수 학 강 의 록 제 5 권

# Complex Hypersurface Singularities with Application in Complex Geometry, Algebraic Geometry and Lie Algebra

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# PREFACE

This is the lecture notes of my serious lectures that I gave in the $1^{st}$ Global Analysis Research Center Symposium on Pure and Applied Mathematics at Seoul National University, Korea.

It represents some part of the materials in the theory of singularities which are interesting to me. I would like to thank organizers for their kind invitation as well as their hospitality while I was in Seoul. I would also like to thank Dr. Chi-Wah Leung for typing the first draft and Professor Changho Keem for finishing the final version of these notes.

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## §0 Introduction

Theory of singularities are becoming ever more the vital subject matter of several main streams of mathematics such as Algebraic geometry, Complex geometry, Differential topology, Lie algebras, etc. In fact, one can view that theory of singularities include complex projective geometry as particular case. In these lectures, we shall only consider isolated hypersurface singularities. We shall discuss some of its connection with other fields.

Let $f(z_0, z_1, \cdots, z_n)$ be a non-constant holomorphic function in $n+1$ variables. Let

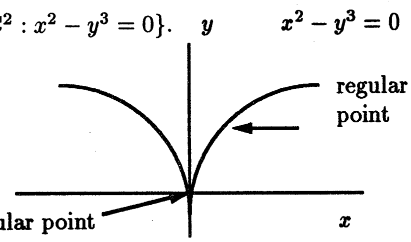
$$V = \{(z_0, z_1, \cdots, z_n) \in \mathbf{C}^{n+1} : f(z_0, \cdots, z_n) = 0\}$$

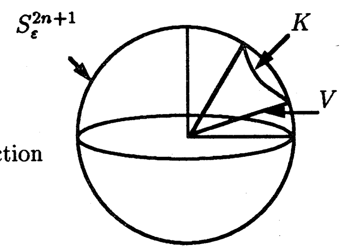
Such a set is called a complex hypersurface.

We want to study the topology and complex structure of $V$ in a neighborhood of some point $p \in V$. As an example, if $p$ is a regular point of $V$ (that is, if some partial derivative $\frac{\partial f}{\partial z_j}$ does not vanish at $p$) then $V$ is a complex manifold of complex dimension $n$ near $p$.

By way of contrast, consider the singular point $p$ of $V$ (that is if $\frac{\partial f}{\partial z_j}(p) = 0 \ \forall \ 0 \le j \le n$). It is the topology and complex structure in a neighborhood of this point that we are going to study. Throughout our lectures, we shall assume that $p$ is an isolated singularity.

Example: $V = \{(x, y) \in \mathbf{C}^2 : x^2 - y^3 = 0\}$.




Let $S_\epsilon^{2n+1}$ be a sphere of radius $\epsilon$ around $p$, and consider the intersection

$$K = V \cap S_\epsilon^{2n+1}$$

If $\epsilon$ is sufficiently small, then $K$ is a compact oriented differentiable manifold of real dimension $2n-1$ and the diffeomorphism type of $K$ does not depend on the choice of $\epsilon$. This basic construction, which associates to every isolated singularity of a complex hypersurface the diffeomorphism type of a certain differentiable manifold, establishes a very interesting connection between complex analysis and differential topology. This construction goes back at least to Brauner (1928) and Zariski (1932).

In 1966 Brieskorn discoverd that the manifold $K$ arising from a singularity $(V, p)$ may be an exotic sphere. Moreover it was found that this may happen even for very simple varieties $V$

$$\{(z_0, z_1, z_2, z_3, z_4) \in \mathbf{C}^5 : z_0^5 + z_1^3 + z_2^2 + z_3^2 + z_4^2 = 0\}$$

for which $K$ is an exotic 7-sphere. More precisely it is the standard generator of the group of the 28 exotic 7-spheres.

In 1968, Milnor proved that the manifold $K = V \cap S^{2n+1}$ is $(n-2)$-connected. He introduced a fibration which is useful in describing the topology of such manifold $K$.

This manifold $K$, being a boundary of complex variety, has some special properties.

## §1 Application in Complex Geometry

In Complex Geometry, one of the natural and fundamental questions is the *classical complex Plateau problem*. Specifically the problem asks which odd-dimensional, real compact connected manifolds in $\mathbf{C}^N$ are boundaries of complex submanifolds in $\mathbf{C}^N$. Let $X$ be a real $2n-1$ dimensionl compact connected submanifold $\subset \mathbf{C}^N$

Federer (1965): $X = \partial V, V$ complex submanifold in $\mathbf{C}^N$.
$\Rightarrow V$ is a unique minimal submanifold having boundary $X$.

As for the existence problem, naturally we do not expect solution for arbitrary submanifold $X$.

For $\dim_{\mathbf{R}} X = 1$, there is a Necessary condition for $X$ to be satisfied:

$$\int_X \omega = 0 \quad \forall \text{ holomorphic } 1-\text{form } \omega \tag{1.1}$$

This can be seen as follows. Suppose $X = \partial V$. Then by Stoke's theorem $\int_X \omega = \int_{\partial V} \omega = \int_V d\omega = \int_V \partial\omega = 0$ because $d = \partial + \bar{\partial}$ and $\partial\omega$ is a (2,0)-form.

There is a corresponding question in uniform algebra: Find complex structure in the maximal ideal space of the subalgebra $P(X)$ of algebra of continuous function $C(X)$ generated by polynomials. i.e. Find the complex structure in $\hat{X} - X$ where $\hat{X}$ is the polynomial hull of $X$.

Wermer (1958): Affirmative answer when $X$ is a simple closed analytic Jordan curve.

Bishop – Stolzenberg (1966): Affirmative answer when $X$ is a continuous differentiable simple closed curve.

Alexander (1971): Affirmative answer when $X$ is a rectifiable simple closed curve.

For $\dim X = 2n-1, n > 1$ Necessary conditions for X to be a boundary of a complex submanifold V in $\mathbf{C}^N$ are as follows:

1) $X$ is orientable, with orientation being induced from $V$.
2) $\forall p \in X \ \dim_{\mathbf{C}}(T_p(X) \cap JT_p(X)) = n-1$, where $J$ is the complex structure of $\mathbf{C}^N$.

Note that $H_p = T_p(X) \cap JT_p(X)$ is invariant under $J$. So $H$ is a complex vector space.

2) is true because $T_p(X) \cap JT_p(X)$ is the orthogonal complement in $T_p(V)$ of the complex line spanned by the normal vector to the boundary $V$ at $p$. We can also see 2) by direct computation

$$\dim_{\mathbf{R}}(T_p(X) \cap JT_p(X)) = \dim_{\mathbf{R}} T_p(X) + \dim_{\mathbf{R}} JT_p(X) - \dim_{\mathbf{R}} T_p(V) = 2n-1+2n-1-2n = 2(n-1)$$

Definition : Let $X$ be a compact, orientable real manifold of dim 2n-1. A CR-structure on $X$ is an (n-1)-dimensional subbundle $S$ of $\mathbf{C}TX$ $(= TX \otimes_{\mathbf{R}} \mathbf{C})$ such that

(1) $S \cap \bar{S} = \{0\}$
(2) If $L, L'$ are local sections of $S$, then so is $[L, L']$.

Such a manifold $X$ is called CR-manifold.

Equivalently, $(X, H, J)$ is a CR-manifold if $H$ is a subbundle of $TX$ with $\dim_{\mathbf{R}} H = 2(n-1)$, $J : H \to H$, and $J^2 = -$Identity. If $X$ and $Y$ are in $H$, then so is $[JX, Y] + [X, JY]$ and $J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y]$.

Let us show that the two definitions of a CR-structure are equivalent. Given $S$ we choose some basis $\{L_1, \cdots, L_{n-1}\}$ and note that $S \cap \bar{S} = \{0\}$ implies that $\{ReL_1, \cdots, ImL_{n-1}\}$ is linearly independent. Set $H$ equal to the linear space over $R$ of this set and define $J$ by $J(ReL_k) = ImL_k$. $J(ImL_k) = -ReL_k$. $H$ clearly does not depend on the choice of basis. The map $J$ extends to a complex linear map of $\mathbf{C} \otimes H$ to itself with $S$ as its $-i$ eigenspace and $\bar{S}$ as its $+i$ eigenspace. So $J$ also is independent of the choice of basis. The integrability condition for $J$ follows from that for $S$.

Conversely, given $H$ and $J$, extend $J$ to a complex linear map of $\mathbf{C} \otimes H$ to itself and let $S$ be the $-i$ eigenspace. Clearly $S \cap \bar{S} = \{0\}$. And now the integrability condition for $J$ implies the one for $V$.

In summary, the condition

$$L \in S$$

is equivalent to

$$L = X + iJX \qquad \text{for some } X \in H$$

Remark: If $X = \partial V$, $V$ complex submanifold, then $S = (\prod_{0,1} \mathbf{C}TV) \cap \mathbf{C}TX$ defines a CR-structure on $X$, where $\prod_{0,1} \mathbf{C}TV$ denotes the (0,1)-vector on $V$.

Definition: Let $X$ be a CR-manifold of dimendion 2n-1.
Let $L_1, L_2, \cdots, L_{n-1}$ be a local basis for $\bar{S}$ over $U \subset X$.
$\bar{L}_1, \bar{L}_2, \cdots, \bar{L}_{n-1}$ be a local basis for $S$ over $U \subset X$.
Choose $N$ purely imaginary (i.e. $\bar{N} = -N$) s.t. $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}, N$ span $\mathbf{C}TX$. Then the matrix $(c_{ij})$ defined by

$$[L_i, \bar{L}_j] = \sum a^k_{ij} L_k + \sum b^k_{ij} \bar{L}_k + c_{ij} N$$

is Hermitian; it is called the Levi form.

Remark: The Levi form is non-invariant; however its essential features are invariant.

**Proposition :** The number of nonzero eigenvalues and the absolute value of the signature $(c_{ij})$ at each point are independent of the choice of $L_1, \cdots, L_{n-1}, N$.

Definition: A CR-manifold $X$ is strongly pseudoconvex if the Levi form is definite.

Rossi (1964): Let $X$ be a strongly pseudoconvex CR-manifold of dim 2n-1,$n \geq 3$. Then $X$ is a boundary of a uniquely determined, bounded complex analytic variety $V$ in $\mathbf{C}^N - X$. Furthermore, there exists boundary regularity i.e.$X \cup V$ is a regular $C^\infty$ submanifold with boundary near $X$.

In 1975 Harvey-Lawson developed an extremely important theory on boundaries of complex analytic varieties. In particular they have proved the following significant theorem.

Harvey – Lawson (1975): Let $X$ be a CR-manifold of $\dim_{\mathbf{R}} = 2n-1, n \geq 2$ in $\mathbf{C}^N$. (no further assumption!) Then $X$ is a boundary (as a current) of a uniquely determined, bounded complex analytic variety $V$ in $\mathbf{C}^N - X$.

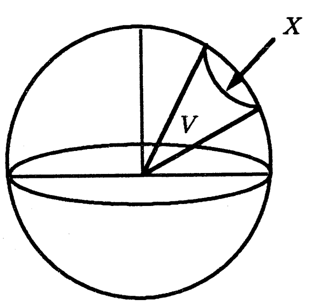
If $\dim_{\mathbf{R}} X = 1$ and $X$ satisfies (1), then $X$ is a boundary of a uniquely determined, bounded complex analytic curve $V$ in $\mathbf{C}^N - X$.

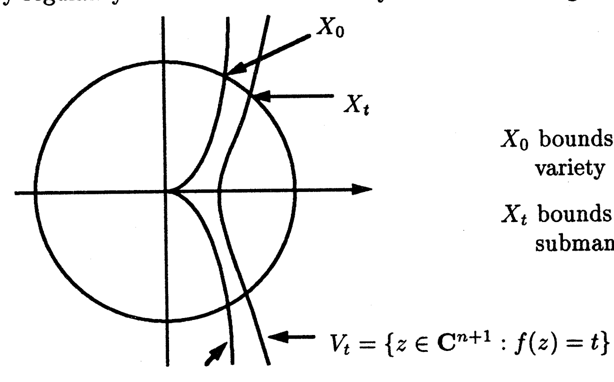
Let us denote $[V] = 2n$ dimensional current in $\mathbf{C}^N$ by integration over manifold points of $V$ (with the canonical orientation) of $\mathbf{C}^\infty$ 2n- forms with compact support in $\mathbf{C}^N$. Then $d[V] = [X]$ means the following
$d[V](\alpha) = [V](d\alpha) = [X](\alpha)$, for any $C^\infty$ (2n-1)-form $\alpha$ with compact support.

With different technique, we have proved the following.

**Theorem 1:** Let $X$ be a connected CR-manifold of dim 2n-1, $n \geq 2$ in $\mathbf{C}^N$. Suppose the Levi form of $X$ is not identically zero at every point of $X$. Then $\exists !$ bounded complex analytic subvariety $V$ of dim n in $\mathbf{C}^N - X$ such that the boundary of $V$ is $X$ in the sense of point set topology. Moreover, outside a set of (2n-1)-measure zero in $X$, $V$ has boundary regularity.

So far the original complex Plateau problem remained unsolved. The strongly pseudoconvexity only guarantees that $V$ has boundary regularity near $X$. However $V$ may have interior singularity.




$X_0$ bounds a complex subvariety with singularity

$X_t$ bounds a complex submanifold.

$V_0 = \{z \in \mathbf{C}^{n+1} : f(z) = 0\}$

Observe that in the above picture, $X_0$ is diffeomorphic to $X_t$. The natural question is whether $X_0$ is CR isomorphic to $X_t$. For this purpose, we need to introduce CR-invariants. In (1965) Kohn-Rossi defined their cohomology on CR-manifolds. It can be defined in the following manner.

Let $M$ be a relatively compact open domain in complex mamifold $M'$ of dimension n and $X = \partial M$. Let r be a $C^\infty$ real valued function defined in a neighborhood of $X$ such that $dr \neq 0$ and $r < 0$ in $M$, $r > 0$ outside $M \cup X$.

$\tilde{\mathbf{A}}^{p,q}$ = sheaf of germs of $C^\infty(p,q)$ forms on $\bar{M}$

$\tilde{\mathbf{C}}^{p,q} = \{\phi \in \tilde{\mathbf{A}}^{p,q} : \phi = \bar{\partial} r \wedge \psi + r\theta$ for some $\psi \in \tilde{\mathbf{A}}^{p,q-1}$ and $\theta \in \tilde{\mathbf{A}}^{p,q}\}$ which is a subsheaf of $\tilde{\mathbf{A}}^{p,q}$

$\tilde{\mathbf{B}}_X^{p,q} := \tilde{\mathbf{A}}^{p,q}/\tilde{\mathbf{C}}^{p,q}$ locally free sheaf supported on $X$.

Since $\bar{\partial}(\psi \wedge \bar{\partial} r + r\theta) = (\bar{\partial}\psi - \theta) \wedge \bar{\partial} r + r\bar{\partial}\theta, \bar{\partial} : \mathbf{C}^{p,q} \to \mathbf{C}^{p,q+1}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathbf{C}}^{p,q} & \longrightarrow & \tilde{\mathbf{A}}^{p,q} & \longrightarrow & \tilde{\mathbf{B}}_X^{p,q} & \longrightarrow & 0 \\ & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial}_b & & \\ 0 & \longrightarrow & \tilde{\mathbf{C}}^{p,q+1} & \longrightarrow & \tilde{\mathbf{A}}^{p,q+1} & \longrightarrow & \tilde{\mathbf{B}}_X^{p,q+1} & \longrightarrow & 0 \end{array}$$

Let $\mathbf{B}_X^{p,q}$ = space of sections of $\tilde{\mathbf{B}}_X^{p,q}$.

Then Kohn-Rossi complex is: $0 \longrightarrow \mathbf{B}_X^{p,0} \xrightarrow{\bar{\partial}_b} \mathbf{B}_X^{p,1} \xrightarrow{\bar{\partial}_b} \cdots \longrightarrow \mathbf{B}_X^{p,n-1} \longrightarrow 0$

and Kohn-Rossi cohomology

$$H_{KR}^{p,q}(X) := \frac{\{\phi \in \mathbf{B}_X^{p,q} : \bar{\partial}_b \phi = 0\}}{\bar{\partial}_b \mathbf{B}_X^{p,q-1}} \quad .$$

As a consequence of Kohn's solution to $\bar{\partial}$-Neumann problem in 1964, Kohn-Rossi proved the following

**Theorem** (Kohn – Rossi 1965):

For strongly pseudoconvex CR-manifold $X, H_{KR}^{p,q}(X)$ is finite dimensional for $1 \leq q \leq n-2$.

Suppose $X$ is a strongly pseudoconvex CR-manifold $\subset \mathbf{C}^N$. Then $X$ is a boundary of a complex analytic subvariety $V$ with isolated singularities $x_1, \cdots, x_m$.

Kohn – Rossi Conjecture (1965): In general, either there is no Kohn-Rossi cohomology of the boundary $X$ of $V$ in degree $(p,q), q \neq 0, n-1$, or it must result from the interior singularities of $V$.

Our following theorem answers the conjecture affirmatively.

**Theorem 2**: Let $X$ be a strongly pseudoconvex CR-manifold of dimension $2n-1, n \geq 3$ which is a boundary of a Stein analytic space $V$ with isolated singularities $x_1, \cdots, x_m$. Then

$$H_{KR}^{p,q}(X) \cong \bigoplus_{i=1}^{m} H_{x_i}^{q+1}(V, \Omega_V^p) \qquad 1 \leq q \leq n-2 \ .$$

Suppose $x_1, \cdots, x_m$ are hypersurface singularities. Then

$$\dim H_{KR}^{p,q} = \begin{cases} 0 & p+q \leq n-2 & 1 \leq q \leq n-2 \\ \sum_{i=1}^{m} \dim A(V, x_i) & p+q = n-1, n & 1 \leq q \leq n-2 \\ 0 & p+q \geq n+1 & 1 \leq q \leq n-2 \end{cases}$$

This is the first explicit computation of Kohn-Rossi's cohomology. Here we denote

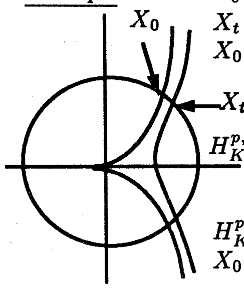
$$A(V, x_i) := \text{moduli algebra of } V \text{ at } x_i$$

Let $\tau_i = \dim A(V, x_i). \tau_i$ is called the number of local moduli of $V$ at $x_i$

Suppose $V = \{z \in \mathbf{C}^{n+1} : f(z) = 0\}$. Then

$$A(V, 0) := \mathbf{C}\{z_0, \cdots, z_n\} \Big/ \left(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n}\right)$$

Example: $X_0 = \{z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0\} \cap \{|z_0|^2 + \cdots + |z_n|^2 = 1\} \subseteq \mathbf{C}^{n+1}$




$X_t = \{z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = t\} \cap \{|z_0|^2 + \cdots + |z_n|^2 = 1\} \subseteq \mathbf{C}^{n+1}$

$X_0$ is diffeomorphic to $X_1$

$$H_{KR}^{p,q}(X_0) = \begin{cases} 0 & p+q \leq n-2 \quad 1 \leq q \leq n-2 \\ (a_0-1)\cdots(a_n-1) & p+q = n-1, n \quad 1 \leq q \leq n-2 \\ 0 & p+q \geq n+1 \quad 1 \leq q \leq n-2 \end{cases}$$

$H_{KR}^{p,q}(X_1) = 0$

$X_0 \ncong X_1$ as CR-manifold.

As a corollary of the previous theorem, we have proved

**Theorem 3** (Solution to complex Plateau Problems) : Let $X$ be a strongly pseudoconvex CR-manifold of real codimension 3 in $\mathbf{C}^{n+1}, n \geq 3$. Then $X$ is a boundary of the complex submanifold $V \subset \mathbf{C}^{n+1} - X$ if and only if Kohn-Rossi's cohomology groups $H_{KR}^{p,q}(X)$ are zero for $1 \leq q \leq n-2$.

The above Theorem 2 actually says that compact CR-manifolds of real codimension three in $\mathbf{C}^{n+1}$ are quite distinquished.

Definition: Let $X$ be a compact (2n-1)-dimensional strongly pseudoconvex CR-manifold in $\mathbf{C}^{n+1}, n \geq 2$. By a theorem of Harvey and Lawson, $X$ is the boundary of a complex variety $V$ in the $C^\infty$ sense. $V$ is smooth except at finitely many isolated singular points $\{x_1, \cdots, x_k\}$. Let $\tau_i$ be the number of local moduli of $V$ at $x_i$. We define $\tau(X)$ to be $\tau_1 + \tau_2 + \cdots + \tau_k$.

**Theorem 4**: Let $X$ be a compact connected (2n-1)-dimensional strongly pseudoconvex CR-manifold in $\mathbf{C}^{n+1}, n \geq 2$, Then $\tau(X)$ defined above is a CR-invariant in the sense that if $X' \subseteq \mathbf{C}^{n+1}$ is another (2n-1)-dimensional CR-manifold which is CR-diffeomorphic to $X$, then $\tau(X) = \tau(X')$. Moreover, $X$ is a boundary of the complex submanifold $V \subseteq \mathbf{C}^{n+1} - X$ if and only if $\tau(X) = 0$. In fact for $n \geq 3, \tau(X) = \dim H_{KR}^{p,q}(X)$ for $p+q = n-1, n$ and $1 \leq q \leq n-2$.

Open Problem : Given an intrinsic interpretation to $\tau(M)$ for $n = 2$.

Definition : Let $X$ be a CR-submanifold in $\mathbf{C}^N$. Set $H_x = T_xX \cap JT_xX$ for $x \in X$, where $J$ denotes the complex structure of $\mathbf{C}^N$. A smooth $S^1$-action on $X$ is said to be holomorphic if it preserves the family of subspaces $H_x \subset TX$ and commutes with $J$. It is said to be transversal if, in addition, the vector field which generates the action, is transversal to $H_x$ for all $x \in X$.

We have the following solution to equivariant Complex Plateau Problem.

**Theorem 5** : (Lawson-Yau) Let $X \subseteq \mathbf{C}^{n+1}$ be a CR-manifold of dimension $2n-1 > 1$. Suppose that $X$ admits a transversal holomorphic $S^1$-action. Then after a biholomorphic change of coordinates in $\mathbf{C}^{n+1}$, $X$ is contained in an affine algebraic hypersurface $Y \subseteq \mathbf{C}^{n+1}$. The hypersurface $Y$ has at most one singular point. It also has a $\mathbf{C}^*$-action and the embedding $X \subseteq Y$ is $S^1$-equivariant.

In Theorem 2, we have actually established a natural vector space isomorphism from $H_{KR}^{p,q}(X)$ to $\oplus_{i=1}^m A(V, x_i)$ for $p+q = n-1$ and $1 \leq q \leq n-2$. Since each $A(V, x_i)$ has a natural Artinian algebra structure, we can use this isomorphism to put algebra structure on $H_{KR}^{p,q}(X)$. Then an algebraic condition can be given as follows, which is powerful enough to determine when two CR-submanifolds in $\mathbf{C}^{n+1}$ of dimension 2n-1 are diffeomorphic to each other.

**Theorem 6** : (Lawson-Yau) Let $X, X' \subseteq \mathbf{C}^{n+1}$ be strongly pseudoconvex CR-manifolds of dimension 2n-1 with transversal $S^1$ actions. Suppose that there exists an algebra isomorphism

$$H_{KR}^{p,q}(X) \cong H_{KR}^{p,q}(X')$$

for some $(p,q)$ with $p+q = n-1, 1 \leq q \leq n-2$. Then there exists a diffeomorphism $f : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ with $f(X) = X'$

The proof of this theorem depends on Theorem 5 and a theorem of Mather and Yau which will be discussed below in §2.

Let $X$ be a compact CR manifold of real dimension 2n-1. One of the fundamental questions in the theory of CR geometry is to decide when $X$ is CR embeddable in $\mathbf{C}^N$, and what is the minimal embedding dimension. These problems have attracted many distinguished mathematicians. In 1974, Boutet de Monvel (and Kohn 1985) proved that if $X$ is a compact $C^\infty$ strongly pseudoconvex CR manifold of dimension 2n-1 and $n \geq 3$, then $X$ is CR-embeddable in $\mathbf{C}^N$. In a series of deep papers published in 1982, Kuranishi developed the theory of harmonic integrals on strongly pseudoconvex CR structures over small balls along the line developed by D.C. Spencer, C.B. Morrey, J.J. Kohn and L. Nirenberg. As a significant application of his deep theory, he proved that for strongly pseudoconvex CR manifold of real dimension 2n-1, there exists local CR embedding in $\mathbf{C}^n$ as long as $n \geq 5$. Later Akahori proved that for $n = 4$, Kuranishi local embedding theorem is still true. On the other hand L. Nirenberg gave an example of a strongly pseudoconvex CR structure which cannot be induced by such an embedding.

Recently, Kohn has studied CR manifolds which are boundaries of bounded pseudoconvex domains in n-dimensional complex manifolds. He used the following notation: the restriction of $\overline{\partial}$ will be denoted by $\overline{\partial}_b$. He proved that the range of $\overline{\partial}_b$ in $L_2$ is closed. For a compact strongly pseudoconvex manifold, the closed range property on functions implies that the manifold is embeddable in $\mathbf{C}^N$. H. Grauert has constructed compact 3-dimensional, strongly pseudoconvex CR manifolds which are not embeddable. Such examples were also studied by H. Rossi and by D. Burns.

In the rest of this section, we shall assume that the compact CR manifold $X$ of real dimension 2n-1 is already embeddable in $\mathbf{C}^N$. This hypothesis is automatically satisfied when $n \geq 3$. The natural important question is to find the minimal embedding dimension. As a Corollary of Theorem 2, we can see that for $n \geq 3$, certain (2n-1)-dimensional CR manifolds which are embedded in $\mathbf{C}^{n+1}$, cannot be embedded in $\mathbf{C}^n$ and there are obstructions for embedding (2n-1)-dimensional CR-manifolds in $\mathbf{C}^{n+1}$ as well.

Example: $X = \{z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0\} \bigcap \{|z_1|^2 + \cdots + |z_n|^2 = 1\} \subseteq \mathbf{C}^{n+1}$ is a (2n-1)-dimensional CR manifold which is not CR-embeddable in $\mathbf{C}^n$.

**Theorem 7 :** Let $X$ be a compact connected real (2n-1)-dimensional strongly pseudoconvex CR-manifold with $n \geq 3$. Then $X$ is not CR-embeddable in $\mathbf{C}^{n+1}$ if one of the following does *not* hold.

(1) $H^{p,q}_{KR}(X) = 0$ for $p+q \leq n-2, 1 \leq q \leq n-2$

(2) $\dim H^{p,q}_{KR}(X) = \dim H^{p',q'}_{KR}(X)$ for $p+q = n-1$ or $n, 1 \leq q \leq n-2$ and $p'+q' = n-1$ or $n, 1 \leq q' \leq n-2$

(3) $H^{p,q}_{KR}(X) = 0$ for $p+q \geq n+1, 1 \leq q \leq n-2$

Now let $X$ be a compact connected strongly pseudoconvex 3-dimensional CR manifold which is embedded in $\mathbf{C}^N$. By a theorem of Harvey and Lawson, $X$ is the boundary of a complex variety $V$ in the $C^\infty$ sense. $V$ is smooth except at finitely many isolated points $\{x_1, \cdots, x_k\}$. Let $\pi : M \to V$ be a resolution of singularities of $V$. Denote $A = \cup_{i=1}^k A_i$, where $A_i = \pi^{-1}(x_i)$. By successive blowing up at points, we may assume that $A$ has normal crossings, i.e. irreducible components of $A_i$ are nonsingular, they intersect transversely and no three meet at a point. By a theorem of Grauert and Mumford, one can define a divisor $K$ supported in $A$ with coefficients in rational numbers such that

$$A_i^j \cdot K = -A_i^j \cdot A_i^j + 2g_i^j - 2$$

where $A_i^j$ is an irreducible component of $A_i$ and $g_i^j$ is the genus of $A_i^j$. Then we have the following Theorem.

**Theorem 8** (Luk-Yau) Let $X$ be a compact connected 3-dimensional strongly pseudoconvex CR manifold embeddable in complex Euclidean space. Denote $M, A, K$ as above. Then

(1) $p_g = \dim H^1(M, \mathcal{O}), \chi = \chi_T(A) + K \cdot K$ and $\omega = \dim H^1(M, \Omega^1) + K \cdot K$ depend only on $X$ itself, where $\mathcal{O}$ is the structure sheaf, $\Omega^1$ is the sheaf of germs of holomorphic 1-forms on $M$ and $\chi_T(A)$ is the topological Euler characteristic of $A$.

(2) $X$ is not CR-embeddable in $\mathbf{C}^3$ if either
   (a) $\chi = \chi_T(A) + K \cdot K$ is not integral, or
   (b) $\omega = \dim H^1(M, \Omega^1) + K \cdot K$ is not integral, or
   (c) $10\, p_g + \omega = 10 \dim H^1(M, \mathcal{O}) + \dim H^1(M, \Omega^1) + K \cdot K < 0$

Actually part (1) of Theorem 8 follows immediately from the following diagram :

$$\begin{array}{ccccccc}
\textit{point modification of } M & = & \overline{M} & \overset{=}{\longleftrightarrow} & \overline{M} & = & \textit{point modification of } M' \\
 & & \downarrow & & \downarrow & & \\
\textit{resolution of} \tilde{V} & = & M & & M' & = & \textit{resolution of} \tilde{V}' \\
 & & \downarrow & & \downarrow & & \\
\textit{normalization of } V & = & \tilde{V} & \overset{\tilde{\varphi}}{\longrightarrow} & \tilde{V}' & = & \textit{normalization of } V' \\
 & & \downarrow & & \downarrow & & \\
X = \partial V & \subseteq & V & \subseteq \mathbf{C}^N, \mathbf{C}^{N'} \supseteq & V' & \supseteq & \partial V' = X'
\end{array}$$

(where $\tilde{\varphi} : V \to \tilde{V}'$ is biholomorphic map and $\varphi : X \to X'$ is CR-isomorphism.) and the fact that $\chi_T(A)$ increases by one after one blow up, and $K \cdot K$ decreases by one after one blow up, while $p_g$ remains the same after one blow up.

We next consider a compact connected 3-dimensional CR manifold $X$ which admits a transversal holomorphic action of $S^1$ and which is embeddable in $\mathbf{C}^N$. Then by a Theorem of Lawson-Yau (Theorem 6), $X$ is the boundary of a complex variety $V$ (with at most one isolated singularity) in the $C^\infty$ sense. The following Theorem 9 is deeper than the part(2)(c) of Theorem 8.

**Theorem 9** : Let $X$ be a compact connected 3-dimensional CR-manifold which admits a transversal holomorphic $S^1$-action and which is embeddable in $\mathbf{C}^N$. Denote $M, A$ and $K$ as before. Then $X$ is not CR-embeddable in $\mathbf{C}^3$ if

$$6\, p_g + \chi \leq 0$$

Before we can prove Theorem 9, we need to recall the Durfee conjecture. Let $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ be the germ of complex analytic function with an isolated critical point at the origin. For $\epsilon > 0$ suitable small and $\delta$ yet smaller, the space $V' = f^{-1}(\delta) \cap B_\epsilon$ (where $B_\epsilon = \{(x,y,z) : |x|^2 + |y|^2 + |z|^2 \leq \epsilon^2\}$) is a real oriented four manifold with boundary whose diffeomorphic type depends only on $f$. $V'$ has the homotopy type of a wedge of two-spheres; the number $\mu$ of two-spheres is precisely $\dim \mathbf{C}\{x,y,z\}/(f_x, f_y, f_z)$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of $V = \{(x,y,z) : f(x,y,z) = 0\}$ with exceptional set $A = \pi^{-1}(0)$. Let

$p_g = \dim H^1(M, \mathcal{O})$

$\chi_T(A) =$ topological Euler characteristic of $A$

$K^2 =$ self intersection number of the canonical divisor on $M$

Recall that Laufer's formula says that $1 + \mu = \chi_T(A) + K^2 + 12 p_g$. However the formula does not provide a direct comparison between $\mu$ and $p_g$, which are two important numerical measures of the complexity of the singularity. In 1978, Durfee has made the following spectacular conjecture.

Durfee Conjecture : $6p_g \leq \mu$ with equality only when $\mu = 0$

Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a holomorphic function with an isolated critical point at 0. $f$ can be developed in a convergent Taylor series $\sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbf{R}^+)^{n+1}\}$ for $\lambda$ such that $a_\lambda \neq 0$. $\Gamma_-(f)$, the Newton polyhedron of $f$, is the cone over $\Gamma(f)$ with vertex 0. For any closed face $\Delta$ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that $f$ is non-degenerate if $f_\Delta$ has no critical point in $(\mathbf{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$.

The geometric genus of the isolated hypersurface singularity $(V, 0)$ defined by $f(z_0, \cdots, z_n) = 0$ is $p_g = \dim H^{n-1}(M, \mathcal{O})$ where $M$ is a resolution of the singularity $(V, 0)$.

**Theorem** (Hodge $n = 2$, Bernstein, Khovanski, Kouchnirenko $n \geq 2$, unpublished but announced by Arnold (1975), Merle-Teissier published 1980)

Let $(V,0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$. Then the geometric genus $p_g=\#\{p\in\mathbf{Z}_+^{n+1}\cap\Gamma_-(f): p \text{ is positive i.e. } p=(p_0,\cdots,p_n), p_i>0\ \forall i\}$.

A polynomial $f(z_0,z_1,\cdots,z_n)$ is weighted homogeneous of type $(w_0,w_1,\cdots,w_n)$, where $(w_0,w_1,\cdots,w_n)$ are fixed rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0}z_1^{i_1}\cdots z_n^{i_n}$ for which $\frac{i_0}{w_0}+\cdots+\frac{i_n}{w_n}=1$.

**Theorem** (Milnor-Orlik)

Let $f(z_0,z_1,\cdots,z_n)$ be a weighted homogeneous polynomial of type $(w_0,w_1,\cdots,w_n)$ with isolated singularity at origin. Then $\mu=(w_0-1)(w_1-1)\cdots(w_n-1)$ where $\mu$ is the Milnor number of $f$.

The general problem of counting the number of positive integral points satisfying

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1 \tag{1.2}$$

where $a,b,c$ are positive numbers, has been a challenging problem for many years. The difficulty lies from the facts that $a,b,c$ are *not* necessarily integers or even rational numbers and it is very hard to estimate the number of positive integral points satisfying $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. The corresponding problem with (1.2) being a strict inequality has been discussed by Lehmer, Lochs and Ehrhart. There are extensive references in Reviews in Number Theory 1940-1972, Vol.4. However, we know of no definite results. Perhaps the most beautiful formula is due to Mordell who actually gave an exact formula for the number of positive integral solutions of (1.2) when $a,b,c$ are relatively positive integers. Recently by using the technique of toridal embedding, James Pommersheim was able to give an exact formula for the number of positive integral solutions of (1.2) when $a,b,c$ are arbitrary positive integers. However this formula is not useful as far as the Durfee conjecture is concerned.

**Theorem 10** (Xu-Yau) Let $a\ge b\ge c\ge 2$ be real numbers. Let $P$ be the number of positive integral solutions of (1.2) i.e. $p=\#\{(x,y,z)\in\mathbf{Z}_+^3:\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\}$. Then

$$6P\le(a-1)(b-1)(c-1)-c+1$$

and the equality is attained if and only if $a=b=c=$ integer.

The following theorem solves the Durfee conjecture affirmatively.

**Theorem 11** (Xu-Yau) Let $(V,0)$ be a two dimensional isolated singularity defined by a weighted homogeneous polynomial $f(z_0,z_1,z_2)=0$. Let $\mu$ be the Milnor $\#$, $p_g$ be the geometric genus and $\nu$ be the multiplicity of the singularity. Then

$$\mu-\nu+1\ge 6p_g$$

with equality if and only if $(V,0)$ is defined by homogeneous polynomial.

In view of the Theorem of Hodge, Bernstein-Khovanski-Kouchnirenko, Merle-Teissier and the Theorem of Milnor-Orlik, one may think that Theorem 11 is a direct consequence of Theorem 10. Actually this is not quite so because as we shall see later §3 $\nu=\min\{m\in\mathbf{Z}:m\ge\min\{w_0,w_1,w_2\}\}$ where $w_0,w_1$ and $w_2$ are weights of $z_0,z_1$ and $z_2$ respectively. The problem here is that $\min\{w_0,w_1,w_2\}$ may not be an integer. So we have to improve the inequality of Theorem 10 slightly in a special case. The following observation is a consequence of results in §3. Suppose $w_0\ge w_1\ge w_2$ and $w_2$ is not an integer. Let $\omega_2=[w_2]+\beta$ with $0<\beta<1$. Then $\beta$ is either $\frac{w_2}{w_0}$ or $\frac{w_2}{w_1}$. The sharper inequality that we needed is the following

**Theorem 11′** (Xu-Yau) Let $a\ge b\ge c\ge 2$ be real numbers. Consider

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1$$

Let $P$ be the number of positive integral solutions of the above inequality i.e. $P=\#\{(x,y,z)\in\mathbf{Z}_+^3:\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\le 1\}$. Suppose $c$ is not an integer and $c=[c]+\beta$ where $\beta$ is either $\frac{c}{a}$ or $\frac{c}{b}$. Then

$$6P<(a-1)(b-1)(c-1)-c+\beta$$

Now we are ready to prove Theorem 11.

Let $w_0, w_1$ and $w_2$ be the weights of $z_0, z_1$ and $z_2$ respectively so that $f(z_0, z_1, z_2)$ is a weighted homogeneous polynomial. By a theorem of Saito, we may assume without loss of generality that $w_0 \geq w_1 \geq w_2 \geq 2$. Since $p_g = \#\{(x,y,z) \in \mathbf{Z}_+{}^3 : \frac{x}{w_0} + \frac{y}{w_1} + \frac{z}{w_2} \leq 1\}$ and $\mu = (w_0 - 1)(w_1 - 1)(w_2 - 1)$, therefore theorem 10 implies

$$6p_g \leq \mu - w_2 + 1 \tag{1.3}$$

with the equality if and only if $w_0 = w_1 = w_2 =$ integer. Recall that $\nu = \inf\ \{n \in \mathbf{Z}_+ : n \geq \inf\ (w_0, w_1, w_2)\}$ $= [w_2] + 1$. If $w_2$ is an integer, then $\nu = w_2$ and Theorem 11 follows directly from (1.3). If $w_2$ is not an integer, then necessarily $w_2 = [w_2] + \beta$ with $0 < \beta < 1$, and $\beta$ is either $\frac{w_2}{w_0}$ or $\frac{w_2}{w_1}$.

In view of Theorem 11′, we have

$$\begin{aligned} 6p_g &< (w_0 - 1)(w_1 - 1)(w_2 - 1) - w_2 + \beta \\ &= \mu - [w_2] \\ &= \mu - \nu + 1 \end{aligned}$$

So we have proved $6p_g \leq \mu - \nu + 1$ and the equality holds only if $(V, 0)$ is defined by homogeneous polynomial.

It remains to prove that if $(V, 0)$ is defined by homogeneous polynomial of degree $\nu$, then $\mu - \nu + 1 = 6p_g$. One checks easily that $\mu = (\nu - 1)^3$ and $p_g = \frac{1}{6}\nu(\nu - 1)(\nu - 2)$. So the desired equality follows immediately.

Let us now give a proof of Theorem 9. For this purpose, it is sufficient to prove that if $X$ is embeddable in $\mathbf{C}^3$, then $6p_g + \chi > 0$. By Theorem 5, if $X$ is embeddable in $\mathbf{C}^3$, then $X$ is the boundary of a subvariety $V \subset \mathbf{C}^3$. The hypersurface $V$ has at most one singular point. In fact it is a quasi-homogeneous isolated hypersurface singularity. In view of Theorem 11 $\mu \geq 6p_g$ and Laufer's formula $1 + \mu = 12p_g + \chi$, we have $6p_g + \chi \geq 1$ as desired.

Given a function $f$ with an isolated singularity at origin. It is an important question to know whether $f$ is a weighted homogeneous polynomial or homogeneous polynomial after a biholomorphic change of variables. The former question was answered by Saito in 1973. However the latter question remains open ever since. In case $f$ is a holomorphic function of three variables, the problem is solved by the following theorem

**Theorem 12** (Xu-Yau) Let $(V, 0)$ be a two-dimensional isolated hypersurface singularity defined by $f(x,y,z) = 0$. Let $\mu$ be the Milnor number, $p_g$ be the geometric genus, $\nu$ be the multiplicity of the singularity and $\tau =$ dimension of the semiuniversal deformation space of coordinates $(V, 0)$=dim $\mathbf{C}\{x,y,z\}/(f, f_x, f_y, f_z)$. Then after a biholomorphic change of coordinate $f$ is a homogeneous polynomial if and only if $\mu - \nu + 1 = 6p_g$ and $\mu = \tau$.

## §2. Equivalence of Singularities: Holomorphic case

Let $\mathcal{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $f : (\mathbf{C}^{n+1}, 0) \to \mathbf{C}$. $\mathcal{O}_{n+1}$ has a unique maximal ideal $m_{n+1}$ consisting of the germs of holomorphic functions which vanish at the origin. Let $G_{n+1}$ be the set of germs at the origin of biholomorphism $\phi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$. $G_{n+1}$ can be made into a group by using composition of map germs for the group operation.

<u>Definition</u> Two germs of holomorphic functions $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are called <u>right equivalent</u> if there exists a $\phi \in G_{n+1}$ such that $f = g \circ \phi$. We use the notation $f \overset{R}{\sim} g$ to denote right equivalence

The group $\mathcal{R} = G_{n+1}$ acts on $m_{n+1}$ by composition on the right. The right equivalence classes are the orbits of this group. The orbit of $f \in m_{n+1}$ is denoted by

$$\mathcal{R}(f) = \{g \in m_{n+1} : g \overset{R}{\sim} f\}$$

<u>Definition</u> Two germs of holomorphic functions $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are called right-left equivalence if there exists $\phi \in G_{n+1}$ and $\psi \in G_1$ such that $f = \psi \circ g \circ \phi$. The notation $f \overset{RL}{\sim} g$ is used to indicate right-left equivalence.

Right-left equivalence also arises from a group action. The group $\mathcal{RL} = G_1 \times G_{n+1}$ acts on $m_{n+1}$ by composing on the left with the $G_1$ component and on the right with the component from $G_{n+1}$. These orbits are denoted by

$$\mathcal{RL}(f) = \{g \in m_{n+1} : g \overset{RL}{\sim} f\}.$$

<u>Definition</u> Suppose $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are holomorphic map germs. $f$ and $g$ are called contact equivalent if and only if there exists a germ of a biholomorphism $H : (\mathbf{C}^{n+2}, 0) \to (\mathbf{C}^{n+2}, 0)$ such that

(a) $H(\mathbf{C}^{n+1} \times \{0\}, 0) = (\mathbf{C}^{n+1} \times \{0\}, 0)$

(b) $H(\text{graph } f) = \text{graph } g$

The notation $f \overset{c}{\sim} g$ is used to indicate contact equivalence.

<u>Definition</u> The contact group $\mathcal{K}$ consists of those germs of biholomorphisms $H : \mathbf{C}^{n+2} \to \mathbf{C}^{n+2}$ for which there exists a holomorphism $h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ such that the following diagram commutes.

$$\begin{array}{ccccc}
(\mathbf{C}^{n+1}, 0) & \xrightarrow{\iota} & (\mathbf{C}^{n+2}, 0) & \xrightarrow{\pi} & (\mathbf{C}^{n+1}, 0) \\
\downarrow h & & \downarrow H & & \downarrow h \\
(\mathbf{C}^{n+1}, 0) & \xrightarrow{\iota} & (\mathbf{C}^{n+2}, 0) & \xrightarrow{\pi} & (\mathbf{C}^{n+1}, 0)
\end{array}$$

where $\iota(z_0, \cdots, z_n) = (z_0, \cdots, z_n, 0)$ and $\pi(z_0, \cdots, z_n, w) = (z_0, \cdots, z_n)$. The group operation is composition.

This condition can be stated alternately. It says that $H(z_0, \cdots, z_{n+1})$ can be written in the form $(h(z_0, \cdots, z_n), k(z_0, \cdots, z_{n+1}))$ where $h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ is the germ of a biholomorphism and $k : (\mathbf{C}^{n+2}, 0) \to (\mathbf{C}, 0)$ is the germ of a holomorphic map with the property that $k(z_0, \cdots, z_n, 0) = 0$.

We can now give the action of the group $\mathcal{K}$ on $m_{n+1}$. If $H \in \mathcal{K}$ and $f \in m_{n+1}$, then $g = Hf$ is defined by the equation $g = k \circ (id, f) \circ h^{-1}$. It is easy to check that elements of $\mathcal{O}_{n+1}$ are contact equivalent if and only if they lie in the same $\mathcal{K}$-orbit.

The $\mathcal{K}$-orbits are denoted by

$$\mathcal{K}(f) = \{g \in m_{n+1} : g \overset{c}{\sim} f\}$$

Contact equivalence is important because it turns out to be very geometric in view of the following proposition

**Proposition 2.1 :** Let $(V, 0)$ and $(W, 0)$ be germs of hypersurfaces in $\mathbf{C}^{n+1}$ defined by $f, g \in m_{n+1}$ respectively. Then $f$ and $g$ are in the same $\mathcal{K}$-orbit if and only if the germs $(V, 0)$ and $(W, 0)$ are biholomorphically equivalent.

Proof : First suppose $f$ and $g$ are in the same $\mathcal{K}$-orbit. Let $H$ be an element of $\mathcal{K}$ such that $H(\text{graph}f)=\text{graph}g$. Then the following set germ equalities hold

$$h^{-1}(W) = h^{-1}(\iota^{-1}\text{graph}g) = \iota^{-1}(H^{-1}\text{graph}g) = \iota^{-1}(\text{graph}f) = V$$

This follows that $h$ provides a biholomorphic equivalence between $(V,0)$ and $(W,0)$.

Now suppose that $(V,0)$ and $(W,0)$ are biholomorphically equivalent. Let $h : (\mathbf{C}^{n+1},0) \to (\mathbf{C}^{n+1},0)$ be a germ of a biholomorphic mapping such that $h(V) = W$. Then there is a unit $u \in \mathcal{O}_{n+1}$ for which $f = u(g \circ h)$. Define $H : (\mathbf{C}^{n+2},0) \to (\mathbf{C}^{n+2},0)$ by $H(z,w) = (h(z), u^{-1}(z)w)$ where $z \in \mathbf{C}^{n+1}$ and $w \in \mathbf{C}$. Then $H \in \mathcal{K}$ and $H(z,f(z)) = (h(z), u^{-1}(z)f(z)) = (h(z), g \circ h(z))$ for $z = (z_0, \cdots, z_n)$. So $H(\text{graph}f)=\text{graph}g$.

**Q.E.D.**

For any $f \in m_{n+1}$ we define the Jacobian ideal $\Delta(f) \subset \mathcal{O}_{n+1}$ to be the ideal generated by the partial derivatives of $f$. The $\mathbf{C}$-algebra $\mathcal{O}_{n+1}/\Delta(f)$ will be called the Milnor algebra associated to $f$. When $f = 0$ defines an isolated singularity at the origin, then the dimension of $\mathcal{O}_{n+1}/\Delta(f)$, considered as a $\mathbf{C}$-vector space, is the topological invariant $\mu$, the Milnor number of the singularity.

Definition Two holomorphic germs $f, g : (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ are $\mathcal{Q}$-equivalent if there is a $\mathbf{C}$-algebra isomorphism of Milnor algebras $\mathcal{O}_{n+1}/\Delta(f) \cong \mathcal{O}_{n+1}/\Delta(g)$. We also introduce the notation

$$\mathcal{Q}(f) = \{g \in m_{n+1} : \mathcal{O}_{n+1}/\Delta(f) \cong \mathcal{O}_{n+1}/\Delta(g)\}$$

The $\mathbf{C}$-algebra $\mathcal{O}_{n+1}/(f, \Delta(f))$ is called the moduli algebra. This name is a natural choice because, considered as a $\mathbf{C}$-vector space, it is the base space for the semi-universal deformation of the singularity defined by $f = 0$

Definition Two holomorphic germs $f, g : (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ are $\mathcal{A}-$equivalent if there is a $\mathbf{C}$-algebra isomorphism of moduli algebras $\mathcal{O}_{n+1}/(f,\Delta(f)) \simeq \mathcal{O}_{n+1}/(g,\Delta(g))$. We will use the following notation for the $\mathcal{A}$-equivalence classes

$$\mathcal{A}(f) = \{g \in m_{n+1} : \mathcal{O}_{n+1}/(f, \Delta(f) \cong \mathcal{O}_{n+1}/(g, \Delta(g)\}$$

Definition Two holomorphic germs $f, g : (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ are $\mathcal{B}$-equivalent if there is a $\mathbf{C}$-algebra isomorphism $\mathcal{O}_{n+1}/(f, m_{n+1}\Delta(f)) \cong \mathcal{O}_{n+1}/(g, m_{n+1}\Delta(g))$. The $\mathcal{B}$-equivalent classes are denoted by

$$\mathcal{B}(f) = \{g \in m_{n+1} : \mathcal{O}_{n+1}/(f, m_{n+1}\Delta(f)) \cong \mathcal{O}_{n+1}/(g, m_{n+1}\Delta(g))\}$$

**Proposition 2.2 :** The diagram shown below gives some of the relationships between the different equivalence classes

$$\begin{array}{ccccccc} \mathcal{R}(f) & \subseteq & \mathcal{RL}(f) & \subseteq & \mathcal{K}(f) & \subseteq & \mathcal{A}(f) \\ & & \cap\vert & & \cap\vert & & \\ & & \mathcal{Q}(f) & & \mathcal{B}(f) & & \end{array}$$

Proof : The inclusion $\mathcal{R}(f) \subset \mathcal{RL}(f) \subset \mathcal{K}(f)$ because there are corresponding embeddings of the groups which respect the group actions. The embedding $\mathcal{R} \hookrightarrow \mathcal{RL}$ is given by $g \longmapsto (id, g)$, while $\mathcal{RL} \hookrightarrow \mathcal{K}$ is defined by $(v,h) \longmapsto H$, where $Hf = (id, v \circ f) \circ h$

**Q.E.D.**

To establish that $\mathcal{RL}(f) \subseteq \mathcal{Q}(f)$, we will use the following lemma.

Lemma : Suppose $f : (\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ is a germ of a holomorphic function and $\gamma = (\psi, \phi)$ is an element of $\mathcal{RL}$. Let $\phi^* : \mathcal{O}_{n+1} \to \mathcal{O}_{n+1}$ be the pullback map given by $f \longmapsto f \circ \phi$. Then $\phi^*\Delta(f) = \Delta(\gamma f)$

Suppose that $g \in \mathcal{RL}(f)$. Then there exists $\gamma \in \mathcal{RL}, \gamma = (\psi, \phi)$ for which $g = \gamma f$. Now $\phi$ induces an isomorphism $\phi^* : \mathcal{O}_{n+1} \to \mathcal{O}_{n+1}$. According to the above Lemma, $\phi^*\Delta(f) = \Delta(g)$. This means that $\phi^*$ induces an isomorphism of the quotient rings, so $g \in \mathcal{Q}(f)$. This proves the inclusion $\mathcal{RL}(f) \subseteq \mathcal{Q}(f)$.

The inclusions $\mathcal{K}(f) \subseteq \mathcal{A}(f)$ and $\mathcal{K}(f) \subseteq \mathcal{B}(f)$ follow from the next lemma in a similar manner.

Lemma Suppose $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are germs of holomorphic functions which are contact equivalent, that is, $g = u(f \circ \phi)$ for some $u$ a unit and $\phi$ a biholomorphic change of coordinates. Then the following equations hold.

(a) $\phi^*(f, \Delta(f)) = (g, \Delta(g))$

(b) $\phi^*(f, m_{n+1}\Delta(f)) = (g, m_{n+1}\Delta(g))$

For any $f, g \in \mathcal{O}_{n+1}$, we say that $f$ and $g$ have the same k-jets at the origin if their derivatives at the origin agree up to order $\leq k$. The k-jet $f^{(k)}$ is the equivalence class of all $g \in \mathcal{O}_{n+1}$ which have the same k-jets as $f$.

Definition Let $f \in \mathcal{O}_{n+1}$ and let $\mathcal{G}$ be a group which acts on $\mathcal{O}_{n+1}$. $f$ is $\underline{\text{k} - \text{determined}}$ relative to $\mathcal{G}$ if for any $g \in \mathcal{O}_{n+1}$ such that $g^{(k)} = f^{(k)}$, the $\mathcal{G}$-orbit of $f$ containing $g$. We say that $f$ is finitely determined relative to $\mathcal{G}$ if $f$ is k-determined for some positive integer k.

The following theorem shows that the notion of finite determinancy can be expressed in both algebraic and geometric terms. We will use the notation $f^{-1}m_1$ to represent the *module* consisting of all elements of the form $\sum_{i=1}^{\infty} a_i f^i$ where $\sum_{i=1}^{\infty} a_i t^i$ is a convergent power series vanishing at zero.

**Theorem** (Mather) Let $(V, 0)$ be the germ of a hypersurface in $\mathbf{C}^{n+1}$ defined by $f = 0$. The following conditions are equivalent.

a) $V \setminus \{0\}$ is nonsingular.
b) $\mathcal{O}_{n+1}/(f, \Delta(f))$ is a finite dimensional $\mathbf{C}$-vector space.
c) $\mathcal{O}_{n+1}/(f, m_{n+1}\Delta(f))$ is a finite dimensional $\mathbf{C}$-vector space.
d) $\mathcal{O}_{n+1}/f^{-1}m_1 + m_{n+1}\Delta(f)$ is a finite dimensional $\mathbf{C}$-vector space.
e) $\mathcal{O}_{n+1}/\Delta(f)$ is a finite dimensional $\mathbf{C}$-vector space.
f) $\mathcal{O}_{n+1}/m_{n+1}\Delta(f)$ is a finite dimensional $\mathbf{C}$-vector space.
g) $f$ is finitely determined relative to $\mathcal{K}$.
h) $f$ is finitely determined relative to $\mathcal{RL}$.
i) $f$ is finitely determined relative to $\mathcal{R}$.

The hypothesis that $f$ is finitely determined simplifies the diagram in Proposition 2.2 showing the relationship between the different types of germ equivalence. The notions of $\mathcal{K}-, \mathcal{A}-$, and $\mathcal{B}-$equivalence turn out to be exactly the same. This is the content of the following theorem of Mather and Yau.

**Theorem 2.3** (Mather-Yau) Suppose $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are holomorphic function with isolated critical points at the origin. The following statements are equivalent.

a) $f, g$ are $\mathcal{K}-$equivalent.
b) $f, g$ are $\mathcal{A}-$equivalent.
c) $f, g$ are $\mathcal{B}-$equivalent.

Let $J^k$ be the set of k-jets at the origin of elements of $\mathcal{O}_{n+1}$. $J^k$ has a natural complex analytic structure obtained by using the Taylor series coefficients as coordinates. For each of the groups $\mathcal{R}, \mathcal{RL}$, and $\mathcal{K}$, let $\mathcal{R}^k, \mathcal{RL}^k$, and $\mathcal{K}^k$ denote the respective sets of k-jets at the origin. They are complex Lie groups which act on $J^k$.

For $f \in \mathcal{O}_{n+1}$ we use the notations $\mathcal{R}^k(f), \mathcal{RL}^k(f)$, and $\mathcal{K}^k(f)$ to stand for the orbits of $f^{(k)}$ with respect to $\mathcal{R}^k, \mathcal{RL}^k$, and $\mathcal{K}^k$.

**Theorem** Let $f : (\mathbf{C}^{n+1}, 0) \to \mathbf{C}$ be the germ of a holomorphic function with an isolated singularity at the origin. Then $\mathcal{R}^k(f), \mathcal{RL}^k(f)$, and $\mathcal{K}^k(f)$ are complex analytic manifolds. The following $\mathbf{C}$-vector space isomorphisms exist between their tangent spaces at $f^{(k)}$ and subspaces of $J^k$.

a) $T_f(\mathcal{R}^k(f)) \cong m_{n+1}\Delta(f)J^k$
b) $T_f(\mathcal{RL}^k(f)) \cong (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$
c) $T_f(\mathcal{K}^k(f)) \cong (f, m_{n+1}\Delta(f))J^k$

We want to look at the jet version of $\mathcal{Q}$-equivalence as well. Let $\mathcal{Q}^k(f) = \{g^{(k)} | \mathcal{O}_{n+1}/\Delta(f) + m_{n+1}^k \cong \mathcal{O}_{n+1}/\Delta(g) + m_{n+1}^k\}$, $a^k(f) = \{g^{(k)} | \Delta(g) \subseteq \Delta(f) + m_{n+1}^k\}$, and $A^k(f) = (a^k(f) + m_{n+1}\Delta(f))J^k$. The following result, due to Shoshitaishvili, gives the structure of the $\mathcal{Q}^k$-equivalence classes.

**Theorem** (Shoshitaishvili) Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be the germ of a holomorphic function with an isolated singularity at the origin. Then $\mathcal{Q}^k(f)$ is a complex analytic manifold and its tangent space at $f^{(k)}$ is isomorphic to the vector space $A^k(f)$.

**R-orbit equivalence**

We now investigate the conditions when the $\mathcal{R}$-orbit of a holomorphic function with an isolated critical point at the origin is the same as the $\mathcal{RL}-, \mathcal{K}-$, and $\mathcal{Q}-$orbits. It turns out that these orbits coincide precisely when the function is analytically equivalent to a weighted homogeneous polynomial.

The following lemma will be very useful for the results to come. We want to emphasize that this lemma is very general and does not require that the singularity be isolated. A similar result proved by Shoshitaishvili is somewhat stronger, but is restricted to the case of an isolated singularity. Our lemma is powerful enough for our applications. Moreover the proof is extremely elementary

**Lemma 2.4** (Benson-Yau) Suppose $f, g \in \mathcal{O}_{n+1}, f$ is weighted homogeneous, and $\Delta(f) = \Delta(g)$. Then $g \in m_{n+1}\Delta(g)$.

*Proof.* Suppose that $f$ is weighted homogeneous of degree $d$ with weights $a_0, \cdots, a_n$. By definition $f(t^{a_0}z_0, \cdots, t^{a_n}z_n) = t^d f(z_0, \cdots, z_n)$ for all $t$. It is easy to check that $\frac{\partial f}{\partial z_j}$ either vanishes or is weighted homogeneous of degree $d - a_j$. And, since $\Delta(f) = \Delta(g)$ there exist elements $\alpha_{ij}, \beta_{ij} \in \mathcal{O}_{n+1}$ for which

$$\frac{\partial f}{\partial z_i} = \sum_{j=0}^{n} \alpha_{ij} \frac{\partial g}{\partial z_j}$$

$$\frac{\partial g}{\partial z_i} = \sum_{j=0}^{n} \beta_{ij} \frac{\partial f}{\partial z_j}$$

We are going to use these facts in the computation below

$$\begin{aligned}
\frac{d}{dt} g(t^{a_0}z_0, \cdots, t^{a_n}z_n) &= \sum_{i=0}^{n} a_i t^{a_i - 1} z_i \frac{\partial g}{\partial z_i}(t^{a_0}z_0, \cdots, t^{a_n}z_n) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} a_i z_i t^{a_i-1} \beta_{ij}(t^{a_0}z_0, \cdots, t^{a_n}z_n) \frac{\partial f}{\partial z_j}(t^{a_0}z_0, \cdots, t^{a_n}z_n) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} a_i z_i t^{d-a_j+a_i-1} \beta_{ij}(t^{a_0}z_0, \cdots, t^{a_n}z_n) \frac{\partial f}{\partial z_j}(z_0, \cdots, z_n) \\
&= \sum_{k=0}^{n}\left[\sum_{i=0}^{n}\sum_{j=0}^{n} a_i z_i t^{d-a_j+a_i-1} \beta_{ij}(t^{a_0}z_0, \cdots, t^{a_n}z_n)\alpha_{jk}(z_0, \cdots, z_n)\right] \frac{\partial g}{\partial z_k}(z_0, \cdots, z_n)
\end{aligned}$$

Then integrate back to find that

$$\begin{aligned}
g(z_0, \cdots, z_n) &= \int_0^1 \frac{d}{dt} g(t^{a_0}z_0, \cdots, t^{a_n}z_n)dt \\
&= \sum_{k=0}^{n} b_k(z_0, \cdots, z_n) \frac{\partial g}{\partial z_k}(z_0, \cdots, z_n)
\end{aligned}$$

where

$$b_k(z_0, \cdots, z_n) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i z_i \alpha_{jk}(z_0, \cdots, z_n) \int_0^1 t^{t-a_j+a_i-1} \beta_{ij}(t^{a_0}z_0, \cdots, t^{a_n}z_n)dt \in m_{n+1}$$

This proves that $g \in m_{n+1}\Delta(g)$. **Q.E.D.**

We can now begin examining the conditions for when the $\mathcal{R}$-orbits coincide with other orbits. The first result is originally due to Shoshitaishvili. The proof given here is easier than the original proof.

**Theorem** (Shoshitaishvili) Suppose $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is a holomorphic function with an isolated critical point at the origin. The following statements are equivalent.

a) $\mathcal{Q}(f) = \mathcal{R}(f)$
b) $m_{n+1}\Delta(f) = a(f) + m_{n+1}\Delta(f)$
c) $a(f) \subseteq m_{n+1}\Delta(f)$
d) $f$ is right equivalent to a weighted homogeneous polynomial.

*Proof.* We start by showing that $a) \Rightarrow b)$. Because $f$ defines an isolated singularity, $m_{n+1}^k \subset \Delta(f)$ for all large $k$. This means that $\mathcal{Q}^k(f)$ is precisely $\mathcal{Q}(f)J^k$. It follows that $\mathcal{Q}^k(f) = \mathcal{R}^k(f)$ for all $k$ large enough. Therefore their tangent spaces must coincide. So we see that $(a(f) + m_{n+1}\Delta(f))J^k = m_{n+1}\Delta(f)J^k$ for all large $k$. This means that $a(f) + m_{n+1}\Delta(f) = m_{n+1}\Delta(f)$.

The implication $b) \Rightarrow c)$ is obvious. As for $c) \Rightarrow d), f \in a(f)$ implies $f \in m_{n+1}\Delta(f)$. By Saito's theorem(cf.[Sa1]), $f$ is right equivalent to a weighted homogeneous polynomial.

The final implication, $d) \Rightarrow a)$, takes more proof. Assume that $f$ is right equivalent to a weighted homogeneous polynomial $f'$ and $g \in \mathcal{Q}(f)$. Then we only need to show that $g \in \mathcal{R}(f)$.

Since $\mathcal{R}(f) = \mathcal{R}(f')$, we can assume without loss of generality that $f$ is a weighted homogeneous polynomial. The following lemma allows us to also assume that $\Delta(f) = \Delta(g)$.

**Lemma** (Mather-Yau) Suppose $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are holomorphic functions with isolated critical points at the origin and $\mathcal{O}_{n+1}/\Delta(f) \simeq \mathcal{O}_{n+1}/\Delta(g)$. Then there exists a $g' \in \mathcal{R}(g)$ such that $\Delta(f) = \Delta(g')$.

*Proof.* Suppose $\phi : \mathcal{O}_{n+1}/\Delta(f) \to \mathcal{O}_{n+1}/\Delta(g)$ is a $\mathbf{C}$-algebra isomorphism. We are going to construct a local system $z_0, \cdots, z_n$ of holomorphic coordinates on $\mathbf{C}^{n+1}$, centered at the origin. Let $k = \dim_{\mathbf{C}}(\Delta(f) \cap m_{n+1} + m_{n+1}^2)/m_{n+1}^2$. Choose elements $z_0, \cdots, z_{k-1} \in \Delta(f) \cap m_{n+1}$ which are linearly independent modulo $m_{n+1}^2$. Then pick $n - k$ more functions $z_k, \cdots, z_n \in m_{n+1}$ to form a basis modulo $m_{n+1}^2$. By the inverse function theorem, $z_0, \cdots, z_n$ form a holomorphic local system of coordinates. We can now define a lifting $\tilde{\phi} : \mathcal{O}_{n+1} \to \mathcal{O}_{n+1}$ of $\phi$ by specifying its image on each of the coordinate functions. For each $i = k, \cdots, n$ pick $w_i = \tilde{\phi}(z_i) \in \mathcal{O}_{n+1}$ so that its projection in $\mathcal{O}_{n+1}/\Delta(g)$ is $\phi(\bar{z}_i)$. Since $\phi$ is an isomorphism of the quotient rings, the $w_k, \cdots, w_n$ must be linearly independent modulo $m_{n+1}^2$. Then choose $w_0, \cdots, w_{k-1} \in \Delta(g) \cap m_{n+1}$ so that the $w_0, \cdots, w_n$ complete a basis modulo $m_{n+1}^2$. By its construction this map makes the diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{\tilde{\phi}} & \mathcal{O}_{n+1} \\ \downarrow & & \downarrow \\ \mathcal{O}_{n+1}/\Delta(f) & \xrightarrow{\phi} & \mathcal{O}_{n+1}/\Delta(g) \end{array}$$

commute. Further, the $w_i$ form a local system of coordinates so that $\tilde{\phi}$ must be biholomorphic at the origin. According to the lemma right after Proposition 2.2, $\Delta(f) = \tilde{\phi}^*\Delta(g) = \Delta(g \circ \tilde{\phi})$. Let $g' = g \circ \tilde{\phi} \in \mathcal{R}(g)$ and the proof of our lemma is complete.

**Q.E.D.**

We will assume from now on that

$$\Delta(f) = \Delta(g) \tag{2.1}$$

where $f$ is a weighted homogeneous polynomial defining an isolated critical point at the origin. It follows from Lemma 2.4 that

$$g \in m_{n+1}\Delta(g) \tag{2.2}$$

$$f \in m_{n+1}\Delta(f) \tag{2.3}$$

We will also assume that $f \neq g$, because otherwise there is nothing more to prove.

Let $L$ be the complex line in $\mathcal{O}_{n+1}$ joining $f$ to $g$. Every element of $L$ is of the form $h = (1-w)f + wg$ for some $w \in \mathbf{C}$. Because of (2.1) $m_{n+1}\Delta(h) \subseteq m_{n+1}\Delta(f)$. Let $L_0$ be the set of $h \in L$ for which

$$m_{n+1}\Delta(h) = m_{n+1}\Delta(f) \tag{2.4}$$

**Lemma** $L_0$ is a connected complex manifold.

*Proof.* Since $f$ defines an isolated critical point at the origin, there exists an integer $k$ such that $m_{n+1}^k \subseteq m_{n+1}\Delta(f)$. For any such $k$

$$m_{n+1}\Delta(h)J^k = m_{n+1}\Delta(f)J^k \tag{2.5}$$

holds if and only if (2.4) holds.

The $\mathbf{C}$-vector space $m_{n+1}\Delta(h)J^k$ is generated by the elements $v_i(h) = (z^p \frac{\partial h}{\partial z_q})^{(k)}, i = (p, q)$, where $p$ runs through the non-negative multi-indices with degree between 1 and $k$ and $q = 0, 1, \cdots, k$. Let $d$ be the dimension of the $\mathbf{C}$-vector space $m_{n+1}\Delta(f)J^k$. By choosing a basis of this space, we may represent each $v_i(h)$ as a row vector of length $d$.

Together the $v_i(h)$ form a matrix with $d$ columns. Because $v_i(h) = (1-w)v_i(f) + wv_i(g)$, each coefficient of the matrix is a linear function of $w$. Equation (2.5) will hold if and only if at least one of the $d \times d$ minors has a nonzero determinant. Since it holds for $w = 0$, at least one of the minors must have a determinant which does not vanish identically. Therefore it is a polynomial in $w$ of degree $\leq d$. Hence there are at most $d$ values at which (2.5) fails to hold.

Therefore we have shown that $L_0$ is equal to $L$ with at most a finite number of points deleted. Since $L$ is a complex, this implies that $L_0$ must be connected.

**Q.E.D.**

Since $f$ has an isolated point at the origin, $f$ is finitely determined with respect to $\mathcal{R}$. Therefore it is enough to show that $g^{(k)} \in \mathcal{R}^k(f)$ for every positive integer $k$. We are going to show $L_0J^k \subset \mathcal{R}^k(f)$ by using the following result proved by Mather. This lemma will be used repeatedly, so for convenience, we will give the proof here.

**Lemma** (Mather) Let $\alpha : G \times U \to U$ be a $C^\infty$ action of a Lie group on a $C^\infty$-manifold $U$, and let $V$ be a connected $C^\infty$-submanifold of $U$. Then necessary and sufficient conditions for $V$ to be contained in a single orbit of $\alpha$ are that

a) $T_v(Gv) \supseteq T_vV$, if $v \in V$

b) $\dim T_v(Gv)$ is independent of the choice of $v \in V$

*Proof.* Necessity is trivial. Now we prove sufficiency. For each $v \in U$, let $\alpha_v : G \to U$ be the mapping defined by $g \mapsto \alpha(g, v)$. Then $T_v(Gv) = \alpha_{v*}(T_{id}G)$. Provide $T_{id}G$ with a Hilbert norm and for each $v \in V$, let $L_v$ be the orthogonal complement of $\ker \alpha_{v*}$ in $T_{id}G$. Define $L = \bigcup_{v\in V}(v \times L_v) \subset V \times T_{id}G$. Condition b) implies that $L$ is a subvector bundle over $V$ of $V \times T_{id}G$. Let $L_0 = \bigcup_{v\in V}(\alpha_{v*}^{-1}(T_vV) \cap L_v)$. Condition a) shows that $L_0$ is a subvector bundle of $L$ and the mapping $\bigcup_{v\in V}\alpha_{v*} : L_0 \to TV$ is an isomorphism of $C^\infty$-vector bundles. Let $\beta : TV \to L_0$ be the inverse of this mapping and let $\pi : V \times T_{id}G \to T_{id}G$ denote the projection map. Then $\pi \circ \beta : TV \to T_{id}G$ is a $C^\infty$-mapping, and $\alpha_{v*}(\pi \circ \beta(\eta)) = \eta$ for any $\eta \in T_vV$.

To prove that $V$ is contained in a single orbit of $\alpha$, it is enough to show that any two points $v_1, v_2$ of $V$ are contained in the same orbit. Since $V$ is connected, there is a smooth curve $\gamma : [0,1] \to V$ joining $v_1$ to $v_2$. We only need to show that for any $t_0 \in [0,1]$, there is an $\epsilon > 0$ such that if $t_0 - \epsilon < t < t_0 + \epsilon$, then $\gamma(t)$ is contained in the same orbit as $\gamma(t_0)$.

Let $\gamma'(t) \in T_{\gamma(t)}V$ denote the derivative of $\gamma(t)$ with respect to $t$, and define $X(t) = \pi \circ \beta(\gamma'(t)) \in T_{id}G$. $X(t)$ is a $C^\infty$ function of $t$ and

$$\alpha_{\gamma(t)*}(X(t)) = \gamma'(t) \tag{2.6}$$

¿From the existence theory for ordinary differential equations, it follows that there exists a curve $t \mapsto \mu(t)$ in $G$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ for a suitable $\epsilon > 0$ such that $\mu(t_0) = I$ and

$$\frac{d\mu(t)}{dt} = \tilde{X}_t(\mu(t)) \tag{2.7}$$

where $\tilde{X}_t$ is the unique right invariant vector field on $G$ which extends $X(t)$.

We now show that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \epsilon < t < t_0 + \epsilon$. This will imply that $\gamma(t)$ is in the same orbit as $\gamma(t_0)$ for all $t$ within this range and finish the proof of the lemma. The derivative with respect to $t$ is

$$\begin{aligned}\frac{d}{dt}\mu(t)^{-1}\gamma(t) &= \frac{d\mu(t)^{-1}}{dt}\gamma(t) + \mu(t)^{-1}\frac{d\gamma(t)}{dt} \\ &= \mu(t)^{-1}\left(-\frac{d\mu(t)}{dt}\mu(t)^{-1}\gamma(t) + \frac{d\gamma(t)}{dt}\right)\end{aligned}$$

By (2.7) and the fact that $\tilde{X}_t$ is right invariant, the quantity inside the brackets becomes $-X(t)\gamma(t) + \gamma'(t)$. According to (2.6), this is zero. Since $\mu(t_0) = I$, this shows that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \epsilon < t < t_0 + \epsilon$. This completes the proof of the lemma.

**Q.E.D.**

We will now apply this lemma. Take the action of $\alpha$ to be the action of $G = \mathcal{R}^k$ on $U = J^k$. We can deduce from the lemma before Mather's lemma that $V = L_0 J^k$ is a connected submanifold of $U = J^k$. Recall, $T_h(\mathcal{R}^k h) = m_{n+1}\Delta(h)J^k$, for any $h \in \mathcal{O}_{n+1}$. If $h^{(k)} \in L_0 J^k$, then (2.4) holds, and we obtain

$$T_h(\mathcal{R}^k h) = m_{n+1}\Delta(f)J^k \tag{2.8}$$

which verifies condition b) of Mather's lemma. The tangent space $T_h(L_0 J^k)$ is the one dimensional complex subspace of $J^k$ spanned by $g - f$. By (2.2) and (2.3), $g - f \in m_{n+1}\Delta(f)J^k$. Hence $T_h(L_0 J^k) \subset T_h(\mathcal{R}^k h)$, which shows that condition a) holds as well.

Therefore we may apply Mather's lemma to conclude that $L_0 J^k$ is contained in a single orbit of the action of $\mathcal{R}^k$ of $J^k$. This proves our result.

**Q.E.D.**

The proofs of the following two theorems are quite easy.

**Theorem 2.5** Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent

a) $\mathcal{R}(f) = \mathcal{K}(f)$
b) $m_{n+1}\Delta(f) = (f, m_{n+1}\Delta(f))$
c) $f$ is right equivalent to a weighted homogeneous polynomial.

**Theorem 2.6** Let $f$:(**Let** $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a holomorphic function with an isolated point at the origin. Then the following statements are equivalent

a) $\mathcal{R}(f) = \mathcal{RL}(f)$
b) $m_{n+1}\Delta(f) = f^{-1}m_1 + m_{n+1}\Delta(f)$
c) $f$ is right equivalent to a weighted homogeneous polynomial.

### $\mathcal{RL}$–orbit equivalence

we now investigate the conditions when the $\mathcal{RL}$-orbit of a holomorphic function with an isolated point at the origin is the same as the $\mathcal{K}-$ and $\mathcal{Q}-$orbits.

**Theorem 2.7** Suppose $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent.

a) $\mathcal{RL}(f) = \mathcal{K}(f)$

b) $f^{-1}m_1 + m_{n+1}\Delta(f) = (f, m_{n+1}\Delta(f))$

c) $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$

*Proof.* a)⇒b) is proved by using the computation of the tangent spaces. Since $\mathcal{RL}(f) = \mathcal{K}(f), \mathcal{RL}^k(f) = \mathcal{K}^k(f)$ for all $k$. We can equate their tangent spaces, getting $(f^{-1}m_1 + m_{n+1}\Delta(f))J^k = (f, m_{n+1}\Delta(f))J^k$ for all $k$. But th $f^{-1}m_1 + m_{n+1}\Delta(f) = (f, m_{n+1}\Delta(f))$.

Assume that b) holds. Then $z_j f \in f^{-1}m_1 + m_{n+1}\Delta(f)$, so there exists a convergent power series $a(t) = \sum_{i=1}^{\infty} a_i t^i, a_i \in \mathbf{C}$ such that $z_j f = \sum_{i=1}^{\infty} a_i f^i + \sum_{i=0}^{n} b_i \frac{\partial f}{\partial z_i}$ where $b_i \in m_{n+1}$ for $0 \le i \le n$. There are two cases to consider, depending on whether or not $a_1$ is nonzero. If $a_1 \ne 0$, then $u = a_1 - z_j + \sum_{i=2}^{\infty} a_i f^{i-1}$ is a unit element in $\mathcal{O}_{n+1}$ and $f = u^{-1}(-\sum_{i=0}^{n} b_i \frac{\partial f}{\partial z_i}) \in m_{n+1}\Delta(f)$. In particular $m_{n+1}(f) \subset m_{n+1}\Delta(f)$ when $a_1 \ne 0$.

On the other hand, if $a_1 = 0$, then we have $z_j f = (\sum_{i=2}^{\infty} a_i f^{i-1})f + \sum_{i=0}^{n} b_i \frac{\partial f}{\partial z_i}$. Since $f$ has a critical point at the origin, $f \in m_{n+1}^2$. Therefore $m_{n+1}(f) \subseteq m_{n+1}^2(f) + m_{n+1}\Delta(f)$. Using Nakayama's Lemma, it follows that $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$. Therefore we have proved in either case that b)⇒c).

Finally, to prove c)⇒a), it is sufficient to prove that $\mathcal{K}(f) \subseteq \mathcal{RL}(f)$. Suppose $g \in \mathcal{K}(f)$. Then there exists $u \in \mathcal{O}_{n+1}, u(0) \ne 0$, such that $g = u(f \circ h)$ where $h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ is a germ at the origin of a biholomorphic mapping. Now $f' = u(0)^{-1}(f \circ h)$ is holomorphic function with the property that $\mathcal{K}(f) = \mathcal{K}(f')$ and $\mathcal{RL}(f) = \mathcal{RL}(f')$. Thus by replacing $f$ by $f'$, we may assume without loss of generality that $g = uf$ where $u(0) = 1$.

It is clear that

$$(f, m_{n+1}\Delta(f)) = (g, m_{n+1}\Delta(g)) \tag{2.9}$$

We will also assume that $f \ne g$, because otherwise there is nothing more to prove.

Let $L$ be the complex line in $\mathcal{O}_{n+1}$ joining $f$ to $g$. Since every $h \in L$ can be written in the form $h = (1 - w)f + wg$ for some $w \in \mathbf{C}$, we have $(h, m_{n+1}\Delta(h)) \subseteq (f, m_{n+1}\Delta(f))$. Let $L_0$ be the set of $h \in L$ for which the two ideals are equal. Using an argument similar to the lemma before Mather's lemma, we find that $L_0$ is a connected manifold.

The hypothesis that $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ has an isolated critical at the origin implies that $f$ is finitely determined with respect to $\mathcal{RL}$. Hence it is enough to prove that $g^{(k)} \in \mathcal{RL}^k(f)$ for every positive integer $k$. In what follows let $k$ be a fixed positive integer.

We want to apply Mather's lemma. In this case $G = \mathcal{RL}^k, U = J^k$, and $V = L_0$. We have to check that conditions a) and b) of the lemma are applicable.

Suppose $h \in L_0$. Then $h = (1 - w)f + wg = (1 - w + wu)f$ for some $w \in \mathbf{C}$. Since $u(0) = 1, 1 - w + wu$ is a unit in $\mathcal{O}_{n+1}$. The following lemma can be applied to $h$.

**Lemma 2.8** Suppose $f, h : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are holomorphic functions with isolated critical points at the origin and $h = uf$ where $u \in \mathcal{O}_{n+1}$ is a unit. If $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$, then $m_{n+1}(h) \subseteq m_{n+1}\Delta(h)$.

*Proof.* Using the hypothesis that $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$, it is easy to see that

$$m_{n+1}\Delta(h) \subseteq m_{n+1}\Delta(f) \tag{2.10}$$

Our first step is to show that these ideals are actually equal. We can do this by proving that $\dim_{\mathbf{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(f) = \dim_{\mathbf{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(h)$.

The exact sequence

$$0 \to \Delta(f)/m_{n+1}\Delta(f) \to \mathcal{O}_{n+1}/m_{n+1}\Delta(f) \to \mathcal{O}_{n+1}/\Delta(f) \to 0$$

shows that

$$\dim_{\mathbf{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(f) = \dim_{\mathbf{C}}\mathcal{O}_{n+1}/\Delta(f) + \dim_{\mathbf{C}}\Delta(f)/m_{n+1}\Delta(f)$$

We are going to show that the right hand side of this equation depends only on the analytic type of the singularity, and not on the defining equation $f = 0$. The first term on the right hand side is the Milnor number, which is a topological invariant of the singularity. We will now prove that the second term is equal to $n+1$.

Consider the map $\phi : \mathbf{C}^{n+1} \to \Delta(f)/m_{n+1}\Delta(f)$ defined by $(a_0, a_1, \cdots, a_n) \mapsto \sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i} + m_{n+1}\Delta(f)$. This map is obviously surjective. Suppose that $\phi$ is not injective, then there exists a nonzero vector $(a_0, a_1, \cdots, a_n)$ in $\mathbf{C}^{n+1}$ such that $\sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i} \in m_{n+1}\Delta(f)$. Without loss of generality, we shall assume $a_0 \neq 0$. Then there exist $b_0, \cdots, b_n \in m_{n+1}$ such that $\sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i} = \sum_{j=0}^{n} b_j \frac{\partial f}{\partial z_j}$. Rearranging terms we find that $\frac{\partial f}{\partial z_0} = (a_0 - b_0)^{-1} \sum_{j=1}^{n} (-a_j + b_j) \frac{\partial f}{\partial z_j}$. This means that $\Delta(f)$ is generated by less than $n+1$ elements, so the critical point of $f$ at the origin cannot be isolated. We have shown that $\phi$ is an isomorphism, and $\dim_{\mathbf{C}} \Delta(f)/m_{n+1}\Delta(f) = n+1$.

This proves that $\dim_{\mathbf{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(f)$ depends only on the singularity and not on $f$. Since $f = 0$ and $h = 0$ define the same singularity, $\dim_{\mathbf{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(f) = \dim_{\mathbf{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(h)$. Combined with (2.10), we see that $m_{n+1}\Delta(h) = m_{n+1}\Delta(f)$.

Since $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$ and $h = uf$, we have $m_{n+1}(h) \subseteq m_{n+1}\Delta(h)$. This completes the proof of the lemma.

We can now use Lemma 2.8 and (2.9) to show

$$\begin{aligned} f^{-1}m_1 + m_{n+1}\Delta(f) &= (f, m_{n+1}\Delta(f)) \\ &= (h, m_{n+1}\Delta(h)) \\ &= h^{-1}m_1 + m_{n+1}\Delta(h) \end{aligned}$$

In particular we can see that

$$(f^{-1}m_1 + m_{n+1}\Delta(f))J^k = (h^{-1}m_1 + m_{n+1}\Delta(h))J^k \tag{2.11}$$

for any $h \in L_0$. Combining this with the computation of the tangent space, $T_h(\mathcal{RL}^k h) = (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$ for any $h \in L_0$. This shows that condition b) holds.

The tangebt space of $L_0$ at any $h$ is the one dimensional complex subspace of $J^k$ spanned by $(g-f)^{(k)}$. According to (2.11), $(g-f)^{(k)} \in (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$, proving that $T_h(L_0) \subset T_h(\mathcal{RL}^k h)$. Thus condition a) holds as well.

We can now apply Mather's lemma. We deduce that $L_0$ is contained in a single orbit of the action of $\mathcal{RL}^k$ on $J^k$, and so in particular $g^{(k)} \in \mathcal{RL}^k(f)$.

**Q.E.D.**

In [Sa1] Saito proved for any $f$ with an isolated critical point at the origin, $f \in m_{n+1}\Delta(f)$ if and only if $uf \in m_{n+1}\Delta(f)$ if and only if up to a biholomorphic change of coordinates $f$ is a weighted homogeneous polynomial. Any $f$ satisfying $f \in m_{n+1}\Delta(f)$ is called a *quasi-homogeneous* function. We have shown that the following conditions are equivalent: $f$ is quasi-homogeneous, $\mathcal{R}(f) = \mathcal{RL}(f), \mathcal{R}(f) = \mathcal{K}(f)$, and $\mathcal{R}(f) = \mathcal{Q}(f)$. Theorem 2.7 suggests the following definition

**Definition** Suppose $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is a function with an isolated critical point at the origin. $f$ is said to be an *almost quasi-homogeneous* function if $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$.

The previous theorem leads us to expect that the singularities defined by almost quasi-honogeneous functions may form a distinguished class of singularities which have some special properties.

We can also give a criterion for when the $\mathcal{RL}$ and $\mathcal{Q}$ orbits coincide. This result is originally due to Shoshitaishvili. One can also use the mehtod developed by Mather-Yau to prove the following theorem.

**Theorem** (Shoshitaishvili) Suppose $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent.

a) $\mathcal{RL}(f) = \mathcal{Q}(f)$

b) $f^{-1}m_1 + m_{n+1}\Delta(f) = a(f) + m_{n+1}\Delta(f)$

We shall give an example of a function which is almost quasi-homogeneous, but not quasi-homogeneous and also an example of a function which is not almost quasi-homogeneous.

*Example 1* Let $f(x,y) = x^5 + y^5 + x^3y^3$. Then

a) $f$ is not quasi-homogeneous.

b) $f$ is almost quasi-homogeneous.

c) $(f, m_{n+1}\Delta(f)) = f^{-1}m_1 + m_{n+1}\Delta(f) = a(f) + m_{n+1}\Delta(f)$

In particular, we have

$$\begin{array}{ccccc} \mathcal{R}(f) & \subsetneq & \mathcal{RL}(f) & = & \mathcal{K}(f) \\ & & \| & & \\ & & \mathcal{Q}(f) & & \end{array}$$

The relations in the diagram follow from Theorems 2.6, 2.7, and Theorem of Shoshitaishvili.

*Example 2* Let $f(x,y) = (y + x^4)(y^2 + x^9)$. Then $f$ is not almost quasi-homogeneous. In particular

$$\mathcal{R}(f) \subsetneq \mathcal{RL}(f) \subsetneq \mathcal{K}(f)$$

The above relation follows from Theorems 2.6, 2.7.

**Relation between $\mathcal{Q}$ and $\mathcal{K}$ equivalence**

There are still two more natural questions. The first is whether $\mathcal{K}(f) \subseteq \mathcal{Q}(f)$, that is, whether the Milnor algebra isomorphism type is an invariant of the corresponding singularity. The second is whether $\mathcal{Q}(f) \subseteq \mathcal{K}(f)$, that is, whether the analytic type of an isolated singularity is determined by the Milnor albegras which are associated to it. The following proposition gives an answer to the first question.

**Proposition 2.9** Suppose $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is a holomorphic function with an isolated critical point at the origin with $\mathcal{K}(f) \subseteq \mathcal{Q}(f)$. Then $f \in \Delta(f) + m_{n+1}\Delta^2(f)$, where $\Delta^2(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by all second partial derivatives of $f$.

*Proof.* Using the computation of the tangent spaces to the manifolds $\mathcal{K}(f)$ and $\mathcal{Q}(f)$, $\mathcal{K}(f) \subseteq \mathcal{Q}(f)$ implies that $(f, m_{n+1}\Delta(f))J^k \subseteq (a(f) + m_{n+1}\Delta(f))J^k$ for all $k$. Since both ideals contain some power of the maximal ideal $m_{n+1}$, we have

$$(f, m_{n+1}\Delta(f)) \subseteq a(f) + m_{n+1}\Delta(f) \tag{2.12}$$

Then $(1 + z_0)f \in a(f) + m_{n+1}\Delta(f)$ and there exist $g \in a(f)$ and $\xi_j \in m_{n+1}$ such that

$$(1 + z_0)f = g + \sum_{j=0}^{n} \xi_j \frac{\partial f}{\partial z_j}$$

Differentiating with respect to $z_0$,

$$f + (1 + z_0)\frac{\partial f}{\partial z_0} = \frac{\partial g}{\partial z_0} + \sum_{j=0}^{n} \frac{\partial \xi_j}{\partial z_0}\frac{\partial f}{\partial z_j} + \sum_{j=0}^{n} \xi_j \frac{\partial^2 f}{\partial z_0 \partial z_j}$$

By definition of $a(f)$, $\frac{\partial g}{\partial z_0} \in \Delta(f)$. Therfore, $f \in \Delta(f) + m_{n+1}\Delta^2(f)$.

**Q.E.D.**

The following remark, due to Mather, shows that it is *not* true in general that $\mathcal{K}(f) \subseteq \mathcal{Q}(f)$.

*Remark.* There exists a polynomial $f(x,y)$ such that $f \notin \Delta(f) + \Delta^2(f)$. In particular $\mathcal{K}(f) \not\subseteq \mathcal{Q}(f)$.

*Example 3* Let $f(x,y) = x^{15} + x^{12}y^3 + x^9y^6 + x^6y^{10} + x^3y^{13} + y^{17}$. Then the following relationships hold

$$\begin{array}{ccccc} \mathcal{R}(f) & \subsetneq & \mathcal{RL}(f) & \subsetneq & \mathcal{K}(f) \\ & & \rotatebox{90}{$\subsetneq$} & & \\ & & \mathcal{Q}(f) & & \end{array}$$

with $\mathcal{Q}(f) \not\subseteq \mathcal{K}(f)$ and $\mathcal{K}(f) \not\subseteq \mathcal{Q}(f)$.

*Proof.* The polynomial $f$ was the lowest degree example that we could find such that $f \notin \Delta(f) + \Delta^2(f)$. Our selection procedure guarantees that $\mathcal{K}(f) \not\subseteq \mathcal{Q}(f)$, and it follows that $\mathcal{RL}(f) \subsetneq \mathcal{K}(f)$ as well. $f$ is not quasi-homogeneous because $f \notin \Delta(f) + \Delta^2(f)$. This means that $\mathcal{R}(f) \subsetneq \mathcal{RL}(f)$.

We used computer programs to check the remaining inclusions. It was found that $a(f) \not\subseteq (f, m\Delta(f))$. This shows that $\mathcal{RL} \subsetneq \mathcal{Q}(f)$ and $\mathcal{Q}(f) \not\subseteq \mathcal{K}(f)$.

The computations in this example are complex. The Milnor number of the singularity is 209, and the smallest power of the maximal ideal contained within $\Delta(f)$ is $m^{30}$. $a(f) + m\Delta(f)$ modulo $m^{31}$ has dimension 317, while $(f, m\Delta(f))$ modulo $m^{31}$ has dimension 329. All of the generators we found for $a(f)$ which were not contained in $(f, m\Delta(f))$ were extremely complicated. Some of their coefficients were rational numbers with over 30 digits in both the numerator and denominator.

**Q.E.D.**

We now turn to the second question and give a general method for constructing functions $F$ for which $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$.

**Theorem 2.10** Suppose $F(x_1, \cdots, x_n, y_1, \cdots, y_n) = f(x_1, \cdots, x_n) + f(y_1, \cdots, y_n)$ where $n > 1$ and $f(x_1, \cdots, x_n)$ is a function with an isolated critical point at the origin which is not quasi-homogeneous. Then $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$.

*Proof.* Suppose $\mathcal{Q}(F) \subseteq \mathcal{K}(F)$. Then using the computation of the tangent spaces to the manifolds $\mathcal{Q}(F)$ and $\mathcal{K}(F)$, $\mathcal{Q}(F) \subseteq \mathcal{K}(F)$ implies that $(a(F) + m_{2n}\Delta(F))J^k \subseteq (F, m_{2n}\Delta(F))J^k$ for all $k$. Since both ideals contain some power of the maximal ideal $m_{2n}$, we have

$$a(F) + m_{2n}\Delta(F) \subseteq (F, m_{2n}\Delta(F)) \tag{2.13}$$

In the following, let $x$ stand for $x_1, \cdots, x_n$ and $y$ for $y_1, \cdots, y_n$. According to the defintion of $a(F)$, $f(x) \in a(F)$. Using (2.13), $f(x) \in (F, m_{2n}\Delta(F))$, so there exist $b(x,y), c_j(x,y), d_j(x,y) \in \mathbf{C}\{x,y\}$ with $c_j(0,0) = d_j(0,0) = 0$ such that

$$f(x) = b(x,y)(f(x) + f(y)) + \sum_{j=1}^{n} c_j(x,y)\frac{\partial f}{\partial x_j}(x) + \sum_{j=1}^{n} d_j(x,y)\frac{\partial f}{\partial y_j}(y) \tag{2.14}$$

Now $b(x,y)$ must be a unit in $\mathbf{C}\{x,y\}$. Otherwise we can rearrange the terms in (2.14) and set $y = 0$ to find that

$$(1 - b(x,0))f(x) = \sum_{j=1}^{n} c_j(x,0)\frac{\partial f}{\partial x_j}(x) \tag{2.15}$$

Here we have used the fact that $f(y)$ has a singularity at the origin. This equation implies that $f(x) \in m_{n+1}\Delta(f)$. Since $f = 0$ defines an isolated singularity, Saito's theorem implies that $f$ is quasi-homogeneous. This is a contradiction to our hypothesis, so it must be true that $b(x,y)$ is a unit.

Next rearrange the terms in (2Next rearrange the terms in (2.14) and set $x = 0$. We get

$$-b(0,y)f(y) = \sum_{j=1}^{n} d_j(0,y)\frac{\partial f}{\partial y_j}(y) \tag{2.16}$$

where we have again used the fact that $f(x)$ has a singularity at the origin. Since $b(0,y)$ is a unit in $\mathbf{C}\{y\}$, it follows that $f(y) \in m_{n+1}\Delta(f)$. As before, this contradicts our hypothesis that $f$ is not quasi-homogeneous. Therefore we conclude that $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$.

**Q.E.D.**

**Corollary 2.11** Suppose $F(x_1,\cdots,x_n,y_1,\cdots,y_n) = f(x_1,\cdots,x_n) + f(y_1,\cdots,y_n)$ where $n > 1$ and $f(x_1,\cdots,x_n)$ is a function with an isolated critical point at the origin which is not quasi-homogeneous. Then there exists a $G \in \mathbf{C}\{x,y\}$ such that $\Delta(G) = \Delta(F)$ but $G \notin \mathcal{K}(F)$.

*Proof.* According to Theorem 2.10, there exists $H \in \mathbf{C}\{x,y\}$ such that $\mathcal{O}_{n+1}/\Delta(H) \simeq \mathcal{O}_{n+1}/\Delta(F)$ but $H \notin \mathcal{K}(F)$. Using Lemma due to Mather-Yau, we can find $G \in \mathcal{R}(H)$ such that $\Delta(F) = \Delta(G)$. Since $H$ is not in $\mathcal{R}(F)$, it follows that $G$ is not in $\mathcal{K}(F)$.

These arguments can be modified to work in $C^\infty$ category as well. The following remark summerizes this extension to the $C^\infty$ case.

*Remark.* Suppose $F(x_1,\cdots,x_n,y_1,\cdots,y_n) = f(x_1,\cdots,x_n) + f(y_1,\cdots,y_n)$ where $n > 1$ and $f(x_1,\cdots,x_n)$ is a function with an isolted critical point at the origin which is not quasi-homogeneous. Then there exists $G(x_1,\cdots,x_n,y_1,\cdots,y_n) \in \mathbf{C}\{x_1,\cdots,x_n,y_1,\cdots,y_n\}$ such that $\Delta(F) = \Delta(G)$ and the zero set $V(F)$ defined by $F = 0$ is not $C^\infty$-diffeomorphic equivalent to the zero set $V(G)$ defined by $G = 0$ although the two sets are homeomorphic.

When $F$ has real coefficients this is also a consequence of Ephraim's Theorem and Corollary 2.11 .

**Corollary 2.12** For any $n > 1$, there exists a one parameter family of non-quasi-homogeneous isolated singularities in which the Milnor algebras corresponding to each singularity are the same, but in which the diffeomorphism types are different.

*Example 4* Let $F(x,y,z,w) = x^5 + y^5 + z^5 + w^5 + x^3y^3 + z^3w^3$. Then the following relationships hold

$$\begin{array}{ccccc} \mathcal{R}(F) & \subsetneq & \mathcal{RL}(F) & \subsetneq & \mathcal{K}(F) \\ & & \rotatebox{90}{$\subsetneq$} & & \\ & & \mathcal{Q}(F) & & \end{array}$$

and $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$.

*Proof.* Observe that $F(x,y,z,w) = f(x,y)+f(z,w)$ where $f(x,y) = x^5+y^5+x^3y^3$. We have already shown in Example 1 that $f(x,y)$ is not quasi-homogeneous function. By Theorem 2.10, we have $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$.

We now claim that $F$ is also not almost quasi-homogeneous, that is, $m_{n+1}(F) \not\subseteq m_{n+1}\Delta(F)$. We are going to show that $xF \notin m_{n+1}\Delta(F)$. Assume the opposite is true. Then there exist power series $a(x,y,z,w), b(x,y,z,w), c(x,y,z,w)$ and $d(x,y,z,w)$ in $m_{n+1}$ such that

$$xF = a(x,y,z,w)\frac{\partial F}{\partial y} + b(x,y,z,w)\frac{\partial F}{\partial y} + c(x,y,z,w)\frac{\partial F}{\partial z} + d(x,y,z,w)\frac{\partial F}{\partial w} \tag{2.17}$$

Comparing the coefficients of $xz^5, xw^5, xz^3w^3$ on both sides, we get the following equations respectively.

$$\begin{aligned} 5c_{1010} &= 1 \\ 5d_{0101} &= 1 \\ 3c_{1010} + 3d_{0101} &= 1. \end{aligned}$$

It turns out that these linear equations form an inconsistent system. This means that $xF \notin m_{n+1}\Delta(F)$ and so $F$ is not almost quasi-homogeneous. It follows from Theorem 2.6 and 2.7 that $\mathcal{R}(F) \subsetneq \mathcal{RL}(F) \subsetneq \mathcal{K}(F)$.
It is also clear that $\mathcal{RL}(F) \subsetneq \mathcal{Q}(F)$, because otherwi $\mathcal{Q}(F) = \mathcal{RL}(F) \subsetneq \mathcal{K}(F)$ which contradicts the fact that $\mathcal{Q}(F) \not\subseteq \mathcal{K}(F)$

**Q.E.D.**

We will give one more example which we have computed.

*Example 5* Let $f(x,y) = (y+x^4)(y^2+x^9)$. Then the following relationships hold

$$\begin{array}{ccccc} \mathcal{R}(f) & \subsetneq & \mathcal{RL}(f) & \subsetneq & \mathcal{K}(f) \\ & & \not\cup \, \cap & & \\ & & \mathcal{Q}(F) & & \end{array}$$

and $\mathcal{Q}(f) = \mathcal{K}(f)$.

*Proof.* This example was discussed before, but we did not consider the inclusions involving $\mathcal{Q}(f)$.

We have used computer program to check these inclusions and have found that $a(f) \not\subseteq f^{-1}m_1 + m\Delta(f)$. One generator of $a(f)$ that is not contained in $f^{-1}m_1 + m\Delta(f)$ is $x^2y^3 + x^6y^2 + x^{11}y + \frac{91}{90}x^{15} + \frac{35}{768}x^{16}$. On the other hand the programs showed that $a(f) + m\Delta(f) = (f, m\Delta(f))$.

Despite the simple form of the polynomial $f$, the computing problem was still fairly complex. The Milnor number of this singularity is 23, but the smallest power of the maximal ideal contained in $\Delta(f)$ is $m^{16}$. The dimension of the **C**-vector space $a(f)$ modulo $m^{17}$ is 113 and the dimension of $f^{-1}m_1 + m\Delta(f)$ modulo $m^{17}$ is 129.

**Q.E.D.**

## §3 Equivalence of Singularities : topological case

No matter whether you are a topologist, algebraist or geometer, one of the fundamental goals is to find a necessary and sufficient condition for two given objects to be isomorphic in the given category. In the theory of isolated hypersurface singularities, the two fundamental problems are as follows: Let $(V, 0)$ and $(W, 0)$ be two isolated hypersurface singularities in $\mathbf{C}^{n+1}$.

**Problem 1.** Give a simple algebraic criterion for $(\mathbf{C}^{n+1}, V, 0)$ to be homeomorphic to $(\mathbf{C}^{n+1}, W, 0)$.

**Problem 2.** Give a simple algebraic criterion for $(\mathbf{C}^{n+1}, V, 0)$ to be biholomorphic to $(\mathbf{C}^{n+1}, W, 0)$.

One supposes that the first problem would be easier than the second one, but it turned out to be contrary. In 1982, Mather and the author solved the second problem completely. We showed that two isolated hypersurface singularities in $\mathbf{C}^{n+1}$ are biholomorphically equivalent iff their corresponding moduli algebra (a finite dimensional commutative local $\mathbf{C}$-algebra) are isomorphic . On the other hand, the progress on the first problem was not as fast as one wants although many well known mathematicians including Milnor and Zariski worked on it. Actually even the Zariski multiplicity problem whether multiplicity of isolated hypersurface singularity is an invariant of topological type, was solved completely only for $n = 1$ case. Recently there are some progress in this problem for $n = 2$ case.

### Topological Types of Isolated Hypersurface Singularities

**Definition** Let $(V_1, 0)$ and $(V_2, 0)$ be two isolated hypersurface singularities in $\mathbf{C}^{n+1}$. We say that $(V_1, 0)$ and $(V_2, 0)$ have the same topological type if $(\mathbf{C}^{n+1}, V_1, 0)$ is homeomorphic to $(\mathbf{C}^{n+1}, V_2, 0)$.

Even for $n = 1$, it took more than forty years for people to completely understand the topological type of plane curve singularities. Let $f$ be the defining function of the plane curve singularity $(V, 0)$. Then $f$ is reduced, i.e. in its decomposition in irreducible analytic functions in $\mathbf{C}\{X, Y\}$, it is square free. Suppose now that $f$ is irreducible in $\mathbf{C}\{X, Y\}$, i.e. the analytic local ring $\mathcal{O} = \mathbf{C}\{X, Y\}/(f)$ is an integral domain. Then we have

**Theorem 3.1** (Puiseux) The normalization $\overline{\mathcal{O}}$ of $\mathcal{O}$ is a regular local ring and $\overline{\mathcal{O}}$ is a finite $\mathcal{O}$-module.

Let $x$ and $y$ be the images of $X$ and $Y$ in $\mathcal{O}$. The maximal ideal $(x, y) = \mathcal{M}$ of $\mathcal{O}$ generates a principal ideal $\mathcal{M}\overline{\mathcal{O}}$, because $\overline{\mathcal{O}} \cong \mathbf{C}\{t\}$. Suppose that $\mathcal{M}\overline{\mathcal{O}} = x\overline{\mathcal{O}}$, i.e. by defintion $x$ is a transversal parameter. Then we may choose the uniformizing parameter $t$ of $\overline{\mathcal{O}}$ so that

$$\begin{cases} x = t^m \\ y = \sum_{\nu \geq m} a_\nu t^\nu . \end{cases} \tag{3.1}$$

We call (3.1) a Puiseux expansion of $f$ at 0.

Puiseux expansion of $f$ can also be rewritten in the form

$$y = \sum a_\kappa x^\kappa \quad a_\kappa \neq 0, \kappa \in \mathbf{Q} \quad \kappa \geq 1.$$

$f$ is regular in case all $\kappa$ are integers. In this case no Puiseux pairs are defined

$$\kappa_1 = \text{ smallest noninteger } \kappa$$

$$\kappa_1 = \frac{n_1}{m_1} \quad (n_1 > m_1) \quad (n_1, m_1) = 1.$$

The number pair $(m_1, n_1)$ is the $1^{st}$ Puiseux pair of $f$

$$\kappa_2 = \text{ smallest exponent after } \kappa_1 \text{ which is not of the form } \frac{q}{m_1} (q > m_1)$$

$$\kappa_2 = \frac{n_2}{m_1 m_2} \quad g.c.d.(n_2, m_2) = 1, \quad m_2 > 1$$

(If necessary we must multiple the fraction for $\kappa_2$ on top and bottom by a divisor of $m_1$). The number $n_2$ and $m_2$ are uniquely determined and the pair $(m_2, n_2)$ is the second Puiseux pair of $f$.

In general, if the Puiseux pairs $(m_1, n_1), \cdots, (m_j, n_j)$ are already defined, let $\kappa_{j+1}$ be the smallest exponent for which the proceding exponents are all expressible in the form

$$\kappa = \frac{q}{m_1 \cdots m_j}$$

while $\kappa_{j+1}$ itself is not. Then let

$$\kappa_{j+1} = \frac{n_{j+1}}{m_1 \cdots m_{j+1}} \quad \text{with} \quad g.c.d.$$

Then $(m_{j+1}, n_{j+1})$ is the next Puiseux pair. Eventually this process terminates i.e. there is a $g$ such that $m_1 \cdots m_g$ Hence we obtain in this way a finite sequence $(m_1, n_1), \cdots, (m_g, n_g)$

**Definition** The pairs $(m_1, n_1), \cdots, (m_g, n_g)$ defined in this way are called the Puiseux pairs of $f$ and $\kappa_1, \cdots, \kappa_g$ the Puiseux e

Since the exponents $\kappa_j$ are monotonically incresaing and greater than 1, the Puiseux pairs satisfy the following conditions:

$$\left.\begin{array}{ll} m_1 < n_1 & \\ n_{j-1} m_j < n_j & \text{for} \quad j \geq 2 \\ g.c.d.(n_j, m_j) = 1 & \text{for} \quad j = 1, \cdots, g \end{array}\right\} (*)$$

Conversely any given sequence of pairs of natural numbers $(m_1, n_1), \cdots, (m_g, n_g)$, which satisfies the conditions (*) is the sequence of Puiseux pairs of a certain Puiseux expansion, say the "standard expansion"

$$\begin{aligned} y(x) &= x^{\frac{n_1}{m_1}} + x^{\frac{n_2}{m_1 m_2}} + \cdots + x^{n_g} \\ &= x^{\kappa_1} + x^{\kappa_2} + \cdots + x^{\kappa_g}. \end{aligned}$$

In 1929, K. Brauner proved the following theorem

**Theorem 3.2** Let $f(X, Y)$ be analytically irreducible at 0 and $f(0) = 0$. Let $n$ be its multiplicity at 0 and $\kappa_1, \cdots, \kappa_g$ be the Puiseux exponents of $f$ at 0. Then the plane curve singularity defined by $f$ has the same topological type as the curve singularity defined by

$$\begin{cases} x = t^n \\ y = t^{\kappa_1} + \cdots + t^{\kappa_g} \end{cases}$$

In 1932, W. Burau and O. Zariski proved that the converse of the above Theorem is also true.

**Theorem 3.3** Let $f(X, Y)$ be analytically irreducible at 0 and $f(0) = 0$. Then the Puiseux exponents of $f$ at 0 are invariants of topological type of $(V, 0)$ where $V = \{f(X, Y) = 0\}$.

Finally, M. Lejeune and O. Zariski proved the following theorem.

**Theorem 3.4** Let $f(X, Y)$ be reduced at 0 and $f(0) = 0$. Then the topology type of the plane curve singularity defined by $f$ is determined by the topology type of every irreducible component of $f$ at 0 and all the pairs of intersection multiplicity of these components.

These together with the theorem of J. Reeve, which asserts that the intersection multiplicity of two plane curves is the same as the linking number of the corresponding knots, give a complete understanding of the topological type of plane curve singularities.

In 1968, Milnor made fundamental contribution in understanding the topology of isolated complex hypersurface singularities. Let us recall his beautiful theory briefly as below.

**Theorem 3.5** Let $V$ be a complex algebraic subvariety in $\mathbf{C}^{n+1}$ and $S(V)$ be the singular set of $V$. Let $f : V \to \mathbf{C}$ be an algebraic function on $V$. Then the restriction of $f$ to $V - S(V)$ has only a finite number of critical values.

**Corollary 3.6** Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ be a polynomial function. Then there exist $t_1, \cdots, t_r \in \mathbf{C}$ such that for all $t \in \mathbf{C} - \{t_1, \cdots, t_r\}$ the hypersurface defined by $f = t$ is nonsingular.

**Corollary 3.7** Let $V$ be complex algebraic subvariety of $\mathbf{C}^{n+1}$. Let $x_0$ be either a simple point of $V$ or an isolated point of the singular set $S(V)$. Then every sufficient samll sphere $S_\epsilon$ centered at $x$ intersects $V$ transversely in a smooth manifold.

Let $B_\epsilon$ (resp. $B_\epsilon^0$) denote the closed (resp. open) ball consisting of all $x$ with $\|x - x_0\| \leq \epsilon$(resp. $< \epsilon$). Again let $x_0$ be either a simple point or an isolated singular point of $V$.

**Proposition 3.8** For all sufficiently small strictly positive real numbers $\epsilon_1, \epsilon_2$, $(S_{\epsilon_1}, S_{\epsilon_1} \cap V)$ is diffeomorphic to $(S_{\epsilon_2}, S_{\epsilon_2} \cap V)$ as pair. Moreover, $(B_{\epsilon_1}, B_{\epsilon_1} \cap V)$ is homeomorphic to $(B_{\epsilon_1}, C(K_{\epsilon_1}))$ as pair, where $K_{\epsilon_1} = S_{\epsilon_1} \cap V$ and $C(K_{\epsilon_1})$ is the real cone over $K_{\epsilon_1}$ which is the union of all line segments joining points $k \in K_{\epsilon_1}$ to the base point $x_0$.

We are now ready to state the Milnor's fibration theorem.

**Theorem 3.9** Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a complex polynomial. Let $V$ be the hypersurface defined by $f = 0$. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon$ with $0 < \epsilon \leq \epsilon_0$, the differentiable mapping $\varphi_\epsilon : S_\epsilon - V \cap S_\epsilon \to S^1$ defined by $\varphi_\epsilon(z) = f(z)/|f(z)|$ for all $x \in S_\epsilon - V$, is a locally trivial differentible fibration.

Milnor gave another presentation of this fibration.

**Theorem 3.10** For $\epsilon > 0$ sufficiently small and $\epsilon \gg \eta > 0$, the mapping $\psi_{\epsilon,\eta} : B_\epsilon^0 \cap f^{-1}(\partial D_\eta) \to \partial D_\eta$ induced by $f$, where $B_\epsilon^0$ is the interior of $B_\epsilon$ and $\partial D_\eta = \{z \in \mathbf{C} : |z| = \eta\}$, is a smooth fibration isomorphic to $\varphi_\epsilon$ in Theorem 3.9 by an isomorphism which preserves the arguments.

**Corollary 3.11** Let $\epsilon_0 > 0$ as in Theorem 3.9. Fix $\epsilon$ with $0 < \epsilon \leq \epsilon_0$. Then the Milnor Fiber $F_\theta = \varphi_\epsilon^{-1}(e^{i\theta})$ is parallelizable and has the homotopy type of an $n$-dimensional finite CW-complex.

Milnor also proved the following.

**Theorem 3.12** The topological space $K_\epsilon = S_\epsilon \cap V$ is $n - 2$ connected.

Given any locally trivial fibration $\phi : E \to S^1$ over the circle, the natural action of a generator of $\pi_1(S^1)$ on the homology of the fiber is described by automorphism $h_* : H_*F_0 \to H_*F_0$. Here $h$ denotes the characteristic homeomorphism o$h$ denotes the characteristic homeomorphism of the fibre $F_0 = \phi^{-1}(1)$. It is obtained, using the covering homotopy theorem, by choosing a continuous one-parameter family of homeomorphisms. $h_t : F_0 \to F_t$ for $0 \leq t \leq 2\pi$, where $h_0$ is the identity and $h = h_{2\pi}$ is the required characteristic homeomorphism. $h$ induces on the homology group of $F_0$ the isomorphisms which is by definition the local monodromy of $V$ at 0.

By a theorem of Milnor and a theorem of Palamodov we have the following theorem .

**Theorem 3.13** If 0 is an isolated critical point of $f$, for $\epsilon > 0$ small enough, the fibers of $\varphi_\epsilon$ have the homotopy type of a bouquet of $\mu$ spheres of dimension $n$ with

$$\mu = \dim_{\mathbf{C}} \mathbf{C}\{z_0, \cdots, z_n\} \Big/ \left( \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n} \right).$$

**Remark.** A bouquet of spheres is the topological space union of spheres having a single point in common. The $\mu$ above is called Milnor number.

**Invariant of the Topological Types**

We shall give necessary conditions for two isolated hypersurface singularities which have the same topological type. Let us first recall the important notion of Whitehead product in algebraic topology. Consider a given space $X$ and a given basic point $x_0$ in $X$. Let $m \geq 1$ and $n \geq 1$ be given integers. For any two given elements $\alpha \in \pi_m(X, x_0), \beta \in \pi_n(X, x_0)$, the Whitehead product of $\alpha$ and $\beta$ is an element $[\alpha, \beta]$ of $\pi_{m+n-1}(X, x_0)$, which is defined as follows.

Let us choose representative maps $f : (I^m, \partial I^m) \to (X, x_0), g : (I^n, \partial I^n) \to (X, x_0)$ for $\alpha, \beta$ respectively. Since $I^{m+n} = I^m \times I^n$, we have $\partial I^{m+n} = (I^m \times \partial I^n) \cup (\partial I^m \times I^n)$. Here $I^n$ is the $n$-cube. Hence we define a map $h : \partial I^{m+n} \to X$ by taking for each point $(s, t)$ in $\partial I^{m+n}$

$$h(s,t) = \begin{cases} f(s) & if\ t \in \partial I^n \\ g(t) & if\ s \in \partial I^m \end{cases}$$

Since the point $r_0 = (0, \cdots, 0)$ of $\partial I^{m+n}$ is in $\partial I^m \times \partial I^n$, we have $h(r_0) = x_0$. Since $\partial I^{m+n}$ is homeomorphic to $S^{m+n-1}, h$ represents an element $\gamma$ of $\pi_{m+n-1}(X, x_0)$. It can be shown that $\gamma$ depends only on the elements $\alpha$ and $\beta$. So we may define $[\alpha, \beta] = \gamma$. We shall list some properties of Whitehead products:

[W1] If $\alpha \in \pi_1(X, x_0)$ and $\beta \in \pi_1(X, x_0)$, then $[\alpha, \beta]$ is the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ of $\pi_1(X, x_0)$.

[W2] If $\alpha \in \pi_m(X, x_0)$ and $\beta \in \pi_i(X, x_0)$ with $m > 1$, then $[\alpha, \beta]$ is the element $\beta\alpha - \alpha$ of $\pi_m(X, x_0)$ where $\beta : \pi_m(X, x_0) \to \pi_m(X, x_0)$ is a group automorphism.

[W3] If $m > 1$, then the assignment $\alpha \to [\alpha, \beta]$ for a given $\beta \in \pi_n(X, x_0)$ defines a homomorphism

$$\beta_* : \pi_m(X, x_0) \to \pi_{m+n-1}(X, x_0).$$

[W4] If $m + n > 2$, then, for every $\alpha \in \pi_m(X, x_0)$ and $\beta \in \pi_n(X, x_0)$ we have $[\beta, \alpha] = (-1)^{mn}[\alpha, \beta]$.

[W5] If $\sigma : I \to X$ is a path joining $x_0$ to $x_1$, then, for every $\alpha \in \pi_m(X, x_1)$ and $\beta \in \pi_n(X, x_1)$, we have $\sigma_{m+n-1}[\alpha, \beta] = [\sigma_m(\alpha), \sigma_n(\beta)]$.

[W6] If $\phi : (X, x_0) \to (Y, y_0)$ is a map, then, for every $\alpha \in \pi_m(X, x_0)$ and $\beta \in \pi_n(X, x_0)$, we have $\phi_*[\alpha, \beta] = [\phi_*(\alpha), \phi_*(\beta)]$.

[W7] For any $\alpha \in \pi_m(X, x_0), \beta \in \pi_n(X, x_0), \gamma \in \pi_q(X, x_0)$, the following Jacobi identity holds:

$$(-1)^{mq}[[\alpha, \beta], \gamma] + (-1)^{nm}[[\beta, \gamma]\alpha] + (-1)^{qn}[[\gamma, \alpha], \beta] = 0$$

Milnor's theory indeed allows us to understand the topological types of isolated hypersurface singularities a lot better than before. In fact the following important theorem was first proved by Lê Dung Tráng, although the proof given here is slightly different from his.

**Theorem 3.14** Suppose that the two isolated hypersurface singularities $(V, 0)$ and $(\tilde{V}, 0)$ have the same topological types. Then they have the same Milnor number $\mu$ and their local monodromy are conjugated to each other.

*Proof* By Proposition 3.8, there exists $\epsilon_0 > 0$ with the following properties: If $\epsilon_0 > \epsilon > 0$, then $B_\epsilon - V \cap B_\epsilon$ is homotopy equivalent to $S_\epsilon - V \cap S_\epsilon$ and $B_\epsilon - \tilde{V} \cap B_\epsilon$ is homotopy equivalent to $S_\epsilon - \tilde{V} \cap S_\epsilon$. Moreover for any $\epsilon_0 > \epsilon_1 > \epsilon_2 > 0, S_{\epsilon_1} - S_{\epsilon_1} \cap V$ and $S_{\epsilon_1} - S_{\epsilon_1} \cap \tilde{V}$ are diffeomorphic to $S_{\epsilon_2} - S_{\epsilon_2} \cap V$ and $S_{\epsilon_2} - S_{\epsilon_2} \cap \tilde{V}$ respectively. We shall also assume that $\epsilon_0$ is so chosen such that Theorem 3.9 is applicable for both $(V, 0)$

and $(\tilde{V},0)$. Since $(V,0)$ and $(\tilde{V},0)$ have the same topological type, there exist neighborhoods $U,\tilde{U}$ of 0 and homeomorphism $\psi: U \to \tilde{U}$ such that $\psi(U\cap V) = \tilde{U}\cap\tilde{V}$ and $\psi(0)=0$. We shall assume that $B_{\epsilon_0} \subseteq U\cap\tilde{U}$.

Let $\epsilon_0 > \epsilon_4 > 0$. Since $\psi$ is continuous, there is $\epsilon_0 > \epsilon_3 > 0$ such that $\psi(B_{\epsilon_3}) \subseteq B_{\epsilon_4}$. $\psi(B_{\epsilon_3})$ is an open neighborhood of 0. So we can find $\epsilon_0 > \epsilon_2 > 0$ such that $B_{\epsilon_2} \subset \psi(B_{\epsilon_3})$. Since $\psi$ is a homeomorphism, we can find $\epsilon_0 > \epsilon_1 > 0$ such that $\psi(B_\epsilon) \subset B_{\epsilon_2} \subset \psi(B_{\epsilon_3})$. Let $x$ be a point in $B_{\epsilon_1} - V$ and $y = \psi(x) \in B_{\epsilon_4} - \tilde{V}$. We have

$$\begin{array}{ccccccc}
\pi_i(\psi(B_{\epsilon_1}) - \tilde{V}, y) & \to & \pi_i(B_{\epsilon_2} - \tilde{V}, y) & \to & \pi_i(\psi(B_{\epsilon_3}) - \tilde{V}, y) & \to & \pi_i(B_{\epsilon_4} - \tilde{V}, y) \\
\uparrow \cong & & & & \uparrow \cong & & \\
\pi_i(B_{\epsilon_1} - V, x) & & \xrightarrow{\cong} & & \pi_i(B_{\epsilon_3} - V, x). & &
\end{array}$$

Since $\pi_i(B_{\epsilon_1} - V, x) \to \pi_i(B_{\epsilon_3} - V, x)$ and $\pi(B_{\epsilon_2} - \tilde{V}, y) \to \pi_i(B_{\epsilon_4} - \tilde{V}, y)$ are isomorphisms, we see easily that $\pi_i(B_{\epsilon_1} - V, x) \to \pi_i(B_{\epsilon_2} - \tilde{V}, y)$ is an isomorphism for all $i$.

Since $S_{\epsilon_1} - S_{\epsilon_1}\cap V \to S^1$ is a locally trivial fibration with fiber $F_\theta$, we have the following exact sequence

$$\begin{array}{ccccccccc}
\pi_{n+1}(S^1) & \to & \pi_n(F_\theta) & \to & \pi_n(S_{\epsilon_1} - S_{\epsilon_1}\cap V) & \to & \pi_n(S^1) & \to & \\
\cdots & \to & \pi_1(F_\theta) & \to & \pi_1(S_{\epsilon_1} - S_{\epsilon_1}\cap V) & \to & \pi_1(S^1) & \to & 0.
\end{array}$$

It follows that $\pi_n(S_{\epsilon_1} - S_{\epsilon_1}\cap V) \cong \pi_n(F_\theta)$ and $\pi_1(S_{\epsilon_1} - S_{\epsilon_1}\cap V) \cong \pi_1(S^1)$. Since $B_{\epsilon_1} - B_{\epsilon_1}\cap V$ is homotopy equivalent to $S_{\epsilon_1} - S_{\epsilon_1}\cap V$, $\pi_n(B_{\epsilon_1} - B_{\epsilon_1}\cap V) \cong \pi_n(S_{\epsilon_1} - S_{\epsilon_1}\cap V) \cong \pi_n(F_\theta)$ and $\pi_1(B_{\epsilon_1} - B_{\epsilon_1}\cap V) \cong \pi(S_{\epsilon_1} - S_{\epsilon_1}\cap V) \cong \pi_1(S^1)$. By Hurewicz theorem $\pi_n(F_\theta)$ is naturally isomorphic to $H_n(F_\theta)$ because $F_\theta$ is $(n-1)$-connected. Therefore by [W3] the generator $h_*$ of $\pi_1(S^1)$ acts on $H_n(F_\theta)$ as homomorphism. Since Whitehead product is functorial by [W6], we have the following commutative diagram

$$\begin{array}{ccc}
H_n(F_\theta) & \xrightarrow{h_*} & H_n(F_\theta) \\
\downarrow & & \downarrow \\
H_n(\tilde{F}_\theta) & \xrightarrow{\tilde{h}_*} & H_n(\tilde{F}_\theta).
\end{array}$$

However by [W2], $h_*$ is precisely the monodromy automorphism minus the identity map on $H_n(F_\theta)$. The theorem follows immediately.

**Q.E.D.**

**Remark** The fact that Milnor number is an invariant of topological type was first observed by Teissier.

**Definition** Let $(V,0) \subseteq (\mathbf{C}^{n+1},0)$ be an isolated hypersurface singularities .. .T.h.e. .g.e.n.e.r.a.t.o.r. ..$p.i.$.1.(.S.·1.).. .i.n.d.u.c.e.s. .t.h.e. .m.o.n.o.d.r.o.m.y. .a.u.t.o.m.o.r.p.h.i.s.m $h^*: H^n(F_\theta,\mathbf{C}) \to H^n(F_\theta,\mathbf{C})$. Then the characteristic polynomial $\Delta_V(z)$ of the singularity $(V,0)$ is $\det(zI - h^*)$.

**Corollary 3.15** Let $(V,0) \subseteq (\mathbf{C}^{n+1},0)$ be an isolated hypersurface singularities. Then the characteristic polynomial $\Delta_V(z)$ is an invariant of topological type of $(V,0)$.

**Definition** Let $(V,0)$ be an isolated singularity in $(\mathbf{C}^{n+1},0)$. Denote $K_\epsilon = V\cap S_\epsilon$. By proposition 3.8, $K_\epsilon$ is independent of $\epsilon$ as a differentiable manifold. We shall denote it by $K_V$ from now on. $K_V$ is called the link of the singularity $(V,0)$.

**Theorem 3.16** Let $(V,0)$ and $(\tilde{V},0)$ be two isolated hypersurface singularities. If $(V,0)$ and $(\tilde{V},0)$ have the same topological type, then $\pi_i(K_V) \cong \pi_i(K_{\tilde{V}})$ for all $i$.

*Proof.* Since $(V,0)$ and $(\tilde{V},0)$ have the same topological type, there exist neighborhoods $U,\tilde{U}$ of 0 and homeomorphism $\psi: U\to\tilde{U}$ such that $\psi(U\cap V) = \tilde{U}\cap\tilde{V}$ and $\psi(0) = 0$. Let $\epsilon_0 > 0$ be sufficiently small so that $B_{\epsilon_0} \subset U\cap\tilde{U}$ and Proposition 3.8 and Theorem 3.9 are applicable for both $(V,0)$ and $(\tilde{V},0)$.

Let $\epsilon_0 > \epsilon_4 > 0$ be given. Since $\psi$ is continuous, there is $\epsilon_0 > \epsilon_3 > 0$ such that $\psi(B_{\epsilon_3}) \subseteq B_{\epsilon_4}$. $\psi(B_{\epsilon_3})$ is an open neighborhood of 0. We can find $\epsilon_0 > \epsilon_2 > 0$ such that $B_{\epsilon_2} \subset \psi(B_{\epsilon_3})$. Since $\psi$ is a homeomorphism, we can find $\epsilon_0 > \epsilon_1 > 0$ such that $\psi(B_{\epsilon_1}) \subset B_{\epsilon_2} \subset \psi(B_{\epsilon_3})$. Let $x$ be a point in $B_{\epsilon_1} \cap V - \{0\}$ and $y = \psi(x)$ in $\psi(B_{\epsilon_1} \cap V - \{0\})$. We have the following commutative diagram

$$\begin{array}{ccccccc} \pi_i(\psi(B_\epsilon) \cap \tilde{V} - \{0\}, y) & \to & \pi_i(B_{\epsilon_2} \cap \tilde{V} - \{0\}, y) & \to & \pi_i(\psi(B_{\epsilon_3}) \cap \tilde{V} - \{0\}, y) & \to & \pi_i(B_{\epsilon_4} \cap \tilde{V} - \{0\}, y) \\ \uparrow \cong & & & & & & \uparrow \cong \\ \pi_i(B_{\epsilon_1} \cap V - \{0\}, x) & & & \xrightarrow{\cong} & & & \pi_i(B_{\epsilon_3} \cap V - \{0\}, x). \end{array}$$

Since $\pi_i(B_{\epsilon_1} \cap V - \{0\}, x) \to \pi_i(B_{\epsilon_3} \cap V - \{0\}, x)$ and $\pi_i(B_{\epsilon_2} \cap \tilde{V} - \{0\}, y) \to \pi_i(B_{\epsilon_4} \cap \tilde{V} - \{0\}, y)$ are isomorphisms, we see easily that $\pi_i(B_{\epsilon_1} \cap V - \{0\}, x) \to \pi_i(B_{\epsilon_2} \cap \tilde{V} - \{0\}, y)$ is an isomorphism. Observe that $B_{\epsilon_1} \cap V - \{0\}$ and $B_{\epsilon_2} \cap \tilde{V} - \{0\}$ are homotopy equivalent to $K_V$ and $K_{\tilde{V}}$ respectively. Thus we have shown $\pi_i(K_V)$ is isomorphic to $\pi_i(K_{\tilde{V}})$ for all $i$.

**Q.E.D.**

## Classification of Topological Types for Surface Singularities

As we saw before, the topological types of plane curve singularities were completely understood by the end of sixties. However, after almost twenty years, there was no progress in understanding the topological types of surface singularities. In fact, there was not even a conjecture what the result should be. Recently, by working on Zariski multiplicity problem, we come up with the following conjecture.

**Conjecture.** Let $(V,0)$ be an isolated hypersurface singularity in $(\mathbf{C}^3,0)$. Then the topological type of $(V,0)$ determines and is determined by the characteristic polynomial $\Delta_V(z)$ of $(V,0)$ and the fundamental group $\pi_1(K)$ of the link of $(V,0)$.

**Remark** We know that the topological type of $(V,0)$ determines $\Delta_V(z)$ and $\pi_1(K)$ by Theorem 3.13 and Theorem 3.15.

Recall that a hypersurface singularity $(V,0) = \{(z_0,\cdots,z_n) : f(z_0,\cdots,z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ is quasi-homogeneous if $f$ is in the Jacobian ideal of $f$, i.e. $f \in (\frac{\partial f}{\partial z_0},\cdots,\frac{\partial f}{\partial z_n})$. Recently Xu and the author proved that the above conjecture is true for quasi-homogeneous surface singularities. Namely we proved the following theorem.

**Theorem 3.17** Let $(V,0)$ and $(W,0)$ be two isolated quasi-homogeneous surface singularities in $\mathbf{C}^3$. Then $(\mathbf{C}^3,V,0)$ is homeomorphic to $(\mathbf{C}^3,W,0)$ if and only if $\pi_1(K_V) \cong \pi_1(K_W)$ and $\Delta_V(z) = \Delta_W(z)$.

In fact Xu and the author have also proved the following theorem which is of independent interest.

**Theorem 3.18** Let $(V,0)$ and $(W,0)$ be two isolated quasi-homogeneous surface singularities having the same topological type. Then $(V,0)$ is connected to $(W,0)$ by a family of constant topological type. In fact $(V,0)$ is connected to one of the following seven class by a family of constant topological type:

| | |
|---|---|
| class I | $V(I) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$ |
| class II | $V(II) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$ $a_1 > 1$ |
| class III | $V(III) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_1 = 0\}$ $a_1 > 1, a_2 > 1$ |
| class IV | $V(IV) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} = 0\}$ |
| class V | $V(V) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$ |
| class VI | $V(VI) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$ where $(a_0-1)(a_1 b_2 + a_2 b_1) = a_0 a_1 a_2$ |
| class VII | $V(VII) = \{z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\}$ where $(a_0-1)(a_1 b_2 + a_2 b_1) = a_2(a_0 a_1 - 1)$ |

**Definition** A polynomial $h(z_0,\cdots,z_n)$ is weighted homogeneous of type $(w_0,\cdots,w_n)$, where $(w_0,\cdots,w_n)$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0/w_0 + \cdots + i_n/w_n = 1$. $(w_0, w_1,\cdots,w_n)$ is called the weights of polynomials $h$. Let $w_i = u_i/v_i$ be the reduced fraction of $w_i$ i.e. $u_i$ and $v_i$ are integers with $(u_i,v_i) = 1$.

By the theorem of Saito, we may assume from now on that $w_i \geq 2$ for $i = 0,\cdots,n$. Saito also proved that quasi-homogeneous function with isolated singularity at 0 can be put into weighted homogeneous polynomial by a biholomorphic change of coordinates. The following proposition which is a consequence of Milnor and Orlik is due to Yoshinaga.

**Proposition 3.19** Let $f(x_0,\cdots,x_n)$ (respectively $g(x_0,\cdots,x_n)$) be a weighted homogeneous polynomial with weights $(\frac{u_0}{v_0},\cdots,\frac{u_n}{v_n})$ (respectively $(u_0'/v_0',\cdots,u_n'/v_n')$) where $\frac{u_i}{v_i}$ is the reduced fraction of $w_i$ (respectively $w_i'$). Assume that $f$ (respectively $g$) has an isolated singularity at origin. Then $\Delta_f(z) = \Delta_g(z)$ if and only if the following two conditions are satisfied:

(1) $\{2,u_0,\cdots,u_n\} = \{2,u_0',\cdots,u_n'\}$

(2) For any $u \in \{2,u_0,\cdots,u_n\}$ $\prod_{u_i=u}(1-\frac{u_i}{v_i}) = \prod_{u_i'=u}(1-\frac{u_i'}{v_i'})$ where the product over an empty set is assumed to be one.

**Definition** Suppose given a real manifold B of dimension $m$, and a family $\{(M_t, N_t) : t \in B, N_t$ is a closed submanifold of compact differentiable manifold $M_t\}$. We say that $(M_t, N_t)$ depends $C^\infty$ on $t$ and that $\{(M_t, N_t) : t \in B\}$ is a $C^\infty$ family of compact manifolds with submanifolds if there is a $C^\infty$ manifold $M$, closed submanifold $N$ and a $C^\infty$ map $\omega$ from $M$ onto $B$ such that $\overline{\omega} = \omega/N$ is also a $C^\infty$ map from $N$ onto $B$ satisfying the following condition

(i) $M_t = \omega^{-1}(t) \supseteq N_t = \overline{\omega}^{-1}(t)$

(ii) The rank of the Jacobian of $\omega$ (respectively $\overline{\omega}$) is equal to $m$ at every point of $M$ (respectively $N$)

**Theorem 3.20** $((M,N),(\omega,\overline{\omega}),B)$ be a $C^\infty$ family of compact manifolds with submanifolds, with $B$ connected. Then $(M_t, N_t) = (\omega^{-1}(t), \overline{\omega}^{-1}(t))$ is diffeomorphic to $(M_{t_0}, N_{t_0}) = (\omega^{-1}(t_0), \overline{\omega}^{-1}(t_0))$ for any $t \in B$.

The proof of Theorem 3.17 and Theorem 3.18 made use of the fundamental results of Neumann and Orlik-Wagreich, the proposition 3.19, Theorem 3.20 and the deep theory below due to Varchenko.

Let $\mathbf{N} \subset \mathbf{R}_+$ be the set of all nonnegative integers and of all nonnegative real numbers. Let $f = \sum a_k x^k, a_k \in \mathbf{C}, k \in \mathbf{N}^{n+1}$ , be an element in $\mathbf{C}\{x_0, \cdots, x_n\}$ and supp $f$ be the set $\{k \in \mathbf{N}^{n+1} : a_k \neq 0\}$. We denote by $\Gamma_+(f)$, the convex hull of the set $\bigcup_{k \in \mathrm{supp} f}(k + \mathbf{R}_+^{n+1})$, in $\mathbf{R}_+^{n+1}$. The polyhedron $\Gamma_+(f)$ which is the union of all compact facets of $\Gamma_+(f)$ will be called Newton's diagram of the power series $f$. The polynomial $\sum_{k \in \Gamma(f)} a_k x^k$ will be called the main part of the power series $f$. Let $\gamma$ be a closed facet of $\Gamma(f)$. Let us denote the polynomial $\sum_{k \in \gamma} a_k x^k$ by $f_\gamma$. The main part of the power series $f$ will be called nondegenerate if for any closed facet $\gamma \in \Gamma(f)$ the polynomials $(x_0 \frac{\partial f_\gamma}{\partial x_0}), \cdots, (x_n \frac{\partial f_\gamma}{\partial x_n})$ have no common zero in $\{(x_0, \cdots, x_n) \in \mathbf{C}^{n+1} : x_0 \cdots x_n \neq 0\}$.

We shall define the notion of characteristic polynomial $\Delta_\Gamma(z)$ associated with the Newton's diagram $\Gamma(f)$. Let

$$\Delta_\Gamma(z) = \left[\prod_{l=1}^{n+1} \Delta^l(z)^{(-1)^{n+l+1}}\right](z-1)^{(-1)^{n+1}}$$

where $\Delta^l$ is a polynomial defined as below. $\Delta^l$ is defined by the $(l-1)$ dimensional facets of the intersections of $\Gamma(f)$ with all possible $l$-dimensional coordinate planes.

Let $L$ be $l$-dimensional affine subspace of $\mathbf{R}^{n+1}$ such that $L \cap \mathbf{Z}^{n+1}$ is $l$-dimensional lattice. By definition let the $l$-dimensional volume of the cube (spanned by any basis of $L \cap \mathbf{Z}^{n+1}$) be equal to one.

Now we shall define $\Delta^l$. Let $I \subseteq \{0, 1, \cdots, n\}$ and $|I| = l$ where $|I|$ is the number of the elements of $I$. Let us consider the pair $L_I, L_I \cap \Gamma(f)$, where $L_I = \{k \in \mathbf{R}^{n+1} : k_i = 0 \ \forall i \notin I\}$. Let $\Gamma_1(I), \cdots, \Gamma_{j(I)}(I)$ be all $(l-1)$-dimensional facets of $L_I \cap \Gamma(f)$ and $L_1, \cdots, L_{j(I)}$ be the $(l-1)$-dimensional affine subspaces, containing them respectively.

Let $\sum_{i \in I} a_i^j k_i = m_j(I)$ be the equation of $L_j$ in $L_I$ where $a_i^j, m_j(I) \in N$ and the greatest common divisor of the numbers $a_i^j, i \in I$, is equal to one. The numbers $a_i^j, m_j(I)$ are defined by these conditions uniquely. The numbers $m_j(I)$ will take part in the definition of $\Delta^l$. Another definition of $m_j(I)$ is the following. Consider the quotient of the lattice $\mathbf{Z}^{n+1} \cap \mathbf{L}_I$ by the subgroup generated by vectors of $\mathbf{Z}^{n+1} \cap \mathbf{L}_j$. This is a cyclic group of order $m_j(I)$. Let $V(\Gamma_j(I))$ be the $(l-1)$-dimensional volume of $\Gamma_j(I)$ in $L_j$. Let

$$\Delta^l(z) = \prod_{I, |I|=l} \prod_{j=1}^{j(I)} (z^{m_j(I)} - 1)^{(l-1)! V(\Gamma_j(I))}.$$

It was observed by Varchenko that $m_j(I)(l-1)!V(\Gamma_j(I))$ is equal to $l!$ multiplied by the $l$-dimensional volume of the cone over $\Gamma_j(I)$ with vertex at origin. According to this remark $\deg\Delta^l$ contains the following geometric meaning. Let $\Gamma_-(f)$ be the cone over $\Gamma(f)$ with vertex at the origin. Then $\deg\Delta^l$ is the sum of $l$-dimensional volumes of the intersections of $\Gamma_-(f)$ with all possible $l$-dimensional coordinate planes, multiplied by $l!$. The following theorem due to Varchenko is of fundamental importance.

**Theorem 3.21** Let $f$ belong to the square of the maximal ideal of $\mathbf{C}\{x_0, x_1, \cdots, x_n\}$ and let the main part of the power series $f$ be nondegenerate. Then the characteristic polynomial of the monodromy of $f$ at the origin is equal to the characteristic polynomial of the Newton diagram of $f$.

In fact in our original proof of Theorem 3.17 and Theorem 3.18, we did not make use of Proposition 3.19, we only need Theorem 3.21.

Let $f(z_0, \cdots, z_n)$ be a weighted homogeneous function with weights $(w_0, \cdots, w_n)$. Then there exist non-zero integers $q_0, \cdots, q_n$ and a positive integer $d$ so that

$$f(t^{q_0} z_0, \cdots, t^{q_n} z_n) = t^d f(z_0, \cdots, z_n).$$

In fact let $\langle w_0, \cdots, w_n \rangle$ denote the smallest positive integers $d$ such that there exists, for each $i$, an integer $q_i$, so that $q_i w_i = d$. These are the $q_i$ and $d$ above.

**Definition** A function $f$ is semi-quasihomogeneous if $f = f_0 + f'$, where $f_0$ is quasi-homogeneous of degree $d$ and has an isolated singularity at 0 and all the monomials of $f'$ are of degree greater than $d$.

Arnold gave normal forms for semi-quasi-homogeneous function in the following manner.

**Theorem 3.22** A semi-quasi-homogeneous function $f$ with weighted homogeneous part $f_0$ is biholomorphically equivalent to the normal form $f_0 + c_1 e_1 + \cdots + c_r e_r$ where the $c_i$ are numbers and the $e_i$ are basis monomials of the Milnor algebra

$$\mathbf{C}\{z_0, \cdots, z_n\} \Big/ \left( \frac{\partial f_0}{\partial z_0}, \cdots, \frac{\partial f_0}{\partial z_n} \right)$$

of the function $f_0$ of degree greater than $d$=degree of $f_0$.

Lê-Ramanujan(cf. [Lê-Ra]) remarked that, the singulariti 0 have the same topological type. Consequently we have the following theorem.

**Theorem 3.23** Theorem 3.17 and Theorem 3.18 are true for semi-quasi-homogeneous singularities.

### Zariski Multiplicity Problem

In his retiring presidential address to the American Mathematical Society in 1971, Zariski asked whether $(V, 0)$ and $(W, 0)$ have the same multiplicity if they have the same topological type. He expected that topologists would be able to answer his question in relatively short order. However the question appears much harder than what Zariski thought. Even special cases of Zariski's problem have proved to be extremely difficult. Only recently Greuel and O'shea proved independently that topological type constant family of isolated quasi-homogeneous singularities are equi-multiple. For quasi-homogeneous surface singularities, Laufer explained the constant multiplicity for topological type constant family of singularities from a different viewpoint. However, it was not known that whether two quasi-homogeneous singularities having the same topological type can be put into a topological type constant family. Xu and the author proved Theorem 3.17. Thus Zariski problem is solved affirmatively in this case. Actually we solved the problem directly without using the result of Greuel and O'shea.

In fact, in view of Theorem 3.23 above, we deduce the following theorem.

**Theorem 3.24** Let $(V, 0)$ and $(W, 0)$ be two isolated semi-quasi-homogeneous singularities in $\mathbf{C}^3$. If $(\mathbf{C}^3, V, 0)$ is homeomorphic to $(\mathbf{C}^3, W, 0)$ as germs, then $V$ and $W$ have the same multiplicity at the origin.

In fact the proof of Theorem 3.24 goes as follows. By the remark of Lê-Ramanujan, we may assume that $(V, 0)$ and $(W, 0)$ are isolated weighted homogeneous singularities.

**Lemma 3.25** Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial with weights $(w_0, w_1, w_2)$. Suppose that $f$ has an isolated singularity at origin. Denote the multiplicity of $f(z_0, z_1, z_2)$ at the origin by $m_f$. Then $m_f \geq min\{w_0, w_1, w_2\}$.

*Proof.* By definition of the multiplicity, there is a monomial $z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}$ in $f$ such that $m_f = \alpha_0 + \alpha_1 + \alpha_2$. Since $f$ is weighted homogeneous with weights $(w_0, w_1, w_2)$, we have $\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2} = 1$. It follows easily that $m_f \geq min\{w_0, w_1, w_2\}$.

**Q.E.D.**

**Proposition 3.26** Suppose that $f$ is one of the seven classes. Let $m = min\{n \in \mathbf{Z} | n \geq min\{w_0, w_1, w_2\}\}$. Then the multiplicity $m_f$ of $f$ is $m$.

*Proof.* Easy exercise.

**Corollary 3.27** Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial with weights $(w_0, w_1, w_2)$. Suppose $f$ has an isolated singularity at origin. Then the multiplicity $m_f$ of $f$ at origin equals to $m = min\{n \in \mathbf{Z} | n \geq min\{w_0, w_1, w_2\}\}$. Hence the multiplicity of $f$ is an invariant of topological type.

*Proof.* By a result due to Orlik-Wagreich-Arnold, $f = g + h$ where $g$ is one of the seven classes having the same weights $(w_0, w_1, w_2)$ and $g$ and $h$ have no monomial in common. It is clear that $m_g \geq m_f$. By Proposition 3.26 we have $m_g \leq min\{n \in \mathbf{Z} | n \geq min\{w_0, w_1, w_2\}\}$. Hence we have $m_f \leq min\{n \in \mathbf{Z} | n \geq min\{w_0, w_1, w_2\}\}$. Conversely by Lemma 3.25, we see that $m_f \geq min\{w_0, w_1, w_2\}$. Thus $m_f \geq min\{n \in \mathbf{Z} | n \geq min\{w_0, w_1, w_2\}\}$.

**Q.E.D.**

Let $(V, 0)$ be a dimension two isolated hypersurface singularity. Lê and Teissier observed that A'Campo's work can often be used to give positive results towards Zariski's question. Let $C(V, 0)$ be the reduced tangent cone. Let $\mathbf{P}C(V, 0)$ denote the hypersurface in $\mathbf{CP}^2$ over which $C(V, 0)$ is a cone. Then, the work of A'Campo's shows that the multiplicity of $(V, 0)$ is determined by the topological type of $(V, 0)$ in case the topological Euler number $\chi(\mathbf{P}C(V, 0))$ is non-zero. The same arguments also show that, for isolated hypersurface two-dimensional singularities, the embedded topology and the multiplicity determine $\chi(\mathbf{P}C(V, 0))$. However, so far, by using A'Campo's result, one can only prove that a surface in $\mathbf{C}^3$ having at 0 a singularity of multiplicity 2 cannot have the same topological type at 0 as another surface singularity of multiplicity different from 2.

For plane curve singularities, the Zariski question was known to be true. The reason that the Zariski question could be answered was that the topological types of plane curve singularities were well understood. If $(m_1, n_1), \cdots, (m_g, n_g)$ are the Puiseux pairs for plane irreducible curve singularity, then one knows that $n_1 \cdots n_g$ is the multiplicity of the singularity.

**Definition** Let $(V, 0)$ be a normal two dimensional singularity. Let $\pi : (M, A) \to (V, 0)$ be a resolution with exceptional set $A$. The geometric genus of a normal two dimensional singularity $(V, 0)$ is the integer

$$p_g(V, 0) = \dim_{\mathbf{C}} R^1 \pi_*(\mathcal{O}_M)_0.$$

The arithmetic genus of a normal two dimensional singularity $(V, 0)$ is the integer

$$p_a(V, 0) = \sup_D p_a(D),$$

where $D$ is a positive cycle and $p_a(D)$ is the virtual genus of $D$ on $M$.

In [Ya10] we first observed the following theorem.

**Theorem 3.28** Let $(V, 0)$ be an isolated hypersurface two dimensional singularities. Then $p_g(V, 0)$ and $p_a(V, 0)$ are invariants of topological type of $(V, 0)$.

As a result of the above observation, we have proved the following special case of Zariski's multiplicity conjecture.

**Theorem 3.29** Let $(V, 0)$ and $(W, 0)$ be two isolated two-dimensional hypersurface singularities in $\mathbf{C}^3$ having the same topological type. If $p_a(V, 0) \leq 2$, then $\nu(V, 0) = \nu(W, 0)$ where $\nu(V, 0)$ and $\nu(W, 0)$ are the multiplicities of $(V, 0)$ and $(W, 0)$ respectively.

## §4 Lie algebras arising from isolated hypersurface singularities and its application to algebraic geometry

The original motivation of this section goes back to the following theorem

**Theorem** (Mather-Yau) Suppose $(V, 0)$ and $(W, 0)$ are germs of hypersurface singularities in $\mathbf{C}^{n+1}$, and $V - \{0\}$ is nonsingular. Then the following conditions are equivalent.

(i) $(V, 0)$ is biholomorphically equivalent to $(W, 0)$

(ii) $A(V)$ is isomorphic to $A(W)$ as $\mathbf{C}$-algebra. Here $A(V)$ and $A(W)$ denote the moduli algebras of $(V, 0)$ and $(W, 0)$ respectively.

The natural question to ask here is the following

**Recognition problem:** When is a commutative local Artinian algebra a moduli algebra of an isolated hypersurface singularity.

Consider $L(V)$, the algebra of derivations of $A(V)$. Since $A(V)$ is finite dimensional as $\mathbf{C}$-vector space and $L(V)$ is contained in the endomorphism algebra of $A(V)$; consequently $L(V)$ is a finite dimensional Lie algebra (with Lie bracket: $[D_1, D_2] = D_1D_2 - D_2D_1$). Thus we have the following natural mapping.

$$\begin{array}{ccc}\{\text{isolated hypersurface singularities}\} & \longrightarrow & \{\text{finite dimensional Lie algebra}\}\\ (V,0) & \longmapsto & L(V)\end{array}$$

By Levi theorem, any finite dimensional Lie algebra is equal to a semi-direct product of a solvable Lie algebra and a semi-simple Lie algebra.

In 1970, Brieskorn established a connection between a very special kind of surface singularities i.e. 2 dimensional rational double points and simple Lie algebras. Unfortunately his approach is extremely difficult, if not impossible at all, to be generalized to arbitrary singularity.

Our approach is very different from his. In fact we have

**Theorem 4.1** If $(V, 0)$ is a hypersurface isolated singularity, then $L(V)$ is a solvable Lie algebra.

Thus the theory of isolated singularities and the theory of finite dimensional solvable Lie algebras are linked together in the first time. The above theorem also gives a necessary condition for the recognition problem.

$$\begin{aligned} V &= \{(z_0, z_1, \cdots, z_n) \in \mathbf{C}^{n+1} : f(z_0, z_1, \cdots, z_n) = 0\}\\ \mathcal{O}_{n+1} &= \mathbf{C}\{z_0, \cdots, z_n\}\\ m &= (z_0, \cdots, z_n)\mathcal{O}_{n+1}\\ \Delta(f) &= \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}_{n+1}. \end{aligned}$$

**Lemma 4.2** Let $D = \sum_{i=0}^{n} a_i(z_0, z_1, \cdots, z_n)\frac{\partial}{\partial z_i}$ be an element in $L(V)$ where $a_i(z_0, \cdots, z_n) \in \mathcal{O}_{n+1}$, $0 \le i \le n$. Then $a_i \in m$. In particular $L(V)$ acts on $A(V)$ and preserves $m$-adic filtration, i.e.

$$L(V)(m^k) \subseteq m^k.$$

*Proof.* Since $V$ has only isolated singularity at 0, there exists an integer $r$ such that

$$(f) + \Delta(f) \supseteq m^r.$$

Let $k$ be the smallest integer $k$ such that $z_0^k \in (f) + \Delta(f)$

$$\begin{aligned} D \in L(V) &\Rightarrow D \text{ leaves the ideal } (f) + \Delta(f) \text{ invariant}\\ &\Rightarrow ka_0(z) \cdot z_0^{k-1} = D(z_0^k) \in (f) + \Delta(f)\\ &\Rightarrow a_0(0, \cdots, 0) = 0 \quad (\text{since k is the smallest integer such that } z^k \in (f) + \Delta(f)). \end{aligned}$$

**Q.E.D.**

Recall that a simple Lie algebra $A_1 = sl(2, \mathbf{C})$ is the complex Lie algebra with basis

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and relations $[\tau, X_+] = 2X_+, [\tau, X_-] = -2X_-, [X_+, X_-] = \tau$.

To prove the theorem, it suffices to prove that $L(V)$ does not contain $sl(2, \mathbf{C})$ as Lie-subalgebra. For this purpose we have to understand all possible $sl(2, \mathbf{C})$ actions on $\mathbf{C}[[z_1, \cdots, z_n]]$ as derivations which preserves $m$-adic filtration.

**Theorem 4.3** Let $L = sl(2, \mathbf{C})$ act on $\mathbf{C}[[x_1, \cdots, x_n]]$ via derivations preserving the $m$-adic filtration i.e., $L(m^k) \subseteq m^k$ where $m$ is the maximal ideal in $\mathbf{C}[[x_1, \cdots, x_n]]$. Then there exists a coordinate change $y_1, \cdots, y_n$ with respect to which $sl(2, \mathbf{C})$ is spanned by

$$\tau = \sum_{j=1}^{n} a_{1j} \frac{\partial}{\partial y_j}$$
$$X_+ = \sum_{j=1}^{n} a_{2j} \frac{\partial}{\partial y_j}$$
$$X_- = \sum_{j=1}^{n} a_{3j} \frac{\partial}{\partial y_j}$$

where $a_{ij}$ is a linear function in $y_1, \cdots, y_n$ variables for all $1 \leq i \leq 3$ and $1 \leq j \leq n$. Here $\{\tau, X_+, X_-\}$ is a standard basis for $sl(2, \mathbf{C})$ i.e., $[\tau, X_+] = 2X_+, [\tau, X_-] = -2X_-$ and $[X_+, X_-] = \tau$.

**Theorem 4.4** Let $sl(2, \mathbf{C})$ act on the formal power series ring $\mathbf{C}[[x_1, \cdots, x_n]]$ preserving the $m$-adic filtration where $m$ is the maximal ideal in $\mathbf{C}[[x_1, \cdots, x_n]]$. Then there exists a coordinate system

$$x_1, x_2, \cdots, x_{l_1}, x_{l_1+1}, x_{l_1+2}, \cdots, x_{l_1+l_2}, \cdots, x_{l_1+l_2+\cdots+l_{s-1}+1}, \cdots, x_{l_1+l_2+\cdots+l_s}$$

such that

$$\tau = D_{\tau,1} + \cdots + D_{\tau,j} + \cdots + D_{\tau,r}$$
$$X_+ = D_{X_+,1} + \cdots + D_{X_+,j} + \cdots + D_{X_+,r}$$
$$X_- = D_{X_-,1} + \cdots + D_{X_-,j} + \cdots + D_{X_-,r}$$

where $r \leq s$ and

$$\begin{aligned} D_{\tau,j} &= (l_j - 1)x_{l_1+\cdots+l_{j-1}+1} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}} + (l_j - 3)x_{l_1+\cdots+l_{j-1}+2} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots \\ &\quad + (-(l_j - 3))x_{l_1+\cdots+l_j-1} \frac{\partial}{\partial x_{l_1+\cdots+l_j-1}} + (-(l_j - 1))x_{l_1+\cdots+l_j} \frac{\partial}{\partial x_{l_1+\cdots+l_j}} \\ D_{X_+,j} &= (l_j - 1)x_{l_1+\cdots+l_{j-1}+1} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots + i(l_j - i)x_{l_1+\cdots+l_{j-1}+i} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i+1}} \\ &\quad + \cdots + (l_j - 1)x_{l_1+\cdots+l_j-1} \frac{\partial}{\partial x_{l_1+\cdots+l_j}} \\ D_{X_-,j} &= x_{l_1+\cdots+l_{j-1}+2} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}} + \cdots + x_{l_1+\cdots+l_{j-1}+i+1} \frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i}} \\ &\quad + \cdots + x_{l_1+\cdots+l_j} \frac{\partial}{\partial x_{l_1+\cdots+l_j-1}} \end{aligned}$$

*Proof.* According to Theorem 4.3 we can choose a coordinate system $\{y_1, \cdots, y_n\}$ such that the coefficient of $\partial/\partial y_i, 1 \le i \le n$, of every element in $sl(2, \mathbf{C})$ are linear functions in $y_1, \cdots, y_n$ variables. In view of the proof of complete classification of representations of $sl(2, \mathbf{C})$, by further change of coordinate we obtain a coordinate system $\{x_1, x_2, \cdots, x_n\}$ such that $sl(2, \mathbf{C})$ takes the form as stated in the theorem.

**Q.E.D.**

**Theorem 4.5** (Yau-Yu) Let $sl(2, \mathbf{C})$ act on the formal power series ring $\mathbf{C}[[x_1, \cdots, x_n]]$ preserving the $m$-adic filtration where $m$ is the maximal ideal in $\mathbf{C}[[x_1, \cdots, x_n]]$. Let $I(f)$ be the complex vector space spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}$. Then there is a complete classification of $I(f)$ as an $sl(2, \mathbf{C})$-submodule.

**Corollary 4.6** (Sampson-Yau-Yu) In case $sl(2, \mathbf{C})$ acts on $m/m^2$ irreducibly, then $I(f)$ is a $sl(2, \mathbf{C})$ module if and only if $f$ is an invariant polynomial and $I(f) = (n)$ where $(n)$ is an irreducible module of dimension $n$.

**Corollary 4.7** (Yau-Yu, Kempf) I(f) is a $sl(2, \mathbf{C})$ submodule if and only if $I(f) = I(g)$ for some $sl(2, \mathbf{C})$ invariant polynomial $g$.

*Example* : $n = 5$

*Case I:* $\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$ $\quad X_+ = x_1 \frac{\partial}{\partial x_2}$ $\quad X_- = x_2 \frac{\partial}{\partial x_1}$

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2, \mathbf{C})$ submodule if and only if $f$ is a polynomial in $x_3, x_4$ and $x_5$ variables.

*Case II:* $\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_1}$ $\quad X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$ $\quad X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$

$$\tau = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad X_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2, \mathbf{C})$ submodule of and only if one of the following occurs

(a) (1) $f$ is a $sl(2, \mathbf{C})$ invariant polynomial and $I(f) = (3) \oplus (1) \oplus (1)$.

(2) $f = g(x_1, x_2, x_3, x_4, x_5) + c_1 x_1 (x_4 + r x_5)^k + c_2 x_2 (x_4 + r x_5)^k + c_3 x_3 (x_4 + r x_5)^k$
where $g(x_1, x_2, x_3, x_4, x_5) = d_1 (x_2^2 - 2x_1 x_3)(x_4 + r x_5)^{k-1} + d_2 x_5 (x_4 + r x_5)^k + d_3 (x_4 + r x_5)^{k+1}$
is a $sl(2, \mathbf{C})$ invariant polynomial with $d_1 \neq 0$ and $d_2 \neq 0$
$I(f) = < \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} > = (3) \oplus (1) \oplus (1)$.

(3) $f = g(x_1, x_2, x_3, x_4, x_5) + c_1 x_1 (r x_4 + x_5)^k + c_2 x_2 (r x_4 + x_5)^k + c_3 x_3 (r x_4 + x_5)^k$
where $g(x_1, x_2, x_3, x_4, x_5) = d_1 (x_2^2 - 2x_1 x_3)(r x_4 + x_5)^{k-1} + d_2 x_4 (r x_4 + x_5)^k + d_3 (r x_4 + x_5)^{k+1}$
is a $sl(2, \mathbf{C})$ invariant polynomial, with $d_1 \neq 0$ and $d_2 \neq 0$.
$I(f) = < \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} > = (3) \oplus (1) \oplus (1)$.

(b) $f$ is a $sl(2, \mathbf{C})$ invariant polynomial in $x_1, x_2, x_3$ and $x_4$ variables and $I(f) = (3) \oplus (1)$.

(c) $f$ is a $sl(2, \mathbf{C})$ invariant polynomial in $x_1, x_2$ and $x_3$ variables and $I(f) = (3)$.

(d) $f$ is a $sl(2, \mathbf{C})$ invariant polynomial in $x_4$ and $x_5$ variables and $I(f) = (1) \oplus (1)$.

(e) $f = (c_1 x_4 + c_2 x_5)^{k+1}$ either $c_1 \neq 0$ or $c_2 \neq 0$. $I(f) = (1)$.

*Case III:* $\tau = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4}$ $\quad X_+ = x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4}$ $\quad X_- = x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3}$

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X_+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2,\mathbf{C})$ submodule if and only if one of the following occurs.

(a) $f$ is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, x_3, x_4$ and $x_5$ variables and $I(f) = (2) \oplus (2) \oplus (1)$.
(b) $f$ is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, x_3$ and $x_4$ variables and $I(f) = (2) \oplus (2)$.
(c) $f = cx_5^{k+1}$ where $c$ is a nonzero constant and $I(f) = (1)$.

*Case IV:* $\tau = 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4}$ $\quad X_+ = 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4}$
$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}$.

$$\tau = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2,\mathbf{C})$ submodule if and only if one of the following occurs.

(a) $f$ is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, x_3, x_4$ and $x_5$ variables and $I(f) = (4) \oplus (1)$.
(b) $f$ is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, x_3$ and $x_4$ variables and $I(f) = (4)$.
(c) $f = cx_5^{k+1}$ where $c$ is a nonzero constant and $I(f) = (1)$.

*Case V:* $\tau = 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}$ $\quad X_+ = 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5}$
$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4}$.

$$\tau = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2,\mathbf{C})$ submodule if and only if one of the following occurs.

(a) (1) $f$ is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, x_3, x_4$ and $x_5$ variables and $I(f) = (3) \oplus (2)$.
(2) $f = g(x_1, x_2, x_3, x_4, x_5) + c_1x_4^3 + c_2x_4^2x_5 + c_3x_4x_5^2 + c_4x_5^3$
where $g(x_1, x_2, x_3, x_4, x_5) = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2$ is a $sl(2,\mathbf{C})$ invariant polynomial and $I(f) =< \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} >= (3) \oplus (2)$.
(b) $f$is a $sl(2,\mathbf{C})$ invariant polynomial in $x_1, x_2$ and $x_3$ variables, and $I(f) = (3)$.

*Case VI:* $\tau = 4x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_4} - 4x_5\frac{\partial}{\partial x_5}$ $\quad X_+ = 4x_1\frac{\partial}{\partial x_2} + 6x_2\frac{\partial}{\partial x_3} + 6x_3\frac{\partial}{\partial x_4} + 4x_4\frac{\partial}{\partial x_5}$

$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4}$

$$\tau = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad X_+ = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad X_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$I(f)$ is a $sl(2, \mathbf{C})$ submodule if and only if $f$ is a $sl(2, \mathbf{C})$ invariant polynomial in $x_1, x_2, x_3, x_4$ and $x_5$ and $I(f) = (5)$.

The proof of Theorem 4.1 follows from the following observation. Once we fix a $sl(2, \mathbf{C})$ action, the singular sets of varieties defined by $sl(2, \mathbf{C})$ invariant polynomials of degree $\geq 3$ have to contain a one dimensional set which depends only on the $sl(2, \mathbf{C})$ action.

**Definition** Let $(X_1, 0)$ and $(X_2, 0)$ be two isolated hypersurface singularities in $(\mathbf{C}^n, 0)$. We say that $(X_1, 0)$ and $(X_2, 0)$ have the same analytic type (respectively topological type) if there exists a germ of biholomorphism (resp. homeomorphism) from $(\mathbf{C}^n, X_1, 0)$ to $(\mathbf{C}^n, X_2, 0)$.

The following question was pointed out to me by Lê Dũng Tráng.

**Question.** Let $f(z_1, \cdots, z_n) = 0$ and $h(w_1, \cdots, w_m) = 0$ be the defining equations for isolated hypersurface singularities $(X_f, 0) \subset (\mathbf{C}^n, 0)$ and $(X_h, 0) \subseteq (\mathbf{C}^m, 0)$. Does the topological type of the hypersurface $X_{f+h}$ defined by $f(z_1, \cdots, z_n) + h(w_1, \cdots, w_m) = 0$ (addition of Thom-Sebastiani) in $(\mathbf{C}^{n+m}, 0)$ depend only on the topological type of $(X_f, 0)$ and $(X_h, 0)$?

In his 1977 paper, Teissier introduced the concept for two isolated hypersurface singularities being $(c)$-cosécantes. He showed that the $(c)$-cosécantes class of the hypersurface defined by $f(z_1, \cdots, z_n) + h(w_1, \cdots, w_m) = 0$ depends only on the $(c)$-cosécantes class of $(X_f, 0)$ and $(X_h, 0)$. He remarked that the analytic type of the hypersurface $X_{f+h}$ defined by $f(z_1, \cdots, z_n) + h(w_1, \cdots, w_m) = 0$ depends not only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$, but also in general on the choice of the equation for $f$ and $h$. However the following theorem says that in case $h$ is quasi-homogeneous, then the analytic type of $X_{f+h}$ indeed depends only on the analytic types of $(X_f, 0)$ and of $(X_h, 0)$. In fact, a "subtraction" theorem holds!

**Theorem 4.8** Let $f(z_1, \cdots, z_n)$ and $g(z_1, \cdots, z_n)$ be holomorphic functions with isolated singularity at origin in $\mathbf{C}^n$, and $h(w_1, \cdots, w_m)$ be a quasi-homogeneous holomorphic function with an isolated singularity at origin. Then $(X_f, 0)$ is biholomorphically equivalent to $(X_g, 0)$ if and only if $(X_{f+h}, 0)$ is biholomorphically equivalent to $(X_{g+h}, 0)$.

As a typical application of the above theorem, we have the following examples.

*Example 1.* Let $\mathcal{M}_{d,n}$ be the moduli space of nonsingular hypersurfaces of degree $d$ in $\mathbf{P}^n$. Then there is a canonical injection from the moduli space $\mathcal{M}_{d,2}$ of nonsingular curves of degree $d$ in $\mathbf{P}^2$ into $\mathcal{M}_{d,n}$. In particular, $\mathcal{M}_{4,2}$, which is a Zariski dense open subset of the moduli space $\mathcal{M}_3$ of complete curves of genus 3 is mapped injectively into $\mathcal{M}_{4,n}$ for $n \geq 3$.

*Example 2.* Let $V_t = \{(z_0, z_1, z_2, \cdots, z_n) \in \mathbf{C}^{n+1} : z_0^4 + tz_0^2 z_1^2 + z_1^4 + g(z_2, \cdots, z_n) = 0$ where $t^2 \neq 4$ and $g(z_2, \cdots, z_n)$ is a quasi-homogeneous holomorphic function with isolated singular point at $(z_2, \cdots, z_n) = (0, \cdots, 0)\}$. Then the complete continuous invariant of this one parameter family is given by $c(t) = (t^2 + 12)^3/(t^2 - 4)^2$ i.e. $V_{t_1}$ is not biholomorphically equivalent to $V_{t_2}$ if and only if $c(t_1) \neq c(t_2)$.

*Example 3.* Let $V_t = \{(z_0, z_1, z_2, \cdots, z_n) \in \mathbf{C}^{n+1} : z_0^3 + z_1^3 + z_2^3 + tz_0 z_1 z_2 + g(z_3, \cdots, z_n) = 0$ where $t^3 + 27 \neq 0$ and $g(z_3, \cdots, z_n)$ are quasi-homogeneous holomorphic functions with isolated singular points at $(z_3, z_4, \cdots, z_n) = (0, 0, \cdots, 0)\}$. Then the complete invariant of this one parameter family is given by $c(t) = [t(t^3 - 216)/(t^3 + 27)]^3$.

*Example 4.* Let $V_t = \{(z_0, z_1, z_2, \cdots, z_n) \in \mathbf{C}^{n+1} : z_0^6 + z_1^3 + tz_0^4 z_1 + g(z_2, \cdots, z_n) = 0$ where $4t^3 + 27 \neq 0$ and $g(z_2, \cdots, z_n)$ is a quasi-homogeneous holomorphic function with isolated points at $(z_2, \cdots, z_n) = (0, \cdots, 0)\}$. Then the complete invariant of this one parameter family is given by $c(t) = t^3$.

Theorem 4.8 is a consequence of Theorem of Mather-Yau and the cancellation theorem for Artinian algebra

Given a family of complex projective hypersurfaces in $\mathbf{CP}^n$, the Torelli problem studied by P.Griffiths and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbf{CP}^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbf{C}^{n+1}$. Let $V = \{z \in \mathbf{C}^{n+1} : f(z) = 0\}$ be a complex hypersurface with isolated singularity at the origin. The moduli algebra of $(V, 0)$ is $A(V) := \mathbf{C}\{z_0, z_1, \cdots, z_n\}/(f, \frac{\partial f}{\partial z_0}, \cdots, \frac{\partial f}{\partial z_n})$. It is a finite dimensional commutative local algebra. Mather and Yau proved that the complex structures of $(V, 0)$ determines and is determined by its moduli algebra. Subsequently we introduced the Lie algebra $L(V)$ to $(V, 0)$, which is the Lie algebra of derivations of $A(V)$. We proved that $L(V)$ is solvable. The natural question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their Lie algebras. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are really studying the complex structures of an isolated hypersurface singularity. In view of the theorem of Lê and Ramanujan, we require that the Milnor number $\mu$ is constant along this family. Recall that the dimension of the moduli algebra (denoted by $\tau$) is a complex analytic invariant. So it suffices to consider only a $(\mu, \tau)$-constant family of isolated complex hypersurface singularities.

### $(\mu, \tau)$Constant deformation for hypersurface singularities with $\mathbf{C}^*$-action

Let $(V, 0) \subseteq (\mathbf{C}^n, 0)$ be an isolated hypersurface singularity with the local defining equation $f(x_1, \cdots, x_n) = 0$. Then the semi-universal deformation of $(V, 0)$ is given by

$$\begin{array}{ccccc} (\mathcal{V}, 0) & \longrightarrow & (\mathbf{C}^n \times \mathbf{C}^k, 0) & & (x_1, \cdots, x_n, t_1, \cdots, t_k) \\ \downarrow & & \downarrow & & \downarrow \\ (S, 0) & = & (\mathbf{C}^k, 0) & & (t_1, \cdots, t_k) \end{array}$$

Here $\mathcal{V} = \{(x_1, \cdots, x_n, t_1, \cdots, t_k) : f(x) + \sum_{i=1}^k t_i g_i(x) = 0\}$, where $g_1(x), \cdots, g_k(x)$ are monomials in $x_1, \cdots, x_n$ which represent a linear basis of the complex vector space $A(V) = \mathbf{C}[\![x_1, \cdots, x_n]\!]/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. We are particularly interested in the case when $f$ is weighted-homogeneous, i.e. when there exist positive integers $q_1, \cdots, q_n$ and $d$ such that for any $t \in \mathbf{C}^* = \mathbf{C} - \{0\}$

$$f(t^{q_1} x_1, t^{q_2} x_2, \cdots, t^{q_n} x_n) = t^d f(x_1, \cdots, x_n). \tag{4.1}$$

In the sequel, we shall always assume that $f$ is weighted homogeneous.

Let us give the variable $x_i$ the weight $q_i$. Then each monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ which appears in $f$ has total weight $d = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n$.

**Theorem 4.9** (Seeley-Yau) Let $f$ be a weighted homogeneous polynomial with isolated singularity at the origin as above. Then

$$\{(x_1, \cdots, x_n, t_1, \cdots, t_m) : f(x) + t_1 g_1(x) + \cdots + t_m g_m(x) = 0\},$$

where $g_1, \cdots, g_m$ are monomials in a basis for the moduli algebra

$$A(V) = \mathbf{C}[\![x_1, \cdots, x_n]\!] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right),$$

is a $(\mu, \tau)$-constant deformation of $V = \{x : f(x) = 0\}$ if and only if weight $(g_i)$ = weight $(f)$ for all $1 \leq i \leq m$.

It is well known that the $(\mu,\tau)$-constant strata $S_E = \{t \in \mathbf{C}^k : (\mu(V_t), \tau(V_t)) = (\mu,\tau)\}$ forms a subvariety in the parameter space of the semi-universal deformation of $(V,0)$. In case $(V,0)$ has a $\mathbf{C}^*$-action, we shall show that $(S_E,0)$ is isomorphic to $(\mathbf{C}^m,0)$, where $m$ is the dimension of $A_d$ (elements in the moduli algebra $A(V)$ of weight $d$).

**Theorem 4.10** (Seeley-Yau) Let $f$ be a weighted homogeneous polynomial with isolated singularity at the origin as in (4.1). Let $g_1,\cdots,g_m$ be elements in a monomial basis of the moduli algebra $A(V) = \mathbf{C}[\![x_1,\cdots,x_n]\!]/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$ such that weight $(g_i)$ = weight $(f)$ for all $1 \le i \le m$. Then the $(\mu,\tau)$-constant strata $S_E$ is $\mathbf{C}^m$ and the equitopological deformation $(\mathcal{W},0) \to (S_E,0)$ of $(V,0)$ is given by

$$\begin{array}{ccccc} (\mathcal{W},0) & \longrightarrow & (\mathcal{V},0) & = & (\mathbf{C}^n \times \mathbf{C}^k, 0) \\ \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle Pr_2} \\ (S_E,0) & \longrightarrow & (S,0) & = & (\mathbf{C}^k,0) \end{array}$$

where $\mathcal{W} = \{(x_1,\cdots,x_n,t_1,\cdots,t_m) : f(x)+t_1g_1(x)+\cdots+t_mg_m(x) = 0\}$, ($Pr_2$ = projection onto the second factor) and $(S_E,0) = (\mathbf{C}^m,0)$.

**Construction of a family of solvable Lie algebras over the $(\mu,\tau)$-constant strata $\mathbf{S_E}$**

Let $(V,0)$ be a hypersurface singularity defined by a weighted homogeneous polynomial $f(x_1,\cdots,x_n)$. We have shown that $(S_E,0) = (\mathbf{C}^m,0)$ and the equitopological deformation is given by

$$\begin{array}{ccc} \{(x_1,\cdots,x_n,t_1,\cdots,t_m) : f(x)+t_1g_1(x)+\cdots+t_mg_m(x) = 0\} & \hookrightarrow & \mathbf{C}^n \times \mathbf{C}^m \\ \big\downarrow & & \big\downarrow {\scriptstyle pr_2} \\ S_E & = & \mathbf{C}^m \end{array}$$

where $g_i$ are those monomials in a monomial basis of $\mathbf{C}\{x_1,\cdots,x_n\}/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$ such that $wt(g_i) = wt(f)$. Now we shall construct a family of solvable Lie algebras over $S_E$. Recall that we have associated to an isolated singularity $(V,0)$ a finite dimensional Lie algebra $L(V)$, which is defined to be the algebra of derivations of the moduli algebra $A(V)$. $L(V)$ is solvable. We shall define a Lie subalgebra $\tilde{L}(V)$ of $L(V)$. This Lie subalgebra $\tilde{L}(V)$ admits a natural deformation over the parameter space $S_E$. Recall that by Theorem 4.10, $S_E$ is isomorphic to $\mathbf{C}^m$ with coordinates $t_1,\cdots,t_m$.

**Definition.** A derivation $D_0 \in L(V)$ is liftable to $S_E$ if there exist differential operators $D_{\tau_i}$ such that

$$D = D_0 + \sum_{|\tau_1|=1} t^{\tau_1} D_{\tau_1} + \sum_{|\tau_2|=2} t^{\tau_2} D_{\tau_2} + \cdots$$

leaves the ideal $(f_{x_1}+t_1g_{1x_1}+\cdots+t_mg_{mx_1}, f_{x_2}+t_1g_{1x_2}+\cdots+t_mg_{mx_2},\cdots,f_{x_n}+t_1g_{1x_n}+\cdots+t_mg_{mx_n})$ in $\mathbf{C}\{x_1,\cdots,x_n,t_1,\cdots,t_m\}$ invariant. (By differential operator, we mean operator of the form $d_1(x)\frac{\partial}{\partial x_1}+\cdots+d_n(x)\frac{\partial}{\partial x_n}$ with $d_j(x)$ a linear combination of monomial basis elements of the moduli algebra $A(V)$.)

Here we use the standard notation for multi-indices. For example if $\alpha = (\alpha_1,\cdots,\alpha_n)$, then $|\alpha| = \alpha_1+\cdots+\alpha_n$ and

$$t^\alpha D_\alpha = t_1^{\alpha_1}\cdots t_n^{\alpha_n}\left(d_1^\alpha(x)\frac{\partial}{\partial x_1}+\cdots+d_n^\alpha(x)\frac{\partial}{\partial x_n}\right).$$

**Definition** The Liftable Lie algebra $\tilde{L}(V)$ is defined to be the set of those $D_0 \in L(V)$ such that $D_0$ is liftable to $S_E$.

Clealy $\tilde{L}(V)$ is a Lie subalgebra of $L(V)$ and has a natural deformation over the parameter space $S_E$. Restricting ourselves to the three variable case with $m = 1$ (i.e. to 1-parameter family of deformations), $D_0$ is liftable to $S_E$ if there exists $a_0^i, b_0^i, c_0^i$ such that (4.2) are satisfied and there exist $D_1, a_1^i, b_1^i, c_1^i$ such that (4.3) are satisfied and there exist $D_2, a_2^i, b_2^i, c_2^i$ such that (4.4) are satisfied etc.

$$\begin{cases} D_0(f_x) = a_0^1 f_x + a_0^2 f_y + a_0^3 f_z \\ D_0(f_y) = b_0^1 f_x + b_0^2 f_y + b_0^3 f_z \\ D_0(f_y) = c_0^1 f_x + c_0^2 f_y + b_c^3 f_z \end{cases} \tag{4.2}$$

$$\begin{cases} D_1(f_x) - (a_1^1 f_x + a_1^2 f_y + a_1^3 f_z) = -D_0 g_x + (a_0^1 g_x + a_0^2 g_y + a_0^3 g_z) \\ D_1(f_y) - (b_1^1 f_x + b_1^2 f_y + b_1^3 f_z) = -D_0 g_y + (b_0^1 g_x + b_0^2 g_y + b_0^3 g_z) \\ D_1(f_z) - (c_1^1 f_x + c_1^2 f_y + c_1^3 f_z) = -D_0 g_z + (c_0^1 g_x + c_0^2 g_y + c_0^3 g_z) \end{cases} \tag{4.3}$$

$$\begin{cases} D_2(f_x) - (a_2^1 f_x + a_2^2 f_y + a_2^3 f_z) = -D_1 g_x + (a_1^1 g_x + a_1^2 g_y + a_1^3 g_z) \\ D_2(f_y) - (b_2^1 f_x + b_2^2 f_y + b_2^3 f_z) = -D_1 g_y + (b_1^1 g_x + b_1^2 g_y + b_1^3 g_z) \\ D_2(f_z) - (c_2^1 f_x + c_2^2 f_y + c_2^3 f_z) = -D_1 g_z + (c_1^1 g_x + c_1^2 g_y + c_1^3 g_z) \end{cases} \tag{4.4}$$

*Example* Let $V = \{(x, y, z) : f(x, y, z) = x^3 + y^3 + z^3 = 0\}$. Then the moduli algebra is given by the vector space spanned by $1, x, y, z, xy, yz, zx$ and $xyz$.

The weight of $g(x, y, z) = xyz$ is three, which is exactly the weight of $f$. In view of Theorem 4.10, the equitopological deformation of $V$ is given by

$$V_t = \{(x, y, z) : x^3 + y^3 + z^3 + txyz = 0\} \qquad t^3 + 27 \neq 0.$$

It is easy to see that the Lie algebra $L(V_0)$ associated to $V = V_0$ is given by

$$L(V_0) = \left\langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x}, zx\frac{\partial}{\partial x}, xy\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial z}, zx\frac{\partial}{\partial z}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial z} \right\rangle.$$

We claim that $x\frac{\partial}{\partial x}$ is not liftable. To see this, we observe that

$$x\frac{\partial}{\partial x}(f_x) = 2f_x + 0f_y + 0f_z \qquad \Rightarrow \qquad a_0^1 = 2, a_0^2 = 0, a_0^3 = 0$$

Suppose that there exist $a_1^1, a_1^2, a_1^3$ and

$$D_1 = (\alpha_1 x + \alpha_2 y + \alpha_3 z)\frac{\partial}{\partial x} + (\beta_1 x + \beta_2 y + \beta_3 z)\frac{\partial}{\partial y} + (\gamma_1 x + \gamma_2 y + \gamma_3 z)\frac{\partial}{\partial z}$$

such that (4.3) is satisfied. Then

$$\begin{aligned} a_1^1 f_x + a_1^2 f_y + a_1^3 f_z &= D_1 f_x + D_0 g_x - (a_0^1 g_x + a_0^2 g_y + a_0^3 g_z) \\ &= 6\alpha_1 x^2 + 6\alpha_2 xy + 6\alpha_3 xz - 2yz. \end{aligned}$$

Because of the appearance of $-2yz$ on the right hand side, there is no choice of $a_1^1, a_1^2, a_1^3, \alpha_1, \alpha_2$ and $\alpha_3$ which makes above equations true.

On the other hand, we claim that $D_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is liftable. In fact, $D_0$ itself preserves the ideal generated by $3x^2 + tyz, 3y^2 + txz$, and $3z^2 + txy$. We can see that

$$\tilde{L}(V_0) = \left\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x}, zx\frac{\partial}{\partial x}, xy\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial z}, zx\frac{\partial}{\partial z}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial z} \right\rangle.$$

Indeed, the equitopological deformation $\{V_t\}$ gives the following deformation of $\tilde{L}(V_0)$.

$$\tilde{L}(V_t) = \Big\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} - \frac{t}{6}zx\frac{\partial}{\partial y}, zx\frac{\partial}{\partial x} - \frac{t}{6}xy\frac{\partial}{\partial z}, xy\frac{\partial}{\partial y} - \frac{t}{6}yz\frac{\partial}{\partial x}, yz\frac{\partial}{\partial y} - \frac{t}{6}xy\frac{\partial}{\partial z},$$
$$yz\frac{\partial}{\partial z} - \frac{t}{6}zx\frac{\partial}{\partial y}, zx\frac{\partial}{\partial z} - \frac{t}{6}yz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}xyz\frac{\partial}{\partial z}\Big\rangle.$$

This deformation is actually a trivial family (as a family of Lie algebras).

**Torelli type problems**

We have constructed a family of Lie algebras $\tilde{L}(V_t)$ over $S_E$. It is natural to study the following Torelli type problem: If $\tilde{L}(V_{t_1}) \simeq \tilde{L}(V_{t_2})$ as Lie algebras, $t_1, t_2$ in $S_E$, is $V_{t_1}$ biholomorphically equivalent to $V_{t_2}$. In what follows, we shall study this problem for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$.

Let $\tilde{E}_7$ be a simple elliptic singularity defined by $\{(x, y, z) \in \mathbf{C} : x^4 + y^4 + z^2 = 0\}$. It is clear from Theorem 4.10 that the $(\mu, \tau)$-constant family is given by

$$V_t = \{(x, y, z) : f(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\} \qquad t^2 \neq 4 \tag{4.5}$$

Hence $S_E = \mathbf{C} - \{\pm 2\}$.

**Theorem 4.11** (Seeley-Yau) A Torelli type theorem holds for simple elliptic singularities $\tilde{E}_7$. i.e., $\tilde{L}(V_{t_1}) \cong \tilde{L}(V_{t_2})$ as Lie algebras for $t_1 \neq t_2$ in $S_E$ if and only if $V_{t_1}$ is biholomorphically equivalent to $V_{t_2}$

Let $\tilde{E}_8$ be a simple elliptic singularity defined by $\{(x, y, z) \in \mathbf{C}^3 : x^6 + y^3 + z^2 = 0\}$. It is clear from Theorem 4.10 that the $(\mu, \tau)$ constant family is given by

$$V_1 = \{(x, y, z) \in \mathbf{C}^3 : f_t(x, y, z) = x^6 + y^3 + z^2 + tx^4y = 0\}$$

with $4t^3 + 27 \neq 0$. Hence $S_E = \mathbf{C} - \{t \in \mathbf{C} : 4t^3 + 27 = 0\}$.

**Theorem 4.12** (Seeley-Yau) A Torelli type theorem holds for simple elliptic singularities $\tilde{E}_8$. i.e., $\tilde{L}(V_{t_1}) \cong \tilde{L}(V_{t_2})$ as Lie algebra for $t_1 \neq t_2$ in $S_E$ if and only if $V_{t_1}$ is biholomorphically equivalent to $V_{t_2}$.

*Proof.* By the theorem of Mather-Yau there is a one to one correspondence between the complex structure of the singularity $V_t$ and its moduli algebra

$$\begin{aligned} A_t &= \mathbf{C}\{x, y, z\} \Big/ \left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right) \\ &= < 1, x, x^2, y, x^3, xy, x^4, x^2y, x^3y, x^4y > \end{aligned}$$

with multiplication rules

$$\begin{aligned} y^2 &= -\frac{t}{3}x^4 \\ x^5 &= -\frac{2t}{3}x^3y \end{aligned}$$

$A_t$ is a graded algebra with $\deg x = 1$ and $\deg y = 2$. Observe that $A_t = \mathbf{C}\{x, y\}/\mathbf{I}_t$ where $\mathbf{I}_t = (\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}) = (3x^5 + 2tx^3y, 3y^2 + tx^4)$. Any element $D \in \mathrm{Der}_{\mathbf{C}}(A_t)$ can be written as

$$\begin{aligned} D =& (a_0 + a_1x + a_2x^2 + a_2^1y + a_3x^3 + a_3^1xy + a_4x^4 + a_4^1x^2y + a_5^1x^3y + a_6^1x^4y)\frac{\partial}{\partial x} \\ &+ (b_0 + b_1x + b_2x^2 + b_2^1y + b_3x^3 + b_3^1xy + b_4x^4 + b_4^1x^2y + b_5^1x^3y + b_6^1x^4y)\frac{\partial}{\partial y}. \end{aligned}$$

The subscripts refer to degrees of monomials, with $a_i^1, b_i^1$ the coefficients of a monomial containing $y$.

A basis for $L_t$ is, for $t \neq 0$, the following:

| | |
|---|---|
| deg0 | $e_0 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ |
| deg1 | $e_1 = x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \quad e_2 = 2ty\frac{\partial}{\partial x} + (2t^2x^3 - 15xy)\frac{\partial}{\partial y}$ |
| deg2 | $e_3 = (2t^2x^4 - 9x^2y)\frac{\partial}{\partial y}, \quad e_4 = 9x^3\frac{\partial}{\partial x} + 4t^2x^4\frac{\partial}{\partial y}, \quad e_5 = -3xy\frac{\partial}{\partial x} + 2tx^4\frac{\partial}{\partial y}$ |
| deg3 | $e_6 = x^4\frac{\partial}{\partial x}, \quad e_7 = x^2y\frac{\partial}{\partial z}, \quad e_8 = x^3y\frac{\partial}{\partial y}$ |
| deg4 | $e_9 = x^3y\frac{\partial}{\partial x}, \quad e_{10} = x^4y\frac{\partial}{\partial y}$ |
| deg5 | $e_{11} = x^4y\frac{\partial}{\partial x}.$ |

For $t = 0$, $\{e_0\}$ is replaced by $\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$. The Lie algebra $\tilde{L}_t$ defined in the previous sectioThe Lie algebra $\tilde{L}_t$ defined in the previous section is spanned by $< e_0, e_1, \cdots, e_{11} >$. The nilradical $N_t$ of $\tilde{L}_t$ is of dimension 11, spanned by $< e_1, \cdots, e_{11} >$. We shall show that the mapping $\{V_t\} \to \{\tilde{L}_t\}$ gives a one-to-one correspondence between the complex structures of $V_t$ and the isomorphism classes of the solvable Lie algebras $\tilde{L}_t$. Again, we will do this by studying the nilradical $N_t$.

$[e_1, e_2] = \frac{5}{3}e_3$

$[e_1, e_3] = -\frac{(8t+54)}{3}e_8$

$[e_1, e_4] = 9e_6 - \frac{54+16t^3}{3}e_8$

$[e_1, e_5] = -3e_7 - \frac{4t^2}{3}e_8$

$[e_1, e_6] = -\frac{4t}{3}e_9 - 2e_{10}$

$[e_1, e_7] = 2e_9 - \frac{4t^2}{9}e_{10}$

$[e_1, e_8] = 3e_{10}$

$[e_1, e_9] = 3e_{11}$

$[e_3, e_4] = (24t^3 + 162)e_{10}$

$[e_3, e_5] = (4t^3 + 27)e_9$

$[e_3, e_7] = -\frac{4t^3+27}{3}e_{11}$

$[e_2, e_3] = -4t^3e_6 + 18te_7$

$[e_2, e_4] = -8t^2e_6 + 54te_7 + (135 + 28t^3)e_8$

$[e_2, e_5] = -8t^2e_6 + 45e_7 + 4t^2e_8$

$[e_2, e_6] = 8te_9 + (15 + 4t^3)e_{10}$

$[e_2, e_7] = -\frac{135 + 4t^3}{9}e_9 - \frac{8t^2}{3}e_{10}$

$[e_2, e_8] = -2te_9$

$[e_2, e_9] = -15e_{11}$

$[e_2, e_{10}] = -2te_{11}$

$[e_4, e_5] = (8t^3 + 54)e_9$

$[e_4, e_6] = -6te_{11}$

$[e_4, e_7] = -\frac{27 + 8t^3}{3}e_{11}$

$[e_5, e_6] = -9e_{11}$

$[e_5, e_7] = -2t^2e_{11}$

$[e_5, e_8] = 3e_{11}$

Other brackets $[e_i, e_j], i < j$ are zero. There are some invariant subspaces which show explicitly the structure of $N_t$. Let

$$Z := center(N_t) =< e_{11} >$$
$$Z^2 := \{x \in N_t : Image(ad_x) \subseteq Z\} =< e_9, e_{10}, e_{11} >$$
$$Z^3 := \{x \in N_t : Image(ad_x) \subseteq Z^2\} =< e_6, e_7, e_8, e_9, e_{10}, e_{11} >$$
$$Z^4 := \{x \in N_t : Image(ad_x) \subseteq Z^3\} =< e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} >$$
$$Z^5 := \{x \in N_t : Image(ad_x) \subseteq Z^4\} = N_t$$
$$N^{(1)} = [N, N] =< e_3, e_6, e_7, e_8, e_9, e_{10}, e_{11} >$$
$$N^{(2)} = [N, N^{(1)}] = \left\langle e_7 - \frac{2t^2}{9}e_6, e_8, e_9, e_{10}, e_{11} \right\rangle$$
$$N^{(3)} = [N, N^{(2)}] =< e_9, e_{10}, e_{11} >$$
$$N^{(4)} = [N, N^{(3)}] =< e_{11} > .$$

The quotient space $Z^2/Z = N^{(3)}/N^{(4)}$ is two dimensional, spanned by the images of $e_9$ and $e_{10}$. There are four invariant lines in this space (i.e. each is preserved under all automprphism of $N$). Their ordered cross-ratio is a complex number which is also invariant under all automorphisms, and will therefore distinguish $N_t$ from $N_s$ unless $N_t \cong N_s$. Let

$$
\begin{aligned}
&l_1 = Z^4/Z^3 \cap N^{(1)}/Z^3 = \mathbf{C}\overline{e}_3 \subseteq Z^4/Z^3 \\
&P_2 = ker(ad_{l_1}) = \mathbf{C}\overline{e}_6 \oplus \mathbf{C}\overline{e}_8 \subseteq Z^3/Z^2 \qquad where\ ad_{l_1} : Z^3/Z^2 \to Z \\
&P_3 = Image(ad_{l_1}) = \mathbf{C}\overline{e}_8 \oplus \mathbf{C}\overline{(9e_7 - 2t^2 e_6)} \subseteq Z^3/Z^2 \qquad where\ ad_{l_1} : Z^5/Z^4 \to Z^3/Z^2 \\
&l_4 = P_2 \cap P_3 = \mathbf{C}\overline{e}_8 \subseteq Z^3/Z^2 \\
&l_5 = \{\overline{x} \in Z^5/Z^4 : ad_{l_1}(\overline{x}) \subseteq l_4\} = \mathbf{C}\overline{e}_1 \subseteq Z^5/Z^4 \qquad where\ ad_{l_1} : Z^5/Z^4 \to Z^3/Z^2 \\
&l_6 = [l_4, l_5] = \mathbf{C}\overline{e}_{10} \subseteq Z^2/Z \\
&l_7 = ker(ad_{l_5}) = \mathbf{C}\overline{\left(e_6 + \frac{2t}{3}e_7 + \frac{8t^3+54}{81}e_8\right)} \subseteq Z^3/Z^2 \qquad where\ ad_{l_5} : Z^3/Z^2 \to Z^2/Z \\
&l_8 = \{\overline{x} \in Z^4/Z^3 : ad_{l_5}(\overline{x}) \subseteq l_7\} = \mathbf{C}\overline{(4e_3 - 3e_4 + 6te_5)} \qquad where\ ad_{l_5} : Z^4/Z^3 \to Z^3/Z^2 \\
&l_9 = [l_1, l_8] = \mathbf{C}\overline{(te_9 - 3e_{10})} \subseteq Z^2/Z \\
&l_{10} = ker(ad_{l_9}) = \mathbf{C}\overline{(3e_1 + e_2)} \subseteq Z^5/Z^4 \qquad where\ ad_{l_9} : Z^5/Z^4 \to Z \\
&l_{11} = [l_{10}, l_4] = \mathbf{C}\overline{(-2te_9 + 9e_{10})} \subseteq Z^2/Z \\
&l_{12} = [l_1, l_{10}] = \mathbf{C}\overline{(2t^3 e_6 - 9te_7 + (4t^3+27)e_8)} \subseteq Z^3/Z^2 \\
&l_{13} = [l_{10}, l_{12}] = \mathbf{C}\overline{(te_9 + (2t^3+9)e_{10})} \subseteq Z^2/Z.
\end{aligned}
$$

It is clear that any Lie algebra isomorphism from $N_t$ to $N_s$ induces an isomorphism from $Z_t^2/Z_t$ to $Z_s^2/Z_s$ which sends the ordered set $\{l_6(t), l_9(t), l_{11}(t), l_{13}(t)\}$ to the ordered set $\{l_6(s), l_9(s), l_{11}(s), l_{13}(s)\}$. The cross ratios of these two ordered sets are $\frac{2}{3}(2t^3+12)$ and $\frac{2}{3}(2s^3+12)$. Consequently $N_t \cong N_s$ implies that $s^3 = t^3$, Conversely if $s^3 = t^3$, then $s = \rho t$ for some $\rho$ with $\rho^3 = 1$ and $V_t$ is biholomorphically equivalent to $V_s$. The biholomorphism is given by $f_t(x, \rho y, z) = f_s(x, y, z)$. In particular $N_s$ is isomorphic to $N_t$ as a Lie algebra if $s^3 = t^3$. Thus $t^3$ can be considered as the modulus of the analytic type of the $\tilde{E}_8$ singularities.

**Q.E.D.**

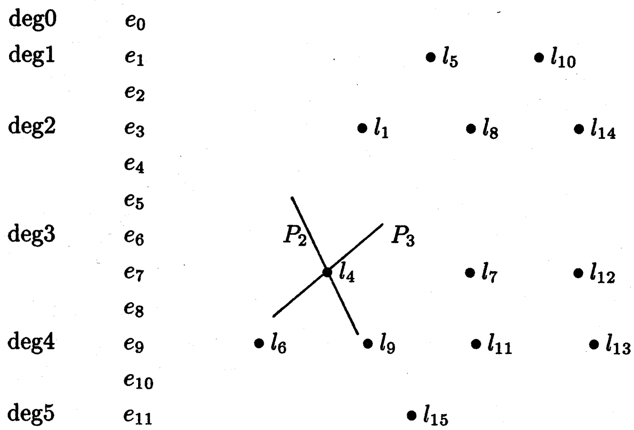
Again this agrees with the work of Saito[Sa2]. He found that $j(t) = \frac{4t^3}{4t^3+27}$, which is a one-to-one function of our modulus $t^3$.

**Remark.** A complete basis for $N_t$ of $\tilde{E}_8$ can be obtained (for $t^3 \neq 0, -\frac{27}{4}$) by defining two more lines $l_{14}$, and $l_{15}$ below, and choosing representative vectors for $l_5, l_{10}, l_1, l_8, l_{14}, l_4, l_7, l_{12}, l_6, l_9, l_{15}$. These vectors will be unique, up to scalar multiples, modulo higher centers $Z^i$.

$$
\begin{aligned}
l_{14} &= \{\overline{x} \in Z^4/Z^3 : ad_{l_5}(\overline{x}) \subseteq l_{12}\} \\
&= \mathbf{C}\overline{\left(\left(\frac{32t^6}{9} + 36t^3 + 81\right)e_3 - \frac{16t^6+108t^3}{9}e_4 - (24t^4 + 162t)e_5\right)} \\
&\subseteq Z^4/Z^3 \qquad where\ ad_{l_5} : Z^4/Z^3 \to Z^3/Z^2 \\
l_{15} &= \mathbf{C}e_{11} = Z.
\end{aligned}
$$

Two sets of Lie algebra generators are easily seen to be $\{e_1, e_2, e_4, e_5\}$ and $\{e_1, 3e_1+e_2, 4e_3-3e_4+6te_5, (\frac{32}{9}t^6 + 36t^3+81)e_3 - (\frac{16}{9}t^6 + 12t^3)e^4 - (24t^4+162t)e_5\}$. The second set represents $\{l_5, l_{10}, l_8, l_{14}\}$.

Notice that $\tilde{L}_t$ is a graded Lie algebra. In fact each $e_i$ is of pure degree acting on $A_t$. For example $e_{11} = x^4 y \frac{\partial}{\partial x}$ raises degree by 5=5deg$x$+deg$y$−deg$x$.




A • represents a complex line and a segment —— represents a complex plane. Notice that, for instance, $Z^2$ is the span of the degree 4 and degree 5 derivations, although the degree 4 subspace is not invariant under all automorphisms of $N_t$.

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