolds, 1993
11. P. J. McKenna, Topological methods for asymmetric boundary value problems, 1993
12. P. B. Gilkey, Applications of spectral geometry to geometry and topology, 1993
13. K.-T. Kim, Geometry of bounded domains and the scaling techniques in several complex variables, 1993
14. L. Volevich, The Cauchy problem for convolution equations, 1994
15. L. Elden and H. S. Park, Numerical linear algebra algorithms on vector and parallel computers, 1993
16. H. J. Choe, Degenerate elliptic and parabolic equations and variational inequalities, 1993
17. S. K. Kim and H. J. Choe (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part I, The first Korea-Japan conference of partial differential equations, 1993

    J. S. Bae and S. G. Lee (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part II, 1993

    D. P. Chi, H. Kim and C.-H. Kang (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part III, 1993
18. H.-J. Kim (ed.), Proceedings of GARC Workshop on geometry and topology ' 93, 1993
19. S. Wassermann, Exact C*-algebras and related topics, 1994
20. S.-H. Kye, Notes on abstract harmonic analysis, 1994
21. K. T. Hahn, Bloch-Besov spaces and the boundary behavior of their functions, 1994
22. H. C. Myung, Non-unital composition algebras, 1994
23. P. B. Dubovskii, Mathematical theory of coagulation, 1994
24. J. C. Migliore, An introduction to deficiency modules and Liaison theory for subschemes of projective space, 1994
25. I. V. Dolgachev, Introduction to geometric invariant theory, 1994
26. D. McCullough, 3-Manifolds and their mappings, 1995
27. S. Matsumoto, Codimension one Anosov flows, 1995
28. J. Jaworowski, W. A. Kirk and S. Park, Antipodal points and fixed points, 1995
29. J. Oprea, Gottlieb groups, group actions, fixed points and rational homotopy, 1995
30. A. Vesnin, On volumes of some hyperbolic 3-manifolds, 1996
31. D. H. Lee, Complex Lie groups and observability, 1996
32. X. Xu, On vertex operator algebras, 1996
33. M. H. Kwack, Families of normal maps in several variables and classical theorems in complex analysis, 1996
34. A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, 1996
35. Y. W. Lee, Introduction to knot theory, 1996
36. H. Kitahara, Some topics on Carnot-Caratheodory metrics, 1996 ; 2nd Printing (revised), 1998
37. D. Auckly, Homotopy K3 surfaces and gluing results in seiberg-witten theory, 1996
38. D. H. Chae (ed.), Proceedings of Miniconference of Partial Differential Equations and Applications, 1997
39. H. J. Choe and H. O. Bae (ed.), Proceedings of Korea-Japan Partial Differential Equations Conference, 1997
40. P. B. Gilkey, J. V. Leahy and J. G. Park, Spinors, spectral geometry, and Riemannian submersions, 1998
41. D.-P. Chi and G. J. Yun, Gromov-Hausdorff topology and its applications to Riemannian manifolds, 1998
42. D. H. Chae and S.-K. Kim (ed.), Proceedings of international workshop on mathematical and physical aspects of nonlinear field theories, 1998
43. H. Kosaki, Type *III* Factors and Index Theory, 1998
44. A. V. Kim and V. G. Pimenov, Numerical methods for delay differential equations - Application of i-smooth calculus-, 1999
45. J. M. Landsberg, Algebraic Geometry and projective differential geometry, 1999
46. S. Y. Choi, H. Kim and H. K. Lee(ed.), The Proceedings of the Conference on Geometric Structures on Manifolds, 1999
47. M. W. Wong, Localization Operators, 1999
48. A. M. Kytmanov, Some Applications of the Bochner-Martinelli Integral, 1999
49. R. D. Carmichael, A. Kamiński and S. Pilipović, Notes on Boundary Values in Ultradistribution Spaces, 1999
50. S. Hutt, Introduction to the Classification of Manifolds, 2000