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# SOME APPLICATIONS OF THE BOCHNER-MARTINELLI INTEGRAL

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This course of lectures devoted entirely to the Bochner-Martinelli integral representation for holomorphic functions in several complex variables and its applications to the $\overline{\partial}$-Neumann problem, holomorphic extension of functions, removable singularities of $CR$ functions and others.

## Introduction

The Bochner-Martinelli integral representation for holomorphic functions of several complex variables appeared in the works of Martinelli (1938) and Bochner (1943). It was the first essentially multidimensional representation in which the integration takes place over the whole boundary of the domain. This integral representation has a universal kernel (not depending on the form of the domain), like the Cauchy kernel in $\mathbb{C}^1$. However, in $\mathbb{C}^n$ when $n > 1$, the Bochner-Martinelli kernel is harmonic, but not holomorphic. For a long time, this circumstance prevented the wide application of the Bochner-Martinelli integral in multidimensional complex analysis.

Interest in the Bochner-Martinelli representation grew in the 1970's in connection with the increased attention to integral methods in multidimensional complex analysis. Moreover, it turned out that the very general Cauchy-Fantappiè representation found by Leray is easily obtained from the Bochner-Martinelli representation (Khenkin). Koppelman's representation for exterior differential forms, which has the Bochner-Martinelli representation as a special case, appeared at the same time.

The Cauchy-Fantappiè and Koppelman representations found significant applications in multidimensional complex analysis: constructing good integral representations for holomorphic functions, an explicit solution of the $\overline{\partial}$-equation and estimates of this solution, uniform approximation of holomorphic functions on compact sets, etc.

At the beginning of the 1970's, it was shown that, notwithstanding the non-holomorphicity of the kernel, the Bochner-Martinelli representation holds only for holomorphic functions. In 1975, Harvey and Lawson obtained a result for odd-dimensional manifolds on spanning by complex chaines; the Bochner-Martinelli formula lies at its foundations. In the 1980's and 1990's, the Bochner-Martinelli formula was successfully exploited in the theory of function of several complex variables: in multidimensional residues, in complex (algebraic) geometry, in questions of boundary regularity of holomorphic mappings, in finding analogues of Carleman's formula, etc.

In sum, one may say that the Bochner-Martinelli formula gives the connection between complex and harmonic analysis in $\mathbb{C}^n$. This becomes especially apparent in the solution of the $\overline{\partial}$-Neumann problem: any function that is orthogonal to the holomorphic functions is the $\overline{\partial}$-normal derivative of a harmonic function.

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# Contents

## 1. Green's Formula in Complex Form

We consider $n$-dimensional complex space $\mathbb{C}^n$ with variables $z = (z_1, \dots, z_n)$. If $z$ and $w$ are points in $\mathbb{C}^n$, then we write $\langle z, w\rangle = z_1w_1 + \cdots + z_nw_n$, and $|z| = \sqrt{\langle z, \bar{z}\rangle}$, where $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$. The topology in $\mathbb{C}^n$ is given by the metric $\langle z, w\rangle \mapsto |z - w|$. If $z \in \mathbb{C}^n$, then $\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n) \in \mathbb{R}^n$, where we write $\operatorname{Re} z_j = x_j$, and $\operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$ with $\operatorname{Im} z_j = y_j$; that is, $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Thus $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The orientation of $\mathbb{C}^n$ is determined by the coordinate order $(x_1, \dots, x_n, y_1, \dots, y_n)$. Accordingly, the volume form $dv$ is given by $dv = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n = dx \wedge dy = (i/2)^n dz \wedge d\bar{z} = (-i/2)^n d\bar{z} \wedge dz$.

As usual, a function $f$ on an open set $U \subset \mathbb{C}^n$ belongs to the space $\mathcal{C}^k(U)$ if $f$ is $k$ times continuously differentiable in $U$. (Here $0 \leqslant k \leqslant \infty$, and $\mathcal{C}^0(U) = \mathcal{C}(U)$). If $M$ is a closed set in $\mathbb{C}^n$, then $f$ belongs to $\mathcal{C}^k(M)$ when $f$ extends to some neighborhood $U$ of $M$ as a function of class $\mathcal{C}^k(U)$. We will also consider the space $C^r(U)$ (or $C^r(M)$) when $r \geqslant 0$ is not necessarily an integer. A function $f$ belongs to $C^r(U)$ if it lies in the class $\mathcal{C}^{[r]}(U)$ (where $[r]$ is the integral part of $r$), and all its derivatives of order $[r]$ satisfy a Hölder condition on $U$ with exponent $r - [r]$.

The space $\mathcal{O}(U)$ consists of those functions $f$ that are holomorphic on the open set $U$; when $M$ is a closed set, $\mathcal{O}(M)$ consists of those functions $f$ that are holomorphic in some neighborhood of $M$ (a different neighborhood for each function). A function $f$ belongs to $\mathcal{A}(U)$ if $f$ is holomorphic in $U$ and continuous on the closure $\overline{U}$ (that is, $f \in \mathcal{O}(U) \cap \mathcal{C}(\overline{U})$).

A domain $D$ in $\mathbb{C}^n$ has boundary of class $\mathcal{C}^k$ (we write $\partial D \in \mathcal{C}^k$) if $D = \{z : \rho(z) < 0\}$, where $\rho$ is a real-valued function of class $\mathcal{C}^k$ on some neighborhood of the closure of $D$, and the differential $d\rho \neq 0$ on $\partial D$. If $k = 1$, then we say that $D$ is a domain with *smooth* boundary. We will call the function $\rho$ a *defining* function for the domain $D$. The orientation of the boundary $\partial D$ is induced by the orientation of $D$.

By a domain with *piecewise-smooth* boundary $\partial D$ we will understand a smooth polyhedron, that is, a domain of the form $D = \{z : \rho_j(z) < 0, j = 1, \dots, m\}$, where the real-valued functions $\rho_j$ are class $\mathcal{C}^1$ in some neighborhood of the closure $\overline{D}$, and for every set of distinct indices $j_1, \dots, j_s$ we have $d\rho_{j_1} \wedge \cdots \wedge d\rho_{j_s} \neq 0$ on the set $\{z : \rho_{j_1}(z) = \cdots = \rho_{j_s}(z) = 0\}$. It is well known that Stokes's formula holds for such domains $D$ and surfaces $\partial D$.

We denote the ball of radius $\varepsilon > 0$ with center at the point $z \in \mathbb{C}^n$ by $B(z, \varepsilon)$, and we denote its boundary by $S(z, \varepsilon)$ (that is, $S(z, \varepsilon) = \partial B(z, \varepsilon)$).

Consider the exterior differential form $U(\zeta, z)$ of type $(n, n-1)$ given by

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} d\bar{\zeta}[k] \wedge d\zeta,$$

where $d\bar{\zeta}[k] = d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \cdots \wedge d\bar{\zeta}_n$. When $n = 1$, the form $U(\zeta, z)$ reduces to the Cauchy kernel $\frac{1}{2\pi i} \cdot \frac{1}{\zeta - z} d\zeta$. The form $U(\zeta, z)$ clearly has coefficients that are harmonic in $\mathbb{C}^n \setminus \{z\}$, and it is closed with respect to $\zeta$ (that is, $d_\zeta U(\zeta, z) = 0$).

Let $g(\zeta, z)$ be the fundamental solution to the Laplace equation:

$$g(\zeta, z) = \begin{cases} -\dfrac{(n-2)!}{(2\pi i)^n} \cdot \dfrac{1}{|\zeta - z|^{2n-2}} & \text{for} \quad n > 1, \\ \dfrac{1}{2\pi i} \ln |\zeta - z|^2 & \text{for} \quad n = 1. \end{cases}$$

Then

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta$$

$$= (-1)^{n-1} \partial_\zeta g \wedge \sum_{k=1}^{n} d\bar{\zeta}[k] \wedge d\zeta[k],$$

where the operator $\partial = \sum\limits_{k=1}^{n} (d\zeta_k) \left( \dfrac{\partial}{\partial \zeta_k} \right)$. We will write the Laplace operator $\Delta$ in the following form:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial \zeta_k \partial \bar{\zeta}_k} = \frac{1}{4} \sum_{k=1}^{n} \left( \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right).$$

If $\zeta_k = x_k + i y_k$, then $\dfrac{\partial}{\partial \zeta_k} = \dfrac{1}{2} \left( \dfrac{\partial}{\partial x_k} - i \dfrac{\partial}{\partial y_k} \right)$, and $\dfrac{\partial}{\partial \bar{\zeta}_k} = \overline{\dfrac{\partial}{\partial \zeta_k}}$.

When $f \in \mathcal{C}^1(U)$, we define the differential form $\mu_f$ via

$$\mu_f = \sum_{k=1}^{n} (-1)^{n+k-1} \frac{\partial f}{\partial \bar{\zeta}_k} d\zeta[k] \wedge d\bar{\zeta}.$$

THEOREM 1.1 (Green's formula in complex form). *Let $D$ be a bounded domain in $\mathbb{C}^n$ with piecewise-smooth boundary, and let $f \in \mathcal{C}^2(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta)U(\zeta,z) - \int_{\partial D} g(\zeta,z)\mu_f(\zeta)$$

$$+ \int_D g(\zeta,z)\Delta f(\zeta)\, d\bar{\zeta} \wedge d\zeta = \begin{cases} f(z), & \text{if} \quad z \in D, \\ 0, & \text{if} \quad z \notin \overline{D}. \end{cases} \tag{1.1}$$

*(The integral in (1.1) converges absolutely.)*

PROOF. Since

$$d_\zeta(f(\zeta)U(\zeta,z) - g(\zeta,z)\mu_f(\zeta)) + g(\zeta,z)\Delta f\, d\bar{\zeta} \wedge d\zeta = 0, \tag{1.2}$$

Stokes's formula implies that (1.1) holds for $z \notin \overline{D}$.

If $z \in D$, then for sufficiently small positive $\varepsilon$, we obtain from (1.2) and Stokes's formula that

$$\int_{\partial D} f(\zeta)U(\zeta,z) - \int_{\partial D} g(\zeta,z)\mu_f(\zeta) + \int_{D\setminus B(z,\varepsilon)} g(\zeta,z)\Delta f(\zeta)\, d\bar{\zeta} \wedge d\zeta$$

$$= \int_{S(z,\varepsilon)} f(\zeta)U(\zeta,z) - \int_{S(z,\varepsilon)} g(\zeta,z)\mu_f(\zeta).$$

When $n > 1$,

$$\left| \int_{S(z,\varepsilon)} g(\zeta,z)\mu_f(\zeta) \right| \leqslant \frac{(n-2)!}{(2\pi)^n \varepsilon^{2n-2}} \int_{S(z,\varepsilon)} |\mu_f| \leqslant C\varepsilon,$$

that is,

$$\lim_{\varepsilon \to 0^+} \int_{S(z,\varepsilon)} g(\zeta,z)\mu_f(\zeta) = 0.$$

(The argument for $n = 1$ is analogous.) However,

$$\int_{S(z,\varepsilon)} f(\zeta)U(\zeta,z) = \frac{(n-1)!}{(2\pi i)^n \varepsilon^{2n}} \int_{S(z,\varepsilon)} f(\zeta) \sum_{k=1}^{n} (-1)^{k-1} (\bar{\zeta}_k - \bar{z}_k) d\bar{\zeta}[k] \wedge d\zeta$$

$$= \frac{(n-1)!}{(2\pi i)^n \varepsilon^{2n}} \int_{B(z,\varepsilon)} \left[ n f(\zeta) + \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{\zeta}_k} (\bar{\zeta}_k - \bar{z}_k) \right] d\bar{\zeta} \wedge d\zeta.$$

Since

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon^{2n}} \int_{B(z,\varepsilon)} \sum_{k=1}^{n} \left( \frac{\partial f}{\partial \bar{\zeta}_k} (\bar{\zeta}_k - \bar{z}_k) \right) d\bar{\zeta} \wedge d\zeta = 0,$$

we have

$$\lim_{\varepsilon \to 0+} \int_{S(z,\varepsilon)} f(\zeta) U(\zeta, z) = \lim_{\varepsilon \to 0+} \frac{n!}{(2\pi i)^n \varepsilon^{2n}} \int_{B(z,\varepsilon)} f(\zeta) d\bar{\zeta} \wedge d\zeta$$

$$= \lim_{\varepsilon \to 0+} \frac{n!}{\pi^n \varepsilon^{2n}} \int_{B(z,\varepsilon)} f(\zeta) dv = f(z)$$

(by the mean-value theorem). □

## 2. Corollaries of Green's Formula

COROLLARY 2.1 (Bochner [**12**]). *Let $D$ be a bounded domain with piecewise-smooth boundary, and let $f$ be a harmonic function in $D$ of class $C^1(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \mu_f(\zeta) = \begin{cases} f(z), & \text{if } z \in D, \\ 0, & \text{if } z \notin \overline{D}. \end{cases} \tag{2.1}$$

COROLLARY 2.2 (Koppelman [**30**]). *Let $D$ be a bounded domain with piecewise-smooth boundary, and let $f$ be a function in $C^1(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{D} \bar{\partial} f(\zeta) \wedge U(\zeta, z) = \begin{cases} f(z), & \text{if } z \in D, \\ 0, & \text{if } z \notin \overline{D}, \end{cases} \tag{2.2}$$

*where*

$$\bar{\partial} = \sum_{k=1}^{n} d\bar{\zeta}_k \frac{\partial}{\partial \bar{\zeta}_k},$$

*and integral in (2.2) converges absolutely.*

Formula (2.2) is the *Bochner-Martinelli formula for smooth functions.*

PROOF. Supposing at first that $f \in C^2(\overline{D})$, we transform the integral

$$\int_D \bar{\partial} f(\zeta) \wedge U(\zeta, z) = \int_D \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{\zeta}_k} \frac{\partial g}{\partial \zeta_k} d\bar{\zeta} \wedge d\zeta = \int_D \partial_\zeta g \wedge \mu_f$$

$$= \int_D d_\zeta(g\mu_f) - \int_D g\Delta f\, d\bar{\zeta} \wedge d\zeta = \int_{\partial D} g\mu_f - \int_D g\Delta f\, d\bar{\zeta} \wedge d\zeta$$

(here we have applied Stokes's formula, since all the integrals converges absolutely). Then for $z \in D$, formula (1.1) implies that

$$\int_D \bar{\partial} f(\zeta) \wedge U(\zeta, z) = \int_{\partial D} f(\zeta) U(\zeta, z) - f(z).$$

Now if $f \in \mathcal{C}^1(\overline{D})$, we obtain (2.2) by approximating $f$ (in the metric of $\mathcal{C}^1(\overline{D})$) by functions of class $\mathcal{C}^2(\overline{D})$. □

COROLLARY 2.3 (Bochner [**12**], Martinelli [**58**]). *If $D$ is a bounded domain in $\mathbb{C}^n$ with piecewise-smooth boundary, and $f$ is a holomorphic function in $D$ of class $\mathcal{C}(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) = \begin{cases} f(z), & \text{if } z \in D, \\ 0, & \text{if } z \notin \overline{D}. \end{cases} \tag{2.3}$$

Formula (2.3) was obtained by Martinelli, and then by Bochner independently and by different methods. It is the first integral representation for holomorphic functions in $\mathbb{C}^n$ in which the integration is carried out over the whole boundary of the domain. This formula is by now classical and has found a place in many textbooks on multidimensional complex analysis (see, for example, [**68**, **79**]).

Formula (2.3) reduces to Cauchy's formula when $n = 1$, but in contrast to Cauchy's formula, the kernel in (2.3) is not holomorphic (in $z$ and $\zeta$) when $n > 1$. By splitting the kernel $U(\zeta, z)$ into real and imaginary parts, it is easy to show that $\int_{\partial D} f(\zeta) U(\zeta, z)$ is the sum of a double-layer potential and a tangential derivative of a single-layer potential; consequently, the Bochner-Martinelli integral inherits some of the properties of the Cauchy integral and some of the properties of the double-layer potential. It differs from the Cauchy integral in not being a holomorphic function, and it differs from the double-layer potential in having somewhat worse boundary behavior. At the same time, it establishes a connection between harmonic and holomorphic functions in $\mathbb{C}^n$ when $n > 1$.

Formula (2.2) implies the jump theorem for the Bochner-Martinelli integral.

Let $D$ be a bounded domain with piecewise-smooth boundary, and let $f$ be a function in $\mathcal{C}^1(\overline{D})$. Denote

$$Mf(z) = \int_{\partial D} f(\zeta)U(\zeta, z), \quad z \notin \partial D.$$

We shall write $M^+f(z)$ for $z \in D$ and $M^-f(z)$ for $z \notin \overline{D}$.

Function $Mf(z)$ is harmonic function for $z \notin \partial D$ and $Mf(z) \to 0$ as $|z| \to \infty$.

COROLLARY 2.4. *Under these conditions function $M^+f$ has a continuous extension on $\overline{D}$, function $M^-f$ has a continuous extension on $\mathbb{C}^n \setminus D$, and*

$$M^+f(z) - M^-f(z) = f(z), \quad z \in \partial D. \tag{2.4}$$

Formula (2.4) is a simplest jump formula for the Bochner-Martinelli integral. There are exist many jump theorems for different classes of functions: for Hölder functions [**50**], for continuous functions [**16, 21**], for integrable functions [**34, 35**], for distributions [**13**], for hyperfunctions [**45**] (see also [**36**, Chapter 1]).

Later on we shall need formula (2.3) for the Hardy spaces $\mathcal{H}^p(D)$, so we now recall some definitions (see, for example, [**25, 74**]). Let $D$ be a bounded domain, and suppose that $\partial D$ is a connected Lyapunov surface, that is, $\partial D \in \mathcal{C}^{1+\alpha}$, $\alpha > 0$. It is known that in such domains, the Green function (for the Laplace equation) $G(\zeta, z)$ has a good boundary behavior: for fixed $z \in D$, the function $G(\zeta, z) \in \mathcal{C}^{1+\alpha}(\overline{D})$.

We say that a holomorphic function $f$ belongs to $\mathcal{H}^p(D)$ (where $p > 0$) if

$$\sup_{\varepsilon>0} \int_{\partial D} |f(\zeta - \varepsilon\nu(\zeta))|^p d\sigma < \infty$$

(here $d\sigma$ is the surface area element on $\partial D$ and $\nu(\zeta)$ is the outer unit normal vector to the surface $\partial D$). A holomorphic function $f$ belongs to $\mathcal{H}^\infty(D)$ if $\sup_D |f(z)| < \infty$.

The class $\mathcal{H}^p(D)$ may also be defined in the following way. Let $D = \{z : \rho(z) < 0\}$ for defining function $\rho$, and let $D_\varepsilon = \{z : \rho(z) < -\varepsilon\}$ for $\varepsilon > 0$. A holomorphic function $f \in \mathcal{H}^p(D)$ if

$$\sup_{\varepsilon>0} \int_{\partial D_\varepsilon} |f(\zeta)|^p d\sigma_\varepsilon < \infty.$$

As is shown in [**74**], this definition does not depend on the choice of the smooth defining function $\rho$.

COROLLARY 2.5. *If $p \geqslant 1$ then for function $f \in \mathcal{H}^p(D)$ formula (2.3) holds.*

PROOF. If $p \geqslant 1$ and $f \in \mathcal{H}^p(D)$, then $f$ has normal boundary values almost everywhere on $\partial D$ (see [**25, 74**]) that form a function of class $\mathcal{L}^p(\partial D)$ (we denote these boundary values again by $f$). Moreover, the function $f$ can be reconstructed in $D$ from its boundary values by Poisson's formula

$$f(z) = \int_{\partial D} f(\zeta) P(\zeta, z)\, d\sigma$$

(where $P(\zeta, z)$ is the Poisson kernel for $D$). Since the Green function $G(\zeta, z) = g(\zeta, z) + h(\zeta, z)$, where for fixed $z \in D$ the function $h(\zeta, z)$ is harmonic in $D$ of class $\mathcal{C}^{1+\alpha}(\overline{D})$, we have

$$P(\zeta, z)\, d\sigma = U(\zeta, z)|_{\partial D} + \sum_{k=1} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta|_{\partial D}.$$

Since the differential form

$$\sum_{k=1} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta$$

is closed, we have

$$\int_{\partial D} f(\zeta) \sum_{k=1} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta$$

$$= \int_D f d \left( \sum_{k=1} (-1)^{k-1} \frac{\partial h}{\partial \zeta_k} d\bar{\zeta}[k] \wedge d\zeta \right) = 0.$$

Consequently, formula (2.3) holds for $f \in \mathcal{H}^p(D)$. □

We remark that it is possible to derive from the Bochner-Martinelli formula (2.3) the Cauchy-Fantappiè formula that was obtained by Leray [**48, 49**].

Let $D$ be a bounded domain with piecewise-smooth boundary, and suppose that for a point $z \in D$ there is defined on $\partial D$ a continuously differentiable vector-valued function $\eta(\zeta) = (\eta_1(\zeta), \dots, \eta_n(\zeta))$ such that

$$\sum_{k=1}^{n} (\zeta_k - z_k)\eta_k(\zeta) = 1, \quad \zeta \in \partial D.$$

THEOREM 2.1 (Leray). *Every function $f \in \mathcal{A}(D)$ satisfies the equation*

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} f(\zeta)\omega'(\eta) \wedge d\zeta, \quad z \in D, \tag{2.5}$$

*where* $\omega'(\eta) = \sum_{k=1}^{n} (-1)^{k-1} \eta_k \, d\eta[k]$.

PROOF. Khenkin's proof is as follows. Consider in the space $\mathbb{C}^{2n}$ of variables $(\eta, \zeta) = (\eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_n)$ the analytic hypersurface $M_z = \{(\eta, \zeta) = \sum_{k=1}^{n} (\zeta_k - z_k)\eta_k = 1\}$, on which the form $\omega'(\eta) \wedge d\zeta$ is closed. The two cycles

$$\Gamma_1 = \{(\eta, \zeta) : \zeta \in \partial D, \eta_j = (\bar{\zeta}_j - \bar{z}_j)|\zeta - z|^{-2}, j = 1, \dots, n\}$$

and

$$\Gamma_2 = \{(\eta, \zeta) : \zeta \in \partial D, \eta_j = \eta_j(\zeta), j = 1, \dots, n\}$$

in $M_z$ are homotopic in $M_z$. The homotopy being given by the formula

$$\tilde{\eta}_j = t \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2} + (1-t)\eta_j(\zeta), \quad 0 \leqslant t \leqslant 1.$$

That is, they are homologous cycles. Consequently,

$$\int_{\Gamma_1} f(\zeta)\omega'(\eta) \wedge d\zeta = \int_{\Gamma_2} f(\zeta)\omega'(\eta) \wedge d\zeta$$

when $f$ is a holomorphic function. But

$$\omega'\left(\frac{\bar{\zeta}_1 - \bar{z}_1}{|\zeta - z|^2}, \dots, \frac{\bar{\zeta}_n - \bar{z}_n}{|\zeta - z|^2}\right) = \frac{(2\pi i)^n}{(n-1)!} U(\zeta, z).$$

Hence (2.5) follows. □

The Cauchy-Fantappiè representation has turned out to be very useful, and it has many applications in multidimensional complex analysis.

Khenkin and Leiterer [**26**, Chapter 4] extended formula (2.3) to domains $D$ in Stein manifolds.

Analogues of the Bochner-Martinelli formula have also been considered in quaternionic analysis [**78**] and in Clifford analysis [**72**].

## 3. The Hodge Operator

Subsequently we shall need some properties of the Hodge operator. We first consider the space $\mathbb{R}^m$ with the usual Euclidean metric. Let $I = (i_1, \dots, i_p)$ be an increasing multi-index, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, and $dx[I] = dx_{j_1} \wedge \cdots \wedge dx_{j_{m-p}}$, where $j_1 < \cdots < j_{m-p}$, and $j_k \neq i_l$ for $k = 1, \dots, m-p$, $l = 1, \dots, p$. The symbol $\sigma(I)$ is defined by

$$dx_I \wedge dx[I] = \sigma(I)\, dx.$$

The *Hodge star operator* $*$ acts on the form $dx_I$ in the following way: $*dx_I = \sigma(I)\, dx[I]$. We extend it to an arbitrary form $\varphi = \sum_I' \varphi_I dx_I$ by linearity (the prime on the summation sign indicates that the sum is taken over increasing multi-indices $I$):

$$*\varphi = \sum_I{}' \varphi_I \sigma(I)\, dx[I].$$

We now give the main properties of the Hodge operator.

(1) $dx_I \wedge *dx_I = dx = dx_1 \wedge \cdots \wedge dx_m$.

(2) $**\, dx_I = (-1)^{mp+p} dx_I$.

We obtain from (1) and (2) that if $\varphi$ and $\psi$ are two $p$-forms, then

$$\varphi \wedge *\bar{\psi} = \sum_I{}' \varphi_I dx_I \wedge * \sum_J{}' \bar{\psi}_J dx_J = \left( \sum_I{}' \varphi_I \bar{\psi}_I \right) dx.$$

Consequently, a scalar product $(\varphi, \psi)$ may be defined for $p$-forms $\varphi$ and $\psi$ with coefficients of class $\mathcal{L}^2$ in a domain $D \subset \mathbb{R}^m$ by

$$(\varphi, \psi) = \int_D \varphi \wedge *\bar{\psi}.$$

Then

$$(\varphi, \varphi) = \|\varphi\|^2 = \int_D \sum_I{}' |\varphi_I|^2 dx.$$

This scalar product is called the *Hodge product.*

Now consider $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ with coordinates $z = (z_1, \dots, z_n)$ and $z_j = x_j + iy_j$ for $j = 1, \dots, n$. We have defined the volume form $dv$ in $\mathbb{C}^n$ as (see Sec. 1)

$$dv = dx \wedge dy = (i/2)^n dz \wedge d\bar{z}.$$

If $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are increasing multi-indices, and we write the $(p, q)$-form $dz_I \wedge d\bar{z}_J$ in terms of the forms $dx_j$ and $dy_k$, apply the Hodge operator, and then express $dx_j$ and $dy_k$ in terms of $dz_j$ and $d\bar{z}_k$, we obtain the following equality:

(3) $*(dz_I \wedge d\bar{z}_J) = 2^{p+q-n}(-1)^{pn} i^n \sigma(I)\sigma(J)\, dz[J] \wedge d\bar{z}[I]$ (a detailed calculation may be found in [**80**, Chapter V, Lemma 1.2]).

The Hodge operator extends to $(p, q)$-forms

$$\varphi = \sum_{I,J}{}' \varphi_{I,J} dz_I \wedge d\bar{z}_J$$

by linearity. Thus, $*\varphi$ is a form of type $(n-q, n-p)$. Properties (1) and (2) carry over to the following:

(4) $dz_I \wedge d\bar{z}_J \wedge \overline{*(dz_I \wedge d\bar{z}_J)} = 2^{p+q} dv$;

(5) $** (dz_I \wedge d\bar{z}_J) = (-1)^{p+q} dz_I \wedge d\bar{z}_J$.

EXAMPLE 3.1. Let $F$ be a smooth function, then

$$*\bar{\partial} F = * \left( \sum_{k=1}^{n} \frac{\partial F}{\partial \bar{z}_k} d\bar{z}_k \right)$$

$$= 2^{1-n} i^n \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial F}{\partial \bar{z}_k} dz[k] \wedge d\bar{z} = 2^{1-n} i^n (-1)^n \mu_F.$$

So what $\mu_F = i^n 2^{n-1} (*\bar{\partial} F)$.

EXAMPLE 3.2. $U(\zeta, z) = \dfrac{(n-1)!}{2\pi^n} (*\partial g(\zeta, z))$.

By using the Hodge operator, it is easy to find the operators formally dual to $d$, $\bar{\partial}$, and $\partial$. For example, we find $\bar{\partial}^*$. If $\varphi$ is a $(p, q-1)$-form and $\psi$ is a $(p,q)$-form, $\varphi$ and $\psi$ have smooth coefficients of class $\mathcal{L}^2(D)$, and $\psi$ has compact support in $D$, then $(\bar{\partial}\varphi, \psi) = (\varphi, \bar{\partial}^* \psi)$, and

$$(\bar{\partial}\varphi, \psi) = \int_D \bar{\partial}\varphi \wedge *\bar{\psi} = \int_D d\varphi \wedge *\bar{\psi} = \int_D d(\varphi \wedge *\bar{\psi}) + (-1)^{p+q} \int_D \varphi \wedge d * \bar{\psi}$$

$$= (-1)^{p+q} \int_D \varphi \wedge \bar{\partial} * \bar{\psi} = -\int_D \varphi \wedge *(\overline{*\partial * \psi}),$$

so $\bar{\partial}^* = - * \partial *$.

In just the same way, we see that $\partial^* = - * \bar{\partial} *$. The operator $\bar{\partial}^*$ carries forms of type $(p,q)$ into forms of type $(p, q-1)$. By definition, $\bar{\partial}^* = 0$ for forms of type $(p, 0)$.

We consider the operator $\Box = \bar{\partial}^*\bar{\partial}+\bar{\partial}\bar{\partial}^*$, which is known as the *complex Laplacian.* If $\varphi$ is a function, then

$$\Box\varphi = \overline{\partial}^*\overline{\partial} = \overline{\partial}^* \sum_{k=1}^{n} \frac{\partial\varphi}{\partial\bar{z}_k} d\bar{z}_k = - * \partial \sum_{k=1}^{n} 2^{1-n} i^n (-1)^{k-1} \frac{\partial\varphi}{\partial\bar{z}_k} dz[k] \wedge d\bar{z}$$

$$= - * 2^{1-n} i^n \sum_{k=1}^{n} \frac{\partial^2\varphi}{\partial\bar{z}_k\partial z_k} dz \wedge d\bar{z} = -2 \sum_{k=1}^{n} \frac{\partial^2\varphi}{\partial\bar{z}_k\partial z_k} = -2\Delta\varphi,$$

that is, $\Box = -2\Delta$ for functions, and this identity continues to hold for forms (see, for example, [**20**, p. 106]). Thus, in $\mathbb{C}^n$, harmonic forms in the sense of $\Box$ are forms with harmonic coefficients. It is also easy to show that $\Box = \partial\partial^* + \partial^*\partial$.

## 4. The $\bar{\partial}$-Neumann Problem for Functions

Suppose $n > 1$, and $D = \{z : \rho(z) < 0\}$ is a bounded domain in $\mathbb{C}^n$ with boundary of class $C^1$, where $\rho$ is defining function. If $F \in C^1(\overline{D})$, then denote

$$\bar{\partial}_n F = \sum_{k=1}^{n} \frac{\partial F}{\partial\bar{z}_k} \rho_k,$$

where $\rho_k = \dfrac{\partial\rho}{\partial z_k} \cdot \dfrac{1}{|\partial\rho|}$.

$\bar{\partial}_n F$ is $\bar{\partial}$-*normal derivative* of function $F$.

If we write the form

$$\bar{\partial}F = \bar{\partial}_b F + \lambda \frac{\bar{\partial}\rho}{|\bar{\partial}\rho|}$$

(where $\bar{\partial}_b F$ is a tangential part of the form $\bar{\partial}F$), then $\lambda = \bar{\partial}_n F$, that is $\bar{\partial}_n F$ is the coefficient of the normal part of the form $\bar{\partial}F$.

If we denote the outer unit normal to $\partial D$ at $z$ by $\nu(z)$, and $s(z) = i\nu(z)$, then

$$\bar{\partial}_n F = \frac{1}{2}\left(\frac{\partial F}{\partial\nu} + i\frac{\partial F}{\partial s}\right).$$

On the other hand from Example 3.1 and from equalities

$$\begin{aligned} d\bar{\zeta}[k] \wedge d\zeta|_{\partial D} &= 2^{n-1} i^n (-1)^{k-1} \bar{\rho}_k \, d\sigma, \\ d\zeta[k] \wedge d\bar{\zeta}|_{\partial D} &= 2^{n-1} i^n (-1)^{n+k-1} \rho_k \, d\sigma, \end{aligned} \tag{4.1}$$

we have

$$\overline{\partial}_n F\, d\sigma = *\overline{\partial}F|_{\partial D} = 2^{1-n}i^n(-1)^n\mu_F|_{\partial D}.$$

If we consider function

$$\tilde{g}(\zeta, z) = 2^{n-1}i^n g(\zeta, z) = -\frac{(n-2)!}{2\pi^n}\cdot\frac{1}{|\zeta - z|^{2n-2}},$$

then formula (2.1) can be written by the following way.

COROLLARY 4.1. *Let $D$ be a bounded domain with piecewise-smooth boundary, and let $F$ be a harmonic function in $D$ of class $C^1(\overline{D})$. Then*

$$\int_{\partial D} F(\zeta)U(\zeta, z) - \int_{\partial D} \tilde{g}(\zeta, z)\overline{\partial}_n F(\zeta)d\sigma = \begin{cases} F(z), & \text{if } z \in D, \\ 0, & \text{if } z \notin \overline{D}. \end{cases} \tag{4.2}$$

We consider the following problem ($\overline{\partial}$-Neumann problem for functions): for given function $\varphi$ on $\partial D$, find a function $F$ on $\overline{D}$ such that

$$\begin{cases} \overline{\partial}_n F = \varphi & \text{on } \partial D, \\ \Box F = 0 & \text{in } D. \end{cases} \tag{4.3}$$

This problem is an exact analogue of the usual Neumann problem for harmonic functions.

Just as for the usual Neumann problem, the problem (4.3) is not always solvable. There is a necessary orthogonality condition. Indeed, if $F$ is a harmonic function of class $C^1(\overline{D})$, then $*\overline{\partial}F$ is a $\partial$-closed form in $D$, since

$$0 = \Box F = \overline{\partial}^*\overline{\partial}F = - * \partial(*\overline{\partial}F),$$

that is, $\partial(*\overline{\partial}F) = 0$. Hence, if $\varphi = \overline{\partial}_n F$ on $\partial D$, and $f$ is a holomorphic function on $\overline{D}$, then

$$\int_{\partial D} \varphi \bar{f}\, d\sigma = \int_{\partial D} \bar{f}(*\overline{\partial}F) = \int_D \partial(\bar{f} * \overline{\partial}F) = \int_D \bar{f}\partial(*\overline{\partial}F) = 0.$$

Thus, a necessary condition for solvability of (4.3) is the orthogonality condition

$$\int_D \varphi\bar{f}\, d\sigma = 0 \quad \text{for all} \quad f \in \mathcal{O}(\overline{D}). \tag{4.4}$$

Compare problem (4.3) with the $\overline{\partial}$-Neumann problem for forms (see [**18**]): given a form $\psi$ of type $(p, q)$ in $D$, find a form $F$ for which $\Box F = \psi$

in $D$, and the normal parts of the forms $F$ and $\bar\partial^* F$ are zero on $\partial D$. If $F$ and $\psi$ are functions, then we need to find a function $F$ such that

$$\begin{cases} \bar\partial_n F = 0 & \text{on} \quad \partial D, \\ \Box F = \psi & \text{in} \quad D. \end{cases} \tag{4.5}$$

If we do not pay attention to the smoothness of the functions, then (4.3) and (4.5) are equivalent. Indeed, one solution of the second equation in (4.5) is the volume potential $F_\psi$. Subtracting it from the solution of (4.5), we obtain $\Box(F-F_\psi)=0$, and $\bar\partial_n(F-F_\psi)=\varphi$ on $\partial D$, that is we have (4.3). Conversely, given (4.3), we take the single-layer potential $F_\varphi^\pm$ for $\varphi$ and extend $F_\varphi^-$ into $D$ as a smooth function to obtain that $\bar\partial_n(F-F_\varphi^+ + F_\varphi^-)=0$ on $\partial D$, and $\Box(F-F_\varphi^+ + F_\varphi^-)=\psi$, that is, we have (4.5).

Problem (4.3) is more natural for studying the boundary properties of holomorphic functions.

The $\bar\partial$-Neumann problem for forms arose in the works of Spencer and then was studied by many authors. An especially large role was played by Kohn.

The following problem is a generalization of (4.3) to differential forms. Suppose $\varphi$ is a form of type $(p, q+1)$ given on $\partial D$, where $0 \leqslant q \leqslant n-1$ and $0 \leqslant p \leqslant n$. We wish to find a form $\alpha$ of type $(p,q)$ in $D$ such that

$$\begin{cases} (\bar\partial\alpha)_n = \varphi_n & \text{on} \quad \partial D, \\ \bar\partial^*\bar\partial\alpha = 0 & \text{in} \quad D. \end{cases} \tag{4.6}$$

Here $\varphi_n$ is a normal part of $\varphi$.

When $p=q=0$, we obtain (4.3), since if $\alpha$ is a function, then $\Box\alpha = \bar\partial^*\bar\partial\alpha$, and $(\bar\partial\alpha)_n = \bar\partial_n\alpha\bar\partial\rho/|\bar\partial\rho|$.

We first find a necessary condition for solvability of (4.6). We consider the form $*\varphi$, which has type $(n-q-1, n-p)$, and we integrate it against a form $\bar\beta$, where $\beta$ is a form of type $(p,q)$ with smooth in $\overline{D}$ coefficients such that $\bar\partial\beta = 0$ in $\overline{D}$:

$$\int_{\partial D} (*\varphi)\wedge\bar\beta = \int_{\partial D} (*\bar\partial\alpha)\wedge\bar\beta = \int_D \partial(*\bar\partial\alpha\wedge\bar\beta) = 0$$

in view of (4.6) and the fact that $\overline{\partial}\beta = 0$. Thus, we find that a necessary condition for solvability of (4.6) is that

$$\int_{\partial D} (*\varphi) \wedge \bar{\beta} = 0 \tag{4.7}$$

for all forms $\beta$ of type $(p, q)$ with $\overline{\partial}\beta = 0$ on $\overline{D}$ and coefficients of class $\mathcal{C}^\infty(\overline{D})$.

## 5. The Homogeneous $\overline{\partial}$-Neumann Problem

We first consider the homogeneous $\overline{\partial}$-Neumann problem

$$\begin{cases} \overline{\partial}_n F = 0 & \text{on} \quad \partial D, \\ \Box F = 0 & \text{in} \quad D. \end{cases} \tag{5.1}$$

It is clear that holomorphic functions $F$ satisfy (5.1). We will show that the converse is also true. First we reformulate the problem. Recall $MF$ is the Bochner-Martinelli integral (see Sec. 1)

$$MF(z) = \int_{\partial D} F(\zeta)U(\zeta, z), \quad z \notin \partial D.$$

THEOREM 5.1 (Aronov [**8**], Kytmanov [**33**]). *Let $F$ be a harmonic function in $D$. The following conditions are equivalent:*
*1. $\overline{\partial}_n F = 0$ on $\partial D$;*
*2. $M^+ F = F$ in $D$;*
*3. $M^- F = 0$ in $\mathbb{C}^n \setminus \overline{D}$.*

PROOF. We consider only the case that $F \in \mathcal{C}^1(\overline{D})$. Conditions 2 and 3 are equivalent by the jump theorem for the Bochner-Martinelli integral (see Corollary 2.4) and by the uniqueness theorem for harmonic functions.

If $\overline{\partial}_n F = 0$ on $\partial D$ then formula (4.2) gives that $M^+ F = F$ in $D$.

If $M^+ F = F$ in $D$, then $M^- F = 0$ outside $\overline{D}$. Thus we obtain from (4.2) that

$$\int_{\partial D} \frac{\overline{\partial}_n F(\zeta)}{|\zeta - z|^{2n-2}} d\sigma(\zeta) = 0 \quad \text{for all} \quad z \notin \partial D.$$

Applying the theorem of Keldysh-Lavrent'ev (see, for example, [**47**, p. 418]) on the density of fractions of the form $\dfrac{1}{|\zeta - z|^{2n-2}}$ in the space $\mathcal{C}(\partial D)$, we obtain that $\overline{\partial}_n F = 0$ on $\partial D$. $\Box$

THEOREM 5.2 (Folland, Kohn [**18**]; Aronov, Kytmanov [**9**]). *Let $F$ be a harmonic function in $D$ of class $\mathcal{C}^1(\overline{D})$. The following conditions are equivalent:*

1. *$\overline{\partial}_n F = 0$ on $\partial D$;*
2. *$M^+F = F$ in $D$;*
3. *$M^-F = 0$ in $\mathbb{C}^n \setminus \overline{D}$;*
4. *$F$ is holomorphic in $D$.*

PROOF. It is sufficient to prove that condition 1 implies condition 4. Since the form $*\overline{\partial} F$ is $\partial$-closed, then

$$0 = \int_{\partial D} \overline{F}(*\overline{\partial} F) = \int_D \partial \overline{F} \wedge *\overline{\partial} F$$
$$= 2^{1-n} i^n \int_D |\overline{\partial} F|^2 dz \wedge d\bar{z} = 2 \int_D |\overline{\partial} F|^2 dv.$$

Hence $\dfrac{\partial F}{\partial \bar{z}_k} = 0$ in $D$ for all $k = 1, \dots, n$, so $F \in \mathcal{O}(D)$. □

The conditions 2, 3, 4 are equivalent without requirement that $F$ is harmonic in $D$.

Let $n > 1$.

THEOREM 5.3 (Kytmanov [**31**]). *If $M^+f$ is holomorphic in $D$, $f \in \mathcal{C}^1(\partial D)$, and $\partial D \in \mathcal{C}^1$ is connected, then the boundary value of $M^+f$ coincides with $f$.*

It is clear that Theorem 5.3 is not true when $n = 1$. Also, it is not true if $\partial D$ is not connected: it suffices to set $f = 1$ on one connected component of $\partial D$ and $f = 0$ on the remaining components.

Theorem 5.2 is proven for continuous functions $F$ by Kytmanov and Aizenberg [**37**], for integrable functions $F$ by Romanov [**65**], for distributions by Kytmanov [**36**, Chapter 4], for hyperfunctions by Kytmanov and Yakimenko [**45**] (see also [**36**, Chapter 4]).

Consider homogeneous $\overline{\partial}$-Neumann problem for differential forms.

Let $\partial D \in \mathcal{C}^1$. We wish to find a form $\gamma$ of type $(p, q)$ in $D$ such that

$$\begin{cases} (\overline{\partial}\gamma)_n = 0 & \text{on} \quad \partial D, \\ \overline{\partial}^* \overline{\partial} \gamma = 0 & \text{in} \quad D. \end{cases} \tag{5.2}$$

Evidently, if $\gamma$ is $\overline{\partial}$-closed then $\gamma$ satisfies condition (5.2).

THEOREM 5.4 (Kytmanov [**36**]). *If the form $\gamma$ has coefficients of class $\mathcal{C}^2(\overline{D})$ and satisfies condition (5.2) then $\overline{\partial}\gamma = 0$ in $D$.*

PROOF. We may assume that $\gamma$ is a form of type $(0, q)$. Then from (5.2) $\partial(*\overline{\partial}\gamma) = 0$ in $D$, and the tangential part of the form $*\overline{\partial}\gamma$ is zero on $\partial D$. (Hodge operator takes the normal part of $\gamma$ into the tangential part of $*\gamma$, and conversely).

If $\gamma = \sum_J' \gamma_J(\zeta) d\bar{\zeta}_J$, then the restriction of the form $(*\overline{\partial}\gamma) \wedge \bar{\gamma}_J d\zeta_J = 0$ on $\partial D$. Therefore

$$0 = \sum_J{}' \int_{\partial D} (*\overline{\partial}\gamma) \wedge \bar{\gamma}_J d\zeta_J = \sum_J{}' \int_D d((*\overline{\partial}\gamma) \wedge \bar{\gamma}_J d\zeta_J)$$

$$= (-1)^{q+1} \sum_J{}' \int_D (*\overline{\partial}\gamma) \wedge \partial\bar{\gamma}_J \wedge d\zeta_J$$

$$= (-1)^{q+1} \sum_J{}' \int_D * \sum_K{}' \sum_{m \notin K} \frac{\partial \gamma}{\partial \bar{\zeta}_m} d\bar{\zeta}_m \wedge d\bar{\zeta}_K \wedge \sum_{k \notin J} \frac{\partial \bar{\gamma}_J}{\partial \zeta_k} d\zeta_k \wedge d\zeta_J$$

$$= (-1)^{q+1} 2^{q+1-n} i^n \times$$

$$\times \sum_J{}' \int_D \sum_K{}' \sum_{m \notin K} \sigma(K, m) \frac{\partial \gamma}{\partial \bar{\zeta}_m} d\zeta[K, m] \wedge d\bar{\zeta} \wedge \sum_{k \notin J} \frac{\partial \bar{\gamma}_J}{\partial \zeta_k} d\zeta_k \wedge d\zeta_J$$

$$= (-1)^{(q+1)(n+1)} 2^{q+1-n} i^n \sum_J{}' \int_D \sum_{K \cup m = I} \sum_{J \cup k = I} \sigma(K, m) \frac{\partial \gamma}{\partial \bar{\zeta}_m} \sigma(J, k) \frac{\partial \bar{\gamma}_J}{\partial \zeta_k} d\zeta \wedge d\bar{\zeta}$$

$$= (-1)^{(q+1)(n+1)} 2^{q+1-n} i^n \sum_J{}' \int_D \sum_{K \cup m = I} \left| \sigma(K, m) \frac{\partial \gamma}{\partial \bar{\zeta}_m} \right|^2 d\zeta \wedge d\bar{\zeta}$$

$$= (-1)^{(q+1)(n+1)} 2^{q+1-n} i^n \int_D |\overline{\partial}\gamma|^2 d\zeta \wedge d\bar{\zeta},$$

that is, $\overline{\partial}\gamma = 0$ (here $d\zeta_m \wedge d\zeta_K \wedge d\zeta[K, m] = \sigma(K, m) d\zeta$). □

## 6. Solvability of the $\overline{\partial}$-Neumann Problem

We now turn to solvability of the $\overline{\partial}$-Neumann problem (4.3). Let $D$ be a bounded domain in $\mathbb{C}^n$ with $\partial D \in \mathcal{C}^\infty$. We consider the Sobolev space $\mathcal{W}_2^s = \mathcal{W}_2^s(D)$, where $s$ is a natural number. This space consists of the functions $f \in \mathcal{L}^2(D)$ such that all derivatives $\partial^\alpha f$ through order $s$ lie in $\mathcal{L}^2(D)$. The topology in $\mathcal{W}_2^s$ is usual. We will need the space $\mathcal{W}_2^{s+\lambda}(\partial D)$

for $0 < \lambda < 1$. It consists of the functions $f \in \mathcal{W}_2^s(\partial D)$ for which the integral

$$\int_{\partial D}\int_{\partial D}\sum_{\|\alpha\|=s}\frac{|\partial^\alpha f(z) - \partial^\alpha f(\zeta)|^2}{|\zeta - z|^{2n+2\lambda-1}}dv_\zeta dv_z$$

converges.

Properties of these spaces may be found, for example, in the survey [**17**]. We will need the following properties.

1. When $s \geqslant 1$, the restriction of a function $f \in \mathcal{W}_2^s(D)$ to $\partial D$ lies in the space $f \in \mathcal{W}_2^{s-1/2}(\partial D)$, and the restriction operator is continuous.

2. If we denote the subspace of harmonic functions in $\mathcal{W}_2^s(D)$ by $\mathcal{G}_2^s = \mathcal{G}_2^s(D)$, then the restriction operator from $\mathcal{G}_2^s$ to $\mathcal{W}_2^{s-1/2}(\partial D)$ is a linear topological isomorphism. Then $\mathcal{W}_2^s = \mathcal{G}_2^s \oplus \mathcal{N}_2^s$, where $\mathcal{N}_2^s$ consists of the functions in $\mathcal{W}_2^s$ that are equal to zero on $\partial D$.

THEOREM 6.1 (Kytmanov [**33, 36**]). *Suppose $D$ is a strongly pseudoconvex domain. If the function $\varphi \in \mathcal{G}_2^s(D)$, where $s \geqslant 2$, satisfies condition (4.4), then there exists a harmonic function $F \in \mathcal{W}_2^{s-1}(D)$ (that is $F \in \mathcal{G}_2^{s-1}(D)$) such that $\overline{\partial}_n F = \varphi$ on $\partial D$, and $F$ may be chosen so that it also satisfies (4.4). The function $F$ defined in this way is unique; we denote it by $N\varphi$; and the Neumann operator $N$ is bounded.*

For proof we use the solvability $\overline{\partial}$-problem in $D$ and $\overline{\partial}$-Neumann problem for form [**18, 28**], the solvability of $\overline{\partial}_b$-problem [**11**].

The analogous theorem is true for the problem (4.6).

Let $s \geqslant 1$. If $F \in \mathcal{W}_2^s(D)$ then the Bochner-Martinelli integral $MF \in \mathcal{G}_2^s(D)$. Consider formula (4.2). The first integral in that formula is $MF$ and the second integral we denote $T\overline{\partial}_n F$. Then for $F \in \mathcal{G}_2^s$ and $z \in D$ we have

$$F = MF + T\overline{\partial}_n F = M^2F + MT\overline{\partial}_n F + T\overline{\partial}_n F = \ldots = M^kF + \sum_{l=0}^{k-1} M^l T\overline{\partial}_n F.$$

So that, if $\overline{\partial}_n F = \varphi$ then we have

$$F = M^kF + \sum_{l=0}^{k-1} M^l T\varphi. \tag{6.1}$$

THEOREM 6.2 (Romanov [**65**]). *For the space $\mathcal{W}_2^1(D)$ we have*

$$\lim_{k\to\infty} M^k = P_A, \quad \lim_{k\to\infty} T^k = P_N$$

*in the strong operator topology of $\mathcal{W}_2^1(D)$, where $P_A$ is a projection from $\mathcal{W}_2^1(D)$ onto $\mathcal{A}_2^1(D) = \mathcal{W}_2^1(D) \cap \mathcal{O}(D)$, and $P_N$ is a projection from $\mathcal{W}_2^1(D)$ onto $\mathcal{N}_2^1(D)$.*

In the paper [**33**], Theorem 6.2 was stated for the space $\mathcal{W}_2^s(D)$ for $s \geqslant 1$. Professor Straube gave an example showing that Theorem 6.2 cannot be true for all domains $D$ and all spaces $\mathcal{W}_2^s(D)$ (see [**36**, p. 172]).

EXAMPLE 6.1. Suppose $\lim_{k\to\infty} M^k = P_A$ in the strong operator topology of $\mathcal{W}_2^s(D)$ for all $s \geqslant 1$. Then $P_A : \mathcal{W}_2^s \to \mathcal{A}_2^s$. Consequently, $P_A$ is a projection operator from $\mathcal{W}_2^s$ onto $\mathcal{A}_2^s$ for all $s$. This implies that the space $\mathcal{C}^\infty(\overline{D}) \cap \mathcal{O}(D)$ is dense in $\mathcal{A}_2^s$ (since $\mathcal{C}^\infty(\overline{D})$ is dense in $\mathcal{W}_2^s$, and this property is preserved under application of the operator $P_A$). But such a density does not hold for every domain $D$ according to an example in [**10**].

PROBLEM 6.1. *For which domains $D$ Theorem 6.2 is true for all spaces $\mathcal{W}_2^s(D)$?*

As long as it has a positive answer in the ball in $\mathbb{C}^n$ (see Sec. 7).

So (6.1) and Theorem 6.2 imply

COROLLARY 6.1. *Let $D$ be a strongly pseudoconvex domain. Suppose (4.4) holds for a function $\varphi \in \mathcal{G}_2^2(D)$. A solution to the $\overline{\partial}$-Neumann problem is given by the series*

$$F = \sum_{l=0}^{\infty} M^l T\varphi, \tag{6.2}$$

*which converges to $F$ in the metric of $\mathcal{G}_2^1(D)$.*

COROLLARY 6.2. *If $D$ is an arbitrary domain (with boundary of class $\mathcal{C}^\infty$). Suppose the series (6.2) converges in $\mathcal{G}_2^1(D)$ for a function $\varphi \in \mathcal{G}_2^2(D)$, then it determines a solution $F$ of the $\overline{\partial}$-Neumann problem (4.3).*

## 7. The Bochner-Martinelli Integral in the Ball

Let $B = B(0,1)$ be the unit ball in $\mathbb{C}^n$ with center at the origin, and let $S = S(0,1)$ be its boundary. We consider in this section the scalar product $(f, g)$ of two functions $f$ and $g$ in $\mathcal{L}^2(S)$ is given by the integral $(f, g) = \int_S f\bar{g}\, d\sigma$.

We will identify the space $\mathcal{L}^2(S)$ with the space of harmonic extensions of such functions from $S$ into $B$, that is, with the space of harmonic functions $f$ in $B$ for which

$$\sup_{0\leqslant r<1} \int_S |f(rz)|^2 d\sigma(z) < \infty.$$

Recall that the Poisson kernel $P(\zeta, z)$ for the ball $B$ has the form

$$P(\zeta, z) = \frac{(n-1)!}{2\pi^n} \cdot \frac{1-|z|^2}{|\zeta - z|^{2n}}, \quad z \in B, \quad \zeta \in S.$$

LEMMA 7.1. *The restriction of the kernel $U(\zeta, z)$ to $S$ equals*

$$\frac{1-\langle \zeta, \bar{z}\rangle}{1-|z|^2} P(\zeta, z)\, d\sigma, \quad z \in B, \quad \zeta \in S.$$

PROOF. It is not difficult to show (using (4.1)) that

$$U(\zeta, z)|_S = \frac{(n-1)!}{2\pi^n} \frac{1-\langle \zeta, \bar{z}\rangle}{|\zeta - z|^{2n}} d\sigma. \quad \square$$

Let $P_{k,s}(z)$ be a harmonic polynomial, homogeneous of degree $k$ in $z$ and degree $s$ in $\bar{z}$ of the form

$$P_{k,s}(z) = \sum_{\|\alpha\|=k} \sum_{\|\beta\|=s} a_{\alpha,\beta} z^\alpha \bar{z}^\beta,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ multi-indices, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $\bar{z}^\beta$ are monomials, and $\|\alpha\| = \alpha_1 + \cdots + \alpha_n$.

We denote the set of the homogeneous harmonic polynomials $P_{k,s}$ by $\mathcal{P}_{k,s}$. It is known (see, for example, [**67**, Chapter 12]) that

$$\mathcal{L}^2(S) = \bigoplus_{k,s=1}^{\infty} \mathcal{P}_{k,s}. \tag{7.1}$$

We are interesting in the operator giving the Bochner-Martinelli integral $Mf$ for $f \in \mathcal{L}^2(S)$.

LEMMA 7.2. *If $P_{k,s} \in \mathcal{P}_{k,s}$, then*

$$MP_{k,s} = \frac{n+k-1}{n+k+s-1} P_{k,s}.$$

PROOF. It is not difficult to check that the harmonic extension of the polynomial $\zeta_j P_{k,s}$ from $S$ to $B$ is given by formula

$$f(\zeta) = \zeta_j P_{k,s} + \frac{1-|\zeta|^2}{n+k+s-1}\frac{\partial P_{k,s}}{\partial \bar{\zeta}_j}, \quad \zeta \in \overline{B}.$$

By Lemma 7.1,

$$MP_{k,s} = \int_S P_{k,s}(\zeta)\frac{1-\langle\zeta,\bar{z}\rangle}{1-|z|^2}P(\zeta,z)\,d\sigma$$

$$= \frac{P_{k,s}(z)}{1-|z|^2} - \sum_{j=1}^{n}\frac{\bar{z}_j}{1-|z|^2}\left(z_j P_{k,s} + \frac{1-|z|^2}{n+k+s-1}\frac{\partial P_{k,s}}{\partial \bar{z}_j}\right)$$

$$= P_{k,s}(z) - \frac{s}{n+k+s-1}P_{k,s}(z) = \frac{n+k-1}{n+k+s-1}P_{k,s}(z). \quad \square$$

THEOREM 7.1 (Romanov [**64**]). *When $n > 1$, the operator $M$ is a bounded self-adjoint operator $\mathcal{L}^2(S) \to \mathcal{L}^2(S)$ with $\|M\| = 1$. Every rational number in the interval $(0,1]$ is an eigenvalue of $M$ of infinite multiplicity. The spectrum of $M$ coincides with the interval $[0,1]$.*

PROOF. The proof follows from Lemma 7.2. □

For $n = 1$ $Mf$ is the Cauchy integral and it has only two eigenvalues: 0 and 1. Thus, the operator $M$ is essentially different when $n = 1$ and $n > 1$.

THEOREM 7.2 (Romanov [**64**]). *Suppose $n > 1$, and let $P_A$ be the operator of projection from $\mathcal{L}^2(S)$ onto the subspace of holomorphic functions in $\mathcal{L}^2(S)$. Then $M^k \to P_A$ as $k \to \infty$ in the strong operator topology of $\mathcal{L}^2(S)$.*

Theorems 7.1 and 7.2 also true for the space $\mathcal{W}_2^s(S)$ for any $s$ since the decomposition (7.1) is valid for this space.

PROBLEM 7.1. *Prove Theorem 7.2 for the space $\mathcal{L}^p(S)$ for $1 < p < \infty$.*

It is known that for this case if $f \in \mathcal{L}^p(S)$ then $Mf \in \mathcal{L}^p(S)$ (see [**70**]). But $\|M\|_{\mathcal{L}^p} \to \infty$ as $p \to \infty$ or $p \to 1$.

In the ball it is possible to give the integral representation for solution of $\overline{\partial}$-Neumann problem.

THEOREM 7.3 (Kytmanov [**33**]). *Suppose $n > 1$ and $\varphi \in \mathcal{W}_2^s(S)$ satisfies (4.4). Then $N\varphi \in \mathcal{W}_2^s(S)$, and*

$$(N\varphi)(z) = \int_S \varphi(\zeta) K(\zeta, z)\, d\sigma(\zeta) \quad z \in B,$$

*where*

$$K(\zeta, z) = \frac{(n-1)!}{2\pi^n} \left[ \frac{1}{n-1} \cdot \frac{1}{|\zeta - z|^{2n-2}} - \frac{1}{n-1} \cdot \frac{1}{(1 - \langle \bar\zeta, z\rangle)^{n-1}} \right.$$
$$+ \frac{1}{n-1} \sum_{j=0}^{n-2} \frac{n - 1 - (j+1)\langle \bar z, \zeta\rangle}{(j+1)|\zeta - z|^{2(j+1)}(1 - \langle \bar\zeta, z\rangle)^{n-j-1}}$$
$$\left. - \frac{1}{(1 - \langle \bar\zeta, z\rangle)^n} \sum_{j=0}^{n-2} \frac{1}{j+1} - \frac{1}{(1 - \langle \bar\zeta, z\rangle)^n} \ln \frac{|\zeta - z|^2}{1 - \langle \bar\zeta, z\rangle} \right].$$

## 8. Generalizations of the $\bar\partial$-Neumann Problem

As before, suppose $D$ is a bounded domain in $\mathbb{C}^n$ with smooth boundary $\partial D$, and $D = \{z : \rho(z) < 0\}$ with defining function $\rho$.

Consider the following elliptic equation ($f \in \mathcal{C}^2(\overline{D})$):

$$L(f) = \sum_{j,k=1}^{n} \frac{\partial}{\partial z_j} \left( a_{j,k}(z) \frac{\partial f}{\partial \bar z_k} \right) = 0, \tag{8.1}$$

where the matrix $A = \|a_{j,k}(z)\|_{j,k=1}^n$ is Hermitian and positive definite on $\overline{D}$, and $a_{j,k} \in \mathcal{C}^1(\overline{D})$, $j, k = 1, \dots n$.

THEOREM 8.1 (Kytmanov [**36**]). *If the following boundary condition holds for a function $f \in \mathcal{C}^2(\overline{D})$:*

$$\sum_{j,k} a_{j,k}(z) \frac{\partial f}{\partial \bar z_k} \rho_j(z) = 0 \quad on \quad \partial D,$$

*and if $f$ satisfies (8.1) in $D$, then $f$ is holomorphic in $D$.*

This theorem generalizes items 1 and 4 of Theorem 5.2.

PROOF. Consider the differential form

$$\omega_f = \sum_{j,k} a_{j,k} \frac{\partial f}{\partial \bar z_k} (-1)^{j-1} dz[j] \wedge d\bar z.$$

The restriction of $\omega_f$ to $\partial D$ equals zero by the hypothesis of the theorem. By Stokes's formula

$$0=\int_{\partial D}\bar f\omega_f=\int_D \bar f d\omega_f+\int_D d\bar f\wedge\omega_f$$

$$=\int_D \bar f\sum_{j,k=1}^n \frac{\partial}{\partial z_j}\left(a_{j,k}(z)\frac{\partial f}{\partial \bar z_k}\right)dz\wedge d\bar z+\int_D\sum_{j,k=1}^n a_{j,k}(z)\frac{\partial \bar f}{\partial z_j}\frac{\partial f}{\partial \bar z_k}dz\wedge d\bar z$$

$$=\int_D\sum_{j,k=1}^n a_{j,k}(z)\frac{\partial \bar f}{\partial z_j}\frac{\partial f}{\partial \bar z_k}dz\wedge d\bar z;$$

since the hermitian matrix $A$ is positive definite, it follows from this that

$$\sum_{j,k=1}^n a_{j,k}(z)\overline{\frac{\partial f}{\partial \bar z_j}}\frac{\partial f}{\partial \bar z_k}=0$$

in $D$, and hence $\dfrac{\partial f}{\partial \bar z_j}=0$, $j=1,\dots,n$, that is, $f\in\mathcal{O}(D)$. □

We remark that the vector field

$$w=\sum_{j,k}\bar a_{j,k}\bar\rho_j\frac{\partial}{\partial z_k}$$

does not lie in the complex tangent space $T_z^c(\partial D)$, since

$$w(\rho)=\sum_{j,k}\bar a_{j,k}\bar\rho_j\rho_k|\partial\rho|^{-1}>0\quad\text{on}\quad\partial D.$$

PROBLEM 8.1. *Prove Theorem 8.1 for another classes of functions $f$ (continuous, integrable, etc.).*

Consider the next problem:

PROBLEM 8.2. *For a function $\varphi$ given on $\partial D$ find a function $F$ on $\partial D$ such that*

$$\begin{cases}\sum\limits_{j,k=1}^n a_{j,k}(z)\dfrac{\partial F}{\partial \bar z_k}\rho_j(z)=\varphi & on\quad \partial D,\\ L(F)=0 & in\quad D.\end{cases}\tag{8.2}$$

This problem is generalization of the $\bar\partial$-Neumann problem (4.3) for functions and Theorem 8.1 gives the solution of homogeneous Problem 8.2.

A necessary condition for solvability of Problem 8.2 is that $\varphi$ be orthogonal to the holomorphic functions. Indeed, if $f \in \mathcal{O}(\overline{D})$, then

$$\int_{\partial D} \varphi \bar f \, d\sigma = \int_{\partial D} \bar f \sum_{j,k=1}^{n} a_{j,k}(z) \frac{\partial F}{\partial \bar z_k} \rho_j(z)\, d\sigma$$

$$= C \int_{\partial D} \bar f \sum_{j,k=1}^{n} a_{j,k}(z)(-1)^{j-1} \frac{\partial F}{\partial \bar z_k} dz[j] \wedge d\bar z$$

$$= C \int_{D} \bar f \partial \left( \sum_{j,k=1}^{n} a_{j,k}(z)(-1)^{j-1} \frac{\partial F}{\partial \bar z_k} dz[j] \wedge d\bar z \right) = C \int_{D} \bar f LF \, dz \wedge d\bar z = 0.$$

Apparently this condition is also sufficient (for strongly pseudoconvex domains).

Problem 8.3. *Give an integral representation for solution of Problem 8.2 in the ball.*

Consider the following problem. Suppose given a vector field $w = w(z) = \sum_{k=1}^{n} w_k(z) \frac{\partial}{\partial z_k}$, $w_k \in C(\partial D)$, such that

$$w(\rho) = \sum_{k=1} w_k(z) \frac{\partial \rho}{\partial z_k} \neq 0 \quad \text{on} \quad \partial D, \tag{8.3}$$

that is, for every point $z \in \partial D$ the vector $w$ does not lie in the complex tangent space $T_z^c(\partial D)$.

Problem 8.4. *Suppose $f \in C^1(\overline{D})$ and $f$ is harmonic in $D$. If*

$$\bar w(f) = \sum_{k=1} \bar w_k \frac{\partial f}{\partial \bar z_k} = 0 \quad \textit{on} \quad \partial D, \tag{8.4}$$

*will $f$ be holomorphic in $D$?*

In contrast to the tangential Cauchy-Riemann conditions, here we require the vanishing of the action of a nontangential vector field $\bar w$ on $f$. Problem 8.4 is an analogue of the oblique derivative problem for real-valued harmonic functions.

If (8.3) does not hold, then it is easy to give an example where (8.4) holds for a function $f$ that is not holomorphic in $D$. It is enough to consider the ball $B(0,1)$ in $\mathbb{C}^2$ and the function $f = \bar{z}_1$, with $w = \partial/\partial z_2$. Therefore inequality (8.3) is violated on the circle $\{z \in \mathbb{C}^2 : |z| = 1, z_2 = 0\}$.

If $w = \sum\limits_{k=1} \bar{\rho}_k \dfrac{\partial}{\partial z_k}$, then Problem 8.4 becomes the homogeneous $\overline{\partial}$-Neumann problem (5.1).

We give a number of equivalent formulations of Problem 8.4. We decompose the field $w$ into a normal and a tangential component:

$$w(z) = \alpha(z) \sum_{k=1}^{n} \bar{\rho}_k \frac{\partial}{\partial z_k} + b(z).$$

By hypothesis, $\alpha \neq 0$ on $\partial D$. The vector field $b(z) \in T_z^c(\partial D)$. As generators of the space $T_z^c(\partial D$, we may take the vectors

$$\rho_m \frac{\partial}{\partial z_k} - \rho_k \frac{\partial}{\partial z_m}, \quad k \neq m, \quad k, m = 1, \dots, n.$$

By decomposing $b(z)$ in terms of these vectors, it is easy to obtain from (8.4) the equality

$$\overline{\partial}_n f = \sum_{k>m} \alpha_{m,k}(z) \left[ \bar{\rho}_m \frac{\partial f}{\partial \bar{z}_k} - \bar{\rho}_k \frac{\partial f}{\partial \bar{z}_m} \right],$$

where $\alpha_{k,m}(z)$ are certain continuous functions on $\partial D$. Multiplying both sides of this equality by $d\sigma$ and applying equality (4.1), we obtain

$$\mu_f|_{\partial D} = \sum_{k>m} a_{k,m}(z)\, df \wedge d\bar{z}[k,m] \wedge dz|_{\partial D}. \tag{8.5}$$

PROBLEM 8.5. *If $f \in C^1(\overline{D})$ is harmonic in $D$, and $a_{k,m} \in C(\partial D)$ for $k, m = 1, \dots, n$ does (8.5) imply that $f$ is holomorphic in $D$?*

When $n = 2$, then (8.5) can be rewritten in the form

$$\mu_f|_{\partial D} = a(z)\, df \wedge dz|_{\partial D}. \tag{8.6}$$

THEOREM 8.2 (Kytmanov, Yakiminko [**46**]). *Suppose $D \subset \mathbb{C}^2$ is simply connected domain, harmonic function $f \in C^1(\overline{D})$ satisfies condition (8.6), and $\operatorname{Re} a \neq k \operatorname{Im} a$ on $\partial D$ for some real constant $k$, then $f$ is holomorphic in $D$.*

For $n > 2$ Problem 8.5 have been solved only for the ball and for holomorphic functions $a_{k,m}(z)$ (see [**36**, Chapter 5]).

Suppose given a function $\varphi$ on $\partial D$.

PROBLEM 8.6. *Find a harmonic function $F$ in $D$ such that $w(F) = \varphi$ on $\partial D$.*

This problem is analogous to the oblique derivative problem for real-valued harmonic functions. What is required is to determine necessary conditions for solvability of Problem 8.6 and to construct a solution of this problem given by an integral representation, for example, in the ball.

## 9. Functions Representable by the Integral Formulas

Suppose $n > 1$. We rewrite (8.5) in integral form by multiplying (8.5) by the fundamental solution $g(\zeta, z)$ of Laplace's equation and integrating over $\partial D$. We obtain

$$\int_{\partial D} g(\zeta, z)\mu_f(\zeta) = \int_{\partial D} g(\zeta, z) \sum_{k>m} a_{k,m}(\zeta) df \wedge d\bar{\zeta}[k, m] \wedge d\zeta, \quad z \notin \partial D.$$

Using Green's formula (2.1) and Stokes's formula, we get

$$f(z) = \int_{\partial D} f(\zeta) \left[ U(\zeta, z) + \sum_{k>m} d(ga_{k,m}) \wedge d\bar{\zeta}[k, m] \wedge d\zeta \right], \quad z \in D. \quad (9.1)$$

PROBLEM 9.1. *If $f \in \mathcal{C}(\overline{D})$ and harmonic in $D$, and $a_{k,m} \in \mathcal{C}^1(\partial D)$, does (9.1) imply the holomorphicity of $f$ in $D$?*

When $a_{k,m} = 0$, we obtain the problem about functions representable by the Bochner-Martinelli integral. If $f \in \mathcal{A}(D)$, then (9.1) is an integral representation of $f$, since the component $d(ga_{k,m}) \wedge d\bar{\zeta}[k, m] \wedge d\zeta$ is a $\bar{\partial}$-exact form.

More general there is a problem on functions representable by the Cauchy-Fantappiè formula (2.5). Consider for a domain $D$ continuously differentiable (in $\zeta$) vector function $\eta = \eta(\zeta, z) = (\eta_1, \dots, \eta_n)$, $\zeta \in \partial D$, $z \in D$, such that

$$\sum_{k=1}^{n} \eta_k(\zeta, z)(\zeta_k - z_k) = 1 \quad \text{for} \quad \zeta \in \partial D, \quad z \in D.$$

Then $\omega'(\eta) \wedge d\zeta$ is the Cauchy-Fantappiè kernel. Concerning formula (2.5), we can pose a problem analogous to the problem for the Bochner-Martinelli integral: namely, if (2.5) holds for $f \in \mathcal{C}(\overline{D})$, will $f$ be holomorphic in $D$?

If the functions $\eta_k$ depend holomorphically on $z$ (for example, in the case of the Khenkin-Ramirez kernel, or convex domains), then the answer is obvious. If the $\eta_k$ are arbitrary, then it is easy to give an example where the answer to this question is negative. Let $P(\zeta, z)$ be the Poisson kernel for unit ball (see Sec. 7). As Dautov showed (see Sec. 10), the kernel $P(\zeta, z)$ is a Cauchy-Fantappiè kernel for $n = 2$. But (2.5) holds with the kernel $P(\zeta, z)$ for all harmonic functions. Therefore our questions must be formulated as follows.

PROBLEM 9.2. *For which Cauchy-Fantappiè kernels $\omega'(\eta)\wedge d\zeta$ does the equality (2.5) for a function $f \in \mathcal{C}(\overline{D})$ imply that $f \in \mathcal{A}(D)$?*

One class of Cauchy-Fantappiè kernels can be specified in the following way. The kernel $\omega'(\eta) \wedge d\zeta$ has the form

$$\omega'(\eta) \wedge d\zeta = \sum_{k=1} \delta_k(\zeta, z)\, d\bar{\zeta}[k] \wedge d\zeta.$$

THEOREM 9.1 (Kytmanov [**36**]). *Let*

$$\delta_k(\zeta, z) = (-1)^{k-1} h_k(\zeta) \frac{\partial h(\zeta, z)}{\partial \bar{\zeta}_k}, \quad k = 1, \ldots, n,$$

*for $\zeta \in D$ and $z \in D$, $\zeta \neq z$, where $h$ and $\partial h(\zeta, z)/\partial \bar{\zeta}_k$ are integrable functions on $\overline{D}$, and $h_k \in \mathcal{C}(\overline{D})$, $h_k > 0$ in $\overline{D}$. If (2.5) holds for a function $f \in \mathcal{C}^1(\overline{D})$, then $f$ is holomorphic in $D$.*

The second class of the Cauchy-Fantappiè kernels defines by the vector function $\eta$ such that

$$\eta_k(\zeta, z) = \frac{|\zeta_k - z_k|^{2\alpha_k - 2}(\bar{\zeta}_k - \bar{z}_k)}{\sum\limits_{m=1}^{n} |\zeta_m - z_m|^{2\alpha_m}},$$

where $\alpha_k \in \mathbb{N}$, $k = 1, \ldots, n$. The kernel $\omega'(\eta) \wedge d\zeta$ has the following form

$$U_\alpha(\zeta, z) = \omega'(\eta) \wedge d\zeta = \frac{(n-1)!}{(2\pi i)^n} \prod_{j=1}^{n} \alpha_j |\zeta_j - z_j|^{2\alpha_j - 2} \times$$

$$\times \sum_{k=1}^{n}(-1)^{k-1}\frac{\bar{\zeta}_k - \bar{z}_k}{\alpha_k\left(\sum\limits_{m=1}^{n}|\zeta_m - z_m|^{2\alpha_m}\right)^n} d\bar{\zeta}[k] \wedge d\zeta.$$

For $\alpha_1 = \cdots = \alpha_n = 1$, we have that $U_\alpha(\zeta, z) = U(\zeta, z)$ the Bochner-Martinelli kernel. For $\alpha_k \neq 1$ this kernel is the $\bar{\partial}$-closed differential form with real-analytic (not harmonic) coefficients.

THEOREM 9.2 (Kytmanov, Myslivets [**39, 41**]). *If for the function $f \in C^\lambda(\partial D)$ $(\lambda > 0)$ the Cauchy-Fantappiè integral*

$$F(z) = \int_{\partial D} f(\zeta)U_\alpha(\zeta, z)$$

*gives a continuous extension $F$ of $f$ into $D$, then the function $F$ is holomorphic in $D$.*

For proof of this theorem we need the jump theorem and maximum modulus theorem for the Cauchy-Fantappiè integral $F$, the Cauchy-Fantappiè formula for smooth functions and so on.

## 10. The General Form of Integral Representations of Holomorphic Functions

The $\bar{\partial}$-Neumann problem is closely related to that of describing the general form of integral representations. By an *integral representation of holomorphic functions* we means a formula of the type

$$f(z) = \int_{\partial D} f(\zeta)\mu_z(\zeta), \tag{10.1}$$

valid for all $z \in D$ and all $f \in \mathcal{O}(\overline{D})$. Here $D$ is a bounded domain with smooth boundary, and $\mu_z(\zeta)$ is a differential form of type $2n-1$ with continuous on $\partial D$ coefficients for any fixed $z \in D$. A form $\mu_z(\zeta)$ satisfying (10.1) will be called a *reproducing kernel.*

Any reproducing kernel can be obtained from a fixed one (for example, from the Bochner-Martinelli kernel) by the addition of forms orthogonal to holomorphic functions (of type (4.4)), and conversely, every form which is orthogonal to holomorphic functions can be obtained as the difference between a fixed reproducing kernel and another suitable one.

Observe that, if a reproducing kernel is multiplied by a function $\varphi_z(\zeta)$, $\zeta \in \overline{D}$, $z \in D$, satisfying

$$\varphi_z(\zeta) \in \mathcal{A}(D), \quad \varphi_z(z) \equiv 1, \quad z \in D, \tag{10.2}$$

then we again get a reproducing kernel. Indeed, suppose $\mu_z(\zeta)$ satisfies (10.1). Then

$$\int_{\partial D} f(\zeta)\varphi_z(\zeta)\mu_z(\zeta) = f(z)\varphi_z(z) = f(z).$$

It must be noted that the class of kernels of the form $\varphi_z(\zeta)\omega'(\eta) \wedge d\zeta$ is invariant under biholomorphic mappings ([**3**]), though it is not known whether a similar assertion holds with regard to the Cauchy-Fantappiè kernel $\omega'(\eta) \wedge d\zeta$.

For $n = 1$, the Cauchy-Fantappiè kernel goes over into the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\, d\zeta}{\zeta - z}, \quad z \in D.$$

THEOREM 10.1 (Aizenberg [**3**]). *Let $D$ be a bounded domain in $\mathbb{C}^1$ with smooth boundary, and $\mu_z(\zeta)$ a continuous reproducing kernel. Then*

$$\mu_z(\zeta) = \frac{1}{2\pi i} \cdot \frac{\varphi_z(\zeta)\, d\zeta}{\zeta - z},$$

*for some $\varphi_z(\zeta)$ satisfying (10.2).*

PROOF. Since $d\zeta$ is non-degenerate on $\partial D$ and continuous, we have

$$\mu_z(\zeta) = g_z(\zeta)\, d\zeta,$$

where $g_z \in \mathcal{C}(\partial D)$ for fixed $z \in D$.

Consider the function

$$h_z(\zeta) = g_z(\zeta) - \frac{1}{2\pi i(\zeta - z)}.$$

For every $f \in \mathcal{O}(\overline{D})$, we have

$$\int_{\partial D} f(\zeta)h_z(\zeta)\, d\zeta = 0.$$

It is well-known that $h_z$ can be extended to $D$, to each fixed $z \in D$, as a function in $\mathcal{A}(D)$. Set

$$\varphi_z(\zeta) = 1 + 2\pi i(\zeta - z)h_z(\zeta). \quad \square$$

THEOREM 10.2 (Dautov [**4, 15**]). *Let $D$ be a bounded domain in $\mathbb{C}^2$ with boundary of class $C^\infty$. Suppose $\mu_z(\zeta)$ is a reproducing kernel with coefficients of class $C^\infty$. In order that each reproducing kernel $\mu_z(\zeta)$ be a Cauchy-Fantappiè kernel, it is necessary and sufficient that $D$ be a domain of holomorphy.*

For $n > 2$, no such description of reproducing kernels has been obtained. It is known (Dautov [**4**, §13]) that in strongly pseudoconvex domains $D$ any reproducing kernel is the linear combination of the Cauchy-Fantappiè kernels.

PROBLEM 10.1. *Prove (or disprove) the sufficiency or necessity in Theorem 10.2 when $n > 2$.*

Theorem 6.1 shows that in strongly pseudoconvex domains $D$, every reproducing kernel $\mu_z(\zeta)$ has the form

$$\mu_z(\zeta) = U(\zeta, z) + *\partial h_z(\zeta),$$

where $h_z(\zeta)$ is a harmonic function in $D$ of class $C^\infty(\overline{D})$.

## 11. The Functions Representable by Logarithmic Residue Formula

Consider one more class of reproducing kernels, so called logarithmic differentials. Let $\psi(\zeta) = (\psi_1(\zeta), \dots, \psi_n(\zeta))$ be a holomorphic mapping such that $\psi_j$ are entire functions in $\mathbb{C}^n$, $j = 1, \dots, n$. Suppose $\psi(\zeta) = 0$ if and only if $\zeta = 0$. We denote the multiplicity of zero at $\zeta = 0$ by $\mu$.

If we denote $U(w) = U(w, 0)$ then

$$U(\psi(\zeta - z)) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{\psi_k(\zeta - z)}}{|\psi(\zeta - z)|^{2n}} \overline{\partial}_\zeta \overline{\psi(\zeta - z)}[k] \wedge \partial_\zeta \psi(\zeta - z).$$

In this formula the point $z$ is fixed. The kernel $U(\psi(\zeta - z))$ is $\overline{\partial}$-closed differential form of the type $(n, n-1)$ with real-analytic coefficients and with the point singularity $\zeta = z$.

If $D$ is a bounded domain with smooth boundary and function $f \in \mathcal{A}(D)$, then

$$\mu f(z) = \int_{\partial D} f(\zeta) U(\psi(\zeta - z)), \quad z \in D. \tag{11.1}$$

Formula (11.1) is the partial case of the *logarithmic residue formula* proved by Yuzhakov [**81**] and Roos [**66**] (see also [**7**]).

So, $\frac{1}{\mu}U(\psi(\zeta - z))$ is a reproducing kernel.

If $\psi_j(\zeta) = \zeta_j^{\alpha_j}$ then $\frac{1}{\mu}U(\psi(\zeta - z)) = U_\alpha(\zeta, z)$, where $U_\alpha(\zeta, z)$ was defined in Sec. 9 and $\mu = \alpha_1 \cdots \alpha_n$.

THEOREM 11.1 (Kytmanov, Myslivets [**40**]). *If $\partial D$ is connected and for $f \in \mathcal{C}^1(\overline{D})$ formula (11.1) holds then $f$ is holomorphic in $D$.*

Scheme of the proof. First of all we have to proof the analogue of the Bochner-Martinelli formula for smooth functions (see (2.2)):

If a function $f \in \mathcal{C}^1(\overline{D})$ then

$$\int_{\partial D} f(\zeta)U(\psi(\zeta - z)) - \int_D \bar{\partial} f(\zeta) \wedge U(\psi(\zeta - z)) = \begin{cases} \mu f(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases}$$

the second integral converges absolutely for $z \in D$.

As a corollary we get a jump theorem for the integral

$$Lf(z) = \int_{\partial D} f(\zeta)U(\psi(\zeta - z)), \quad z \notin \partial D.$$

Namely, the integral $Lf$ has the continuous extension on $\partial D$ from the inside $D$ and from the outside $D$, and

$$L^+(z) - L^-(z) = \mu f(z), \quad z \in \partial D.$$

The second step is the following:

$$\frac{\partial}{\partial \bar{z}_j} U(\psi(\zeta - z)) = \bar{\partial}_\zeta U_j(\zeta, z),$$

moreover, $U_j(\zeta, z)$ is a $(n, n-2)$-differential form with real-analytic coefficients and with the point singularity $\zeta = z$. From here we get

$$\frac{\partial}{\partial \bar{z}_j} Lf = \int_{\partial D} f(\zeta)\bar{\partial}_\zeta U_j(\zeta, z) = \sum_{s=1}^{n} \frac{\partial}{\partial z_s} \int_{\partial D} f(\zeta)\bar{\partial}_\zeta U_{js}(\zeta, z)$$

for some $(n, n-2)$-differential forms $U_{js}$ with real-analytic coefficients and with the point singularity $\zeta = z$.

If we denote

$$\beta_{js}(z) = \int_{\partial D} f(\zeta)\bar{\partial}_\zeta U_{js}(\zeta, z), \quad z \notin \partial D,$$

then from conditions of theorem and from jump formula we have that differential form

$$\beta_j = \sum_{s=1}^{n} (-1)^{s-1} \beta_{js}(z) dz[s] \wedge d\bar{z}$$

is $\partial$-closed outside $\overline{D}$.

We solve the equation $\beta_j = \partial \alpha_j$ outside $\overline{D}$ and prove then this equality is true into $D$. From here we get then $Lf$ is holomorphic in $D$. □

PROBLEM 11.1. *Prove Theorem 11.1 for continuous functions $f$ (and for other classes of functions).*

## 12. Functions with the Property of One-dimensional Holomorphic Continuation along Complex Lines

Consider complex lines $l_{z,b}$ of the form

$$l_{z,b} = \{\zeta : \zeta_k = z_k + tb_k, k = 1, \dots, n, t \in \mathbb{C}\}.$$

The point $z \in \mathbb{C}^n$ and the point $b \in \mathbb{CP}^{n-1}$ ($b$ is defined to within of multiplication on a complex number $\lambda \neq 0$).

We write the Bochner-Martinelli kernel $U(\zeta, z)$ in variables $t$ and $b$. We have $|\zeta - z|^2 = |t|^2 |b|^2$. Then

$$d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n = (b_1 dt + t db_1) \wedge \dots \wedge (b_n dt + t db_n)$$

$$= t^{n-1} \sum_{j=1}^{n} (-1)^{j-1} b_j dt \wedge db[j],$$

since $db = db_1 \wedge \dots \wedge db_n = 0$ in $\mathbb{CP}^{n-1}$.

In exactly the same way

$$\sum_{k=1}^{n} (-1)^{k-1} (\bar{\zeta}_k - \bar{z}_k) d\bar{\zeta}[k] = \bar{t}^{n-1} \sum_{j=1}^{n} (-1)^{j-1} \bar{b}_j d\bar{b}[j].$$

From here we have

LEMMA 12.1. *The Bochner-Martinelli kernel in variables $t$ and $b$ has the form*

$$U(\zeta, z) = \frac{dt}{t} \wedge \lambda(b),$$

*where*

$$\lambda(b) = \frac{(n-1)!(-1)^{n-1}}{(2\pi i)^n} \frac{\sum_{j=1}^n (-1)^{j-1}\bar{b}_j d\bar{b}[j] \wedge \sum_{j=1}^n (-1)^{j-1} b_j db[j]}{|b|^{2n}}.$$

Let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Give the following definition (Stout [**75**]). The function $f \in \mathcal{C}(\partial D)$ has a *property of one-dimensional holomorphic continuation along complex lines* if for any complex lines $l_{z,b}$ (meeting $\overline{D}$) there exists a function $F_{z,b}$ with the following properties:

a) $F_{z,b} \in \mathcal{C}(\overline{D} \cap l_{z,b})$;

b) $F_{z,b} = f$ on the set $\partial D \cap l_{z,b}$;

c) $F_{z,b}$ is holomorphic in interior (with respect to topology of $l_{z,b}$) points of the set $\overline{D} \cap l_{z,b}$.

THEOREM 12.1 (Stout [**75**]). *If $\partial D \in \mathcal{C}^2$ and a function $f \in \mathcal{C}(\partial D)$ has a property of one-dimensional holomorphic continuation along all complex lines then $f$ has a holomorphic extension into $D$ as a function of several complex variables.*

PROOF. Consider the integral

$$M^- f(z) = \int_{\partial D} f(\zeta) U(\zeta, z), \quad z \notin \overline{D}.$$

The Fubini theorem, Lemma 12.1 and the conditions of the theorem imply

$$M^- f(z) = \int_{\mathbb{CP}^{n-1}} \lambda(b) \int_{\partial D \cap l_{z,b}} f(\zeta) \frac{dt}{t} = 0.$$

Applying Theorem 5.2 for continuous functions (see [**36**, Theorem 15.4]) we have that $M^+ f$ gives holomorphic extension of $f$ into $D$. □

For the ball Theorem 12.1 was proved in [**1, 61**].

PROBLEM 12.1 (Stout). *Which families $\mathcal{L}$ of the complex lines are sufficient for holomorphic extension?*

A family $\mathcal{L}$ is *sufficient* for holomorphic extension if any function $f \in \mathcal{C}(\partial D)$ with a property of one-dimensional holomorphic continuation along complex lines $l_{z,b} \in \mathcal{L}$ has a holomorphic extension into $D$.

Consider the following family.

Let $V$ be an open set in $\mathbb{C}^n$. We denote

$$\mathcal{L}_V = \{l_{z,b} : l_{z,b} \cap V \neq \varnothing\}.$$

If $\partial D$ is connected and $V \cap \overline{D} = \varnothing$ then $\mathcal{L}_V$ is sufficient family since in this case $M^- f = 0$ in $V$ then $M^- f = 0$ outside $\overline{D}$ (by uniqueness theorem for harmonic functions).

THEOREM 12.2 (Agranovskiĭ, Semenov [**2**]). *Let $\partial D$ be connected and an open set $V \subset D$. Then $\mathcal{L}_V$ is sufficient family for a holomorphic extension.*

PROOF. Consider the integral

$$\int_{\partial D} (\zeta_k - z_k) f(\zeta) U(\zeta, z) = \int_{\mathbb{CP}^{n-1}} \lambda(b) \int_{\partial D \cap l_{z,b}} f(\zeta) dt = 0, \quad z \in V.$$

Then this integral vanishes into $D$. Apply to this integral the Laplace operator:

$$\Delta \int_{\partial D} \zeta_k f(\zeta) U(\zeta, z) = \Delta \left( z_k \int_{\partial D} f(\zeta) U(\zeta, z) \right).$$

From here

$$0 = \frac{\partial M^+ f}{\partial \bar{z}_k}, \quad z \in D.$$

Then $M^+ f$ is a holomorphic function in $D$. Applying Theorem 5.3 for continuous functions (see [**36**, Corollary 15.6]) we have that $M^+ f$ gives holomorphic continuation of $f$ into $D$. □

The proof of this theorem shows that holomorphic extension of $f$ into $D$ is possible under more weak limitations on $f$, so-called Morera conditions.

THEOREM 12.3 (Globevnik and Stout [**19**]). *Let $n > 1$, and let $D$ be a bounded domain in $\mathbb{C}^n$ with connected, smooth (of class $C^2$) boundary. If a function $f \in C(\partial D)$ and for almost all $z \in \mathbb{C}^n$, for almost all $b \in \mathbb{CP}^{n-1}$*

$$\int_{\partial D \cap l_{z,b}} f(\zeta) dt = 0,$$

*then the function $f$ has a holomorphic continuation into $D$.*

PROOF is the same as for Theorem 12.2.

Some modification of this proof shows that it is true the next assertion.

THEOREM 12.4 (Kytmanov, Myslivets [**38, 41**]). *Under conditions of Theorem 12.3 for a fixed nonnegative integer $k$ the integral*

$$\int_{\partial D \cap l_{z,b}} f(\zeta) t^k dt = 0$$

*for all $l_{z,b}$, then the function $f$ has a holomorphic continuation into $D$.*

Theorems 12.3 and 12.4 are true for the family $\mathcal{L}_V$ for any open set $V$.

Consider two holomorphic mappings $\Phi = (\varphi_1, \dots, \varphi_n)$ and $\psi = (\psi_1, \dots, \psi_n)$ consisting on entire functions in $\mathbb{C}^n$, and such that $\Phi(z) = 0$, $\psi(z) = 0$ if and only if $z = 0$, $(\psi \circ \Phi)(z) = (z_1^{p_1}, \dots, z_n^{p_n})$ for some $p_j \in \mathbb{N}$, $j = 1, \dots, n$.

Let

$$m_{z,b} = \{\zeta : \zeta_j = z_j + \varphi_j(b_1 t^{k_1}, \dots, b_n t^{k_n}), j = 1, \dots, n, t \in \mathbb{C}\}$$

be a complex curve. The points $z \in \mathbb{C}^n$, $b \in \mathbb{CP}^{n-1}$ and $k_j \in \mathbb{N}$ such that $k_1 p_1 = \dots = k_n p_n$.

THEOREM 12.5 (Myslivets (1999)). *If $\partial D \in C^2$ and connected and a function $f \in C(\partial D)$ has a property of one-dimensional holomorphic continuation along all complex curves $m_{z,b}$ then $f$ holomorphically extends into $D$.*

PROOF uses the theorem of type Theorem 11.1 and the representation of a kernel $U(\psi(\zeta - z))$ in variables $t$ and $b$ of the next form

$$U(\psi(\zeta - z)) = \frac{dt}{t} \wedge \gamma(b),$$

where $\gamma(b)$ is a $(n-1, n-1)$-differential form independent of $t$.

PROBLEM 12.2. *Which families of complex curves $m_{z,b}$ are sufficient for a holomorphic continuation?*

## 13. Holomorphic Extension from a Part of the Boundary

In the next two sections, we assume that the dimension $n > 1$. We will be interested in two questions:

PROBLEM 13.1. *If $\Gamma$ is the boundary of a bounded domain $D$, and $f$ is a CR-function on $\Gamma \setminus K$, where $K$ is a compact subset of $\Gamma$, what condition must be imposed on $K$ so that $f$ extends into $D$ as a holomorphic function $F$?*

PROBLEM 13.2. *Under what conditions on $K$ and $f$ does the extension $F$ have "good" boundary behavior near $K$? In other words, when does $F$ determine on $\Gamma$ a CR-function $\tilde{f}$? This function also will serve as an extension of $f$ from $\Gamma \setminus K$ to $\Gamma$.*

Recall a function $f \in \mathcal{C}^1(\Gamma)$ is *CR-function* if $\bar{w}(f) = 0$ for any vector field $w \in T^c(\Gamma)$. A function $f \in \mathcal{C}(\Gamma)$ (or $f \in \mathcal{L}^1_{\text{loc}}(\Gamma)$) is *CR-function* if

$$\int_\Gamma f(\zeta)\overline{\partial}\varphi(\zeta) = 0$$

for any $(n, n-2)$-differential forms $\varphi$ with $\mathcal{C}^\infty$ coefficients and with compact support.

Well-known Hartogs-Bochner theorem says that any $CR$-function given on connected smooth boundary $\partial D$ of a bounded domain $D$ holomorphically extends into $D$. G.Lupacciolu considered Problem 13.1 for continuous $CR$-functions given on the part of a boundary.

Let $D$ be a bounded domain in $\mathbb{C}^n$ such that $\overline{D}$ has a schlicht envelope of holomorphy. For any compact set $K \subset \overline{D}$ we set

$$\widehat{K}_{\overline{D}} = \left\{ z \in \overline{D} : |h(z)| \leqslant \max_K |h|, h \in \mathcal{O}(\overline{D}) \right\}.$$

We assume that the hypersurface $\Gamma = \partial D \setminus K$ is a smooth (class $\mathcal{C}^1$), connected manifold in $\mathbb{C}^n \setminus K$.

Suppose $K = \widehat{K}_{\overline{D}}$; for example, $K$ could be a polynomially convex, or $K = \widehat{K}_\Omega$, where $\widehat{K}_\Omega$ is the envelope of $K$ with respect to $\mathcal{O}(\Omega)$, with $\Omega$ an open set containing $\overline{D}$.

THEOREM 13.1 (Lupacciolu [**51**]). *If $K = \widehat{K}_{\overline{D}}$, $\Gamma = \partial D \setminus K$, and $f$ is a continuous $CR$-function on $\Gamma$, then there exists a holomorphic function $F$ in $D \setminus K$ continuous up to $\Gamma$ and such that boundary values of $F$ on $\Gamma$ coincides with $f$.*

PROOF. First of all we discuss some preliminaries. Let $V$ be an open neighborhood of $K$ in $\mathbb{C}^n$, and let $\beta$ be a function of class $\mathcal{C}^\infty$ in $\mathbb{C}^n$ that is equal to 1 on $K$, satisfies $0 \leqslant \beta < 1$ off $K$, and has compact support in $V$. For each positive $\varepsilon$, we write $D_\varepsilon = D \cap \{z : \beta(z) < 1 - \varepsilon\}$, and $\Gamma_\varepsilon = \partial D \cap \{z : \beta(z) < 1 - \varepsilon\}$. By Sard's theorem, the set $\Gamma_\varepsilon$ is a smooth manifold with smooth boundary $\partial\Gamma_\varepsilon$, for almost all positive $\varepsilon$. Now $D \setminus K = \cup_{\varepsilon>0} D_\varepsilon$, and $\Gamma = \partial D \setminus K = \cup_{\varepsilon>0}\Gamma_\varepsilon$.

If $f$ is a continuous $CR$-function on $\Gamma$ then the approximation Baouendi-Treves theorem shows that for almost all positive $\varepsilon$

$$\int_{\Gamma_\varepsilon} f\overline{\partial}\varphi = \int_{\partial\Gamma_\varepsilon} f\varphi \tag{13.1}$$

for any $(n, n-2)$-differential form $\varphi$ with smooth coefficients and with compact support in $\mathbb{C}^n \setminus K$.

Subsequently we will assume that the sequence $\varepsilon_s \to 0$ is decreasing as $s \to \infty$ and is chosen so that $\partial\Gamma_{\varepsilon_s}$ is a smooth manifold, and (13.1) holds for $\Gamma_{\varepsilon_s}$. We write $D_{\varepsilon_s} = D_s$ and $\Gamma_{\varepsilon_s} = \Gamma_s$.

Suppose $G$ is an open neighborhood of $\overline{D}$, and $h \in \mathcal{O}(G)$ (we may assume that $G$ is a domain of holomorphy). For each positive $\varepsilon$, we consider the set

$$G_\varepsilon(h) = \left\{ z \in G : |h(z)| > \max_{D \setminus D_\varepsilon} |h| \right\}.$$

Then $G_\varepsilon(h) \subset G \setminus (\overline{D \setminus D_\varepsilon})$, and for every $z \in G_\varepsilon(h)$ the level set

$$L_z(h) = \{\zeta \in G : h(\zeta) = h(z)\} \subset G_\varepsilon(h).$$

We write

$$G(h) = \{z \in G : |h(z)| > \max_K |h|\}.$$

Since $K = \bigcap_{\varepsilon>0} \overline{D \setminus D_\varepsilon}$, we have $G(h) = \bigcup_{\varepsilon>0} G_\varepsilon(h)$. Since $K = \widehat{K}_{\overline{D}}$, we have

$$\overline{D} \setminus K \subset \bigcup_{G \supset \overline{D}} \bigcup_{h \in \mathcal{O}(G)} G(h).$$

By Hefer's theorem, for each $h \in \mathcal{O}(G)$ there are holomorphic functions $h_1(\zeta, z), \dots, h_n(\zeta, z)$ on $G \times G$ such that

$$h(z) - h(\zeta) = \sum_{k=1}^{n} h_k(\zeta, z)(z_k - \zeta_k). \tag{13.2}$$

We can explicitly compute a solution of the $\overline{\partial}$-problem $\overline{\partial}\Phi_h(\zeta) = U(\zeta, z)$ on the set $G \setminus L_z(h)$. Let

$$U_k(\zeta.z) = \frac{(-1)^k (n-2)!}{(\zeta_k - z_k)(2\pi i)^n} \left[ \sum_{j=1}^{k-1} (-1)^j \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n-2}} d\bar{\zeta}[j,k] \right.$$

$$\left. + \sum_{j=k+1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n-2}} d\bar{\zeta}[k,j] \right] \wedge d\zeta.$$

(The forms $U_k(\zeta, z)$ were first considered by Martinelli [**59**] in the proof of the Hartogs extension theorem.)

LEMMA 13.1. *Set*

$$\Phi_h(\zeta) = \frac{1}{h(\zeta) - h(z)} \sum_{k=1}^{n} h_k(\zeta, z)(\zeta_k - z_k)U_k(\zeta, z).$$

*Then* $\Phi_h(\zeta)$ *is defined in* $G \setminus L_z(h)$, *and* $\overline{\partial}_\zeta \Phi_h(\zeta) = U(\zeta, z)$.

PROOF. It easy to check that $\overline{\partial}_\zeta U_k(\zeta, z) = U(\zeta, z)$ outside $L_z(\zeta_k)$. Therefore

$$\overline{\partial}\Phi_h(\zeta) = \frac{1}{h(\zeta) - h(z)} \sum_{k=1}^{n} h_k(\zeta, z)\overline{\partial}_\zeta U_k(\zeta, z)$$

$$= \frac{1}{h(\zeta) - h(z)} \sum_{k=1}^{n} h_k(\zeta, z)(\zeta_k - z_k)U(\zeta, z) = U(\zeta, z)$$

by (13.2). □

Now suppose that $G$ and $G'$ are open sets in $\mathbb{C}^n$ with nontrivial intersection, $h \in \mathcal{O}(G)$, and $h' \in \mathcal{O}(G')$. We consider the Hefer decomposition (13.2) for $h$ and $h'$ and a point $z \in G \cap G'$.

If $n \geqslant 3$, we consider the forms $U_{k,j}(\zeta, z)$ given for $1 \leqslant k < j \leqslant n$ by

$$U_{k,j}(\zeta, z) = \frac{(-1)^{n+j+k}(n-3)!}{(2\pi i)^n(\zeta_k - z_k)(\zeta_j - z_j)} \left[\sum_{m=1}^{k-1} \frac{(-1)^m(\bar{\zeta}_m - \bar{z}_m)}{|\zeta - z|^{2n-4}} d\bar{\zeta}[m, k, j] \right.$$

$$+ \sum_{m=k+1}^{j-1} \frac{(-1)^{m-1}(\bar{\zeta}_m - \bar{z}_m)}{|\zeta - z|^{2n-4}} d\bar{\zeta}[k, m, j]$$

$$\left. + \sum_{m=j+1}^{n} \frac{(-1)^m(\bar{\zeta}_m - \bar{z}_m)}{|\zeta - z|^{2n-4}} d\bar{\zeta}[k, j, m] \right] \wedge d\zeta.$$

For $k > j$, we set $U_{k,j} = -U_{j,k}$. We further denote

$$\chi_{h,h'}(\zeta) = \sum_{1 \leqslant k < j \leqslant n} \frac{(h_k h'_j - h'_k h_j)(\zeta_k - z_k)(\zeta_j - z_j)U_{k,j}(\zeta, z)}{(h(\zeta) - h(z))(h'(\zeta) - h'(z))}.$$

LEMMA 13.2. *The following equalities hold on the set* $(G \setminus L_z(h)) \cap (G' \setminus L_z(h'))$:

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) = \overline{\partial}\chi_{h,h'}(\zeta), \quad \textit{if} \quad n \geqslant 3, \tag{13.3}$$

$$\Phi_h(\zeta) - \Phi_{h'}(\zeta) = \frac{(h_1 h_2' - h_2 h_1')d\zeta_1 \wedge d\zeta_2}{(2\pi i)^2 (h(\zeta) - h(z))(h'(\zeta) - h'(z))}, \quad \text{if} \quad n = 2. \quad (13.4)$$

LEMMA 13.3. *On the set $G \setminus L_z(h)$, we have*

$$\frac{\partial \Phi_h}{\partial \bar{z}_k} = \frac{\partial U_k}{\partial \bar{z}_k} - \bar{\partial}_\zeta \Psi_h^k, \quad k = 1, \dots, n, \quad \text{for} \quad n \geqslant 3,$$

*and*

$$\frac{\partial \Phi_h}{\partial \bar{z}_k} = \frac{\partial U_k}{\partial \bar{z}_k}, \quad k = 1, 2, \quad \text{for} \quad n = 2,$$

*where*

$$\Psi_h^k(\zeta) = \frac{1}{h(\zeta) - h(z)} \sum_{j \neq k} h_j(\zeta, z)(\zeta_j - z_j) \frac{\partial U_{j,k}(\zeta, z)}{\partial \bar{z}_k}.$$

Suppose now $G \supset \overline{D}$, $h \in \mathcal{O}(G)$, and $G(h) = \bigcup_{s=1}^{\infty} G_s(h)$, where

$$G_s(h) = \{z \in G : |h(z)| > \max_{\overline{D \setminus D_s}} |h|\}.$$

Consider the function

$$F_h^s(z) = \int_{\Gamma_s} f(\zeta) U(\zeta, z) - \int_{\partial \Gamma_s} f(\zeta) \Phi_h(\zeta). \quad (13.5)$$

The idea of the proof is to show that $F_h^s$ is holomorphic in $G_s(h) \setminus \Gamma$ and independent of $s$ and $h$, while outside $\overline{D}$ it equals zero, and so (13.5) gives a holomorphic extension of $f$ into $D \setminus K$ (by the jump theorem of the Bochner-Martinelli integral).

The function (13.5) is defined for $z \notin \Gamma$ and $z \in G_s(h)$, since $\partial \Gamma_s \subset \overline{D \setminus D_s}$.

We first show that it does not depend on $s$. Indeed, if $s' > s$, then the function $F_h^{s'}$ also is defined in $G_s(h) \setminus \Gamma$, so

$$F_h^{s'}(z) - F_h^s(z) = \int_{\Gamma_{s'} \setminus \Gamma_s} f(\zeta) U(\zeta, z)$$

$$- \int_{\partial \Gamma_{s'}} f(\zeta) \Phi_h(\zeta) + \int_{\partial \Gamma_s} f(\zeta) \Phi_h(\zeta).$$

Since $|h(z)| > |h(\zeta)|$ if $z \in G_s(h)$, and $\zeta \in \overline{\Gamma_{s'} \setminus \Gamma_s}$ (because $\Gamma_{s'} \setminus \Gamma_s \subset \overline{D \setminus D_s}$), the form $\Phi_h$ has no singularities on $\overline{\Gamma_{s'} \setminus \Gamma_s}$, and $\bar{\partial} \Phi_h = U(\zeta, z)$ by Lemma 13.1. The coefficients of $\Phi_h$ are class $C^\infty$, and we can extend them outside $\overline{\Gamma_{s'} \setminus \Gamma_s}$ as functions of class $C^\infty$ with compact support, so (13.1) is applicable. Consequently, $F_h^{s'}(z) = F_h^s(z)$. From now on we shall omit the symbol $s$ on the function $F_h^s$.

LEMMA 13.4. *If $F$ satisfies the hypotheses of Theorem 13.1, then for $z \in G_s(h)$ we have*

$$F(z) = F_h(z), \quad z \in D \setminus K.$$

PROOF. The Bochner-Martinelli formula can be applied to the function $F$ in the domain $D_s$. The boundary $\partial D_s = \Gamma_s \cup K_s$, where $K_s = D \cap \{z : 1 - \beta(z) = \varepsilon_s\}$, and we may assume that $K_s$ is also a smooth manifold. For $z \in G_s(h) \cap D$, we then have

$$F(z) = \int_{\partial D_s} F(\zeta)U(\zeta, z) = \int_{\Gamma_s} f(\zeta)U(\zeta, z) + \int_{K_s} F(\zeta)U(\zeta, z).$$

But $|h(z)| > |h(\zeta)|$ for $\zeta \in K_s \subset \overline{D \setminus D_s}$, so $U(\zeta, z) = \bar{\partial}\Phi_h(\zeta)$, and by Stokes's formula

$$\int_{K_s} F(\zeta)U(\zeta, z) = \int_{\partial K_s} F(\zeta)\Phi_h(\zeta) = -\int_{\partial \Gamma_s} f(\zeta)\Phi_h(\zeta). \quad \square$$

Lemma 13.4 shows why we have to choose the function $F_h^s$ in the form (13.5).

LEMMA 13.5. *The function $F_h(z)$ is holomorphic in $G(h) \setminus \Gamma$.*

PROOF. If $z \in G_s(h) \setminus \Gamma$, then by Lemma 13.3 and by (13.1) we have

$$\frac{\partial F_h}{\partial \bar{z}_k} = \int_{\Gamma_s} f \frac{\partial U}{\partial \bar{z}_k} - \int_{\partial \Gamma_s} f \frac{\partial \Phi_h}{\partial \bar{z}_k}$$

$$= \int_{\Gamma_s} f \frac{\partial U}{\partial \bar{z}_k} - \int_{\partial \Gamma_s} f \left( \frac{\partial U_k}{\partial \bar{z}_k} - \bar{\partial}\Psi_h^k \right) = \int_{\Gamma_s} f \bar{\partial}_\zeta \left( \frac{\partial U_k}{\partial \bar{z}_k} \right) - \int_{\partial \Gamma_s} f \frac{\partial U_k}{\partial \bar{z}_k}.$$

But since the form $\dfrac{\partial U_k}{\partial \bar{z}_k}$ has no singularities on $\Gamma_s$ (if $z \notin \Gamma$), we obtain that $\dfrac{\partial F_h}{\partial \bar{z}_k} = 0$ for $k = 1, \dots, n$ by again applying (13.1). $\square$

LEMMA 13.6. *If $h' \in \mathcal{O}(G')$, where $G' \supset \overline{D}$, then $F_h = F_{h'}$ for $z \in G'_s(h') \cap G_s(h) \setminus \Gamma$.*

PROOF. The case $n \geqslant 3$ is proved using (13.3) and (13.1).

If $n = 2$, then using (13.4) we obtain

$$F_h(z) - F_{h'}(z) = \frac{1}{(2\pi i)^2} \int_{\partial\Gamma_s} \frac{f(\zeta)(h_2 h_1' - h_1 h_2')d\zeta}{(h(\zeta) - h(z))(h'(\zeta) - h'(z))}. \tag{13.6}$$

Since $z \in G_s'(h') \cap G_s(h) \setminus \Gamma$, we have $|h(\zeta)| < |h(z)|$ and $|h'(\zeta)| < |h'(z)|$ for $\zeta \in \partial\Gamma_s$, so

$$\frac{1}{(h(\zeta) - h(z))(h'(\zeta) - h'(z))} = \sum_{k,j \geqslant 0} \frac{(h(\zeta))^k (h'(\zeta))^j}{(h(z))^{1+k}(h'(z))^{1+j}},$$

and this series converges absolutely and uniformly on $\partial\Gamma_s$. Substituting it into (13.6) and integrating term by term, we find

$$F_h(z) - F_{h'}(z) = \frac{1}{(2\pi i)^2} \sum_{k,j \geqslant 0} \frac{1}{(h(z))^{1+k}(h'(z))^{1+j}} \int_{\partial\Gamma_s} f\mu_{k,j},$$

where

$$\mu_{k,j} = (h_2 h_1' - h_1 h_2')(h(\zeta))^k (h'(\zeta))^j d\zeta.$$

The form $\mu_{k,j}$ is $\overline{\partial}$-closed on $\Gamma_s$, so by (13.1),

$$\int_{\partial\Gamma_s} f\mu_{k,j} = \int_{\Gamma_s} f\overline{\partial}\mu_{k,j} = 0. \quad \square$$

Thus, the integral (13.5) defines a function $F$ that is holomorphic in $D \setminus K$ and holomorphic outside $\overline{D}$. Let $W$ be a neighborhood of $\partial D \setminus K = \Gamma$ contained in

$$\bigcup_{G \supset \overline{D}} \bigcup_{h \in \mathcal{O}(G)} G(h)$$

such that $W \setminus \Gamma = W_+ \cup W_-$, where $W_+ \subset D$, while $W_- \subset \mathbb{C}^n \setminus \overline{D}$ and $W_+$, $W_-$ are connected sets. We will show that $F = 0$ in $W_-$.

Consider a neighborhood $G \supset \overline{D}$ and $h \in \mathcal{O}(G)$, and suppose $z \in G$ is a point such that $|h(z)| > \max_{\overline{D}} |h|$. Such a point exists since $\overline{D} \subset G$ and $|h|$ does not attain a maximum inside $G$. This point $z \notin \overline{D}$. Let

$$G_1(h) = \{z : |h(z)| > \max_{\overline{D}} |h|\}.$$

Then

$$F(z) = F_h(z) = \int_{\Gamma_s} f(\zeta)U(\zeta, z) - \int_{\partial\Gamma_s} f(\zeta)\Phi_h(\zeta).$$

But $\Phi_h$ does not have singularities on $\Gamma_s$ because $L_z(h)$ does not intersect $\partial D$, while $\overline{\partial}\Phi_h = U(\zeta, z)$. By (13.1), $F(z) = 0$.

Since $F = 0$ in $G_1(h) \subset G(h)$, while $G_1(h) \cap W_- \neq \varnothing$ (so $G_1$ will abut $\partial D \setminus K$) and $W_-$ is a connected set, $F \equiv 0$ in $W_-$. □

Theorem 13.1 generalizes results from [**76**].

As can be seen from the proof of this theorem, we do not need to require connectedness of $\Gamma$, but only, for example, the following: the compact set $K \subset \overline{D}$, and $K$ is convex with respect to the class $\mathcal{O}(G)$, where $G$ is a domain of holomorphy containing $\overline{D}$, while the complement $\mathbb{C}^n \setminus \overline{D}$ is connected. Then every $CR$-function $f \in \mathcal{C}(\Gamma)$ extends holomorphically into $D \setminus K$. Indeed, in this case $F$ will be holomorphic in $G \setminus \overline{D}$, and consequently will extend holomorphically into $D$ by Hartogs's theorem.

For domains in $\mathbb{C}^2$, the $\mathcal{O}(\overline{D})$-convexity of $K$ is also a necessary condition for the existence of a holomorphic extension.

THEOREM 13.2 (Stout [**77**]). *Suppose $D$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^2$ with $\partial D \in C^2$. If $K$ is a compact subset of $\partial D$, then every $CR$-function in $\partial D \setminus K$ extends holomorphically into $D$ if and only if $K = \widehat{K}_{\overline{D}}$.*

After Theorem 13.1 it was appeared a lot of results in this direction. Lupacciolu [**52**]–[**55**] considered different classes of compact sets: meromorphically convex sets, $p$-meromorphically convex sets; Kytmanov [**34**] considered the case of integrable $CR$-functions; other classes compact sets and $CR$-functions were considered by Lupacciolu and Stout [**56**]. The surveys of different results can be found in [**77, 14**].

Theorem 13.1 was considered on generic manifolds in [**42, 43**].

Consider a local version of Theorem 13.1. Let $\Omega$ be a domain of holomorphy in $\mathbb{C}^n$. Let $K \subset \Omega$ be a compact set, and suppose the hypersurface $\Gamma$ is a smooth, oriented, relatively closed manifold in $\Omega \setminus K$. We will assume that $\Gamma = \{z \in \Omega \setminus K : \rho(z) = 0\}$, $\rho \in C^1(\Omega \setminus K)$ and real-valued, $d\rho \neq 0$ on $\Gamma$. Then $\Omega \setminus (\Gamma \cup K) = \Omega^+ \cup \Omega^-$, where $\Omega^+ = \{z \in \Omega \setminus K : \rho(z) > 0\}$ and $\Omega^- = \{z \in \Omega \setminus K : \rho(z) < 0\}$.

We are interested in the question:

PROBLEM 13.3. *Whether every $CR$-function $f$ on $\Gamma$ can be represented as the difference of boundary values of functions $h^\pm$ that are holomorphic in $\Omega^\pm$.*

If $n = 2$, then even in the simplest situations the answer is negative. Consider the following example.

EXAMPLE 13.1. Suppose $\Omega$ is the unit bidisk, that is $\Omega = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$, let $K = \{(0,0)\}$, and let $\Gamma = \{z \in \Omega \setminus K : |z_1| = |z_2|\}$. Then $\Gamma$ is a smooth hypersurface in $\Omega \setminus K$. The set $\Omega \setminus (\Gamma \cup K) = \Omega^+ \cup \Omega^-$, where $\Omega^+ = \{z \in \Omega : |z_1| < |z_2|\}$, and $\Omega^- = \{z \in \Omega : |z_1| > |z_2|\}$. We take the function $1/(z_1 z_2)$ as the $CR$-function $f$ on $\Gamma$. If $f = h^+ - h^-$ on $\Gamma$, where $h^\pm \in \mathcal{O}(\Omega^\pm)$, then

$$\int_{\substack{|z_1|=a-\varepsilon \\ |z_2|=a}} h^+ dz_1 \wedge dz_2 = 0,$$

since for fixed $z_2$ the function $h^+$ is holomorphic in the disk $\{|z_1| < |z_2|\}$. For precisely the same reason,

$$\int_{\substack{|z_1|=a+\varepsilon \\ |z_2|=a}} h^- dz_1 \wedge dz_2 = 0,$$

and then

$$\int_{\substack{|z_1|=a \\ |z_2|=a}} f dz_1 \wedge dz_2$$

must be zero; but it equals $(2\pi i)^2$.

This example shows that in $\mathbb{C}^2$, we cannot remove even a point from $\Omega$.

THEOREM 13.3 (Kytmanov [35]). *Let $\Omega$ be a domain of holomorphy in $\mathbb{C}^n$, where $n \geqslant 3$. Consider a compact set $K = \widehat{K}_\Omega \subset \Omega$ and a smooth, oriented, relatively closed hypersurface $\Gamma$ in $\Omega \setminus K$. If $f$ is a $CR$-function on $\Gamma$, then $f$ can be represented on $\Gamma$ as the difference of boundary values of functions $h^\pm \in \mathcal{O}(\Omega^\pm)$.*

## 14. Removable Singularities of $CR$-functions

THEOREM 14.1 (Kytmanov [34]). *Under the hypotheses of Theorem 13.1, if the $CR$-function $f \in \mathcal{L}^\infty(\Gamma)$, then its holomorphic extension $F$ also is bounded, and*

$$\|f\|_{\mathcal{L}^\infty} = \sup_{D \setminus K} |F|.$$

*In particular, if $\partial D \in C^1$ and $K \subset \partial D$, then $F \in \mathcal{H}^\infty(D)$ and, therefore, its boundary values gives a $CR$-extension of $f$ on the whole boundary $\partial D$.*

PROOF. Suppose $F(z^0) = 1$ for some point $z^0 \in D\setminus K$, yet $\|f\|_{\mathcal{L}^\infty} < 1$. Since the $f^k$ are also $CR$-functions, and their holomorphic extensions given by the functions $F^k$, we have by (13.5) that

$$1 = F^k(z^0) = \int_{\Gamma_s} f^k(\zeta)U(\zeta, z) - \int_{\partial\Gamma_s} f^k(\zeta)\Phi_h(\zeta).$$

The integrals on the right-hand side tends to zero as $k \in \infty$ since $f^k(\zeta) \to 0$ as $k \to \infty$, and we can apply the Lebesgue dominated convergence theorem. □

COROLLARY 14.1 (Analogue of Riemann's theorem). *Suppose $\overline{D}$ is a holomorphically convex compact set, $\partial D \in C^1$, and $\mathcal{O}(\overline{D})$ is dense in the class of functions $\mathcal{A}(D)$. If $h \in \mathcal{A}(D)$, $f$ is a $CR$-function on $\partial D\setminus K$, where $K = \{z \in \partial D : h(z) = 0\}$, and $f \in \mathcal{L}^\infty(\partial D)$, then $f$ is a $CR$-function on $\partial D$.*

Since $\sigma(K) = 0$, we may assume here that $f$ is defined on whole boundary $\partial D$.

PROOF. Let $S = \{z \in \overline{D} : h(z) = 0\}$. Since $\mathcal{O}(\overline{D})$ is dense in $\mathcal{A}(D)$, we have $\widehat{K}_{\overline{D}} = \widehat{K}_{\mathcal{A}(D)}$, so $\widehat{K}_{\overline{D}} = S$. In view of Theorem 14.1, the function $f$ extends into $D\setminus S$ as a holomorphic function $F$ that is bounded in $D\setminus S$. By Riemann's theorem, $F$ extends holomorphically into $D$. Consequently, $f$ is a $CR$-function on $\partial D$. □

If, in the situation of Corollary14.1, the $CR$-function $f \in \mathcal{L}^1(\partial D \setminus K)$, then it might not be a $CR$-function on $\partial D$.

EXAMPLE 14.1. Suppose $f = 1/z_1$, $D = B(0,1)$, and $K = \{z \in S(0,1) : z_1 = 0\}$. Then $f$ is a $CR$-function on $S(0,1) \setminus K$, and we can show that $f \in \mathcal{L}^1(S(0,1))$. Indeed,

$$\int_{S(0,1)} \frac{1}{|z_1|} d\sigma \leqslant C \int_{\{|z_1|\leqslant 1\}} \frac{1}{|z_1|} dx_1 \wedge dy_1 \int_{\{|z_2|^2+\cdots+|z_n|^2 = 1-|z_1|^2\}} d\sigma'$$
$$\leqslant C_1 \int_{\{|z_1|\leqslant 1\}} \frac{1}{|z_1|}(1-|z_1|^2)^{(2n-3)/2} dx_1 \wedge dy_1$$
$$= C_1 2\pi \int_0^1 (1-r^2)^{2n-3)/2} dr < \infty.$$

But $f$ is not a $CR$-function on $S(0,1)$, for otherwise it would be extend by the Hartogs-Bochner theorem to $B(0,1)$ as a function $F \in \mathcal{H}^1(D)$, and by the uniqueness theorem would be coincide with $1/z_1$ in $B(0,1)$.

However, if we replace the zero set $K$ in Corollary 14.1 by a peak set, then the corollary remains true for functions of class $\mathcal{L}^1(\partial D)$.

THEOREM 14.2 (Kytmanov [**32**]). *Let $D$ be a bounded domain in $\mathbb{C}^n$ such that $\partial D$ is a Lyapunov surface. Suppose that $K \subset \partial D$ is a peak set for the class of holomorphic functions in $D$ of class $C^\alpha(\overline{D})$, $\alpha > 0$ (that is, $K = \{z \in \partial D : \psi(z) = 1\}$, where $\psi$ is a holomorphic function in $D$ of class $C^\alpha(\overline{D})$, and $|\psi| < 1$ on $\partial D \setminus K$). If $f \in \mathcal{L}^1(\partial D)$, and $f$ is a $CR$-function on $\partial D \setminus K$, then $f$ is a $CR$-function on $\partial D$.*

The generalizations of this theorem are given in [**44, 60**].

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