수 학 강 의 록 제 49 권

NOTES ON BOUNDARY VALUES IN ULTRADISTRIBUTION SPACES

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Notes of the Series of Lectures
held at the Seoul National University

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펴낸날 : 1999년 12월 30일
지은이 : R. D. Carmichael, Andrzej Kamiński and Stevan Pilipović
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터 [TEL : 82-2-880-6562]

Notes on Boundary Values in Ultradistribution Spaces

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Contents

Introduction

These lecture notes are an extended version of the material presented by the third author in four lectures given in the fall of 1998 during his short visit to Seul National University.

There exists a vast literature concerning ultradistribution spaces (see [52] [3], [75], [48]-[50], [51] and references therein), but there is no monograph concerning both the general theory and applications within ultradistribution spaces. Important results in the frame of ultradistribution spaces which will not be included in these notes were obained by D. Kim, S. Y. Chung and their collaborators ([27]-[30], [47]), Matsuzawa ([58]-[60]), Vogt, Meise, Taylor, Petzsche and their collaborators ([61], [62], [63], [64]), the Italian school with Rodino, Gramchev ([73], [39]) and many others. A list of papers with results on various problems within ultradistribution spaces is given in the references which is, however, far from being complete.

These notes are concerned with the analysis of boundary values of holomorphic functions having appropriate growth estimates and with the Cauchy and Poisson integrals in the weighted ultradistribution spaces $\mathcal{D}'(*, L^s)$.

The problems of characterizing holomorphic functions whose boundary values are elements of the spaces of distributions, ultradistributions, infra-hyperfunctions and, vice versa, of finding boundary value representations of elements of the quoted spaces of generalized functions by holomorphic functions have a long history; for references see e.g. [55], [80], [73], [83], [80], [2], [82], [19] and references therein.

Carmichael and his co-workers ([8]-[19], [20]) have studied the Cauchy and Poisson kernels in appropriate tube domains. By considering the Cauchy and Poisson integrals of distributions in appropriate subspaces of the Schwartz space $\mathcal{D}'$, they obtained characterizations of these subspaces by the a priori estimates of the corresponding analytic or harmonic functions in tube domains.

The complete boundary value characterizations for the spaces $\mathcal{D}'((M_p), \Omega)$, $\mathcal{D}'(\{M_p\}), \Omega)$ of ultradistributions and the spaces $\mathcal{E}'((M_p), \Omega)$, $\mathcal{E}'(\{M_p\}, \Omega)$ of infra-hyperfunctions, related to a non-quasianalytic and quasianalytic sequence (M_p), respectively, are given in [75], [48], [64], [74].

The spaces $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'((\{M_p\}), L^s)$ for $s \geq 1$ related to a non-quasianalytic sequence (M_p) are studied in papers by Carmichael and Pilipović. In these notes, we investigate classes of analytic functions having boundary values in these spaces. For the analysis of Hardy type spaces of holomorphic functions, with bounds given by appropriate associated functions corresponding to the sequences (M_p), we apply the Cauchy and Poisson integrals as well as the Fourier transforms. The geometry of tube domains also is considered in the notes.

A complete boundary value characterization for the spaces $\mathcal{D}'(*, L^s)$, on $\mathbf{R}^n$ with $s \in (1, \infty)$, is given by means of almost analytic extensions, while in cases $s = \infty$ and $s = 1$ only partial results are obtained.

The paper is organized as follows.

In Chapter 1 we define some notions connected with cones in $\mathbf{R}^n$ as well as the Cauchy and Poisson kernels corresponding to tube domains. We present there results which will be used later in proving boundary value representations.

Chapter 2 contains the definitions and main properties of the spaces of ultradifferentiable test functions of Beurling and Roumieu type as well as of the corresponding spaces of ultradistributions. We are mainly interested in the spaces $\mathcal{D}(*, L^s)$ and $\mathcal{S}^*$ and their strong duals. After presenting basic properties of the sequences (M_p) and ultradifferential operators generating the respective ultradistribution spaces, we prove structural theorems for these spaces. We also give the definitions of the Fourier and Laplace transforms.

Chapter 3 is devoted to characterizations of bounded sets in the spaces $\mathcal{D}'(*, L^p)$ of L^p ultradistributions and $\mathcal{S}^*$ of tempered ultradistributions.

In Chapter 4, the Cauchy and Poisson kernels are studied as elements of ultradifferentiable spaces $\mathcal{D}(*, L^r)$ (in Section 4.1) and then the Cauchy and Poisson integrals are treated as elements of ultradistribution spaces $\mathcal{D}'(*, L^s)$ (in Sections 4.2 and 4.3). For $s \geq 2$ the use of Cauchy integrals gives a complete boundary value characterization of elements in $\mathcal{D}'(*, L^s)$; notice that the Poisson integral of an element of the space $\mathcal{D}'(*, L^s)$, $s > 1$, converges to this element in the corresponding general ultradistribution space.

In Chapter 5, we deal with the boundary values of analytic functions in appropriate tube domains. Section 5.1 concerns the Fourier transform and suitable generalizations of Hardy spaces within ultradistribution classes for $r \in (1, 2]$. In Section 5.2, we show that elements of such spaces have boundary values in $\mathcal{D}'((M_p), L^1)$, while appropriate L^s bounds for $s \geq 2$ lead to boundary values in $\mathcal{D}'((M_p), L^r)$ for $r \in (1, 2]$. The extension of the results of Section 5.2 to the case $r > 2$ is given in Section 5.3 for appropriate cones. By means of almost analytic extensions and Stokes' theorem, we give in Section 5.4 the complete boundary value characterization for the spaces $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ with $s > 1$. The results given in Section 5.4 for ultradistributions on the real line, are true also in the multidimensional case. In Section 5.5, the cases $s = \infty$ and $s = 1$ are considered. Due to the method of Komatsu (see [48]) appropriate L^∞ and L^1 estimates are obtained for the corresponding boundary values in the respective ultradistribution spaces.

Chapter 1

Cones in $\mathbf{R}^n$ and kernels

1.1 Notation

We present the n-dimensional notation which will be used throughout.

For the origin in $\mathbf{R}^n$, the n-dimensional Euclidean space, we use the standard symbol 0 and it follows easily from the context if 0 denotes the number or the vector. Thus $0 = (0, \ldots, 0) \in \mathbf{R}^n$. The operations on vectors in $\mathbf{R}^n$ (in particular, in $\mathbf{N}^n$ and $\mathbf{N}_0^n$) and inequalities between them are meant coordinatewise which, in particular, simplifies summation symbols involving indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbf{N}_0^n$:

$$\sum_{0 \le \beta \le \alpha} a_\beta = \sum_{\beta_1 = 1}^{\alpha_1} \cdots \sum_{\beta_n = 1}^{\alpha_n} a_{\beta_1, \ldots, \beta_n}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of arbitrary reals (in particular, arbitrary integers). If $t = (t_1, t_2, \ldots, t_n) \in \mathbf{R}^n$, we define $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$, in particular, $t^\alpha = t^{\alpha_1 + \ldots + \alpha_n}$ for $t \in \mathbf{R}$, whenever the symbols t^{α_j} make sense. The symbol z^α for $z \in \mathbf{C}^n$ is defined analogously. For $\alpha, \beta \in \mathbf{N}_0^n$ with $\alpha \le \beta$ we define $\bar{\alpha} = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$ and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_1}{\beta_1}.$$

Given two vectors $t = (t_1, t_2, \ldots, t_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbf{R}^n$ we use the symbol $\langle t, y \rangle$ for their scalar product, i.e.,

$$\langle t, y \rangle = t_1 y_1 + t_2 y_2 + \ldots + t_n y_n.$$

The scalar product $\langle t, z \rangle$ for $t \in \mathbf{C}^n$ (in particular, for $t \in \mathbf{R}^n$) is defined similarly.

Let α denote an n-tuple of nonnegative integers, i.e. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}_0^n$. The symbol $D^\alpha = D_t^\alpha$ with $t = (t_1, t_2, \ldots, t_n) \in \mathbf{R}^n$ denotes the differential operator given by

$$D_t^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} \quad \text{with} \quad D_j = -\frac{1}{2\pi i} \frac{\partial}{\partial t_j} \quad \text{for} \quad j = 1, \ldots, n. \tag{1.1}$$

On the other hand, the symbol $\frac{\partial^\alpha}{\partial t^\alpha}$ with $t \in \mathbf{R}^n$ denotes the partial differential operator defined analogously as in (1.1), but with the constant 1 instead of $-(2\pi i)^{-1}$. We also write $\varphi^{(\alpha)}(t)$ instead of $\frac{\partial^\alpha \varphi(t)}{\partial t^\alpha}$ for functions φ on $\mathbf{R}^n$. A similar convention is applied to the symbols D_z^α, $\frac{\partial^\alpha}{\partial z^\alpha}$ and $\varphi^{(\alpha)}(z)$ for $z \in \mathbf{C}^n$ and functions φ on $\mathbf{C}^n$.

For $z \in \mathbf{C}^n$, we denote

$$|z| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2},$$

the Euclidean norm of z in $\mathbf{C}^n$.

It will be convenient to apply constantly the following notation χ^α, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0^n$, for the function given by

$$\chi^\alpha(x) = x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}, \quad x = (x_1, \ldots, \_n) \in \mathbf{R}^n.$$

Moreover, we put

$$\langle\chi\rangle(x) = \langle x\rangle = (1 + |x|^2)^{1/2}.$$

We shall also apply the following notation for exponents with two variables z, $\zeta \in \mathbf{C}^n$:

$$E_z(\zeta) = \exp(2\pi i\langle z, \zeta\rangle), \quad z, \zeta \in \mathbf{C}^n \quad (z \in \mathbf{C}^n, \ t \in \mathbf{R}^n).$$

We also denote

$$e_y(t) = \exp(-2\pi\langle y, t\rangle), \quad y \in \mathbf{R}^n, \ t \in \mathbf{R}^n,$$

i.e. we have

$$E_{iy} = e_y \quad \text{for} \quad y \in \mathbf{R}^n.$$

In particular, $e_y(t) = \exp(-2\pi yt)$ for $y, t \in \mathbf{R}$.

The Fourier transform of an L^1-function φ, denoted by $\mathcal{F}[\varphi]$ or by $\hat{\varphi}$, is defined by

$$\mathcal{F}[\varphi](x) = \hat{\varphi}(x) = \int_{\mathbf{R}^n} \varphi(t) e^{2\pi i\langle x,t\rangle}\, dt = \int_{\mathbf{R}^n} \varphi(t) E_(t)\, dt$$

and the inverse Fourier transform of an L^1-function φ, denoted by $\mathcal{F}^{-1}[\varphi]$ or by $\check{\varphi}$ is defined by

$$\mathcal{F}^{-1}[\varphi](x) = \check{\varphi}(x) = \int_{\mathbf{R}^n} \varphi(t) e^{-2\pi i\langle x,t\rangle}\, dt = \int_{\mathbf{R}^n} \varphi(t) E_{-x}(t)\, dt.$$

We assume familiarity on the part of the reader with the Fourier transform on L^r, $1 \le r \le 2$, the corresponding inverse Fourier transform, and the associated Plancherel theory for the Fourier transform.

The symbol supp g will mean the support of a given function or ultradistribution g.

1.2 Cones in $\mathbf{R}^n$

We introduce the definitions and notation associated with cones in $\mathbf{R}^n$ and tubes in $\mathbf{C}^n$ (cf. [83], [84]).

A set $C \subseteq \mathbf{R}^n$ is a **cone** (with the vertex at zero) if $y \in C$ implies $\lambda y \in C$ for all positive reals λ. The intersection of the cone C with the unit sphere $\{y \in \mathbf{R}^n \colon\ |y| = 1\}$ is called the **projection** of C and is denoted by $pr(C)$. If C_1 and C_2 are cones such that $pr(\overline{C_1}) \subset pr(C_2)$, the cone C_1 will be called a **compact subcone** of C_2 and we will write then $C_1 \subset\subset C_2$. An open convex cone C such that $\overline{C}$ does not contain any entire straight line will be called a **regular cone**. The set

$$C^* = \{t \in \mathbf{R}^n \colon\ \langle t, y\rangle \geq 0 \text{ for all } y \in C\}$$

is the **dual cone** of the cone C. A cone is called **self dual** if $C^* = \overline{C}$. For any cone C, the dual cone C^* is closed and convex. We have $C^* = \overline{C}^* = (O(C))^*$ and $C^{**} = \overline{O(C)}$, where $O(C)$ denotes the convex hull of C.

The function

$$u_C(t) = \sup_{y \in pr(C)} (-\langle t, y\rangle)$$

is said to be the **indicatrix** of the cone C.

We have $C^* = \{t \in \mathbf{R}^n \colon\ u_C(t) \leq 0\}$. Moreover, $u_C(t) \leq u_{O(C)}(t)$ for all $t \in \mathbf{R}^n$ and $u_C(t) = u_{O(C)}(t)$ for $t \in C^*$.

Given a cone C, put $C_* = \mathbf{R}^n \setminus C^*$. The number

$$\rho_C = \sup_{t \in C_*} u_{O(C)}(t)/u_C(t)$$

characterizes the convexity of C. Notice that a cone C is convex if and only if $\rho_C = 1$. Further, if a cone is open and consists of a finite number of components, then $\rho_C < +\infty$.

We give some examples of cones and their dual cones. If $C = (0, \infty)$, then $C^* = [0, \infty)$, $u_C(t) = -t$ and $\rho_C = 1$. The case $C = (-\infty, 0)$ is analogous. If $C = \mathbf{R}^n$, then $C^* = \{0\}$, $u_C(t) = |t|$ and $\rho_C = 1$. Let $u = (u_1, \ldots, u_n)$ be any of the 2^n n-tuples whose entries are 0 or 1. Then

$$C_n = \{y \in \mathbf{R}^n \colon\ (-1)^{u_j} y_j > 0,\ \ j = 1, \ldots, n\}$$

is a self dual cone in $\mathbf{R}^n$ and we call it a **n-rant**.

Each of the 2^n n-rants C_n in $\mathbf{R}^n$ is an example of a regular cone. The forward and backward light cones, defined by

$$\Gamma^+ = \{y \in \mathbf{R}^n \colon\ y_1 > (y_2^2 + \ldots + y_n^2)^{1/2}\}$$

and

$$\Gamma^- = \{y \in \mathbf{R}^n \colon\ y_1 < -(y_2^2 + \ldots + y_n^2)^{1/2}\},$$

respectively, are important self dual cones in mathematical physics.

For an arbitrary cone C in $\mathbf{R}^n$ the set

$$T^C = \mathbf{R}^n + iC = \{z = x + iy \colon\ x \in \mathbf{R}^n,\ y \in C\}$$

will be called a **tube** in $\mathbf{C}^n$. The set $\{z = x + iy\colon\ x \in \mathbf{R}^n,\ y = 0\}$ is called the **distinguished boundary** of the tube T^C, while the boundary of the cone C will be referred to as the **topological boundary** of T^C.

We now present two important lemmas concerning cones and dual cones which will be of particular use in the construction and analysis of the Cauchy and Poisson kernel functions below. The lemmas are proved in [83], Section 25; we give here a separate proof of the second lemma.

Lemma 1.2.1 *Let C be an open connected cone in $\mathbf{R}^n$. The closure $\overline{O(C)}$ of $O(C)$ contains an entire straight line if and only if the dual cone C^* lies in some $(n-1)$-dimensional plane.*

Lemma 1.2.2 *Let C be an open (not nesessarily connected) cone in $\mathbf{R}^n$. For every $y \in O(C)$ there exists a positive δ (depending on y) such that*

$$\langle y, t\rangle \geq \delta\, |y|\, |t|, \qquad t \in C^*. \tag{1.2}$$

Further, if C' is an arbitrary compact subcone of $O(C)$, then there exists a $\delta > 0$ (depending only on C' and not on $y \in C'$) such that (1.2) holds for all $y \in C'$ and all $t \in C^*$.

Proof. Since $u_C(t) = u_{O(C)}(t)$ for $t \in C^*$, we have $\langle y, t\rangle \geq 0$ for all $y \in O(C)$ and all $t \in C^*$. For an arbitrary $y \in O(C)$, we have

$$\tilde{y} = y/|y| \in pr(O(C)) \subset O(C),$$

since $O(C)$ is a cone. Moreover, $O(C)$ is open, because C is open. Thus there exists a $\delta = \delta_y > 0$ such that

$$B(\tilde{y}, 2\delta) = \{y'\colon\ |y' - \tilde{y}| < 2\delta\} \subset O(C).$$

Hence

$$\tilde{y} - (t/|t|)\,\delta \in B(\tilde{y}, 2\delta) \subset O(C)$$

and thus

$$\langle \tilde{y} - (t/|t|)\,\delta, t\rangle \geq 0,$$

for every $t \in C^*$, but this implies (1.2). Now, let C' be an arbitrary compact subcone of $O(C)$. Let d be the distance d from $pr(C')$ to the complement of $O(C)$ in $\mathbf{R}^n$, i.e.

$$d = \inf\{|y_1 - y_2|\colon\ y_1 \in pr(C'),\ y_2 \notin O(C)\}.$$

Obviously, d is positive and depends only on C' and not on $y \in C'$. Define now $\delta = d/2$. The preceding considerations show that (1.2) holds for all $y \in C'$ and $t \in C^*$. The proof is complete. □

For C being an open connected cone in $\mathbf{R}^n$, we denote the distance from $y \in C$ to the topological boundary γC of C by

$$d(y) = \inf\{|y - y_1|\colon\ y_1 \in \gamma C\}.$$

It has been shown in [84], p. 159 that

$$d(y) = \inf_{t \in pr(C^*)} \langle t, y \rangle, \quad y \in C. \tag{1.3}$$

Let C' be an arbitrary compact subcone of C. It follows from Lemma 1.2.2 and (1.3) that there exists a $\delta = \delta(C') > 0$, depending only on C' and not on $y \in C'$, such that

$$0 < \delta|y| \leq d(y) \leq |y|, \qquad y \in C' \subset\subset C. \tag{1.4}$$

Let C be an open connected cone in $\mathbf{R}^n$. We make the following convention concerning the notation $y \to 0$, $y \in C$, which normally means that y varies arbitrarily within C while $y \to 0$. But frequently the above symbol will mean that $y \to 0, y \in C'$ for every compact subcone C' of C. We shall distinguish between these two convergences only when necessary; in most relevant situations the analysis clearly shows which of the interpretations of the symbol $y \to 0, y \in C$, is used in a given case.

Let V be an ultradistribution (distribution, generalized function) and let f be a function of variable $z = x + iy \in T^C$ for a given cone C. By $f(x+iy) \to V$ in the weak topology of the ultradistribution space as $y \to 0, y \in C$, we mean the convergence:

$$\langle f(x+iy), \varphi(x) \rangle \to \langle V, \varphi \rangle$$

as $y \to 0, y \in C$, for each fixed element φ in the corresponding test function space. By $f(x+iy) \to V$ in the strong topology of the ultradistribution space as $y \to 0, y \in C$, we mean

$$\langle f(x+iy), \varphi(x) \rangle \to \langle V, \varphi \rangle$$

as $y \to 0$, $y \in C$, where the convergence is uniform for an arbitrary bounded set in the corresponding test function space. Then V is called the weak or strong, respectively, ultradistributional boundary value of f and is defined on the distinguished boundary of the tube T^C.

1.3 Cauchy and Poisson kernels

Let C be a regular cone in $\mathbf{R}^n$, that is C is an open convex cone such that $\overline{C}$ does not contain any entire straight line. The **Cauchy kernel** $K(z-t)$, $z \in T^C = \mathbf{R}^n + iC$, $t \in \mathbf{R}^n$, corresponding to the tube T^C, is defined by

$$K(z-t) = \int_{C^*} \exp(2\pi i \langle z-t, u \rangle)\, du, \quad z \in T^C, \quad t \in \mathbf{R}^n. \tag{1.5}$$

In case $C = C_n$ is any of the 2^n n-rants in $\mathbf{R}^n$, the Cauchy kernel $K(z-t) = K_n(z-t)$ takes the classical form

$$K(z-t) = \frac{(-1)^n}{(2\pi i)^n} \prod_{j=1}^{n} (t_j - z_j)^{-1}, \quad z \in \mathbf{R}^n + iC_n, \quad t \in \mathbf{R}^n,$$

since $C_n^* = \overline{C}_n$ in this case.

The **Poisson kernel** corresponding to the tube T^C is the function

$$Q(z;t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)}, \quad z = x+iy \in T^C, \quad t \in \mathbf{R}^n. \tag{1.6}$$

In case $C = C_n$ is any of the n-rants, the Poisson kernel $Q(z;t) = Q_n(z;t)$ reduces to the classical form

$$Q(z;t) = \frac{(-1)^n}{\pi^n} \prod_{j=1}^{n} \frac{y_j}{(t_j - x_j)^2 + y_j^2}, \quad z = x+iy \in \mathbf{R}^n + iC_n, \quad t \in \mathbf{R}^n.$$

If the cone C above had been assumed to be open and connected but not necessarily convex, we would have defined the kernels $K(z-t)$ and $Q(z;t)$ for $z \in T^{O(C)}$ and would obtain all the properties concerning the kernels for $z \in T^{O(C)}$. Thus we have assumed that C is convex without loss of generality. From Lemma 1.2.1, the dual cone C^* will lie in an $(n-1)$-dimensional plane if $\overline{C}$ contains an entire straight line, i.e. in this case the Lebesgue measure of C^* would be zero, so the Cauchy kernel $K(z-t)$ would be zero and the Poisson kermnel $Q(z;t)$ would be undefined. To avoid this situation we must have guaranteed that $\overline{C}$ does not contain any entire straight line. Therefore we consider regular cones unless explicitely stated otherwise.

We conclude this section with several technical lemmas which will be used in our analysis concerning the Cauchy and Poisson kernels.

Lemma 1.3.1 *Let C be an open connected cone in $\mathbf{R}^n$.*

I. Fix arbitrarily $z \in T^C = \mathbf{R}^n + iC$ and denote by I_{C^*} the characteristic function of C^*. Then $E_z I_{C^*} \in L^p$ for all p, $1 \le p \le \infty$.

II. Assume that g is a continuous function on $\mathbf{R}^n$ with support in C^* such that, for arbitrary $m > 0$ and compact subcone C' of C,

$$|g(t)| \le M(C', m)\exp(2\pi(\langle w, t\rangle + \sigma|w|)), \quad t \in \mathbf{R}^n, \tag{1.7}$$

whenever $\sigma > 0$ and $w \in C' \setminus (C' \cap \overline{B}(0,m))$, where $\overline{B}(0,m)$ is the closure of the ball with the center at 0 and the radius m and $M(C',m)$ is a constant. Then, for an arbitrary y in C, $y \neq 0$, we have $e_y g \in L^p$, whenever $1 \le p < \infty$.

Proof. To prove part I fix z in T^C and let $y = \operatorname{Im} z$. Applying Lemma 1.2.2, we find a $\delta = \delta_y > 0$ such that

$$|E_z(t)| I_{C^*}(t) = e_y(t) I_{C^*}(t) \le e_{\delta|y|}(|t|) I_{C^*}(t) \le 1 \tag{1.8}$$

for all $z = x+iy \in T^C$ and all $t \in \mathbf{R}^n$, since $I_{C^*}(t) = 0$ for $t \notin C^*$. Part I of the lemma for $p = \infty$ follows from (1.8). For $1 \le p < \infty$, we use (1.8) and integration by parts $n-1$ times (or the gamma function after the change of variable for $v = 2\pi\delta p|y|r$)) to get

$$\int_{\mathbf{R}^n} |E_z(t) I_{C^*}(t)|^p \, dt \le \int_{\mathbf{R}^n} e_{p\delta|y|}(|t|)\, dt$$

$$= \Omega_n \int_0^\infty r^{n-1} e_{p\delta|y|}(r)\, dr = (n-1)!\,\Omega_n (2\pi\delta p|y|)^{-n}, \tag{1.9}$$

where Ω_n is the surface area of the unit sphere in $\mathbf{R}^n$. The estimate in (1.9) proves part I of the lemma for $1 \le p < \infty$.

To prove part II fix a point y in C. Since C is open, there exists a compact subcone C' of C and an $m > 0$ such that $y \in C' \setminus (C' \cap \overline{B}(0,m))$. Since $y \notin \overline{B}(0,m))$, we have $|y| > m$. Choose $w = \lambda y$, where λ is an arbitrary number such that $m/|y| < \lambda < 1$. Since C' is a cone, $y \in C'$ and $\lambda|y| > m$, we have $w = \lambda y \in C' \setminus (C' \cap \overline{B}(0,m))$, i.e. the estimate given by (1.7) is true for w just chosen. Since $C' \subset\subset C$, it follows from Lemma 1.2.2 that there is a $\delta = \delta(C') > 0$, not depending on $y \in C'$, such that (1.2) holds for all $t \in C^*$. Hence, denoting $A(\sigma, \lambda, y) = M(C', m)\exp(2\pi\sigma\lambda|y|)$, we have

$$|e_y(t)g(t)| \le A(\sigma,\lambda,y)e_{(1-\lambda)y}(t) \le A(\sigma,\lambda,y)e_{(1-\lambda)\delta|y|}(|t|)$$

for $t \in C^*$. Integrating by parts (or using the gamma function) yields

$$\int_{\mathbf{R}^n} |E_{iy}(t)g(t)|^p\,dt = \int_{C^*} e_{p(1-\lambda)\delta|y|}(|t|)\,dt$$

$$= \Omega_n(A(\sigma,\lambda,y))^p \int_0^\infty r^{n-1} e_{p(1-\lambda)\delta|y|}(r)\,dr$$

$$= (n-1)!\,\Omega_n(A(\sigma,\lambda,y))^p(2\pi p(1-\lambda)\delta|y|)^{-n} < \infty,$$

since supp $g \subseteq C^*$. This completes the proof of part II and the lemma. □

Lemma 1.3.2 *Let C be a regular cone. The Cauchy kernel $K(z-t)$ is an analytic function of variable $z \in T^C$ for each fixed $t \in \mathbf{R}^n$.*

Proof. Let I_{C^*} denote the characteristic function of C^*. By the proof of Lemma 1.3.1, $I_{C^*}E_{z-t} \in L^1$ for fixed $z \in T^C$ and $t \in \mathbf{R}^n$. Let K be an arbitrary compact subset of T^C and let $z \in K \subset T^C$, $z \neq 0$. There exists a compact subcone C' of C such that $y = \operatorname{Im} z \in C'$ and y is in a positive distance (say k) from 0. By Lemma 1.2.2, there is a $\delta = \delta(C') > 0$, depending only on C', such that

$$|I_{C^*}(u)E_{z-t}(u)| = I_{C^*}(u)\exp(-2\pi\langle y,u\rangle) \le I_{C_*}(u)\exp(-2\pi\delta k|u|) \qquad (1.10)$$

for $t \in \mathbf{R}^n$ and $u \in C^*$. The right hand side of (1.10) is an L^1-function of variable $u \in \mathbf{R}^n$ for arbitrary $z \in K$ and $t \in \mathbf{R}^n$, by the proof of Lemma 1.3.1, and the function $I_{C^*}(u)E_{z-t}(u)$ is analytic in $z \in T^C$ for each fixed $t \in \mathbf{R}^n$ and $u \in \mathbf{R}^n$. To conclude the assertion it remains to use a well known theorem concerning integrals involving a parameter (see e.g. [6], pp. 295-296). □

Lemma 1.3.3 *Let C be a regular cone and fix $w = u + iv \in T^C$. The function*

$$K(z+w) = \int_{C^*} E_{z+w}(u)\,du, \quad z \in T^C,$$

is analytic in $z \in T^C$ and

$$|K(z+w)| \le M_v < \infty, \quad z \in T^C,$$

where M_v is a constant which depends only on $v = \operatorname{Im} w$.

Proof. The proof that $K(z+w)$ is analytic in $z \in T^{C'}$ is the same as in the proof of Lemma 1.3.2. We have $\langle y, u\rangle \geq 0$ for $y \in C$ and $u \in C^*$. By Lemma 1.2.2, there is a $\delta = \delta_v > 0$ such that $\langle v, u\rangle \geq \delta |v||u|$ for $v \in C$ and $u \in C^*$. The assertion now follows by a similar analysis as in (1.9). □

Lemma 1.3.4 *Let $h \in L^p$, $1 \leq p \leq 2$ and let $g(u) = \mathcal{F}^{-1}[h;u]$ in the sense of the space L^p. Assume that $gE_z \in L^1$ for $z \in T^C$ and* supp $g \subseteq C^*$. *We have*

$$\int_{C^*} g(u)E_z(u)\,du = \int_{\mathbf{R}^n} h(t)K(z-t)\,dt, \quad z \in T^C. \tag{1.11}$$

Proof. Let $z \in T^C$. Let $1 \leq p \leq 2$ and $1/p + 1/q = 1$. As a result of the remarks below, $K(z-t)$ as a function of $t \in \mathbf{R}^n$ belongs to L^q for every $z \in T^C$. Therefore the integral on the right hand side of (1.11) is well defined. First consider $p = 1$. By Lemma 1.3.3 and Fubini's theorem, we have

$$\int_{\mathbf{R}^n} h(t)K(z-t)\,dt = \int_{\mathbf{R}^n} h(t)\,dt \int_{C^*} E_{z-t}(u)\,du$$

$$= \int_{C^*} E_z(u)\,du \int_{\mathbf{R}^n} h(t)E_{-u}(t)\,dt = \int_{C^*} g(u)E_z(u)\,du,$$

which proves (1.11) for $p = 1$. In case $1 < p \leq 2$, the function g is the limit in the L^q-form of the sequence of functions

$$g_k(u) = \int_{|t|\leq k} h(t)E_{-u}(t)\,dt, \quad k = 1, 2, \dots .$$

Using Fubini's theorem, we have now

$$\int_{C^*} g(u)E_z(u)\,du = \lim_{k\to\infty} \int_{C^*} g_k(u)E_z(u)\,du$$

$$= \lim_{k\to\infty} \int_{|t|\leq k} h(t)\,dt \int_{C^*} E_{z-t}(u)\,du = \int_{\mathbf{R}^n} h(t)K(z-t)\,dt$$

for $z \in T^C$, which proves (1.11) in case $1 < p \leq 2$. □

The Poisson kernel defined in (1.6) has been known for some time to be an approximate identity. We state this in the following lemma (see [78], p. 105):

Lemma 1.3.5 *Let C be a regular cone, let $z \in T^C$ and $t \in \mathbf{R}^n$. The Poisson kernel $Q(z;t)$ has the following properties:*

(*i*) $Q(z;t) \geq 0, \quad z \in T^C, \quad t \in \mathbf{R}^n$;

(*ii*) $\int_{\mathbf{R}^n} Q(z;t)\,dt = 1, \quad z \in T^C$;

(*iii*) $\lim_{z\to t_0, z\in T^C} \int_{|t-t_0|>\delta} Q(z;t)\,dt = 0, \quad \delta > 0$

uniformly for all $t_0 \in \mathbf{R}^n$.

We shall prove later that the Cauchy and Poisson kernel are in certain ultradifferentiable function spaces.

Chapter 2

Ultradifferentiable functions and ultradistributions

2.1 Sequences (M_p)

We define subspaces of some of the Schwartz test spaces through the use of sequences of positive real numbers which satisfy certain conditions. The corresponding dual spaces then contain the generalized functions of Schwartz.

By $(M_p) = (M_p)_{p\in\mathbf{N}_0}$ we will denote a sequence of positive numbers which satisfies some of the following conditions:

$(M.1)$ $M_p^2 \le M_{p-1}M_{p+1}, \qquad p \in \mathbf{N};$

$(M.2)$ there are positive constants A and H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q}, \qquad p \in \mathbf{N}_0;$$

$(M.3)$ there is a constant $A > 0$ such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le ApM_p/M_{p+1}, \qquad p \in \mathbf{N}.$$

Sometimes $(M.2)$ and $(M.3)$ will be replaced by the following weaker conditions:

$(M.2)'$ there are constants A and H such that

$$M_{p+1} \le AH^p M_p, \qquad p \in \mathbf{N}_0;$$

$(M.3)'$ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$.

Sequences (M_p) satisfying some or all of these properties are the basis for the ultradistributions to be studied here. The paper of Komatsu [48] serves as a basic reference for these sequences. If $s > 1$, the Gevrey sequences (M_p) given by $M_p = (p!)^s$, $M_p = p^{ps}$ and $M_p = \Gamma(1+ps)$, where Γ denotes the gamma function, are basic examples of sequences satisfying some of the above stated conditions.

We will prove some properties of sequences (M_p). It will be convenient to consider together with a given sequence (M_p) also its multi-dimensional variant M_α with multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ defined by

$$M_\alpha = M_{\alpha_1 + \dots \alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$$

Lemma 2.1.1 *Let (M_p) be an arbitrary sequence of positive numbers.*

(*i*) *If the sequence (M_p) satisfies $(M.1)$, then*

$$M_p M_q \le M_0 M_{p+q}, \quad p, q \in \mathbf{N}_0. \tag{2.1}$$

(*ii*) *If the sequence (M_p) satisfies condition $(M.2)$, then*

$$M_p \ge \frac{M_{p+q}}{(AM_1)^q H^{p+1} H^{p+2} \dots H^{p+q}}, \quad p \in \mathbf{N}_0, \ q \in \mathbf{N}, \tag{2.2}$$

where the positive constants A and H are from $(M.2)$, and

$$M_\alpha \le BE^\alpha M_{\alpha_1} M_{\alpha_2} \dots M_{\alpha_n} \tag{2.3}$$

for every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$, where B and E are positive constants.

Proof. Applying $(M.1)$ repeatedly, we get

$$\frac{M_p}{M_{p+1}} \le \frac{M_{p-1}}{M_p} \le \frac{M_{p-2}}{M_{p-1}} \le \dots \le \frac{M_0}{M_1}.$$

Using this and similar arguments, we have

$$\begin{aligned} \frac{M_p}{M_{p+q}} &= \frac{M_p}{M_{p+1}} \frac{M_{p+1}}{M_{p+2}} \dots \frac{M_{p+q-1}}{M_{p+q}} \\ &\le \frac{M_0}{M_1} \frac{M_1}{M_2} \dots \frac{M_{q-1}}{M_q} = \frac{M_0}{M_q}, \end{aligned}$$

from which (2.1) follows. Inequalities (2.2) and (2.3) follow by direct repeating applications of $(M.2)$. □

Let (M_p) and (N_p) be sequences of positive numbers which (always) satisfy $(M.1)$. Following Komatsu [48], Definition 3.1, p. 52, we write

$$M_p \subset N_p \tag{2.4}$$

if there exist constants $L > 0$ and $B > 0$, independent of p, such that

$$M_p \le BL^p N_p, \quad p \in \mathbf{N}_0. \tag{2.5}$$

Following [48], Definition 3.9, p. 53, we write

$$M_p \prec N_p \tag{2.6}$$

if for any $L > 0$ there is a constant $B > 0$, independent of p, such that (2.5) holds.

Komatsu has proved in [48], p. 74, that $p! \prec M_p$ for every sequence (M_p) satisfying $(M.1)$ and $(M.3)'$. This and Stirling's formula imply $p^p \prec M_p$ (with the convention $0^0 = 1$ and the assumption $M_0 = 1$), as noticed by Pilipović in [66], p. 209. Moreover, $\Gamma(s+p) \prec M_p,\ s > 0$, by Lemma 4.1 in [48], p. 56, and an analysis made in [48], p. 74. We summarize these facts in the following lemma.

Lemma 2.1.2 *Let the sequence (M_p) satisfy conditions $(M.1)$ and $(M.3)'$. We have $p! \prec M_p$, $p^p \prec M_p$ and $\Gamma(s+p) \prec M_p$ for $s > 0$.*

For a sequence (M_p) the associated functions M and M^* of Komatsu, are defined by

$$M(\rho) = \sup_{p \in \mathbf{N}_0} \log(\rho^p M_0 / M_p), \quad 0 < \rho < \infty, \tag{2.7}$$

and

$$M^*(\rho) = \sup_{p \in \mathbf{N}_0} \log(\rho^p p! M_0 / M_p), \quad 0 < \rho < \infty. \tag{2.8}$$

Some properties of the associated function M are collected in the following lemma.

Lemma 2.1.3 *If the sequence (M_p) satisfies $(M.1)$, then*

$$M(\rho + \alpha) \le M(2\rho) + M(2\alpha), \qquad \rho > 0,\ \alpha > 0, \tag{2.9}$$

If the sequence (M_p) satisfies $(M.1)$ and $(M.2)$, then

$$2M(\rho) \le M(H\rho) + \log(AM_0), \quad \rho > 0, \tag{2.10}$$

where A and H are the constants in $(M.2)$; if $L \ge 1$, then there is a constant $K > 0$ such that

$$M(L\rho) \le (3/2)LM(\rho) + K, \quad \rho > 0; \tag{2.11}$$

if $L \ge 1$, then there is a constant $B > 0$ and a constant $E_L > 0$ depending on L such that

$$LM(\rho) \le M(B^{L-1}\rho) + E_L, \quad \rho > 0. \tag{2.12}$$

Proof. Petzsche obtained (2.9) in [64], p. 142 (Lemma 1.10), under the assumption that (M_p) satisfies $(M.1)$. Inequality (2.10) are shown by Komatsu in [48], p. 51 (Proposition 3.6) and by Petzsche in [64], p. 138 (Lemma 1.4), under conditions $(M.1)$ and $(M.2)$ on the sequence (M_p) and inequalities (2.11) and (2.12) are proved in [64], Lemma 1.7, p. 140. □

2.2 Ultradifferential operators

We denote by $\mathcal{R}$ the family of all sequences (r_p) of positive numbers which increase to infinity. This set is partially ordered and directed by the relation $(r_p) \preceq (s_p)$, which means that there exists p_0 such that $r_p \leq s_p$ for every $p > p_0$.

Let $x \in \mathbf{R}^n$ and $\beta \in \mathbf{N}_0^n$. We define then

$$< x >^\beta = \prod_{j=1}^{n} (1 + |x_j|^2)^{\beta_j/2}.$$

An operator of the form $P(D) = \sum_{\alpha \in \mathbf{N}_0^n} a_\alpha D^\alpha$, $a_\alpha \in \mathbf{C}$, is an ultradifferential operator of class (M_p) (resp. of class $\{M_p\}$) if there are constants $A > 0$, $h > 0$ (resp. for every $h > 0$ there is an $A > 0$) such that

$$|a_\alpha| \leq Ah^\alpha / M_\alpha, \quad \alpha \in \mathbf{N}_0^n.$$

Special classes of entire functions will be needed. We recall some facts from [48], [51].

Let $r > 0$ and $m_p = M_p/M_{p-1}$ for $p \in \mathbf{N}$. Put

$$P_r(\zeta) = (1 + \zeta_1^2 + \ldots + \zeta_n^2)^{n'} \prod_{j=1}^{\infty} \left(1 + \frac{\zeta_1^2 + \ldots + \zeta_n^2}{r^2 m_j^2}\right) \tag{2.13}$$

for $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$. If conditions $(M.1)$, $(M.2)$ and $(M.3)$ hold, then $P_r(D)$ is an ultradifferentiable operator of class (M_p); it maps $\mathcal{D}((M_p), \mathbf{R}^n)$ (cf. the next paragraph) into itself and

$$\mathcal{F}(P_r(D)\phi)(\xi) = P_r(\xi)\widehat{\varphi}(\xi), \quad \xi \in \mathbf{R}^n \tag{2.14}$$

for $\varphi \in \mathcal{D}((M_p, \mathbf{R}^n)$. Put, for a given (r_p),

$$P_{(r_p)}(\zeta) = (1 + \zeta_1^2 + \ldots + \zeta_n^2)^{n'} \prod_{j=1}^{\infty} \left(1 + \frac{\zeta_1^2 + \ldots + \zeta_n^2}{r_j^2 m_j^2}\right), \quad \zeta \in \mathbf{C}^n, \tag{2.15}$$

where n' is an integer greater than $n/2$.

If conditions $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied, the function $P_{(r_p)}$ is of class $\{M_p\}$. For elements of $\mathcal{D}(\{M_p\}, \mathbf{R}^n)$ and the ultradifferential operator $P_{(r_p)}(D)$ equation (2.14) holds, as well.

For a given sequence (M_p) and $(r_p) \in \mathcal{R}$ we consider the corresponding sequence (N_p), defined by

$$N_p = M_p \prod_{j=1}^{p} r_j, \; p \in \mathbf{N}.$$

If the associated function corresponding to the sequence (M_p), given by (2.7), is denoted by M, then the associated function corresponding to the sequence (N_p) defined above is denoted by N, i.e. given by the formula

$$N(\rho) = \sup\{\log(\rho^p M_0/N_p) : \; p \in \mathbf{N}_0\}, \quad \rho > 0.$$

If an element of $\mathcal{R}$ is denoted by $(\tilde{r}_p)$, the corresponding associated function is denoted by $\tilde{N}$. It follows from the definition that for every $(r_p) \in \mathcal{R}$ and constants $C > 0$ and $c > 0$ there are $(\tilde{r}_p) \in \mathcal{R}$, and $\rho_0 > 0$ such that

$$CN(c\rho) \le \tilde{N}(\rho), \quad \rho > \rho_0. \tag{2.16}$$

Assume the conditions $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied. From [48], Proposition 4.5 and p. 91, it follows that there exist constants $D > 0$ and $c > 0$ such that

$$D \exp(-N(c|\xi|)) \le |1/P_{(r_p)}(\xi)| \le \exp(-N(\xi)), \quad \xi \in \mathbf{R}^n. \tag{2.17}$$

Using the Cauchy formula

$$\partial^k (1/P_{(r_p)}(\xi)) = \frac{k!}{(2\pi i)^n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{[1/P_{(r_p)}(\zeta)]\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - \xi_1)^{k_1+1} \ldots (\zeta_n - \xi_n)^{k_n+1}},$$

for $k \in \mathbf{N}_0^n$ and $\xi \in \mathbf{R}^n$, where $\Gamma_j = \{\zeta_j : \ |\zeta_j - \xi_j| = d\}$ with $d > 0$ for $j = 1, \ldots, n$, we see that there exists a $C > 0$ such that

$$|\partial^k (1/P_{(r_p)}(\xi))| \le Ck! d^{-k} \exp(-N(\xi)/C), \quad \xi \in \mathbf{R}^n. \tag{2.18}$$

The two-dimensional version of Lemma 3.4 from [51] will be needed.

Lemma 2.2.1 *Let* $a_{p,q} > 0, \ \ p, q \in \mathbf{N}_0$.
(i) *There are* $h > 0$ *and* $C > 0$ *such that*

$$\sup\{\frac{a_{p,q}}{h^{p+q}} : \quad p, q \in \mathbf{N}_0\} \le C \tag{2.19}$$

if and only if

$$\sup\{\frac{a_{p,q}}{R_p S_q} : \ p, q \in \mathbf{N}_0\} < \infty \tag{2.20}$$

for arbitrary sequences $(r_j), (s_j)$ *in* $\mathcal{R}$, *where*

$$R_p = \prod_{j=1}^{p} r_j, \quad S_q = \prod_{j=1}^{q} s_j, \qquad p, q \in \mathbf{N}. \tag{2.21}$$

(ii) *There are sequences* $(r_j), (s_j) \in \mathcal{R}$ *and a constant* $C > 0$ *such that*

$$\sup\{R_p S_q a_{p,q} : \quad p, q \in \mathbf{N}\} \le C,$$

with R_p *and* S_q *given by* (2.21), *if and only if*

$$\sup\{h^{p+q} a_{p,q} : \ p, q \in \mathbf{N}_0\} < \infty$$

for every $h > 0$.

Proof. One has only to prove the if parts.

(i) Assume that (2.20) holds for arbitrary $(r_j),(s_j)\in\mathcal{R}$, but (2.19) does not hold for every $h>0$ and $C>0$. Let (h_k) be a sequence which strictly increases to ∞. There exists a sequence (p_k,q_k) in $\mathbf{N}_0^2$ such that

$$p_{k+1}+q_{k+1}>p_k+q_k,\quad h^{-(p_k+q_k)}a_{p_k,q_k}>k,\qquad k\in\mathbf{N}.$$

The following cases may appear:

(a) there is a $p_0\in\mathbf{N}_0$ such that (p_0,q_m) is a subsequence of (p_k,q_k) and (q_m) is strictly increasing;

(b) symmetric case to previous one;

(c) there is a subsequence (p_m,q_m) of the sequence (p_k,q_k) such that both (p_m) and (q_m) are strictly increasing sequences.

Let us prove that case (c) leads to the contradiction. The reasoning in the other two cases is similar. Define

$$r_j=h_1\quad\text{for}\quad 1\le j\le p_1,\qquad s_j=h_1\quad\text{for}\quad 1\le j\le q_1,$$

$$r_j=(h_m^{p_m}h_{m-1}^{-p_{m-1}})^{1/(p_m-p_{m-1})}\quad\text{for}\quad p_{m-1}<j\le p_m$$

and

$$s_j=(h_m^{q_m}h_{m-1}^{-q_{m-1}})^{1/(q_m-q_{m-1})}\quad\text{for}\quad q_{m-1}<j<q_m,$$

where $m=2,3,\ldots$. The constructed sequences (r_j) and (s_j) do not satisfy (2.20), which contradicts the assumption.

(ii) For a given $h\ge1$ put

$$C_h=\sup\{h^{p+q}a_{p,q}:\ p,q\in\mathbf{N}_0\};\qquad \tilde{C}_h=\sup\{h^kb_k:\ k\in\mathbf{N}_0\},$$

where

$$b_k=\sup\{a_{p,q}:\ p+q=k,\ p,q\in\mathbf{N}_0\},\qquad k\in\mathbf{N}_0,$$

and let

$$H_k=\sup\{h^kC_h^{-1}:\ h\ge1\}$$

for $k\in\mathbf{N}_0$. Clearly $C_h\le\tilde{C}_h$. Fix $p,q\in\mathbf{N}_0$ and set $k=p+q$. For every $h\ge1$, we have

$$H_{p+q}a_{p,q}\le\sup\{\frac{h^k}{\tilde{C}_h}a_{p,q}\}\le\frac{C_h}{\tilde{C}_h}\le1$$

and so

$$\sup\{H_{p+q}a_{p,q}:\ p,q\in\mathbf{N}_0\}\le1.$$

The sequence (h_j), where $h_j=H_j/H_{j-1}$ for $j\in\mathbf{N}$, is increasing and, moreover, $H_j/h^j\to\infty$ as $j\to\infty$ for every $h>0$.

Since

$$\prod_{j=1}^{p}h_j\prod_{j=1}^{q}h_ja_{p,q}\le\prod_{j=1}^{p+q}h_ja_{p,q},$$

by taking $r_j=h_j$ and $s_j=h_j$ we obtain

$$R_pS_qa_{p,q}<\infty$$

and this implies the assertion. □

2.3 Functions and ultradistributions of Beurling and Roumieu type

In this section we define and obtain properties of the ultradifferentiable functions and ultradistributions that are considered here of type L^s.

Let (M_p), $p \in \mathbf{N}_0$, be a sequence of positive numbers. We define $\mathcal{D}((M_p),\ \Omega)$ (respectively, $\mathcal{D}(\{M_p\},\Omega)$), where Ω is an open set in $\mathbf{R}^n$ to be the set of all complex valued infinitely differentiable functions $\varphi(t)$ with compact support in Ω such that there exists an $N > 0$ for which

$$\sup_{t\in R^n} |D_t^\alpha \varphi(t)| \leq N h^\alpha M_\alpha, \quad \alpha \in \mathbf{N}_0^n \tag{2.22}$$

for all $h > 0$ (respectively, for some $h > 0$). Here the positive constants N and h depend only on φ: they do not depend on α. The topologies of $\mathcal{D}((M_p),\Omega)$ and $\mathcal{D}(\{M_p\},\Omega)$ are given in Komatsu [48], p. 44, which is a good source of information concerning these spaces. Let $\mathcal{D}(h, K)$ denote the space of smooth functions supported by a compact set K for which (2.22) holds and $\mathcal{D}((M_p), K)$ and $\mathcal{D}(\{M_p\}, K)$ denote subspaces of $\mathcal{D}((M_p),\Omega)$ and $\mathcal{D}(\{M_p\},\Omega)$ consisting of elements supported by K, respectively. Recall that

$$\begin{aligned}
\mathcal{D}^{(M_p)}(\Omega) = \mathcal{D}((M_p),\Omega) &= \text{ind } \lim_{K\subset\subset\Omega}\text{proj } \lim_{h\to 0}\mathcal{D}(h,K)\\
&= \text{ind } \lim_{K\subset\subset\Omega}\mathcal{D}((M_p),K);\\
\mathcal{D}^{\{M_p\}}(\Omega) = \mathcal{D}(\{M_p\},\Omega) &= \text{ind } \lim_{K\subset\subset\Omega}\text{proj } \lim_{h\to 0}\mathcal{D}(h,K)\\
&= \text{ind } \lim_{K\subset\subset\Omega}\mathcal{D}(\{M_p\},K).
\end{aligned}$$

Notation $\mathcal{D}_K^{(M_p)} = \mathcal{D}((M_p), K)$ and $\mathcal{D}_K^{\{M_p\}} = \mathcal{D}(\{M_p\}, K)$ is also used.

The strong duals of the above spaces, denoted by $\mathcal{D}'^{(M_p)}(\Omega) = \mathcal{D}'((M_p),\Omega)$ and $\mathcal{D}'^{\{M_p\}}(\Omega) = \mathcal{D}'(\{M_p\},\Omega)$ are called the spaces of Beurling and Roumieu ultradistributions, respectively.

The spaces of test functions and ultradistributions which correspond to the spaces $\mathcal{D}_{L^s}$ and $\mathcal{D}'_{L^s}$ of L. Schwartz ([77], pp. 199-205) will be basic for our work. The space $\mathcal{D}((M_p), L^s)$ (respectively, $\mathcal{D}(\{M_p\}, L^s)$), $1 \leq s \leq \infty$, is defined to be the set of all complex valued infinitely differentiable functions φ such that there is a constant $N > 0$ for which

$$|D^\alpha \varphi|_{L^s} \leq N h^\alpha M_\alpha, \quad \alpha \in \mathbf{N}_0^n \tag{2.23}$$

for all $h > 0$ (respectively, for some $h > 0$). We have

$$\mathcal{D}((M_p), L^s) \subset \mathcal{D}(\{M_p\}, L^s), \quad 1 \leq s \leq \infty.$$

Further,

$$\mathcal{D}((M_p), \mathbf{R}^n) \subset \mathcal{D}((M_p), L^s), \quad 1 \leq s \leq \infty,$$

and

$$\mathcal{D}(\{M_p\}, \mathbf{R}^n) \subset \mathcal{D}(\{M_p\}, L^s), \quad 1 \leq s \leq \infty.$$

A natural topology is defined on $\mathcal{D}((M_p), L^s)$, $1 \leq s \leq \infty$, as follows. First put

$$\|\varphi\|_{s,h} = \sup_{\alpha} \frac{\|D^\alpha \varphi\|_{L^s}}{h^\alpha M_\alpha}, \quad h > 0, \tag{2.24}$$

and

$$\mathcal{D}((M_p), h, L^s) = \{\varphi \in C^\infty : \ \|\varphi\|_{s,h} < \infty\}, \quad h > 0. \tag{2.25}$$

Now, since $\mathcal{D}(M_p, h_1, L^s) \subset \mathcal{D}(M_p, h_2, L^s)$ whenever $0 < h_1 < h_2$, we may equip the set $\mathcal{D}((M_p), L^s)$ with the projective limit topology by putting

$$\mathcal{D}((M_p), L^s) = \text{proj } \lim\nolimits_{h \to 0} \mathcal{D}((M_p), h, L^s). \tag{2.26}$$

A net (φ_λ) of elements of $\mathcal{D}((M_p), L^s)$ converges to $\varphi \in \mathcal{D}((M_p), L^s)$ as $\lambda \to \infty$ in this topology (we write then $\varphi_\lambda \to \varphi$ in $\mathcal{D}((M_p), L^s)$ as $\lambda \to \infty$), if

$$\lim_{\lambda \to \infty} \|D^\alpha(\varphi_\lambda - \varphi)\|_{L^s} = 0, \quad \alpha \in \mathbf{N}_0^n \tag{2.27}$$

and, in addition, there is a constant $N > 0$, independent of λ and α, such that

$$\|D^\alpha(\varphi_\lambda - \varphi)\|_{L^s} \leq N h^\alpha M_\alpha, \quad \alpha \in \mathbf{N}_0^n, \tag{2.28}$$

for all $h > 0$.

In $\mathcal{D}(\{M_p\}, L^s)$, $1 \leq s \leq \infty$, we define the inductive limit topology in the following way:

$$\mathcal{D}(\{M_p\}, L^s) = \text{ind } \lim\nolimits_{h \to \infty} \mathcal{D}((M_p), h, L^s), \tag{2.29}$$

where the spaces $\mathcal{D}(M_p, h, L^s)$ are defined in (2.25) with the topolgy defined by the family of norms given in (2.24). In this topology a net (φ_λ) of elements of $\mathcal{D}(\{M_p\}, L^s)$ converges to $\varphi \in \mathcal{D}(\{M_p\}, L^s)$ as $\lambda \to \infty$ (and we write then $\varphi_\lambda \to \varphi$ in $\mathcal{D}(\{M_p\}, L^s)$ as $\lambda \to \infty$, if (2.27) holds and, in addition, there are constants $N > 0$ and $h > 0$, independent of λ and α, such that (2.28) holds.

The spaces $\mathcal{D}((M_p), L^s)$ defined above and the spaces $\mathcal{D}_{L^s}^{(M_p)}(R^n)$, defined by Pilipović in [65], §3, coincide for $1 \leq s \leq \infty$; it is easy to verify that the norms in (2.24) are equivalent to the norms $\gamma_{s,h}(\varphi)$ in the sense of [65], §3. Various important properties of the spaces $\mathcal{D}((M_p), L^s)$ are proved in [65], §3; among them the fact that $\mathcal{D}((M_p), \mathbf{R}^n)$ is dense in $\mathcal{D}((M_p), L^s)$, whenever $1 \leq s \leq \infty$.

Throughout we assume that the sequence (M_p) will satisfy at least conditions $(M.1)$ and $(M.3)'$ so that $\mathcal{D}((M_p), L^s)$ and $\mathcal{D}(\{M_p\}, L^s)$ contain sufficiently many functions (see Komatsu [48], p. 26).

We denote by $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ the spaces of continuous linear forms on $\mathcal{D}((M_p), L^s)$ and $\mathcal{D}(\{M_p\}, L^s)$, respectively. Following several authors, we call $\mathcal{D}'((M_p), L^s)$ (resp. $\mathcal{D}'(\{M_p\}, L^s)$ the space of **ultradistributions** of class (M_p) or **of Beurling type** (resp. of class $\{M_p\}$) or **of Roumieu type**). Following Komatsu ([48], pp. 47 and 61), we use the notation $\mathcal{D}(*, L^s)$ and $\mathcal{D}'(*, L^s)$, where $*$ is the common notation for the symbols (M_p) and $\{M_p\}$.

An additional function space is $\dot{\mathcal{B}}(*, \mathbf{R}^n)$, corresponding to the Schwartz space $\dot{\mathcal{B}}$. The space $\dot{\mathcal{B}}(*, \mathbf{R}^n)$ is defined to be the completion of in $\mathcal{D}(*, L^\infty)$.

We now present characterization results for $\mathcal{D}'(*, L^s)$. We prove the result for $\mathcal{D}'(\{M_p\}, L^s)$ here. The proof for $\mathcal{D}'((M_p), L^s)$ is similar and can be found in Pilipović [65] (Theorem 5).

Theorem 2.3.1 *Let $1 \leq s < \infty$. Let $\{g_\alpha\}_{0\leq\alpha<\infty}$ be a sequence of functions in L^r, $1/r + 1/s = 1$, such that for all $k > 0$*

$$\|g_\alpha\|_{L^r} = \mathcal{O}\Big(\frac{1}{k^\alpha M_\alpha}\Big) \quad \text{as } \alpha \to \infty. \tag{2.30}$$

Then

$$V = \sum_{0\leq\alpha<\infty} D^\alpha g_\alpha \tag{2.31}$$

is an element of $\mathcal{D}'(\{M_p\}, L^s)$. Conversely, if $V \in \mathcal{D}'(\{M_p\}, L^s)$, then V has the form (2.31) where $\{g_\alpha\}_{0\leq\alpha<\infty}$ is a sequence of functions in L^r satisfying (2.30) for all $k > 0$.

Proof. Let $\varphi \in \mathcal{D}(\{M_p\}, L^s)$ and let V be given by (2.31) with $\{g_\alpha\}$ satisfying (2.30). From the definition of $\mathcal{D}(\{M_p\}, L^s)$ there exist constants $N > 0$ and $H > 0$ such that

$$\begin{aligned}\Big|\sum_{0\leq\alpha<\infty}(-1)^\alpha \int_{R^n} g_\alpha(t) D^\alpha\varphi(t)\,dt\Big| &\leq \sum_{0\leq\alpha<\infty} \|g_\alpha\|_{L^r}\|D^\alpha\varphi\|_{L^s} \\ &\leq \sum_{0\leq\alpha<\infty} NH^\alpha M_\alpha \|g_\alpha\|_{L^r}.\end{aligned} \tag{2.32}$$

By (2.30), there exist $P > 0$ and $P' > 0$ such that

$$\|g_\alpha\|_{L^r} \leq P/(k^\alpha M_\alpha), \quad \alpha \geq P',$$

for all $k > 0$. Choosing $k = 2H$ we have

$$\sum_{P'\leq\alpha\infty} NH^\alpha M_\alpha \|g_\alpha\|_{L^r} \leq NP \sum_{P'\leq\alpha\infty} (1/2)^\alpha < \infty, \tag{2.33}$$

which proves that the series on the right of (2.32) converges. Hence the series

$$\sum_{0\leq\alpha<\infty}(-1)^\alpha \int_{R^n} g_\alpha(t) D^\alpha\varphi(t)\,dt$$

converges absolutely and $\langle V, \varphi\rangle$ is a well defined complex number. Let (φ_j) be a sequence in $\mathcal{D}(\{M_p\}, L^s)$ such that $\varphi_j \to 0$ in $\mathcal{D}(\{M_p\}, L^s)$ as $j \to \infty$. We have

$$|\langle V, \varphi_j\rangle| \leq \sum_{0\leq\alpha<\infty} \|g_\alpha\|_{L^r}\|D^\alpha\varphi_j\|_{L^s}. \tag{2.34}$$

From the convergence in $\mathcal{D}(\{M_p\}, L^r)$ it follows that there exist $N > 0$ and $H > 0$, which are independent of α and j, such that

$$\|D^\alpha\varphi_j\|_{L^s} \leq NH^\alpha M_\alpha, \quad \alpha \in \mathbf{N}_0^n.$$

This together with (2.33) and (2.30) shows that the series on the right of (2.34) converges uniformly in j. This in turn yields $|\langle V, \varphi_j\rangle| \to 0$ as $j \to \infty$, since

$\varphi_j \to 0$ in $\mathcal{D}(\{M_p\}, L^s)$ as $j \to \infty$, i.e. V is continuous on $\mathcal{D}(\{M_p\}, L^s)$. The linearity of V on $\mathcal{D}(\{M_p\}, L^s)$ is obvious. consequently, $V \in \mathcal{D}'(\{M_p\}, L^s)$.

We now prove the converse. In Roumieu ([75], p. 43) we put $\mathcal{F} = L^s$ and consider the space $E(L^s, \{M_p\})$ of Roumieu ([75], p. 43). Set

$$\Phi(\{M_p\}, L^s) = \{\{(-1)^\alpha D^\alpha \varphi\}_{\alpha \in \mathbf{N}_0^n} : \ \varphi \in \mathcal{D}(\{M_p\}, L^s)\},$$

where $1 \leq s < \infty$. From the defining properties of $\mathcal{D}(\{M_p\}, L^s)$ we conclude

$$\Phi(\{M_p\}, L^s) \subset E(L^s, \{M_p\}), \quad 1 \leq s < \infty,$$

and the topology of the subspace $\Phi(\{M_p\}, L^s)$ is induced by the topology of $E(L^s, \{M_p\})$. Let $V \in \mathcal{D}'(\{M_p\}, L^s)$. We define now an element $V_1 \in \Phi'(\{M_p\}, L^s)$, corresponding to V, by

$$\langle V_1, \{(-1)^\alpha D^\alpha \varphi\}_{\alpha \in \mathbf{N}_0^n}\rangle = \langle V, \varphi\rangle, \qquad \varphi \in \mathcal{D}(\{M_p\}, L^s). \tag{2.35}$$

By the Hahn - Banach theorem there is an element $V_2 \in E'(L^s, \{M_p\})$ such that $V_1 = V_2$ on $\Phi(\{M_p\}, L^s)$. Thus, by the characterization of $E'(L^s, \{M_p\})$ given in Roumieu ([75], Proposition 3, p. 45), we can find a sequence $\{g_\alpha\}_{\alpha \in \mathbf{N}_0^n}$ such that $g_\alpha \in L^r$ with $1/r + 1/s = 1$ for $\alpha \in \mathbf{N}_0^n$ and (2.30) holds for all $k > 0$ such that

$$\langle V_1, \{(-1)^\alpha D^\alpha \varphi\}_{\alpha \in \mathbf{N}_0^n}\rangle = \sum_{0 \leq \alpha < \infty} \langle g_\alpha, (-1)^\alpha D^\alpha \varphi\rangle \tag{2.36}$$

for $\varphi \in \mathcal{D}(\{M_p\}, L^s)$. Notice that (2.35) and (2.36) yield (2.31). The proof is thus complete. □

As previously noted, a similar characterization result is true for $\mathcal{D}'((M_p), L^s)$ and we now present it. The proof is similar to that of Theorem 2.3.1 and can be found in Pilipović [65] (Theorem 5).

Theorem 2.3.2 *Let $1 < s < \infty$. Let $\{g_\alpha\}_{\alpha \in \mathbf{N}_0^n}$ be a sequence of functions in L^r with $1/r + 1/s = 1$ such that, for some $k > 0$,*

$$\|g_\alpha\|_{L^r} = O\Big(\frac{1}{k^\alpha M_\alpha}\Big) \quad \text{as } \alpha \to \infty. \tag{2.37}$$

Then

$$V = \sum_{0 \leq \alpha < \infty} D^\alpha g_\alpha \tag{2.38}$$

is an element of $\mathcal{D}'((M_p), L^s)$. Conversely, if $V \in \mathcal{D}'((M_p), L^s)$, then V has the form (2.38), where $\{g_\alpha\}_{\alpha \in \mathbf{N}_0^n}$ is a sequence of functions in L^r satisfying (2.37) for some $k > 0$.

Condition (2.37) on the sequence $\{g_\alpha\}$ is equivalent to

$$\sup_\alpha (k^\alpha M_\alpha \|g_\alpha\|_{L^r}) < \infty \tag{2.39}$$

for some $k > 0$. The derivatives in (2.31) and (2.38) are to be taken in the usual ultradistribution sense.

Notice that $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ are not distribution spaces in the sense of Schwartz but are ultradistribution spaces in the sense of Komatsu and Roumieu. These spaces are generalizations of the Schwartz spaces $\mathcal{D}'_{L^s}$. Theorems 2.3.1 and 2.3.2 show, why elements of $\mathcal{D}'_{L^s}$ are finite sums of distributional derivatives of L^r-functions, while the ultradistributions in $\mathcal{D}'(*, L^s)$ are infinite sums of ultradistribution derivatives of L^r-functions which satisfy (2.30) or (2.37).

2.4 Fourier transform on $\mathcal{D}(*, L^s)$ and $\mathcal{D}'(*, L^s)$

We now consider the Fourier transform acting on $\mathcal{D}(*, L^s)$ and study the resulting spaces. Using this analysis we are able to define an inverse Fourier transform on the dual spaces of these Fourier transform spaces which will map the dual spaces to $\mathcal{D}'(*, L^s)$. We use these results in some of our ultradistributional boundary value analysis presented later.

Consider the spaces $\mathcal{D}(*, L^r)$, $1 \le r \le 2$, where $*$ is either (M_p) or $\{M_p\}$. Put

$$\mathcal{FD}(*, L^r) = \{\psi\colon \ \psi = \hat{\varphi}, \quad \varphi \in \mathcal{D}(*, L^r)\}, \quad 1 \le r \le 2.$$

We have

$$\mathcal{FD}(*, L^r) \subset L^s, \quad 1/r + 1/s = 1,$$

and the Fourier transform is a one to one mapping of $\mathcal{D}(*, L^r)$ onto $\mathcal{FD}(*, L^r)$. To determine a topology on $\mathcal{FD}(*, L^r)$ let $\varphi \in \mathcal{D}(*, L^r)$ and recall from the Fourier transform theory that

$$\mathcal{F}[D^\alpha \varphi] = \chi^\alpha \hat{\varphi} \in L^s$$

with $1/r + 1/s = 1$ for any n-typle α of nonnegative integers.

For an arbitrary $\varphi \in \mathcal{D}((M_p), L^r)$ (respectively, $\varphi \in \mathcal{D}(\{M_p\}, L^r)$) and $\psi = \hat{\varphi} \in \mathcal{FD}((M_p), L^r)$, (respectively, $\psi = \hat{\varphi} \in \mathcal{D}(\{M_p\}, L^r))))$, we have

$$\sup_\alpha \frac{\|\chi^\alpha \psi\|_{L^s}}{h^\alpha M_\alpha} = \sup_\alpha \frac{\|\mathcal{F}[D^\alpha \varphi]\|_{L^s}}{h^\alpha M_\alpha} \le \sup_\alpha \frac{\|D^\alpha \varphi\|_{L^r}}{h^\alpha M_\alpha} < \infty \tag{2.40}$$

for all (resp. for some) $h > 0$, in view of the Parseval inequality and (2.14).

On the space $\mathcal{FD}((M_p), L^r)$ (resp. on the space $(\mathcal{FD}(\{M_p\}, L^r))$) define the family $\{\tau_h^s\}_{h>0}$ of norms as follows:

$$\tau_h^s(\psi) = \sup_\alpha \frac{\|\chi^\alpha \psi\|_{L^s}}{h^\alpha M_\alpha} \tag{2.41}$$

for $\psi \in \mathcal{FD}((M_p), L^r)$ (resp. for $(\in \mathcal{FD}(\{M_p\}, L^r))$. We endow the space $\mathcal{FD}((M_p), L^r)$ (resp. the space $(\mathcal{FD}(\{M_p\}, L^r)))$ with the projective (resp. inductive) limit topology with respect to this family of norms.

A net of elements ψ_λ in $\mathcal{FD}((M_p), L^r)$ (resp., $\mathcal{FD}(\{M_p\}, L^r)$) converges to zero as $\lambda \to \infty$ in this topology in $\mathcal{FD}((M_p), L^r)$ (resp. in $\mathcal{FD}(\{M_p\}, L^r)$) if

$$\lim_{\lambda \to \infty} \|\chi^\alpha \psi_\lambda\|_{L^s} = 0$$

for all n-tuples α of nonnegative integers and for every $h > 0$ there is a constant $N > 0$ which is independent of α and λ (resp. there are constants $N > 0$ and $h > 0$ which are independent of α and λ) such that

$$\sup_{\alpha} \frac{\|\chi^{\alpha}\psi_{\lambda}\|_{L^s}}{h^{\alpha}M_{\alpha}} \leq N$$

for all $h > 0$ (resp. for the given $h > 0$).

Using this meaning of convergence, we have the following lemma.

Lemma 2.4.1 *The Fourier transform is an isomorphism from $\mathcal{D}(*, L^r)$ onto $\mathcal{FD}(*, L^r)$ for $1 \leq r \leq 2$.*

Proof. We have previously noted that the Fourier transform is a one to one mapping of $\mathcal{D}(*, L^r)$ onto $\mathcal{FD}(*, L^r)$, $1 \leq r \leq 2$. Now let (φ_λ) be a net in $\mathcal{D}(*, L^r)$ which converges to zero in $\mathcal{D}(*, L^r)$ as $\lambda \to \infty$. Since

$$\mathcal{F}[D^{\alpha}\varphi] = (\chi)^{\alpha}\hat{\varphi} \in L^s,$$

with $1/r + 1/s = 1$, for every $\varphi \in \mathcal{D}(*, L^r)$ and $\alpha \in \mathbf{N}_0$ we conclude from the Parseval inequality, (2.40), and the definition of convergence in $\mathcal{D}(*, L^r)$, given in Section 2.3, that $\psi_\lambda = \hat{\varphi}_\lambda$ converges to zero in $\mathcal{FD}(*, L^r)$ as $\lambda \to \infty$. Consequently, the Fourier transform is a continuous mapping from $\mathcal{D}(*, L^r)$ to $\mathcal{FD}(*, L^r)$. The proof is complete. □

Let $\mathcal{F}'\mathcal{D}(*, L^r)$ for $1 \leq r \leq 2$ denote the space of all continuous linear forms on $\mathcal{FD}(*, L^r)$. We now define the inverse Fourier transform on the space $\mathcal{F}'\mathcal{D}(*, \mathrm{L}^r)$ in case $1 \leq r \leq 2$. For $V \in \mathcal{F}'\mathcal{D}(*, L^r)$, we define the inverse Fourier transform $v = \mathcal{F}^{-1}[V]$ by the Parseval formula

$$\langle v, \varphi\rangle = \langle V, \tilde{\psi}\rangle, \qquad \varphi \in \mathcal{D}(*, L^r), \quad \psi = \hat{\varphi} \in \mathcal{FD}(*, L^r), \tag{2.42}$$

where $\tilde{\psi}(x) = \psi(-x)$ for $x \in \mathbf{R}^n$. For $V \in \mathcal{F}'\mathcal{D}(*, L^r)$, we have $v = \mathcal{F}^{-1}[V] \in \mathcal{D}'(*, L^r)$, i.e. v is a continuous linear form on $\mathcal{D}(*, L^r)$. Linearity is obvious and continuity of v on $\mathcal{D}(*, L^r)$ follows, because the convergence of a net (φ_λ) to zero in $\mathcal{D}(*, L^r)$ implies the convergence of the net $(\tilde{\psi}_\lambda)$ to zero in $\mathcal{FD}(*, L^r)$, according to inequality (2.40). In this way, we have proved the following assertion:

Lemma 2.4.2 *The inverse Fourier transform defined on $\mathcal{F}'\mathcal{D}(*, L^r)$ by formula (2.42) maps $\mathcal{F}'\mathcal{D}(*, L^r)$ to $\mathcal{D}'(*, L^r)$ for $1 \leq r \leq 2$.*

We shall use the construction (2.42) in boundary value results subsequently.

2.5 Ultradifferentiable functions of ultrapolynomial growth

We assume that $(M.1)$ and $(M.3)'$ hold. The spaces of Gelfand-Shilov type s whose elements are ultradifferential functions of ultrapolynomial growth are the test spaces for spaces of tempered ultradistributions. These spaces are studied in [40], [68], [53], [27], [28], and many other papers. Here we follow the preprint [45]. Let $m > 0$ and $r \in [1, \infty)$ be given.

Let $\mathcal{S}_r^{(M_p),m} = \mathcal{S}_r^{(M_p),m}(\mathbf{R}^n)$ and $\mathcal{S}_\infty^{(M_p),m} = \mathcal{S}_\infty^{(M_p),m}(\mathbf{R}^n)$ be the spaces of smooth functions φ on $\mathbf{R}^n$ such that

$$\sigma_{m,r}(\varphi) = \left[\sum_{\alpha,\beta\in\mathbf{N}_0^n} \int_{\mathbf{R}^n} \mid \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} < x >^\beta \varphi^{(\alpha)}(x) \mid^r dx\right]^{1/r}$$

$$= \left[\sum_{\alpha,\beta\in\mathbf{N}_0^n} \left(\|\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} < \chi >^\beta \varphi^{(\alpha)}\|_r\right)^r\right]^{1/r} < \infty,$$

and

$$\sigma_{m,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0^n} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \| < x >^\beta \varphi^{(\alpha)} \|_\infty,$$

equipped with the topologies induced by the norms $\sigma_{m,r}$ and $\sigma_{m,\infty}$, respectively.

The space $S_r^{(M_p),m}$ is a Banach space and especially, $S_2^{(M_p),m}$ is a Hilbert space where the scalar product is defined by

$$(\varphi, \psi) = \sum_{\alpha,\beta\in N_0} \int_{\mathbf{R}^n} \left(\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\right)^2 \langle x\rangle^{2\beta} \varphi^{(\alpha)}(x)\overline{\psi^{(\alpha)}}(x)\, dx,$$

for $\varphi, \psi \in S_2^{(M_p),m}$.

Let $\mathcal{S}^{(M_p)} = \mathcal{S}^{(M_p)}(\mathbf{R}^n)$ and $\mathcal{S}^{\{M_p\}} = \mathcal{S}^{\{M_p\}}(\mathbf{R}^n)$ be the projective (as $m \to \infty$) and the inductive (as $m \to 0$) limits of the spaces $\mathcal{S}_2^{(M_p),m}$, respectively.

The dual spaces of $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are denoted by $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$, respectively. These are the spaces of tempered ultradistributions of Beurling and Roumieu type, respectively.

The structure of the test spaces is described in the following two theorems. A simple consequence will be, if $(M.2)'$ is fullfiled, that $\mathcal{S}^{(M_p)})$ and $\mathcal{S}^{\{M_p\}}$ are the projective (as $m \to \infty$) and the inductive (as $m \to 0$) limits not only of the spaces of the spaces $\mathcal{S}_2^{(M_p),m}$ but also of the spaces $\mathcal{S}_r^{(M_p),m}$ respectively, where $r \in [1, \infty]$.

In the theorem below and further on we shall use the convention, analogous to that applied already in (2.21), which will simplify the notation of products of subsequent elements of sequences belonging to the family $\mathcal{R}$ described in section . Namely for a given sequence $(r_p) \in \mathcal{R}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0$ we define

$$R_\infty = \prod_{j=1}^{\infty} r_j, \qquad R_\alpha = \prod_{j=1}^{\bar\alpha} r_j$$

in case $\alpha \neq 0$, where $\bar\alpha = \alpha_1 + \ldots + \alpha_n$, and, additionally, $R_0 = 1$. Analogous products of subsequent elements of sequences $(s_j), (a_j), (b_j)$ etc. in $\mathcal{R}$ will be denoted by S_α, A_α, B_α etc., respectively, for a given $_\alpha \in \mathbf{N}_0$.

Theorem 2.5.1 *Let (M_p) satisfy $(M.1)$ and $(M.3)'$. Then*

$$\mathcal{S}^{\{M_p\}} = \text{proj } \lim_{(r_j),(s_j)\in\mathcal{R}} \mathcal{S}^{(M_p)}_{(r_j),(s_j)},$$

where $\mathcal{S}^{(M_p)}_{(r_j),(s_j)}$ is the space of functions $\varphi \in C^\infty$ such that

$$\gamma_{(r_j),(s_j)}(\varphi) = \sup\{\frac{\|(1+|\chi|^2)^{\beta/2}\partial^\alpha \varphi\|_{L^2}}{M_\alpha R_\alpha M_\beta S_\beta}: \ \alpha,\beta \in \mathbf{N}_0^n\} < \infty.$$

Proof. From Lemma 2.2.1 it follows that $\varphi \in C^\infty(\mathbf{R}^n)$ belongs to $\mathcal{S}^{\{M_p\}}$ if and only if $\gamma_{(r_j),(s_j)}(\varphi) < \infty$ for every $(r_j),(s_j) \in \mathcal{R}$.

Every norm $\gamma_{(r_j),(s_j)}$, where $(r_j),(s_j) \in \mathcal{R}$, is continuous on the space $\mathcal{S}^{(M_p)}_h$, $h > 0$, and so on the space $\mathcal{S}^{\{M_p\}}$.

Since $\mathcal{S}^{\{M_p\}}$ is reflexive, every continuous seminorm p is bounded by the seminorm p^B, where B is a bounded set in $\mathcal{S}'^{\{M_p\}}$, defined by

$$p^B(\varphi) = \sup\{| < f,\varphi > |: \ f \in B\}.$$

We have

$$p^B(\varphi) \le \sup_{f\in B} \sum_{\alpha\beta\in\mathbf{N}_0^n} \|(1+|\chi|^2)^{\beta/2} D^\alpha \varphi\|_{L^2}.$$

By Lemma 2.2.1, it follows that there exist (r_j) and (s_j) from $\mathcal{R}$ such that for some $C > 0$ we have

$$p^B(\varphi) \le C\gamma_{(r_j),(s_j)}(\varphi).$$

The proof is completed. □

Let $(a_p),(b_p) \in \mathcal{R}$ and let $\mathcal{S}^{(M_p)}_{(a_p),(b_p),\infty}$ be the space of smooth functions φ on $\mathbf{R}^n$ such that

$$\wp_{a_p,b_p,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0^n} \frac{\|\langle\chi\rangle^\beta \varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha \, M_\beta B_\beta,} < \infty,$$

equipped with the topology induced by the norm $\wp_{(a_p),(b_p),\infty}$.

Theorem 2.5.2 *The following families of norms in the space $\mathcal{S}^{(M_p)}_{(a_p),(b_p),\infty}$ are equivalent:*

1. The families $\{\sigma_{m,\infty}, m > 0\}$ (respectively, $\{\wp_{(a_p),(b_p),\infty}, (a_p),(b_p) \in \mathcal{R}\}$) and $\{s_{m,\infty}, m > 0\}$ (respectively, $\{S_{(a_p),(b_p),\infty}, (a_p),(b_p) \in \mathcal{R}\}$) of norms, where

$$s_{m,\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0^n} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\chi^\beta \varphi^{(\alpha)}\|_\infty,$$

$$\Big(\textit{respectively,} \ \ S_{(a_p),(b_p),\infty}(\varphi) = \sup_{\alpha,\beta\in\mathbf{N}_0^n} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_\infty}{A_\alpha M_\alpha B_\beta M_\beta}\Big).$$

are equivalent.

2. If condition $(M.2)'$ is satisfied, then the following families of norms $\{\sigma_{m,r}: \ m > 0\}$, $r \in [1,\infty]$, $\{s_{m,r}; m > 0\}$, $r \in [1,\infty]$, and $\{\varsigma_m; \ m > 0\}$ (

respectively, $\{\wp_{(a_p),(b_p),r} : (a_p),(b_p) \in \mathcal{R}\}$, $\{S_{(a_p),(b_p),r}, (a_p),(b_p) \in \mathcal{R}\}$, $r \in [1,\infty]$, *and* $\{\Lambda_{(a_p),(b_p),r}, (a_p),(b_p) \in \mathcal{R}\}, r \in [1,\infty]$) *are mutually equivalent, where*

$$s_{m,p}(\varphi) = \sum_{\alpha,\beta \in \mathbf{N}_0^n} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\chi^\beta \varphi^{(\alpha)}\|_p,$$

$$\varsigma_m(\varphi) = \sup_{\alpha \in \mathbf{N}_0^n} \frac{m^\alpha}{M_\alpha} \|\varphi^{(\alpha)} \exp[M(m|\cdot|)]\|_\infty,$$

$$\left(\textit{respectively,} \quad \wp_{(a_p),(b_p),r}(\varphi) = \sum_{\alpha,\beta \in \mathbf{N}_0^n} \frac{\|\langle\chi\rangle^\beta \varphi^{(\alpha)}\|_r}{A_\alpha M_\alpha B_\beta M_\beta},\right.$$

$$S_{(a_p),(b_p),r}(\varphi) = \sum_{\alpha,\beta \in \mathbf{N}_0^n} \frac{\|\chi^\beta \varphi^{(\alpha)}\|_r}{A_\infty M_\alpha B_\infty M_\beta},$$

$$\left.\Lambda_{(a_p),(b_p),r}(\varphi) = \sup_{\alpha \in \mathbf{N}_0^n} \frac{1}{A_\alpha M_\alpha} \|\varphi^{(\alpha)} \exp[N_{(b_p)}(|\cdot|)]\|_r\right).$$

Proof. For the sake of simplicity we will prove the assertions in the case $n = 1$. Parts of respective assertions given in parentheses can be proved in a similar way.

1. Obviously, $s_{m,\infty}(\varphi) \le \sigma_{m,\infty}(\varphi)$ for every smooth function φ and $m > 0$. Condition $(M.3)'$ implies that, for every $L > 0$,

$$\frac{L^k k!}{M_k} \longrightarrow 0 \text{ as } k \to \infty \tag{2.43}$$

(see [48], (4.6)). Since

$$\langle x\rangle^\beta \le 2^{\beta/2} \max\left(1, |x|^\beta\right), \qquad x \in \mathbf{R},\ \beta \in \mathbf{N}_0,$$

for every $m > 0$ there exists a $C > 0$ such that, for every smooth function φ and $\alpha, \beta \in \mathbf{N}_0$, we have

$$\begin{aligned}
\frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\langle\chi\rangle^\beta \varphi^{(\alpha)}\|_\infty &\le \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} 2^\beta \max\left(\|\varphi^{(\alpha)}\|_\infty, \|\chi^\beta \varphi^{(\alpha)}\|_\infty\right) \\
&\le \max\left(C\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_\infty, \frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta}\|\chi^\beta \varphi^{(\alpha)}\|_\infty\right) \\
&\le \le C \sup_{\beta \in \mathbf{N}_0} \frac{(2m)^{\alpha+\beta}}{M_\alpha M_\beta} \|\chi^\beta \varphi^\alpha\|_\infty = C s_{m,\infty}(\varphi).
\end{aligned}$$

Therefore, for every $m > 0$ there exists a $C > 0$ such that $\sigma_{m,\infty}(\varphi) \le C s_{m,\infty}(\varphi)$ for every smooth function φ.

2. For a given function ψ denote by $\|\psi\|'_t$, $\|\psi\|''_t$ and $\|\psi\|_t$ its L^t-norms on the sets $[-1,1]$, $\mathbf{R}\setminus[-1,1]$ and $\mathbf{R}$, respectively.

Let $\alpha,\beta,\gamma\in\mathbf{N}_0$ and $t\in[1,\infty)$. For a given smooth function φ denote

$$a_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|'_\infty;\quad b_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|''_\infty;\quad c_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|_\infty;$$

$$A^t_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|'_t;\quad B^t_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|''_t;\quad C^t_{\alpha,\beta}(\varphi)=\|\chi^\beta\varphi^{(\alpha)}\|_t.$$

Moreover, denote by $I_{\gamma,t}$ the L^t-norm of the function $\tau(x)=x^{-\gamma}$ on $\mathbf{R}\setminus[-1,1]$ and $\mathbf{R}$.

Due to $(M.2)'$, for every $m>0$ there exists a constant $D>0$ such that

$$\begin{aligned}
s_{m,t}(\varphi) &\le \sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(a_{\alpha,\beta}(\varphi)+b_{\alpha,\beta+\gamma}(\varphi)I_{\gamma,1}\right)\\
&\le \sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}c_{\alpha,\beta}(\varphi)+D\sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}}c_{\alpha,\beta+\gamma}(\varphi)\\
&\le Ds_{m(1+H^\gamma),\infty}(\varphi)
\end{aligned}\tag{2.44}$$

for every smooth function φ.

Clearly,

$$|x^\beta\varphi^{(\alpha)}(x)|\le\beta C_{\alpha,\beta,1}(\varphi)+C_{\alpha+1,\beta,1}(\varphi)$$

for α, $\beta\in\mathbf{N}_0$ and $x\in\mathbf{R}$. Hence, by condition $(M.2)'$, for every $m>0$ there exists a $D>0$ such that

$$\begin{aligned}
s_{m,\infty}(\varphi) &\le D\sup_{\alpha,\beta\in\mathbf{N}_0}\frac{H^\alpha m^{\alpha+\beta}}{M_{\alpha+1}M_\beta}(2^\beta C_{\alpha,\beta,1}(\varphi)+Hm^{\alpha+2}C_{\alpha+1,\beta,1}(\varphi))\\
&\le Ds_{2m(1+H),1}(\varphi)
\end{aligned}\tag{2.45}$$

for every smooth function φ.

Let now $t\in(1,\infty)$, $q=t/(t-1)$ and $\gamma=[1/q]+1$. The Hölder inequality, (2.43) and $(M.2)'$ imply that for every $m>0$ there exists $D>0$ such that, for every smooth function φ,

$$\begin{aligned}
s_{m,1}(\varphi) &= \sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(A_{\alpha,0,1}(\varphi)+B_{\alpha,\beta,1}(\varphi)\right)\\
&\le \sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left[DA_{\alpha,0,t}(\varphi)+B_{\alpha,\beta+\gamma,t}(\varphi)I_{\gamma,q}\right]\\
&\le D\sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\left(\|\varphi^{(\alpha)}\|_{L^t}+\|\chi^{\beta+\gamma}\varphi^{(\alpha)}\|_{L^t}\right)\\
&\le D\left(\sum_{\alpha\in\mathbf{N}_0}\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_{L^t}+\sum_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}H^{\gamma\beta}}{M_\alpha M_{\beta+\gamma}}\|\chi^{\beta+\gamma}\varphi^{(\alpha)}\|_{L^t}\right)\\
&\le Cs_{m(1+H^\gamma),t}(\varphi).
\end{aligned}\tag{2.46}$$

The equivalence of the families $\{s_{m,r},\ m>0\}$ and $\{s_{m,p},\ m>0\}$ for $r,p\in[1,\infty]$ follows from (2.44), (2.45) and (2.46). The proof of the equivalence of $\{\sigma_{m,p}, m>0\}$ and $\{\sigma_{m,r}, m>0\}$, where $r,p\in[1,\infty]$, is analogous.

Condition $(M.2)'$ implies that for every $\varphi\in\mathcal{S}^{(M_p)}$ and every $m>0$ there exists $D>0$ such that for every $\alpha,\beta\in\mathbf{N}_0$ and $|x|>k>1$, we have

$$\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\,|x^\beta\varphi^{(\alpha)}(x)|\le D\,\frac{m^\alpha(mH)^{\beta+1}}{M_\alpha M_{\beta+1}}\,|x^\beta\varphi^{(\alpha)}(x)|$$
$$\le\frac{D}{k}\,\frac{m^\alpha(mH)^{\beta+1}}{M_\alpha M_{\beta+1}}\,|x^{\beta+1}\varphi^{(\alpha)}(x)|\ \le\ \frac{C}{k}.$$

Therefore, for every $m>0$ and $\varphi\in\mathcal{S}^{(M_p)}$, $(m^{\alpha+\beta}/(M_\alpha M_\beta))\,|x^\beta\varphi^{(\alpha)}(x)|$ converges to zero as $|x|$ tends to infinity, uniformly in $\alpha,\beta\in\mathbf{N}_0$. From the definition of the space $\mathcal{S}^{(M_p)}$ it follows that $\{(m^{\alpha+\beta}/(M_\alpha M_\beta))\,|x^\beta\varphi^{(\alpha)}(x)|\}_{\alpha,\beta}$, $m>0$, converges to zero uniformly in $x\in\mathbf{R}$ as $(\alpha+\beta)$ tends to infinity. Hence, for a given element φ of $\mathcal{S}^{(M_p)}$ and every $m>0$ there are $\alpha_0,\beta_0\in\mathbf{N}_0$ and $x_0\in\mathbf{R}$ such that

$$\sup_{\alpha,\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|\chi^\beta\varphi^{(\alpha)}\|_\infty=\frac{m^{\alpha_0+\beta_0}}{M_{\beta_0}M_{\alpha_0}}|x_0^{\beta_0}\varphi^{(\alpha_0)}(x_0)|$$
$$=\|\sup_{\beta\in\mathbf{N}_0}\left[\sup_{\alpha\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|\chi^\beta\varphi^{(\alpha)}|\right]\|_\infty=\|\sup_{\alpha\in\mathbf{N}_0}\left[\sup_{\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|\chi^\beta\varphi^{(\alpha)}|\right]\|_\infty$$
$$=\sup_{\alpha\in\mathbf{N}_0}\left[\|\sup_{\beta\in\mathbf{N}_0}\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}|\chi^\beta\varphi^{(\alpha)}|\|_\infty\right]=\sup_{\alpha\in\mathbf{N}_0}(\frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\exp[M(m|\cdot|)]\|_\infty).$$

The proof is completed. □

Remark It is easy to verify that the proofs of the theorems of this section hold in the n-dimensional case. In Particular, if $(M.2)$ holds, they can be presented in the same way.

Corollary 2.5.1 $\mathcal{S}^{\{M_p\}}=\text{proj}\lim_{(a_p),(b_p)\in\mathcal{R}}\in\mathcal{S}^{(M_p)}_{(a_p),(b_p);\infty}$.

Theorem 2.5.3 *1. The spaces $\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are $(F\tilde{S})$ and (LS)- spaces respectively.*

2. If $(M.2)'$ is fulfilled then

$$\mathcal{D}^*\hookrightarrow\mathcal{S}^*\hookrightarrow\mathcal{E}^*,\quad \mathcal{S}^*\hookrightarrow\mathcal{S}.$$
$$\mathcal{E}'^*\hookrightarrow\mathcal{S}'^*\hookrightarrow\mathcal{D}'^*,\quad \mathcal{S}'\hookrightarrow\mathcal{S}'^*.$$

Proof. Again, we give the proof for the case $n=1$.

Recall that a locally convex topological vector space is an (FS)-space (resp. an (LS)-space) if it is a projective limit (resp. an inductive limit) of a countable, compact specter of spaces. If the mentioned specter is also nuclear, the space is called an (FN)-space (resp. an (LN)-space); for more detailes see [36].

1. In order to prove the first part of the assertion it is enough to show that the inclusion mapping

$$i: \mathcal{S}_2^{(M_p),\tilde{m}} \to \mathcal{S}_2^{(M_p),m}, \quad m < \tilde{m},$$

is compact. Since $\mathcal{S}_2^{(M_p),\tilde{m}}$ and $\mathcal{S}_2^{(M_p),m}$ are Banach spaces, it safices to prove that the unit ball B of the space $\mathcal{S}_2^{(M_p),\tilde{m}}$ is a relatively compact set in $\mathcal{S}_2^{(M_p),m}$. Using the analogous idea as in the proof of Theorem 1 in [36] (p. 29, Satz 1) one can prove the next assertion.

A set B is relatively compact in $\mathcal{S}_2^{(M_p),m}$ if and only if

(i) the set $B_\beta^\alpha = \{\langle x\rangle^\beta \varphi^{(\alpha)},\ \varphi \in B\}$ is a relatively compact subset of L^2 for each $\alpha, \beta \in \mathbf{N}_0$ and

(ii) the series $\sum\limits_{\alpha,\beta\in\mathbf{N}_0} \int\limits_{\mathbf{R}} \Big|\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\langle x\rangle^\beta \varphi^{(\alpha)}(x)\Big|^2 dx$ converges uniformly for $\varphi \in B$.

Let us prove that B fulfills (i) by checking whether the set B_β^α, $\alpha, \beta \in \mathbf{N}_0$, fulfills the assumptions of Kolmogoroff's theorem ([36]). It is obvious that for each $\alpha, \beta \in \mathbf{N}_0$ the set $B_\beta^\alpha = \{\langle x\rangle^\beta \varphi^{(\alpha)},\ \varphi \in B\}$ is bounded in the space L^2.

Applying the Cauchy-Schwarz inequality and the Fubini-Tonelli theorem we see that, for arbitrary $\varphi \in B$ and $\alpha, \beta \in \mathbf{N}_0$,

$$\int\limits_{\mathbf{R}} \Big|\langle x+h\rangle^\beta \varphi^{(\alpha)}(x+h) - \langle x\rangle^\beta \varphi^{(\alpha)}(x)\Big|^2 dx$$

$$\leq \int\limits_{\mathbf{R}} \Big(\int\limits_0^1 \Big|\frac{d}{dt}(\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th))\Big|\, dt\Big)^2 dx$$

$$\leq \int\limits_{\mathbf{R}} \Big(\int\limits_0^1 \Big|\frac{d}{dt}[\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th)]\Big|^2 dt\Big) dx$$

$$\leq 2\beta^2 h^2 \int\limits_0^1 \Big(\int\limits_{\mathbf{R}} \Big|(\langle x+th\rangle^\beta \varphi^{(\alpha)}(x+th))\Big|^2 dx\Big) dt$$

$$+2h^2 \int\limits_0^1 \Big(\int\limits_{\mathbf{R}} \Big|(\langle x+th\rangle^\beta \varphi^{(\alpha+1)}(x+th))\Big|^2 dx\Big) dt$$

$$\leq 2\beta^2 h^2 \Big(\int\limits_{\mathbf{R}} \Big|\langle\xi\rangle^\beta \varphi^{(\alpha)}(\xi)\Big|^2 d\xi\Big) + h^2 \Big(\int\limits_{\mathbf{R}} \Big|\langle\xi\rangle^\beta \varphi^{(\alpha+1)}(\xi)\Big|^2 d\xi\Big)$$

$$\leq 2h^2 \Big(\beta^2 2 \frac{M_\alpha M_\beta}{\tilde{m}^\alpha + \beta} + 2\frac{M_{\alpha+1} M_\beta}{\tilde{m}^\alpha + \beta + 1}\Big).$$

Hence, the integral

$$\int\limits_{\mathbf{R}} |\langle x+h\rangle^\beta \varphi^{(\alpha)}(x+h) - \langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2 dx$$

converges to zero as $h \to 0$ uniformly for $\varphi \in B$.

For each $\varphi \in B$ and $k > 0$

$$\langle k\rangle^2 \int\limits_{\mathbf{R}\setminus[-k,k]} |\langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2\,dx \le \int\limits_{\mathbf{R}\setminus[-k,k]} |\langle x\rangle^{\beta+1}\varphi^{(\alpha)}(x)|^2\,dx \le \frac{M_\alpha M_{\beta+1}}{\tilde{m}^{\alpha+\beta+1}}.$$

Therefore

$$\int\limits_{\mathbf{R}\setminus[-k,k]} |\langle x\rangle^\beta \varphi^{(\alpha)}(x)|^2\,dx \le \langle k\rangle^{-2}\frac{M_\alpha M_{\beta+1}}{\tilde{m}^{\alpha+\beta+1}}, \qquad \varphi \in B.$$

According to the theorem of Kolmogoroff, it follows that the set B_α^β, $\alpha, \beta \in \mathbf{N}_0$, is relatively compact in L^2.

Let us prove that B fulfills condition (ii). For each $\varepsilon > 0$ there exists $\mu \in \mathbf{N}_0$ such that $m^\alpha \le \varepsilon \tilde{m}^\alpha$ for all $\alpha \ge \mu$. Hence, for each $\varphi \in B$

$$\sum_{\substack{\alpha\ge\mu\\ \beta\in\mathbf{N}_0}} \int\limits_{\mathbf{R}} \left|\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\langle x\rangle^\beta \varphi^{(\alpha)}(x)\right|^2 dx \le$$

$$\le \varepsilon^2 \sum_{\substack{\alpha\ge\mu\\ \beta\in\mathbf{N}_0}} \int\limits_{\mathbf{R}} \left|\frac{\tilde{m}^{\alpha+\beta}}{M_\alpha M_\beta}\langle x\rangle^\beta \varphi^{(\alpha)}(x)\right|^2 dx \le \varepsilon^2.$$

Thus we have completed the proof of assertion 1.

2. Since the proofs of the second assertion in the cases (M_p) and $\{M_p\}$ are analogous, we will prove the assertion only in the first case. Let $\varphi \in \mathcal{D}^{(M_p)}$ and $\mathrm{supp}\varphi \subset [-k,k]$, $k > 1$. Condition $(M.3)'$ implies that for each $m > 0$ there exists $C > 0$ such that

$$\sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\|\langle x\rangle^\beta \varphi^{(\alpha)}\|_\infty = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{(mk)^\beta m^\alpha}{M_\beta M_\alpha}\|\varphi^{(\alpha)}\|_\infty \le C \sup_{\alpha\in\mathbf{N}_0} \frac{m^\alpha}{M_\alpha}\|\varphi^{(\alpha)}\|_\infty.$$

It follows that the inclusion mapping $i : \mathcal{D}^{(M_p)} \to \mathcal{S}^{(M_p)}$ is continuous.

The sequence $\varphi_j(x) = \rho(x/j)\rho(x)$, $j \in \mathbf{N}$, where ρ is a function defined by (1.2) converges to φ in the space $\mathcal{S}^{(M_\alpha)}$, since for fixed $\varphi \in \mathcal{S}^{(M_p)}$ and $m > 0$, $\dfrac{m^{\alpha+\beta}}{M_\alpha M_\beta}|x^\beta \varphi^{(\alpha)}(x)|$ converges uniformly in $\alpha, \beta \in \mathbf{N}_0$ as $|x|$ tends to infinity. It follows that $\mathcal{D}^{(M_p)}$ is dense in $\mathcal{S}^{(M_p)}$. □

2.6 Tempered ultradistributions

A non-trivial example, in case $n = 1$, of an element of the space $\mathcal{S}'^*$ is

$$\langle f, \varphi\rangle = \int\limits_{\mathbf{R}} f\varphi\,dx, \qquad \varphi \in \mathcal{S}^*,$$

where f is a locally integrable function of the ultrapolynomial growth of the class $*$, i.e.

$$|f(x)| \le P(x), \qquad x \in \mathbf{R}.$$

where P is an ultrapolynomial of the class $*$. Note that if $(M.2)'$ is fulfilled the function f is of the ultrapolynomial growth of the class (M_α) (respectively, $\{M_p\}$) if and only if for some $m > 0$ and some $C > 0$ (respectively, for every $m > 0$ there exists $C > 0$) such that

$$|f(x)| \le C \exp M(m|x|), \qquad x \in \mathbf{R}.$$

Let us now give the structure theorems for the space $\mathcal{S}'^*$.

Theorem 2.6.1 *Let $(M.2)'$ hold, $r \in (1,\infty]$, and $f \in \mathcal{D}'^{(M_p)}$ (resp., $f \in \mathcal{D}'^{\{M_p\}}$). Then*

1. $f \in \mathcal{S}'^{(M_p)}$ (resp. $f \in \mathcal{S}^{\{M_p\}}$) if and only if f is of the form

$$f = \sum_{\alpha,\beta \in \mathbf{N}_0} (\langle x\rangle^\beta F_{\alpha,\beta})^{(\alpha)}, \tag{2.47}$$

in the sense of convergence in $\mathcal{S}'^{(M_p)}$ (resp. $\mathcal{S}'^{\{M_p\}}$), where $(F_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}_0}$ is a sequence of elements of L^r such that for some (resp. each) $m > 0$ we have

$$\Big(\sum_{\alpha,\beta\in\mathbf{N}_0} \int_{\mathbf{R}} \Big| \frac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x)\Big|^r\Big)^{1/r} < \infty, \tag{2.48}$$

in case $r \in (1,\infty)$, and

$$\sup_{\substack{\alpha,\beta\in\mathbf{N}_0\\ x\in\mathbf{R}}} \Big(\frac{M_\alpha M_\beta}{m^{\alpha+\beta}} |F_{\alpha,\beta}(x)|\Big) < \infty, \tag{2.49}$$

in case $r = \infty$.

2. Let $(M.2)$ and $(M.3)$ be fulfilled. $f \in \mathcal{S}'^*$ if and only if f is of the form

$$f = P(D)F, \tag{2.50}$$

where P is an ultradifferentiable operator of the class $*$ and F is a continuous function of $\mathbf{R}$ of the ultrapolynomial growth of the class $*$.

Note that the weak and the strong sequential convergence are equivalent in $\mathcal{S}'^*$.

Proof. 1. In case (M_p)), the proof of assertion 1 is quite analogous to the proof given in [68].

In case $\{M_p\}$), it follows easily that (2.47) determines an element of $\mathcal{S}'^{\{M_p\}}$. To prove the converse we will use the dual Mittag-Leffler lemma ([48], Lemma 1.4) similarly as in the proof of [48], Proposition 8.6.

Let $X_m = \mathcal{S}_q^{(M_p),m}$, let $Y_m = \{(\varphi_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}_0};\ \|(\varphi_{\alpha,\beta})\|_{Y_m} < \infty\}$, where $q = r/(r-1)$, and let

$$\|(\varphi_{\alpha,\beta})\|_{Y_m} = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \|\varphi_{\alpha,\beta}\|_q.$$

The space Y_m is a reflexive Banach space. According to Alaoglu's theorem, a bounded set in Y_m is weakly compact in Y_m. Therefore the inclusion mapping

$i : Y_{m'} \to Y_m$, $m' > m$ is weakly compact. We will identify X_m with a closed subspace of Y_m in which X_m is mapped by

$$X_m \to Y_m, \quad \langle x\rangle^\beta D^\alpha : \varphi \mapsto (\langle x\rangle^\beta \varphi^{(\alpha)})_{\alpha,\beta}.$$

Clearly (X_m) and (Y_m) are injective sequences of Banach spaces and if $m' > m$, then $X_{m'} \cap Y_m = X_m$. It follows that the quotient space $Z_m = Y_m/X_m$ (with the quotient topology) is also an injective weakly compact sequence of Banach spaces.

It follows from the dual Mittag-Leffler lemma that

$$0 \leftarrow (\lim \mathrm{ind}_{m\to 0} X_m)' \overset{\sum(-1)^\alpha D^\alpha \langle x\rangle^\beta}{\longleftarrow} (\lim \mathrm{ind}_{m\to 0} Y_m)'$$

is topologically exact (see [48]). The above fact together with the identities:

$$\lim \mathrm{proj}_{m\to 0} X'_m = (\lim \mathrm{ind}_{m\to 0} X_m)'$$

and

$$(\lim \mathrm{proj}_{m\to 0} Y'_m) = (\lim \mathrm{ind}_{m\to 0} Y_m)'$$

imply that the space $\lim \mathrm{ind}_{m\to 0} X_m$ has the same strong dual as the closed subspace $\lim \mathrm{ind}_{m\to 0} X_m$ of $\lim \mathrm{ind}_{m\to 0} Y_m$. Since Y'_m is the Banach space of all $F = (F_{\alpha,\beta})$, $F_{\alpha,\beta} \in L^r$, with

$$\|f\|_{Y'_m} = \begin{cases} \Big(\sum\limits_{\alpha,\beta\in\mathbf{N}_0} \int\limits_{\mathbf{R}} \Big| \dfrac{M_\alpha M_\beta}{m^{\alpha+\beta}} F_{\alpha,\beta}(x) \Big|^r \Big)^{1/r} < \infty, & r \in (1,\infty), \\ \sup\limits_{\substack{\alpha,\beta\in\mathbf{N}_0 \\ x\in\mathbf{R}}} \Big(\dfrac{M_\alpha M_\beta}{m^{\alpha+\beta}} |F_{\alpha,\beta}(x)| \Big) < \infty, & r = \infty. \end{cases}$$

The assertion is proved. □

In [45] we characterize spaces $\mathcal{S}^*$ by using Hermite expansions. The following theorem is based on this expansion.

Theorem 2.6.2 *If condition (M.2) is fulfilled, then $\mathcal{S}^{(M_p)}$, $\mathcal{S}'^{\{M_p\}}$ are (FN)-spaces and $\mathcal{S}^{\{M_p\}}$, $\mathcal{S}'^{(M_p)}$ are (LN)-spaces, respectively.*

2.7 Laplace transform

Suppose that conditions $(M.1)$, $(M.2)$ and $(M.3)'$ are fulfilled.

We will give the definition of the Laplace transform in case $n = 1$. For $n > 1$ the definitions can be extended easily.

Denote by $\mathcal{S}'^*_+(\mathbf{R})$ the subspace of $\mathcal{S}'^*$ consisting of elements supported by $[0,\infty)$. Let $g \in \mathcal{S}'^*_+$. For fixed $y > 0$, we define $g\exp(-y\,\cdot)$ as an element of $\mathcal{S}'^*$ by

$$\langle g\exp(-y\,\cdot), \varphi\rangle = \langle g, \varrho\exp(-y\,\cdot)\varphi\rangle, \quad \varphi \in \mathcal{S}^*(\mathbf{R}),$$

where ϱ is an element of $\mathcal{E}^*(\mathbf{R})$ such that, for some $\varepsilon > 0$, $\varrho(x) = 1$ if $x \in (-\varepsilon, \infty)$ and $\varrho(x) = 0$ if $x \in (-\infty, -2\varepsilon)$. It is easy to see that the definition does not depend on the choice of ϱ.

An example of such a function is $\varrho = f * \omega$, where $\omega \in \mathcal{D}^*$, $\int \omega = 1$ and $\operatorname{supp} \omega \subset [-\varepsilon/2, \varepsilon/2]$ (for the existence of such a function see [48], Theorem 4.2) and f is a function such that $f(x) = 1$ for $x \geq -3\varepsilon/2$ and $f(x) = 0$ for $x < -3\varepsilon/2$. Clearly, the function ϱ so defined belongs to $\mathcal{E}^*$.

As in the case of $\mathcal{S}'_+$ (see e.g. [84]), we define now the Laplace transform of $g \in \mathcal{S}'^*_+$ by

$$(\mathcal{L}g)(\zeta) = \mathcal{F}(g \exp(-y \cdot))(x), \quad \zeta = x + iy \in \mathbf{C}_+.$$

Clearly, if $y > 0$ is fixed, $\mathcal{L}g$ is an element of $\mathcal{S}'^*$.

Let

$$G(\zeta) = \langle g, \eta \exp(i\zeta \cdot)\rangle, \quad \zeta = x + iy \in \mathbf{C}_+,$$

where η is as chosen above. The function G is holomorphic on $\mathbf{C}_+$ and its definition does not depend on η.

Chapter 3

Boundedness

3.1 Boundedness in $\mathcal{D}'(*, L^s)$

Denote by $\mathcal{C}_0$ the space of continuous functions f on $\mathbf{R}^n$ vanishing at ∞, i.e. such that $\lim_{|x|\to\infty} f(x) = 0$, equipped with the norm $\|\cdot\|_{L^\infty}$. Its dual space, the space of measures, is denoted by $\mathcal{M}^1$ (as in [42]) and we denote the dual norm in $\mathcal{M}^1$ by $\|\cdot\|_{\mathcal{M}^1}$. Note that under conditions $(M.1)$ and $(M.3)'$ $\mathcal{D}(*, \mathbf{R}^n)$ is dense in $\mathcal{C}_0$.

Theorem 3.1.1 *Let (M_p) satisfy $(M.1)$ and $(M.3)'$. Then*

(i) A set $B \subset \mathcal{D}'((M_p), L^t)$, $t \in (1, \infty]$, is bounded if and only if every $f \in B$ can be represented in the form

$$f = \sum_{\alpha=0}^{\infty} D^\alpha f_\alpha, \qquad f_\alpha \in L^t, \quad \alpha \in \mathbf{N}^n,$$

and, moreover, there exist $d > 0$ and $C > 0$, independent of $f \in B$, such that

$$\sum_{\alpha=0}^{\infty} d^\alpha M_\alpha \|f_\alpha\|_{L^t} < C; \tag{3.1}$$

(ii) A set $B \subset \mathcal{D}'((M_p), L^1)$ is bounded if and only if the representation of f in (i) holds with $f_\alpha \in \mathcal{M}^1$ and the condition in (i) holds with the norm $\|f_\alpha\|_{\mathcal{M}^1}$;

(iii) An element f of $\mathcal{D}'(\{M_p\}, \mathbf{R}^n)$ belongs to $\mathcal{D}'(\{M_p\}, L^t)$ for $t \in [1, \infty]$ if and only if f is of the form

$$f = \sum_{0 \le \alpha < \infty} D^\alpha f_\alpha,$$

where $f_\alpha \in L^t$ if $t \in (1, \infty]$ and $f_\alpha \in \mathcal{M}^1$ if $t = 1$ for $\alpha \in \mathbf{N}_0^n$, moreover, for every $d > 0$, we have

$$\sum_{0 \le \alpha < \infty} d^\alpha M_\alpha \|f_\alpha\|_{L^t} < \infty \quad \text{in case} \quad t \in (1, \infty];$$

$$\sum_{0 \le \alpha < \infty} d^\alpha M_\alpha \|f_\alpha\|_{\mathcal{M}^1} < \infty \quad \text{in case} \quad t = 1.$$

Proof. Clearly, the conditions given in (i) - (iii) are sufficient. We will prove that they are necessary.

(i) Notice that $\mathcal{D}((M_p), L^s)$ with $s = t/(t-1) \in [1, \infty)$ is barrelled , the set B is equicontinuous in $\mathcal{D}'((M_p), L^t)$ and, for some $d > 0$ and $C > 0$,

$$|\langle f, \varphi \rangle| \le C \|\varphi\|_{L^s, d}, \quad f \in B, \quad \varphi \in \mathcal{D}((M_p), L^s).$$

Hence, by the Hahn-Banach theorem, elements of B can be extended to constitute an equicontinuous set B_1 on $\mathcal{D}^{(M_p)}_{L^s, d}$. Let $Y_{s,d}$ be the space of all sequences (φ_α) in L^s such that

$$\|(\varphi_\alpha)\|_{L^s, d} = \sup\{\frac{\|\varphi_\alpha\|_{L^s}}{d^\alpha M_\alpha} : \ \alpha \in \mathbf{N}_0^n\} < \infty$$

equipped with this norm. Again by the Hahn-Banach theorem, elements of B_1 can be extended to constitute an equicontinuous set B_2 on $Y_{s,d}$. An equicontinuous set on $Y_{s,d}$ consists of all sequences (f_α) from L^t for which (3.1) holds and this implies assertion (i).

(ii) Let $X_{\infty,h}$ be the space of all smooth functions φ such that $\varphi^{(\alpha)} \in \mathcal{C}_0$ $\|\varphi\|_{L^\infty, h} < \infty$ for every $\alpha \in \mathbf{N}_0^n$, equipped with the norm $\| \cdot \|_{L^\infty, h}$. We have

$$\dot{\mathcal{B}}((M_p), \mathbf{R}^n) = \operatorname{proj} \lim_{h \to 0} X_{\infty,h},$$

which implies that $\dot{\mathcal{B}}((M_p), \mathbf{R}^n)$ is barrelled. Thus, using the same reasoning as in (i), the proof of (ii) follows.

(iii) Let $Y_{s,h}$ with $s \in [1, \infty]$ and $h > 0$ be the space of all sequences $(\varphi_\alpha) = (\varphi_\alpha)_{\alpha \in \mathbf{N}_0^n}$ in L^s, for $s \in [1, \infty)$, and in $\mathcal{C}_0$, for $s = \infty$, such that

$$\|(\varphi_\alpha)\|_{L^s, h} = \sup\{\frac{\|\varphi_\alpha\|_{L^s}}{h^\alpha M_\alpha} : \ \alpha \in \mathbf{N}_0^n\} < \infty,$$

equipped with the so defined norm.

For a given $h > 0$, let $X_{s,h} = \mathcal{D}((M_p), h, L^s)$, for $s \in [1, \infty)$, and $X_{\infty,h}$ be as in the proof of (ii). We identify $X_{s,h}$ with the corresponding subspace of $Y_{s,h}$, for $s \in [1, \infty]$ and $h > 0$ via the mapping $\varphi \to (\varphi^{(\alpha)})$. Notice that $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n) = \operatorname{ind} \lim_{h \to \infty} X_{s,h}$. According to this identification, we have

$$\mathcal{D}(\{M_p\}, L^s) \subset Y_s = \operatorname{ind} \lim_{h \to \infty} Y_{s,h}, \qquad s \in [1, \infty),$$

and

$$\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n) \subset Y_\infty = \operatorname{ind} \lim_{h \to \infty} Y_{\infty,h}.$$

Since the inclusion mappings are continuous, every continuous linear functional on $\mathcal{D}(\{M_p\}, L^s)$ or on $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n)$ is continuous on this space, equipped with the induced topology from $\operatorname{ind} \lim_{h \to \infty} Y_{s,h}$ for $s \in [1, \infty]$. Thus the Hahn-Banach theorem implies the assertion (iii), because

$$(\operatorname{ind} \lim_{h \to \infty} Y_{s,h})' = \operatorname{proj} \lim_{h \to \infty} Y'_{s,h}, \qquad s \in [1, \infty]$$

in the set theoretical sense. □

Remark. With the notation as in (*iii*) for $s \in (1,\infty)$, the sequence $(Y_{s,h})_{h\in \mathbf{N}^n}$ is weakly compact. This implies that $(X_{s,h})_{h\in \mathbf{N}^n}$ and $(Z_{s,h})_{h\in \mathbf{N}^n}$, where $Z_{s,h} = Y_{s,h}/X_{s,h}$, are weakly compact, as well. Thus the dual Mittag - Leffler lemma (see [48], Lemma 1.4) implies that the sequence

$$O \longleftarrow \operatorname{proj} \lim_{h\to\infty} X'_{s,h} \longleftarrow \operatorname{proj} \lim_{h\to\infty} Y'_{s,h}$$

is exact, where (in the topological sense)

$$\operatorname{proj} \lim_{h\to\infty} X'_{s,h} = X'_s = (\operatorname{ind} \lim_{h\to\infty} X_{s,h})'$$

and

$$\operatorname{proj} \lim_{h\to\infty} Y'_{s,h} = Y'_s = (\operatorname{ind} \lim_{h\to\infty} Y_{s,h})'.$$

This implies that $\mathcal{D}(\{M_p\}, L^s)$ and X_s, equipped with the induced topology from Y_s, have the same strong duals (see [48], Lemma 1.4, (*iii*)). We do not know whether the space X_s with the induced topology is quasi-barrelled and, consequently, we do not have a characterization of bounded sets in $\mathcal{D}'(\{M_p\}, L^t)$ for $t \in (1,\infty)$.

Denote by $\mathcal{D}(\{M_p\},(r_p), L^s)$, with $(r_p) \in \mathcal{R}$ and $s \in [1,\infty]$, the space of all smooth functions φ such that

$$\|\varphi\|_{L^s,(r_p)} = \sup\{\frac{\|\partial^\alpha \varphi\|_{L^s}}{M_\alpha(\prod_{j=1}^{\alpha} r_j)} : \quad \alpha \in \mathbf{N}_0^n\} < \infty,$$

equipped with the norm so defined, and let

$$\tilde{\mathcal{D}}(\{M_p\}, L^s) = \operatorname{proj} \lim_{(r_p)\in\mathcal{R}} \mathcal{D}(\{M_p\},(r_p), L^s).$$

For the completion of $\mathcal{D}(\{M_p\}, \mathbf{R}^n)$ in the space $\tilde{\mathcal{D}}(\{M_p\}, L^\infty)$ we use the symbol $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n)$. The corresponding dual spaces are denoted by $\tilde{\mathcal{D}}'(\{M_p\}, L^t)$ for $t = s/(s-1) \in (1,\infty]$ and by $\tilde{\mathcal{D}}'(\{M_p\}, L^1)$, respectively.

Theorem 3.1.2 *Let (M_p) satisfy $(M.1)$ and $(M.3)'$. Then*

(*i*) *$\tilde{\mathcal{D}}(\{M_p\}, L^s) = \mathcal{D}(\{M_p\}, L^s)$ for $s \in [1,\infty)$ in the set theoretical sense; the same is true for $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n)$ and $\dot{\tilde{\mathcal{B}}}(\{M_p\}, \mathbf{R}^n)$;*

(*ii*) *the inclusion mappings $i : \mathcal{D}(\{M_p\}, L^s) \to \tilde{\mathcal{D}}(\{M_p\}, L^s)$ for $s \in [1,\infty)$ and $i : \dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n) \to \dot{\tilde{\mathcal{B}}}(\{M_p\}, \mathbf{R}^n)$ are continuous.*

Proof. Note that $\dot{\tilde{\mathcal{B}}}(\{M_p\}, \mathbf{R}^n) = \operatorname{proj}\lim_{(r_p)\in\mathcal{R}} X_{\infty,(r_p)}$, where $X_{\infty,(r_p)}$ is the space of all smooth functions φ such that $\varphi^{(\alpha)} \in C_0$ for $\alpha \in \mathbf{N}_0^n$ and $\|\varphi\|_{L^\infty,(r_p)} < \infty$, equipped with this norm.

The proof of (*i*) follows from Lemma 3.4 in [48] and (*ii*) follows from the inequality

$$\|\varphi\|_{L^s,(r_p)} \le C_{(r_p),h} \|\varphi\|_{L^s,h}, \quad \varphi \in \mathcal{D}(\{M_p\}, h, L^s),$$

where $(r_p) \in \mathcal{R}$, $h > 0$ and $C_{(r_p),h} > 0$ is a suitable constant.

From here to the end of this section we shall assume that conditions $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied.

The following assertion of Komatsu will be used. Note that the first part of this assertion is also proved in [32].

Lemma 3.1.1 *(see [49]) Let K be a compact neighbourhood of zero, $r > 0$ and $(r_p) \in \mathcal{R}$. Then*

(i) There is an $u \in \mathcal{D}((M_p), r/2, K)$ and $\psi \in \mathcal{D}((M_p), K)$ such that

$$P_r(D)u = \delta + \psi, \tag{3.2}$$

where P_r is of the form (2.13).

(ii) There are $u \in C^\infty$ and $\psi \in \mathcal{D}(\{M_p\}, K)$ such that

$$P_{(r_p)}(D)u = \delta + \psi, \quad \operatorname{supp} u \subset K \tag{3.3}$$

and

$$\sup_{x \in K} \frac{|\partial^\alpha u(x)|}{R_\alpha M_\alpha} \to 0, \quad as \quad \alpha \to \infty, \tag{3.4}$$

where $P_{(r_p)}$ is of the form (2.15).

Theorem 3.1.3 *Let $A \subset \mathcal{D}'(*, \mathbf{R}^n)$. Then*

(i) A is a bounded subset of $\mathcal{D}'((M_p), L^t)$ for $t \in [1, \infty]$ if and only if there are an $r > 0$ and bounded sets A_1 and A_2 in L^t such that every $f \in A$ is of the form

$$f = P_r(D)F_1 + F_2, \qquad F_1 \in A_1, \quad F_2 \in A_2;$$

(ii) A is an equicontinuos subset of $\tilde{\mathcal{D}}'(\{M_p\}, L^t)$ for $t \in [1, \infty]$ if and only if there are an $(r_p) \in \mathcal{R}$ and bounded sets A_1 and A_2 in L^t such that every $f \in A$ is of the form

$$f = P_{(r_p)}(D)F_1 + F_2, \qquad F_1 \in A_1, \quad F_2 \in A_2. \tag{3.5}$$

Proof. Notice that we do not know whether the basic space is quasi-barrelled and therefore we assume in (ii) that A is equicontinuous. We shall prove only assertion (ii), because it is more complicated. Since $P_{(r_p)}$ maps continuously the spaces $\mathcal{D}(\{M_p\}, L^s)$, $s \in [1, \infty)$ and $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R}^n)$ into themselves, (3.5) implies that A is bounded in $\mathcal{D}'(\{M_p\}, L^t)$.

We shall now prove the converse for $t = s/(s-1)$ with $s \geq 1$. For $t = 1$ (i.e. $s = \infty$) the proof is similar. Let Ω be a bounded open set in $\mathbf{R}^n$ containing zero, $K = \bar{\Omega}$ and $\varphi \in \mathcal{D}(\{M_p\}, K)$.

First we show that, for every $f \in A$, the functional T defined by $T(\varphi) = f * \varphi$ is a continuous linear functional on the space $\mathcal{D}(\{M_p\}, \mathbf{R}^n)$ endowed with the topology of L^s. Since A is equicontinuous, there are a constant $C > 0$ (which does not depend on $f \in A$) and an $(r_p) \in \mathcal{R}$ such that

$$\begin{aligned} |< f * \varphi, \psi >| = |< f, \check{\varphi} * \psi >| &\leq C \|\check{\varphi} * \psi\|_{L^s, (r_p)} \\ &\leq C \|\varphi\|_{K, (r_p)} \|\psi\|_{L^s} \leq C_1 \|\psi\|_{L^s}, \end{aligned} \tag{3.6}$$

for every $\psi \in \mathcal{D}(\{M_p\}, \mathbf{R}^n)$, where $\tilde{\varphi}(-x) = \varphi(x)$.

Since $\mathcal{D}(\{M_p\}, \mathbf{R}^n)$ is dense in L^s, it follows that $\{f * \varphi\colon\ f \in A\}$ is a set of (continuous) functions bounded in L^t. Moreover, (3.6) implies that

$$\sup\{\|f * \varphi\|_{L^t}\colon\ f \in A\} \leq C\|\varphi\|_{K,(r_p)}.$$

Consequently, if B is a bounded set of $\mathcal{D}(\{M_p\}, K)$ then

$$\sup\{\|f * \varphi\|_{L^t}\colon\ f \in A,\ \varphi \in B,\} < \infty.$$

Next, we show that there is (another) $(r_p) \in \mathcal{R}$ such that $\{f * \theta\colon\ f \in A\}$ is bounded set in L^t for every $\theta \in \mathcal{D}(\{M_p\}, r_p, \Omega)$. Let B_1 be the unit ball in L^s and B be a bounded subset of $\mathcal{D}(\{M_p\}, K)$. Then

$$|< f * \check{\psi}, \check{\varphi} >| = |< f * \varphi, \psi >| \leq \|f * \varphi\|_{L^t}\|\psi\|_{L^s} = \|f * \varphi\|_{L^t} \leq D < \infty$$

for all $f \in A$, $\psi \in B_1 \cap \mathcal{D}(\{M_p\}, \mathbf{R}^n)$ and $\varphi \in B$, where D does not depend on φ and f. This implies that the set

$$\{f * \check{\psi}\colon\ f \in A,\ \psi \in B_1 \cap \mathcal{D}(\{M_p\}, \mathbf{R}^n)\}$$

is bounded in $\mathcal{D}'(\{M_p\}, K)$. Since $\mathcal{D}(\{M_p\}, K)$ is barrelled, this family is equicontinuous in $\mathcal{D}'(\{M_p\}, K)$. This implies that there exists a neighbourhood $V_{(r_p)}(\varepsilon)$ of zero in $\mathcal{D}(\{M_p\}, K)$ of the form:

$$V_{(r_p)}(\varepsilon) = \{\theta \in \mathcal{D}(\{M_p\}, K)\colon\ \|\theta\|_{K,(r_p)} \leq \varepsilon\},\ \varepsilon > 0,$$

with $\varepsilon > 0$, such that $\theta \in V_{(r_p)}(\varepsilon)$ implies that

$$|< f * \check{\psi}, \check{\theta} >| = |< f * \theta, \psi >| \leq 1,$$

for $f \in A$ and $\psi \in B_1 \cap \mathcal{D}(\{M_p\}, \mathbf{R}^n)$. The same is true for the closure of $V_{(r_p)}(\varepsilon)$ in $\mathcal{D}(\{M_p\}, (r_p), K)$.

Let now $\delta_k(t) = k\omega(kt)$ for $t \in \mathbf{R}$ and $k \in \mathbf{N}$, where $\omega \in \mathcal{D}(\{M_p\}, \mathbf{R}^n)$, $0 \leq \omega \leq 1$ and

$$\int_{\mathbf{R}} \omega(t)\, dt = 1,$$

and let $\delta_k(x) = \delta_k(x_1) \cdot \ldots \cdot \delta_k(x_n)$ for $x \in \mathbf{R}^n$ and $k \in \mathbf{N}$.

One can easily prove that, for every $\mu \in \mathcal{D}(\{M_p\}, r_p, \Omega)$, the sequence $(\mu * \delta_k)$ of elements of $\mathcal{D}(\{M_p\}, K)$ converges to μ in the norm $\|\cdot\|_{K,(r_p)}$. For an arbitrary $\theta \in \mathcal{D}(\{M_p\}(r_p, \Omega)$, there is an $N > 0$ such that $\|\theta/N\|_{K,(r_p)} < \varepsilon$ and there exists a sequence in $\overline{V_{(r_p)}(\varepsilon)}$ which converges to θ/N in the norm $\|\cdot\|_{K,(r_p)}$. This implies that, for every $\theta \in \mathcal{D}(\{M_p\}, r_p, \Omega)$, there is a constant $C > 0$ such that

$$|< f * \check{\psi}, \check{\theta} >| = |< f * \theta, \psi >| \leq C,$$

for $f \in A$ and $\psi \in B_1 \cap \mathcal{D}(\{M_p\}, \mathbf{R}^n)$. Consequently,

$$|< f * \theta, \psi >| \leq C\|\psi\|_{L^s}$$

for $f \in A$ and $\psi \in \mathcal{D}(\{M_p\}, \mathbf{R}^n)$. This proves that $\{f * \theta\colon\ f \in A\}$ is a bounded set in L^t for every $\theta \in \mathcal{D}(\{M_p\}, r_p, \Omega)$.

Lemma 2.6.1 implies that there are $u \in \mathcal{D}(\{M_p\}, r_p, \Omega)$ and $\psi \in \mathcal{D}(\{M_p\}, \Omega)$ such that

$$f = P_{(r_p)}(u * f) - \psi * f$$

for every $f \in A$. Since $\{u * f\colon\ f \in A\}$ and $\{\psi * f\colon\ f \in A\}$ are bounded sets in L^t, the proof is completed. □

Let $r > 0$ (resp. $(r_p) \in \mathcal{R}$) be given. There is a $\tilde{r} > 0$ (resp. $(\tilde{r}_p) \in \mathcal{R}$) such that the function $P_r\varphi$ (resp. $P_{(r_p)}\varphi$) is continuous for $\varphi \in \mathcal{D}((M_p)\tilde{r}/2, K)$ (resp. $\varphi \in \mathcal{D}(\{M_p\}, (\tilde{r}_p), K)$). This and the preceding theorem imply the following corollary.

Corollary 3.1.1 *An ultradistribution $f \in \mathcal{D}'(*, \mathbf{R}^n)$ is an element of the space $\mathcal{D}'((M_p), L^t)$ (resp. $\tilde{\mathcal{D}}'(\{M_p\}, L^t)$) for $t \in [1, \infty]$ if and only if for every compact set K there is an $r > 0$ (resp. $(r_p) \in \mathcal{R}$) such that $f * \varphi \in L^t$ for every $\varphi \in \mathcal{D}((M_p)r/2, K)$ (resp. $\varphi \in \mathcal{D}(\{M_p\}, (r_p), K)$).*

3.2 Boundedness in $\mathcal{S}'^*$

The following structural theorem is true for tempered ultradistributions.

Theorem 3.2.1 *Let (M_p) satisfy conditions $(M.1)$ and $(M.3)'$. Then a set $B \subset \mathcal{S}'^{(M_p)}$ (resp. $B \subset \mathcal{S}'^{\{M_p\}}$ is bounded if and only if every $f \in B$ can be represented in the form*

$$f = \sum_{\alpha,\beta \in \mathbf{N}_0^n} D^\alpha((1 + |x|^2)^{\beta/2} f_{\alpha,\beta}),$$

where $f_{\alpha,\beta} \in L^2$ for $\alpha, \beta \in \mathbf{N}_0^n$ are functions with the following property: for some $d > 0$ (resp. for every $d > 0$) there exists a $D > 0$, independent of $f \in B$, such that

$$\sum_{\alpha,\beta \in \mathbf{N}_0^n} d^{\alpha+\beta} M_\alpha M_\beta \|f_{\alpha,\beta}\|_{L^2} < D.$$

Proof. We shall prove the assertion only in the more difficult case $* = \{M_p\}$.

Note that the space $\mathcal{S}^{\{M_p\}}$ is barrelled and thus B is an equicontinuous subset of $\mathcal{S}'^{\{M_p\}}$.

Let W_h, $h > 0$, be the space of all sequences $(\varphi_{\alpha,\beta}) = (\varphi_{\alpha,\beta})_{\alpha,\beta \in \mathbf{N}_0^n}$ in L^2 such that

$$\|(\varphi_{\alpha,\beta})\|_{L^2,h} = \sup\{\frac{\|\varphi_{\alpha,\beta}\|_{L^2}}{h^{\alpha+\beta} M_\alpha M_\beta}\colon\ \alpha, \beta \in \mathbf{N}_0^n\} < \infty,$$

equipped with the above defined norm. We identify $\mathcal{S}_h^{(M_p)}$ with the corresponding subspace of W_h. Since the inductive sequences W_h, $h \in \mathbf{N}$ and

$\mathcal{S}_h^{(M_p)}$ $h \in \mathbf{N}$ are weakly compact and compact, respectively, it follows from Lemma 1.4, *(iii)* in ([48], that the sequence

$$0 \leftarrow \text{proj} \lim_{h\to\infty} (\mathcal{S}_h^{(M_p)})' \leftarrow \text{proj} \lim_{h\to\infty} W_h'$$

is exact, where

$$\text{proj} \lim_{h\to\infty} (\mathcal{S}_h^{(M_p)})' = \mathcal{S}'^{\{M_p\}} = (\text{ind} \lim_{h\to\infty} \mathcal{S}_h^{(M_p)})'$$

and

$$\text{proj} \lim_{h\to\infty} W_h' = W' = (\text{ind} \lim_{h\to\infty} W_h)'.$$

Since the space $\mathcal{S}^{\{M_p\}}$ is Montel, by Lemma 1.4, (v) in ([48], it is a closed subspace of W and, by the Hahn-Banach theorem, the equicontinuous set $B \subset \mathcal{S}'^{\{M_p\}}$ can be extended to the equicontinuous set $\tilde{B}$ in W'. Thus $\tilde{B}$ consists of all sequences $(f_{\alpha,\beta}) = (f_{\alpha,\beta})_{\alpha,\beta\in\mathbf{N}_0^n} \in L^2$ from L^2 such that for every $d \in \mathbf{N}$ there is a constant $C > 0$, the same for all the elements of $\tilde{B}$, such that

$$\sum_{\alpha,\beta\in\mathbf{N}_0^n} d^{\alpha+\beta} M_\alpha M_\beta \|f_{\alpha,\beta}\|_{L^2} < C.$$

The mapping

$$\sum_{\alpha,\beta\in\mathbf{N}_0^n} (-1)^\alpha D^\alpha (1+|\chi|^2)^{\beta/2}$$

maps W' onto $\mathcal{S}'^{\{M_p\}}$ and

$$B \subset (\sum_{\alpha,\beta\in\mathbf{N}_0^n} (-1)^\alpha D^\alpha (1+|\chi|)^{\beta/2})\tilde{B},$$

which implies the assertion. □

Theorem 3.2.2 *Assume that satisfies conditions $(M.1)$, $(M.2)$ and $(M.3)$. Let $B \subset \mathcal{D}'((M_p), \mathbf{R}^n)$ (resp. $B \subset \mathcal{D}'(\{M_p\}, \mathbf{R}^n)$). Then B is a bounded subset of $S'^{(M_p)}$ (resp. B is a bounded subset of $S'^{\{M_p\}}$) if and only if every $f \in B$ is of the form*

$$f = P(D)F, \quad F \in B_1, \tag{3.7}$$

where P is an operator of class (M_p) (resp. of class $\{M_p\}$) and B_1 is the set of all continuous functions on $\mathbf{R}^n$ such that for some $k > 0$ and some $C > 0$ (resp. for every $k > 0$ there is a $C > 0$) the estimate

$$|F(x)| \le C \exp(M(k|x|)), \qquad x \in \mathbf{R}^n. \tag{3.8}$$

holds for all $F \in B_1$.

Proof. We shall prove again the case $* = \{M_p\}$, since the idea of the proof is similar in both cases and this case is more complicated.

Clearly, (3.8) implies that the set of all elements f of the form given by (3.7) is a bounded set in $\mathcal{S}'^{\{M_p\}}$.

Let B be a bounded set in $S'^{\{M_p\}}$. For the Fourier transform $\hat{f}$ of the ultradistribution $f \in B$ there are $(r_j), (s_j) \in \mathcal{R}$ and a constant $A > 0$ (which do not depend on $f \in B$) such that

$$|<\hat{f}, \varphi>| < A\gamma_{(r_j),(s_j)}(\varphi), \qquad \varphi \in S^{\{M_p\}}. \tag{3.9}$$

To simplify the notation, for given sequences $(r_j), (s_j) \in \mathcal{R}$ and a given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0^n$, denote

$$R_\alpha = \prod_{j=1}^{\bar{\alpha}} r_j, \quad S_\alpha = \prod_{j=1}^{\bar{\alpha}} s_j, \quad S'_\alpha = \prod_{j=1}^{\bar{\alpha}} r_j/2,$$

where $\bar{\alpha} = \alpha_1 + \ldots + \alpha$. For some $D > 0$ and $c > 0$, we have

$$\sup_{\alpha \in \mathbf{N}_0^n} \frac{(1+|x|^2)^{\alpha/2}}{M_\alpha R_\alpha} \leq D \exp(N(c|x|)), \quad x \in \mathbf{R}^n. \tag{3.10}$$

Let $(\tilde{r}_p)$ and ρ_0 correspond to (r_p), c and C in (2.16), where C is given by (2.18) and c by (3.10). If $\varphi \in \mathcal{D}^{\{M_p\}}$, it follows from (2.16) - (2.18) that

$$\gamma_{(r_j),(s_j)}(\varphi/P_{(\tilde{r}_p)}) \leq \sup_{\alpha,\beta \in \mathbf{N}_0^n} \frac{\|(1+|\chi|^2)^{\alpha/2} \sum_{0 \leq k \leq \beta} \binom{\beta}{k} \partial^{\beta-k}\varphi \partial^k (1/P_{(\tilde{r}_p)})\|_{L^2}}{M_\alpha R_\alpha M_{\beta-k} S_{\beta-k} M_k S_k}$$

$$\leq \sup_{\substack{k,\beta \in \mathbf{N}_0^n \\ k \leq \beta}} \left\| \sup_{\alpha \in \mathbf{N}_0} \frac{(1+|\chi|^2)^{\alpha/2}}{M_\alpha R_\alpha} 2^{-\beta} \sum_{0 \leq k \leq \beta} \binom{\beta}{k} \sup_{0 \leq k \leq \beta} \frac{|\partial^k (1/P_{(\tilde{r}_p)}) \partial^{\beta-k}\varphi|}{M_k S'_k M_{k-\beta} S'_{k-\beta}} \right\|_{L^2}$$

$$\leq D \sup_{\substack{k,\beta \in \mathbf{N}_0^n \\ k \leq \beta}} \sup_{0 \leq k \leq \beta} \frac{k! d^{-k}}{M_k S'_k} \| e^{N(c|\cdot|-N(|\cdot|)/C} 2^{-\beta} \sum_{0 \leq k \leq \beta} \binom{\beta}{k} \frac{|\partial^{\beta-k}\varphi|}{M_{\beta-k} S'_{\beta-k}} \|_{L^2}$$

$$\leq C_1 \sup_{\substack{k,\beta \in \mathbf{N}_0^n \\ k \leq \beta}} 2^{-\beta} \sum_{0 \leq k \leq \beta} \binom{\beta}{k} \frac{\|\partial^{\beta-k}\varphi\|_{L^2}}{M_{\beta-k} S'_{\beta-k}} \leq C_1 \|\varphi\|_{L^2,(s_j/2)}$$

for a certain constant $C_1 > 0$. Thus, (3.9) implies that

$$|\langle \hat{f}/P_{(\tilde{r}_p)}, \varphi \rangle| = |\langle \hat{f}, \varphi/P_{(\tilde{r}_p)} \rangle| \leq C_1 \|\varphi\|_{L^2,(s_p/2)}$$

for a suitable constant $C_1 > 0$, all $f \in B$ and all $\varphi \in \mathcal{D}(\{M_p\}, L^2)$. This implies that the set $\{\hat{f}/P_{(\tilde{r}_p)} : f \in B\}$ is equicontinuous in $\tilde{\mathcal{D}}'(\{M_p\}, L^2)$. Hence, by Theorem 3.1.3, every $\hat{f}$ for $f \in B$ is of the form

$$\hat{f}(\xi) = P_{(\tilde{r}_p)}(\xi)[P_{(\tilde{\tilde{r}}_p)}(D)\tilde{F}_1(\xi) + \tilde{F}_2(\xi)], \qquad \tilde{F}_1 \in \tilde{B}_1, \quad \tilde{F}_2 \in \tilde{B}_2,$$

where $\tilde{B}_1$ and $\tilde{B}_2$ are bounded subsets of L^2. Using the inverse Fourier transform, we obtain

$$f(x) = P_{(\tilde{r}_p)}(D)[P_{(\tilde{\tilde{r}}_p)}(x)F_1(x) + F_2(x)], \qquad F_1 \in B_1, \quad F_2 \in B_2,$$

where B_1 and B_2 are bounded subsets of L^2. Put

$$F(x) = \int_0^{x_1} \dots \int_0^{x_n} [P_{(\tilde{r}_p)}(t) F_1(t) + F_2(t)]\, dt_1 \dots dt_n,$$

where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $t = (t_1, \dots, t_n) \in \mathbf{R}^n$, with $F_1 \in B_1$ and $F_2 \in B_2$ and

$$P(D) = P_{(\tilde{r}_p)}(D) \frac{\partial^n}{\partial x_1 \dots \partial x_n}.$$

From (2.14) it follows that

$$\begin{aligned} |F(x)| &\leq C \exp(\tilde{\tilde{N}}(|x|)(1 + |x|^2)^n \int_0^x \frac{F_1(t) + F_2(t)}{(1 + |t|^2)^n}\, dt \\ &\leq C_1(\|F_1\|_{L^2} + \|F_2\|_{L^2})(1 + |x|^2)^n \exp\tilde{\tilde{N}}(x) \end{aligned}$$

for $x \in \mathbf{R}^n$. Since for every $k > 0$ there is a $\rho_k > 0$ such that

$$\tilde{\tilde{N}}(x) \leq M(k|x|),$$

whenever $|x| > \rho_k$ (see [48]), (3.8) follows and the theorem is proved. □

Chapter 4

Cauchy and Poisson integrals

4.1 Cauchy and Poisson kernels as ultradifferentiable functions

The first authors who have studied the representations of the Schwartz distributions $\mathcal{D}'_{L^r}$ as boundary values of analytic functions were Tillmann [80] and Łuszczki and Zieleźny [57]. In [57], the one-dimensional case of functions analytic in half planes was studied. In [80], the n-dimensional case was analyzed for functions analytic in the 2^n tubes of the form $\mathbf{R}^n + iC_n$ with n-rants C_n in $\mathbf{R}^n$ defined as follows:

$$C_n = \{y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n : \; u_j y_j > 0 \; (j = 1, \ldots, n) \quad \text{for} \quad u \in \Theta, \}$$

where

$$\Theta = \{u = (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n : \; u_j = \pm 1 \; (j = 1, \ldots, n)\}.$$

The values of r considered in these papers were $1 < r < \infty$ and fundamental to the analysis was the property that the Cauchy kernel function corresponding to half planes or tubes $\mathbf{R}^n + iC_n$ is an element of $\mathcal{D}_{L^s}$, $1 < s < \infty$. In [8], Carmichael noticed that the Cauchy kernel for tubes $\mathbf{R}^n + iC_n$ is an element of $\dot{\mathcal{B}} \cap \mathcal{D}_{L^\infty}$, too.

In this section we will prove that the general Cauchy kernel defined in Section 1.3 corresponding to a regular cone C is an element of $\mathcal{D}(*, L^s)$, $1 < s \le \infty$, where $*$ is both (M_p) and $\{M_p\}$, and hence is in $\mathcal{D}_{L^s}$, $1 < s \le \infty$. Additionally, we will prove that the Poisson kernel corresponding to the tube $\mathbf{R}^n + iC_n$ is in $\mathcal{D}(*, L^s)$, $1 \le s \le \infty$, and hence is in $\mathcal{D}_{L^s}$, $1 \le s \le \infty$.

In later sections we will use the Cauchy and Poisson kernels to construct the Cauchy and Poisson integrals of ultradistributions in the corresponding spaces $\mathcal{D}'(*, L^s)$ and we will prove some results about these integrals. We will conclude that an analysis similar to that of Tillmann and of Łuszczki and Zieleźny can be obtained in the more general tube setting of $\mathbf{R}^n + iC$ for all values of r $(1 < r < \infty)$ for which their results were obtained in the special cases of $\mathbf{R}^n + iC_n$ and half planes, respectively.

Let C be a regular cone in $\mathbf{R}^n$. We shall consider now the Cauchy and Poisson kernels, coresponding to $\mathbf{R}^n + iC$, defined in Section 1.3.

Theorem 4.1.1 *Let the sequence (M_p) of positive numbers satisfy conditions $(M.1)$ and $(M.3)$. We have $K(z-\cdot) \in \mathcal{D}(*, L^s), 1 < s \leq \infty$, for $z \in T^C$, where the symbol * means either (M_p) or $\{M_p\}$.*

Proof of Theorem 4.1.1 for dimension $n = 1$. First let C be the cone $C = (0, \infty)$ in $\mathbf{R}^1$. We have $C^* = [0, \infty)$ and $K(z-t) = 1/2\pi i(t-z)$ as usual for $z = x+iy \in \mathbf{R}^1 + i(0,\infty)$ and $t \in \mathbf{R}^1$. Let α be a nonnegative integer. We have

$$\frac{d^\alpha K(z-t)}{dt^\alpha} = (-1)^\alpha K^{(\alpha)}(z-t) = (2\pi i)^{-1}(-1)^\alpha \alpha!(t-z)^{-\alpha-1}.$$

For $1 < s < \infty$, we have

$$\left\|K^{(\alpha)}(z-\cdot)\right\|_{L^s} = \Big(\int_{-\infty}^{\infty} |\alpha!/2\pi i(t-z)^{\alpha+1}|^s\, dt\Big)^{1/s}$$

$$\leq (\alpha!/2\pi y^\alpha)\Big(\int_{-\infty}^{\infty} ((t-x)^2+y^2)^{-s/2}\, dt)^{1/s} \leq K(s,x,y)(\alpha!/y^\alpha), \quad (4.1)$$

where $K(s,x,y)$ is a constant depending on s, x, and y (we recall that $y > 0$ here). Let $h > 0$ be arbitrary and apply definition (2.6). Since (M_p) satisfies $(M.1)$ and $(M.3)'$, we have

$$\alpha! \prec M_\alpha,$$

by Lemma 2.1.2; thus for $L = hy$ there is a constant $B > 0$ which is independent of α such that

$$\alpha! \leq B(hy)^\alpha M_\alpha, \qquad \alpha \in \mathbf{N}_0. \quad (4.2)$$

Using (4.2) in (4.1) we obtain

$$\left\|K^{(\alpha)}(z-\cdot)\right\|_{L^s} \leq BK(s,x,y)h^\alpha M_\alpha, \qquad \alpha \in \mathbf{N}_0, \quad (4.3)$$

for all $h > 0$ which proves that $K(z-\cdot) \in \mathcal{D}((M_p), L^s), 1 < s < \infty$, for $z \in \mathbf{R}^1 + i(0,\infty)$.

For $s = \infty$, and $z \in \mathbf{R}^1 + i(0,\infty)$, we have

$$\left|\frac{d^\alpha K(z-t)}{dt^\alpha}\right| \leq \frac{\alpha!}{2\pi}((t-x)^2+y^2)^{-(\alpha+1)/2} \leq \frac{\alpha!}{2\pi y^{\alpha+1}}.$$

Let again $h > 0$ be arbitrary. Using (4.2) where $L = hy$ in (2.5), we have

$$\left|\frac{d^\alpha K(z-t)}{dt^\alpha}\right| \leq (B/2\pi y)h^\alpha M_\alpha, \qquad \alpha \in \mathbf{N}_0,$$

for all $h > 0$ which proves that $K(z-\cdot) \in \mathcal{D}((M_p), L^\infty)$ and, combining our results, we have $K(z-\cdot) \in \mathcal{D}((M_p), L^s), 1 < s \leq \infty$, for $z \in \mathbf{R}^1 + iC$ with $C = (0, \infty \subset \mathbf{R}^1$.

If $C = (-\infty, 0) \subset \mathbf{R}^1$ then $C^* = (-\infty, 0]$; and $K(z-\cdot) \in \mathcal{D}((M_p), L^s), 1 < s \leq \infty$, as a function of $t \in \mathbf{R}^1$ for $z = x+iy \in \mathbf{R}^1 + i(-\infty, 0)$ is proved like that for $C = (0,\infty)$ with $|y|$ in place of y in the proof. Since $\mathcal{D}((M_p), L^s) \subset$

$\mathcal{D}(\{M_p\}, L^s)$ we thus have $K(z - \cdot) \in \mathcal{D}(\{M_p\}, L^s)$ also for both cases $C = (0, \infty)$ and $C = (-\infty, 0)$. This completes the proof of Theorem 4.1.1 for dimension $n = 1$. □

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ such that $\mu_j = \pm 1, j = 1, \ldots, n$. Recall the $2^n n-$rants C_n defined above. Each C_n is a regular cone in $\mathbf{R}^n$ and $C_n^* = \bar{C}_\mu$. The Cauchy kernel corresponding to the tube $T^{C_\mu} = \mathbf{R}^n + iC_\mu$ takes the form

$$K(z-t) = (2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{t_j - z_j}, \qquad z = x + iy \in T^{C_\mu}, \quad t \in \mathbf{R}^n, \tag{4.4}$$

where

$$\operatorname{sgn}(y_j) = \begin{cases} 1, & y_j > 0, \\ -1, & y_j < 0, \end{cases}$$

for $j = 1, \ldots, n$. Thus for the tubes $\mathbf{R}^n + iC_\mu$ in $\mathbf{C}^n$, it is clear from the form of $K(z-t)$ that a proof like that in the one dimensional case will yield the desired result of Theorem 4.1.1. This case for the tubes $\mathbf{R}^n + iC_\mu$ also follows as a special case of the general proof of Theorem 4.1.1 for dimension $n \geq 2$ given below.

Before giving the proof of Theorem 4.1.1 for dimension $n \geq 2$ we first adopt some notation and prove a needed lemma.

Let $a > 0$ be arbitrary but fixed. Let C be a regular cone in $\mathbf{R}^n$ and C^* be its dual cone. Let $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^n$ be arbitrary but fixed. Put

$$C_a^* = \{\eta \in C^* \colon\ |\eta| \leq a\}; \tag{4.5}$$

$$C_{a,x,t}^* = \{\zeta \colon\ \zeta = \eta + i|\eta|(x-t)\eta \in C_a^*\}; \tag{4.6}$$

$$J_{a,x,t}^* = \{\zeta \colon\ \zeta\eta + iau(x-t), \eta \in C^*, |\eta| = a, 0 \leq u \leq 1.\}. \tag{4.7}$$

The differential form properties used in the following lemma can be found in [1] and [35]. This lemma will be used in the proof of Theorem 4.1.1 for dimension $n \geq 2$ given below; in particular it will be used to obtain equation (4.15) below.

Lemma 4.1.1 *For every $\alpha \in \mathbf{N}_0^n$, we have*

$$\lim_{a \to +\infty} \int_{J_{a,x,t}^*} \zeta^\alpha E_{z-t}(\zeta)\, d\zeta = 0$$

for $z = x + iy \in T^C$ and $t \in \mathbf{R}^n$.

Proof. The form $d\eta = d\eta_1 d\eta_2 \ldots d\eta_n$ is an n-form and the set $\{\eta \colon\ |\eta| = a\}$ has dimension less than n for any $a > 0$. Thus

$$d\eta_1 d\eta_2 \ldots d\eta_n = 0,$$

whenever $|\eta| = a$. Since $du\, du = du \wedge du = 0$, we obtain

$$d\zeta = d\zeta_1 d\zeta_2 \ldots d\zeta_n = ia \sum_{j=1}^{n} (x_j - t_j) d_j,$$

where

$$d_j = d\eta_1 \dots d\eta_{j-1}\, du\, d\eta_{j+1} \dots d\eta_n.$$

Letting $E(C^*, a) = \{\eta \in C^*: \ |\eta| = a\}$, we thus have

$$\begin{aligned} I_a &= \int\limits_{J^*_{\alpha,x,t}} \zeta^a E_{z-t}(\zeta)\, d\zeta \\ &= ia \int\limits_{E(C^*,a)} \int\limits_0^1 (\eta + iau(x-t))^\alpha E_{z-t}(\eta + iau(x-t)) \langle x - t, d\rangle \\ &= ia \int\limits_{E(C^*,a)} \int\limits_0^1 (\eta + iau(x-t))^\alpha \Phi_{x,y,t}(\eta, u) \langle x - t, d\rangle, \end{aligned} \tag{4.8}$$

where

$$\langle x - t, d\rangle = \sum_{j=1}^n (x_j - t_j) d_j$$

and

$$\Phi_{x,y,t}(\eta, u) = \exp[2\pi i(< x + iy - t, \eta > + aui < x + iy - t, x - t >)].$$

From the last term in (4.8), we have $I_a = 0$ for $x = t$ and every $a > 0$. Thus the remainder of the proof proceeds under the assumption that $x \neq t$. For an arbitrary $a > 0$, it follows from (4.8) that

$$\begin{aligned} |I_a| \leq a \int\limits_{E(C^*,a)} \int\limits_0^1 \Big(\prod_{j=1}^n (|\eta_j| + au|x_j - t_j|)^{\alpha_j}\Big) \\ \cdot \exp[-2\pi(au|x - t|^2 + < y, \eta >)] \sum_{j=1}^n |x_j - t_j| d_j. \end{aligned} \tag{4.9}$$

Now $y \in C$ is fixed; by Lemma 1.2.2. there exists a $\delta = \delta_y > 0$ depending on y such that

$$< y, \eta > \geq \ \delta |y||\eta|, \eta \in C^*. \tag{4.10}$$

Using (4.10) and the estimate

$$\prod_{j=1}^n (|\eta_j| + au|x_j - t_j|)^{\alpha_j} \leq (|\eta| + au|x - t|)^\alpha$$

we have

$$\begin{aligned} |I_a| \leq a \int_{E(C^*,a)} \int_0^1 (|\eta| + au|x - t|)^\alpha \exp(-2\pi au|x - t|^2) \\ \cdot \exp(-2\pi\delta|y||\eta|) \sum_{j=1}^n |x_j - t_j| d_j \end{aligned}$$

$$= a\exp(-2\pi\delta a|y|)\int_0^1 (a+au|x-t|)^\alpha \exp(-2\pi au|x-t|^2)\,du$$
$$\cdot\sum_{j=1}^{n}|x_j-t_j| \int_{E(C^*,a)} 1\,d\eta_1\dots d\eta_{j-1}\,d\eta_{j+1}\dots d\eta_n$$
$$\le a^{\alpha+1}S(a)\exp(-2\pi\delta a|y|)\Big(\sum_{j=1}^{n}|x_j-t_j|\Big)$$
$$\cdot\int_0^1 (1+u|x-t|)^\alpha \exp(-2\pi au|x-t|^2)\,du \tag{4.11}$$

where $S(a) = (2\pi^{n/2}a^{n-1}/\Gamma(n/2))$ is the surface area of the sphere of radius $a > 0$ in $\mathbf{R}^n$. For $x \ne t$ in (4.11) we notice that

$$(1+u|x-t|)^\alpha \le (1+|x-t|)^\alpha, 0 \le u \le 1.$$

Using this fact and then integrating, for $x \ne t$ we continue (4.11) as

$$|I_a| \le a^{\alpha+1}S(a)\exp(-2\pi\delta a|y|)(1+|x-t|)^\alpha$$
$$\cdot(1-\exp(-2\pi a|x-t|^2))(2\pi a|x-t|^2)^{-1}\sum_{j=1}^{n}|x_j-t_j|. \tag{4.12}$$

For $x \ne t$, the right side of (4.12) approaches 0 as $a \to +\infty$. Combining this fact with the previously noted fact that $I_a = 0$ for any $a > 0$ if $x = t$, the proof of Lemma 4.1.1 is completed. □

Proof of Theorem 4.1.1 for dimension $n \ge 2$.

Let α be any n-tuple of nonnegative integers. For fixed $t \in \mathbf{R}^n$ and $z = x + iy \in T^C$, the function

$$\zeta^\alpha \exp(2\pi i < z-t, \zeta >)$$

is an entire analytic function of ζ in $\mathbf{C}^n$. Thus by the discussion made in [83] (Section IV.22.6, p. 198), the form

$$\zeta^\alpha \exp(2\pi i < z-t, \zeta >)d\zeta_1 d\zeta_2 \dots d\zeta_n$$

is a closed differential form; that is

$$d(\zeta^\alpha \exp(2\pi i < z-t, \zeta >)d\zeta_1 d\zeta_2 \dots d\zeta_n = 0$$

and

$$\int_{E^*_{a,x,t}} \zeta^\alpha \exp(2\pi i < z-t, \zeta >)d\zeta_1 d\zeta_2 \dots d\zeta_n = 0 \tag{4.13}$$

for each $a > 0, x \in \mathbf{R}^n$, and $t \in \mathbf{R}^n$ where

$$E^*_{a,x,t} = C^*_a \cup \overline{C^*_{a,x,t}} \cup J^*_{a,x,t} \tag{4.14}$$

with $\overline{C^*_{a,x,t}}$ denoting $C^*_{a,x,t}$, as defined in (4.6), suitably oriented. Now recall the definition of the Cauchy kernel function in (1.5), the sets defined in (4.5) - (4.7), and Lemma 4.1.1. Using these we let $a \to +\infty$ in (4.13) and obtain

$$\begin{aligned} D^\alpha K(z-t) &= \int_{C^*} \eta^\alpha \exp(2\pi i < z-t, \eta >)\, d\eta \\ &= \int_{C^*_{x,t}} (\eta + i|\eta|(x-t))^\alpha \\ &\cdot \exp\left(2\pi i(x-t+iy)(\eta+i|\eta|(x-t))\right)\, d\zeta, \end{aligned} \tag{4.15}$$

where

$$C^*_{x,t} = \{\zeta :\ \zeta = \eta + i|\eta|(x-t), \eta \in C^*\}. \tag{4.16}$$

Between $d\eta = d\eta_1 \ldots d\eta_n)$ in the first integral and $d\zeta = d\zeta_1 \ldots d\zeta_n)$ in the second integral in (4.15), we have the following relationship:

$$\begin{aligned} d\zeta_j &= d\eta_j + i(x_j - t_j)\Big(\frac{\partial|\eta|}{\partial\eta_1} d\eta_1 + \ldots + \frac{\partial|\eta|}{\partial\eta_n} d\eta_n\Big) \\ &= d\eta_j + i(x_j - t_j)\Big(\frac{\eta_1}{|\eta|} d\eta_1 + \ldots + \frac{\eta_n}{|\eta|} d\eta_n\Big). \end{aligned} \tag{4.17}$$

Using the differential properties

$$d\eta_j d\eta_k = d\eta_j \wedge d\eta_k = -d\eta_k \wedge d\eta_j = -d\eta_k d\eta_j, \quad j \neq k,$$

and

$$d\eta_j d\eta_j = d\eta_j \wedge d\eta_j = 0$$

(see e.g. [1] or [35]), we obtain, exactly as in [42] (p. 359, the last line), the relation

$$d\zeta = \Big(1 + i\frac{< x-t, \eta >}{|\eta|}\Big) d\eta \tag{4.18}$$

from (4.17). Thus from (4.15) and (4.18) we get

$$\begin{aligned} D^\alpha K(z-t) &= \int_{C^*} (\eta + i|\eta|(x-t))^\alpha \\ &\cdot \exp\left(2\pi i(x-t+iy)(\eta+i|\eta|(x-t))\right) \Big(1 + i\frac{< x-t, \eta >}{|\eta|}\Big) d\eta \end{aligned} \tag{4.19}$$

Now

$$|(\eta + i|\eta|(x-t))^\alpha| = \Big|\prod_{j=1}^{n} (\eta_j + i|\eta|(x_j - t_j))^{\alpha_j}\Big| \le |\eta|^\alpha (1 + |x-t|)^\alpha \tag{4.20}$$

and

$$|1 + i\langle x-t, \eta\rangle/|\eta|| \le 1 + |\langle x-t, \eta\rangle|/|\eta| \le 1 + |x-t||\eta|/|\eta| = 1 + |x-t|. \tag{4.21}$$

Using (4.10), (4.20) and (4.21) in (4.19), we have

$$|D^\alpha K(z-t)| \le (1 + |x-t|)^{\alpha+1} \int_{C^*} |\eta|^\alpha \exp(-2\pi|\eta|(\delta|y| + |x-t|^2)\, d\eta. \tag{4.22}$$

Letting $pr(C^*)$ denote the projection of C^*, which is the intersection of C^* with the unit sphere in $\mathbf{R}^n$, we change variables in the integral in (4.22) by letting $\psi \in pr(C^*)$ and $0 < r < \infty$ and obtain

$$|D_t^\alpha K(z-t)| \leq (1+|x-t|)^{\alpha+1} \cdot \int_{pr(C^*)} \int_0^\infty \exp(-2\pi r(\delta|y|) + |x-t|^2)) r^{\alpha+n-1}\, dr\, d\psi \leq S(C^*)(1+|x-t|)^{\alpha+1} \int_0^\infty \exp(-2\pi r(\delta|y|) + |x-t|^2)) r^{\alpha+n-1}\, dr \tag{4.23}$$

where $S(C^*)$ is the surface area of $pr(C^*)$. It follows (see [23], p.93, (4.6) or [24] p. 60, (3.5)) that

$$\sup_{r\geq 0}[r^\alpha \exp(-\pi r(\delta|y|+|x-t|^2))] \leq \begin{cases} 1, & \alpha = 0, \\ (\pi(\delta|y|+|x-t|^2)/\bar{\alpha})^{-\alpha}, & \alpha \neq 0, \end{cases} \tag{4.24}$$

where $\bar{\alpha} = \alpha_1 + \ldots + \alpha_n$). Using (4.24) and integrating by parts we continue (4.23) as

$$|D_t^\alpha K(z-t)| \leq S(C^*)\Gamma(n)(1+|x-t|)^{\alpha+1}(\pi)^{-n-\alpha} \cdot (\delta|y|+|x-t|^2)^{-n-\alpha}\bar{\alpha}^\alpha \tag{4.25}$$

with the convention that $0^0 = 1$. Thus for $1 < s < \infty$ we have

$$\int_{\mathbf{R}^n} |D_t^\alpha K(z-t)|^s\, dt \leq \left(\frac{S(C^*)\Gamma(n)}{\pi^n}\right)^s \pi^{-s\alpha}\bar{\alpha}^{s\alpha} \cdot \int_{\mathbf{R}^n} \frac{(1+|x-t|)^{s(\alpha+1)}}{(\delta|y|+|x-t|^2)^{s(\alpha+n)}}\, dt. \tag{4.26}$$

Here $z + iy \in T^C$ is arbitrary but fixed, $\delta = \delta_y > 0$ is fixed, and $n \geq 2$. Using (4.26), straightforward estimating shows

$$\int_{\mathbf{R}^n} |D_t^\alpha K(z-t)|^s\, dt \leq (S(C^*)\Gamma(n)/\pi^n)^s \pi^{-s\alpha}\bar{\alpha}^{s\alpha} \cdot \begin{cases} N(s,\delta,y,n)(\frac{2}{\delta|y|})^{s\alpha} + N'(s,n), & \text{if } \delta|y| \geq 1, \\ N"(s,\delta,y,n)(\frac{2}{\delta|y|})^{s\alpha}, & \text{if } \delta|y| < 1. \end{cases} \tag{4.27}$$

Here $N(s,\delta,y,n), N'(s,n)$, and $N"(s,\delta,y,n)$ are positive constants which depend on the parameters listed. Using (4.27) and further estimates we have

$$\|D_t^\alpha K(z-\cdot)\|_{L^s} \leq (S(C^*)\Gamma(n)/\pi^n)\pi^{-\alpha}\bar{\alpha}^\alpha \cdot \begin{cases} M(s,\delta,y,n)(\frac{2}{\delta|y|})^\alpha, & \text{if } \delta|y| \leq 2, \\ M'(s,\delta,y,n), & \text{if } \delta|y| > 2. \end{cases} \tag{4.28}$$

for $1 < s < \infty$. Since the sequence (M_p) satisfies $(M.1)$ and $(M.3)'$ we have $\bar{\alpha}^\alpha \prec M_\alpha$, by Lemma 2.1.2.

Let $h > 0$ be arbitrary. For the fixed $y = \text{Im } z \in C$ and $\delta = \delta_y > 0$ put $L = (h\pi\delta|y|/2)$ if $\delta|y| \leq 2$ or $L = h\pi$ if $\delta|y| > 2$. From (2.5) and (4.28) there is a constant $B > 0$ which is independent of α such that

$$\|D^\alpha K(z - \cdot)\|_{L^s} \leq (S(C^*)\Gamma(n)/\pi^n)R(s, \delta, y, n)Bh^\alpha M_\alpha \tag{4.29}$$

for all $h > 0$, where $R(s, \delta, y, n)$ is a constant depending on the stated parameters. Now (4.29) proves that $K(z - \cdot) \in \mathcal{D}((M_p), L^s)$, $1 < s < \infty$, for $z \in T^C$ and $n \geq 2$.

For the case $s = \infty$ and $n \geq 2$ we return to (4.25) where $z = x + iy \in T^C$ and $\delta = \delta_y > 0$ are fixed. We have

$$\frac{(1 + |x - t|)^{\alpha+1}}{(\delta|y| + |x - t|^2)^{\alpha+n}} \leq \max\left\{\frac{2^{\alpha+1}}{(\delta|y|)^{\alpha+n}}, 2^{\alpha+1}\right\} \tag{4.30}$$

for all $t \in \mathbf{R}^n$ and for fixed $y \in C$ and $\delta = \delta_y > 0$. Let $h > 0$ be arbitrary and put $L = (h\pi\delta|y|)/2$ if $\delta|y| < 1$ or $L = h\pi/2$ if $\delta|y| \geq 1$. From (4.25), (4.30) and the fact that $\alpha^\alpha \prec M_\alpha$ here, we obtain

$$|D_t^\alpha K(z - t)| \leq (S(C^*)\Gamma(n)/\pi^n) \max\{2/(\delta|y|)^n, 2\}Bh^\alpha M_\alpha \tag{4.31}$$

where the constant $B > 0$ is independent of α, which proves that $K(z - \cdot) \in \mathcal{D}((M_p), L^\infty)$ as a function of $t \in \mathbf{R}^n$ for $z \in T^C$ and $n \geq 2$.

Thus we have $K(z - \cdot) \in \mathcal{D}((M_p), L^s) \subset \mathcal{D}(\{M_p\}, L^s)$, $1 < s \leq \infty$, as a function of $t \in \mathbf{R}^n$ for $z \in T^C$ and $n \geq 2$. The proof is complete. □

We recall that $K(z - \cdot)$ cannot be in $\mathcal{D}(*, L^1)$ or $\mathcal{D}_{L^1}$ since $K(z - \cdot)$ is not in L^1 as a function of $t \in \mathbf{R}^n$ for $z \in T^C$. Theorem 4.1.1 additionally proves that $K(z - \cdot) \in \mathcal{D}_{L^s}$, $1 < s \leq \infty$ as a function of $t \in \mathbf{R}^n$ for $z \in T^C$ since $\mathcal{D}(*, L^s) \subset \mathcal{D}_{L^s}$, $1 < s \leq \infty$.

For a regular cone C we now consider the Poisson kernel $Q(z; t)$, $z \in T^C = \mathbf{R}^n + iC$, $t \in \mathbf{R}^n$, defined in equation (1.6.

Theorem 4.1.2 *Let the sequence of positive real numbers (M_p), $p = 0, 1, 2, \ldots$, satisfy $(M.1)$ and $(M.3)'$. We have $Q(z; t) \in \mathcal{D}(*, L^s)$, $1 \leq s \leq \infty$, as a function of $t \in R^n$ for $z \in T^C$ where $*$ is either (M_p) or $\{M_p\}$.*

Proof. From (1.5) and Lemma 1.2.1 we just note that $K(2iy) > 0$, $y \in C$, in (1.6) and Lemma 1.3.5 holds since C is a regular cone by assumption. Now let $s = 1$. Let $z = x + iy \in T^C$ be arbitrary but fixed, and let α be any n-tuple of nonnegative integers. By the generalized Leibniz rule

$$D_t^\alpha Q(z; t) = \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta K(z - t) D_t^\gamma \overline{K(z - t)}, \tag{4.32}$$

and $Q(z;t)$ is infinitely differentiable as a function of $t \in \mathbf{R}^n$. Using (4.32) and Hölder's inequality we have

$$\int_{\mathbf{R}^n} |D_t^\alpha Q(z;t)|\,dt$$
$$\le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \int_{\mathbf{R}^n} |D_t^\beta K(z-t) D_t^\gamma \overline{K(z-t)}|\,dt$$
$$\le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \|D_t^\beta K(z-\cdot)\|_{L^2} \|D_t^\gamma \overline{K(z-\cdot)}\|_{L^2} \tag{4.33}$$

for any n-tuple α of nonnegative integers. By the proof of Theorem 4.1.1, $\overline{K(z-\cdot)} \in \mathcal{D}(*, L^s)$, $1 < s \le \infty$, as a function of $t \in \mathbf{R}^n$ for $z \in T^C$. Let $h > 0$ be arbitrary. Thus by Theorem 4.1.1 there exist constants N and N' which are both independent of h and α such that

$$\|D_t^\beta K(z-\cdot)\|_{L^s} \le N h^\beta M_\beta,\ \beta \in \mathbf{N}_0^n, \tag{4.34}$$

and

$$\|D_t^\gamma \overline{K(z-\cdot)}\|_{L^2} \le N' h^\gamma M_\gamma,\ \gamma \in \mathbf{N}_0^n\,. \tag{4.35}$$

Combining (4.33), (4.34) and (4.35) and using (2.1), we have

$$\int_{R^n} |D_t^\alpha Q(z;t)|\,dt \le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} N N' h^{\beta+\gamma} M_\beta M_\gamma$$
$$\le \frac{NN'M_0}{K(2iy)} \Big(\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big) h^\alpha M_\alpha, \tag{4.36}$$

(4.36) holds for such α, $\alpha \in \mathbf{N}_0^n$. Thus

$$Q(z;t) \in \mathcal{D}((M_p), L^1) \subset \mathcal{D}(\{M_p\}, L^1).$$

Now let $1 < s \le \infty$, and let $h > 0$ be arbitrary. Let α be any n-tuple of nonnegative integers. By Theorem 4.1.1 (see (4.29) and (4.31)) there exist constants N and N' which are independent of h and of $\alpha \in \mathbf{N}_0^n$, such that

$$|D_t^\gamma \overline{K(z-t)}| \le N h^\gamma M_\gamma,\ \ \gamma \in \mathbf{N}_0^n, \tag{4.37}$$

and

$$\|D_t^\beta K(z-\cdot)\|_{L^s} \le N' h^\beta M_\beta,\ \ \beta \in \mathbf{N}_0^n, \tag{4.38}$$

where $\beta+\gamma=\alpha$. Using (4.32), (4.37), (4.38), and (2.1), we proceed as in (4.36) to obtain

$$\|D_t^\alpha Q(z;\cdot)\|_{L^s} \le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \|D^\beta K(z-\cdot) \overline{D^\gamma K(z-\cdot)}\|_{L^s}$$
$$\le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (N h^\gamma M_\gamma) \|D_t^\beta K(z-\cdot)\|_{L^s}$$
$$\le \frac{1}{K(2iy)} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (N h^\gamma M_\gamma)(N' h^\beta M_\beta)$$
$$\le \frac{NN'M_0}{K(2iy)} \Big(\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big) h^\alpha M_\alpha, \tag{4.39}$$

(4.39) holds for each α, $\alpha \in \mathbf{N}_0^n$. Thus $Q(z;t) \in \mathcal{D}((M_p), L^s) \subset \mathcal{D}(\{M_p\}, L^s)$ for $1 < s \leq \infty$. The proof is complete. □

For the n-rant C_n, defined in the introduction to this section, the Cauchy kernel takes the form

$$K(z-t) = (2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{t_j - z_j}$$

for $z = x + iy \in T^{C_n}$ and $t \in \mathbf{R}^n$ where

$$\operatorname{sgn}(y_j) = \begin{cases} 1, & y_j > 0, \\ -1, & y_j < 0, \end{cases}$$

for $j = 1, \ldots, n$. According to (4.4), the Poisson kernel takes the form

$$Q(z;t) = (\pi)^{-n} \prod_{j=1}^{n} \frac{(\operatorname{sgn}(y_j))y_j}{(t_j - x_j)^2 + y_j^2}$$

for $z = x + iy \in T^{C_n}$ and $t \in \mathbf{R}^n$, where $\operatorname{sgn}(y_j)$ is as above. These forms of $K(z-t)$ and $Q(z;t)$ are special cases to which Theorem 4.1.1 and 4.1.2, respectively, are applicable.

4.2 Cauchy integral of ultradistributions

Let C be a regular cone in $\mathbf{R}^n$, and let the sequence of positive real numbers (M_p) satisfy $(M.1)$ and $(M.3)'$. Let $U \in \mathcal{D}'(*, L^s)$, $1 < s \leq \infty$, where we recall that $*$ means either (M_p) or $\{M_p\}$. Because of Theorem 4.1.1 we can form

$$C(U;z) = \langle U_t, K(z-t) \rangle, \quad z \in T^C, \tag{4.40}$$

which is the Cauchy integral of U. In this section we show that this Cauchy integral is an analytic function in T^C, has both pointwise and norm growth properties, and has boundary value properties.

Theorem 4.2.1 *Let C be a regular cone in $\mathbf{R}^n$ and let the sequence (M_p), $p = 0, 1, 2, \ldots$, satisfy $(M.1)$ and $(M.3)'$. Let $U \in \mathcal{D}'(*, L^s)$, $1 < s < \infty$. The Cauchy integral $C(U;z)$ is analytic in T^C.*

Proof. We first give the proof for $\mathcal{D}'((M_p), L^s)$. From Theorem 2.3.2 we have

$$C(U;z) = \sum_{0 \leq \alpha < \infty} (-1)^{\alpha} \int_{\mathbf{R}^n} g_{\alpha}(t) D_t^{\alpha} K(z-t)\, dt, \quad z \in T^C, \tag{4.41}$$

where the $g_{\alpha}(t) \in L^r$, $1/r + 1/s = 1$, such that (2.37) holds for some $k > 0$. Let Q be a compact subset of T^C. From the proof of Theorem 4.1.1 (recall (4.3) and (4.29)) there is a constant $B(s,Q,n)$ depending on s, the compact set Q, and the dimension n such that

$$\|D_t^\alpha K(z-\cdot)\|_{L^s} \le B(s,Q,n)h^\alpha M_\alpha \tag{4.42}$$

for all $h>0$, where $z \in Q \subset T^C$ and $\alpha \in \mathbf{N}_0^n$. For each $g_\alpha(t)$ in (4.41) we see from (4.42) and Hölder's inequality that

$$\begin{aligned}\int_{\mathbf{R}^n} |g_\alpha(t) D_t^\alpha K(z-t)|\,dt &\le \|g_\alpha\|_{L^r}\|D_t^\alpha K(z-\cdot)\|_{L^s} \\ &\le B(s,Q,n)h^\alpha M_\alpha \|g_\alpha\|_{L^r}\end{aligned} \tag{4.43}$$

for all $h>0$, where $z \in Q \subset T^C$. Recall that (2.37) is equivalent to (2.39) on the functions $g_\alpha(t)$ in (4.41). Using (4.43) and (2.39) and choosing the h in (4.43) to be $k/2$ for the k in (2.39) we have the existence of a constant $D>0$ such that

$$\int_{\mathbf{R}^n} |g_\alpha(t) D_t^\alpha K(z-t)|\,dt \le B(s,Q,n)D(1/2)^\alpha$$

which we use to obtain

$$\sum_{0\le\alpha<\infty} \int_{\mathbf{R}^n} |g_\alpha(t) D_t^\alpha K(z-t)|\,dt \le B(s,Q,n)D \sum_{0\le\alpha<\infty}(1/2)^\alpha < \infty. \tag{4.44}$$

The bound on the right of (4.44) is independent of $z \in Q \subset T^C$. The resulting uniform and absolute convergence of the series in (4.41) for $z \in Q$, where Q is any compact subset of T^C, proves that $C(U;z)$ is analytic in T^C.

Using Theorem 2.3.1, the proof that $C(U;z)$ is analytic in T^C for $U \in \mathcal{D}'(\{M_p\},L^s)$, $1<s<\infty$, can similarly be proved. This completes the proof of Theorem 4.2.1. □

We obtain a pointwise growth estimate for the Cauchy integral after proving a needed lemma.

Lemma 4.2.1 *Let C be a regular cone in $\mathbf{R}^n$. For any n-tuple α of nonnegative integers,*

$$h_{y,\alpha} \in L^r \tag{4.45}$$

for all r, $1 \le r \le \infty$ and for $y \in C$, where

$$h_{y,\alpha}(t) = t^\alpha I_{C^*}(t) e^{-2\pi\langle y,t\rangle}, \quad t \in \mathbf{R}^n,$$

and I_{C^*} is the characteristic function of the dual cone C^* of C.

Proof. Let $y \in C$. By Lemma 1.2.2 there is a $\delta = \delta_y > 0$ such that (1.2) holds for $t \in C^*$. Let α be an arbitrary n-tuple of nonnegative integres. For $r=\infty$ we apply (1.2) and obtain

$$\begin{aligned}|t^\alpha I_{C^*}(t)e^{-2\pi\langle y,t\rangle}| &\le \sup_{t\in C^*}|t^\alpha e^{-2\pi\langle y,t\rangle}| \le \sup_{t\in C^*}|t|^\alpha e^{-2\pi\delta|y||t|} \\ &\le \sup_{\substack{\rho>0\\ u\in pr(C^*)}} |\rho u|^\alpha \exp(-2\pi\delta|y||\rho u|) = \sup_{\rho>0}\rho^\alpha e^{-2\pi\delta\rho|y|}\end{aligned} \tag{4.46}$$

for all $t \in \mathbf{R}^n$ and for $y \in C$. From the above calculations we conclude that

$$\sup_{\rho \geq 0} \rho^{\alpha} e^{-2\pi\delta\rho|y|} \leq \begin{cases} 1, & \alpha = 0, \\ \left(\dfrac{\alpha}{2\pi\delta|y|}\right)^{\alpha}, & \alpha \neq 0. \end{cases} \tag{4.47}$$

From (4.46) and (4.47) we conclude (4.45) for $r = \infty$, where $y \in C$.

Now, let $1 \leq r < \infty$. From (1.2) and a calculation as in (1.9) we have

$$\int_{\mathbf{R}^n} |t^{\alpha} I_{C^*}(t) e_y(t)|^r \, dt \leq \int_{\mathbf{R}^n} I_{C^*}(t) |t|^{r\alpha} e_{r\delta|y|}(|t|) \, dt$$

$$\leq \Omega_n \int_0^{\infty} w^{r\alpha+n-1} e_{r\delta|y|}(w) \, dw = \Omega_n \Gamma(r\alpha + n)(2\pi r\delta|y|)^{-r\alpha-n} \tag{4.48}$$

for $y \in C$, where Ω_n is the surface area of the unit sphere in $\mathbf{R}^n$ and the change of variable for $u = 2\pi r\delta w|y|$ was used to obtain the gamma function Γ. Now (4.48) proves (4.45) for $1 \leq r < \infty$. □

Theorem 4.2.2 *Let the cone C and the sequence (M_p) satisfy the hypotheses of Theorem 4.2.1. If $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, for each compact subcone $C' \subset\subset C$ there are constants $A = A(n, C', s) > 0$ and $T = T(C') > 0$ such that*

$$|C(U; z)| \leq A|y|^{-n/r} \exp(M^*(T/|y|)), \quad z = x + iy \in \mathbf{R}^n + iC', \tag{4.49}$$

where n is the dimension, $1/r + 1/s = 1$, and M^* is the function defined in (2.8). If $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 \leq s < \infty$, for each compact subcone $C' \subset\subset C$ and arbitrary constant $T > 0$, which is independent of $C' \subset\subset C$, there is a constant $A = A(n, C', s) > 0$ such that (4.49) holds.

Proof. Let $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, and C' be an arbitrary compact subcone of C. From (4.41) we have

$$|C(U; z)| \leq \sum_{0 \leq \alpha < \infty} \|D_t^{\alpha} K(z - \cdot)\|_{L^s} \|g_{\alpha}\|_{L^r}, \quad z \in T^C, \tag{4.50}$$

where the $g_{\alpha} \in L^r$, $1/r + 1/s = 1$, such that (2.37) holds for some $k > 0$. For $2 \leq s < \infty$ we have $1 < r \leq 2$, $1/r + 1/s = 1$. For these s and subsequent r we note Lemma 4.2.1 and can write

$$D_t^{\alpha} K(z - t) = \mathcal{F}^{-1}[I_{C^*}(\eta)\eta^{\alpha} e^{2\pi i \langle z, \eta \rangle} : \, t], \quad z \in T^C, \quad t \in \mathbf{R}^n, \tag{4.51}$$

where the inverse Fourier transform can be interpreted in both the L^1 and L^r sense. By the Plancherel theory of Fourier analysis and the analysis of (4.46), (4.47), and (4.48), we see for $z = x + iy \in \mathbf{R}^n + iC'$, $C' \subset\subset C$, from (4.51) that

$$\|D_t^{\alpha} K(z - \cdot)\|_{L^s} \leq \Big(\int_{C^*} |\eta|^{r\alpha} e^{-2\pi r \langle y, \eta \rangle} \, d\eta \Big)^{1/r}$$

$$\leq (\Omega_n)^{1/r}\Big(\int_0^\infty w^{r\alpha+n-1}\exp(-2\pi\delta r w|y|)\,dw\Big)^{1/r}$$

$$\leq (\Omega_n)^{1/r}(\sup_{w\geq 0}(w^{r\alpha}e^{-\pi\delta r w|y|}))^{1/r}\Big(\int_0^\infty u^{n-1}e^{-\pi\delta w|y|}\,dw\Big)^{1/r}$$

$$= (\Omega_n(n-1)!)^{1/r}(\pi\delta r|y|)^{-n/r}\sup_{w\geq 0}(w^{\alpha}e^{-\pi\delta w|y|})$$

$$\leq \begin{cases} (\Omega_n(n-1)!)^{1/r}(\pi\delta r|y|)^{-n/r}, & \alpha = 0, \\ (\Omega_n(n-1)!)^{1/r}(\pi\delta r|y|)^{-n/r}\Big(\dfrac{\alpha}{\pi\delta|y|}\Big)^{\alpha}, & \alpha\in\mathbf{N}^n. \end{cases} \tag{4.52}$$

Using (4.52) in (4.50) we have

$$|C(v;z)| \leq \tilde{A}(n,C',s)|y|^{-n/r}\sum_{0\leq\alpha<\infty}\Big(\frac{\alpha}{\pi\delta|y|}\Big)^{\alpha}\|g_\alpha\|_{L^r} \tag{4.53}$$

for $z = x+iy \in T^{C'} = \mathbf{R}^n + iC'$, where

$$\tilde{A}(n,C',s) = (\Omega_n(n-1)!)^{1/r}(\pi\delta r)^{-n/r}$$

with $1/r+1/s=1,\ \ 2\leq s<\infty$. Using (2.39), the fact that

$$\alpha^\alpha \leq e^\alpha\alpha!, \quad \alpha = 1,2,3,\ldots, \tag{4.54}$$

from the proof of Stirling's formula, and our convention that $\alpha^\alpha = 1$ if $\alpha = 0$, and putting $T = T(C') = (2e/k\pi\delta)$ for the k in (2.39), we continue (4.53) as

$$|C(U;z)| \leq B\tilde{A}(n,C',s)|y|^{-n/r}\sum_{0\leq\alpha<\infty}\Big(\frac{1}{2}\Big)^{\alpha}\Big(\frac{T}{|y|}\Big)^{\alpha}\frac{\alpha!}{M_\alpha} \tag{4.55}$$

for some $B>0$. Recalling the definition of the associated function $M^*(\rho)$ in (2.8), we have for each $\alpha = 1,2,3,\ldots$ that

$$\Big(\frac{T}{|y|}\Big)^{\alpha}\frac{\alpha!}{M_\alpha} = \frac{1}{M_0}\Big(\Big(\frac{T}{|y|}\Big)^{\alpha}\alpha!\frac{M_0}{M_\alpha}\Big)$$
$$= \frac{1}{M_0}\exp(\log\Big(\Big(\frac{T}{|y|}\Big)^{\alpha}\alpha!\frac{M_0}{M_\alpha}\Big)) \leq \frac{1}{M_0}\exp(M^*(T/|y|)). \tag{4.56}$$

The constant $T = T(C') = (2e/k\pi\delta)$ depends on $C' \subset\subset C$ because δ depends on C' but not on $y\in C'$. Using (4.56) in (4.55) and putting

$$A = A(n,C',s) = (B/M_0)\tilde{A}(n,C',s)\sum_{0\leq\alpha<\infty}(1/2)^{\alpha}$$

the desired estimate (4.49) follows in case $U\in\mathcal{D}'((M_p),L^s),\ \ 2\leq s<\infty$.

For $U\in\mathcal{D}'(\{M_p\},L^s),\ \ 2\leq s<\infty$, we use Theorem 2.3.1 and the analysis of (4.50) and (4.52) to obtain the conclusion (4.49) for this case similarly as we did for the case $U\in\mathcal{D}'((M_p),L^s)$ above. For the case of $\mathcal{D}'(\{M_p\},L^s)$ the constant $T > 0$ in (4.49) is arbitrary and independent of $C' \subset\subset C$ because of its dependence on arbitrary $k > 0$ is (2.30) of Theorem 2.3.1 which does not depend on $C' \subset\subset C$. The proof of Theorem 4.2.2 is completed. □

We now obtain a norm growth estimate for the Cauchy integral of ultradistributions.

Theorem 4.2.3 *Let the cone C and the sequence (M_p) satisfy the hypotheses of Theorem 4.2.1.*

If $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, then for each compact subcone $C' \subset\subset C$ there exists a constant $T = T(C') > 0$ depending on C' such that, in case $s = 2$,

$$\|C(U;z)\|_{L^s} = \Big(\int_{\mathbf{R}^n} |C(U;x+iy)|^s\,dx\Big)^{1/s} \leq K(U)\exp(M^*(T/|y|)) \tag{4.57}$$

for $y \in C' \subset\subset C$ and, in case $2 < s < \infty$,

$$\|C(U;z)\|_{L^s} \leq K(U,C',s,r,n)|y|^{-n(s-r)/rs}\exp(M^*(T/|y|)) \tag{4.58}$$

for $y \in C' \subset\subset C$ with $1/r + 1/s = 1$, where $K(U)$ in (4.57), i.e. in case $s = 2$, is a constant depending on U, and $K(U,C',s,r,n)$ in (4.58), i.e. in case $2 < s < \infty$, is a constant depending on U, C', s, r and n.

If $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 \leq s < \infty$, then for each compact subcone $C' \subset\subset C$ and an arbitrary constant $T > 0$, which may or may not depend on $C' \subset\subset C$, there is a constant $K(U)$ if $s = 2$ or a constant $K(U,C',s,r,n)$ if $2 < s < \infty$ such that (4.57) and (4.58) hold, respectively.

Proof. For $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, we use Theorem 2.3.2 and Fubini's theorem to obtain

$$\begin{aligned} C(U;z) &= \sum_{0\leq\alpha<\infty} \langle D_t^\alpha g_\alpha(t), K(z-t)\rangle \\ &= \sum_{0\leq\alpha<\infty} (-1)^\alpha \langle g_\alpha(t), D_t^\alpha K(z-t)\rangle \\ &= \sum_{0\leq\alpha<\infty} \int_{\mathbf{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\mathcal{F}^{-1}[g_\alpha(t);\eta]\,d\eta \end{aligned} \tag{4.59}$$

where each $g_\alpha(t) \in L^r$, $1/r + 1/s = 1$, and $I_{C^*}(\eta)$ is the characteristic function of C^*. Since each $g_\alpha(t) \in L^r$, $1 < r \leq 2$, for each s, $2 \leq s < \infty$, $1/r + 1/s = 1$, then $\mathcal{F}^{-1}[g_\alpha(t);\eta] \in L^s$, $2 \leq s < \infty$. If $s = 2$ there $r = 2$, and by Lemma 4.2.1 each of the integrands in the last term in (4.59) is in $L^1 \cap L^2$. In the case $s = 2$, we use Parseval's equality to get

$$\begin{aligned} &\Big\|\int_{\mathbf{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\mathcal{F}^{-1}[g_\alpha(t):\ \eta]\,d\eta\Big\|_{L^2} \\ &= \|\mathcal{F}[\eta^\alpha I_{C^*}(\eta)e^{-2\pi\langle y,\eta\rangle}\mathcal{F}^{-1}[g_\alpha(t):\ \eta];x]\|_{L^2} \\ &= \|\eta^\alpha I_{C^*}(\eta)e^{-2\pi\langle y,\eta\rangle}\mathcal{F}^{-1}[g_\alpha(t):\ \eta]\|_{L^2} \\ &\leq \Big(\sup_{\eta\in\mathbf{R}^n}\big|\eta^\alpha I_{C^*}(\eta)e^{-2\pi\langle y,\eta\rangle}\big|^2 \int_{\mathbf{R}^n}\big|\mathcal{F}^{-1}[g_\alpha(t);\eta\big|^2 d\eta\Big)^{1/2}. \end{aligned} \tag{4.60}$$

Now let $s > 2$ and note that by Hölder's inequality

$$\int_{\mathbf{R}^n} \left| I_{C^*}(\eta)\eta^\alpha e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t);\eta]\right|^r d\eta$$

$$\leq \Big(\int_{\mathbf{R}^n} (|\mathcal{F}^{-1}[g_\alpha(t):\ \eta]|^r)^{s/r}\, d\eta\Big)^{r/s} \|(\chi)^{\alpha r} I_{C^*} e_y^r\|_{L^{rs/(s-r)}}, \qquad (4.61)$$

where both terms on the right of (4.61) are finite, since $\mathcal{F}^{-1}[g_\alpha(t);\eta] \in L^s$ and because of Lemma 4.2.1. Thus, by (4.61) and Lemma 4.2.1, each of the integrands in the last term in (3.9) is in $L^1 \cap L^r$ for the case $s > 2$, $1/r + 1/s = 1$. By the Parseval inequality and (4.61), we have, in the case $s > 2$,

$$\|\int_{\mathbf{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t):\ \eta] d\eta\|_{L^s}$$

$$= \|\mathcal{F}[\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t);\eta]:\ x]\|_{L^s}$$

$$\leq \|\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t);\eta]\|_{L^r}$$

$$\leq \Big(\int_{\mathbf{R}^n} (|\mathcal{F}^{-1}[g_\alpha(t):\ \eta]|^r)^{s/r}\, d\eta\Big)^{1/s}$$

$$\cdot \Big(\int_{\mathbf{R}^n} |I_{C^*}(\eta)\eta^\alpha e^- 2\pi\langle y,\eta\rangle|^{rs/(s-r)}\, d\eta\Big)^{(s-r)/sr}. \qquad (4.62)$$

Recalling Lemma 1.2.2, given a compact subcone $C' \subset\subset C$ there is a $\delta = \delta(C') > 0$ such that (1.2) holds for all $y \in C'$ and all $t \in C^*$, using (1.2) and the estimates (4.46) and (4.47) we have

$$\sup_{\eta\in\mathbf{R}^n} |\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle}|$$

$$\leq \sup_{\eta\in C^*} (|\eta|^\alpha \exp(-2\pi\delta|y||\eta|) \leq \begin{cases} 1, & \alpha = 0, \\ \left(\dfrac{\alpha}{2\pi\delta|y|}\right)^\alpha, & \alpha \neq 0, \end{cases} \qquad (4.63)$$

for $y \in C' \subset\subset C$ and $\delta = \delta(C') > 0$.

For the case $s = 2$ we use (4.59), (4.60), (4.63), and the Parseval equality to obtain for $y \in C' \subset C$ that

$$\|C(U;z)\|_{L^2} \leq \sum_{0\leq\alpha<\infty} \left\| \int_{\mathbf{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t);\eta] d\eta \right\|_{L^2}$$

$$\leq \sum_{0\leq\alpha<\infty} \Big(\sup_{\eta\in\mathbf{R}^n} |\eta^\alpha + I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle}|^2 \int_{\mathbf{R}^n} |\mathcal{F}^{-1}[g_\alpha(t):\ \eta]|^2\, d\eta\Big)^{1/2}$$

$$\leq \sum_{0\leq\alpha<\infty} \left(\frac{\alpha}{2\pi\delta|y|}\right)^\alpha \|g_\alpha\|_{L^2}, \qquad (4.64)$$

where the convention $\alpha^\alpha = 1$ if $\alpha = 0$ is used. For $s > 2$ and $C' \subset C$ let us write $\delta = \delta(C') > 0$ in (1.2) as $\delta = \delta_1 + \delta_2$, where $\delta_1 = \delta_1(C') > 0$ and $\delta_2 = \delta_2(C') > 0$.

Now using (4.59), (4.62), Parseval's inequality, (1.2) with $\delta = \delta_1 + \delta_2$, and (4.63), we have for $y \in C' \subset C$

$$\|C(U;z)\|_{L^s} \leq \sum_{0\leq\alpha<\infty} \Big\| \int_{\mathbf{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\mathcal{F}^{-1}[g_\alpha;\eta]d\eta \Big\|_{L^s}$$
$$\leq \sum_{0\leq\alpha<\infty} \|\mathcal{F}^{-1}[g_\alpha;\eta]\|_{L^s}\|I_{C^*}(\eta)\eta^\alpha e^{-2\pi\langle y,\eta\rangle}\|_{L^{rs(s-r)}}$$
$$\leq \sum_{0\leq\alpha<\infty} \|g_\alpha\|_{L^r}\Big(\sup_{\eta\in\mathbf{R}^n}|I_{C^*}(\eta)\eta^\alpha \exp(-2\pi\delta_1|y||\eta|)|\Big)$$
$$\cdot\Big(\int_{C^*} \exp(-2\pi\delta_2|y||\eta|rs/(s-r))\,d\eta\Big)^{(s-r)/rs}$$
$$\leq \Omega_n\Big(\int_0^\infty w^{n-1}\exp(-2\pi\delta_2|y|wrs/(s-r))\,dw\Big)^{(s-r)/rs}$$
$$\cdot\sum_{0\leq\alpha<\infty} \|g_\alpha\|_{L^r}\Big(\frac{\alpha}{2\pi\delta_1|y|}\Big)^\alpha$$
$$= \Omega_n((n-1)!(2\pi\delta_2|y|rs/(s-r))^{-n})^{(s-r)/rs}$$
$$\cdot\sum_{0\leq\alpha<\infty} \|g_\alpha\|_{L^r}\Big(\frac{\alpha}{2\pi\delta_1|y|}\Big)^\alpha \tag{4.65}$$

(where Ω_n is the surface area of the unit sphere in $\mathbf{R}^n$ as in (4.48)).

For the fixed $k > 0$ in the converse part of Theorem 2.3.2 we know that (2.39) holds; that is

$$\sup_\alpha(k^\alpha M_\alpha\|g_\alpha\|_{L^r}) < \infty \tag{4.66}$$

where $2 \leq s < \infty$, $1/r + 1/s = 1$, here.

For the case $s = 2$ and this fixed $k > 0$ we return to (4.64) to obtain

$$\|C(U;z)\|_{L^2} \leq \sup_\alpha(k^\alpha M_\alpha\|g_\alpha\|_{L^2}) \sum_{0\leq\alpha<\infty}\Big(\frac{\bar{\alpha}}{2k\pi\delta|y|}\Big)^{-\alpha}\frac{1}{M_\alpha}. \tag{4.67}$$

Recall (4.54) and our convention that $\alpha^\alpha = 1$ if $\alpha = 0$; using this in (4.67) and putting $T = e/k\pi\delta$ we have from (4.67)

$$\|C(U;z)\|_{L^2} \leq \sup_\alpha(k^\alpha M_\alpha\|g_\alpha\|_{L^2})$$
$$\cdot\sum_{0\leq\alpha<\infty}\Big(\frac{1}{2}\Big)^\alpha\Big(\frac{T}{|y|}\Big)^\alpha\frac{\alpha!}{M_\alpha}$$
$$\leq \sup_\alpha(k^\alpha M_\alpha\|g_\alpha\|_{L^2})\Big(\sum_{0\leq\alpha<\infty}\Big(\frac{1}{2}\Big)^\alpha\Big)(M_0)^{-1}\exp(M^*(T/|y|)) \tag{4.68}$$

for $y \in C' \subset C$ where we have used the calculation from (4.56). This proves (4.57), i.e. the assertion in case $U \in \mathcal{D}'((M_p), L^2)$, where

$$K(U) = (M_0)^{-1}\sup_\alpha(k^\alpha M_\alpha\|g_\alpha\|_{L^2})\Big(\sum_{0\leq\alpha<\infty}\Big(\frac{1}{2}\Big)^\alpha\Big).$$

For $s > 2$ we return to (4.65) and proceed using (4.66) and (4.54) to obtain (4.58) for the case $U \in \mathcal{D}'((M_p), L^s)$, $2 < s < \infty$, similarly as we did for the case $U \in \mathcal{D}'((M_p), L^2)$, where

$$\begin{aligned} K(U, C', s, r, n) &= \Omega_n((n-1)!(2\pi\delta_2 rs/(s-r))^{-n})^{(s-r)/rs}(1/M_0) \\ &\quad \cdot \sup_{\alpha}(k^\alpha M_\alpha \|g_\alpha\|_{L^r}\Big(\sum_{0\leq\alpha<\infty}\Big(\frac{1}{2}\Big)^\alpha\Big) \end{aligned}$$

and $T = e/k\pi\delta_1$.

Under the assumption $U \in \mathcal{D}'(\{M_p\}, L^s)$ with $2 \leq s < \infty$, the assersions in (4.57) - (4.58) are obtained due to the characterization given in Theorem 2.3.1 and an analysis similar to that in case $U \in \mathcal{D}'((M_p), L^s)$, given previously in this proof. For $U \in \mathcal{D}'(\{M_p\}, L^s)$, $T > 0$ in (4.57) - (4.58) is an arbitrary constant, because it depends on an arbitrary $k > 0$ in (2.30), and T may or may not depend on C' depending on the choice of an arbitrary $k > 0$. A complete proof of inequalities (4.57) - (4.58) for $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 \leq s < \infty$, is given in [25]. The proof of Theorem 4.2.3 is complete. □

We desire to extend the growth results of Theorem 4.2.2 and 4.2.3 to the case $1 < s < 2$. The proofs of Theorem 4.2.2 and 4.2.3 given above depend considerably on properties of the Fourier transform for elements in L^s, $2 \leq s < \infty$, properties which are not available in general for the cases $1 < s < 2$. A detailed analysis of integrals, as in the proof of Theorem 4.1.1, yields these growth results for $1 < s < 2$. We consider this in future research.

Throughout the remainder of this section the sequence (M_p) will satisfy $(M.1)$, $(M.2)$ and $(M.3)'$.

We now proceed to investigate the boundary value properties of the analytic function $C(U; z)$, $z \in T^C$, for $U \in \mathcal{D}'(*, L^s)$. We first define a convolution which corresponds to that given in the definition in [48], p. 71. Let $U \in \mathcal{D}'(*, L^s)$ and $\varphi \in \mathcal{D}(*, L^s)$, $1 < s < \infty$: the convolution of U with φ is given by

$$(U * \varphi)(x) = \langle U_t, \varphi(x-t)\rangle, \quad x \in \mathbf{R}^n. \tag{4.69}$$

Lemma 4.2.2 *Let $U \in \mathcal{D}'(*, L^s)$ and $\varphi \in \mathcal{D}(*, L^s)$, $1 < s < \infty$. Then $(U * \varphi)(x) \in \mathcal{D}(*, L^\infty)$.*

Proof. Let $U \in \mathcal{D}'((M_p), L^s)$ and $\varphi \in \mathcal{D}((M_p), L^s)$. By (4.69) and Theorem 2.3.2, we have

$$(U * \varphi)(x) = \sum_{0\leq\alpha<\infty}(-1)^\alpha \int_{\mathbf{R}^n} g_\alpha(t) D_t^\alpha \varphi(x-t)\, dt. \tag{4.70}$$

Let β be an arbitrary n-tuple of nonnegative integers. The βth derivative of the sum on the right of (4.70) can be taken under the summation and the integral sign. Hence $U * \varphi \in C^\infty(\mathbf{R}^n)$. With the aid of the chain rule and the definition of $\mathcal{D}((M_p), L^s)$ we find a constant $N > 0$ such that

$$|D_x^\beta(U * \varphi)(x)| = \Big|\sum_{0\leq\alpha<\infty}(-1)^\alpha \int_{\mathbf{R}^n} g_\alpha(t) D_x^\beta D_t^\alpha(x-t)\, dt\Big|$$

$$= \Big| \sum_{0 \le \alpha < \infty} (-1)^{2\alpha} \int_{\mathbf{R}^n} g_\alpha(t) D^{\alpha+\beta}\varphi(x-t)\,dt \Big| \le \sum_{0\le\alpha<\infty} \|g_\alpha\|_{L^r} \|D^{\alpha+\beta}\varphi\|_{L^s}$$

$$\le N \sum_{0\le\alpha<\infty} h^{\alpha+\beta} M_{\alpha+\beta} \|g_\alpha\|_{L^r} \tag{4.71}$$

for every $h > 0$. Due to condition $(M.2)$, there exist positive constants A and H such that

$$M_{\alpha+\beta} \le AH^{\alpha+\beta} M_\alpha M_\beta = AH^{\alpha+\beta} M_\alpha M_\beta$$

Taking this into account in (4.71), we obtain

$$|D_x^\beta(U * \varphi)(x)| \le AN \sum_{0\le\alpha<\infty} h^{\alpha+\beta} H^{\alpha+\beta} M_\alpha M_\beta \|g_\alpha\|_{L^r}$$

and, consequently,

$$\frac{|D_x^\beta(U*\varphi)(x)|}{N(hH)^\beta M_\beta} \le A \sum_{0\le\alpha<\infty} (jH)^\alpha M_\alpha \|g_\alpha\|_{L^r}, \tag{4.72}$$

where $h > 0$ and $j > 0$ are arbitrary and A and H are positive constants. note that we have renamed h to be j on the right side of (4.72). Recalling that the g_α satisfy (2.37) or (2.39) for some $k > 0$, we can appropriately choose the arbitrary $j > 0$ on the right of (4.72) to show, as in the proof of Theorem 4.2.1, that the series on the right side of (4.72) converges. Since $h > 0$ on the left of (4.72) is arbitrary, (4.72) proves the desired growth for $(U * \varphi)$ to be in $\mathcal{D}((M_p), L^\infty)$. The proof is therefore complete for the (M_p) case.

The proof for the $\{M_p\}$ case is obtained by a similar analysis with the use of (4.69) and Theorem 2.3.1. □

The following two results lead us directly to the calculation of the boundary value of the Cauchy integral $C(U; z)$, $z \in T^C$.

Theorem 4.2.4 *Let C be a regular cone in $\mathbf{R}^n$ and let $U \in \mathcal{D}'(*, L^s)$ with $1 < s < \infty$. Let $\varphi \in \mathcal{D}(*, L^1)$. For a fixed $y = \operatorname{Im} z \in C$ we have*

$$\langle C(U; x+iy), \varphi(x)\rangle = \langle U, \langle K(x+iy-t), \varphi(x)\rangle\rangle. \tag{4.73}$$

Proof. By Theorem 4.1.1 we have $K(z - \cdot) \in \mathcal{D}(*, L^s)$ for all s, $1 < s \le \infty$, as a function of $t \in \mathbf{R}^n$ for $z \in T^C$. Put

$$K_y(x) = K(x+iy) = \int_{C^*} \exp(2\pi i\langle x+iy, \eta\rangle)\,d\eta, \qquad y \in C.$$

The proof of Theorem 4.1.1 can be adapted to show that $K_y \in \mathcal{D}(*, L^s)$ for all s, $1 < s \le \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C$. By Lemma 4.2.2,

$$(U * K_y)(x) = \langle U, K(x-t+iy)\rangle = C(U; z) \in \mathcal{D}(*, L^\infty)$$

for $y \in C$ and $1 < s < \infty$: thus

$$\langle C(U:\ z), \varphi(x)\rangle = \langle\langle U, K(z-t)\rangle, \varphi(x)\rangle$$

is well defined for $\varphi \in \mathcal{D}(*, L^1)$ with $y \in C$. Using the representation of $U \in \mathcal{D}'(*, L^s)$ given in either Theorem 2.3.1 or Theorem 2.3.2 and Fubini's theorem we have

$$\begin{aligned}
&\langle C(U; z), \varphi(x)\rangle = \langle\langle U, K(z-t)\rangle, \varphi(x)\rangle \\
&= \sum_{0\leq\alpha<\infty} (-1)^{\alpha} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} g_{\alpha}(t) D_t^{\alpha} K(z-t)\, dt \varphi(x)\, dx \\
&= \sum_{0\leq\alpha<\infty} (-1)^{\alpha} \int_{\mathbf{R}^n} g_{\alpha}(t) \int_{\mathbf{R}^n} D_t^{\alpha} K(z-t)\varphi(x)\, dx\, dt \\
&= \sum_{0\leq\alpha<\infty} (-1)^{\alpha}\langle g_{\alpha}(t), D_t^{\alpha}\langle K(z-t), \varphi(x)\rangle\rangle \\
&= \langle U, \langle K(z-t), \varphi(x)\rangle\rangle
\end{aligned}$$

for $y = \operatorname{Im} z \in C$, and the proof is complete. □

We now show that the term $\langle K(x+iy-t), \varphi(x)\rangle$, $t \in \mathbf{R}^n$, converges in the topology of $\mathcal{D}(*, L^s)$, $2 \leq s < \infty$, to a certain limit function as $y \to 0$, $y \in C$. The choice of the space to which φ belongs depends on whether $*$ is (M_p) or $\{M_p\}$. For $\mathcal{D}((M_p), L^s)$ we have the following result.

Theorem 4.2.5 *Let C be a regular cone and let $I_{C^*}(\eta)$ be the characteristic function of the dual cone C^* of C. Let $\varphi \in \mathcal{D}((M_p), \mathbf{R}^n)$. We have*

$$\lim_{y\to 0, y\in C} \langle K(x+iy-t), \varphi(x)\rangle = \mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t] \tag{4.74}$$

in $\mathcal{D}((M_p), L^s)$, $2 \leq s < \infty$.

Proof. Recall the definition of convergence in $\mathcal{D}((M_p), L^s)$ from Section 2.3. For $\varphi \in \mathcal{D}((M_p), \mathbf{R}^n)$, $\widehat{\varphi}(\eta)$ is an element of the Schwartz space $\mathcal{S}$ and hence $(I_{C^*}(\eta)\widehat{\varphi}(\eta)) \in L^r$ for all r, $1 \leq r \leq \infty$. Thus $\mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta); t]$ can be interpreted as both the L^1 and L^r inverse Fourier transform, $1 < r \leq 2$. For any n-tuple α of nonnegative integers we have

$$D_t^{\alpha}\mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t] = \mathcal{F}^{-1}[I_{C^*}(\eta)\eta^{\alpha}\widehat{\varphi}(\eta):\ t]. \tag{4.75}$$

For all α, $(I_{C^*}(\eta)\eta^{\alpha}\widehat{\varphi}(\eta)) \in L^r$ for all r, $1 \leq r \leq \infty$, since $\widehat{\varphi}(\eta) \in \mathcal{S}$. The Parseval theory for the Fourier transform now yields that the left side of (4.75) is an element of L^s, $1/r + 1/s = 1$, $1 \leq r \leq 2$, for all s and all α. That the left side of (4.75) satisfies the required boundedness condition for elements in $\mathcal{D}((M_p), L^s)$ for each α will follow from techniques that we present later in this proof. The preceding points yield that the right side of (4.74) is an element of $\mathcal{D}((M_p), L^s)$, $2 \leq s < \infty$. Similarly, details in the remaining analysis in this proof will imply that for each $y \in C$,

$$\langle K(x+iy-t), \varphi(x)\rangle \in \mathcal{D}((M_p), L^s),\ 2 \leq s < \infty,$$

as a function of $t \in \mathbf{R}^n$.

We now prove the desired convergence in L^s in order to obtain (4.74). Let $z = x + iy \in T^C$. By a change of order of integration

$$\langle K(x+iy-t), \varphi(x)\rangle = \int\limits_{\mathbf{R}^n}\int\limits_{C^*} e^{2\pi i\langle x+iy-t,\eta\rangle} d\eta \varphi(x)\, dx$$

$$= \int\limits_{\mathbf{R}^n} I_{C^*}(\eta) e^{-2\pi i\langle t,\eta\rangle} e^{-2\pi\langle y,\eta\rangle} \int\limits_{\mathbf{R}^n} \varphi(x) e^{2\pi i\langle x,\eta\rangle}\, dx\, d\eta$$

$$= \int\limits_{\mathbf{R}^n} I_{C^*}(\eta)\widehat{\varphi}(\eta) e^{-2\pi\langle y,\eta\rangle} e^{-2\pi i\langle t,\eta\rangle}\, d\eta. \tag{4.76}$$

For any n-tuple α of nonnegative integers we have

$$\|D_t^\alpha\langle K(z-t), \varphi(x)\rangle - D_t^\alpha \mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t]\|_{L^s}$$
$$= \|\mathcal{F}^{-1}[I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(e^{-2\pi\langle y,\eta\rangle} - 1); t]\|_{L^s}. \tag{4.77}$$

Since $\langle y, \eta\rangle \geq 0$ for $y \in C$ and $\eta \in C^*$, we have

$$|I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(e^{-2\pi\langle y,\eta\rangle} - 1)| \leq 2|\eta^\alpha\widehat{\varphi}(\eta)|, \qquad y \in C. \tag{4.78}$$

Since $\widehat{\varphi}(\eta) \in \mathcal{S}^{(M_p)}$, we conclude from the above inequality that

$$|I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(\exp(-2\pi\langle y,\eta\rangle) - 1)) \in L^r$$

for all r, $1 \leq r \leq \infty$. Thus if $1 < r \leq 2$ and $1/r + 1/s = 1$, we have, by the Parseval inequality,

$$\|\mathcal{F}^{-1}[I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(e^{-2\pi\langle y,\eta\rangle} - 1); t]\|_{L^s}$$
$$\leq \|I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(e^{-2\pi\langle y,\eta\rangle} - 1)\|_{L^r}. \tag{4.79}$$

By (4.78) and the Lebesgue dominated convergence theorem, we have

$$\lim_{y\to 0, y\in C} \int\limits_{\mathbf{R}^n} |I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)(e^{-2\pi\langle y,\eta\rangle} - 1)|^r\, d\eta = 0 \tag{4.80}$$

for $1 < r \leq 2$. Combining (4.77), (4.79), and (4.80) we have

$$\lim_{y\to 0, y\in C} \|D_t^\alpha\langle K(x+iy-t), \varphi(x)\rangle - D_t^\alpha\mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t]\|_{L^s} = 0 \tag{4.81}$$

for all n-tuples α of nonnegative integers and for $2 \leq s < \infty$ as desired.

To complete the proof it remains to show that there is a constant $N > 0$, which is independent of α and $y \in C$, such that for all $h > 0$

$$\|D_t^\alpha\langle K(x+iy-t), \varphi(x)\rangle - D_t^\alpha\mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta); t]\|_{L^s} \leq N h^\alpha M_\alpha \tag{4.82}$$

for each α. As in (4.77), (4.78) and (4.79) we have for $y \in C$ that

$$\|D_t^\alpha\langle K(z-t), \varphi(x)\rangle\|_{L^s} = \|\mathcal{F}^{-1}[I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)e^{-2\pi\langle y,\eta\rangle}; t]\|_{L^s}$$
$$\leq \|I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta)e^{-2\pi\langle y,\eta\rangle}\|_{L^r} \leq \|\eta^\alpha\widehat{\varphi}(\eta)\|_{L^r} \tag{4.83}$$

for $1 < r \le 2$ and $1/r+1/s=1$. Recall the definition of $\mathcal{D}((M_p),\mathbf{R}^n)$ in Section 2.3 and let $\varphi \in \mathcal{D}((M_p),\mathbf{R}^n)$. Put

$$R = \int_{\mathbf{R}^n} \Big|1+\sum_{j=1}^{n} \eta_j^{2n}\Big|^{-r} d\eta, \quad R' = \Big(\int_{\operatorname{supp}\varphi} 1\, dt\Big)^r.$$

Using Fourier transform properties, we have the existence of a constant $B > 0$ from (2.22) such that

$$\begin{aligned}
\int_{\mathbf{R}^n} |\eta^\alpha \widehat{\varphi}(\eta)|^r\, d\eta &= \int_{\mathbf{R}^n} \Big|\eta^\alpha\Big(1+\sum_{j=1}^{n}\eta_j^{2n}\Big)\widehat{\varphi}(\eta)\Big/\Big(1+\sum_{j=1}^{n}\eta_j^{2n}\Big)\Big|^r d\eta \\
&= \int_{\mathbf{R}^n} \Big|\mathcal{F}\Big[\Big(\Big(1+\sum_{j=1}^{n} D_j^{2n}\Big)(-1)^\alpha D^\alpha\Big)\varphi(t);\eta\Big]\Big/\Big(1+\sum_{j=1}^{n}\eta_j^{2n}\Big)\Big|^r d\eta \\
&\le R \sup_{\eta\in\mathbf{R}^n} \Big|\mathcal{F}\Big[\Big(\Big(1+\sum_{j=1}^{n} D_j^{2n}\Big)(-1)^\alpha D^\alpha\Big)\varphi(t);\eta\Big]\Big|^r \\
&\le R\Big(\int_{\mathbf{R}^n} \Big|\Big(\Big(1+\sum_{j=1}^{n} D_j^{2n}\Big)(-1)^\alpha D^\alpha\Big)\varphi(t)\Big|\, dt\Big)^r \\
&\le R\Big(\int_{\mathbf{R}^n} (|D^\alpha\varphi(t)| + |D_1^{2n}D^\alpha\varphi(t)| + \ldots + |D_n^{2n}D^\alpha\varphi(t)|)\, dt\Big)^r \\
&\le RR'(Bk^\alpha M_\alpha + Bk^{\alpha+2n}M_{\alpha+2n} + \ldots + Bk^{\alpha+2n}M_{\alpha+2n})^r \\
&\le RR'(Bn)^r(k^\alpha M_\alpha + k^{\alpha+2n}M_{\alpha+2n})^r \qquad (4.84)
\end{aligned}$$

for all $k > 0$. Using property $(M.2)$ of the sequence (M_p), $p = 0,1,2,\ldots$, we have constants $A > 0$ and $H > 0$ such that

$$M_{\alpha+2n} \le AH^{\alpha+2n}M_\alpha M_{2n}.$$

Putting $H' = \max\{1,H\}$ we continue (4.84) as

$$\begin{aligned}
\int_{\mathbf{R}^n} |\eta^\alpha\widehat{\varphi}(\eta)|^r\, d\eta &\le RR'(Bn)^r(k^\alpha M_\alpha + Ak^{\alpha+2n}H^{\alpha+2n}M_\alpha M_{2n})^r \\
&\le RR'(Bn)^r((kH')^\alpha M_\alpha + A(kH')^{2n}(kH')^\alpha M_\alpha M_{2n})^r \\
&\le RR'(Bn)^r(1+(A(kH')^{2n}M_{2n}))^r((kH')^\alpha M_\alpha)^r. \qquad (4.85)
\end{aligned}$$

Thus, by (4.83) and (4.85), we have

$$\|D_t^\alpha\langle K(z-t),\varphi(x)\rangle\|_{L^s} \le (RR')^{1/r}Bn(1+A(kH)^{2n}M_{2n})h^\alpha M_\alpha \qquad (4.86)$$

for all $h = (kH') > 0$ (recall that $k > 0$ is arbitrary and that $H' > 0$ is fixed). By the same analysis as in (4.86), we get

$$\begin{aligned}
&\|D_t^\alpha\mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t]\|_{L^s} = \|\mathcal{F}^{-1}[I_{C^*}(\eta)\eta^\alpha\widehat{\varphi}(\eta):\ t]\|_{L^s} \\
&\le \|\eta^\alpha\widehat{\varphi}(\eta)\|_{L^r} \le (RR')^{1/r}Bn(1+A(kH')^{2n}M_{2n})h^\alpha M_\alpha \qquad (4.87)
\end{aligned}$$

for all $h > 0$. Combining (4.86) and (4.87) and using the Minkowski inequality, we have (4.82) for each α, where N can be chosen to be

$$N = (RR')^{1/r} Bn(1 + A(kH')^{2n} M_{2n})$$

and $k > 0$ can be chosen arbitrarily. The proof of Theorem 4.2.5 is complete. □

Let us now consider the space $S_\infty(\{N_p\}, \{M_p\}) \subset \mathcal{S}^{(M_p)}$ of Roumieu [75] p. 70, where the sequences (M_p) and (N_p) satisfy conditions $(M.1)$, $(M.2)$ and $(M.3)'$. Under these conditions, the sequences (M_p) and (N_p) satisfy the conditions of Gel'fand and Shilov in [37], (9), p. 245; thus the space $S_\infty(\{N_p\}, \{M_p\}) = S_{N_p}^{(M_p)}$ has the Fourier transform property

$$\mathcal{F}(S_{(M_p)}^{(N_p)}) = S_{(M_p)}^{(N_p)} \subset \mathcal{S}^{(M_p)} \tag{4.88}$$

in the notation of Gel'fand and Shilov (see [37], (11), p. 254). Because of the Fourier transform theory for the spaces $S_\infty(\{N_p\},\{M_p\})$, we may take $\varphi \in S_\infty(\{N_p\},\{M_p\})$ in the below result, corresponding to Theorem 4.2.5.

Theorem 4.2.6 *Let C be a regular cone and let $I_{C^*}(\eta)$ be the characteristic function of the dual cone C^* of C. Let $\varphi \in S_\infty(\{N_p\},\{M_p\})$ (or $\varphi \in \mathcal{D}(\{M_p\},\mathbf{R}^n)$), where both sequences (M_p) and (N_p) satisfy conditions $(M.1)$, $(M.2)$ and $(M.3)$. We have*

$$\lim_{y\to 0, y\in C} \langle K(x+iy-t), \varphi(x)\rangle = \mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta):\ t] \tag{4.89}$$

in $\mathcal{D}(\{M_p\}, L^s)$, $2 \le s < \infty$.

Proof. The proof Theorem 4.2.6 is very similar to that of Theorem 4.2.5. The difference is in the technique to estimate the term on the right side of (4.83) when we take $\varphi \in S_\infty(\{N_p\},\{M_p\})$. We have

$$\int_{\mathbf{R}^n} |\eta^\alpha \widehat{\varphi}(\eta)|^r\, d\eta = \int_{|\eta|\le 1} |\eta^\alpha \widehat{\varphi}(\eta)|^r\, d\eta + \int_{|\eta|>1} |\eta^{\alpha+\overline{2}} \widehat{\varphi}(\eta)|^r \eta^{-\overline{2}r}\, d\eta,$$

where $\overline{2}$ is the n-tuple $(2,2,\ldots,2)$. Now the boundedness condition for convergence in $\mathcal{D}(\{M_p\}, L^s)$ with $2 \le s < \infty$ follows by using (4.88) and the defining growth of $S_\infty(\{N_p\},\{M_p\})$. Of course, (4.89) in Theorem 4.2.6 also holds for $\varphi \in \mathcal{D}(\{M_p\},\mathbf{R}^n)$, by a similar reasoning as in the proof of Theorem 4.2.5. □

Now let $\varphi \in \mathcal{D}(*,\mathbf{R}^n)$. Since $\mathcal{D}(*,\mathbf{R}^n) \subset \mathcal{D}(*,L^1)$, we have (4.73) for $U \in \mathcal{D}'(*,L^s,\ 1 < s < \infty$ and $\varphi \in \mathcal{D}(*,\mathbf{R}^n)$. This fact combined with Theorem 4.2.4, Theorem 4.2.5 and the continuity of U prove the following result:

Corollary 4.2.1 *Let C be a regular cone in $\mathbf{R}^n$. Let $U \in \mathcal{D}'(*,L^s)$, $2 \le s < \infty$. Let $\varphi \in \mathcal{D}(*,\mathbf{R}^n)$. We have*

$$\lim_{y\to 0, y\in C} \langle C(U; x+iy), \varphi(x)\rangle = \langle U, \mathcal{F}^{-1}[I_{C^*}(\eta)\widehat{\varphi}(\eta); t]\rangle. \tag{4.90}$$

Using Theorems 4.2.1, 4.2.2 and Corollary 4.2.1, we obtain an analytic decomposition theorem for elements in $\mathcal{D}'(*, L^s)$, $2 \le s < \infty$.

Theorem 4.2.7 *Let m be a positive integer and let C_j, $j = 1, \ldots, m$, be regular cones such that*

$$\mathbf{R}^n \setminus \sum_{j=1}^{m} C_j^* \quad \text{and} \quad C_j^* \cap C_k^* = \emptyset \ \text{for} \ j,k = 1,\ldots,m; \ j \neq k, \tag{4.91}$$

are sets of the Lebesgue measure zero. Let $U \in \mathcal{D}'(*, L^s)$, $2 \le s < \infty$, and $\varphi \in \mathcal{D}(*, \mathbf{R}^n)$. Then there exist functions f_j which are analytic in $\mathbf{R}^n + iC_j$, $j = 1, \ldots, m$, such that

$$\langle U, \varphi \rangle = \sum_{j=1}^{m} \lim_{y \to 0, y \in C_j} \langle f_j(x+iy), \varphi(x) \rangle. \tag{4.92}$$

If $U \in \mathcal{D}'((M_p),\ L^s)$, then for each $j = 1, \ldots, m$ and for each compact subcone $C_j' \subset C_j$ there are constants $A_j = A_j(n, C_j', s) > 0$ and $T_j = T_j(C_j') > 0$ such that

$$|f_j(x+iy)| \le A_j |y|^{-n/r} \exp(M^*(T_j/|y|)), \qquad z = x+iy \in \mathbf{R}^n + iC_j', \tag{4.93}$$

where n is the dimension, $1/r + 1/s = 1$ and M^* is the function defined in (2.8).

If $U \in \mathcal{D}'(\{M_p\}, L^s)$, then for each $j = 1, \ldots, m$. each compact subcone $C_j' \subset C_j$ and arbitrary constant $T_j > 0$, which is independent of $C_j' \subset C_j$, there is a constant $A_j = A_j(n, C_j', s) > 0$ such that (4.93) holds.

Proof. For each $j = 1, \ldots, m$ put

$$f_j(x+iy) = \langle U_t, \int_{C^*} \exp(2\pi i \langle z - t, \eta \rangle)\, d\eta, \quad z = x+iy \in \mathbf{R}^n + iC_j.$$

By Theorems 4.2.1 and 4.2.2 each $f_j(z)$ is holomorphic in $\mathbf{R}^n + iC_j$ and satisfies the relevant version of (4.93) for $*$ being (M_p) or $\{M_p\}$. To prove (4.92) first note that

$$\lim_{y \to 0, y \in C_j} \langle f_j(x+iy), \varphi(x) \rangle = \langle U, \mathcal{F}^{-1}[I_{C_j^*}(\eta)\widehat{\varphi}(\eta) \colon\ t] \rangle, \quad j = 1, \ldots, m, \tag{4.94}$$

by Theorem 4.2.5. Since $\varphi \in \mathcal{D}(*, \mathbf{R}^n) \subset \mathcal{S}^{(M_p)}$ then $\widehat{\varphi} \in \mathcal{S}^{(M_p)}$, and we know that $\mathcal{F}^{-1}[\widehat{\varphi}(\eta) \colon\ t] = \varphi(t)$. We now use the linearity of U, the assumptions (4.91) on the dual cones C_j^*, $j = 1, \ldots, m$, and (4.94) to obtain

$$\sum_{j=1}^{m} \lim_{y \to 0, y \in C_j} \langle f_j(x+iy), \varphi(x) \rangle = \sum_{j=1}^{m} \langle U, \mathcal{F}^{-1}[I_{C_j^*}(\eta)\widehat{\varphi}(\eta) \colon\ t] \rangle$$

$$= \langle U, \sum_{j=1}^{m} \mathcal{F}^{-1}[I_{C_j^*}(\eta)\widehat{\varphi}(\eta); t] \rangle = \langle U, \mathcal{F}^{-1}[\widehat{\varphi}(\eta); t] \rangle = \langle U, \varphi \rangle.$$

The proof is complete. □

The 2^n n-rants in $\mathbf{R}^n$ are an example of a finite number of regular cones for which (4.91) holds. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be any of the 2^n n-tuples whose components are 0 or 1. $C_\mu = \{y \in \mathbf{R}^n : (-1)^{\mu_j} y_j > 0,\ j = 1, \ldots, n\}$ is n-rant in $\mathbf{R}^n$ and is a regular cone with the property that $C^*_\mu = \overline{C}_\mu$. Thus Theorem 4.2.9 holds, in particular, for $m = 2^n$ with each C_j being an n-rant C_μ in $\mathbf{R}^n$.

Of course we can also state a L^s norm growth estimate on each $f_j(z)$, $z \in \mathbf{R}^n + iC'_j$, $C'_j \subset\subset C_j$, $j = 1, \ldots, m$ because of Theorem 4.2.3.

We desire to obtain Theorem 4.2.5, 4.2.6, 4.2.7, and Corollary 4.2.1 for $1 < s < 2$ as well. We consider this in future research.

4.3 Poisson integral of ultradistributions

As in Section 4.2 we let C be a regular cone in $\mathbf{R}^n$ and let the sequence of positive real numbers (M_p), $p = 0, 1, 2, \ldots$, satisfy the cinditions $(M.1)$ and $(M.3)'$.

Let $U \in \mathcal{D}'(*, L^s)$, $1 \le s \le \infty$ (where $*$ represents either (M_p) or $\{M_p\}$). Because of Theorem 4.1.2 we can form

$$P(U; z) = \langle U_t, Q(z; t)\rangle, \quad z \in T^C, \tag{4.95}$$

where $Q(z; t)$, $z \in T^C$, $t \in \mathbf{R}^n$, is the Poisson kernel coressponding to the tube T^C. $P(U; z)$ defined in (4.95) is the Poisson integral of the ultradistribution U with respect to the tube T^C. In contrast to the Cauchy integral, the Poisson integral $P(U; z)$ is not a holomorphic function of $z \in T^C$ in general. If the cone C is a half line $(0, \infty)$ or $(\infty, 0)$ in $\mathbf{R}^1$, $P(U; z)$ is a harmonic function, if C is a n-rant C_μ in $\mathbf{R}^n$, $P(U; z)$ is an n-harmonic function.

Throughout the remainder of this section the sequence (M_p) satisfies the cinditions $(M.1)$, $(M.2)$, and $(M.3)'$.

We will prove that the Poisson integral $P(U; z)$ defined in (4.95) obtains v as its boundary value as $y \to 0$, $y \in C$, for an arbitrary regular cone C; thus the Poisson integral $P(U; z)$ has boundary values in arbitrary with Poisson integrals pf L^p functions in tubes. To obtain the desired boundary value result we need two preliminary theorems.

Theorem 4.3.1 *Let C be a regular cone in $\mathbf{R}^n$. Let $U \in \mathcal{D}'(*, L^s)$, $1 < s < \infty$. Let $\varphi \in \mathcal{D}(*, L^1)$. We have*

$$\langle P(U; x + iy), \varphi(x)\rangle = \langle U_t, \langle Q(x + iy; t), \varphi(x)\rangle, \ y \in C. \tag{4.96}$$

Proof.

$$K_y(x) = K(x + iy) = \int_{C^*} \exp(2\pi i\langle x + iy, \eta\rangle)\, d\eta, \quad y \in C,$$

as in the proof of Theorem 4.2.4; and put

$$Q_y(x) = \frac{K(x + iy)\overline{K(x + iy)}}{K(2iy)}, \quad x \in \mathbf{R}^n,\ y \in C. \tag{4.97}$$

As in Lemma 1.3.5 we have $Q_y(x) \geq 0,\ x \in \mathbf{R}^n,\ y \in C$, and $Q(z;y) \geq 0,\ z = x+iy \in T^C,\ t \in \mathbf{R}^n$. Further, we have

$$\int_{\mathbf{R}^n} Q_y(x)\, dx = 1, \qquad y \in C,$$

and if $\delta > 0$,

$$\lim_{y \to 0, y \in C} \int_{|x| \geq \delta} Q_y(x)\, dx = 0.$$

The proof of Theorem 4.1.2 can be adapted to show that $Q_y(x) \in \mathcal{D}(*, L^s),\ 1 \leq s \leq \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C$. By Lemma 4.2.4

$$(U * Q_y)(x) = \langle U_t, Q_y(x-t)\rangle = \langle U_t, Q(x+iy;t)\rangle = P(U; x+iy)$$

is an element of $\mathcal{D}(*, L^\infty)$; hence $\langle P(U; x+iy), \varphi(x)\rangle$ is a well defined function of $y \in C$ for $\varphi \in \mathcal{D}(*, L^1)$. The proof of (4.96) is now completed by using the representation of $U \in \mathcal{D}'(*, L^s)$ gives in Theorem 2.3.1 and 2.3.2 and Fubini's theorem as in the proof of Lemma 4.2.2. □

Theorem 4.3.2 *Let C be a regular cone in $\mathbf{R}^n$ and $\varphi \in \mathcal{D}(*, L^s)$, $1 \leq s < \infty$. We have*

$$\lim_{y \to 0, y \in C} \langle Q(x+iy;t), \varphi(x)\rangle = \varphi(t) \tag{4.98}$$

in $\mathcal{D}(*, L^s)$.

Proof. We first consider the case that $*$ is (M_p). Let $\varphi \in \mathcal{D}((M_p), L^s)$. Differentiation under the integral sign shows that $\langle Q(x+iy;t), \varphi(x)\rangle$ is an infinitely differentiable function of $t \in \mathbf{R}^n$ for $y \in C$. Boundedness techniques which will be used later in this proof show that any derivative of $\langle Q(x+iy;t), \varphi(x)\rangle$ is an element of L^s as a function of $t \in \mathbf{R}^n$ for each $y \in C$ and satisfies the defining growth (2.23) of elements in $\mathcal{D}((M_p), L^s)$ (see (4.102) below); thus $\langle Q(x+iy;t), \varphi(x)\rangle \in \mathcal{D}((M_p), L^s)$ for each $y \in C$. For any n-tuple α of non-negative integers, we make a change of variable and obtain

$$\begin{aligned} &\|D_t^\alpha \int_{\mathbf{R}^n} Q(x+iy:\ t)\varphi(x)\, dx - D_t^\alpha \varphi\|_{L^s} \\ &= \|D_t^\alpha \int_{\mathbf{R}^n} \varphi(x+t) Q_y(x)\, dx - D_t^\alpha \varphi(t)\|_{L^s} \\ &= \|\int_{\mathbf{R}^n} \psi(x+t) Q_y(x)\, dx - \psi(t)\|_{L^s}, \end{aligned} \tag{4.99}$$

where $\psi(t) = D_t^\alpha \varphi(t)$ and $Q_y(x)$ is given in (4.97). Using the approximate identity properties of the Poisson kernel as given in the proof of Theorem 4.3.1, the proof of [19], Lemma 7, p. 213, shows that the right side of (4.99) approaches zero as $y \to 0,\ y \in C$. Thus

$$\lim_{y \to 0, y \in C} D_t^\alpha \langle Q(x+iy;t), \varphi(x)\rangle = D_t^\alpha \varphi(t) \tag{4.100}$$

in L^s for any n-tuple α of nonnegative integers.

To complete the proof of the result for $*$ being (M_p), the boundedness condition (2.28) for convergence in $\mathcal{D}((M_p), L^s)$ remain to be shown. Proceeding as in (4.99) we use a change of variable and the chain rule to obtain

$$\|D_t^\alpha \int_{\mathbf{R}^n} Q(x+iy:\ t)\varphi(x)\,dx\|_{L^s}$$
$$= \| \int_{\mathbf{R}^n} D_t^\alpha \varphi(x+t) Q_y(x)\,dx\|_{L^s} = \| \int_{\mathbf{R}^n} D_t^\alpha \varphi(u) Q(u+iy;t)\,du\|_{L^s} \tag{4.101}$$

By Jensen's inequality [34], 2.4.19, p. 91, Fubini's theorem and the approximate identity properties of the Poisson kernel stated in the proof of Theorem 4.3.1 we have

$$\int_{\mathbf{R}^n} \Big| \int_{\mathbf{R}^n} D_u^\alpha \varphi(u) Q(u+iy;t)\,du\Big|^s dt \le \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |D_u^\alpha\varphi(u)|^s Q(u+iy;t)\,du\,dt$$
$$= \int_{\mathbf{R}^n} |D_u^\alpha \varphi(u)|^s \int_{\mathbf{R}^n} Q(u+iy;t)\,dt\,du = \int_{\mathbf{R}^n} |D_u^\alpha\varphi(u)|^s\,du. \tag{4.102}$$

Combining (4.101), (4.102), and the fact that $\varphi \in \mathcal{D}((M_p), L^s)$ we have the existence of a constant $N > 0$ from (2.23) such that

$$\|D_t^\alpha \int_{\mathbf{R}^n} Q(x+iy;t)\varphi(x)\,dx\|_{L^s} \le \|D_u^\alpha \varphi(u)\|_{L^s} \le N h^\alpha M_\alpha \tag{4.103}$$

for all $h > 0$. By (4.103) and the Minkowski inequality we have

$$\|D_t^\alpha \int_{\mathbf{R}^n} Q(x+iy;t)\varphi(x)\,dx - D_t^\alpha \varphi\|_{L^s} \le 2N h^\alpha M_\alpha \tag{4.104}$$

for all $h > 0$ and all $y \in C$. Now we combine formulae (4.100) and (4.104), which hold for all α, to prove (4.98) in $\mathcal{D}((M_p), L^s)$ for $1 \le s < \infty$.

The proof of (4.98) for $\mathcal{D}(\{M_p\}, L^s)$ is similar. The proof is completed. □

We can now obtain the desired boundary value property of the Poisson integral $P(U; z)$.

Theorem 4.3.3 *Let C be a regular cone in $\mathbf{R}^n$. Let $U \in \mathcal{D}'(*, L^s)$, $1 \le s < \infty$, and let $\varphi \in \mathcal{D}(*, \mathbf{R}^n)$. We have*

$$\lim_{y \to 0, y \in C} \langle P(U; x+iy), \varphi(x)\rangle = \langle U, \varphi\rangle. \tag{4.105}$$

Proof. First recall that $\mathcal{D}(*, \mathbf{R}^n) \subset \mathcal{D}(*, L^s)$ for all s, $1 \le s \le \infty$. Formula (4.105) now follows by combining Theorems 4.3.1, 4.3.2 and the continuity of U. □

It is interesting to note that the boundary value of the Cauchy integral $C(U;z)$ as obtained in equation (4.90) of Theorem 4.2.8 depends on the cone C, whereas the boundary value of the Poisson integral $P(U;z)$ in (4.105) is always U independently of the cone C.

Let us note that for $*$ being $\{M_p\}$, Theorem 4.3.3 can be slightly generalized by taking $\varphi \in \mathcal{S}_\infty(\{N_p\},\{M_p\}) = \mathcal{S}^{(N_q)}_{(M_p)}$, the space of Roumieu [75], p. 70 and Gel'fand and Shilov [37], p. 245, as was done in Theorem 4.2.7. Of course both sequences (M_p) and (N_p) are taken to satisfy the conditions $(M.1)$, $(M.2)$ and $(M.3)'$. We write $\mathcal{S}_\infty(\{M_p\},\{N_p\})$ here, instead of $\mathcal{S}_\infty(\{N_p\},\{M_p\})$ as in Theorem 4.2.7, because the Fourier transform is not introduced in the proof here; recall (4.88).

Chapter 5

Boundary values of analytic functions

Analytic functions in tubes which satisfy certain growth conditions involving the (M_p) sequences are shown in this chapter to obtain ultradistribution boundary values. As a basis for the boundary value results we define and study generalizations of the Hardy spaces H^r corresponding to tubes in C^n. For analytic functions considered in the chapter, representations are obtained in terms of the Fourier-Laplace and Cauchy integrals.

5.1 Generalizations of H^r functions in tubes

Let B be a proper open subset of $\mathbf{R}^n$. The set of analytic functions f in $T^B = \mathbf{R}^n + iB$ of variavble $z = x + iy$ $(x \in \mathbf{R}^n, y \in B)$, which satisfy the estimate

$$\|f(\cdot + iy)\|_{L^r} \leq M, \qquad y \in B,$$

where the constant M is independent of $y \in B$, is called the Hardy space $H^r(T^B)$, $r > 0$. Stein and Weiss in [78] have obtained representation and boundary value results for the Hardy spaces (see [78] for additional references concerning H^r functions). Generalizations of the spaces $H^r(T^B)$ have been considered and analyzed by several authors including Vladimirov [85], Carmichael and Hayashi [20], and Carmichael [9] - [18].

As in [78], let B denote a proper open subset of $\mathbf{R}^n$, the base of the tube T^B. Let $d(y)$ denote the distance from $y \in B$ to the complement of B in $\mathbf{R}^n$. The space $S_A^r(T^B)$, where $0 < r < \infty$ and $A \geq 0$, is the set of all analytic functions f (of variable $z = x + iy$) in $T^B = \mathbf{R}^n + iB$, which satisfy the inequality

$$\|f(\cdot + iy)\|_{L^r} \leq M(1 + (d(y))^{-m})^q \exp(2\pi A|y|), \; y \in B,$$

for some constants $m \geq 0$, $q \geq 0$ and $M > 0$ which can depend on f, r, and A but not on $y \in B$. If $B = C$, i.e. B is a cone, $d(y)$ is interpreted to be the distance from $y \in C$ to the boundary of C. The spaces $S_A^r(T^B)$ were defined and studied by Carmichael in [9] - [17]. For various values of r and various bases B of the tube T^B, Carmichael has obtained Cauchy, Poisson, and

Fourier-Laplace integral representations of the $S_A^r(T^B)$ functions and boundary value results. The $\dot{H}^r$ functions, the functions of Vladimirov [85], and the functions of Carmichael and Hayashi [20] are all special cases of the $S_A^r(T^B)$ functions.

As an example let us consider the cone $(0,\infty)$ in $\mathbf{R}^1$ and the corresponding tube $T^{(0,\infty)}$ which is the upper half plane in C^1. The function g defined by

$$g(z) = \exp\left(\frac{-2\pi i z}{(i+z)z}\right)$$

is an element of the space $S_1^2(T^{(0,\infty)})$, but is not in the Hardy space $H^2(T^{(0,\infty)})$.

In this section we wish to consider other generalizations of H^r-functions, generalizations associated with sequences (M_p). They are introduced by the norm growth (4.57) - (4.58) obtained on the Cauchy integral of ultradistributions in $\mathcal{D}'(*, L^s)$.

The results of this section will be useful in obtaining the ultradistributional boundary value results in the next section.

Again let B be a proper open subset of $\mathbf{R}^n$ and let $d(y)$ denote the distance from $y \in B$ to the complement of B in $\mathbf{R}^n$. Now let f satisfy the inequality

$$\|f(\cdot + iy)\|_{L^r} \leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B, \tag{5.1}$$

where $K > 0$, $T > 0$, $m \geq 0$, and $q \geq 0$ are all independent of $y \in B$ and M^* is the associated function of the sequence (M_p) defined in (2.8). The space of analytic functions in T^B which satisfy (5.1) will be denoted by $H^r_{(M_p)}(T^B)$. The spaces $H^r_{(M_p)}(T^B)$ for $0 < r < \infty$ are a generalization of the Hardy spaces.

If $B = C$, where c is an open connected cone in $\mathbf{R}^n$, we get from the formula

$$d(y) = \inf_{t \in pr(C^*)} \langle t, y \rangle, \quad y \in C,$$

given in [84], p. 159] the estimate $d(y) \leq |y|$, $y \in C$. From this inequality for $B = C$ we see that the term $(1 + (d(y))^{-m})^q$ in (5.1) is a generalizing factor of the growth in which the standard term $|y|$ is replaced by $d(y)$. Further, the right hand side of (5.1) allows for divergence to ∞ as y approaches any point on the boundary of C and not only as y approaches just 0 as would be in the standard case (i.e. if $|y|$ were in place of $d(y)$ in (5.1)).

In this section we shall obtain necessary and sufficient conditions that elements of $H^r_{(M_p)}(T^B)$ spaces, where B is a proper open connected subset of $\mathbf{R}^n$, are representable by Fourier-Laplace integrals for certain values of r. We also obtain a Cauchy integral representation. As indicated previously, these results will lead us to boundary value and related results in the following section.

Througout this section we assume that the sequence (M_p) satisfies conditions $(M.1)$ and $(M.3)'$.

We begin with proving some lemmas.

Lemma 5.1.1 *Let B be a proper open connected subset of $\mathbf{R}^n$. Suppose that $1 \leq s < \infty$ and g is a measurable function on $\mathbf{R}^n$ which satisfies the estimate*

$$\|e^{-2\pi\langle y,\cdot\rangle} g\|_{L^s} \leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B, \tag{5.2}$$

where $K > 0$, $T > 0$, $m \geq 0$, and $q \geq 0$ are independent of $y \in B$. The function F given by

$$F(z) = \int_{\mathbf{R}^n} g(t)e^{2\pi i\langle z,t\rangle}\,dt, \quad z \in T^B, \tag{5.3}$$

is analytic in the tube T^B.

Proof. The proof is an extension of that in [[20], Theorem 2.1]. Let $y_0 \in B$ be arbitrary. Choose an open neighbourhood $N(y_0)$ of y_0 such that $\overline{N(y_0)} \subset B$. There exists a $\delta > 0$ such that $\{y; |y - y_0| = \delta\} \subset N(y_0)$. Decompose $\mathbf{R}^n$ into a finite union of nonoverlapping cones $C_1, C_2, \ldots, C_k$, each having vertex at the origin, such that $\langle u, v\rangle \geq (2^{1/2}/2)|u||v|$, whenever u and v are two points in a certain C_j for $j = 1, \ldots, k$.

For each $j = 1, \ldots, k$ choose a fixed y_j such that $y_0 - y_j \in C_j$ and $|y - y_0| = \delta$. Then, taking $\varepsilon = 2^{1/2}\pi s\delta > 0$, we have

$$-2\pi s\langle y_j - y_0, t\rangle \geq \varepsilon|t| = 2\pi s(2^{1/2}/2)|y_0 - y_j|, \tag{5.4}$$

whenever $1 \leq s < \infty$ and $t \in C_j (j = 1, \ldots, k)$. For each $j = 1, \ldots, k$, (5.4) and (5.2) yield

$$\begin{aligned} &\int_{C_j} |g(t)|^s \exp(-2\pi s\langle y_0, t\rangle)\exp(\varepsilon|t|)\,dt \\ &\leq \int_{\mathbf{R}^n} |g(t)|^s \exp(-2\pi s\langle y_j, t\rangle)\,dt \\ &\leq K^s(1 + (d(y_j))^{-m})^{sq}\exp(sM^*(T/|y_j|), \end{aligned} \tag{5.5}$$

since $y_j \in \{y : |y - y_0| = \delta\} \subset N(y_0) \subset B$ for $j = 1, \ldots, k$. Now, (5.5) yields

$$\begin{aligned} &\int_{\mathbf{R}^n} |g(t)|^s \exp(-2\pi s\langle y_0, t\rangle)\exp(\varepsilon|t|)\,dt \\ &\leq K^s \sum_{j=1}^{k} (1 + (d(y_j))^{-m})^{sq}\exp(sM^*(T/|y_j|)). \end{aligned} \tag{5.6}$$

For $s = 1$, (5.6) and the fact that $(\varepsilon|t|/2) \leq \varepsilon|t|$, $t \in \mathbf{R}^n$, yield

$$\begin{aligned} &\int_{\mathbf{R}^n} |g(t)|\exp(-2\pi\langle y_0, t\rangle)\exp(\varepsilon|t|/2)\,dt \\ &\leq K \sum_{j=1}^{k} (1 + (d(y_j))^{-m})^{q}\exp(M^*(T/|y_j|)). \end{aligned} \tag{5.7}$$

For $1 < s < \infty$, *by* Hölder's inequality, the identity

$$\exp(\varepsilon|t|/2s) = \exp(\varepsilon|t|/s)\exp(-\varepsilon|t|/2s)$$

and by (5.6), we get

$$\begin{aligned}
&\int_{\mathbf{R}^n} |g(t)| \exp(-2\pi\langle y_0, t\rangle) \exp(\varepsilon|t|/2s)\, dt \\
&\le A\Big(\int_{\mathbf{R}^n} |g(t)|^s \exp(-2\pi s\langle y_0, t\rangle) \exp(\varepsilon|t|)\, dt\Big)^{1/s} \\
&\le AK\Big(\sum_{j=1}^{k}(1+(d(y_j))^{-m}\Big)^{sq} \exp(sM^*(T/|y_j|))\Big)^{1/s} \qquad (5.8)
\end{aligned}$$

where $1/s + 1/r = 1$ and

$$A = \Big(\int_{\mathbf{R}^n} \exp(-\varepsilon r|t|/2s)\, dt\Big)^{1/r}.$$

If $|y - y_0| < \varepsilon/4\pi s$, $y = \operatorname{Im} z$ and $1 \le s < \infty$, then

$$\begin{aligned}
\left|g(t)e^{2\pi i\langle z,t\rangle}\right| &= |g(t)| \exp(-2\pi\langle y - y_0, t\rangle) \exp(-2\pi\langle y_0, t\rangle) \\
&\le |g(t)| \exp(2\pi|y - y_0||t|) \exp(-2\pi\langle y_0, t\rangle) \\
&\le |g(t)| \exp(\varepsilon|t|/2s) \exp(-2\pi\langle y_0, t\rangle) \qquad (5.9)
\end{aligned}$$

for all $t \in \mathbf{R}^n$. Estimates (5.7) and (5.8) now show that the right side of (5.9) is an L^1-function, independent of y such that $|y - y_0| < \varepsilon/4\pi s$, whenever $1 \le s < \infty$. Since $y_0 \in B$ is arbitrary, we conclude from (5.9) that $F(z)$ defined by (5.3) is an analytic function of $z \in T^B$. Estimate (5.9) also proves that $\exp(-2\pi\langle y, t\rangle)g(t) \in L^1$, $y \in B$, whenever $1 \le s < \infty$. The proof is complete. □

In the following lemma, supp(g) denotes the support of g.

Lemma 5.1.2 *Let C be an open connected cone in $\mathbf{R}^n$. Let $1 \le s < \infty$. Let $g(t)$ be a measurable function on $\mathbf{R}^n$ such that (5.2) holds for $y \in C$. We have* supp $g \subseteq C^*$ *almost everywhere.*

Proof. Assume that $g(t) \ne 0$ on a set of positive measure in $\mathbf{R}^n \setminus C^* = \{t : \langle y, t\rangle < 0 \quad \text{for some } y \in C\}$: then there is a point $t_0 \in \mathbf{R}^n \setminus C^*$ such that $g(t) \ne 0$ on a set of positive measure in the neighbourhoods $N(t_0 : \eta) = \{t : |t - t_0| < \eta\}$ for arbitrary $\eta > 0$. Since $t_0 \in \mathbf{R}^n \setminus C^*$ there is a point $y_0 \in pr(C) \subset C$ such that $\langle t_0, y_0\rangle < 0$, where $pr(C)$ denotes the projection of C which is $\{y \in C : |y| = 1\}$. Using the continuity of $\langle t, y_0\rangle$ as a function t, there is a fixed $\sigma > 0$ and a fixed neighbourhood $N(t_0; \eta')$ such that

$$\langle t, y_0\rangle < -\sigma < 0 \text{ for all } t \in N(t_0 : \eta').$$

Choose η above to be η'. For any $\lambda > 0$ we thus have

$$-\langle\lambda y_0, t\rangle = -\lambda\langle y_0, t\rangle > \lambda\sigma > 0, \quad t \in N(t_0 : \eta'), \quad \lambda > 0. \qquad (5.10)$$

Since $y_0 \in pr(C) \subset C$ and C is a cone then $\lambda y_0 \in C$, $\lambda > 0$. Using (5.10) and (5.2) with $y = \lambda y_0$ we have, for all $\lambda > 0$,

$$\begin{aligned} e^{2r\pi\lambda\sigma} \int_{N(t_0;\eta')} |g(t)|^s \, dt &\leq \int_{N(t_0:\, \eta')} |g(t)|^s \exp(-2r\pi\langle\lambda y_0, t\rangle) \, dt \\ &\leq \int_{\mathbf{R}^n} |g(t)|^s \exp(-2\pi s\langle\lambda y_0, t\rangle) \, dt \\ &\leq K^s(1 + (d(\lambda y_0))^{-m})^{qs} \exp(sM^*(T/|\lambda y_0|)) \\ &= K^s(1 + \lambda^{-m}(d(y_0))^{-m})^{qs} \exp(sM^*(T/\lambda)), \end{aligned} \qquad (5.11)$$

since $d(\lambda y_0) = \lambda d(y_0)$ and $y_0 \in pr(C)$ imply $|y_0| = 1$. The integral on the left of (5.11) is finite. We let $\lambda \to \infty$ in (5.11); thus for the fixed $T > 0$, which is independent of $y \in C$, we can consider $\lambda > 2T$. For the sequence (M_p) which defines M^* we have

$$(T/\lambda)^p p!(M_0/M_p) < (1/2)^p p!(M_0/M_p), \quad \lambda > 2t,$$

for $p = 0, 1, 2, \ldots$: and

$$M^*(T/\lambda) < M^*(1/2) < \infty, \quad \lambda > 2T, \qquad (5.12)$$

from the definition of M^* and the fact that $(p!M_0/M_p) \to 0$ as $p \to \infty$ (see [48], p. 74), since (M_p) satisfies $(M.1)$ and $(M.3)'$. By (5.11) and (5.12),

$$e^{2\pi s\lambda\sigma} \int_{N(t_0:\, \eta')} |g(t)|^s \, dt \leq K^s[1 + (\lambda d(y_0))^{-m}]^{qs} \exp(sM^*(1/2))$$

and thus

$$e^{2\pi s\lambda\sigma}[1 + (\lambda d(y_0))^{-m}]^{-qs} \int_{N(t_0:\, \eta')} |g(t)|^s \, dt \leq K^s \exp(sM^*(1/2)) \qquad (5.13)$$

for $\lambda > 2T$. Since all in (5.13) are independent of λ, we let $\lambda \to \infty$. We conclude that g must be zero almost everywhere in $N(t_0:\ \eta')$, since $\exp(2\pi s\lambda\sigma) \to \infty$ for $\sigma > 0$ and $[1 + (\lambda d(y_0))^{-m}]^{-qs} \to 1$ as $\lambda \to \infty$.

But this contradicts the fact that $g(t) \neq 0$ on a set of positive measure in $N(t_0:\ \eta')$. Thus g must be zero almost everywhere in $\mathbf{R}^n \setminus C^*$ and supp $g \subseteq C^*$ almost everywhere, since the dual cone C^* of C is a closed set in $\mathbf{R}^n$. The proof is complete. □

The next two lemmas concern the spaces $H^r_{(M_p)}(T^B)$, $0 < r < \infty$, defined in the paragraph containing (5.1).

In the following lemma B^c denotes the complement of B in $\mathbf{R}^n$, and

$$d(B',\ B^c) = \inf\{|y_1 - y_2| :\ y_1 \in B',\ y_2 \notin B\}$$

is the distance from $B' \subset B$ to B^c.

Lemma 5.1.3 *Let B denote on open connected subset of $\mathbf{R}^n$ which does not contain 0. Let $0 < r < \infty$ and $f(z) \in H^r_{(M_p)}(T^B)$. Let B' be a subset of B which satisfies $d(B', B^c) \geq 2\delta > 0$ for some $\delta > 0$. There is a constant K' depending on the $\delta > 0$, on the dimension n, and on $f(z)$, but not on $z + x + iy \in T^B$, such that*

$$|f(x+iy)| \leq K' \exp(M^*(T/(|y| - \delta)), \quad x + iy \in T^{B'}. \tag{5.14}$$

Proof. Let $z_0 = x_0 + iy_0$ be a arbitrary point in $T^{B'}$. Put

$$R_\delta = \{z \in \mathbf{C}^n : \ |z - z_0| < \delta\} \text{ and } N(y_0 : \ \delta) = \{y \in \mathbf{R}^n : \ |y - y_0| < \delta\}.$$

Then $R_\delta \subset \mathbf{R}^n + iN(y_0; \delta) \subset T^B$ and

$$\Big(\int_{R_\delta} |f(x+iy)|^r \, dx \, dy\Big)^{1/r} \leq \Big(\int_{N(y_0: \ \delta)} \int_{\mathbf{R}^n} |f(x+iy)|^r \, dx \, dy\Big)^{1/r}$$
$$\leq K\Big(\int_{N(y_0: \ \delta)} \Big(1 + (d(y))^{-m}\Big)^{rq} \exp(rM^*(T/|y|)) \, dy\Big)^{1/r} \tag{5.15}$$

by using (5.1). Now

$$\exp(rM^*(T/|y|)) \leq \exp(rM^*(T/(|y_0| - \delta))), \ y \in N(y_0; \delta). \tag{5.16}$$

Recall that $\inf\{|y_1 - y_2| : y_1 \in B', \ y_2 \notin B\} \geq 2\delta > 0$. Hence $y \in N(y_0 : \ \delta)$ implies $d(y) \geq \delta$. Thus

$$(1 + (d(y))^{-m})^{rq} \leq (1 + \delta^{-m})^{rq}, \ \ y \in N(y_0 : \ \delta). \tag{5.17}$$

Combining (5.15), (5.16), and (5.17) we get

$$\Big(\int_{R_\delta} |f(x+iy)|^r dx \, dy\Big)^{1/r} \leq KN(1 + \delta^{-m})^q \exp(M^*(T/(|y| - \delta))) \tag{5.18}$$

where $N = (\text{measure}(N(y_0 : \ \delta))^{1/r}$ is a number that is actually independent of any given $y_0 \in B'$ because N has the same value for each $y_0 \in B'$: hence N depends only on δ and the dimension n and not on $y_0 \in B'$.

Since $f(z)$ is analytic in T^B then $|f(z)|^r$, $0 < r < \infty$, is a subharmonic function of the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, $z = x + iy \in T^B$ [78], p. 79. Thus [78], Chapter 2, Section 4] yields

$$|f(z_0)|^r \leq (\Omega_{2n} \delta^{2n})^{-1} \int_{R_\delta} |f(x+iy)|^r \, dx \, dy \tag{5.19}$$

where Ω_{2n} is the volume of the unit sphere in $\mathbf{R}^{2n}$. The desired estimate (5.14) now follows by combining (5.18) and (5.19) and observing that $z_0 = x_0 + iy_0 \in T^{B'}$ was arbitrary. □

Lemma 5.1.4 *Let B denote an open connected subset of $\mathbf{R}^n$ which does not contain 0. Let $1 < r \le 2$. Let $f(z) \in H^r_{(M_p)}(T^B)$. For all y and y' in B we have*

$$e^{2\pi\langle y,t\rangle}h_y(t) = e^{2\pi\langle y',t\rangle}h_{y'}(t) \tag{5.20}$$

for almost every $t \in \mathbf{R}^n$, where

$$h_y(t) = \mathcal{F}^{-1}[f(x+iy);t], \quad y \in B,$$

is the L^s, $1/r + 1/s = 1$, inverse Fourier transform of $f(x+iy)$, $y \in B$.

The proof of Lemma 5.1.4 follows from the growth (5.14) by analysis similar to that in [78], pp. 99-101, with Lemma 5.1.3 taking the place of [78], Lemma 2.12, p. 99, in the proof of Lemma 5.1.4. Further, in the proof of Lemma 5.1.4 the Parseval equlity in the case $r = 2$ as in [78], top of p. 101. We thus leave the details of the proof of Lemma 5.1.4 to the interested reader.

We can now give Fourier-Laplace and, in certain instances, Cauchy integral representations of the $H^r_{(M_p)}(T^B)$ functions for certain values of r.

Theorem 5.1.1 *Let B denote an open connected subset of $\mathbf{R}^n$ which does not contain $0 \in \mathbf{R}^n$. Let $f(z) \in H^r_{(M_p)}(T^B)$, $1 < r \le 2$. There exists a measurable function $g(t)$, $t \in \mathbf{R}^n$, such that (5.2) holds for $y \in B$, $1/r + 1/s = 1$, where $K > 0$, $T > 0$, $m \ge 0$ and $q \ge 0$ are independent of $y \in B$; and*

$$f(z) = \int\limits_{\mathbf{R}^n} g(t)e^{2\pi i\langle z,t\rangle}\,dt, \quad z \in T^B. \tag{5.21}$$

Proof. Put

$$g(t) = e^{2\pi\langle y,t\rangle}h_y, \quad y \in B, \tag{5.22}$$

where $h_y(t) = F^{-1}[f(x+iy);t]$, $y \in B$, is the L^s, $1/r+1/s = 1$, inverse Fourier transform of $f(x+iy)$, $y \in B$; by the Plancherel-Fourier transform theory $h_y(t)$ is an element of L^s since by (5.1) $f(x+iy) \in L^r$ as a function of $x \in \mathbf{R}^n$ for $y \in B$. By Lemma 5.1.4 $g(t)$ is independent of $y \in B$. From (5.22) we have

$$e^{-2\pi\langle y,t\rangle}g(t) = F^{-1}[f(x+iy);t], \quad y \in B. \tag{5.23}$$

Since $f(x+iy) \in L^r$, $1 < r \le 2$, as a function of $x \in \mathbf{R}^n$ for $y \in B$ as previously noted, then

$$(e^{-2\pi\langle y,t\rangle}g(t)) \in L^s, \; 1/r + 1/s = 1, \; y \in B,$$

by the Plancherel-Fourier transform theory, and

$$\|e^{-2\pi\langle y,t\rangle}g\|_{L^s} \le \|f(x+iy)\|_{L^r} \le K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)),$$

$y \in B$, by the Parseval inequality and (5.1); thus the growth is obtained. From (5.23), by the Plancherel-Fourier transform theory, we also have

$$f(x+iy) = F[e^{-2\pi\langle y,t\rangle}g(t)](x), \quad z = x+iy \in T^B, \tag{5.24}$$

with the Fourier transform $F[e^{-2\pi\langle y,t\rangle}g(t)](x)$ being in L^r. But by (5.2) and Lemma 5.1.1 the integral on the right side of (5.21), which is the L^1 transform of $(e^{-2\pi\langle y,t\rangle}g(t))$, $y \in B$, is a holomorphic function of $z \in T^B$. (Recall from the proof of Lemma 5.1.1 that

$$(e^{-2\pi\langle y,t\rangle}g(t)) \in L^1,\ y \in B,$$

since (5.2) is satisfied here.) From the Plancherel-Fourier transform theory we know that the L^1 and L^r, $1 < r \leq 2$, Fourier transforms of the same function are equal when both of these transforms exists; hence the desired equality (5.21) follows from (5.24). The proof is complete. □

Corollary 5.1.1 *Let C be an open connected cone in $\mathbf{R}^n$ and assume that $f \in H^r_{(M_p)}(T^C)$ with $1 < r \leq 2$. There exists a measurable function $g(t)$, $t \in \mathbf{R}^n$, such that (5.2) holds for $y \in C$, $\text{supp}\ g \subseteq C^*$ almost everywhere, and (5.21) holds for $z \in T^C$.*

Proof. The proof is obtained by combining Theorem 5.1.1 and Lemma 5.1.2. Here $1/r + 1/s = 1$, $1 < r \leq 2$, for the value of s in (5.2) and the constants K, T, m, and q are as in (5.2) . □

In Theorem 5.1.1 and Corollary 5.1.1 we do not know that $g(t) \in L^s$. However, if $g(t)$ is an element of L^s and if it is the inverse Fourier transform of some function $G(t) \in L^r$, $1 < r \leq 2$, we can obtain an additional representation of $f(z)$ in Corollary 5.1.1 in terms of the Cauchy integral as noted in the following result.

Corollary 5.1.2 *Let C be a regular cone in $\mathbf{R}^n$. Let $f(z) \in H^r_{(M_p)}(T^C)$, $1 < r \leq 2$. Let the function $g(t)$ of Corollary 5.1.1 be the inverse Fourier transform of a function $G(t) \in L^r$, $1 < r \leq 2$. We have $g(t) \in L^s$, $1/r + 1/s = 1$, $\text{sipp}(g) \subseteq C^*$ almost everywhere, (5.21) holds for $z \in T^C$, and*

$$f(z) = \int_{\mathbf{R}^n} G(t)K(z-t)\,dt, \quad z \in T^C. \tag{5.25}$$

Proof. By assumption the obtained functiom $g(t)$ of Corollary 5.1.1 satisfies

$$g(u) = F^{-1}[G(t); u], \quad u \in \mathbf{R}^n, \text{ for } G(t) \in L^r,$$

$1 < r \leq 2$, here. By the Plancherel theory we then have that $g(u) \in L^s$, $1/r + 1/s = 1$. Note that the Cauchy integral in (5.25) is well defined here because of the properties of $K(z-t)$ as given in Theorem 4.1.1. Now using the definition of the Cauchy kernel $K(z-t)$, $t \in \mathbf{R}^n$, $z \in T^C$, Fubini's theorem, and the representation (5.21), which holds here, we obtain

$$\int_{\mathbf{R}^n} G(t)K(z-t)\,dt = \lim_{k\to\infty} \int_{|t|\leq k} G(t) \int_{C^*} \exp(2\pi i\langle z-t, u\rangle)\,du\,dt$$

$$= \lim_{k\to\infty} \int_{C^*} \exp(2\pi i\langle z, u\rangle) \int_{|t|\leq k} G(y)\exp(-2\pi i\langle t, u\rangle)\,dt\,du$$

$$= \int_{C^*} g(u)\exp(2\pi i\langle z, u\rangle)\,du = f(z) \tag{5.26}$$

for $z \in T^C$, and the Cauchy integral representation (5.25) is obtained. □

As a dual theorem to Theorem 5.1.1 we have the following result.

Theorem 5.1.2 *Let B be a proper open connected subset of $\mathbf{R}^n$. Let $1 < r \le 2$. Let $g(t)$ be a measurable function on $\mathbf{R}^n$ which satisfies*

$$\|e^{-2\pi\langle y,t\rangle} g\|_{L^r} \le K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B, \tag{5.27}$$

where $K > 0$, $T > 0$, $m \ge 0$, and $q \ge 0$ are independent of $y \in B$. Then

$$f(z) = \int_{\mathbf{R}^n} g(t) e^{2\pi i\langle z,t\rangle}\, dt, \quad z \in T^B, \tag{5.28}$$

is analytic in T^B and satisfies

$$\|f(x + iy)\|_{L^s} \le K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \quad y \in B: \tag{5.29}$$

that is $f(z) \in H^s_{(M_p)}(T^B)$, $1/r + 1/s = 1$, $1 < r \le 2$.

Proof. By the proof of Lemma 5.1.1, $f(z)$ is analytic in T^B and

$$(e^{-2\pi\langle y,t\rangle} g(t)) \in L^1, \ y \in B.$$

By (5.27), $(e^{-2\pi\langle y,t\rangle} g(t)) \in L^r$, $1 < r \le 2$, as a function of $t \in \mathbf{R}^n$ for $y \in B$. Thus the right side of (5.28) can be viewed as both the L^1 and the L^r Fourier transform of $(e^{-2\pi\langle y,t\rangle} g(t))$, $y \in B$, and by the Parseval inequality

$$\|f(x + iy)\|_{L^s} = \|F[e^{-2\pi\langle y,t\rangle} g(t):\ x]\|_{L^s} \le \|e^{-2\pi\langle y,t\rangle} g\|_{L^r} \tag{5.30}$$

for $y \in B$ and $1/r + 1/s = 1$. The estimate in (5.29) now follows from (5.30) and (5.27).

In case B is an open connected subset of $\mathbf{R}^n$ which does not contain $0 \in \mathbf{R}^n$ and $r = 2$, Theorem 5.1.2 is a converse result to Theorem 5.1.1.

5.2 Boundary values in $\mathcal{D}'((M_p), L^s)$ for analytic functions in tubes

In this section we will consider analytic function of the type $H^r_{(M_p)}(T^c)$ defined in Section 5.1 and obtain boundary values of the functions in $\mathcal{D}'((M_p), L^s)$. Before doing this e will state precisely conditions that we need on the sequence (M_p), $p = 0, 1, 2, \ldots$, is order to obtain the boundary value results, and we need to define new associated functions corresponding to there sequences. We then will prove a number of lammas which form a basis our boundary value results.

If the sequence (M_p), satisfies condition $(M.1)$ of Section 2.1 we have

$$\frac{M_p}{M_{p-1}} \le \frac{M_{p+1}}{M_p}, \quad p = 1, 2, 3, \ldots . \tag{5.31}$$

Put ([48], (3.10), p. 50)

$$m_p = \frac{M_p}{M_{p-1}}, \quad p = 1, 2, 3, \ldots . \tag{5.32}$$

(5.3) implies that the sequence m_p is nondecreasing if (M_p), $p \in \mathbf{N}_0^n$, satisfies $(M.1)$.

Put

$$m_p^* = \frac{m_p}{p}, \quad p = 1, 2, 3, \ldots, \tag{5.33}$$

and we will assume in several results in this section that the sequence $\{m_p^*\}$ is nondecreasing as, for example, Petzsche [63], p. 394, has done for his analysis. An example of a sequence (M_p), for which the sequence m_p^* is nondecreasing is $M_p = (p!)^s$, $p \in \mathbf{N}_0^n$, $s > 1$. If the sequence m_p^* is nondecreasing, we immediately obtain that the sequence m_p defined in (5.32) is a structly increasing sequence; this follows directly from the definition (5.33).

Put ([48], p. 50)

$$m(\lambda) = \text{(the number of } m_p \leq \lambda), \tag{5.34}$$

and note that $m(\lambda)$ is finite for all $\lambda > 0$ if $m_p \to \infty$ as $p \to \infty$. For (M_p), $p = 0, 1, 2, 3, \ldots$, satisfying $(M.1)$, m_p is nondecreasing, hence

$$m(\lambda) = \sup_p \{p; m_p \leq \lambda\} \tag{5.35}$$

and $m(\lambda)$ is a nondecreasing function of λ. Now put

$$M_p^* = \frac{M_p}{p!}, \qquad p = 1, 2, 3, \ldots, \tag{5.36}$$

and note from (5.33) that

$$m_p^* = \frac{M_p^*}{M_{p-1}^*}. \tag{5.37}$$

Lemma 5.2.1 *The sequence m_p^* is nondecreasing if and only if the sequence M_p^* satisfies $(M.1)$.*

Proof. Let m_p^* be nondecreasing. From (5.37) we have

$$\frac{M_{p+1}}{M_p(p+1)} \geq \frac{M_p}{M_{p-1}(p)}, \qquad p = 1, 2, 3, \ldots .$$

Hence

$$M_{p-1}M_{p+1} \geq (M_p)^2 \frac{(p+1)!}{p!} \frac{(p-1)!}{p!}$$

and

$$\frac{M_{p-1}}{(p-1)!} \frac{M_{p+1}}{(p+1)!} \geq \left(\frac{M_p}{q!}\right)^2.$$

From the definition (5.36) we thus have obtained

$$M_{p-1}^* M_{p+1}^* \geq (M_p^*)^2, \qquad p = 1, 2, 3, \ldots,$$

which is $(M.1)$ of Section 2.1.

For the converse, let us assume that M_p^* satisfies $(M.1)$. For $p = 1, 2, 3, \ldots$, we have from (5.36) that

$$\frac{(M_p)^2}{(p!)^2} \leq \frac{M_{p-1}}{(p-1)!}\frac{M_{p+1}}{(p+1)!}$$

and

$$(M_p)^2 \leq M_{p-1}M_{p+1}\frac{p!}{(p-1)!}\frac{p!}{(p+1)!} = M_{p-1}M_{p+1}\Big(\frac{p}{p+1}\Big).$$

This implies

$$\frac{M_p}{M_{p-1}(p)} \leq \frac{M_{p+1}}{M_p(p+1)}$$

or, from (5.32) and (5.33),

$$m_p^* \leq m_{p+1}^*. \qquad p = 1, 2, 3, \ldots .$$

Thus m_p^* is nondecreasing, and the proof is complete. □

We define

$$m^*(\lambda) = \text{ (the number of } m_p^* \leq \lambda); \tag{5.38}$$

if m_p^* is nondecreasing we have

$$m^*(\lambda) = \sup_p\{p : m_p^* \leq \lambda\}. \tag{5.39}$$

Recall the associated functions $M(\rho)$ and $M^*(\rho)$ defined in (2.7) and (2.8), respectively. If the sequence (M_p), $p = 0, 1, 2, 3, \ldots$, satisfies $(M.1)$ we have

$$M(\rho) = \int_0^\rho m(\lambda)/\lambda \, d\lambda, \qquad 0 < \rho < \infty, \tag{5.40}$$

from [48], (3.11), p. 50. Similarly, since M_p^* satisfies $(M.1)$ when m_p^* nondecreasing by Lemma 5.2.1, we have

$$M^*(\rho) = \int_0^\rho m^*(\lambda)/\lambda \, d\lambda, \qquad 0 < \rho < \infty, \tag{5.41}$$

if m_p^* is nondecreasing, In (5.40) and (5.41), $m(\lambda)$ and $m^*(\lambda)$ are defined in (5.34) and (5.38), respectively.

Using the above information on sequences (M_p), we now prove four needed lemmas.

Lemma 5.2.2 *Let the sequence (M_p) satisfy $(M.1)$ and let m_p^* be nondecreasing. We have*

$$m^*(t/2m(t)) \leq m(t), \quad t \geq m_1. \tag{5.42}$$

Proof. Recall that $m(\lambda)$ and $m^*(\lambda)$ are given by (5.35) and (5.39), respectively, under the assumptions here. Let $t \geq m_1 = M_1/M_0$ be arbitrary but fixed trought this proof; and denote $p_0 = m(t)$ for this fixed t. We have

$$m_{p_0} \leq t, \; m_{p_0+1} > t,$$

and

$$m_{p_0+1}/p_0 > t/m(t). \tag{5.43}$$

For $t \geq m_1$ and $p_0 = m(t)$ we have $p_0 \geq 1$ and $2p_0 \geq p_0 + 1$; hence

$$\begin{aligned} &2(m_{p_0+1}/(p_0+1)) \geq ((p_0+1)/p_0)(m_{p_0+1}/(p_0+1)) \\ &= (m_{p_0+1}/p_0) > t/m(t), \end{aligned}$$

which implies

$$m^*_{p_0+1} = m_{p_0}/(p_0+1) > t/2m(t)$$

and

$$m^*(t/2m(t)) = \sup_p\{p : m^*_p \leq (t/2m(t))\} < p_0 + 1.$$

Inequality (5.42) follows immediately from this inequality for $t \geq m_1$ and $p_0 = m(t)$. □

Recall that m_p is strictly increasing if m^*_p is nondecreasing. In fact we can say more. Recalling the definition (5.33) of m^*_p, we have for each $p = 1, 2, 3, \ldots$ that $m^*_{p+1} \geq m^*_p$ implies $m_{p+1}/m_p \geq (p+1)/p$ for m^*_p, nondecreasing. Thus for each $p = 2, 3, 4, \ldots$

$$\frac{m_p}{m_1} = \frac{m_p}{m_{p-1}}\frac{m_{p-1}}{m_{p-2}} \cdots \frac{m_3}{m_2}\frac{m_2}{m_1} \geq \frac{p}{p-1}\frac{p-1}{p-2} \cdots \frac{2}{1} = p.$$

Thus, if m^*_p is nondecreasing we have $m_p \geq pm_1$, $p = 2, 3, 4, \ldots$, which yields $m_p \to \infty$ as $p \to \infty$. Using these facts we prove the following.

Lemma 5.2.3 *Let the sequence* (M_p), $p = 0, 1, 2, 3, \ldots$, *satisfy* $(M.1)$ *and let* m^*_p *be nondecreasing. For any* $t > m_1$ *we have*

$$\frac{m(\lambda)}{\lambda} > \frac{m(t)}{2t} \cdot m_1 \leq \lambda < t. \tag{5.44}$$

Proof. Since m^*_p is nondecreasing, from (5.33) we have

$$(p+1)/m_{p+1} \leq p/m_p, \;\; p = 1, 2, \ldots . \tag{5.45}$$

Let $t > m_1$ be arbitrary but fixed. There is a $p \geq 1$ such that $m_p \leq t < m_{p+1}$, and therefore $m(t) = p$. We first assume $m_p < t < m_{p+1}$ and λ satisfies $mp \leq \lambda < t$ in which case $m(\lambda) = p$, and

$$\frac{m(\lambda)}{\lambda} = \frac{p}{\lambda} > \frac{p}{t} = \frac{m(t)}{t} > \frac{m(t)}{2t}. \tag{5.46}$$

This is the desired result for the case that $m_p \leq \lambda < t < m_{p+1}$ for the value of p such that $m_p < t < m_{p+1}$.

The remaining cases are covered by considering λ satisfying $m_1 \leq \lambda < m_p$ for the p for which $m_p \leq t < m_{p+1}$. In this case $m_k \leq \lambda < m_{k+1}$ for some $k = 1, \ldots, p-1$. Using the fact that $k \geq (k+1)/2$, $k = 1, 2, \ldots$, using (5.45) respectively, and using the present facts that $m_1 \leq \lambda < m_p$ for the p such that $m_p \leq t < m_{p+1}$ we have

$$\frac{m(\lambda)}{\lambda} = \frac{k}{m_{k+1}} \geq \frac{k+1}{2m_{k+1}} \geq \frac{k+2}{2m_{k+2}} \geq \ldots \geq \frac{p}{2m_p} \tag{5.47}$$

and

$$\frac{m(t)}{t} = \frac{p}{t} \leq \frac{p}{m_p}. \tag{5.48}$$

Combining (5.47) and (5.48) we have

$$\frac{m(\lambda)}{\lambda} > \frac{p}{2m_p} \geq \frac{m(t)}{2t} \tag{5.49}$$

for the satisfying $m_1 \leq \lambda < m_p$ for the p for which $m_p \leq t < m_{p+1}$. Combining the conclusions (5.46) and (5.49) we have obtained (5.44). □

From Lemma 5.2.1, m_p^* is nondecreasing if and only if M_p^* satisfies $(M.1)$. Thus in Lemmas 5.2.2 and 5.2.3 the hypothesis that m_p^* be nondecreasing can be replaced by the assumption that M_p^* satisfies $(M.1)$. This the case also in the statements of Lemmas 5.2.4 and 5.2.5.

Lemma 5.2.4 *Let the sequence* (M_p), *satisfy* $(M.1)$ *and let* m_p^* *be nondecreasing. For* $s = 2(m_1 + 1)$ *we have*

$$M(t) > \frac{m(t)}{s}, \quad t \geq m_1 + 1. \tag{5.50}$$

Proof. Since $m(t) = 0$ for $0 \leq t < m_1$, from (5.40) we have

$$M(t) = \int_{m_1}^{t} m(\lambda)/\lambda d\lambda, \quad t > m_1. \tag{5.51}$$

From (5.51) and the result (5.44) of Lemma 5.2.3 we have

$$M(t) > m(t)(t - m_1)/2t = \frac{m(t)}{2} - \frac{m(t)}{2t} m_1, \quad t > m_1.$$

For $t \geq m_1 + 1$ we thus have

$$M(t) > \frac{m(t)}{2} - \frac{m(t)}{2(m_1+1)} m_1 = \frac{m(t)}{2}(1 - m_1/(m_1+1)) = m(t)/2(m_1+1)$$

which is (5.50) for $s = 2(m_1 + 1)$. □

Lemma 5.2.5 *Let the sequence (M_p), satisfy $(M.1)$ and let m_p^* be nondecreasing. For $s = 2(m_1 + 1)$ we have*

$$M^*\Big(\frac{t}{2sM(t)}\Big) \leq M(t) + A, \;\; t \geq m_1 + 1. \tag{5.52}$$

for some constant A.

Proof. From (5.50) of Lemma 5.2.4 we have

$$(sM(t)) > m(t), \; t \geq m_1 + 1,$$

where $s = 2(m_1 + 1)$. Thus for $t \geq m_1 + 1$,

$$1/(sM(t)) < 1/m(t) \text{ and } t/2sM(t) < t/2m(t).$$

By the fact that $m^*(\lambda)$ is a nondecreasing function of λ and (5.42) of Lemma 5.2.2 we have

$$m^*(t/2sM(t)) \leq m^*(t/2m(t)) \leq m(t), \;\; t \geq m_1 + 1. \tag{5.53}$$

Recalling (5.39) we have $m^*(\lambda) = 0, \;\; 0 < \lambda M m_1$: hence from (5.41)

$$M^*(t) = \int_{m_1}^{t} m^*(\lambda)/\lambda d\lambda, \;\; t \geq m_1 + 1. \tag{5.54}$$

By a straightforward chain rule calculation and using (5.51) we have

$$\frac{dM^*(\frac{t}{2sM(t)})}{dt} = \frac{m^*(\frac{t}{2sM(t)})}{\frac{t}{2sM(t)}}\Big(\frac{1}{2s}\Big)\Big(\frac{1}{m(t)} - \frac{m(t)}{(m(t))^2}\Big) \tag{5.55}$$

for $t \geq m_1 + 1$. Using (5.55), (5.53) and (5.40) we have for $t \geq m_1 + 1$ that

$$M^*\Big(\frac{t}{2sM(t)}\Big) - M^*\Big(\frac{m_1+1}{2sM(m_1+1)}\Big) = \int_{m_1+1}^{t} \frac{dM^*(\frac{\lambda}{2sM(\lambda)})}{d\lambda} d\lambda$$

$$= \int_{m_1+1}^{t} \frac{m^*(\frac{\lambda}{2sM(\lambda)})}{\frac{\lambda}{2sM(\lambda)}}\Big(\frac{1}{2s}\Big)\Big(\frac{1}{M(\lambda)} - \frac{m(\lambda)}{(M(\lambda))^2}\Big)d\lambda$$

$$\leq \int_{m_1+1}^{t} \frac{m^*(\frac{\lambda}{2sM(\lambda)})}{\frac{\lambda}{2sM(\lambda)}}\Big(\frac{1}{2s}\Big)\Big(\frac{1}{M(\lambda)}\Big)d\lambda$$

$$= \int_{m_1+1}^{t} \frac{1}{\lambda} m^*\Big(\frac{\lambda}{2sM(\lambda)}\Big)d\lambda \leq \int_{m_1+1}^{t} \frac{m(\lambda)}{\lambda} d\lambda$$

$$= \int_{m_1}^{t} \frac{m(\lambda)}{\lambda} d\lambda - \int_{m_1}^{t} \frac{m(\lambda)}{\lambda} d\lambda = M(t) - M(m_1+1). \tag{5.56}$$

Now (5.52) follows from (5.56) with

$$A = M^*\Big(\frac{m_1+1}{2sM(m_1+1)}\Big) - M(m_1+1)$$

and the proof is finished. □

We now proceed to prove several other lemmas which are needed to prove our boundaru value results. The proof of the following lemma is similar to that of [66], Lamma 10.

Lemma 5.2.6 *Let the sequence (M_p), satisfy $(M.1)$ and $(M.2)$ and let m_p^* be nondecreasing. Let C be an open connected cone in $\mathbf{R}^n$. Let $\nu > 0$ and $k > 0$ be constants. There exist positive constants K and u such that*

$$\int_C \exp(-\nu\langle y,t\rangle - M^*(k/|y|))\,dy \geq K\exp(-M(u|t|)),\ t \in C^* \setminus \partial C^*, \quad (5.57)$$

where ∂C^* is the boundary of the dual cone C^* of C.

Proof. Let $s = 2(m_1+1)$. Let $t \in \mathbf{R}^n$ be arbitrary but fixed such that $t \in C^* \setminus \partial C^*$ and $|t| > m_1 + 1$. For such t put

$$A_1(t) = \Big\{y \in \mathbf{R}^n\colon\ |y| \in [a_1(|t|), b_1(|t|)],$$

where

$$a(\theta) = a_{s,k}(\theta) = 2skM(\theta)/\theta;\quad b(\theta) = b_{s,k}(\theta) = (2skM(\theta)+1)/\theta \qquad (5.58)$$

for $\theta > 0$, and

$$C_1(t) = A_1(t) \cap C.$$

For $y \in C_1(t)$ and $|t| > m_1 + 1$, we use the fact that $M^*(\rho)$ in (5.41) is nondecreasing and Lemma 5.2.5 to obtain

$$M^*(k/|y|) \leq M^*(|t|/2sM(|t|))$$

and

$$\begin{aligned} \exp(-M^*(k/|y|)) &\geq \exp(-M^*(|t|/2sM(|t|))) \\ &\geq \exp(-M(|t|) - A) \end{aligned} \qquad (5.59)$$

where the constant A from (5.52) depends on the sequence (M_p) through the functions $M(\cdot)$ and $M^*(\cdot)$. For $y \in C_1(t)$ we have

$$\exp(-\nu|y||t|) \geq \exp(-2sk\nu M(|t|) - \nu). \qquad (5.60)$$

Recall that $\langle y,t\rangle \geq 0,\ y\in C,\ t\in C^*$. Combining (5.59) and (5.60) we have for $t\in C^*\setminus \partial C^*$ and $|t|>m_1+1$ that

$$\begin{aligned} I(t) &= \int_C \exp(-\nu\langle y,t\rangle - M^*(k/|y|)\,dy \\ &\geq \int_{C_1(t)} \exp(-\nu|y||t| - M^*(k/|y|))\,dy \\ &\geq \exp(-2sk\nu M(|t|) - \nu)\exp(M(|t|)-A)\int_{C_1(t)} 1\,dy \\ &= \exp(-(\nu+A))\exp(-(1+2sk\nu)M(|t|))\int_{C_1(t)} 1\,dy. \end{aligned} \tag{5.61}$$

Let $pr(C)$ denote the projection C which is the intersection of C with the unit sphere in $\mathbf{R}^n$. We have

$$\begin{aligned} \int_{C_1(t)} 1\,dy &= \int_{pr(C)}\int_{a(|t|)}^{b(|t|)} r^{n-1}dr\,d\sigma \\ &= \left(\frac{1}{n}\right)\left(\int_{pr(C)} 1 d\sigma\right)[(b(|t|))^n - (a(|t|))^n] \\ &\geq n^{-1}|t|^{-n}S(pr(C)), \end{aligned} \tag{5.62}$$

where $S(pr(C))$ is the surface area of $pr(C)$. Recall from (5.40) that $M(\rho)$ is an increasing function of ρ. For $t\in C^*\setminus\partial C^*$ and $|t|>m_1+1$ we can choose a constant $Q>0$ independent of t such that

$$|t|^{-n}\geq Q\exp(-M(|t|)),\ t\in C^*\setminus\partial C^*,\quad |t|>m_1+1,$$

which we use in (5.62) to obtain

$$\int_{C_1(t)} 1\,dy \geq Qn^{-1}S(pr(C))\exp(-M(|t|)),\ t\in C^*\setminus\partial C^*,\quad |t|>m_1+1. \tag{5.63}$$

(Equivalently, we have

$$\exp(M(|t|))\geq |t|^n M_0/M_n$$

directly from the definition of the function $M(\cdot)$, which can be used in (5.62) to obtain (5.63) as well with $Q=M_0/M_n$). Combining (5.61) and (5.63) we have for $t\in C^*\setminus\partial C^*$ and $|t|>m_1+1$ that

$$\begin{aligned} I(t) &\geq Qn^{-1}S(pr(C))\exp(-(\nu+A))\exp(-(2+2sk\nu)M(|t|)) \\ &= B\exp(-(2+2sk\nu)M(|t|)) \end{aligned} \tag{5.64}$$

where the constant

$$B=(Qn^{-1}S(pr(C))\exp(-(\nu+A))$$

depends on ν, m_1, n, and the cone C but not on t.

Now let $t \in C^* \setminus \partial C^*$ such that $|t| \leq m_1 + 1$. Put

$$C_2(t) = A_2(t) \cap C,$$

where

$$A_2(t) = \{y \in \mathbf{R}^n: \ |y| \in [a_2(t), b_2(t)]\}$$

with

$$a_2(t) = a_1(|t| + m_1 + 1); \quad b_2(t) = b_1(|t| + m_1 + 1)$$

according to the previous notation adopted in (5.58). Proceeding similarly as in obtaining (5.59), (5.60) and (5.62) we now obtain for $y \in C_2(t)$ and $|t| \leq m_1+1$ that

$$\begin{aligned} \exp(-M^*(k/|y|)) &\geq \exp\Big(- M^*\Big(\frac{|t| + m_1 + 1}{2sM(|t| + m_1 + 1)}\Big)\Big) \\ &\geq \exp\Big(- M^*\Big(\frac{2m_1 + 2}{2sM(m_1 + 1)}\Big)\Big); \end{aligned} \tag{5.65}$$

$$\exp(-\nu|y||t|) \geq \exp(-\nu(2skM(2m_1 + 2) + 1)); \tag{5.66}$$

$$\int_{C_2(t)} 1\, dy \geq n^{-1}S(pr(C))(2m_1 + 2)^{-n}, \tag{5.67}$$

where again $S(pr(C))$ is the surface area of $pr(C)$. Combining (5.65), (5.66) and (5.67) we obtain for $t \in C^* \setminus \partial C^*$ and $|t| \leq m_1 + 1$ that

$$\begin{aligned} I(t) &= \int_C \exp(-\nu\langle y, t\rangle - M^*(k/|y|)\, dy \\ &\geq \int_{C_2(t)} \exp(-\nu|y||t| - M^*(k/|y|))\, dy \geq \frac{S(pr(C))}{n(2m_1 + 2)^n} \\ &\cdot \exp\left(-\nu(2skM(2m_1 + 2) + 1)) - M^*\left(\frac{2m_1 + 2}{2skM(m_1 + 1)}\right)\right) = R \end{aligned} \tag{5.68}$$

where the constant R depends on n, the sequence (M_p), $p = 0, 1, 2, 3, \ldots$, trough the functions $M(\cdot)$, and $M^*(\cdot)$, and the cone C as well as the given constants $\nu > 0$ and $k > 0$ but not on t.

Now (5.64) holds for $t \in C^* \setminus \partial C^*$ such that $|t| > m_1 + 1$ and (5.68) holds for $t \in C^* \setminus \partial C^*$ such that $|t| \leq m_1 + 1$. Combining (5.64) and (5.68) we obtain a constant K_1 such that

$$I(t) \geq K_1 \exp(-(2 + 2sk\nu)M(|t|)), \quad t \in C^* \setminus \partial C^*. \tag{5.69}$$

The sequence (M_p), $p = 0, 1, 2, 3, \ldots$, is assumed to satisfy $(M.1)$ and $(M.2)$ here. Thus (2.12) holds for $L = (2 + 2sk\nu) > 1$ there. Hence the existence of constants $K > 0$ and $u > 0$ such that (5.57) holds follows from (5.69) and (2.12). The constants are independent of $t \in C^* \setminus \partial C^*$ but are dependent upon the dimension n, k, ν, the sequence (M_p), $p = 0, 1, 2, 3, \ldots$, trough the functions $M(\cdot)$ and $M^*(\cdot)$, and the cone C. The proof is complete. □

The sequence $(M_p) = (p!)^s$, $p \in \mathbf{N}_0$, $s > 1$ is an example of a sequence which satisfies the hypothesis of the previous lemma.

We use Lemma 5.2.6 to prove the next lemma.

Lemma 5.2.7 *Let the sequence (M_p), satisfy $(M.1)$ and $(M.2)$ and let m_p^* be nondecresing . Let C, be a regular cone in $\mathbf{R}^n$. Let $1 \leq s < \infty$. Let $g(t)$ be a measurable function on $\mathbf{R}^n$ with* $\text{supp}(g) \subseteq C^*$ *almost everywhere such that*

$$\int_{\mathbf{R}^n} |e^{-2\pi\langle y,t\rangle} g(t)|^s \, dt \leq R \exp(M^*(k|y|)), \quad y \in C, \tag{5.70}$$

for some conatants $R > 0$ and $k > 0$. Then there exists a constant $d > 0$ such that

$$\|g(t) \exp(-M(d|t|))\|_{L^s} < \infty. \tag{5.71}$$

Proof. From (5.70) and the fact that supp $g \subseteq C^*$ almost everywhere we have

$$\int_{C^*} |g(t)|^s e^{-2\pi s\langle y,t\rangle} \, dt \exp(-M^*(k/|y|)) \leq R, \quad y \in C. \tag{5.72}$$

Let $\overline{\varepsilon} \in pr(C^*) \setminus \partial C^*$ be fixed with ∂C^* denoting the boundary of C^*. Multiplying (5.72) by $\exp(-2\pi s\langle y, \overline{\varepsilon}\rangle)$, $y \in C$, and integrating over C we get

$$\begin{aligned} &\int_C \left(\int_{C^*} |g(t)|^s e^{-2\pi\langle y,t\rangle} \, dt \right) \exp(-M^*(k/|y|) - 2\pi s\langle y, \overline{\varepsilon}\rangle) \, dy \\ &\leq R \int_C \exp(-2\pi s\langle y, \overline{\varepsilon}\rangle) \, dy. \end{aligned} \tag{5.73}$$

Recalling the proof of Lemma 5.2.6, the integral on the right of (5.73) is finite for $\overline{\varepsilon} \in pr(C^*) \setminus \partial C^*$. By Fubini's theorem we interchange the order of integration on the left of (5.73) and obtain

$$\int_{C^*} |g(t)|^s \int_C \exp(-2\pi s\langle y, t + \overline{\varepsilon}\rangle) \exp(-M^*(k/|y|)) \, dy \, dt < \infty. \tag{5.74}$$

Using the fact $1 \leq s < \infty$, Lemma 5.2.6 and (5.74) we now have the existence of positive constants K ans m such that

$$\begin{aligned} &K \int_{C^*} |g(t)|^s \exp(-sM(m|t + \overline{\varepsilon}|)) \, dt \\ &\leq K \int_{C^*} |g(t)|^s \exp(-M(m|t + \overline{\varepsilon}|)) \, dt \qquad (5.75) \\ &= K \int_{C^* \setminus \partial C^*} |g(t)|^s \exp(-M(m|t + \overline{\varepsilon}|)) \, dt \\ &\leq \int_{C^* \setminus \partial C^*} |g(t)|^s \int_C \exp(-2\pi s\langle y, t + \overline{\varepsilon}\rangle - M^*(k/|y|)) \, dy \, dt < \infty (5.76) \end{aligned}$$

which proves that the integer on the left of (5.76) is finite; in applying Lemma 5.2.6 in (5.76) we have use the fact that since C is regular here (and hence convex) then $t \in C^* \setminus \partial C^*$ and $\overline{\varepsilon} \in pr(C^*) \setminus \partial C^*$ imply that $(t + \overline{\varepsilon}) \in C^* \setminus \partial C^*$. Since $\overline{\varepsilon} \in pr(C^*) \setminus \partial C^*$ we have $|t + \overline{\varepsilon}| \leq |t| + |\overline{\varepsilon}| = 1 + |t|$. Using this, the fact that $M(\rho)$ is a nondecreasing function of $\rho > 0$ [11, p. 65], and the property (2.9) we have

$$M(m|t + \overline{\varepsilon}) \leq M(|t| + m) \leq M(2m|t|) + M(2m). \tag{5.77}$$

Using (5.77) in (5.76) we thus obtain that

$$Ke^{-sM(2m)}\int_{C^*}|g(t)|^s e^{-sM(2m|t|)}\,dt<\infty$$

since $\exp(sM(2m))$ is finite; this proves (5.71) with $d=2m$. □

Lemma 5.2.7 is used to prove the next lemma which, in turn, will be used later in this section to obtain the ultradistribution boundary value results.

Lemma 5.2.8 *Let the sequence (M_p), satisfy $(M.1)$ and $(M.2)$ and be such that m_p^* is nondecreasing. Let C be a regular cone in $\mathbf{R}^n$. Let $1\le s<\infty$. Let $g(t)$ be a measurable functionon $\mathbf{R}^n$ with* supp $\subseteq C^*$ *almost everywhere such that*

$$\|e^{-2\pi\langle y,t\rangle}g\|_{L^s}\le K\exp(M^*(k/|y|)),\quad y\in C, \tag{5.78}$$

for some constants $K>0$ and $k>0$. Then there is a constant $b>0$ such that

$$\|g(t)\exp(-M(b|t|))\|_{L^s}<\infty. \tag{5.79}$$

Proof. From (5.78) we have

$$\int_{\mathbf{R}^n}|e^{-2\pi\langle y,t\rangle}g(t)|^s\,dt\le K^s\exp(sM^*(k/|y|)),\quad y\in C. \tag{5.80}$$

Under the assumptions $(M.1)$ and $(M.2)$ on the sequence (M_p), it follows that $M_p/p!$ satisfies $(M.2)$; and M_p/p satisfies $(M.1)$ by Lemma 5.2.1. Note that

$$M^*(\rho)=\sup_p\log(\rho^p p!M_0/M_p)=\sup_p\log(\rho^p M_0/p!)); \tag{5.81}$$

thus applying prof of [64], Lemma 1.7 (b), pp. 140-141, corresponding to the sequence $M_p/p!$, which satisfies $(M.1)$ and $(M.2)$, (see also the proof of [48], Proposition 3.6, p. 51), we have the existence of a positive real number B and a constant $Q_s>0$ depending on s such that

$$sM^*(k/|y|)\le M^*(B^{s-1}k/|y|)+Q_s,\quad 1\le s<\infty. \tag{5.82}$$

(Recall (2.12) for $M(\rho)$.) Using (5.82) in (5.80) we have

$$\int_{\mathbf{R}^n}|e^{-2\pi\langle,t\rangle}g(t)\le K^s\exp(Q_s)\exp(M^*(B^{s-1}k?|y|)),\quad y\in C. \tag{5.83}$$

The conclusion (5.79) now follows from (5.83) and the assumptions on $g(t)$ and on (M_p) by applying Lemma 5.2.7. The proof is complete. □

Recall the spaces $\mathcal{FD}(*,L^r)$ of Section 2.4.

Lemma 5.2.9 *Let the sequence (M_p), satisfy $(M.1)$ and $(M.2)$. Let $\varphi\in\mathcal{D}((M_p),L^1)$. Then*

$$\sup_{x\in R^n}|\widehat{\varphi}(x)\exp(M(h|x|))|<\infty \tag{5.84}$$

for every $h>0$.

Proof. $\varphi \in \mathcal{D}((M_p), L^1)$ implies $\widehat{\varphi} \in \mathcal{FD}((M_p, L^1)$; hence for every $k > 0$, every n-tuple α of nonnegative integers, and all $x \in \mathbf{R}^n$ we have

$$\frac{|x^\alpha \widehat{\varphi}(x)|}{k^\alpha M_\alpha} \leq N \tag{5.85}$$

where N is a positive constant which is independent of k, α, and $x \in \mathbf{R}^n$. Using (2.3) in (5.85) we get

$$\frac{1}{(M_0)^n B} \sup_{\alpha_1} \frac{(1/kE)^{\alpha_1} |x_1|^{\alpha_1} M_0}{M_{\alpha_1}} \dots \sup_{\alpha_n} \frac{(1/kE)^{\alpha_n} |x_n|^{\alpha_n} M_0}{M_{\alpha_n}} |\widehat{\varphi}(x)| \leq N$$

for positive constants B and E, and by the definition of $M(\rho)$ in (2.7) we have

$$\frac{1}{(M_0)^n B} \exp(M(|x_1|/kE) + \dots + M(|x_n|/kE))|\widehat{\varphi}(x)| \leq N, \quad x \in \mathbf{R}^n. \tag{5.86}$$

Now $|x| \leq n(|x_1| + \dots + |x_n|)$, and

$$\begin{aligned} M(|x|/knE) &\leq M((|x_1|/kE) + \dots + (|x_n|/kE)) \\ &\leq M(n|x_1|/kE) + \dots + M(n|x_n|/kE) \\ &\leq (3n/2)(M(|x_1|/kE) + \dots + M(|x_n|/kE)) + K \end{aligned} \tag{5.87}$$

with the last inequality being obtained from (2.11) where $K > 0$ is a constant. From (5.87) we have

$$(2/3n)M(|x|/knE) \leq M(|x_1|/kE) + \dots + M(|x_n|/kE) + K_1, \quad x \in \mathbf{R}^n, \tag{5.88}$$

where $K_1 > 0$ is a constant. Using (5.88) in (5.86) we have

$$\frac{1}{(M_0)^n B} \exp(-K_1) \exp((2/3n)M(|x|/knE))|\widehat{\varphi}(x)| \leq N, \quad x \in \mathbf{R}^n,$$

and hence

$$\frac{1}{(M_0)^n B} \exp(-K_1) \sup_{x \in \mathbf{R}^n} (\exp((2/3n)M(|x|/knE))|\widehat{\varphi}(x)| \leq N, \tag{5.89}$$

for all $k > 0$. For arbitrary $h > 0$ and a positive integer q chosen such that $(2/3n)2^q > 1$, choose k such that

$$\frac{1}{knE} = H^q h, \tag{5.90}$$

where H is the constant from condition $(M.2)$. Using (2.10) repeatedly we have the existence of a constant G such that

$$\begin{aligned} M(H^q h|x|) &= M(H(H^{q-1}h|x|)) \geq 2M(H^{q-1}h|x|) - G \\ &\leq 2^2 M(H^{q-2}h|x|) - 2G - G \\ &\leq 2^q M(h|x|) - (1 + 2 + \dots + 2^{q-1})G. \end{aligned} \tag{5.91}$$

From the choice of q, (5.90) and (5.91), we have

$$\begin{aligned}&\exp((2/3n)M(|x|/knE)) = \exp((2/3n)M(H^q h|x|))\\ &\geq \exp((2/3n)2^q M(h|x|))\exp(-(2/3n)(1+2+\ldots+2^{q-1})G)\\ &\geq \exp(-(2/3n)(1+2+\ldots+2^{q-1})G)\exp(M(h|x|)) \qquad (5.92)\end{aligned}$$

for all $h > 0$. (5.89) and (5.92) now combine to prove (5.84). The proof is complete. □

The proof of the following result is obtained as a corollary to the proof of Lemma 5.7.9 and is omitted; the details of the proof of the following corollary are essentially those of the proof of Lemma 5.7.9.

Corollary 5.2.1 *If φ_λ, is a net of elements in $\mathcal{D}((M_p), L^1)$ which converges to zero in $\mathcal{D}((M_p), L^1)$ as $\lambda \to \infty$, where the sequence (M_p), satisfies $(M.1)$ and $(M.2)$, then*

$$\lim_{\lambda\to\infty}\sup_{x\in\mathbf{R}^n}|\exp(M(h|x|))\widehat{\varphi}_\lambda(x)| = 0$$

for all $h > 0$.

The following result will be used in the boundary value analysis.

Lemma 5.2.10 *Let the sequence (M_p), satisfy $(M.1)$ and $(M.2)$. Let $\psi \in \mathcal{FD}((M_p), L^r)$, $1 \leq r \leq 2$. Then for any $k > 0$*

$$\|\psi(t)\exp(M(k|t|))\|_{L^s} < \infty, \qquad (5.93)$$

$1/r + 1/s = 1$, where M is the associated function in (2.7).

Proof. The proof is obtained by the calculation in [66], p. 205. For $k > 0$ and $t \in \mathbf{R}^n$ we have

$$(M_0)^{-1}\exp(M(k|t|)) \leq \sup_{p\in\mathbf{N}_0}\frac{(kn)^p(|t_1|+\ldots+|t_n|)^p}{(M_p)} \leq \sup_\alpha \frac{(kn^2)^\alpha|t^\alpha|}{M_\alpha}$$

where α is in n-tuple of nonnegative integers. Thus for $\psi \in \mathcal{FD}((M_p), L^r)$, $1 \leq r \leq 2$, and $1/r + 1/s = 1$, we have

$$\begin{aligned}&(M_0)^{-1}\|\psi(t)\exp(M(k|t|))\|_{L^s}\\ &\leq \Big\|\Big(\sup_\alpha\frac{(kn)^2)^\alpha|t^\alpha|}{M_\alpha}\Big)\psi(t)\Big\|_{L^s} \leq \sum_\alpha \frac{(kn^2)^\alpha}{M_\alpha}\|\chi^\alpha\psi\|_{L^s}. \qquad (5.94)\end{aligned}$$

For $\psi \in \mathcal{FD}((M_p), L^r)$ we have from (2.40)

$$\sup_\alpha \frac{\|\chi^\alpha\psi\|_{L^s}}{h^\alpha M_\alpha} < \infty$$

for all $h > 0$; putting $h = (2kn^2)^{-1}$, we have from (5.94)

$$\begin{aligned}&(M_0)^{-1}\|\psi(t)\exp(M(k|t|))\|_{L^s}\\ &\leq \sum_\alpha \frac{1}{h^\alpha M_\alpha}\Big(\frac{1}{2}\Big)^\alpha\|\chi^\alpha\psi\|_{L^s} \leq \Big(\sup_\alpha\frac{\|\chi^\alpha\psi\|_{L^s}}{h^\alpha M_\alpha}\Big)\sum_\alpha\Big(\frac{1}{2}\Big)^\alpha < \infty,\end{aligned}$$

which proves (5.93) as desired. □

The following result is proved using the details of the proof of Lemma 5.2.10 just as the proof of Corollary 5.2.1 followed from the details of the proof of Lemma 5.2.9.

Corollary 5.2.2 *If ψ_λ, is a net of elements in $\mathcal{FD}((M_p), L^r)$, $1 \le r \le 2$, which converges to zero in $\mathcal{FD}((M_p), L^r)$ as $\lambda \to \infty$, where the sequence (M_p) satisfies $(M.1)$ and $(M.2)$, then*

$$\lim_{\lambda \to \infty} \|\psi_\lambda(t) \exp(M(k|t|))\|_{L^s} = 0,$$

$1/ + 1/s = 1$, *for all $k > 0$.*

Using the lemmas and corollaries proved to this point in this section, we can now obtain boundary value results. In our proofs we also use properties obtained in Section 5.1.

Thus, throughout the remainder of this section we will assume that the sequence (M_p), $p = 0, 1, 2, 3, \ldots$, satisfies conditions $(M.1)$, $(M.2)$, and $(M.3)'$ of Section 2.1 and is such that m_p^* is nondecreasing, where m_p^* is defined in (5.33).

Let C be a regular cone in $\mathbf{R}^n$. W consider functions $f(z)$ which are analytic in $T^C = \mathbf{R}^n + iC$ and which satisfy

$$\|f(x+iy)\|_{L^r} \le K \exp(M^*(T/|y|)), \quad y \in C, \tag{5.95}$$

where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$ and M^* is the associated function of the sequence (M_p), defined in (2.8). Thus the norm growth (5.95) which we are considering in this section is that in (5.1) with $m = 0$ or $q = 0$ there, and the functions $f(z)$ are certain elements in $H^r_{(M_p)}(T^C)$ as defined in Section 5.1.

We first prove that elements in $H^r_{(M_p)}(T^C)$, $1 < r \le 2$, which satisfy (5.95) obtain ultradistribution boundary values in $\mathcal{D}'((M_p), L^1)$.

Theorem 5.2.1 *Let $f(z)$ be analytic in T^C and satisfy (5.95) with $1 < r \le 2$. There exists an element $U \in \mathcal{D}'((M_p), L^1)$ such that*

$$\lim_{y \to 0} f(x+iy) = U \tag{5.96}$$

in $\mathcal{D}'((M_p), L^1)$.

Proof. Let $\varphi \in \mathcal{D}((M_p), L^1)$. From Section 2.3 we see that $\mathcal{D}((M_p), L^1) \subseteq \mathcal{D}((M_p), L^s)$, $1 \le s < \infty$.

By (5.95), we have $f(\cdot + iy) \in L^r$, $1 < r \le 2$ for $y \in C$. Thus $\langle f(\cdot + iy), \varphi\rangle$ is well defined for $y \in C$. By Corollary 5.1.1 (see the proof of Theorem 5.1.1) and the assumption in (5.95), there exists a measurable function g on $\mathbf{R}^n$ with supp $g \subseteq C^*$ such that

$$\|e_y g\|_{L^s} \le K \exp(M^*(T/|y|)), \quad y \in C, \tag{5.97}$$

and

$$f(x+iy) = \mathcal{F}[e_y g](x), \quad z = x+iy \in T^C, \tag{5.98}$$

where $e_y(t) = \exp(-2\pi\langle y,t\rangle)$ for $y,t \in \mathbf{R}^n$. The above Fourier transform is meant both in the sense of L^1 and L^r (see (5.24)). By (5.98) and the Parseval equality we have

$$\langle f(\cdot+iy),\varphi\rangle = \langle \mathcal{F}[e_y,g],\varphi\rangle = \langle e_y g, g\hat{\varphi}\rangle \tag{5.99}$$

for $\varphi \in \mathcal{D}((M_p),L^1)$ and $y \in C$.

We now want to show that $g\widehat{\varphi} \in L^1$. To do so note first that for given constants $k_1 \geq 1$ and $k_2 \geq 1$ there exist, due to (2.11) and (2.12), constants $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that

$$\begin{aligned}\exp[M(k_1|t|)+M(k_2|t|)] &\leq K_1 \exp[(3/2)(k_1+k_2)M(|t|)] \\ &\leq K_2 \exp[M(K_3|t|)]. \end{aligned} \tag{5.100}$$

Denoting $e_M^j(t) = \exp[-M(k_j|t|)]$ for $j=1,2$ and $e_M^3(t) = \exp[M(K_3|t|)]$, we conclude from (5.100) and the Hölder inequality, that

$$\begin{aligned}\int_{\mathbf{R}^n} |g(t)\widehat{\varphi}(t)|\,dt &\leq K_2 \int_{\mathbf{R}^n} |g(t)\widehat{\varphi})(t)| e_M^1(t)e_M^2(t)e_M^3(t)\,dt \\ &\leq K_2\|e_M^3\widehat{\varphi}\|_{L^\infty}\|g e_M^1 e_M^2\|_{L^1} \leq \|e_M^3\widehat{\varphi}\|_{L^\infty}\|g e_M^1\|_{L^s}\|e_M^2\|_{L^r}. \end{aligned} \tag{5.101}$$

Clearly, $e_M^2 \in L^r$ for $1 < r \leq 2$. Moreover, since $k_1 \geq 1$ was taken arbitrary in (5.101), we may choose k_1 to be equal to b in (5.79). Hence $\|g e_M^1\|_{L^s} < \infty$ and the right side of (5.101) is finite, by Lemma 5.29 (inequality (5.97) and the proofs of Lemmas 5.2.8, 5.2.7, 5.2.6. Consequently, $g\widehat{\varphi} \in L^1$, by virtue of (5.101).

Since $\langle y,t\rangle \geq 0$, $y \in C$, $t \in C^*$, and supp $g \subseteq C^*$ almost everywhere we have

$$|e^{-2\pi\langle y,t\rangle} g(t)\widehat{\varphi}(t)| \leq |g(t)\widehat{\varphi}(t)|$$

for almost all $t \in \mathbf{R}^n$; and from the preceding paragraph $(g(t)\widehat{\varphi}(t)) \in L^1$. Thus by the Lebesgue dominated convergence theorem

$$\lim_{\substack{y\to 0\\ y\in C}} \int_{\mathbf{R}^n} e^{-2\pi\langle y,t\rangle} g(t)\widehat{\varphi}(t)\,dt = \int_{\mathbf{R}^n} g(t)\widehat{\varphi}(t)\,dt \tag{5.102}$$

We now define U by

$$\langle U,\varphi\rangle = \langle g(t),\varphi(t)\rangle, \quad \varphi \in \mathcal{D}((M_p),L^1). \tag{5.103}$$

If φ_λ is a net in $\mathcal{D}((M_p),L^1)$ which converges to zero in $\mathcal{D}((M_p),L^1)$ as $\lambda \to \infty$ then analysis as in (5.11) and the Corollary 5.2.1 prove that

$$\lim_{\lambda\to\infty}\langle U,\varphi_\lambda\rangle = \lim_{\lambda\to\infty}\langle g(t),\varphi_\lambda(t)\rangle = 0.$$

Hence U is continuous on $\mathcal{D}((M_p), L^1)$ and the linearity of U is obvious. Thus $U \in \mathcal{D}'((M_p), L^1)$. Returning now to (5.99) and using (5.102) and the definition (5.103) we obtain for $\varphi \in \mathcal{D}((M_p), L^1)$ that

$$\begin{aligned}\lim_{\substack{y\to 0\\ y\in C}} \langle f(x+iy), \varphi\rangle &= \lim_{\substack{y\to 0\\ y\in C}} \langle e^{-2\pi\langle y,t\rangle} g(t), \varphi(t)\rangle \\ &= \langle g(t), \varphi(t)\rangle = \langle U, \varphi\rangle \end{aligned} \tag{5.104}$$

which proves (5.96), and the proof is complete. □

The following result is a dual theorem to Theorem 5.2.1, and boundary value results are obtained in $\mathcal{D}'((M_p), L^r)$, $1 < r \le 2$.

Recall the spaces $\mathcal{F}\mathcal{D}((M_p), L^r)$ and $\mathcal{F}'\mathcal{D}((M_p), L^r)$ which were defined in Section 2.4.

Theorem 5.2.2 *Let $1 < r \le 2$. Let $g(t)$ be a measurable function on $\mathbf{R}^n$ such that*

$$\|e^{-2\pi\langle y,t\rangle} g\|_{L^r} \le K \exp(M^*(T/|y|)), \quad y \in C, \tag{5.105}$$

where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$. Then

$$f(z) = \int_{\mathbf{R}^n} g(t) e^{2\pi i\langle z,y\rangle}\, dt, \quad z \in T^C, \tag{5.106}$$

is analytic in T^C, satisfies (5.95) with L^r replaced by L^s, $1/r + 1/s = 1$, and there is an element $U \in \mathcal{D}'((M_p), L^r)$ such that

$$\lim_{\substack{y\to 0\\ y\in C}} f(x+iy) = U \tag{5.107}$$

in $\mathcal{D}'((M_p), L^r)$.

Proof. By the proof of Theorem 5.1.2, $f(z)$ is analytic in T^C and satisfies (5.95) with L^r replaced by L^s, $1/r + 1/s = 1$; and by the proof of Lemma 5.1.1, the Fourier transform $\mathcal{F}[e^{-2\pi\langle y,t\rangle} g(t)](x)$ on the right of (5.106) can be interpreted in both the L^1 and L^r sense.

Further, $\text{supp}(g) \subseteq C^*$ almost everywhere by Lemma 5.1.2. Now let $\varphi \in \mathcal{D}((M_p), L^r)$, $1 < r \le 2$. Let $\psi(t) = \mathcal{F}[\varphi(t)](x)$ and $\check{\psi}(t) = \psi(-t)$; hence $\psi \in \mathcal{F}\mathcal{D}((M_p), L^r)$. By the proof of Lemma 5.2.10 we have that

$$\|\check{\psi}(t) \exp(M(h|t|))\|_{L^s} < \infty, \tag{5.108}$$

$1/r + 1/s = 1$, for any $h > 0$. From Hölder's inequality, (5.108) and the proof of Lemma 5.2.8 we have the existence of a constant $b > 0$ such that

$$\begin{aligned}&\int_{\mathbf{R}^n} |g(t)\check{\psi}(t)|\, dt \\ &\le \|g(t)\exp(-M(b|t|))\|_{L^s} \|\check{\psi}(t)\exp(M(b|t|))\|_{L^r} < \infty \end{aligned} \tag{5.109}$$

and by this inequality and Corollary 5.2.2 we have that $g \in \mathcal{F}'\mathcal{D}((M_p), L^r)$, $1 < r \le 2$.

Now define $U = \mathcal{F}^{-1}[g]$ by (2.42). Since $g \in \mathcal{F}'\mathcal{D}((M_p), L^r)$, $1 < r \leq 2$, we have $U \in \mathcal{D}'((M_p), L^r)$ by Lemma 2.4.2. Thus if $\varphi \in \mathcal{D}((M_p), L^r)$ we have from (5.106) and the drfinition of U that

$$\langle f(x+iy) - U, \varphi\rangle = \langle g(t)(e^{-2\pi\langle y,y\rangle} - 1)\check{\psi}(t)\rangle. \tag{5.110}$$

Since $\text{supp}(g) \subseteq C^*$ almost everywhere then

$$|g(t)(e^{-2\pi\langle y,t\rangle} - 1)\check{\psi}(t)| \leq 2|g(t)\check{\psi}(t)|$$

for almost all $t \in \mathbf{R}^n$; and from (5.109), $(g(t)\check{\psi}(t)) \in L^1$. By the Lebesgue dominated convergence theorem

$$\lim_{\substack{y\to 0\\ y\in C}} \int_{\mathbf{R}^n} g(t)(e^{-2\pi\langle y,t\rangle} - 1)\check{\psi}(t)\, dt = 0. \tag{5.111}$$

Now (5.107) is obtained by combining (5.110) and (5.111). The proof is complete. □

Corollary 5.2.3 *Let $f(z)$ be analytic in T^C and satisfy (5.95) with $r = 2$. There is an element $U \in \mathcal{D}'((M_p), L^2)$ such that*

$$\lim_{\substack{y\to 0\\ y\in C}} f(x+iy) = U \tag{5.112}$$

in $\mathcal{D}'((M_p), L^2)$ and

$$f(z) = \langle U, K(z-t)\rangle, \quad z \in T^C. \tag{5.113}$$

Proof. From Theorem 5.1.1 and is proof, Corollary 5.1.1, and (5.95) we have the existence of a measurable function $g(t)$ with $\text{supp}\, g \subseteq C^*$ almost everywhere such that (5.105) holds with $r = 2$, and $f(z)$ has the representation (5.21) (i.e. (5.106)).

The existence of $U \in \mathcal{D}'((M_p), L^2)$ for which (5.112) holds follows from Theorem 5.2.2. The Cauchy kernel $K(z-t)$, $z \in T^C$, $t \in \mathbf{R}^n$, defined in (1.5) satisfies $K(z-\cdot) \in \mathcal{D}((M_p), L^2)$ as a function of $t \in \mathbf{R}^n$ for $z \in T^C$ by Theorem 4.1.1. Thus $\langle U, K(z-t)\rangle$, $z \in T^C$, is well defined.

Recall from the proof of Theorem 5.2.2 that $U = \mathcal{F}^{-1}[g]$ here is as defined in (2.42). Thus from the representation (5.21) (i.e. (5.106)) where $\text{supp}\, g \subseteq C^*$ almost everywhere, the fact that

$$K(z-t) = \int_{C^*} \exp(2\pi i\langle z-t, \eta\rangle)\, d\eta = \mathcal{F}^{-1}[I_{C^*}(\eta)e^{2\pi\langle z,t\rangle} :\ t], \quad z \in T^C, \tag{5.114}$$

where $I_{C^*}(\eta)$ is the characteristic function of C^*, and (2.42) we have

$$\begin{aligned} f(z) &= \langle g(t), e^{-2\pi\langle z,t\rangle} g\rangle = \langle g(t), I_{C^*}(t)e^{2\pi i\langle z,t\rangle}\rangle \\ &= \langle U, K(z-t)\rangle \end{aligned} \tag{5.115}$$

for $z \in T^c$; and (5.113) is obtained. The proof is complete. □

If C_1 is any regular cone such that $C^* \cap C_1^*$ is a set of Lebesgue measure zero for the cone C of Corollary 5.2.3, then the calculation in (5.115) shows that

$$\langle U, K(z-t)\rangle = 0, \quad z \in T^{C_1},$$

in Corollary 5.2.3 where U is the boundary value of $f(z)$ in (5.112) and $K(z-t)$ is the Cauchy kernel in (5.114).

The following is a companion theorem to the boundary value results presented previously in this Section. The proof techniques for the following result are the same as those for these previous results; hence the proof is omitted.

Theorem 5.2.3 *Let $f(z)$ be analytic in T^C and be the Fourier transform of a function in L^r, $1 < r \leq 2$, for $y = Im(z) \in C$. Let there exist constants $K > 0$ and $T > 0$ which are independent of $y \in C$ such that*

$$\|\mathcal{F}^{-1}[f(x+iy); t]\|_{L^r} \leq K \exp(M^*(T/|y|)), \quad y \in C.$$

There is an element $U \in \mathcal{D}'((M_p), L^r)$ such that

$$\lim_{\substack{y \to 0 \\ y \in C}} f(x+iy) = U$$

in $\mathcal{D}'((M_p), L^r)$.

Boundary value results for $\mathcal{D}'(\{(M_p)\}, L^r)$ similar to those contained in this section need to be proved; we leave this for future investigations. We also desire in the future to extend the boundary value results of this section to analytic functions in tubes T^C for which (5.95) holds for $1 < r < \infty$.

5.3 Case $2 < r < \infty$

We will extend results of Section 5.1 and 5.2, where possible, for values of r in (5.1) for which $2 < r < \infty$. The results of this section will concern tubes T^C defined by special types of cones C which we now define.

Let $u = (u_1, u_2, \ldots, u_n)$ be any of the 2^n n-tuples whose components are 0 or 1. We have previously defined the n-rant

$$C_u = \{y \in \mathbf{R}^n : \ (-1)^{u_j} y_j > 0, \ j = 1, \ldots, n\}$$

in $\mathbf{R}^n$. The n-rant C_u are open convex cones with the property that $C_u^* = \overline{C}_u$. We will call C_0 the first n-rant.

Now let C be the interior of the convex hull of n linearly independent rays meeting at $0 \in \mathbf{R}^n$ [78], p. 118. Select vectors $a_1, a_2, \ldots, a_n$ in the direction of these rays; then C can be written as

$$C = \{y \in \mathbf{R}^n : \ y = y_1 a_1 + y_2 a_2 + \ldots + y_n a_n, \ y_1 > 0, \ y_2 > 0 \ldots y_n > 0\}.$$

C is an open convex cone in $\mathbf{R}^n$. There is a nonsingular linear transform L mapping the standard basis vectors e_j, $j = 1, \ldots, n$ (i.e. the basis vectors of

$\mathbf{R}^n$ with 1 in the j-th component and 0 in the other $n-1$ components) one-one and onto the a_j, $j = 1, \dots, n$. L is then a one-one mapping from C_0 onto C and the boundary of C_0 is mapped to the boundary of C.

Further any cone contained in C_0 is mapped a one-one and onto fashion to a cone contained in C with boundary being mapped to boundary.

We extend L to $\mathbf{C}^n$ by putting

$$L(u + iv) = L(u) + iL(v), \ u + iv \in \mathbf{C}^n:$$

then L maps the tube $\mathbf{R}^n + iC_0$ one-one and onto the tube $\mathbf{R}^n + iC$. If $f(x + iy)$ is holomorphic in the tube $T^C = \mathbf{R}^n + iC$ then the function $g(u + iv) = f(L(u) + iL(v))$ is holomorphic in the tube T^{C_0} [78], p. 118, with the same being true of corresponding open convex cones which are proper subset of C_0 and C and which are mapped to one another by L and L^{-1}: that is holomorphicity in there tubes is preserved under the transform L.

Similar statements to the above can be made for L^{-1}, the inverse of L, since L is consingular.

We call a cone C as described in the first sentence of this paragraph a n-rant cone because of its identification with C_0 by the linear transform L and L^{-1}.

We note that Rudin has used the analytic invariance of tubes under nonsingular linear transforms to prove edge of the wedge theorems in [76].

C is a polygonal cone [78], p. 118 if it is the interior of the convex hull of a finite number of rays meeting the origin $0 \in \mathbf{R}^n$ among which there are n (at least n) that are linearly independent. Thus a polygonal cone C is a finite union of n-rant cones C_j, $j = 1, \dots, m$ [78], p. 118.

Recall that any n-rant cone is an open convex as is any polygonal cone. Thus the n-rant cones C_j, $j = 1, \dots, m$, whose union is C possess a very important intersection property which we describe now. No boundary point of any of the C_j, $j = 1, \dots, m$, is an element of that C_j. Thus if $y \in C$ such that y is on the boundary of some C_j then y must be in one or more of the other C_j. Because of this property (i.e. because of the convexity of the polygonal cone C) the n-rant cones C_j, $j = 1, \dots, m$, whose union is the polygonal cone C must overlap as they cover C in the following sense:

given C_1 there is another one of the C_j, $j = 1 \neq 1$, (call it C_2) such that $C_1 \cap C_2 \neq \emptyset$:

given $C_1 \cup C_2$ there is another one of the $C_j, j \neq 1$, $j \neq 2$(call it C_3) such that at least one of $C_1 \cap C_3$ and $C_2 \cap C_3$ is not empty;

given $C_1 \cup C_2 \cup C_3$ there is another one of the $C_j, j \neq 1, j \neq 2, j \neq 3$, (call it C_4) such that at least one of $C_1 \cap C_4$, $C_2 \cap C_4$ and $C_3 \cap C_4$ is not empty;

given $\bigcup_{j=1}^{m-1} C_j$, the remaining quadrant cone C_m intersects at least one of the C_j, $j = 1, \dots, m-1$.

These intersection properties of the n-rant cones whose union is a given polygonal cone will be important in our proof of a result below we collectively refer to the above described intersections of the n-rant cones $C_j, j = 1, \dots, m$, whose union is a given polygonal cone C as the intersection property of the n-rant cones.

Let us also note that the intersection of two open convex cones is an open convex cone; thus each of the nonempty intersections in the intersection property of the n-rant cones above is itself an open convex cone which is contained in n-rant cone.

The notion of polygonal cone of Stein and Weiss is closely associated with the notion of a cone containing an admissible set of vectors in the sense of Vladimirov [86], p. 930. A cone with an admissible set of vectors can be a polygonal cone.

A regular cone is an open convex cone C in $\mathbf{R}^n$ such that $\overline{C}$ does not contain any entire straight line. Any regular cone C is property contained in an open convex cone $\Gamma \subset \mathbf{R}^n$ such that $\overline{C} \subset \Gamma \cup \{0\}$.

The proof of [78], Theorem 5.5, p. 118, shows that there are a finite number of polygonal cones Λ_j, $j = 1, \ldots, k$, and the polygonal cone Λ which is the convex hull of $\bigcup_{j=1}^{k} \Lambda_j$ such that

$$C \subset \bigcup_{j=1}^{k} \Lambda_j \subset \Lambda \subset \Gamma \text{ and } \overline{C} \subset \bigcup_{j=1}^{k} \Lambda_j \cup \{0\} \subset \Lambda \cup \{0\} \subset \Gamma \cup \{0\}.$$

Recalling the preceding paragraph each polygonal cone Λ and Λ_j, $j = 1, \ldots, k$, is the union of a finite number of n-rant cones; we then have from $C \subset \bigcup_{j=1}^{k} \Lambda_j \subset \Lambda$ that $C \subset \bigcup_{j=1}^{m} C_j$, $m \geq k$, where the $C_j, j = 1, \ldots, m$, are n-rant cones.

If $C \cap C_j = \emptyset$ for any $j = 1, \ldots, m$, we delete this from the $\bigcup_{j=1}^{m} C_j$ and obtain $C \subset \bigcup_{j=1}^{r} C_j, r \leq m$, where $C \cap C_j \neq \emptyset, j = 1, \ldots, r$. (We can obtain $C \subset \bigcup_{j=1}^{r} C_j$ equally well from the inclusion $C \subset \Lambda$ since Λ is a polygonal cone.) We can now write $C = \bigcup_{j=1}^{r} (C \cap C_j)$. Now each $C \cap C_j$, $j = 1, \ldots, r$, is a n-rant cone or is contained in a n-rant cone, namely C_j. Since C and the $C_j, j = 1, \ldots, r$, are open convex cones in $\mathbf{R}^n$ then the $C \cap C_j, j = 1, \ldots, r$, are open convex cones. Thus the set equality $C = \bigcup_{j=1}^{r} (C \cap C_j)$ and the same argument as in the intersection property for the polygonal cone case in the preceding paragraph yield that the $(C \cap C_j)$, $j = 1, \ldots, r$, such that $C = \bigcup_{j=1}^{r} (C \cap C_j)$ satisfy the intersection property descibed in the preceding paragraph. It is precisely this representation $C = \bigcup_{j=1}^{r} (C \cap C_j)$ of a regular cone C in terms of the open convex cones $(C \cap C_j)$, $j = 1, \ldots, r$, which satisfy the stand intersection property and each which in a n-rant cone or is contained in a n-rant cone that we need in one of the results below.

Using analysis similar to that in [15] we obtain Fourier-Laplace intehral representation of analytic functions in tubes T^C which satisfy (5.1) for $2 < r < \infty$ for C being a n-rant, a n-rant cone, and a polygonal cone. We begin with the following result.

Theorem 5.3.1 *Let C be an open convex which contained in or is any of the 2^n n-rants C_n in $\mathbf{R}^n$. Let $f(z)$ be analytic in T^C and satisfy (5.1) for $2 < r < \infty$. There exists a measurable function $g(t)$, $t \in \mathbf{R}^n$, with* $\operatorname{supp}(g) \subseteq C^*$ *almost everywhere such that*

$$\|\varepsilon^{-2\pi\langle y,t\rangle} g\|_{L^2} \le M(1+(d(y))^{-m})^q \exp(M * T/|y|)), \; y \in C, \tag{5.116}$$

for constants $M > 0$. $T > 0$, $m \ge 0$, and $q \ge 0$ which are independent of $y \in C$, and

$$f(z) = X(z) \int_{\mathbf{R}^n} g(t)\varepsilon^{2\pi i\langle z,t\rangle}\,dt, \quad z \in T^C. \tag{5.117}$$

where $X(z)$ is a polynomial in $z \in T^C$.

Proof. For C being contained in or being the n-rant C_μ put

$$X(z) = \prod_{j=1}^{n} (1 - i(-1)^{\mu_1} z_j)^{n+2}, \quad z = x + iy \in T^C. \tag{5.118}$$

We have

$$|1/X(z+iy)| \le \prod_{j=1}^{n} (1 + x_j^2)^{-1-n/2}, \quad z \in T^C. \tag{5.119}$$

Put

$$F(z) = f(z)/X(z), \quad z \in T^C, \tag{5.120}$$

which is analytic in T^C. We have

$$\begin{aligned}
&\int_{\mathbf{R}^n} |F(x+iy)|^2\,dx = \int_{\mathbf{R}^n} |f(x+iy)/X(x+iy)|^2\,dx \\
&\le \| |f(x+iy)|^2 \|_{L^{r/2}} \| \, 1/X(x+iy)|^2 \|_{L^{r/(r-2)}} \\
&= \Big(\int_{\mathbf{R}^n} |f(x+iy)|^r\,dx \Big)^{2/r} \Big(\int_{\mathbf{R}^n} |1/X(x+iy)|^{2r/(r-2)}\,dx \Big)^{(r-2)/r} \\
&\le \Big(K(1+(d(y))^{-m})^q \exp(M^*(T/|y|)) \Big)^2 \\
&\cdot \Big(\int_{\mathbf{R}^n} \prod_{j=1}^{n} (1+x_j^2)^{-2r/(r-2)-nr/(r-2)}\,dx \Big)^{(r-2)/r} \\
&\le (M(1+(d(y))^{-m})^q \exp(M^*(T/|y|)))^2,
\end{aligned} \tag{5.121}$$

where

$$M = K \Big(\int_{\mathbf{R}^n} \prod_{j=1}^{n} (1+x_j^2)^{-2r/(r-2)-nr/(r-2)}\,dx \Big)^{(r-2)/2r}. \tag{5.122}$$

Thus from (5.121) and the fact that $F(z)$ is analytic in T^C we can apply Corollary 5.1.1 to obtain a function $g(t)$, $t \in \mathbf{R}^n$, with supp $g \subseteq C^*$ almost everywhere such that

$$\|e^{-2\pi\langle y,t\rangle} g\|_{L^2} \leq M(1+(d(y))^{-m})^q \exp(M^*T/|y|)), \quad y \in C, \tag{5.123}$$

which is (5.116) and

$$F(z) = \int_{\mathbf{R}^n} g(t) e^{2\pi i \langle z,t\rangle}\, dt, \quad z \in T^C. \tag{5.124}$$

(5.117) is now obtained from (5.120) and (5.124) with $X(z)$ being given in (5.118). The proof is complete. □

We ask if the representation (5.117) can be rewritten in the form $f(z) = \langle V, \exp(2\pi i\langle z,t\rangle)\rangle$, $z \in T^C$, for some ultradistribution V? If so, $g(t)$ will have to possess sufficient properties to allow for $\langle V, \exp(2\pi i \langle z,t\rangle)\rangle$ to be well defined.

If $m = 0$ or $q = 0$ in (5.1) the Fourier-Laplace integral in (5.117) obtains an ultradistributional boundary value as $y = \text{Im } z \to 0$, $y \in C$, by Theorem 5.2.2 since $g(t)$ satisfies (5.123) with $m = 0$ or $q = 0$. Can this fact be used along with (5.117) to prove that $f(z)$ also obtains an ultradistributional boundary value?

Recall the concept of n-rant cone given above. We extend Theorem 5.3.1 to the case that C is in or is a n-rant cone.

Theorem 5.3.2 *Let C be an open convex cone that is contained in or is a n-rant cone in $\mathbf{R}^n$. Let f be an analytic function in T^C and satisfy (5.1) for $2 < r < \infty$. There exists a measurable function g on $\mathbf{R}^n$ and a nonsingular linear transform L such that* supp $g \subseteq (L^{-1}(C))^*$,

$$\|e^{-2\pi\langle L^{-1}(y),\cdot\rangle} g\|_{L^r} \leq M(k+(d(L^{-1}(y)))^{-m})^q \exp(M^*(R/|L^{-1}(y)|)) \tag{5.125}$$

for $y \in C$, with constants $M > 0, R > 0, k > 0, m \geq 0$, and $q \geq 0$ which are independent of $y \in C$, and

$$f(z) = X(L^{-1}(x) + iL^{-1}(y)) \int_{\mathbf{R}^n} g(t) e^{2\pi i\langle L^{-1}(x)+iL^{-1}(y),t\rangle}\, dt, \tag{5.126}$$

for $x + iy \in \mathbf{R}^n + iC$, where X is a polynomial of variable $u + iv = L^{-1}(x) + iL^{-1}(y)$.

Proof. Let Γ denote the n-rant cone that C is contained in or is. There exists a nonsingular linear transform L (with domain and range being $\mathbf{R}^n$) which maps the first n-rant C_0 onto Γ in a one to one manner such that the boundary of C_0 is mapped to the boundary of Γ. Further, if C is properly contained in Γ then $L^{-1}(C)$ is an open convex cone which is contained in C_0 and L maps $L^{-1}(C)$ one to one and onto C with the boundary of $L^{-1}(C)$ being mapped to the boundary of C. ($L^{-1}(C) = C_0$ if $C = \Gamma$). For $u + iv \in \mathbf{R}^n + iL^{-1}(C)$ put

$$G(u+iv) = f(L(u) + iL(v)) = f(x+iy), \tag{5.127}$$

where $u+iv \in \mathbf{R}^n + iL^{-1}(C)$ and $x+iy \in \mathbf{R}^n + iC$. The function $G(u+iv)$ is analytic in $\mathbf{R}^n + iL^{-1}(C)$. We have using (5.1) that

$$\int\limits_{\mathbf{R}^n} |G(u+iv)|^r\,du = \frac{1}{|\det(L)|}\int\limits_{\mathbf{R}^n} |f(x+iL(v))|^r\,dx$$
$$\le \frac{1}{|\det(L)|}(K(1+(d(L(v)))^{-m})^q \exp(M^*(T/|L(v)|)))^r \quad (5.128)$$

for $y = L(v) \in C$. Recalling that the boundary of $L^{-1}(C)$ is mapped to the boundary of C by L we have for $v \in L^{-1}(C)$ that

$$d(L(v)) = \inf_{y' \in \partial L^1(C)} |L(v) - y'|$$
$$= \inf_{v' \in \partial L^{-1}(C)} |L(v) - L(v')| = \inf_{v' \in \partial L^{-1}(C)} |L(v-v')|, \quad (5.129)$$

where ∂C denotes the boundary of C. Corresponding to the nonsingular linear transform L there exist constant $a > 0$ and $b > 0$ (see [16], p. 93) such that

$$a|w| \le |L(w)| \le b|w|, \quad w \in \mathbf{R}^n, \quad (5.130)$$

with a and b being independent of $w \in \mathbf{R}^n$. Using (5.130) in (5.129) we have

$$d(L(v)) = \inf_{v' \in \partial L^{-1}(C)} |L(v-v')|$$
$$\ge \inf_{v' \in \partial L^{-1}(C)} a|v-v'| = ad(v), \quad v \in L^{-1}(c). \quad (5.131)$$

Using (5.131) and (5.130) in (5.128) we obtain

$$\int_{\mathbf{R}^n} |G(u+iv)|^r\,du \le \frac{1}{|\det(L)|}(K(1+(ad(v))^{-m})^q \exp(M^*(T/a|v|)))^r$$
$$= \frac{1}{|\det(L)|}((K/a^{mq})(a^m + (d(v))^{-m})^q \exp(M^*(T/a|v|)))^r \quad (5.132)$$

for $v \in L^{-1}(C)$ since M^* is an increasing function. From (5.132) and the analyticity of $G(u+iv)$ in $\mathbf{R}^n + iL^{-1}(C)$ we have by Theorem 5.3.1 the existence of a measurable function $g(t)$, $t \in \mathbf{R}^n$, with $\text{supp}(g) \subseteq (L^{-1}(C))^*$ almost everywhere such that

$$\|e^{-2\pi\langle v,t\rangle} g\|_{L^2} \le M(k + (d(v))^{-m})^q \exp(M^*(R/|v|)),\ v \in L^{-1}(C), \quad (5.133)$$

for some $M > 0$, $k > 0$, $m \ge 0$, and $R > 0$; and

$$G(u+iv) = X(u+iv)\int\limits_{\mathbf{R}^n} g(t)e^{2\pi i\langle u+iv,t\rangle}\,dt,\ u+iv \in \mathbf{R}^n + iL^{-1}(C), \quad (5.134)$$

where $X(u+iv)$ is a polynomial. Thus from (5.127) and (5.134) we have

$$f(x+iy) = X(L^{-1}(x) + iL^{-1}(y))\int\limits_{\mathbf{R}^n} e^{2\pi i\langle L^{-1}(x)+iL^{-1}(y),t\rangle}\,dt,\ x+iy \in T^C,$$

which is (5.126) and (5.133) is (5.125). The proof is complete. □

If $m = 0$ or $q = 0$ in (5.1), can we obtain an ultradistributional boundary value for $f(z)$ in Theorem 5.3.2 using (5.127)? We could if we know an ultradistributional boundary value existed in Theorem 5.3.1 for $m = 0$ or $q = 0$ there.

Now recall the concept of a polygonal cone given above and the fact of the intersection property for the n-rant cones C_j, $j = 1, \ldots, m$, whose union is the polygonal cone. Let us extend Theorem 5.3.1 and 5.3.2 to the case that C can be a polygonal cone.

For C being a polygonal cone, $C = \bigcup_{j=1}^{m} C_j$ where the C_j are n-rant cones which have the intersection noted above. Let $f(z)$ be analytic in T^C and satisfy (5.1) for $2 < r < \infty$. Now $y \in C$ implies $y \in C_j$ for some $j = 1, \ldots, m$; and note that the distance from y to the boundary of C_j. Thus $f(z)$ is analytic in $\mathbf{R}^n + iC_j$ and satisfies (5.1) for $y \in C_j$ for each $j = 1, \ldots, m$. Thus by Theorem 5.3.2 for each n-rant cone C_j there is a nonsingular linear transform L_j mapping C_0 one to one and onto C_j and a function $g_j(t)$ with $\operatorname{supp}(g_j) \subseteq (L_j^{-1}(C_j))^* = C_0^*$ almost everywhere and a polynomial X_j such that

$$\begin{aligned} &\|e^{-2\pi\langle L_j^{-1}(y), t\rangle} g_j\|_{L^2} \\ &\leq M(k + (d(L_j^{-1}(y)))^{-m})^q \exp(M^*(R/|L_j^{-1}(y)|)) \end{aligned} \tag{5.135}$$

for $y \in C_j$ and

$$f(x + iy) = X_j(L_j^{-1}(x) + iL_j^{-1}(y)) \int_{\mathbf{R}^n} g_j(t) e^{2\pi i\langle L_j^{-1}(x) + iL_j^{-1}(y), t\rangle}\, dt, \tag{5.136}$$

$x + iy \in \mathbf{R}^n + iC_j$.

In this way, we have proved the following result:

Theorem 5.3.3 *Let C be a polygonal cone in $\mathbf{R}^n$. Let $f(z)$ be analytic in T^C and satisfy (5.1) for $2 < r < \infty$. There exist n-rant cones C_j, $j = 1, \ldots, m$; nonsingular linear transforms L_j mapping C_0 one to one and onto C_j; functions g_j having* $\operatorname{supp}(g_j) \subseteq \overline{C_0}$ *almost everywhere and satisfying (5.135); and polynomials X_j such that (5.136) holds where $C = \bigcup_{j=1}^{m} C_j$.*

A similar result to Theorem 5.3.3 can be proved for C being a regular cone.

Can an ultradistributional boundary value be obtained for $f(z)$ in Theorem 5.3.3 and for the corresponding result for C being a regular cone?

5.4 Boundary values via almost analytic extensions

In this section we give another approach which is based on the almost analytic extensions. This concept gives for $\mathbf{R}^n$ the most general results although cases $p = \infty$, $p = 1$ are still open. Here, we will consider the case when the space dimension is $n = 1$; case $n > 1$ is considered in [22]. We refer to papers [21], [22], [66] - [68] and [72].

We continue to assume conditions $(M.1)$, $(M.2)$, $(M.3)'$ and that m_p^* is nondecreasing sequence.

Using the same method as in [66], 2.2. Proposition, and the Minkovski inequality one can prove the following lemma:

Lemma 5.4.1 *Let $r > 1$ and $h > 0$ be given. There is $H > 0$such that for every $(\varphi \in \mathcal{D}((M_p), h, L^r)$ there are $(\varphi \in C^1(\mathbf{C})$ and $C > 0$ such that $\varphi|\mathbf{R} = \varphi$ and*

$$\sup_{y\in\mathbf{R}}\{e^{M^*(hH/|y|)}\|\frac{\partial}{\partial\bar{z}}\varphi(\cdot+iy)\|_{L^r},\ \|\varphi^{(j)}(\cdot+iy)\|_{L^r},\ j=0,1\} < C\|\varphi\|_{L^r,h}.$$

(If $y = 0$, then $\frac{\partial}{\partial\bar{z}}\varphi(x) = 0$).

We remark that in Lemma 5.4.1, we add the estimate for $\varphi'(\cdot + iy)$ in order to have a symmetric assertion to the assertion of Lemma 5.4.4 below.

For the main assertions we need the following three lemmas.

Lemma 5.4.2 *Let F be a holomorphic function on $\mathbf{C}\backslash\mathbf{R}$ such that, in (M_p) case, there are $k > 0$ and $C > 0$, resp. in $\{M_p\}$ case, for every $k > 0$ there is $C > 0$, such that*

$$\|F(\cdot+iy)\|_{L^s} \le Ce^{M^*(k/|y|)},\quad y \ne 0.$$

Then, for every compact set $K \subset \mathbf{R}$ there are $p > 0$ and $B > 0$, resp. for every $p > 0$ there is $B > 0$, such that

$$\sup_{x\in K}\{|F(x+iy)|\} \le Be^{M^*(p/|y|)},\quad y \ne 0.$$

Proof. We shall prove the assertion only for (M_p)–case since the proof for $\{M_p\}$–case is similar.

Let $\alpha \in \mathcal{D}((M_p), \mathbf{R})$, supp$\alpha \subset [-a, a]$ and $\alpha \equiv 1$ in a neighbourhood of K. For $x \in K$ and $y \ne 0$ we have

$$F(x+iy) = \alpha(x)F(x+iy) = \int_{-\infty}^{x}(\alpha(t)F(t+iy))'\,dt.$$

Let $K_t = \{z\colon\ |z - t - iy| = \frac{|y|}{4}\}$, $x \in K$ and $s = r/(r-1)$. By using Cauchy's formula and Hölders inequality we have (with suitable constants)

$$|F(x+iy)| \le C_1\Big(\int_{-a}^{a}|\alpha'(t)|\Big|\int_{z\in K_t}\frac{F(z)}{z-t-iy}\,dz\Big|\,dt$$

$$+\int_{-a}^{a}|\alpha(t)|\,\Big|\int_{z\in K_t}\frac{F(z)\,dz}{(z-t-iy)^2}\Big|\,dt\Big)$$

$$\le C_1\Big[\Big(\int_{-a}^{a}|\alpha'(t)|^r\,dt\Big)^{1/r}\Big(\int_{-a}^{a}\Big|\int_{z\in K_t}\frac{F(z)}{z-t-iy}\,dz\Big|^s\,dt\Big)^{1/s}$$

$$+(\int_{-a}^{a}|\alpha(t)|^r\,dt)^{1/r}(\int_{-a}^{a}|\int_{z\in K_t}\frac{F(z)}{(z-t-iy)^2}\,dz|^s\,dt)^{1/s}]$$

$$\leq C_2[(\int_{-a}^{a}|\int_0^{2\pi}F(t+iy+\frac{|y|}{4}e^{i\varphi})d\varphi|^s\,dt)^{1/s}$$

$$+\frac{1}{|y|}(\int_{-a}^{a}|\int_0^{2\pi}\frac{F(t+iy+\frac{|y|}{4}e^{i\varphi})}{e^{i\varphi}}d\varphi|^s\,dt)^{1/s}]$$

$$\leq C_2[(\int_{-a}^{a}(\int_0^{2\pi}1d\varphi)^{s/r}\int_0^{2\pi}|F(t+iy+\frac{|y|}{4}e^{i\varphi}|^s d\varphi\,dt)^{1/s}$$

$$+\frac{1}{|y|}(\int_{-a}^{a}(\int_0^{2\pi}1d\varphi)^{s/r}\int_0^{2\pi}|\frac{F(t+iy+\frac{|y|}{4}e^{i\varphi})}{e^{i\varphi}}|^s d\varphi\,dt)^{1/s}]$$

$$\leq C_3[(\int_{-a}^{a}\int_0^{2\pi}|F(t+iy+\frac{|y|}{4}e^{i\varphi})|^s d\varphi\,dt)^{1/s}$$

$$+\frac{1}{|y|}(\int_{-a}^{a}\int_0^{2\pi}|F(t+iy+\frac{|y|}{4}e^{i\varphi})|^s d\varphi\,dt)^{1/s}]$$

$$\leq C_4(1+\frac{1}{|y|})(\int_0^{2\pi}\|F(\cdot+iy+\frac{|y|}{4}e^{i\varphi})\|_s^s d\varphi)^{1/s}$$

$$\leq C_5(1+\frac{1}{|y|})\exp(M^*(\frac{k}{|y+\frac{|y|}{4}\sin\varphi|}))\leq C_5(1+\frac{1}{|y|})\exp(M^*(\frac{k}{|y|-|y|/4}))$$

$$\leq C_5(1+\frac{1}{|y|})\exp(M^*(4k/|y|))\leq C_6\exp(M^*(5k/|y|)).$$

The lemma is proved. □

By using Sobolev's lemma one can easily prove the following one.

Lemma 5.4.3 *Let $r>1$ and $\varphi\in\mathcal{D}((M_p),L^r)$, resp. $\varphi\in\mathcal{D}(\{M_p\},L^r)$. Then for every compact set $K\subset\mathbf{R}$ and every $h>0$, resp. for some $h>0$ there are $C>0$ and $k>0$, such that*

$$\sup_{\substack{x\in K\\ p\in\mathbf{N}_0}}\{\frac{h^p}{(M_p)}|\varphi^{(p)}(x)|\}\leq C\|\varphi\|_{k,L^r}.$$

Lemma 5.4.4 *Let $\psi_0=\{z;|Imz|<\delta_0\}$, $\delta_0>0$, $\varphi^{(j)}(\cdot+iy)\in L^r$, with $r>1$ for $j=0,1$ and $|y|<\delta_0$, and suppose that $\varphi\in C^1(\psi_0)$. Assume that for every $h>0$, resp. some $h>0$,*

$$D_h=\sup_{0<|y|<\delta_0}\{\|\frac{\partial}{\partial\bar z}\varphi(\cdot+iy)\|_{L^r}e^{M^*(h/|y|)},\ \|\varphi^{(j)}(\cdot+iy)\|_{L^r},\ j=0,1\}<\infty. \tag{5.137}$$

Then, $\varphi=\psi_{|\mathbf{R}}$ is in $\mathcal{D}((M_p),L^r)$, resp. $\mathcal{D}(\{M_p\},L^r)$, and for every $h>0$, resp. for some $h>0$, there is $C>0$ such that

$$\|\varphi\|_{L^r,h}\leq CD_h.$$

Proof. We denote $\Gamma_{a,\delta\pm} = \{\zeta\colon\ \zeta = t \pm i\delta, |t| < a\}$, $\psi_a = \{\zeta; |Im\zeta| < \delta,\ |Re\zeta| < a\}$, $\gamma_{a,\pm} = \{\zeta; \zeta = \pm a + it,\quad |t| < \delta\}$, $\Gamma_{\delta\pm} = \{\zeta; \zeta = t \pm i\delta,\quad t \in \mathbf{R}\}$, $\psi = \{\zeta; |Im\zeta| < \delta\}$. This notation will be used later, as well.

Let $x \in \mathbf{R}$, $p \in \mathbf{N}$. By Cauchy's formula, for sufficiently large a, we have

$$\varphi^{(p)}(x) = \frac{p!}{2\pi i}\Big(\int_{\Gamma_{a,\delta-}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\Gamma_{a,\delta+}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}}$$

$$+ \int_{\gamma_{a,+}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\gamma_{a,-}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} + \int_{\psi_a} \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)\,d\zeta \wedge d\bar\zeta}{(\zeta - x)^{p+1}}\Big).$$

Since

$$|\varphi(x+iy)| = |\int_0^x \varphi'(t+iy)\,dt| \le |x|^{1/s}\Big(\int_{-\infty}^{\infty} |\varphi'(t+iy)|^r\,dt\Big)^{1/r},$$

(5.137) implies that for every $p \in \mathbf{N}$,

$$\int_{\gamma_{a\pm}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} \to 0 \text{ as } a \to 0.$$

This implies

$$\varphi^{(p)}(x) = \frac{p!}{2\pi i}\Big(\int_{\Gamma_{\delta-}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}} - \int_{\Gamma_{\delta+}} \frac{\varphi(\zeta)\,d\zeta}{(\zeta - x)^{p+1}}$$

$$+ \int_{\psi} \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)\,d\zeta \wedge d\bar\zeta}{(\zeta - x)^{p+1}}\Big) = \frac{1}{2\pi i}(I_1 - I_2 + I_3).$$

Let us estimate I_1, I_2 and I_3.

$$|I_1|^r \le p!^r\Big(\int_{-\infty}^{\infty} \frac{|\varphi(t+x-i\delta)|\,dt}{|t-i\delta|^{p+1}}\Big)^r \le$$

$$\le p!^r \int_{-\infty}^{\infty} \frac{|\varphi(t+x-i\delta)|^r\,dt}{|t-i\delta|^{pr}}\Big(\int_{-\infty}^{\infty} \frac{dt}{(t^2+\delta^2)^{s/2}}\Big)^{r/s} \le$$

$$\le \frac{Ap!^r}{\delta} \int_{-\infty}^{\infty} \frac{|\varphi(t+x-i\delta)|^r\,dt}{|t-i\delta|^{pr}}, \quad \text{where } A = \Big(\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{s/2}}\Big)^{r/s}.$$

By Hölder's inequality and Fubini's theorem we have

$$\int_{-\infty}^{\infty} |I_1|^r\,dx \le \frac{Ap!^r}{\delta} \int_{-\infty}^{\infty} \frac{dt}{|t-i\delta|^2} \int_{-\infty}^{\infty} \frac{|\varphi(t+x-i\delta)|^r}{|t-i\delta|^{pr-2}}\,dx$$

$$\le \frac{Ap!^r}{\delta} \int_{-\infty}^{\infty} \frac{dt}{t^2+\delta^2} \int_{-\infty}^{\infty} \frac{|\varphi(t+x-i\delta)|^r\,dx}{\delta^{pr-2}} \le \frac{A\pi p!^r}{2\delta^2}\frac{1}{\delta^{pr-2}}D_h^r.$$

Thus, by using $p! \prec M_p$ we obtain for suitable $\tilde{A} > 0$,

$$\Big(\int_{-\infty}^{\infty} |I_1|^r\,dx\Big)^{1/r} \le \tilde{A} D_h \frac{p!}{\delta^p e^{M^*(h/\delta)}} \le \tilde{A} D_h h^{-p} M_p.$$

The same inequality holds for $\|I_2\|_{L^r}$. Let us estimate $\|I_3\|_{L^r}$. We have

$$\left|\int\int_{\psi}\frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)\,d\zeta\wedge d\bar\zeta}{(\zeta-x)^{p+1}}\right|^r\le\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|\,d\eta\,d\xi}{|\xi+i\eta-x|^{p+1}}\right)^r$$

$$=\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|}{|\eta|^{1/s}|\xi+i\eta-x|^{p+1-(2/s)}}\frac{|\eta|^{1/s}d\eta d\xi}{|\xi+i\eta-x|^{2/s}}\right)^r$$

$$\le\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|^r\,d\eta\,d\xi}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-(2/s))r}}\right)\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|d\eta d\xi}{|\xi+i\eta-x|^2}\right)^{r/s}$$

$$=p!^r\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|^r\,d\eta\,d\xi}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-(2/s))r}}\right)\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|d\eta d\xi}{|\xi+i\eta-x|^2}\right)^{r/s}.$$

We will use the fact that

$$\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}\frac{|\eta|d\eta\,d\xi}{(\xi-x)^2+\eta^2}=2\int_0^{\delta}\left(\int_{-\infty}^{\infty}\frac{d\xi/\eta}{((\xi-x)/\eta)^2+1}\right)d\eta=2\pi\delta.$$

This implies

$$p!\left|\int_{-\infty}^{\infty}\right|\int\int_{\psi}\frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)}{(\zeta-x)^{p+1}}\,d\zeta\wedge d\bar\zeta|^r\,dx)^{1/r}$$

$$\le(2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|^r}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-2/s)r}}\,d\xi\,d\eta\right)dx\right)^{1/r}$$

$$=(2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+i\eta)|^r}{|\eta|^{r/s}|\xi+i\eta-x|^{(p+1-2/s)r-2}}\frac{d\xi d\eta}{|\xi+i\eta-x|^2}\right)dx\right)^{1/r}$$

$$\le(2\pi\delta)^{1/s}p!\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\left(\int_{-\infty}^{\infty}\left(\frac{|\frac{\partial}{\partial\bar\zeta}\varphi(\xi+x+i\eta)|^r\,dx}{|\eta|^{(p+1-1/s-2/r)r}}\right)\frac{d\xi d\eta}{\xi^2+\eta^2}\right)^{1/r}$$

$$\le(2\pi\delta)^{1/s}p!D_h\left(\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}\left(\frac{|\eta|^{1/r}}{|\eta|^pe^{M^*(h/|\eta|)}}\right)^r\frac{d\xi d\eta}{\xi^2+\eta^2}\right)^{1/r}$$

$$\le D_h(2\pi\delta)^{1/s}h^{-p}M_p\left(\int_{-\infty}^{\infty}\int_{-\delta}^{\delta}\frac{|\eta|d\xi d\eta}{\xi^2+\eta^2}\right)^{1/r}\le AD_h\delta h^{-p}M_p.$$

Minkowski's inequality implies that for every $h>0$,resp. some $h>0$, there is a constant $C>0$ such that

$$\|\varphi^{(p)}\|_{L^r}\le CD_hh^{-p}M_p,\ \ p\in\mathbf{N}_0.$$

This implies the assertion. □

Let $s\in[1,\infty]$. Denote by $\mathcal{H}((M_p),L^s)$, resp. $\mathcal{H}(\{M_p\},L^s)$, the space of functions f holomorphic in $\psi_0\setminus\mathbf{R}$, where

$$\psi_0=\{x+iy;x\in\mathbf{R}\,|y|<\delta_0\},\ \delta_0=\delta_0(f),$$

which satisfy the following estimate: For some $k > 0$ and some $C > 0$, resp. for every $k > 0$ there exists $C > 0$, such that

$$||f(\cdot + iy)||_{L^s} \leq Ce^{M^*(k/|y|)}, \ |y| < \delta_0, \ y \neq 0.$$

The common notation for both spaces is $\mathcal{H}(*, L^s)$.

We denote by $\mathcal{H}(L^s, \mathbf{R})$ the space of functions f holomorphic in corresponding ψ_0 and satisfying

$$||f(\cdot + iy)||_{L^s} < C_f, \ |y| < \delta_0.$$

Let f be a holomorphic function in $\psi_0 \setminus \mathbf{R}$ ($\psi_0 = \{z|(Imz) < \delta_0\}$, $\delta_0 = \delta_0(f)$). If for every $\varphi \in \mathcal{D}(*, L^r)$ there exists the limit

$$\langle Tf, \varphi\rangle := \lim_{\varepsilon\to 0} \int_{\mathbf{R}} \varphi(x)(f(x + i\varepsilon) - f(x - i\varepsilon))\, dx,$$

then we call Tf the boundary value of f in $\mathcal{D}'(*, L^r)$.

Theorem 5.4.1 *Let $r > 1$ and let $f \in \mathcal{H}^*_{L^s}$. Then for every $\varphi \in \mathcal{D}'(*, L^r)$.*

$$\langle Tf, \varphi\rangle = \int_\psi f(z)\frac{\partial}{\partial z}\varphi(z)\, dz \wedge d\bar{z} - \int_{\Gamma_{\delta-}} f(z)\varphi(z)\, dz + \int_{\Gamma_{\delta-}} f(z)\varphi(z)\, dz,$$

where φ is defined in Lemma 5.4.1. Moreover, Tf belongs to $\mathcal{D}'(*, L^r)$.

Proof. Let

$$\psi_{a^+} = \{z; Imz \in (0, \delta), \ |Rez| < a\},$$

$$\psi_{a^-} = \{z; Imz \in (-\delta, 0), \ |Rez| < a\}, \ \delta \in (0, \delta_0),$$

and let $\varepsilon < (\delta_0 - \delta)/2$. Lemmas 5.4.2 and 5.4.3 enable us to apply Stokes' theorem which implies

$$\int\int_{\psi_{a+}} f(x + i(y + \varepsilon))\frac{\partial}{\partial \bar{z}}\varphi(z) d\bar{z} \wedge dz = \int_{\partial\psi_{a+}} f(x + i(y + \varepsilon))\varphi(z)\, dz.$$

Since $y + \varepsilon \in (\varepsilon, \varepsilon + \delta) \subset (0, \delta_0)$ for $y \in (0, \delta)$, we obtain

$$||f(\cdot + i(y + \varepsilon))||_{L^s} \leq Ce^{M^*(k/\varepsilon)}.$$

This fact and [81], p.125, Lemma, imply $f(x + i(y + \varepsilon)) \to 0$ as $|x| \to \infty$, uniformly for $y \in (0, \delta)$. Thus, by Lemma 5.4.3 and by letting $a \to \infty$ we obtain

$$\int\int_{\psi_+} f(x + i(y + \varepsilon))\frac{\partial}{\partial \bar{z}}\varphi(z) d\bar{z} \wedge dz$$
$$= \int_{-\infty}^{\infty} f(x + i\varepsilon)\varphi(x)\, dx - \int_{-\infty}^{\infty} f(x + i(\varepsilon + \delta))\varphi(x + i\delta)\, dx. \quad (5.138)$$

Similarly,

$$\int\int_{\psi_-} f(x+i(y-\varepsilon))\frac{\partial}{\partial z}\varphi(z)d\bar{z}\wedge dz$$
$$\int_{-\infty}^{\infty} f(x-i(\varepsilon+\delta))f(x-i\delta)\,dx - \int_{-\infty}^{\infty} f(x-i\varepsilon)\varphi(x)\,dx. \quad (5.139)$$

We have (with suitable $C > 0$)

$$|\int\int_{\psi_+} f(x+i(y+\varepsilon))\frac{\partial}{\partial \bar{z}}\varphi(z)d\bar{z}\wedge dz|$$
$$= 2|\int_0^\delta dy(\int_{-\infty}^{\infty} f(x+i(y+\varepsilon))\frac{\partial}{\partial \bar{z}}\varphi(x+iy)\,dx)|$$
$$\leq 2\int_0^\delta dy(\int_{-\infty}^{\infty}|f(x+i(y+\varepsilon))|^s\,dx)^{1/s}(\int_{-\infty}^{\infty}(|\frac{\partial}{\partial \bar{z}}\varphi(x+iy)|^r\,dx)^{1/r}$$
$$\leq C\int_0^\delta e^{M^*(k/(y+\varepsilon))-M^*(k/y)}\,dy < \infty.$$

The same holds for the integral over ψ_-. Since the integrands in (5.138) and (5.139) pointwise converge to the corresponding integrable functions, as $\varepsilon \to 0$, we obtain

$$\langle Tf,\varphi\rangle = \int\int_{\psi} f(z)\frac{\partial}{\partial \bar{z}}\varphi(z)d\bar{z}\wedge dz - \int_{\partial\psi} f(z)\varphi(z)\,dz,$$

which proves the first part of the assertion.

By using Hölder's inequality, the estimate for f and Lemma 5.4.1 we obtain that for some $h > 0$ and some $C > 0$, resp. for every $h > 0$ there is $C > 0$, such that

$$|\langle Tf,\varphi\rangle| \leq C\|\varphi\|_{h,L^r}$$

which completes the proof. □

For the proof of the next theorem we need the following estimate: There is $B > 0$ such that for every $y > 0$ and every $g \in L^s$ $(s > 1)$

$$\int_{-\infty}^{\infty}|\int_{-\infty}^{\infty}\frac{g(t)\,dt}{t-x-iy}|^s\,dx)^{1/s} \leq B\|g\|_s. \quad (5.140)$$

This estimate is obtained by combining Theorem 1.4, Lemma 1.5 (Ch. IV), Theorem 3.10 and 3.7 (Ch. II) in [81].

Theorem 5.4.2 *Let $r > 1$ and $s = r/(r-1)$. The mapping $T : \mathcal{H}(*, L^s) \to \mathcal{D}'(*, L^s)$ is surjective. Its kernel is $\mathcal{H}_{L^s}$.*

Proof. We shall prove the assertion only for (M_p)–case because the $\{M_p\}$–case can be proved simillarly.

Let $f \in \mathcal{D}'((M_p), L^s)$ be of the form

$$f=\sum_{p=0}^{\infty}(-1)^p f_p^{(p)},\ f_p\in L^s \text{ such that } \sum_{p=0}^{\infty}\frac{M_p}{K^p}\|f_p\|_{L^s}<\infty.$$

One can easily prove that the function $t\mapsto \frac{1}{t-z}$, $t\in\mathbf{R}$, $z=x+iy$, $x\in\mathbf{R}$, $y\neq 0$, belongs to $\mathcal{D}((M_p),L^r)$. We shall prove that

$$z\mapsto g(z)=-\langle f(t),\frac{1}{t-x-iy}\rangle, x\in\mathbf{R},\ y\neq 0,$$

belongs to $\mathcal{H}((M_p),L^s)$. By Minkowski's inequality and (5.140) we have

$$\|\langle f(t),\frac{1}{(t-\cdot)-iy)}\rangle\|_{L^s}\le\|\sum_{p=0}^{\infty}\langle f_p(t),\frac{p!}{(t-z)^{p+1}}\rangle\|_{L^s}$$

$$\le\|\langle f_0(t),\frac{1}{(t-\cdot)-iy)}\rangle\|_{L^s}+\sum_{p=1}^{\infty}\|\langle f_p(t),\frac{p!}{(t-z)^{p+1}}\rangle\|_{L^s}$$

$$\le B\|f_0\|_{L^s}+\sum_{p=1}^{\infty}\frac{p!}{y^{p-1}}\|\int_{\mathbf{R}}\frac{|f_p(t)|}{|t-z|^2}\,dt\|_{L^s}.$$

Since

$$\Big(\int_{\mathbf{R}}\frac{dt}{|t-x-iy|^{1+r/2}}\Big)^{s/r}=\frac{1}{|y|^{s/2}}\Big(\int_{\mathbf{R}}\frac{du}{|u-i|^{1+r/2}}\Big)^{s/r}=\frac{1}{y^{s/2}}A,$$

$(r=s/(s-1))$, and for $p\ge 1$,

$$\Big|\int_{\mathbf{R}}\frac{|f_p(t)|\,dt}{|t-z|^{3/2-1/r}|t-z|^{1/2+1/r}}\Big|^s\le\int_{\mathbf{R}}\frac{|f_p(t)|^s}{|t-z|^{1+s/2}}\Big(\int_{\mathbf{R}}\frac{dt}{|t-z|^{1+r/2}}\Big)^{s/r},$$

we obtain

$$\|\langle f(t),\frac{1}{t-z}\rangle\|_{L^s}\le B\|f_0\|_{L^s}+A^{s/r}\sum_{p=1}^{\infty}\frac{p!}{|y|^{p-1+s/2}}\Big(\int_{\mathbf{R}}\Big(\int_{\mathbf{R}}\frac{|f_p(t)|^s}{|t-z|^{1+s/2}}\,dt\Big)\,dx\Big)^{1/s}$$

$$\le B\|f_0\|_{L^s}+A^{s/r}\sum_{p=0}^{\infty}\frac{p!}{|y|^{p-1+s/2}}\Big(\int_{\mathbf{R}}|f_p(t)|^s\Big(\int_{\mathbf{R}}\frac{dx}{|t-x-iy|^{1+s/2}}\Big)\,dt\Big)^{1/s}$$

$$\le B\|f_0\|_{L^s}+A^{s/r+1/s}\sum_{p=0}^{\infty}\frac{p!}{|y|^{p-1+s/2+1/2}}\|f_p\|_{L^s}$$

$$\le A_1\sum_{p=0}^{\infty}\frac{p!}{|y|^p}\|f_p\|_{L^s},$$

where $A_1=B+A^{s/r+1/s}\,|y|^{(1-s)/2}$. This implies that for $y\neq 0$

$$\|\langle f(t),\frac{1}{t-z}\rangle\|_{L^s}\le A_1\sup_p\{\frac{k^p p!}{M_p|y|^p}\}\sum_{p=0}^{\infty}\frac{M_p}{k^p}\|f_p\|_{L^s}\le\tilde{A}_1e^{M^*(k/|y|)},$$

and that $g \in \mathcal{H}((M_p), L^s)$. We shall show that $f = Tg$. Let $\varphi \in \mathcal{D}((M_p), L^r)$ and φ be its almost analytic extension. For $z \in \mathbf{C}$. put

$$\varphi_1(z) = \frac{1}{2\pi i}\int_\psi \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta,$$

$$\varphi_2(z) = \frac{1}{2\pi i}\int_{\Gamma_{\delta-}} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta,\ \varphi_3(z) = -\frac{1}{2\pi i}\int_{\Gamma_{\delta+}} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta.$$

We have $\varphi(x) = \varphi_1(x) + \varphi_2(x) + \varphi_3(x)$, $x \in \mathbf{R}$. By the same arguments as in Lemma 5.4.4 it follows that $x \mapsto \varphi_2(x)$, $x \mapsto \varphi_3(x)$, $x \in \mathbf{R}$ are in $\mathcal{D}((M_p), L^r)$. Thus $x \mapsto \varphi_1(x)$, $x \in \mathbf{R}$, is in $\mathcal{D}((M_p), L^r)$. We have

$$\langle f, \varphi\rangle = \frac{1}{2\pi i}(\langle f(x), \int_\psi \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)}{\zeta - x}\, d\zeta \wedge d\bar\zeta\rangle +$$

$$+\langle f(x), \int_{\Gamma_{\delta-}} \frac{\varphi(\zeta)}{\zeta - x}\, d\zeta\rangle - \langle f(x), \int_{\Gamma_{\delta+}} \frac{\varphi(\zeta)}{\zeta - x}\, d\zeta\rangle) =$$

$$= \frac{1}{2\pi i}(\int_\psi \langle f(x), \frac{1}{\zeta - x}\rangle \frac{\partial}{\partial\bar\zeta}\varphi(\zeta)\, d\zeta \wedge d\bar\zeta + \int_{\Gamma_{\delta-}} \langle f(x), \frac{1}{\zeta - x}\rangle \varphi(\zeta)\, d\zeta$$

$$-\int_{\Gamma_{\delta+}} \langle f(x), \frac{1}{\zeta - x}\rangle \varphi(\zeta)\, d\zeta)$$

$$= \int_\psi g(\zeta)\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)\, d\zeta \wedge d\bar\zeta - \int_{\Gamma_{\delta-}} g(\zeta)\varphi(\zeta)\, d\zeta + \int_{\Gamma_{\delta+}} g(\zeta)\varphi(\zeta)\, d\zeta = \langle Tg, \varphi\rangle.$$

The interchange of f and integrals given above is allowed because one can prove that it is allowed if $\int_\psi$ and $\int_{\Gamma_{\delta\pm}}$ are replaced by $\int_{\psi_a}$ and $\int_{\Gamma_{a\delta\pm}}$, $a > 0$, and because

$$\int_{\psi_a} \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)}{\zeta - \cdot}\, d\zeta \wedge d\bar\zeta \to \int_\psi \frac{\frac{\partial}{\partial\bar\zeta}\varphi(\zeta)}{\zeta - \cdot}\, d\zeta \wedge d\bar\zeta,$$

$$\int_{\Gamma_{a\delta\pm}} \frac{\varphi(\zeta)}{\zeta - \cdot}\, d\zeta \to \int_{\Gamma_{\delta\pm}} \frac{\varphi(\zeta)}{\zeta - \cdot}\, d\zeta \quad a \to \infty, \text{ in } \mathcal{D}((M_p), L^r).$$

By similar arguments as in the proof of Theorem 3.3. in [63] one can prove that Ker $T = \mathcal{H}(L^s, \mathbf{R})$, i.e. the assertion of Theorem 5.4.1 is proved. □

5.5 Cases $s = \infty$ and $s = 1$

The method used in previous section could not be applied for $s = \infty$ and $s = 1$ because the function

$$\mathbf{R} \ni t \mapsto \frac{1}{t - x - iy},\quad x + iy \in \mathbf{C},\ \ y \neq 0,$$

is not in L^1. Note that this function belongs to $\dot{\mathcal{B}}((M_p), \mathbf{R})$ but we did not succeed to prove that for an $f \in \mathcal{D}'(*, L^\infty)$ or $f \in \mathcal{D}'(*, L^1)$ there exists the corresponding $F(z)$ in $\mathcal{H}(*, L^\infty)$ or $\mathcal{H}(*, L^1)$ which converes to f in $\mathcal{D}'(*, L^\infty)$ or $\mathcal{D}'(*, L^1)$. We shall prove the converse assertion i.e. that elements in $\mathcal{H}(*, L^\infty)$ and $\mathcal{H}(*, L^1)$ determine elements in $\mathcal{D}'(*, L^\infty)$ and $\mathcal{D}'(*, L^1)$ as boundary values but assuming the stronger condition (5.140) instead of (5.139). This condition enables us to follow the method of Komatsu [48], proof of Theorem 11.5. The following lemma from [48] is needed.

Lemma 5.5.1 *Let* (N_p) *satisfies* $(M.1)$, $(M.2)$, $(M.3)'$ $n_p = N_p/N_{p-1}$, *and let*

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} (1 + \frac{\zeta}{n_p}), \quad \zeta \in \mathbf{C},$$

$$G(z) = \frac{1}{2\pi} \int_0^\infty P(\zeta)^{-1} e^{iz\zeta}\, d\zeta, \quad z \in \mathbf{C}.$$

Then $G(z)$ *is a holomorphic function which can be continued analytically to the Riemann domain* $\{z;\ -\pi < \arg z < 2\pi\}$ *on which we have* $P(D)G(z) = -(2\pi i z)^{-1}$. $G(z)$ *is bounded on the domain* $\{z;\ -\frac{-\pi}{2} \le \arg z \le \frac{3\pi}{2}\}$. *Furthermore, set for* $y > 0$

$$g(y) = G_+(-iy) - G_-(-iy),$$

where G_+ *is the branch of* G *on* $\{z;\ -\pi < \arg z \le 0\}$ *and* G_- *is that on* $\{z;\ \pi \le \arg z < 2\pi\}$. *Then for some* $A > 0$

$$|g(y)| \le A\sqrt{y} e^{M^*(L/y)}, \quad y > 0.$$

Theorem 5.5.1 *Assume that conditions* $(M.1)$, $(M.2)$, $(M.3)'$ *are satisfied and the sequence* (m_p^*) *is nondecreasing. Let* $F \in \mathcal{H}(*, L^\infty)$ *(resp.* $F \in \mathcal{H}(*, L^1)$*). Then*

$$F(\cdot + iy) \to F(\cdot + i0) \in \mathcal{D}'(*, L^\infty) \ \text{as}\ y \to 0^+,$$

$$(\text{resp.}\ F(\cdot + iy) \to F(\cdot + i0) \in \mathcal{D}'(*, L^1) \ \text{as}\ y \to 0^+)$$

in the sense of convergence in $\mathcal{D}'(*, L^\infty)$, *(resp.*$\mathcal{D}'(*, L^1)$*).*

Proof. We shall prove the theorem only for the $\{(M_p)\}$–case which is more complicated. We shall use the construction from [48], Theorem 5.4.3 (see also [67]). Our aim is to prove that for every $\varphi \in \mathcal{D}(\{M_p\}, L^1)$, (resp. $\varphi \in \dot{\mathcal{B}}(\{M_p\}, \mathbf{R})$) the set

$$\{\langle F(\cdot + iy), \varphi\rangle:\ 0 < y < \delta_0\}$$

is bounded and, moreover, $\langle F(\cdot + iy), \varphi\rangle$ converges as $y \to 0$ for every $\varphi \in \mathcal{D}(\{M_p\}, \mathbf{R})$. Since $\mathcal{D}(\{M_p\}, \mathbf{R})$ is dense in $\mathcal{D}(\{M_p\}), L^1)$, resp. $\dot{\mathcal{B}}(\{M_p\}, \mathbf{R})$, this will imply the assertion in Theorem 5.5.1.

Assume first that $F \in \mathcal{H}(\{M_p\}, L^\infty)$ and that $\varphi \in \mathcal{D}(\{M_p\}, L^1)$ such that for $h_0 > 0$, $||\varphi||_{L^1, h_0} < \infty$.

Let $I_k = (k-2, k+2)$, $k \in \mathbf{Z}$, (the set of all integers) and ψ_k, $k \in \mathbf{Z}$, be a partition of unity in $\mathcal{D}(\{M_p\}, \mathbf{R})$ such that for some $R > 0$, which does not depend on k,

$$\operatorname{supp} \psi_k \subset I_k, \quad ||\psi_k||_{L^1, h_0} \le R, \quad k \in \mathbf{Z}.$$

We have

$$\int_{-\infty}^{\infty} F(x+iy)\varphi(x)\,dx = \sum_{k\in\mathbf{Z}} \int_{I_k} F(x+iy)\varphi(x)\psi_k(x)\,dx,\quad 0<y<\delta_0.$$

We shall construct an ultradifferential operator of class $\{M_p\}$ of the form

$$P(D) = (1+D)^2 \prod_{p=1}^{\infty}(1+\frac{D}{n_p}), \tag{5.141}$$

such that the equations

$$P(D)H_k(x+iy) = F(x+iy),\quad k\in\mathbf{Z},$$

have the solutions $H_k(x+iy)$ which are holomorphic in

$$\Pi_k = \{x+iy;\ x\in I_k, 0<y<\frac{\delta_0}{2}\}$$

and bounded in some neighbourhood of I_k, $k\in\mathbf{Z}$.

As in [48], pp.98–99, one can show that there is a sequence (n_p) such that the operator (5.141) is of class $\{M_p\}$, $M_p \prec N_p$, and

$$||F(\cdot+iy)||_{L^\infty} < Ce^{N^*(\frac{1}{y})},\quad |y|<\delta_0.$$

Note that conditions $(M.1)$, $(M.2)$, $(M.3)'$ imply that if $P(D)$ of the form (5.141) then it is of the class $\{M_p\}$ and, for this, condition that m_p^* is nondecreasing could not be replaced by $(M.1)$.

Fix k and denote by z_k^0 the point $k+i\delta$ ($\frac{\delta_0}{2}<\delta<\delta_0$). Let

$$H_k(z) = \int_\Gamma G(z-\omega)F(\omega)\,d\omega,\quad z=x+iy,\ x\in I_k,\ 0<y<\frac{\delta_0}{2},$$

where $G(z)$ is the Green kernel of $P(D)$ given in Lemma 5.5.1 and Γ_k is a simple closed curve laying in $\{x+iy;\ x\in I_k, y\in(0,\delta)\}$ starting at z_k^0 and encircling counterclockwise a slit connecting z_k^0 and z. We deform the path Γ_k to the union of segments joining z_k^0 and $z_k^1 = x+i\frac{\delta_0}{2}$, a segment joining z_k^1 and z, a segment joining z and z_k^1 and a segment joining z_k^1 and z_k^0. This is possible because $G(z)$ is bounded for $\frac{-\pi}{2}\le \arg z\le \frac{3\pi}{2}$. By the same arguments as in [48] we have $P(D)H_k(z)=F(z)$, $z\in\Pi_k$, and thus, we obtain

$$|\int_{-\infty}^{\infty} F(x+iy)\varphi(x)\,dx| \le \sum_{k\in\mathbf{Z}} \int_{I_k} |F(x+iy)\psi_k(x)\varphi(x)|\,dx$$

$$\le \sum_{k\in\mathbf{Z}} \int_{I_k} |\int_{\Gamma_k} G(z-\omega)F(\omega)\,d\omega P(D)(\psi_k(x)\varphi(x))|\,dx,\quad 0<y<\frac{\delta_0}{2}.$$

Denote the part of γ_k from z_k^1 to z and z to z_k^1 by Γ_k^1 and the rest by Γ_k^0, $k\in\mathbf{Z}$. We have

$$\int_{I_k} |(\int_{\Gamma_k^1} G(z-\omega)F(\omega)\,d\omega)P(D)(\varphi(x)\psi_k(x))|\,dx \le$$

$$\sup_{x\in I_k}\{|\int_{\Gamma_k^1} G(z-\omega)F(\omega)\,d\omega|\} \int_{I_k} |P(D)(\varphi(x)\psi_k(x))|\,dx. \tag{5.142}$$

Denote by A_k the first and by B_k the second factor on the right side of (5.142). Since $P(D) = \sum_\alpha D^\alpha$, is of class $\{M_p\}$, from (5.141) with $2r < h_0^2$, and from $M_{\alpha-j}M_j \le M_\alpha$, $j \le \alpha$, $j, \alpha \in \mathbf{N}_0^n$, we have

$$\sum_{k\in\mathbf{Z}} B_k \le \sum_{k\in\mathbf{Z}}\sum_{\alpha=0}^{\infty} a_\alpha \sum_{j=0}^{\alpha}\binom{\alpha}{j}\int_{I_k}|\varphi^{(\alpha-j)}(x)\psi_k^{(j)}(x)|\,dx$$

$$\le C\sum_{\alpha=0}^{\infty}\frac{r^\alpha}{(h_0^2)^\alpha}a_\alpha\sum_{j=0}^{\alpha}\binom{\alpha}{j}\|\varphi\|_{L^1,h_0}\|\psi_k\|_{L^\infty,h_0}$$

$$\le CR\|\varphi\|_{L^1,h_0}\sum_{\alpha=0}^{\infty}\frac{r^\alpha}{(h_0^2)^\alpha} < \infty.$$

For A_k we have

$$A_k = \sup_{x\in I_k}\{|\int_0^{\delta-y} g(t)F(x+iy+it)\,dt|\}$$

$$\le A\sqrt{y}\sup_{x\in I_k}\int_0^{\delta-y} e^{-N^*(\frac{1}{t+y})}e^{N^*(\frac{1}{t})}\,dt < \infty.$$

This implies that $\sum_{k\in\mathbf{Z}} A_k B_k < \infty$. Consider the path Γ_k^0. We have

$$|\int_{I_k}|(\int_{\Gamma_k^0} G(z-\omega)F(\omega)\,d\omega)P(D)(\varphi(x)\psi_k(x))|\,dx \le$$

$$\sup_{x\in I_k}\{|\int_{\Gamma_k^0} G(z-\omega)F(\omega)\,d\omega|\}\int_{I_k}|P(D)(\varphi(x)\psi_k(x))|\,dx = D_k B_k.$$

Since for $z \in \Pi_k$, $\omega \in \Gamma_k^0$, $G(z-\omega)$ is uniformly bounded by a constant which does not depend on k, we obtain $\sum_{k\in\mathbf{Z}} D_k B_k < \infty$. This implies

$$|\int F(x+iy)\varphi(x)\,dx| \le \sum_{k\in\mathbf{Z}}(D_k + A_k)B_k < \infty.$$

The proof that there is $F(x+i0) \in \mathcal{D}'(\{M_p\},\mathbf{R})$ such that for every $\varphi \in \mathcal{D}(\{M_p\},\mathbf{R})$

$$\langle F(x+iy),\varphi\rangle \to \langle F(x+i0),\varphi\rangle, \quad y \to 0,$$

is given in [48] and [63]. Thus, we conclude that

$$F(x+iy) \to F(x+i0) \in \mathcal{D}(\{M_p\},L^\infty), \quad y \to 0,$$

which finishes the proof in case $F \in \mathcal{H}(\{M_p\},L^\infty)$.

The proof of Theorem 5.5.1 for $F \in \mathcal{H}(\{M_p\},L^1)$ is analogous to the previous case. The partition of unity ψ_k and the constructed sequence $H_k(z)$, $z \in \Pi_k$, leads us to the proof that for every $\varphi \in \mathcal{D}(\{M_p\},L^\infty)$

$$\{\langle F(x+iy),\varphi\rangle, 0 < y < \frac{\delta_0}{2}\}$$

is bounded. So we have to prove that for any $\varphi \in \mathcal{D}(\{M_p\}.\mathbf{R})$, $\langle F(x+iy), \varphi\rangle$ converges as $y \to 0^+$.

Let I be a bounded open interval and $\Pi_I = \{x+iy; x \in I, y \in (0, \frac{\delta_0}{2})\}$. As in the first part of of the proof we construct $P(D)$ of the form (5.141) and of $\{M_p\}$-class and H_I such that $P(D)H_I(x+iy) = F(x+iy), x+iy \in \Pi_I$. We put

$$H_I(x+iy) = \int_{\Gamma^1} G(z-\omega)F(\omega)d\omega + \int_{\Gamma^0} G(z-\omega)F(\omega)d\omega,$$

where $\Gamma = \Gamma^1 \cup \Gamma^2$ is a path constructed in the same way as Γ_k with I instead of I_k and $z^0 = x^0 + i\delta$ (x^0 is the midle point of I) instead of z_k^0. By using Hölder's inequality we obtain that $H(\cdot + iy) \in L^1(I)$ for every $0 < y < \delta_0/2$, and that

$$\|H(\cdot + iy)\|_{L^1} < C, 0 < y < \delta_0/2.$$

This implies that $H_I(x+iy) \to H_I(x+i0) \in L^1, \quad y \to 0^+$, and thus, $H_I(x+iy) \to H_I(x+i0)$ in $\mathcal{D}'(\{M_p\}, \mathbf{R})$.

This means that

$$\langle F(x+iy), \varphi\rangle \to \langle F(x+i0), \varphi\rangle, \quad y \to 0,$$

for every $\varphi \in \mathcal{D}(\{M_p\}, \mathbf{R})$ and the proof is completed. □

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