수 학 강 의 록 제 47 권

LOCALIZATION OPERATORS

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Preface

This is an expanded version of the lectures given at the Global Analysis Research Center (GARC) of the Seoul National University in June, 1999, and at Peking University in July, 1999, at the invitation by Professor Dohan Kim and Professor Lizhong Peng respectively.

A localization operator L_F on a complex and separable Hilbert space X is a bounded linear operator on X defined in terms of a square-integrable representation of a locally compact and Hausdorff group G on X, and a function F on G. A major result that we prove is that the linear operator L_F is in the Schatten-von Neumann class S_p if the function F is in $L^p(G)$, $1 \leq p \leq \infty$. A formula for the trace of L_F is given in terms of F when F is in $L^1(G)$. We then look at localization operators on two specific groups, i.e., the Weyl-Heisenberg group and the affine group. In the case of the Weyl-Heisenberg group, we are led to the class of time-frequency localization operators studied by Daubechies in the paper [1].

If we let $G = \mathbb{R}^n$ and $X = L^2(\mathbb{R}^n)$, and the representation of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ is taken to be modulation of a function in $L^2(\mathbb{R}^n)$, then, of course, the representation is no longer irreducible and hence cannot be square-integrable. The "localization operator" L_F in this setting is called a wavelet multiplier and is also in the Schatten-von Neumann class S_p when F is in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. An interesting fact that we show in Chapter 8 is that the Landau-Pollak-Slepian operator arising in signal analysis is in fact a wavelet multiplier.

Using the basic theory of the Weyl transform which we recall without proof from the book [22] by Wong, we give two product formulas for the wavelet multipliers and a product formula for the Daubechies operators.

The product formulas for wavelet multipliers given in Chapters 9 and 10 are new, while all the other results have recently been published or accepted for publication.

Contents

1 Square-Integrable Representations

Let G be a locally compact and Hausdorff group on which the left Haar measure is denoted by μ, i.e.,

$$\int_G f(gh)d\mu(h) = \int_G f(h)d\mu(g^{-1}h) = \int_G f(h)d\mu(h)$$

for all f in $L^1(G)$ and all g in G. Let X be a separable and complex Hilbert space of which the dimension is infinite. We denote the inner product and norm in X by $(\ ,\ )$ and $\|\ \|$ respectively. Let $B(X)$ be the C^*-algebra of all bounded linear operators on X, and let $\|\ \|_*$ denote the norm in $B(X)$. An irreducible and unitary representation $\pi : G \to B(X)$ is said to be square-integrable if there exists a nonzero element φ in X such that

$$\int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) < \infty. \tag{1.1}$$

We call any element φ in X for which $\|\varphi\| = 1$ and (1.1) is valid an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$, and we define the constant c_φ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g). \tag{1.2}$$

Theorem 1.1 *Let φ be an admissible wavelet for the square-integrable representation $\pi : G \to B(X)$. Then*

$$(x, y) = \frac{1}{c_\varphi} \int_G (x, \pi(g)\varphi)(\pi(g)\varphi, y)d\mu(g) \tag{1.3}$$

for all x and y in X.

Remark 1.2 The formula (1.3) is known as the resolution of the identity formula. The mapping from X into functions on G given by $x \mapsto c_\varphi^{-1/2}(x, \pi(g)\varphi)$, $x \in X$, $g \in G$, is known as the wavelet transform associated to the admissible wavelet φ.

To prove Theorem 1.1, we need a lemma.

Lemma 1.3 *The subspace M of X defined by*

$$M = \left\{ x \in X : \int_G |(x, \pi(g)\varphi)|^2 d\mu(g) < \infty \right\} \tag{1.4}$$

is a closed subspace of X.

Proof of Theorem 1.1: For $x \in M$ and $h \in G$, we obtain, by (1.4),

$$\begin{aligned} \int_G |(\pi(h)x, \pi(g)\varphi)|^2 d\mu(g) &= \int_G |(x, \pi(h^{-1}g)\varphi)|^2 d\mu(g) \\ &= \int_G |(x, \pi(g)\varphi)|^2 d\mu(g) < \infty, \end{aligned}$$

and hence, by (1.4), $\pi(h)x \in M$ for all x in M and h in G. Therefore, using the fact that $\varphi \in M$, M is a nonzero subspace of X which is invariant with respect to the square-integrable representation $\pi : G \to B(X)$. Thus, $\overline{M} = X$, and hence, by Lemma 1.3, $M = X$. Now, we define the linear operator $A_\varphi : X \to L^2(G)$ by

$$(A_\varphi x)(g) = (x, \pi(g)\varphi), \quad x \in X, \ g \in G. \tag{1.5}$$

Then, for $x \in X$, and g, $h \in G$, by (1.5), we have

$$\begin{aligned} (A_\varphi \pi(h)x)(g) &= (\pi(h)x, \pi(g)\varphi) \\ &= (x, \pi(h^{-1}g)\varphi) \\ &= (A_\varphi x)(h^{-1}g), \end{aligned}$$

and so,

$$A_\varphi \pi(h) = L(h) A_\varphi, \tag{1.6}$$

where

$$(L(h)f)(g) = f(h^{-1}g), \quad g, h \in G, \tag{1.7}$$

for all f in $L^2(G)$. Let $\{x_k\}$ be a sequence of elements in X such that $x_k \to x$ in X and $A_\varphi x_k \to f$ in $L^2(G)$ as $k \to \infty$. Then there is a subsequence of $\{A_\varphi x_k\}$, again denoted by $\{A_\varphi x_k\}$, such that

$$A_\varphi x_k \to f \tag{1.8}$$

a.e. on G as $k \to \infty$. Since

$$(x_k, \pi(g)\varphi) \to (x, \pi(g)\varphi), \quad g \in G,$$

as $k \to \infty$, it follows from (1.5) that

$$(A_\varphi x_k)(g) \to (A_\varphi x)(g), \quad g \in G. \tag{1.9}$$

Thus, by (1.8) and (1.9), $A_\varphi x = f$. Hence $A_\varphi : X \to L^2(G)$ is a closed linear operator, and, by the closed graph theorem, $A_\varphi : X \to L^2(G)$ is a bounded linear operator. Finally, for all x and y in X, we get, by (1.5)–(1.7),

$$\begin{aligned}
(A_\varphi^* L(g) A_\varphi x, y) &= (L(g)A_\varphi x, A_\varphi y)_{L^2(G)} \\
&= \int_G (A_\varphi x)(g^{-1}h)\overline{(A_\varphi y)(h)}d\mu(h) \\
&= \int_G (A_\varphi x)(h)\overline{(A_\varphi y)(gh)}d\mu(h) \\
&= (A_\varphi x, L(g^{-1})A_\varphi y)_{L^2(G)} \\
&= (A_\varphi x, A_\varphi \pi(g^{-1})y)_{L^2(G)} \\
&= (\pi(g)A_\varphi^* A_\varphi x, y), \quad g \in G,
\end{aligned}$$

where A_φ^* is the adjoint of A_φ, and hence

$$A_\varphi^* L(g) A_\varphi = \pi(g) A_\varphi^* A_\varphi, \quad g \in G. \tag{1.10}$$

Moreover, by (1.6),

$$A_\varphi^* L(g) A_\varphi = A_\varphi^* A_\varphi \pi(g), \quad g \in G. \tag{1.11}$$

Thus, by (1.10), (1.11) and the fact that $\pi : G \to B(X)$ is irreducible, we conclude that there exists a constant c such that

$$A_\varphi^* A_\varphi = cI, \tag{1.12}$$

where I is the identity operator on X. Thus, for all x and y in X, we get, by (1.12),

$$\begin{aligned}
c(x,y) &= (A_\varphi^* A_\varphi x, y) \\
&= (A_\varphi x, A_\varphi y)_{L^2(G)} \\
&= \int_G (x, \pi(g)\varphi)\overline{(y, \pi(g)\varphi)}d\mu(g) \\
&= \int_G (x, \pi(g)\varphi)(\pi(g)\varphi, y)d\mu(g).
\end{aligned} \tag{1.13}$$

Thus, by (1.2) and (1.13),

$$\begin{aligned} c &= c(\varphi,\varphi) \\ &= \int_G |(\varphi,\pi(g)\varphi)|^2 d\mu(g) \\ &= c_\varphi. \end{aligned} \tag{1.14}$$

Hence, by (1.13) and (1.14), the proof is complete provided that we can prove Lemma 1.3.

Proof of Lemma 1.3: Let us begin with the observation that if we equip M with the inner product $(\ ,\ )_\varphi$ and norm $||\ ||_\varphi$ given, respectively, by

$$(x,y)_\varphi = (x,y) + (A_\varphi x, A_\varphi y) \tag{1.15}$$

and

$$||x||_\varphi^2 = (x,x)_\varphi \tag{1.16}$$

for all x and y in M, then M is a Hilbert space which we denote by M_φ and $A_\varphi : M_\varphi \to L^2(G)$ is a bounded linear operator. Using (1.6), (1.7), (1.15), (1.16) and the fact that $\pi : G \to B(X)$ is a unitary representation, we get

$$\begin{aligned} ||\pi(g)x||_\varphi^2 &= ||\pi(g)x||^2 + ||A_\varphi \pi(g)x||^2 \\ &= ||x||^2 + ||L(g)A_\varphi x||^2 \\ &= ||x||^2 + ||A_\varphi x||^2 = ||x||_\varphi^2 \end{aligned}$$

for all g in G and x in M_φ. Moreover, for all g in G, $\pi(g) : M_\varphi \to M_\varphi$ is onto. Indeed, using the fact that $\pi : G \to B(X)$ is a representation and the invariance of M with respect to $\pi : G \to B(X)$, we get

$$y = \pi(g)\pi(g^{-1})y$$

for all g in G and y in M_φ. Thus, $\pi : G \to M_\varphi$ is a unitary representation of G on M_φ, and consequently A_φ is a scalar multiple of an isometry from M_φ into X. So, by (1.15) and (1.16), there is a positive constant λ such that

$$||A_\varphi x||^2 = \lambda ||x||_\varphi^2 = \lambda ||x||^2 + \lambda ||A_\varphi x||^2, \quad x \in M. \tag{1.17}$$

Hence, by (1.17), $\lambda < 1$ and

$$||A_\varphi x||^2 = \frac{\lambda}{1-\lambda} ||x||^2, \quad x \in M. \tag{1.18}$$

Using (1.18) and the density of M in X, we can extend $A_\varphi : M \to X$ to a bounded linear operator from X into X, which we denote by $\tilde{A}_\varphi : X \to X$. So, if $\{x_k\}$ is a sequence of elements in M such that $x_k \to x$ in X as $k \to \infty$, then $A_\varphi x_k \to \tilde{A}_\varphi x$ in $L^2(G)$ as $k \to \infty$. Since A_φ is a closed linear operator from X into X with domain M, it follows that $x \in M$. Therefore M is a closed subspace of X.

Remark 1.4 Theorem 1.1 is a simplified version of Theorem 3.1 in the paper [8] by Grossmann, Morlet and Paul. By Remark 1.2, the mapping $c_\varphi^{-1/2} A_\varphi$, where A_φ is constructed in the proof of Theorem 1.1, is the wavelet transform associated to the admissible wavelet φ.

As an immediate consequence of Theorem 1.1 and Remark 1.4, we give the following corollary.

Corollary 1.5 *The wavelet transform associated to an admissible wavelet φ given by $X \ni x \mapsto c_\varphi^{-1/2} A_\varphi x \in L^2(G)$ is an isometry of X into $L^2(G)$.*

We can give some information on the set $AW(\pi)$ of admissible wavelets associated to an irreducible and unitary representation $\pi : G \to B(X)$ for an important class of locally compact and Hausdorff groups G. A locally compact and Hausdorff group G is said to be unimodular if on G the left Haar measure μ is also the right Haar measure, i.e.,

$$\int_G f(gh)d\mu(h) = \int_G f(h)d\mu(g^{-1}h) = \int_G f(h)d\mu(h)$$

and

$$\int_G f(hg)d\mu(h) = \int_G f(h)d\mu(hg^{-1}) = \int_G f(h)d\mu(h)$$

for all f in $L^1(G)$ and all g in G.

Theorem 1.6 *Let G be a unimodular group and let $\pi : G \to B(X)$ be an irreducible and unitary representation of G on X. Then $AW(\pi) = \phi$ or $AW(\pi) = \{x \in X : \|x\| = 1\}$.*

Proof: Let $\mathcal{D}$ be the subspace of X defined by

$$\mathcal{D} = \left\{ x \in X : \int_G |(x, \pi(g)x)|^2 d\mu(g) < \infty \right\}. \tag{1.19}$$

Suppose that $\mathcal{D} \neq \{0\}$. Let $x \in \mathcal{D}$ and $h \in G$. Then, using (1.19) and the unimodularity of G,

$$\begin{aligned}\int_G |(\pi(h)x, \pi(g)\pi(h)x)|^2 d\mu(g) &= \int_g |(x, \pi(h^{-1}gh)x)|^2 d\mu(g) \\ &= \int_G |(x, \pi(g)x)|^2 d\mu(g) < \infty,\end{aligned}$$

and hence, by (1.19), $\pi(h)x \in \mathcal{D}$. Therefore $\mathcal{D}$ is an invariant subspace of X with respect to $\pi : G \to B(X)$. Thus, $\overline{\mathcal{D}} = X$. Let A be the linear operator from X into X with domain $\mathcal{D}$ defined by

$$(Ax)(g) = (x, \pi(g)x), \quad x \in \mathcal{D},\ g \in G. \tag{1.20}$$

Then, for $x \in X$, and $g,\ h \in G$, using (1.20) and the fact that $\pi : G \to B(X)$ is a representation, we get

$$\begin{aligned}(A\pi(h)x)(g) &= (\pi(h)x, \pi(g)\pi(h)x) \\ &= (x, \pi(h^{-1}gh)x) \\ &= (Ax)(h^{-1}gh),\end{aligned}$$

and so,

$$A\pi(h) = L(h)R(h)A, \tag{1.21}$$

where

$$(L(h)f)(g) = f(h^{-1}g), \quad g,\ h \in G, \tag{1.22}$$

and

$$(R(h)f)(g) = f(gh), \quad g,\ h \in G, \tag{1.23}$$

for all f in $L^2(G)$. Let $\{x_k\}$ be a sequence of elements in $\mathcal{D}$ such that $x_k \to x$ in X and $Ax_k \to f$ in $L^2(G)$ ad $k \to \infty$. Then there is a subsequence of $\{Ax_k\}$, again denoted by $\{Ax_k\}$, such that

$$Ax_k \to f \tag{1.24}$$

a.e. on G as $k \to \infty$. Since

$$(x_k, \pi(g)x_k) \to (x, \pi(g)x), \quad g \in G,$$

it follows from (1.20) and (1.24) that

$$(Ax_k)(g) \to (x, \pi(g)x), \quad g \in G. \tag{1.25}$$

Thus, by (1.24) and (1.25), $(x, \pi(\cdot)x) \in L^2(G)$. Hence

$$\int_G |(x, \pi(g)x)|^2 d\mu(g) < \infty. \tag{1.26}$$

Therefore, by (1.19) and (1.26),

$$x \in \mathcal{D}. \tag{1.27}$$

By (1.20), (1.24), (1.25) and (1.27), $Ax = f$. Thus, A is a closed linear operator from X into $L^2(G)$ with domain $\mathcal{D}$. Let $\mathcal{D}$ be the Hilbert space in which the inner product $(\ ,\ )_A$ and norm $||\ ||_A$ are given, respectively, by

$$(x, y)_A = (x, y) + (Ax, Ay) \tag{1.28}$$

and

$$||x||_A^2 = (x, x)_A \tag{1.29}$$

for all x and y in $\mathcal{D}$. Then $A : \mathcal{D} \to L^2(G)$ is a bounded linear operator. Using (1.21)–(1.23), (1.28), (1.29), the fact that $\pi : G \to B(X)$ is a unitary representation and the unimodularity of G,

$$\begin{aligned} ||\pi(g)x||_A^2 &= ||\pi(g)x||^2 + ||A\pi(g)x||^2 \\ &= ||x||^2 + ||L(g)R(g^{-1})Ax||^2 \\ &= ||x||^2 + ||Ax||^2 = ||x||_A^2 \end{aligned}$$

for all g in G and x in $\mathcal{D}$. Moreover, for all g in G, $\pi(g) : \mathcal{D} \to \mathcal{D}$ is onto. Indeed, using the fact that $\pi : G \to B(X)$ is a representation and the invariance of $\mathcal{D}$ with respect to $\pi : G \to B(X)$, we get

$$y = \pi(g)\pi(g^{-1})y$$

for all g in G and y in $\mathcal{D}$. Thus, $\pi : G \to \mathcal{D}$ is a unitary representation of G on $\mathcal{D}$, and consequently A is a scalar multiple of an isometry from $\mathcal{D}$ into X. So, by (1.28) and (1.29), we can find a positive number λ such that

$$||Ax||^2 = \lambda||x||_A^2 = \lambda||x||^2 + \lambda||Ax||^2, \quad x \in \mathcal{D}. \tag{1.30}$$

Hence, by (1.30), $\lambda < 1$ and

$$\|Ax\|^2 = \frac{\lambda}{1-\lambda}\|x\|^2, \quad x \in \mathcal{D}. \tag{1.31}$$

Using (1.31) and the density of $\mathcal{D}$ in X, we can extend $A : \mathcal{D} \to X$ to a bounded linear operator from X into X, which we denote by $\tilde{A} : X \to X$. So, if $\{x_k\}$ is a sequence of elements in $\mathcal{D}$ such that $x_k \to x$ in X as $k \to \infty$, then $Ax_k \to \tilde{A}x$ in $L^2(G)$ as $k \to \infty$. Since A is a closed linear operator from X into X with domain $\mathcal{D}$. it follows that $x \in \mathcal{D}$. Therefore $\mathcal{D}$ is a closed subspace of X. Thus, using the irreducibility of $\pi : G \to B(X)$, we conclude that $\mathcal{D} = X$ and the proof is complete. □

Remark 1.7 We give in Chapter 4 a unimodular group G, and an irreducible and unitary representation $\pi : G \to B(X)$ of G on X for which $AW(\pi) = \{x \in X : \|x\| = 1\}$. A different unimodular group G', and a new irreducible and unitary representation $\pi' : G' \to B(X)$ for which $AW(\pi) = \phi$ are also given. It is worth emphasizing the fact that Theorem 1.6 is false, in general, for non-unimodular groups, and Chapter 5 is devoted to a study of a non-unimodular group for which the conclusion of Theorem 1.6 is not true.

2 Localization Operators

In this chapter we address the problem of associating to every function F in $L^p(G)$, $1 \le p \le \infty$, a bounded localization operator $L_F : X \to X$. The problem of associating a bounded localization operator $L_F : X \to X$ to a function F in $L^1(G)$ or $L^\infty(G)$ is relatively easy and this is tackled first.

Let $F \in L^1(G) \cup L^\infty(G)$. Then, for any x in X, we define $L_F x$ by

$$(L_F x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y) d\mu(g) \tag{2.1}$$

for all y in X. Then we have the following proposition.

Proposition 2.1 *Let $F \in L^1(G)$. Then $L_F : X \to X$ is a bounded linear operator and*

$$\|L_F\|_* \le \frac{1}{c_\varphi} \|F\|_{L^1(G)}.$$

Proof: Let x, $y \in X$. Then using (2.1), the Schwarz inequality, $\|\varphi\| = 1$ and the fact that $\pi(g) : X \to X$ is unitary for all g in G, we have

$$|(x, \pi(g)\varphi)(\pi(g)\varphi, y)| \le \|x\| \, \|y\|. \tag{2.2}$$

Since $F \in L^1(G)$, it follows from (2.1) and (2.2) that

$$|(L_F x, y)| \le \frac{1}{c_\varphi} \|F\|_{L^1(G)} \|x\| \, \|y\|$$

and the proof of the proposition is complete. □

We also have the following proposition.

Proposition 2.2 *Let $F \in L^\infty(G)$. Then $L_F : X \to X$ is a bounded linear operator and*

$$\|L_F\|_* \le \|F\|_{L^\infty(G)}.$$

Proof: Let x, $y \in X$. Then, using (2.1), the Schwarz inequality and the assumption that $F \in L^\infty(G)$, we have

$$|(L_F x, y)|$$
$$\leq \frac{1}{c_\varphi}||F||_{L^\infty(G)} \left\{\int_G |(x, \pi(g)\varphi)|^2 d\mu(g)\right\}^{\frac{1}{2}} \left\{\int_G |(\pi(g)\varphi, y)|^2 d\mu(g)\right\}^{\frac{1}{2}}. \quad (2.3)$$

Now, by (1.3),

$$||x||^2 = \frac{1}{c_\varphi}\int_G |(x, \pi(g)\varphi)|^2 d\mu(g) \quad (2.4)$$

and

$$||y||^2 = \frac{1}{c_\varphi}\int_G |(\pi(g)\varphi, y)|^2 d\mu(g). \quad (2.5)$$

Hence, by (2.3)–(2.5),

$$|(L_F x, y)| \leq ||f||_{L^\infty(G)}||x||\ ||y||,$$

and this completes the proof of the proposition. □

We can now associate a localization operator $L_F : X \to X$ to every function F in $L^p(G)$, $1 < p < \infty$, and prove that $L_F : X \to X$ is a bounded linear operator. The precise result is the following theorem.

Theorem 2.3 *Let $F \in L^p(G)$, $1 < p < \infty$. Then there exists a unique bounded linear operator $L_F : X \to X$ such that*

$$||L_F||_* \leq c_\varphi^{-\frac{1}{p}}||F||_{L^p(G)}, \quad (2.6)$$

and $L_F x$ is given by (2.1) for all x in X and all simple functions F on G for which

$$\mu\{g \in G : F(g) \neq 0\} < \infty.$$

To prove Theorem 2.3, we need a recall of the Riesz-Thorin theorem given in, e.g., Chapter 10 of the book [22] by Wong.

Theorem 2.4 (The Riesz-Thorin Theorem) *Let (X, μ) be a measure space and (Y, ν) a σ-finite measure space. Let T be a linear transformation with domain $\mathcal{D}$ consisting of all μ-simple functions f on X such that*

$$\mu\{s \in X : f(s) \neq 0\} < \infty$$

and such that the range of T is contained in the set of all ν-measurable functions on Y. Suppose that α_1, α_2, β_1 and β_2 are numbers in $[0,1]$ and there exist positive constants M_1 and M_2 such that

$$\|Tf\|_{L^{\frac{1}{\beta_j}}(Y)} \leq M_j \|f\|_{L^{\frac{1}{\alpha_j}}(X)}, \quad f \in \mathcal{D},\ j = 1, 2.$$

Then, for $0 < \theta < 1$,

$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$$

and

$$\beta = (1-\theta)\beta_1 + \theta\beta_2,$$

we have

$$\|Tf\|_{L^{\frac{1}{\beta}}(Y)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^{\frac{1}{\alpha}}(X)}, \quad f \in \mathcal{D}.$$

Proof of Theorem 2.3: Since X is a separable and complex Hilbert space of which the dimension is infinite, it follows that there exists a unitary operator $U : X \to L^2(\mathbb{R}^n)$. Let $F \in L^1(G)$. Then, by Proposition 2.1, the linear operator $\tilde{L}_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined by

$$\tilde{L}_F = U L_F U^{-1}, \tag{2.7}$$

is bounded and

$$|\tilde{L}_F|_* \leq \frac{1}{c_\varphi} \|F\|_{L^1(G)}, \tag{2.8}$$

where $|\ |_*$ is the norm in the C^*-algebra $B(L^2(\mathbb{R}^n))$ of all bounded linear operators from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. If $F \in L^\infty(G)$, then, by Proposition 2.2, the linear operator $\tilde{L}_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined by (2.7), is also bounded and

$$|\tilde{L}_F|_* \leq \|F\|_{L^\infty(G)}. \tag{2.9}$$

Let $\mathcal{D}$ be the set of all simple functions F on G such that

$$\mu\{g \in G : F(g) \neq 0\} < \infty.$$

Let $f \in L^2(\mathbb{R}^n)$ and T be the linear transformation from $\mathcal{D}$ into the set of all Lebesgue measurable functions on $\mathbb{R}^n$ defined by

$$TF = \tilde{L}_F f, \quad F \in \mathcal{D}. \tag{2.10}$$

Then, by (2.8) and (2.9),

$$\|TF\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{c_\varphi}\|F\|_{L^1(G)}\|f\|_{L^2(\mathbb{R}^n)}$$

and

$$\|TF\|_{L^2(\mathbb{R}^n)} \leq \|F\|_{L^\infty(G)}\|f\|_{L^2(\mathbb{R}^n)}$$

for all functions F in $\mathcal{D}$. Thus, by Theorem 2.4,

$$\|TF\|_{L^2(\mathbb{R}^n)} \leq c_\varphi^{-\frac{1}{p}}\|F\|_{L^p(G)}\|f\|_{L^2(\mathbb{R}^n)}, \quad F \in \mathcal{D}. \tag{2.11}$$

Therefore, by (2.10) and (2.11),

$$\|\tilde{L}_F f\|_{L^2(\mathbb{R}^n)} \leq c_\varphi^{-\frac{1}{p}}\|F\|_{L^p(G)}\|f\|_{L^2(\mathbb{R}^n)}, \quad F \in \mathcal{D}. \tag{2.12}$$

Since (2.12) is true for arbitrary functions f in $L^2(\mathbb{R}^n)$, it follows that

$$|\tilde{L}_F|_* \leq c_\varphi^{-\frac{1}{p}}\|F\|_{L^p(G)}, \quad F \in \mathcal{D}. \tag{2.13}$$

Let $F \in L^p(G)$, $1 < p < \infty$. Then there exists a sequence $\{F_k\}$ of functions in $\mathcal{D}$ such that $F_k \to F$ in $L^p(G)$ as $k \to \infty$. By (2.13),

$$|\tilde{L}_{F_k} - \tilde{L}_{F_j}|_* \leq c_\varphi^{-\frac{1}{p}}\|F_k - F_j\|_{L^p(G)} \to 0$$

as k, $j \to \infty$. Therefore $\{\tilde{L}_{F_k}\}$ is a Cauchy sequence in $B(L^2(\mathbb{R}^n))$. Using the completeness of $B(L^2(\mathbb{R}^n))$, we can find a bounded linear operator $\tilde{L}_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\tilde{L}_{F_k} \to \tilde{L}_F$ as $k \to \infty$. Since each $\tilde{L}_{F_k}$ satisfies (2.13), it follows that $\tilde{L}_F$ also satisfies (2.13). Thus, the linear operator $L_F : X \to X$, where

$$L_F = U^{-1}\tilde{L}_F U,$$

is a bounded linear operator satisfying the conclusions of the theorem if $F \in L^p(G)$, $1 < p < \infty$. To prove uniqueness, let $F \in L^p(G)$, $1 < p < \infty$, and suppose that $P_F : X \to X$ is another bounded linear operator satisfying the conclusions of the theorem. Let $Q : L^p(G) \to B(X)$ be the linear operator defined by

$$QF = L_F - P_F, \quad F \in L^p(G).$$

Then, by (2.6),

$$\|QF\|_* \leq 2c_\varphi^{-\frac{1}{p}}\|F\|_{L^p(G)}, \quad F \in L^p(G).$$

Furthermore, QF is equal to the zero operator on X for all F in $\mathcal{D}$. Thus, $Q : L^p(G) \to B(X)$ is a bounded linear operator that is equal to zero on the dense subspace $\mathcal{D}$ of $L^p(G)$. Therefore $P_F = L_F$ for all functions F in $L^p(G)$.

Remark 2.5 The bounded linear operators $L_F : X \to X$ introduced in this chapter are dubbed localization operators. The impetus for the terminology stems from the simple observation that if $F(g) = 1$ for all g in G, then the resolution of the identity formula, i.e., (1.3), implies that the corresponding linear operator is simply the identity operator on X. Thus, in general, the function F is there to localize on G so as to produce a nontrivial bounded linear operator on X with various applications in mathematical sciences. The results in this chapter can be found in Sections 2 and 3 of the paper [10] by He and Wong.

3 The Schatten-von Neumann Property

We prove in this chapter that a localization operator $L_F : X \to X$ associated to a function F in $L^p(G)$, $1 \leq p \leq \infty$, is in the Schatten-von Neumann class S_p, $1 \leq p \leq \infty$. We begin with a recall of the definition of the Schatten-von Neumann class S_p, $1 \leq p \leq \infty$, and some of its basic properties that we need in this chapter.

Let A be a compact operator from the separable and complex Hilbert space X into X. If we denote by $A^* : X \to X$ the adjoint of $A : X \to X$, then the linear operator $(A^*A)^{\frac{1}{2}} : X \to X$ is positive and compact. Let $\{\psi_k : k = 1, 2, \cdots\}$ be an orthonormal basis for X consisting of eigenvectors of $(A^*A)^{\frac{1}{2}} : X \to X$, and let $s_k(A)$ be the eigenvalue of $(A^*A)^{\frac{1}{2}} : X \to X$ corresponding to the eigenvector ψ_k, $k = 1, 2, \cdots$. We say that the compact operator $A : X \to X$ is in the Schatten-von Neumann class S_p, $1 \leq p < \infty$, if

$$\sum_{k=1}^{\infty} s_k(A)^p < \infty,$$

and we call $s_k(A)$, $k = 1, 2, \cdots$, the singular values of A. It can be shown that S_p, $1 \leq p < \infty$, is a Banach space in which the norm $\| \, \|_{S_p}$ is given by

$$\|A\|_{S_p} = \left\{ \sum_{k=1}^{\infty} s_k(A)^p \right\}^{\frac{1}{p}}, \quad A \in S_p.$$

We let S_∞ be the C^*-algebra $B(X)$ of all bounded linear operators from X into X.

The following properties of S_1 and S_∞ are well-known.

Proposition 3.1 *Let $A : X \to X$ be a bounded linear operator such that*

$$\sum_{k=1}^{\infty} |(A\varphi_k, \varphi_k)| < \infty$$

for all orthonormal bases $\{\varphi_k : k = 1, 2, \cdots\}$ *for* X. *Then* $A : X \to X$ *is in* S_1.

Proposition 3.2 *Let* $A : X \to X$ *be a compact operator and let* $s_k(A)$, $k = 1, 2, \cdots$, *be its singular values. Then*

$$\sup_{1 \le k \le \infty} |s_k(A)| = \|A\|_*.$$

Further properties of S_p, $1 \le p \le \infty$, can be found in Reed and Simon [16], Simon [17] and Zhu [24], among others.

The first and foremost result on the Schatten-von Neumann property of localization operators is given in the following proposition.

Proposition 3.3 *Let* $F \in L^1(G)$. *Then the localization operator* $L_F : X \to X$ *is in* S_1 *and*

$$\|L_F\|_{S_1} \le \frac{4}{c_\varphi} \|F\|_{L^1(G)}. \tag{3.1}$$

Proof: Let $\{\varphi_k : k = 1, 2, \cdots\}$ be any orthonormal basis for X. Then, using (2.1) and the fact that $\pi : G \to B(X)$ is a representation, we get

$$\begin{aligned} \sum_{k=1}^{\infty} |(L_F \varphi_k, \varphi_k)| &= \sum_{k=1}^{\infty} \left| \frac{1}{c_\varphi} \int_G F(g)(\varphi_k, \pi(g)\varphi)(\pi(g)\varphi, \varphi_k) d\mu(g) \right| \\ &\le \sum_{k=1}^{\infty} \frac{1}{c_\varphi} \int_G |F(g)||(\varphi_k, \pi(g)\varphi)|^2 d\mu(g). \end{aligned} \tag{3.2}$$

Hence, using (3.2), Fubini's theorem, the Parseval identity, $\|\varphi\| = 1$ and the fact that $\pi(g) : X \to X$ is unitary for all g in G, we get

$$\sum_{k=1}^{\infty} |(L_F \varphi_k, \varphi_k)| \le \frac{1}{c_\varphi} \|F\|_{L^1(G)} < \infty. \tag{3.3}$$

Hence, by Proposition 3.1, the localization operator $L_F : X \to X$ is in S_1. To prove the estimate (3.1), let $F \in L^1(G)$ be a nonnegative function. Then

$$(L_F^* L_F)^{\frac{1}{2}} = L_F. \tag{3.4}$$

Thus, if $\{\psi_k : k = 1, 2, \cdots\}$ is an orthonormal basis for X consisting of eigenvectors of $(L_F^* L_F)^{\frac{1}{2}} : X \to X$, we have, by (3.3) and (3.4),

$$\begin{aligned} \|L_F\|_{S_1} &= \sum_{k=1}^{\infty} ((L_F^* L_F)^{\frac{1}{2}} \psi_k, \psi_k) \\ &= \sum_{k=1}^{\infty} (L_F \psi_k, \psi_k) \\ &\leq \frac{1}{c_\varphi} \|F\|_{L^1(G)}. \end{aligned} \tag{3.5}$$

Now, if $F \in L^1(G)$ is a real-valued function, then we write $F = F_+ - F_-$, where

$$F_+(g) = \max(F(g), 0)$$

and

$$F_-(g) = -\min(F(g), 0)$$

for all g in G. Then, by (3.5),

$$\begin{aligned} \|L_F\|_{S_1} &= \|L_{F_+} - L_{F_-}\|_{S_1} \leq \|L_{F_+}\|_{S_1} + \|L_{F_-}\|_{S_1} \\ &\leq \frac{1}{c_\varphi} \left(\|F_+\|_{L^1(G)} + \|F_-\|_{L^1(G)} \right) \\ &\leq \frac{2}{c_\varphi} \|F\|_{L^1(G)}. \end{aligned} \tag{3.6}$$

Finally, let $F \in L^1(G)$ be a complex-valued function. Then we write $F = F_1 + iF_2$, where F_1 and F_2 are the real and imaginary parts of F respectively. Then, by (3.6),

$$\begin{aligned} \|L_F\|_{S_1} &= \|L_{F_1} + iL_{F_2}\|_{S_1} \leq \|L_{F_1}\|_{S_1} + \|L_{F_2}\|_{S_1} \\ &\leq \frac{2}{c_\varphi} \left(\|F_1\|_{L^1(G)} + \|F_2\|_{L^1(G)} \right) \\ &\leq \frac{4}{c_\varphi} \|F\|_{L^1(G)}, \end{aligned}$$

and the proof of Proposition 3.3 is complete. □

A consequence of Proposition 3.3 is the following result.

Proposition 3.4 *Let $F \in L^p(G)$, $1 \leq p < \infty$. Then the localization operator $L_F : X \to X$ is compact.*

Proof: We again denote by $\mathcal{D}$ the set of all simple functions F on G such that

$$\mu\{g \in G : F(g) \neq 0\} < \infty.$$

Let $\{F_k\}$ be a sequence of functions in $\mathcal{D}$ such that $F_k \to F$ in $L^p(G)$ as $k \to \infty$. Then, by (2.8),

$$||L_{F_k} - L_F||_* \leq c_\varphi^{-\frac{1}{p}} ||F_k - F||_{L^p(G)} \to 0$$

as $k \to \infty$, i.e., $L_{F_k} \to L_F$ in $B(X)$ as $k \to \infty$. Since, by Proposition 3.3, $L_{F_k} : X \to X$ is in S_1 and hence compact, it follows that $L_F : X \to X$ is compact. □

Remark 3.5 That Proposition 3.4 is false for $p = \infty$ can be seen easily by taking the function F on G to be such that

$$F(g) = 1, \quad g \in G.$$

For then, by the resolution of the identity formula (1.3), $L_F : X \to X$ is the identity operator on X. In view of the hypothesis that X is infinite dimensional, $L_F : X \to X$ is not compact.

The next task is to prove that a localization operator $L_F : X \to X$, where $F \in L^p(G)$, $1 \leq p \leq \infty$, is in the Schatten-von Neumann class S_p, $1 \leq p \leq \infty$. To do this, we need the theory of complex interpolation, which we now recall. We omit the proofs, which can be found in Sections 2.1 and 2.2 of the book [24] by Zhu.

Let B_0 and B_1 be Banach spaces in which the norms, respectively, are denoted by $|| \; ||_{B_0}$ and $|| \; ||_{B_1}$. We say that B_0 and B_1 are compatible if there is a vector space V such that $B_0 \subseteq V$ and $B_1 \subseteq V$. If this is the case, then the subspaces $B_0 \cap B_1$ and $B_0 + B_1$ of V are Banach spaces when equipped with the norms $|| \; ||_{B_0 \cap B_1}$ and $|| \; ||_{B_0+B_1}$ given by

$$||v||_{B_0 \cap B_1} = \max_{k=0,1} ||v||_{B_k}$$

for all v in $B_0 \cap B_1$, and

$$\|v\|_{B_0+B_1} = \inf\{\|b_0\|_{B_0} + \|b_1\|_{B_1} : v = b_0 + b_1,\ b_0 \in B_0,\ b_1 \in B_1\}$$

for all v in $B_0 + B_1$, respectively.

Let B_0 and B_1 be compatible Banach spaces. A Banach space B is called an intermediate space between B_0 and B_1 if

$$B_0 \cap B_1 \subseteq B \subseteq B_0 + B_1,$$

where the inclusions are continuous. An intermediate space B between B_0 and B_1 is said to be an interpolation space between B_0 and B_1 if any bounded linear operator on $B_0 + B_1$ which is bounded from B_k into B_k, $k = 0, 1$, is also bounded from B into B.

Let $S = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and let B be any Banach space. A function $f : S \to B$ is said to be analytic on S if for any bounded linear functional b' on B, the complex-valued function $b' \circ f : S \to \mathbb{C}$ is analytic on S.

Let B_0 and B_1 be compatible Banach spaces. Then we define $\mathcal{F}(B_0, B_1)$ to be the set of all bounded and continuous functions f from the closure $\bar{S}$ of S into $B_0 + B_1$ such that f is analytic on S, and the mappings

$$y \mapsto f(k + iy), \quad k = 0, 1,$$

are continuous from $\mathbb{R}$ into B_k, $k = 0, 1$. Then it can be shown that $\mathcal{F}(B_0, B_1)$ is a Banach space with respect to the norm $\|\ \|_{\mathcal{F}}$ given by

$$\|f\|_{\mathcal{F}} = \max_{k=0,1} \sup_{y \in \mathbb{R}} \|f(k + iy)\|_{B_k}, \quad f \in \mathcal{F}(B_0, B_1).$$

For any number θ in $[0, 1]$, we let B_θ be the subspace of B_0+B_1 consisting of all elements b in $B_0 + B_1$ such that $b = f(\theta)$ for some f in $\mathcal{F}(B_0, B_1)$. Then we can show that B_θ is a Banach space with respect to the norm $\|\ \|_{B_\theta}$ given by

$$\|b\|_{B_\theta} = \inf_{b=f(\theta)} \|f\|_{\mathcal{F}}, \quad b \in B_\theta,$$

and B_θ is an interpolation space between B_0 and B_1. We denote B_θ by $[B_0, B_1]_\theta$.

The following two results on interpolation spaces will be useful to us.

Theorem 3.6 *Let B_0, B_1 and $\tilde{B}_0$, $\tilde{B}_1$ be two pairs of compatible Banach spaces. Let A be a bounded linear operator from $B_0 + B_1$ into $\tilde{B}_0 + \tilde{B}_1$ such that A is a bounded linear operator from B_k into $\tilde{B}_k$ with norm $\leq M_k$, $k = 0, 1$. Then, for any number θ in $(0, 1)$, A is a bounded linear operator from $[B_0, B_1]_\theta$ into $[\tilde{B}_0, \tilde{B}_1]_\theta$ with norm $\leq M_0^{1-\theta} M_1^\theta$.*

Theorem 3.7 *For $1 \leq p \leq \infty$,*

$$[L^1(G), L^\infty(G)]_{\frac{1}{p'}} = L^p(G)$$

and

$$[S_1, S_\infty]_{\frac{1}{p'}} = S_p,$$

where p' is the conjugate index of p.

Now, we can come to the main result on the Schatten-von Neumann property of localization operators.

Theorem 3.8 *Let $F \in L^p(G)$, $1 \leq p \leq \infty$. Then the localization operator $L_F : X \to X$ is in S_p and*

$$||L_F||_{S_p} \leq \left(\frac{4}{c_\varphi}\right)^{\frac{1}{p}} ||F||_{L^p(G)}.$$

Proof: By Proposition 3.3,

$$||L_F||_{S_1} \leq \frac{4}{c_\varphi} ||F||_{L^1(G)}, \quad F \in L^1(G), \tag{3.7}$$

and, by Propositions 2.2 and 3.2,

$$||L_F||_{S_\infty} = ||L_F||_* \leq ||F||_{L^\infty(G)}, \quad F \in L^\infty(G). \tag{3.8}$$

So, by (3.7), (3.8), Theorems 3.6 and 3.7, the proof is complete. □

Remark 3.9 This chapter is an exposition of the results in Sections 4–6 of the paper [10] by He and Wong.

4 The Trace

In this chapter, we compute the trace of a localization operator $L_F : X \to X$ and estimate the trace of any integral power of $L_F : X \to X$ for any function F in $L^1(G)$.

The starting point is the following proposition, which supplements Proposition 3.1.

Proposition 4.1 *Let $A : X \to X$ be a bounded linear operator in S_1 and let $\{\varphi_k : k = 1, 2, \cdots\}$ be any orthonormal basis for X. Then the series $\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k)$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\varphi_k : k = 1, 2, \cdots\}$.*

A proof of Proposition 4.1 can be found on, say, page 211 of the book [16] by Reed and Simon.

In view of Proposition 4.1, we can define the trace $\mathrm{tr}(A)$ of any linear operator $A : X \to X$ in S_1 by

$$\mathrm{tr}(A) = \sum_{k=1}^{\infty}(A\varphi_k, \varphi_k), \tag{4.1}$$

where $\{\varphi_k : k = 1, 2, \cdots\}$ is any orthonormal basis for X.

Let $F \in L^1(G)$. Then the localization operator $L_F : X \to X$ is in S_1 and we have the following interesting result on its trace.

Theorem 4.2 $\mathrm{tr}(L_F) = \frac{1}{c_\varphi}\int_G F(g)d\mu(g)$.

Proof: Let $\{\varphi_k : k = 1, 2, \cdots\}$ be an orthonormal basis for X. Then, using (2.1), (4.1), Fubini's theorem, the Parseval identity, $||\varphi|| = 1$ and the fact that $\pi(g) : X \to X$ is a unitary operator for all g in G, we get

$$\mathrm{tr}(L_F) = \sum_{k=1}^{\infty}(L_F\varphi_k, \varphi_k)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{c_\varphi} \int_G F(g) |(\varphi_k, \pi(g)\varphi)|^2 d\mu(g) \\
&= \frac{1}{c_\varphi} \int_G F(g) \sum_{k=1}^{\infty} |(\varphi_k, \pi(g)\varphi)|^2 d\mu(g) \\
&= \frac{1}{c_\varphi} \int_G F(g) \|\pi(g)\varphi\|^2 d\mu(g) \\
&= \frac{1}{c_\varphi} \int_G F(g) d\mu(g).
\end{aligned}
$$

□

Since S_1 is an ideal in the C^*-algebra $B(X)$ of all bounded linear operators from X into X, it follows that, for $m = 1, 2, \cdots$, the linear operator $L_F^m : X \to X$ is in S_1. The following theorem gives an estimate for the trace $\mathrm{tr}(L_F^m)$ of $L_F^m : X \to X$ in terms of the trace $\mathrm{tr}(L_{|F|})$ of the localization operator $L_{|F|} : X \to X$.

Theorem 4.3 *For* $m = 1, 2, \cdots$,

$$|\mathrm{tr}(L_F^m)| \leq (\mathrm{tr}(L_{|F|}))^m.$$

To prove Theorem 4.3, we need the following lemma, which is an immediate consequence of (2.1).

Lemma 4.4 $L_F^* = L_{\bar{F}}$.

Proof of Theorem 4.3: Let $\{\varphi_k : k = 1, 2, \cdots\}$ be an orthonormal basis for X. Then, by (2.1), Proposition 2.1, (4.1), Lemma 4.4, Fubini's theorem, the Schwarz inequality, the Parseval identity, $\|\varphi\| = 1$ and the fact that $\pi(g) : X \to X$ is a unitary operator for all g in G, we get

$$
\begin{aligned}
&|\mathrm{tr}(L_F^m)| \\
= & \left| \sum_{k=1}^{\infty} (L_F^m \varphi_k, \varphi_k) \right| \\
= & \left| \sum_{k=1}^{\infty} (L_F \varphi_k, L_{\bar{F}}^{m-1} \varphi_k) \right|
\end{aligned}
$$

$$\begin{aligned}
&= \left|\sum_{k=1}^{\infty}\frac{1}{c_\varphi}\int_G F(g)(\varphi_k,\pi(g)\varphi)(\pi(g)\varphi, L_F^{m-1}\varphi_k)d\mu(g)\right| \\
&= \left|\sum_{k=1}^{\infty}\frac{1}{c_\varphi}\int_G F(g)(\varphi_k,\pi(g)\varphi)(L_F^{m-1}\pi(g)\varphi, \varphi_k)d\mu(g)\right| \\
&\leq \frac{1}{c_\varphi}\int_G |F(g)|\left(\sum_{k=1}^{\infty}|(\varphi_k,\pi(g)\varphi)|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}|(L_F^{m-1}\pi(g)\varphi,\varphi_k)|^2\right)^{\frac{1}{2}} d\mu(g) \\
&= \frac{1}{c_\varphi}\int_G |F(g)|\,\|\pi(g)\varphi\|\,\|L_F^{m-1}\pi(g)\varphi\| d\mu(g) \\
&\leq \left(\frac{1}{c_\varphi}\int_G |F(g)|d\mu(g)\right)^m . \qquad (4.2)
\end{aligned}$$

Thus, by Theorem 4.2 and (4.2), the proof is complete.

A consequence of Theorem 4.3 is the following result on the norm of the localization operator $L_F : X \to X$ in the Schatten-von Neumann class S_p when F is a nonnegative function in $L^1(G)$ and p is a positive integer.

Corollary 4.5 *Let F be a nonnegative function in $L^1(G)$. Then, for $p = 1, 2, \cdots$,*

$$\|L_F\|_{S_p} \leq \operatorname{tr}(L_F).$$

Moreover,

$$\|L_F\|_{S_1} = \operatorname{tr}(L_F).$$

Proof: Since F is a nonnegative function in $L^1(G)$, it follows from (2.1) that $L_F : X \to X$ is a positive operator. Thus, the singular values of $L_F : X \to X$ coincide with the eigenvalues of $L_F : X \to X$. So,

$$\|L_F\|_{S_p} = \left(\sum_{k=1}^{\infty}(L_F^p\psi_k,\psi_k)\right)^{\frac{1}{p}}, \qquad (4.3)$$

where $\{\psi_k : k = 1, 2, \cdots\}$ is an orthonormal basis for X consisting of eigenvectors of $L_F : X \to X$. By (4.1), (4.3) and Theorem 4.3,

$$\|L_F\|_{S_p} = (\operatorname{tr}(L_F^p))^{\frac{1}{p}} \leq \operatorname{tr}(L_F)$$

for $p = 1, 2, \cdots$. If $p = 1$, then, by (4.1) and (4.3),

$$\|L_F\|_{S_1} = \text{tr}(L_F).$$

$\square$

Remark 4.6 The results in this chapter are taken from the paper [4] by Du and Wong

5 The Weyl-Heisenberg Group

We show in this chapter that localization operators on the Weyl-Heisenberg group are the same as the linear operators studied by Daubechies in the paper [1] on signal analysis. We begin with a detailed study of the Weyl-Heisenberg group.

Let $\mathbb{R}^n \times \mathbb{R}^n = \{(q,p) : q,\, p \in \mathbb{R}^n\}$ and let $\mathbb{Z}$ be the set of all integers. Let $(WH)^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}/2\pi\mathbb{Z}$. Then we define the binary operation $\cdot$ on $(WH)^n$ by

$$(q_1,\, p_1,\, t_1) \cdot (q_2,\, p_2,\, t_2) = (q_1 + q_2,\, p_1 + p_2,\, t_1 + t_2 + q_1 \cdot p_2) \qquad (5.1)$$

for all points (q_1, p_1, t_1) and (q_2, p_2, t_2) in $(WH)^n$, where $q_1 \cdot p_2$ is the Euclidean inner product of q_1 and p_2 in $\mathbb{R}^n$; t_1, t_2 and $t_1 + t_2 + q_1 \cdot p_2$ are cosets in the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ in which the group law is addition modulo 2π. It is easy to prove the following proposition and we omit the proof.

Proposition 5.1 *With respect to the multiplication $\cdot$ defined by (5.1), $(WH)^n$ is a non-abelian group in which $(0,0,0)$ is the identity element and the inverse element of $(q,\, p,\, t)$ is $(-q,\, -p,\, -t + q \cdot p)$ for all $(q,\, p,\, t)$ in $(WH)^n$.*

Remark 5.2 To simplify the notation a little bit, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$. Thus, $(WH)^n = \mathbb{C}^n \times \mathbb{R}/2\pi\mathbb{Z}$ which can also be identified with $\mathbb{C}^n \times [0, 2\pi] = \mathbb{R}^n \times \mathbb{R}^n \times [0, 2\pi]$.

Proposition 5.3 *The Lebesgue measure $dqdpdt$ on $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2\pi]$ is the left (and right) Haar measure on $(WH)^n$.*

Proof: To prove left invariance, let f be an integrable function on $(WH)^n$. It is helpful to think of f as a function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that $f(q, p, \cdot)$

is a periodic function with period 2π for fixed but arbitrary q and p in $\mathbb{R}^n$. Then, for all (z', t') in $(WH)^n$, we get

$$\begin{aligned}
&\int_{(WH)^n} f((z',t')\cdot(z,t))dzdt \\
= &\int_0^{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(q'+q,\, p'+p,\, t'+t+q'\cdot p)dqdpdt \\
= &\int_{t'+q'\cdot p}^{2\pi+t'+q'\cdot p}\int_{\mathbb{C}^n} f(z,s)dzds \\
= &\int_0^{2\pi}\int_{\mathbb{C}^n} f(z,s)dzds \\
= &\int_{(WH)^n} f(z,t)dzdt.
\end{aligned}$$

The proof for right invariance is similar. □

Remark 5.4 With respect to the multiplication $\cdot$ defined by (5.1), $(WH)^n$ is a locally compact and Hausdorff group in which the left (and right) Haar measure is the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n \times [0, 2\pi]$. We call $(WH)^n$ the Weyl-Heisenberg group. In light of the existence of a left (and right) Haar measure on $(WH)^n$, $(WH)^n$ is unimodular.

Let $\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ be the mapping defined by

$$(\pi(q,p,t)f)(x) = e^{i(p\cdot x - q\cdot p + t)} f(x-q), \quad x \in \mathbb{R}^n, \tag{5.2}$$

for all points (q, p, t) in $(WH)^n$ and all functions f in $L^2(\mathbb{R}^n)$.

Proposition 5.5 *$\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ is a representation of $(WH)^n$ on $L^2(\mathbb{R}^n)$.*

Proof: Let (q_1, p_1, t_1) and (q_2, p_2, t_2) be points in $(WH)^n$. Then, for all functions f in $L^2(\mathbb{R}^n)$, by (5.2),

$$\begin{aligned}
&(\pi(q_1,p_1,t_1)(\pi(q_2,p_2,t_2)f))(x) \\
= \;& e^{i(p_1\cdot x - q_1\cdot p_1 + t_1)}(\pi(q_2,p_2,t_2)f)(x-q_1) \\
= \;& e^{i(p_1\cdot x - q_1\cdot p_1 + t_1)} e^{i(p_2\cdot(x-q_1) - q_2\cdot p_2 + t_2)} f(x-q_2-q_1) \\
= \;& e^{i((p_1+p_2)\cdot x - (p_1+p_2)\cdot q_1 + t_1 + t_2 - q_2\cdot p_2)} f(x-(q_1+q_2))
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&(\pi((q_1,p_1,t_1)\cdot(q_2,p_2,t_2))f)(x)\\
&= (\pi(q_1+q_2,p_1+p_2,t_1+t_2+q_1\cdot p_2)f)(x)\\
&= e^{i((p_1+p_2)\cdot x-q_1\cdot p_1-q_2\cdot p_1-q_2\cdot p_2+t_1+t_2)}f(x-(q_1+q_2)) \qquad (5.4)
\end{aligned}$$

for all x in $\mathbb{R}^n$. Hence, by (5.3) and (5.4),

$$\pi(q_1,p_1,t_1)\pi(q_2,p_2,t_2)=\pi((q_1,p_1,t_1)\cdot(q_2,p_2,t_2))$$

for all points (q_1,p_1,t_1) and (q_2,p_2,t_2) in $(WH)^n$. It remains to prove that $\pi(q,p,t)f\to f$ in $L^2(\mathbb{R}^n)$ as $(q,p,t)\to(0,0,0)$ for all functions f in $L^2(\mathbb{R}^n)$. But

$$\begin{aligned}
&\|\pi(q,p,t)f-f\|^2_{L^2(\mathbb{R}^n)}\\
&= \int_{\mathbb{R}^n}\left|e^{i(p\cdot x-q\cdot p+t)}f(x-q)-f(x)\right|^2dx\\
&= \int_{\mathbb{R}^n}\left|e^{i(p\cdot x-q\cdot p+t)}\{f(x-q)-f(x)\}+\left\{e^{i(p\cdot x-q\cdot p+t)}-1\right\}f(x)\right|^2dx\\
&\le 2\int_{\mathbb{R}^n}|f(x-q)-f(x)|^2dx+2\int_{\mathbb{R}^n}\left|\left(e^{i(p\cdot x-q\cdot p+t)}-1\right)f(x)\right|^2dx \qquad (5.5)
\end{aligned}$$

for all (q,p,t) in $(WH)^n$ and all functions f in $L^2(\mathbb{R}^n)$. By the L^2-continuity of translations,

$$\int_{\mathbb{R}^n}|f(x-q)-f(x)|^2dx\longrightarrow 0 \qquad (5.6)$$

as $q\to 0$. For almost all x in $\mathbb{R}^n$,

$$\left|\left(e^{i(p\cdot x-q\cdot p+t)}-1\right)f(x)\right|^2\longrightarrow 0 \qquad (5.7)$$

as $(q,p,t)\to(0,0,0)$, and

$$\left|\left(e^{i(p\cdot x-q\cdot p+t)}-1\right)f(x)\right|^2\le 4|f(x)|^2. \qquad (5.8)$$

Hence, by (5.7), (5.8) and the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^n}\left|\left(e^{i(p\cdot x-q\cdot p+t)}-1\right)f(x)\right|^2dx\longrightarrow 0 \qquad (5.9)$$

as $(q, p, t) \to (0, 0, 0)$. Hence, by (5.5), (5.6) and (5.9),

$$\|\pi(q, p, t)f - f\|_{L^2(\mathbb{R}^n)} \longrightarrow 0$$

as $(q, p, t) \to (0, 0, 0)$, and the proof is complete. □

The following theorem gives us all the information that we want to know about the representation $\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$.

Theorem 5.6 *For all functions f and g in $L^2(\mathbb{R}^n)$, we have*

$$\int_{(WH)^n} |\langle f, \pi(z,t)g\rangle|^2 dz dt = (2\pi)^{n+1} \|f\|^2_{L^2(\mathbb{R}^n)} \|g\|^2_{L^2(\mathbb{R}^n)}, \tag{5.10}$$

where $\langle\ ,\ \rangle$ is the inner product in $L^2(\mathbb{R}^n)$.

Remark 5.7 The proof of Theorem 5.6 requires some basic knowledge of Fourier analysis, which we assume. Standard references include the books by Goldberg [6], Stein and Weiss [21] and Wong [23]. Notwithstanding these comments, it is essential to make note that the Fourier transform $\hat{f}$ of a function f in $L^1(\mathbb{R}^n)$ that we adopt is the one given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

Proof of Theorem 5.6: We begin with the case when both f and g are in the Schwartz space $\mathcal{S}$ of functions on $\mathbb{R}^n$. If we denote the left hand side of (5.10) by $I(f, g)$, then

$$\begin{aligned}
I(f,g) &= \int_0^{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x) e^{-i(p\cdot x - q\cdot p + t)} \overline{g(x-q)} dx \right|^2 dq dp dt \\
&= 2\pi \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-ip\cdot x} f(x) \overline{g(x-q)} dx \right|^2 dq dp \\
&= (2\pi)^{n+1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ip\cdot x} f(x) (T_{-q}\bar{g})(x) dx \right|^2 dq dp \\
&= (2\pi)^{n+1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(fT_{-q}\bar{g})^\wedge(p) dx|^2 dq dp, \qquad (5.11)
\end{aligned}$$

where $(T_{-q}\bar{g}) = \bar{g}(x-q)$, x, $q \in \mathbb{R}^n$. So, by (5.11), Plancherel's theorem and Fubini's theorem,

$$\begin{aligned} I(f,g) &= (2\pi)^{n+1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)g(x-q)|^2 dx dp \\ &= (2\pi)^{n+1} \int_{\mathbb{R}^n} |f(x)|^2 \left(\int_{\mathbb{R}^n} |g(x-q)|^2 dp \right) dx \\ &= (2\pi)^{n+1} \|f\|^2_{L^2(\mathbb{R}^n)} \|g\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Now, for f, $g \in L^2(\mathbb{R}^n)$, let $\{f_k\}$ and $\{g_k\}$ be sequences in $\mathcal{S}$ such that $f_k \to f$ in $L^2(\mathbb{R}^n)$ and $g_k \to g$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. Then, by what we have just shown,

$$I(f_k, g_k) \longrightarrow (2\pi)^{n+1} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \tag{5.12}$$

as $k \to \infty$. Also,

$$\langle f_k, \pi(z,t) g_k \rangle \longrightarrow \langle f, \pi(z,t) g \rangle \tag{5.13}$$

for all (z,t) in $(WH)^n$ as $k \to \infty$. Furthermore, for all k and j,

$$\begin{aligned} & |\langle f_k, \pi(x,t) g_k \rangle - \langle f_j, \pi(z,t) g_j \rangle|^2 \\ = \; & |\langle f_k - f_j, \pi(z,t) g_k \rangle + \langle f_j, \pi(z,t)(g_k - g_j) \rangle|^2 \\ \leq \; & 2|\langle f_k - f_j, \pi(z,t) g_k \rangle|^2 + 2|\langle f_j, \pi(z,t)(g_k - g_j) \rangle|^2, \end{aligned}$$

and hence, using (5.10) for functions in $\mathcal{S}$, we get a positive constant C such that

$$\begin{aligned} & \int_{(WH)^n} |\langle f_k, \pi(z,t) g_k \rangle - \langle f_j, \pi(z,t) g_j \rangle|^2 dz dt \\ \leq \; & C \left(\|f_k - f_j\|^2_{L^2(\mathbb{R}^n)} + \|g_k - g_j\|^2_{L^2(\mathbb{R}^n)} \right) \longrightarrow 0 \end{aligned}$$

as k, $j \to \infty$. So,

$$\langle f_k, \pi(\cdot\,,\cdot) g_k \rangle \longrightarrow h \tag{5.14}$$

for some h in $L^2((WH)^n)$ as $k \to \infty$. Therefore there exists a subsequence of $\{\langle f_k, \pi(\cdot\,,\cdot) g_k \rangle\}$, again denoted by $\{\langle f_k, \pi(\cdot\,,\cdot) g_k \rangle\}$, such that

$$\langle f_k, \pi(\cdot\,,\cdot) g_k \rangle \longrightarrow h \tag{5.15}$$

a.e. on $(WH)^n$ as $k \to \infty$. Thus, by (5.13)–(5.15),

$$I(f_k, g_k) \longrightarrow \int_{(WH)^n} |\langle f, \pi(z,t)g\rangle|^2 dzdt \tag{5.16}$$

as $k \to \infty$. Hence, by (5.12) and (5.16), the proof is complete.

Corollary 5.8 *$\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ is an irreducible and unitary representation of $(WH)^n$ on $L^2(\mathbb{R}^n)$.*

Proof: That $\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ is a unitary representation of $(WH)^n$ on $L^2(\mathbb{R}^n)$ is an immediate consequence of (5.2). Let M be a nonzero and closed subspace of $L^2(\mathbb{R}^n)$ which is invariant with respect to the representation $\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$. Let g be a nonzero function in M. Then

$$\{\pi(z,t)g : (z,t) \in (WH)^n\} \subseteq M. \tag{5.17}$$

Let $f \in L^2(\mathbb{R}^n)$ be such that f is orthogonal to M. Then, by (5.17),

$$\langle f, \pi(z,t)g\rangle = 0, \quad (z,t) \in (WH)^n. \tag{5.18}$$

Then, by Theorem 5.6 and (5.18),

$$||f||_{L^2(\mathbb{R}^n)}||g||_{L^2(\mathbb{R}^n)} = 0$$

and hence $f = 0$. So, M is a dense subspace of $L^2(\mathbb{R}^n)$. Since M is also a closed subspace of $L^2(\mathbb{R}^n)$, it follows that $M = L^2(\mathbb{R}^n)$ and the proof is complete. □

Corollary 5.9 *$\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ is a square-integrable representation of $(WH)^n$ on $L^2(\mathbb{R}^n)$.*

Proof: Let φ be any nonzero function in $L^2(\mathbb{R}^n)$. Then, by Theorem 5.6,

$$\int_{(WH)^n} |\langle \varphi, \pi(z,t)\varphi\rangle|^2 dzdt = (2\pi)^{n+1}||\varphi||^4_{L^2(\mathbb{R}^n)} < \infty, \tag{5.19}$$

and this completes the proof. □

Corollary 5.10 *Every function φ in $L^2(\mathbb{R}^n)$ with $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$ is an admissible wavelet for the representation $\pi : (WH)^n \to B(L^2(\mathbb{R}^n))$ of $(WH)^n$ on $L^2(\mathbb{R}^n)$ and*

$$c_\varphi = (2\pi)^{n+1}. \tag{5.20}$$

Corollary 5.10 is an immediate consequence of (5.19).

We can now study localization operators on the Weyl-Heisenberg group $(WH)^n$. To this end, let φ be any function in $L^2(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$, and let F be any function in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ or in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then, by (5.2) and (5.20), the localization operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is given by

$$\begin{aligned} \langle L_F f, g\rangle &= \frac{1}{c_\varphi}\int_0^{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(q,p)\langle f, \pi(q,p,t)\varphi\rangle\langle \pi(q,p,t)\varphi, g\rangle dq dp dt \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(q,p)\langle f, \varphi_{q,p}\rangle\langle \varphi_{q,p}, g\rangle dq dp \end{aligned} \tag{5.21}$$

for all functions f and g in $L^2(\mathbb{R}^n)$, where $\varphi_{q,p}$ is the function on $\mathbb{R}^n$ given by

$$\varphi_{q,p}(x) = e^{ip\cdot x}\varphi(x-q), \quad x \in \mathbb{R}^n, \tag{5.22}$$

for all q and p in $\mathbb{R}^n$. The localization operator $L_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is then exactly the same as the linear operator $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$\langle D_F f, g\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(q,p)\langle f, \varphi_{q,p}\rangle\langle \varphi_{q,p}, g\rangle dq dp \tag{5.23}$$

for all functions f and g in $L^2(\mathbb{R}^n)$, where $\varphi_{q,p}$ is the function defined by (5.22). The linear operator $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the localization operator first studied in the paper [1] by Daubechies in the context of signal analysis, and hence we call $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the Daubechies operator associated to the function F. See also Section 2.8 of the book [2] by Daubechies in this connection. By (5.21), (5.23) and Theorem 3.8, we have the following result.

Theorem 5.11 *Let $F \in L^p(\mathbb{R}^n \times \mathbb{R}^n)$, $1 \le p \le \infty$. Then there exists a unique linear operator $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in S_p such that*

$$\|D_F\|_{S_p} \le 4^{\frac{1}{p}}(2\pi)^{-\frac{n}{p}}\|F\|_{L^p(\mathbb{R}^n\times\mathbb{R}^n)} \tag{5.24}$$

and, for all functions f and g in $L^2(\mathbb{R}^n)$, $\langle D_F f, g\rangle$ is given by (5.23) for all simple functions F on $\mathbb{R}^n \times \mathbb{R}^n$ such that the Lebesgue measure of the set $\{(q,p) \in \mathbb{R}^n \times \mathbb{R}^n : F(q,p) \neq 0\}$ is finite.

Proof: We only need to check the inequality (5.24). But, by (5.20), (5.21), (5.23) and Theorem 3.8,

$$\|D_F\|_{S_p} = \|L_F\|_{S_p} \leq 4^{\frac{1}{p}}(2\pi)^{-\frac{n+1}{p}}\|F\|_{L^p((WH)^n)}. \tag{5.25}$$

But, by a simple computation,

$$\|F\|_{L^p((WH)^n)} = (2\pi)^{\frac{1}{p}}\|F\|_{L^p(\mathbb{R}^n\times\mathbb{R}^n)}. \tag{5.26}$$

Thus, by (5.25) and (5.26), (5.24) follows. □

As a sharp contrast to the Weyl-Heisenberg group $(WH)^n$, we end this chapter by showing that the Heisenberg group H^n introduced in Chapter 8 of the book [21] by Wong is one on which every irreducible and unitary representation of H^n on $L^2(\mathbb{R}^n)$ is not square-integrable, or equivalently, the set $AW(\pi)$ of all admissible wavelets for any irreducible and unitary representation $\pi : H^n \to B(L^2(\mathbb{R}^n))$ of H^n on $L^2(\mathbb{R}^n)$ is empty.

The Heisenberg group H^n is the non-abelian group $\mathbb{C}^n \times \mathbb{R}$ in which the group law $\cdot$ is given by

$$(z,t)\cdot(w,s) = (z+w, t+s+2\mathrm{Im} z\cdot\bar{w})$$

for all (z,t) and (w,s) in $\mathbb{C}^n \times \mathbb{R}$, where

$$z\cdot\bar{w} = \sum_{j=1}^{n} z_j\bar{w}_j.$$

The Heisenberg group H^n is a unimodular group on which the left (and right) Haar measure is the Lebesgue measure $dzdt$ on $\mathbb{C}^n \times \mathbb{R}$.

According to the Stone-von Neumann theorem, every irreducible and unitary representation $\pi : H^n \to B(L^2(\mathbb{R}^n))$ of H^n on $L^2(\mathbb{R}^n)$ is, up to unitary equivalence, given by

$$(\pi(z,t)f)(x) = e^{i\lambda(q\cdot x+\frac{1}{2}q\cdot p+\frac{1}{4}t)}f(x+p), \quad x \in \mathbb{R}^n, \tag{5.27}$$

or

$$(\pi(z,t)f)(x) = e^{i(\alpha\cdot q+\beta\cdot p)}f(x), \quad x \in \mathbb{R}^n, \tag{5.28}$$

for all functions f in $L^2(\mathbb{R}^n)$, where $\lambda \in \mathbb{R}$, $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(z,t) = (q,p,t)$.

Theorem 5.12 *Every irreducible and unitary representation of H^n on $L^2(\mathbb{R}^n)$ is not square-integrable.*

Proof: Let $\pi : H^n \to B(L^2(\mathbb{R}^n))$ be an irreducible and unitary representation of H^n on $L^2(\mathbb{R}^n)$. Suppose $\pi : H^n \to B(L^2(\mathbb{R}^n))$ is given by (5.27). Then, for all φ in $L^2(\mathbb{R}^n)$,

$$\int_{-\infty}^{\infty}\int_{\mathbb{C}^n} |\langle\varphi,\pi(z,t)\varphi\rangle|^2 dz dt = \int_{-\infty}^{\infty}\left(\int_{\mathbb{C}^n} |\langle\varphi,\varphi_{\lambda,q,p}\rangle|^2 dq dp\right) dt,$$

where

$$\varphi_{\lambda,q,p}(x) = e^{i\lambda q\cdot x}\varphi(x+p), \quad x \in \mathbb{R}^n.$$

Thus,

$$\int_{-\infty}^{\infty}\int_{\mathbb{C}^n} |\langle\varphi,\pi(z,t)\varphi\rangle|^2 dz dt = \infty$$

unless

$$\langle\varphi,\varphi_{\lambda,q,p}\rangle = 0, \quad q,\, p \in \mathbb{R}^n,$$

or equivalently,

$$\int_{-\infty}^{\infty} \varphi(x)e^{-i\lambda q\cdot x}\bar{\varphi}(x+p)dx = 0, \quad q,\, p \in \mathbb{R}^n. \tag{5.29}$$

But (5.29) is valid if and only if

$$\varphi(x)\bar{\varphi}(x+p) = 0$$

for almost all x and p in $\mathbb{R}^n$. Thus, $\varphi(x) = 0$ for almost all x in $\mathbb{R}^n$, Indeed, if $\varphi(x) \neq 0$ for all x in a set S with positive measure. Then, for all x in S, $\varphi(x+p) = 0$ for almost all p in $\mathbb{R}^n$, and this is a contradiction. Hence the representation $\pi : H^n \to B(L^2(\mathbb{R}^n))$ of H^n on $L^2(\mathbb{R}^n)$ is not square-integrable. If $\pi : H^n \to B(L^2(\mathbb{R}^n))$ is given by (5.28), then, for all φ in $L^2(\mathbb{R}^n)$,

$$\int_{-\infty}^{\infty}\int_{\mathbb{C}^n} |\langle\varphi,\pi(z,t)\varphi\rangle|^2 dz dt = \int_{-\infty}^{\infty}\int_{\mathbb{C}^n} |\langle\varphi,\varphi\rangle|^2 dx dt = \infty$$

unless $\varphi = 0$. Thus, $\pi : H^n \to B(L^2(\mathbb{R}^n))$ is not a square-integrable representation of H^n on $L^2(\mathbb{R}^n)$. □

Remark 5.13 This chapter is a detailed, expanded and improved account of the results in Chapter 15 of the book [22] by Wong. An account of the Weyl-Heisenberg group can be found on the paper [12] by Heil and Walnut.

6 The Affine Group

We study in this chapter the affine group U, the Hardy space $H^2_+(\mathbb{R})$, and an irreducible and unitary representation $\pi : U \to B(H^2_+(\mathbb{R}))$ of U on $H^2_+(\mathbb{R})$ for which the set $AW(\pi)$ of all admissible wavelets for the representation $\pi : U \to B(H^2_+(\mathbb{R}))$ is a proper subset of the unit sphere with center at the origin in $H^2_+(\mathbb{R})$.

Let U be the upper half plane given by

$$U = \{(b, a) : b \in \mathbb{R}, \ a > 0\}.$$

Then we define the binary operation $\cdot$ on U by

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2) \tag{6.1}$$

for all points (b_1, a_1) and (b_2, a_2) in U.

Proposition 6.1 *With respect to the multiplication $\cdot$ defined by (6.1), U is a non-abelian group in which $(0, 1)$ is the identity element and the inverse element of (b, a) is $\left(-\frac{b}{a}, \frac{1}{a}\right)$ for all (b, a) in U.*

Proof: Let (b_1, a_1) and (b_2, a_2) be points in U. Then, by (6.1),

$$\begin{aligned} ((b_1, a_1) \cdot (b_2, a_2)) \cdot (b_3, a_3) &= (b_1 + a_1 b_2, a_1 a_2) \cdot (b_3, a_3) \\ &= (b_1 + a_1 b_2 + a_1 a_2 b_3, a_1 a_2 a_3) \end{aligned}$$

and

$$\begin{aligned} (b_1, a_1) \cdot ((b_2, a_2) \cdot (b_3, a_3)) &= (b_1, a_1) \cdot (b_2 + a_2 b_3, a_2 a_3) \\ &= (b_1 + a_1 b_2 + a_1 a_2 b_3, a_1 a_2 a_3). \end{aligned}$$

Thus, the associative law is valid. For all (b, a) in U, by (6.1),

$$(b, a) \cdot (0, 1) = (b, a)$$

and

$$(0,1)\cdot(b,a)=(b,a).$$

Thus, $(0,1)$ is the identity element. Finally, let $(b,a)\in U$. Then, by (6.1),

$$(b,a)\cdot\left(-\frac{b}{a},\frac{1}{a}\right)=(0,1)$$

and

$$\left(-\frac{b}{a},\frac{1}{a}\right)\cdot(b,a)=(0,1).$$

Hence the inverse element of (b,a) is $\left(-\frac{b}{a},\frac{1}{a}\right)$. Therefore U is a group with respect to the multiplication $\cdot$ defined by (6.1). That the group U is non-abelian is easy to check and hence omitted. □

Proposition 6.2 *The left and right Haar measures on U are given by*

$$d\mu=\frac{dbda}{a^2}$$

and

$$d\nu=\frac{dbda}{a}$$

respectively.

Proof: To prove left invariance, let f be an integrable function on U with respect to $d\mu$. Then, for all (b',a') in U, we get

$$\int_U f((b',a')\cdot(b,a))d\mu=\int_0^\infty\int_{-\infty}^\infty f(b'+a'b,a'a)\frac{dbda}{a^2}. \qquad (6.2)$$

Let $\beta=b'+a'b$ and $\alpha=a'a$. Then, by (6.2),

$$\int_U f((b',a')\cdot(b,a))d\mu=\int_0^\infty\int_{-\infty}^\infty f(\beta,\alpha)\frac{d\beta d\alpha}{\alpha^2}=\int_U f(b,a)d\mu.$$

To prove right invariance, let f be an integrable function on U with respect to $d\nu$. Then, for all (b',a') in U, we get

$$\int_U f((b,a)\cdot(b',a'))d\nu=\int_0^\infty\int_{-\infty}^\infty f(b+ab',aa')\frac{dbda}{a}. \qquad (6.3)$$

Let $\beta = b + ab'$ and $\alpha = aa'$. Then, by (6.3),

$$\int_U f((b,a)\cdot(b',a'))d\nu = \int_0^\infty \int_{-\infty}^\infty f(\beta,\alpha)\frac{d\beta d\alpha}{\alpha} = \int_U f(b,a)d\nu.$$

□

Remark 6.3 With respect to the multiplication $\cdot$ defined by (6.1), U is a locally compact and Hausdorff group on which the left Haar measure is different from the right Haar measure. Thus, U is a non-unimodular group, which we call the affine group.

Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq [0,\infty)\},$$

where $\mathrm{supp}(\hat{f})$ is the set of every x in $\mathbb{R}$ for which there is no neighborhood of x on which $\hat{f}$ is equal to zero almost everywhere. Similarly, we define $H^2_-(\mathbb{R})$ to be the subspace of $L^2(\mathbb{R})$ by

$$H^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq (-\infty,0]\}.$$

We call $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ the Hardy space and the conjugate Hardy space respectively.

Proposition 6.4 *$H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are closed subspaces of $L^2(\mathbb{R})$.*

Proof: That $H^2_+(\mathbb{R})$ is a subspace of $L^2(\mathbb{R})$ is obvious. Let $\{f_k\}_{k=1}^\infty$ be a sequence in $H^2_+(\mathbb{R})$ such that $f_k \to f$ in $L^2(\mathbb{R})$ as $k \to \infty$. Then, by Plancherel's theorem,

$$\hat{f}_k \to \hat{f}$$

in $L^2(\mathbb{R})$ as $k \to \infty$. Then there exists a subsequence of $\{f_k\}_{k=1}^\infty$, again denoted by $\{f_k\}_{k=1}^\infty$, such that

$$\hat{f}_k \to \hat{f} \tag{6.4}$$

a.e. on $\mathbb{R}$ as $k \to \infty$. Using the definition of $H^2_+(\mathbb{R})$ and the definition of $\mathrm{supp}(f_k)$, we get $\hat{f}_k = 0$ a.e. on $(-\infty, 0]$ for $k = 1, 2, \cdots$. Thus, by (6.4),

$\hat{f} = 0$ a.e. on $(-\infty, 0]$. Hence $f \in H_+^2(\mathbb{R})$. Therefore $H_+^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$. The proof that $H_-^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$ is similar. □

To be specific, only the Hardy space $H_+^2(\mathbb{R})$ is considered. The discussion is equally valid for the conjugate Hardy space $H_-^2(\mathbb{R})$.

Let $\pi : U \to B(H_+^2(\mathbb{R}))$ be the mapping defined by

$$(\pi(b,a)f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}, \tag{6.5}$$

for all points (b, a) in U and all functions f in $H_+^2(\mathbb{R})$.

Proposition 6.5 $\pi : U \to B(H_+^2(\mathbb{R}))$ *is a representation of* U *on* $H_+^2(\mathbb{R})$.

To prove Proposition 6.5, we use the subspace W of $H_+^2(\mathbb{R})$ defined by

$$W = \{f \in H_+^2(\mathbb{R}) : \hat{f} \in C_0^\infty(0, \infty)\}.$$

Lemma 6.6 *W is a dense subspace of* $H_+^2(\mathbb{R})$.

Proof: Let $f \in H_+^2(\mathbb{R})$. Then $\operatorname{supp}(\hat{f}) \subseteq [0, \infty)$. Let $\{\varphi_k\}_{k=1}^\infty$ be a sequence of functions in $C_0^\infty(0, \infty)$ such that

$$\varphi_k \to f \tag{6.6}$$

in $L^2(\mathbb{R})$ as $k \to \infty$. For $k = 1, 2, \cdots$, let f_k be the function in $L^2(\mathbb{R})$ such that

$$\hat{f}_k = \varphi_k. \tag{6.7}$$

Then $f_k \in W$, $k = 1, 2, \cdots$, and, by (6.6), (6.7) and Plancherel's theorem, $f_k \to f$ in $L^2(\mathbb{R})$ as $k \to \infty$. Therefore W is a dense subspace of $H_+^2(\mathbb{R})$.

Proof of Proposition 6.5: Let (b_1, a_1), and (b_2, a_2) be points in U. Then, by (6.5), we get, for all functions f in $H_+^2(\mathbb{R})$,

$$\begin{aligned}(\pi(b_1,a_1)\pi(b_2,a_2)f)(x) &= \frac{1}{\sqrt{a_1}}(\pi(b_2,a_2)f)\left(\frac{x-b_1}{a_1}\right) \\ &= \frac{1}{\sqrt{a_1 a_2}} f\left(\frac{x-b_1-a_1 b_2}{a_1 a_2}\right)\end{aligned} \tag{6.8}$$

and

$$
\begin{aligned}
(\pi((b_1, a_1) \cdot (b_2, a_2))f)(x) &= (\pi(b_1 + a_1 b_2, a_1 a_2)f)(x) \\
&= \frac{1}{\sqrt{a_1 a_2}} f\left(\frac{x - b_1 - a_1 b_2}{a_1 a_2}\right) \qquad (6.9)
\end{aligned}
$$

for all x in $\mathbb{R}$. Hence, by (6.8) and (6.9), $\pi : U \to B(H_+^2(\mathbb{R}))$ is a group homomorphism. It remains to prove that $\pi(b,a)f \to f$ in $L^2(\mathbb{R})$ as $(b,a) \to (0,1)$ for all functions f in $H_+^2(\mathbb{R})$. But, by Plancherel's theorem and the elementary properties of the Fourier transform, we get, for all functions f in W,

$$
\begin{aligned}
\|\pi(b,a)f - f\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right) - f(x) \right|^2 dx \\
&= \int_{-\infty}^{\infty} |\sqrt{a} e^{-ib\xi} \hat{f}(a\xi) - \hat{f}(\xi)|^2 d\xi \\
&\leq 2 \int_{-\infty}^{\infty} |\sqrt{a} e^{-ib\xi} (\hat{f}(a\xi) - \hat{f}(\xi))|^2 d\xi \\
&\quad + 2 \int_{-\infty}^{\infty} |(\sqrt{a} e^{-ib\xi} - 1)\hat{f}(\xi)|^2 d\xi. \qquad (6.10)
\end{aligned}
$$

For all ξ in $\mathbb{R}$,

$$
|(\sqrt{a} e^{-ib\xi} - 1)\hat{f}(\xi)|^2 \longrightarrow 0 \qquad (6.11)
$$

as $(b,a) \to (0,1)$ and

$$
|(\sqrt{a} e^{-ib\xi} - 1)\hat{f}(\xi)|^2 \leq 9|\hat{f}(\xi)|^2 \qquad (6.12)
$$

for all b in $\mathbb{R}$ and all a in $(0,2)$. By (6.11), (6.12) and the Lebesgue dominated convergence theorem,

$$
\int_{-\infty}^{\infty} |(\sqrt{a} e^{-ib\xi} - 1)\hat{f}(\xi)|^2 d\xi \longrightarrow 0 \qquad (6.13)
$$

as $(b,a) \to (0,1)$. For all ξ in $\mathbb{R}$,

$$
|\hat{f}(a\xi) - \hat{f}(\xi)|^2 \longrightarrow 0 \qquad (6.14)
$$

as $a \to 1$, and

$$
|\hat{f}(a\xi) - \hat{f}(\xi)| \leq 2 \sup_{\xi \in \mathbb{R}} |\hat{f}(\xi)| \chi_R(\xi) \qquad (6.15)
$$

for all a in $\left(\frac{1}{2}, 2\right)$, where R is a fixed positive number such that

$$\hat{f}(\xi) = 0, \quad \xi > R,$$

and χ_R is the characteristic function on $[0, 2R]$. Thus, by (6.14), (6.15) and the Lebesgue dominated convergence theorem,

$$\int_{-\infty}^{\infty} |\sqrt{a}e^{-ib\xi}(\hat{f}(a\xi) - \hat{f}(\xi))|^2 d\xi \longrightarrow 0 \tag{6.16}$$

as $(b, a) \to (0, 1)$. So, by (6.10), (6.13) and (6.16),

$$\pi(b, a)f \to f \tag{6.17}$$

in $L^2(\mathbb{R})$ as $(b, a) \to (0, 1)$ for all f in W. Let $f \in H_+^2(\mathbb{R})$. Then, by Lemma 6.6, we can find a sequence $\{f_k\}$ of functions in W such that $f_k \to f$ in $L^2(\mathbb{R})$ as $k \to \infty$. Then, for any positive number ε, let k_0 be the positive integer such that

$$\|f_{k_0} - f\|_{L^2(\mathbb{R})} < \frac{2\varepsilon}{3}. \tag{6.18}$$

So, by (6.17), (6.18) and the obvious fact that $\pi(b, a) : H_+^2(\mathbb{R}) \to H_+^2(\mathbb{R})$ is a unitary operator for all (b, a) in U, there exists a positive number δ such that

$$\begin{aligned} \|\pi(b, a)f - f\|_{L^2(\mathbb{R})} &\leq \|\pi(b, a)(f - f_{k_0})\|_{L^2(\mathbb{R})} + \|\pi(b, a)f_{k_0} - f_{k_0}\|_{L^2(\mathbb{R})} \\ &\quad + \|f_{k_0} - f\|_{L^2(\mathbb{R})} \\ &< \varepsilon \end{aligned}$$

whenever (b, a) is within δ-distance of $(0, 1)$. Thus, $\pi(b, a)f \to f$ in $L^2(\mathbb{R})$ for all f in $H_+^2(\mathbb{R})$ as $(b, a) \to (0, 1)$ and the proof is complete. □

Proposition 6.7 $\pi : U \to B(H_+^2(\mathbb{R}))$ *is an irreducible and unitary representation of* U *on* $H_+^2(\mathbb{R})$.

Proof: That $\pi(b, a) : H_+^2(\mathbb{R}) \to H_+^2(\mathbb{R})$ is a unitary operator for all (b, a) in U is easy to check and has actually been used in the proof of Proposition 6.5. Let M be a nonzero and closed subspace of $H_+^2(\mathbb{R})$ such that M is invariant with respect to $\pi : U \to B(H^2_+(\mathbb{R}))$. Let g be a nonzero function in M. Then

$$\{\pi(b,a)g : (b,a) \in U\} \subseteq M.$$

Let $f \in H^2_+(\mathbb{R})$ be such that f is orthogonal to M. Then, for all points $(b,a) \in U$,

$$\int_{-\infty}^{\infty} f(x)\bar{g}\left(\frac{x-b}{a}\right) dx = 0,$$

and hence, by Plancherel's theorem,

$$\int_{-\infty}^{\infty} e^{ib\xi}\hat{f}(\xi)\overline{\hat{g}(a\xi)}d\xi = 0. \tag{6.19}$$

Thus, by (6.19),

$$\hat{f}(\xi)\overline{\hat{g}(a\xi)} = 0 \tag{6.20}$$

for almost all ξ in $\mathbb{R}$. Suppose $\hat{f}(\xi) \neq 0$ for all ξ in a set S with positive measure. Then, for all ξ in S, by (6.20), we get

$$\hat{g}(a\xi) = 0$$

for all positive numbers a. Thus, $\hat{g} = 0$ and hence $g = 0$. This is a contradiction. □

To get more information on the irreducible and unitary representation $\pi : U \to B(H^2_+(\mathbb{R}))$, we need the following subspace A of $H^2_+(\mathbb{R})$ given by

$$A = \left\{ f \in H^2_+(\mathbb{R}) : \int_0^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi} d\xi < \infty \right\}.$$

Theorem 6.8 *For all f in $H^2_+(\mathbb{R})$ and all g in A,*

$$\int_0^{\infty}\int_{-\infty}^{\infty} |\langle f, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} = 2\pi \int_0^{\infty} |\hat{f}(\xi)|^2 d\xi \int_0^{\infty} \frac{|\hat{g}(\xi)|^2}{\xi} d\xi.$$

Proof: Let $f \in W$ and $g \in A$. Then, by (6.5), Plancherel's theorem and the elementary properties of the Fourier transform, we get

$$\int_0^{\infty}\int_{-\infty}^{\infty} |\langle f, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{-\infty}^\infty \left| \int_{-\infty}^\infty f(x)\bar{g}\left(\frac{x-b}{a}\right) dx \right|^2 \frac{dbda}{a^3} \\
&= \int_0^\infty \int_{-\infty}^\infty \left| \int_{-\infty}^\infty e^{ib\xi}\hat{f}(\xi)\overline{\hat{g}(a\xi)}d\xi \right|^2 \frac{dbda}{a} \\
&= 2\pi \int_0^\infty \int_{-\infty}^\infty \left| (2\pi)^{-\frac{1}{2}} \int_{-\infty}^\infty e^{ib\xi}\hat{f}(\xi)\overline{\hat{g}(a\xi)}d\xi \right|^2 \frac{dbda}{a} \\
&= 2\pi \int_0^\infty \int_{-\infty}^\infty \left| \left(\hat{f}\left(D_a\bar{\hat{g}}\right)\right)^\vee (b) \right|^2 \frac{dbda}{a} \\
&= 2\pi \int_0^\infty \int_{-\infty}^\infty \left| \hat{f}(\xi)\left(D_a\bar{\hat{g}}\right)(\xi) \right|^2 \frac{d\xi da}{a}, \qquad (6.21)
\end{aligned}
$$

where

$$\left(D_a\bar{\hat{g}}\right)(\xi) = \overline{\hat{g}(a\xi)}. \quad \xi \in \mathbb{R}. \qquad (6.22)$$

Thus, by (6.21), (6.22) and Fubini's theorem,

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty |\langle f, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} &= 2\pi \int_0^\infty \left(\int_0^\infty |\hat{f}(\xi)|^2 d\xi\right) |\hat{g}(a\xi)|^2 \frac{da}{a} \\
&= 2\pi \int_0^\infty |\hat{f}(\xi)|^2 d\xi \int_0^\infty \frac{|\hat{g}(\xi)|^2}{\xi} d\xi \qquad (6.23)
\end{aligned}
$$

for all f in W and all g in A. Now, let $f \in H_+^2(\mathbb{R})$ and $g \in A$. Then, by Lemma 6.6, there exists a sequence $\{f_k\}_{k=1}^\infty$ of functions in W such that

$$f_k \to f \qquad (6.24)$$

in $L^2(\mathbb{R})$ as $k \to \infty$. For $k = 1, 2, \cdots$, we get, by (6.23), (6.24) and Plancherel's theorem,

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty |\langle f_k, \pi(b,a)g\rangle - \langle f_j, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} \\
&= 2\pi \int_0^\infty |\hat{f}_k(\xi) - \hat{f}_j(\xi)|^2 d\xi \int_0^\infty \frac{|\hat{g}(\xi)|^2}{\xi} d\xi \longrightarrow 0
\end{aligned}
$$

as $k, j \to \infty$. So, $\{\langle f_k, \pi(\cdot,\cdot)g\rangle\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(U, d\mu)$. Hence there exists a function h in $L^2(U, d\mu)$ such that

$$\langle f_k, \pi(\cdot,\cdot)g\rangle \to h \qquad (6.25)$$

in $L^2(U, d\mu)$ as $k \to \infty$. Therefore there exists a subsequence of $\{\langle f_k, \pi(\cdot,\cdot)g\rangle\}_{k=1}^{\infty}$, again denoted by $\{\langle f_k, \pi(\cdot,\cdot)g\rangle\}_{k=1}^{\infty}$, such that

$$\langle f_k, \pi(\cdot,\cdot)g\rangle \to h \tag{6.26}$$

a.e. on U as $k \to \infty$. But, by (6.24),

$$\langle f_k, \pi(b,a)g\rangle \to \langle f, \pi(b,a)g\rangle \tag{6.27}$$

for all (b,a) in U as $k \to \infty$. So, by (6.25)–(6.27),

$$\int_0^\infty \int_{-\infty}^\infty |\langle f_k, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} \longrightarrow \int_0^\infty \int_{-\infty}^\infty |\langle f, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} \tag{6.28}$$

as $k \to \infty$. But, by (6.23), (6.24) and Plancherel's theorem,

$$\int_0^\infty \int_{-\infty}^\infty |\langle f_k, \pi(b,a)g\rangle|^2 \frac{dbda}{a^2} \longrightarrow 2\pi \int_0^\infty |\hat{f}(\xi)|^2 d\xi \int_0^\infty \frac{|\hat{g}(\xi)|^2}{\xi} d\xi. \tag{6.29}$$

Hence, by (6.28) and (6.29), the proof is complete. □

Corollary 6.9 $\pi : U \to B(H_+^2(\mathbb{R}))$ *is a square-integrable representation of* U *on* $H_+^2(\mathbb{R})$.

Proof: Let $\varphi \in A$. Then, by Theorem 6.8,

$$\int_0^\infty \int_{-\infty}^\infty |\langle \varphi, \pi(b,a)\varphi\rangle|^2 \frac{dbda}{a^2} = 2\pi \int_0^\infty |\hat{\varphi}(\xi)|^2 d\xi \int_0^\infty \frac{|\hat{\varphi}(\xi)|^2}{\xi} d\xi < \infty \tag{6.30}$$

and this completes the proof. □

Corollary 6.10 *Every function* φ *in* A *with* $\|\varphi\|_{L^2(\mathbb{R})} = 1$ *is an admissible wavelet for the representation* $\pi : U \to B(H_+^2(\mathbb{R}))$ *of* U *on* $H_+^2(\mathbb{R})$ *and*

$$c_\varphi = 2\pi \int_0^\infty \frac{|\hat{\varphi}(\xi)|^2}{\xi} d\xi.$$

Corollary 6.10 is an immediate consequence of (6.30).

Remark 6.11 Corollary 6.10 tells us that the set $AW(\pi)$ of all admissible wavelets for the representation $\pi : U \to B(H^2_+(\mathbb{R}))$ of U of $H^2_+(\mathbb{R})$ is nonempty. That $AW(\pi)$ is a proper subset of $\{f \in H^2_+(\mathbb{R}) : \|f\|_{L^2(\mathbb{R})} = 1\}$ is illustrated by the following example.

Example 6.12 Let χ be the characteristic function on $[0,1]$ and let f_0 be the function in $L^2(\mathbb{R})$ such that $\hat{f}_0 = \chi$. Then $f_0 \in H^2_+(\mathbb{R})$. Using the same calculations in the derivation of (6.23), we get

$$\int_0^\infty \int_{-\infty}^\infty |\langle f_0, \pi(b,a) f_0\rangle|^2 \frac{dbda}{a^2} = \int_0^\infty |\hat{f}_0(\xi)|^2 d\xi \int_0^\infty \frac{|\hat{f}_0(\xi)|^2}{\xi} d\xi. \tag{6.31}$$

But

$$\int_0^\infty \frac{|\hat{f}_0(\xi)|^2}{\xi} d\xi = \int_0^1 \frac{1}{\xi} d\xi = \infty. \tag{6.32}$$

Thus, by (6.31) and (6.32), the function φ on $\mathbb{R}$ defined by

$$\varphi = f_0 / \|f_0\|_{L^2(\mathbb{R})}$$

is in $\{f \in H^2_+(\mathbb{R}) : \|f\|_{L^2(\mathbb{R})} = 1\}$, but is not in $AW(\pi)$.

Using Theorem 3.8 and Corollary 6.10, the following result on localization operators on the affine group is immediate.

Theorem 6.13 *Let $\varphi \in A$ and $F \in L^p(U, d\mu)$, $1 \le p \le \infty$. Then the localization operator $L_F : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R})$ given by*

$$\langle L_F f, g\rangle = \frac{1}{c_\varphi} \int_0^\infty \int_{-\infty}^\infty F(b,a) \langle f, \pi(b,a)\varphi\rangle \langle \pi(b,a)\varphi, g\rangle \frac{dbda}{a^2}$$

for all f and g in $H^2_+(\mathbb{R})$, where

$$c_\varphi = 2\pi \int_0^\infty \frac{|\hat{\varphi}(\xi)|^2}{\xi} d\xi,$$

is in S_p and

$$\|L_F\|_{S_p} \le \left(\frac{4}{c_\varphi}\right)^{\frac{1}{p}} \|F\|_{L^p(U, d\mu)}.$$

7 Wavelet Multipliers

Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_\sigma u = \mathcal{F}^{-1}\sigma\mathcal{F}u, \quad u \in L^2(\mathbb{R}^n),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transformation and inverse Fourier transformation respectively. The Fourier transform $\mathcal{F}u$, sometimes denoted by $\hat{u}$, of a function u in $L^2(\mathbb{R}^n)$, is given by

$$\mathcal{F}u = \lim_{R\to\infty} (\chi_R u)^\wedge,$$

where χ_R is the characteristic function of the ball with center at the origin and radius R,

$$(\chi_R u)^\wedge(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\chi_R(x)u(x)dx, \quad \xi \in \mathbb{R}^n,$$

and the convergence of $(\chi_R u)^\wedge$ to $\mathcal{F}u$ is understood to be in $L^2(\mathbb{R}^n)$. It is a consequence of Plancherel's theorem that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.

Let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. The aim of this chapter is to make precise the definition of the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where σ is a function in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and to prove that the resulting bounded linear operator is in the Schatten-von Neumann class S_p. To this end, we first prove that if $\sigma \in L^\infty(\mathbb{R}^n)$, then the bounded linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ can be realized as a wavelet multiplier (to be explained) associated to a unitary representation $\pi : \mathbb{R}^n \to B(L^2(\mathbb{R}^n))$ of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$. This connection explains the impetus for the study of the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and also reveals that the techniques developed in Chapters 1–3 can be exploited.

Let $\pi : \mathbb{R}^n \to B(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\cdot\xi}u(x), \quad x, \xi \in \mathbb{R}^n, \tag{7.1}$$

for all functions u in $L^2(\mathbb{R}^n)$.

Proposition 7.1 *Let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then, for all functions u and v in $\mathcal{S}$,*

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \langle u, \pi(\xi)\varphi\rangle \langle \pi(\xi)\varphi, v\rangle d\xi = \langle \varphi u, \varphi v\rangle. \tag{7.2}$$

Proof: Using Plancherel's theorem and the fact that

$$(\pi(\xi)\varphi)^\wedge = T_{-\xi}\hat{\varphi}, \quad \xi \in \mathbb{R}^n,$$

where

$$(T_{-\xi}f)(x) = f(x-\xi), \quad x \in \mathbb{R}^n,$$

for any measurable function f on $\mathbb{R}^n$, we get

$$\langle u, \pi(\xi)\varphi\rangle = (\hat{u} * \hat{\psi})(\xi) \tag{7.3}$$

and

$$\langle \pi(\xi)\varphi, v\rangle = \overline{(\hat{v} * \hat{\psi})(\xi)} \tag{7.4}$$

for all ξ in $\mathbb{R}^n$, where

$$\psi(x) = \overline{\varphi(x)}, \quad x \in \mathbb{R}^n, \tag{7.5}$$

and

$$(\hat{f} * \hat{\psi})(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi - \eta)\hat{\psi}(\eta) d\eta, \quad \xi \in \mathbb{R}^n,$$

for all functions f in $\mathcal{S}$. Thus, by (7.3)–(7.5), Plancherel's theorem and the fact that

$$(f\psi)^\wedge = (2\pi)^{-n/2}(\hat{f} * \hat{\psi}), \quad f \in \mathcal{S}, \tag{7.6}$$

we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \langle u, \pi(\xi)\varphi\rangle \langle \pi(\xi)\varphi, v\rangle d\xi &= \int_{\mathbb{R}^n} (\hat{u} * \hat{\psi})(\xi)\overline{(\hat{v} * \hat{\psi})(\xi)} d\xi \\
&= (2\pi)^n \int_{\mathbb{R}^n} u(\xi)\psi(\xi)\overline{v(\xi)\psi(\xi)} d\xi \\
&= (2\pi)^n \langle \varphi u, \varphi v\rangle,
\end{aligned}$$

and the proof is complete. □

Remark 7.2 Formula (7.2) can be considered as an analogue of the resolution of the identity formula (1.3) for the unitary representation $\pi : \mathbb{R}^n \to B(L^2(\mathbb{R}^n))$ of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$.

Proposition 7.3 *Let $\sigma \in L^\infty(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. If, for any function u in $\mathcal{S}$, we define $P_\sigma u$ by*

$$\langle P_\sigma u, v\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)\langle u, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, v\rangle d\xi \tag{7.7}$$

for all functions v in $\mathcal{S}$, then

$$\langle P_\sigma u, v\rangle = \langle(\varphi T_\sigma \bar{\varphi})u, v\rangle, \quad u,\, v \in \mathcal{S}.$$

Proof: By (7.3)–(7.5), we get

$$\langle P_\sigma u, v\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(\hat{u} * \hat{\psi})(\xi)\overline{(\hat{v} * \hat{\psi})(\xi)} d\xi \tag{7.8}$$

for all functions u and v in $\mathcal{S}$. But, by (7.6) and (7.8),

$$\langle P_\sigma u, v\rangle = \int_{\mathbb{R}^n} \sigma(\xi)(u\psi)^\wedge(\xi)\overline{(v\psi)^\wedge(\xi)} d\xi, \quad u,\, v \in \mathcal{S}. \tag{7.9}$$

Thus, by (7.5), (7.9), Plancherel's theorem and the definition of $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, we get

$$\langle P_\sigma u, v\rangle = \langle T_\sigma(\psi u), \psi v\rangle = \langle(\bar{\psi} T_\sigma \psi)u, v\rangle = \langle(\varphi T_\sigma \bar{\varphi})u, v\rangle$$

for all functions u and v in $\mathcal{S}$. □

Remark 7.4 By Proposition 7.3, the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ associated to σ in $L^\infty(\mathbb{R}^n)$ and φ in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the condition that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$ is a variant of a localization operator studied in Chapter 2. See in particular formula (2.1) for the analogy. Had the "admissible wavelet" φ in (7.7) been replaced by the function φ_0 on $\mathbb{R}^n$ given by

$$\varphi_0(x) = 1, \quad x \in \mathbb{R}^n,$$

we would have obtained

$$\langle P_\sigma u, v\rangle = \langle T_\sigma u, v\rangle, \quad u,\, v \in \mathcal{S},$$

i.e., P_σ would have been a "constant coefficient" pseudo-differential operator, or a Fourier multiplier studied in, say, the book [23] by Wong. In view of the fact that the function φ in the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ plays the role of the admissible wavelet in the linear operator $P_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, it is reasonable to call the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a wavelet multiplier.

In order to define the linear operator $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where σ is a function in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and φ is a function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$, we need some preparations.

Proposition 7.5 *Let $\sigma \in L^1(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. If, for any function u in $L^2(\mathbb{R}^n)$, we define $P_\sigma u$ by (7.7) for all functions v in $L^2(\mathbb{R}^n)$, then $P_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and*

$$|P_\sigma|_* \le (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)}. \tag{7.10}$$

Proof: By (7.1), (7.7), the Schwarz inequality and the assumption that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$,

$$\begin{aligned} |\langle P_\sigma u, v\rangle| &\le (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)||\langle u, \pi(\xi)\varphi\rangle||\langle \pi(\xi)\varphi, v\rangle| d\xi \\ &\le (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

for all functions u and v in $L^2(\mathbb{R}^n)$. □

Proposition 7.6 *Let $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then there exists a unique bounded linear operator $P_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that*

$$|P_\sigma|_* \le (2\pi)^{-n/p} \|\varphi\|^{2/p'}_{L^\infty(\mathbb{R}^n)} \|\sigma\|_{L^p(\mathbb{R}^n)},$$

and for all functions u and v in $L^2(\mathbb{R}^n)$, $\langle P_\sigma u, v\rangle$ is given by (7.7) for all simple functions σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n : \sigma(\xi) \ne 0\}$ is finite.

Proof: Suppose $\sigma \in L^\infty(\mathbb{R}^n)$. Then, by (7.1), (7.3)–(7.5), (7.7), the Schwarz inequality and the assumption that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$, we get

$$|\langle P_\sigma u, v\rangle| \leq (2\pi)^{-n}\|\sigma\|_{L^\infty(\mathbb{R}^n)} \left\{\int_{\mathbb{R}^n} |(\hat{u} * \hat{\psi})(\xi)|^2 d\xi\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}^n} |(\hat{v} * \hat{\psi})(\xi)|^2 d\xi\right\}^{\frac{1}{2}} \quad (7.11)$$

for all functions u and v in $L^2(\mathbb{R}^n)$. By (7.5), (7.6), (7.11) and Plancherel's theorem,

$$\begin{aligned} |\langle P_\sigma u, v\rangle| &\leq \|\sigma\|_{L^\infty(\mathbb{R}^n)} \left\{\int_{\mathbb{R}^n} |u(x)\psi(x)|^2 dx\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{R}^n} |v(x)\psi(x)|^2 dx\right\}^{\frac{1}{2}} \\ &\leq \|\sigma\|_{L^\infty(\mathbb{R}^n)} \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} \end{aligned} \quad (7.12)$$

for all functions u and v in $L^2(\mathbb{R}^n)$. So, by (7.12),

$$|P_\sigma|_* \leq \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \|\sigma\|_{L^\infty(\mathbb{R}^n)}, \quad \sigma \in L^\infty(\mathbb{R}^n). \quad (7.13)$$

Thus, by (7.10), (7.13) and the interpolation argument used in the proof of Theorem 2.3, the proof is complete. □

Remark 7.7 Propositions 7.5 and 7.6 allow us to define the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for any function σ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and any function φ in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$, as the bounded linear operator $P_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

We can now give the Schatten-von Neumann property of the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $\sigma \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and φ is a function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. We begin with the case when $p = 1$.

Proposition 7.8 *Let $\sigma \in L^1(\mathbb{R}^n)$ and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in S_1 and*

$$\|\varphi T_\sigma \bar{\varphi}\|_{S_1} \leq \frac{4}{(2\pi)^n} \|\sigma\|_{L^1(\mathbb{R}^n)}. \quad (7.14)$$

Proof: Let $\{\varphi_k : k = 1, 2, \cdots\}$ be any orthonormal basis for $L^2(\mathbb{R}^n)$. Then, by (7.7),

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle P_\sigma \varphi_k, \varphi_k\rangle| &= \sum_{k=1}^{\infty} \left|(2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)\langle \varphi_k, \pi(\xi)\varphi\rangle\langle \pi(\xi)\varphi, \varphi_k\rangle d\xi\right| \\ &\le \sum_{k=1}^{\infty} (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)||\langle \varphi_k, \pi(\xi)\varphi\rangle|^2 d\xi. \qquad (7.15)\end{aligned}$$

So, by (7.1), (7.15), Fubini's theorem, the Parseval identity and the assumption that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$,

$$\sum_{k=1}^{\infty} |\langle P_\sigma \varphi_k, \varphi_k\rangle| \le (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)} < \infty.$$

Hence, by Proposition 3.1, the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in S_1. The proof of Proposition 3.3 can then be used to prove (7.14). □

Theorem 7.9 *Let $\sigma \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in S_p and*

$$\|\varphi T_\sigma \bar{\varphi}\|_{S_p} \le 4^{\frac{1}{p}} \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{\frac{2}{p'}} (2\pi)^{-\frac{n}{p}} \|\sigma\|_{L^p(\mathbb{R}^n)}.$$

Proof: If $p = 1$, then Theorem 7.9 follows from Proposition 7.8. If $p = \infty$, then Theorem 7.9 follows from (7.13). Thus, for $1 < p < \infty$, Theorem 7.9 follows from Theorems 3.6 and 3.7, and the endpoint cases $p = 1$ and $p = \infty$. □

Remark 7.10 The contents of this chapter can be found in Sections 1–4 of the paper [11] by He and Wong. The results in this chapter in the setting of a locally compact, Hausdorff and abelian group G instead of $\mathbb{R}^n$ can be found in the Ph.D. dissertation [9] by He.

8 The Landau-Pollak-Slepian Operator

We show in this chapter that the Landau-Pollak-Slepian operator arising in signal analysis is in fact a wavelet multiplier. We begin with a discussion of the Landau-Pollak-Slepian operator.

Let Ω and T be positive numbers. Then we define the linear operators $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(P_\Omega f)^\wedge(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \leq \Omega, \\ 0, & |\xi| > \Omega, \end{cases} \tag{8.1}$$

and

$$(Q_T f)(\xi) = \begin{cases} f(x), & |x| \leq T, \\ 0, & |x| > T, \end{cases} \tag{8.2}$$

for all functions f in $L^2(\mathbb{R})$.

Proposition 8.1 *$P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are self-adjoint projections.*

Proof: By (8.1) and Plancherel's theorem,

$$\begin{aligned} \langle P_\Omega f, g\rangle &= \langle (P_\Omega f)^\wedge, \hat{g}\rangle = \int_{-\infty}^{\infty} (P_\Omega f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi \\ &= \int_{-\Omega}^{\Omega} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \int_{-\Omega}^{\Omega} \hat{f}(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi \\ &= \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi = \langle \hat{f}, (P_\Omega g)^\wedge\rangle \\ &= \langle f, P_\Omega g\rangle, \quad f,\, g \in L^2(\mathbb{R}). \end{aligned}$$

Therefore $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint. By (8.2),

$$\begin{aligned} \langle Q_T f, g\rangle &= \int_{-\infty}^{\infty} (Q_T f)(x)\overline{g(x)}dx = \int_{-T}^{T} f(x)\overline{g(x)}dx \\ &= \int_{-\infty}^{\infty} f(x)\overline{(Q_T g)(x)}dx \\ &= \langle f, Q_T g\rangle, \quad f,\, g \in L^2(\mathbb{R}). \end{aligned}$$

Therefore $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint. By (8.1), the fact that $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and Plancherel's theorem,

$$\begin{aligned}
\langle P_\Omega^2 f, g\rangle &= \langle P_\Omega f, P_\Omega g\rangle = \langle (P_\Omega f)^\wedge, (P_\Omega g)^\wedge\rangle \\
&= \int_{-\infty}^{\infty} (P_\Omega f)^\wedge(\xi)\overline{(P_\Omega g)^\wedge(\xi)}d\xi = \int_{-\Omega}^{\Omega} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\
&= \int_{-\infty}^{\infty} (P_\Omega f)^\wedge(\xi)\overline{\hat{g}(\xi)}d\xi = \langle (P_\Omega f)^\wedge, \hat{g}\rangle \\
&= \langle P_\Omega f, g\rangle, \quad f,\, g \in L^2(\mathbb{R}).
\end{aligned}$$

Thus, $P_\Omega^2 = P_\Omega$ and hence $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a projection. Finally, by (8.2) and the fact that $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint,

$$\begin{aligned}
\langle Q_T^2 f, g\rangle &= \langle Q_T f, Q_T g\rangle = \int_{-\infty}^{\infty} (Q_T f)(x)\overline{(Q_T g)(x)}dx \\
&= \int_{-T}^{T} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} (Q_T f)(x)\overline{g(x)}dx \\
&= \langle Q_T f, g\rangle, \quad f,\, g \in L^2(\mathbb{R}).
\end{aligned}$$

Thus, $Q_T^2 = Q_T$ and hence $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a projection. □

In signal analysis, a signal is a function f in $L^2(\mathbb{R})$. Thus, for any function f in $L^2(\mathbb{R})$, the function $Q_T P_\Omega f$ can be considered to be a time and band-limited signal. Therefore it is of interest to compare the energy $\|Q_T P_\Omega f\|_{L^2(\mathbb{R})}^2$ of the time and band-limited signal $Q_T P_\Omega f$ with the energy $\|f\|_{L^2(\mathbb{R})}^2$ of the original signal f. Using the fact that $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are self-adjoint and the fact that $Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a projection, we get

$$\begin{aligned}
&\sup\left\{\frac{\|Q_T P_\Omega f\|_{L^2(\mathbb{R})}^2}{\|f\|_{L^2(\mathbb{R})}^2} : \quad f \in L^2(\mathbb{R}),\, f \neq 0\right\} \\
=\ &\sup\left\{\frac{\langle Q_T P_\Omega f, Q_T P_\Omega f\rangle}{\|f\|_{L^2(\mathbb{R})}^2} : \quad f \in L^2(\mathbb{R}),\, f \neq 0\right\} \\
=\ &\sup\left\{\frac{\langle P_\Omega Q_T P_\Omega f, f\rangle}{\|f\|_{L^2(\mathbb{R})}^2} : \quad f \in L^2(\mathbb{R}),\, f \neq 0\right\} \\
=\ &\sup\left\{\langle P_\Omega Q_T P_\Omega f, f\rangle : \quad f \in L^2(\mathbb{R}),\, \|f\|_{L^2(\mathbb{R})} = 1\right\}. \qquad (8.3)
\end{aligned}$$

Since $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint, it follows from (8.3) that

$$\sup\left\{\frac{\|Q_T P_\Omega f\|^2_{L^2(\mathbb{R})}}{\|f\|^2_{L^2(\mathbb{R})}} : \quad f \in L^2(\mathbb{R}),\, f \neq 0\right\} = |P_\Omega Q_T P_\Omega|_*.$$

The bounded linear operator $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that we have just seen in the context of time and band-limited signals is called the Landau-Pollak-Slepian operator. See the fundamental papers [13, 14] by Landau and Pollak, [18, 19] by Slepian and [20] by Slepian and Pollak for more detailed information.

That the Landau-Pollak-Slepian operator is in fact a wavelet multiplier studied in Chapter 7 is the content of the following theorem.

Theorem 8.2 *Let φ be the function on $\mathbb{R}$ defined by*

$$\varphi(x) = \begin{cases} \frac{1}{\sqrt{2\Omega}}, & |x| \leq \Omega, \\ 0, & |x| > \Omega, \end{cases} \tag{8.4}$$

and let σ be the characteristic function on $[-T, T]$, i.e.,

$$\sigma(\xi) = \begin{cases} 1, & |\xi| \leq T, \\ 0, & |\xi| > T. \end{cases} \tag{8.5}$$

Then the Landau-Pollak-Slepian operator $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitarily equivalent to a scalar multiple of the wavelet multiplier $\varphi T_\sigma \varphi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. In fact,

$$P_\Omega Q_T P_\Omega = 2\Omega \mathcal{F}^{-1}(\varphi T_\sigma \varphi)\mathcal{F}. \tag{8.6}$$

Proof: By (8.4), φ is a function in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\|\varphi\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\varphi(x)|^2 dx = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} dx = 1.$$

So, by Proposition 7.3,

$$((\varphi T_\sigma \varphi)u, v) = (2\pi)^{-1}\int_{-\infty}^{\infty} \sigma(\xi)(u, \pi(\xi)\varphi)(\pi(\xi)\varphi, v)d\xi \tag{8.7}$$

for all functions u and v in $\mathcal{S}$. By (7.1) and (8.4),

$$\begin{aligned}\langle u, \pi(\xi)\varphi\rangle &= \int_{-\infty}^{\infty} e^{-ix\xi}\varphi(x)u(x)dx \\ &= \frac{1}{\sqrt{2\Omega}}\int_{-\Omega}^{\Omega} e^{-ix\xi}u(x)dx, \quad u \in \mathcal{S}. \end{aligned} \tag{8.8}$$

By (8.1),

$$(P_\Omega \check{u})^\wedge(x) = \begin{cases} u(x), & |x| \le \Omega, \\ 0, & |x| > \Omega, \end{cases} \tag{8.9}$$

for all functions u in $\mathcal{S}$, where $\check{u}$ is the inverse Fourier transform of u. So, by (8.8), (8.9) and the Fourier inversion formula,

$$\begin{aligned}\langle u, \pi(\xi)\varphi\rangle &= \frac{1}{\sqrt{2\Omega}}\int_{-\infty}^{\infty} e^{-ix\xi}(P_\Omega\check{u})^\wedge(x)dx \\ &= \sqrt{\frac{\pi}{\Omega}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}(P_\Omega\check{u})^\wedge(x)dx \\ &= \sqrt{\frac{\pi}{\Omega}}(P_\Omega\check{u})(-\xi), \quad \xi \in \mathbb{R}, \end{aligned} \tag{8.10}$$

for all functions u in $\mathcal{S}$. Hence, by (8.2), (8.5), (8.7), (8.10), Plancherel's theorem and the fact that $P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint,

$$\begin{aligned}\langle(\varphi T_\sigma \varphi)u, v\rangle &= (2\pi)^{-1}\frac{\pi}{\Omega}\int_{-\infty}^{\infty} \sigma(\xi)(P_\Omega\check{u})(\xi)\overline{(P_\Omega\check{v})(\xi)}d\xi \\ &= \frac{1}{2\Omega}\int_{-T}^{T}(P_\Omega\check{u})(\xi)\overline{(P_\Omega\check{v})(\xi)}d\xi \\ &= \frac{1}{2\Omega}\int_{-\infty}^{\infty}(Q_T P_\Omega\check{u})(\xi)\overline{(P_\Omega\check{v})(\xi)}d\xi \\ &= \frac{1}{2\Omega}\langle Q_T P_\Omega \check{u}, P_\Omega \check{v}\rangle = \frac{1}{2\Omega}\langle P_\Omega Q_T P_\Omega \check{u}, \check{v}\rangle \\ &= \frac{1}{2\Omega}\langle \mathcal{F}P_\Omega Q_T P_\Omega \mathcal{F}^{-1}u, v\rangle, \quad u, v \in \mathcal{S}, \end{aligned}$$

and hence (8.6) is proved. □

Remark 8.3 The results in this chapter can be found in Section 5 of the paper [11] by He and Wong.

9 A Product Formula for Wavelet Multipliers

The wisdom of Chapter 8 is that a wavelet multiplier can be considered as a filter which time and band-limits a signal. Thus, if we are interested in finding a filter that has the same effect as two wavelet multipliers arranged in series, we are actually seeking a formula for the product of two wavelet multipliers.

We give in this chapter a formula for the product of two wavelet multipliers $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\tau \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where σ and τ are functions in $L^2(\mathbb{R}^n)$, and φ is a function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. To do this, we need a recall of some basic results without proofs on the Weyl transform from the book [5] by Folland and the book [22] by Wong.

Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform associated to σ is the bounded linear operator $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$\langle W_\sigma f, g \rangle = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi$$

for all functions f and g in $L^2(\mathbb{R}^n)$, where $W(f,g)$ is the Wigner transform of f and g defined by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \quad x,\, \xi \in \mathbb{R}^n. \tag{9.1}$$

It can be proved that

$$W(f,g)(x,\xi) = V(f,g)^\wedge(x,\xi), \quad x,\, \xi \in \mathbb{R}^n, \tag{9.2}$$

where the function $V(f,g)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is the Fourier-Wigner transform of f and g defined by

$$V(f,g)(q,p) = (2\pi)^{-n/2} \langle \rho(q,p) f, g \rangle, \quad q,\, p \in \mathbb{R}^n, \tag{9.3}$$

and

$$(\rho(q,p)f)(x) = e^{iq\cdot x+\frac{1}{2}iq\cdot p}f(x+p), \quad x \in \mathbb{R}^n. \tag{9.4}$$

For all Schwartz functions f and g on $\mathbb{R}^n$, the functions $V(f,g)$ and $W(f,g)$ are Schwartz functions on $\mathbb{R}^n \times \mathbb{R}^n$.

For all functions f and g in $L^2(\mathbb{R}^n)$, the functions $V(f,g)$ and $W(f,g)$ are in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Furthermore, we have

$$||W(f,g)||_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}||g||_{L^2(\mathbb{R}^n)} \tag{9.5}$$

for all functions f and g in $L^2(\mathbb{R}^n)$. That the same identity is true when W is replaced by V follows from (9.2) and Plancherel's theorem.

Let $h \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then, for all functions f in $L^2(\mathbb{R}^n)$, we define the function $K_h f$ on $\mathbb{R}^n$ by

$$(K_h f)(x) = \int_{\mathbb{R}^n} h(x,y)f(y)dy, \quad x \in \mathbb{R}^n.$$

Then $K_h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and we call it the Hilbert-Schmidt operator corresponding to the kernel h. The following result, obtained by Pool in [15], is the main ingredient in the derivation of the product formula for two wavelet multipliers.

Proposition 9.1 *Let $h \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Hilbert-Schmidt operator corresponding to the kernel h is the same as the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and*

$$\sigma = (2\pi)^{n/2}\mathcal{F}_2 Th, \tag{9.6}$$

where $\mathcal{F}_2$ is the Fourier transform on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ with respect to the second variable and T is the linear operator on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ defined by

$$(Tf)(x,y) = f\left(x+\frac{y}{2}, x-\frac{y}{2}\right), \quad x, y \in \mathbb{R}^n, \tag{9.7}$$

for all functions f in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

We can now give a formula for the product of two wavelet multipliers.

Theorem 9.2 *Let σ and τ be functions in $L^2(\mathbb{R}^n)$, and let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then the product of the wavelet multipliers $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\tau \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the linear operator $\varphi W_\lambda \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $W_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Weyl transform associated to λ and*

$$\lambda(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} W(\sigma, \bar{\tau})(\xi, y - x)|\varphi(y)|^2 dy \tag{9.8}$$

for all x and ξ in $\mathbb{R}^n$.

The following lemma will be used in the proof of Theorem 9.2.

Lemma 9.3 *Let f and g be functions in $L^2(\mathbb{R}^n)$. Then*

$$W(\check{f}, \check{g})(x, \xi) = W(f, g)(\xi, -x), \quad x, \xi \in \mathbb{R}^n.$$

Proof: Let f and g be functions in $\mathcal{S}$. Then, by (9.3) and (9.4),

$$V(\check{f}, \check{g})(q, p) = (2\pi)^{-n/2} \langle e^{\frac{1}{2} i q \cdot p} M_q T_p \check{f}, \check{g} \rangle \tag{9.9}$$

for all q and p in $\mathbb{R}^n$, where

$$(T_p u)(x) = u(x + p), \quad x \in \mathbb{R}^n,$$

and

$$(M_q u)(x) = e^{iq \cdot x} u(x), \quad x \in \mathbb{R}^n,$$

for all measurable functions u on $\mathbb{R}^n$. So, by (9.3), (9.4), (9.9) and Plancherel's theorem,

$$\begin{aligned} V(\check{f}, \check{g})(q, p) &= (2\pi)^{-n/2} \langle e^{\frac{1}{2} i q \cdot p} (T_{-q} M_p f)^\vee, \check{g} \rangle \\ &= (2\pi)^{-n/2} \langle e^{\frac{1}{2} i q \cdot p} T_{-q} M_p f, g \rangle \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{1}{2} i q \cdot p} e^{i p \cdot (x - q)} f(x - q) \overline{g(x)} dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i p \cdot x - \frac{1}{2} i q \cdot p} f(x - q) \overline{g(x)} dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (\rho(p, -q) f)(x) \overline{g(x)} dx \\ &= V(f, g)(p, -q), \quad q, p \in \mathbb{R}^n. \end{aligned} \tag{9.10}$$

So, by (9.1) and (9.9),

$$\begin{aligned} W(\check{f},\check{g})(x,\xi) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iq\cdot x-ip\cdot\xi}V(\check{f},\check{g})(q,p)dqdp \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iq\cdot x-ip\cdot\xi}V(f,g)(p,-q)dqdp \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{iq\cdot x-ip\cdot\xi}V(f,g)(p,q)dqdp \\ &= W(f,g)(\xi,-x), \quad x,\,\xi\in\mathbb{R}^n. \end{aligned} \tag{9.11}$$

Thus, by (9.5), (9.10), (9.11), Plancherel's theorem and a limiting argument, the proof is complete. □

Proof of Theorem 9.2: We begin with the observation that for all functions f in $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$,

$$\begin{aligned} (T_\sigma f)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(\xi)\hat{f}(\xi)d\xi \\ &= (2\pi)^{-n/2}(\check{\sigma}*f)(x), \quad x\in\mathbb{R}^n. \end{aligned} \tag{9.12}$$

Now,

$$(\varphi T_\sigma\bar{\varphi})(\varphi T_\tau\bar{\varphi}) = \varphi T_\sigma\omega T_\tau\bar{\varphi}, \tag{9.13}$$

where

$$\omega = |\varphi|^2, \tag{9.14}$$

and for all functions f in $\mathcal{S}$, we get, by (9.12) and Fubini's theorem,

$$\begin{aligned} (T_\sigma\omega T_\tau f)(x) &= (2\pi)^{-n/2}(\check{\sigma}*\omega T_\tau f)(x) \\ &= (2\pi)^{-n}(\check{\sigma}*\omega(\check{\tau}*f))(x) \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\check{\sigma}(x-y)\omega(y)(\check{\tau}*f)(y)dy \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\check{\sigma}(x-y)\omega(y)\left(\int_{\mathbb{R}^n}\check{\tau}(y-z)f(z)dz\right)dy \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\check{\sigma}(x-y)\omega(y)\check{\tau}(y-z)dy\right)f(z)dz \\ &= \int_{\mathbb{R}^n}h(x,z)f(z)dz, \quad x\in\mathbb{R}^n, \end{aligned} \tag{9.15}$$

where

$$h(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} \check{\sigma}(x-y)\omega(y)\check{\tau}(y-z)dy, \quad x,\, z\in\mathbb{R}^n. \quad (9.16)$$

By Minkowski's inequality in integral form, Fubini's theorem, Plancherel's theorem and (9.14),

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |h(x,z)|^2 dxdz\right)^{\frac{1}{2}} \\
&= (2\pi)^{-n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} \check{\sigma}(x-y)\omega(y)\check{\tau}(y-z)dy\right|^2 dxdz\right)^{\frac{1}{2}} \\
&\le (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\check{\sigma}(x-y)\omega(y)\check{\tau}(y-z)|^2 dxdz\right)^{\frac{1}{2}} dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}|\omega(y)|\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\check{\sigma}(x-y)\check{\tau}(y-z)|^2 dxdz\right)^{\frac{1}{2}} dy \\
&= (2\pi)^{-n}\|\varphi\|^2_{L^2(\mathbb{R}^n)}\|\sigma\|_{L^2(\mathbb{R}^n)}\|\tau\|_{L^2(\mathbb{R}^n)}. \qquad (9.17)
\end{aligned}$$

So, by (9.15)–(9.17) and Proposition 9.1, $T_\sigma\omega T_\tau : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is a Weyl transform $W_\lambda : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$, and by (9.6),

$$\lambda = (2\pi)^{n/2}\mathcal{F}_2 Th. \quad (9.18)$$

By (9.7) and (9.16),

$$\begin{aligned}
(Th)(x,z) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\check{\sigma}\left(x+\frac{z}{2}-y\right)\omega(y)\check{\tau}\left(y-x+\frac{z}{2}\right)dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\check{\sigma}\left(x-y+\frac{z}{2}\right)\omega(y)\check{\tau}\left(\frac{z}{2}-(x-y)\right)dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\check{\sigma}\left(x-y+\frac{z}{2}\right)\omega(y)\overline{\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)}dy \quad (9.19)
\end{aligned}$$

for all x and z in $\mathbb{R}^n$. For almost all x in $\mathbb{R}^n$, we get, by (9.14), (9.19), Fubini's theorem, the Schwarz inequality and Plancherel's theorem,

$$\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\check{\sigma}\left(x-y+\frac{z}{2}\right)\omega(y)\overline{\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)}dy\right|dz$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\check{\sigma}\left(x-y+\frac{z}{2}\right)\right||\omega(y)|\left|\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)\right|dy\right)dz \\
&= \int_{\mathbb{R}^n}|\omega(y)|\left(\int_{\mathbb{R}^n}\left|\check{\sigma}\left(x-y+\frac{z}{2}\right)\right|\left|\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)\right|dz\right)dy \\
&\leq \int_{\mathbb{R}^n}|\omega(y)|\left(\int_{\mathbb{R}^n}\left|\check{\sigma}\left(x-y+\frac{z}{2}\right)\right|^2 dz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}\left|\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)\right|^2 dz\right)^{\frac{1}{2}}dy \\
&= 2^n\|\varphi\|^2_{L^2(\mathbb{R}^n)}\|\sigma\|_{L^2(\mathbb{R}^n)}\|\tau\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Thus, by (9.1), (9.14), (9.19), Fubini's theorem and Lemma 9.3,

$$\begin{aligned}
&(\mathcal{F}_2Th)(x,\xi) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\omega(y)\left\{(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-iz\cdot\xi}\check{\sigma}\left(x-y+\frac{z}{2}\right)\overline{\check{\bar{\tau}}\left(x-y-\frac{z}{2}\right)}dz\right\}dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\omega(y)W(\check{\sigma},\check{\bar{\tau}})(x-y,\xi)dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}W(\sigma,\bar{\tau})(\xi,y-x)\omega(y)dy, \quad x,\ \xi\in\mathbb{R}^n,
\end{aligned}$$

and hence, by (9.13), (9.14) and (9.18), (9.8) follows.

10 Another Product Formula for Wavelet Multipliers

We give in this chapter another formula for the product of two wavelet multipliers. In order to do this, we need a recall of a formula, in the paper [7] by Grossmann, Loupias and Stein, for the product of two Weyl transforms associated to functions in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. To this end, we need the notion of a twisted convolution.

As usual, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$ and any point (q,p) in $\mathbb{R}^n \times \mathbb{R}^n$ with the point $z = q + ip$ in $\mathbb{C}^n$, and we define the symplectic form $[\ ,\ ]$ on $\mathbb{C}^n$ by

$$[z, w] = 2\mathrm{Im}(z \cdot \bar{w}), \quad z,\, w \in \mathbb{C}^n,$$

where

$$z = (z_1, z_2, \cdots, z_n),$$

$$w = (w_1, w_2, \cdots, w_n),$$

and

$$z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w}_j.$$

Now, for any fixed real number λ, we define the twisted convolution $f *_\lambda g$ of two measurable functions f and g on $\mathbb{C}^n$ by

$$(f *_\lambda g)(z) = \int_{\mathbb{C}^n} f(z - w) g(w) e^{i\lambda[z,w]} dw, \quad z \in \mathbb{C}^n, \tag{10.1}$$

where dw is the Lebesgue measure on $\mathbb{C}^n$, provided that the integral exists. The following theorem can be found in the paper [7] by Grossmann, Loupias and Stein.

Theorem 10.1 *Let σ and τ be functions in $L^2(\mathbb{C}^n)$. Then the product of the Weyl transforms $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $W_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*

is the same as the Weyl transform $W_\omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, *where* ω *is the function in* $L^2(\mathbb{C}^n)$ *given by*

$$\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_{\frac{1}{4}} \hat{\tau}).$$

A proof of Theorem 10.1 can be found in Chapter 9 of the book [22] by Wong.

Another ingredient in the derivation of another formula for the product of two wavelet multipliers is given in the following theorem.

Theorem 10.2 *Let* $\sigma \in L^2(\mathbb{R}^n)$, *and let* φ *be any function in* $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *such that* $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. *Then the wavelet multiplier* $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is the same as the Weyl transform* $W_{\sigma_\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, *where*

$$\sigma_\varphi(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} W(\varphi,\varphi)(x,\xi-\eta)\sigma(\eta)d\eta, \quad x,\, \xi \in \mathbb{R}^n. \tag{10.2}$$

Proof: By (9.12), we get, for all functions f in $\mathcal{S}$,

$$\begin{aligned}
((\varphi T_\sigma \bar{\varphi})f)(x) &= (2\pi)^{-n/2}\varphi(x)(\check{\sigma} * \bar{\varphi}f)(x) \\
&= (2\pi)^{-n/2}\varphi(x) \int_{\mathbb{R}^n} \check{\sigma}(x-y)\bar{\varphi}(y)f(y)dy \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x)\check{\sigma}(x-y)\bar{\varphi}(y)f(y)dy \\
&= \int_{\mathbb{R}^n} h(x,y)f(y)dy, \quad x \in \mathbb{R}^n,
\end{aligned} \tag{10.3}$$

where

$$h(x,y) = (2\pi)^{-n/2}\varphi(x)\check{\sigma}(x-y)\bar{\varphi}(y), \quad x,\, y \in \mathbb{R}^n. \tag{10.4}$$

Now, by (10.4), Fubini's theorem and Plancherel's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |h(x,y)|^2 dxdy \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x)|^2|\check{\sigma}(x-y)|^2|\varphi(y)|^2 dxdy \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} |\varphi(y)|^2 \left(\int_{\mathbb{R}^n} |\varphi(x)|^2|\check{\sigma}(x-y)|^2 dx\right) dy
\end{aligned}$$

$$\begin{aligned}
&\leq (2\pi)^{-n}\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n}|\varphi(y)|^2dy\right)\|\check{\sigma}\|^2_{L^2(\mathbb{R}^n)}\\
&= (2\pi)^{-n}\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}\|\varphi\|^2_{L^2(\mathbb{R}^n)}\|\sigma\|^2_{L^2(\mathbb{R}^n)} < \infty. \qquad (10.5)
\end{aligned}$$

So, by (10.3)–(10.5), $\varphi T_\sigma\bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert-Schmidt operator with kernel h, and hence, by Proposition 9.1, is the same as the Weyl transform $W_{\sigma_\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where

$$\sigma_\varphi(x,\xi) = (2\pi)^{n/2}(\mathcal{F}_2Th)(x,\xi), \quad x,\, \xi \in \mathbb{R}^n. \qquad (10.6)$$

But, by (9.7) and (10.4),

$$\begin{aligned}
(Th)(x,y) &= h\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\\
&= (2\pi)^{-n/2}\varphi\left(x+\frac{y}{2}\right)\check{\sigma}(y)\bar{\varphi}\left(x-\frac{y}{2}\right), \quad x,\, y \in \mathbb{R}^n,
\end{aligned}$$

and hence by (7.6),

$$\begin{aligned}
&(\mathcal{F}_2Th)(x,\xi)\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\xi\cdot y}\varphi\left(x+\frac{y}{2}\right)\check{\sigma}(y)\bar{\varphi}\left(x-\frac{y}{2}\right)dy\\
&= (2\pi)^{-n}(W(\varphi,\varphi)(x,\cdot)*\sigma)(\xi), \quad x,\, \xi \in \mathbb{R}^n. \qquad (10.7)
\end{aligned}$$

Hence, by (10.6) and (10.7), the proof is complete. □

We can now give another formula for the product of two wavelet multipliers.

Theorem 10.3 *Let σ and τ be functions in $L^2(\mathbb{R}^n)$, and let φ be any function in $L^2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then the product of the wavelet multipliers $\varphi T_\sigma\bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\tau\bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and λ is the function in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ given by*

$$\hat{\lambda} = (2\pi)^{-n}(\hat{\sigma}_\varphi *_{\frac{1}{4}} \hat{\tau}_\varphi),$$

where σ_φ and τ_φ are defined by (10.2).

Theorem 10.3 is an immediate consequence of Theorems 10.1 and 10.2.

11 A Product Formula for Daubechies Operators

Let $F \in L^2(\mathbb{C}^n)$. Then the Daubechies operator is the bounded linear operator $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by (5.23) for all functions f and g in $L^2(\mathbb{R}^n)$. We are interested in obtaining a formula for the product of two Daubechies operators in this chapter.

The starting point is the following theorem.

Theorem 11.1 *Let Λ be the function on $\mathbb{C}^n$ defined by*

$$\Lambda(z) = \pi^{-n} e^{-|z|^2}, \quad z \in \mathbb{C}^n. \tag{11.1}$$

Then, for all functions F in $L^2(\mathbb{C}^n)$, the Daubechies operator $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Weyl transform $W_{F*\Lambda} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Theorem 11.1 is Theorem 17.1 in the book [22] by Wong.

For any fixed real number λ, we define the λ-convolution $f *^\lambda g$ of two measurable functions f and g on $\mathbb{C}^n$ by

$$(f *^\lambda g)(z) = \int_{\mathbb{C}^n} f(z-\omega) g(\omega) e^{\lambda(z\cdot\bar{\omega} - |\omega|^2)} d\omega, \quad z \in \mathbb{C}^n, \tag{11.2}$$

provided that the integral exists. We have the following result.

Theorem 11.2 *Let F and G be functions in $L^2(\mathbb{C}^n)$. If there exists a function H in $L^2(\mathbb{C}^n)$ such that the Daubechies operator $D_H : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the product of the Daubechies operators $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $D_G : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, then*

$$\hat{H} = (2\pi)^{-n} (\hat{F} *^{\frac{1}{2}} \hat{G}). \tag{11.3}$$

Proof: By Theorem 11.1,

$$W_{H*\Lambda} = W_{F*\Lambda}W_{G*\Lambda}. \tag{11.4}$$

It follows from Theorem 10.1 and (11.4) that for all ζ in $\mathbb{C}^n$,

$$(H*\Lambda)^\wedge(\zeta) = (2\pi)^{-n}((F*\Lambda)^\wedge *_{\frac{1}{4}} (G*\Lambda)^\wedge(\zeta). \tag{11.5}$$

By (11.1) and an easy computation, we get

$$\hat{\Lambda}(\zeta) = (2\pi)^{-n}e^{-\frac{|\zeta|^2}{4}}, \quad \zeta \in \mathbb{C}^n. \tag{11.6}$$

Thus, by (11.4)–(11.6) and the definition of a twisted convolution given in (10.1), we get

$$\begin{aligned}\hat{H}(\zeta)e^{-\frac{|\zeta|^2}{4}} &= (2\pi)^n\{(\hat{F}\hat{\Lambda}) *_{\frac{1}{4}} (\hat{G}\hat{\Lambda})\}(\zeta)\\ &= (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)e^{-\frac{1}{4}|\zeta-\omega|^2}\hat{G}(\omega)e^{-\frac{1}{4}|\omega|^2}e^{\frac{1}{4}i[\zeta,\omega]}d\omega\\ &= (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)\hat{G}(\omega)e^{\frac{1}{4}\{-|\zeta-\omega|^2-|\omega|^2+i[\zeta,\omega]\}}d\omega. \quad (11.7)\end{aligned}$$

So, by (11.7),

$$\hat{H}(\zeta) = (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)\hat{G}(\omega)e^{\frac{1}{4}\{|\zeta|^2-|\zeta-\omega|^2-|\omega|^2+i[\zeta,\omega]\}}d\omega, \quad \zeta \in \mathbb{C}^n. \tag{11.8}$$

Now, for all ζ and ω in $\mathbb{C}^n$,

$$\begin{aligned}&|\zeta|^2 - |\zeta-\omega|^2 - |\omega|^2 + i[\zeta,\omega]\\ =\ &|\zeta|^2 - |\zeta|^2 + 2\mathrm{Re}(\zeta\cdot\bar{\omega}) - |\omega|^2 - |\omega|^2 + 2i\mathrm{Im}(\zeta\cdot\bar{\omega})\\ =\ &2(\zeta\cdot\bar{\omega}) - 2|\omega|^2. \end{aligned}\tag{11.9}$$

Therefore, by (11.2), (11.8) and (11.9), we get, for all ζ in $\mathbb{C}^n$,

$$\hat{H}(\zeta) = (2\pi)^{-n}\int_{\mathbb{C}^n} \hat{F}(\zeta-\omega)\hat{G}(\omega)e^{\frac{1}{2}(\zeta\cdot\bar{\omega}-|\omega|^2)}d\omega,$$

and hence (11.3). □

From the proof of Theorem 11.2, we get the following corollary.

Corollary 11.3 *Let F and G be functions in $L^2(\mathbb{C}^n)$ such that $\hat{F} *^{\frac{1}{2}} \hat{G} \in L^2(\mathbb{C}^n)$. Then there exists a function H in $L^2(\mathbb{C}^n)$ such that $\hat{H} = (2\pi)^{-n} (\hat{F} *^{\frac{1}{2}} \hat{G})$ and the Daubechies operator $D_H : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the product of the Daubechies operators $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $D_G : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Remark 11.4 In general, for functions F and G in $L^2(\mathbb{C}^n)$, it is not true that $\hat{F} *^{\frac{1}{2}} \hat{G} \in L^2(\mathbb{C}^n)$. So, the product of two Daubechies operators associated to functions in $L^2(\mathbb{C}^n)$ need not be a Daubechies operator associated to a function in $L^2(\mathbb{C}^n)$. This can best be seen from the following example.

Example 11.5 Let W be the subset of $\mathbb{R} \times \mathbb{R}$ defined by

$$W = \{(q,p) \in \mathbb{R} \times \mathbb{R} : 0 \leq q,\, p \leq 1\}. \tag{11.10}$$

We identify points ω and ζ in $\mathbb{C}$ with points (q,p) and (x,ξ) in $\mathbb{R} \times \mathbb{R}$ respectively. Let $F \in L^2(\mathbb{C})$ be defined by

$$\hat{F}(q,p) = e^{-\frac{1}{4}|q|}\chi(p), \quad q,\, p \in \mathbb{R}, \tag{11.11}$$

where χ is the characteristic function on $[-1,1]$, and let $G \in L^2(\mathbb{C})$ be defined by

$$\hat{G}(\omega) = \begin{cases} e^{\frac{1}{2}|\omega|^2}, & \omega \in W, \\ 0, & \omega \notin W. \end{cases} \tag{11.12}$$

Then, by (11.10)–(11.12),

$$\begin{aligned} & (\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta) \\ = & \int_W \hat{F}(\zeta - \omega)\hat{G}(\omega)e^{-\frac{1}{2}|\omega|^2}e^{\frac{1}{2}\zeta\bar{\omega}}d\omega \\ = & \int_0^1 \int_0^1 e^{-\frac{1}{4}|x-q|}\chi(\xi - p)e^{\frac{1}{2}(qx+p\xi)}e^{\frac{1}{2}i(q\xi - px)}dqdp \\ = & \left(\int_0^1 e^{-\frac{1}{4}|x-q|}e^{\frac{1}{2}qx+\frac{1}{2}iq\xi}dq\right)\left(\int_0^1 \chi(\xi - p)e^{\frac{1}{2}p\xi-\frac{1}{2}ipx}dp\right) \end{aligned} \tag{11.13}$$

for all ζ in $\mathbb{C}$. But for $x > 1$ and $0 < \xi < 1$, we get from (11.13)

$$(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta) = \left(\int_0^1 e^{-\frac{1}{4}x}e^{\frac{1}{4}q(1+2\zeta)}dq\right)\left(\int_0^1 e^{-\frac{1}{2}ip\zeta}dp\right)$$

$$
\begin{aligned}
&= \frac{4e^{-\frac{1}{4}x}}{1+2\zeta}\left(e^{\frac{1}{4}(1+2\zeta)}-1\right)\frac{2i}{\zeta}\left(e^{-\frac{1}{2}i\zeta}-1\right) \\
&= \frac{4e^{\frac{1}{4}x}}{1+2\zeta}\left(e^{\frac{1}{4}+\frac{1}{2}i\xi}-e^{-\frac{1}{2}x}\right)\frac{2i}{\zeta}\left(e^{\frac{1}{2}\xi}e^{-\frac{1}{2}ix}-1\right)
\end{aligned}
$$

and hence $\hat{F} *^{\frac{1}{2}} \hat{G} \notin L^2(\mathbb{C})$.

In view of Remark 11.4 and Example 11.5, it is a natural problem to seek some subspace of $L^2(\mathbb{C}^n)$ such that the product of two Daubechies operators associated to functions in the subspace is indeed a Daubechies operator associated to a function in $L^2(\mathbb{C}^n)$.

For any nonnegative real number c, we denote by $\mathcal{S}_c$ the set of all measurable functions F on $\mathbb{C}^n$ such that

$$|\hat{F}(\zeta)| \leq e^{-c|\zeta|^2}|f(\zeta)|, \quad \zeta \in \mathbb{C}^n,$$

for some function f in $L^2(\mathbb{C}^n)$. It is clear that $\mathcal{S}_c$ is a subspace of $L^2(\mathbb{C}^n)$ for all $c \geq 0$. It is also clear that if $c \leq d$, then $\mathcal{S}_d \subseteq \mathcal{S}_c$.

We can now give a formula for the product of two Daubechies operators associated to functions in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$.

Theorem 11.6 *Let F and G be functions in $\mathcal{S}_c$, where $c > \frac{1+\sqrt{5}}{8}$. Then the product of the Daubechies operators $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $D_G : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Daubechies operator $D_H : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $H \in \bigcap_{0<d<c'} \mathcal{S}_d$, $c' = c - \frac{1}{4} - \frac{4c^2}{8c+1} > 0$, and*

$$\hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G}).$$

Proof: Let f and g be functions in $L^2(\mathbb{C}^n)$ such that

$$|\hat{F}(\zeta)| \leq e^{-c|\zeta|^2}|f(\zeta)| \tag{11.14}$$

and

$$|\hat{G}(\zeta)| \leq e^{-c|\zeta|^2}|g(\zeta)| \tag{11.15}$$

for all ζ in $\mathbb{C}^n$. Then, by (11.2), (11.14) and (11.15), we get, for all ζ in $\mathbb{C}^n$,

$$\begin{aligned}
&|(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \\
= &\left| \int_{\mathbb{C}^n} \hat{F}(\zeta - \omega)\hat{G}(\omega)\, e^{\frac{1}{2}(\zeta\cdot\bar{\omega} - |\omega|^2)} d\omega \right| \\
\leq &\int_{\mathbb{C}^n} |\hat{F}(\zeta-\omega)||\hat{G}(\omega)|\, e^{\frac{1}{2}|\zeta||\omega|}\, e^{-\frac{1}{2}|\omega|^2} d\omega \\
\leq &\int_{\mathbb{C}^n} e^{-c|\zeta-\omega|^2} |f(\zeta-\omega)|\, e^{-c|\omega|^2} |g(\omega)|\, e^{\frac{1}{4}(|\zeta|^2+|\omega|^2)}\, e^{-\frac{1}{2}|\omega|^2} d\omega \\
\leq &\; e^{-(c-\frac{1}{4})|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta-\omega)||g(\omega)|\, e^{2c\mathrm{Re}(\zeta\cdot\bar{\omega})}\, e^{-(2c+\frac{1}{4})|\omega|^2} d\omega. \quad (11.16)
\end{aligned}$$

But, for any positive number ε, we have

$$\begin{aligned}
2c\mathrm{Re}(\zeta\cdot\bar{\omega}) &\leq 2c|\zeta\cdot\bar{\omega}| \leq 2c|\zeta||\omega| \\
&= 2c\sqrt{\varepsilon}\,|\zeta|\frac{|\omega|}{\sqrt{\varepsilon}} \\
&\leq c\left(\varepsilon|\zeta|^2 + \frac{1}{\varepsilon}|\omega|^2\right) \quad (11.17)
\end{aligned}$$

for all ζ and ω in $\mathbb{C}^n$. So, by (11.16) and (11.17), we have, for all ζ in $\mathbb{C}^n$,

$$|(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-(c-\frac{1}{4}-c\varepsilon)|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta-\omega)||g(\omega)|\, e^{-(2c+\frac{1}{4}-\frac{c}{\varepsilon})|\omega|^2} d\omega. \quad (11.18)$$

Since $c > \frac{1+\sqrt{5}}{8}$, it follows from (11.18) that for any positive number ε such that

$$\frac{c}{2c+\frac{1}{4}} < \varepsilon < 1 - \frac{1}{4c}, \quad (11.19)$$

there exists a positive constant d_ε such that

$$|(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2} \int_{\mathbb{C}^n} |f(\zeta-\omega)||g(\omega)|\, e^{-d_\varepsilon|\omega|^2} d\omega, \quad \zeta \in \mathbb{C}^n, \quad (11.20)$$

where

$$c_\varepsilon = c - \frac{1}{4} - c\varepsilon. \quad (11.21)$$

Since, for any ε satisfying (11.19), the function $|g|e^{-d_\varepsilon|\cdot|^2}$ is in $L^1(\mathbb{C}^n)$, it follows from Young's inequality that the function h_ε on $\mathbb{C}^n$ defined by

$$h_\varepsilon(\zeta) = \int_{\mathbb{C}^n} |f(\zeta - \omega)||g(\omega)|\, e^{-d_\varepsilon|\omega|^2} d\omega, \quad \zeta \in \mathbb{C}^n, \tag{11.22}$$

is in $L^2(\mathbb{C}^n)$. Thus, by (11.20) and (11.22),

$$|(\hat{F} *^{\frac{1}{2}} \hat{G})(\zeta)| \leq e^{-c_\varepsilon|\zeta|^2} h_\varepsilon(\zeta), \quad \zeta \in \mathbb{C}^n, \tag{11.23}$$

for any ε satisfying (11.19). Now, by Plancherel's theorem, let $H \in L^2(\mathbb{C}^n)$ be such that

$$\hat{H} = (2\pi)^{-n}(\hat{F} *^{\frac{1}{2}} \hat{G}). \tag{11.24}$$

Then, by (11.23) and (11.24), $H \in \mathcal{S}_{c_\varepsilon}$, and hence, by (11.19) and (11.21), $H \in \bigcap_{0<d<c'} \mathcal{S}_d$. That the Daubechies operator $D_H : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the product of the Daubechies operators $D_F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $D_G : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is then a consequence of (11.24) and Corollary 11.3. □

Remark 11.7 The results in this chapter can be found in the paper [3] by Du and Wong.

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Index

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