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THE PROCEEDINGS OF THE CONFERENCE ON GEOMETRIC STRUCTURES ON MANIFOLDS

Edited by
SUHYOUNG CHOI
HYUK KIM
HYUNKOO LEE

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
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Suhyoung Choi, Hyuk Kim and Hyunkoo Lee
Department of Mathematics
Seoul National University
Seoul, 151-742
Korea

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Preface

This collection of survey articles are from two conferences and three series of lectures given in the second half of 1997: the Workshop on Gauge Theory (July 28 – August 2), the Conference on Geometric Structures on Manifolds (September 29 – October 2) and the series of lectures given by Professors Boris Apanasov, Steven Kerckhoff, and Sadayoshi Kojima. The concentration on this topic during the second half of 1997 exposed the graduate students and the researchers in Korea to many aspects of manifold topology from geometric perspectives.

Several aspects of the current state of the manifold topology theory were presented and discussed. The first is the gauge theory on 3- and 4-manifolds involving Casson and Seiberg-Witten invariants. The second is that of singular hyperbolic 3-manifolds which recently advanced by efforts of Hodgson, Kerckhoff, and Kojima who all gave presentations here. The third is that of affine and projective structures on manifolds. Barbot and Choi in particular gave presentations of the classification of radiant affine 3-manifolds and the resolution of the Carriére conjecture. A significant tie between affine differential geometry and the representations of the fundamental groups of surfaces was presented by Labourie. The fourth was exposed by Apanasov, Kamishima, and Inkang Kim who made progress in rigidity questions of discrete groups. There are other significant developments exposed during this period but we end with the above inadequate mention.

The Workshop on Gauge theory was supported by the Research Institute of Mathematics (RIM, SNU) and the conference and the series of talks were supported by the Global Analysis Research Center (GARC, SNU) funded by the Korea Science and Engineering Foundation. We are very thankful of the generous financial support and the encouragement and guidance of Professor Sang-Moon Kim, the director of GARC and Professor Sung-Ki Kim, the director of RIM. We also thank many graduate students in our department for their help in setting up and running the workshops smoothly.

Suhyoung Choi, Hyuk Kim, and Hyunkoo Lee

September 1999

The List of Speakers

- Workshop on Gauge Theory (July 28–August 2):
  Yongseung Cho (Ewha Womans Univ.), Christopher Herald (Swarthmore College), Jongsu Kim (Sogang Univ.), Thomas Leness (Michigan State Univ.), Jongil Park (UC, Irvine), Myong-Hee Sung (SUNY), Jinsung Park (GARC), Yunhi Cho (GARC), Yoonhi Hong (Ewha Womans Univ.), Misung Cho (Ewha Womans Univ.)
- Lectures (September 8–12):
  Steven Kerckhoff (Stanford Univ.)
- Conference on Geometric Structures on Manifolds (September 29–October 2):
  Thierry Barbot (ENS de Lyon), Yunhi Cho (GARC), Suhyoung Choi (Seoul N. Univ.), Craig Hodgson (Univ. Melbourne), Sungbok Hong (Korea Univ.), Yoshinobu Kamishima (Kumamoto Univ.), Ann-Chi Kim (Pusan N. Univ.), Inkang Kim (KAIST), François Labourie (Univ. Paris-Sud), Shigenori Matsumoto (Nihon Univ.), Ken'ichi Ohshika (Univ. Tokyo), Kyungsoo Park (Seoul N. Univ.), Teruhiko Soma (Tokyo Denki Univ.), A. Zeghib (ENS de Lyon)
- Lectures (November 3–7):
  Boris Apanasov (Univ. Oklahoma), Sadayoshi Kojima (Tokyo Institute of Tech.)

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COMPLEX HYPERBOLIC MANIFOLDS: RIGIDITY VERSUS FLEXIBILITY AND INSTABILITY OF DEFORMATIONS

BORIS APANASOV†

ABSTRACT. The paper studies deformations of noncompact complex hyperbolic manifolds (with locally Bergman metric), varieties of discrete representations of their fundamental groups into $PU(n,1)$ and the problem of (quasiconformal) stability of deformations of such groups and manifolds in the sense of L. Bers and D. Sullivan.

1. INTRODUCTION

This paper presents a recent progress in the theory of deformations of noncompact complex hyperbolic manifolds M (of infinite volume, with variable sectional curvature) and spherical Cauchy-Riemannian manifolds at their infinity M_∞, varieties of discrete faithful representations of the fundamental groups $\pi_1 M$ into $PU(n,1)$, and the problem of (quasiconformal) stability of deformations of such groups and manifolds whose geometry makes them surprisingly different from those in the real hyperbolic geometry with constant negative sectional curvature.

Geometry of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ is the geometry of the unit ball $\mathbb{B}^n_{\mathbb{C}}$ in $\mathbb{C}^n$ with the Kähler structure given by the Bergman metric whose automorphisms are biholomorphic automorphisms of the ball, i.e., elements of $PU(n,1)$. We notice that complex hyperbolic manifolds (modeled on $\mathbb{H}^n_{\mathbb{C}}$) with non-elementary fundamental groups are complex hyperbolic in the sense of S. Kobayashi [Kob]; we refer the reader to [AX1, CG, G4] for general information on such manifolds, in particular for several equivalent descriptions of the basic class of geometrically finite complex hyperbolic manifolds and for a discussion on surprising differences between such manifolds and real hyperbolic manifolds with constant negative sectional curvature. Here we study deformations of complex hyperbolic manifolds and their fundamental groups by using the spherical Cauchy-Riemannian geometry at infinity. This CR-geometry is modeled on the one point compactification of the (nilpotent) Heisenberg group $\mathcal{H}_n$, which appears as the sphere at infinity of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. Since any complex hyperbolic manifold can be represented as the quotient $M = \mathbb{H}^n_{\mathbb{C}}/G$ by a discrete torsion free isometry action of the fundamental group of M, $\pi_1(M) \cong G \subset PU(n,1)$, its boundary at infinity $\partial_\infty M$ is naturally identified as the quotient $\Omega(G)/G$ of the discontinuity set of G at infinity. Here the discontinuity set $\Omega(G)$ is the maximal subset of $\partial\mathbb{H}^n_{\mathbb{C}}$ where G acts

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Key words and phrases. Negative curvature, complex hyperbolic geometry, Cauchy-Riemannian manifolds, discrete subgroups of $PU(n,1)$, disk and circle bundles over surfaces, equivariant homeomorphisms, geometric isomorphisms, quasiconformal maps, deformations of geometric structures, stability, Teichmüller spaces.

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discretely; its complement $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}} \backslash \Omega(G)$ is the limit set of G, $\Lambda(G) = \overline{G(x)} \cap \partial\mathbb{H}^n_{\mathbb{C}}$ for any $x \in \mathbb{H}^n_{\mathbb{C}}$.

One can reduce the study of deformations of complex hyperbolic manifold M, or equivalently the Teichmüller space $\mathcal{T}(M)$ of isotopy classes of complex hyperbolic structures on M, to studying the variety $\mathcal{T}(G)$ of conjugacy classes of discrete faithful representations $\rho : G \to PU(n,1)$ (involving the space $\mathcal{D}(M)$ of the developing maps, see [G2, FG]). Here $\mathcal{T}(G) = \mathcal{R}_0(G)/PU(n,1)$, and the variety $\mathcal{R}_0(G) \subset \operatorname{Hom}(G, PU(n,1))$ consists of discrete faithful representations ρ of the group G whose co-volume $\operatorname{Vol}(\mathbb{H}^n_{\mathbb{C}}/G)$ may be infinite.

Due to the Mostow rigidity theorem [Mo1], hyperbolic structures of finite volume and (real) dimension at least three are uniquely determined by their topology, and one has no continuous deformations of them. Despite that, real hyperbolic manifolds N can be deformed as conformal manifolds, or equivalently as higher-dimensional hyperbolic manifolds $M = N \times (0,1)$ of infinite volume. First such deformations were given by the author [A2] and, after Thurston's "Mickey Mouse" example [T], they were called bendings of N along its totally geodesic hypersurfaces, see also [A1, A3-A5, JM, Ko]. Furthermore such a flexibility of the real hyperbolic geometry is emphasized by the fact that all those deformations can be induced by continuous families of G-equivariant quasiconformal self-homeomorphisms $f_t : \overline{\mathbb{H}^n_{\mathbb{R}}} \to \overline{\mathbb{H}^n_{\mathbb{R}}}$ of the closure of the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. In particular, these G-equivariant quasiconformal homeomorphisms deform continuously the limit set $\Lambda(G) \subset \partial\mathbb{H}^n_{\mathbb{R}}$ (of a "Fuchsian group" $G \subset \operatorname{Isom}\mathbb{H}^{n-1}_{\mathbb{R}} \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{R}}$) from a round sphere $\partial\mathbb{H}^{n-1}_{\mathbb{R}} = S^{n-2} \subset S^{n-1} = \partial\mathbb{H}^n_{\mathbb{R}}$ into nondifferentiably embedded (nonrectifiable) topological $(n-2)$-spheres in $\partial\mathbb{H}^n_{\mathbb{R}}$ which are the limit sets $\Lambda(G_t)$ of "quasi-Fuchsian groups" $G_t = f_t G f_t^{-1} \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{R}}$, and obviously the restrictions $f_t|_{\Lambda(G)} : \Lambda(G) \to \Lambda(G_t)$ are quasisymmetric maps.

Such a geometric realization of isomorphisms of discrete groups became the start point in our study of deformations of discrete groups of isometries of negatively curved spaces X, see [A7]:

Problem. *Given an isomorphism $\varphi : G \to H$ of geometrically finite discrete groups $G, H \subset \operatorname{Isom} X$, find subsets $X_G, X_H \subset \overline{X}$ invariant for the action of groups G and H, respectively, and an equivariant homeomorphism:*

$$f_\varphi : X_G \to X_H \quad \varphi(g) \circ f_\varphi = f_\varphi \circ g \quad \text{for all } g \in G,$$

which induces the isomorphism φ. Determine metric properties of f_φ, in particular whether it is either quasisymmetric or quasiconformal with respect to the given negatively curved metric d in X (or the induced sub-Riemannian Carnot-Carathéodory structure at infinity ∂X).

If the groups $G, H \subset \operatorname{Isom} X$ are neither lattices nor trivial and have parabolic elements, the only known geometric realization of their isomorphisms in dimension $\dim X \geq 3$ is due to P. Tukia's [Tu] isomorphism theorem for real hyperbolic spaces $X = \mathbb{H}^n_{\mathbb{R}}$. However, the Tukia's construction (based on geometry of convex hulls of the limit sets $\Lambda(G)$ and $\Lambda(H)$) cannot be used in the case of variable negative curvature due to lack of control over convex hulls (where the convex hull of three points may be 4-dimensional), especially nearby parabolic fixed points. However, as a first step in solving the above geometrization Problem, we have the following isomorphism theorem [A8, A10] in the complex hyperbolic space:

Theorem 3.2. *Let $\varphi : G \to H$ be a type preserving isomorphism of two non-elementary geometrically finite discrete subgroups $G, H \subset \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$. Then there exists a unique equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ of their limit sets that induces the isomorphism ϕ.*

However, in contrast to the real hyperbolic case where such geometric realizations of type preserving isomorphisms of geometrically finite groups are always quasisymmetric maps [Tu], it is doubtful that the (unique) equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ constructed in Theorem 3.2 is always CR-quasisymmetric (with respect to the CR-structure on the Heisenberg group $\mathcal{H}_n = \partial\mathbb{H}^n_{\mathbb{C}}\backslash\{\infty\}$). Namely, a possible obstruction to quasisymmetricity directly appear from the following corollary of our construction in Section 4:

Corollary 4.2. *Let $M = \mathbb{H}^2_{\mathbb{C}}/G$ be a complex hyperbolic surface with the holonomy group $G \subset PU(1,1) \subset PU(2,1)$ that represents the total space of a non-trivial disk bundle over a Riemann surface of genus $p \geq 0$ with at least four punctures (hyperbolic 2-orbifold with at least four punctures). Then the Teichmüller space $\mathcal{T}(M)$ contains a smooth simple curve $\alpha : [0, \pi/2) \hookrightarrow \mathcal{T}(M)$ with the following properties:*

1. *the curve α passes through the surface $M = \alpha(0)$;*
2. *each complex hyperbolic surface $M_t = \alpha(t) = \mathbb{H}^2_{\mathbb{C}}/G_t$, $t \in [0, \pi/2)$, with the holonomy group $G_t \subset PU(2,1)$ is homeomorphic to the surface M;*
3. *for any parameter t, $0 < t < \pi/2$, the complex hyperbolic surface M_t is not quasiconformally equivalent to the surface M.*

Besides the claims in this Corollary, it follows also from the construction of the above complex hyperbolic surfaces M and M_t that their boundaries, the spherical CR-manifolds $N = \partial M = \Omega(G)/G$ and $N_t = \partial M_t = \Omega(G_t)/G_t$ have similar properties. Namely these 3-dimensional CR-manifolds $\{N_t\}$, $0 \leq t < \pi/2$, represent points of a smooth simple curve $\alpha_\infty : [0, \pi/2) \hookrightarrow \mathcal{T}(N)$ in the Teichmüller space $\mathcal{T}(N)$ of the manifold $N = N_0 = \partial M$, are mutually homeomorphic total spaces of non-trivial circle bundles over a Riemann surface of genus $p \geq 0$ with at least four punctures, however none of $\{N_t\}$ with $t > 0$ is quasiconformally equivalent to $N = N_0$. We note that, for the simplest case of manifolds with cyclic fundamental groups, a similar (though based on different ideas) effect of homeomorphic but not quasiconformally equivalent spherical CR-manifolds has been also recently presented by R. Miner [Mi].

It is quite natural that the result in Corollary 4.2 is related to the classical problem of quasiconformal stability of deformations from the theory of Kleinian groups, in particular to well known results by L. Bers [Be1, Be2] and D. Sullivan [Su1]. Following to L. Bers [Be2], a finitely generated Kleinian group $G \subset \mathrm{PSL}(2,\mathbb{C})$ is said to be quasiconformally stable if every homomorphism $\chi : G \to \mathrm{PSL}(2,\mathbb{C})$ preserving the square traces of parabolic and elliptic elements (hence type-preserving) and sufficiently close to the identity is induced by an equivarint quasiconformal mapping $w : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $\chi(g) = wgw^{-1}$ for all $g \in G$. Due to Bers's criterion (involving the quadratic differentials for G, see [Be1]), it follows that Fuchsian groups, Schottky groups, groups of Schottky type and ceratin non-degenerate B-groups are all quasiconformally stable [Be2]. Changing the condition on homomorphisms χ in terms of the trace of elements $g \in G$ to the condition that χ preserves the type of elements of a discrete groups G, one has a natural generalization of quasiconformal stability for discrete groups $G \subset PU(n,1)$. In that sense,

B. Aebisher and R. Miner [AM] recently proved that (classical) Schottky subgroups $G \subset PU(n,1)$ are quasiconformally stable. Nevertheless, as our construction in Section 4 shows, Fuchsian groups $G \subset PU(2,1)$ are quasiconformally unstable:

Theorem 4.1. *There are co-finite Fuchsian groups $G \subset PU(1,1) \subset PU(2,1)$ with signatures $(g, r; m_1, \ldots, m_r)$, where genus $g \geq 0$ and there are at least four cusps (with branching orders $m_i = \infty$), such that:*

1. *the Teichmüller space $\mathcal{T}(G)$ of such a group G contains a smooth simple curve α, $\alpha : [0, \pi/2) \hookrightarrow \mathcal{T}(G)$, that passes through the Fuchsian group $G = \alpha(0)$, and whose points $\alpha(t) = G_t \subset PU(2,1)$, $0 < t < \pi/2$, are all non-trivial quasi-Fuchsian groups;*
2. *each isomorphism $\chi : G \to G_t$, $0 < t < \pi/2$, is induced by a G-equivariant homeomorphism $f_t : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$ of the closure $\overline{\mathbb{H}^2_{\mathbb{C}}} = \mathbb{H}^2_{\mathbb{C}} \cup \partial \mathbb{H}^2_{\mathbb{C}}$ of the complex hyperbolic space;*
3. *for any parameter t, $0 < t < \pi/2$, the action of the quasi-Fuchsian group G_t is not quasiconformally conjugate to the action of the Fuchsian group $G = \alpha(0)$ (in both CR-structure at infinity $\partial \mathbb{H}^2_{\mathbb{C}} = \mathcal{H}_2 \cup \{\infty\}$ and the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$).*

However it is still an open question whether the actions of the constructed quasi-Fuchsian groups G_t and $G_{t'}$ on their limit sets $\Lambda(G_t)$ and $\Lambda(G_{t'})$ could be "quasiconformally" conjugate, in other words, whether the canonical G-equivariant homeomorphism $f_{\chi_t} : \Lambda(G) \to \Lambda(G_t)$ of the limit sets (constructed in Theorem 3.2) that induces the isomorphism $\phi : G \to G_t$, $0 < t < \pi/2$, is in fact quasisymmetric.

In Sections 4 and 5, we address the basic problem of existence of non-trivial deformations of "non-real" hyperbolic manifolds (in particular, complex hyperbolic ones) and their (discrete) holonomy groups which, in contrast to the described flexibility in the real hyperbolic case, seem much more rigid. Indeed, due to Pansu [P], quasiconformal maps in the sphere at infinity of quaternionic/octonionic hyperbolic spaces induced by hyperbolic quasi-isometries are necessarily automorphisms, and thus there cannot be interesting quasiconformal deformations of corresponding structures (even any topological conjugation of two different (free) Schottky groups in those spaces cannot be quasiconformal, cf. [AM]). Secondly, due to Corlette's rigidity theorem [C3], such closed manifolds of (quaternionic or octonionic) dimension at least two and corresponding uniform lattices are even super-rigid – analogously to Margulis super-rigidity in higher rank [M, A13]. The last fact and our joint work with Inkang Kim [AK] imply impossibility of quasi-Fuchsian deformations of quaternionic/octonionic manifolds of infinite volume homotopy equivalent to their closed analytic submanifolds, for quaternionic manifolds of dimension at least three, see also [Ka]. Furthermore, complex hyperbolic manifolds share the above rigidity of quaternionic/octonionic hyperbolic manifolds. Namely, due to the Goldman's local rigidity theorem in dimension $n = 2$ [G1], every nearby discrete representation $\rho : G \to PU(2,1)$ of a cocompact lattice $G \subset PU(1,1)$ stabilizes a complex geodesic in the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ (which is also true for small deformations of cocompact lattices $G \subset PU(n-1,1)$ in higher dimensions $n \geq 3$ [GM]), and thus the limit set $\Lambda(\rho G) \subset \partial \mathbb{H}^n_{\mathbb{C}}$ is always a round sphere S^{2n-3}. In higher dimensions $n \geq 3$, this local rigidity of complex hyperbolic n-manifolds M homotopy equivalent to their closed complex totally geodesic hypersurfaces is even global (at least in the connected component of representation variety [C1, BCG, Y1]).

Our goal here is to show that, in contrast to rigidity of complex hyperbolic n-manifolds M from the above class, the Stein spaces from the following two classes of complex hyperbolic surfaces M are not rigid (it seems to us that the property of a complex hyperbolic manifold to be a Stein space is crucial for its flexibility). Such a flexibility has two independent aspects related to both conditions in the Goldman's local rigidity theorem, firstly, the existence of a complex analytic subspace homotopy equivalent to the manifold M and, secondly, compactness of that subspace.

Namely, as it follows from the above Theorem 4.1 and Corollary 4.2, the first class of non-rigid complex hyperbolic manifolds consists of complex Stein surfaces M homotopy equivalent to their *non-compact* complex analytic subspaces (Riemann surfaces of genus $p \geq 0$ with finite hyperbolic area, with at least four punctures).

The second class of non-rigid manifolds consists of Stein spaces represented by complex hyperbolic manifolds M homotopy equivalent to their closed totally *real* geodesic submanifolds. Namely, in complex dimension two, we provide a canonical construction of continuous non-trivial quasi-Fuchsian deformations of complex surfaces fibered over closed Riemannian surfaces of genus $g > 1$ depending on $3(g-1)$ continuous parameters (in addition to "Fuchsian" deformations, where in particular, the Teichmuller space of the base surface has dimension $6(g-1)$). This is the first such non-trivial deformations of fibrations with compact base (for non-compact base, see a different Goldman-Parker' deformation [GP] of ideal triangle groups $G \subset PO(2,1)$). The obtained flexibility of such holomorphic fibrations and the number of its parameters (at least $9(g-1)$) provide the first advance toward a conjecture on dimension $16(g-1)$ of the Teichmuller space of such complex surfaces. It is related to A. Weil's theorem [W] (see also [G3, p.43]), that the variety of conjugacy classes of all (not necessarily discrete) representations $G \to PU(2,1)$ near the embedding $G \subset PO(2,1)$ is a real-analytic manifold of dimension $16(g-1)$. We remark that discreteness of representations of $G \cong \pi_1 M$ is an essential condition for deformation of a complex manifold M which does not follow from the mentioned Weil's result.

Our construction here is inspired by the well know bending deformations of real hyperbolic (conformal) manifolds along totally geodesic hypersurfaces. In the case of complex hyperbolic (and Cauchy-Riemannian) structures, it works however in a different way than that in the real hyperbolic case. Namely our complex bending deformations involve simultaneous bending of the base of the fibration of the complex surface M as well as bendings of each of its totally geodesic fibers (see Remark 5.4). Such bending deformations of complex surfaces are associated to their real simple closed geodesics (of real codimension 3), but have nothing common with cone deformations of real hyperbolic 3-manifolds along closed geodesics (see [A4, A5]).

Furthermore, there are well known complications (cf. [KR3, P, Cap]) in constructing equivariant quasiconformal homeomorphisms in the complex hyperbolic space and in Cauchy-Riemannian geometry, which are due to necessary conditions for such maps to preserve the Kähler and contact structures (correspondingly in the complex hyperbolic space and at its infinity, the one-point compactification of the Heisenberg group$\mathcal{H}_n$). Despite that, as it follows from our construction, the complex bending deformations are induced by equivariant homeomorphisms which are in addition quasiconformal with respect to the corresponding metrics. One of our main results in this direction may be formulated as follows.

Theorem 5.1. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a given non-elementary discrete group. Then, for any simple closed geodesic α in the Riemann 2-surface $S = H^2_{\mathbb{R}}/G$ and a sufficiently small $\eta_0 > 0$, there is a holomorphic family of G-equivariant quasiconformal homeomorphisms $F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$, $-\eta_0 < \eta < \eta_0$, which defines a bending (quasi-Fuchsian) deformation $\mathcal{B}_\alpha : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ of the group G along the geodesic α, $\mathcal{B}_\alpha(\eta) = F^*_\eta$.*

We notice that such complex bending deformations depend on many independent parameters, as it is shown by application of our construction and Élie Cartan [Car] angular invariant in Cauchy-Riemannian geometry:

Corollary 5.2. *Let $S_p = \mathbb{H}^2_{\mathbb{R}}/G$ be a closed totally real geodesic surface of genus $p > 1$ in a given complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, $G \subset PO(2,1) \subset PU(2,1)$. Then there is a real analytic embedding $\pi \circ \mathcal{B} : B^{3p-3} \hookrightarrow \mathcal{T}(M)$ of a real $(3p-3)$-ball into the Teichmüller space of M, defined by bending deformations along disjoint closed geodesics in M and by the projection $\pi : \mathcal{D}(M) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ in the development space $\mathcal{D}(M)$.*

The above embedding and the fact that the Teichmuller space of the base surface S_p (totally geodesically) embedded in the complex surface M is a complex manifold of dimension $3(g-1)$ show that we have in fact a real analytic embedding $B^{9p-9} \hookrightarrow \mathcal{T}(M)$ of a real $9(p-1)$-ball into the Teichmüller space of the complex hyperbolic surface M.

In our subsequent work, we apply the constructed bending deformations to answer a well known question about cusp groups on the boundary of the Teichmüller space of $\mathcal{T}(M)$ of a Stein complex surface M fibering over a compact Riemann surface of genus $p > 1$:

Theorem 5.6. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then, for any simple closed geodesic $\alpha \subset S_p = H^2_{\mathbb{R}}/G$, there is a continuous deformation $\rho_t = f^*_t$ induced by G-equivariant quasiconformal homeomorphisms $f_t : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$ whose limit representation ρ_∞ corresponds to a boundary cusp point of the Teichmüller space $\mathcal{T}(G)$, that is the boundary group $\rho_\infty(G)$ has an accidental parabolic element $\rho_\infty(g_\alpha)$ where $g_\alpha \in G$ represents the geodesic $\alpha \subset S_p$.*

We note that, due to our construction of such continuous quasiconformal deformations, they are independent if the corresponding geodesics $\alpha_i \subset S_p$ are disjoint. It implies the existence of a boundary group in $\partial\mathcal{T}(G)$ with "maximal" number of non-conjugate accidental parabolic subgroups:

Corollary 5.7. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then there is a continuous deformation $R : \mathbb{R}^{3p-3} \to \mathcal{T}(G)$ whose boundary group $G_\infty = R(\infty)(G)$ has $3p-3$ non-conjugate accidental parabolic subgroups.*

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2. Complex hyperbolic geometry and geometrical finiteness

Here we recall some known facts (see, for example, [AX1, GP1, G4, KR1]) concerning the Kähler geometry of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, its link with the nilpotent geometry of the Heisenberg group $\mathcal{H}_n$ induced on each horosphere in $\mathbb{H}^n_{\mathbb{C}}$, and the Cauchy-Riemannian geometry (and contact structure) in the $(2n-1)$-sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}}$ which can be identified with the one-point compactification $\overline{\mathcal{H}_n} = \mathcal{H}_n \cup \{\infty\}$ of the Heisenberg group.

One can realize the complex hyperbolic space,

$$\mathbb{H}^n_{\mathbb{C}} = \{[z] \in \mathbb{CP}^n \ : \ z \in \mathbb{C}^{n,1}\, ,\ \langle z, z\rangle < 0\}\,,$$

as the set of negative lines in the Hermitian vector space $\mathbb{C}^{n,1}$, with Hermitian structure given by the indefinite $(n,1)$-form $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n - z_{n+1}\overline{w}_{n+1}$. Its boundary $\partial\mathbb{H}^n_{\mathbb{C}} = \{[z] \in \mathbb{CP}^{n,1} \ : \ \langle z, z\rangle = 0\}$ consists of all null lines in $\mathbb{CP}^n$ and is homeomorphic to the $(2n-1)$-sphere S^{2n-1}.

There are two common models of complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ as domains in $\mathbb{C}^n$, the unit ball $\mathbb{B}^n_{\mathbb{C}}$ and the Siegel domain $\mathfrak{S}_n$. They arise from two affine patches in the projective space $\mathbb{CP}^n$ related to $\mathbb{H}^n_{\mathbb{C}}$ and its boundary. Namely, embedding $\mathbb{C}^n$ onto the affine patch of $\mathbb{CP}^{n,1}$ defined by $z_{n+1} \neq 0$ (in homogeneous coordinates) as $A : \mathbb{C}^n \to \mathbb{CP}^n, z \mapsto [(z,1)]$, we may identify the unit ball $\mathbb{B}^n_{\mathbb{C}}(0,1) \subset \mathbb{C}^n$ with $\mathbb{H}^n_{\mathbb{C}} = A(\mathbb{B}^n_{\mathbb{C}})$. Here the metric in $\mathbb{C}^n$ is defined by the standard Hermitian form $\langle\langle\, , \rangle\rangle$, and the induced metric on $\mathbb{B}^n_{\mathbb{C}}$ is the Bergman metric (with constant holomorphic curvature -1) whose sectional curvature is between -1 and -1/4.

The full group $\operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ of isometries of (the ball model of) $\mathbb{H}^n_{\mathbb{C}}$ is generated by the group of holomorphic automorphisms of the ball $\mathbb{B}^n_{\mathbb{C}}$ (=the projective unitary group $PU(n,1)$ defined by the group $U(n,1)$ of unitary automorphisms of $\mathbb{C}^{n,1}$ that preserve $\mathbb{H}^n_{\mathbb{C}}$), together with the antiholomorphic automorphism of $\mathbb{H}^n_{\mathbb{C}}$ defined by the $\mathbb{C}$-antilinear unitary automorphism of $\mathbb{C}^{n,1}$ given by the complex conjugation $z \mapsto \bar{z}$. The group $PU(n,1)$ can be embedded in a linear group due to A. Borel [Bor] (cf. [AX1, L.2.1]), hence any finitely generated group $G \subset PU(n,1)$ is residually finite and has a finite index torsion free subgroup. Elements $g \in PU(n,1)$ are of the following three types. If g fixes a point in $\mathbb{H}^n_{\mathbb{C}}$, it is called *elliptic*. If g has exactly one fixed point in the closure $\overline{\mathbb{H}^n_{\mathbb{C}}} \cong \overline{\mathbb{B}^n_{\mathbb{C}}}$, and it lies in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *parabolic*. If g has exactly two fixed points, and they lie in $\partial\mathbb{H}^n_{\mathbb{C}}$, g is called *loxodromic*. These three types exhaust all the possibilities.

The second model of $\mathbb{H}^n_{\mathbb{C}}$, as the Siegel domain, arises from the affine patch complimentary to a projective hyperplane H_∞ which is tangent to $\partial\mathbb{H}^n_{\mathbb{C}}$ at a point $\infty \in \partial\mathbb{H}^n_{\mathbb{C}}$. For example, taking that point ∞ as $(0', -1, 1)$ with $0' \in \mathbb{C}^{n-1}$ and $H_\infty = \{[z] \in \mathbb{CP}^n : z_n + z_{n+1} = 0\}$, one has the map $\mathbf{S} : \mathbb{C}^n \to \mathbb{CP}^n \backslash H_\infty$ such that

$$\begin{pmatrix} z' \\ z_n \end{pmatrix} \longmapsto \begin{bmatrix} z' \\ \frac{1}{2} - z_n \\ \frac{1}{2} + z_n \end{bmatrix} \quad \text{where} \quad z' = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1}\,.$$

In the obtained affine coordinates, the complex hyperbolic space is identified with the *Siegel domain*

$$\mathfrak{S}_n = \mathbf{S}^{-1}(\mathbb{H}^n_{\mathbb{C}}) = \{z \in \mathbb{C}^n \ : \ z_n + \overline{z}_n > \langle\langle z', z'\rangle\rangle\}\,,$$

where the Hermitian form is $\langle \mathbf{S}(z), \mathbf{S}(w)\rangle = \langle\langle z', w'\rangle\rangle - z_n - \overline{w}_n$. The automorphism group of this affine model of $\mathbb{H}^n_{\mathbb{C}}$ is the group of affine transformations of $\mathbb{C}^n$ preserving $\mathfrak{S}_n$. Its stabilizer of the point ∞ is $\mathcal{H}_n \rtimes U(n-1)\cdot \exp(t)$ where $\mathcal{H}_n$ is its unipotent radical, the *Heisenberg group* that consists of all *Heisenberg translations*

$$T_{\xi,v} : (w', w_n) \mapsto \left(w' + \xi, w_n + \langle\langle \xi, w'\rangle\rangle + \frac{1}{2}(\langle\langle \xi, \xi\rangle\rangle - iv)\right),$$

where $w', \xi \in \mathbb{C}^{n-1}$ and $v \in \mathbb{R}$.

In particular, $\mathcal{H}_n$ acts simply transitively on $\partial\mathfrak{S}_n\backslash\{\infty\}$ and on each horosphere H_t (in the complex hyperbolic space) centered at ∞, which in fact has the form:

$$H_t = \{(z', z_n) \in \mathfrak{S}_n : z_n + \overline{z}_n - \langle\langle z', z'\rangle\rangle = t\}, \quad t > 0.$$

On the base of that, one obtains the *upper half space model* for the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ by identifying $\mathbb{C}^{n-1}\times\mathbb{R}\times[0,\infty)$ and $\overline{\mathbb{H}^n_{\mathbb{C}}}\backslash\{\infty\}$ as

$$(\xi, v, u) \longmapsto \begin{bmatrix} \xi \\ \frac{1}{2}(1 - \langle\langle \xi, \xi\rangle\rangle - u + iv) \\ \frac{1}{2}(1 + \langle\langle \xi, \xi\rangle\rangle + u - iv) \end{bmatrix} \in \partial\mathfrak{S}_n\backslash\{\infty\},$$

where $(\xi, v, u) \in \mathbb{C}^{n-1}\times\mathbb{R}\times[0,\infty)$ are the horospherical coordinates of the corresponding point in $\overline{\mathbb{H}^n_{\mathbb{C}}}\backslash\{\infty\}$ (with respect to the point $\infty \in \partial\mathbb{H}^n_{\mathbb{C}}$, see [GP1]).

We notice that, under this identification, the geodesics running to ∞ are the vertical lines $c_{\xi,v}(t) = (\xi, v, e^{2t})$ passing through points $(\xi, v) \in \mathbb{C}^{n-1}\times\mathbb{R}$. Also we see that, via the geodesic perspective from ∞, the "boundary plane" $H_0 = \mathbb{C}^{n-1}\times\mathbb{R}\times\{0\} = \partial\mathbb{H}^n_{\mathbb{C}}\backslash\{\infty\}$ and various horospheres correspond as $H_t \to H_u$ with $(\xi, v, t) \mapsto (\xi, v, u)$. Each of them can be identified with the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1}\times\mathbb{R}$. It is a 2-step nilpotent group with center $\{0\}\times\mathbb{R} \subset \mathbb{C}^{n-1}\times\mathbb{R}$, with the isometric action on itself and on $\mathbb{H}^n_{\mathbb{C}}$ by left translations:

$$T_{(\xi_0,v_0)} : (\xi, v, u) \longmapsto (\xi_0 + \xi, v_0 + v + 2\,\mathrm{Im}\langle\langle \xi_0, \xi\rangle\rangle, u),$$

and the inverse of (ξ, v) is $(\xi, v)^{-1} = (-\xi, -v)$. The unitary group $U(n-1)$ acts on $\mathcal{H}_n$ and $\mathbb{H}^n_{\mathbb{C}}$ by rotations: $A(\xi, v, u) = (A\xi, v, u)$ for $A \in U(n-1)$. The semidirect product $\mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$ is naturally embedded in $U(n,1)$ as follows:

$$A \longmapsto \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(n,1) \quad \text{for} \quad A \in U(n-1),$$

$$(\xi, v) \longmapsto \begin{pmatrix} I_{n-1} & \xi & \xi \\ -\bar{\xi}^t & 1 - \frac{1}{2}(|\xi|^2 - iv) & -\frac{1}{2}(|\xi|^2 - iv) \\ \bar{\xi}^t & \frac{1}{2}(|\xi|^2 - iv) & 1 + \frac{1}{2}(|\xi|^2 - iv) \end{pmatrix} \in U(n,1)$$

where $(\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1}\times\mathbb{R}$ and $\bar{\xi}^t$ is the conjugate transpose of ξ.

The action of $\mathcal{H}(n)$ on $\overline{\mathbb{H}^n_{\mathbb{C}}}\backslash\{\infty\}$ also preserves the Cygan metric ρ_c there, which plays the same role as the Euclidean metric does on the upper half-space model of the real hyperbolic space $\mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$ and is induced by the following norm:

$$||(\xi, v, u)||_c = |\,||\xi||^2 + u - iv|^{1/2}, \quad (\xi, v, u) \in \mathbb{C}^{n-1}\times\mathbb{R}\times[0,\infty). \qquad (2.1)$$

The relevant geometry on each horosphere $H_u \subset \mathbb{H}^n_{\mathbb{C}}$, $H_u \cong \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$, is the spherical CR-geometry induced by the complex hyperbolic structure. The geodesic perspective from ∞ defines CR-maps between horospheres, which extend to CR-maps between the one-point compactifications $H_u \cup \infty \approx S^{2n-1}$. In the limit, the induced metrics on horospheres fail to converge but the CR-structure remains fixed. In this way, the complex hyperbolic geometry induces CR-geometry on the sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}} \approx S^{2n-1}$, naturally identified with the one-point compactification of the Heisenberg group $\mathcal{H}_n$.

Our main assumption on a complex hyperbolic n-manifold M is the geometrical finiteness of its fundamental group $\pi_1(M) = G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$, which in particular implies that the discrete group G is finitely presented [AX1].

Here a subgroup $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ is called *discrete* if it is a discrete subset of $\operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$. The *limit set* $\Lambda(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ of a discrete group G is the set of accumulation points of (any) orbit $G(y)$, $y \in \mathbb{H}^n_{\mathbb{C}}$. The complement of $\Lambda(G)$ in $\partial\mathbb{H}^n_{\mathbb{C}}$ is called the *discontinuity set* $\Omega(G)$. A discrete group G is called *elementary* if its limit set $\Lambda(G)$ consists of at most two points. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\operatorname{fix}(G)$; then $\Lambda(G) = \operatorname{fix}(G)$, and G consists of either parabolic or elliptic elements. As it was observed by many authors, parabolicity in the variable curvature case is not as easy a condition to deal with as it is in the constant curvature space. Even the notion of a parabolic cusp point become somewhat complicated. Namely, following to [Bow], a parabolic fixed point p of a discrete group $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ is called a *cusp point* if the quotient $(\Lambda(G) \setminus \{p\})/G_p$ of the limit set of G by the action of the parabolic stabilizer $G_p = \{g \in G : g(p) = p\}$ is compact. However our approach [AX1-AX3] makes this notion and geometrical finiteness in pinched negative curvature itself much more transparent.

Geometrical finiteness has been essentially used for real hyperbolic manifolds, where geometric analysis and ideas of Thurston provided powerful tools for understanding of their structure. Due to the absence of totally geodesic hypersurfaces in a space of variable negative curvature and recent results [AX1, GP1] on Dirichlet polyhedra for simplest parabolic groups in $\mathbb{H}^n_{\mathbb{C}}$, we cannot use the original definition of geometrical finiteness which came from an assumption that the corresponding real hyperbolic manifold $M = \mathbb{H}^n/G$ may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces, that is the group G should possess a finite-sided fundamental polyhedron, see [Ah]. However, we can use many other (equivalent) definitions of geometrical finitiness.

The first one, **GF1** (originally due to A. Beardon and B. Maskit [BM]) defines *geometrically finite* discrete groups $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ (and their complex hyperbolic orbifolds $M = \mathbb{H}^n_{\mathbb{C}}/G$) as those whose limit set $\Lambda(G)$ entirely consists of parabolic cusps and conical limit points. Here a limit point $z \in \Lambda(G)$ is called a *conical limit point* of G if, for some (and hence every) geodesic ray $\ell \subset \mathbb{H}^n_{\mathbb{C}}$ ending at z, there is a compact set $K \subset \mathbb{H}^n_{\mathbb{C}}$ such that $g(\ell) \cap K \neq \emptyset$ for infinitely many elements $g \in G$. The last condition is equivalent to the following [BM, AX3]:

For every geodesic ray $\ell \subset \mathbb{H}^n_{\mathbb{C}}$ ending at z and for every $\delta > 0$, there is a point $x \in \mathbb{H}^n_{\mathbb{C}}$ and a sequence of distinct elements $g_i \in G$ such that the orbit $\{g_i(x)\}$ approximates z inside the δ-neighborhood $N_\delta(\ell)$ of the ray ℓ.

Our study of geometrical finiteness in variable curvature [AX1-AX3] is based on analysis of geometry and topology of thin (parabolic) ends of corresponding manifolds and parabolic cusps of discrete isometry groups $G \subset PU(n,1)$. Namely, suppose a point $p \in \partial \mathbb{H}^n_{\mathbb{C}}$ is fixed by some parabolic element of a given discrete group $G \subset \mathrm{Isom}\, \mathbb{H}^n_{\mathbb{C}}$, and G_p is the stabilizer of p in G. Conjugating G by an element $h_p \in PU(n,1)\,, h_p(p) = \infty$, we may assume that the stabilizer G_p is a subgroup $G_\infty \subset \mathcal{H}(n) = \mathcal{H}_n \rtimes U(n-1)$. In particular, if p is the origin $0 \in \mathcal{H}_n$, the transformation h_0 can be taken as the Heisenberg inversion $\mathcal{I}$ in the hyperchain $\partial \mathbb{H}^{n-1}_{\mathbb{C}}$. It preserves the unit Heisenberg sphere $S_c(0,1) = \{(\xi, v) \in \mathcal{H}_n : ||(\xi, v)||_c = 1\}$ and acts in $\mathcal{H}_n$ as follows:

$$\mathcal{I}(\xi, v) = \left(\frac{\xi}{|\xi|^2 - iv}, \frac{-v}{v^2 + |\xi|^4} \right) \quad \text{where } (\xi, v) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}. \tag{2.1}$$

For any other point p, we may take h_p as the Heisenberg inversion $\mathcal{I}_p$ which preserves the unit Heisenberg sphere $S_c(p,1) = \{(\xi, v) : \rho_c(p, (\xi, v)) = 1\}$ centered at p. The inversion $\mathcal{I}_p$ is the conjugate of $\mathcal{I}$ by the Heisenberg translation T_p; it maps p to ∞.

After such a conjugation, we can regard the parabolic stabilizer G_p as a discrete isometry group acting in the (nilpotent) Heisenberg group $\mathcal{H}_n$. This action is completely described by our following result [AX1-AX3]:

Theorem 2.1. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, C be a compact group of automorphisms of $\mathcal{N}$, and $\Gamma \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\mathcal{N}$ and a finite index subgroup Γ^* of Γ with the following properties:*

(1) *There exists $b \in \mathcal{N}$ such that $b\Gamma b^{-1}$ preserves $\mathcal{N}_\Gamma$;*
(2) *$\mathcal{N}_\Gamma / b\Gamma b^{-1}$ is compact;*
(3) *$b\Gamma^* b^{-1}$ acts on $\mathcal{N}_\Gamma$ by left translations and the action of $b\Gamma^* b^{-1}$ on $\mathcal{N}_\Gamma$ is free.*

Due to this Theorem, there is a connected Lie subgroup $\mathcal{H}_\infty \subseteq \mathcal{H}_n$ preserved by G_∞ (up to changing the origin). So we can make the following definition.

Definition 2.2. A set $U_{p,r} \subset \overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{p\}$ is called a *standard cusp neighborhood of radius* $r > 0$ at a parabolic fixed point $p \in \partial \mathbb{H}^n_{\mathbb{C}}$ of a discrete group $G \subset PU(n,1)$ if, for the Heisenberg inversion $\mathcal{I}_p \in PU(n,1)$ with respect to the unit sphere $S_c(p,1)$, $\mathcal{I}_p(p) = \infty$, the following conditions hold:

(1) $U_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in \mathbb{H}^n_{\mathbb{C}} \cup \mathcal{H}_n : \rho_c(x, \mathcal{H}_\infty) \geq 1/r\}\right)$;
(2) $U_{p,r}$ is precisely invariant with respect to $G_p \subset G$, that is:

$$\gamma(U_{p,r}) = U_{p,r} \quad \text{for } \gamma \in G_p \quad \text{and} \quad g(U_{p,r}) \cap U_{p,r} = \emptyset \quad \text{for } g \in G \backslash G_p\,.$$

Now, due to [AX1], we can give a geometric definition of a cusp point. Namely, a parabolic fixed point $p \in \partial \mathbb{H}^n_{\mathbb{C}}$ of a discrete group $G \subset \mathrm{Isom}\, \mathbb{H}^n_{\mathbb{C}}$ is a cusp point if and only if it has a standard cusp neighborhood $U_{p,r} \subset \overline{\mathbb{H}^n_{\mathbb{C}}} \backslash \{p\}$.

This fact and [Bow] allow us to give another equivalent definitions of geometrical finiteness which is originally due to A. Marden [Ma]. In particular it follows that a discrete subgroup G in $PU(n,1)$ is *geometrically finite* (**GF2**) if and only if its quotient space

$$M(G) = [\mathbb{H}^n_{\mathbb{C}} \cup \Omega(G)]/G \tag{2.2}$$

has finitely many ends, and each of them is a cusp end, that is an end whose neighborhoods can be taken (for an appropriate $r > 0$) in the form:

$$U_{p,r}/G_p \approx (S_{p,r}/G_p) \times (0,1]\,, \tag{2.3}$$

where

$$S_{p,r} = \partial_H U_{p,r} = \mathcal{I}_p^{-1}\left(\{x \in H^n_{\mathbb{C}} \cup \mathcal{H}_n \,:\, \rho_c(x, \mathcal{H}_\infty) = 1/r\}\right) .$$

Now we see that a geometrically finite manifold can be decomposed into a compact submanifold and finitely many cusp submanifolds of the form (2.3). Clearly, each of such cusp ends is homotopy equivalent to a Heisenberg $(2n-1)$-manifold which can be described as follows [AX1]:

Theorem 2.3. *Let $\Gamma \subset \mathcal{H}_n \rtimes U(n-1)$ be a torsion-free discrete group acting on the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$ with non-compact quotient. Then the quotient $\mathcal{H}_n/\Gamma$ has zero Euler characteristic and is a vector bundle over a compact manifold. Furthermore, this compact manifold is finitely covered by a nil-manifold which is either a torus or the total space of a circle bundle over a torus.*

Now it follows that each cusp end is homotopy equivalent to a compact k-manifold, $k \leq 2n-1$, finitely covered by a nil-manifold which is either a (flat) torus or the total space of a circle bundle over a torus. It implies that the fundamental groups of cusp ends are finitely presented, and we get the following finiteness [AX1]:

Corollary 2.4. *Geometrically finite groups $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ are finitely presented.*

Another application of our geometric approach shows that cusp ends of a geometrically finite complex hyperbolic orbifolds M have, up to a finite covering of M, a very simple structure [AX1]:

Theorem 2.5. *Let $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ be a geometrically finite discrete group. Then G has a subgroup G_0 of finite index such that every parabolic subgroup of G_0 is isomorphic to a discrete subgroup of the Heisenberg group $\mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$. In particular, each parabolic subgroup of G_0 is free Abelian or 2-step nilpotent.*

In terms of finite coverings, this result has the following sense:

Corollary 2.6. *For a given geometrically finite orbifold $M(G) = \overline{\mathbb{H}^n_{\mathbb{C}}}\backslash\Lambda(G)/G$, there is its finite covering $\hat{M}$ such that neighborhoods of each (cusp) end of $\hat{M}$ are homeomorphic to the product of infinite interval $[0,\infty)$, a closed k-dimensional ball B^k and a closed $(2n-k-1)$-dimensional manifold which is either the (flat) torus T^{2n-k-1} or the total space of a (non-trivial) circle bundle over the torus T^{2n-k-2}.*

Finally we formulate two additional definitions of geometrical finiteness which are originally due to W. Thurston [T]):

(GF3): The thick part of the minimal convex retract (=convex core) $C(G)$ of $\mathbb{H}^n_{\mathbb{C}}/G$ is compact.

(GF4): For some $\epsilon > 0$, the uniform ϵ-neighborhood of the convex core $C(G) \subset \mathbb{H}^n_{\mathbb{C}}/G$ has finite volume, and there is a bound on the orders of finite subgroups of G.

Due to Bowditch [Bow], the four definitions **GF1, GF2, GF3** and **GF4** of geometrical finiteness for a discrete group $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ are all equivalent, see also [AX1, AX3].

Now we would like to define the above terms of "convex core" and "thick part" of a complex hyperbolic orbifold M. Namely, the convex core $C(G)$ of a complex hyperbolic orbifold $M = \mathbb{H}^n_{\mathbb{C}}/G$ can be obtained as the G-quotient of the complex hyperbolic convex hull $C(\Lambda(G))$ of the limit set $\Lambda(G)$. Here the convex hull $C(\Lambda(G)) \subset \mathbb{H}^n_{\mathbb{C}}$ is the minimal convex subset in $\mathbb{H}^n_{\mathbb{C}}$ whose closure in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ contains the limit set $\Lambda(G)$. Clearly, it is G-invariant, and its quotient $C(G) = C(\Lambda(G))/G$ is the minimal convex retract of $\mathbb{H}^n_{\mathbb{C}}/G$; we call it the *convex core* of $M = \mathbb{H}^n_{\mathbb{C}}/G$.

Now let ϵ be any positive number less than the Margulis constant in dimension n, $\epsilon(n)$. Then for a given discrete group $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ and its orbifold $M = \mathbb{H}^n_{\mathbb{C}}/G$, the ϵ-thin part $\operatorname{thin}_\epsilon(M)$ is defined as:

$$\operatorname{thin}_\epsilon(M) = \{x \in \mathbb{H}^n_{\mathbb{C}} \,:\, G_\epsilon(x) = \langle g \in G \,:\, d(x, g(x)) < \epsilon\rangle \text{ is infinite}\}/G. \tag{2.4}$$

The ϵ-thick part $\operatorname{thick}_\epsilon(M)$ of an orbifold M is defined as the closure of the complement to the ϵ-thin part $\operatorname{thin}_\epsilon(M) \subset M$.

As a consequence of the Margulis Lemma [M, BGS], there is the following description [BGS, Bow] of the thin part of a negatively curved orbifold which we formulate for complex hyperbolic geometry:

Theorem 2.7. *Let $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ be a discrete group and ϵ, $0 < \epsilon < \epsilon(n)$, be chosen. Then the ϵ-thin part $\operatorname{thin}_\epsilon(M)$ of $M = \mathbb{H}^n_{\mathbb{C}}/G$ is a disjoint union of its connected components, and each such component has the form $T_\epsilon(\Gamma)/\Gamma$ where Γ is a maximal infinite elementary subgroup of G. Here, for each such elementary subgroup $\Gamma \subset G$, the connected component (Margulis region)*

$$T_\epsilon = \{x \in \mathbb{H}^n_{\mathbb{C}} \,:\, \Gamma_\epsilon(x) = \langle g \in \Gamma \,:\, d(x, \gamma(x)) < \epsilon\rangle \textit{ is infinite}\} \tag{2.5}$$

is precisely invariant for Γ in G:

$$\Gamma(T_\epsilon) = T_\epsilon\,, \quad g(T_\epsilon) \cap T_\epsilon = \emptyset \quad \textit{for any } g \in G\backslash\Gamma\,. \tag{2.6}$$

We note that in the real hyperbolic case of dimension 2 and 3, a Margulis region T_ϵ in (2.5) with parabolic stabilizer $\Gamma \subset G$ can be taken as a horoball neighborhood centered at the parabolic fixed point p, $\Gamma(p) = p$. It is not true in general because of Apanasov's construction [A3] in real hyperbolic spaces of dimension at least 4. As we discussed it in [AX1], this construction works in complex hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}$, $n \geq 2$, as well.

However, applying the structural Theorem 2.1 to actions of parabolic groups nearby their fixed points, we obtain a description of parabolic Margulis regions for any discrete groups $G \subset \operatorname{Isom}\mathbb{H}^n_{\mathbb{C}}$ (even in more general situation of pinched Hadamard manifolds, see [AX3, Lemma 5.2]). Namely, let $\Gamma \subset G$ be such a discrete parabolic subgroup. Without loss of generality, we may assume that its fixed point $p \in \partial\mathbb{H}^n_{\mathbb{C}}$ is ∞ in the Siegel domain, or equivalently, in the upper half-space model of $\mathbb{H}^n_{\mathbb{C}}$. Then, on each horosphere $H_t \subset \mathbb{H}^n_{\mathbb{C}}$ centered at $p = \infty$, the parabolic group Γ acts as a discrete subgroup of $\mathcal{H}_n \rtimes U(n-1)$. Hence, applying Theorem 2.1, we have a Γ-invariant connected subspace $\mathcal{H}_\Gamma \subset \partial\mathbb{H}^n_{\mathbb{C}}\backslash\{p\}$ where Γ acts co-compactly, and on which a finite index subgroup $\Gamma^* \subset \Gamma$ acts freely by left translations. We define the subspace $\tau_\Gamma \subset \mathbb{H}^n_{\mathbb{C}}$ to be spanned by $\mathcal{H}_\Gamma$ and all geodesics $(z, p) \subset \mathbb{H}^n_{\mathbb{C}}$ that connect $z \in \mathcal{H}_\Gamma$ to the parabolic fixed point p. Let τ^t_Γ be the "half-plane" in τ_Γ of a height $t > 0$, that is the part of τ_Γ whose last horosperical coordinate is at least t. Then, due to [AX3, Lemma 5.2], we have:

Lemma 2.8. *Let $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ be a discrete group and p a parabolic fixed point of G. Let T_ϵ be a Margulis region for p as given in (2.5) and let τ^t_Γ be the half-plane defined as above. Then for any δ, $0 < \delta < \epsilon/2$, there exists a large enough number $t > 0$ such that the Margulis region T_ϵ contains the δ-neighborhood $N_\delta(\tau^t_\Gamma)$ of the half-plane τ^t_Γ.*

This fact, Theorem 2.7 and **GF3**-characterization of geometrical finiteness (compactness of the ϵ-thick part of the convex core $C(G)$ with sufficiently small $\epsilon > 0$) imply the following (equivalent) description of geometrically finite complex hyperbolic orbifolds $M = \mathbb{H}^n_{\mathbb{C}}/G$. Namely it follows that the action of a geometrically finite discrete group $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ on the convex hull $C(\Lambda(G))$ has a G-invariant family of precisely invariant disjoint horoballs centered at parabolic fixed points (their sufficiently small sizes are determined by Lemma 2.8). In other words, we have:

Corollary 2.9. *A discrete group $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ is geometrically finite if and only if there is a G-invariant family of disjoint open horoballs $B_i \subset \mathbb{H}^n_{\mathbb{C}}$ centered at parabolic fixed points $p_i \in \partial \mathbb{H}^n_{\mathbb{C}}$ of the group G such that the orbifold*

$$C_0(G) = \big(C(\Lambda(G)) \setminus \cup_i B_i\big)/G \tag{2.7}$$

is compact (and homotopy equivalent to $M = \mathbb{H}^n_{\mathbb{C}}/G$).

3. Geometric isomorphisms

Here we would like to discuss the well known problem of geometric realizations of isomorphisms of discrete groups. Adapting its formulation in §1 for discrete groups $G, H \subset PU(n,1)$, we have:

Problem 3.1. *Given a type preserving isomorphism $\varphi : G \to H$ of discrete groups $G, H \subset PU(n,1)$, find subsets $X_G, X_H \subset \overline{\mathbb{H}^n_{\mathbb{C}}}$ invariant for the action of groups G and H, respectively, and an equivariant homeomorphism $f_\varphi : X_G \to X_H$ which induces the isomorphism φ. Determine metric properties of f_φ, in particular, whether it is either quasisymmetric or quasiconformal with respect to either the Bergman metric in $\mathbb{H}^n_{\mathbb{C}}$ or the induced Cauchy-Riemannian structure at infinity $\partial \mathbb{H}^n_{\mathbb{C}}$.*

Such type problems were studied by several authors. In the case of lattices G and H in rank 1 symmetric spaces X, G. Mostow [Mo1] proved in his celebrated rigidity theorem that such isomorphisms $\varphi : G \to H$ can be extended to inner isomorphisms of X, provided that there is no analytic homomorphism of X onto $PSL(2,\mathbb{R})$. For that proof, it was essential to prove that φ can be induced by a quasiconformal homeomorphism of the sphere at infinity ∂X which is the one point compactification of a (nilpotent) Carnot group N (for quasiconformal mappings in Heisenberg and Carnot groups, see [KR1, KR2, P]).

If geometrically finite groups $G, H \subset PU(n,1)$ have parabolic elements and are neither lattices nor trivial, the only known geometric realization of their isomorphisms in dimension $\dim X \geq 3$ is due to P. Tukia's [Tu] isomorphism theorem for real hyperbolic spaces $X = \mathbb{H}^n_{\mathbb{R}}$. However, that Tukia's construction (based on geometry of convex hulls of the limit sets $\Lambda(G)$ and $\Lambda(H)$) cannot be used in the case of variable negative curvature due to lack of control over convex hulls (where the convex hull of three points may be 4-dimensional), especially nearby parabolic fixed points. Another (dynamical) approach due to C. Yue [Y2, Cor. B] (and the Anosov-Smale stability theorem for hyperbolic flows) can be used only for convex cocompact groups G and H [Y3]. As a first step in solving the above geometrization Problem 3.1, we have the following isomorphism theorem [A9, A11] in the complex hyperbolic space:

Theorem 3.2. *Let $\phi : G \to H$ be a type preserving isomorphism of two non-elementary geometrically finite groups $G, H \subset \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$. Then there exists a unique equivariant homeomorphism $f_\phi : \Lambda(G) \to \Lambda(H)$ of their limit sets that induces the isomorphism ϕ. Moreover, if $\Lambda(G) = \partial\mathbb{H}^n_{\mathbb{C}}$, the homeomorphism f_ϕ is the restriction of a hyperbolic isometry $h \in \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$.*

Proof. For completeness, we prove this fact (following to [A9, A11]). We consider the Cayley graph $K(G,\sigma)$ of a group G with a given finite set σ of generators. This is a 1-complex whose vertices are elements of G, and such that two vertices $a, b \in G$ are joined by an edge if and only if $a = bg^{\pm 1}$ for some generator $g \in \sigma$. Let $|*|$ be the word norm on the graph $K(G,\sigma)$, that is the norm $|g|$ equals the minimal length of words in the alphabet σ representing a given element $g \in G$. Choosing a function ρ such that $\rho(r) = 1/r^2$ for $r > 0$ and $\rho(0) = 1$, one can define the length of an edge $[a,b] \subset K(G,\sigma)$ as $d_\rho(a,b) = \min\{\rho(|a|), \rho(|b|)\}$. Considering paths of minimal length in the sense of the function $d_\rho(a,b)$, one can extend it to a metric on the Cayley graph $K(G,\sigma)$. So taking the Cauchy completion $\overline{K(G,\sigma)}$ of that metric space, we have the definition of the group completion $\overline{G}$ as the compact metric space $\overline{K(G,\sigma)}\backslash K(G,\sigma)$, see [Fl]. Up to a Lipschitz equivalence, this definition does not depend on σ. It is also clear that, for a cyclic group $\mathbb{Z}$, its completion $\overline{\mathbb{Z}}$ consists of two points. Nevertheless, for a nilpotent group G with one end, its completion $\overline{G}$ is a one-point set [Fl].

Now we can define a proper equivariant embedding $F : K(G,\sigma) \hookrightarrow \mathbb{H}^n_{\mathbb{C}}$ of the Cayley graph of a given geometrically finite group $G \subset PU(n,1)$. To do that we may assume that the stabilizer of a base point, say $0 \in \mathbb{B}^n_{\mathbb{C}} \cong \mathbb{H}^n_{\mathbb{C}}$, is trivial. Then we set $F(g) = g(0)$ for any vertex $g \in K(G,\sigma)$, and F maps any edge $[a,b] \subset K(G,\sigma)$ to the geodesic segment $[a(0), b(0)] \subset \mathbb{H}^n_{\mathbb{C}}$ joining the points $a(0)$ and $b(0)$.

Proposition 3.3. *For a geometrically finite discrete group $G \subset \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$, there are constants $K, K' > 0$ such that the following bounds hold for all elements $g \in G$ with $|g| \geq K'$:*

$$\ln(2|g| - K)^2 - \ln K^2 \leq d(0, g(0)) \leq K|g|\,. \tag{3.4}$$

The proof of this claim is based on a comparison of the Bergman metric $d(*,*)$ and the path metric $d_0(*,*)$ on the following subset $\mathbb{H}_0 \subset \mathbb{H}^n_{\mathbb{C}}$. As in \$2, let $C(\Lambda(G)) \subset \mathbb{H}^n_{\mathbb{C}}$ be the convex hull of the limit set $\Lambda(G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ of the group G. Since G is geometrically finite, the complement in $M(G)$ to neighbourhoods of (finitely many) cusp ends is compact, and its retract can be taken as the compact suborbifold $C_0(G)$ in the convex core $C(G)$, see (2.7) and Corollary 2.9. Its universal cover $\mathbb{H}_0 \subset C(\Lambda(G))$ is the complement in the convex hull $C(\Lambda(G))$ to a G-invariant family of disjoint open horoballs $B_i \subset \mathbb{H}^n_{\mathbb{C}}$ centered at parabolic fixed points $p_i \in \partial\mathbb{H}^n_{\mathbb{C}}$ of the group G, and each of which is precisely invariant with respect to its (parabolic) stabilizer $G_i \subset G$.

Now, having a co-compact action of the group G on the domain $\mathbb{H}_0 \subset \mathbb{H}^n_{\mathbb{C}}$ whose boundary includes some horospheres, we can reduce our comparison of distance functions $d = d(x,x')$ and $d_0 = d_0(x,x')$ to their comparison on a horosphere. So we can take points $x = (0,0,u)$ and $x' = (\xi, v, u)$ on a "horizontal" horosphere $H_u =$

$\mathbb{C}^{n-1} \times \mathbb{R} \times \{u\} \subset \mathbb{H}^n_{\mathbb{C}}$. Then the distances d and d_0 are as follows [Pr2]:

$$\cosh^2 \frac{d}{2} = \frac{1}{4u^2}\left(|\xi|^4 + 4u|\xi|^2 + 4u^2 + v^2\right), \quad d_0^2 = \frac{|\xi|^2}{u} + \frac{v^2}{4u^2}. \tag{3.5}$$

This comparison and the basic fact due to Cannon [Can] that, for a co-compact action of a group G in a metric space X, its Cayley graph can be quasi-isometrically embedded into X, finish our proof of (3.4), compare [A7].

Now we apply Proposition 3.3 to define a G-equivariant extension of the map F from the Cayley graph $K(G,\sigma)$ to the group completion $\overline{G}$. Since the group completion of any parabolic subgroup $G_p \subset G$ is either a point or a two-point set (depending on whether G_p is a finite extension of cyclic or a nilpotent group with one end), we get

Theorem 3.4. *For a geometrically finite discrete group $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$, there is a continuous G-equivariant map $\Phi_G : \overline{G} \to \Lambda(G)$. Moreover, the map Φ_G is bijective everywhere but the set of parabolic fixed points $p \in \Lambda(G)$ whose stabilizers $G_p \subset G$ have rank one. On this set, the map Φ_G is two-to-one.*

Now we can finish our proof of Theorem 3.2 by looking at the following diagram of maps:

$$\Lambda(G) \xleftarrow{\Phi_G} \overline{G} \xrightarrow{\overline{\phi}} \overline{H} \xrightarrow{\Phi_H} \Lambda(H),$$

where the homeomorphism $\overline{\phi}$ is induced by the isomorphism ϕ, and the continuous maps Φ_G and Φ_H are defined by Theorem 3.4. Namely, one can define a map $f_\phi = \Phi_H \overline{\phi} \Phi_G^{-1}$. Here the map Φ_G^{-1} is the right inverse to Φ_G, which exists due to Theorem 3.4. Furthermore, the map Φ_G^{-1} is bijective everywhere but the set of parabolic fixed points $p \in \Lambda(G)$ whose stabilizers $G_p \subset G$ have rank one, where the map Φ_G^{-1} is 2-to-1. Hence the composition map f_ϕ is bijective and G-equivariant. Its uniqueness follows from its continuity and the fact that the image of the attractive fixed point of a loxodromic element $g \in G$ must be the attractive fixed point of the loxodromic element $\phi(g) \in H$ (such loxodromic fixed points are dense in the limit set, see [A1]).

The last claim of the Theorem 3.2 directly follows from the Mostow rigidity theorem [Mo1] because a geometrically finite group $G \subset \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ with $\Lambda(G) = \partial \mathbb{H}^n_{\mathbb{C}}$ is co-finite: $\operatorname{Vol}(\mathbb{H}^n_{\mathbb{C}}/G) < \infty$.

□

Remark 3.5. Our proof of Theorem 3.2 can be easily extended to the general situation of type preserving isomorphisms of geometrically finite discrete groups in pinched Hadamard manifolds due to recent results in [AX3]. Namely, it is possible to construct equivariant homeomorphisms $f_\phi : \Lambda(G) \to \Lambda(H)$ conjugating the actions (on the limit sets) of isomorphic geometrically finite groups $G, H \subset \operatorname{Isom} X$ in a (symmetric) space X with pinched negative curvature K, $-b^2 \leq K \leq -a^2 < 0$. Actually, bounds similar to (3.4) in Prop. 3.3 (crucial for our argument) can be obtained from a result due to Heintze and Im Hof [HI, Th.4.6] which compares the geometry of horospheres $H_u \subset X$ with that in the spaces of constant curvature $-a^2$ and $-b^2$, respectively. It gives, that for all $x, y \in H_u$ and their distances $d = d(x,y)$ and $d_u = d_u(x,y)$ in the space X and in the horosphere H_u, respectively, one has that

$$\frac{2}{a}\sinh(a \cdot d/2) \leq d_u \leq \frac{2}{b}\sinh(b \cdot d/2).$$

Upon existence of such a (canonical) homeomorphisms f_φ that induces a given type-preserving isomorphisms φ of discrete subgroups of $\mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$, the geometric realization Problem 3.1 can be reduced to the questions whether f_φ is quasisymmetric with respect to the Carnot-Carathéodory (or Cygan) metric, and whether there exists its G-equivariant extension to a bigger set (in particular to the sphere at infinity ∂X or even to the whole space $\overline{\mathbb{H}^n_{\mathbb{C}}}$, cf. [KR2]) inducing the isomorphism φ. For convex cocompact groups obtained by nearby representations, this may be seen as a generalization of D.Sullivan stability theorem [Su2], see also [A7]. We shall discuss that question in the next Section. Also we note that, besides the metrical (quasisymmetric) part of the geometrization Problem 3.1, there are some topological obstructions for extensions of equivariant homeomorphisms f_φ, $f_\varphi : \Lambda(G) \to \Lambda(H)$. It follows from the next example.

Example 3.6. *Let $G \subset PU(1,1) \subset PU(2,1)$ and $H \subset PO(2,1) \subset PU(2,1)$ be two geometrically finite (loxodromic) groups isomorphic to the fundamental group $\pi_1(S_g)$ of a compact oriented surface S_g of genus $g > 1$. Then the equivariant homeomorphism $f_\varphi : \Lambda(G) \to \Lambda(H)$ cannot be homeomorphically extended to the whole sphere $\partial\mathbb{H}^2_{\mathbb{C}} \approx S^3$.*

Proof. The obstruction in this example is topological and is due to the fact that the quotient manifolds $M_1 = \mathbb{H}^2_{\mathbb{C}}/G$ and $M_2 = \mathbb{H}^2_{\mathbb{C}}/H$ are not homeomorphic. Namely, these complex surfaces are disk bundles over the Riemann surface S_g and have different Toledo invariants: $\tau(\mathbb{H}^2_{\mathbb{C}}/G) = 2g-2$ and $\tau(\mathbb{H}^2_{\mathbb{C}}/H) = 0$, see [To].

□

The complex structures of the complex surfaces M_1 and M_2 are quite different, too. The first manifold M_1 has a natural embedding of the Riemann surface S_g as a closed analytic totally geodesic submanifold, and hence M_1 cannot be a Stein manifold. The second manifold M_2, a disk bundle over the Riemann surface S_g has a totally geodesic real section and is a Stein manifold due to a result by Burns–Shnider [BS].

Moreover due to Goldman [G1], since the surface $S_g \subset M_1$ is a closed analytic submanifold, the manifold M_1 is locally rigid in the sense that every nearby representation $G \to PU(2,1)$ stabilizes a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ and is conjugate to a representation $G \to PU(1,1) \subset PU(2,1)$. In other words, there are no non-trivial "quasi-Fuchsian" deformations of the group G and the complex surface M_1. On the other hand, as we show in Section 5 (cf. Theorem 5.1), the second manifold M_2 has plentiful enough Teichmüller space of different "quasi-Fuchsian" complex hyperbolic structures.

4. Deformations of holomorphic bundles: flexibility versus rigidity

Due to the natural inclusion $PO(n,1) \subset PU(n,1)$, any real hyperbolic n-manifold $M_{\mathbb{R}} = \mathbb{H}^n_{\mathbb{R}}/G$ can be (totally geodesically) embedded into a complex hyperbolic n-manifold $M_{\mathbb{C}} = \mathbb{H}^n_{\mathbb{C}}/G$ which is the total space of n-disk bundle over $M_{\mathbb{R}}$. Similarly, due to the inclusion $PU(n-1,1) \subset PU(n,1)$, any discrete torsion free group $G \subset PU(n-1,1)$ defines a holomorphic 2-disk bundle (with the total space $\mathbb{H}^n_{\mathbb{C}}/G$) over its totally geodesic complex analytic submanifolds $\mathbb{H}^{n-1}_{\mathbb{C}}/G$. In particular, one can consider both types of disk bundles over a Riemann surface S. Then a flexibility of such bundles becomes evident starting with hyperbolic structures on a Riemann surface S of genus $g > 1$, which form the Teichmüller space $\mathcal{T}_g$, a complex analytic $(3g-3)$-manifold. And though, due to the Mostow rigidity theorem [Mo1], hyperbolic structures of finite volume and (real) dimension at least three are uniquely determined by their topology, so one has no continuous deformations of them, we still have some flexibility.

Firstly, real hyperbolic 3-manifolds have plentiful enough infinitesimal deformations and, according to Thurston's hyperbolic Dehn surgery theorem [T], noncompact hyperbolic 3-manifolds of finite volume can be approximated by compact hyperbolic 3-manifolds. Secondly, despite their hyperbolic rigidity, real hyperbolic manifolds M can be deformed as conformal manifolds, or equivalently as higher-dimensional hyperbolic manifolds $M \times (0,1)$ of infinite volume. First such quasi-Fuchsian deformations were given by the author [A2] and, after Thurston's "Mickey Mouse" example [T], they were called bendings of M along its totally geodesic hypersurfaces, see also [A1, A2, A4-A6, JM, Ko, Su1]. Furthermore, all these deformations are quasiconformally equivalent showing a rich supply of quasiconformal G-equivariant homeomorphisms in the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. In particular, the limit set $\Lambda(G) \subset \partial\mathbb{H}^{n+1}_{\mathbb{R}}$ deforms continuously from a round sphere $\partial\mathbb{H}^n_{\mathbb{R}} = S^{n-1} \subset S^n = \mathbb{H}^{n+1}_{\mathbb{R}}$ into nondifferentiably embedded topological $(n-1)$-spheres quasiconformally equivalent to S^{n-1}.

Contrasting to the above flexibility, "non-real" hyperbolic manifolds (locally symmetric spaces of rank one) seem much more rigid. In particular, due to P. Pansu [P], quasiconformal maps in the sphere at infinity of quaternionic/octonionic hyperbolic spaces that are induced by hyperbolic quasi-isometries are necessarily CR-automorphisms, and thus there cannot be interesting quasiconformal deformations of corresponding structures. Secondly, due to Corlette's rigidity theorem [C3], such closed manifolds of (quaternionic or octonionic) dimension at least two and corresponding uniform lattices are even super-rigid – analogously to Margulis super-rigidity in higher rank [M, A13]. The last fact and our joint work with Inkang Kim [AK] imply impossibility of quasi-Fuchsian deformations of quaternionic/octonionic manifolds of infinite volume homotopy equivalent to their closed analytic submanifolds, for quaternionic manifolds of dimension at least three see also [Ka]. Furthermore, complex hyperbolic manifolds share the above rigidity of quaternionic/octonionic hyperbolic manifolds. Namely, due to the Goldman's local rigidity theorem in dimension $n = 2$ [G1] and its extension for $n \geq 3$ [GM], every nearby discrete representation $\rho : G \to PU(n,1)$ of a cocompact lattice $G \subset PU(n-1,1)$ stabilizes a complex totally geodesic subspace $\mathbb{H}^{n-1}_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$. Thus the limit set $\Lambda(\rho G) \subset \partial\mathbb{H}^n_{\mathbb{C}}$ is always a round sphere S^{2n-3}. Moreover, in higher dimensions $n \geq 3$, this local rigidity of complex hyperbolic n-manifolds M homotopy equivalent to their closed complex totally geodesic hypersurfaces is even global (at least in the connected component of representation variety [C1, BCG, Y1]). These facts may be viewed as some arguments in favor of general rigidity and stability of deformations of complex hyperbolic structures.

To the contrary, our goal here and in the next section is to show that the opposite situation nevertheless holds: there are non-rigid complex hyperbolic manifolds which are disk bundles over their totally geodesic (both complex and real) submanifolds, and deformations of such manifolds may be quasiconformally unstable. The complex hyperbolic manifolds that are so flexible appear to be Stein spaces, so we expect that all Stein (complex hyperbolic) manifolds with "big" ends at infinity may have such nontrivial deformations.

The flexibility of complex hyperbolic 2-manifolds we deal with in this Section (and their property to be Stein spaces) is related to noncompactness of the (finite area) fibration base of these holomorphic disk bundles. In addition to constructing (quasi-

Fuchsian) deformations of such bundles, we shall also show that they are quasiconformally unstable. Here we use the notion of quasiconformal stability that has its roots in the classical problem of quasiconformal stability of deformations from the theory of Kleinian groups, in particular in well known stability theorems by L. Bers [B1, B2] and D. Sullivan [Su1]. Let us recall that definition by following to L. Bers [B2]. Namely, a homomorphism $\chi : G \to \mathrm{PSL}(2,\mathbb{C})$ of a finitely generated group $G \subset \mathrm{PSL}(2,\mathbb{C})$ will be called allowable if it preserves the square traces of parabolic and elliptic elements (hence χ is type-preserving). A finitely generated Kleinian group $G \subset \mathrm{PSL}(2,\mathbb{C})$ is said to be *quasiconformally stable* if every allowable homomorphism $\chi : G \to \mathrm{PSL}(2,\mathbb{C})$ sufficiently close to the identity is induced by an equivarint quasiconformal mapping $w : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, that is $\chi(g) = wgw^{-1}$ for all $g \in G$. It is clear that degenerate Kleinian groups are quasiconformally unstable. However, due to a Bers's [B1] criterion (which involves quadratic differentials for the group G), it follows that Fuchsian groups, Schottky groups, groups of Schottky type and certain non-degenerate B-groups are all quasiconformally stable [B2].

We obtain a natural generalization of quasiconformal stability for discrete groups $G \subset \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$ by changing the condition on homomorphisms χ in terms of the trace of elements $g \in G$ to the condition that such a homomorphism $\chi : G \to \mathrm{Isom}\,\mathbb{H}^n_{\mathbb{C}}$ preserves the type of elements of a given discrete group G. In that sense, B. Aebisher and R. Miner [AM] recently proved that (classical) Schottky groups $G \subset PU(n,1)$ are quasiconformally stable. Here a finitely generated discrete group $G = \langle g_1, \dots, g_k \rangle \subset PU(n,1)$ is called a classical Schottky group of rank k in the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ if the sides of its Dirichlet polyhedron $D_y(G) \subset \mathbb{H}^n_{\mathbb{C}}$,

$$D_z(G) = \{z \in \mathbb{H}^n_{\mathbb{C}} : d(z,y) < d(z,g(y)) \text{ for any } g \in G\backslash\{\mathrm{id}\}\}, \tag{4.1}$$

centered at some point $y \in \mathbb{H}^n_{\mathbb{C}}$ are disjoint and non-asymptotic.

Nevertheless, as we shall show below, Fuchsian groups $G \subset PU(2,1)$ are quasiconformally unstable:

Theorem 4.1. *There are co-finite Fuchsian groups $G \subset PU(1,1) \subset PU(2,1)$ with signatures $(g, r; m_1, \dots, m_r)$, where genus $g \geq 0$ and there are at least four cusps (with branching orders $m_i = \infty$), such that:*

(1) *the Teichmüller space $\mathcal{T}(G)$ contains a smooth simple curve $\alpha : [0, \pi/2) \hookrightarrow \mathcal{T}(g)$ which passes through the Fuchsian group $G = \alpha(0)$ and whose points $\alpha(t) = G_t \subset PU(2,1)$, $0 < t < \pi/2$, are all non-trivial quasi-Fuchsian groups;*
(2) *each isomorphism $\chi : G \to G_t$, $0 < t < \pi/2$, is induced by a G-equivariant homeomorphism $f_t : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$ of the closure $\overline{\mathbb{H}^2_{\mathbb{C}}} = \mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$ of the complex hyperbolic space;*
(3) *for any parameter t, $0 < t < \pi/2$, the action of the quasi-Fuchsian group G_t is not quasiconformally conjugate to the action of the Fuchsian group $G = \alpha(0)$ (in both CR-structure at infinity $\partial\mathbb{H}^2_{\mathbb{C}} = \mathcal{H}_2 \cup \{\infty\}$ and the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$).*

Before we go on with the (constructive) proof of this Theorem, we note that though the constructed unstable Fuchsian groups $G \subset PU(1,1) \subset PU(2,1)$ of finite co-volume may have finite order elements, their finite index torsion free subgroups (and Riemann-Hurvitz formula for genus of a branching covering, see [KAG, (41)]) immediate imply the following:

Corollary 4.2. *Let $M = \mathbb{H}^2_{\mathbb{C}}/G$ be a complex hyperbolic surface with the holonomy group $G \subset PU(1,1) \subset PU(2,1)$ that represents the total space of a non-trivial disk bundle over a Riemann surface of genus $p \geq 0$ with at least four punctures (hyperbolic 2-orbifold with at least four punctures). Then the Teichmüller space $\mathcal{T}(M)$ contains a smooth simple curve $\alpha : [0, \pi/2) \hookrightarrow \mathcal{T}(M)$ with the following properties:*

1. *the curve α passes through the surface $M = \alpha(0)$;*
2. *each complex hyperbolic surface $M_t = \alpha(t) = \mathbb{H}^2_{\mathbb{C}}/G_t$, $t \in [0, \pi/2)$, with the holonomy group $G_t \subset PU(2,1)$ is homeomorphic to the surface M;*
3. *for any parameter t, $0 < t < \pi/2$, the complex hyperbolic surface M_t is not quasiconformally equivalent to the surface M.*

Besides the claims in this Corollary, it follows also from the construction of the above complex hyperbolic surfaces M and M_t that their boundaries, the spherical CR-manifolds $N = \partial M = \Omega(G)/G$ and $N_t = \partial M_t = \Omega(G_t)/G_t$ have similar properties:

Corollary 4.3. *Let $N = N_0 = \partial M$ be the 3-dimensional spherical CR-manifold with Fuchsian holonomy group $G \subset PU(1,1) \subset PU(2,1)$ that is the boundary at infinity of the complex hyperbolic surface M from Corollary 4.2 (and which is the total space of a non-trivial circle bundle over a Riemann surface with at least four punctures). Then the Teichmüller space $\mathcal{T}(N)$ of the CR-manifold N contains a smooth simple curve $\alpha_N : [0, \pi/2) \hookrightarrow \mathcal{T}(N)$ with the following properties:*

1. *the curve α_N passes through the CR-manifold $N = \alpha_N(0)$;*
2. *each CR-manifold $N_t = \alpha_N(t) = \mathbb{H}^2_{\mathbb{C}}/G_t\}$, $t \in [0, \pi/2)$, with the holonomy group $G_t \subset PU(2,1)$ is the total space of a non-trivial circle bundle over the Riemann surface with at least four punctures and is homeomorphic to the manifold N;*
3. *for any parameter t, $0 < t < \pi/2$, the CR-manifold N_t is not quasiconformally equivalent to the manifold N.*

Remarks 4.4.

1. As a corollary of Theorem 4.1 and an Yue's [Y2] result on Hausdorff dimension, we have that there are deformations of a co-finite Fuchsian group $G \subset PU(1,1)$ into quasi-Fuchsian groups $G_\alpha = f_\alpha G f_\alpha^{-1} \subset PU(2,1)$ with Hausdorff dimension of the limit set $\Lambda(G_\alpha)$ strictly bigger than one, see also [C2]. Moreover, the deformed groups G_α are Zariski dense in $PU(2,1)$.
2. We note that, for the simplest case of manifolds with cyclic fundamental groups, a similar to Corollary 4.3 (though based on different ideas) effect of homeomorphic but not quasiconformally equivalent spherical CR-manifolds N and N' has been also recently observed by R. Miner [Mi]. In fact, among his Cauchy-Riemannian 3-manifolds (homeomorphic to $\mathbb{R}^2 \times S^1$), there are exactly two quasiconformal equivalence classes whose representatives have the cyclic holonomy groups generated correspondingly by a vertical Heisenberg translation by $(0,1) \in \mathbb{C} \times \mathbb{R}$ and a horizontal translation by $(1,0) \in \mathbb{C} \times \mathbb{R}$.
3. The existence of non-trivial quasi-Fuchsian representations of Fuchsian groups with signatures $(g, r; m_1, \dots, m_r)$ in the above Theorem 4.1 has its origin in an example (for genus $g = 0$ and $r = 4$ singular points with all four branching indices $m_i = \infty$) constructed by M. Carneiro and N. Gusevskii, see [Gu], who deal with a group $G \subset \operatorname{Isom} \mathbb{H}^1_{\mathbb{C}}$ generated by four involutions and acting in the

invariant complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ as the group generated by reflections in sides of an ideal 4-gon.

(4) Despite the impossibility of quasiconformal conjugation of the constructed actions of quasi-Fuchsian groups G_t and $G = G_0$ in the sphere at infinity $\partial\mathbb{H}^2_{\mathbb{C}}$, it is still an open question whether the actions of these groups on their limit sets $\Lambda(G_t)$ and $\Lambda(G)$ could be "quasiconformally" conjugate, in other words, whether the canonical G-equivariant homeomorphism f_{χ_t}, $f_{\chi_t} : \Lambda(G) \to \Lambda(G_t)$ of the limit sets (constructed in Theorem 3.1) that induces the isomorphism $\chi_t : G \to G_t$, $0 < t < \pi/2$, is in fact quasisymmetric.

Proof of Theorem 4.1. Here we present a construction of the deformation (for full details, see [A12, A14]). We shall start with the lattice $G \subset PU(1,1)$ in the claim as a subgroup of index 2 in a discrete group $\Gamma \subset \operatorname{Isom}\mathbb{H}^1_{\mathbb{C}}$ generated by reflections in sides of a finite area hyperbolic polygon $F \subset \mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^2_{\mathbb{C}}$.

Namely, let $\Gamma \subset \operatorname{Isom}\mathbb{H}^1_{\mathbb{C}}$, $\Gamma = \Gamma_1 * \Gamma_2$, be a *co-finite* (free) lattice which is the free product of a dihedral parabolic subgroup Γ_1 and another subgroup Γ_2 which has at least one parabolic subgroup and whose quotient $[\overline{\mathbb{H}^1_{\mathbb{C}}}\backslash\Lambda(\Gamma_2)]/\Gamma_2$ has one boundary component at infinity, see Fig. 1. Since the subgroup Γ_2 has at least one parabolic subgroup, it can also be decomposed as the free product of its subgroups, $\Gamma_2 = \Gamma_3 * \Gamma_4$. In the simplest case, each of these subgroups Γ_3 and Γ_4 may have order two.

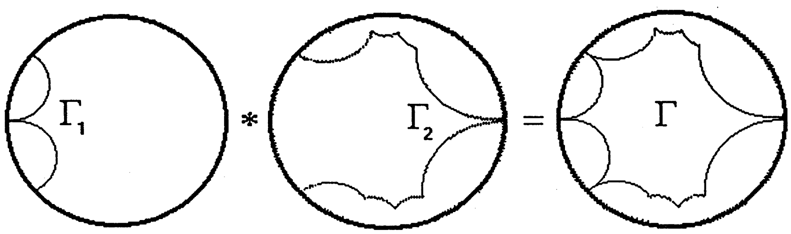

FIGURE 1. $\Gamma = \Gamma_1 * \Gamma_2$.

Here, for each reflection $g \subset \operatorname{Isom}\mathbb{H}^1_{\mathbb{C}}$ in a geodesic $\ell = (a,b) \subset \mathbb{H}^1_{\mathbb{C}}$, we have a uniquely defined (up to unitary rotation around the complex geodesic $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^n_{\mathbb{C}}$) action of g in the whole space $\mathbb{H}^n_{\mathbb{C}}$ as the anti-holomorphic involution whose fixed set is a real hyperbolic n-subspace in $\mathbb{H}^n_{\mathbb{C}}$ intersecting the complex geodesic $\mathbb{H}^1_{\mathbb{C}}$ along the geodesic $\ell = (a,b)$. In particular, assuming that $\mathbb{H}^1_{\mathbb{C}} = \{0\} \times \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+ = \mathbb{H}^n_{\mathbb{C}}$ and that the geodesic ℓ ends at ideal points $a = (0,1), b = (0,-1) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$, we have that the reflection g in ℓ acts in $\mathbb{H}^n_{\mathbb{C}}$ as the following anti-holomorphic involution (we call it a *real involution*):

$$i_\ell(\xi, v, u) = \left(\frac{A\bar{\xi}}{||\xi||^2 + u + iv}, \frac{v}{\left|||\xi||^2 + u + iv\right|^2}, \frac{u}{\left|||\xi||^2 + u + iv\right|^2} \right), \tag{4.2}$$

where $A \in U(n-1)$ and $(\xi, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times [0, \infty)$; compare with the Heisenberg inversion $\mathcal{I}$ in (2.1). The action of the real involution $i_\ell \in \operatorname{Isom} \mathbb{H}^n_{\mathbb{C}}$ at infinity $\partial \mathbb{H}^n_{\mathbb{C}} = \mathbb{C}^{n-1} \times \mathbb{R} \times \{0\} \cup \{\infty\}$ preserves the unit Heisenberg sphere $S_c(0,1) = \{(\xi, v) \in \mathcal{H}_n : ||(\xi, v)||_c = 1\}$, swaps the origin $(0,0) \in \mathcal{H}_n$ and ∞, and pointwise fixes an $\mathbb{R}$-sphere (of dimension $(n-1)$) that lies in the Heisenberg sphere $S_c(0,1)$ and passes through its poli $a = (0,1), b = (0,-1) \in \mathcal{H}_n = \mathbb{C}^{n-1} \times \mathbb{R}$. In particular, for $n = 2$ and $A\xi = -\xi$, the pointwise fixed $\mathbb{R}$-circle has the following equation in cylindrical coordinates (cf. [G4]):

$$\{(re^{i\theta}, v) \in \mathcal{H} = \mathbb{C} \times \mathbb{R} : r^2 + iv = -e^{2i\theta}, \tag{4.3}$$

and its vertical projection to the horizontal plane $\mathbb{C} \times \{0\} \subset \mathcal{H}$ is the lemniscate of Bernoulli:

$$\{\xi = x + iy : (x^2 + y^2)^2 + x^2 - y^2 = 0\}.$$

Now we may assume that our group $\Gamma \subset \operatorname{Isom} \mathbb{H}^1_{\mathbb{C}}$ is generated by real involutions whose restrictions to $\mathbb{H}^1_{\mathbb{C}}$ are reflections in sides of the fundamental polygon F, and its limit set is the (vertical) chain $\{0\} \times \mathbb{R} \cup \{\infty\} \subset \partial \mathbb{H}^2_{\mathbb{C}}$. Furthermore, assuming that the (parabolic) fixed point of the dihedral subgroup $\Gamma_1 \subset \Gamma$ is ∞ and deforming the action of Γ in $\mathbb{H}^1_{\mathbb{C}}$ (in Teichmüller space $\mathcal{T}(\Gamma)$), we may take a fundamental polyhedron $D \subset \mathbb{H}^2_{\mathbb{C}}$ of the group Γ as the polyhedron bounded by bisectors Σ_i whose poli lie in the one point compactification of the vertical line $\{0\} \times \mathbb{R} \subset \mathcal{H}$ and whose boundaries S_i at infinity are as follows. Two of them (corresponding to the generators of Γ_1) are the extended horizontal planes in $\mathcal{H}$,

$$S_1 = \mathbb{C} \times \{s_0\} \cup \{\infty\}, \ S_2 = \mathbb{C} \times \{-s_0\} \cup \{\infty\} \subset \mathbb{C} \times \mathbb{R} \cup \{\infty\}. \tag{4.4}$$

The spheres at infinity of all other bisectors are Heisenberg spheres S_i, $i \geq 3$, that lie between the planes (4.4) and whose centers lie in the vertical line. Furthermore, we may assume that two such spheres, S_3 and S_4, are unit Heisenberg spheres tangent to the spheres S_1 and S_2, correspondingly at the points $p_1 = (0, s_0)$ and $p_2 = (0, -s_0)$. Also, due to our condition on one more parabolic subgroup (in Γ_2), we have that there is a pair of spheres, S_j and S_{j+1}, $j \geq 3$, tangent to each other at some point $p_3 = (0, s_1)$.

We have a choice of $\mathbb{R}$-circles $m_i \subset S_i$ that are fixed by the corresponding real involutions $\gamma_i \in \operatorname{Isom} \mathbb{H}^2_{\mathbb{C}}$ that generate the group Γ. We note that those involutions γ_i are lifts of reflections in sides of the polygon F in the complex geodesic $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^2_{\mathbb{C}}$. So those lifts should be compatible in the sense that the union of closures of real arcs $m_i \cap \overline{D}$ in their pointwise fixed $\mathbb{R}$-circles m_i is a closed loop α on the boundary of the 3-dimensional polyhedron $P = \overline{D} \cap \partial \mathbb{H}^2_{\mathbb{C}}$.

As such $\mathbb{R}$-circles $m_i \subset S_i$, $i = 1, 2$, we take two real (extended) lines that are parallel to each other and pass the corresponding poli $p_i = (0, \pm s_0)$ in S_i. In the adjacent spheres S_j, $j = 3, 4$, we take $\mathbb{R}$-circles m_j as those unique circles that are tangent at the points $(0, \pm s_0)$ to to corresponding $\mathbb{R}$-circles m_1 and m_2. Continuing this process in the spheres S_l adjacent to S_3 and S_4, we take those $\mathbb{R}$-circles $m_l \subset S_l$ that intersect the corresponding $\mathbb{R}$-circles $m_i \subset S_i$, $i = 3, 4$. Continuing this process, we will reach next tangent points, etc. Finally, at a tangent point $(0, s_1)$, our $\mathbb{R}$-circles m_j and $m_j + 1$ will meet at some angle $\theta_{j,j+1}$. Then our group Γ,

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k \rangle = \langle \gamma_1 \rangle * \langle \gamma_2 \rangle * \Gamma_3 * \Gamma_4 , \tag{4.5}$$

is generated by the real involutions γ_i pointwise fixing the corresponding $\mathbb{R}$-circles m_i, and acts in the invariant complex geodesic $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^2_{\mathbb{C}}$ as the group generated by reflections in sides of hyperbolic k-gon P_0 of finite area.

Obviously, we have that the subgroup $G \subset \Gamma$ of index two with the fundamental polyhedron $\overline{D} \cup \gamma_1(\overline{D})$ is a subgroup of $PU(1,1) \subset PU(2,1)$. Topologically, the quotient $\mathbb{H}^2_{\mathbb{C}}/G$ is a 2-disk bundle over 2-dimensional sphere with at least four punctures; geometrically, the base of this bundle is its totally geodesic complex analytic orbifold of finite hyperbolic volume (due to Riemann-Hurvitz formula for genus of a branching covering over 2-sphere, see [KAG, (41)], it is covered by a Riemann surface of a genus $p \geq 0$ with at least four punctures).

Also we note that tangent points of bisectors bounding the fundamental polyhedron $D \subset \mathbb{H}^2_{\mathbb{C}}$, in particular the points $p_0 = \infty$ and $p_i = (0, \pm s_0) \in \mathcal{H}$, $i = 1, 2$, are parabolic fixed points of Γ. Moreover, due to tangency of the corresponding $\mathbb{R}$-circles in the definition of Γ, all elements $g \in G_{p_i}$ in the stabilizer subgroups $G_{p_i} \subset G \subset PU(2,1)$ of those last three points are (conjugated to) pure vertical translations, that is (after conjugation) they act in $\overline{\mathbb{H}^2_{\mathbb{C}}}$ as Heisenberg translations $(\xi, v, u) \mapsto (\xi, v + v_i, u)$, $i = 0, 1, 2$.

Deformation of groups Γ and $G \subset \Gamma$.

To deform the groups Γ and $G \subset \Gamma$, we define a family of discrete faithfull representations $\rho_t : \Gamma \to \operatorname{Isom} \mathbb{H}^2_{\mathbb{C}}$, $0 \leq t < \pi/2$, with the images $\rho(t) = \Gamma^t$ where $\rho(0) = \Gamma^0 = \Gamma$.

Namely, all these representations ρ_t coincide (up to conjugations by unitary rotations A in (4.2)) on the subgroup $\Gamma_2 \subset \Gamma$ and only differ on the dihedral subgroup $\Gamma_1 \subset \Gamma$ in the following way:

$$\Gamma^t = \langle \gamma_{1,t}, \gamma_{2,t}, \gamma_{3,t} \ldots, \gamma_{k,t} \rangle = \langle \gamma_{1,t} \rangle * \langle \gamma_{2,t} \rangle * \Gamma_3 * A_t \Gamma_4 A_t^{-1} , \tag{4.6}$$

where $A_t \in U(1)$ acts in $\mathbb{H}^2_{\mathbb{C}}$ by unitary rotation about the complex geodesic $\mathbb{H}^1_{\mathbb{C}} \subset \mathbb{H}^2_{\mathbb{C}}$, and the generators $\gamma_{i,t}$ are anti-holomorphic (real) involutions with pointwise fixed $\mathbb{R}$-circles $m_{i,t} \subset \partial \mathbb{H}^2_{\mathbb{C}}$. In particular, $\gamma_{1,t}$ and $\gamma_{2,t}$ generate the new dihedral subgroup $\Gamma_{1,t} = \rho_t(\Gamma_1)$ of the group $\Gamma^t \subset \operatorname{Isom} \mathbb{H}^2_{\mathbb{C}}$.

As before, we define the real involutions $\gamma_{i,t}$ by determining their fixed $\mathbb{R}$-circles $m_{i,t} \subset \partial \mathbb{H}^2_{\mathbb{C}}$. Namely, for each t, $0 \leq t < \pi/2$, let $p_{1,t}$ be a point on the $\mathbb{R}$-circle m_3 that is seen from the center of the sphere S_3 at the angle t, and let $p_{2,t}$ be the point on the Heisenberg sphere S_4 that is symmetric to the point $p_{1,t}$, $p_{2,t} = -p_{1,t}$. Then let $m_{4,t}$ be the $\mathbb{R}$-circle in the sphere S_4 that contains the point $p_{2,t}$, and $A_t \in U(1)$ be the unitary rotation of S_4 that maps m_4 to $m_{4,t}$. We note that $A_t \neq \mathrm{id}$ if $t \neq 0$, and its rotation angle monotonically increases from 0 to $\pi/2$ as t tends from 0 to $\pi/2$. Now we keep all generators in the subgroup Γ_3 and conjugate generators of the group Γ_4 by A_t. The remaining $\mathbb{R}$-circles $m_{1,t}, m_{2,t} \subset \partial \mathbb{H}^2_{\mathbb{C}}$ are given as the (Euclidean) lines in $\mathbb{C} \times \mathbb{R}$ tangent to the corresponding $\mathbb{R}$-circles $m_{3,t} = m_3$ and $m_{4,t} = A_t(m_4)$ at the points $p_{1,t}$ and $p_{2,t}$, respectively.

In other words, we replace the bisectors Σ_1 and Σ_2 by bisectors Σ_1^t and Σ_2^t whose boundaries at infinity are the (extended) parallel planes S_1^t and S_2^t tangent to the spheres S_3 and S_4 at the points $p_{1,t}$ and $p_{2,t}$, respectively. Then the previously defined $\mathbb{R}$-circles $m_{1,t}$ and $m_{2,t}$ are the intersection lines of the planes S_1^t and S_2^t with the contact planes at these tangent points $p_{1,t}$ and $p_{2,t}$, correspondingly. Clearly, these lines are not parallel if $t \neq 0$, and the angle between them (at infinity) monotonically increases from 0 to $\pi/2$ as t tends from 0 to $\pi/2$. It is also worth to mention that, for $t \neq 0$, the finite poli of the spheres S_1^t and S_2^t lie somewhere in the $\mathbb{R}$-circles $m_{1,t}$ and $m_{2,t}$ and differ from the tangent points $p_{1,t}$ and $p_{2,t}$, respectively.

We note that as in the group Γ, all tangent points of bisectors Σ_i^t and Σ_j^t (and in particular, the points p_1^t, p_2^t and ∞) are parabolic fixed points of the deformed group Γ^t. Moreover, though parabolic elements fixing the points p_1^t and p_2^t are still conjugate to vertical Heisenberg translations, the index two subgroup of projective unitary transformations in the stabilizer of ∞ in Γ^t is generated by screw vertical translation if $t \neq 0$. The rotation angle of that screw translation monotonically increases from 0 to π as t tends from 0 to $\pi/2$.

The above property of tangent points implies that the polyhedron $D^t \subset \mathbb{H}^2_{\mathbb{C}}$ bounded by bisectors Σ_i^t is a fundamental polyhedron of the discrete group Γ^t, and its intersection with the sphere at infinity, the polyhedron $P^t = \overline{D^t} \cap \partial\mathbb{H}^2_{\mathbb{C}}$ bounded by the spheres S_i^t, is a fundamental polyhedron for the Γ^t-action at infinity.

As another implication of properties of parabolic subgroups in Γ and Γ^t, we obtain non-triviality of our deformation given by the family of (faithful discrete) representations $\rho_t \in \mathrm{Hom}(\Gamma, \mathrm{Isom}\,\mathbb{H}^2_{\mathbb{C}}$, $\rho_t\Gamma = \Gamma^t$. Namely, for any two different parameters t and t', the groups Γ^t and $\Gamma^{t'}$ cannot be conjugated in $\mathrm{Isom}\,\mathbb{H}^2_{\mathbb{C}}$ because the corresponding parabolic transformations in their stabilizers of ∞,

$$\rho_t(\gamma) \in \Gamma^t_\infty \text{ and } \rho_{t'}(\gamma) \in \Gamma^{t'}_\infty, \tag{4.7}$$

have rotational parts (unitary rotations) with different angles. We note that one may also derive this fact by using the Cartan angular invariant for triples of points $((0,-s_0),(0,s_1),(0,s_0))$ and $(p_{2,t},(0,s_1),p_{1,t})$. This Cartan invariant (see the next Section) is different from $\pm\pi/2$ for any $t \neq 0$, which also shows that the groups Γ^t are non-trivial quasi-Fuchsian groups (with the limit topological circles $\Lambda(\Gamma^t)$ different from "round" circles in $\partial\mathbb{H}^2_{\mathbb{C}}$). Nevertheless, these quasi-Fuchsian groups Γ^t are not quasiconformal conjugates of the Fuchsian group Γ:

Lemma 4.5. *For any parameter t, $0 < t < \pi/2$, the quasi-Fuchsian representation ρ_t cannot be conjugate to the (Fuchsian) inclusion $\rho_0 : \Gamma \subset PU(2,1)$ by any quasiconformal homeomorphism in neither $\mathbb{H}^2_{\mathbb{C}}$ nor the Heisenberg group $\mathcal{H}$.*

We can prove this fact by using the above observation that the (type preserving) isomorphism $\rho_t : \Gamma \to \Gamma^t$ strictly increases the angles of rotational parts of parabolic elements in (4.7), see [A12, A14].

We shall finish the proof of Theorem 4.1 by showing that our deformaton is nevertheless topological, that is it could be induced by a continuous family of equivariant homeomorphisms:

Proposition 4.6. *For any two parameters t and t', $0 \le t < t', \pi/2$, the faithful discrete (quasi-Fuchsian) representations $\rho_t : \Gamma \to \Gamma^t$ and $\rho_{t'} : \Gamma \to \Gamma^{t'}$ are conjugate by an* *equivariant homeomorphism (that continuously depends on parameters t and t′),*

$$f_{t,t'} : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}} . \tag{4.8}$$

Proof. It is enough to construct such an equivariant homeomorphism $f_t = f_{0,t}$ in (4.8) that conjugates the groups Γ and Γ^t for any $t \in [0, \pi/2)$ and continuously depends on t. To do that, we need to define such an equivariant homeomorphism f_t on the closure $\overline{D} \subset \overline{\mathbb{H}^2_{\mathbb{C}}}$ of the fundamental polyhedron $D \subset \mathbb{H}^2_{\mathbb{C}}$ of the group Γ, whose image $f_t(\overline{D}) = \overline{D^t}$ is the closure of the fundamental polyhedron of the deformed group Γ^t. Having such a map of closed polyhedra, we can immediately extend it equivariantly to the whole discontinuity domain, that is to a Γ-equivariant homeomorphism

$$f_t : \mathbb{H}^2_{\mathbb{C}} \cup \Omega(\Gamma) \to \mathbb{H}^2_{\mathbb{C}} \cup \Omega(\Gamma^t) , \tag{4.9}$$

whose extension by continuity to the limit set is the unique Γ-equivariant homeomorphism of the limit sets, $\Lambda(\Gamma) \to \Lambda(\Gamma^t)$, induced by the type preserving isomorphism of the groups Γ and Γ^t, see Theorem 3.2.

Now we start a construction of our Γ-equivariant homeomorphism $f_t : \overline{D} \to \overline{D^t}$ on the closure of the fundamental polyhedron D, where it maps k-sides of D to the corresponding k-sides of D^t, $0 \le k \le 3$, in particular tangent points p_i to tangent points $p_{i,t}$, and the $\mathbb{R}$-circles m_j to the corresponding $\mathbb{R}$-circles $m_{j,t}$.

First, we define our homeomorphism f_t in the 3-polyhedron $P \subset \mathcal{H} \subset \partial\mathbb{H}^2_{\mathbb{C}}$ at infinity. Fixing an orientation on the $\mathbb{R}$-circles m_j and $m_{j,t}$ (preserved by their identifications), we consider corresponding (positive) semispheres $\hat{S}_j \subset S_j$ and $\hat{S}^t_j \subset S^t_j$ bounded by those $\mathbb{R}$-circles. Since, up to $PU(2,1)$, each such semisphere may be considered as a halfplane in an extended 2-plane, we define homeomorphisms $f_t|_{\hat{S}_j}$ as restrictions of either Euclidean isometries in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ that map $\hat{S}_j \to \hat{S}^t_j$ or conjugations of such isometries by elements of $PU(2,1)$. On the complements $S_j \backslash \hat{S}_j$, we define f_t so that they are compatible with the generators of the group Γ:

$$f_t|_{S_j \backslash \hat{S}_j}(x) = \gamma_{j,t} \circ f_t|_{S_j} \circ \gamma_j(x) , \quad \text{for } x \in S_j \backslash \hat{S}_j . \tag{4.10}$$

It defines our Γ-equivariant homeomorphism f_t on the boundary ∂P of the fundamental polyhedron $P \subset \mathcal{H}$, $P = \partial_\infty D$, which is not simply-connected. However, spliting P into a cell along an embedded topological 2-disk $\sigma \subset P$, $\partial\sigma = \alpha$, where a closed curve $\alpha \subset \partial P$ is the union of arcs of our $\mathbb{R}$-circles m_j, we can extend our Γ-equivariant homeomorphism to the whole polyhedron P, $f_t|_P : P \to P^t$.

In addition to 3-polyhedra P and P^t, the boundaries of the fundamental 4-dimensional polyhedra D and D^t each have $2k$ more sides. Those sides lie on bisectors Σ_j and Σ^t_j and are pairwise identified by the generators γ_j and $\gamma_{j,t}$, respectively. Each of those bisectors is split into two halves, Σ^+_j and Σ^-_j, or $\Sigma^{t,+}_j$ and $\Sigma^{t,-}_j$, along the corresponding totally real geodesic 2-plane spanned by the corresponding $\mathbb{R}$-circle, either m_j or $m_{j,t}$, and those halves are pairwise identified by our generators.

Since we have already defined our homeomorphism f_t on the boundary of each bisector Σ_j, we can extend it to the whole bisector by using natural foliations of Σ_j and Σ^t_j by disjoint real geodesics with ends x, $\gamma_j(x)$ and $f_t(x)$, $\gamma_{j,t} f_t(x)$, respectively.

Now, having our Γ-equivariant homeomorphism f_t defined on the topological boundary 3-sphere ∂D, we can extend this homeomorphism to a homeomorphism of the closed 4-balls, $\overline{D} \to \overline{D^t}$, that conjugates the dynamics of the groups Γ and Γ^t on the closures of our fundamental polyhedra. Clearly, all the steps in our construction of those homeomorphisms continuously depend on t. Hence, the equivariant extension (4.9) of that map to the whole 4-space completes the proof of our Proposition as well as of the whole Theorem 4.1.

□

We can use our construction of deformations in the above proof to give a lower bound for the number of independent parameters of nontrivial quasi-Fuchsian deformations in the variety of of representations $\mathrm{Hom}(G, PU(2,1))/PU(2,1)$ of a co-finite lattice $G \subset PU(1,1)$ (in addition to the dimension of the Teichmüller space of the real hyperbolic 2-orbifold $S = \mathbb{H}^1_{\mathbb{C}}/G$ of finite volume). Here $G \subset PU(1,1)$ is index two subgroup in a lattice $\Gamma\, \mathrm{Isom}\, \mathbb{H}^2_{\mathbb{R}}$, and our lower bound is based on the number of conjugacy classes of maximal parabolic subgroups in G. Namely, taking disjoint hyperbolic geodesics in the G-invariant complex geodesic whose ends are at parabolic vertices of a fundamental hyperbolic polygon $F \subset \mathbb{H}^1_{\mathbb{C}}$, we may extend them to disjoint bisectors in $\mathbb{H}^2_{\mathbb{C}}$ whose boundary spinal spheres at infinity are mutually tangent at those parabolic vertices. Moving those adjacent spinal spheres along their real meridians (in the same way as as we did it above) allows us to obtain the following estimate.

Corollary 4.7. *Let $\Gamma\, \mathrm{Isom}\, \mathbb{H}^2_{\mathbb{R}}$ be a co-finite lattice generated by reflections in sides of an ideal hyperbolic polygon in $\mathbb{H}^2_{\mathbb{R}} = \mathbb{H}^1_{\mathbb{C}} \subset ch2$, $G \subset PU(1,1)$ its index two subgroup, and N be the number of parabolic cusps in $S = \mathbb{H}^1_{\mathbb{C}}/G$. Then nontrivial quasi-Fuchsian deformations in the representation variety $\mathrm{Hom}(G, PU(2,1))/PU(2,1)$ have at least $[(N-1)/3]$ independent parameters where $[a]$ is the integer part of a number a.*

5. Bending deformations of complex hyperbolic structures

Here we present another class of non-rigid complex hyperbolic manifolds fibering over a Riemann surface. In fact we point out that the non-compactness condition for the base of fibration in non-rigidity results in §4 is not essential, either. Namely, complex hyperbolic Stein manifolds M homotopy equivalent to their closed totally *real* geodesic surfaces are not rigid, too. To prove that, we shall present a canonical construction of continuous non-trivial quasi-Fuchsian deformations of complex surfaces fibered over closed Riemannian surfaces of genus $g > 1$. Such deformations depend on $3(g-1)$ real-analytic parameters (in addition to "Fuchsian" deformations, where in particular, the Teichmuller space of the base surface has dimension $6(g-1)$). This provides the first such non-trivial deformations of fibrations with compact base (for non-compact base, see a different Goldman-Parker' deformation [GP2] of ideal triangle groups $G \subset PO(2,1)$). The obtained flexibility of such holomorphic fibrations and the number of its parameters provide a partial confirmation of a conjecture on dimension $16(g-1)$ of the Teichmuller space of such complex surfaces. It is related to A.Weil's theorem [W] (see also [G3, p.43]), that the variety of conjugacy classes of all (not necessarily discrete) representations $G \to PU(2,1)$ near the embedding $G \subset PO(2,1)$ is a real-analytic manifold of dimension $16(g-1)$. We remark that discreteness of representations of $G \cong \pi_1 M$ is an essential condition for deformation of a complex manifold M which does not follow from the mentioned Weil's result.

Our construction is inspired by the approach the author used for bending deformations of real hyperbolic (conformal) manifolds along totally geodesic hypersurfaces ([A2, A4]). In the case of complex hyperbolic (and Cauchy-Riemannian) structures, the constructed "bendings" work however in a different way than in the real hyperbolic case. Namely our complex bending deformations involve simultaneous bending of the base of the fibration of the complex surface M as well as bendings of each of its totally geodesic fibers (see Remark 5.9). Such bending deformations of complex surfaces are associated to their real simple closed geodesics (of real codimension 3), but have nothing common with the so called cone deformations of real hyperbolic 3-manifolds along closed geodesics, see [A6, A9].

Furthermore, despite well known complications in constructing equivariant homeomorphisms in the complex hyperbolic space and in Cauchy-Riemannian geometry (which should preserve Kähler and contact structures in $\mathbb{H}^n_{\mathbb{C}}$ and at its infinity $\overline{\mathcal{H}_n}$, respectively), the constructed complex bending deformations are induced by equivariant homeomorphisms of $\overline{\mathbb{H}^n_{\mathbb{C}}}$. Moreover, in contrast to the situation with deformations in §4, those equivariant homeomorphisms are in addition quasiconformal:

Theorem 5.1. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a given non-elementary discrete group. Then, for any simple closed geodesic α in the Riemann 2-surface $S = H^2_{\mathbb{R}}/G$ and a sufficiently small $\eta_0 > 0$, there is a holomorphic family of G-equivariant quasiconformal homeomorphisms $F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$, $-\eta_0 < \eta < \eta_0$, which defines the bending (quasi-Fuchsian) deformation $\mathcal{B}_\alpha : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ of the group G along the geodesic α, with $\mathcal{B}_\alpha(\eta) = F^*_\eta$.*

We notice that deformations of a complex hyperbolic manifold M may depend on many parameters described by the Teichmüller space $\mathcal{T}(M)$ of isotopy classes of complex hyperbolic structures on M. One can reduce the study of this space $\mathcal{T}(M)$ to studying the variety $\mathcal{T}(G)$ of conjugacy classes of discrete faithful representations $\rho : G \to PU(n,1)$ (involving the space $\mathcal{D}(M)$ of the developing maps, see [G2, FG]). Here $\mathcal{T}(G) = \mathcal{R}_0(G)/PU(n,1)$, and the variety $\mathcal{R}_0(G) \subset \mathrm{Hom}(G, PU(n,1))$ consists of discrete faithful representations ρ of the group G with infinite co-volume, $\mathrm{Vol}(\mathbb{H}^n_{\mathbb{C}}/G) = \infty$. In particular, our complex bending deformations depend on many independent parameters as it can be shown by applying our construction and Élie Cartan [Car] angular invariant in Cauchy-Riemannian geometry:

Corollary 5.2. *Let $S_p = \mathbb{H}^2_{\mathbb{R}}/G$ be a closed totally real geodesic surface of genus $p > 1$ in a given complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, $G \subset PO(2,1) \subset PU(2,1)$. Then there is a real-analytic embedding $\pi \circ \mathcal{B} : B^{3p-3} \hookrightarrow \mathcal{T}(M)$ of a real $(3p-3)$-ball into the Teichmüller space of M, defined by bending deformations along disjoint closed geodesics in M and by the projection $\pi : \mathcal{D}(M) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ in the development space $\mathcal{D}(M)$.*

Bending Construction (Proof of Theorem 5.1). Now we start with a totally real geodesic surface $S = \mathbb{H}^2_{\mathbb{R}}/G$ in the complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$, where $G \subset PO(2,1) \subset PU(2,1)$ is a given discrete group, and fix a simple closed geodesic α on S. We may assume that the loop α is covered by a geodesic $A \subset \mathbb{H}^2_{\mathbb{R}} \subset \mathbb{H}^2_{\mathbb{C}}$ whose ends at infinity are ∞ and the origin of the Heisenberg group $\mathcal{H} = \mathbb{C} \times \mathbb{R}$, $\overline{\mathcal{H}} = \partial\mathbb{H}^2_{\mathbb{C}}$. Furthermore, using quasiconformal deformations of the Riemann surface S (in the Teichmüller space $\mathcal{T}(S)$, that is, by deforming the inclusion $G \subset PO(2,1)$ in $PO(2,1)$ by bendings along the loop α, see Corollary 3.3 in [A10]), we can assume that the hyperbolic length of α is sufficiently small and the radius of its tubular neighborhood is big enough:

Lemma 5.3. *Let g_α be a hyperbolic element of a non-elementary discrete group $G \subset PO(2,1) \subset PU(2,1)$ with translation length ℓ along its axis $A \subset \mathbb{H}^2_{\mathbb{R}}$. Then any tubular neighborhood $U_\delta(A)$ of the axis A of radius $\delta > 0$ is precisely invariant with respect to its stabilizer $G_0 \subset G$ if $\sinh(\ell/4) \cdot \sinh(\delta/2) \leq 1/2$. Furthermore, for sufficiently small ℓ, $\ell < 4\delta$, the Dirichlet polyhedron $D_z(G) \subset \mathbb{H}^2_{\mathbb{C}}$ of the group G centered at a point $z \in A$ has two sides a and a' intersecting the axis A and such that $g_\alpha(a) = a'$.*

Then the group G and its subgroups G_0, G_1, G_2 in the free amalgamated (or HNN-extension) decomposition of G have Dirichlet polyhedra $D_z(G_i) \subset \mathbb{H}^2_{\mathbb{C}}$, $i = 0,1,2$, centered at a point $z \in A = (0,\infty)$, whose intersections with the hyperbolic 2-plane $\mathbb{H}^2_{\mathbb{R}}$ have the shapes indicated in Figures 2-5.

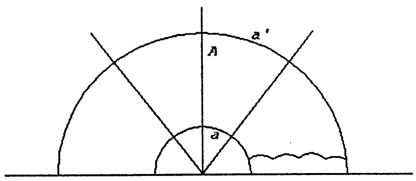

FIGURE 2. $G_1 \subset G = G_1 *_{G_0} G_2$

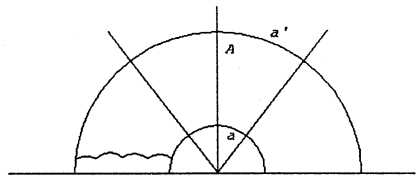

FIGURE 3. $G_2 \subset G = G_1 *_{G_0} G_2$

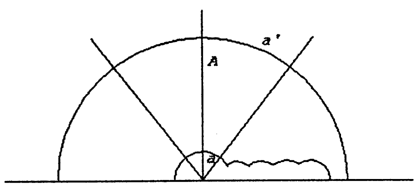

FIGURE 4. $G_1 \subset G = G_1 *_{G_0}$

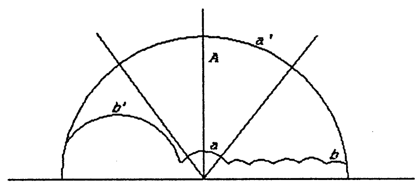

FIGURE 5. $G = G_1 *_{G_0}$

In particular we have that, except of two bisectors $\mathfrak{S}$ and $\mathfrak{S}'$ that are equivalent under the hyperbolic translation g_α (that generates the stabilizer $G_0 \subset G$ of the axis A), all other bisectors bounding such a Dirichlet polyhedron lie in sufficiently small "cone neighborhoods" C_+ and C_- of the arcs (infinite rays) $\mathbb{R}_+$ and $\mathbb{R}_-$ of the real circle $\mathbb{R} \times \{0\} \subset \mathbb{C} \times \mathbb{R} = \mathcal{H}$.

Actually, we may assume that the Heisenberg spheres at infinity of the bisectors $\mathfrak{S}$ and $\mathfrak{S}'$ have radii 1 and $r_0 > 1$, correspondingly. Then, for a sufficiently small ϵ, $0 < \epsilon << r_0 - 1$, the cone neighborhoods $C_+, C_- \subset \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \{\infty\} = \mathbb{C} \times \mathbb{R} \times [0, +\infty)$ are correspondingly the cones of the ϵ-neighborhoods of the points $(1,0,0), (-1,0,0) \in \mathbb{C} \times \mathbb{R} \times [0,+\infty)$ with respect to the Cygan metric ρ_c in $\overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \{\infty\}$, see (2.1).

Clearly, we may consider the length ℓ of the geodesic α so small that closures of all equidistant halfspaces in $\overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \{\infty\}$ bounded by those bisectors (and whose interiors are disjoint from the Dirichlet polyhedron $D_z(G)$) do not intersect the co-vertical bisector whose infinity is $i\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$. It follows from the fact [G4, Thm VII.4.0.3] that equidistant half-spaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in $\mathbb{H}^2_{\mathbb{C}}$ are disjoint if and only if the half-planes $\mathfrak{S}_1 \cap \mathbb{H}^2_{\mathbb{R}}$ and $\mathfrak{S}_2 \cap \mathbb{H}^2_{\mathbb{R}}$ are disjoint, see Figures 2-5.

Now we are ready to define a quasiconformal bending deformation of the group G along the geodesic A, which defines a bending deformation of the complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ along the given closed geodesic $\alpha \subset S \subset M$.

We specify numbers η and ζ such that $0 < \zeta < \pi/2$, $0 \leq \eta < \pi - 2\zeta$ and the intersection $C_+ \cap (\mathbb{C} \times \{0\})$ is contained in the angle $\{z \in \mathbb{C} : |\arg z| \leq \zeta\}$. Then we define a bending homeomorphism $\phi = \phi_{\eta,\zeta} : \mathbb{C} \to \mathbb{C}$ which bends the real axis $\mathbb{R} \subset \mathbb{C}$ at the origin by the angle η, see Fig. 6:

$$\phi_{\eta,\zeta}(z) = \begin{cases} z & \text{if } |\arg z| \geq \pi - \zeta \\ z \cdot \exp(i\eta) & \text{if } |\arg z| \leq \zeta \\ z \cdot \exp(i\eta(1 - (\arg z - \zeta)/(\pi - 2\zeta))) & \text{if } \zeta < \arg z < \pi - \zeta \\ z \cdot \exp(i\eta(1 + (\arg z + \zeta)/(\pi - 2\zeta))) & \text{if } \zeta - \pi < \arg z < -\zeta \,. \end{cases} \tag{5.1}$$

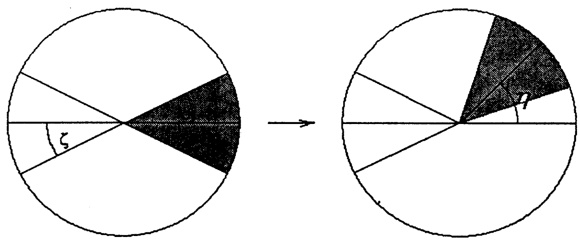

FIGURE 6

For negative η, $2\zeta - \pi < \eta < 0$, we set $\phi_{\eta,\zeta}(z) = \overline{\phi_{-\eta,\zeta}(\overline{z})}$. Clearly, $\phi_{\eta,\zeta}$ is quasiconformal with respect to the Cygan norm (2.1) and is an isometry in the ζ-cone neighborhood of the real axis $\mathbb{R}$ because its linear distortion is given by

$$K(\phi_{\eta,\zeta}, z) = \begin{cases} 1 & \text{if } |\arg z| \geq \pi - \zeta \\ 1 & \text{if } |\arg z| \leq \zeta \\ (\pi - 2\zeta)/(\pi - 2\zeta - \eta) & \text{if } \zeta < \arg z < \pi - \zeta \\ (\pi - 2\zeta + \eta)/(\pi - 2\zeta) & \text{if } \zeta - \pi < \arg z < -\zeta \,. \end{cases} \tag{5.2}$$

Foliating the punctured Heisenberg group $\mathcal{H}\backslash\{0\}$ by Heisenberg spheres $S(0,r)$ of radii $r > 0$, we can extend the bending homeomorphism $\phi_{\eta,\zeta}$ to an elementary bending homeomorphism $\varphi = \varphi_{\eta,\zeta} : \mathcal{H} \to \mathcal{H}$, $\varphi(0) = 0$, $\varphi(\infty) = \infty$, of the whole sphere $S^3 = \overline{\mathcal{H}}$ at infinity.

Namely, for the "dihedral angles" $W_+, W_- \subset \mathcal{H}$ with the common vertical axis $\{0\} \times \mathbb{R}$ and which are foliated by arcs of real circles connecting points $(0, v)$ and $(0, -v)$ on the vertical axis and intersecting the the ζ-cone neighborhoods of infinite rays $\mathbb{R}_+, \mathbb{R}_- \subset \mathbb{C}$, correspondingly, the restrictions $\varphi|_{W_-}$ and $\varphi|_{W_+}$ of the bending homeomorphism $\varphi =$

$\varphi_{\eta,\zeta}$ are correspondingly the identity and the unitary rotation $U_\eta \in PU(2,1)$ by angle η about the vertical axis $\{0\} \times \mathbb{R} \subset \mathcal{H}$, see also [A10, (4.4)]. Then it follows from (5.2) that $\varphi_{\eta,\zeta}$ is a G_0-equivariant quasiconformal homeomorphism in $\overline{\mathcal{H}}$.

We can naturally extend the foliation of the punctured Heisenberg group $\mathcal{H}\backslash\{0\}$ by Heisenberg spheres $S(0,r)$ to a foliation of the hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ by bisectors $\mathfrak{S}_r$ having those $S(0,r)$ as the spheres at infinity. It is well known (see [Mo2]) that each bisector $\mathfrak{S}_r$ contains a geodesic γ_r which connects points $(0,-r^2)$ and $(0,r^2)$ of the Heisenberg group $\mathcal{H}$ at infinity, and furthermore $\mathfrak{S}_r$ fibers over γ_r by complex geodesics Y whose circles at infinity are complex circles foliating the sphere $S(0,r)$.

Using those foliations of the hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ and bisectors $\mathfrak{S}_r$, we extend the elementary bending homeomorphism $\varphi_{\eta,\zeta} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ at infinity to an elementary bending homeomorphism $\Phi_{\eta,\zeta} : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$. Namely, the map $\Phi_{\eta,\zeta}$ preserves each of bisectors $\mathfrak{S}_r$, each complex geodesic fiber Y in such bisectors, and fixes the intersection points y of those complex geodesic fibers and the complex geodesic connecting the origin and ∞ of the Heisenberg group $\mathcal{H}$ at infinity. We complete our extension $\Phi_{\eta,\zeta}$ by defining its restriction to a given (invariant) complex geodesic fiber Y with the fixed point $y \in Y$. This map is obtained by radiating the circle homeomorphism $\varphi_{\eta,\zeta}|_{\partial Y}$ to the whole (Poincaré) hyperbolic 2-plane Y along geodesic rays $[y,\infty) \subset Y$, so that it preserves circles in Y centered at y and bends (at y, by the angle η) the geodesic in Y connecting the central points of the corresponding arcs of the complex circle ∂Y, see Fig.6.

Due to the construction, the elementary bending (quasiconformal) homeomorphism $\Phi_{\eta,\zeta}$ commutes with elements of the cyclic loxodromic group $G_0 \subset G$. Another most important property of the homeomorphism $\Phi_{\eta,\zeta}$ is the following.

Let $D_z(G)$ be the Dirichlet fundamental polyhedron of the group G centered at a given point z on the axis A of the cyclic loxodromic group $G_0 \subset G$, and $\mathfrak{S}^+ \subset \mathbb{H}^2_{\mathbb{C}}$ be a "half-space" disjoint from $D_z(G)$ and bounded by a bisector $\mathfrak{S} \subset \mathbb{H}^2_{\mathbb{C}}$ which is different from bisectors $\mathfrak{S}_r, r > 0$, and contains a side $\mathfrak{s}$ of the polyhedron $D_z(G)$. Then there is an open neighborhood $U(\overline{\mathfrak{S}^+}) \subset \overline{\mathbb{H}^2_{\mathbb{C}}}$ such that the restriction of the elementary bending homeomorphism $\Phi_{\eta,\zeta}$ to it either is the identity or coincides with the unitary rotation $U_\eta \subset PU(2,1)$ by the angle η about the "vertical" complex geodesic (containing the vertical axis $\{0\} \times \mathbb{R} \subset \mathcal{H}$ at infinity).

The above properties of quasiconformal homeomorphism $\Phi = \Phi_{\eta,\zeta}$ show that the image $D_\eta = \Phi_{\eta,\zeta}(D_z(G))$ is a polyhedron in $\mathbb{H}^2_{\mathbb{C}}$ bounded by bisectors. Furthermore, there is a natural identification of its sides induced by $\Phi_{\eta,\zeta}$. Namely, the pairs of sides preserved by Φ are identified by the original generators of the group $G_1 \subset G$. For other sides $\mathfrak{s}_\eta$ of D_η, which are images of corresponding sides $\mathfrak{s} \subset D_z(G)$ under the unitary rotation U_η, we define side pairings by using the group G decomposition (see Fig. 2-5).

Actually, if $G = G_1 *_{G_0} G_2$, we change the original side pairings $g \in G_2$ of $D_z(G)$-sides to the hyperbolic isometries $U_\eta g U_\eta^{-1} \in PU(2,1)$. In the case of HNN-extension, $G = G_1 *_{G_0} = \langle G_1, g_2 \rangle$, we change the original side pairing $g_2 \in G$ of $D_z(G)$-sides to the hyperbolic isometry $U_\eta g_2 \in PU(2,1)$. In other words, we define deformed groups $G_\eta \subset PU(2,1)$ correspondingly as

$$G_\eta = G_1 *_{G_0} U_\eta G_2 U_\eta^{-1} \quad \text{or} \quad G_\eta = \langle G_1, U_\eta g_2 \rangle = G_1 *_{G_0} . \tag{5.3}$$

This shows that the family of representations $G \to G_\eta \subset PU(2,1)$ does not depend on angles ζ and holomorphically depends on the angle parameter η. Let us also observe that, for small enough angles η, the behavior of neighboring polyhedra $g'(D_\eta)$, $g' \in G_\eta$ is the same as of those $g(D_z(G))$, $g \in G$, around the Dirichlet fundamental polyhedron $D_z(G)$. This is because the new polyhedron $D_\eta \subset \mathbb{H}^2_{\mathbb{C}}$ has isometrically the same (tesselations of) neighborhoods of its side-intersections as $D_z(G)$ had. This implies that the polyhedra $g'(D_\eta)$, $g' \in G_\eta$, form a tesselation of $\mathbb{H}^2_{\mathbb{C}}$ (with non-overlapping interiors). Hence the deformed group $G_\eta \subset PU(2,1)$ is a discrete group, and D_η is its fundamental polyhedron bounded by bisectors.

Using G-compatibility of the restriction of the elementary bending homeomorphism $\Phi = \Phi_{\eta,\zeta}$ to the closure $\overline{D_z(G)} \subset \overline{\mathbb{H}^2_{\mathbb{C}}}$, we equivariantly extend it from the polyhedron $\overline{D_z(G)}$ to the whole space $\mathbb{H}^2_{\mathbb{C}} \cup \Omega(G)$ accordingly to the G-action. In fact, in terms of the natural isomorphism $\chi : G \to G_\eta$ which is identical on the subgroup $G_1 \subset G$, we can write the obtained G-equivariant homeomorphism $F = F_\eta : \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G) \to \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G_\eta)$ in the following form:

$$\begin{aligned} F_\eta(x) &= \Phi_\eta(x) \quad \text{for} \quad x \in \overline{D_z(G)}, \\ F_\eta \circ g(x) &= g_\eta \circ F_\eta(x) \quad \text{for} \quad x \in \overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \Lambda(G),\ g \in G,\ g_\eta = \chi(g) \in G_\eta\,. \end{aligned} \tag{5.4}$$

Due to quasiconformality of Φ_η, the extended G-equivariant homeomorphism F_η is quasiconformal. Furthermore, its extension by continuity to the limit (real) circle $\Lambda(G)$ coincides with the canonical equivariant homeomorphism $f_\chi : \Lambda(G) \to \Lambda(G_\eta)$ given by the isomorphism Theorem 3.2. Hence we have a G-equivariant quasiconformal self-homeomorphism of the whole space $\overline{\mathbb{H}^2_{\mathbb{C}}}$, which we denote as before by F_η.

The family of G-equivariant quasiconformal homeomorphisms F_η induces representations $F^*_\eta : G \to G_\eta = F_\eta G_2 F_\eta^{-1}$, $\eta \in (-\eta_0, \eta_0)$. In other words, we have a curve $\mathcal{B} : (-\eta_0, \eta_0) \to \mathcal{R}_0(G)$ in the variety $\mathcal{R}_0(G)$ of faithful discrete representations of G into $PU(2,1)$, which covers a nontrivial curve in the Teichmüller space $\mathcal{T}(G)$ represented by conjugacy classes $[\mathcal{B}(\eta)] = [F^*_\eta]$. We call the constructed deformation $\mathcal{B}$ the bending deformation of a given lattice $G \subset PO(2,1) \subset PU(2,1)$ along a bending geodesic $A \subset \mathbb{H}^2_{\mathbb{C}}$ with loxodromic stabilizer $G_0 \subset G$. In terms of manifolds, $\mathcal{B}$ is the bending deformation of a given complex surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ homotopy equivalent to its totally real geodesic surface $S_g \subset M$, along a given simple geodesic α.

□

Remark 5.4. It follows from the above construction of the bending homeomorphism $F_{\eta,\zeta}$, that the deformed complex hyperbolic surface $M_\eta = \mathbb{H}^2_{\mathbb{C}}/G_\eta$ fibers over the pleated hyperbolic surface $S_\eta = F_\eta(\mathbb{H}^2_{\mathbb{R}})/G_\eta$ (with the closed geodesic α as the singular locus). The fibers of this fibration are "singular real planes" obtained from totally real geodesic 2-planes by bending them by angle η along complete real geodesics. These (singular) real geodesics are the intersections of the complex geodesic connecting the axis A of the cyclic group $G_0 \subset G$ and the totally real geodesic planes that represent fibers of the original fibration in $M = \mathbb{H}^2_{\mathbb{C}}/G$.

Proof of Corollary 5.2. Since, due to (5.3), bendings along disjoint closed geodesics are independent, we need to show that our bending deformation is not trivial, and $[\mathcal{B}(\eta)] \neq [\mathcal{B}(\eta')]$ for any $\eta \neq \eta'$.

The non-triviality of our deformation follows directly from (5.3), compare [A9]. Namely, the restrictions $\rho_\eta|_{G_1}$ of bending representations to a non-elementary subgroup $G_1 \subset G$ (in general, to a "real" subgroup $G_r \subset G$ corresponding to a totally real geodesic piece in the homotopy equivalent surface $S \simeq M$) are identical. So if the deformation $\mathcal{B}$ were trivial then it would be conjugation of the group G by projective transformations that commute with the non-trivial real subgroup $G_r \subset G$ and pointwise fix the totally real geodesic plane $\mathbb{H}^2_{\mathbb{R}}$. This contradicts to the fact that the limit set of any deformed group G_η, $\eta \neq 0$, does not belong to the real circle containing the limit Cantor set $\Lambda(G_r)$.

The injectivity of the map $\mathcal{B}$ can be obtained by using Élie Cartan [Car] angular invariant $\mathbb{A}(x)$, $-\pi/2 \leq \mathbb{A}(x) \leq \pi/2$, for a triple $x = (x^0, x^1, x^2)$ of points in $\partial\mathbb{H}^2_{\mathbb{C}}$. It is known (see [G4]) that, for two triples x and y, $\mathbb{A}(x) = \mathbb{A}(y)$ if and only if there exists $g \in PU(2,1)$ such that $y = g(x)$; furthermore, such a g is unique provided that $\mathbb{A}(x)$ is neither zero nor $\pm\pi/2$. Here $\mathbb{A}(x) = 0$ if and only if x^0, x^1 and x^2 lie on an $\mathbb{R}$-circle, and $\mathbb{A}(x) = \pm\pi/2$ if and only if x^0, x^1 and x^2 lie on a chain ($\mathbb{C}$-circle).

Namely, let $g_2 \in G \backslash G_1$ be a generator of the group G in (4.5) whose fixed point $x^2 \in \Lambda(G)$ lies in $\mathbb{R}_+ \times \{0\} \subset \mathcal{H}$, and $x^2_\eta \in \Lambda(G_\eta)$ the corresponding fixed point of the element $\chi_\eta(g_2) \in G_\eta$ under the free-product isomorphism $\chi_\eta : G \to G_\eta$. Due to our construction, one can see that the orbit $\gamma(x^2_\eta)$, $\gamma \in G_0$, under the loxodromic (dilation) subgroup $G_0 \subset G \cap G_\eta$ approximates the origin along a ray $(0, \infty)$ which has a non-zero angle η with the ray $\mathbb{R}_- \times \{0\} \subset \mathcal{H}$. The latter ray also contains an orbit $\gamma(x^1)$, $\gamma \in G_0$, of a limit point x^1 of G_1 which approximates the origin from the other side. Taking triples $x = (x^1, 0, x^2)$ and $x_\eta = (x^1, 0, x^2_\eta)$ of points which lie correspondingly in the limit sets $\Lambda(G)$ and $\Lambda(G_\eta)$, we have that $\mathbb{A}(x) = 0$ and $\mathbb{A}(x_\eta) \neq 0, \pm\pi/2$. Due to Theorem 3.2, both limit sets are topological circles which however cannot be equivalent under a hyperbolic isometry because of different Cartan invariants (and hence, again, our deformation is not trivial).

Similarly, for two different values η and η', we have triples x_η and $x_{\eta'}$ with different (non-trivial) Cartan angular invariants $\mathbb{A}(x_\eta) \neq \mathbb{A}(x_{\eta'})$. Hence $\Lambda(G_\eta)$ and $\Lambda(G_{\eta'})$ are not $PU(2,1)$-equivalent.

□

One can apply the above proof to a general situation of bending deformations of a complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/G$ whose holonomy group $G \subset PU(2,1)$ has a non-elementary subgroup G_r preserving a totally real geodesic plane $\mathbb{H}^2_{\mathbb{R}}$. In other words, such a complex surfaces M has an embedded totally real geodesic surface with geodesic boundary. So we immediately have:

Corollary 5.5. *Let $M = \mathbb{H}^2_{\mathbb{C}}/G$ be a complex hyperbolic surface with embedded totally real geodesic surface $S_r \subset M$ with geodesic boundary, and $\mathcal{B} : (-\eta, \eta) \to \mathcal{D}(M)$ be the bending deformation of M along a simple closed geodesic $\alpha \subset S_r$. Then the map $\pi \circ \mathcal{B} : (-\eta, \eta) \to \mathcal{T}(M) = \mathcal{D}(M)/PU(2,1)$ is a smooth embedding provided that the limit set $\Lambda(G)$ of the holonomy group G does not belong to the G-orbit of the real circle $S^1_{\mathbb{R}}$ and the chain $S^1_{\mathbb{C}}$, where the latter is the infinity of the complex geodesic containing a lift $\tilde{\alpha} \subset \mathbb{H}^2_{\mathbb{C}}$ of the closed geodesic α, and the former one contains the limit set of the holonomy group $G_r \subset G$ of the geodesic surface S_r.*

□

As an application of the constructed bending deformations, we can answer a well known question about cusp groups on the boundary of the Teichmüller space $\mathcal{T}(M)$ of a Stein complex hyperbolic surface M fibering over a compact Riemann surface of genus $p > 1$. It is a direct corollary of the following our result which will be published elsewhere:

Theorem 5.6. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then, for any simple closed geodesic $\alpha \subset S_p = H^2_{\mathbb{R}}/G$, there is a continuous deformation $\rho_t = f_t^*$ induced by G-equivariant quasiconformal homeomorphisms $f_t : \overline{\mathbb{H}^2_{\mathbb{C}}} \to \overline{\mathbb{H}^2_{\mathbb{C}}}$ whose limit representation ρ_∞ corresponds to a boundary cusp point of the Teichmüller space $\mathcal{T}(G)$, that is the boundary group $\rho_\infty(G)$ has an accidental parabolic element $\rho_\infty(g_\alpha)$ where $g_\alpha \in G$ represents the geodesic $\alpha \subset S_p$.*

We note that such continuous quasiconformal deformations corresponding to simple closed geodesics $\alpha, \alpha' \subset S_p$ are independent if the geodesics α and α' are disjoint. It implies the existence of a boundary group in $\partial\mathcal{T}(G)$ with "maximal" number of non-conjugate accidental parabolic subgroups:

Corollary 5.7. *Let $G \subset PO(2,1) \subset PU(2,1)$ be a uniform lattice isomorphic to the fundamental group of a closed surface S_p of genus $p \geq 2$. Then there is a continuous deformation $R : \mathbb{R}^{3p-3} \to \mathcal{T}(G)$ whose boundary group $G_\infty = R(\infty)(G)$ has $(3p-3)$ non-conjugate accidental parabolic subgroups.*

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Dept. of Math., University of Oklahoma, Norman, OK 73019, USA
E-mail address: apanasov@ou.edu

Mathematical Sciences Research Institute, Berkeley, CA 94720, USA

Grad. School of Math. Sci., Univ. of Tokyo, Tokyo 153, Japan

RADIANT AFFINE 3-MANIFOLDS

THIERRY BARBOT

The aim of these notes is to give a very concise version of [2] and [3]. The (sketchy) proofs, if not omitted, are oversimplified; in case-by-case arguments we will only consider the simplest situations.

We are interested in radiant affine closed 3-manifolds, i.e. closed 3-manifolds locally modelled on the linear space R^3, and with coordinate change maps represented by linear transformations. The terminology 'radiant' comes from the existence of a special flow, the so-called *radiant affine flow.* In each affine chart of the radiant affine manifold, the flow is locally the homothetic flow, i.e.:

$$(t, u) \mapsto e^t u$$

The radiant flow is well-defined since the flow above is preserved by the action of the linear group $GL(3, R)$. We denote it by Φ^t. Observe that it is naturally transversely real projective. Our work can be viewed in the spirit of [15]: we are studying a very special class of transversely projective flows (note that many transversely projective flows are not radiant flows of affine structures, for example, the geodesic flows of negatively curved surfaces).

The classification of affine structures is very easy in dimension one and is completely known in dimension two thanks to the work of J.P. Benzécri and T. Nagano, K. Yagi (see [4], [16]; see also [1]). It remains essentially open in dimension greater than three. In dimension three, some results are known in some particular cases. For example:

- the complete case ([12]) (it includes the Lorentzian case ([6])),
- it is known that circle bundles over hyperbolic surfaces do not admit unimodular affine structures ([8]), and that for every affine structure on such a manifold, the monodromies of the fibers are homotheties (Fried, [11]).

Here, we are only interested in the radiant affine case. The only known closed radiant affine 3-manifolds are those obtained by 'suspending automorphisms of real projective orbifolds' (we give a precise definition of this construction in the next section). *Benzécri's examples* are the affine suspensions of the identity map of a projective surface. Y. Carrière arose the following question ([7]): are the affine suspensions the only examples of closed radiant affine 3-manifolds? S. Choi is announcing a positive answer to this question, using the results discussed here ([10]).

According to Fried's theorem quoted above, a corollary of our study is:

Theorem *Let M be a Seifert bundle over a hyperbolic surface (i.e. with negative Euler characteristic). Assume that M is equipped with an affine structure (not necesseraly radiant) for which the monodromy of the fibers is not trivial. Then, M is affinely isomorphic to a Benzécri's example.*

(This corollary was the main motivation of our work).

Let us present the results discussed in this text: an *affinely tangent surface* (abbreviation a.t.s) is a closed embedded flat surface in the radiant affine manifold tangent to the

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radiant flow. Such a surface is a finite union of tori or Klein bottles since it is tangent to a nonsingular vector field.

Theorem A *Assume that the radiant flow of a radiant affine closed 3-manifold M admits a periodic orbit θ. Then, one of the following (non-exclusive) situations occurs:*

- M is affinely isomorphic to a affine suspension, or

- θ is of saddle type, i.e. admits stable and unstable manifolds, or

- θ is contained in a affinely tangent surface which is a union of periodic orbits of the radiant flow.

The following theorem elucidates the third case of theorem A:

Theorem B *If a radiant affine closed 3-manifold admits an affinely tangent surface, then it is an affine suspension.*

As a nearly immediate corollary of Theorem B, we obtain that every radiant affine closed 3-manifold with virtually solvable monodromy group is an affine suspension. Indeed, in this case, we have two subcases:

1. Either a finite index subgroup of the holonomy group is conjugated in the set of matrices of the form:
$$\begin{pmatrix} a & b & c \\ 0 & r\cos\theta & r\sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{pmatrix}$$
In this case, the 1-dimensional foliation $dy = dz = 0$ is preserved by the holonomy group. Therefore, it defines on M a 1-dimensional foliation with a transverse holomorphic structure. The proof then follows from the classification of transversally holomorphic flows ([13, 5]).
2. or, up to a finite covering, and after conjugacy, we can assume that the holonomy group preserves the plane $P = \{z = 0\}$. If P does not meet the image of the developing map, then $\frac{dz}{z}$ induces on the affine manifold a closed nonsingular one form transverse to the radiant flow. We then obtain by Tischler's argument that the radiant flow admits a section, which is enough for the conclusion (see Section 1). On the other hand, if P meets the image of the developing map, then any connected component of its preimage by the developing map projects in the closed manifold over an affinely tangent surface. We conclude then by Theorem B.

The following theorem is exactly the result needed for proving the theorem about Seifert manifolds announced above:

Theorem C *If M is a Seifert bundle over a hyperbolic orbifold equipped with a radiant affine structure, then it is isomorphic to a Benzécri's example.*

1. Suspensions of real projective structures

Let Σ be a closed surface equipped with a real projective structure (the construction can be generalized to the case of orbifolds). Let $f_i : U_i \to V_i \subset S^2$ be a family of projective charts covering Σ. When U_i meets U_j, we have an element $\bar{g}_{ij}$ of $P^+GL(3, R)$ such that on $U_i \cap U_j$:
$$f_i = \bar{g}_{ij} \circ f_j$$
Here $P^+GL(3, R)$ is the quotient of $GL(3, R)$ by the homotheties with a positive factor, with its natural action on the sphere S^2 of linear half lines. Let's choose representatives g_{ij} of the $\bar{g}_{ij}$ in $GL(3, R)$. We impose the condition $g_{ij}g_{jk}g_{ki} = id$ if $U_i \cap U_j \cap U_k$ is not empty. Such a choice is always possible: take for example the unique representative of $\bar{g}_{ij}$ with determinant ± 1. The set of the possible choices is parametrized by $H^1(\Sigma, R)$. For every i, let W_i be the open cone in R^3 with vertex at 0, union of the half lines belonging to V_i.

Let us denote by W the quotient of the disjoint union of the W_i by the relation identifying each element x_j of W_j with the element $g_{ij}(x_j)$ of W_i (when $g_{ij}(x_j)$ belongs effectively to W_i, of course). This quotient is a noncompact radiant affine manifold, equipped with a complete radiant flow $\hat{\Phi}^t$. The quotient of W by the relation 'being on the same orbit of $\hat{\Phi}^t$' is homeomorphic to Σ. The quotient map is a fibration. Let Σ_0 be any section of this fibration. W is diffeomorphic to $\Sigma \times R$.

Let φ be a projective transformation of Σ. It lifts[1] to an affine diffeomorphism $\hat{\varphi}$ of W well-defined up to composition by $\hat{\Phi}^t$. If T is big enough, $\hat{\Phi}^T\hat{\varphi}(\Sigma_0)$ is a section of $\hat{\Phi}^t$ disjoint from Σ_0. Therefore, $\hat{\Phi}^T\hat{\varphi}$ acts freely and properly discontinuously on W. The quotient of this action is a closed radiant affine manifold homeomorphic to the topological suspension Σ_φ of $\varphi : \Sigma \to \Sigma$. We call it *affine suspension of* φ. Observe that the suspension is not uniquely defined: we made some choices. These choices are parametrized by an open subset of $H^1(\Sigma_\varphi, R)$. These parameters are the morphisms $H_1(\Sigma_\varphi, R) \to R$ represented by $det \circ \rho$, where ρ is the morphism of monodromy (see below) and det the determinant map.

When φ is the identity map, we call the radiant affine manifolds obtained as above *Benzécri's examples.* It follows from Choi's classification of real projective structures on closed surfaces (cf. [9]) that if the Euler characteristic of Σ is negative, then the automorphism group of the projective surface is finite. Therefore, all the affine suspensions obtained from Σ are finitely covered by Benzécri's examples. Actually, it is easy to see that more precisely, they are Benzécri's examples over projective orbifolds.

By construction, the radiant affine flow of an affine suspension admits a closed cross-section homeomorphic to Σ. Note that this section, equipped with the projective structure induced by the transverse projective structure of the radiant flow is isomorphic to the initial projective surface Σ. It is quite obvious that the converse is true:

Proposition 1.1. *A radiant affine closed manifold is an affine suspension if and only if its radiant flow admits a closed cross-section.* ∎

Therefore, the set of affine suspensions is closed under finite coverings. We will use very often this observation: in the problems we will have to consider, we can always restrict the study to any more suitable finite covering.

Proposition 1.2. *A radiant affine closed manifold is a Benzécri's example if and only if all the orbits of its radiant flow are periodic.*

Proof This is a consequence of Epstein's theorem about foliations with all leaves closed (see [2]). ∎

2. Notations and preliminary remarks

Let M be a radiant affine closed 3-manifold. Let $p : \widetilde{M} \to M$ be a universal covering of M. Let Γ be the group of automorphisms of this covering: it is isomorphic to the fundamental group. Let $\mathcal{D} : \widetilde{M} \to R^3$ be the developing map, and $\rho : \Gamma \to GL(3, R)$ be the monodromy morphism. They satisfy the relation:

$$\forall \gamma \in \Gamma \quad \mathcal{D} \circ \gamma = \rho(\gamma) \circ \mathcal{D}$$

[1]Actually, such a lifting does not always exist for any choice of W, but for many of W above the given Σ, we can perform such liftings. The condition is: let $\bar{\rho} : \pi_1(W) \to GL(3, R)$ be the monodromy morphism of W. Observe that $\pi_1(W)$ is isomorphic to $\pi_1(\Sigma)$. Let φ_* be the automorphism of $\pi_1(\Sigma)$ induced by φ. Then, φ lifts if and only if $det \circ \bar{\rho}$ is constant on the orbits of φ_*. For example, the choice of the g_{ij}'s of determinant ± 1 works. We don't want to go into further details.

Taking a double covering if necessary, we assume that all the elements of the monodromy group $\rho(\Gamma)$ have positive determinant.

A *screen* of $\mathcal{D}$ is an open connected subset of $\widetilde{M}$ on which $\mathcal{D}$ is injective. More generally, we extend this notion to any local homeomorphism $f : X \to Y$ between manifolds. The following lemmas are easily checked:

Lemma (inserted screens) *Let U and V be two screens of a local homeomorphism $f : X \to Y$. Assume that $U \cap V$ is not empty, and that $f(U)$ is contained in $f(V)$. Then U is contained in V.* ∎

Lemma (closing the screen) *Let U be a screen of a local homeomorphism $f : X \to Y$. Assume that the boundary of $f(U)$ in Y is a finite union of points or of smooth hypersurfaces disconnecting Y. Then, the restriction of f to the closure of U in X is injective.* ∎

Lemma (filling the disk) *Let D be a closed embedded disc in X of the same dimension n as X ($n \geq 2$). Assume that the image of ∂D by f is the boundary of a closed embedded disc D' in Y, and that for all elements x of D near ∂D, the image $f(x)$ belongs to D'. Then, the restriction of f to D is a homeomorphism between D and D'.*

Let $\widetilde{\Phi}^t$ the lifting of the radiant flow to $\widetilde{M}$. Let Q be the quotient of $\widetilde{M}$ by the relation identifying the points of $\widetilde{M}$ belonging to the same orbit of $\widetilde{\Phi}^t$. We denote the projection map by $\pi : \widetilde{M} \to Q$. An *affine Hopf manifold* is a finite quotient of the quotient of $R^3 \setminus \{0\}$ by a linear contraction. An equivalent definition is that an affine Hopf manifold is a affine suspension of some automorphism of a real projective elliptic orbifold.

Proposition 2.1. *Q equipped with the quotient topology is homeomorphic to the plane R^2 or to the sphere S^2. The map π is a (locally) trivial fibration. If Q is homeomorphic to the sphere, then M is an affine Hopf manifold.* ∎

The action of Γ on $\widetilde{M}$ induces another action on the quotient Q such that π is equivariant. Q has a natural real projective structure preserved by this action. The developing map $\bar{\mathcal{D}} : Q \to S^2$ of this structure is just the induced map by $\mathcal{D}$. We will denote by $\bar{\mathcal{D}}$ the composition of $\bar{\mathcal{D}}$ with the projection map $S^2 \to RP^2$ too. The 'monodromy' $\bar{\rho} : \Gamma \to P^+GL(3,R)$ is the morphism induced by ρ. Since we assume that the determinants of the $\rho(\gamma)$ are positive, we can consider $\bar{\rho}$ as a map from Γ to $P^+GL^+(3,R) \simeq SL(3,R)$.

There is a natural correspondence between the dynamics of $\widetilde{\Phi}^t$ on M and the dynamics of Γ on Q. For example, an orbit of $\widetilde{\Phi}^t$ corresponds to an orbit of Γ on Q. An orbit of $\widetilde{\Phi}^t$ is periodic if and only if the corresponding Γ-orbit is closed and discrete, or equivalently if and only if some (and therefore all) element of the corresponding Γ-orbit is fixed by a non-trivial element. Moreover, since periodic orbits of $\widetilde{\Phi}^t$ are homeomorphic to the circle, the stabilizer of a point in Q is either cyclic, or trivial.

Consider an affinely tangent surface Σ in M. Let $\tilde{\Sigma}$ be some connected component of the lifting of Σ in $\widetilde{M}$. Let $\pi_1(\Sigma)$ be the subgroup of Γ consisting of the elements globally preserving $\tilde{\Sigma}$ (the terminology is quite ambiguous, since $\pi_1(\Sigma)$ depends on the choice of the lifting). The group $\pi_1(\Sigma)$ is conjugate in Γ to the image of the fundamental group of Σ by the morphism induced by the inclusion of Σ in M. Let c be the projection of $\tilde{\Sigma}$ in Q by π. Clearly, $\bar{\mathcal{D}}(c)$ is contained in a 'projective line of S^2' (i.e., a great circle). We call c a *transverse shadow of* Σ. We will see later that, if M is not an affine suspension, then the restriction of $\bar{\mathcal{D}}$ to c is injective, and that c is a line (i.e. Σ is incompressible) (Lemmas 3.1, 3.3).

We denote by Δ the subgroup of the diagonal matrices in $GL(3,R)$, and Δ^+ the subgroup of diagonal matrices with positive coefficients. We will use the same notation for the subgroups of diagonal matrices in $SL(3,R)$. Observe that Δ^+ is a maximal connected abelian subgroup of $SL(3,R)$. An abusive simplification we will use in the proofs is the following: when we will deal with a maximal connected abelian subgroup of $SL(3,R)$, we will only consider the case where this subgroup is Δ^+.

3. Closed flat submanifolds preserved by the radiant flow

In this section, we give partial proofs of Theorems A and B. We begin with Theorem A: assume that $\tilde{\Phi}^t$ admits a periodic orbit θ. Let $\tilde{\theta}$ be a lifting of θ: it is an element of Q preserved by an element γ_0 of Γ. We can assume the only elements of Γ preserving $\tilde{\theta}$ are the powers of γ_0. There is an element x of $\tilde{\theta}$ and a real T such that x is a fixed point of $\gamma_0\tilde{\Phi}^T$. The image of the fixed points of $\gamma_0\tilde{\Phi}^T$ by p form a closed subset of M that we denote by $F(\gamma_0, T)$.

Observe that if $\bar{\rho}(\gamma_0)$ is the identity, then M has to be a Benzécri's example. Indeed, in this case, the set of fixed points of γ_0 is a closed open subset of Q, thus is all of Q. Then, all the orbits of $\tilde{\Phi}^t$ are periodic. We conclude by 'Epstein's theorem'.

Consider now the case where $\bar{\mathcal{D}}(\tilde{\theta})$ belongs to a line $\bar{l}_0$ of fixed points of $\bar{\rho}(\gamma_0)$. Let l_0 be the set of points of $\widetilde{M}$ whose image by $\bar{\mathcal{D}}$ belongs to $\bar{l}_0$, and which are fixed by γ_0. It is a closed codimension one subset of Q. Let $L_0 = \pi^{-1}(l_0)$. Its projection by p is an immersed surface Σ of M. For every element γ of Γ, we have three possibilities:

- γl_0 is disjoint from l_0, or
- $\gamma l_0 = l_0$, or
- γl_0 intersects transversely l_0 at some point $\tilde{\theta}$.

Let's prove that the third case is impossible. Assume it occurs. Let $\theta = p(\tilde{\theta})$. Let D be a local cross section to $\tilde{\Phi}^t$ through θ. Let f be the first return map along $\tilde{\Phi}^t$ on D. Then, it is locally conjugate to the action of $\bar{\rho}(\gamma_0)^{\pm 1}$ near $\bar{\mathcal{D}}(\tilde{\theta})$. On the other hand, it has two lines of fixed points: one corresponding to L_0 near $\tilde{\theta}$, and the other to γL_0 near $\tilde{\theta}$. This is a contradiction.

Since the third case is impossible, we obtain that $p(L_0)$ is an injectively immersed surface in M. It is a closed subset of $F(\gamma_0, T)$: therefore, $p(L_0)$ is embedded. We have proven that $p(L_0)$ is an affinely tangent surface containing θ and union of periodic orbits. This proves Theorem A in this case.

There remains the case where θ is an isolated periodic orbit of $\tilde{\Phi}^t$. A rigorous way to go on with the proof is to consider all the possible conjugacy classes of $\bar{\rho}(\gamma_0)$ in $SL(3,R)$, showing that the only possibility, if M is not a suspension, is that $\bar{\rho}(\gamma_0)$ is hyperbolic (i.e. all its eigenvalues are real and pairwise different in absolute value) and θ is of saddle type (i.e. $\mathcal{D}(\tilde{\theta})$ is the proper direction of $\rho(\gamma_0)$ associated to the intermediate eigenvalue).

We will just prove here that $\bar{\mathcal{D}}(\tilde{\theta})$ is not a repulsive (or attractive) fixed point of $\bar{\rho}(\gamma_0)$.

Suppose not. Consider the domain of repulsion U of γ_0 in Q. It is easily seen that U is a screen of $\bar{\mathcal{D}}$, and that $\bar{\mathcal{D}}(U)$ is an open hemisphere. Thus, $W = p(\pi^{-1}(U))$ is the domain of attraction of θ in M (we assume here that T is negative). It is homeomorphic to the open solid torus. Therefore, for all elements γ of Γ, γU intersects U if and only if γ is a power of γ_0, in which case we have $\gamma U = U$. Consider an element x of ∂W: let $\tilde{x}$ be a preimage of x by p. Let V be a convex screen in Q containing $\pi(\tilde{x})$. There is a sequence x_n of elements of U and a sequence γ_n of elements of Γ such that $\gamma_n x_n$ converges towards $\pi(\tilde{x})$. Since V is a convex screen, and since the $\gamma_n U$ are open hemispheres pairwise disjoint or equal, we obtain that $\pi(\tilde{x})$ itself belongs to some $\gamma_n U$. This proves that ∂W

is the projection by p of $\pi^{-1}(\partial U)$. It follows that ∂W is an embedded surface. Since it is an a.t.s (affinely tangent surface), it is a finite union of tori. Moreover, each element γ of $\pi_1(\partial W) \subset \Gamma$ (see notations) preserves ∂U. Therefore, γ^2 preserves U. Thus, γ^2 is a power of γ_0. It follows that $\pi_1(\partial U)$ is cyclic up to a finite index. Thus, ∂U is a single compressible torus. The transverse trace of this torus must be a circle contained in ∂U. By the 'filling the disc lemma ', we obtain that the closure of U in Q is a closed embedded disc on which $\bar{\mathcal{D}}$ is injective. Let V be a screen of $\bar{\mathcal{D}}$ containing the closure of U. The union of the γ_0 iterates of V is a γ_0-invariant screen V_∞. The image $\bar{\mathcal{D}}(V_\infty)$ is the sphere minus a point $\bar{y}$ (the opposite of $\bar{\mathcal{D}}(\tilde{\theta})$). By the 'closing the screen lemma', we see that either Q is homeomorphic to the sphere, or V_∞ is closed in Q, i.e. is exactly Q. In the former case, M is a Hopf manifold. In the latter case, the image of the developing map is $S^2 \setminus \{\bar{y}\}$. Then, $\bar{y}$ is $\bar{\rho}(\Gamma)$-invariant. This implies that $\tilde{\theta}$ is fixed by every element of Γ. Thus, Γ is a cyclic group. This is impossible since M is an irreducible closed 3-manifold. We have concluded our partial proof of Theorem A.

Let us now consider the proof of Theorem B. It means that we have to prove that if M admits an a.t.s, then it is an affine suspension. Let Σ be such an a.t.s. Let $\tilde{\Sigma}$ be a lifting of Σ in $\widetilde{M}$, i.e. a connected component of $p^{-1}(\Sigma)$. Remember that Σ is a torus or a Klein bottle.

Lemma 3.1. *If Σ is compressible, then M is an affine suspension.*

Proof If Σ is compressible, its transverse shadow is a circle. Since Q is simply connected, this shadow is the boundary of some disc D. Let γ be any element of $\pi_1(\Sigma)$ preserving $\tilde{\Sigma}$, and therefore D. According to the 'filling the disc lemma', D is a screen of $\bar{\mathcal{D}}$. On the other hand, $\bar{\rho}(\gamma)$ surely has a fixed point in the closure of the hemisphere $\bar{\mathcal{D}}(D)$ which is not of saddle type. Thus, γ admits a fixed point in the closure of D which is not of saddle type. Thanks to Theorem A, we see that, if M is not an affine suspension, then all the boundary of D in Q is fixed by γ, and that γ has no fixed point in the interior of D. Up to conjugacy in $SL(3, R)$, $\bar{\rho}(\gamma)$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

In other words, there are two points x_1 and x_2 in ∂D, which are respectively the α-limit point and ω-limit point of every element of D under the action of γ. Let $\mathcal{F}$ be the singular foliation of Q induced by the foliation of S^2 whose leaves are the great circles containing $\bar{\mathcal{D}}(x_1)$ and $\bar{\mathcal{D}}(x_2)$. Note that all the leaves of $\mathcal{F}$ contained in D are γ-invariant. Let Ω be the union of the regular leaves of $\mathcal{F}$ which are γ-invariant and contain x_1 and x_2 in their closure. It is a screen $\bar{\mathcal{D}}$, containing D, which is γ-invariant and saturated by $\mathcal{F}$. If Ω is all of Q, then the x_i are Γ-invariant, which implies that M is a Hopf manifold. Therefore, we can assume that $\partial\Omega$ is not empty. We obtain then that $\partial\Omega$ contains two different leaves f_1 and f_2 of $\mathcal{F}$, with closures containing respectively x_1 and x_2, and such that $\bar{\mathcal{D}}(f_1)$ and $\bar{\mathcal{D}}(f_2)$ are contained in the same half line l. Moreover, the extremities of l are $\bar{\mathcal{D}}(x_1)$ and $\bar{\mathcal{D}}(x_2)$, and no point of l is fixed by $\bar{\rho}(\gamma)$. But the leaves f_1 and f_2 are γ-invariant. This implies that $\bar{\mathcal{D}}(f_1) = l = \bar{\mathcal{D}}(f_2)$, which is a contradiction to the 'closing the screen' lemma. ∎

According to the previous lemma, there is a subgroup H of $\pi_1(\Sigma) \subset \Gamma$ of index at most two isomorphic to $Z \oplus Z$. The following lemma establishes a converse of this fact:

Lemma 3.2. *Let c' be a closed line in Q preserved by a subgroup H' of Γ isomorphic to $Z \oplus Z$. Then, c' is the transverse shadow of an a.t.s in M.*

Proof By 'line', we mean that $\bar{\mathcal{D}}(c')$ is contained in a great circle of S^2. Since $\pi^{-1}(c')$ is homeomorphic to $R \oplus R$, and since H' acts freely and properly discontinuously on $\widetilde{M}$, the quotient of $\pi^{-1}(c')$ by H' is a torus. Thus, $p(\pi^{-1}(c'))$ is an immersed torus in M. The arguments used in the proof of Theorem A imply that this torus has no self-intersection, i.e., is an embedded torus.: indeed, such a self-intersection would be a periodic orbit of Φ^t. According to Theorem A, this periodic orbit is of saddle type (or the torus is a union of periodic orbits, but we gave previously an argument excluding this case). Let x_0 be an element of c' corresponding to a lifting of this periodic orbit: it is a saddle-type fixed point of some hyperbolic element γ_0 of Γ. Moreover, c' is tangent to the (un)stable line of γ_0 through x_0. Since γ_0 has no attractive or repulsive fixed point in Q, c' is contained in this (un)stable line. Therefore, this (un)stable line is preserved by H'. It follows that $H' \approx Z \oplus Z$ fix x_0. Contradiction. ∎

Lemma 3.3. *The restriction of $\bar{\mathcal{D}}$ to c is injective.*

Proof We refer to [2]. ∎

A corollary of the previous lemma is that the restriction of $\mathcal{D}$ to $\tilde{\Sigma}$ is injective. Hence, $\rho(H)$ acts freely and properly discontinuously on the cone $\mathcal{D}(\tilde{\Sigma})$, and the restriction of ρ to H is injective. Moreover, thanks to Theorem A, we can assume that the projection $\Lambda = \bar{\rho}(H) \subset SL(3, R)$ is isomorphic to $Z \oplus Z$ (if not, M is a Benzécri's example). Let G be the neutral component of the Zariski closure of Λ. Exchanging Λ with a finite index subgroup of itself if necessary, we can assume that Λ is contained in G. Observe that $\rho(H)$ preserves a plane in R^3 (the plane containing $\mathcal{D}(\tilde{\Sigma})$) and a line included in this plane (the line containing one of the boundary components of $\mathcal{D}(\tilde{\Sigma})$). Thus, G is isomorphic to $R \oplus R$. It is a maximal connected abelian subgroup of $SL(3, R)$, and Λ is a cocompact lattice of G.

For the sake of simplicity, we assume here that G is (conjugate to) the diagonal group Δ^+. All the other cases can be treated in a more or less similar way. We won't discuss them here.

The action of G on the sphere S^2 is generated by a Lie homomorphism from the Lie algebra $\mathcal{G}$ of G to the Lie algebra of vector fields on the sphere. Pulling back by $\bar{\mathcal{D}}$, we obtain a Lie algebra $\mathcal{V}$ of vector fields on Q, each of them H-invariant. Each element of $\mathcal{V}$ generates a local flow on Q, but, since Q is not complete, there is no *a priori* reason for these local flows to be complete. The following proposition shows that it is actually true:

Proposition 3.4. *The algebra $\mathcal{V}$ generates an action of G on Q. Moreover, the action of H on Q through covering automorphisms coincide with its action through $\rho : H \to G$.*

Before proving this proposition, we explain briefly how it is used to prove Theorem B. The action of G on Q cannot have fixed points (indeed, a fixed point of G on Q would correspond to an orbit of $\tilde{\Phi}^t$ with fundamental group containing $Z \oplus Z$!) Therefore, the image of the developing map $\bar{\mathcal{D}}$ is a union of orbits of dimension 1 and 2. Its boundary is a union of G-fixed points and of one-dimensional G-orbits. Since this boundary is $\rho(\Gamma)$-invariant, it follows, taking a finite index subgroup if necessary, that every G-fixed point is fixed by $\rho(\Gamma)$. In other words, $\rho(\Gamma)$ is contained (up to a finite index) in Δ^+. The monodromy group $\rho(\Gamma)$ commutes with the action of the diagonal matrices of $GL(3, R)$.

We obtain then by pull-back an affine action of R^3 on M. This action is not locally free in general (if it is, then M is homogeneous. The conclusion of Theorem B is then easy). There are many ways to conclude from this point. For example: this action has no orbit of dimension zero or one. The two-dimensional orbits of this action are tori. Moreover, the fundamental groups of these tori admit the common factor H: they are thus isotopic one another. The complement of these tori is a union of open orbits diffeomorphic to $]0,1[\times T^2$. These open orbits correspond to the G-invariant triangles in S^2 admitting as vertices fixed points of G. Therefore, M is a torus bundle over the circle. It is easy to identify the affine nature of the homogeneous components $]0,1[\times T^2$. Since M is obtained by gluing these components along the tori, we recover the affine nature of M itself. It follows that T is an affine suspension.

Proof of 3.4 We will define the domain of definition of G on Q, and then prove that this domain is all of Q. For every nontrivial element g of G, let g^t be the one parameter subgroup of G such that $g^1 = g$. We say that g is defined on a point x of Q if the curve $t \mapsto g_i^t\bar{\mathcal{D}}(x)$ for t between 0 and 1 can be lifted as a curve c_t starting at x such that $c_s = g^s x$ for every time s for which g^s belongs to H. We denote then the extremity at time 1 of this lifted curve by $g * x$. Let $\mathcal{U}'$ be the set of points where all the elements of G are defined. The operation $*$ defines an action of G on $\mathcal{U}'$. Finally, let $\mathcal{U}$ be the set of points x of $\mathcal{U}'$ where the two actions of H coincide, i.e. such that $hx = h * x$ for every element h of H. Observe that the transverse shadow c is contained in $\mathcal{U}$. Note also that $\mathcal{U}$ is H-invariant.

Let g_1^t and g_2^s be two one-parameter subgroups of G generating G and such that g_1^1 and g_2^1 both belong to Λ. By an abuse of notation, we denote by g_1^1 and g_2^1 the elements of H which are mapped by $\bar{\rho}$ to g_1^1 and g_2^1. Let $\mathcal{U}_1$ be the set of the elements of Q where $g_1^t g_2^s$ are defined for every $(t,s) \in [0,1] \times [0,1]$. Since every element g of G is of the form $\rho(h)g_1^t g_2^s$ $(t,s) \in [0,1] \times [0,1]$, $h \in H$, and since G is abelian, we see that $\mathcal{U}$ and $\mathcal{U}_1$ are equal (actually, it is not so obvious, but there is no difficulty in proving this claim. The proof we have in mind uses the topological properties of the plane Q).

Here we give some immediate properties of $\mathcal{U}$:

- $\mathcal{U}$ contains the transverse shadow c,
- $\bar{\mathcal{D}}(\mathcal{U})$ does not contain any fixed point of G (indeed, such a fixed point would induce a fixed point of H in Q),
- $\bar{\mathcal{D}}(\mathcal{U})$ is a union of G-orbits,
- every one-dimensional orbit of G in Q is the transverse shadow of some a.t.s (this follows from lemma 3.2),
- the open orbits of G in Q are screens of $\bar{\mathcal{D}}$,
- $\mathcal{U}$ is open (this follows from $\mathcal{U} = \mathcal{U}_1$).

In order to prove the proposition, we have to prove that $\mathcal{U}$ is the whole Q. By connectedness of Q, it is enough to show that $\mathcal{U}$ is closed.

Let x be an element in the closure of $\mathcal{U}$: we want to show that it belongs to $\mathcal{U}$. Note that $g(x)$ is defined for every sufficiently small elements g of G. Therefore, $\bar{\mathcal{D}}(x)$ is not a fixed point of G. If $\bar{\mathcal{D}}(x)$ belongs to an open orbit of G, we get easily from the inserted screen lemma that x belongs to $\mathcal{U}$, and we are done in this case. There remains the case where $\bar{\mathcal{D}}(x)$ belongs to some one-dimensional orbit l of G. Let c' be the connected component of $\bar{\mathcal{D}}^{-1}(l)$ containing x. Let $U' \subset \mathcal{U}$ be an open G-orbit whose boundary meet c'. It is a screen of $\bar{\mathcal{D}}$ which is H-invariant. It follows from the 'closing the screen' lemma that the restriction of $\bar{\mathcal{D}}$ to the union of the H-iterates of c' is injective. A corollary is that if H preserves c', then c' is contained in $\mathcal{U}$. In other words, we can assume that $\bar{\mathcal{D}}(c')$ is strictly contained in l.

Assume first that no element of H fixes x: in this case, $\bar{\mathcal{D}}(x)$ is a non-trivial accumulation point of its Λ-orbit. It follows that there are two elements h_1 and h_2 in H such that:

- $h_1(x)$ and $h_2(x)$ belong both to c',
- h_1 and h_2 have no non-trivial common power in H, i.e. $\bar{\rho}(h_1)$ and $\bar{\rho}(h_2)$ generate a cocompact lattice of G.

By the first property, we obtain that c' is preserved by the group generated by h_1 and h_2. Since this group is a lattice of G (by the second property), this implies $\bar{\mathcal{D}}(c') = l$. As observed before, it follows that x belongs to $\mathcal{U}$.

Now, we have to deal with the case where some element h of H fixes x. The elements of c' near x are of the form gx, where g is an element of G defined at x. Since g and $\bar{\rho}(h)$ commute, this implies that c' is a line of fixed points of h. According to Theorem A, c' is the transverse shadow of some a.t.s Σ' in M, i.e. $c' = \pi(\tilde{\Sigma}')$ with $p(\tilde{\Sigma}') = \Sigma'$. Let $H' = \pi_1(\Sigma')$, where H' is chosen so that $\tilde{\Sigma}'$ and c' are H'-invariant. Let $\Lambda' = \bar{\rho}(H')$ and let G' be the neutral component of the Zariski closure of Λ' (we will assume that Λ' is contained in G'). Let $\mathcal{U}'$ be the set of the elements of Q where the action of G' is defined: it is the anolog of $\mathcal{U}$ with respect to G. The curve c' is contained in $\mathcal{U}'$, and it is a one-dimensional orbit of G'. (Note that there is no reason for G' and G being the same subgroup. If it is the case, $G' = G$ is defined at x, and thus, x belongs to $\mathcal{U}$ as desired.)

We introduce another simplification: *we assume that G' is conjugate in $SL(3,R)$ to $G = \Delta^+$.* Then, there are two different open orbits of G' in $\mathcal{U}'$ both containing c' in their boundary. One of them, that we call U'', meets the open G-orbit $U' \subset \mathcal{U}$. Then, c' is a boundary component of U'. Since $\bar{\mathcal{D}}(c')$ is strictly contained in l, there is a one-dimensional orbit $\bar{c}''$ of G' in S^2 meeting $\bar{\mathcal{D}}(U')$ and contained in the boundary of $\bar{\mathcal{D}}(U'')$. By the inserted screen lemma, we see that $\partial U''$ meets U'. Let c'' be a connected component of $U' \cap \partial U''$. We have $\bar{\mathcal{D}}(c'') \subset \bar{c}''$. Applying once more the preceding arguments, we see that c'' is a transverse shadow of some a.t.s. But, since it meets U', there is a sequence h_n of elements of H such that the $h_n c''$ has an accumulation point on c. Moreover, this accumulation is not trivial, i.e. no $h_n c''$ is equal to c. This is impossible, since c and c'' are transverse shadows of embedded surfaces!

This contradiction completes the proof of Proposition 3.4, and thus, of Theorem B. ∎

4. Affine Seifert manifolds

In this section, we discuss the (quite intricate) proof of Theorem C. Here, M is a closed Seifert manifold of dimension 3 over a hyperbolic orbifold, equipped with a radiant affine structure. We want to prove that M is a Benzécri's example. Up to a finite covering, M is a circle bundle over a hyperbolic surface Σ. We denote by H the center of the fundamental group Γ: it is cyclic. The quotient of Γ by H is isomorphic to the fundamental group $\bar{\Gamma}$ of Σ. If M is a bundle over the circle, then it is diffeomorphic to $\Sigma \times S^1$. Therefore, if $\tilde{\Phi}^t$ admits a cross section, this cross-section is a real projective surface with negative Euler characteristic. As we observed when we defined affine suspensions, this implies that M is a Benzécri's example. Therefore, we will have proven Theorem C as soon as we have established that M is an affine suspension.

4.1. **Preliminary observations.** The following proposition is a corollary of a theorem of D. Fried ([11]), but its proof is quite easy in our case:

Proposition 4.1. *For every element h of H, the monodromy $\rho(H)$ is a homothety (possibly trivial).*

Proof Let h be a generator of H. Its monodromy $\bar{\rho}(h)$ commutes with every element of $\rho(\Gamma)$. Assume that $\rho(h)$ has three different eigenvalues (two of them possibly complex): then the sum of the proper spaces associated to two of them is $\rho(\Gamma)$-invariant. As observed in the introduction, this implies that M is an affine suspension, and therefore a Benzécri's example. The result is true in this case.

Assume now that $\rho(h)$ has exactly two different eigenvalues. One of them is double: the characteristic subspace associated to it is a hyperplane. This plane is $\rho(\Gamma)$-invariant, and we conclude in this case as above.

There remains the case where $\rho(h)$ has one and only one eigenvalue. Assume that it is not a homothety, but that it admits two independent eigenvectors. Then, the plane generated by these eigenvectors is $\rho(\Gamma)$-invariant: we are done.

Thus, to prove the proposition, we just have to prove that $\rho(h)$ is not conjugate to:

$$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

where a is a real number. But in this case the centralizator of $\rho(h)$, which must contain $\rho(\Gamma)$, is solvable. We conclude in this case too, by exhibiting an invariant plane. ∎

Therefore, H is contained in the kernel N of $\bar{\rho} : \Gamma \to SL(3, R)$. In other words, it induces a map $\hat{\rho}$ from $\bar{\Gamma}$ to $SL(3, R)$. *We assume from now that M is not a suspension.* Our aim is to obtain a contradiction.

Lemma 4.2. *N is equal to H, i.e. the map $\hat{\rho} : \bar{\Gamma} \to SL(3, R)$ is injective.*

Proof For every element n of N, the set of fixed points of n is closed and open. If not empty, it must be the whole of Q. In other words, the orbits of $\widetilde{\Phi}^t$ are all periodic. This implies that M is a Benzécri's example, but we have excluded it by hypothesis. Thus, the action of N on Q is free. It is easy to prove, thanks to the inserted screen lemma, that the action is properly discontinuous. The quotient of Q by this action is a surface S. Since N has a non-trivial center (it contains H), this surface is either an annulus, or a torus. The latter case is impossible, since $\bar{\mathcal{D}}$ induces a local homeomorphism from S to S^2. The former case implies that $N = H$, since H is not strictly contained in any cyclic subgroup of Γ. ∎

Denote by A the annulus which is the quotient of Q by the action of H. We have a projective map $\hat{\mathcal{D}}$ from A to S^2. The group $\bar{\Gamma}$ acts projectively on A, the projective monodromy being $\hat{\rho}$. We can assume, up to finite coverings, that $\bar{\Gamma}$ preserves the orientation, and preserves each end of A. We give a name for the ends of A: one is called *North,* the other *South.* Call *a line* any embedding l of the real line in A such that $\hat{\mathcal{D}}(l)$ is contained in a great circle of S^2. Call it *vertical* if it does not disconnect the annulus. A vertical line joins the two ends of the annulus, and one of its ends is called North, and the other South according to an obvious convention. Another obvious convention is the following: two disjoint vertical lines l_1 and l_2 cut the annulus into two domains homeomorphic to the disc: one is called *to the east of l_1 with respect to l_2* and the other *to the west of l_1 with respect to l_2*. There is no confusion for anybody with a basic knowledge of geography.

Proposition 4.3. *No line in R^3 passing through the origin is globally preserved by a finite index subgroup of $\rho(\Gamma)$.*

Proof Up to finite coverings, and up to conjugacy, we are led to study the case where $\rho(\Gamma)$ preserves the line $\{x = y = 0\}$. Let $\mathcal{F}_0$ be the (singular) foliation of R^3 whose leaves are the planes containing $\{x = y = 0\}$. It is $\rho(\Gamma)$-invariant. Therefore, it induces a foliation $\mathcal{F}$ on M. The first observation is that $\mathcal{F}$ is not singular. The proof goes as follows: it is easy to see that regular leaves are planes, annuli, Möbius bands, tori or Klein bottles. They have at most two ends. By Theorem B, we can assume that the two latter cases do not occur. We now apply the following theorem of G. Duminy ([14]): *for any codimension one foliation of class C^2 of a closed manifold admitting an exceptional minimal set, there is a leaf of the minimal set admitting an infinite number of ends.* In our case, since the leaves of $\mathcal{F}_0$ have at most two ends, there is no minimal exceptional set. Since $\mathcal{F}_0$ has no compact leaf, we obtain that every regular leaf is dense in M. But such a leaf, having at most two ends, cannot pass twice near a singular leaf! This contradiction proves the claim.

Since we are on a circle bundle, and since $\mathcal{F}$ has no compact leaves by Theorem B, we obtain from a theorem of Thurston that $\mathcal{F}$ is a suspension foliation (see [17]). Let $\tilde{\mathcal{F}} = \mathcal{D}^*(\mathcal{F}_0)$ be the lifting of $\mathcal{F}$ to $\widetilde{M}$, and $\mathcal{L}$ the leaf space of $\tilde{\mathcal{F}}$. It is homeomorphic to the real line since $\mathcal{F}$ is a suspension. Let $\mathcal{S}$ be the quotient of $\mathcal{L}$ by H: it is homeomorphic to the circle. $\mathcal{D}$ induces a submersion of $\mathcal{S}$ into the leaf space of $\mathcal{F}_0$ restricted to $R^3 \setminus \{x = y = 0\}$, which is the double covering of the projective line. This submersion is thus a finite covering. Since $[\Gamma, \Gamma]$ is not solvable, we can exhibit an element γ of Γ, fixing a point in $\mathcal{S}$, and such that $\rho(\gamma)$ is (up to conjugacy) of the form:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the eigenvector $(0,0,1)$ is not in the image of the developing map. Therefore, according to Theorem A, γ has no fixed point in Q. On the other hand, since $\mathcal{S} \to RP^1$ is a finite covering, every connected component of $\mathcal{D}^{-1}(\{x = 0\})$ or of $\mathcal{D}^{-1}(\{y = 0\})$ is globally preserved by γ. We deduce that all the fixed points of $\bar{\rho}(\gamma)$ in S^2 belong to the boundary of the image of $\bar{\mathcal{D}}$. Consider two successive fixed points of γ on $\mathcal{L}$: they correspond to leaves F_1 and F_2 of $\tilde{\mathcal{F}}$. Up to conjugacy, and choosing F_1 adequately, we can assume that $\mathcal{D}(F_1)$ is $\{y = 0, x > 0, z > 0\}$. Then, $\mathcal{D}(F_2)$ is either $\{x = 0, y > 0, z > 0\}$ or $\{x = 0, y > 0, z < 0\}$. In the first case, we can find a curve τ connecting F_1 and F_2 such that $\mathcal{D}(\tau)$ is contained in some plane $\{z = z_0\}$. But, this property would imply that every iterate $\gamma^n\tau$ intersects τ. This is impossible since γ acts properly discontinuously on $\widetilde{M}$. Hence, $\mathcal{D}(F_2) = \{x = 0, y > 0, z < 0\}$. It follows that the image of domain between F_1 and F_2 is mapped by $\mathcal{D}$ over $\{x > 0, y > 0\}$. Indeed, the unique connected open $\bar{\rho}(\gamma)$-invariant subset of $\{x > 0, y > 0\}$ is $\{x > 0, y > 0\}$ itself. Applying this argument inductively to the γ-fixed leaves following F_2, we obtain that the image of the developing map is the whole S^2 minus some half-lines connecting $(0,0,\pm 1)$ to other fixed points of $\bar{\rho}(\gamma)$. It follows that some finite index subgroup of $\rho(\Gamma)$ preserves the planes $\{x = 0\}$ and $\{y = 0\}$. Thus, M admits some a.t.s: this contradicts Theorem B. A more detailed proof is given in [3]. ∎

A corollary of Proposition 4.3 and of Theorem B is the following:

Proposition 4.4. *Let x_1, x_2, x_3 be three different points in the projective plane. Let l_1, l_2, l_3 be the three projective lines containing x_2 and x_3, x_1 and x_3, x_1 and x_2 respectively. Then, there is an element γ of Γ such that no x_i is mapped by $\bar{\rho}(\gamma)$ into some l_j.* ∎

We make the following assumption, which will greatly simplify the future arguments:

Assumption 1: If γ_0 is an element of Γ such that $\rho(\gamma_0)$ is a regular hyperbolic matrix (i.e. with three real eigenvalues of different absolute values), and if γ is an element of Γ that does not commute with γ_0, then no fixed point of $\bar{\rho}(\gamma_0)$ in S^2 is mapped by $\bar{\rho}(\gamma)$ into one of the three great circles globally preserved by $\bar{\rho}(\gamma_0)$.

The proposition above suggest that this assumption is not too much abusive.

Proposition 4.5. *Every eigenvalue of any element of the monodromy group is real.*

Proof Let γ be an element of Γ. We have to show that the eigenvalues of $\rho(\gamma)$ are all real. Assume not. Then, up to conjugacy, $\rho(\gamma)$ is a rotation, or a matrix of the form:

$$\begin{pmatrix} \lambda\cos\theta & -\lambda\sin\theta & 0 \\ \lambda\sin\theta & \lambda\cos\theta & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

We consider here only the easiest case: the case where $\rho(\gamma)$ is a rotation. Then, it preserves the conformal structure of the sphere: there is a conformal structure on A preserved by γ. The sphere S^2 can be identified with the complex projective plane $\hat{C} = C \cup \{\infty\}$, where $\rho(\gamma)$ acts by $z \mapsto e^{i\theta}z$.

It follows from Theorem B that γ has no periodic orbit on A. Hence, A is homeomorphic to the annulus $\{a <\mid u \mid< b\}$ in the complex plane (b can be infinite), and γ is conjugate by this homeomorphism to $u \mapsto re^{i\alpha}u$, where α has irrational quotient with π. If r is not 1, the quotient of A by γ is a torus. The pullback of the foliation $Argz = Cte$ on $S^2 \simeq \hat{C}$ by $\hat{\mathcal{D}}$ induces a foliation on this torus. By Poincaré-Hopf, this foliation has no singularity, which means that 0 and ∞ are not in the image of the developing map. Hence, the map associating to $u \in A$ the norm in $C \cup \{\infty\}$ of $\hat{\mathcal{D}}(u)$ induces a submersion of the torus into R. This contradiction shows that $r = 1$. Then, all the orbits of γ on A are non-discrete: it follows in this case too that 0 and ∞ are not in the image of $\hat{\mathcal{D}}$. Being $\bar{\rho}(\gamma)$-invariant, and since the rotation number of $\bar{\rho}(\gamma)$ is irrational (since $\hat{\rho}$ is injective), the image of $\hat{\mathcal{D}}$ is an annulus bounded by the intersection of S^2 with two planes $\{z = z_0\}$ and $\{z = z_1\}$. According to Proposition 4.3, we have $-1 < z_0 < z_1 < 1$. Since $\rho(\Gamma)$ has no invariant plane, z_0 and z_1 are not null. Up to finite index, $\rho(\Gamma)$ preserves some quadratic Lorentzian forms $Q_1 = x^2 + y^2 - \epsilon_1 z^2$ and $Q_2 = x^2 + y^2 - \epsilon_2 z^2$ (they are the forms Q_i such that $\{Q_i = 0\} \cap S^2 = \{z = z_i\} \cap S^2$). Therefore, M is a Lorentzian affine manifold. According to a theorem of Y. Carrière ([6]), it must be complete. This is a contradiction since radiant affine manifolds are not complete (0 is not in the developing image)! ∎

Corollary 4.6. *$\bar{\rho}(\Gamma)$ is a discrete subgroup of $SL(3, R)$.*

Proof Let G be the neutral component of the closure of $\bar{\rho}(\Gamma)$ in $SL(3, R)$. Since $\bar{\rho}$ is injective on $\bar{\Gamma}$, $\bar{\rho}(\Gamma)$ is a surface group: it has no non-trivial solvable normal subgroup. Therefore, if G is not trivial, it is not solvable. Its Levi decomposition has a non-trivial semisimple part S. Consider a Cartan decomposition of this semisimple part. According to the proposition above, the eigenvalues of every element of the compact part K of this decomposition are real. It follows that K is trivial, thus S is trivial too. This contradiction shows that G is trivial, i.e. $\rho(\Gamma)$ is discrete. ∎

4.2. **Hyperbolic monodromy elements.** The goal of this section is to prove the following:

Proposition 4.7. *No element of $\rho(\Gamma)$ is positive hyperbolic, i.e. has three distinct real positive eigenvalues.*

Let us see how to conclude the proof of Theorem C using this proposition. This proposition, with Proposition 4.5, implies the following: for every element γ of Γ, $\bar{\rho}(\gamma)^2$ has a double eigenvalue. This property is expressible by a polynomial equation. In particular, the Zariski closure G of $\bar{\rho}(\Gamma)$ satisfies the same property. Let G_0 be the neutral component of G: every element of G_0 has only real eigenvalues. The arguments of the proof of Corollary 4.6 lead to a contradiction.

We are now concerned with the proof of Proposition 4.7. Let's assume to the contrary that there is an element γ_0 of Γ such that, in a good coordinate system, $\rho(\gamma_0)$ is:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

with $0 < a < b < c$. The action of $\bar{\rho}(\gamma_0)$ on the projective plane has three fixed points: a repulsive point $x_1 = [1;0;0]$, a saddle point $x_2 = [0;1;0]$, and a contracting point $x_3 = [0;0;1]$. There are three invariant projective lines: the *unstable line* $\{x = 0\}$, the *stable line* $\{z = 0\}$, and the *principal line* $\{y = 0\}$. Let Ψ_0 be the (singular) foliation of RP^2 whose leaves are the orbits of the action of the following one-parameter group:

$$\begin{pmatrix} a^t & 0 & 0 \\ 0 & b^t & 0 \\ 0 & 0 & c^t \end{pmatrix}$$

Every leaf is $\bar{\rho}(\gamma_0)$-invariant. Let Ψ be the pullback of Ψ_0 on A by the developing map: it is a γ_0-invariant foliation.

An *invariant affine line* is an invariant projective line minus a fixed point. An *invariant half line* is an invariant projective line minus the two fixed points contained in it. A (γ_0-) invariant line in A is a curve in A with image by $\hat{\mathcal{D}}$ contained in an invariant affine line of $\bar{\rho}(\gamma_0)$ that is globally preserved by γ_0. Observe that according to Theorem A, no γ_0-invariant line contains a preimage of x_1 or of x_2. An invariant line is of principal, stable or unstable type, according to the type of the $\rho(\gamma_0)$-invariant line containing its image by $\hat{\mathcal{D}}$.

The proof of Proposition 4.7 is based on a combinatorial study of the γ_0-invariant lines. The first step is:

Proposition 4.8. *There is an integer k such that there is a γ_0^k-invariant line in Q.*

Sketch of proof: The basic idea is the following: prove that if every γ_0^k has no invariant line, then there is a subannulus $A' \subset A$ which is γ_0-invariant, on which γ_0 acts freely and properly discontinuously. Then, the quotient of A' by the action of γ_0 would be a projective torus with cyclic monodromy generated by a hyperbolic element. From the classification of projective tori (e.g. [1]) we see that such a torus must admit a closed projective line, i.e. that γ_0 must admit an invariant line in A'. This is a contradiction. ∎

Replace γ_0 by γ_0^k: we know that there are γ_0-invariant lines. As usual, we make a very abusive assumption:

Assumption 2: We assume that γ_0 has no fixed point in A.

We collect here some easy facts about γ_0-invariant lines:

- a γ_0-invariant line is a closed embedding of the real line in A (a point in the closure would be a fixed point of γ_0),

- the restriction of $\hat{\mathcal{D}}$ to a γ_0-invariant line is injective, and the image is a $\bar{\rho}(\gamma_0)$-invariant half line,

- the intersection of a γ_0-invariant line l with some iterate γl consists of at most one point, except if γ commutes with γ_0 (this follows from the preceding property, and from Assumption 1 made after Proposition 4.4),

- for any γ_0-invariant line l, and for any element γ of Γ that does not commute with γ_0, γl meets at most one γ_0-invariant line of a given type.

In order to better understand the relative positions of γ_0-invariant lines, we introduce the notion of *generalized triangle:* It is a screen of $\hat{\mathcal{D}}$ in A with the following properties:

- its boundary contain two different γ_0-invariant lines,

- it is a union of γ_0-invariant leaves of the foliation Ψ, and

- its image is contained in a γ_0-invariant triangle (i.e. a connected component of the projective plane minus the $\bar{\rho}(\gamma_0)$-invariant lines),

An *edge* is a γ_0-invariant line contained in the boundary of the generalized triangle. A generalized γ_0-triangle is *convex* if its image by $\hat{\mathcal{D}}$ is an entire $\bar{\rho}(\gamma_0)$-triangle. It is easily verified that a generalized triangle is convex as soon as it admits an edge which is a principal γ_0-invariant line.

A basic fact is the following:

Proposition 4.9. *There is a sequence $T_0, T_1, \ldots, T_n = T_0$ of successive generalized triangles such that:*

- they are pairwise disjoint,

- for every index i, T_i and T_{i-1} have a common edge d_i (and only one).

Proof See [3]. ∎

The union of the T_i with the common edges d_i is an open subset Ω of A, homeomorphic to the annulus, such that the inclusion $\Omega \subset A$ is a homotopy equivalence. In particular, every vertical line intersect Ω. Moreover, the edges d_i are all vertical lines. The d_i are the unique γ_0-invariant lines intersecting Ω. Observe also that every connected component of the preimage of a $\bar{\rho}(\gamma_0)$-invariant line meeting Ω has to be one of the edges d_i.

For any given generalized triangle T_i, its edges d_{i-1} and d_i project by $\hat{\mathcal{D}}$ onto half invariant lines issuing from one fixed point x_j. By the closing the screen lemma, we see that there is an end of T in A such that every curve τ in T converging to this end projects by $\hat{\mathcal{D}}$ onto a curve which converges to x_j. This particular end of T_i is called *the angle* of T_i. It is said to be *of type x_j*. This angle is said to be North or South, depending on to which end of A the curve τ above is converging.

Lemma 4.10. *For every edge d_i, and for any element γ in Γ not commuting with γ_0, γd_i meet some edge d_j.*

Proof Suppose not. The vertical line γd_i must intersect Ω, and therefore, one of the triangles T_j (if not, we have $\gamma d_i = d_j$ for some j. Then, $\bar{\rho}(\gamma_0)$ and $\bar{\rho}(\gamma\gamma_0\gamma^{-1})$ have two common fixed points in RP^2. So, they generate a solvable subgroup of $\bar{\rho}(\Gamma)$. Since this last group is a surface group, this is possible if and only if γ commutes with γ_0, but this is excluded by hypothesis). Assume that the angle of T_j is at the South, of type x_k. If γd_i does not intersect the edges of T_j, then its Southern end must accumulate on the Southern angle of T_j. It follows that the fixed point x_k is an extremity of $\hat{\mathcal{D}}(\gamma d_i)$. But this extremity is on the other hand the image by $\bar{\rho}(\gamma)$ of an extremity of $\hat{\mathcal{D}}(d_i)$, which is a fixed point x_p of $\hat{\mathcal{D}}(\gamma_0)$. In other words, $\bar{\rho}(\gamma)$ maps a fixed point of $\bar{\rho}(\gamma_0)$ on another (or the same!) fixed point. This contradicts Assumption 1. ∎

The generalized triangles T_i have very special intersections with their iterates γT_j.

Lemma 4.11. *There is at least one edge d_i of each type.*

Proof Assume that no edge d_i is of principal type (or stable type, or unstable type...). Then, all the d_i are of stable and unstable type. It follows that the angles of the T_i are all on the same end of A, say, at the South. Let τ be any vertical line. Then, the Southern extremity of $\hat{\mathcal{D}}(\tau)$ is the fixed point x_2. Moreover, for every element γ of Γ, the Southern extremity of $\hat{\mathcal{D}}(\gamma\tau)$ is also x_2. Therefore, $\bar{\rho}(\Gamma)$ fixes x_2. This contradicts Proposition 4.3. ∎

Lemma 4.12. *Let d_1, d_2, and d_3 be three successive edges. Assume that d_1 is of principal type. Then, d_3 is not of principal type.*

Proof Assume that d_1 and d_3 are both of principal type. Inverting γ_0 if necessary, we can assume that d_2 is of stable type. Since d_1 and d_2 are principal, all the triangles T_0, T_1, T_2 and T_3 are convex. Let U be the union of these triangles and of the three edges d_1, d_2 and d_3. It is a screen of $\hat{\mathcal{D}}$, whose image is a $\bar{\rho}(\gamma_0)$-invariant hemisphere minus the closure of a stable half line. According to Lemma 4.10, there is an element γ of Γ and an edge d_j such that γd_j meets d_2 transversely. The intersection of γd_j with U is closed. Its projection is contained in the intersection of $\hat{\mathcal{D}}(U)$ and a great circle cutting $\hat{\mathcal{D}}(d_2)$ transversely. This intersection is an affine line l. Since $U \cap \gamma d_j$ is closed in U, $\hat{\mathcal{D}}(U \cap \gamma d_j)$ must be the whole affine line l. But this is impossible: since $\hat{\mathcal{D}}(\gamma d_j)$ is a half-line, it cannot contain a complete affine line! ∎

Lemma 4.13. *Let d_i be an edge of principal type. Then, d_{i+3} is also of principal type.*

Proof There is no loss of generality in assuming that $i = 1$, and that d_2 is of stable type. Then, according to the preceding lemma, d_3 is of unstable type. If Lemma 4.13 is false, d_4 has to be of stable type. Note that T_0 and T_1 are convex. Let B be the union of T_0, of T_1 and of d_1: it is a screen, and $\bar{B} = \hat{\mathcal{D}}(B)$ is a half-plane bounded by stable and unstable affine lines. We can assume that the angle of T_1 is at the North. Then, the angles of T_2 and T_3 are both at the South. Let γ be an element of Γ and let d_j be an edge such that $d' = \gamma d_j$ intersects d_3 (cf. Lemma 4.10). Assume that d' intersects d_2 too. Then, it enters B. Since $d' \cap B$ is closed, its projection by $\hat{\mathcal{D}}$, which enters in $\bar{B}$ on the stable side, must leave $\bar{B}$ on the unstable side. In other words, the closure of $\hat{\mathcal{D}}(d')$ intersects the unstable line twice: once at $\hat{\mathcal{D}}(d' \cap d_3)$, and also when leaving $\bar{B}$. This is impossible, because $\hat{\mathcal{D}}(d')$ is a affine half-line: its closure cannot intersect a great circle twice. Therefore, d' does not intersect d_2. Consider now two cases, according to the type of d_5 (observe that d_5 cannot be of the same type as d_4):

- if d_5 is of principal type: then, the situation is symmetric with respect to d_3. The argument above shows that d' does not intersect d_4. Therefore, d' is contained in the domain between d_2 and d_4 containing d_3. But T_2 and T_3 both have angle at the South, and d' is a vertical line: it follows that the Southern end of d' is at the angle of T_2 or T_3, i.e. the Southern extremity of $\hat{\mathcal{D}}(d')$ is x_2. Contradiction.
- if d_5 is of unstable type: since d' intersects the unstable edge d_3, it cannot intersect d_5. But T_2, T_3 and T_4 have all angles at the South. A version of the argument in the previous case leads to a contradiction. ∎

Lemma 4.14. *The type of the edge d_i only depends on the rest modulo 3 of i.*

Proof According to the previous lemmas, we just have to show that if d_2 is of principal type, then d_1 and d_3 have different types. We prove this by contradiction. Assume that d_1 and d_3 have the same type, say stable type. Assume that T_1 and T_2 have angles at the South (there is no loss of generality). Let $U_{1,3}$ be the domain in A bounded by d_1 and d_3 containing d_2. Let $U_{0,1}$ (respectively $U_{1,2}$, $U_{2,3}$) be the domain between d_0 and d_1 (respectively d_1 and d_2, d_2 and d_3) at the East of d_0 (respectively d_1, d_2). Let $d'_i = \gamma d_i$ be some edge intersecting d_2. Since the Southern angle of d'_i cannot be the Southern angle of T_1 or T_2, it must intersect d_1 or d_3. Note that it cannot intersect both, and therefore the Northern part of d'_i is contained in $U_{1,3}$. By symmetry, we can assume that d'_i intersects d_3. Thus, the Northern part of d'_i is contained in $U_{1,2}$. In what follows, we assume that each T_{j+1} is at the East at d_j with respect to d_{j+1}.

Let k be the smallest integer such that $d'_{i-k} = \gamma d_{i-k}$ does not intersect d_2. Since d_1 intersects at most three d'_j, k is less than three. The edge d'_{i-k+1} intersects d_2. Since it cannot intersect either d'_i, and since its Southern end is not the Southern angle of T_2, it intersects d_3. Replacing d'_i by d'_{j+1-k}, we can thus assume that d'_{i-1} does not intersect d_2.

Let $T'_{i-1} = \gamma T_{i-1}$, and $U'_{i-1,i}$ be the domain between d'_{i-1} and d'_i. Observe that the Southern part of d_2 is contained in $U'_{i-1,i}$. It follows that the angle of T'_{i-1} is not at the South, but at the North. Since this angle is not the Northern end of d_1, it follows that the Northern part of d'_{i-1} is contained in $U_{1,2}$. Thus, d'_{i-1} intersects d_1. The triangle T'_{i-1} intersects two different γ_0-invariant lines of stable type (d_1 and d_3). Hence, T'_{i-1} is not convex. According to the previous lemmas, it follows that d'_{i-2} and d'_{i+1} are both of principal type.

Assumption 1 implies that $\hat{\mathcal{D}}(d'_{i-1})$ has no extremity on the unstable $\bar{\rho}(\gamma_0)$-invariant line. So, its Northern part must intersect this unstable line. As a corollary, we see that d'_{i-1} does not intersect γ_0-edges of unstable type. Therefore, d'_{i-1} does not intersect d_0. The Southern part of d'_{i-1} is contained in $U_{0,1}$. The angle of T'_{i-2} is at the South (since the angle of T'_{i-1} is at the North, and since d'_{i-2}, d'_{i-1} and d'_i are of different types). It follows that the Southern part of d'_{i-2} is contained in $U_{0,1}$. Moreover, T'_{i-2} is convex (since d'_{i-2} is principal): d'_{i-2} cannot intersect d_0, since we have seen before that d'_{i-1} intersects the preimage of the unstable $\bar{\rho}(\gamma_0)$-invariant line. The North of d'_{i-2} cannot be the Northern angle of T_0. The unique possibility is that d'_{i-2} intersects d_1. As above, we get that the unstable $\bar{\rho}(\gamma_0)$-invariant line, which still intersects $\hat{\mathcal{D}}(d'_i)$ and $\hat{\mathcal{D}}(d'_{i-1})$, intersects $\hat{\mathcal{D}}(d'_{i-2})$. It follows that d'_{i-3} does not meet $U_{1,2}$. Thus, the angle of T'_{i-3} is at the South. The Southern part of d'_{i-3} is at the South of $U_{0,1}$. d'_{i-3} does not intersect d_1, and its Northern end is not the angle of T_0. Thus, d'_{i-3} intersect d_0. But T'_{i-3} is convex since d'_{i-2} is principal. This is impossible since its two edges intersect different connected components of the preimage of the unstable leaf: d'_{i-3} intersects d_0, and $\hat{\mathcal{D}}(d'_{i-2})$ intersects the unstable boundary of $\hat{\mathcal{D}}(T_1)$. This contradiction completes the proof of the lemma. ∎

Thanks to the preceding lemma, we have a precise notion of the picture of the sequence $T_1, \dots, T_n$. In particular, it follows that the convex triangle containing the image $\hat{\mathcal{D}}(T_i)$ depends only on the rest of i modulo 3. If we fix the index i such that d_0 is principal and d_1 of stable type, then the generalized triangle T_i is convex unless the i is 1 modulo 3. Replacing each T_{3i+1} by some $T'_{3i+1} \subset T_{3i+1}$, we can assume that that the restriction of $\hat{\mathcal{D}}$ to Ω is a finite covering over the real projective plane minus a compact convex subset $C(\gamma_0)$. The fixed points of $\bar{\rho}(\gamma_0)$ belong to the boundary of $C(\gamma_0)$. Moreover, all the iterates $\bar{\rho}(\gamma)(x_k)$ belong to the $C(\gamma_0)$. Indeed, if not, Ω would contain a preimage x' by

$\hat{\mathcal{D}}$ of some $\bar{\rho}(\gamma)(x_k)$. Let l' be the connected component through x' of some $\bar{\rho}(\gamma\gamma_0\gamma^{-1})$-invariant line. Its intersection with Ω is connected. It follows that l' is vertical. Thus, l' intersects $\gamma\Omega$, and thus meets the closure of some $\gamma\gamma_0\gamma^{-1}$-invariant generalized triangle. Therefore, l' is $\gamma\gamma_0\gamma^{-1}$ invariant, and x' is a fixed point of $\gamma\gamma_0\gamma^{-1}$. This contradicts Assumption 2.

Consider the intersection C of all the $\bar{\rho}(\gamma)C(\gamma_0)$. It is a $\bar{\rho}(\Gamma)$-invariant convex subset, containing all the $\bar{\rho}(\gamma)(x_k)$. It follows that its interior is not empty.

We have exhibited a convex open set that is $\hat{\rho}(\Gamma)$-invariant: the interior $Int(C)$ of C. Since $\hat{\rho}(\Gamma)$ is discrete, preserves the Hilbert metric of $Int(C)$, and has no elliptic element (because of Proposition 4.5), its action over $Int(C)$ is free and properly discontinuous. The quotient of this action is a surface with fundamental group the surface group $\bar{\rho}(\hat{\Gamma})$. Hence it is compact. But the compact projective quotients of convex domains of RP^2 are well-known. In particular, no hyperbolic element can have all its fixed points in the closure of the convex domain. We get a contradiction since by construction $\bar{\rho}(\gamma_0)$ has all its fixed points in C.

This final contradiction gives the proof of Theorem C.

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ENS LYON, UMPA, UMR 5669, 46, ALLÉE D'ITALIE, 69364 LYON, FRANCE

REAL PROJECTIVE STRUCTURES AND RADIANT AFFINE STRUCTURES ON MANIFOLDS.

SUHYOUNG CHOI

ABSTRACT. In this note, we briefly survey recent results on real projective and radiant affine structures on manifolds.

1. INTRODUCTION

Given a space X with an action of a Lie group G, the invariant properties of X under the action is said to be the (X, G)-geometry according to Felix Klein's Erlangen program. We require X to be a homogenous manifold and the local action of an element g of G to determine the global action of g on X. We wish to put geometric structures modelled on (X, G)-geometry to a manifold M. An obvious way to do this to identify open sets of M to open sets in X by charts and require the transition functions to lie in G, i.e., we create an (X, G)-atlas for M. A maximal (X, G)-atlas is said to be an (X, G)-*structure* on M. Simplest examples are euclidean, hyperbolic, and spherical structures on manifolds. A *euclidean* structure is a geometric structure modelled on the euclidean space and the isometry group acting on it; a *hyperbolic* structure is one modelled on the hyperbolic space and the group of isometries on the hyperbolic space. A *spherical* structure is one modelled on the sphere $\mathbf{S}^n$ with the group $O(n+1)$ of isometries of the sphere. (When M has nonempty boundary we allow the charts to map to a half open sets in X.)

An important subclass is composed of homogeneous Riemannian structures, i.e., (X, G)-geometric structures where X has a Riemannian metric and G acts transitively on X as an isometry group of X. The above euclidean, hyperbolic, and spherical structures are examples, as well as (X, G) where X is a symmetric space and G the group of isometries generated by reflections. Homogeneous Riemannian (X, G)-structures on a manifold can be defined by requiring that each point has an open neighborhood isometric to an open subset of X.

We concentrate on affine and real projective structures which are not homogeneous Riemannian structures. Real projective geometry is defined by the pair

$$(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R})),$$

where $\mathbf{R}P^n$ is the usual projective space of rays in $\mathbf{R}^{n+1}$ and $\mathrm{PGL}(n+1, \mathbf{R})$ the projective general linear group acting on $\mathbf{R}P^n$ in the obvious manner. An affine geometry is given by the pair $(\mathbf{R}^n, \mathrm{Aff}(\mathbf{R}^n))$ where $\mathbf{R}^n$ is the usual affine space of dimension n and $\mathrm{Aff}(\mathbf{R}^n)$ the group of affine transformations of $\mathbf{R}^n$, i.e., transformations of form $x \mapsto Ax + b$ where A is a nonsingular linear map and b a vector. A *real projective structure* on a manifold is simply a geometric structure modelled on real projective geometry and an *affine structure* is one modelled on affine geometry.

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A manifold with an affine structure is said to be an *affine* manifold; one with a real projective structure a *real projective* manifold. Affine manifolds are manifolds with torsion free flat affine connections in differentio-geometric sense. Also, real projective manifolds can be described as manifolds with torsion free projectively flat affine connections. Given an affine or real projective structure on a manifold, we have a well-defined notion of geodesics given by the connection. It is easy to see that an arc on the manifold is geodesic if and only if the composition with charts are geodesics in the affine space or the real projective space respectively.

It turns out to be central in this field to understand the notions of developing maps and holonomy for a given (X, G)-structure on a manifold M. An (X, G)-structure on M determines charts for an open cover O of M, which induces charts for an open cover O' of $\tilde{M}$. For two overlapping open sets of O', the charts differ by a post-composition with an element of G. So by starting from one open set in O', we may patch the charts by post-composing with elements of G. By continuing this process over more and more open sets in O', we can consistently define an immersion from $\tilde{M}$ to X. This is called a developing map. Note that by an initial choice of the chart, the developing map is defined uniquely up to post-composition with an element of G.

Given a deck transformation ϑ, $\mathbf{dev}\circ\vartheta$ is another developing map. Hence it follows that there exists an element $h(\vartheta) \in G$ such that $h(\vartheta)\circ\mathbf{dev} = \mathbf{dev}\circ\vartheta$. Here $h : \pi_1(M) \to G$ is a homomorphism where $\pi_1(M)$ is considered the group of deck transformations. It follows that given another pair $(\mathbf{dev}', h')$ of M, we have $\mathbf{dev}' = g\circ\mathbf{dev}$ and $h'(\cdot) = gh(\cdot)g^{-1}$ for an element g in G.

Let M and N be (X, G)-manifolds. A map $f : M \to N$ is said to be an *(X, G)-map* if for each point p of M, there exists a neighborhood U with a chart ϕ and a neighborhood V of $f(p)$ with chart ψ so that $f(U) \subset V$ and $\psi\circ f\circ\phi^{-1}$ is a restriction of an element of G. We say two (X, G)-structures are (X, G)-equivalent if there exists an (X, G)-diffeomorphism, i.e., a diffeomorphism which is an (X, G)-map.

A simplest example of a real projective structure on a surface can be obtained as follows. Let $\mathbf{R}P^2$ be the real projective plane where we denote by $[v]$ the point on $\mathbf{R}P^2$ corresponding to the vector $v \in \mathbf{R}^3$. Let ϑ be an element of $\mathrm{PGL}(3, \mathbf{R})$ acting on $\mathbf{R}P^2$ which is represented by a diagonal matrix with positive diagonal elements $\lambda_1, \lambda_2, \lambda_3$ in the strictly decreasing order. Then points $[e_1], [e_2]$, and $[e_3]$ are fixed points. There are four ϑ-invariant triangles with vertices $[e_1], [e_2]$, and $[e_3]$. Choose one of them, and denote it by $\triangle$. Then $\triangle^o/ < \vartheta >$ has an obvious real projective structure as ϑ is real projective. Let us denote by l_{ij} the open line segment connecting $[e_i]$ and $[e_j]$ in this triangle. Then we easily see that $\triangle^o \cup l_{12} \cup l_{13}/ < \vartheta >$ is a compact annulus with a projective structure. $\triangle^o \cup l_{13} \cup l_{23}/ < \vartheta >$ is another such annulus. These two are called elementary annuli. However $\triangle^o \cup l_{12} \cup l_{23}$ is not a Hausdorff space. Such annuli and ones equivalent to them are called *elementary* annuli.

The hyperbolic space H^n is often identified with a positive part of hyperboloid given by $x_1^2 - x_2^2 - \cdots - x_n^2 = 1$ in the flat Lorentz space with the induced metric from the Lorentz metric. The group $\mathrm{PSO}(1, n)$ acts on the positive part as an isometry group. The projection map from $\mathbf{R}^{n+1} - \{O\}$ to $\mathbf{R}P^n$ maps the positive part to an open ball B in $\mathbf{R}P^n$. The group $\mathrm{PSO}(1, n)$ identifies with its natural copy in $\mathrm{PGL}(n + 1, \mathbf{R})$ and acts on B. The pair $(B, \mathrm{PSO}(1, n))$ is a model of hyperbolic geometry. As B is a subset of $\mathbf{R}P^n$ and $\mathrm{PSO}(1, n)$ a subgroup of $\mathrm{PGL}(n + 1, \mathbf{R})$, it follows that an atlas of charts to B with transition functions in $\mathrm{PSO}(1, n)$ is an atlas of charts to $\mathbf{R}P^n$ with transition functions projective. Hence, a hyperbolic structure naturally induces a unique canonical real projective structure. We call such a structure projectively equivalent to one of these a *hyperbolic real projective structure.* Conversely, we may require that the real projective structure to have a developing map a diffeomorphism onto B or any ball of form $g(B)$ for $g \in \mathrm{PGL}(n+1, \mathbf{R})$ and the structure ought to be hyperbolic.

The most easy examples of affine structures are from euclidean structures. This follows since euclidean transformations or motions are affine transformations. Hence, it is easy to see that tori admit affine structures. $\mathbf{R}^n$ paired with the group of similarities of $\mathbf{R}^n$ determines a similarity geometry. As the group of similarities is a subgroup of the affine group, it follows that any similarity structure uniquely determine an affine structure. The most famous examples are Hopf manifolds: Remove the origin from $\mathbf{R}^n$ and let ϑ be the similarity map $x \mapsto 2x$. Then $\mathbf{R}^n - \{O\}/ < \vartheta >$ is homeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^1$. A *complete* affine manifold is an affine manifold such that its developing map is a homeomorphism to $\mathbf{R}^n$. It is the same as requiring that the manifold is equivalent to quotient manifolds of the affine space by a properly discontinuous and free action of a group of affine transformations. Euclidean structures always induce complete affine structures, but the converse is not true [14].

Manifolds with affine structures have canonical real projective structures. The complement of a codimension-one subspace of $\mathbf{R}P^n$ has a natural identification to an affine space and is called an *affine patch* (see [5]) in such a way that an arc is an affine geodesic if and only if it is a projective geodesic. The identification is unique up to post-composition with an affine transformation $\mathbf{R}^n \to \mathbf{R}^n$ and is such that the group of projective transformations acting on the affine patch are precisely the group of affine transformation of $\mathbf{R}^n$. Hence, by considering $\mathbf{R}^n$ as an affine patch, we see that an affine structure induces a canonical real projective structure. A real projective structure equivalent to such is said to be an *affine real-projective* structure. Conversely, if a real projective manifold has a developing map into an affine patch and the associated holonomy are affine with respect to the affine patch, then its real projective structure is an affine one.

2. Previous results

In this section, we will go over some old results. Benzécri showed that affine surfaces must have Euler characteristic zero. (Later this was generalized to the famous Milnor-Benzécri inequality.) Kostant-Sullivan [16] showed that the complete compact affine manifolds have Euler characteristic equal to zero.

Nagano-Yagi [18] classified all affine structures on tori. Goldman generalized this and classified all real projective structures on annuli and tori. In particular, he showed that a real projective annulus with geodesic boundary or a real projective torus has a number of disjoint simple closed geodesics, the closures of the components of complement of which are elementary annuli if a linear map in $\mathrm{GL}(3, \mathbf{R})$ corresponding to the holonomy of a generator is diagonalizable with distinct positive diagonal entries. That is, such real projective surfaces decompose into elementary annuli along simple closed geodesics. Recently, Benoist [4] showed that closed nilmanifolds with real projective structure admit such a similar type of decompositions generalizing the above results with clean ingenious proofs.

A manifold with a real projective structure is said to be *convex* if the developing map is a diffeomorphism onto an affinely convex domain in an affine patch in $\mathbf{R}P^n$. There is a well-known construction of *grafting* defined by Goldman and Thurston. Let Σ be a convex orientable closed real projective surface. Take a collection of disjoint simple closed geodesics α_i no two of which are homotopic. (Any closed curve in Σ has a closed geodesic homotopic to it. See Choi [10].) We take the closure of the components of the complement of these geodesics. Then they form disjoint real projective surfaces Σ_j, $j = 1, \ldots, m$, with totally geodesic boundary. We denote by $\alpha_i^{\pm}$ for $i = 1, \ldots, n$ the corresponding boundary components corresponding to α_i. By choosing base points for each Σ_j, we choose a development pair $(\mathbf{dev}_j, h_j)$ for each Σ_j. Suppose that α_i^+ lies in Σ_j and α_i^- lie in Σ_k. A projective automorphism of $\mathbf{S}^2$ is *positive hyperbolic* if it is represented by a matrix with mutually distinct positive eigenvalues. We see that $h_j(\alpha_i^+)$ and $h_k(\alpha_i^-)$ have conjugate matrices and are positive hyperbolic as simple closed geodesics in closed surfaces are positive hyperbolic (see Goldman [15]).

Given a projective transformation ϑ with a diagonal matrix with positive strictly decreasing diagonal entries, we can form an annulus with totally geodesic boundary (see Sullivan-Thurston [20]). Loosely speaking, their construction asks you to choose a path in $\mathbf{R}P^2 - \{[e_1], [e_2], [e_3]\}$ starting and ending in one of the open line segments connecting $[e_i]$s satisfying certain rules, and gives you an annulus A with geodesic boundary. The holonomy of a generator of the fundamental group has a conjugate matrix to that of ϑ. We may assume that the boundary components of A are principal geodesics (see below) since our chosen arc can start and end on arbitrary line segments.

Let c be a two-sided closed geodesic in a real projective surface Σ with negative Euler characteristic so that the deck transformation ϑ corresponding to c has positive hyperbolic holonomy. As $h(c)$ has a matrix conjugate to a diagonal matrix with positive diagonal entries in strictly decreasing order, $h(c)$ has three fixed points s, m, and w associated with the diagonal elements respectively. We see easily that s is an attracting fixed point, and w is a repelling one.

Let $\tilde{c}$ be a lift of c so that a deck transformation ϑ corresponding to c acts on it. We say that c is *principal* if $\mathbf{dev} \circ \tilde{c}$ connects the fixed point s and w. This is equivalent to saying that $\mathbf{dev} \circ \tilde{c}$ is an imbedding to a line segment contained in a convex $h(\vartheta)$-invariant neighborhood. Thus, one can always find a convex annulus in a thickened Σ containing c as a neighborhood.

Lemma 2.1 (Goldman). *Let Σ_1 and Σ_2 be two real projective surfaces with geodesic boundary. Let α_1 and α_2 be distinct principal geodesic boundary components of Σ_1 and Σ_2 respectively. Suppose that the holonomy of α_1 are conjugate to that of α_2. Then there exists a real projective surface Σ with a simple closed geodesic α so that the closures of the components of $\Sigma - \alpha$ are equivalent to Σ_1 and Σ_2. Furthermore, we may assume that $\Sigma_1 = \Sigma_2$.*

Proof. This is in Goldman [15]. The idea is to thicken Σ_1 and Σ_2 and find convex annuli neighborhoods A_1 and A_2 of α_1 and α_2 respectively. Then we can find smaller annuli in A_1 and A_2 respectively which are equivalent since the holonomies of the core curves are conjugate. By identifying these annuli and throwing away suitable irrelevant subsets, we obtain a surface Σ. □

For each i, we have an annulus A_i with principal boundary components with holonomy of the core curve conjugate to that of $h(\alpha_i^{\pm})$. By Lemma 2.1 it follows that we may obtain a real projective surface Σ' with a collection of disjoint closed geodesics the closure of whose complements are equivalent to Σ_j, $j = 1, \ldots, m$, and A_i, $i = 1, \ldots, n$. The real projective annulus obtained in this manner is said to be constructed by grafting A_i along α_i.

Since A_i are not convex for any choice of A_i by Sullivan-Thurston construction, we see easily that the grafted real projective surfaces are never convex. Goldman [15] constructed examples of real projective surfaces whose developing map is onto $\mathbf{R}P^2$.

We can also do a similar construction called a grafting construction for a one-sided simple closed geodesic in a nonorientable closed convex surface by first cutting along the curve and making a boundary component and attaching a Möbius band with the principal boundary component to the surface. This construction is also called *grafting.*

Conversely, Thurston and Goldman asked around 1979 whether all real projective structures on closed surfaces with negative Euler characteristic are obtained in such a manner. The answer is yes. We say that a real projective surface Σ *decomposes* into real projective surfaces $\Sigma_1, \ldots, \Sigma_n$ along a collection of disjoint simple closed geodesics if they are the closures of components of the complement of the union of the geodesics.

Theorem 2.1. *Let Σ be a closed real projective surface with negative Euler characteristic. Then there exists a canonical collection of disjoint simple closed geodesics $\alpha_1, \ldots, \alpha_n$ so that Σ decomposes into maximal convex real projective surfaces with principal boundary and maximal annuli and maximal Möbius bands.*

Proof. It follows from Theorem 2 in [9]. □

A *maximal annulus* in Σ is a compact annulus with geodesic boundary which is not contained properly in any other such annulus or compact Möbius band with totally geodesic boundary. A *maximal Möbius band* is defined similarly. A *maximal principal boundary convex real projective surface* in Σ is a convex real projective subsurface in Σ with principal boundary components which is not contained properly in any other such subsurface.

The above results show that if Σ is an orientable closed surface, then Σ can be obtained from a convex real projective surface by grafting: We may remove all maximal annuli from Σ and obtain convex real projective surfaces with principal geodesic boundary. Then we glue these together in an obvious manner to obtain a closed real projective surface Σ'. Now it is easy to see that Σ can be obtained from Σ' by grafting the removed annuli. Furthermore Σ' is convex by Theorem 3.7 of Goldman [15] since we glued convex real projective surfaces along principal geodesic boundary components.

A similar consideration easily shows that all real projective structures on a closed surface of negative Euler characteristic are obtained by grafting from convex real projective structures.

Using this and the result of Goldman [15], Choi and Goldman showed that the deformation space of real projective structures on a closed surfaces is a countable disjoint union of cells. (see [13] for further details).

Similar results hold for real projective structures on nonorientable closed surfaces of negative Euler characteristic (see [9]) and compact surfaces with boundary.

3. Kuiper completion and convex subsets

It turns out that to study real projective structures or affine structures it is useful to complete the universal cover or holonomy cover as Kuiper [17] did in his study of conformally flat manifolds.

A holonomy cover is a cover of M corresponding to the kernel of a holonomy homomorphism $h : \pi_1(M) \to G$. It is obviously unique up to covering space isomorphisms. A developing map obviously descends into an immersion $M_h \to X$, which we still call a developing map. Let $\mathbf{dev} : M_h \to X$ be a developing map. Then any other developing map differs from it by a post-composition with an element of G. Thus, it follows that given any deck transformation ϑ, we have an element $h'(\vartheta)$ of G such that $h'(\vartheta) \circ \mathbf{dev} = \mathbf{dev} \circ \vartheta$.

Moreover, the map $\Gamma \to G$ from the group of deck transformations of M_h is a homomorphism $\pi_1(M)/\pi_1(M_h) \to G$ which is induced from $h : \pi_1(M) \to G$. Moreover if $(\mathbf{dev}'', h'')$ is another such pair, then $\mathbf{dev}'' = g \circ \mathbf{dev}$ and $h''(\cdot) = gh'(\cdot)g^{-1}$ for an element g of G.

Let X have a metric which is not necessarily invariant under the action of G but one where the group action is quasi-isometric. Given a developing map $\mathbf{dev} : \tilde{M} \to X$, we induce the metric of X on $\tilde{M}$. Now the deck transformation will not be isometries in general. We complete the metric in the Cauchy sense. This gives us a complete metric space $\check{M}$, which is called *Kuiper completion.* The set of *ideal points* $\tilde{M}_\infty$ is defined as $\check{M} - \tilde{M}$. As $\mathbf{dev}$ is distance nonincreasing, $\mathbf{dev}$ extends to a distance nonincreasing map on $\check{M}$, which we still call a developing map and denote by the same notation $\mathbf{dev}$. As deck transformations are quasi-isometries, they also extend to self-diffeomorphisms of $\check{M}$, which we call deck transformations also. But now, they may have fixed points. For any deck transformation $\vartheta : \check{M} \to \check{M}$, we have $h(\vartheta) \circ \mathbf{dev} = \mathbf{dev} \circ \vartheta$.

The same discussion holds for M_h as well. Thus, we obtain a Kuiper completion $\check{M}_h$, the set of *ideal points* $\tilde{M}_\infty$ and $\mathbf{dev} : \check{M}_h \to X$ the extension of $\mathbf{dev} : M_h \to X$, and deck transformations $\check{M}_h \to \check{M}_h$.

Going back to real projective and affine structures, we will now set up some conventions: Often, we will want to lift our map $\mathbf{dev}$ to the double cover $\mathbf{S}^n$ over $\mathbf{R}P^n$. As $\tilde{M}$ is simply connected this is always possible. Since $\mathbf{S}^n$ is a double cover, $\mathbf{S}^n$ has an induced real projective structure. We may consider $\mathbf{S}^n$ as a space of rays from the origin in $\mathbf{R}^{n+1}$. $\mathrm{GL}(n+1,\mathbf{R})$ acts on the rays, and hence on $\mathbf{S}^n$ as real projective automorphisms. The group of real projective automorphisms of $\mathbf{S}^n$ is easily seen to equal to the group $\mathrm{SL}_\pm(n+1,\mathbf{R})$ of linear maps with determinant ± 1.

We see easily that M admits an $(\mathbf{S}^n, \mathrm{SL}_\pm(n+1,\mathbf{R}))$-structure. Since $\mathbf{dev} : \tilde{M} \to \mathbf{S}^n$ can be considered a developing map for this structure, we have a holonomy homomorphism $h : \pi_1(M) \to \mathrm{SL}_\pm(n+1,\mathbf{R})$ satisfying $h(\vartheta) \circ \mathbf{dev} = \mathbf{dev} \circ \vartheta$ for each deck transformation ϑ. From now on, we will assume that we alway lift to $\mathbf{S}^n$ as here.

The sphere $\mathbf{S}^n$ has the standard Riemannian metric μ as the standard sphere in $\mathbf{R}^{n+1}$. A real projective automorphism act as a quasi-isometry here. We define the Kuiper completion $\check{M}$ using this metric. Also, the holonomy cover M_h and its Kuiper completion $\check{M}_h$ are defined using the metric.

We regard affine manifolds as real projective manifolds with the unique compatible real projective structure. Hence, we define Kuiper completions in this manner.

We want to define convexity for real projective manifolds. A *convex segment* in $\mathbf{S}^n$ is a segment which does not contain an antipodal points in the interior, i.e., its μ-length is $\leq \pi$. A subset of $\mathbf{S}^n$ is called *convex* if any two points can be connected by a convex segment. A *zero-dimensional great sphere* $\mathbf{S}^0$ is the pair of antipodal points, which is not convex. A great sphere $\mathbf{S}^i$ for $i = 1, \ldots, n$ is a convex subset of $\mathbf{S}^n$ under this definition.

An *i-hemisphere* is the closure of a component of $\mathbf{S}^i$ removed with a great $(i-1)$-sphere in $\mathbf{S}^i$. An *i-bihedron* is the closure of a component of $\mathbf{S}^i$ removed with two distinct great $(i-1)$-spheres. They are convex.

Given a convex subset, we can define its dimension as the least dimension of a great sphere $\mathbf{S}^i$ containing it. Obviously, the closure of a convex subset is closed and the dimension does not change. A *simply convex* subset is a convex subset whose closure is a compact convex subset of an open hemisphere.

We can classify closed i-dimensional convex subsets as follows: $\mathbf{S}^i$; an i-dimensional hemisphere; a proper convex subset of an i-hemisphere which is not simply convex; and simply convex sets.

A subset of a Kuiper completion K of $\tilde{M}$ or M_h is called a *convex segment* if **dev** restricted to it is an embedding onto a convex segment in $\mathbf{S}^n$.

A subset of K is called *convex* if any two points can be conneced by a convex segment. We can also classify compact convex subsets of K as **dev** restricted to them are imbeddings onto their images. A *tame* subset is a convex subset of a compact convex subset of K or a convex subset of M_h or $\tilde{M}_h$. **dev** restricted to a tame set is always an imbedding.

Theorem 3.1. *The following are equivalent:*

(a) *$\tilde{M}$ is convex.*
(b) *M_h is convex.*
(c) **dev**$|\tilde{M}$ *is an imbedding onto a convex subset of* $\mathbf{S}^n$.
(d) *M is projectively equivalent to a convex domain in* $\mathbf{S}^n$.

Remark 3.1. We say that M is *convex* if $\tilde{M}$ is convex. Notice that for closed manifolds this definition agrees with the definition given in Section 2 unless $\tilde{M}$ is equivalent to $\mathbf{S}^n$ as we can easily see from (c) and (d) of above and the fact that an open convex subset of $\mathbf{S}^n$ is either a subset of an open hemisphere which is an affine space, or $\mathbf{S}^n$. (This fact follows from the classification of closed convex subsets of $\mathbf{S}^n$.) From now on, we will use this more general second definition only.

4. Real projective n-manifolds

Now we wish to generalize the above results [12], [11], [9] to general dimensions. A real projective manifold M is *i-convex* for $1 \leq i \leq n-1$ if given an affine $(i+1)$-simplex T with a side F, every nondegenerate real projective map $f : T - F^o \to M$ extends to one from T. Note that 1-convexity is equivalent to convexity (see Theorem A.2 of [8]).

Lemma 4.1. *The followings are equivalent:*

(a) *M is $(n-1)$-convex.*
(b) *Given an n-simplex T embedded in $\check{M}_h$ with faces $F_1, \dots, F_{n+1}$, if $T \cap F_2 \cup \cdots \cup F_{n+1}$ is in M_h, then T is a subset of M_h.*
(c) *Given an n-simplex T embedded in $\check{M}$ with faces $F_1, \dots, F_{n+1}$, if $T \cap F_2 \cup \cdots \cup F_{n+1}$ is in M_h, then T is a subset of M_h*

A real projective manifold has *convex* boundary if each point has a neighborhood with a chart to $\mathbf{R}P^n$ whose image is a convex set in an affine patch. It has *concave* boundary if each point has a neighborhood with a chart to $\mathbf{R}P^n$ whose image is a simply convex open ball removed with a convex open set meeting the ball.

An *n-crescent* R is a tame n-ball in $\check{M}$ (resp. $\check{M}_h$) so that its images under the developing map is an n-hemisphere or n-bihedron such that an $(n-1)$-hemisphere in the manifold-boundary lies in the ideal set but the boundary itself is not in the ideal set entirely. If R is a bihedron, then the interior of its side in the ideal set is denoted by α_R and the other side itself by ν_R. If R is an n-hemisphere, then the union of all open $(n-1)$-hemisphere in $\delta R \cap \check{M}_\infty$ (resp. $\delta R \cap M_{h,\infty}$) is denoted by α_R. ν_R equals the compact convex $(n-1)$-ball which is the complement of α_R in δR.

Theorem 4.1 (Main[8]). *Let M be a compact real projective n-manifold with convex boundary. If M is not $(n-1)$-convex, then $\check{M}$ and $\check{M}_h$ include n-crescents respectively.*

From now on, we will discuss only about $\check{M}_h$, but readers can easily figure out that most of what follows hold for $\check{M}$ as well. The reason that crescents are useful is that the union of certain collections are equivariant and is topologically nice and hence covers a submanifold in M.

The essential properties of crescents come from how they meet. If R_1 and R_2 are hemispheric n-crescent, and their interiors meet, then $R_1 = R_2$. If R_1 and R_2 are bihedral n-crescents and their interiors meet, then R_1 and R_2 meet transversally or $R_1 = R_2$. The definition of transversal intersection is quite long (see [8]). However, we characterise it by the following:

(a) The interiors of R_1 and R_2 meet.
(b) $\mathbf{dev}|R_1 \cup R_2$ is a diffeomorphism onto $\mathbf{dev}(R_1) \cup \mathbf{dev}(R_2)$ which is a subset of an n-hemisphere H.
(c) $\mathbf{dev}(\alpha_{R_1})$ and $\mathbf{dev}(\alpha_{R_2})$ are subsets of δH.
(d) $\mathbf{dev}(\nu_{R_1})$ and $\mathbf{dev}(R_2)$ meet transversally at an $(n-1)$-hemisphere.

The set $\nu_{R_i} \cap R_j$ is an $(n-1)$-bihedron and $R_i \cap R_j$ is the closure of a component of $R_j - \nu_{R_i}$ for $i, j = 1, 2$.

We will now assume that all crescents in $\check{M}_h$ are bihedral. If some crescents are hemispheric, the discussion becomes simpler (see [8] for details). We say that two n-crescents R and S are equivalent if there exists a chain of n-crescents $R_0, R_1, \ldots, R_m$ such that $R_0 = R$ and $R_m = S$ and $R_i^o \cap R_{i+1}^o \neq \emptyset$.

We define for a given n-crescent R in $\check{M}_h$

$$\Lambda(R) = \bigcup_{S\sim R} S, \quad \Lambda_1(R) = \bigcup_{S\sim R} (S - \nu_S), \quad \delta_\infty\Lambda(R) = \bigcup_{S\sim R} \alpha_S. \tag{1}$$

First, $\Lambda(R)$ is a closed set. $\mathrm{bd}\Lambda(R) \cap M_h$ is a properly imbedded submanifold. $\Lambda_1(R)$ is an n-manifold with boundary $\delta_\infty\Lambda(R)$. $\mathbf{dev}|\Lambda(R)$ maps into an n-hemisphere H where $\mathbf{dev}(\delta_\infty\Lambda(R)) \subset \delta H$, and $\mathbf{dev}(\Lambda(R) - \delta_\infty\Lambda(R)) \subset H^o$. Given a deck transformation ϑ, we have $\Lambda(\vartheta(R)) = \vartheta(\Lambda(R)), \Lambda_1(\vartheta(R)) = \vartheta(\Lambda_1(R))$, and $\delta_\infty\Lambda(\vartheta(R)) = \vartheta(\delta_\infty\Lambda(R))$. We also have that given two n-crescents if $\Lambda(R) \cap \Lambda_1(S) \cap M_h$ is not empty, then $R \sim S$ and $\Lambda(R) = \Lambda(S)$. Hence, by this property we get an equivariance property; that is, if $\Lambda(R)$ and $\vartheta(\Lambda(R))$ meet in interior in M_h, then $\Lambda(R) = \vartheta(\Lambda(R))$.

Sometimes $\Lambda(R)$ and $\Lambda(S)$ may not meet in the interior. Then we have $\Lambda(R) \cap \Lambda(S) \cap M_h$ is a subset of $\mathrm{bd}\Lambda(R) \cap \mathrm{bd}\Lambda(S) \cap M_h$ and furthermore a totally geodesic submanifold in M_h. The union of all such submanifolds is called the *pre-two-faced submanifold*. We can show that they cover a compact closed $(n-1)$-submanifold in M^o under the covering map $M_h \to M$. This submanifold is called a *two-faced submanifold arising from bihedral crescents*. One notes that they are canonical as the pre-two-faced submanifolds are and so are all of the sets of form $\Lambda(R)$ for some n-crescents R. (We define two-faced submanifolds arising from hemispheric n-crescents in a similar manner.)

Suppose that there are no two-faced submanifolds. Then given R and S, we have either $\Lambda(R)$ and $\Lambda(S)$ are disjoint or they coincide. As we can show that the union of all the sets of form $\Lambda(R) \cap M_h$ is a properly imbedded submanifold of codimension zero in M_h, and as the union must be acted upon by the deck transformation group, it follows that the union covers a compact submanifold, which is a concave affine manifold.

A *concave affine manifold of type II* is a real projective manifold N such that the Kuiper completion $\check{N}_h$ of its holonomy cover N_h is a subset of $\Lambda(R)$ for some n-crescent R.

Remark 4.1. Here, we required that there are no hemispheric n-crescent in $\check{N}_h$. For a benefit of the readers, a *concave affine manifold of type I* is a real projective manifold N such that $\check{N}_h$ is a subset of a hemispheric n-crescent.

An affine patch $\mathbf{R}^n$ in $\mathbf{R}P^n$ correspond to an open hemisphere H in $\mathbf{S}^n$ under the covering map $\mathbf{S}^n \to \mathbf{R}P^n$. The group of affine transformations of $\mathbf{R}^n$ corresponds to the projective automorphisms preserving H. Since H has an affine structure, H is also called an *affine patch.*

By above properties, we see easily that a concave affine manifold of type II is affine as the holonomy must preserve the interior of the hemisphere, which is really an affine patch.

Let A be a properly imbedded $(n-1)$-manifold in M^o, which is not necessarily connected or totally geodesic. The so-called splitting S of M along A is obtained by completing $M - N$ by adding boundary which consists of either the union of two disjoint copies of components of A or a double cover of components of A (see [8] for more details).

A manifold N *decomposes into* manifolds $N_1, N_2, \dots$ if there exists a properly imbedded $(n-1)$-submanifold Σ so that N_i are components of the manifold obtained from splitting M along Σ; $N_1, N_2, \dots$ are said to be the *resulting* manifolds of the decomposition.

Corollary 4.1. *Suppose that M is compact but not $(n-1)$-convex. Then*

1. *after splitting M along the two-faced $(n-1)$-manifold A_1 arising from hemispheric n-crescents, the resulting manifold M^{s} decomposes properly into concave affine manifolds of type I and real projective n-manifolds with totally geodesic boundary which does not include any concave affine manifolds of type I.*
2. *We let N be the disjoint union of the resulting manifolds of the above decomposition other than concave affine ones. After cutting N along the two-faced $(n-1)$-manifold A_2 arising from bihedral n-crescents, the resulting manifold N^{s} decomposes into maximal concave affine manifolds of type II and real projective n-manifolds with convex boundary which is $(n-1)$-convex and includes no concave affine manifold of type II.*

Furthermore, A_1 and A_2 are canonically defined and the decompositions are also canonical in the following sense: If M^{s} equals $N \cup K$ for K the union of concave affine manifolds of type I in M^{s} and N the closure of the complement of K includes no concave affine manifolds of type I, then the above decomposition agrees with the decomposition into components of submanifolds in (1). *If N^{s} equals $S \cup T$ for T the union of maximal concave affine manifold of type II in N^{s} and S the closure of the complement of T that is $(n-1)$-convex and includes no concave affine manifold of type II, then the decomposition agree with the decomposition into components of submanifolds in* (2).

If $A_1 = \emptyset$, then we define $M^{\mathrm{s}} = M$ and if $A_2 = \emptyset$, then we define $N^{\mathrm{s}} = N$.

We note that M, M^{s}, N, N^{s} have totally geodesic or empty boundary. The final decomposed pieces of N^{s} are not so in general. Concave affine manifolds of type II have in general boundary concave seen from its inside and the $(n-1)$-convex real projective manifolds have convex boundary seen from inside.

5. Radiant affine 3-manifolds

We will apply the result of the previous section to study radiant affine 3-manifolds seen as real projective manifolds.

A *radiant* affine manifold is an affine manifold such that its holonomy group, i.e., $h(\pi_1(M))$, fixes a point in the affine space. We may regard this point as the origin O always by changing the coordinates or developing maps by an affine map. Now, $h(\pi_1(M))$ form a group of linear transformations. Let v be the vector field given by $\sum x_i \frac{\partial}{\partial x_i}$. Then v is invariant under the linear group action, and the lift v' of v on $\tilde{M}$ by a developing map $\mathbf{dev}$ is deck-transformation invariant. Hence, v' descends to a nowhere zero vector field on M. We call this vector field the *radiant vector field*. It is obvious that the radiant vector field is canonically defined on a radiant affine manifold. The vector field generates the *radiant flow*.

We recall the Benzécri construction of radiant affine $(n+1)$-manifolds from real projective n-manifolds. We now define affine suspensions over real projective surfaces and orbifolds (see Carrière [6] and Barbot [1] [2]). A *real projective orbifold* is simply an orbifold with geometric structure modelled on $(\mathbf{R}P^n, \mathrm{PGL}(n+1, \mathbf{R}))$. (For definition of geometric structures on orbifolds, see Ratcliff [19].)

Let Σ be a compact real projective n-manifold. Choosing an arbitrary Euclidean metric in $\mathbf{R}^{n+1}$, we define an immersion $\mathbf{dev}' : \Sigma_h \times \mathbf{R} \to \mathbf{R}^{n+1}$ by simply mapping (x,t) to $e^t u(x)$ where $u(x)$ is the unit vector at the origin in the direction of $\mathbf{dev}(x)$ in $\mathbf{R}^{n+1}$. Recalling that there is a natural quotient map $\mathrm{GL}(n+1, \mathbf{R}) \to \mathrm{Aut}(\mathbf{S}^n)$, we choose any lift $h' : \pi_1(\Sigma)/\pi_1(\Sigma_h) \to \mathrm{GL}(n+1, \mathbf{R})$ of h, and define a corresponding action of $\pi_1(\Sigma)/\pi_1(\Sigma_h)$ on $\Sigma_h \times \mathbf{R}$ by $\vartheta(x,t) = (\vartheta(x), t + \log ||h'(\vartheta)(u(x))||)$.

Letting $\Sigma_h \times \mathbf{R}$ have the affine structure induced from the immersion $\mathbf{dev}'$, we see that $\pi_1(\Sigma)/\pi_1(\Sigma_h)$ defines a properly discontinuous and free affine action of $\Sigma_h \times \mathbf{R}$ preserving each fiber homeomorphic to $\mathbf{R}$, and the quotient space is homeomorphic to $\Sigma \times \mathbf{R}$, i.e., a trivial $\mathbf{R}$-fiber bundle over Σ. We identify the quotient space with $\Sigma \times \mathbf{R}$, and choose any section $s : \Sigma \to \Sigma \times \mathbf{R}$ so that $s(\Sigma)$ becomes a compact imbedded surface.

Let ϕ be any orientation-preserving projective automorphism of Σ. Then ϕ lifts to a projective automorphism ϕ_h of Σ_h. Since ϕ_h is a projective automorphism, there exists an element ρ in $\mathrm{Aut}(\mathbf{S}^n)$ satisfying $\mathbf{dev} \circ \phi_h = \rho \circ \mathbf{dev}$. We may choose any element ρ' of $\mathrm{GL}(n+1, \mathbf{R})$ which induces ρ, and ρ' defines an affine automorphism ρ'' of $\Sigma_h \times \mathbf{R}$ given by

$$\rho''(x,t) = (\phi_h(x), t + \log ||\rho'(u(x))||).$$

We now require the lift h' to satisfy

$$\rho'^{-1} \circ h'(\vartheta) \circ \rho' = h'(\phi_h^{-1} \circ \vartheta \circ \phi_h)$$

for ϑ in the deck transformation group $\pi_1(M)/\pi_1(M_h)$. This is equivalent to requiring that $\det \circ h'$ be invariant under the action of ϕ_h on $\pi_1(\Sigma)/\pi_1(\Sigma_h)$ as the above equation is already true for $\mathrm{Aut}(\mathbf{S}^n)$. It is easy to see that there are many lifts h' satisfying this condition as we can consider this question on the abelianization $H_1(\Sigma; \mathbf{R})$, the homomorphism induced by $\det \circ h$, and the associated action of ϕ.

Since given a deck transformation ϑ of $\Sigma_h \times \mathbf{R}$, $\varphi = \phi_h^{-1} \circ \vartheta \circ \phi_h$ satisfies $\rho'' \circ \vartheta = \varphi \circ \rho''$, it follows that ρ'' induces an affine automorphism $a_{\rho'}$ of $\Sigma \times \mathbf{R}$.

As we let $e^r \mathrm{I}$ denote the *dilatation*; that is, it multiplies each vector in $\mathbf{R}^{n+1}$ by a factor e^r, it induces an affine automorphism D_r on $\Sigma_h \times \mathbf{R}$ also called a *dilatation* given by $D_r(x,t) = (x, t+r)$. Since D_r commutes with any deck transformation of $\Sigma_h \times \mathbf{R}$, it follows that D_r defines an affine fiber-preserving automorphism D'_r of $\Sigma_h \times \mathbf{R}$.

Any other choice ρ'_1 in $\mathrm{GL}(n+1, \mathbf{R})$ of ρ' equals $e^r \mathrm{I} \circ \rho'$ for some r, and given the affine automorphism $a_{\rho'_1}$ of $\Sigma \times \mathbf{R}$ corresponding to ρ'_1, we see that $a_{\rho'_1}$ equals $D'_r \circ \rho'$ for some r. Hence by choosing r sufficiently large > 1, and positive, we can make $a_{\rho'_1}(s(\Sigma))$ and $s(\Sigma)$ disjoint and $a_{\rho'_1}(s(\Sigma))$ to lie in the radially outer-component of $\Sigma \times \mathbf{R} - s(\Sigma)$. Since $a_{\rho'_1}$ is a fiber-preserving diffeomorphism, $a_{\rho'_1}(s(\Sigma))$ is another cross section. We let N denote the compact $(n+1)$-manifold in $\Sigma \times \mathbf{R}$ bounded by $s(\Sigma)$ and $a_{\rho'_1}(s(\Sigma))$, and identify $s(\Sigma)$ and $a_{\rho'_1}(s(\Sigma))$ by $a_{\rho'_1}$ to obtain a compact radiant affine $(n+1)$-manifold homeomorphic to the mapping torus $\Sigma \times_\phi \mathbf{S}^1$, i.e., $\Sigma \times I/\sim$ where $\sim$ is defined by $(x,0) \sim (\phi(x), 1)$. We call the resulting affine $(n+1)$-manifold the *affine suspension* over Σ *using* the projective automorphism ϕ.

If we let ϕ be the identity automorphism of Σ, then the affine suspension is a so-called Benzécri suspension. But even when ρ is of finite order, we will also call it *Benzécri suspension* over the projective orbifold $\Sigma/ < \phi >$.

Affine suspensions over a real projective surfaces of negative Euler characterists are Benzécri suspensions due to the following theorem.

Theorem 5.1 ([8]). *Let $\rho : \Sigma \to \Sigma$ be a projective automorphism of a real projective surface Σ of negative Euler characteristic. Then ρ is of finite order.*

Theorem 5.2 ([8]). *A compact radiant affine 3-manifold M with totally geodesic or empty boundary admits a total cross section Σ to the radiant flow if and only if it is affinely diffeomorphic to an affine suspension over a compact real projective surface Σ' with totally geodesic or empty boundary. Moreover, in the above case, Σ with the induced real projective structure from the affine space by the radiant flow is real projectively diffeomorphic to Σ'.*

Fried produced an example of a radiant affine 6-manifold which is not a suspension. Carrière [6] conjectured that radiant affine 3-manifolds must be affine suspensions.

Let M be a radiant affine 3-manifold. Regarding it as a real projective manifold, we apply the results of the previous section to M, and we obtain a sharper result.

Theorem 5.3. *Let M be a compact radiant affine 3-manifold with empty or totally geodesic boundary. Then M decomposes into 2-convex radiant affine 3-manifolds and radiant concave affine 3-manifolds along closed totally geodesic tori or Klein bottles tangent to the radiant vector field.*

To get the totally geodesic submanifolds, we note that obviously two-faced submanifolds are totally geodesic. Also, the boundary of concave affine manifolds of type I are also totally geodesic as the boundary of a hemisphere is totally geodesic. The boundary of concave affine manifolds of type II are not totally geodesic in general. However, it is clear that they are also invariant under the radial flow in the radiant affine manifold. Then using the radial invariance, we can prove the total geodesity of the boundary.

We can show [7, Section 9] that a radiant concave affine 3-manifold with empty or totally geodesic boundary is an affine suspensions of the real projective sphere, the real projective plane, a hemisphere, or a π-annulus (or π-Möbius band) of type II (see [11] for definition).

We will obtain even sharper theorem by looking at the resulting 2-convex radiant affine 3-manifolds; the manifold also decomposes into nice pieces. The main idea is to use radial invariance and repeat the same arguments as in [8] to obtain crescent-cones instead of 3-crescents.

Let M be a compact radiant affine 3-manifold with totally geodesic or empty boundary. We consider M as a real projective manifold. Let M_h be the holonomy cover of M and $\check{M}_h$ the Kuiper completion with a developing map $\mathbf{dev} : \check{M}_h \to \mathbf{S}^n$ and the associated holonomy homomorphism $h : \pi_1(M)/\pi_1(M_h) \to \mathrm{Aut}(\mathbf{S}^3)$. Then $\mathbf{dev}$ maps into the hemisphere in $\mathbf{S}^3$, which is the closure of an affine patch. The boundary is a great 2-sphere, called the *sphere at infinity*.

We assume that $\check{M}_h$ does not include any 3-crescents as M is assumed to have undergone the above decomposition process (Theorem 5.3).

A *trihedron* is a 3-ball in $\check{M}_h$ which maps to a 3-dimensional polyhedron which is the closure of a component of the complement of three great 2-spheres in general position.

It has three sides. A *crescent-cone* is a trihedron in $\check{M}_h$ with two sides in $M_{h,\infty}$ and its intersection with M_h is invariant under the radial flow.

We show that if M is a non-convex but 2-convex, then $\check{M}_h$ includes a crescent-cone. We sketch the argument without much regard to the rigor. First, we obtain a triangle detacting the nonconvexity of M_h. Then by radial extension, we obtain a radiant tetrahedron F in $\check{M}_h$. F has four faces F_1, F_2, F_3, and F_4 where F_4 maps to the sphere at infinity under **dev**. One of F_1, F_2, or F_3 meets the ideal set in its interior as the triangle detacts the nonconvexity. We assume that F_3 does and the interior of F_1 and F_2 are disjoint from the ideal set. The ideal points of F_3 form radial segments. There are more than two components in $F_3 \cap M_h$. We choose two segments in F_1 and F_2 respectively and divide $F_1 \cup F_2$ into two parts, the upper one and the lower one. We choose a sequence of points in $F_3 \cap M_h$ converging to an ideal point equidistant from the upper and lower parts in the metric induced from an arbitrary Riemannian one on M (i.e., not the one used for Kuiper completion). Using deck transformations, we pull back these points to a compact neighborhood of a fundamental domain.

The major step is to show that the sequence of the images of $F_1 \cup F_2$ converges an ideal set, i.e., that the sequences of images of $F_1 \cup F_2$ leave every compact subset of M_h or go infinitely far away. This is proved by contradiction: If the images stay bounded, then we can show by tabulating how the sequence of the images of the segments behaves that the holonomy of the deck transformation used to pull-back blows up in certain way so that we obtain 3-crescents by blowing up certain domains near F. However, we removed all 3-crescents in the beginning to obtain M.

Once we know that the sequence of the images of $F_1 \cup F_2$ leave every compact set, then the sequence of the images of F is shown to converge to a radiant compact 3-ball in $\check{M}_h$, which is shown to be a radiant tetrahedron or a crescent-cone. The former case is ruled out with a help of 3-manifold topology.

The crescent-cones will play the same role as n-crescents to give us equivariant objects which cover compact submanifolds of codimension zero with totally geodesic boundary, which are called *concave-cone affine manifolds.* We can classify concave-cone affine manifolds to be affine suspensions over affine tori, affine Klein bottles, affine annuli with geodesic boundary or affine Möbius bands with geodesic boundary.

Theorem 5.4 ([7]). *Let M be a compact radiant affine 3-manifold with empty or totally geodesic boundary. Then M decomposes along the union of finitely many disjoint totally geodesic tori or Klein bottles, tangent to the radial flow, into*

1. *convex radiant affine 3-manifolds,*
2. *affine suspensions of real projective spheres, real projective planes, real projective hemispheres, or π-annuli (or Möbius bands) of type C; or affine tori, affine Klein bottles, or affine annuli (or Möbius bands) with geodesic boundary.*

Theorem 5.5 (Barbot [1] [3]). *Suppose that M is a closed radiant affine 3-manifold. If M has a totally geodesic torus or Klein bottle tangent to the radiant flow, then M is an affine suspension.*

Theorem 5.6 (Barbot-Choi [7]). *Suppose that M is a compact radiant affine 3-manifold with nonempty totally-geodesic boundary. Assume that each boundary component is convex or has a cover affinely isomorphic to $\mathbf{R}^2 - \{O\}$. Then M is an affine suspension.*

Using above two theorems, it is easy to see that if M admits a decomposition, then M is an affine suspension. If M admits no decomposition, and M is convex, then again Barbot showed that M is an affine suspension. Hence, we showed that M is always an affine suspension over a surface. If the Euler characteristic of the surface is negative, then M is a Benzécri suspension.

We now have a topological characterization of radiant affine 3-manifolds. If M is an affine suspension over a surface Σ of negative Euler characteristic, then since the projective monodromy must be of finite order, we see easily that M is a Seifert space with the base orbifold $\Sigma / < t >$ where t is a finite order automorphism of Σ. Here the Euler number of the Seifert bundles is zero. If M is an affine suspension over a tori, or Klein bottle, then M has a structure of a bundle over a circle with fiber homeomorphic to a tori or Klein bottle.

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Department of Mathematics, College of Natural Sciences, Seoul National University, 151–742 Seoul, Korea

E-mail address: shchoi@math.snu.ac.kr

GROUP PRESENTATIONS AND 3-MANIFOLDS

SANG-JEUNG HA AND ANN-CHI KIM

§0. Introduction and Preliminaries

Fact 1. For $n \geq 4$, every finitely presented group

$$G =< x_1, x_2, \ldots x_m; r_1, r_2, \ldots, r_q >$$

is isomorphic to the fundamental group of a closed orientable n-manifold.

Question 1. What about $n = 3$? That is, given a finitely presented group G, is G the fundamental group of a closed orientable 3-manifold ? Moreover, to what extent does the fundamental group characterize the manifold ?

Fact 2. Every closed orientable 3-dimensional manifold has a spine that is a 2-dimensional cell complex with just one 3-cell. Such a cell complex determines a group presentation in a natural way, i.e. the 1-cells corresponding to the generators and 2-cells relators.

Theorem (Stallings 1962). *For every finitely presented group G, no algorithm exists to answer the question: Is G the fundamental group of a closed orientable 3-dimensional manifold ?*

In order to study 3-dimensional manifolds with some particular groups as its fundamental groups we like to have examples.

§1. Cyclically presented groups and 3-manifolds

Examples. *Minkus Manifolds* $M_n(k,h), (k = 2g+1, h = 1)$

$$\pi_1(M_n(k,1)) =< x_1, x_2, \ldots x_n; R_j, j = 1, 2, 3, \ldots n >,$$

where $R_j = \{x_j x_{j+1}^{-1} x_{j+2} x_{j+3}^{-1} \cdots x_{j+2g} = 1, \text{indices mod } n\}$, cyclically presented groups.

Theorem 1. *The $M_n(k,1)$ is the n-sheeted cyclic branched covering over the torus knot or link of type $(k,2)$.*

Theorem 2. *The manifold $M_n(k,1)/r$ is homeomorphic to the 3-sphere S^3.*

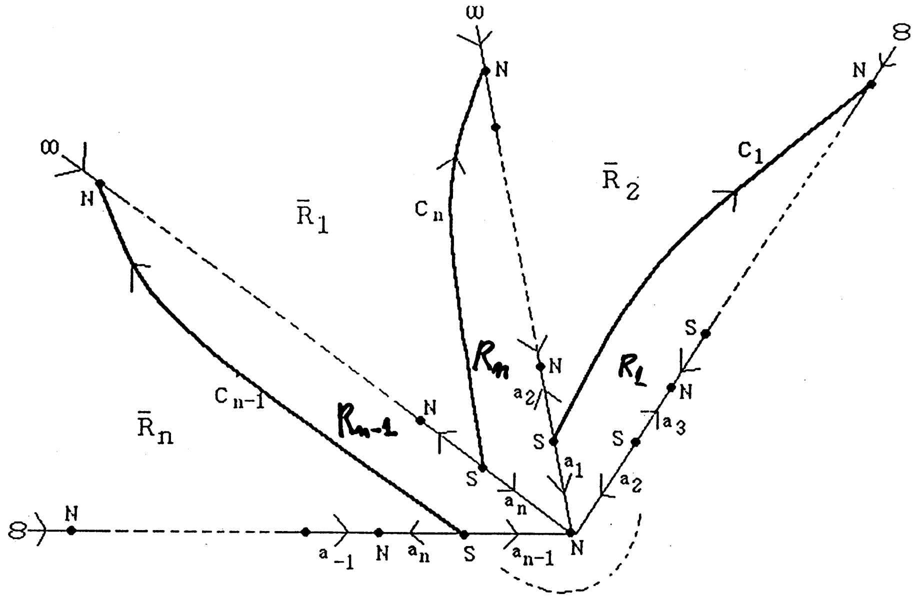

FIGURE 1. Minkus Complex $M_n(k,1)$

Definition. *Let F_n be the free group on n generators $x_0, x_1, x_2, \dots, x_{n-1}$ and θ denote the automorphism of F_n such that $\theta(x_i) = x_{i+1}$, indices mod n. For any reduced word $w \in F_n$ we shall define $G_n(w) = F_n/R$, where R is the normal closure in F_n of the set $\{w, \theta(w), \theta^2(w), \dots, \theta^{n-1}(w)\}$. Then a group G is said to have a cyclic presentation if G is isomorphic to $G_n(w)$ for some n and w.*

For example, if we take $w = x_0x_1x_2^{-1}$, then the groups $G_n(w)$ are, so called, *Fibonacci groups*

$$F(2,m) = \langle x_0, x_1, \dots, x_{m-1} : x_ix_{i+1}x_{i+2}^{-1} = 1, \text{ indices mod } m\rangle.$$

Note that $F(2,m)$ are the link between certain objects in 3-dimensional topology, in 3-dimensional hyperbolic geometry, in the theory of discontinuous transformation groups in rank one Lie groups, and in knot theory.

By combining the Minkus Manifold Construction Methods and Fibonacci Groups Presentations, we may consider a combinatorial polyhedron, that is, *the generalized icosahedron*:

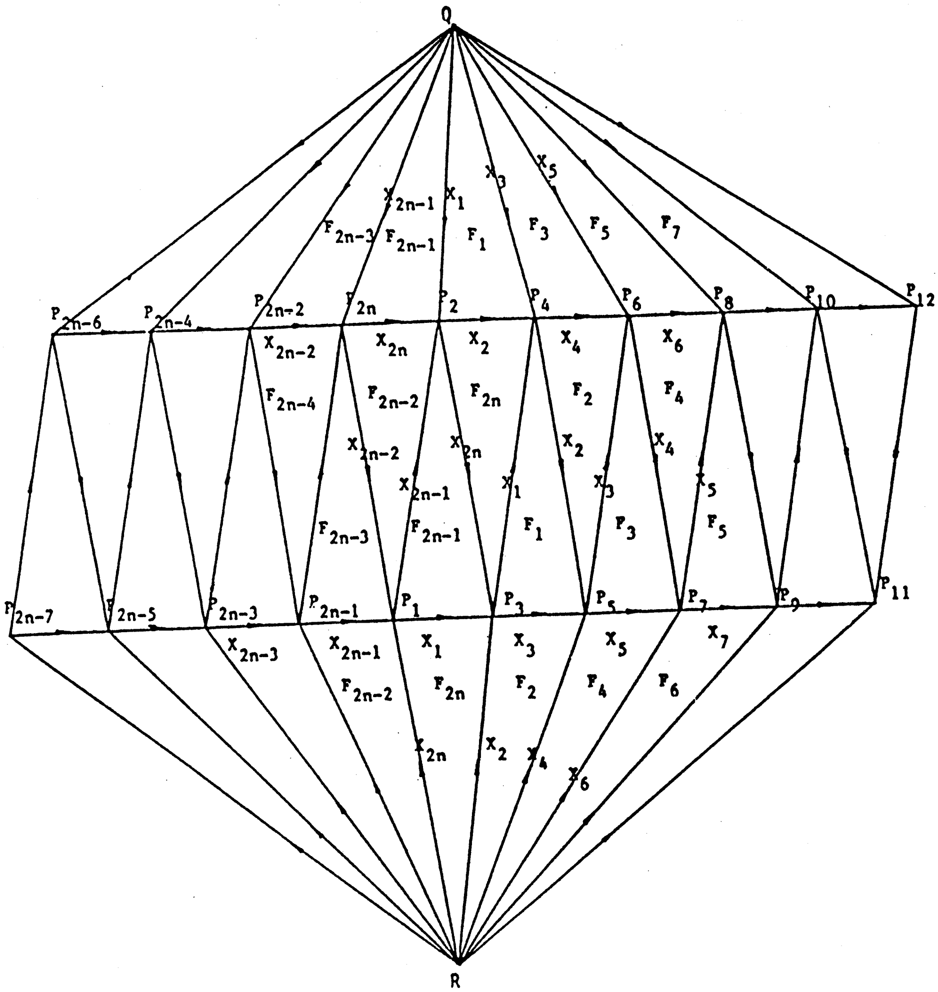

FIGURE 2. The combinatorial polyhedron

Helling-Kim-Mennicke. *In fact for* $m = 2n$, *there is a closed orientable* 3*-dimensional manifold* M_n *such that*

(1) *The fundamental group* $\pi_1(M_n)$ *is isomorphic to* $F(2, 2n)$.

(2) *The manifold* M_n *is hyperbolic for* $n \geq 4$. *That is to say,* $F(2, 2n)$ *acts as a discrete group of isometries on hyperbolic* 3*-space* H^3, *with quotient space* M_n.

(3) *The group* $F(2, 2n)$, *as a subgroup of* $SL(2, C)$ *is arithmetic for values* $n = 4$, *5, 6, 8, 12 or* ∞.

Hilden-Lozano-Montesinos; Howie. *For* $n \geq 4$, $F(2, 2n)$ *corresponds to a spine of the* n*-fold cyclic covering of the* 3*-sphere branched over the figure-8-knot.* (See: Topology '90, Walter de Gruyter)

Mednykh-Vesnin.

(1) *For any $n \geq 2$, the volumes of the compact hyperbolic Fibonacci manifolds M_n are equal to the volumes of non-compact manifolds $S^3 \backslash Th_n$.*
(2) *For any $n \geq 2$ the Fibonacci manifolds M_n are two-fold coverings of S_3 branched over the Turk's headlink Th_n.*

(See: Siberian Math. J. vol.36, no.2(1995); vol.37, no.3(1996))

Maclachlan. *It is impossible to realize a 3-dimensional manifold whose fundamental groups are isomorphic to $F(2, 2n+1)$.* (See London Math. Soc. Lecture Notes Series no.204)

(For more informations, see: B. Apanasov's Book, Conformal Geometry of Discrete Groups and Manifolds, Chapter III, §11 - Fibonacci Manifolds, to appear, de Gruyter.)

On the other hands, if we take $w = x_0 x_2 x_1^{-1} = 1$ (from the Fibonacci word $w = x_0 x_1 x_2^{-1}$), then we have, so called, the *Sieradski groups* $S(n)$, which is the fundamental groups of the Sieradski Manifolds.

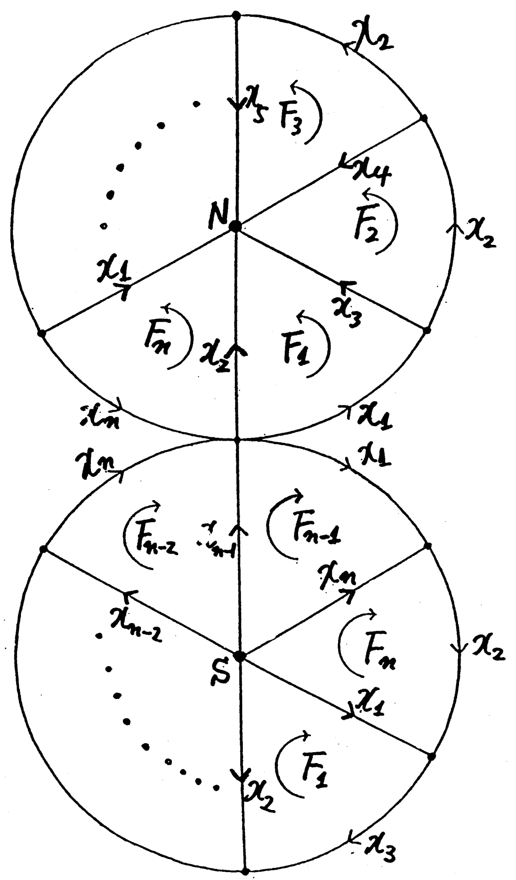

FIGURE 3. The Sieradski complex

Sieradski. *The squashable complex C_n constructed above yields a closed orientable 3-manifold M_n. Furthermore, M_n admits a spine K_n modelled on the cyclic presentation*

$$S(n) =< x_1, x_2, \dots, x_n; x_i x_{i+2} = x_{i+1}, \textit{indices mod } n > .$$

Cavicchioli-Hagenbarth-Kim; Howie. *The manifold M_n is homeomorphic to the n-fold cyclic covering of the 3-sphere branched over the trefoil knot $K(3,2)$, i.e., M_n is the Brieskorn manifold $M(2,3,n)$ in the sense of Milnor.*

Question 2. What cyclically presented groups correspond to the spine of closed orientable 3-manifolds ?

Question 3. Dunwoody (1994) describes an algorithm to enumerate the all spines with a cyclic symmetry, and proposes an open question: *what class of knots / links arise from the 3-manifolds ?* (see: Proc. Groups-Korea'94, Walter de Gruyter,1995))

§2. A construction of 3-manifolds based on the Fibonacci Manifolds

(1) On the cyclic coverings of the knot 5_2 (By Michele Mulazzani)

The manifold $M_n (n \geq 1)$ is defined by pairwise identification of the 2-faces of a polyhedron P_n, which is homeomorphic to a 3-ball, whose boundary complex provides a tessellation of the 2-sphere as depicted in Figure 4. The tessellation consists of $4n$ quadrilaterals, $8n$ edges and $4n + 2$ vertices. The n quadrilaterals around the north pole N are labelled by $Q_1, Q_2, \dots, Q_n$. The n quadrilaterals around the south pole S - which is the point at infinity in Figure 4 - are labelled by $R_1, R_2, \dots, R_n$, and the other quadrilaterals are labelled by $Q'_1, R'_1, Q'_2, R'_2, \dots, Q'_n, R'_n$, as indicated in Figure 4. To obtain M_n, we glue Q_i with Q'_i (resp. R_i with R'_i,) for each $i = 1, 2, \dots, n$, by an orientation reversing identification which matches N with A_i (resp. S with B_i).

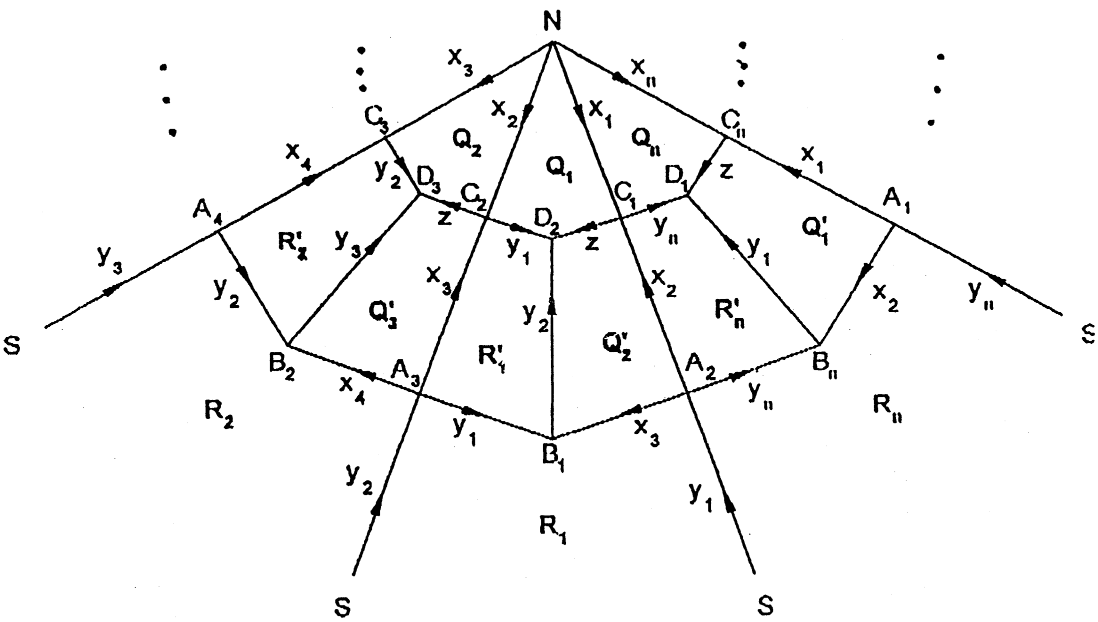

FIGURE 4

Via this gluing we get, for each $i = 1, 2, \ldots, n$, the following identifications on the edges: $NC_i \cong A_iC_{i-1} \cong A_{i-1}B_{i-2}$ (which we shall call x_i), $SA_{i+1} \cong B_iA_{i+2} \cong B_{i-1}D_i \cong C_{i+1}D_{i+1}$ (which we shall call y_i), and $C_1D_2 \cong C_2D_3 \cong \cdots C_nD_1$ (which we shall call z). As a consequence the vertices match as follows: $N \cong A_i \cong D_i$ and $S \cong B_i \cong C_i$, for each $i = 1, 2, \ldots, n$. Observe that, here and in the following, subscripts are considered mod n. Thus, we obtain a 3-dimensional cellular complex K_n, having one 3-cell, $2n$ quadrilaterals, $2n+1$ edges and two vertices. Since its Euler characteristic is $\chi(K_n) = 2 - (2n+1) + 2n - 1 = 0$, the space $M_n = |K_n|$ is a genuine closed, connected, orientable 3-manifold according to the Seifert-Threlfall criterion (see [ST], p. 216).

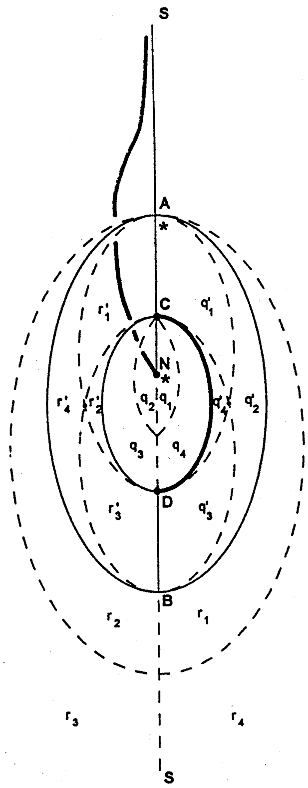

FIGURE 5

Let θ_n be the clockwise rotation of $2\pi/n$ radiants around the polar axis of the 3-ball P_n. It is easy to see that all the above defined identifications are invariant with respect to this rotation; therefore θ_n induces an orientation preserving homeomorphism g_n on M_n. The set $Fix(g_n)$ consists of the points of the polar diameter NS and the points of the edges z. Let G_n be the group of homeomorphisms of M_n generated by g_n. Of course, G_n has order n and $\text{Fix}(g_n)$ for each $k = 1, 2, \ldots, n-1$. The quotient space M_n/G_n is homeomorphic to M_1 and the canonical quotient map

$$p_n : M_n/G_n \longrightarrow M_1$$

is an n-fold branched cyclic covering, whose branching set is the 1-subcomplex of M_1 composed of NS and z (see Figure 5, where the branching set is shown by a thick line and each of the boundary quadrilateral Q, Q', R, R' is subdivided into four triangles).

Figures 5-10 depict, in detail, the identifications performed on the 2-sphere of Figure 5 to obtain M_1, showing the development of the branching set. More precisely, we have successively performed the identifications between the following regions: q_1 and q_4 with q'_1 and q'_4 (Fig. 7 $\longrightarrow$ Fig. 8), q_2 and q_3 with q'_2 and q'_3 (Fig. 6 $\longrightarrow$ Fig. 7), r_1 and r'_1 (Fig. 7 $\longrightarrow$ Fig. 8), r_2 and r'_2 (Fig. 8 $\longrightarrow$ Fig. 9). Notice that the complex is a three-ball at each of these stages. As a final step we identify r_3 and r_4 with r'_3 and r'_4 obtaining a three-sphere, where the branching set is a knot embedded as in Figure 10.

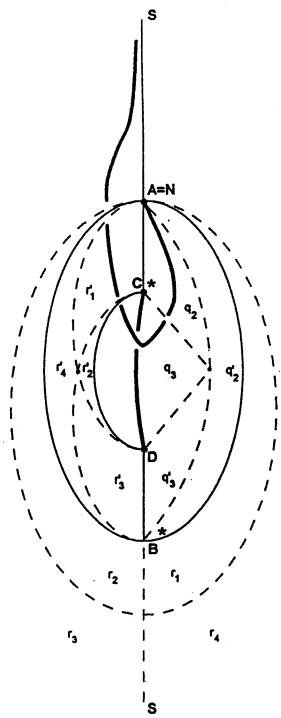

FIGURE 6

Hence, M_1 is homeomorphic to a 3-sphere and the branching set is the two-bridge knot $b(7,3)$, according to Schubert's notation (see [BZ], p.181), which is the knot 5_2 of the Alexander, Briggs, Reidemeister table ([BZ], p.312).

So we have proved the following:

Theorem 3. *The manifold M_n is the n-fold cyclic covering of S^3, branched over the two-bridge knot* $b(7,3)$.

As already known, the 2-fold branched coverings of the two-bridge knot or link $b(p,q)$ is the lens space $L(p,q)$ (see [Sc]). Therefore, we immediately have:

Corollary 4. *The manifold M_2 is the lens space* $L(7,3)$.

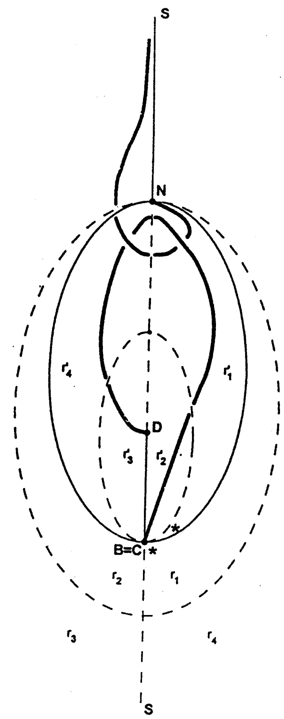

FIGURE 7

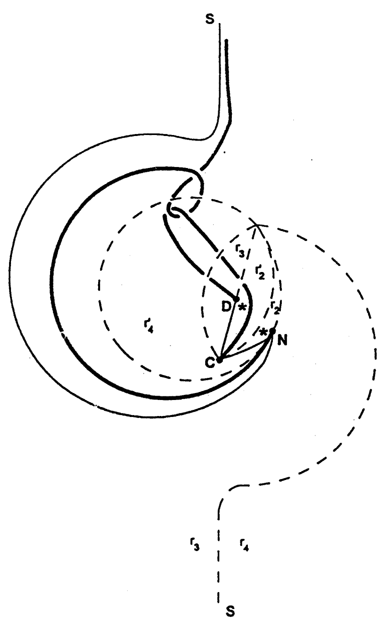

FIGURE 8

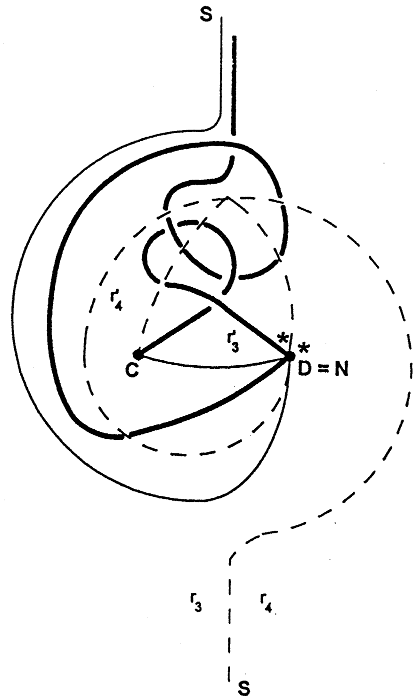

FIGURE 9

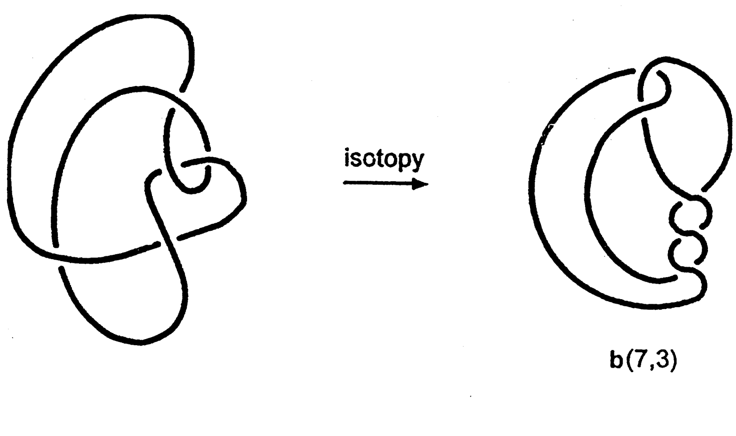

FIGURE 10

Remark. From a result of [Mu], each M_n turns out to be an element of a certain class of manifolds $S(b, l, t, c)$, depending on four integer parameters, introduced in [LM]. In particular, M_n is homeomorphic to the Lins-Mandel space $S(n, 7, 3, n-1) = S(n, 7, 4, 1)$.

Question 4. (Open Problem) Consider a combinatorial polyhedron consisting of n pentagons in the northern and southern hemisphere, and of $2n$ pentagons in the equator band. *Then how regularly can this combinatorial polyhedron be embedded in hyperbolic 3-space in order to obtain a tessellation of hyperbolic 3-space ? What is the group of isometries of such tessellation ?* (see: Fig. 11)

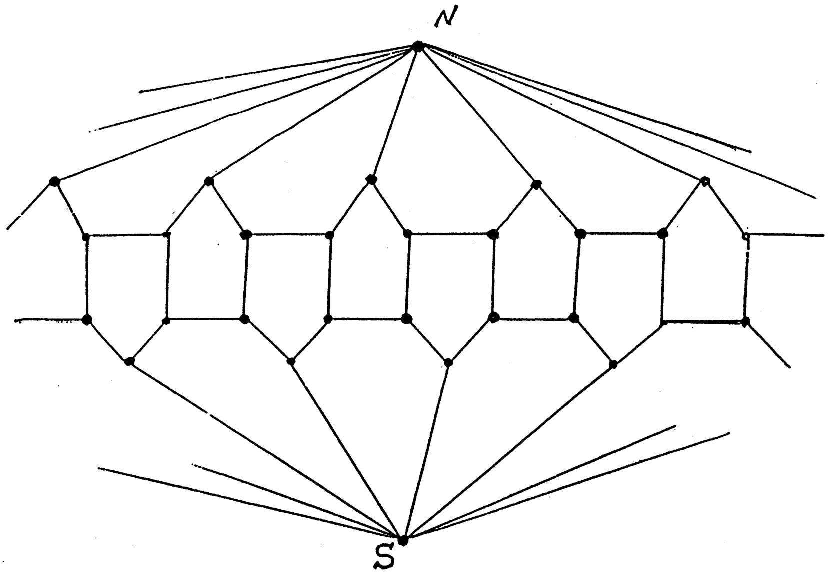

FIGURE 11

§3. A generalized Dodecahedron spaces

Consider a tessellation of the 2-sphere consisting of two m-gons and $2m$ pentagons satisfying certain properties shown in the following figure.

Under the the identifications R and S, all vertices become equivalent. We shall consider the oriented edges

$$w_j = (A_j, A_{j+1}), \qquad u_j = (B_{j-1}, C_j).$$

Then R and S imply the following identifications of edges:

$$\begin{aligned} (A_j, B_j) &= w_{j-k} \\ (C_j, B_j) &= w_j \\ (G_{j-1}, G_j) &= w_j \\ (C_j, G_j) &= w_{j+k} \\ u_0 = u_1 = u_2 = \ldots u_{m-1} &= u, \text{say.} \end{aligned}$$

Hence we obtain the Euler characteristic:

$$\alpha^0 = 1, \alpha^1 = m + 1, \alpha^2 = m + 1, \alpha^3 = 1.$$

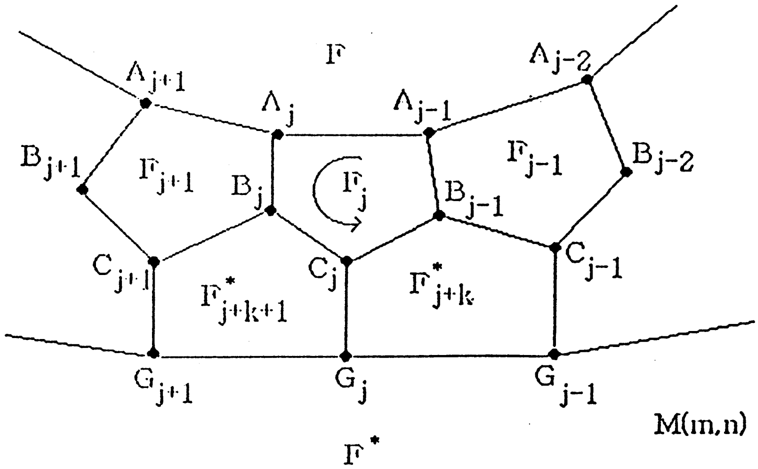

FIGURE 12

Properties $\mathbf{P_1}$. All edges except $B_jC_j(1 \leq j \leq m)$ have the same length, say x, and the edges B_jC_j have the same length, say y.

$\mathbf{P_2}$. In the pentagon $A_jB_jC_jB_{j-1}A_{j-1}$, the length of the segments $A_jB_{j-1}, A_{j-1}B_j$ and $A_{j-1}C_j$ are all same, denoted by z. Also A_jC_j and B_jB_{j-1} have the same length, say w.

$\mathbf{P_3}$. Denote the dihedral angle of the edge AB by $\angle AB$. Then

(1) $\angle B_jC_j = 2\pi/m \quad (1 \leq j \leq m)$.
(2)

$$\begin{aligned}\angle A_jA_{j+1} &= \angle A_{j+1}A_{j+2} = \ldots = \angle A_mA_1 \\ &= \angle G_jG_{j+1} = \angle G_{j+1}G_{j+2} = \cdots = \angle G_mG_1 = \epsilon.\end{aligned}$$

(3) the dihedral angle at edges not in (i) and (ii) are all the same, denoted by μ.
(4) $2\epsilon + 3\mu = 2\pi$.

Helling-Kim-Mennicke (1) For any $m \geq 3$ there are $\varphi(m)$ hyperbolic manifolds $M(m,k)$, where k runs through the relatively prime integers mod m.

(2) For a fixed $m \geq 3$ and $(m,k) = 1$, two manifolds $M(m,k)$ and $M(m,k')$ are homeomorphic if

$$k \cong k' \bmod m \text{ or } kk' \cong \pm 1 \bmod m.$$

For example, if $m = 5$, then we have $k = 1,2,3,4$. The $k = 2$ and $k = 4$ are the first and the second manifolds in the paper of Best (1971), and $k = 3$ was studied by Seifert and Weber (1939). Note J. Rubinstein's unpublished paper, *Hyperbolic manifolds from a regular polyhedron.* He suggested that there are eight such manifolds arising from the dodecahedron with dihedral angle $2\pi/5$.

The first homology groups:

$$H_1(M(m,k),Z) = \begin{cases} Z_{m/6} \times Z_{m/2} \times Z_{12m}, & m \cong 0 \text{ mod } 6 \\ Z_{m/2} \times Z_{m/2} \times Z_{4m}, & m = \pm 2\text{mod}6 \\ Z_{m/3} \times Z_m \times Z_{3m}, & m \cong 3 \text{ mod } 6 \\ Z_m \times Z_m \times Z_m, & (m,6) = 1. \end{cases}$$

Derevnine (1994). *If two manifolds $M(m,k)$ and $M(m,k')$ are homeomorphic, then $k \equiv k'$ mod m or $kk' \equiv \pm 1$ mod m.*

Note that for $(m,k) = 1$ the manifolds $M(m,k)$ are cyclic branched coverings of 3-sphere over the Whitehead link W.

Thus the fundamental group $\pi_1(S^3 \backslash W)$ is obtained by the Wirtinger's group presentation process. Then we have

$$(1)\ s_1 s_4 s_2^{-1} s_4^{-1} = 1 \qquad (2)\ s_2 s_5^{-1} s_3^{-1} s_5 = 1$$
$$(3)\ s_3 s_2^{-1} s_1^{-1} s_2 = 1 \qquad (4)\ s_4 s_1 s_5^{-1} s_1^{-1} = 1$$
$$(5)\ s_5 s_3^{-1} s_4^{-1} s_3 = 1.$$

Eliminating s_2, s_3, s_5, we get

$$\pi_1(S^3 \backslash W) = \langle s_1, s_4 | s_4^{-1} s_1^{-1} s_4 s_1^{-1} s_4^{-1} s_1 s_4 s_1^{-1} s_4 s_1 s_4^{-1} s_1 s_4 s_1^{-1} s_4^{-1} s_1 = 1 \rangle.$$

Some Combinatorial Group Theory attached to the Whitehead Link can be discussed. But we omit it here.

§4. More generalized Dodecahedron and Open Problems

From §3 we shall consider the case of $(m,k) = d (d \neq 1)$ i.e., m and k are not relatively prime. Then the infinite family of 3-manifolds has a number of interesting problems, including some open problems: For example, if $m = 4$ and $k = 2$, *then the manifold $M_{(4,2)}$ is the 2-fold covering of 3-sphere branched over the link 8^3_9.*

However, the the manifolds $M(m,k)$ has not yet completed. We now shall describe some open problems and conjectures as follows:

Conjecture 1. *The polyhedron $P_{(m,k)}$ can be realizable as the polyhedron in H^3 for almost all m and k.*

Conjecture 2. (*On the branched coverings*) *Any manifold $M_{(m,k)}$ is m-fold covering over the orbifold $W(m, m/d)), d = (m,k), d$ is the number of vertices.*

Conjecture 3. (*The geometric structures*) *The manifold $M_{(m,k)}$ is d-fold covering over the orbifold $M_{(m/d,m/d)}(d)$, where (d) is the circle from top to bottom of $M_{(m/d,m/d)}$ labelled by "d."*

The geometrical structures of the manifolds $M_{(m,k)}$ is the same as ones of the orbifolds $W(m, m/d)$. The orbifold $W(m,n)$ belongs to the following geometries:

$$\begin{array}{ll} W(2,2) & \text{spherical} \\ W(2,3) & \text{spherical} \\ W(2,4) & \text{nil} \\ W(2,n), n>2 & H^2 \times R \\ W(m,n), m>2, n>2, & \text{hyperbolic.} \end{array}$$

Problem 1. (*The volumes for* $M_{(m,k)}$) *Compute the hyperbolic volumes.*

Problem 2. (*Arithmeticy*) *According to Hilden-Lazona-Montesinos*(1995)*, the orbifold* $W(m,n)$ *is* arithmetic *for* $m = 3,4,5,6,12$ *or* ∞. *The manifold* $M_{(m,k)}$ *is arithmetic if and only if the orbifold* $W(m, m/d)$ *is arithmetic. Find all arithmetic orbifolds* $W(m,n)$ *for* $m > n > 2$.

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Department of Mathematics, Pusan National University, 609-735 Pusan, Korea
E-mail address: ackim@hyowon.cc.pusan.ac.kr

CHERN-SIMONS GAUGE THEORY ON 3-MANIFOLDS

CHRISTOPHER M. HERALD

ABSTRACT. Invariants of 3-dimensional manifolds derived from gauge theory will be discussed, including the Casson invariant and Floer homology. These lectures outline the results of research by the author ([27]-[29]) and joint research by Hans Boden and the author ([5]), as well as many others.

A good deal of gauge theory on 3-dimensional manifolds is motivated by an analogy with finite dimensional Morse theory. In the first lecture, we recall the derivation of the Euler characteristic and homology from Morse theory and then describe in general terms the analogous constructions in gauge theory. In the second lecture the definitions of the Casson invariant and Floer homology are outlined. Some approaches toward calculation of these invariants are explained in the third. The fourth lecture outlines the adaptation of the gauge theory constructions to 3-dimensional manifolds with boundary. Several gauge theoretic knot invariants are discussed. The final lecture discusses recent work by Hans Boden and the author which defines an $SU(3)$ Casson invariant. An equivariant generalization of the Morse theoretic definition of the Euler characteristic is described and provides an analogy for the $SU(3)$ invariant. Included also in this lecture are a few comments on the difficulties in generalizing Floer homology to larger structure groups than $SU(2)$.

1. Topological invariants from a Morse function

To set the stage for the definitions of the gauge theoretic 3-manifold invariants that follow, we will briefly recall a means of constructing the Euler characteristic and the homology groups of a smooth manifold from critical point set and gradient flow information for a Morse function. The emphasis will be not on the fact that the objects so constructed agree with other definitions of Euler characteristic and homology, but rather an independent proof (using only Morse theory) that the objects are independent of the choice of Morse function. The reason for this peculiar emphasis is that, in the gauge theory picture, there are no classical invariants with which to compare the gauge theory invariants.

1.1. **Morse theory and the Euler characteristic.** Let M be a smooth, closed, compact manifold. Choose a smooth function $f : M \to \mathbb{R}$ with the property that Hessian $\operatorname{Hess} f_p : T_pM \otimes T_pM \to \mathbb{R}$ is a nondegenerate (symmetric bilinear) pairing at each critical point p. This condition insures that the set of critical points is isolated. Such a function is called a *Morse function* on M.

The critical points of a Morse function may be classified by their *Morse index* as follows. Let $\mu(p)$ denote the number of negative eigenvalues of $\operatorname{Hess} f_p$, counted with multiplicities. See Figure 1.

The Poincaré-Hopf theorem relates the Euler characteristic of M to the signed count of critical points:

Theorem 1.1. *The Euler characteristic of M equals $\sum_{p\in \operatorname{Crit} f}(-1)^{\mu(p)}$.*

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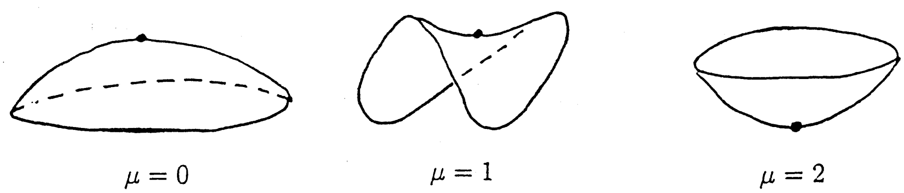

FIGURE 1. The Morse index of local maximum, saddle points, and local minimum critical points of a Morse function on a surface.

One can prove that the sum is independent of f as follows. Suppose that f_0, f_1 are Morse functions. They can be connected by a path of functions $f_t, t \in [0,1]$. One can show that for a generic choice of path between them, although it will not be the case that all intermediate functions are Morse functions, the parameterized critical set

$$W = \{(p,t) \in M \times [0,1] | p \in \operatorname{Crit} f_t\}$$

will be a 1-dimensional manifold, a cobordism between $\operatorname{Crit} f_0 \times \{0\}$ and $\operatorname{Crit} f_1 \times \{1\}$. Thus it only remains to check that one can define an orientation on the 1-manifold such that the boundary orientation at $(p,1)$, $p \in \operatorname{Crit} f_1$ coincides with the sign $(-1)^{\mu(p)}$ and the boundary orientation at $(p,0)$, $p \in \operatorname{Crit} f_0$ coincides with the sign $-(-1)^{\mu(p)}$.

Figure 2 illustrates a path of functions on the 2-sphere, interpreted as height functions. Sketched below the functions is the parameterized critical set.

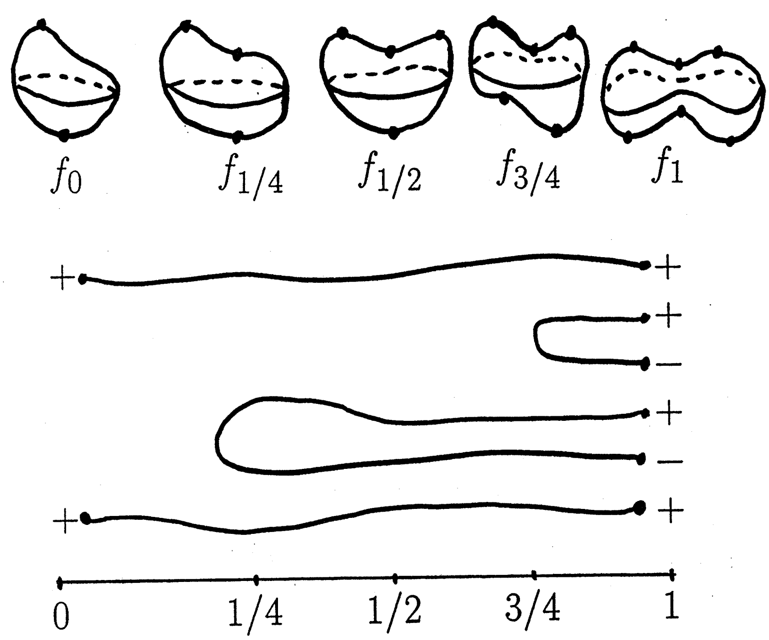

FIGURE 2. A 1-parameter family of height functions on the 2-sphere and, below it, the parameterized critical set for the family.

1.2. **Morse theory and homology.** By incorporating gradient flow information in addition to the critical points, one obtains a refinement of the above invariant, namely the homology of M. We begin by choosing a metric g on M, which allows us to obtain from f a gradient vector field ∇f on M. A *gradient flow* is defined to be a parameterized curve $\phi : (-\infty, \infty) \to M$ satisfying the equation

$$\tfrac{\partial \phi(t)}{\partial t} = -\nabla f(\phi(t))$$

for all t. This equation is symmetric under 'time translation' $\phi(t) \mapsto \phi(t+T)$, which gives an action of $\mathbb{R}$ on the space of gradient flows. For any two critical points $p, p' \in \operatorname{Crit} f$, we set

$$\mathcal{N}(p,p') = \{\phi \text{ gradient flow} |\lim_{t\to\infty} \phi(t) = p', \lim_{t\to-\infty} \phi(t) = p\}/\mathbb{R}.$$

Theorem 1.2. *For a Morse function f and a generic metric, the spaces $\mathcal{N}(p,p')$ are smooth manifolds with dimension given by*

$$\dim \mathcal{N}(p,p') = \mu(p) - \mu(p') - 1.$$

Furthermore, once certain orientation data are fixed for each critical point p, the spaces $\mathcal{N}(p,p')$ are canonically oriented.

Define a chain complex by letting $C(M,f)$ be the free $\mathbb{Z}$ module generated by $\operatorname{Crit} f$, with grading given by μ. Define an operator $\partial : C_i(M) \to C_{i-1}(M)$ by

$$\partial p = \sum_{\mu(p')=\mu(p)-1} \#\mathcal{N}(p,p')p'.$$

That this operator satisfies $\partial \circ \partial = 0$ follows from a compactness result about the 1-dimensional gradient flow spaces $\mathcal{N}(p,p')$, $\mu(p') = \mu(p) - 2$.

Theorem 1.3. *If $\mu(p') = \mu(p) - 2$, then $\mathcal{N}(p,p')$ is compact except for ends which may be identified with*

$$\bigcup_{\substack{q\in \operatorname{Crit} f\\ \mu(q)=\mu(p)-1}} \mathcal{N}(p,q) \times \mathbb{R}^+ \times \mathcal{N}(q,p').$$

These identifications are orientation preserving.

A careful analysis of orientations shows that the p' coefficient of $\partial\partial p$ is the number of 'broken trajectory' ends of the one-dimensional flow space from p to p', counted with boundary orientation, and hence is zero. The noncompactness is illustrated in Figure 3.

Theorem 1.4. *The homology of the complex defined above is $H_*(M)$.*

The proof that the homology is independent of the choice of f and g runs roughly as follows. Choose a path of functions f_t and metrics g_t, $t \in \mathbb{R}$ connecting two choices, i.e., with $(f_t, g_t) = (f_0, g_0)$ for $t < 0$ and $(f_t, g_t) = (f_1, g_1)$ for $t > 1$. Consider the space of time-dependent flows, satisfying

$$\tfrac{\partial \phi(t)}{\partial t} = -\nabla^{g_t} f_t(\phi(t)).$$

Solutions to the time-dependent equation are no longer translation invariant. We now define a homomorphism $F_* : C_*(M, f_0) \to C_*(M, f_1)$ by counting 0-dimensional components of the space of time-dependent flows connecting $p \in \operatorname{Crit} f_0$ to $p' \in \operatorname{Crit} f_1$ where the critical points are of the same index. One can show by a compactness argument for 1-dimensional components of the space of time-dependent flows that $\partial \circ F_* - F_* \circ \partial = 0$, and this homomorphism of complexes is shown to be an isomorphism on homology.

Remark 1.5. *It is possible to obtain from f a cell decomposition for M whose cells correspond to the critical points, but we will not pursue this here. See* [6], *for example, for more details.*

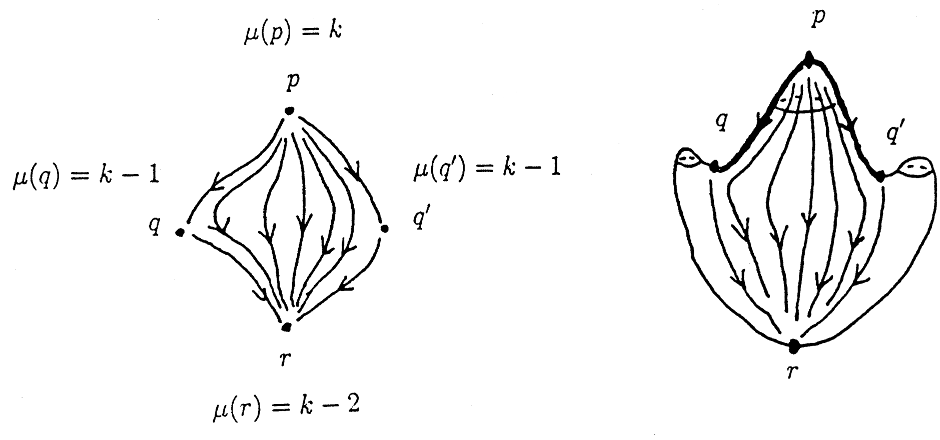

FIGURE 3. The component of the 1-dimensional gradient flow space $\mathcal{N}(p,p')$ on the front of the 2-sphere may be compactified by including broken trajectories through q and q'.

1.3. Overview of Morse theory in 3-dimensional gauge theory. Finally, we sketch how this approach to defining invariants is related to gauge theory on a 3-manifold. Let X be a closed 3-manifold with the same homology as S^3. $\mathcal{A}$ will denote the space of $SU(2)$ connections on X, and $\mathcal{G}$ the gauge group. Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ be the quotient. We look for a function $f : \mathcal{B} \to \mathbb{R}$ sharing enough properties with a Morse function that we can mimic the same constructions. The basic problem is that $\mathcal{B}$ is infinite dimensional and contains singularities. Despite this difficulty, we shall obtain invariants of X, known as the Casson invariant and Floer homology, which are analogous to the Euler characteristic and Morse homology. These will be the subjects of the next lecture. In the third lecture, we shall discuss further properties of these invariants, and the methods developed so far for calculating them.

The fourth lecture will be concerned with 3-manifolds with boundary. In this case, there is a much more tenuous analogy with Morse theory and an intriguing new symplectic ingredient thrusts itself into the theory. We will discuss gauge theoretic invariants of knots, outlining work by the author and subsequent work by several others.

The fifth lecture will discuss the recent generalization of the Casson invariant to $SU(3)$ structure group, joint work by the H. Boden and the author. In the $SU(3)$ situation, the singularities of $\mathcal{B}$ are not so easily avoided as in the $SU(2)$ case, and must be taken into account in the formula for the Casson invariant.

2. The Casson invariant and Floer homology

In this lecture we give a more detailed account of $SU(2)$ gauge theory and discuss the work of Taubes [49] and Floer [20]. These are the gauge theory analogues of the construction of the Euler characteristic and homology from a Morse function.

2.1. Connections, Holonomy, and Gauge Transformations. To begin with, let X be a smooth, compact 3-dimensional manifold, and consider an $SU(2)$ bundle P over X. For cohomological reasons, P is necessarily trivial, and it will simplify our presentation somewhat if we fix a trivialization $P = X \times SU(2)$. The space of connections on P is denoted by $\mathcal{A}$; using the fixed trivialization, $\mathcal{A}$ is identified with $\Omega^1(X; su(2)) = \Omega^1_X \otimes su(2)$, the set of 1-forms on X with values in $su(2)$.

A connection $A \in \mathcal{A}$ gives a means of lifting any curve $\gamma : [0,1] \to X$ to a *horizontal* or *covariantly constant* curve $\tilde{\gamma} : [0,1] \to P$, with the lift uniquely determined by the lift $\tilde{\gamma}(0)$ of $\gamma(0)$. Two different horizontal lifts differ by the action of an $SU(2)$ element. If $\gamma(0) = \gamma(1)$, then the $SU(2)$ element which takes $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ is called the *holonomy* of A around γ, denoted by $\mathrm{hol}_\gamma(A)$. Holonomy gives a map from based loops into $SU(2)$.

The group of automorphisms $\mathcal{G} \cong Map(X, SU(2))$ acts on $\mathcal{A}$ by $gA = g^{-1}dg + g^{-1}Ag$. This group is called the gauge group, and its elements are called gauge transformations. Denote the quotient $\mathcal{A}/\mathcal{G}$ by $\mathcal{B}$.

2.2. **Reducible connections.** The action of the gauge group is not free. The stabilizer of a connection is isomorphic to the stabilizer of its holonomy group in $SU(2)$. There are three isomorphism types. If $\mathrm{Stab}\, A \cong \{\pm \mathrm{Id}\} = Z(SU(2))$, we call the connection A *irreducible.* If $\mathrm{Stab}\, A \cong U(1)$, A is said to be *abelian*, and if $\mathrm{Stab}\, A \cong SU(2)$, A is *central.* Abelian and central connections are called *reducible.* Let θ denote the trivial connection coming from the fixed trivialization P. The trivial connection is central, of course.

We denote the space of irreducible connections by $\mathcal{A}^*$. Then $\mathcal{G}/\{\pm \mathrm{Id}\}$ acts freely on $\mathcal{A}^*$. Define $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ and $\mathcal{B}^r = \mathcal{B} - \mathcal{B}^*$. If $\mathcal{A}$ and $\mathcal{G}$ are completed with appropriate Sobolev norms, then the quotient space $\mathcal{B}^*$ becomes a smooth Banach manifold. At the reducible orbits $\mathcal{B}$ has singularities due to the group action.

2.3. **Curvature and flat connections.** To each connection A is associated a curvature 2-form $F_A \in \Omega^2(X; su(2))$, given by $F_A = dA + A \wedge A$. A connection A is called *flat* if $F(A) = 0$. We denote the set of orbits of flat connections by $\mathcal{M} = F^{-1}(0)/\mathcal{G}$, $\mathcal{M}^* = \mathcal{M} \cap \mathcal{B}^*$, and $\mathcal{M}^r = \mathcal{M} \cap \mathcal{B}^r$.

Flatness is equivalent to the condition that the holonomy map from based loops in X to $SU(2)$ descends to a map on $\pi_1 X$. One obtains a well-known identification

$$\mathcal{M} \cong \mathrm{Hom}(\pi_1 X, SU(2))/\text{conjugation},$$

which associates to the orbit of a flat connection the conjugation orbit of its holonomy representation. The object on the right is sometimes called the $SU(2)$ character variety.

To identify the flat moduli space in specific examples, the character variety interpretation is easier often to work with. It is also useful to note that $SU(2)$ can be identified with the group of unit quaternions. Explicit calculations can be found, for example, in [34], [35], [9], and [31].

2.4. **The deformation complex.** Fixing a Riemannian metric on X and adopting the standard inner product $\langle \alpha, \beta \rangle = -\mathrm{Tr}(\alpha\beta)$ on $su(2)$ provides an L^2 metric on $\Omega^p(X; su(2))$. If we fix in addition an orientation on X, we obtain the Hodge star operator

$$* : \Omega^p(X; su(2)) \to \Omega^{3-p}(X; su(2)),$$

which expresses Poincaré duality on the level of forms. The relationship between the metric and the star operator is

$$-\int_X \mathrm{Tr}(a \wedge *b) = \langle a, b \rangle.$$

A connection A on P gives an associated operator d_A on $su(2)$ valued forms,

$$d_A : \Omega^p(X; su(2)) \to \Omega^{p+1}(X; su(2)),$$

given by $d_A\eta = d\eta + [A \wedge \eta]$, where $[\cdot \wedge \cdot]$ denotes a combination of the wedge product on forms and the Lie bracket on $su(2)$. If A is a flat connection, then

$$\Omega^0(X; su(2)) \xrightarrow{d_A} \Omega^1(X; su(2)) \xrightarrow{d_A} \Omega^2(X; su(2)) \xrightarrow{d_A} \Omega^3(X; su(2))$$

defines an elliptic complex. Its cohomology, denoted by $H^*_A(X; su(2))$, is the cohomology of X with coefficients in the twisted flat bundle $ad_A P$. $H^0_A(X; su(2))$ is naturally identified with the Lie algebra of the stabilizer of A.

The Kuranishi model provides a local, finite dimensional model for the flat moduli space near $[A]$ as the zero set of a map $\Phi : H^1_A(X; su(2)) \to H^2_A(X; su(2))$, modulo the stabilizer of A. Notice that Poincaré duality implies that $H^1_A(X; su(2)) \cong H^2_A(X; su(2))$. In particular, if $H^1_A(X; su(2)) = 0$ for each flat connection A, then the flat moduli space is a smooth 0-dimensional manifold. See [27], for example, for details.

2.5. **The Chern-Simons functional.** To do anything Morse theoretical, we need a function on the space of connections. The primary function we will consider is the Chern-Simons functional, defined by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A).$$

This function arises naturally in the study of connections on 4-manifolds as follows. Chern-Weyl theory implies that for a connection $\mathbf{A}$ on a principal $SU(2)$ bundle Q over a closed 4-manifold M, the integral

$$\frac{1}{8\pi^2} \int_M \mathrm{Tr}(F_{\mathbf{A}} \wedge F_{\mathbf{A}})$$

equals $c_2(Q)[M]$. If M is a 4-manifold with $\partial M = X$, and the connection $\mathbf{A}$ restricts to A on X, then

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \text{ modulo } \mathbb{Z}.$$

The most important properties of the Chern-Simons functional for us are

(i) the L^2 gradient of CS is $\nabla \mathrm{CS} = -\frac{1}{4\pi^2} * F_A$, and
(ii) $\mathrm{CS}(gA) = \deg g + \mathrm{CS}(A)$.

The first property means that the set of critical points of CS is the set of flat connections. Since the orbit of a flat connection under the identity component of the gauge group is infinite dimensional, CS is certainly not a Morse function on $\mathcal{A}$. To take care of this, consider the induced function on $\mathcal{B}$. By (ii), $\mathrm{CS} : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$ is well-defined. This is the function to which we apply the ideas of Morse theory discussed in the first lecture.

As an aside, the set of values of the Chern-Simons functional on the flat connections on a 3-manifold is an intriguing invariant. Methods for computing the Chern-Simons invariants in many cases were developed by P. Kirk and E. Klassen [32] and D. Auckly [2]. In all known cases, the values of Chern-Simons is a rational number. Whether this holds in general is a very interesting question.

The next issue we face is making sense of the index of a critical point. The space $\mathcal{B}^*$ is infinite dimensional, and the spectrum of the Hessian of CS is unbounded in both directions, so we cannot simply count the number of negative eigenvalues. We can, however, compare the Hessians at two critical orbits by seeing how many eigenvalues change sign along a path connecting them. This gives a relative index. The problem is then how to pick an overall normalization.

The natural starting point with which to compare all other critical points is the trivial connection, which unfortunately is a singular point in $\mathcal{B}$. Because of this, it is useful to replace the hessian operator Hess CS : $T\mathcal{B}^* \to T\mathcal{B}^*$ by a self-adjoint Fredholm operator on $(0+1)$-forms which gives the same relative index between irreducible critical connections but which also makes sense at the reducibles. We summarize the ideas below; for details, see [49].

For each connection A, define an operator $K_A : \Omega^{0+1}(X; su(2)) \to \Omega^{0+1}(X; su(2))$ by

$$K_A(\sigma, \tau) = (d_A^* \tau, d_A \sigma + *d_A \tau).$$

Here d_A^* denotes the L^2 adjoint of d_A. When A is flat, the operator is simply the deformation complex folded up using adjoints. K defines a family of essentially self-adjoint Fredholm operators, parameterized by $A \in \mathcal{A}$, on the L^2 completion of the space of $(0+1)$-forms. This means that the spectrum of K_A near zero looks like that of a finite dimensional self-adjoint operator.

If K_t, $0 \leq t \leq 1$, is a path of self-adjoint Fredholm operators, then the spectral flow $\mathrm{SF}(K_0, K_1)$ counts how many eigenvalues (with multiplicities) cross zero from negative to positive minus the number that cross positive to negative. We need, however, a convention for how to count zero eigenvalues at the ends of the path. Choose a $\delta > 0$ smaller than the magnitude of every nonzero eigenvalue at either end, and count intersections (with signs and multiplicities) of the graphs of the eigenvalues with the line from $(0, -\delta)$ to $(1, \delta)$. See Figure 4.

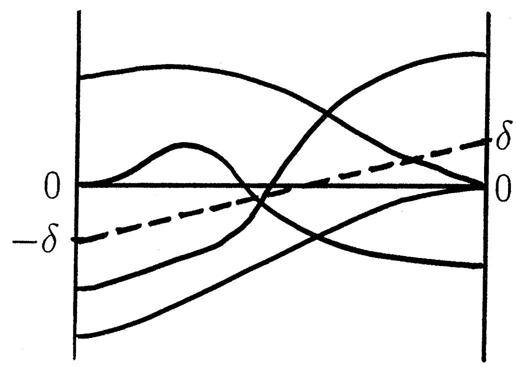

FIGURE 4. The spectral flow convention is to calculate intersections with the line connecting $(0, -\delta)$ with $(1, \delta)$.

The spectral flow along a path is a homotopy invariant of the path. The way it varies when one endpoint is changed by a gauge transformation is $\mathrm{SF}(\theta, gA) - \mathrm{SF}(\theta, A) = 8 \deg g$. For the purposes of defining an Euler characteristic, we only need the parity to be independent of the gauge representative, so this is sufficient. We define a $\mathbb{Z}_8$ grading on the set of flat orbits by

$$\mu([A]) = \mathrm{SF}(K_\theta, K_A) \pmod 8.$$

2.6. **Perturbations.** Finally we come to the issue of perturbations. We first describe the class of admissible perturbation functions following [49], [20]. Roughly speaking, one takes a sum of invariant functions (for example, any function composed with the trace $\mathrm{Tr} : SU(2) \to \mathbb{R}$) of the holonomy around a finite collection of closed loops in X. For analytical reasons, it is better to average these by integrating over the normal disk of a tubular neighborhood of each loop.

To begin with, we fix a collection of embeddings $\gamma_i : S^1 \times D^2 \to X$. (It is necessary for this collection to be sufficiently large. See [27], [5] for details.) Then fix a radially symmetric bump function η on the disk, and select functions $f_1, \dots, f_n$. Define a function $h : \mathcal{A} \to \mathbb{R}$ by

$$h(A) = \sum_{i=1}^{n} \int_{D^2} f_i \left(\mathrm{Tr}(\mathrm{hol}_{\gamma_i}(x, A))\right) \eta(x) d^2x.$$

A function of this form is called an *admissible function.* It is gauge invariant since a gauge tranformation changes holonomy by a conjugation. Once a sufficiently large collection of solid tori is fixed, the space $\mathcal{F}$ of admissible functions may then be identified with $C^3(\mathbb{R}, \mathbb{R})^{\times n}$, and we give it this topology.

Now fix an admissible perturbation function h. A connection A is called *perturbed flat* if it satisfies the equation

$$*F_A - 4\pi^2 \nabla h(A) = 0,$$

in other words A is a critical point of $\mathrm{CS} + h$. The *perturbed flat moduli space* is the set of gauge orbits of perturbed flat connections

$$\mathcal{M}_h = \{A | \; * F_A - 4\pi^2 \nabla h(A) = 0\} / \mathcal{G}.$$

The deformation complex for $\mathcal{M}_h$ and the cohomology groups thereof, denoted by $H^*_{A,h}(X; su(2))$, are defined just as before but with $d_A - *4\pi^2 \operatorname{Hess} h(A)$ in place of d_A for the middle operator. We can similarly extend our earlier definition of K_A to a family of self-adjoint operators parameterized by $\mathcal{A} \times \mathcal{F}$ by

$$K_{A,h}(\sigma, \tau) = (d_A^* \tau, d_A \sigma + *d_{A,h} \tau)$$

where $*d_{A,h} = *d_A - 4\pi^2 \operatorname{Hess} h(A)$. Define a $\mathbb{Z}_8$ grading on $\mathcal{M}_h$ by

$$\mu([A]) = \mathrm{SF}(K_\theta, K_{A,h}).$$

2.7. **Casson's invariant from gauge theory.** We are now in a position to give Taubes' gauge theoretic definition of the Casson invariant.

Theorem 2.1. ([49]) *Let X be an oriented integral homology sphere. For any $h \in \mathcal{F}$, $\mathcal{M}_h$ is compact. There is a neighborhood U of $0 \in \mathcal{F}$ and a Baire subset $U' \subset U$ such that, for $h \in U'$, $\mathcal{M}^*_h$ is a compact 0-dimensional manifold. The quantity*

$$\sum_{[A] \in \mathcal{M}^*_h} (-1)^{\mu([A])}$$

is independent of the perturbation $h \in U'$ and the metric on X, and equals -2 times the Casson invariant $\lambda(X)$.

Comments. Compactness is proven by standard gauge theoretic techniques, using the Uhlenbeck compactness theorem. Note, however, that compactness of the unperturbed flat moduli space is easier. The space of representations of a finitely presented group into a compact Lie group is compact. Thus $\mathrm{Hom}(\pi_1 X, SU(2))/\text{conjugation}$ is compact.

The restriction that $H^1(X; \mathbb{Z}) = 0$ precludes the existence of nontrivial reducible perturbed flat connections for small perturbations. This allows one to show that for a generic path $h(t), t \in [0,1]$ near $h = 0$, the irreducible parameterized moduli space

$$W^* = \bigcup_{t \in [0,1]} \mathcal{M}^*_{h(t)} \times \{t\}$$

gives a compact cobordism between $\mathcal{M}^*_{h(0)}$ and $\mathcal{M}^*_{h(1)}$. This allows one to conclude, after considering orientations, that the number of signed points in $\mathcal{M}^*_h$ is independent of h.

If the assumption that $H_1(X; \mathbb{Z}) = 0$ is relaxed, so that there exist flat abelian connections, W^* is no longer compact. Noncompact ends limit to the points of the abelian portion W^r of the parameterized moduli space, as illustrated in Figure 5. We will discuss this phenomenon in greater detail in the final lecture.

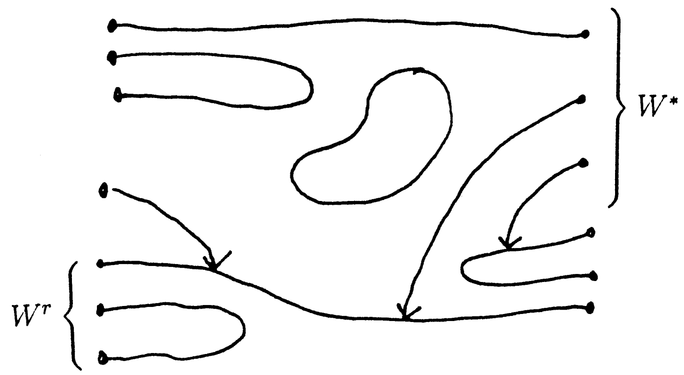

FIGURE 5. The irreducible stratum may have noncompact ends limiting to points in the abelian stratum.

2.8. **Floer homology.** Following Floer [20], we now use the critical orbits of CS as generators of a chain complex and define a boundary operator by counting gradient flows between critical orbits in $\mathcal{B}^*$ with sign.

We begin by discussing the relationship between gradient flows in $\mathcal{B}^*$ and anti-self-dual connections on $X \times \mathbb{R}$. For any oriented, 4-dimensional Riemannian manifold M, a connection $\mathbf{A}$ on a bundle over M is called *anti-self-dual* (ASD) if it satisfies the equation

$$*_M F_{\mathbf{A}} = -F_{\mathbf{A}}.$$

ASD connections are also called instantons.

Given a path of connections $A_t, t \in \mathbb{R}$, the corresponding 4-manifold connection on $X \times \mathbb{R}$ with no dt component is anti-self-dual with respect to the product metric if and only if A_t solves the gradient flow equation

$$\frac{\partial}{\partial t} A_t = *F_{A_t} = -4\pi^2 \nabla \operatorname{CS}(A_t).$$

Conversely, any anti-self-dual connection on $X \times \mathbb{R}$ which limits to flat connections as $t \to \pm\infty$ can be gauge transformed into this form, by a gauge transformation which is unique up to gauge transformations constant in t.

For $[A_-], [A_+] \in \mathcal{M}^*$, define

$$\tilde{\mathcal{N}}([A_-],[A_+]) = \\ \{\mathbf{A} \in \mathcal{A}(X \times \mathbb{R}) \mid \mathbf{A} \text{ is ASD}, \lim_{t \to \pm\infty} [\mathbf{A}|_t] = [A_\pm]\} / \mathcal{G}(X \times \mathbb{R})$$

and

$$\mathcal{N}([A_-],[A_+]) = \tilde{\mathcal{N}}([A_-],[A_+])/\mathbb{R},$$

where $\mathbb{R}$ acts by translation $r : \mathbf{A}(x,t) \mapsto \mathbf{A}(x, t+r)$. (We are suppressing necessary technical details, such as weighted Sobolev norms. Also, if necessary, the ASD equation is perturbed to be the gradient flow equation for the perturbed Chern-Simons function.)

Proposition 2.2. *For generic perturbations, $\mathcal{N}([A_-],[A_+])$ is a (possibly empty) union of smooth, canonically oriented manifolds, each of which has dimension equal to $\mu([A_+]) - \mu([A_-])$ (mod 8). In each of the possible dimensions, there are finitely many components. When the dimension is less than 8, the components are compact.*

(As in the finite dimensional case, the orientations depend on a choice of orientation data at each critical point, this time in the form of a fixed 4-manifold M with $\partial M = X$ and, for each $[A] \in \mathcal{M}^*(X)$, an orientation of the determinant line bundle arising from the deformation complex for instantons on M asymptotic to $[A]$.)

When $\mu([A_+]) - \mu([A_-]) \equiv 1 \pmod 8$, let $n([A_+],[A_-])$ denote the number of points in $\mathcal{N}([A_-],[A_+])$, counted with orientation.

Now we can define the Floer homology (instanton homology) of X. Let $FC_*(X)$ denote the $\mathbb{Z}_8$ graded Z-module generated by the points of $\mathcal{M}_h^*$, $h \in U'$. Then define $\partial : FC_p(X) \to FC_{p-1}(X)$ by the following formula. If $\mu([A]) \equiv p \pmod 8$, then

$$\partial[A] = \sum_{\mu([A'])\equiv p-1 \pmod 8} n([A],[A'])[A'] \in FC_{p-1}(X).$$

Theorem 2.3. *([20]) $(FC_*(X),\partial)$ defines a complex (that is, $\partial\circ\partial = 0$) and its $\mathbb{Z}_8$ graded homology group $FH_*(X)$ is an invariant of the oriented homology sphere X.*

The proof that $\partial \circ \partial = 0$, as in finite dimensions, is based on the following compactness result about the space of gradient trajectories between critical orbits of index difference two.

Proposition 2.4. *Suppose that $\mu([A'']) - \mu([A]) \cong 2 \pmod 8$. Then $\mathcal{N}([A],[A''])$ is compact except for ends diffeomorphic to $\mathcal{N}([A],[A']) \times [0,\infty) \times \mathcal{N}([A'],[A''])$ for orbits $[A']$ of index $\mu([A]) + 1$.*

The proof that $FH_*(X)$ is independent of the perturbation of the Chern-Simons function and independent of the metric on X involves counting time-dependent gradient flows for a 1-parameter family of perturbations and showing that the homomorphism of complexes thus obtained induces an isomorphism on Floer homology.

Note that the Euler characteristic of the Floer homology of X is, by Theorem 2.1, minus 2 times the Casson invariant. (See [33] for a clarification of the sign.)

Floer homology is related to Donaldson invariants of 4-manifolds in the following way. There are relative Donaldson invariants for manifolds M with $\partial M = X$ which take values in $FH_*(X)$. If $M = M_1 \cup M_2$ with $\partial M_1 = X$ and $\partial M_2 = -X$, as indicated in Figure 6, then the Donaldson invariants of M can be obtained from the relative invariants of the pieces by a pairing $FH_*(X) \times FH_*(-X) \to \mathbb{Z}$.

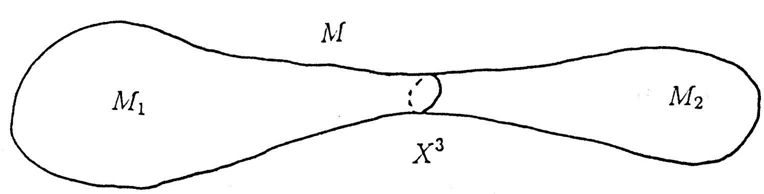

FIGURE 6. A 4-manifold decomposed into two pieces along a 3-dimensional submanifold.

3. Casson invariant and Floer homology–properties and calculation

In this section we discuss further properties of the Casson invariant and Floer homology, and the tools available for computation of these invariants. Throughout the section, X is an oriented homology 3-sphere.

3.1. Casson's definition of $\lambda(X)$. Casson's definition of the invariant $\lambda(X)$ (see [1]) involved topology rather than gauge theory. We sketch the topological definition in this subsection.

Choose a Heegard decomposition for X, i.e., a decomposition $X = X_1 \cup_\Sigma X_2$ where X_i are solid handle bodies with $\partial X_1 = \Sigma$, $\partial X_2 = -\Sigma$. Let g be the genus of the surface Σ. The Seifert Van Kampen theorem allows the computation of $\pi_1(X)$ from the identification maps by the following diagram.

$$\begin{array}{ccccc} & & \pi_1(X_1) & & \\ & \swarrow & & \nwarrow & \\ \pi_1(X) & & & & \pi_1(\Sigma) \\ & \nwarrow & & \swarrow & \\ & & \pi_1(X_2) & & \end{array}$$

For $M = X, X_1, X_2, \Sigma$, let $\mathcal{R}(M) = \mathrm{Hom}(\pi_1(M), SU(2))$. Applying the functor $\mathrm{Hom}(\cdot, SU(2))$ to the previous diagram reverses all the arrows.

$$\begin{array}{ccccc} & & \mathcal{R}(X_1) & & \\ & \nearrow & & \searrow & \\ \mathcal{R}(X) & & & & \mathcal{R}(\Sigma) \\ & \searrow & & \nearrow & \\ & & \mathcal{R}(X_2) & & \end{array}$$

Dividing by the conjugation action of $SU(2)$ gives an "intersection picture" for $\mathcal{M}(X)$.

$$\begin{array}{ccccc} & & \mathcal{M}(X_1) & & \\ & \nearrow & & \searrow & \\ \mathcal{M}(X) & & & & \mathcal{M}(\Sigma) \\ & \searrow & & \nearrow & \\ & & \mathcal{M}(X_2) & & \end{array}, \quad \mathcal{M}(X) = \mathcal{M}(X_1) \times_{\mathcal{M}(\Sigma)} \mathcal{M}(X_2)$$

$\mathcal{M}(X_i)$ is a manifold of dimension $3g-3$ with singularities from the reducible orbits. $\mathcal{M}(\Sigma)$ is a $(6g-6)$-dimensional manifold with singularities, and with a symplectic structure (first studied by Goldman [26]). $\mathcal{M}(X)$ may be thought of as the set of intersections of the two maps $r_i : \mathcal{M}(X_i) \to \mathcal{M}(\Sigma)$.

The irreducible strata of $\mathcal{M}(X_i)$ and $\mathcal{M}(\Sigma)$ can be canonically oriented. Fortunately, $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$ do not intersect along the singular strata except at the trivial orbit, and the remaining intersections form a compact set. Thus one can choose a perturbation of the maps r_i, compactly supported on $\mathcal{M}^*(\Sigma)$, so that the intersections are transverse and thereby obtain a well-defined oriented intersection number.

3.2. The Atiyah-Floer Conjecture. Floer also defined a homology theory in the symplectic category. One considers two Lagrangian submanifolds L^n, L'^n of a symplectic manifold M^{2n}. The intersections of L and L' form the generators of the Floer complex, and the boundary operator counts pseudoholomorphic curves with boundary in $L \cup L'$ "connecting" intersection points. See Figure 7.

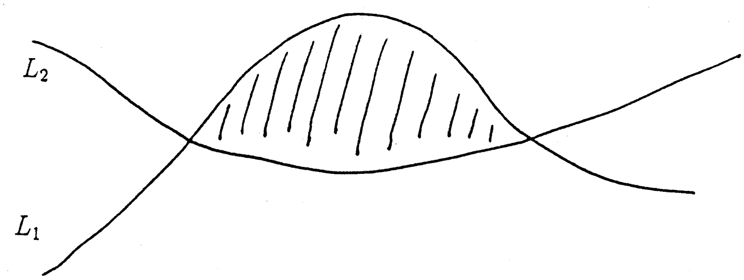

FIGURE 7. A pseudoholomorphic disk bounded by the Lagrangian submanifolds L_1 and L_2.

In the intersection description of $\lambda(X)$ used by Casson, the maps $r_i : \mathcal{M}(X_i) \to \mathcal{M}(\Sigma)$ are Lagrangian submanifolds with respect to the symplectic structure on $\mathcal{M}(\Sigma)$ (at least on the nonsingular part). This naturally raises the following question:

Conjecture 3.1 (Atiyah-Floer). *Symplectic Floer homology can be adapted to handle the singularities in the various moduli spaces from a Heegard decomposition. Then symplectic Floer homology will equal instanton Floer homology in this case.*

This conjecture was recently proven by Lee and Li ([36] and [37]).

3.3. **Properties of $\lambda(X)$.** The Casson invariant satisfies the following properties.

(i) $\lambda(X) = 0$ if $\pi_1(X) = 0$.
(ii) $\lambda(-X) = -\lambda(X)$.
(iii) $\lambda(X_1 \# X_2) = \lambda(X_1) + \lambda(X_2)$.
(iv) $\lambda(K_n) = \lambda(X) + n\Delta_K''(1)$.

where K_n is $\frac{1}{n}$ surgery on the knot $K \subset X$ and $\Delta_K(t)$ is the Alexander polynomial of K.

A consequence of (i) and (iv) is that any knot in S^3 with $\Delta_K''(1) \neq 0$ has Property P. A knot has Property P if every nontrivial surgery on it yields a manifold with nontrivial fundamental group. Surgery on a knot with Property P cannot yield a counterexample to the Poincaré conjecture.

Another consequence of (iv) is that $\lambda(X)$ is calculable from a surgery diagram for X. This raises a question about Floer homology: is there also a surgery formula describing the how FH_* changes during surgery, a formula which will make Floer homology computable? The following subsection describes a partial answer to this question.

3.4. **The exact triangle for FH_*.** Recall that the boundary operator in Floer homology is defined by counting instantons on $X \times \mathbb{R}$ which connect flat orbits. Floer observed that one could make a similar count for any 4-manifold M with $\partial M = X \cup -X'$. One thereby obtains a homomorphism from $FH_*(X)$ to $FH_*(X')$.

By a clever application of these ideas to certain 4-dimensional cobordisms that arise in 3-manifold surgery, Floer showed that there is an instanton homology group $FH_*(X, K)$ of a knot K in X such that there is an exact triangle (see [8])

$$\begin{array}{ccccc} & & FH_*(X,K) & & \\ & \swarrow & & \nwarrow & \\ FH_*(X_n) & & \longrightarrow & & FH_*(X_{n+1}) \end{array}$$

where $X_n(K)$ is the 3-manifold obtained by $\frac{1}{n}$-surgery on K.

Unfortunately, what is (so far) missing from Floer theory is an identification of $FH_*(X, K)$ with something easily computable. Clearly the Euler characteristic of $FH_*(X, K)$ is $2\Delta''_K(1)$, but there has so far not been much progress in understanding the groups themselves. Saveliev has, however, used the exact triangle to do some computations ([48] and [?]).

3.5. **Calculations of Floer homology.** The first calculations of Floer homology were by Fintushel and Stern for Seifert fibered homology spheres [19]. They were able to compute not only the flat connections on these 3-manifolds, but also the $\mathbb{Z}_8$ gradings of those flat connections. Using this information they showed that the chain complex has generators only in every other dimension. Therefore the boundary operators must be trivial.

Since then, the technology for computing spectral flow (and hence gradings) has improved considerably. There is a general splitting theorem that allows one to calculate $\mathrm{SF}([A],[A'])$ roughly whenever

(i) X is a union of X_1 and X_2 along their boundary Σ.
(ii) $[A|_{X_i}]$ and $[A'|_{X_i}]$ can be connected by a path in $\mathcal{M}(X_i)$ for $i = 1, 2$.
(iii) the restriction maps $r_i : \mathcal{M}(X_i) \to \mathcal{M}(\Sigma)$ are understood.

Then the spectral flow is a sum of a relative term from each X_i and a Maslov index computed from the maps r_i (see [51], [43], [11], [33]). This has led to more examples where the index of all flat orbits have the same parity and so the boundary operator is necessarily trivial.

The first cases shown to have nontrivial boundary operators were calculated independently by Li [39] and Fukaya [23]. They derived a general spectral sequence which computes $FH_*(X \# X')$ in terms of instanton data (both instantons connecting irreducible orbits and instantons connecting irreducible orbits to the trivial orbit) on X and X'. With the exception of a few cases, the necessary instanton data remains to be calculated.

There are also known cases where, for small perturbations there are generators in the complex in both even and odd dimensions, and by judiciously choosing perturbations, one can cancel pairs of flat orbits. One such example is $+1$ surgery on the granny knot (the composite of two right trefoils) (see Figure 8). Unfortunately, no such examples are known where sufficiently many flat orbits can be cancelled to determine the Floer homology.

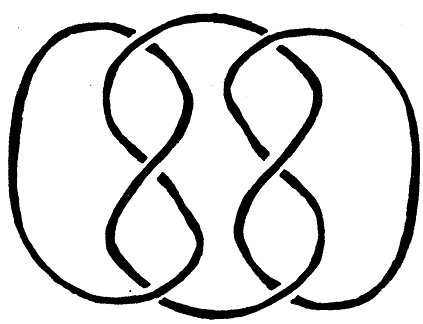

FIGURE 8. The granny knot.

4. GAUGE THEORY ON KNOT COMPLEMENTS

In this lecture we discuss flat moduli spaces of 3-manifolds with torus boundary. Like a Heegard decomposition, a decomposition of a closed 3-manifold $X = X_1 \cup_{T^2} X_2$ along a torus gives a description of $\mathcal{M}(X)$ in terms of $\mathcal{M}(X_1)$ and $\mathcal{M}(X_2)$ and the restriction maps $r_i : \mathcal{M}(X_i) \to \mathcal{M}(T^2)$.

$$\begin{array}{ccccc} & & \mathcal{M}(X_1) & & \\ & \nearrow & & \searrow & \\ \mathcal{M}(X) & & & & \mathcal{M}(\Sigma) \\ & \searrow & & \nearrow & \\ & & \mathcal{M}(X_{2)} & & \end{array} , \quad \mathcal{M}(X) = \mathcal{M}(X_1) \times_{\mathcal{M}(T^2)} \mathcal{M}(X_2)$$

We begin with the problem of generalizing the Morse theory description of the flat moduli to the case $\partial X = T^2$. A treatment of the general case $\partial X \neq \emptyset$ is given in [27], and we refer the reader there for details. With the generic structure of the moduli space (when $\partial X = T^2$) established, we discuss some results about flat moduli spaces of knot complements.

4.1. **The flat moduli space of T^2.** We begin with a simple calculation to identify $\mathcal{M}(T^2)$. Let μ be a meridian for the knot K, i.e., a curve in X which generates $H_1(X)$ and bounds a normal disk of K in Y. Also, choose a longitude λ, parallel copy of the knot which is null homologous in the complement of K. The homotopy classes of μ and λ generate $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. Any representation $\rho : \pi_1 T^2 \to SU(2)$ is conjugate to one taking the pair (λ, μ) to a pair of diagonal $SU(2)$ matrices

$$\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \right).$$

By conjugating to interchange eigenvalues, if necessary, θ may be taken to be in $[0, \pi]$. Thus $\mathcal{M}(T^2)$ may be identified with the rectangle $[0, \pi] \times [0, 2\pi]$ with the edge identifications illustrated in Figure 9. Topologically, this is S^2, but it is more natural to consider it a "pillowcase" with singular points at the four corners (central orbits) .

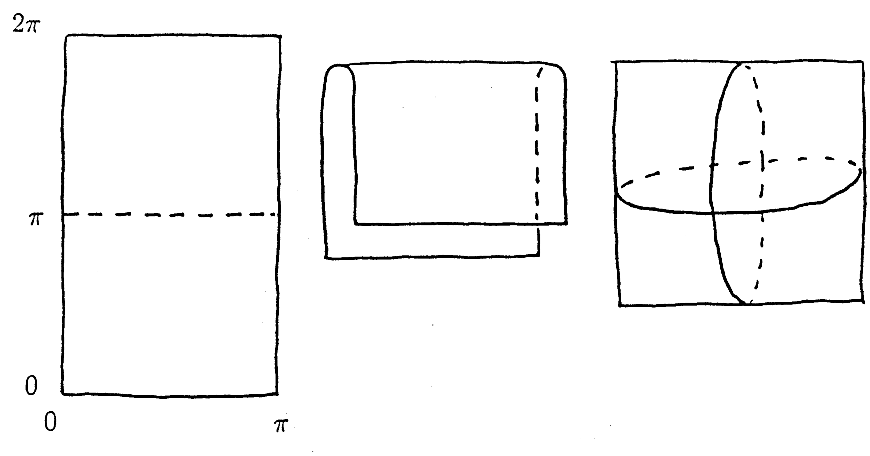

FIGURE 9. Constructing the pillowcase from a fundamental domain $[0, \pi] \times [0, 2\pi]$.

4.2. **Chern-Simons theory when $\partial X = T^2$.** There are two main difficulties in pushing the Morse theory analogy through when $\partial X \neq \emptyset$.

(i) CS(A) is not gauge invariant, even modulo the integers.
(ii) Flat connections are not the critical points of CS.

The proofs in the closed case involve integration by parts, and a boundary term prevents those arguments from working here.

A construction of Ramadas, Singer and Weitsman [46] solves both problems. The function $e^{2\pi i\,\mathrm{CS}(A)}$ is interpreted as a section of a certain circle bundle over $\mathcal{B}(X)$, a bundle carrying a natural connection, and the critical points of this section with respect to that connection are the flat connections.

The construction of a bundle over $\mathcal{B}(X)$ of which $e^{2\pi i\,\mathrm{CS}(A)}$ induces a section is simple. We simply divide $\mathcal{A}(X) \times U(1)$ by the $\mathcal{G}(X)$ action given by

$$g(A, \exp(i\phi)) = (gA, \exp(i\phi)\exp\left(i2\pi(\mathrm{CS}(gA) - \mathrm{CS}(A))\right)).$$

Observe that the twisting factor $\exp\left(i2\pi(\mathrm{CS}(gA) - \mathrm{CS}(A))\right)$ only depends on the restrictions of A and g to ∂X. (This follows from the fact that $\mathrm{CS} : \mathcal{A} \to \mathbb{R}/\mathbb{Z}$ *is* well-defined on closed 3-manifolds.) Thus the bundle over $\mathcal{B}(X)$ so constructed is the pull-back of a bundle, which we denote by $\mathcal{L}$, over $\mathcal{B}(\partial X)$.

$$\begin{array}{ccc} r^*(\mathcal{L}) & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{B}(X) & \longrightarrow & \mathcal{B}(\partial X) \end{array}$$

In [46], a connection ω on $\mathcal{L}$ is constructed with the property that its curvature 2-form, when restricted to $\mathcal{M}(\partial X) \subset \mathcal{B}(\partial X)$, is the symplectic structure on $\mathcal{M}(\partial X)$. As in the nondegenerate case on a 3-manifold, $T_{[A]}\mathcal{M}(\partial X) \cong H^1_A(\partial X; su(2))$. The symplectic structure is given by the cup product pairing on cohomology:

$$\Omega(\alpha, \beta) = \int_{\partial X} \mathrm{Tr}(\alpha \wedge \beta).$$

The connection ω gives a way to differentiate the section $e^{2\pi i\,\mathrm{CS}(A)}$ of the bundle $r^*\mathcal{L}$ over $\mathcal{B}(X)$. A critical point, i.e., a point where this derivative is zero, is an orbit of flat connections. This immediately implies that the lift of $r : \mathcal{M}(X) \to \mathcal{M}(\partial X)$ given by $e^{2\pi i\,\mathrm{CS}(A)}$ is horizontal. This fact gives as a corollary an integrality condition on the immersion $r : \mathcal{M}(X) \to \mathcal{M}(\partial X)$. We state the result below for the case $\partial X = T^2$.

Corollary 4.1. ([27]) *Suppose $\partial X = T^2$ and $\gamma : S^1 \to \mathcal{M}(X)$ is a loop. Then the signed symplectic area bounded by the loop $r \circ \gamma : S^1 \to \mathcal{M}(T^2)$ is zero modulo the area of the pillowcase.*

Proof. The proof consists of applying Stokes theorem to the form $d\omega = \Omega$. The holonomy of ω around $r \circ \gamma$ must be a multiple of 2π, since it has a closed horizontal lift, which means, by Stokes' theorem, that $r \circ \gamma$ bounds symplectic area $2\pi n$ for some $n \in \mathbb{Z}$. The symplectic area of the pillowcase is -2π. □

The corollary puts constraints on what the image of $r : \mathcal{M}(X) \to \mathcal{M}(T^2)$ can look like, as illustrated in Figure 10.

In addition, there is another constraint on the image of a smooth 1-dimensional component of $\mathcal{M}(X)$ in $\mathcal{M}(T^2)$, coming from the splitting formula for spectral flow ([11], [12], [43]), which roughly says that

$$\#\{\text{horizontal tangencies}\} + 2\#\{\text{corners enclosed}\} \equiv 0 \pmod 8.$$

The first term is a count of places where the tangent direction is parallel to the $\rho(\lambda) =$ constant direction, with signs indicated in Figure 11. The second term is also a signed count. Figure 12 shows some images allowed and forbidden by this second constraint.

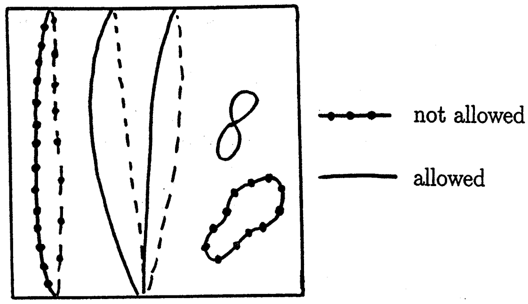

FIGURE 10. The images of closed components in $\mathcal{M}(X)$ must satisfy the integrality condition of Corollary 4.1.

FIGURE 11. In the Maslov index term of the splitting formula horizontal tangencies are counted with sign.

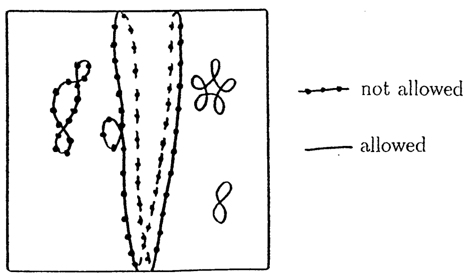

FIGURE 12. The images of closed components in $\mathcal{M}(X)$ must also satisfy a Maslov index condition.

4.3. **Structure of the moduli space.** The following theorem is a special case of Theorem 15 of [27], which covers arbitrary genus boundary.

Theorem 4.2. *If $\partial X = T^2$, then for generic perturbations h, the perturbed flat moduli space consists of*

(i) *finitely many central orbits,*
(ii) *a smooth 1-dimensional manifold of abelian orbits with one noncompact end limiting to each central orbit, and*
(iii) *a smooth 1-dimensional manifold of irreducible orbits with finitely many ends, each limiting to a different perturbed flat abelian orbit.*

The restriction map $r : \mathcal{M}_h(X) \to \mathcal{M}(T^2)$ is an immersion taking the 1-dimensional strata of $\mathcal{M}(X)$ into the smooth part of $\mathcal{M}(T^2)$.

We shall call the (perturbed) flat moduli space *nondegenerate* if conditions (i), (ii), and (iii) of Theorem 4.2 hold, and call it *degenerate* otherwise. We will denote the reducible part of the moduli space by $\mathcal{M}_h^r$. The special points of $\mathcal{M}_h^r$ which are in the closure of

$\mathcal{M}^r_h$ are called *bifurcation points.* Note that if $H_*(X) = H_*(S^1)$, then before perturbation there are two central flat orbits and a smooth 1-dimensional arc of abelian flat orbits connecting them.

Two examples of flat moduli spaces of knot complements and their images on the pillowcase are depicted in Figure 13.

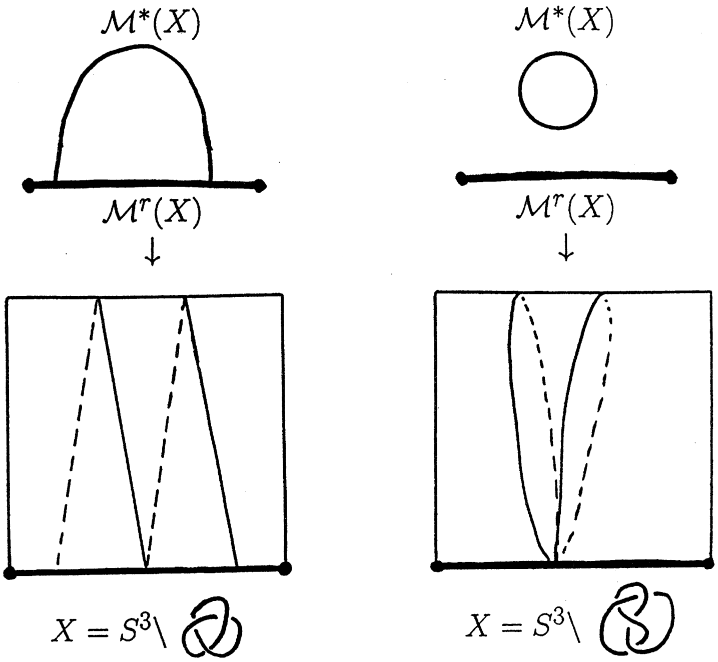

FIGURE 13. The flat moduli spaces of two knot complements and their images in the pillowcase.

4.4. **Structure near a bifurcation point.** A reducible $SU(2)$ connection on the complement of a knot in a homology 3-sphere reduces to a $U(1)$ connection on a trivial $\mathbb{C}$ bundle and hence can be gauge transformed so that the connection 1-form takes values in the 1-dimensional sub Lie algebra of diagonal $su(2)$ elements (which we henceforth identify with $u(1)$). The orthogonal complement of $u(1)$ in $su(2)$ is isomorphic to $\mathbb{C}$, with $U(1)$, the maximal torus of $SU(2)$ acting on the $\mathbb{C}$ with weight 2. The deformation complex and the operator $K_{A,h}$ respect a splitting of $su(2)$-valued into $u(1)$ and $\mathbb{C}$ components.

Proposition 4.3. *If h is generic and $[A] \in \mathcal{M}^r_h$ is a bifurcation point, then*
(i) $H^1_{A,h}(X; u(1)) = T_{[A]}\mathcal{M}^r_h \cong \mathbb{R}$ *and* $H^2_{A,h}(X; u(1)) = 0$.
(ii) $H^1_{A,h}(X; \mathbb{C}) \cong H^2_{A,h}(X; \mathbb{C}) \cong \mathbb{C}$.

The Kuranishi map gives a model for the moduli space near A, namely $\Phi^{-1}(0)/\operatorname{Stab} A$, for a map

$$\Phi : H^1_{A,h}(X; u(1) \oplus \mathbb{C}) \to H^2_{A,h}(X; u(1) \oplus \mathbb{C}).$$

See Figure 14. Under the assumptions of Proposition 4.3, Φ has the form $\Phi(t, z) = \pm tz$ up to change of coordinates.

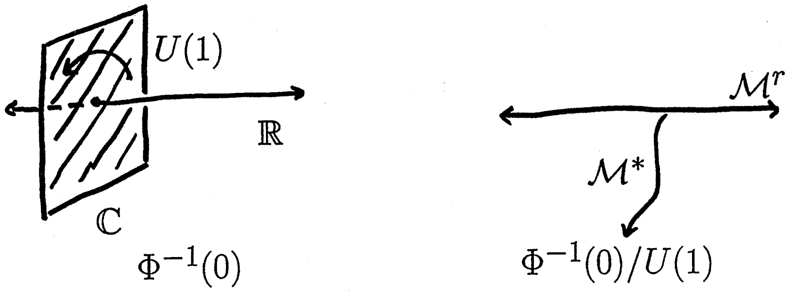

FIGURE 14. A local model for the flat moduli space near a bifurcation point.

The bifurcation points in $\mathcal{M}_h^r$ are characterized by the property that the cohomology $H^1_{A,h}(X;\mathbb{C}) = \ker K^{\mathbb{C}}_{A,h}$ jumps from 0 to $\mathbb{C}$ at these points. One can show that $K^{\mathbb{C}}_{A,h}$ is a Hermitian Fredholm operator, and at these points a real eigenvalue (of complex multiplicity one) crosses zero.

4.5. **The Alexander matrix and equivariant signature.** In this subsection we recall the definition of the Tristam-Levine equivariant knot signature, the Alexander matrix, and the Alexander polynomial. In the next subsection we will describe some results relating these invariants to the flat moduli space.

Let Y be a homology 3-sphere, K be a knot in Y, and X be the complement of an open tubular neighborhood of K in Y. Let F be a Seifert surface for K, and choose an orientation of the normal bundle of F in Y. If $\{x_i\}$ denotes a basis for $H_1(F;\mathbb{Z})$, then let x_i^+ denote the push-off of x_i in the positive normal direction. We then define the linking matrix V by $V_{ij} = \text{link}(x_i, x_j^+)$.

The symmetrized Alexander matrix of $K \subset Y$ is $A(t) = t^{\frac{1}{2}}V - t^{-\frac{1}{2}}V^T$. The Alexander polynomial $\Delta_K(t)$ equals the determinant of $A(t)$. Define another matrix $B(t)$ by

$$B(t) = (1-t)V + (1-t^{-1})V^T = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})A(t).$$

Clearly the complex values $z \neq \pm 1$ for which $B(z)$ is singular are exactly the roots of the Alexander polynomial.

For unit complex numbers z, $B(z)$ is a Hermitian matrix. The equivariant knot signature is $\text{Sign}B(z)$, the number of positive eigenvalues minus the number of negative.

Let $A_t, t \in [0, 2\pi]$ be a path of flat reducible connections with

$$\text{hol}_\mu(A_t) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}.$$

Then one can identify the spectral flow of K_{A_t} with that of the curve of finite dimensional Hermitian matrices $B(e^{2it})$. In [33], Kirk, Klassen and Ruberman use this fact to compute spectral flow in several interesting cases. In [28] and [29], this fact is used to derive further relationships between the flat moduli space and more classical knot invariants, which we discuss below.

4.6. **A fixed trace Casson-type invariant of knots.** The first result involves counting the (perturbed) orbits which have meridinal holonomy of fixed trace. For $\alpha \in [0, \pi]$, set

$$S_\alpha = \{[A] \in \mathcal{M}(T^2) |\, \text{Tr}(\text{hol}_\mu(A)) = 2\cos\alpha\}.$$

One could count the fixed-trace flat orbits on X by counting intersections of $r(\mathcal{M}_h(X))$ with S_α. To count points with orientations, however, one must consider double cover $\widetilde{\mathcal{M}}_h(X)$, which has a canonical orientation, intersecting with a lift $\widetilde{S_\alpha}$ of S_α. The double covers have a restriction map to the torus, the double branched cover of $\mathcal{M}(T^2)$. See Figure 15.

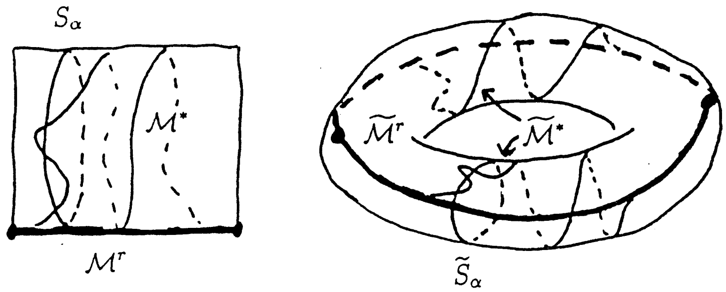

FIGURE 15. The double cover of the pillowcase, branched at the corners, is a torus. Here, the two copies of $\mathcal{M}(X)$ can be canonically oriented.

Theorem 4.4 ([28]). *Suppose $\mathcal{M}(X)$ is nondegenerate.*

(i) *Then* $\tilde{r}(\widetilde{\mathcal{M}}^*(X)) \cdot \tilde{S}_0 = -4\lambda(Y)$.
(ii) *For any* $0 < \alpha \leq \pi$ *with* $\Delta_K(e^{i2\alpha}) \neq 0$,

$$\tilde{r}(\widetilde{\mathcal{M}}^*(X)) \cdot \tilde{S}_0 = -4\lambda(Y) - \frac{1}{2}\operatorname{Sign} B_K(e^{i2\alpha}).$$

If $\mathcal{M}(X)$ is degenerate, then these properties hold for generic small perturbations.

The first theorem of this type appeared in an interesting paper by Lin [41], which used topological methods. It covered the case $\alpha = \frac{\pi}{2}$ and $Y = S^3$. Austin has informed me that he has independently found a proof of the theorem (again a topological version) in the case $Y = S^3$ for arbitrary α.

Another approach by Cappell, Lee, and Miller [13] using branched covers has led to a similar result for $e^{i\alpha}$ a root of unity.

One natural question is whether there is a fixed-trace Floer homology of a knot, the Euler characteristic of which is the signed number of fixed-trace flat connections. Various people have worked or are currently working (independently) on versions of such a homology theory, including Austin, Collin, Fukaya, Gerard, Li and Salamon. Salamon described an outline of a very general framework for symplectic Floer homology for any 3-manifold with boundary [47]. Austin and Li have independently taken symplectic approaches to (fixed-trace) Floer homology of knots [40].

On the gauge theoretic side, Collin is currently attacking the problem differently by considering orbifold connections on a branched cover, and has partially constructed a theory for $e^{i\alpha}$ a root of unity [15]. Gerard has independently succeded in finding an analogue of the Chern-Simons function on a suitable space of fixed trace connections (with no root of unity restriction), but has yet to sort out the gradient flows.

4.7. **Unperturbed moduli spaces.** While all the results so far have concerned perturbed flat moduli spaces for generic perturbations, one might well ask what can be said about the character variety (the unperturbed flat moduli space) from Chern-Simons gauge theory. Frohman and Klassen showed, by topological means, that if $e^{2i\alpha}$ is a simple root of the Alexander polynomial of a knot in S^3, then the abelian representation ρ_α taking μ to the diagonal matrix with entries $e^{i\alpha}, e^{-i\alpha}$ can be deformed to a family of irreducible representations [22].

This result was generalized as follows, by considering perturbed flat moduli spaces and letting h limit back to zero.

Theorem 4.5. ([29]) *If K is a knot in a homology sphere, then for any root of unity $e^{2i\alpha}$ with $\Delta_K(e^{2i\alpha}) = 0$ for which the right and left hand limits $\lim_{\beta\to\alpha^\pm} \operatorname{Sign} B_K(e^{2i\beta})$ do not agree, there is a continuous family of irreducible $SU(2)$ limiting to ρ_α.*

Note that if $e^{2i\alpha}$ is an odd multiplicity root of $\Delta_K(z)$ then the hypothesis of the theorem applies. This theorem implies the existence of irreducible representations for some homology 3-spheres with trivial Casson's invariant (see [29]).

5. Coping with reducible orbits in the moduli space

In this lecture, we discuss the difficulties encountered when one attempts to generalize the Casson invariant and Floer homology to situations when there are reducible flat connections to contend with.

The problem with generalizing the Casson invariant to either 3-manifolds other than homology 3-spheres or structure group $SU(n)$, $n > 2$, is that the irreducible portion of the parameterized moduli space is not compact, and hence the cobordism argument fails to show that the count of irreducible perturbed flat orbits is independent of perturbation. Noncompact ends limit to reducible flat orbits, i.e., singularities in the quotient space $\mathcal{B}$.

To illustrate this problem, consider the critical set of an invariant function on S^2, represented in Figure 16 as a height function, invariant under equatorial rotations. (Fixed points correspond to reducible orbits.) Clearly the number of free critical orbits counted with sign is not independent of the function. Thus an invariant cannot be constructed by simply counting critical points of a Morse function on $S^2/S^1 \setminus \{\text{singularities}\}$.

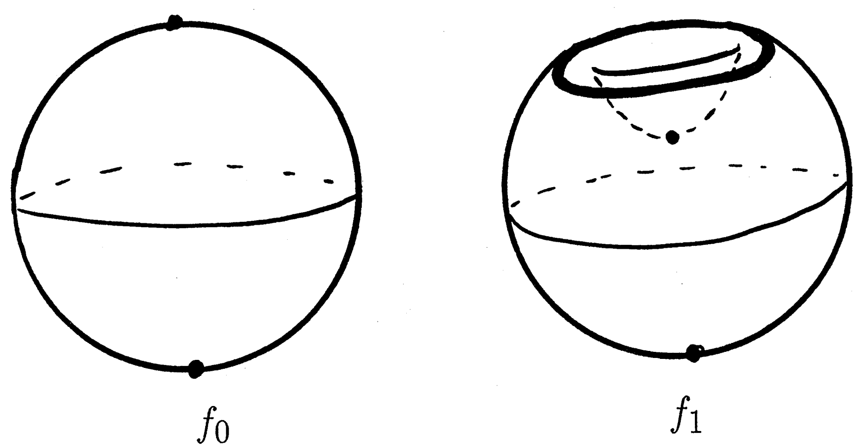

FIGURE 16. Two functions on the 2-sphere which are invariant under equatorial rotations.

After mentioning related work by several others, we will generalize our discussion in Lecture 1 of the Euler characteristic from Morse theory to an equivariant setting. We discuss the correction term needed to change the count of free critical orbits into an invariant independent of the function. We then discuss an application of these ideas to gauge theory. Specifically, we will describe joint work with Hans Boden defining an $SU(3)$ Casson invariant. We will end with some comments about generalizing Floer homology to situations when there are reducible flat orbits.

5.1. **Other work on generalizations.** Various approaches to generalizing the Casson invariant to situations where there are reducible flat orbits have been explored.

(i) In [7], Boyer and Nicas consider only compact components of the irreducible flat moduli space. This allows them to define an invariant without a correction term. Unfortunately, they had difficulty establishing connect sum and surgery formulae for their invariant.
(ii) In [50], Walker adopted Casson's intersection picture for the character variety of a 3-manifold. By a delicate analysis of the intersections at the singularities, he generalized the Casson invariant to rational homology spheres by adding a correction term from the reducible stratum. He furthermore recovered all the nice properties of $\lambda(X)$.
(iii) In a research announcement [10], Cappell, Lee and Miller proposed a generalization to $SU(n), n > 2$ using symplectic methods.

5.2. **Equivariant Morse theory in finite dimensions.** The flat moduli space of $SU(3)$ connections on a homology sphere contains singular points, the orbits of reducible (non-abelian) connections. Only one singular stratum which enters into the analysis, consisting of orbits with circle stabilizer. To illustrate the difficulties and the form of the correction term needed to define a Casson invariant, we consider a finite dimensional version first.

Let M be a compact manifold with a semifree S^1 action (each orbit is either a free circle orbit or a point). The fixed point set is a submanifold $L \subset M$, with normal bundle $N(L)$. $N(L)$ has the structure of a complex vector bundle. If $f : M \to \mathbb{R}$ is smooth, invariant and generic within the invariant functions, then $\mathrm{Crit}(f)$ consists of isolated points of L and isolated S^1 orbits. In the tangent directions normal to a critical orbit, $\mathrm{Hess}\, f$ is nondegenerate. $\mathrm{Hess}\, f|_{N(L)}$ is Hermitian, but we consider it as a real operator (so its eigenvalues have even multiplicity).

For critical fixed points p and critical free orbits c, we define the following gradings:

(i) $\mu(c) = \dim\{\text{negative eigenspace of } \mathrm{Hess}\, f(x), x \in c\}$
(ii) $\lambda(p) = \dim\{\text{negative eigenspace of } \mathrm{Hess}\, f|_L(p)\}$
(iii) $\nu(p) = \dim\{\text{negative eigenspace of } \mathrm{Hess}\, f(p)|_{N(L)}\}$

We make the following definition:

$$\mathcal{X}_f^{S^1}(M) = \sum_c (-1)^{\mu(c)} - \sum_p (-1)^{\lambda(p)} \left(\frac{\nu(p)}{2}\right)$$

Remark. Note that the first term counts critical points in the smooth part of M/S^1. The second term is a correction term.

Theorem 5.1. *$\mathcal{X}_f^{S^1}(M)$ is independent of f.*

<u>Properties of $\mathcal{X}^{S^1}(M)$</u>

(i) For a free action, $\mathcal{X}^{S^1}(M) = \mathcal{X}(M/S^1)$.

(ii) If the S^1 action is trivial, then $\mathcal{X}^{S^1}(M) = 0$.
(iii) $\mathcal{X}^{S^1}(M)$ is the relative Euler class of the pair $(M/S^1, L/S^1)$.

The first property is clear. For the second, note that in this case $\nu(p) = 0$ for all critical points. To see why the third is true, choose a generic function with the property that

$$\max_{x\in L} f(x) < \min_{x\in \mathrm{Crit}(f)\setminus L} f(x).$$

The correction term vanishes for this function, and the formula simply counts (with sign) the cells required to build up the relative manifold $(M/S^1, L/S^1)$ from $(L/S^1, L/S^1)$.

5.3. **Sketch of proof of Thm. 5.1.** Let f_0, f_1 be generic invariant functions. Connect f_0 to f_1 by a generic path f_t of invariant functions. The parameterized critical set is

$$W = \bigcup_{t\in[0,1]} \mathrm{Crit}\, f_t \times \{t\} \subset M \times [0,1].$$

$W^r = W \cap (L \times [0,1])$ is a compact 1-manifold with boundary. $W^* = W \cap ((M \setminus L) \times [0,1])$ is a 2-manifold with free $U(1)$ action. Thus $W^*/U(1)$ is a 1-manifold. $W^*/U(1)$ is not typically compact, however. Its closure in $M/U(1)$ includes a finite number of *bifurcation points* in the interior W^r. The structure of $W/U(1)$ near a bifurcation point, and the preimage in W, are illustrated in Figure 17. The bifurcation points are exactly the points of W^r at which $\nu(p)$ changes, and at a bifurcation point $\nu(p)$ always changes by ± 2.

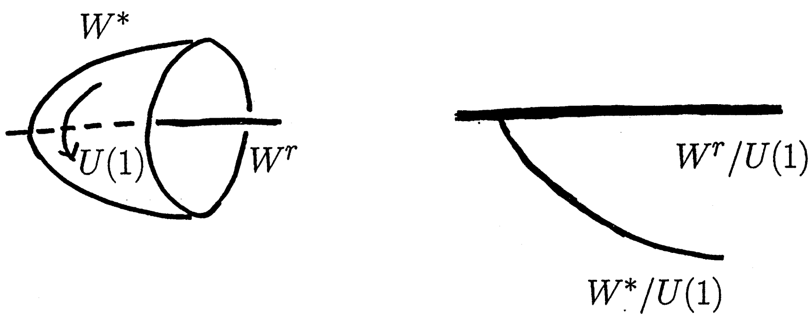

FIGURE 17. A local model for the singularities in the parameterized critical set of a generic 1-parameter family of S^1-invariant functions.

We can obtain canonical orientations on the W^*/S^1 and W^r as follows. Define a family of linear operators parameterized by $M \times [0,1]$ as follows.

$$J : u(1) \times TM \times T[0,1] \to u(1) \times TM$$

by the formula

$$J(x,t)(u\hat{\imath}, \vec{v}, \tau) =$$
$$\left(\left\langle \vec{v}, \tfrac{d}{ds} \exp(s\hat{\imath})x\big|_{s=0} \right\rangle, u\tfrac{d}{ds} \exp(s\hat{\imath})x\big|_{s=0} + \mathrm{Hess}\, f_t(x)\vec{v} - \tau \tfrac{\partial}{\partial t} \nabla f_t(x) \right).$$

The relevance of the operator J to our parameterized moduli space is this. If $(x,t) \in W$, then $\ker J(x,t)$ contains the tangent space to W/S^1. Consider the cases:
If $(x,t) \in W^r$, then

$$\ker J(x,t) = u(1) \oplus T_{(x,t)} W^r \ (\oplus \mathbb{C} \text{ at bifurcations})$$

and

$$\mathrm{coker}\, J(x,t) = u(1) \ (\oplus \mathbb{C} \text{ at bifurcations}).$$

If $(x,t) \in W^*$, then $\ker J(x,t) = T_{[(x,t)]}W^*/S^1$ and $\operatorname{coker} J(x,t) = 0$.

There is a canonical orientation on the virtual index bundle Ind J, and from it we obtain canonical orientations on the W^*/S^1 and W^r. These have the following properties:

(i) for any regular point p of projection proj : $W^r \to [0,1]$,
$\mathcal{O}_c = (-1)^{\lambda(c)}\text{proj}^*\mathcal{O}_{[0,1]}$.

(ii) for any regular point $[C]$ of projection proj : $W^*/S^1 \to [0,1]$,
$\mathcal{O}_{[C]} = (-1)^{\mu(C)}\text{proj}^*\mathcal{O}_{[0,1]}$.

(iii) the boundary orientation at a bifurcation point $\partial(\overline{W^*/S^1})$ is $\frac{1}{2}(\nu(b_+) - \nu(b_-))$, where b_+ and b_- are points of W^r as shown in Figure 18.

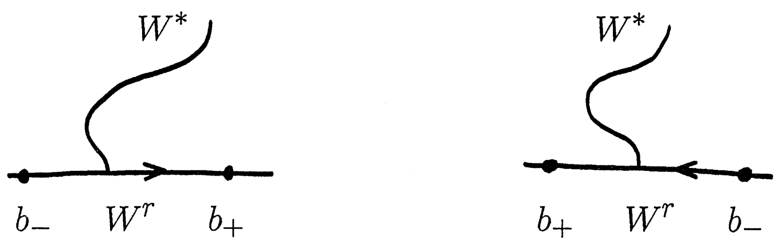

FIGURE 18. The relationship between b_-, b_+, and the orientation on W^r near a bifurcation point in W/S^1.

Note that since f_0 and f_1 are generic, 0 and 1 are regular values of proj : $\overline{W^*/S^1} \to [0,1]$. By (ii), the difference

$$\sum_{p \subset \mathrm{Crit}(f_1)} (-1)^{\mu(p)} - \sum_{p \subset \mathrm{Crit}(f_0)} (-1)^{\mu(p)}$$

equals the number of boundary points of $\overline{W^*/S^1}$ in the ends of $M \times [0,1]$, counted with the boundary orientation.

By (i) and (iii), the difference in the correction terms calculates minus the number of endpoints of $\overline{W^*/S^1}$ which are bifurcation points, also with boundary orientation. Now the theorem follows from the fact that the total number of boundary points of $\overline{W^*/S^1}$, counted with boundary orientation, is zero.

5.4. **Reducible $SU(3)$ connections.** As in Taubes' description of the Casson invariant, we replace Morse index by spectral flow of the operator K_A (on $su(3)$ valued forms, of course). As in our finite dimensional model, we must split this into the component tangent to the reducible stratum and the component normal to the reducible stratum.

On a homology sphere, the only nontrivial reducible flat $SU(3)$ connections (or perturbed flat connections for small perturbations) have stabilizer isomorphic to $U(1)$. Any reducible connection with $U(1)$ stabilizer can be assumed, after gauge transformation, to take values in $s(u(2) \times u(1))$. Then the stabilizer of A consists of gauge transformations consisting of constant diagonal matrices with entries $e^{i\theta}, e^{i\theta}, e^{-2i\theta}$. The decomposition of $T_A\mathcal{A} = \Omega^1(X; su(3))$ into tangent vectors tangent to and normal to the reducible connections is obtained by decomposing $su(3)$ as $su(3) = s(u(2) \times u(1)) \oplus \mathbb{C}^2$.

The first summand consists of matrices of the form

$$\begin{pmatrix} i(a+r) & b+ic & 0 \\ -b+ic & i(-a+r) & 0 \\ 0 & 0 & -2ir \end{pmatrix},$$

where a, b, c, and r are real. The second summand consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\overline{z_1} & -\overline{z_2} & 0 \end{pmatrix}.$$

The operator K_A (when A is an $S(U(2) \times U(1))$ connection) respects the corresponding splitting of $\Omega^{0+1}(X; su(3))$, and thus it makes sense to refer to the $\mathbb{C}^2$ spectral flow of K_A along a path of connections. This quantity replaces the quantity $\nu(p)$ in the finite dimensional model.

5.5. **The SU(3) Casson invariant formula.** Let X be a $\mathbb{Z}$-homology 3-sphere. Choose a generic perturbation function $h : \mathcal{A}_{SU(3)} \to \mathbb{R}$, and consider the perturbed flat moduli space $\mathcal{M}_h = \mathcal{M}_h^* \cup \mathcal{M}_h^r \cup [\theta]$.

Choose representatives B for each orbit in $\mathcal{M}_h^r$, and for each such B, choose a nearby flat connection $\hat{B}$. Then set

$$\lambda_{SU(3)}(X, h) =$$
$$\sum_{[A]\in\mathcal{M}_h^*} (-1)^{SF(\theta,A)} - \frac{1}{2} \sum_{[B]\in\mathcal{M}_h^r} (-1)^{SF_{s(u(2)\times u(1))}(\theta,B)} \left(SF_{\mathbb{C}^2}(\theta, b) + 4CS(\hat{B}) + 2\right).$$

Theorem 5.2. *([5]) $\lambda_{SU(3)}(X, h)$ is independent of h and of the choices of representatives $b \in [b]$.*

Notation and comments:

(i) The Chern-Simons term is necessary because the $\mathbb{C}^2$ spectral flow in the correction term depends on the representative B for $[B]$.
(ii) If $\lambda_{SU(3)}(X) \neq 0$, then $\pi_1 X$ admits irreducible representations to either $SU(2)$ or $SU(3)$.
(iii) Our correction term

$$\frac{1}{2}(-1)^{SF_{s(u(2)\times u(1))}(\theta,B)} \left(SF_{\mathbb{C}^2(\theta,B)} - 4\,\mathrm{CS}(\hat{B})\right)$$

generalizes Walker's. I.e., his can be written as a difference of spectral flow and Chern-Simons terms.
(iv) Adding in the constant part of the correction term only adds a multiple of Casson's SU(2) invariant. It makes our invariant satisfy $\lambda_{SU(3)}(-X) = \lambda_{SU(3)}(X)$.
(v) The conjectured rationality of the $SU(2)$ Chern-Simons functional on flat connections would imply that $\lambda_{SU(3)}$ is rational.

There are many interesting questions raised by Theorem 5.2. The most intriguing is what sort of surgery relations (if any) does this new invariant satisfy. A related question is this: is $\lambda_{SU(3)}$ a finite type invariant [44, 25]? By [42], the Casson-Walker invariant equals 6 times λ_1, the first Ohtsuki invariant [45], so one is especially interested in any relationship between $\lambda_{SU(3)}$ and λ_2, the second Ohtsuki invariant. Positive results would be interesting for two reasons: (i) they would render $\lambda_{SU(3)}$ computable by algebraic means, and (ii) they would clarify what geometric information the finite type invariants carry. There are still, of course, the problems of defining the generalized Casson $SU(n)$ invariants for $n > 3$ and of extending $\lambda_{SU(3)}$ to rational homology 3-spheres. Both of these involve multiple reducible strata.

5.6. **Generalizing Floer homology.** There has been some progress in generalizing Floer homology to situations where there are nontrivial reducible flat connections. Lee and Li showed in [38] that one can define Floer homology for a rational homology sphere for each "chamber" in the space of perturbations. One crosses a "chamber wall" each time one the parameterized moduli space of a path of perturbations has a bifurcation point. They are not able, unfortunately, to describe how the Floer homology changes as they cross a chamber wall.

In a different direction, Austin and Braam considered the space of connections modulo based gauge transformations (gauge transformations which are the identity at one fixed basepoint). There is an $SO(3)$ action on the quotient space, and they study the infinite dimensional analogue of equivariant cohomology using the Chern-Simons function as a Bott-Morse function. They arrive at an equivariant Floer cohomology for rational homology spheres [4], one which has proved useful in computing Donaldson invariants. Again their perturbations must be carefully chosen to stay within a particular chamber.

It would be desirable to have a more general theory which allowed one to understand how the Floer homology (equivariant or not) changes as one crosses a chamber wall. Such an understanding would allow one, for an infinite family of rational homology spheres, to perturb away flat connections (like cancelling critical points of a Morse function in finite dimensions) until the equivariant Floer cohomology is clear from the chain complex. In other words, one could determine exactly the (nontrivial) boundary operator in these cases [30]. For a more general perturbation, the structure of noncompact ends of the instanton moduli space connecting flat connections of index difference two (the key to showing $\partial^2 = 0$), is more complicated, and the boundary operator will be more complicated. This is a subject of ongoing research by the author.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, SWARTHMORE COLLEGE, SWARTHMORE, PA, 19081

GEOMETRIC SUPERRIGIDITY OF SPHERICAL PSEUDO-QUATERNIONIC MANIFOLDS

YOSHINOBU KAMISHIMA

INTRODUCTION

We study a geometric structure on $(4n+3)$-dimensional smooth manifolds. A *spherical pseudo-quaternionic structure* is a geometric structure on a $(4n+3)$-manifold locally modelled on the sphere S^{4n+3} with coordinate changes lying in the Lorentz group $\mathrm{PSp}(n+1,1)$. Here $\mathrm{PSp}(n+1,1)$ is isomorphic to the isometry group $\mathrm{Iso}(\mathbb{H}_{\mathbb{F}}^{n+1})$ of the quaternionic hyperbolic space $\mathbb{H}_{\mathbb{F}}^{n+1}$ where $\mathbb{F}$ stands for the noncommutative field of quaternions. The space $\mathbb{H}_{\mathbb{F}}^{n+1}$ has the projective compactification whose boundary is the sphere S^{4n+3} on which $\mathrm{PSp}(n+1,1)$ acts as projective transformations. The pair $(\mathrm{PSp}(n+1,1), S^{4n+3})$ is said to be *spherical pseudo-quaternionic geometry* (cf. [20], [5]). A $(4n+3)$-manifold locally modelled on this geometry is said to be a *spherical pseudo-quaternionic manifold.*

In this paper we shall classify compact spherical pseudo-quaternionic manifolds with *amenable holonomy* groups, and show a *geometric rigidity* of compact spherical pseudo-quaternionic manifolds with quaternionic hyperbolic fundamental groups. Given a pseudo-quaternionic structure on M, we have a holonomy representation

$$\rho : \pi_1(M) \to \mathrm{PSp}(n+1,1).$$

As a spherical pseudo-quaternionic structure, the sphere with one point removed, $S^{4n+3} - \{\infty\}$, is identified with the Heisenberg nilpotent Lie group $\mathcal{M}$. Here $\mathcal{M}$ lies in the central extension $1 \to \mathbb{R}^3 \to \mathcal{M} \xrightarrow{\nu} \mathbb{F}^n \to 1$ for which $\mathbb{F}^n$ is the n-dimensional quaternionic vector space (cf. §1). Let $\mathrm{Sim}(\mathcal{M})$ be the subgroup of $\mathrm{PSp}(n+1,1)$ whose elements leave $\mathcal{M}$ invariant (equivalently, each element stabilizes the point at infinity ∞). The automorphism group $\mathrm{Sim}(\mathcal{M})$ is isomorphic to the semidirect product $\mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. Recall that an *amenable* closed subgroup of $\mathrm{PSp}(n+1,1)$ is conjugate to a subgroup of either $\mathrm{Sim}(\mathcal{M})$ or a maximal compact subgroup $\mathrm{Sp}(n+1) \cdot \mathrm{Sp}(1)$. We obtain the following classification concerning amenable groups (*e.g.*, virtually solvable groups).

Theorem A. *Let M be a compact spherical pseudo-quaternionic $(4n+3)$-manifold. If the holonomy group is amenable, then M is the spherical space form S^{4n+3}/F, an infraHopf manifold $S^1 \times S^{4n+2}/F$ or an infranilmanifold $\mathcal{M}/\Gamma$.*

A typical example of spherical pseudo-quaternioinic manifolds is a compact locally homogeneous space M. That is, the group $\mathrm{Aut}_{PQ}(\tilde{M})$ of spherical pseudo-quaternionic

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transformations of the universal covering space $\tilde{M}$ acts transitively on $\tilde{M}$. More specifically, let $S^{4n+3} - S^{4m-1}$ be the sphere complement where S^{4m-1} is the boundary of the quaternionic hyperbolic space $\mathbb{H}^m_{\mathbb{F}}$. Then the subgroup $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$ of $\mathrm{PSp}(n+1,1)$ acts transitively on $S^{4n+3} - S^{4m-1}$ with stabilizer isomorphic to the compact subgroup. We have the compact locally homogeneous spherical pseudo-quaternionic manifold

$$\mathrm{Sp}(m)\times\Delta\mathrm{Sp}(1)\times\mathrm{Sp}(n-m)\backslash\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)/\pi = S^{4n+3} - S^{4m-1}/\pi,$$

where $1 \leqq m \leqq n$ and the fundamental group is isomorphic to a discrete uniform subgroup of $\mathrm{PSp}(m,1)$. (Compare §1.) Applying the Corlette's superrigidity [10] to the case of the isometry group $\mathrm{PSp}(n,1)$ $(n \geqq 2)$, we obtain the following geometric rigidity concerning quaternionic hyperbolic groups.

Theorem B. *Let M be a compact spherical pseudo-quaternionic $(4n+3)$-manifold whose fundamental group $\pi_1(M)$ is isomorphic to a discrete uniform subgroup of* $\mathrm{PSp}(m,1)$ *for some m where $2 \leqq m \leqq n$. Then M is pseudo-quaternionically isomorphic to the locally homogeneous space*

$$\mathrm{Sp}(m)\times\Delta\mathrm{Sp}(1)\times\mathrm{Sp}(n-m)\backslash\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)/\Gamma$$

where $m = 2,\cdots,n$.

Our method to prove the above theorems is to study a *Carnot-Carathéodory structure* on spherical pseudo-quaternionic manifolds. A quaternionic Carnot-Carathéodory structure on a $(4n+3)$-manifold M consists of a nondegenerate codimension 3 subbundle B of the tangent bundle TM endowed with a family of local complex structures $\{I,J,K\}$ forming a quaternionic structure on B. By the definition, B satisfies Hörmander's condition so that the pair (M,B) will be a Carnot-Carathéodory manifold (cf. [33], [35]). The triad $(M,B,\{I,J,K\})$ is said to be a *quaternionic Carnot-Carathéodory manifold.* (Compare §2.) Corresponding to the fundamental fact in *spherical Cauchy-Riemann geometry* [5], we shall exhibit the following quaternionic structure on the standard sphere S^{4n+3} in §2.

Proposition C. *If* $\mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3})$ *is the group of all quaternionic Carnot-Carathéodory transformations of S^{4n+3} for the structure* $(\mathrm{Null}\ \theta,\{I,J,K\})$*, then*

$$\mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3}) = \mathrm{PSp}(n+1,1).$$

That is, the spherical pseudo-quaternionic geometry $(\mathrm{PSp}(n+1,1),S^{4n+3})$ *coincides with the geometry obtained from the quaternionic Carnot-Carathéodory structure.*

By this proposition, a spherical pseudo-quaternionic $(4n+3)$-manifold M will be a quaternionic Carnot-Carathéodory manifold. For example, the quaternionic Carnot-Carathéodory structure $(B,\{I,J,K\})$ restricted to the domain $S^{4n+3} - S^{4n-1}$ is mapped isometrically onto the quaternionic hyperbolic geometry $(\mathrm{PSp}(n,1),\mathbb{H}^n_{\mathbb{F}})$ at each point of $S^{4n+3} - S^{4n-1}$ and has the automorphism group $\mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3} - S^{4n-1}) = \mathrm{Sp}(n,1)\cdot\mathrm{Sp}(1)$. Also there is an induced quaternionic Carnot-Carathéodory structure $(B,\{I,J,K\})$ on

$\mathcal{M}$ for which we see that the Carnot-Carathéodory metric on B plays the same role as the euclidean metric on $\mathbb{F}^n$. (See §3.) The detail of this paper will appear in [22].

1. Spherical pseudo-quaternionic geometry

Let $\mathbb{F}^{n+2}$ denote the quaternionic vector space, equipped with the Hermitian form

$$\mathcal{B}(z,w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

Consider the following subspaces in $\mathbb{F}^{n+2} - \{0\}$:

$$V_0^{4n+7} = \{z \in \mathbb{F}^{n+2} \mid \mathcal{B}(z,z) = 0\}, \quad V_-^{4n+8} = \{z \in \mathbb{F}^{n+2} \mid \mathcal{B}(z,z) < 0\}.$$

Let $P : \mathbb{F}^{n+2} - \{0\} \longrightarrow \mathbb{FP}^{n+1}$ be the canonical projection onto the quaternionic projective space. By definition [7], the quaternionic hyperbolic space $\mathbb{H}_{\mathbb{F}}^{n+1}$ is defined to be $P(V_-^{4n+8})$. Let $\mathrm{GL}(n+2,\mathbb{F})$ be the group of all invertible $(n+2)\times(n+2)$-matrices with quaternion entries. The group $\mathrm{Sp}(n+1,1)$ is the subgroup of $\mathrm{GL}(n+2,\mathbb{F})$ whose elements preserve the form $\mathcal{B}$. The action of $\mathrm{Sp}(n+1,1)$ on V_-^{4n+8} induces an action on $\mathbb{H}_{\mathbb{F}}^{n+1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ and the quaternionic hyperbolic group $\mathrm{PSp}(n+1,1)$ is defined to be the quotient of $\mathrm{Sp}(n+1,1)$ by the center. It is known that $\mathbb{H}_{\mathbb{F}}^{n+1}$ is a complete simply connected Riemannian manifold of $-1 \leqq$ sectional curvature $\leqq -\frac{1}{4}$, and with the group of isometries $\mathrm{PSp}(n+1,1)$ (cf. [26]).

The projective compactification of $\mathbb{H}_{\mathbb{F}}^{n+1}$ is obtained by taking the closure $\bar{\mathbb{H}}_{\mathbb{F}}^{n+1}$ of $\mathbb{H}_{\mathbb{F}}^{n+1}$ in $\mathbb{FP}^{n+1}$. Then it follows that $\bar{\mathbb{H}}_{\mathbb{F}}^{n+1} = \mathbb{H}_{\mathbb{F}}^{n+1} \cup P(V_0^{4n+7})$. The boundary $P(V_0^{4n+7})$ of $\mathbb{H}_{\mathbb{F}}^{n+1}$ is the standard sphere of dimension $4n+3$. Put $P(V_0^{4n+7}) = S^{4n+3}$. Then the hyperbolic action of $\mathrm{PSp}(n+1,1)$ on $\mathbb{H}_{\mathbb{F}}^{n+1}$ extends to a smooth action on S^{4n+3} acting as projective transformations because the compactification $\mathbb{H}_{\mathbb{F}}^{n+1} \cup S^{4n+3}$ sits inside $\mathbb{FP}^{n+1}$. The action of $\mathrm{PSp}(n+1,1)$ is transitive on S^{4n+3} whose stabilizer at infinity ∞ is isomorphic to $\mathrm{Sim}(\mathcal{M})$. (Compare 1.2.) We then call the pair $(\mathrm{PSp}(n+1,1), S^{4n+3})$ *spherical pseudo-quaternionic geometry.* Notice that the same construction for the real (resp. complex) hyperbolic space is referred to conformally flat geometry $(\mathrm{PO}(n,1), S^n)$, (resp. spherical CR-geometry $(\mathrm{PU}(n+1,1), S^{2n+1})$) (cf. [12], [5]). Let M be a smooth manifold of dimension $4n+3$. Suppose that $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ is a maximal collection of charts of M satisfying that

$M = \bigcup_{\alpha\in\Lambda} U_\alpha$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ $(\subset S^{4n+3})$ is a homeomorphism, and if $U_\alpha \cap U_\beta \neq \emptyset$, then the coordinate change $g_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ extends to an element of $\mathrm{PSp}(n+1,1)$.

Such a collection of charts is said to give a *uniformization* on M. An equivalence class of uniformizations (by refinement) is called a *spherical pseudo-quaternionic structure* on M. A manifold equipped with this structure is said to be a spherical pseudo-quaternionic manifold M. Denote by $\mathrm{Aut}_{\mathrm{PQ}}(M)$ the group of spherical pseudo-quaternionic transformations of M. Using a uniformization on M (cf. [29]), there is a developing pair: $(\rho, \mathrm{dev}) : (\mathrm{Aut}_{\mathrm{PQ}}(\tilde{M}), \tilde{M}) \longrightarrow (\mathrm{PSp}(n+1,1), S^{4n+3})$ where $\tilde{M}$ is the universal covering space of M and $\pi_1(M) \subset \mathrm{Aut}_{\mathrm{PQ}}(\tilde{M})$.

Let $P : (\mathrm{Sp}(n+1,1), V_-^{4n+8} \cup V_0^{4n+7}) \longrightarrow (\mathrm{PSp}(n+1,1), \mathbb{H}_{\mathbb{F}}^{n+1} \cup S^{4n+3})$ be the equivariant projection. If $\{\infty\}$ is the point at infinity of S^{4n+3}, then the stabilizer $\mathrm{PSp}(n+1,1)_\infty$ is a noncompact maximal amenable Lie subgroup of $\mathrm{PSp}(n+1,1)$. Let $\{e_1, \cdots, e_{n+2}\}$ be the standard basis of $\mathbb{F}^{n+2}$ with respect to the Hermitian form $\mathcal{B}$, *i.e.*, $\mathcal{B}(e_1, e_1) = -1$, $\mathcal{B}(e_i, e_j) = \delta_{ij}$ $(i, j = 2, \cdots n+2)$, $\mathcal{B}(e_1, e_j) = 0$ $(j = 2, \cdots n+2)$. Since V_0^{4n+7} is a cone, we can assume that the inverse image $P^{-1}(\infty)$ consists of a quaternionic line passing through the vector $f_1 = (e_1 + e_{n+2})/\sqrt{2}$ (that is, $P(f_1) = \infty$). If H is a subgroup of $\mathrm{Sp}(n+1,1)$ which leaves f_1 invariant, then PH is isomorphic to $\mathrm{PSp}(n+1,1)_\infty$. Put $f_{n+2} = (e_1 - e_{n+2})/\sqrt{2}$. Now each element g of H has the following form with respect to the basis $\{f_1, e_2, \cdots, e_{n+1}, f_{n+2}\}$:

$$g = \begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix}$$

where $\lambda, \mu \in \mathbb{F}^*$, B is an (n,n)-matrix, x is an n-th line vector, and y is an n-th column vector. As $\mathcal{B}(gz, gw) = \mathcal{B}(z, w)$ for arbitrary $z, w \in \mathbb{F}^{n+2}$, we have the following relations (cf. [7]).

$$(*) \qquad \bar{\lambda}\mu = 1, \quad x = \lambda^t\bar{y}B, \quad \bar{z}\mu + \bar{\mu}z = |y|^2, \quad B \in \mathrm{Sp}(n).$$

Let $\mathcal{M}$ be the subgroup consisting of the following matrices;

$$\begin{pmatrix} 1 & x & z \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}$$

satisfying that $\mathrm{Re}\, z = \dfrac{|y|^2}{2}$, $x = {}^t\bar{y}$. Note that this follows from $(*)$. Putting

$$z = \frac{|y|^2}{2} + \boldsymbol{i}\alpha + \boldsymbol{j}\beta + \boldsymbol{k}\gamma,$$

there is a one-to-one correspondence between the product $\mathbb{R}^3 \times \mathbb{F}^n$ and $\mathcal{M}$:

$$((\alpha, \beta, \gamma), y) = \begin{pmatrix} 1 & {}^t\bar{y} & \frac{|y|^2}{2} + \boldsymbol{i}\alpha + \boldsymbol{j}\beta + \boldsymbol{k}\gamma \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the one-to-one correspondence gives a group law on the product $\mathbb{R}^3 \times \mathbb{F}^n$. As a consequence, $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{F}^n$ with group law:

$$(a, y) \cdot (b, z) = (a + b + \mathrm{Im} < y, z >, y + z).$$

Here $< >$ is the Hermitian inner product and $\mathrm{Im} < >$ is the imaginary part. $\mathcal{M}$ is nilpotent because $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ which is the center consisting of the form $((\alpha, \beta, \gamma), 0)$. $\mathcal{M}$ is called the *Heisenberg nilpotent Lie group.* (Compare [12].) Moreover H is isomorphic to the semidirect product $\mathcal{M} \rtimes (\mathrm{Sp}(n) \times \mathbb{F}^*)$. We define the subgroup $\mathrm{Sim}(\mathcal{M})$ of $\mathrm{PSp}(n+1,1)$ to be $PH = \mathrm{PSp}(n+1,1)_\infty$. Then $\mathrm{Sim}(\mathcal{M})$ is isomorphic to the semidirect product

$\mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. The action of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ on $\mathcal{M}$ is given as follows: if $(A \cdot g, t) \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ and $(a, y) \in \mathcal{M}$, then

$$(A \cdot g, t) \circ (a, y) = (t^2 \cdot gag^{-1},\ t \cdot Ayg^{-1}).$$

Choosing $x_0 = [0, \cdots, 0, 1] = P(f_{n+2}) \in S^{4n+3} - \{\infty\}$, $\mathcal{M}$ acts simply transitively on $S^{4n+3} - \{\infty\}$ by $\rho(g) = gx_0$ for $g \in \mathcal{M}$. $S^{4n+3} - \{\infty\}$ is identified with $\mathcal{M}$ as a spherical pseudo-quaternionic structure. The pair $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ is called *quaternionic Heisenberg geometry.*

Choosing a torsionfree discrete cocompact subgroup Γ from $\mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$, we have a principal fibration of an infranilmanifold as a compact spherical pseudo-quaternionic manifold;

$$(i) \qquad T^3 \to \mathcal{M}/\Gamma \longrightarrow \mathbb{F}^n/\hat{\Gamma}$$

where T^3 is the 3-torus and $\mathbb{F}^n/\hat{\Gamma}$ ($\hat{\Gamma} \subset \mathrm{E}(n) = \mathbb{F}^n \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$ is the quaternionic euclidean flat orbifold.

Let $\mathcal{M} - \{0\} (= S^{4n+3} - \{0, \infty\}) \approx \mathbb{R}^+ \times S^{4n+2}$. Then it follows that (cf. [20])

$$\mathrm{Aut}_{\mathrm{PQ}}(\mathbb{R}^+ \times S^{4n+2}) = (\mathrm{O}(1,1)^0 \rtimes \mathbb{Z}/2) \times (\mathrm{O}(3) \times \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)).$$

Choosing a torsion free discrete cocompact subgroup Δ of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ we obtain an infraHopf manifold

$$(ii) \qquad \mathbb{R}^+ \times S^{4n+2}/\Delta \approx S^1 \times S^{4n+2}/G,$$

where G is a finite subgroup of $\mathrm{Aut}_{\mathrm{PQ}}(S^1 \times S^{4n+2}) = (S^1 \rtimes \mathbb{Z}/2) \times (\mathrm{O}(3) \times \mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$. In particular, the Hopf manifold $S^1 \times S^{4n+2}$ is a spherical pseudo-quaternionic manifold. Since there is an orientation reversing involution τ in

$$\mathrm{Aut}_{\mathrm{PQ}}(S^{4n+3} - \{0, \infty\}) = \mathrm{Aut}_{\mathrm{PQ}}(\mathbb{R}^+ \times S^{4n+2}),$$

we can perform an operation of the connected sum which is closed under the spherical pseudo-quaternionic structures. Similarly to the conformal, CR-cases (cf. [29]), we obtain that

Proposition 1.1. *Let M_1, M_2 be spherical pseudo-quaternionic manifolds. Then the connected sum $M_1 \# M_2$ supports a spherical pseudo-quaternionic structure.*

Let $V_{-1}^{4n+3} = \{z \in \mathbb{F}^{n+1} - \{0\} \mid \mathcal{B}(z, z) = -1\}$ be another quadric in $\mathbb{F}^{n+1}$. In the quaternion case, the group $\mathrm{GL}(n+1, \mathbb{F})$ is acting on $\mathbb{F}^{n+1}$ from the left and $\mathbb{F}^* = \mathrm{GL}(1, \mathbb{F})$ acting as the scalar multiplications from the right. We have the following equivariant Hopf bundle over the quaternionic projective space $\mathbb{FP}^n$.

$$(\mathbb{R}^* \cdot \mathbb{F}^*, \mathbb{F}^*) \to (\mathrm{GL}(n+1, \mathbb{F}) \cdot \mathbb{F}^*, \mathbb{F}^{n+1} - \{0\}) \to (\mathrm{PGL}(n+1, \mathbb{F}), \mathbb{FP}^n).$$

It is known that V_{-1}^{4n+3} is a simply connected geodesically complete semi-Riemannian manifold of type $(3, 4n)$ with constant curvature -1 (cf. [30]), and by the definition, the image $P(V_{-1}^{4n+3}) = \mathbb{H}_{\mathbb{F}}^n$ is a quaternionic hyperbolic space of dimension $4n$. We give examples of compact spherical pseudo-quaternionic manifolds which are locally homogeneous spaces. Let $S^{4n+3} - S^{4m-1}$ be the sphere complement $(1 \leqq m \leqq n)$, which is isomorphic to the quotient space $P(V_{-1}^{4m+3} \times S^{4(n-m)+3})$ by chasing the equivariant principal bundle:

$$\begin{array}{ccc} (\mathbb{Z}/2, \mathrm{Sp}(1)) & & \\ \downarrow & & \\ (G, V_{-1}^{4m+3} \times S^{4(n-m)+3}) & & \\ \downarrow P & \searrow & \\ (PG, P(V_{-1}^{4m+3} \times S^{4(n-m)+3})) & = & S^{4n+3} - S^{4m-1} \subset \mathbb{FP}^{n+1}, \end{array}$$

where

$$G = \mathrm{Sp}(m,1) \times \mathrm{Sp}(n-m+1) \xrightarrow{P} \mathrm{Sp}(m,1) \cdot \mathrm{Sp}(n-m+1).$$

If $\mathrm{Aut}_{\mathrm{PQ}}(S^{4n+3} - S^{4m-1})$ is the subgroup of $\mathrm{PSp}(n+1,1)$ preserving S^{4m-1}, then it is isomorphic to $PG = \mathrm{Sp}(m,1) \cdot \mathrm{Sp}(n-m+1)$. Let

$$\Delta\mathrm{Sp}(1) = \{\begin{pmatrix} \lambda & 0 \\ 0 & \mathrm{I}_m \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mathrm{I}_{n-m} \end{pmatrix}\}/\{\pm 1\} \subset \mathrm{Sp}(m,1) \cdot \mathrm{Sp}(n-m+1).$$

Then $\mathrm{Sp}(m) \times \Delta\mathrm{Sp}(1) \times \mathrm{Sp}(n-m)\backslash\mathrm{Sp}(m,1) \cdot \mathrm{Sp}(n-m+1) = S^{4n+3} - S^{4m-1}$, which is a Riemannian homogeneous space because the stabilizer $\mathrm{Sp}(m) \times \Delta\mathrm{Sp}(1) \times \mathrm{Sp}(n-m)$ is compact. Choosing a torsionfree discrete uniform subgroup $\pi \subset \mathrm{Sp}(m,1) \cdot \mathrm{Sp}(n-m+1)$, we obtain a compact locally homogeneous spherical pseudo-quaternionic manifold $S^{4n+3} - S^{4m-1}/\pi$. As π is mapped isomorphically onto a torsionfree discrete uniform subgroup $\Gamma \subset \mathrm{PSp}(m,1)$, there is a fiber bundle over the quaternionic hyperbolic manifold $\mathbb{H}_{\mathbb{F}}^m/\Gamma$.

$$(iii) \qquad S^{4(n-m)+3} \to S^{4n+3} - S^{4m-1}/\pi \longrightarrow \mathbb{H}_{\mathbb{F}}^m/\Gamma.$$

In particular when $m = n$, $S^{4n+3} - S^{4n-1} = P(V_{-1}^{4n+3} \times S^3) = V_{-1}^{4n+3}$. There is a principal bundle over the compact quaternionic hyperbolic manifold: $\mathrm{Sp}(1) \to V_{-1}^{4n+3}/\pi \longrightarrow \mathbb{H}_{\mathbb{F}}^n/\Gamma$.

1.2. **Proper action of subgroups of** $\mathrm{PSp}(n+1,1)$**.** For our later use, we prove the existence of proper actions of connected Lie groups on the sphere complement which is not homogeneous. Let $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ be the Heisenberg geometry for which $\mathrm{Sim}(\mathcal{M}) = \mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. There is the equivariant principal bundle:

$$\mathbb{R}^3 \to (\mathrm{Sim}(\mathcal{M}), \mathcal{M}) \xrightarrow{\nu} (\mathrm{Sim}(\mathbb{F}^n), \mathbb{F}^n).$$

The subgroup $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ acts on $\mathbb{F}^n$ by $A \cdot g(z) = A \cdot z \cdot g^{-1}$. Given an $\mathbb{R}$-vector subspace V of $\mathbb{F}^n$, denote by $\mathrm{Sp}(V)$ the subgroup of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ leaving V invariant. Put $G = \nu^{-1}(V \rtimes (\mathrm{Sp}(V) \times \mathbb{R}^+))$. Then G is a closed subgroup of $\mathrm{Sim}(\mathcal{M})$ which preserves the subspace $\nu^{-1}(V) = \mathbb{R}^3 \times V$. Note that $\mathcal{N} = \nu^{-1}(V)$ is also a nilpotent subgroup of

$\mathcal{M}$. Suppose that $\dim V = k$. As G stabilizes the point at infinity $\{\infty\}$, G preserves the $(k+3)$-sphere $S^{k+3} = (\mathbb{R}^3 \times V) \cup \{\infty\}$. Moreover, G leaves invariant

$$\mathcal{M} - \mathbb{R}^3 \times V = \mathcal{M} \cup \{\infty\} - \mathbb{R}^3 \times V \cup \{\infty\} = S^{4n+3} - S^{k+3}.$$

Let X be the universal covering space of $S^{4n+3} - S^{k+3}$.

Lemma 1.2. *Suppose that* $V \neq \mathbb{F}^n$.

(1) *G acts properly on $S^{4n+3} - S^{k+3}$.*

(2) *There is a G-invariant complete Riemannian metric on $S^{4n+3} - S^{k+3}$.*

(3) *Given a discontinuous subgroup Γ of G, let $(\tilde{\Gamma}, X)$ be any lift of the action $(\Gamma, S^{4n+3} - S^{k+3})$ to X. Then no such group $\tilde{\Gamma}$ acts properly discontinuously with compact quotient $X/\tilde{\Gamma}$.*

Proof. **(1)** Let K be a compact subset of $S^{4n+3} - S^{k+3}$. We prove that the subset of G, $\zeta_G(K) = \{g \in G \mid gK \cap K \neq \emptyset\}$, is compact. There is an equivariant fibration:

$$\mathbb{R}^3 \to (G, S^{4n+3} - S^{k+3}) \xrightarrow{\nu} (V \rtimes (\mathrm{Sp}(V) \times \mathbb{R}^+), S^{4n} - S^k),$$

where $S^k = V \cup \{\infty\}$. Put $\hat{G} = V \rtimes (\mathrm{Sp}(V) \times \mathbb{R}^+)$. Let $\mathrm{Conf}(S^{4n}) = \mathrm{PO}(4n+1,1)$ be the group of conformal transformations of S^{4n}. If $\mathrm{Conf}(S^{4n}, S^k)$ is the subgroup of $\mathrm{Conf}(S^{4n})$ preserving S^k, then we have

$$(\mathrm{Conf}(S^{4n}, S^k), S^{4n} - S^k) = (\mathrm{PO}(k+1,1) \times \mathrm{O}(4n-k), \mathbb{H}_{\mathbb{R}}^{k+1} \times S^{4n-k-1})$$

where the product $\mathrm{PO}(k+1,1) \times \mathrm{O}(4n-k)$ acts as isometries. (Compare [19].) Since $\hat{G}$ is a closed similarity subgroup of $\mathrm{PO}(k+1,1) \times \mathrm{O}(4n-k)$, $\hat{G}$ acts properly on $S^{4n} - S^k$. Let $\{g_i\}$ be a sequence in $\zeta_G(K)$. Given a sequence $\{x_i\} \in K$ with $\lim x_i = x$, suppose that $\lim g_i x_i = y$ for some $y \in K$. Since $\lim \nu(g_i)\nu(x_i) = \nu(y)$, there is an element $h \in \hat{G}$ such that $\lim \nu(g_i) = h$. As $\mathbb{R}^3 \to G \xrightarrow{\nu} \hat{G}$ is a principal fibration with contractible fiber, we choose a section $s : \hat{G} \longrightarrow G$. In particular, $\lim s(\nu(g_i)) = s(h)$. Let $s(h) = g'$ so that $\lim s(\nu(g_i))x_i = g'x$. Since there is a sequence $\{t_i\} \in \mathbb{R}^3$ for which $t_i \cdot s(\nu(g_i)) = g_i$, we have $\lim t_i \cdot s(\nu(g_i))x_i = y$. As $\mathbb{R}^3$ acts properly, there is an element $t \in \mathbb{R}^3$ such that $\lim t_i = t$. Hence $\lim g_i = \lim t_i \cdot s(\nu(g_i)) = t \cdot g'$.

(2) There exists a G-invariant Riemannian metric g on $S^{4n+3} - S^{k+3}$ by (1). (Compare [28].) We prove that g is complete. Let d be the distance function on $S^{4n+3} - S^{k+3}$ and $\{x_n\}_{n \in \mathrm{N}}$ a Cauchy sequence in $S^{4n+3} - S^{k+3}$. Take a simply connected closed subgroup $\nu^{-1}(V \rtimes \mathbb{R}^+) = (\mathbb{R}^3 \times V) \rtimes \mathbb{R}^+$ acting freely on $S^{4n+3} - S^{k+3}$. Moreover, the solvable subgroup $V \rtimes \mathbb{R}^+$ ($\subset \mathrm{Sim}(V)^0$) acts simply transitively on $\mathbb{H}_{\mathbb{R}}^{k+1}$, the above fibration induces the following fibration:

$$(\mathbb{R}^3 \times V) \rtimes \mathbb{R}^+ \to S^{4n+3} - S^{k+3} \xrightarrow{\mu} S^{4n-k-1}.$$

Then the sequence $\{\mu(x_n)\}$ has an accumulation point $z \in S^{4n-k-1}$. Choose $\tilde{z} \in S^{4n+3} - S^{k+3}$ such that $\mu(\tilde{z}) = z$. Let K be a compact neighborhood of $\tilde{z}$ in $S^{4n+3} - S^{k+3}$. There is a number $\varepsilon > 0$ such that the ε-ball $B_\varepsilon(\tilde{z})$ centered at $\tilde{z}$ is contained in K. As

$\lim \mu(x_n) = z \in \mu(K)$, there exists a sufficiently large L such that $\mu(x_n) \in \mu(B_{\frac{\varepsilon}{3}}(\tilde{z}))$ for $n \geqq L$. Choose $\{\tilde{z}_n\} \in B_{\frac{\varepsilon}{3}}(\tilde{z})$ with $\mu(\tilde{z}_n) = \mu(x_n)$ for $n \geqq L$. Then there is a sequence $\{s_n\} \in (\mathbb{R}^3 \times V) \rtimes \mathbb{R}^+$ such that $x_n = s_n \cdot \tilde{z}_n$ for $n \geqq L$. As $\{x_n\}$ is Cauchy, there exists an $M > L$ such that $\mathrm{d}(x_m, x_n) < \dfrac{\varepsilon}{3}$ for $m, n \geqq M$. In particular, $\mathrm{d}(x_M, x_n) = \mathrm{d}(s_M \cdot \tilde{z}_M, s_n \cdot \tilde{z}_n) < \dfrac{\varepsilon}{3}$. As $\mathrm{d}(\tilde{z}_n, \tilde{z}) < \dfrac{\varepsilon}{3}$ for $n \geqq M$,

$$\begin{aligned}
\mathrm{d}(s_n^{-1}s_M \cdot \tilde{z}, \tilde{z}) &= \mathrm{d}(s_M \cdot \tilde{z}, s_n\tilde{z}) \\
&\leqq \mathrm{d}(s_M \cdot \tilde{z}, s_M \cdot \tilde{z}_M) + \mathrm{d}(s_M \cdot \tilde{z}_M, s_n\tilde{z}_n) + \mathrm{d}(s_n\tilde{z}_n, s_n\tilde{z}) \\
&= \mathrm{d}(\tilde{z}, \tilde{z}_M) + \mathrm{d}(x_M, x_n) + \mathrm{d}(\tilde{z}_n, \tilde{z}) < \varepsilon.
\end{aligned}$$

Therefore $s_n^{-1}s_M \cdot \tilde{z} \in K$ for $n \geqq M$. By properness of $(\mathbb{R}^3 \times V) \rtimes R^+$, we have $\lim s_n^{-1}s_M = s'$ or $\lim s_n = s_M s'^{-1}$. As $\{\tilde{z}_n\} \in K$ and K is compact, there is a point $w \in K$ with $\lim \tilde{z}_n = w$ up to a subsequence. Then,

$$\lim x_n = \lim s_n \cdot \tilde{z}_n = s_M s'^{-1} \cdot w.$$

Hence $S^{4n+3} - S^{k+3}$ is complete.

(3) If $k \neq 4n-2$, then $X = S^{4n+3} - S^{k+3}$. The action $(\tilde{\Gamma}, X)$ coincides with the action $(\Gamma, S^{4n+3} - S^{k+3})$. Since $\hat{G}$ acts properly on $S^{4n} - S^k = \mathbb{H}_{\mathbb{R}}^{k+1} \times S^{4n-k-1}$ and transitively on $\mathbb{H}_{\mathbb{R}}^{k+1}$, the quotient $S^{4n} - S^k/\hat{G}$ is compact Hausdorff. Noting the fibration that $G/\Gamma \longrightarrow X/\Gamma \longrightarrow X/G = S^{4n} - S^k/\hat{G}$, if X/Γ is compact, then Γ is a discrete uniform subgroup of G. On the other hand, G has the exact sequence

$$1 \to \mathbb{R}^3 \times V \to G \xrightarrow{\tau} \mathrm{Sp}(V) \times \mathbb{R}^+$$

in which $\mathcal{N} = \mathbb{R}^3 \times V \subset \mathcal{M}$ is a maximal normal nilpotent Lie subgroup of G. If $\Delta = \Gamma \cap \mathcal{N}$, then Δ is a discrete uniform subgroup of $\mathcal{N}$. (See [38].) Thus $\tau(\Gamma)$ is discrete and cocompact in $\mathrm{Sp}(V) \times \mathbb{R}^+$. Since $\mathbb{R}^+$ acts as contraction or expansion on $\mathcal{N} \subset \mathcal{M}$ (acts by different scale factors as in (1.1)), so does $\tau(\Gamma)$ on Δ. Hence Δ cannot be discrete in $\mathcal{N}$, being a contradiction.

Suppose that $k = 4n-2$. Then $V = \mathbb{F}^{n-1} \times \mathbb{R}^2 \subset \mathbb{F}^n$. So, the group $\mathrm{Sp}(V)$ is isomorphic to $\mathrm{Sp}(n-1) \cdot \mathrm{SO}(2)$ where $\mathrm{SO}(2)$ is a circle of $\mathrm{Sp}(1)$. Then the equivariant fibration of (1) induces the following:

$$\mathbb{R}^3 \to (\tilde{G}, X) \xrightarrow{\tilde{\nu}} (V \rtimes (\mathrm{Sp}(n-1) \times \mathbb{R} \times \mathbb{R}^+), \mathbb{H}_{\mathbb{R}}^{4n-1} \times \mathbb{R}),$$

where $\mathbb{Z} \to \mathbb{R} \to \mathrm{SO}(2)$ is a covering group. As again the quotient space

$$X/\tilde{G} = \mathbb{H}_{\mathbb{R}}^{4n-1} \times \mathbb{R}/V \rtimes (\mathrm{Sp}(n-1) \times \mathbb{R} \times \mathbb{R}^+) = S^{4n} - S^{4n-2}/\hat{G}$$

is compact Hausdorff. If $X/\tilde{\Gamma}$ is compact, then the fibration implies that $\tilde{\Gamma}$ is a discrete uniform subgroup of $\tilde{G}$. Since $\tilde{G}$ has the group extension $1 \to \mathbb{R}^3 \times V \to \tilde{G} \xrightarrow{\tau} \mathrm{Sp}(n-1) \times \mathbb{R} \times \mathbb{R}^+ \to 1$ in which $\mathcal{N} = \mathbb{R}^3 \times V$ is a maximal normal nilpotent Lie subgroup of $\tilde{G}$. Thus the intersection $\mathcal{N} \cap \tilde{\Gamma}$ is a discrete uniform subgroup of $\mathcal{N}$. The same argument yields a contradiction. Therefore there is no discrete cocompact subgroup $\tilde{\Gamma}$ of $\tilde{G}$. $\square$

It is easy to see that the closed connected abelian subgroups of $\mathcal{M}$ are only $\mathbb{R}^3 = (\mathbb{R}^3, 0)$, $\mathbb{R}^n = (0, \mathbb{R}^n)$ or $\mathbb{R}^3 \times \mathbb{R}^n$ up to conjugacy. The subgroup of $\mathrm{Sim}(\mathcal{M})$ leaving invariant $\mathbb{R}^n$ is isomorphic to the subgroup $\mathrm{Sim}(\mathbb{R}^n)^0 = \mathbb{R}^n \rtimes (\mathrm{SO}(n) \times \mathbb{R}^+)$. When we take V as the abelian group $\mathbb{R}^n$, $G = \mathbb{R}^3 \rtimes \mathrm{Sim}(\mathbb{R}^n)^0$ which leaves invariant $\mathbb{R}^3 \times \mathbb{R}^n$ and also S^{n+3}.

Corollary 1.3. *The group G acts properly on $\mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n = S^{4n+3} - S^{n+3}$. In particular, $\mathrm{Sim}(\mathbb{R}^n)$ acts properly on it.*

Recall that a proper totally geodesic subspace in $\mathbb{H}_{\mathbb{F}}^{n+1}$ is isometric to $\mathbb{H}_{\mathbb{K}}^m$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{F}$, $1 \leqq m \leqq n$), or $\mathbb{H}_{\mathbb{R}}^{n+1}$, $\mathbb{H}_{\mathbb{C}}^{n+1}$, or a 3-dimensional (hyperbolic) subspace $\mathbb{H}^1(\mathrm{I})$ (which is orthogonal to $\mathbb{H}_{\mathbb{R}}^1$ in $\mathbb{H}_{\mathbb{F}}^1$). (Compare [7].) In order to construct a non-homogeneous but compact example, we need the following.

Proposition 1.4. (1) *The subgroup of $\mathrm{PSp}(n+1,1)$ preserving $\mathbb{H}_{\mathbb{R}}^{n+1}$ in $\mathbb{H}_{\mathbb{F}}^{n+1}$ is isomorphic to $\mathrm{PO}(n+1,1) \times \mathrm{SO}(3)$, which also preserves S^n in S^{4n+3}.*

(2) *$\mathrm{PO}(n+1,1)$ acts properly on $S^{4n+3} - S^n$.*

(3) *There is a complete Riemannian metric on $S^{4n+3} - S^n$ invariant under*

$$\mathrm{PO}(n+1,1) \times \mathrm{SO}(3).$$

(1) follows from the result of [7]; the subgroup of $\mathrm{Sp}(n+1,1)$ which preserves $\mathbb{H}_{\mathbb{R}}^{n+1}$ is isomorphic to $\mathrm{O}(n+1,1) \cdot \mathrm{Sp}(1)$. As before if $\mathrm{Conf}(S^{4n+3}, S^n)$ is the subgroup of $\mathrm{Conf}(S^{4n+3})$ preserving $S^n = \partial H_{\mathbb{R}}^{n+1}$, then

$$(\mathrm{Conf}(S^{4n+3}, S^n), S^{4n+3} - S^n) = (\mathrm{PO}(n+1,1) \times \mathrm{O}(3n+3), \mathbb{H}_{\mathbb{R}}^{n+1} \times S^{3n+2}) \text{ (cf. [19]).}$$

We have an isometric action of $\mathrm{PO}(n+1,1)$ on $S^{4n+3} - S^n$. However it is noted that the above action of (2) on $S^{4n+3} - S^n$ is different from this isometric action.

Proof. (2) Since $\mathbb{R}^n \cup \{\infty\} = S^n \subset \mathcal{M} \cup \{\infty\} = S^{4n+3}$, note that $S^{4n+3} - S^n = \mathcal{M} - \mathbb{R}^n$. Moreover $\mathrm{PO}(n+1,1) = \mathrm{O}(n+1) \cdot \mathrm{Sim}(\mathbb{R}^n)$. It suffices to check that $\mathrm{Sim}(\mathbb{R}^n)$ acts properly on $\mathcal{M} - \mathbb{R}^n$. Let K be a compact set of $\mathcal{M} - \mathbb{R}^n$ and $\zeta_{\mathrm{Sim}(\mathbb{R}^n)}(K) = \{g \in \mathrm{Sim}(\mathbb{R}^n) \mid gK \cap K \neq \emptyset\}$. Let $\{g_i\}$ be a sequence in $\zeta_{\mathrm{Sim}(\mathbb{R}^n)}(K)$. Given a sequence $\{p_i\} \in K$ with $\lim p_i = p$, suppose that $\lim g_i p_i = q$ for some $q \in K$. There is the fibration: $\mathbb{R}^3 \to \mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n \xrightarrow{\nu} \mathbb{F}^n - \mathbb{R}^n$ as above.

Case I. *Suppose that an infinite number of points $\{p_i\}$ satisfy that $\nu(p_i) \in \mathbb{R}^n$.* Then we have $\nu(p) \in \mathbb{R}^n$. Recall that an element g of $\mathrm{Sim}(\mathbb{R}^n)$ has the form:

$$g = (0, x) \cdot (B, t).$$

For

$$\begin{pmatrix} \alpha \\ z \end{pmatrix} \in \mathcal{M},$$

we have

$$g\begin{pmatrix}\alpha\\ z\end{pmatrix}=(0,x)\cdot\begin{pmatrix}t^2\alpha\\ tBz\end{pmatrix}=\begin{pmatrix}t^2\alpha+\mathrm{Im}<x,tBz>\\ x+tBz\end{pmatrix}.$$

Let

$$p_i=\begin{pmatrix}\alpha_i\\ z_i\end{pmatrix}\longrightarrow p=\begin{pmatrix}\alpha\\ z\end{pmatrix}.$$

In our case, $z_i, z\in\mathbb{R}^n$. Note that $\alpha\neq 0=(0,0,0)$ because $p\in K$. In particular we have that $\lim\alpha_i^{-1}=\alpha^{-1}$. In the sequence

$$g_i\begin{pmatrix}\alpha_i\\ z_i\end{pmatrix}=\begin{pmatrix}t_i^2\alpha_i+\mathrm{Im}<x_i,t_iB_iz_i>\\ x_i+t_iB_iz_i\end{pmatrix}\longrightarrow q=\begin{pmatrix}\alpha'\\ z'\end{pmatrix},$$

since $\mathrm{Im}<x_i,t_iB_iz_i>=0$, this reduces to

$$g_ip_i=\begin{pmatrix}t_i^2\alpha_i\\ x_i+t_iB_iz_i\end{pmatrix}.$$

It follows that $\lim t_i^2\alpha_i=\alpha'$. Then $t_i^2=t_i^2\cdot|\alpha_i|\cdot|\alpha_i^{-1}|\longrightarrow|\alpha'|\cdot|\alpha^{-1}|$. Thus we assume that $\lim t_i=t<\infty$. As $\lim x_i+t_iB_iz_i=z'$, $\lim z_i=z$ and $B_i\in\mathrm{SO}(n)$, we obtain that $\lim x_i=x$ for some $x\in\mathbb{R}^n$ (up to a subsequence). Assume $\lim B_i=B\in\mathrm{SO}(n)$. Then it follows that

$$\lim g_i=\lim\,(0,x_i)\cdot(B_i,t_i)=(0,x)\cdot(B,t)\in\mathrm{Sim}(\mathbb{R}^n).$$

This proves Case I.

Case II. *Suppose that an infinite number of points $\{p_i\}$ satisfy that $\nu(p_i)\in\mathbb{F}^n-\mathbb{R}^n$ but $\nu(p)\in\mathbb{R}^n$.* For the point p_i, put $z_i=y_i+\mathrm{Im}(z_i)$ where $y_i=\mathrm{Re}(z_i)$. As $\nu(p_i)=z_i$ converges to $\nu(p)=z$, it follows that $\lim y_i=z$ and $\lim\mathrm{Im}(z_i)=0$. Put

$$p_i'=\begin{pmatrix}\beta_i\\ \mathrm{Im}(z_i)\end{pmatrix}=\begin{pmatrix}\alpha_i-\mathrm{Im}<y_i,z_i>\\ \mathrm{Im}(z_i)\end{pmatrix}.$$

As $\mathrm{Im}<y_i,z_i>=<y_i,\mathrm{Im}(z_i)>\longrightarrow 0$, $\lim p_i'=(\alpha,0)$. Since $\alpha\neq 0$, we may assume that $\beta_i\neq 0$ for infinitely many i. We have $\lim|\beta_i|^{-1}=|\alpha|^{-1}$. Let $L_{y_i}=(0,y_i)\cdot(\mathrm{I},1)\in\mathrm{Sim}(\mathbb{R}^n)$ be the translation. Then $\lim L_{y_i}=(0,z)\cdot(\mathrm{I},1)=L_z\in\mathrm{Sim}(\mathbb{R}^n)$. Consider the sequence $\{g_i\circ L_{y_i}\}\in\mathrm{Sim}(\mathbb{R}^n)$. Then,

$$g_i\circ L_{y_i}(p_i')=g_i\begin{pmatrix}\alpha_i-\mathrm{Im}<y_i,z_i>+<y_i,\mathrm{Im}(z_i)>\\ y_i+\mathrm{Im}(z_i)\end{pmatrix}=g_i\begin{pmatrix}\alpha_i\\ z_i\end{pmatrix}=g_ip_i\longrightarrow q.$$

On the other hand, $g_i\circ L_{y_i}=(0,x_i)\cdot(B_i,t_i)\circ(0,y_i)=(0,x_i+t_iB_iy_i)\cdot(B_i,t_i)$. Put $g_i\circ L_{y_i}=(0,w_i)\cdot(B_i,t_i)$. Then

$$g_i\circ L_{y_i}(p_i')=\begin{pmatrix}t_i^2\cdot\beta_i+<w_i,t_iB_i\mathrm{Im}(z_i)>\\ w_i+t_iB_i\mathrm{Im}(z_i)\end{pmatrix}$$

It follows that

$$\lim \mathrm{Re}(w_i + t_i B_i \mathrm{Im}(z_i)) = \mathrm{Re}(z'), \ i.e., \ w_i \longrightarrow \mathrm{Re}(z').$$

Similarly, $\mathrm{Im}(w_i + t_i B_i \mathrm{Im}(z_i)) = t_i B_i \mathrm{Im}(z_i) \longrightarrow \mathrm{Im}(z')$. Therefore,

$$\lim < w_i, t_i B_i \mathrm{Im}(z_i) > = < \mathrm{Re}(z'), \mathrm{Im}(z') > .$$

Since $\lim(t_i^2 \cdot \beta_i + < w_i, t_i B_i \mathrm{Im}(z_i) >) = \alpha'$,

$$t_i^2 = t_i^2 \cdot |\beta_i| \cdot |\beta_i|^{-1} \leqq (|\alpha'| \ + \ | < \mathrm{Re}(z'), \mathrm{Im}(z') > |) \cdot |\alpha|^{-1} + 1.$$

Thus, $\{t_i\}$ is bounded. Let $\lim t_i = t$ (up to a subsequence). As

$$g_i \circ L_{y_i} = (0, w_i) \cdot (B_i, t_i) \longrightarrow (0, \mathrm{Re}(z')) \cdot (B, t),$$

and $L_{y_i} \longrightarrow L_z$,

$$\lim g_i = \lim g_i \circ L_{y_i} \circ L_{y_i}^{-1} = (0, \mathrm{Re}(z')) \cdot (B, t) \circ L_z^{-1} \in \mathrm{Sim}(\mathbb{R}^n).$$

So $\{g_i\}$ converges.

Case III. *Suppose that an infinite number of points $\{p_i\}$ satisfy that $\nu(p_i) \in \mathbb{F}^n - \mathbb{R}^n$ and $\nu(p) \in \mathbb{F}^n - \mathbb{R}^n$.* Then $\mathrm{Im}(z) = \mathrm{Im}(\nu(p)) \neq 0$. Recall

$$g_i p_i = \begin{pmatrix} t_i^2 \alpha_i + \mathrm{Im} < x_i, t_i B_i z_i > \\ x_i + t_i B_i z_i \end{pmatrix}.$$

If $x_i + t_i B_i z_i \in \mathbb{R}^n$, then $\nu(p_i) = z_i \in \mathbb{R}^n$. In our case $g_i p_i \in \mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n$.

If $\lim g_i p_i = q \in \mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n$, then all points $\{p_i, p, g_i p_i, q\} \in \mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n = S^{4n+3} - S^{n+3}$. By Corollary 4, $\mathrm{Sim}(\mathbb{R}^n)$ acts properly on $\mathcal{M} - \mathbb{R}^3 \times \mathbb{R}^n$. It follows that $\lim g_i = g \in \mathrm{Sim}(\mathbb{R}^n)$.

We show that q does not lie in $\mathbb{R}^3 \times \mathbb{R}^n$. Suppose that $q \in \mathbb{R}^3 \times \mathbb{R}^n$. Then $z' \in \mathbb{R}^n$ or $\mathrm{Im}(z') = 0$. As $q \in K \subset \mathcal{M} - \mathbb{R}^n$, $\alpha' \neq 0$. Since $g_i p_i \longrightarrow q$, note that $t_i^2 \alpha_i + \mathrm{Im} < x_i, t_i B_i z_i > \longrightarrow \alpha'$ and $x_i + t_i B_i z_i \longrightarrow z'$. It follows that $\mathrm{Im}(x_i + t_i B_i z_i) = t_i B_i \mathrm{Im}(z_i) \longrightarrow \mathrm{Im}(z') = 0$. Then, $\lim |t_i B_i \mathrm{Im}(z_i)| = \lim t_i |\mathrm{Im}(z_i)| = 0$. On the other hand, $\nu(p_i) = z_i \longrightarrow \nu(p) = z$, so $\lim \mathrm{Im}(z_i) = \mathrm{Im}(z) \neq 0$ by our case. Thus $\lim |\mathrm{Im}(z_i)|^{-1} = |\mathrm{Im}(z)|^{-1}$. Therefore, $\lim t_i = \lim t_i |\mathrm{Im}(z_i)| \cdot |\mathrm{Im}(z_i)|^{-1} = 0 \cdot |\mathrm{Im}(z)|^{-1} = 0$. Moreover,

$$\begin{aligned} |x_i| &= |x_i + t_i B_i z_i - t_i B_i z_i| \\ &\leqq |x_i + t_i B_i z_i| + t_i |z_i| \\ &\leqq |z'| + 0 \cdot |z| + 1 = |z'| + 1. \end{aligned}$$

So $\{x_i\}$ is bounded, let $\lim x_i = x$. Then

$$t_i^2 \alpha_i + \mathrm{Im} < x_i, t_i B_i z_i > = t_i^2 \alpha_i + < x_i, t_i B_i \mathrm{Im}(z_i) > \longrightarrow 0 \cdot \alpha + < x, 0 > = 0.$$

This contradicts that $t_i^2 \alpha_i + \mathrm{Im} < x_i, t_i B_i z_i > \longrightarrow \alpha' \neq 0$ as above.

(3) There exist a $\mathrm{PO}(n+1,1) \times \mathrm{SO}(3)$-invariant Riemannian metric g on $S^{4n+3} - S^n$ by (2) and a principal fibration $\mathbb{R}^n \rtimes \mathbb{R}^+ \rightarrow S^{4n+3} - S^n \xrightarrow{\mu} S^{4n+3} - S^n / \mathbb{R}^n \rtimes \mathbb{R}^+$. Since $S^{4n+3} - S^n \approx \mathbb{H}_{\mathbb{R}}^{n+1} \times S^{3n+2}$ topologically and $\mathbb{R}^n \rtimes \mathbb{R}^+ \subset \mathrm{Sim}(\mathbb{R}^n)$ is a transitive subgroup

of $\mathbb{H}_{\mathbb{R}}^{n+1}$, $S^{4n+3}-S^n/\mathbb{R}^n \rtimes \mathbb{R}^+$ is compact (which is homeomorphic to S^{3n+2}). Then as in the argument of (2) of Lemma 1.2, we can prove that g is complete. □

The argument of Proposition 1.4 can be applied to the complex case, which yields the following.

Corollary 1.5. (1) *The subgroup of* $\mathrm{PSp}(n+1,1)$ *preserving* $\mathbb{H}_{\mathbb{C}}^{n+1}$ *in* $\mathbb{H}_{\mathbb{F}}^{n+1}$ *is isomorphic to*

$$P(\mathrm{U}(n+1,1)\cdot S^1\{\pm 1,\pm \boldsymbol{j}\}) \approx \mathrm{U}(n+1,1)\times\{\pm 1\},$$

which preserves also S^{2n+1} *in* S^{4n+3}.

(2) *Put* $H = P(\mathrm{U}(n+1,1)\cdot S^1\{\pm 1,\pm \boldsymbol{j}\})$. *Then* H *acts properly on* $S^{4n+3}-S^{2n+1}$. *Moreover,* $S^{4n+3}-S^{2n+1}$ *admits a* H*-invariant complete Riemannian metric.*

(3) *There is a compact spherical pseudo-quaternionic manifold* $S^{4n+3}-S^n/\Gamma$ *for which* $S^{4n+3}-S^n$ *is not homogeneous but* Γ *is a discrete uniform subgroup of* $\mathrm{PO}(n+1,1)$. *Similarly for a discrete uniform subgroup of* $\mathrm{U}(n+1,1)$, *the quotient space* $S^{4n+3}-S^{2n+1}/\Gamma$ *is a compact locally non-homogeneous spherical pseudo-quaternionic manifold.*

2. Quaternionic Carnot-Carathéodory structure

2.1. Sasakian 3-structure on S^{4n+3}. Recall the construction of *Sasakian 3-structure* on S^{4n+3}. (Compare [41].) Denote by $<\ ,\ >$ the Hermitian inner product over $\mathbb{F}^{n+1}$, which is invariant under the standard quaternionic structure $\{I,J,K\}$. Let $<\ ,\ >_p$ be the inner product on $\mathrm{T}_p\mathbb{F}^{n+1}$ obtained from the parallel translation of the inner product $<\ ,\ >_0=<\ ,\ >$ at the origin of $\mathbb{F}^{n+1}$. Letting $g_p(X,Y)=\mathrm{Re}<X,Y>_p$ for $X,\ Y\in \mathrm{T}_p\mathbb{F}^{n+1}$, g is the standard euclidean metric on $\mathbb{F}^{n+1}$ which is invariant under $\{I,\ J,\ K\}$. Let S^{4n+3} be the unit sphere in $\mathbb{F}^{n+1}$. The restriction of g to S^{4n+3} gives the spherical Riemannian metric on S^{4n+3}. There exists a normal vector field N on S^{4n+3} such that $TS^{4n+3}\oplus N = \mathrm{T}\mathbb{F}^{n+1}|S^{4n+3}$. Put

$$\xi_1 = IN,\quad \xi_2 = JN,\quad \xi_3 = KN.$$

Then the subspace generated by $\{\xi_i\}_{i=1,2,3}$ with N forms the tangent plane $\mathrm{T}\mathbb{F}^1$ in $\mathrm{T}\mathbb{F}^{n+1}$. Since $g(\xi_1,N)=\mathrm{Re}<IN,N>=0$, similarly for $J,\ K$, the subspace $\{\xi_i\}_{i=1,2,3}$ belongs to TS^{4n+3}. The full set $(S^{4n+3},g,\{\xi_1,\xi_2,\xi_3\},\{I,J,K\})$ is said to be the *canonical Sasakian 3-structure* on S^{4n+3}. It is easy to check that the isometry group $\mathrm{Iso}(S^{4n+3},g)^0$ is isomorphic to $\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$.

Identify $\mathrm{Im}\,\mathbb{F}=\mathbb{R}\boldsymbol{i}+\mathbb{R}\boldsymbol{j}+\mathbb{R}\boldsymbol{k}$ with the Lie algebra $\mathfrak{sp}(1)$ of $\mathrm{Sp}(1)$. Put $\omega_i(X)=g(\xi_i,X)$. Then ω_i is a (real valued) 1-form on S^{4n+3}. Define an $\mathfrak{sp}(1)$-valued 1-form ω on S^{4n+3} to be

$$\omega(X)=\omega_1(X)\boldsymbol{i}+\omega_2(X)\boldsymbol{j}+\omega_3(X)\boldsymbol{k}.$$

A direct calculation shows that for $X, Y \in TS^{4n+3}$

$$(*) \qquad d\omega(X,Y) = g(X,IY)\boldsymbol{i} + g(X,JY)\boldsymbol{j} + g(X,KY)\boldsymbol{k}.$$

If $R_a : S^{4n+3} \to S^{4n+3}$ is the right translation defined by $R_a(x) = x \cdot a^{-1}$ for $a \in \mathrm{Sp}(1)$, then ω satisfies that $R_a^*\omega = \bar{a} \cdot \theta \cdot a$ (cf. [42]). Thus ω turns out to be a connection form of the Hopf bundle:

$$\mathrm{Sp}(1) \to S^{4n+3} \xrightarrow{\nu} \mathbb{FP}^n.$$

Put

$$B = \{X \in TS^{4n+3} \mid \omega_i(X) = 0 \text{ for } i = 1,\ 2,\ 3\}.$$

Since $g|B \times B$ is invariant under $\{I, J, K\}$, B is a $4n$-dimensional invariant subbundle of TS^{4n+3} such that

$$B \oplus \{\xi_1, \xi_2, \xi_3\} = TS^{4n+3}.$$

By $(*)$, we have $[B, B] = \{\xi_1, \xi_2, \xi_3\}$ so that B is a Carnot-Carathéodory structure on S^{4n+3} (cf. [33]).

Lemma 2.1. [1](nondegenerate): *Let $d\omega \wedge d\omega = d\omega^2$. The form ω satisfies that*

$$\omega \wedge \omega \wedge \omega \wedge \overbrace{d\omega^2 \wedge \cdots \wedge d\omega^2}^{n\ times} \neq 0 \textit{ at every point of } S^{4n+3}.$$

[2](integrable): *There exist quaternion-valued one-forms ω^α $(\alpha = 1, \ldots, n)$ on S^{4n+3} such that*

$$d\omega = -\frac{1}{2}\delta_{\bar{\alpha}\beta}\omega^{\bar{\alpha}} \wedge \omega^\beta \quad (\mathrm{mod}\ \omega).$$

Proof. [1] If $\omega = \omega_1\boldsymbol{i} + \omega_2\boldsymbol{j} + \omega_3\boldsymbol{k}$, then the three-form $\omega \wedge \omega \wedge \omega$ and the four-form $d\omega \wedge d\omega$ are real valued;

$$(1) \qquad \omega^3 = -6\omega_1 \wedge \omega_2 \wedge \omega_3, \quad d\omega^2 = -(d\omega_1 \wedge d\omega_1 + d\omega_2 \wedge d\omega_2 + d\omega_3 \wedge d\omega_3).$$

Choose an orthonormal vector field X_α $(\alpha = 1, \ldots, n)$ from B with respect to g so that $\{X_\alpha, IX_\alpha, JX_\alpha, KX_\alpha\}_{\alpha=1,\ldots,n}$ forms a basis of B. Then the nonzero terms are

$$\begin{aligned} d\omega_1(X_\alpha, IX_\alpha) &= d\omega_1(JX_\alpha, KX_\alpha) = d\omega_2(X_\alpha, JX_\alpha) = d\omega_2(KX_\alpha, IX_\alpha) \\ &= d\omega_3(X_\alpha, KX_\alpha) = d\omega_3(JX_\alpha, KX_\alpha) = -1, \end{aligned}$$

and so $d\omega_1^2(X_\alpha, IX_\alpha, JX_\alpha, KX_\alpha) = -\dfrac{1}{3}$. Then a calculation shows that

$$d\omega_1^{2p}(X_1, IX_1, JX_1, KX_1; \ldots; X_n, IX_n, JX_n, KX_n) = \frac{(2p)! \cdot 2^{2p}}{(4p)!}$$

and $\omega_1 \wedge \omega_2 \wedge \omega_3(\xi_1, \xi_2, \xi_3) = \dfrac{1}{6}$. It is easy to see that $\omega^3 \wedge d\omega^{2n}$ is a positive constant.

[2] Choose a coframe θ^α with $\theta^\alpha(X_\beta)=\delta^\alpha_\beta$ $(\alpha,\beta=1,\dots,n)$. Put

$$\theta^{\alpha+n}=-\theta^\alpha\circ I,\ \theta^{\alpha+2n}=-\theta^\alpha\circ J,\ \theta^{\alpha+3n}=-\theta^\alpha\circ K,$$

and define a quaternion-valued one-form ω_α to be $\omega^\alpha=\theta^\alpha+\theta^{\alpha+n}\boldsymbol{i}+\theta^{\alpha+2n}\boldsymbol{j}+\theta^{\alpha+3n}\boldsymbol{k}$. As $B\oplus\{\xi_1,\xi_2,\xi_3,N\}=B\oplus \mathrm{T}\mathbb{F}^1=\mathrm{T}\mathbb{F}^{n+1}|S^{4n+3}$, we have that $<\ ,\ >|B=\sum_\alpha\omega^{\bar\alpha}\otimes\omega^\alpha$. If we note $g(X,Y)=\mathrm{Re}<X,Y>$, then each $d\omega_1(X,Y)=g(X,IY)$, $d\omega_2(X,Y)=g(X,JY)$ or $d\omega_3(X,Y)=g(X,KY)$ represents the imaginary part of $\sum_\alpha\omega^{\bar\alpha}\otimes\omega^\alpha(X,Y)$ for $X,Y\in B$ respectively. A calculation shows that

$$\begin{aligned}
d\omega_1&=-\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+n}-\theta^{\alpha+2n}\wedge\theta^{\alpha+3n})\\
d\omega_2&=-\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+2n}-\theta^{\alpha+3n}\wedge\theta^{\alpha+n})\\
d\omega_3&=-\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+3n}-\theta^{\alpha+n}\wedge\theta^{\alpha+2n}).
\end{aligned}\tag{2}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2}\delta_{\bar\alpha\beta}\omega^{\bar\alpha}\wedge\omega^\beta&=\frac{1}{2}\sum_\alpha\omega^{\bar\alpha}\wedge\omega^\alpha=\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+n}-\theta^{\alpha+2n}\wedge\theta^{\alpha+3n})\boldsymbol{i}\\
&+\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+2n}-\theta^{\alpha+3n}\wedge\theta^{\alpha+n})\boldsymbol{j}+\sum_\alpha(\theta^\alpha\wedge\theta^{\alpha+3n}-\theta^{\alpha+n}\wedge\theta^{\alpha+2n})\boldsymbol{k}.
\end{aligned}$$

Therefore, we obtain that $d\omega=-\frac{1}{2}\delta_{\bar\alpha\beta}\omega^{\bar\alpha}\wedge\omega^\beta$ (mod ω). □

2.2. **Definition of quaternionic Carnot-Carathéodory structure.** From the viewpoint of (2.1), we introduce the notion of quaternionic Carnot-Carathéodory structure. Let M be a smooth $(4n+3)$-manifold. Suppose that $\theta=(\theta_1\boldsymbol{i},\theta_2\boldsymbol{j},\theta_3\boldsymbol{k})$ is an $\mathfrak{sp}(1)$-valued 1-form on M^{4n+3}. Here $\mathfrak{sp}(1)$ is $\mathrm{Im}\,\mathbb{F}=\mathbb{R}\boldsymbol{i}+\mathbb{R}\boldsymbol{j}+\mathbb{R}\boldsymbol{k}$ as before. Put $\Psi=-(d\theta_1{}^2+d\theta_2{}^2+d\theta_3{}^2)$ so that Ψ is a 4-form. Suppose that

[i] $\theta_1\wedge\theta_2\wedge\theta_3\wedge\Psi^n\neq 0$ at every point of M. In particular,

$$\mathrm{Null}\ \theta=\{X\in TM\mid\theta(X)=0\}$$

is a $4n$-dimensional subbundle of TM.

[ii] Put Null $\theta=B$. Each fiber B_x $(x\in M)$ has the structure of an n-dimensional quaternion vector space. Moreover,

[iii] B admits an integrable $\mathrm{GL}(n,\mathbb{F})$-structure.

More precisely, we shall explain these [i], [ii], [iii]. In view of Lemma 2.1, the condition [i] is equivalent to

$$d\theta=c\cdot\theta^\alpha\wedge g_{\alpha\bar\beta}\theta^{\bar\beta}\quad\mathrm{mod}\ \ \theta,\theta^r.\tag{1}$$

The matrix $g_{\alpha\bar\beta}$ is Hermitian, *i.e.*, $g_{\alpha\bar\beta}=\bar g_{\beta\bar\alpha}=g_{\bar\beta\alpha}$, and $c=\pm1$. The nondegeneracy of the structure requires that $g_{\alpha\bar\beta}$ is invertible. As a consequence, $B=\mathrm{Null}\ \theta$ is a codimension

3 subbundle of TM. The condition [ii] implies the existence of almost complex structures $\{I_x,\ J_x,\ K_x\}$ on each fiber B_x ($x \in M$). Moreover, from the condition [iii] there exist globally defined complex structures $\{I,\ J,\ K\}$ on B. Let $B \otimes \mathbb{F} = T^{1,0} + T^{0,1}$ be the splitting for I. That is, $T^{1,0} = \{X \mid IX = Xi\}$ and $\bar{T}^{1,0} = T^{0,1}$. Similarly, let $B \otimes \mathbb{F} = T'^{1,0} + T'^{0,1}$ be the splitting for J. Put $T^{1,1} = T^{1,0} \cap T'^{1,0}$. Then,

$$T^{1,1} = \{X \mid IX = X\boldsymbol{i},\ JX = X\boldsymbol{j},\ KX = X\boldsymbol{k}\}.$$

There are coframe fields θ^α ($\alpha = 1, \cdots, n$) dual to $T^{1,1}$. The condition [iii] requires that

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha - \varphi \cdot \delta^\alpha{}_\beta \wedge \theta^\beta \mod \theta. \tag{2}$$

Note 2.2. (1) *Notice that the condition [iii] for the case of S^{4n+3} does not violate the quaternionic (Kähler) structure on the quaternionic projective space $\mathbb{FP}^n$. Because the complex structures $\{I, J, K\}$ are not compatible with the action of* Sp(1), *the projection π in the Hopf bundle* $\mathrm{Sp}(1) \to S^{4n+3} \xrightarrow{\pi} \mathbb{FP}^n$ *does not induce (almost) complex structure globally on $\mathbb{FP}^n$.*
(2) *There exist vector fields ξ_1, ξ_2, ξ_3 on M such that $\theta(\xi_1) = \boldsymbol{i}, \theta(\xi_2) = \boldsymbol{j}, \theta(\xi_3) = \boldsymbol{k}$. Then, $B \oplus \{\xi_1, \xi_2, \xi_3\} = TM$ as above.*

To this end, we formulate the following definitions.

Definition 2.3. (1) *The pair $(\theta, \{I, J, K\})$ endowed with the conditions [i], [ii], [iii] is called a quaternionic Carnot-Carathéodory Hermitian structure on M.*
(2) *The pair $(B, \{I, J, K\})$ endowed with the conditions [i], [ii], [iii] is called a quaternionic Carnot-Carathéodory structure on M.*

Two quaternionic Carnot-Carathéodory Hermitian structures $(\theta, \{I, J, K\}), (\theta', \{I', J', K'\})$ represent the same quaternionic Carnot-Carathéodory structure on M if and only if $I = I',\ J = J',\ K = K',\ \theta' = u \cdot \mathrm{a}\theta\mathrm{a}^{-1}$ for some positive function u and some function $\mathrm{a} \in \mathrm{Sp}(1)$.

A manifold equipped with this structure is called a quaternionic Carnot-Carathéodory Hermitian manifold (resp. quaternionic Carnot-Carathéodory manifold).

Definition 2.4. *A diffeomorphism $f : M \to M$ is called a quaternionic Carnot-Carathéodory transformation if f_* preserves B,* i.e.,
(i) $f^*\theta = u \cdot \mathrm{a}\theta\bar{\mathrm{a}}$ *for some positive function u and a function* a *lying in* Sp(1), **(ii)** *If* $\mathrm{a} \mapsto A$ *under the map* $\mathrm{Sp}(1) \longrightarrow \mathrm{Aut}(\mathbb{F}) = SO(3)$, *i.e.,*

$$(\mathrm{a}\boldsymbol{i}\mathrm{a}^{-1}, \mathrm{a}\boldsymbol{j}\mathrm{a}^{-1}, \mathrm{a}\boldsymbol{k}\mathrm{a}^{-1}) = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})A,$$

then f satisfies that

$$f_*(\xi_1,\xi_2,\xi_3) = u^2\cdot(\xi_1,\xi_2,\xi_3)\cdot A(p) \pmod{B}_p$$

at each point $p \in M$. In addition,
(iii) *If $u = 1$ and some $\mathrm{a} \in \mathrm{Sp}(1)$ so that $f^*\theta = \mathrm{a}\cdot\theta\cdot\bar{\mathrm{a}}$, then f is said to be a quaternionic Carnot-Carathéodory Hermitian transformation.. Denote $\mathrm{Aut_{QCC}}^1(M,(\theta,\{I,J,K\}))$ the group of all quaternionic Carnot-Carathéodory Hermitian transformations of M onto itself. Similarly, $\mathrm{Aut_{QCC}}(M,(B,\{I,J,K\}))$ is the group of all quaternionic Carnot-Carathéodory transformations of M onto itself.*

By the definition, we have the quaternionic Carnot-Carathéodory Hermitian structure $(\theta,\{I,\ J,\ K\})$ and the quaternionic Carnot-Carathéodory structure $(\mathrm{Null}\ \theta,\{I,\ J,\ K\})$ on the sphere S^{4n+3}. Let $(\mathrm{PSp}(n+1,1),S^{4n+3})$ be the spherical pseudo-quaternionic geometry. We prove Proposition C of the introduction.

Proposition 2.5. *Let $(\mathrm{Null}\ \theta,\{I,\ J,\ K\})$ be the canonical quaternionic Carnot-Carathéodory structure on S^{4n+3}. Then its geometry coincides with the spherical pseudo-quaternionic geometry on S^{4n+3};*

$$(\mathrm{Aut_{QCC}}(S^{4n+3},(\mathrm{Null}\ \theta,\{I,J,K\})),S^{4n+3}) = (\mathrm{PSp}(n+1,1),S^{4n+3}).$$

First we need the following lemma.

Lemma 2.6. (i) $\mathrm{PSp}(n+1,1)\subset \mathrm{Aut_{QCC}}(S^{4n+3})$.
(ii) *For an arbitrary $a\in\mathbb{F}^*$, there exists an element $g\in \mathrm{Sp}(1)\times\mathbb{R}^+\subset \mathrm{PSp}(n+1,1)_\infty$ such that $g^*\omega = \bar{\chi}\cdot\omega\cdot\chi$ for some function $\chi: S^{4n+3}\to\mathbb{F}^*$ with $\chi(\infty) = a^{-1}$.*

Proof. (i) Let $f\in \mathrm{PSp}(n+1,1)$ be an element. If $\tilde f: V_0^{4n+7}\longrightarrow V_0^{4n+7}$ is a lift of f to $\mathrm{Sp}(n+1,1)$, then $\tilde f$ is represented by a matrix $A\in \mathrm{Sp}(n+1,1)$;

$$\tilde f\begin{pmatrix} z_1\\ \vdots\\ z_{n+2}\end{pmatrix} = A\begin{pmatrix} z_1\\ \vdots\\ z_{n+2}\end{pmatrix}.$$

For brevity, write the form $\tilde\theta$ as follows:

$$\tilde\theta = \bar z_1^{-1}((\bar z_1,\ldots,\bar z_{n+2})\cdot \mathrm{I}_{1,n+1}\cdot\begin{pmatrix} dz_1\\ \vdots\\ dz_{n+2}\end{pmatrix})z_1^{-1}.$$

Here

$$\mathrm{I}_{1,n+1} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & +1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & +1 \end{pmatrix}.$$

Then

$$\tilde{f}^*\tilde{\theta} = \tilde{f}^*(\bar{z}_1^{-1})((\bar{z}_1, \ldots, \bar{z}_{n+2})A^* \cdot \mathrm{I}_{1,n+1} \cdot A \begin{pmatrix} dz_1 \\ \vdots \\ dz_{n+2} \end{pmatrix})\tilde{f}^*(z_1^{-1}).$$

If $\chi' : V_0^{4n+7} \to \mathbb{F}$ is a smooth map defined by $\chi'(z) = z_1 \cdot \tilde{f}^* z_1^{-1}$ for $z = (z_1, \ldots, z_{n+2})$, then by the definition

$$\begin{aligned} \tilde{f}^*\tilde{\theta} &= (\bar{\tilde{f}}^*\bar{z}_1^{-1} \cdot \bar{z}_1)(\bar{z}_1^{-1} \cdot (\bar{z}_1, \ldots, \bar{z}_{n+2}) \cdot \mathrm{I}_{1,n+1} \cdot \begin{pmatrix} dz_1 \\ \vdots \\ dz_{n+2} \end{pmatrix} \cdot z_1^{-1})(z_1 \cdot \tilde{f}^* z_1^{-1}) \\ &= \bar{\chi}' \cdot \tilde{\theta} \cdot \chi'. \end{aligned}$$

On the other hand, for $t \in \mathbb{F}^*$,

$$\begin{aligned} \chi'(z \cdot t) &= z_1(z \cdot t) \cdot z_1^{-1}(\tilde{f}(z \cdot t)) = (z_1 \cdot t) \cdot z_1(\tilde{f}(z) \cdot t)^{-1} \\ &= (z_1 \cdot t) \cdot t^{-1} z_1(\tilde{f}(z))^{-1} = z_1 \cdot (\tilde{f}^* z_1(z))^{-1} = \chi'(z). \end{aligned}$$

Thus, χ' factors through a map $\chi : S^{4n+3} \to \mathbb{F}$ such that $\chi \circ \pi = \chi'$. As $f \circ \pi = \pi \circ \tilde{f}$ and $\pi^*\theta = \tilde{\theta}$, we obtain that

$$f^*\omega = \bar{\chi} \cdot \omega \cdot \chi.$$

By the definition, $f \in \mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3})$.

(ii) Choose $\tilde{g} \in \mathrm{PSp}(n+1,1)_\infty$ such that $\lambda = a$, $\mu = b$, $B = I$, $x = y = z = 0$ from (1.1), then $\tilde{g}$ has the form with respect to the basis $\{e_1, \ldots, e_{n+2}\}$:

$$\tilde{g} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \\ z_{n+2} \end{pmatrix} = \begin{pmatrix} (a+b)z_1 + (a-b)z_{n+2}/2 \\ z_2 \\ \vdots \\ z_{n+1} \\ (a-b)z_1 + (a+b)z_{n+2}/2 \end{pmatrix}.$$

On the other hand, $g^*\omega = \bar{\chi} \cdot \omega \cdot \chi$ from **(i)** where $\chi(\pi(z)) = \chi'(z) = z_1 \cdot z_1(\tilde{g})^{-1} = 2z_1((a+b)z_1 + (a-b)z_{n+2})^{-1}$. As $\chi(\infty) = \chi'(f_1) = \chi'(\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}})$, we have $\chi(\infty) = a^{-1}$. □

Proof of Proposition 2.5. Put $H = \mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3})$. Without loss of generality, H is assumed to be connected. We examine the structure of the Lie group H. Let f be a diffeomorphism in H. By the definition $f^*\omega = \bar{\lambda} \cdot \omega \cdot \lambda$ where $\lambda : S^{4n+3} \to \mathbb{F}^*$. First if the map $\mathrm{Sp}(1)\longrightarrow \mathrm{Aut}(\mathbb{F}) = \mathrm{SO}(3)$ sends $\lambda/|\lambda|$ to A, then $f_*(\xi_1,\xi_2,\xi_3) = |\lambda|^2(\xi_1,\xi_2,\xi_3)\cdot A$ (mod B). (See Definition 2.4.) Let H_∞ be the stabilizer of H at $\{\infty\}$, which contains $\mathrm{PSp}(n+1,1)_\infty$. Consider the tangential representation at $\{\infty\}$, $\tau : H_\infty \rightarrow \mathrm{Aut}(\mathrm{T}_{\{\infty\}}S^{4n+3})$.

Given $h \in H_\infty$ with $h^*\omega = \bar{\mu}\cdot\omega\cdot\mu$, suppose that $\mu(\infty) = 1$. Then $h_*(\xi_1,\xi_2,\xi_3) = (\xi_1,\xi_2,\xi_3)$ (mod B_∞) at the point $\{\infty\}$. Recall the real valued four-form $\Omega = d\omega\wedge d\omega$. Since $h^*d\omega = \bar{\mu}\cdot d\omega\cdot\mu$ (mod ω), we have $h^*\Omega = |\mu|^4\Omega$ (mod ω). Put $h'_* = h_*|B_\infty$. Since h_* maps B onto itself, $h'_* \in \mathrm{Aut}(B_\infty) = \mathrm{GL}(4n,\mathbb{R})$. In particular, $h'^*\Omega_\infty = \Omega_\infty$ on B_∞. Using the consisting relation (2) for Ω_∞, the above formula implies that $h'_* \in \mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. Thus, with respect to the basis ω^α ($\alpha = 1,\dots,n$), there exist $(U^\alpha_\gamma)\in\mathrm{Sp}(n)$ and $b\in\mathrm{Sp}(1)$ such that $h'^*\omega^\alpha = U^\alpha_\gamma\cdot\omega^\gamma\cdot b$. Since $d\omega = -\frac{1}{2}\delta_{\bar{\alpha}\beta}\omega^{\bar{\alpha}}\wedge\omega^\beta$ (mod ω), we have that $h'^*d\omega = \bar{b}\cdot d\omega\cdot b$ (mod ω), which implies $b = \mu(\infty) = 1$. Hence,

$$\tau(h) = \begin{pmatrix} I_3 & V \\ 0 & U \end{pmatrix}$$

with respect to the basis $\{\xi_1,\xi_2,\xi_3\}$ and that of B_∞. Here V is a $(3,4n)$-matrix and $U = (U^\alpha_\gamma)$. If we denote by $\mathrm{M}(3,4n)$ the vector space consisting of $(3,4n)$-matrices, then $\tau(h)\in\mathrm{M}(3,4n)\rtimes\mathrm{Sp}(n)$.

Now suppose that $f\in H_\infty$ and $f^*\omega = \bar{\lambda}\cdot\omega\cdot\lambda$. Put $\lambda(\infty) = a\in\mathbb{F}^*$. From Lemma 2.6, choose $g\in\mathrm{Sp}(1)\times\mathbb{R}^+$ such that $g^*\omega = \bar{\chi}\cdot\omega\cdot\chi$ with $\chi(\infty) = a^{-1}$. Consider the element $f\circ g\in H_\infty$. Since $(f\circ g)^*\omega = \overline{\chi\cdot g^*\lambda}\cdot\omega\cdot\chi\cdot g^*\lambda$ (mod ω), we have that $\chi\cdot g^*\lambda(\infty) = \chi(\infty)\cdot\lambda(g(\infty)) = 1$. Then by the above argument and $\tau(g)\in\mathrm{Sp}(1)\times\mathbb{R}^+$, we conclude that

$$\tau(H_\infty)\subset\mathrm{M}(3,4n)\rtimes(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+).$$

In particular, $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ is a maximal compact subgroup of $\tau(H_\infty)$. If we note that τ maps compact groups of H_∞ monomorphically into its image, the maximal compact subgroup of H_∞ is $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ as well as $\mathrm{PSp}(n+1,1)_\infty$.

Let K be a maximal compact subgroup of H. Since $\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ is the maximal compact subgroup of $\mathrm{PSp}(n+1,1)$, we have that

$$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\subset\mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)\subset K.$$

As $\mathrm{PSp}(n+1,1)$ acts transitively on the simply connected space S^{4n+3}, we have $H/H_\infty = S^{4n+3}$. In particular, H_∞ is connected. By the structure theorem of connected Lie groups, the coset space H/K (resp. $H_\infty/\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$) is diffeomorphic to the euclidean space $\mathbb{R}^m$ (resp. $\mathbb{R}^\ell$) for some m (resp. ℓ). If we note that $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) = H_\infty\cap K$, then there is the fibration:

$$K/\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\rightarrow H/H_\infty\longrightarrow\mathbb{R}^m/\mathbb{R}^\ell.$$

Hence $K/\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) = S^{4n+3}$ and $m = \ell$. We obtain that $K = \mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$.

Let $R\cdot S$ be the decomposition of H where R is the radical, and S is a semisimple Lie group. If $\pi : H\longrightarrow\hat{S}$ is the canonical projection onto the semisimple Lie group $\hat{S}$ without center, then π maps $\mathrm{PSp}(n+1,1)$ isomorphically onto the simple Lie subgroup $\pi(\mathrm{PSp}(n+1,1))$ of $\hat{S}$. Since $\pi(K)$ is a maximal compact subgroup of $\hat{S}$ and $K \subset \mathrm{PSp}(n+1,1)$, we have $\pi(\mathrm{PSp}(n+1,1)) = \hat{S}$. Therefore H is the semidirect product $R \rtimes \mathrm{PSp}(n+1,1)$.

On the other hand, $\pi(H_\infty)$ is a connected subgroup of $\pi(\mathrm{PSp}(n+1,1))$ containing $\pi(\mathrm{PSp}(n+1,1)_\infty)$. The classification theorem 4.4.1 of Chen-Greenberg implies that $\pi(H_\infty) = \pi(\mathrm{PSp}(n+1,1)_\infty)$. Putting $R' = R \cap H_\infty$, similarly we have that $H_\infty = R' \rtimes \mathrm{PSp}(n+1,1)_\infty$. Then

$$S^{4n+3} = H/H_\infty = R/R' \times \mathrm{PSp}(n+1,1)/\mathrm{PSp}(n+1,1)_\infty = R/R' \times S^{4n+3}.$$

Therefore $R = R'$. In particular, it follows that $R = R_\infty$. As $H = R \rtimes \mathrm{PSp}(n+1,1)$ acts effectively and transitively on S^{4n+3}, this implies that $R = \{1\}$. Hence $H = \mathrm{PSp}(n+1,1)$. This completes the proof. □

Corollary 2.7. *The canonical quaternionic Carnot-Carathéodory Hermitian structure coincides with the canonical Sasakian 3-structure on S^{4n+3}.*

$$(\mathrm{Aut}^1_{\mathrm{QCC}}(S^{4n+3}, (\theta, \{I, J, K\})), S^{4n+3}) = (\mathrm{Sp}(n+1)\cdot \mathrm{Sp}(1), S^{4n+3}).$$

Proof. There is a $I = I_1,\ J = I_2,\ K = I_3$-invariant Riemannian metric on S^{4n+3} defined by

$$g(X,Y) = \sum_{i=1}^{3} \theta_i(X)\cdot\theta_i(Y) + \sum_{i=1}^{3} d\theta_i(X, I_iY).$$

Since each element of $\mathrm{Aut}^1_{\mathrm{QCC}}(S^{4n+3}, (\theta, \{I, J, K\}))$ preserves g, $\mathrm{Aut}^1_{\mathrm{QCC}}(S^{4n+3}, (\theta, \{I, J, K\}))$ is a compact group. □

Using the G-structure theory, we have

Corollary 2.8. *If a quaternionic Carnot-Carathéodory Hermitian $(4n+3)$-manifold M is compact, then the group $\mathrm{Aut_{QCC}}^1(M, (\theta, \{I, J, K\}))$ is a Lie group whose dimension is less than or equal to $2n^2+5n+6$ $(=\dim\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(n) + \dim\ M)$. If M is compact, then $\mathrm{Aut_{QCC}}^1(M, (\theta, \{I, J, K\}))$ is compact.*

Proposition 2.9. *A spherical pseudo-quaternionic $(4n+3)$-manifold M admits a quaternionic Carnot-Carathéodory structure.*

Proof. Let B be the canonical quaternionic Carnot-Carathéodory structure on S^{4n+3} where $B = \mathrm{Null}\ \omega$. Given a maximal collection of charts $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ of M (cf. §1), for each chart $\phi_\alpha : U_\alpha \to S^{4n+3}$, we put

$$B_\alpha = \phi_\alpha^* B, \quad \omega_\alpha = \phi_\alpha^*\omega$$

on U_α. (Note that $B_\alpha = \mathrm{Null}\ \omega_\alpha$.) If $U_\alpha \cap U_\beta \neq \emptyset$, then $g_{\alpha\beta}\circ\phi_\alpha = \phi_\beta$ for an element $g_{\alpha\beta} \in \mathrm{PSp}(n+1,1)$. Since $g_{\alpha\beta}^* B = B$ by Proposition 2.5, $B_\alpha = \phi_\alpha^* B = \phi_\beta^* B = B_\beta$ on

$U_\alpha \cap U_\beta$. The union $\{B_\alpha\}_{\alpha\in\Lambda}$ gives rise to a codimension 3 subbundle B' on M. As $B'|U_\alpha = B_\alpha$ is locally equivalent to B (*that is*, each ω_α satisfies [1], [2] of Lemma 2.1), B' is a quaternionic Carnot-Carathéodory structure on M. □

Remark 2.10. (1) On S^{4n+3}, we have obtained a globally defined $\mathfrak{sp}(1)$-valued one-form ω defining B and three independent vector fields $\{\xi_1, \xi_2, \xi_3\}$ (equivalently, there exists the quaternionic structure of complex structures $\{I, J, K\}$ on B). In general, a spherical pseudo-quaternionic manifold M admits a family of $\mathfrak{sp}(1)$-valued one-forms ω_α and three independent vector fields $\{\xi_1^\alpha, \xi_2^\alpha, \xi_3^\alpha\}$ locally defined on each U_α (equivalently, a quaternionic structure $\{I_\alpha, J_\alpha, K_\alpha\}$ on each B_α). If $U_\alpha \cap U_\beta \neq \emptyset$, then $g_{\alpha\beta} \circ \phi_\alpha = \phi_\beta$ with $g_{\alpha\beta} \in \mathrm{Aut}_{\mathrm{QCC}}(S^{4n+3})$, and so $(g_{\alpha\beta})_*(\xi_1, \xi_2, \xi_3) = u^2(\xi_1, \xi_2, \xi_3) \cdot A$ for some $A : U_\alpha \cap U_\beta \to \mathrm{SO}(3)$. As $(\phi_\alpha)_*(\xi_1^\alpha, \xi_2^\alpha, \xi_3^\alpha) = (\xi_1, \xi_2, \xi_3)$, we have

$$(\xi_1^\beta, \xi_2^\beta, \xi_3^\beta) = u^{-2}(\xi_1^\alpha, \xi_2^\alpha, \xi_3^\alpha) \cdot A^{-1}$$

on $U_\alpha \cap U_\beta$. So the union $E = \{\xi_1^\alpha, \xi_2^\alpha, \xi_3^\alpha\}_{\alpha\in\Lambda}$ defines an $\mathrm{SO}(3) \times \mathbb{R}^+$-bundle over M.

A vector field ξ is said to be a quaternionic Carnot-Carathéodory vector field if ξ generates a local one-parameter group $\{\phi_t\}_{|t|<\epsilon}$ of quaternionic Carnot-Carathéodory transformations of M. That is, for each t, ϕ_t is a quaternionic Carnot-Carathéodory transformation on a neighborhood.

Proposition 2.11 (cf. [21]). *Let* $(B_\alpha, U_\alpha, \{I_\alpha, J_\alpha, K_\alpha\})_{\alpha\in\Lambda}$ *be a quaternionic Carnot-Carathéodory structure on a spherical pseudo-quaternionic manifold* M^{4n+3}, *where* $\bigcup_\alpha B_\alpha = B'$ *defines a Carnot-Carathéodory structure on* M. *If* ξ *is a nonzero quaternionic Carnot-Carathéodory vector field, then the set* $\{x \in M \mid \xi_x \in B_x\}$ *is a codimension* 3 *regular submanifold of* M.

Proof. Put $\mathcal{N} = \{x \in M \mid \xi_x \in B'_x\}$. Each $\mathfrak{sp}(1)$-valued 1-form ω_α on U_α can be described as

$$\omega_\alpha = (\omega_\alpha)_1 \boldsymbol{i} + (\omega_\alpha)_2 \boldsymbol{j} + (\omega_\alpha)_3 \boldsymbol{k}$$

such that $\phi_\alpha^*\omega_i = (\omega_\alpha)_i$ $(i = 1, 2, 3)$. Define a smooth map $f_\alpha : U_\alpha \to \mathbb{R}^3$ to be

$$f_\alpha(p) = (\omega_\alpha)_p(\xi_p) = ((\omega_{\alpha 1})_p(\xi_p), (\omega_{\alpha 2})_p(\xi_p), (\omega_{\alpha 3})_p(\xi_p)).$$

As $B_\alpha = B'|U_\alpha = \mathrm{Null}\ \omega_\alpha$, $\mathcal{N} \cap U_\alpha = f_\alpha^{-1}(0)$. It is sufficient to show that Rank $(df_\alpha)_p = 3$ for all $p \in \mathcal{N} \cap U_\alpha$. For this, let $\iota_\xi : A^\ell(M) \to A^{\ell-1}(M)$ be the interior product for each integer ℓ (cf. [26] for example). Then we have that $\iota_\xi\omega_\alpha(p) = (\omega_\alpha)_p(\xi_p) = f_\alpha(p)$. Since $L_\xi\omega_\alpha = \iota_\xi \cdot d\omega_\alpha + d \cdot \iota_\xi\omega_\alpha$, it follows that $df_\alpha = L_\xi\omega_\alpha - \iota_\xi d\omega_\alpha : \mathrm{T}U_\alpha \to \mathrm{T}\mathbb{R}^3$. Let $\{\psi_t\}_{|t|<\epsilon}$ be a local one-parameter group generated by ξ. By Definition 10, $(\psi_t)_*(B_\alpha) = B_\alpha$ for sufficiently small t. If $Y_p \in (B_\alpha)_p$, then

$$(L_\xi)_p\omega_\alpha(Y_p) = \lim_{t\to 0} \frac{(\omega_\alpha)_p(Y_p) - (\omega_\alpha)_{\psi_t(p)}((\psi_t)_*(Y_p))}{t} = 0,$$

which implies that $(df_\alpha)_p(v) = -(\iota_\xi \cdot d\omega_\alpha)_p(v) = -2d\omega_\alpha(\xi, v)$ for $v \in (B_\alpha)_p$.

Put

$$v_1 = -I_\alpha\xi, \quad v_2 = -J_\alpha\xi, \quad v_3 = -K_\alpha\xi.$$

Then v_1, v_2, v_3 belong to $(B_\alpha)_p$ such that

$$d\omega_\alpha(\xi, v_1) = (1,0,0), \ d\omega_\alpha(\xi, v_2) = (0,1,0), \ d\omega_\alpha(\xi, v_3) = (0,0,1).$$

For this, as $\phi_{\alpha*} \circ I_\alpha = I \circ \phi_{\alpha*}$ by the definition, $d\omega_\alpha(\xi, v_1) = d\phi_\alpha^*\omega(\xi, v_1) = d\omega(\phi_{\alpha*}\xi, -I\phi_{\alpha*}\xi)$. On the other hand, from the property $(*)$ of (2.1) and that $g|B' \times B'$ is invariant under I, J, K,

$$\begin{aligned} d\omega(\phi_{\alpha*}\xi, -I\phi_{\alpha*}\xi) &= g(\phi_{\alpha*}\xi, \phi_{\alpha*}\xi)\boldsymbol{i} + g(\phi_{\alpha*}\xi, K\phi_{\alpha*}\xi)\boldsymbol{j} + g(\phi_{\alpha*}\xi, -J\phi_{\alpha*}\xi)\boldsymbol{k} \\ &= g(\phi_{\alpha*}\xi, \phi_{\alpha*}\xi)\boldsymbol{i}. \end{aligned}$$

Normalizing if necessary, we obtain that $d\omega_\alpha(\xi, v_1) = (1,0,0)$. Similarly for v_2, v_3. Therefore $f_\alpha^{-1}(0)$ is a codimension 3 regular submanifold of U_α. If $U_\alpha \cap U_\beta \neq \emptyset$, then $g_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ satisfies that $g_{\alpha\beta}^*\omega = \bar\lambda\omega\lambda$ for some function $\lambda : U_\alpha \cap U_\beta \to \mathbb{F}^*$ by Lemma 2.1. Then,

$$f_\beta(p) = (\omega_\beta)_p(\xi_p) = \overline{\phi_\alpha^*\lambda}(p) \cdot f_\alpha(p) \cdot \phi_\alpha^*\lambda(p).$$

So $f_\alpha^{-1}(0) = f_\beta^{-1}(0)$ on $U_\alpha \cap U_\beta$. Since $\mathcal{N} = \bigcup_{\alpha\in\Lambda} \mathcal{N} \cap U_\alpha = \bigcup_{\alpha\in\Lambda} f_\alpha^{-1}(0)$, $\mathcal{N}$ is a codimension 3 regular submanifold of M. □

3. Amenable holonomy and classification

3.1. Quaternionic Carnot-Carathéodory structure on Heisenberg manifolds. Let $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ be the Heisenberg geometry. We study the quaternionic Carnot-Carathéodory structure $(B, \{I, J, K\})$ on $\mathcal{M}$ induced from Proposition 2.11. This structure is obtained from that of S^{4n+3} restricted to $S^{4n+3} - \{\infty\}$. As before, there is the equivariant principal bundle:

$$\mathbb{R}^3 \to (\mathrm{Sim}(\mathcal{M}), \mathcal{M}) \xrightarrow{\nu} (\mathrm{Sim}(\mathbb{F}^n), \mathbb{F}^n).$$

Lemma 3.1. (1) *The fiber $\mathbb{R}^3$ is transverse to B.*

(2) *The center $\mathbb{R}^3$ is compatible with $\{I, J, K\}$. i.e., $t_* \circ I = I \circ t_*$ for all $t \in \mathbb{R}^3$, and similarly for J, K.*

Proof. (1) Let ξ be a nontrivial vector field induced by a one-parameter subgroup of $\mathbb{R}^3$. So ξ is a quaternionic Carnot-Carathéodory vector field. Suppose that $\xi_p \in B_p$ for some point $p \in \mathcal{M}$. Since $\mathbb{R}^3$ is the center of $\mathcal{M}$, $g_*\xi_p = \xi_{gp}$ for all $g \in \mathcal{M}$. In particular we have $\xi_x \neq 0$ for all $x \in \mathcal{M}$. As B is invariant under the action of $\mathcal{M}$, the subspace $\{x \in \mathcal{M} \mid \xi_x \in B_x\}$ coincides with the whole space $\mathcal{M}$. This contradicts Proposition 2.11. Thus $\mathbb{R}^3$ is transversal to B at each point of $\mathcal{M}$.

(2) Recall that

$$\mathcal{M} = S^{4n+3} - \{\infty\} = \{[z, y, 1]\} \subset \mathbb{FP}^{n+1} \ \text{(cf. (1.1))}.$$

Let $U = \{[z,y,\mu]\ |\mu \neq 0\}$ be an open subset in $\mathbb{FP}^{n+1}$. Each $t \in \mathbb{R}^3$ satisfies that $t[z,y,\mu] = [z+t\mu, y, \mu]$. If $\varphi : \mathcal{M} \subset U \to \mathbb{F}^{n+1}$ is a parametrization defined by $\varphi([z,y,\mu]) = (z\mu^{-1}, y\mu^{-1})$, then the action of $\mathbb{R}^3$ on $\mathcal{M}$ is equivalent to the usual translations of $\mathbb{R}^3$ on $\mathbb{R}^3 \times \mathbb{F}^n \subset \mathbb{F}^{n+1}$:

$$\varphi \cdot t \cdot \varphi^{-1}(w,x) = (w+t, x).$$

Chasing the commutative diagram;

$$\begin{array}{ccccc} \mathrm{T}_pU & \xrightarrow{I_p} & \mathrm{T}_pU & \xrightarrow{t_*} & \mathrm{T}_{tp}U \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{T}_{\varphi(p)}\mathbb{F}^{n+1} = \mathbb{F}^{n+1} & \xrightarrow{i} & \mathrm{T}_{\varphi(p)}\mathbb{F}^{n+1} = \mathbb{F}^{n+1} & \xrightarrow{(\varphi\cdot t\cdot\varphi^{-1})_*} & \mathrm{T}_{\varphi(tp)}\mathbb{F}^{n+1} = \mathbb{F}^{n+1}, \end{array}$$

we obtain that $(\varphi \cdot t \cdot \varphi^{-1})_* = \mathrm{id}$ and hence $t_* \circ I = I \circ t_*$ for all $t \in \mathbb{R}^3$, similarly for J, K. □

Corollary 3.2. *Let $(B, \{I,J,K\})$ be the induced quaternionic Carnot-Carathéodory structure on $\mathcal{M}$. Then ν induces the standard quaternionic structure $\{I_0, J_0, K_0\}$ on $\mathbb{F}^n$,i.e., $\nu_* \circ I = I_0 \circ \nu_*$, etc. In particular, ν_* maps $(B, \{I,J,K\})$ isomorphically onto the tangent bundle $(\mathrm{T}\mathbb{F}^n, \{I_0, J_0, K_0\})$ at each point of $\mathcal{M}$.*

We recall the properties of *dilations* on Heisenberg manifolds. Choose a left invariant metric g on $\mathcal{M}$ with the group of isometries $\mathrm{E}(\mathcal{M}) = \mathcal{M} \rtimes \mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. If we note that ν is a homomorphism of $\mathcal{M}$ onto $\mathbb{F}^n$, then g induces the standard euclidean metric g_0 on $\mathbb{F}^n$. Corollary 3.2 implies that $\nu_* : (B, g,\ \{I,J,K\}) \to (\mathrm{T}\mathbb{F}^n, g_0, \{I_0, J_0, K_0\})$ is a local isometry at each point of $\mathcal{M}$. As g_0 is invariant under $\{I_0, J_0, K_0\}$, $g|B\times B$ is invariant under $\{I,J,K\}$. Let $\lambda : \mathrm{Sim}(\mathcal{M}) \longrightarrow \mathbb{R}^+$ be the scale factor homomorphism as well as $\lambda_0 : \mathrm{Sim}(\mathbb{F}^n) = \mathbb{F}^n \rtimes (\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)\times\mathbb{R}^+) \longrightarrow \mathbb{R}^+$. Since g_0 satisfies that $(g_0)_{hp}(h_*X, h_*Y) = \lambda_0(h)^2 \cdot (g_0)_p(X,Y)$ for each $h \in \mathrm{Sim}(\mathbb{F}^n)$, we have for each $\alpha \in \mathrm{Sim}(\mathcal{M})$, and $X, Y \in B_x$

$$(*) \qquad g_{\alpha x}(\alpha_*X, \alpha_*Y) = \lambda(\alpha)^2 \cdot g_x(X,Y).$$

Therefore $(\mathrm{Sim}(\mathcal{M}), g|B\times B, \mathcal{M})$ plays the same role as the euclidean similarity geometry. The similar property holds for $(\mathrm{Sim}(\mathcal{M}), g|\mathrm{T}\mathbb{R}^3 \times \mathrm{T}\mathbb{R}^3)$. In fact, if $h = ((\alpha,\beta,\gamma),z)\cdot(A\cdot g, t) \in \mathrm{Sim}(\mathcal{M})$, then for $w = (w, 0) \in \mathbb{R}^3$ and $x \in \mathcal{M}$,

$$h(w\cdot x) = t^2 \cdot gwg^{-1} \cdot hx.$$

Since $\mathbb{R}^3$ is the normal subgroup of $\mathrm{Sim}(\mathcal{M})$, each element of $\mathrm{Sim}(\mathcal{M})$ leaves the subbundle $\mathrm{T}\mathbb{R}^3$ invariant. Moreover, if $\{\xi_1, \xi_2, \xi_3\}$ are the vector fields which generate $\mathbb{R}^3$, then $h_*((\xi_i)_x) = t^2(\mathrm{Ad}_g\xi)_{hx} = \lambda(h)^2(\mathrm{Ad}_g\xi)_{hx}$. As Ad acts as isometries with respect to g, we have that for $X, Y \in \mathrm{T}\mathbb{R}^3_x$,

$$(**) \qquad g_{hx}(h_*X, h_*Y) = \lambda(h)^4 \cdot g_x(X,Y).$$

Denote by $\mathcal{F}$ the frame bundle on $\mathcal{M}$ generated by $\{\xi_1, \xi_2, \xi_3\}$. Since $\mathcal{F}^{\perp} = B$ with respect to g and B is invariant under $\mathrm{Sim}(\mathcal{M})$, there is a $\mathrm{Sim}(\mathcal{M})$-invariant direct decomposition: $\mathrm{T}\mathcal{M} = \mathcal{F} \oplus B$, or equivalently $g = g|\mathcal{F} \times \mathcal{F} \oplus g|B \times B$.

3.2. **Classification of compact manifolds with amenable holonomy.** Recall that a representation $\rho : \pi \to \mathrm{PSp}(n+1,1)$ is said to be *amenable* if the closure of the image $\overline{\rho(\pi)}$ in $\mathrm{PSp}(n+1,1)$ lies in a maximal amenable Lie subgroup of $\mathrm{PSp}(n+1,1)$. The rest of this section is spent for the proof of Theorem B of Introduction.

Given a spherical pseudo-quaternionic structure on a compact smooth connected $(4n+3)$-manifold M, there exists a developing pair

$$(\rho, \mathrm{dev}) : (\pi, \tilde{M}) \longrightarrow (\mathrm{PSp}(n+1,1), S^{4n+3}).$$

A maximal amenable subgroup of $\mathrm{PSp}(n+1,1)$ is conjugate to $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ or $\mathrm{Sim}(\mathcal{M})$. If the holonomy group $\rho(\pi)$ is amenable, then we can assume that $\rho(\pi)$ lies in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ or $\mathrm{Sim}(\mathcal{M})$. In the former case, choose a spherical metric on S^{4n+3} such that $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ is a subgroup of isometries. The pullback by the developing map gives a π-invariant Riemannian metric on $\tilde{M}$. This metric induces a Riemannian metric on M. As M is compact by our hypothesis, $\tilde{M}$ is complete. Therefore dev is a local isometry of a complete Riemannian manifold $\tilde{M}$ into S^{4n+3}. Hence dev is a covering map (cf. [6]). Thus dev is homeomorphic so that $M \approx S^{4n+3}/F$ where $F = \rho(\pi) \subset \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ is a finite subgroup acting freely on S^{4n+3}. In the latter case, $\rho(\pi)$ lies in $\mathrm{Sim}(\mathcal{M})$. There is no Riemannian metric invariant under $\mathrm{Sim}(\mathcal{M})$.

First we study the complete similarity manifolds. Given a Heisenberg similarity manifold N, there exists a developing pair

$$(\rho, \mathrm{dev}) : (\pi_1(N), \tilde{N}) \longrightarrow (\mathrm{Sim}(\mathcal{M}), \mathcal{M}).$$

In general, N is said to be *geodesically complete* if the developing map is a homeomorphism of $\tilde{N}$ onto $\mathcal{M}$. Let ∇ be a left invariant affine connection on $\mathcal{M}$ induced by g. Since ∇ is invariant under the automorphism group of $\mathcal{M}$, each element of $\mathrm{Sim}(\mathcal{M})$ preserves ∇. The pullback Riemannian metric $g' = \mathrm{dev}^* g$ defines a $\pi_1(N)$-invariant affine connection ∇' on $\tilde{N}$. Thus ∇' induces an affine connection on N.

In other words, *geodesically completeness* on N is equivalent to that the *exponential map is defined on the entire tangent space $T_x\tilde{N}$ for some point $x \in \tilde{N}$* (cf. [6]). This does not depend on the choice of a point in $\tilde{N}$ because *geodesically completeness* on $\tilde{N}$ is the same as *metric completeness* by g'. (See [47].) However, note that the Riemannian metric g' does not necessarily induce a Riemannian metric on N.

Put $\Gamma = \rho(\pi)$. Assume that Γ is infinite and amenable in $\mathrm{Sim}(\mathcal{M})$. Put $\tilde{M}' = \tilde{M} - \mathrm{dev}^{-1}(\infty)$. Then the developing pair reduces to the following:

$$(\rho, \mathrm{dev}) : (\pi, \tilde{M}') \longrightarrow (\mathrm{Sim}(\mathcal{M}), \mathcal{M}).$$

Recall from Corollary 3.2 that $\mathcal{M}$ supports the Carnot-Carathéodory structure B and the frame bundle $\mathcal{F}$. As dev is an immersion, we have the pullback metric $g' = \mathrm{dev}^* g$, the induced subbundles $B' = \text{dev}^*B$ and $\mathcal{F}' = \text{dev}^*\mathcal{F}$ on $\tilde{M}'$ respectively. There exists a ball $D_r(x)$ about zero of radius r in $\text{T}_x\tilde{M}'$ with respect to g' such that the exponential map $\exp_x : D_r(x) \to \tilde{M}'$ is defined. Obviously there is the commutative diagram: $(\text{dev}(x) = p)$

$$\begin{array}{ccc} \text{T}_x\tilde{M}' & \xrightarrow{\text{dev}_*} & \text{T}_p\mathcal{M} \\ \cup & & \\ D_r(x) & & \downarrow \exp_p \\ \downarrow \exp_x & & \\ \tilde{M}' & \xrightarrow{\text{dev}} & \mathcal{M}. \end{array}$$

If $\tilde{M}'$ is (geodesically) complete, the local isometry dev is a homeomorphism of $\tilde{M}'$ onto $\mathcal{M}$. The holonomy group Γ will be discrete in $\text{Sim}(\mathcal{M})$. If $\text{dev}^{-1}(\infty) \neq \emptyset$, then it is easy to see that $\text{dev} : \tilde{M} \longrightarrow S^{4n+3}$ is homeomorphic. Since Γ is infinite by our hypothesis, we have that $\tilde{M} = \tilde{M}'$. Recall that $\mathbb{R}^+ \subset \text{Sim}(\mathcal{M})$ acts on $\mathcal{M}$ as expansion or contraction. If Γ is discrete in $\text{Sim}(\mathcal{M})$, then Γ is either conjugate to a subgroup of $\mathcal{M} \rtimes \text{Sp}(n)\cdot\text{Sp}(1) = \text{E}(\mathcal{M})$ or $\text{Sp}(n)\cdot\text{Sp}(1) \times \mathbb{R}^+$. As the latter group fixes the origin of $\mathcal{M}$, $\Gamma \subset \text{E}(\mathcal{M})$ in our case. Hence M is isomorphic to an infranilmanifold $\mathcal{M}/\Gamma$.

Next we examine the incomplete similarity manifolds. Suppose that $\tilde{M}'$ is incomplete. Define

$$R(x) = \sup\{r \mid \exp_x : D_r(x) \to \tilde{M}' \text{ is defined}\}.$$

Then $R(x) < \infty$ for all points x of $\tilde{M}'$ by the above remark. Put $r = R(x)$. With respect to the pull-back affine connection ∇', there exists a vector $v \in \partial D_r(x)$ such that $\gamma(t) = \exp_x tv$ is an incomplete geodesic, *i.e.*, $\gamma(t)$ is defined on $0 \leqq t < 1$ but not $t = 1$. Recall that $M = \tilde{M}/\pi$ decomposes into the union of the orbit space $\tilde{M}'/\pi$ and $\text{dev}^{-1}(\infty)/\pi$.

Lemma 3.3. *Let $P : \tilde{M} \longrightarrow M$ be the covering map. Then the geodesic image $\{P(\gamma(t))\}_{0 \leqq t < 1}$ has an accumulation point inside $\tilde{M}'/\pi$.*

Proof. Since M is compact, the geodesic image

$$\{P(\gamma(t))\}_{0 \leqq t < 1} \in \tilde{M}'/\pi$$

has an accumulation point y in M. Since $\text{dev}^{-1}(\infty)/\pi$ consists of finite points, we may consider the case that $\text{dev}^{-1}(\infty)/\pi$ consists of a single point. Suppose that $\text{dev}^{-1}(\infty)/\pi = \{y\}$ in M. Choose an evenly covered neighborhood U of y. When $\tilde{U}$ is a lift of U to $\tilde{M}$, by the definition, $\alpha \cdot \tilde{U} \cap \tilde{U} \neq \emptyset$ for some $\alpha \in \pi$ if and only if $\alpha = 1$. We can assume that $\text{dev} : \tilde{U} \to \text{dev}(\tilde{U})$ is homeomorphic. If $\tilde{y} \in \tilde{U}$ is a point with $P(\tilde{y}) = y$, then $\text{dev}(\tilde{y}) = \{\infty\}$ as above.

As y is an accumulation point of the geodesic image $\{P(\gamma(t))\}_{0 \leqq t < 1}$, there exists a sequence $0 < t_1 < t_2 < \cdots < t_n < \cdots < 1$ such that $\gamma(t_n) \in \alpha_n \cdot \tilde{U}$ for elements $\alpha_n \in \pi$.

On the other hand, as $\mathcal{M}$ is complete, there exists a limit point $\lim_{n\to\infty} \mathrm{dev}(\gamma(t_n)) \in \mathcal{M}$. Since $\mathrm{dev}(\gamma(t_n)) \in \rho(\alpha_n)\,\mathrm{dev}(\tilde{U})$ and each $\rho(\alpha_n)$ stabilizes the point $\{\infty\}$, we have that

$$\lim_{n\to\infty} \mathrm{dev}(\gamma(t_n)) = \bigcap_{n=1}^{\infty} \rho(\alpha_n)\mathrm{dev}(\tilde{U}) = \{\infty\},$$

it is a contradiction. Hence the accumulation point y of the geodesic image $\{P(\gamma(t))\}_{0\leq t<1}$ lies in $\tilde{M}'/\pi$. □

We then apply the same argument of Fried [11], also Miner [32] to the quaternionic case. By Lemma 3.3, the geodesic image in $\tilde{M}'/\pi$ has an accumulation point z inside $\tilde{M}'/\pi$, and it passes by z infinitely many times and arbitrarily close. By the argument of [11],[32], this *recurrent* property gives a family of elements $\{\gamma_{ij}\}$ of $\pi_1(M, z)$ such that γ_{ij} $(j >> i)$ maps $\gamma(t_i)$ very close to $\gamma(t_j)$ $(0 < t_i < t_j < 1)$. Moreover

Lemma 3.4 ([11], Lemma 3.2 [32]). *Denote by $0 = (0,0)$ the origin of $\mathcal{M}$. Suppose that $\exp_p \circ \mathrm{dev}_*(v) = 0 \in \mathcal{M}$ $(\mathrm{dev}(x) = p)$. For sufficiently large i, j, the holonomy image $\rho(\gamma_{ij})$ can be chosen to be a Heisenberg similarity transformation centered arbitrarily close to 0 with arbitrarily small rotation matrix.*

Then, $\exp_x$ can be defined on the half space $H_x = \{X \in \mathrm{T}_x\tilde{M}' \mid g'_x(v, X) < r^2\}$ containing $D_r(x)$. (See the figures of [11], [32].) To see this, note that $\exp_p \mathrm{dev}_*(D_r(x))$ is a maximal metric ball about p of radius r whose boundary contains 0. Let $X \in H_x$. By Lemma 3.3, there exists an element $\rho(\gamma_{ij}) \in \Gamma$ such that for sufficiently large $i < j$, $\rho(\gamma_{ij}) \circ \exp_p \mathrm{dev}_*(X) \in \exp_p \mathrm{dev}_*(D_r(x))$, and $\gamma_{ij}x \in \exp_x(D_r(x))$. Using dev^{-1} locally, we have that $\exp_{\gamma_{ij}x}((\gamma_{ij})_*X) \in \exp_x(D_r(x))$. Define

$$\exp_x X = {\gamma_{ij}}^{-1} \circ \exp_{\gamma_{ij}x}((\gamma_{ij})_*X).$$

Since $\mathrm{dev}({\gamma_{ij}}^{-1} \circ \exp_{\gamma_{ij}x}(\gamma_{ij})_*X) = \exp_p \mathrm{dev}_*(X)$, which is well defined. As a consequence, for each $y \in \tilde{M}'$, there exists a unique vector $V = V_y \in \partial D_r(y)$ for which $\exp_y V$ is not defined.

For each $x \in \tilde{M}'$, we have the half space $H_x = \{X \in \mathrm{T}_x\tilde{M}' \mid g'_x(V, X) < r^2\}$ on which $\exp_x$ is defined. Here $V = V_x$ is a unique vector lying on the boundary $\partial D_r(x)$ such that $\exp_x V$ is undefined, where $r = R(x) = ||V|| = \sqrt{||V^f||^2 + ||V^b||^2}$. Unfortunately, the half spaces H_x are not necessarily translated onto each other by the elements of π.

Let $\partial H_x = \{X \in \mathrm{T}_x\tilde{M}' \mid g'_x(V, X) = r^2\}$ denote the boundary of H_x. The following lemma is obtained from the idea of Fried [11] (also Miner [32]) which has been already used to show the existence of half space H_x on which $\exp_x$ is defined.

Lemma 3.5 (Compare Lemma 1 [11], Proposition 2.6 [32]). *Let 0 be the origin of $\mathcal{M}$ and suppose $\exp_p \circ \mathrm{dev}_*(v) = 0$ as above. Then*

$$0 \in \bigcap_{y\in\tilde{M}'} \exp_q \circ \mathrm{dev}_*(\partial H_y), \quad \mathrm{dev}(y) = q.$$

Proof. Suppose not. Then the origin 0 is not contained in $\exp_q \circ \mathrm{dev}_*(\partial H_y)$ for some $y \in \tilde{M}'$. By Lemma 3.3, there is an element $\gamma \in \pi$ whose holonomy $\rho(\gamma)$ carries the half space $\exp_q \circ \mathrm{dev}_*(H_y)$ arbitrarily close to 0 in $\mathcal{M}$. For the vector $V = V_y \in \partial D_{R(y)}(y)$, $\gamma_* V$ is a unique vector of $\partial D_{R(\gamma y)}(\gamma y)$ such that $\exp_{\gamma y}\gamma_* V$ is undefined. So the geodesic $\exp_{\rho(\gamma)q}(t\cdot \mathrm{dev}_*(\gamma_* V))$ $(0 \leqq t < 1)$ lies in the half space $\exp_{\rho(\gamma)q} \circ \mathrm{dev}_* H_{\gamma y}$, and it meets the boundary $\exp_{\rho(\gamma)q}(\mathrm{dev}_*(\partial H_{\gamma y}))$ perpendicularly at the point $\exp_{\rho(\gamma)q}(\mathrm{dev}_*(\gamma_* V))$. Denote by $\prec_y (V, Y)$ the angle between V and Y at y and let $D^{\epsilon}_{R(y)}(y) = \{Y \in D_{R(y)}(y) \mid \prec_y (V,Y) < \epsilon\}$ be the cone at y of the axis V with angle $\epsilon > 0$. As the geodesic $\exp_{\rho(\gamma)q}(t\cdot \mathrm{dev}_*(\gamma_* V))$ is arbitrarily close to 0, we choose a small ϵ such that

$$\exp_q \circ \mathrm{dev}_*(D^{\epsilon}_{R(y)}(y)) \subset \exp_{\rho(\gamma)q} \circ \mathrm{dev}_*(H_{\gamma y}).$$

By the commutativity,

$$\mathrm{dev} \circ \exp_y(D^{\epsilon}_{R(y)}(y)) \subset \mathrm{dev} \circ \exp_{\gamma y}(H_{\gamma y}),$$

which implies that $\exp_y(D^{\epsilon}_{R(y)}(y))$ is properly embedded in the convex domain $\exp_{\gamma y}(H_{\gamma y})$. Then the closed metric cone $\overline{\exp_y(D^{\epsilon}_{R(y)}(y))}$ sits inside $\exp_{\gamma y}(H_{\gamma y})$ and hence is compact. This contradicts that $\exp_y V$ is undefined. □

Put $J = \bigcap_{y \in \tilde{M}'} \exp_q \circ \mathrm{dev}_*(\partial H_y)$. By the construction of J, the developing image $\mathrm{dev}(\tilde{M}')$ is obviously outside J. Note that $S^{4n+3} = \mathcal{M} \cup \{\infty\}$, and $J \subset \exp_q \circ \mathrm{dev}_*(\partial H_y)$. It is easy to see that

Corollary 3.6. (1) $\mathrm{dev}(\tilde{M}') \subset \mathcal{M} - J$.
(2) *If $\bar{J}$ be the closure of J in S^{4n+3}, then either $\bar{J} = J$ or $\bar{J} = J \cup \{\infty\}$.*

Lemma 3.7. *If $\bar{J} = J \cup \{\infty\}$, then* $\mathrm{dev}^{-1}(\infty) = \emptyset$. *As a consequence, the developing map reduces to the following:* $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$.

Proof. If $x \in \mathrm{dev}^{-1}(\infty)$, then there is a neighborhood U of x in $\tilde{M}$ with $U - \{x\} \subset \tilde{M}'$ such that $\mathrm{dev}(U - \{x\}) = \mathrm{dev}(U) - \{\infty\}$. As $\{\infty\} \in \bar{J} - J$, $(\mathrm{dev}(U) - \{\infty\}) \cap J \neq \emptyset$, but $\mathrm{dev}(U) - \{\infty\} \subset \mathrm{dev}(\tilde{M}')$, which is impossible by Corollary 3.6. In particular, $\tilde{M} = \tilde{M}'$. □

As above, the unique vector $V \in \partial D_r(x)$ has the property that $\exp_x(t\cdot V)$ is defined for $0 \leqq t < 1$, but not $t = 1$. In this case, the image $\exp_p(\mathrm{dev}_* V)$ $(\mathrm{dev}(x) = p)$ is said to be an *invisible* point. In general if $\exp_x(t\cdot X)$ is defined for $t = 1$, then $\exp_p(\mathrm{dev}_* X)$ is called a *visible* point because $\exp_p(\mathrm{dev}_* X) = \mathrm{dev} \circ \exp_x(X)$, otherwise the point $\exp_p(\mathrm{dev}_* X)$ is invisible. Especially every point in the half space $\exp_q(\mathrm{dev}_* H_y)$ is visible for each $y \in \tilde{M}'$, while invisible points lie only on the boundary $\exp_q \circ \mathrm{dev}_*(\partial H_y)$. (Compare [11], [32].)

Lemma 3.8. *J is invariant under Γ. In particular, $\bar{J}$ is a Γ-invariant closed subset in S^{4n+3} contained in a submanifold of dimension at most $4n+2$.*

Proof. Let $m \in J$. There exists a vector $X \in \partial H_y$ such that $m = \exp_q(\mathrm{dev}_*(X))$ for each $y \in \tilde{M}'$ ($\mathrm{dev}(y) = q$). If we note that each boundary $\exp_q \circ \mathrm{dev}_*(\partial H_y)$ contains 0 from Lemma 3.5, then $\exp_q(\mathrm{dev}_* X)$ is an invisible point; otherwise $\exp_q(\mathrm{dev}_*(V))$ would be an visible point. Since each $\gamma \in \pi$ maps the unique vector $V \in \partial D_{R(y)}(y)$ onto the unique vector $\gamma_* V \in \partial D_{R(\gamma y)}(\gamma y)$, the geodesic $\exp_{\rho(\gamma)q}(t \cdot \mathrm{dev}_*(\gamma_* V))$ determines the boundary $C = \exp_{\rho(\gamma)q} \circ \mathrm{dev}_*(\partial H_{\gamma y})$. Suppose that $\rho(\gamma)m = \exp_{\rho(\gamma)q}(\mathrm{dev}_*(\gamma_* X))$ does not lie on C. Then the geodesic $\alpha(t) = \exp_{\rho(\gamma)q}(t \cdot \mathrm{dev}_*(\gamma_* X))$ intersects C at some $s < 1$. Put $n = \alpha(s) = \exp_{\rho(\gamma)q} \circ \mathrm{dev}_*(s \cdot \gamma_* X)$, which lies on C. On the other hand, since $s \cdot X \in H_y$, $\exp_y(s \cdot X)$ is defined. Put $z = \gamma \circ \exp_y(s \cdot X) \in \tilde{M}'$. Then

$$\mathrm{dev}(z) = \mathrm{dev} \circ \exp_{\gamma y}(s \cdot \gamma_* X) = \exp_{\rho(\gamma)q} \circ \mathrm{dev}_*(s \cdot \gamma_* X) = n.$$

We have the maximal disc $\exp_n \mathrm{dev}_* D_{R(z)}(z)$ centered at n, which intersects the boundary C with the middle of the disc. Let $\exp_n \circ \mathrm{dev}_*(H_z)$ be the half space containing $\exp_n \mathrm{dev}_* D_{R(z)}(z)$. Since $\exp_n \circ \mathrm{dev}_*(\partial H_z)$ contains 0 and the point

$$w = \exp_{\rho(\gamma)q}(\underset{*}{\mathrm{dev}}(\gamma_* V)) \in C,$$

we conclude that w sits inside $\exp_n \circ \mathrm{dev}_*(H_z)$. This contradicts that w is an invisible point. Therefore, the point $\rho(\gamma)m \in \exp_{\rho(\gamma)q} \circ \mathrm{dev}_*(\partial H_{\gamma y})$ at each $y \in \tilde{M}'$. Hence, J is invariant under Γ. □

Concerning the structure of the boundary $\exp_q \circ \mathrm{dev}_*(\partial H_y)$, we have the following.

Lemma 3.9. *Let $V = V^f \oplus V^b$ be the decomposition for the unique vector $V = V_y \in \partial D_{R(y)}(y)$.*

1. *If $V^f \neq 0$, then $\exp_q \circ \mathrm{dev}_*(\partial H_y) = \mathbb{R}^2 \times U$ where $U = \mathbb{F}^n$ or U is an affine half space of $\mathbb{F}^n$.*
2. *If $V^f = 0$, then $\exp_q \circ \mathrm{dev}_*(\partial H_y) = \mathbb{R}^3 \times U$ where U is a $(4n-1)$-dimensional affine subspace of $\mathbb{F}^n$.*

Proof. Recall that

$$\partial H_y = \{X \in \mathrm{T}_y\tilde{M}' \mid g'_y(V, X) = g'_y(V^f, X^f) + g'_y(V^b, X^b) = r^2\} \text{ for } X = X^f \oplus X^b \in T_y\tilde{M}'.$$

(1) Suppose that $V^f \neq 0$. Let $\mathrm{dev}(y) = q$. The projection $\mathrm{T}_q\mathbb{R}^3 \to \mathrm{T}_q\mathcal{M} \xrightarrow{\nu_*} \mathrm{T}_{\nu(q)}\mathbb{F}^n$ induces the map $\nu_* : \mathrm{dev}_*(\partial H_y) \longrightarrow \nu_* \circ \mathrm{dev}_*(\partial H_y)$. Put $v_X = \nu_*(\mathrm{dev}_*(X)) = \nu_*(\mathrm{dev}_*(X^b))$ and $s_X = r^2 - g'_y(V^b, X^b)$. Then the inverse image at v_X is a two dimensional affine subspace of $\mathrm{T}_q\mathbb{R}^3$:

$${\nu_*}^{-1}(v_X) = \{\mathrm{dev}_*(Y^f + X^b) \mid g'_y(V^f, Y^f) = s_X\}.$$

Since it is a half space as before, ${\nu_*}^{-1}(v_X)$ is perpendicular to $\mathrm{dev}_*(V^f)$ and so the fibers ${\nu_*}^{-1}(v_X)$ are parallel to each other. Let $\mathrm{T}_q\mathcal{M} = \mathcal{F}_q \oplus B_q$ be the decomposition for $q \in \mathcal{M}$.

Then we note that $\mathcal{F}_0$ is the ideal of the nilpotent Lie algebra $T_0\mathcal{M}$ generated by the center $\mathbb{R}^3$. If L_q is the left translation of $\mathcal{M}$, then there is the commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_0\mathcal{M} & \xrightarrow{dL_q} & \mathrm{T}_q\mathcal{M} \\ \Big\downarrow \exp & & \Big\downarrow \exp_q \\ \mathcal{M} & \xrightarrow{L_q} & \mathcal{M}. \end{array}$$

Let $T_0 \in \mathcal{F}_0$, $S_0 \in B_0$. Since $\mathbb{R}^3$ is the center, $[T_0, S_0] = 0$ and so the product formula (cf. [16]) becomes $\exp(tT_0) \cdot \exp(tS_0) = \exp(t(T_0 + S_0) + o(t^3))$ for small $t > 0$. As $\exp(tT_0) \in \mathbb{R}^3$, we have

$$\exp(T_0) \cdot \exp(S_0) = \exp(T_0 + S_0).$$

Choose T_0, S_0 such that $dL_q(T_0) = \mathrm{dev}_*(Y^f)$, $dL_q(S_0) = \mathrm{dev}_*(X^b)$. By the commutativity,

$$\exp_q(\mathrm{dev}_*(Y^f + X^b)) = \exp T_0 \cdot \exp_q(\mathrm{dev}_*(X^b)).$$

Let $\nu : \exp_q \circ \mathrm{dev}_*(\partial H_y) \longrightarrow \exp_{\nu(q)} \circ \nu_* \circ \mathrm{dev}_*(\partial H_y)$ be the projection. Put $z_X = \exp_{\nu(q)}(v_X)$. Then the inverse image at z_X,

$$\begin{aligned} \nu^{-1}(z_X) &= \exp_q(\nu_*{}^{-1}(v_X)) \\ &= \{\exp T_0 \cdot \exp_q(\mathrm{dev}_*(X^b)) \mid g'_y(V^f, Y^f) = s_X,\ T_0 = dL_{q^{-1}} \circ \mathrm{dev}_*(Y^f)\}. \end{aligned}$$

By the above remark, these inverse images $\nu^{-1}(z_X)$ are parallel two dimensional affine subspaces in $\mathbb{R}^3$. Hence the subset $\{\exp T_0 \mid T_0 = dL_{q^{-1}} \circ \mathrm{dev}_*(Y^f),\ g'_y(V^f, Y^f) = s_X\}$ is a two dimensional vector subgroup $\mathbb{R}^2$ of $\mathbb{R}^3$. Let

$$W' = \{\mathrm{dev}_*(Y^b) \in \mathrm{dev}_*(\partial H_y) \mid g_q(\mathrm{dev}_*(V^b), \mathrm{dev}_*(Y^b)) = r^2 - g_q(\mathrm{dev}_*(V^f), \mathrm{dev}_*(Y^f)) \leqq r^2\}$$

be the subspace of $\mathrm{T}_q\mathcal{M}$. As ν_* is a local isometry of (B_q, g_q) onto $(\mathrm{T}_{\nu(q)}\mathbb{F}^n, (g_0)_{\nu(q)})$, ν_* maps W' isometrically onto

$$\nu_*(W') = \{\nu_* \circ \mathrm{dev}_*(Y^b) \in \nu_* \circ \mathrm{dev}_*(\partial H_y) \mid (g_0)_{\nu(q)}(\nu_* \circ \mathrm{dev}_*(V^b), \nu_* \circ \mathrm{dev}_*(Y^b)) \leqq r^2\}.$$

In particular the submanifold $(\exp_q(W'), g)$ is isometric to the affine subspace $(\exp_{\nu(q)} \circ \nu_*(W'), g_0)$ of $\mathbb{F}^n$. Put $U = \exp_{\nu(q)} \circ \nu_*(W')$. Then U is either $\mathbb{F}^n$ or an affine half space according to whether $V^b = 0$ or $V^b \neq 0$. Since $(\exp_q(W'), g)$ is a flat submanifold of $\mathcal{M}$, we have that $\exp_q(W') = (\alpha, U)$ for some $\alpha \in \mathbb{R}^3$. If we note that $\exp_q \circ \mathrm{dev}_*(\partial H_y) = \mathbb{R}^2 \cdot \exp_q(W')$ which contains 0, then $\alpha \in \mathbb{R}^2$ and hence $\exp_q \circ \mathrm{dev}_*(\partial H_y) = \mathbb{R}^2 \times U$.

For (2), as $V^f = 0$, $g_q(\mathrm{dev}_*(V^b), \mathrm{dev}_*(Y^b)) = (g_0)_{\nu(q)}(\nu_* \circ \mathrm{dev}_*(V^b), \nu_* \circ \mathrm{dev}_*(Y^b)) = r^2$. Then $U = \exp_{\nu(q)} \circ \nu_*(W') = \nu \circ \exp_q \circ \mathrm{dev}_*(\partial H_y)$ is a $(4n-1)$-dimensional affine subspace of $\mathbb{F}^n$. It is easy to see that $\exp_q \circ \mathrm{dev}_*(\partial H_y)$ is a principal $\mathbb{R}^3$-bundle over U. Hence, $\exp_q \circ \mathrm{dev}_*(\partial H_y) = \mathbb{R}^3 \times U$. □

By the definition, $J = \bigcap_{y \in \tilde{M}'} \exp_q \circ \mathrm{dev}_*(\partial H_y)$. Thus J is an affine subspace of $(\mathcal{M}, \nabla)$, *i.e.*, $J = \mathbb{R}^k \times W$ where $\mathbb{R}^k$ is a vector space of $\mathbb{R}^3$ and W is an affine subspace of $\mathbb{R}^{4n} = \mathbb{F}^n$. In particular we have the following.

Corollary 3.10. *(i) If $\bar{J} = J$, then J is a single point 0 in $\mathcal{M}$.*
(ii) If $\bar{J} = J \cup \{\infty\}$, then $\bar{J}$ is an ℓ ($\geqq 1$)-dimensional sphere S^ℓ.

Proof of Theorem A.

(I) $\bar{J} = J$. By Corollary 3.10, $J = 0$, which implies that the holonomy group $\Gamma \subset \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$.

(I)₁ → **$(\mathbf{I})_1$** $\mathrm{dev}^{-1}(\infty) = \emptyset$. Then $\tilde{M} = \tilde{M}'$ and $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - \{0\} = \mathbb{R}^+ \times S^{4n+2}$ by Corollary 3.6. As M is compact, $\mathrm{dev} : \tilde{M} \longrightarrow \mathbb{R}^+ \times S^{4n+2}$ is homeomorphic so that M is finitely covered by a Hopf manifold $S^1 \times S^{4n+2}$.

$(\mathbf{I})_2$ $\mathrm{dev}^{-1}(\infty) \neq \emptyset$. Then the developing map satisfies that $\mathrm{dev} : \tilde{M} \longrightarrow S^{4n+3} - \{0\}$. Replace $S^{4n+3} - \{0\}$ for the role of $\mathcal{M} = S^{4n+3} - \{\infty\}$. Put $\mathcal{M}' = S^{4n+3} - \{0\}$. As a consequence, we show this case does not occur. In fact, if $\tilde{M}$ is complete, then $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M}'$ is homeomorphic. Then Γ acts freely on $\mathcal{M}'$, while Γ has a fixed point $\{\infty\}$ inside $\mathcal{M}'$. So $\tilde{M}$ is incomplete. The same argument as above shows that there is a Γ-invariant affine subspace J' which is outside the developing image $\mathrm{dev}(\tilde{M})$. If $\partial\,\mathrm{dev}(\tilde{M})$ is the boundary of the developing image in S^{4n+3}, then $\partial\,\mathrm{dev}(\tilde{M})$ is a Γ-invariant closed subset containing at least two points. Recall the limit set $L(\Gamma)$ of Γ in S^{4n+3}, which is defined to be the boundary of the closure of the orbit $\Gamma \cdot w$ for a point $w \in \mathbb{H}_{\mathbb{F}}^{n+1}$. (Compare [7].) By minimality, $L(\Gamma) \subset \partial\,\mathrm{dev}(\tilde{M})$. Thus we obtain that $\mathrm{dev} : \tilde{M} \longrightarrow S^{4n+3} - L(\Gamma)$. As $\Gamma \subset \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+$ in our case, $L(\Gamma) = \{0, \infty\}$ and so the developing image $\mathrm{dev}(\tilde{M})$ misses the point $\{\infty\}$. This contradicts the hypothesis $(\mathbf{I}_2)$.

(II) $\bar{J} = J \cup \{\infty\}$. By Lemma 3.7 and Corollary 3.10 *(ii)*, $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$ where J is an ℓ ($\geqq 1$)-dimensional affine flat subspace of $\mathcal{M}$. We define a Γ-invariant Riemannian metric on $\mathcal{M} - J$. Let Z be a vector field which assigns to each $p \in \mathcal{M} - J$ the vector $Z(p)$ from p to J, which is perpendicular to J with respect to the left invariant metric g on $\mathcal{M}$. Such a vector $Z(p)$ is uniquely determined because J is an affine flat subspace $(\mathcal{M}, \nabla)$, $Z(p)$ is the shortest vector which meets J. Moreover, since J is invariant under Γ, the vector field Z is Γ-invariant. Let $X = X^f \oplus X^b$ (resp. $Y = Y^f \oplus Y^b$) be a decomposition for vectors $X, Y \in \mathrm{T}_x(\mathcal{M} - J\tilde{M})$. We define a Riemannian metric:

$$\tilde{h}_p(X, Y) = \frac{g_x(X^f, Y^f)}{||Z^f(p)||^2 + ||Z^b(p)||^4} + \frac{g_x(X^b, Y^b)}{||Z^f(p)|| + ||Z^b(p)||^2}.$$

By the dilation property $(*)$, $(**)$, $\tilde{h}$ is a Riemannian metric on $\mathcal{M} - J$ which is invariant under Γ. The pullback by dev defines a π-invariant Riemannian metric $\mathrm{dev}^* \tilde{h}$ on $\tilde{M}$ such that $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$ is a local isometry. Then M becomes a compact Riemannian manifold induced by $\mathrm{dev}^* \tilde{h}$. As a consequence $\tilde{M}$ is complete so that $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$ is a covering map. Since J is an ℓ-dimensional affine flat subspace of $\mathcal{M}$ with $1 \leqq \ell \leqq 4n+2$, there are the following cases: **(i)** if $\dim J \leqq 4n$, then $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$ is homeomorphic. **(ii)** if $\dim J = 4n+1$, then $\mathrm{dev} : \tilde{M} \longrightarrow \mathcal{M} - J$ is a covering map. **(iii)** if $\dim J = 4n+2$, then dev is a homeomorphism of $\tilde{M}$ onto a connected component of $\mathcal{M} - J$.

For the cases **(i)**, **(iii)**, Γ is discrete in $\mathrm{Sim}(\mathcal{M})$, so either $L(\Gamma) = \{\infty\}$ or $L(\Gamma) = \{0, \infty\}$. Since $\partial\,\mathrm{dev}(\tilde{M})$ contains more than one point, we have that $L(\Gamma) \subset \partial\,\mathrm{dev}(\tilde{M})$. Thus dev : $\tilde{M} \longrightarrow S^{4n+3} - L(\Gamma)$ is homeomorphic for which

$$S^{4n+3} - L(\Gamma) = \mathcal{M} \text{ or } S^{4n+3} - L(\Gamma) = \mathcal{M} - \{0\} \text{ respectively.}$$

Hence $\mathrm{dev}(\tilde{M}) \cap J - \{0\} \neq \emptyset$. This contradicts Lemma 3.7 under the hypothesis **(II)**.

For the case **(ii)**, note that $J = \mathbb{R}^k \times \mathbb{R}^{4n+1-k}$ such that $\bar{J} = S^{4n+1}$ and $1 \leqq k \leqq 3$.

(ii)$_1$ *Suppose that* $k = 3$, *i.e.*, $J = \mathbb{R}^3 \times \mathbb{R}^{4n-2}$. Then $\nu^{-1}(\mathbb{R}^{4n-2}) = J$ from (1.2). Moreover, $\mathcal{M} - J = S^{4n+3} - \bar{J} = S^{4n+3} - S^{4n+1} \approx B^{4n+2} \times S^1$ where B^{4n+2} is a $(4n+2)$-dimensional ball. Let X be the universal covering space of $\mathcal{M} - J$. The developing map dev lifts to a homeomorphism $\tilde{\mathrm{dev}} : \tilde{M} \longrightarrow X$ which maps π onto a subgroup $\tilde{\Gamma}$ acting properly discontinuously and freely on X. In particular, $M \approx X/\tilde{\Gamma}$ is compact. However if we note that the action $(\tilde{\Gamma}, X)$ is a lift of the action $(\Gamma, S^{4n+3} - S^{4n+1})$, then $X/\tilde{\Gamma}$ cannot be compact by (3) of Lemma 1.2.

(ii)$_2$ *Suppose that* $J = \mathbb{R}^2 \times \mathbb{R}^{4n-1}$ *or* $J = \mathbb{R}^1 \times \mathbb{F}^n$. Consider the case $J = \mathbb{R}^2 \times \mathbb{R}^{4n-1}$. Recall that an element g of $\mathrm{Sim}(\mathcal{M})$ has the form: $(\alpha, x) \cdot (A \cdot g, t)$. For

$$\begin{pmatrix} \beta \\ z \end{pmatrix} \in \mathcal{M},$$

the action of g satisfies that

$$g\begin{pmatrix} \beta \\ z \end{pmatrix} = \begin{pmatrix} \alpha + t^2 g \cdot \beta \cdot g^{-1} + \mathrm{Im} < x, tAz \cdot g^{-1} > \\ x + tAz \cdot g^{-1} \end{pmatrix}.$$

The subspace $\mathbb{R}^{4n-1}$ contains at least one quaternionic subspace $\mathbb{F}^1$. As Γ leaves J invariant for which $\beta \in \mathbb{R}^2$ and $z \in \mathbb{R}^{4n-1}$ can be chosen arbitrarily, it follows that $x = 0$. Thus Γ lies in the subgroup $(\mathbb{R}^2, 0) \rtimes \mathrm{Sp}(n) \cdot \mathrm{SO}(2) \times \mathbb{R}^+$. (Note that this group does not preserve $\mathbb{R}^3 \times \mathbb{R}^n$, although Γ acts invariantly on $\mathbb{R}^3 = (\mathbb{R}^3, 0)$ as similarity transformations.)

Let $\gamma = (\alpha, 0) \cdot (A \cdot g, t)$ be an element of $(\mathbb{R}^2, 0) \rtimes \mathrm{Sp}(n) \cdot \mathrm{SO}(2) \times \mathbb{R}^+$. Then, for $(\beta, x) \in \mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^{4n}$

$$\gamma\begin{pmatrix} \beta \\ x \end{pmatrix} = \begin{pmatrix} \alpha + t^2 g \cdot \beta \cdot g^{-1} \\ t \cdot Ax \cdot g^{-1} \end{pmatrix} = \begin{pmatrix} \alpha + t^2 \cdot B_1 \beta \\ t \cdot B_2 x \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} t^2 \cdot B_1 & 0 \\ 0 & t \cdot B_2 \end{pmatrix}\begin{pmatrix} \beta \\ x \end{pmatrix}.$$

Here, a matrix $B_1 \in \mathrm{SO}(2)$ is the conjugate of g and a matrix $B_2 \in \mathrm{SO}(4n)$ is $A \cdot g$.

Let $f : \mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^{4n} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^{4n}$ be the diffeomorphism defined by $f(\alpha, x) = (\alpha, x|x|)$ where $|\cdot|$ is the euclidean norm. Then it is easy to check that

$$f(\gamma\begin{pmatrix} \beta \\ x \end{pmatrix}) = \begin{pmatrix} t^2 \cdot B_1 & 0 \\ 0 & t^2 \cdot B_2 \end{pmatrix} \cdot f(\begin{pmatrix} \beta \\ x \end{pmatrix}) + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

There is a homomorphism $\tau : \Gamma \longrightarrow (\mathbb{R}^2, 0) \rtimes (\mathrm{SO}(2) \times \mathrm{SO}(4n)) \times \mathbb{R}^+ \subset \mathrm{Sim}(\mathbb{R}^{4n+3})$ such that $f \circ \gamma = \tau(\gamma) \circ f$. We have an equivariant diffeomorphism:

$$(\Gamma, \mathcal{M}) \xrightarrow{f} (\tau(\Gamma), \mathbb{R}^{4n+3}).$$

The pair $(\tau \circ \rho, f \circ \text{dev}) : (\pi, \tilde{M}) \longrightarrow (\text{Sim}(\mathbb{R}^{4n+3}), \mathbb{R}^{4n+3})$ give a similarity structure on M in which $f \circ \text{dev}$ misses $f(J)$. The result of Fried says that the developing map $f \circ \text{dev} : \tilde{M} \longrightarrow \mathbb{R}^{4n+3}$ of a compact incomplete similarity manifold M misses exactly one point $\{0\}$ (up to conjugacy). Hence $f(J) = \{0\}$, which is impossible by $\dim J = 4n+1$. The same argument can be applied to the case that $J = \mathbb{R}^1 \times \mathbb{F}^n$. As a consequence, **(II)** does not occur. This completes the proof of Theorem A. □

4. Developing maps on the boundaries and correction

The result of this section will be used for the proof of Theorem A which is proved in the next section. However we treat in a more general manner for our purpose not only in quaternionic case. Recall that a geometric structure on a smooth n-manifold is a maximal collection of charts modeled on a simply connected n-dimensional homogeneous space X of a Lie group $\mathcal{G}$ whose coordinate changes are restrictions of transformations from $\mathcal{G}$. We call such a structure a $(\mathcal{G}, X)$-structure and a manifold with this structure is called a $(\mathcal{G}, X)$-manifold. In the paper [13], we have used the following lemma to show the uniqueness of developing maps in compact conformally flat manifolds.

Lemma *Let A be a Γ-invariant closed subset in X. Suppose that in the complement of A in X there exists a component U which admits a Γ-invariant complete Riemannian metric. Then the developing map* $\text{dev} : V \longrightarrow U$ *on each component V of* $\text{dev}^{-1}(U)$ *is a covering map.*

However we recognized that the statement of the above lemma is not valid in some geometric structure, which is shown by the example by Kapovich (Compare [9]). Choi and Lee [9] have shown that the lemma is true for any geometric structure under some additional condition on X. On the other hand, we have noticed that our results in [13] can be proved more directly without use of the above lemma. So the purpose of this section is to show that the geometric uniqueness of developing maps is true not only in compact conformally flat manifolds, but also in compact spherical CR manifolds and spherical pseudo-quaternionic manifolds. That is, our previous results of [13] will be generalized into the geometry on the boundary of rank one symmetric spaces of noncompact type.

Let $\mathbb{K}$ stand for the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the field of quaternions $\mathbb{F}$. Denote $|K| = 1, 2$, or 4 respectively. Let $\mathbb{K}^{n+2}$ denote the vector space, equipped with the Hermitian pairing over $\mathbb{K}$; $\mathcal{B}(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}$. Define the $(n+2)|K|$-dimensional cone V_- to be the subspace $\{z \in \mathbb{K}^{n+2} |\ \text{Re}(z) > 0,\ \mathcal{B}(z,z) < 0\}$. If $P : \mathbb{K}^{n+2} - \{0\} \longrightarrow \mathbb{KP}^{n+1}$ is the canonical projection onto the $\mathbb{K}$-projective space, then the image $P(V_-)$ is defined to be the $\mathbb{K}$-hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{K}}$ of dimension $(n+1)|K|$ (cf. [7]). Let $\text{O}(n+1, 1; \mathbb{K})$ be the subgroup of $\text{GL}(n+2, \mathbb{K})$ whose elements preserve the Hermitian form $\mathcal{B}$. Since $\text{O}(n+1, 1; \mathbb{K})$ leaves V_- invariant, it induces an action on $\mathbb{H}^{n+1}_{\mathbb{K}}$ whose kernel is the center $\mathcal{Z}(n+1, 1; \mathbb{K})$. It is isomorphic to $\{\pm 1\}$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{F}$, or the circle S^1 if $\mathbb{K} = \mathbb{C}$. Denote by $\mathrm{PO}(n+1,1;\mathbb{K})$ the quotient group $\mathrm{O}(n+1,1;\mathbb{K})/\mathcal{Z}(n+1,1;\mathbb{K})$. We usually write $\mathrm{PO}(n+1,1), \mathrm{PU}(n+1,1)$ or $\mathrm{PSp}(n+1,1)$, which is known as the full group of isometries of the complete simply connected $\mathbb{K}$-hyperbolic space $\mathbb{H}_{\mathbb{K}}^{n+1}$ respectively.

The projective compactification of $\mathbb{H}_{\mathbb{K}}^{n+1}$ is obtained by taking the closure $\bar{\mathbb{H}}_{\mathbb{K}}^{n+1}$ inside $\mathbb{KP}^{n+1}$. If we put an $(n+2)|K|-1$ dimensional subspace $V_0 = \{z \in \mathbb{K}^{n+2} \mid \mathcal{B}(z,z) = 0\}$, then $\bar{\mathbb{H}}_{\mathbb{K}}^{n+1} = \mathbb{H}_{\mathbb{K}}^{n+1} \cup P(V_0)$ so that the boundary $\partial\mathbb{H}_{\mathbb{K}}^{n+1} = P(V_0)$ is the standard sphere of dimension n, $2n+1$, $4n+3$ according to $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{F}$. Put $\partial\mathbb{H}_{\mathbb{K}}^{n+1} = S^{(n+1)|K|-1}$. Then the group of isometries $\mathrm{PO}(n+1,1;\mathbb{K})$ extends to a transitive action of projective transformations of $S^{(n+1)|K|-1}$. Thus we obtain the geometry $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$. In each case we call $(\mathrm{PO}(n+1,1), S^n)$ *conformally flat geometry,* $(\mathrm{PU}(n+1,1), S^{2n+1})$ *spherical CR geometry,* and $(\mathrm{PSp}(n+1,1), S^{4n+3})$ *spherical pseudo-quaternionic geometry.*

If $\mathbb{H}_{\mathbb{K}}^{m+1}$ $(1 \leqq m \leqq n-1)$ is the totally geodesic subspace of $\mathbb{H}_{\mathbb{K}}^{n+1}$, then the geometric subsphere $S^{(m+1)|K|-1}$ of $S^{(n+1)|K|-1}$ is defined to be $\partial\mathbb{H}_{\mathbb{K}}^{m+1}$. Put $\mathbb{Y} = S^{(n+1)|K|-1} - S^{(m+1)|K|-1}$ and denote by $\mathrm{Aut}(\mathbb{Y})$ the subgroup of $\mathrm{PO}(n+1,1;\mathbb{K})$ whose elements preserve $S^{(m+1)|K|-1}$. Then $\mathrm{Aut}(\mathbb{Y})$ is isomorphic to the subgroup $\mathrm{P}(\mathrm{O}(m+1,1;\mathbb{K}) \times \mathrm{O}(n-m;\mathbb{K}))$ (cf. [7], [20], [22]). Moreover $\mathbb{Y}$ is a Riemannian homogeneous space

$$\mathrm{P}(\mathrm{O}(m+1,1;\mathbb{K}) \times \mathrm{O}(n-m;\mathbb{K}))/\mathrm{P}(\mathrm{O}(m+1;\mathbb{K}) \times \mathrm{O}(1;\mathbb{K}) \times \mathrm{O}(n-m-1;\mathbb{K})).$$

Then the homogeneous Riemannian metric h on $\mathbb{Y}$ induces an equivariant Riemannian submersion:

$$S^{(n-m)|K|-1} \to (\mathrm{Aut}(\mathbb{Y}), \mathbb{Y}, h) \xrightarrow{\nu} (\mathrm{PO}(m+1,1;\mathbb{K}), \mathbb{H}_{\mathbb{K}}^{m+1}, h_0).$$

Here h_0 is the hyperbolic metric on $\mathbb{H}_{\mathbb{K}}^{m+1}$. (See [24], [20].)

There is the (equivariant) projection onto the closed ball: $\mathrm{O}(n-m;\mathbb{K}) \to S^{(n+1)|K|-1} \xrightarrow{P} \mathbb{D}_{\mathbb{K}}^{m+1}$. As the fixed point set Fix $(\mathrm{O}(n-m;\mathbb{K}), S^{(n+1)|K|-1}) = S^{(m+1)|K|-1}$, we note that $P|\mathbb{Y} = \nu$, *i.e.*, ν extends to a map identically on the ideal boundary $\partial(S^{(n+1)|K|-1} - S^{(m+1)|K|-1}) = S^{(m+1)|K|-1} = \partial\mathbb{H}_{\mathbb{K}}^{m+1}$.

Recall that if a smooth connected manifold M admits a $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$-structure, then there exists a developing pair (ϕ, dev), where $\mathrm{dev} : \tilde{M} \to S^{(n+1)|K|-1}$ is a *structure-preserving* immersion and $\phi : \pi_1(M) \to \mathrm{PO}(n+1,1;\mathbb{K})$ is a homomorphism whose image $\phi(\pi_1(M))$ is called the holonomy group for M. We prove the following.

Proposition 4.1. *Let M be a compact $(\mathrm{PO}(n+1,1;\mathbb{K}), S^{(n+1)|K|-1})$-manifold in dimension $(n+1)|K|-1$. Suppose that $\phi(\pi_1(M))$ leaves a geometric subsphere $S^{(m+1)|K|-1}$ $(0 \leqq m \leqq n-1)$. Then the restriction of the developing map*

$$\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow S^{(n+1)|K|} - S^{(m+1)|K|-1}$$

is a covering map.

Proof. Put $\pi = \pi_1(M)$ and $\Gamma = \phi(\pi)$. Since the holonomy group Γ leaves invariant a geometric subsphere $S^{(m+1)|K|-1}$, consider the composite of the restriction of the developing map dev and a Riemannian submersion h:

$$(\pi, \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \overset{(\rho,\mathrm{dev})}{\longrightarrow} (\mathrm{Aut}(\mathbb{Y}), \mathbb{Y}, h) \overset{\nu}{\longrightarrow} (\mathrm{PO}(m+1,1;\mathbb{K}), \mathbb{H}_{\mathbb{K}}^{m+1}, h_0).$$

Let $\mathrm{dev}^* h$ be the induced Riemannian metric on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$, which is invariant under π. We prove that $\mathrm{dev}^* h$ on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is complete. Let $\{x_i\}$ be a Cauchy sequence in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ with respect to $\mathrm{dev}^* h$. Assume that $\mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \neq \emptyset$. Let ρ^* (resp. ρ) be the distance function on $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ (resp. $\mathbb{Y}$), and ρ_0 be the (hyperbolic) distance function on $\mathbb{H}_{\mathbb{K}}^{m+1}$. As $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is invariant under π, M decomposes into the union

$$(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}))/\pi$$

and $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ where $\mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ consists of a finite number of compact submanifolds. If $P : \tilde{M} \longrightarrow M$ is a covering map, then the sequence $\{P(x_i)\}$ has an accumulation point y (after passing to a subsequence). Choose $\tilde{y} \in \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ with $P(\tilde{y}) = y$. There exists a neighborhood W of $\tilde{y}$ in $\tilde{M}$ such that the closure $\bar{W}$ is compact. Moreover, $P : \bar{W} \longrightarrow P(\bar{W})$ and $\mathrm{dev} : \bar{W} \longrightarrow \mathrm{dev}(\bar{W})$ are diffeomorphic. As $y \in P(W)$, there exist elements $\{\gamma_i\} \in \pi$ such that $\{\gamma_i \cdot x_i\} \in W$ for $i \geqq L$ where L is a sufficiently large number. We have $\lim \gamma_i \cdot x_i = \tilde{y}$. Since $\{x_i\}$ is Cauchy in $(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}), \rho^*)$, associated with each integer n, there exists an integer $\lambda(n)$ satisfying that if $i, j \geqq \lambda(n)$, $\rho^*(x_i, x_j) < \frac{1}{n}$. Let $B_{\frac{1}{n}}(x_{\lambda(n)})$ be the ball of radius $\frac{1}{n}$ centered at $x_{\lambda(n)}$ in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$. In particular,

$$\{x_i\} \in B_{\frac{1}{n}}(x_{\lambda(n)}) \quad \text{for } i \geqq \lambda(n).$$

As $\lambda(n)$ increases as n does, we can assume that $\lambda(n) \geqq n$ for $n \geqq N$ where N is a sufficiently large number with $N > L$. Note that $\{\gamma_{\lambda(n)} \cdot x_{\lambda(n)}\} \in W$ for $n \geqq N$ as above. Then we show that there is an integer m such that $B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)}) \subset W$. Suppose not. Put $\partial' W = \partial \bar{W} \cap (\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}))$.

Then for each $n \geqq N$, there is a point $z_{\lambda(n)}$ of $B_{\frac{1}{n}}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)})$ lying on $\partial' W$. Thus we have that $\rho^*(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}, z_{\lambda(n)}) \leqq \frac{1}{n}$.

In general, for every $z \in \partial' W \subset \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$, the metrics satisfy

$$\rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z)) \leqq \rho(\mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \mathrm{dev}(z)) \leqq \rho^*(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}, z).$$

By the above,

$$(*) \qquad \rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z_{\lambda(n)})) \leqq \frac{1}{n}.$$

As $\lim \gamma_i \cdot x_i = \tilde{y}$, and ν extends to a map identically on the boundary

$$\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}) \longrightarrow \nu \circ \mathrm{dev}(\tilde{y}) \in \nu(S^{(m+1)|K|-1}) = S^{(m+1)|K|-1} = \partial \mathbb{H}_{\mathbb{K}}^{m+1}.$$

On the other hand, $\bar{W}$ is a compact neighborhood of $\tilde{y}$ in $\tilde{M}$ and dev $|\bar{W}$ is diffeomorphic. In particular, $\nu \circ \mathrm{dev}(\tilde{y}) \in S^{(m+1)|K|-1} - \partial\mathrm{dev}(\bar{W})$. Since $\{z_{\lambda(n)}\} \in \partial' W$, $\{z_{\lambda(n)}\}$ has an accumulation point z in $\partial\bar{W}$. Passing to a subsequence if necessary, it follows that $\mathrm{dev}(z_{\lambda(n)}) \longrightarrow \mathrm{dev}(z) \in \partial\mathrm{dev}(\bar{W})$. Therefore, as $\nu \circ \mathrm{dev}(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})) \subset \mathbb{H}_{\mathbb{K}}^{m+1}$, we obtain either $\nu\circ\mathrm{dev}(z) \in \mathbb{H}_{\mathbb{K}}^{m+1}$ or $\nu\circ\mathrm{dev}(z) \in \partial\mathrm{dev}(\bar{W}) \cap S^{(m+1)|K|-1}$, i.e., $\nu\circ\mathrm{dev}(z) \neq \nu \circ \mathrm{dev}(\tilde{y})$. Moreover, when $\nu \circ \mathrm{dev}(z) \in \mathbb{H}_{\mathbb{K}}^{m+1}$, as $\nu \circ \mathrm{dev}(\tilde{y})$ lies on the boundary $S^{(m+1)|K|-1}$,

$$\lim_{n\to\infty} \rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z_{\lambda(n)})) = \infty,$$

which is impossible by $(*)$.

If $\nu \circ \mathrm{dev}(z) \in \partial\mathrm{dev}(\bar{W}) \cap S^{(m+1)|K|-1}$, then both $\nu \circ \mathrm{dev}(z)$ and $\nu \circ \mathrm{dev}(\tilde{y})$ are lying on the boundary but $\nu \circ \mathrm{dev}(\tilde{y}) \neq \nu \circ \mathrm{dev}(z)$ as above. This implies again

$$\lim_{n\to\infty} \rho_0(\nu \circ \mathrm{dev}(\gamma_{\lambda(n)} \cdot x_{\lambda(n)}), \nu \circ \mathrm{dev}(z_{\lambda(n)})) = \infty,$$

which is impossible. Hence this contradiction yields that $B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)}) \subset W$ for some m. Since $\{x_i\}_{i \geqq \lambda(m)} \in B_{\frac{1}{m}}(x_{\lambda(m)})$, the isometry $\gamma_{\lambda(m)}$ (with respect to ρ^*) shows that $\{\gamma_{\lambda(m)} \cdot x_i\}_{i \geqq \lambda(m)} \in B_{\frac{1}{m}}(\gamma_{\lambda(m)} \cdot x_{\lambda(m)})$. As $\bar{W}$ is compact, there is a point $w \in \bar{W}$ such that $\lim_{i\to\infty} \gamma_{\lambda(m)} \cdot x_i = w$. Therefore $\lim_{i\to\infty} x_i = \gamma_{\lambda(m)}^{-1} \cdot w$ for which $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w) = \lim_{i\to\infty} \mathrm{dev}(x_i)$. Since the sequence of images $\{\mathrm{dev}(x_i)\}$ is also Cauchy in $\mathbb{Y}$, $\{\mathrm{dev}(x_i)\}$ has a limit point in $\mathbb{Y}$, which therefore implies that $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w) \in \mathbb{Y}$. Thus $\mathrm{dev}(\gamma_{\lambda(m)}^{-1} \cdot w)$ is not contained in $S^{(m+1)|K|-1}$, *i.e.*, $\gamma_{\lambda(m)}^{-1} \cdot w \in \tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$. This shows that the Cauchy sequence $\{x_i\}$ converges in $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ so that $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})$ is complete. As a consequence, the local isometry dev : $\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow \mathbb{Y}$ is a covering map. □

Remark 4.2. *For the induced Riemannian metric from an arbitrary geometric structure, the above proof does not work with respect to the argument of* minimal geodesic; *the covering map $P : \tilde{M} \longrightarrow M$ induces a local isometry of $(\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}), \rho^*)$ onto $((\tilde{M} - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi, \hat{\rho}^*)$. Given a Cauchy sequence $\{y_j\}$ lying in $P(W)$, choose a lift of sequence $\{\tilde{y}_j\}$ from W. Since $P : W \longrightarrow P(W)$ is diffeomorphic, P : $W - \mathrm{dev}^{-1}(S^{(m+1)|K|-1}) \longrightarrow P(W) - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$ is an isometry, however, note that given two points y_i, y_j in $P(W)$, the minimal geodesic between y_i and y_j does not necessarily lie in $P(W) - \mathrm{dev}^{-1}(S^{(m+1)|K|-1})/\pi$. So the equality $\hat{\rho}^*(y_i, y_j) = \rho^*(\tilde{y}_i, \tilde{y}_j)$ does not hold in general, which implies that the lift $\{\tilde{y}_j\}$ is not necessarily Cauchy. We did not check this point for an arbitrary geometric structure, which is the mistake of the argument of the proof in Lemma B of* [13] *(also Lemma 4 of* [14]*). Thanks to the example by Kapovich, we verify this phenomenon.*

5. Geometric rigidity on spherical pseudo-quaternionic manifolds

Margulis has shown that: *Let G be a connected semisimple Lie group with trivial center and has no compact factor. Given an irreducible lattice Γ of G and a homomorphism $\rho : \Gamma \longrightarrow G'$ where G' is a semisimple Lie group with trivial center and without compact factor, ρ extends to a homomorphism from G to G' provided that the real rank of G is at least two and $\rho(\Gamma)$ is Zariski dense in G'.* Note that a connected semisimple Lie group with trivial center supports a real algebraic structure. (Compare [48].) This sort of result is called *Margulis' superrigidity* and the question is left to the rank one semisimple Lie groups, namely the real (resp. complex, quaternionic, Cayley) hyperbolic groups. It is known that the Margulis' superrigidity is false for the real hyperbolic case, for instance, because of the existence of *bending* (= a nontrivial deformation of Fuchsian groups in higher dimensions). Kevin Corlette has proved the Margulis' superrigidity affirmatively for the isometry group $\mathrm{PSp}(n, 1)$; *Let Γ be a lattice in $\mathrm{PSp}(n, 1)$ and G any semisimple Lie group with trivial center and without compact factor. If $\rho : \Gamma \longrightarrow G$ is a homomorphism with Zariski dense image and $n \geqq 2$, then ρ extends to a homomorphism $\varphi : \mathrm{PSp}(n, 1) \rightarrow G$.*

Using this fact, we prove Theorem B in Introduction.

Proof of Theorem B.

Given a compact spherical pseudo-quaternionic $(4n + 3)$-manifold M, there exists a developing pair $(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \longrightarrow (\mathrm{PSp}(n + 1, 1), S^{4n+3})$. Put $\pi = \pi_1(M)$, $\rho(\pi) = \Gamma$. Thus we have the holonomy representation $\rho : \pi \longrightarrow \Gamma \subset \mathrm{PSp}(n + 1, 1)$.

By the hypothesis, π is isomorphic to a discrete uniform subgroup of $\mathrm{PSp}(m, 1)$ for some $2 \leqq m \leqq n$. We may assume that $\pi \subset \mathrm{PSp}(m, 1)$. If Γ is virtually amenable, then Classification Theorem A shows that π is virtually nilpotent. This case does not occur by the hypothesis.

Let $\mathcal{A}(\Gamma)$ be the Zariski closure (real algebraic closure) of Γ in $\mathrm{PSp}(n + 1, 1)$. Then by Theorem 4.4.2 of [7] shows that either $\mathcal{A}(\Gamma) = \mathrm{PSp}(n + 1, 1)$ or $\mathcal{A}(\Gamma)$ leaves a proper totally geodesic subspace in $\mathbb{H}_{\mathbb{F}}^{n+1}$ invariant.

Step 1. Suppose that $\mathcal{A}(\Gamma) = \mathrm{PSp}(n+1, 1)$, that is, Γ is Zariski dense in $\mathrm{PSp}(n+1, 1)$. The superrigidity by Corlette implies that ρ extends to a continuous homomorphism $\varphi : \mathrm{PSp}(m, 1) \longrightarrow \mathrm{PSp}(n + 1, 1)$. Since φ is analytic (cf. [16]), the image $\varphi(\mathrm{PSp}(m, 1))$ is a connected Lie subgroup of $\mathrm{PSp}(n + 1, 1)$. As $\Gamma \subset \varphi(\mathrm{PSp}(m, 1))$ does not leave any proper totally geodesic subspace in $\mathbb{H}_{\mathbb{F}}^{n+1}$ invariant, $\varphi(\mathrm{PSp}(m, 1)) = \mathrm{PSp}(n + 1, 1)$ and so $m = n + 1$. Since $m \leqq n$ by our hypothesis, this is impossible.

Step 2. Suppose that $\mathcal{A}(\Gamma)$ leaves a proper totally geodesic subspace in $\mathbb{H}_{\mathbb{F}}^{n+1}$ invariant. A proper totally geodesic subspace in $\mathbb{H}_{\mathbb{F}}^{n+1}$ is isometric to $\mathbb{H}_{\mathbb{K}}^{k}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{F}$, $1 \leqq k \leqq n$), $\mathbb{H}_{\mathbb{R}}^{n+1}$, $\mathbb{H}_{\mathbb{C}}^{n+1}$, or a 3-dimensional $\mathbb{R}$-subspace $\mathbb{H}^1(\mathrm{I})$. (Compare [7].) Note that $\mathbb{H}^1(\mathrm{I})$ is orthogonal to $\mathbb{H}_{\mathbb{R}}^{1}$ in $\mathbb{H}_{\mathbb{F}}^{1}$ and so isometric to $\mathbb{H}_{\mathbb{R}}^{3}$. If $\mathcal{A}(\Gamma)$ leaves invariant a proper subspace $\mathbb{H}_{\mathbb{K}}^{k}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{F}$,
$k \leqq n$), then $\mathcal{A}(\Gamma)$ leaves also $\mathbb{H}_{\mathbb{F}}^{k}$ invariant. Thus $\mathcal{A}(\Gamma)$ preserves $S^{4k-1} = \partial \mathbb{H}_{\mathbb{F}}^{k}$. The subgroup of $\mathrm{PSp}(n + 1, 1)$ preserving S^{4k-1} is isomorphic to $\mathrm{Sp}(k, 1) \cdot \mathrm{Sp}(n - k + 1)$. In

particular, $\mathcal{A}(\Gamma) \subset \mathrm{Sp}(k,1)\cdot\mathrm{Sp}(n-k+1)$ for $k \leqq n$. Recall that there is a Γ-invariant homogeneous Riemannian metric on $S^{4n+3} - S^{4k-1}$ from (1.2).

If $\mathcal{A}(\Gamma)$ leaves invariant $\mathbb{H}_{\mathbb{K}}^{n+1}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), then $\mathcal{A}(\Gamma)$ preserves $S^n = \partial\mathbb{H}_{\mathbb{R}}^{n+1}$ (resp. $S^{2n+1} = \partial\mathbb{H}_{\mathbb{C}}^{n+1}$). By Proposition 1.4, Corollary 1.5, $\mathcal{A}(\Gamma) \subset \mathrm{PO}(n+1,1)\times\mathrm{SO}(3)$ or $\mathcal{A}(\Gamma) \subset \mathrm{P}(\mathrm{U}(n+1,1)\cdot S^1\{\pm1, \pm j\})$ respectively. Moreover $S^{4n+3} - S^n$ (resp. $S^{4n+3} - S^{2n+1}$) admits a Γ-invariant complete Riemannian metric.

If $\mathcal{A}(\Gamma)$ preserves $\mathbb{H}^1(\mathrm{I})$, then it leaves also $\mathbb{H}_{\mathbb{F}}^1$ invariant. As $\mathbb{H}^1(\mathrm{I})$ is isometric to $\mathbb{H}_{\mathbb{R}}^3$, the subgroup of $\mathrm{PSp}(n+1,1)$ preserving $\mathbb{H}^1(\mathrm{I})$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})\times\mathrm{Sp}(n)$ where $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Iso}(\mathbb{H}_{\mathbb{R}}^3)$. Thus $\mathcal{A}(\Gamma) \subset \mathrm{PSL}(2,\mathbb{C})\times\mathrm{Sp}(n)$ which leaves invariant $S^3 = \partial\mathbb{H}_{\mathbb{F}}^1$.

Let $\tau : \mathrm{Sp}(k,1)\cdot\mathrm{Sp}(n-k+1) \longrightarrow \mathrm{PSp}(k,1)$ be the canonical projection; similarly for $\tau : \mathrm{PSL}(2,\mathbb{C})\times\mathrm{Sp}(n)\longrightarrow\mathrm{PSL}(2,\mathbb{C})$, $\tau : \mathrm{PO}(n+1,1)\times\mathrm{SO}(3)\longrightarrow\mathrm{PO}(n+1,1)$ or $\tau : \mathrm{P}(\mathrm{U}(n+1,1)\cdot S^1\{\pm1,\pm j\})\longrightarrow\mathrm{PU}(n+1,1)$ respectively. Suppose that H is one of $\mathrm{PSp}(k,1)$, $\mathrm{PSL}(2,\mathbb{C})$, $\mathrm{PO}(n+1,1)$, or $\mathrm{PU}(n+1,1)$.

Since τ is a proper map and $\mathcal{A}(\Gamma)$ (with finitely many components) leaves the proper totally geodesic subspace invariant, Theorem 4.4.1 of [7] implies that $\tau(\mathcal{A}(\Gamma)) = H$. In particular, $\tau(\Gamma)$ is Zariski dense in H. We obtain a homomorphism $\tau\circ\rho : \pi\longrightarrow H$ with Zariski dense image $\tau(\Gamma)$. As H is a noncompact simple Lie group and without center, applying the Corlette's superrigidity, $\tau\circ\rho$ extends to a continuous homomorphism Ψ : $\mathrm{PSp}(m,1)\longrightarrow H$. As in (i), we obtain that $\Psi(\mathrm{PSp}(m,1)) = H$. Moreover $\mathrm{PSp}(m,1)$ has no normal subgroup, so $\Psi : \mathrm{PSp}(m,1)\longrightarrow H$ is an isomorphism. Therefore $2 \leqq m = k \leqq n$. As a consequence, Γ is a discrete uniform subgroup of $\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)$. In particular we have $\Lambda(\Gamma) = L(\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1)) = S^{4m-1}$. Consider the developing map

$$\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{4m-1})\longrightarrow S^{4n+3} - S^{4m-1}$$

for which the homogeneous Riemannian metric h on $S^{4n+3} - S^{4m-1}$ (cf. 1.2) induces a Riemannian submersion:

$$S^{4(n-m)+3}\rightarrow(\mathrm{Sp}(m,1)\cdot\mathrm{Sp}(n-m+1), S^{4n+3} - S^{4m-1}, h) \xrightarrow{\nu} (\mathrm{PSp}(m,1), \mathbb{H}_{\mathbb{F}}^m, \hat{h}).$$

Here $\hat{h}$ is the hyperbolic metric on $\mathbb{H}_{\mathbb{F}}^m$. Let $\mathrm{dev}^* h$ be the induced Riemannian metric on $\tilde{M} - \mathrm{dev}^{-1}(S^{4m-1})$ such that $\mathrm{dev}^* h$ is invariant under π. We prove that $\mathrm{dev}^* h$ on $\tilde{M} - \mathrm{dev}^{-1}(S^{4m-1})$ is complete. Recall that ν maps the boundary $S^{4m-1} = \partial(S^{4n+3} - S^{4m-1})$ identically onto $S^{4m-1} = \partial\mathbb{H}_{\mathbb{F}}^m$ under the quotient map: $\mathrm{Sp}(n-m)\rightarrow S^{4n+3} \xrightarrow{\nu} D^{4m}$.

Then by Proposition 4.1, the developing map $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(S^{4m-1})\longrightarrow S^{4n+3} - S^{4m-1}$ is a covering map. Since $S^{4n+3} - S^{4m-1}$ is simply connected,

$$\mathrm{dev} : \tilde{M} - \overset{-1}{\mathrm{dev}}(S^{4m-1})\longrightarrow S^{4n+3} - S^{4m-1}$$

is a diffeomorphism. In particular, $\mathrm{dev} : \tilde{M}\longrightarrow\mathrm{dev}(\tilde{M})$ is a diffeomorphism. As Γ is discrete and acts properly discontinuously on $\mathrm{dev}(\tilde{M})$, $\mathrm{dev}(\tilde{M}) \subset S^{4n+3} - L(\Gamma)$. As above $L(\Gamma) = S^{4m-1}$, so $\mathrm{dev} : \tilde{M}\longrightarrow S^{4n+3} - S^{4m-1}$ is diffeomorphic. Thus M is isomorphic to

$S^{4n+3} - S^{4m-1}/\Gamma$ which is the locally homogeneous space

$$\mathrm{Sp}(m) \times \Delta\mathrm{Sp}(1) \times \mathrm{Sp}(n-m)\backslash\mathrm{Sp}(m,1) \times \mathrm{Sp}(n-m+1)/\Gamma$$

where $2 \leqq m \leqq n$. This completes the proof of Theorem B. □

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY,1-1, MINAMI-OHSAWA, HACHIOJI, TOKYO 192-0397, JAPAN

E-mail address: yoshi@comp.metro-u.ac.jp

CONFIGURATION SPACES OF POINTS ON THE CIRCLE AND HYPERBOLIC DEHN FILLINGS

SADAYOSHI KOJIMA, HARUKO NISHI, AND YASUSHI YAMASHITA

1. Configuration Space

Let $X(n)$ be a space of configurations of n distinct points in the real projective line $\mathbf{RP}^1$ up to projective automorphisms. $X(n)$ can be expressed by the point set,

$$X(n) = ((\mathbf{RP}^1)^n - \mathbf{D}) / \operatorname{PGL}(2, \mathbf{R}),$$

where $\mathbf{D}$ is the big diagonal set,

$$\mathbf{D} = \{(\alpha_1, \cdots, \alpha_n) \in (\mathbf{RP})^n \mid \alpha_i = \alpha_j \text{ for some } i \neq j\},$$

and $\operatorname{PGL}(2, \mathbf{R})$ acts on $(\mathbf{RP}^1)^n$ diagonally. We assume that the number of points is at least five.

There are two obvious observations. $X(n)$ is not connected since we are not allowed to have collisions of points. Reading off markings of points in the configurations in cyclic order, each component is labeled by a circular permutation of n letters up to reflection. In particular, the number of connected components is $(n-1)!/2$. Also a configuration can be normalized by sending three consecutive points to $\{0, 1, \infty\}$ so that the other points lie in the open unit interval $(0, 1)$. Thus each component of $X(n)$ is identified with the set of ordered $n-3$ points in $(0, 1)$, and in particular, is homeomorphic to a cell of dimension $n-3$.

Hence $X(n)$ is topologically not quite interesting in fact. However it contains much more rich structures, see for instance [3, 13, 14, 15]. The present article is to briefly describe more geometric aspect of the configuration space $X(n)$ recently developed by the authors, where the details will appear in [7, 12].

2. Hyperbolization and Gluing

Consider Euclidean n-gons with vertices marked by integers from 1 to n, where the marking may not be cyclically monotone. Let $X_{n,c}$ be the set of all marked equiangular n-gons up to mark preserving, possibly orientation reversing, congruence, and X_n a further quotient of $X_{n,c}$ by similarities.

For any $\alpha \in (\mathbf{RP}^1)^n - \mathbf{D}$, we assign the unit disc in $\mathbf{C}$ with n points specified on the boundary. By the Schwarz-Christoffel mapping or its complex conjugate, we can map α to an *equiangular* n-gon up to mark preserving similarity. This induces a map from $X(n)$ to X_n since a projective transformation on the unit disc does not change the image of the map. It is also injective because if two configurations α and β map to the same element of $X_{n,c}$ by f_α and f_β, then $f_\beta \circ f_\alpha^{-1}$ is a mark preserving projective automorphism of the unit disk. By the Carathéodory theorem, this map is surjective. Therefore there is a canonical homeomorphism between $X(n)$ and X_n.

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Each connected component of $X(n)$ consists of the configurations with a fixed circular permutation of markings up to reflection. For the simplicity for the moment, we shall focus on the component U of $X_{n,c}$ which is labeled by $12\cdots n$. Let $U_s \subset X_n$ be the set of its mark preserving similarity classes. Note that U_s corresponds to the component of $X(n)$ labeled by $12\cdots n$ also. We identity U_s with the set U_1 which, by definition, consists of the set of equiangular polygons having area $= 1$.

Each element of U can be described by a vector of side lengths $(x_1, x_2, \dots, x_n)$ where x_j is the length of the edge between the vertices marked by j and $j+1$. Since they represent an equiangular n-gon, they satisfy:

$$x_1 + x_2\zeta_n + \cdots + x_n\zeta_n^{n-1} = 0$$

where $\zeta_n = \exp(2\pi i/n)$. Set

$$\mathcal{E}_n := \left\{(x_1, x_2, \dots, x_n) \mid x_1 + x_2\zeta_n + \cdots + x_n\zeta_n^{n-1} = 0\right\},$$

$$\mathcal{E}_n^+ := \mathcal{E}_n \cap \bigcap_{j=1}^{n} \{x_j > 0\}.$$

Note that U can be identified with $\mathcal{E}_n^+$.

For each element P of $\mathcal{E}_n^+$, we denote by Area(P) the area of P. It is a quite surprising observation by Thurston implicitly in [11] and explicitly in [4, 8] that the "Area" is a quadratic form of signature $(1, n-3)$ on $\mathcal{E}_n$. Now $\mathcal{E}_n$ together with Area becomes a Minkowski space. Let $\mathcal{P}_n$ be a connected component of $\text{Area}^{-1}(1)$ containing polygons with positive side lengths. Then $\mathcal{P}_n$ is the hyperbolic space and U_1 is canonically identified with

$$\text{Area}^{-1}(1) \cap \mathcal{E}_n^+ = \mathcal{P}_n \cap \bigcap_{i=1}^{n} \{x_i > 0\}.$$

The region is bounded by $\mathcal{P}_n \cap \{x_i = 0\}$ for $i = 1, 2, \cdots, n$. Since $\{x_i = 0\}$ represents a hyperplane containing the origin in $\mathcal{E}_n$, the intersection with $\mathcal{P}_n$ is the hyperbolic hyperplane. Then the closure of $\text{Area}^{-1}(1) \cap \mathcal{E}_n^+$ is an $(n-3)$-dimensional hyperbolic polyhedron. We denote it by Δ_n and the face corresponding to $\{x_i = 0\} \cap \Delta_n$ by F_i. Then a computation in [7] shows

Lemma 1. *The faces of Δ_n intersect as follows.*

(1) $|i-j| \geq 2 \Rightarrow F_i \perp F_j$,
(2) *If* $n = 5$ *or* 6, *then* $F_j \cap F_{j+1} = \emptyset$,
(3) *If* $n \geq 7$,

$$\cos(\omega_n) = \frac{1}{2\cos\frac{2\pi}{n}},$$

where ω_n is the dihedral angle between F_j and F_{j+1}.

ω_n is monotone increasing with respect to n, and approaches $\pi/3$ when $n \to \infty$. $\omega_6 = 0$ and $\omega_8 = \pi/4$. These are the only cases when ω_n is a rational multiple of π.

The hyperbolization of U_1 assigns not only a projective class of a configuration to each point in the interior of Δ_n but a degenerate configuration to each point on the boundary. Gluing $(n-1)!/2$ copies of Δ_n together along the faces which represent the same degenerate configurations and we obtain $\overline{X_n}$. $X_n \approx X(n)$ now lives in $\overline{X_n}$ as an open dense subset.

The point lying on a face of codimension one in Δ_n corresponds to a configuration with a collision of two points. The number of polyhedra which share such a face is two according to how nondegenerate configurations approach to the degenerate one. Hence the gluing does not yield any singularity along such a face. This proves that the gluing gives a hyperbolic cone-manifold by the definition of cone-manifolds.

The point lying on a face of codimension two in Δ_n corresponds to a configuration with either a pair of collisions of two points or a collision of three point. In the first case, the number of polyhedra which share such a face is four according to how nondegenerate configurations approach to the degenerate one. On the other hand, the dihedral angle of two faces which share a codimension two face is $\pi/2$ by Lemma 1 (1). Hence again the gluing does not yield singularity along such a face.

When $n = 5$, the above two observations show that $\overline{X_5}$ is nonsingular. Actually, it is a hyperbolic surface which consists of 12 hyperbolic right angle pentagons. Since each vertex belongs to four pentagons, the number of faces, edges, vertices are $12, 30, 15$ respectively and Euler characteristic is -3. Hence it is a nonorientable surface homeomorphic to a connected sum of five copies of $\mathbf{RP}^2$.

When $n = 6$, $\overline{X_6}$ consists of 60 hyperbolic hexahedra. In this case, we are not allowed to have a collision of three successive points and the gluing does not yield any singularity along face of codimension at most two. Consider a point on the face of codimension three. Since $n - 3 = 3$, such a face is a vertex and corresponds to a triple of collisions of two points. The number of hexahedra in $\overline{X_6}$ which share such a vertex is eight. On the other hand, a neighborhood of the vertex of Δ_6 is isometric to a neighborhood of the vertex of the first orthant in the Poincaré model of the hyperbolic space. Hence again the gluing does not yield singularity. Moreover, since a horospherical cut of an ideal vertex in Δ_6 is always square, the gluing yields complete ends. Therefore $\overline{X_6}$ is a complete hyperbolic 3-manifold.

We can derive a few more informations about geometry of $\overline{X_6}$. Since Δ_6 is scissors congruent to a quarter of the regular ideal octahedron, whose volume is $3.66386\cdots$, the volume of $\overline{X_6}$ is $54.957\cdots$. $\overline{X_6}$ admits a natural action of the symmetry group of degree 6 by permuting markings of points. It turns out to be a full isometry group since the quotient is congruent to the smallest nonorientable orbifold with $\{4,4,2\}$-cusp in [1]. We will see in the next section that $\overline{X_6}$ has 10 cusps.

When $n \geq 7$, we must consider a neighborhood of a degenerate configuration with a collision of three successive points on a face of codimension two in $\overline{X_n}$. There are 6 polyhedra which share the degenerate configuration according to the permutations of three markings involved in the collision. By Lemma 1, the dihedral angle of each piece is less than $2\pi/6$, and the singularity appears. Hence we have

Theorem 2. *$\overline{X_n}$ is a hyperbolic cone-manifold.*

- *When $n = 5$ or 6, it is nonsingular.*
- *When $n \geq 7$, the singular set is nonempty.*

Remark 3. Every configuration appeared in $\overline{X_n}$ has at least three marked points which are disjointly placed on the circle. Normalize neighbor configurations of any particular degenerate one by sending such marked points to $\{0, 1, \infty\}$ by a projective automorphism, and we can parameterize its neighborhood in $\overline{X_n}$ by the position of other marked $n - 3$ points. This shows that $\overline{X_n}$ is topologically a manifold.

3. WEIGHTS

The configuration space $X(n)$ was identified with the space of marked equiangular n-gons up to similarity X_n. The identification was given by the Schwarz-Christoffel mapping with constant external angles $2\pi/n$. If we replace a constant $2\pi/n$ by other nonconstant external angles, the images of the Schwarz-Christoffel mapping change and one can expect that the hyperbolic structure of the configuration space will be changed also.

Let Θ_n be the set of n-tuples of real numbers $\theta = (\theta_1, \ldots, \theta_n)$ satisfying the relations

$$\sum_{j=1}^{n} \theta_j = 2\pi \quad \text{and} \quad 0 < \theta_i + \theta_j < \pi \quad (i, j \in \{1, \ldots, n\}).$$

The second condition is to ensure that a collision of any two consecutive points forms a nonempty top dimensional face.

Fix $\theta = (\theta_1, \ldots, \theta_n)$ in Θ_n. Choose $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{RP}^1)^n - \mathbf{D}$, and assign to α the unit disc in $\mathbf{C}$ with n points specified on the boundary. Then we map it by the Schwarz-Christoffel formula to an n-gon P whose vertices are the images of the specified points and the external angle of the image of the jth point α_j is θ_j. P is defined up to mark preserving similarity. Let $X_{n,\theta}$ be the space of mark preserving similarity classes of Euclidean n-gons with external angles $\{\theta_1, \cdots, \theta_n\}$ compatible with markings. Then again, there is a canonical homeomorphism between $X(n)$ and $X_{n,\theta}$.

The hyperbolization for each component of $X_{n,\theta}$ can be worked out in a completely same manner as before. Although the components of $X_{n,\theta}$ are no longer congruent each other, the gluing rule still makes sense, and we obtain a hyperbolic cone-manifold $\overline{X_{n,\theta}}$ as well by identifying the boundary of $X_{n,\theta}$. $\overline{X_{n,\theta}}$ contains $X_{n,\theta} \approx X(n)$ as an open dense subset. $\overline{X_{n,\theta}}$ is a deformation of $\overline{X_n}$ in some sense. However the deformation theory which fits our setting has been developed only when $n = 5, 6$. We will see only these cases more precisely.

4. To Teichmüller Space When $n = 5$

Fixing $\theta = (\theta_1, \ldots, \theta_5)$ in Θ_5, let $p = \langle i_1 i_2 i_3 i_4 i_5 \rangle$ be a circular permutation of $\{1, \ldots, 5\}$ up to refection and $U_{p,\theta}$ a component of $X_{5,\theta}$ which consists of all pentagons whose marking correspond to p. The external angle of the vertex marked by i is θ_i by definition. Then the hyperbolization gives a hyperbolic pentagon $\Delta_{p,\theta}$ whose interior bijectively corresponds to $U_{p,\theta}$. The boundary of $\Delta_{p,\theta}$ can be interpreted as the set of appropriately degenerate configurations.

For each $\theta \in \Theta_5$, the appropriate extension of Lemma 1 (1) shows that $\Delta_{p,\theta}$ is a right angle pentagon, and $\overline{X_{5,\theta}}$ is a hyperbolic surface with the same topology as $\overline{X_5} \approx \#^5\mathbf{RP}^2$. The surface is nonorientable and does not support any complex structure at all. However an analogue of the Teichmüller theory can be established. In fact, if we choose a maximal family of mutually disjoint nonparallel simple closed curves, then the set of hyperbolic structures is parameterized by their lengths and twisting amount for 2-sided ones. Hence the Teichmüller space $\mathcal{T}(\#^5\mathbf{RP}^2)$ is homeomorphic to $\mathbf{R}^9$. We thus obtained a map

$$\Phi_5 : \Theta_5 \to \mathcal{T}(\#^5\mathbf{RP}^2)$$

by assigning to θ a marked isometry class of $\overline{X_{5,\theta}}$.

We may label an edge of $\Delta_{p,\theta}$ by a circular permutations of 3 markings and a group of 2 markings, such as $i_1 i_2 i_3 (i_4 i_5)$, involved in the collision up to reflection. Then three edges with common collision pair, such as $i_1 i_2 i_3 (i_4 i_5)$, $i_2 i_1 i_3 (i_4 i_5)$, $i_2 i_3 i_1 (i_4 i_5)$, define a simple closed geodesic on $\overline{X_{5,\theta}}$. The length of such a curve gives a Teichmüller invariant.

Define a function $N(i_1i_2i_3;\theta)$ by

$$N(i_1i_2i_3;\theta) = \frac{\sin\theta_{i_1}\sin\theta_{i_2}\sin\theta_{i_3} - \sin(\theta_{i_1}+\theta_{i_2}+\theta_{i_3})(\sin\theta_{i_1}\sin\theta_{i_2}+\sin\theta_{i_2}\sin\theta_{i_3}+\sin\theta_{i_3}\sin\theta_{i_1})}{\sin(\theta_{i_1}+\theta_{i_2})\sin(\theta_{i_2}+\theta_{i_3})\sin(\theta_{i_3}+\theta_{i_1})}.$$

A technical computation worked out in [7] shows that the length of a simple closed curve formed by collisions of the vertices i_4 and i_5, which we denote by $L(i_4i_5;\theta)$, satisfies

$$\cosh(L(i_4i_5;\theta)) = N(i_1i_2i_3;\theta).$$

This formula was used to show that Φ_5 is a local embedding at $\theta_0 = (2\pi/5,\dots,2\pi/5)$.

In [12], we choose different Teichmüller invariants to analyze the global property of Φ_5. There are ten simple closed geodesics on $\overline{X_{5,\theta}}$ represented by a pair of markings as above. They are uniquely placed on $\overline{X_{5,\theta}}$ since the geodesic representative of simple closed curves within their homotopy class is unique. Hence a geometric cell decomposition by such curves, which consists of 12 right angle pentagons, is uniquely determined by the hyperbolic structure. In particular, the shapes of such pentagons are Teichmüller invariants.

To see the inverse of Φ_5, we take the shapes of two pentagons with only a common vertex. Remember that the set of marked right angle pentagons is naturally parameterized by a 2-dimensional cell, see [8]. Hence if we ignore the gluing consistency, two pentagons have a four dimensional freedom to change their shapes. In [12], we show by a similar method as in [8] that there is no obstruction from the gluing consistency in fact, and all possible combination of shapes of such a pair is realized by a unique $\theta \in \Theta_5$. We thus obtain the right inverse of Φ_5 and now establish

Theorem 4. *Φ_5 is an embedding.*

5. To Dehn Filling Space When $n = 6$

We continue to use the notations Θ_6, $p = \langle i_1i_2i_3i_4i_5i_6\rangle$, $U_{p,\theta}$, $\Delta_{p,\theta}$ and $\mathcal{P}_6$, etc. Then denote simply by (i_ji_{j+1}) the face of $\Delta_{p,\theta}$ which corresponds to the collisions of the points marked by i_j and i_{j+1}. When $\theta_0 = (2\pi/6,\cdots,2\pi/6)$, Δ_{p,θ_0} has three ideal vertices. The four faces containing an ideal vertex have labels of type (i_ki_{k+1}), $(i_{k+1}i_{k+2})$, $(i_{k+3}i_{k+4})$, $(i_{k+4}i_{k+5})$ for some $k \in \{0,\dots,5\}$. We denote this vertex by $(i_ki_{k+1}i_{k+2})(i_{k+3}i_{k+4}i_{k+5})$.

If $\theta_{i_k}+\theta_{i_{k+1}}+\theta_{i_{k+2}} < \pi$, then three vertices i_k, i_{k+1}, i_{k+2} of the hexagon can collide, and (i_ki_{k+1}) and $(i_{k+1}i_{k+2})$ intersects in $\mathcal{P}_6$. If $\theta_{i_{k+3}}+\theta_{i_{k+4}}+\theta_{i_{k+5}} < \pi$, then $(i_{k+3}i_{k+4})$ and $(i_{k+4}i_{k+5})$ intersects (see figure 1). Let us use the notation $(i_1i_2i_3)i_4i_5i_6$ and $i_1i_2i_3(i_4i_5i_6)$ to indicate such edges.

By Lemma 1 (1), each dihedral angle around the old edges is $\pi/2$, so that they fit together without yielding any singularity after gluing. But the hyperbolic structure at the new edges will be singular in $\overline{X_{6,\theta}}$. A cross section perpendicular to the new edge will be a cone, obtained by taking a 2-dimensional hyperbolic sector of some angle and identifying the two bounding rays emanating from the center. Such a singular structure appears in the hyperbolic Dehn filling theory in [10].

Let $\mathcal{L}$ be the union of singular loci (or ideal vertices). $\overline{X_{6,\theta}} - \mathcal{L}$ is homeomorphic to $\overline{X_6}$ and $\overline{X_{6,\theta}}$ is its Dehn filled resultant. $\overline{X_{6,\theta}} - \mathcal{L}$ carries a nonsingular but incomplete hyperbolic metric. The holonomy representation of $\overline{X_{6,\theta}} - \mathcal{L}$ lifts to $\rho_\theta : \Pi = \pi_1(\overline{X_6}) \to \mathrm{SL}(2,\mathbf{C})$. The algebro geometric quotient of all $\mathrm{SL}(2,\mathbf{C})$-representations of Π is called a character variety and denoted by $X(\Pi)$. Assigning the trace of ρ_θ to each θ, we obtain

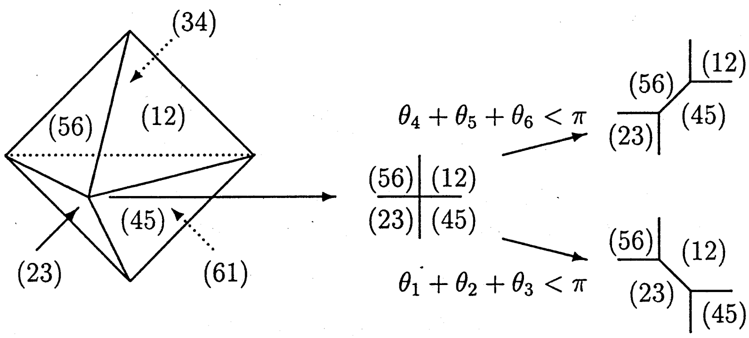

FIGURE 1. Cusp of $\overline{X_6}$.

the map

$$\Phi_6 : \Theta_6 \to X(\pi_1(\Pi)).$$

To find Dehn filling invariants, let us very briefly review foundations of the hyperbolic Dehn filling theory based on Thurston [10], Neumann-Zagier [9], Culler-Shalen [5] and Hodgson-Kerckhoff [6]. Let N be an orientable complete hyperbolic 3-cone-manifold of finite volume with singularity $\mathcal{L}$ and $\rho : \pi_1(N - \mathcal{L}) \to \mathrm{SL}(2, \mathbf{C})$ a lift of the holonomy representation of $N - \mathcal{L}$. If $m_1, \ldots, m_s$ are meridional curves for $\mathcal{L}$, then the map $f : X(\Pi) \to \mathbf{C}^s$ defined by

$$f(\chi) = (\chi(m_1), \cdots, \chi(m_s))$$

is a local diffeomorphism at χ_ρ where χ_ρ is the trace of ρ. In particular, the traces of meridional elements are Dehn filling invariants.

The ideal vertices of Δ_{p,θ_0}'s lie in the same component of cusps in $\overline{X_6}$ iff the labels are identical as a partition of six numbers . Hence the number of cusps is equal to the number of partitions of $\{1, 2, \cdots, 6\}$ into a pair of three numbers, $= \binom{6}{3}/2 = 10$. We may use the notation $(i_1 i_2 i_3)(i_4 i_5 i_6)$ to indicate a component of cusps also.

To define appropriate meridional elements, assume for the moment that $i_1 + i_2 + i_3 < \pi$. Then the cusp labeled by $(i_1 i_2 i_3)(i_4 i_5 i_6)$ becomes a singular locus in $\overline{X_{6,\theta}}$ labeled by $(i_1 i_2 i_3) i_4 i_5 i_6$, and there is a natural meridional element winding once around the singular locus. We denote it by $m_{i_1 i_2 i_3}$. Note that $m_{i_1 i_2 i_3}$ is a meridional element if $i_1 + i_2 + i_3 < \pi$ but no longer meridional if $i_1 + i_2 + i_3 > \pi$. Actually $\rho_\theta(m_{i_1 i_2 i_3})$ is a rotation, a parabolic translation or a hyperbolic translation according to whether $\theta_{i_1} + \theta_{i_2} + \theta_{i_3}$ is less than, equal to or greater than π. Then a computation worked out in [7] shows the identity,

$$\chi_{\rho_\theta}(m_{i_1 i_2 i_3}) = 2N(i_1 i_2 i_3; \theta).$$

This formula was used to show that Φ_6 is a local embedding at $\theta_0 = (2\pi/6, \ldots, 2\pi/6)$.

In [12], we choose different Dehn filling invariants to analyze the global property of Φ_6. There are 15 geodesic surfaces in $\overline{X_{6,\theta}}$ represented by a pair of markings as in the case when $n = 5$. They are uniquely placed on $\overline{X_{6,\theta}}$ since the geodesic representative of a surface within their proper homotopy class is unique if any. Hence a geometric cell decomposition by such surfaces, which consists of 60 hexahedra, is uniquely determined.

To see the inverse of Φ_6, we take the shapes of two hexahedra with only a common edge. Recall that the set of a hexahedra arisen in this context is naturally parameterized by a 3-dimensional cell, see [2]. Hence if we ignore other gluing consistencies than having only a common edge, they has a five dimensional freedom to change their shapes. In [12], we show by a similar method as in [2] that there is no obstruction from gluing consistency in fact, and all possible combination of shapes of such a pair can be realized by a unique $\theta \in \Theta_6$. We thus obtain a right inverse of Φ_6 and now establish

Theorem 5. *Φ_6 is an embedding.*

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Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ohokayama, Meguro, Tokyo 152-8552 Japan
E-mail address: sadayosi@is.titech.ac.jp

Department of Mathematics, Kyushu University, 33, Fukuoka 812-8581 Japan
E-mail address: nishi@math.kyushu-u.ac.jp

Department of Information and Computer Sciences, Nara Women's University, Kita-Uoya Nishimachi, Nara 630-8506, Japan
E-mail address: yamasita@ics.nara-wu.ac.jp

TOPOLOGICAL ASPECTS OF KLEINIAN GROUPS

KEN'ICHI OHSHIKA

This note is based on a series of talks by the author given at Seoul University in September 1997. The aim of the note is to expose recent progress in Kleinian group theory using topological techniques.

Kleinian groups are discrete subgroups of the Lie group $PSL_2\mathbf{C}$. Since the group $PSL_2\mathbf{C}$ is the group of isometries of the hyperbolic space $\mathbf{H}^3$, if G is a torsion-free Kleinian group, the quotient $\mathbf{H}^3/G$ becomes a complete hyperbolic 3-manifold. (This is because a torsion-free Kleinian group acts on $\mathbf{H}^3$ without fixed points.) The study of Kleinian groups is closely related to that of hyperbolic 3-manifolds in this way. On the other hand, $PSL_2\mathbf{C}$ is also the group of conformal automorphisms of the Riemann sphere S^2. (The Riemann sphere can be regarded as the sphere at infinity of $\mathbf{H}^3$. We denote the sphere by S^2_∞ when we regard it as the sphere at infinity.) From this point of view, the Kleinian group theory is related to the complex analysis of one-variable, and to the theory of Teichmüller spaces. What makes studying Kleinian groups interesting mathematically is this Janus-like nature. In this note, Kleinian group we deal with are assumed to be torsion free and finitely generated.

1. Hyperbolic 3-manifolds of infinite volume

The well-known rigidity theorem by Mostow asserts that two homotopically equivalent hyperbolic 3-manifolds of finite volume are in fact isometric.

Theorem 1.1 (Mostow). *Let M_1, M_2 be hyperbolic 3-manifolds of finite volume. Let $f : M_1 \to M_2$ be a homotopy equivalence. Then f can be homotoped to an isometry.*

The non-trivial elements of $PSL_2\mathbf{C}$ are classified into three families. The first is that of loxodromic elements which are conjugate to matrices of the form $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, where $|\lambda| > 1$. As a translation acting on $\mathbf{H}^3$, a loxodromic element leaves a geodesic invariant and rotates $\mathbf{H}^3$ around the invariant geodesic while translating it. The second family is that of parabolic elements which are conjugate to a matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This means that a parabolic element fixes a unique point in the Riemann sphere S^2_∞, which is regarded as the points at infinity of $\mathbf{H}^3$, and that if we set the upper half-space model so that the fixed point corresponds to ∞, the element acts on the upper half-space as a Euclidean translation. The third family is that of elliptic elements which are conjugate to matrices of the form $\begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix}$, where $|\omega| = 1$. An elliptic element fixes a point in $\mathbf{H}^3$. Therefore torsion-free Kleinian groups contain no elliptic elements. For a Kleinian group G, the parabolic elements fixing a point x form an abelian group which is isomorphic to either an infinite cyclic group or $\mathbf{Z} \times \mathbf{Z}$. We can choose a horoball B touching S^2_∞ at the point x so that B is translated to a horoball disjoint from B by elements of G other than the parabolic elements fixing x. For such a horoball B, its quotient by the abelian group formed by the parabolic elements stabilizing it, is contained in $\mathbf{H}^3/G$, and called a cusp.

A hyperbolic 3-manifolds of finite volume may have cusps corresponding to parabolic subgroups isomorphic to $\mathbf{Z} \times \mathbf{Z}$, but not those isomorphic to $\mathbf{Z}$.

Mostow's rigidity theorem says in particular that a Kleinian group with finite-volume quotient cannot be deformed. In contrast to this, Kleinian groups with infinite-volume quotient have non-trivial deformations as depicted below.

Example 1.2. Consider a Fuchsian group $G \subset PSL_2\mathbf{R}$ corresponding to the fundamental group of a closed surface S of genus g greater than 1. The space of faithful discrete representations of $\pi_1(S)$ to $PSL_2\mathbf{R}$ modulo conjugacy is exactly the Teichmüller space of S, which is known to be homeomorphic to the Euclidean space of dimension $6g-6$. This means that the deformation space of G as a Kleinian group contains an at least $(6g-6)$-dimensional Euclidean space inside. For such a group G, the quotient $\mathbf{H}^3/G$ is homeomorphic to $S \times \mathbf{R}$ and contains a totally geodesic surface homeomorphic to S at its centre. This hyperbolic manifold has evidently infinite volume.

Example 1.3. A homeomorphism ω from S^2 to itself is said to be quasi-conformal when ω has an L^2-distributional derivative, and $\|\omega_{\bar{z}}/\omega_z\|_\infty < 1$. By the Ahlfors-Bers theory, for any Fuchsian group G, there is $12g-12$-dimensional space Q of quasi-conformal homeomorphisms such that for $\omega \in Q$, the group $\omega G \omega^{-1}$ is a Kleinian group, and that $\omega G \omega^{-1}$ is conjugate to $\omega' G \omega'^{-1}$ only when $\omega = \omega'$. Such a group $\omega G \omega^{-1}$ is said to be a quasi-Fuchsian group. Also for a quasi-Fuchsian group Γ (isomorphic to $\pi_1(S)$,) the quotient $\mathbf{H}^3/\Gamma$ is homeomorphic to $S \times \mathbf{R}$. More generally, when a Kleinian group is obtained from another Kleinian group G by conjugating it using a quasi-conformal homeomorphism, it is called a quasi-conformal deformation of G.

Example 1.4. Take mutually disjoint $2g$ circles $C_1, \ldots, C_g, C_1', \ldots, C_g'$ bounding disjoint discs on the complex plane $\mathbf{C}$. Let $\phi_i : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be a conformal map which takes the disc bounded by C_i to the complimentary disc of one bounded by C_i'. Then the group generated by $\phi_1, \ldots, \phi_g$ is a discrete free group, which is called a Schottky group of rank g. For a Schottky group of rank g, the quotient manifold $\mathbf{H}^3/G$ is homeomorphic to the interior of a handle body of genus g.

2. Fundamental conjectures in the Kleinian group theory

In this section, we shall state four unsolved conjectures on Kleinian groups, which would be fundamental for a further development in the Kleinian group theory. The first one is called Ahlfors' conjecture. Before stating it, let us define the limit set of a Kleinian group.

Definition 2.1. Let G be a (torsion-free) Kleinian group. The closure of the set $\{x \in S^2 | \exists g \in G \text{ such that } gx = x\}$ is called the limit set of G and denoted by Λ_G.

Conjecture 1 (Ahlfors [1]). Let G be a finitely generated Kleinian group. Then its limit set either has measure 0 or is the entire sphere.

This conjecture looks more related to complex analysis than geometry. We shall see later, however, that Ahlfors' conjecture is closely related to the following Marden's conjecture, which has a rather topological appearance.

Conjecture 2 (Marden [7]). For any finitely generated Kleinian group G, the quotient manifold $\mathbf{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold. (We say that $\mathbf{H}^3/G$ is topologically tame then.)

The remaining two conjectures can be stated only using terms which cannot defined succinctly. Here we defer the definitions and state the conjectures without defining the terms.

Conjecture 3 (Ending lamination conjecture [20]). Kleinian groups are classified completely by the topological types of the quotient manifolds and the end invariants.

Conjecture 4 (Bers-Thurston [20]). Every Kleinian group is an algebraic limit of geometrically finite Kleinian groups.

3. Geometrically finite and infinite groups

In this section, we shall define a category of Kleinian groups relatively well-understood, which are called geometrically finite groups. After that, we shall see examples of geometrically infinite groups.

Definition 3.1. A Kleinian group G is said to be geometrically finite if the quotient manifold $\mathbf{H}^3/G$ has a convex submanifold C_G which is a deformation retract of $\mathbf{H}^3/G$ and has finite volume. The submanifold which is minimal among such convex submanifolds is called the convex core of $\mathbf{H}^3/G$.

Fuchsian groups, quasi-Fuchsian groups, and Schottky groups which appeared in §2 are all geometrically finite. For a Fuchsian group G, the convex core of $\mathbf{H}^3/G$ is degenerated into a two-dimensional submanifold, a totally geodesic surface. For geometrically finite groups, Ahlfors' conjecture is known to be true. This was proved by Ahlfors himself in 1966 ([2]). Marden's conjecture is also known to be true for geometrically finite groups. As a matter of fact, Marden's conjecture was motivated by his own result that it is true for geometrically finite groups ([7]). To understand intuitively that Marden's conjecture is true for geometrically finite groups, the best way is to consider the nearest point map. Let G be a geometrically finite Kleinian group, and suppose for simplicity that G has no parabolic elements. Then the convex core C_G is a compact submanifold in $\mathbf{H}^3/G$. (Except for the case of Fuchsian groups, the convex core is a 3-manifold.) For $x \in \mathbf{H}^3/G$ we define $r(x)$ to be the nearest point of C_G from x. It is easy to see that r is a continuous map and gives a parametrization of $(\mathbf{H}^3/G) \setminus C_G$ as $\partial C_G \times \mathbf{R}$.

On the other hand, there are plenty of examples of geometrically infinite groups. In fact, for any geometrically finite Kleinian group with non-trivial quasi-conformal deformation space, we can construct a geometrically infinite group as an algebraic limit of quasi-conformal deformations. Let us see the simplest example. Let G be a Fuchsian group corresponding to a closed surface S of genus $g \geq 2$. The space $QF(S)$ of quasi-Fuchsian groups obtained by deforming G is homeomorphic to the $(12g-12)$-dimensional Euclidean space. Actually, it can be parametrized by the product of two Teichmüller space $\mathcal{T}(S) \times \mathcal{T}(S)$. This correspondence can be interpreted as follows. Since the limit set of $\Gamma \in QF(S)$ is a Jordan curve in S^2, its complement, which is called the domain of discontinuity of Γ and denoted by Ω_Γ, is the union of two topological open discs. Its quotient Ω_Γ/Γ is a Riemann surface homeomorphic to the disjoint union of two copies of S which are identified by an orientation-reversing homeomorphisms. In this way, Γ determines a pair of two marked conformal structures on S, hence a point in $\mathcal{T}(S) \times \mathcal{T}(S)$. This is exactly the correspondence from $QF(S)$ to $\mathcal{T}(S) \times \mathcal{T}(S)$. Now take a sequence $\{p_i = (m_i, n_i)\} \subset \mathcal{T}(S) \times \mathcal{T}(S)$, such that m_i is constant and $\{n_i\}$ does not accumulate inside $\mathcal{T}(S)$. Let Γ_i be a quasi-Fuchsian group in $QF(S)$ corresponding to p_i. Bers proved in [3] that such a sequence of quasi-Fuchsian group Γ_i converges to a Kleinian group Γ_∞ algebraically. Moreover if Γ_∞ has no parabolic elements, which is generically true, then the group Γ_∞ must be geometrically infinite. Such a group is called totally degenerate b-group without accidental parabolics. Recall that Bers-Thurston conjecture asserts that all Kleinian groups would be obtained by a similar fashion, taking a limit of quasi-conformal deformations of geometrically finite groups.

4. Bonahon's theorem on freely indecomposable Kleinian groups

Peter Scott proved that any 3-manifold M with finitely generated fundamental group contains a compact 3-submanifold such that the inclusion from the submanifold to M induces the isomorphism between the fundamental groups. (See [15].) Such a submanifold is called a core of M. McCullough, Miller and Swarup proved that if M has two cores, there must be a homeomorphism between them inducing the identity on $\pi_1(M)$. (See [8].) It is known that proving that $\mathbf{H}^3/G$ is topologically tame is equivalent to proving that $\mathbf{H}^3/G$ is homeomorphic to the interior of its core. Bonahon proved in [4] that both Ahlfors' and Marden's conjectures are true for a class of Kleinian groups which are called freely indecomposable.

Definition 4.1. A Kleinian group G is said to be freely indecomposable when for any free product decomposition $G = A * B$, there exists a parabolic element of G whose conjugacy class does not contain an element of either factor. In the case when G has no parabolic elements, this means that G cannot be decomposed into a non-trivial free product.

Bonahon proved the following.

Theorem 4.2 (Bonahon [4]). *If a Kleinian group G is freely indecomposable, then $\mathbf{H}^3/G$ is topologically tame. Moreover, the limit set Λ_G of G either has measure 0 or is the entire sphere.*

What Bonahon really proved is that in this case $\mathbf{H}^3/G$ is geometrically tame in the sense of Thurston, which implies the both results above. In the rest of this section, we shall review Bonahon's argument. To review his argument will also illuminate what kind of difficulty we face in the general case. We assume that Kleinian groups have no parabolic elements to make the argument simpler. The condition that G is freely indecomposable is equivalent to one that a core of $\mathbf{H}^3/G$ has incompressible boundary. We generalize the notion of geometric finiteness to one for ends of hyperbolic 3-manifolds.

Definition 4.3. An end e of a hyperbolic 3-manifold M is said to be geometrically finite if e has a neighbourhood which intersects no closed geodesics. Otherwise the end is said to be geometrically infinite.

We can easily see that G is geometrically finite if and only if all ends of $\mathbf{H}^3/G$ are geometrically finite. Let e be a geometrically infinite end of $\mathbf{H}^3/G$. The first step of Bonahon's argument is to prove that there exists a sequence of closed geodesics $\{\gamma_i\}$ which tends to the end e, i.e., such that for any neighbourhood U of e there exists i_0 with the property that if $i \geq i_0$ then $\gamma_i \subset U$. The assumption that e is geometrically infinite assures that for any neighbourhood of e, there exists a closed geodesic intersecting it. Taking a sequence of neighbourhoods U_i forming a base of neighbourhood system of e, we get a sequence of closed geodesics δ_i such that δ_i intersects U_i. From this one can construct a piece-wise geodesic closed curve γ_i' contained in U_i consisting of four geodesic arcs. Either by homotoping γ_i' to a closed geodesic, or by taking an axis of Margulis tube intersecting a homotopy between γ_i and a closed geodesic, we get a sequence of closed geodesic as we wanted.

Take a core C of $\mathbf{H}^3/G$. There is a unique boundary component S of C that faces e. Any closed curve contained in the component of the complement containing e can be homotoped to one on S. Let $\overline{\gamma}_i$ be a closed curve on S homotopic to γ_i in $\mathbf{H}^3/G$. Since S

is incompressible a homotopy can be chosen to be disjoint from the interior of C. (This is one of the points where the assumption of free indecomposability is crucial.) Fix a hyperbolic metric on S and for a closed curve γ on S, denote by $\text{length}_S(\gamma)$ the length of the closed geodesic homotopic to γ. The second step of Bonahon's argument is to prove that

$$\frac{i(\gamma_i, \gamma_i)}{\text{length}_S(\gamma_i)^2} \to 0 \text{ as } i \to \infty. \tag{1}$$

This is proved by considering the ratio of the area of a piece-wise totally geodesic homotopy and the intersection number with the homotopy and the closed geodesic. The fact 1 can be regarded as meaning that a weighted simple closed curve $\gamma_i/\text{length}_S(\gamma_i)$ approaches to an object without self-intersection. Bonahon considered the space of geodesic currents on S to make the intuitive assertion above have a mathematical meaning. We shall not go into defining this space precisely here. The space of geodesic currents is the space of transverse measures for geodesic flows on the unit tangent bundle of S. A measured lamination, which is defined by Thurston to construct a compactification of a Teichmüller space with completing the set of weighted simple closed curves, can be characterized as a geodesic current with null self-intersection. Thus in the space of geodesic currents, we see that $\gamma_i/\text{length}_S(\gamma_i)$ converges to a measured lamination λ. As can be seen in Thurston's construction, the space of measured laminations is a completion of the set of weighted simple closed curves. Therefore for the measured lamination λ, there exist weighted simple closed curves $w_i\alpha_i$ converging to λ. Let α_i^* be the closed geodesic homotopic to α_i in $\mathbf{H}^3/G$. The third step is to prove that the sequence of closed geodesics $\{\alpha_i^*\}$ also tends to the end e. This is the most subtle and complicated part of Bonahon's argument. As it will necessitate many preliminaries to explain this part of the argument, we skip it entirely. Having proved this, one can use Thurston's technique of pleated surfaces to complete the proof. Refer to [16] for details.

Definition 4.4. Let S be a hyperbolic surface, and N a hyperbolic 3-manifold. A continuous map $f : S \to N$ is called a pleated surface when

1. the arc length function induced from N coincides with that of the hyperbolic metric on S, and
2. there exists a closed set ν consisting of mutually disjoint simple geodesics, such that the restriction of f to ν and each component of $S \setminus \nu$ is totally geodesic.

The ν as above is called a geodesic lamination realized by f.

For each closed geodesic α_i^* in the sequence we have above, we can construct easily a pleated surface $f_i : S \to \mathbf{H}^3/G$ homotopic to the inclusion of S, which takes a simple closed curve homotopic to α_i to α_i^*. There is a uniform bound for the diameters of homotopic pleated surfaces outside the Margulis tubes. Hence the sequence of pleated surfaces f_i also tends to the end e. Since S was assumed to be incompressible, by Freedman-Scott-Hass theory, an embedded surface S_i homotopic to S exists in an arbitrarily small neighbourhood of $f_i(S)$. Thus we have a sequence of embedded surfaces S_i homotopic to S going out to e. The incompressibility of S makes it possible to choose a homotopy between S_i and S_{i+1} contained in the submanifold cobounded by them. This implies that e has a neighbourhood homeomorphic to $S \times \mathbf{R}$. As this is the case for every geometrically infinite end, $\mathbf{H}^3/G$ is topologically tame. Also it can be seen that Ahlfors' conjecture is true for such a G roughly by the argument as follows. If Ahlfors' conjecture failed to hold, then there would be a non-trivial harmonic function on $\mathbf{H}^3/G$ defined by the visual measure of the limit set. Consider its value in the convex core. By the maximal principle, its maximum value cannot be attained in the interior of the convex core. On the other hand, the value at the boundary is at most 2π, which cannot be maximal. Thus the gradient flow of the harmonic function must go out to a geometrically infinite end. One can show by a fairly simple argument that the existence of a sequence of pleated surfaces makes this impossible.

5. Ending laminations and Minsky's theorems

In this section, we shall define ending laminations and end invariants so that the ending lamination conjecture stated before will make sense, and describe a partial solution to the conjecture by Minsky and its generalization by the present author. Recall that in the last section, it was proved that for any geometrically infinite end e of $\mathbf{H}^3/G$, where G is a freely indecomposable Kleinian group, there exists a sequence of closed geodesics α_i^* which are homotopic to simple closed curves α_i on the component S of a core C facing e. Evidently there is an ambiguity for the choice of such a sequence of simple closed curves. The argument used to prove that the intersection number of $\gamma_i/\text{length}_S(\gamma_i)$ goes to 0, however, implies that the support of the limit measured lamination of $\alpha_i/\text{length}_S(\alpha_i)$ is independent of the choice of such a sequence. It is possible that by changing a sequence we would get a different transverse measure with the same support. The support of a measured lamination is a geodesic lamination. The support of the limit measured lamination as above is called the ending lamination of e.

Definition 5.1. Let G be a freely indecomposable Kleinian group without parabolic elements. Let $e_1, \ldots, e_m$ be the ends of $\mathbf{H}^3/G$. (If we allow G to have parabolic elements, we need to consider the ends of non-cuspidal part of $\mathbf{H}^3/G$ instead of $\mathbf{H}^3/G$ itself.) We define the invariant of e_i to be the ending lamination of e_i if e_i is geometrically infinite. When e_i is geometrically finite, we define its invariant to be the point of Teichmüller space determined by the conformal structure at infinity corresponding to e_i. The end invariant of $\mathbf{H}^3/G$ is defined to be the n-tuple whose i-th coordinate is the invariant of e_i.

This definition can be generalized to the case when G may be freely decomposable provided that $\mathbf{H}^3/G$ is topologically tame as follows. First consider a geometrically finite end e of $\mathbf{H}^3/G$. The end corresponds to a component Σ of the quotient of the domain of discontinuity. The point which is different from freely indecomposable case is that Σ is compressible in the Kleinian manifold $(\mathbf{H}^3 \cup \Omega_G)/G$. This causes ambiguity in determining the marking of Σ. Thus we cannot determine a point in the Teichmüller space of the boundary component of a core facing e, but one in its quotient space, $\mathcal{T}(S)/\text{Diff}^0(S, C)$, where $\text{Diff}^0(S, C)$ denotes the groups of isotopy classes of auto-diffeomorphisms of S extending to those of C acting on $\pi_1(C)$ by inner-automorphisms. Next consider a geometrically infinite end e, which is assumed to have a neighbourhood homeomorphic to $S \times \mathbf{R}$ for the boundary component S of C facing e. We can define an ending lamination similarly to the case when G is freely indecomposable by virtue of Canary's theorem below. The only point that we have to change is that we have to assume that the simple closed curves α_i converge in the Masur domain. Here Masur domain is defined to be the set of measured laminations which have non-zero intersection numbers with all measured laminations that are limits of weighted compression discs.

Theorem 5.2 (Canary [6]). *Let $\mathbf{H}^3/G$ be a topologically tame hyperbolic 3-manifold. Then for each geometrically infinite end e of $\mathbf{H}^3/G$, we can construct a sequence of pleated surfaces tending to e similarly to the case of freely indecomposable groups, and we can choose a sequence of simple closed curves $\{\alpha_i\}$ as above whose projective classes converge inside* *the projectivized Masur domain. Moreover for such a group G, the limit set Λ_G either is the entire sphere or has measure 0.*

Minsky gave a partial solution to the ending lamination conjecture as below.

Theorem 5.3 (Minsky [9]). *Suppose that G and Γ are freely indecomposable Kleinian groups such that $\mathbf{H}^3/G$ is homeomorphic to $\mathbf{H}^3/\Gamma$. Suppose moreover that there is a lower bound $\epsilon > 0$ for the injectivity radii at all points of $\mathbf{H}^3/G$ and $\mathbf{H}^3/\Gamma$. In this situation, if the end invariants of $\mathbf{H}^3/G$ and $\mathbf{H}^3/\Gamma$ coincide by the correspondence induced by a homeomorphism $h : \mathbf{H}^3/G \to \mathbf{H}^3/\Gamma$, then h is homotopic to an isometry. In particular, G and Γ are conformally conjugate.*

In other words, the ending lamination conjecture is true for freely indecomposable Kleinian groups such that the quotient manifolds have injectivity radii bounded away from 0. The author generalized Minsky's theorem above to topologically tame groups which are possibly freely decomposable, with the same assumption on the injectivity radii. See [13]. It seems quite difficult to remove the assumption on the injectivity radii in general. Yet, Minsky also proved that for Kleinian groups isomorphic to a once-punctured torus group, the ending lamination conjecture is true without assumption on the injectivity radii.

6. Deducing the Bers-Thurston conjecture from the ending lamination conjecture

In this section, we shall briefly see that the ending lamination conjecture implies the Bers-Thurston conjecture in the case of freely indecomposable Kleinian groups without parabolic elements. Suppose that we are given a freely indecomposable Kleinian group Γ without parabolic elements. By Bonahon's theorem, we know that $\mathbf{H}^3/\Gamma$ is topologically tame, i.e., homeomorphic to the interior of a compact 3-manifold K. Thurston's uniformization theorem implies that there is a geometrically finite Kleinian group G without parabolic elements such that $\mathbf{H}^3/G$ is homeomorphic to $\mathrm{Int}K$. (See [17] and [20].) It is known by Ahlfors-Bers theory that the quasi-conformal deformation space of G is parametrized by the Teichmüller space $\mathcal{T}(\Omega_G/G)$, which corresponds exactly to the end invariants in the case of geometrically finite groups. We can take a sequence of quasi-conformal deformations G_i of G so that the end invariants $e(G_i)$ of $\mathbf{H}^3/G_i$ converge to that of $\mathbf{H}^3/\Gamma$. It means that if a coordinate of $e(G_i)$ converges inside the Teichmüller space as $i \to \infty$, then the corresponding coordinate of $e(\Gamma)$ is the limit point in the Teichmüller space, and that if a coordinate diverges in the Teichmüller space, then the support of its limit in the Thurston compactification, which is a projective lamination, is the corresponding coordinate of $e(\Gamma)$. Then by the author's theorem on realization of ending laminations ([11]), we can see that $\{G_i\}$ converges to a Kleinian group with the same end invariant as Γ. If the ending lamination conjecture is true, this implies that Γ is the limit of $\{G_i\}$, which means in particular that Γ is a limit of geometrically finite groups. Thus the Bers-Thurston conjecture would also be solved.

7. Conclusion

As was shown in the last section, for freely indecomposable Kleinian groups, the most important remaining problem is the ending lamination conjecture. By solving it, the picture in this case is nearly complete. In contrast, it remains a far way to go to fully understand freely decomposable Kleinian groups. The first obstacle is Marden's conjecture. The author has been trying to tackle solving the conjecture using topological method, which is still in process. He hopes that he can complete the work in the near future. Also in the general case, it is not so simple as in the case of freely indecomposable groups to deduce the Bers-Thurston conjecture from the ending lamination conjecture. One of the reasons why is that the studies on convergence/divergence of sequences of freely decomposable groups are not well developed. Except for groups with simple algebraic structures, as in [5], [14] and [12], the asymptotic behaviour of quasi-conformal deformations of freely decomposable geometrically finite groups are not well understood.

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, KOMABA, MEGURO-KU, TOKYO 153, JAPAN

NON-ZERO DEGREE MAPS BETWEEN 3-MANIFOLDS

TERUHIKO SOMA

Introduction

This note is a survey for some results on non-zero degree maps between closed 3-manifolds obtained by the author [14], [15], and based on his lectures in the GARC Conference on Geometric Structures on Manifolds at Seoul National University, 1997. Our subject is closely connected with the following problem by Y. Rong; Problem 3.100 in [9]:

Let M be a closed, connected, orientable 3-manifold.

(A) Are there only finitely many irreducible 3-manifolds N with a degree-one map $M \longrightarrow N$?

(B) Does there exist an integer $n(M)$ depending only on M such that if

$$M \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$$

is a sequence of degree-one maps with $n \geq n(M)$ and each M_i $(i = 1, \ldots, n)$ is irreducible, then the sequence must contain a homotopy equivalence?

For two closed, connected, orientable manifolds M, N of the same dimension n, we say that N is *dominated* (resp. *d-dominated*) by M if there exists a continuous map $f : M \longrightarrow N$ with $\deg(f) \neq 0$ (resp. $|\deg(f)| = d$). First, we discuss the case where the dominated manifold N is hyperbolic. By an argument invoking the Gromov invariant, it is shown that the volume of N is bounded by a constant depending only on M (see Thurston [16, Chapter 6]). According to H.C. Wang [19], there are only finitely many hyperbolic n-manifolds with bounded volume if $n > 3$. This shows that the number of mutually non-homeomorphic, hyperbolic n-manifolds dominated by a fixed M is finite if $n \neq 3$. In the case of $n = 3$, a similar argument does not work. In fact, by Thurston's Hyperbolic Dehn Surgery Theorem [16], one can have infinitely many hyperbolic 3-manifolds with bounded volume. However, even in this case, it can be proved that the number of hyperbolic 3-manifolds dominated by any closed, orientable 3-manifold is finite (see Theorem 2.1 in §2). The proof is based on the argument in Thurston [18], where a certain 3-manifold M' is hyperbolized with ideal 3-simplices by using a faithful, discrete representation $\rho : \pi_1(M') \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$. We "hyperbolize" our M similarly by using a non-zero degree map $f : M \longrightarrow N$. Arguments in Boileau-Wang [1, §3] show that this theorem does not hold when M is non-orientable. In [12, Corollary 4.1], Rong proved that, if the M is Seifert-fibered of infinite π_1, then M 1-dominates only finitely many Seifert fibered spaces of infinite π_1.

Next, we consider the case where dominated 3-manifolds are Haken. Any Haken manifold N is decomposed into hyperbolic pieces and Seifert pieces by a certain union of incompressible tori in N. We denote by $\mathcal{H}(N)$ the disjoint union of hyperbolic pieces in the decomposition. A compact, connected 3-manifold H is said to be *dominated by M as a hyperbolic piece* if there exists a Haken manifold N dominated by M and such that H

is homeomorphic to a component of $\mathcal{H}(N)$. Theorem 3.1 in §3 shows that, for any closed, connected, orientable 3-manifold M, there are only finitely many 3-manifolds dominated by M as hyperbolic pieces. By invoking this theorem, one can prove that there exists an integer $n_1(M)$ depending only on M such that, for any family of Haken manifolds N_i $(i = 1, \dots, n)$ with $n \geq n_1(M)$ and dominated by M, at least two of the hyperbolic unions $\mathcal{H}(N_i)$ have the same topological type (see Corollary 3.3). This corollary is crucial in the proof of Theorem 3.4 in §3 which gives a complete answer to Problem 3.100 (B) in [9] "up to the Geometrization Conjecture" by Thurston [17].

§1. Preliminaries

We refer to Hempel [5] and Jaco [6] for fundamental definitions and notation on 3-manifold topology, and to Thurston [16] for those on hyperbolic geometry. Throughout this note, let us assume that all 3-manifolds are oriented.

A non-degenerate, oriented 3-simplex Δ in the hyperbolic 3-space $\mathbf{H}^3$ is *positive* if the orientation is compatible with that of $\mathbf{H}^3$, and otherwise *negative.* If Δ is an ideal 3-simplex in $\mathbf{H}^3$ all whose vertices are contained in the sphere S^2_∞ at infinity, then Δ admits an isometric $\mathbf{Z}_2 \times \mathbf{Z}_2$-action generated by elliptic elements. Let $\{v_1, v_2, v_3, v_4\}$ be the set of vertices of the Δ, e_{ij} the edge of Δ connecting v_i with v_j, and D_i the face of Δ opposite to v_i. We suppose that the vertices are numbered so that the triad $(v_1 - v_4, v_2 - v_4, v_3 - v_4)$ of vectors forms the frame compatible with the orientation of Δ. We direct each e_{ij} from v_i to v_j temporarily. For any even permutation (i, j, k, l) of $(1, 2, 3, 4)$, there exists a unique element $\gamma \in \mathrm{Isom}^+(\mathbf{H}^3)$ taking D_k onto D_l and fixing v_i, v_j. Then, the *edge invariant* $z(e_{ij})$ is the complex number whose modulus is the translation distance of γ with respect to the direction of e_{ij}, and whose argument is the angle of rotation of γ. Clearly, the invariant is independent of the direction of the edge, that is, $z(e_{ij}) = z(e_{ji})$. By the $\mathbf{Z}_2 \times \mathbf{Z}_2$-symmetry of Δ, mutually opposite edges of Δ have the same edge invariant. Moreover, $z(e_{23}) = z(e_{41}) = (z-1)/z$ and $z(e_{13}) = z(e_{24}) = 1/(1-z)$ if $z = z(e_{12}) = z(e_{34})$ (see [16, Chapter 4] for details). Even in the case of Δ degenerated, the edge invariant is defined similarly. Then, for any edge e of the Δ, the invariant $z(e)$ takes the value in $\mathbf{R} - \{0, 1\}$.

Here, we will present two kinds of δ-inner-outer decompositions for any non-degenerate, ideal 3-simplices Δ in $\mathbf{H}^3$ and any small $\delta > 0$. It is an important fact that the diameter of each component of the δ-inner part of Δ is bounded by a constant independent of Δ.

Let D be an ideal, straight 2-simplex in $\mathbf{H}^2$ such that all vertices of D are in the circle S^1_∞ at infinity. If $\delta > 0$ is sufficiently small, then the closure T in D of the complement $D - \mathcal{N}_\delta(\partial D, D)$ is a triangle, where $\mathcal{N}_\delta(\partial D, D)$ is the δ-neighborhood of ∂D in D. The convex hull $D_{\mathrm{inn}(\delta)}$ in D spanned by the three vertices of T is called the *δ-inner part* of D. Note that $D_{\mathrm{inn}(\delta)}$ is a triangle with geodesic edges and containing T. The closure $D_{\mathrm{out}(\delta)}$ of the complement $D - D_{\mathrm{inn}(\delta)}$ is called the *δ-outer part.*

Let Δ be a non-degenerate, ideal 3-simplex in $\mathbf{H}^3$ such that all vertices v_1, v_2, v_3, v_4 of Δ is in S^2_∞, and D_i $(i = 1, 2, 3, 4)$ the face of Δ opposite to v_i. The edge of Δ connecting v_i with v_j is denoted by $e_{ij} = e_{ji}$. Take $\delta > 0$ so that each $D_{i,\mathrm{inn}(\delta)}$ is a triangle. For each v_i, let w_{ik} $(k \neq i)$ be the vertex of $D_{k,\mathrm{inn}(\delta)}$ adjacent to v_i, and let $T_i = T_i(\delta)$ be the totally geodesic triangle in Δ spanned by w_{ik}'s with $k \in \{1, 2, 3, 4\} - \{i\}$. Note that all T_i $(i = 1, 2, 3, 4)$ are isometric to each other. Let u_{ikj} be the foot of the perpendicular from w_{ik} to e_{ij} in D_k. The convex hull $A_{ij} = A_{ji}$ of $w_{ik}, w_{il}, w_{jk}, w_{jl}, u_{ikj}, u_{ilj}, u_{jki}, u_{jli}$ in Δ is called a *δ-arm* of Δ for $\{k, l\} = \{1, 2, 3, 4\} - \{i, j\}$ (see Fig. 1).

For any $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, let J_{ii} (resp. J_{ij}) be the convex hull in Δ spanned by v_i and w_{ik}'s for $k \in \{1, 2, 3, 4\} - \{i\}$ (resp. by v_i and w_{ik}, u_{ikj}'s for $k \in \{1, 2, 3, 4\} - \{i, j\}$). We call the union $J_i = \cup_{l=1}^4 J_{il}$ is a *δ-joint* of Δ and each J_{il} is a *δ-subjoint* of J_i

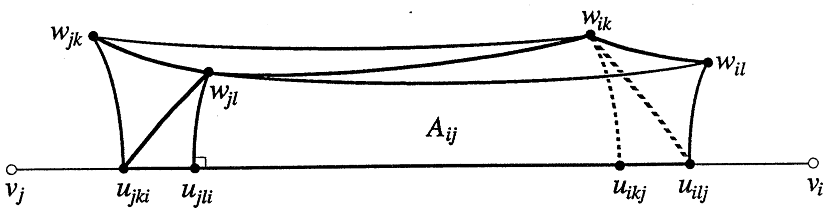

FIGURE 1

(see Fig. 2). The union $\Delta_{\text{out}(\delta)} = (\cup_{1\le i<j\le 4} A_{ij}) \cup (\cup_{k=1}^{4} J_k)$ is called the *δ-outer part* of Δ, and the closure $\Delta_{\text{inn}(\delta)}$ of the complement $\Delta - \Delta_{\text{out}(\delta)}$ is the *δ-inner part.* Note that $\text{diam}(T_i(\delta))$ is bounded by a constant independent of δ. However, $\text{diam}(T_i(\delta))$ diverges to the infinity if $z(e) \to 1$ in C for some edge e of Δ. This is inconvenient for us to analyze a geometric limit of the δ-inner parts of ideal 3-simplices. If $z(e_{ij}) = z(e_{kl})$ is sufficiently close to 1 for some $\{i,j,k,l\} = \{1,2,3,4\}$, then

(1.1) $\quad \text{dist}_\Delta(u_{ijk}, u_{jik}) = \text{dist}_\Delta(u_{ijl}, u_{jil}) = \text{dist}_\Delta(u_{kli}, u_{lki}) = \text{dist}_\Delta(u_{klj}, u_{lkj}) \le \delta.$

We say that Δ is *δ-stretched* if it satisfies (1.1), and otherwise Δ is *δ-normal.*

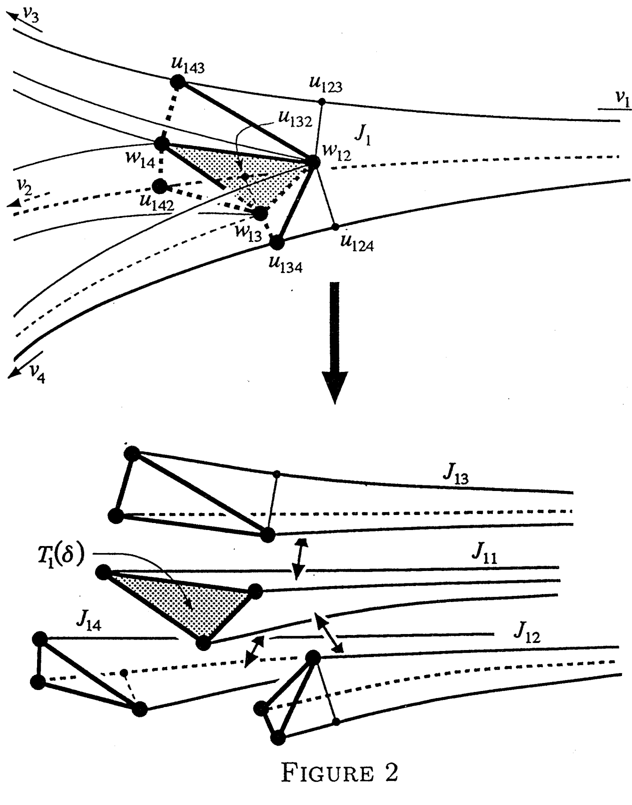

FIGURE 2

Here, we will consider the case where Δ is δ-stretched, and define another δ-inner-outer decomposition for Δ. If necessary renumbering the vertices of the Δ, it may be assumed that $\text{dist}_\Delta(u_{123}, u_{213}) = \text{dist}_\Delta(u_{124}, u_{214}) = \text{dist}_\Delta(u_{341}, u_{431}) = \text{dist}_\Delta(u_{342}, u_{432}) \le \delta$ (see Fig. 3). The δ-arms A'_{12} and A'_{34} here are equal to A_{12} and A_{34} respectively. The δ-arm A'_1 is the convex hull of $w_{13}, w_{14}, w_{34}, w_{43}, u_{134}, u_{143}, u_{341}$, and u_{431}. We set $J'_{12} =$

$J_{12}, J'_{21} = J_{21}, J'_{34} = J_{34}$, and $J'_{43} = J_{43}$. The convex hull of $v_1, w_{13}, w_{14}, u_{134}$, and u_{143} is denoted by J'_{134}. The convex hulls $J'_{234}, J'_{312}, J'_{412}$ are defined similarly. Then, the unions $J'_1 = J'_{12} \cup J'_{134}, J'_2 = J'_{21} \cup J'_{234}, J'_3 = J'_{34} \cup J'_{312}$, and $J'_4 = J'_{43} \cup J'_{412}$ are called *δ-joints* of Δ. The union

$$\Delta_{\mathrm{out}(\delta)'} = (\cup_{i=1}^4 A'_i) \cup A'_{12} \cup A'_{34} \cup (\cup_{j=0}^4 J'_j)$$

is called the *δ-outer part* of Δ, and the closure $\Delta_{\mathrm{inn}(\delta)'}$ of the complement $\Delta - \Delta_{\mathrm{out}(\delta)'}$ is the *δ-inner part* in the δ-stretched case.

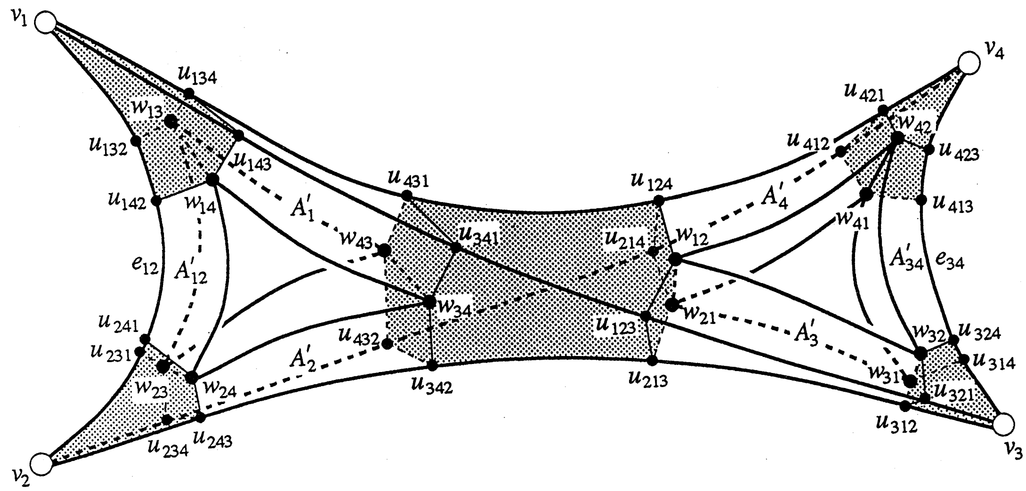

FIGURE 3

A *δ-microchips C* is a compact Riemannian 3-manifold isometric to a convex polyhedron in $\mathbf{H}^3$ of diameter $< 10\delta$. For a set $\mathcal{C} = \{C_\lambda; \lambda \in \Lambda\}$ of δ-microchips, $\partial\mathcal{C}$ is the set $\{\partial C_\lambda; \lambda \in \Lambda\}$ with the total area $\mathrm{Area}(\partial\mathcal{C}) = \sum_{\lambda\in\Lambda} \mathrm{Area}(\partial C_\lambda)$. When the rule of the intersection $C_\lambda \cap C_\mu$ of any two elements $C_\lambda, C_\mu \in \mathcal{C}$ is determined, the union $\cup_{\lambda\in\Lambda} C_\lambda$ with the arcwise metric induced from those of C_λ's is denoted by $\sqcup\mathcal{C}$.

Now, we define a δ-microchip decomposition for $\Delta_{\mathrm{out}(\delta)}$ in the case where Δ is δ-normal. Let $\{P_n; n \in \mathbf{Z}\}$ be the set of totally geodesic planes in $\mathbf{H}^3$ perpendicular to e_{ij} with $\mathrm{dist}_{\mathbf{H}^3}(P_n, P_{n+1}) = 2\delta$ for any $n \in \mathbf{Z}$. These planes decompose the δ-arm A_{ij} into δ-microchips (see Fig. 4). Similarly, J_{ij}'s are decomposed into δ-microchips. For example, 2δ-equidistant, totally geodesic planes in $\mathbf{H}^3$ perpendicular to a longest ray connecting a vertex of T_1 with v_1 separate J_{11} into δ-microchips. The union $\mathcal{C}_\Delta$ of all these microchips defines a δ-microchip decomposition for $\Delta_{\mathrm{out}(\delta)}$, that is, $\sqcup\mathcal{C}_\Delta = \Delta_{\mathrm{out}(\delta)}$.

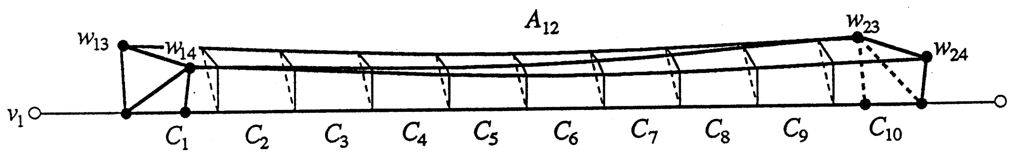

FIGURE 4

We note that there may exist $C_1 \in \mathcal{C}_\Delta$ in A_{ij} and $C_2 \in \mathcal{C}_\Delta$ in J_i such that $\mathrm{int}C_1 \cap \mathrm{int}C_2 \neq \emptyset$. However, it does not cause any problem in our argument. Also in the δ-stretched case, a

δ-microchip decomposition $\mathcal{C}_\Delta$ for $\Delta_{\mathrm{out}(\delta)'}$ is defined similarly.

Lemma 1.1. *In either case, there exists a constant $K > 1$ independent of δ and Δ such that, if $C \in \mathcal{C}_\Delta$ is contained in a δ-arm A (resp. a δ-subjoint J), then* $\mathrm{Area}(\partial C) \leq K\mathrm{Area}(\partial C \cap \partial A)$ *(resp.* $\mathrm{Area}(\partial C) \leq K\mathrm{Area}(\partial C \cap \partial J)$*).* □

The proof is elementary, so it will be left to the reader (cf. the proof of [13, Lemma 2]). Since $\lim_{\delta\to 0} \mathrm{Area}(\partial A) = 0$ and $\lim_{\delta\to 0} \mathrm{Area}(\partial J) = 0$ for any δ-arms A and δ-subjoints J, Lemma 1.1 implies the following.

Corollary 1.2. *Suppose that any $\varepsilon > 0$ and any non-degenerate, ideal 3-simplex Δ in $\mathbf{H}^3$ are given. Then, there exists $\delta_0 > 0$ such that, for any $0 < \delta \leq \delta_0$, there is a δ-microchip decomposition $\mathcal{C}_\Delta$ for $\Delta_{\mathrm{out}(\delta)}$ (or $\Delta_{\mathrm{out}(\delta)'}$) with the total area* $\mathrm{Area}(\partial\mathcal{C}_\Delta) < \varepsilon$.

Let $\Delta_1, ..., \Delta_n$ be non-degenerate, oriented, ideal 3-simplices in $\mathbf{H}^3$ such that all vertices of Δ_i are contained in S^2_∞. Remove all edges from Δ_i and denote the resulting simplex by Δ_i°. We suppose that each face D_{ij}° of Δ_i° has the orientation induced from that of Δ_i°, that is, the combination of a positive frame of D_{ij}° and a normal vector on D_{ij}° to Δ_i° directing outward defines the orientation compatible with that of Δ_i°. Identifying faces of Δ_i°'s suitably by orientation-reversing isometries, one can construct a connected 3-manifold G°. The fundamental group $\pi_1(G^\circ)$ of G° is a free group. Since the attaching maps are orientation-reversing, G° has a unique orientation compatible with that of each Δ_i°. The boundary ∂G° is the disjoint union of all faces not identified with any other faces. We say that G° is an *ideal simplicial complex* obtained from $\Delta_1^\circ, ..., \Delta_n^\circ$.

Let $p : \widetilde{G}^\circ \longrightarrow G^\circ$ be the universal covering, and let $d : \widetilde{G}^\circ \longrightarrow \mathbf{H}^3$ be a developing map defined in a usual manner. The developing map is illustrated in Fig. 5 schematically. The d introduces a holonomy $\rho : \pi_1(G^\circ) \longrightarrow \mathrm{Isom}^+(\mathbf{H}^3)$ with $\rho(\gamma) \circ d = d \circ \tau_\gamma$ for all $\gamma \in \pi_1(G^\circ)$, where τ_γ is the covering transformation on $\widetilde{G}^\circ$ associated to γ. The complex G° can be extended to the union $G = \Delta_1 \cup \cdots \cup \Delta_n$ as a metric space if $\rho(\gamma)$ is either trivial or elliptic for $\gamma \in \pi_1(G^\circ)$ corresponding to the meridians of any edges in the 1-skeleton of G. We say that the G is the *complete ideal simplicial complex* (*obtained by completing* G°) if G is complete as a metric space. For any small $\delta > 0$, the union $G_{\mathrm{inn}(\delta)} = \cup_{i=1}^n \Delta_{i,\mathrm{inn}(\delta)}$ is called the *δ-inner part* of G°, where each $\Delta_{i,\mathrm{inn}(\delta)}$ denotes the δ-inner part of Δ_i in either the δ-normal or δ-stretched case. The closure $G^\circ_{\mathrm{out}(\delta)}$ in G° of the complement $G^\circ - G_{\mathrm{inn}(\delta)}$ is the *δ-outer part* of G°.

§2. Non-zero degree maps to hyperbolic 3-manifolds

In this section, we will sketch the proof of the following finiteness theorem referred in Introduction, which gives a partial answer to Problem 3.100 (A) in [9].

Theorem 2.1 ([14]). *For any closed, connected, orientable 3-manifold M, the number of mutually non-homeomorphic, orientable, hyperbolic 3-manifolds dominated by M is finite.*

We say that a contractible 1-complex Γ is a *star of degree n* if Γ consists of n edges which have a common vertex. The following lemma is used in the proof of Theorem 2.1.

Lemma 2.2. *Let W be a compact 3-manifold, $\mathcal{T} = T_1 \cup \cdots \cup T_n$ a disjoint union of tori, and Γ a star of degree n such that $\Gamma \cap T_i$ is a single end point of Γ for each $i \in \{1, \ldots, n\}$.*

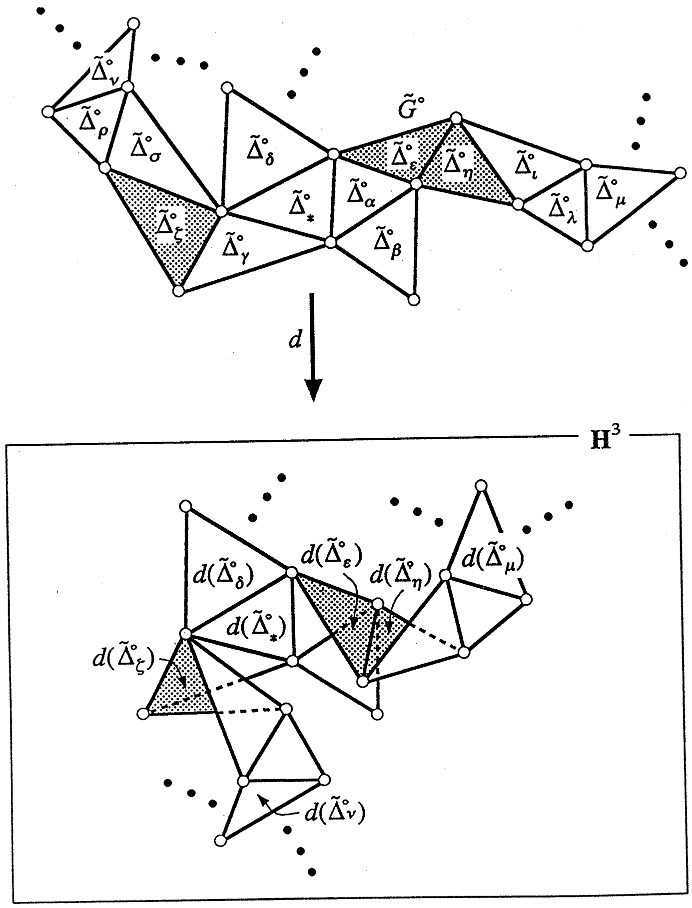

FIGURE 5. The shaded triangles represent negative 3-simplices

Suppose that $\varphi : \partial W \longrightarrow \mathcal{T}$ is a continuous map such that, for each T_i, the degree d_i of $\varphi|\varphi^{-1}(T_i) : \varphi^{-1}(T_i) \longrightarrow T_i$ is non-zero. Then, there is at most one way to extend $\mathcal{T}$ to a disjoint union $\mathcal{V} = V_1 \cup \cdots \cup V_n$ of solid tori with $\partial V_i = T_i$ such that φ extends to a continuous map $\Phi : W \longrightarrow \mathcal{V} \cup \Gamma$.

Proof. Suppose that there exists a continuous map $\Phi : W \longrightarrow \mathcal{V} \cup \Gamma$ extending φ. Consider a meridian disk D_i for V_i with $\partial D_i \cap \Gamma = \emptyset$. If necessary after modifying Φ by a proper homotopy, we may assume that Φ is transverse to $D_1 \cup \cdots \cup D_n$. Then, each $F_i = \Phi^{-1}(D_i)$ is a compact, orientable surface in W with $\partial F_i \subset \partial W$. Orient F_i so that $\varphi_*([\partial F_i]) = d_i[\partial D_i]$ in $H_1(T_i; \mathbf{Z})$. Consider another continuous map $\Phi' : W \longrightarrow \mathcal{V}' \cup \Gamma$ extending φ, where $\mathcal{V} = V_1' \cup \cdots \cup V_n'$ is a disjoint union of solid tori with $\partial V_i' = T_i$. Since $\Phi'(F_i)$ is contained in $\mathcal{V}' \cup \Gamma$, $\varphi_*([\partial F_i]) = d_i[\partial D_i] = 0$ in $H_1(\mathcal{V}' \cup \Gamma; \mathbf{Z})$. Since $d_i \neq 0$ and since the homomorphism $H_1(V_i'; \mathbf{Z}) \longrightarrow H_1(\mathcal{V}' \cup \Gamma; \mathbf{Z})$ induced from the inclusion is injective, $[\partial D_i] = 0$ in $H_1(V_i'; \mathbf{Z})$. Hence, ∂D_i bounds a meridian disk in V_i'. This completes the proof. □

For any topological space X, let $(C_*(X), \partial_*)$ be the singular chain complex with real coefficient. An element c of the k-chain group $C_k(X)$ is a finite linear combination $c =$

$\sum_{i=1}^{n} r_i \sigma_i$ with continuous maps (singular k-simplices) $\sigma_i : \Delta^k \longrightarrow X$ and $r_i \in \mathbf{R}$, where Δ^k is a regular k-simplex of edge length 1 in the Euclidean k-space. *Gromov's norm* $||c||$ of c is defined by $||c|| = \sum_{i=1}^{n} |r_i|$. Moreover, for any $\alpha \in H_k(X; \mathbf{R})$, we set

$$||\alpha|| = \inf\{||c||; c \in Z_k(X) \text{ with } [c] = \alpha\},$$

where $Z_k(X)$ is the k-cycle group of X. When N is a closed, oriented n-manifold, the norm $||[N]||$ of the fundamental class $[N] \in H_n(N; \mathbf{R})$ is called the *Gromov invariant* of N and denoted simply by $||N||$.

The following proposition is immediate from the definition.

Proposition 2.3. (i) *Let $f : X \longrightarrow X'$ be a continuous map between topological spaces. Then, the inequality $||f_*(\alpha)|| \leq ||\alpha||$ holds for any $\alpha \in H_k(X; \mathbf{R})$.*

(ii) *For a continuous map $f : N \longrightarrow N'$ between closed, connected, orientable n-manifolds, we have $|\deg(f)|\,||N'|| \leq ||N||$.*

Sketch of Proof of Theorem 2.1. The proof is done by reduction to absurdity. So, we may assume that there exists a closed, connected 3-manifold M dominating closed, connected, hyperbolic 3-manifolds N_n ($n \in \mathbf{N}$) which are not homeomorphic to each other. Let $f_n : M \longrightarrow N_n$ be a non-zero degree map. According to Thurston [16, Chapter 6], $\mathrm{Vol}(N_n)$ is equal to $||N_n||\mathbf{v}_3$, where $\mathbf{v}_3$ is the volume of a regular, ideal simplex in $\mathbf{H}^3$. By Proposition 2.3, for any $n \in \mathbf{N}$,

$$\mathrm{Vol}(N_n) = ||N_n||\mathbf{v}_3 \leq \frac{||M||\mathbf{v}_3}{|\deg(f_n)|} \leq ||M||\mathbf{v}_3.$$

Thus, the volumes $\mathrm{Vol}(N_n)$ are bounded. By Jϕrgensen's Theorem [16, Chapter 6], if necessary taking a subsequence of $\{N_n\}$ instead, we may assume that there exists a complete, connected, hyperbolic 3-manifold N with $\mathrm{Vol}(N) < \infty$ such that each N_n is obtained by hyperbolic Dehn surgery on N. In particular, we have sequences $\{\varepsilon_n\}$, $\{K_n\}$ with $\varepsilon_n \searrow 0$, $K_n \searrow 1$ so that there exist K_n-quasi-isometric diffeomorphisms $g_n : N_{n,\mathrm{thick}(\varepsilon_n)} \longrightarrow N_{\mathrm{thick}(\varepsilon_n)}$.

Fix a (topologically) simplicial decomposition $\mathcal{D}$ on M, and let $\widehat{\Delta}_1, \dots, \widehat{\Delta}_m$ be the 3-simplices in $\mathcal{D}$. By modifying M and f_n, we will first construct ideal simplicial complexes G_n and continuous maps $f'_n : G_n \longrightarrow N_n$ which are locally isometric on each simplex. In fact, the complex G_n is the union of ideal straight 3-simplices $\Delta_{i,n}$ obtained by straightening singular 3-simplices $f_n|\widehat{\Delta}_i : \widehat{\Delta}_i \longrightarrow N_n$ for any $\widehat{\Delta}_i$. Note that the diameter of each component of $G_{n,\mathrm{inn}(\delta)}$ is bounded, and each ideal 3-simplex in G_n is parametrized by a complex number $z_{i,n}$. In fact, for the edge $e_{i,n}$ of $\Delta_{i,n}$ corresponding to a fixed edge e_i of $\widehat{\Delta}_i$, we set $z_{i,n} = z(e_{i,n})$ as in §1. Since $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is compact, if necessary passing to a subsequence, we may assume that, for all $i \in \{1, \dots, m\}$, $\{z_{i,n}\}_{n=1}^{\infty}$ converges to a point $z_i \in \widehat{\mathbf{C}}$. Let $H_{1,n}, \dots, H_{\nu,n}$ be the components of $G_{n,\mathrm{inn}(\delta)}$. These $H_{\alpha,n}$'s are renumbered so that, if necessary passing to a subsequence, the following (2.1) and (2.2) hold.

(2.1): There exists an $\varepsilon > 0$ such that $f'_n(H_{\alpha,n}) \cap N_{n,\mathrm{thin}(\varepsilon)} = \emptyset$ for all sufficiently large $n \in \mathbf{N}$ and $\alpha \in \{1, \dots, \mu\}$.

(2.2): There exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ such that $f'_n(H_{\beta,n}) \cap N_{n,\mathrm{thin}(\varepsilon_n)} \neq \emptyset$ for all sufficiently large $n \in \mathbf{N}$ and $\beta \in \{\mu+1, \dots, \nu\}$.

From now on, we set $\mathcal{I}_n = \cup_{\alpha=1}^{\mu} H_{\alpha,n}$, $\mathcal{Z}_n = \cup_{\beta=\mu+1}^{\nu} H_{\alpha,n}$ and $\mathcal{O}_n = G_{n,\mathrm{out}(\delta)}$. Since the diameters of $\mathcal{I}_n$'s are bounded, the sequence $\{\mathcal{I}_n\}$ converges geometrically to the δ-inner part $\mathcal{I}$ of a certain simplicial complex. In particular, there exist L_n-quasi-isometries

$h_n : \mathcal{I} \longrightarrow \mathcal{I}_n$ with $L_n \searrow 1$. Note that the images $g_n \circ f'_n \circ h_n(\mathcal{I})$ are contained in the compact set $N_{\text{thick}(\varepsilon)}$. By Ascoli-Arzelà's Theorem, we may assume that the sequence $\{g_n \circ f'_n \circ h_n : \mathcal{I} \longrightarrow N_{\text{thick}(\varepsilon)}\}$ converges uniformly to a continuous map $f' : \mathcal{I} \longrightarrow N_{\text{thick}(\varepsilon)} \subset N$. Let H_α be the component of $\mathcal{I}$ which is the geometric limit of the sequence $\{H_{\alpha,n}\}$. Since $K_n, L_n \searrow 1$ and since f'_n is a locally isometric immersion in each ideal 3-simplex of G_n, for each simplex $\Delta_{i,\text{inn}(\delta)}$ in $\mathcal{I}$, $f'|\Delta_{i,\text{inn}(\delta)}$ is a locally isometric immersion. This implies that a holonomy $\rho_\alpha : \pi_1(H_\alpha) \longrightarrow \text{Isom}^+(\mathbf{H}^3)$ for $\alpha \in \{1, \dots, \mu\}$ is the composition $\rho_N \circ (f'|H_\alpha)_*$ of a holonomy $\rho_N : \pi_1(N) \longrightarrow \text{Isom}^+(\mathbf{H}^3)$ of N and the induced homomorphism $(f'|H_\alpha)_* : \pi_1(H_\alpha) \longrightarrow \pi_1(N)$. Note that the sequence $\{\rho_{\alpha,n} \circ (h_n|H_\alpha)_*\}$ converges algebraically to ρ_α, where $\rho_{\alpha,n}$ is a holonomy of $H_{\alpha,n}$.

The $\mathcal{I}_n$ is the "essential" δ-inner part of $G_{n,\text{inn}(\delta)}$. We note that $\mathcal{I}_n$, $\mathcal{O}_n$, $\mathcal{Z}_n$ have pairwise disjoint interiors and $\mathcal{I}_n \cap \mathcal{Z}_n = \emptyset$, and that the topological type of $(G_n; \mathcal{I}_n, \mathcal{O}_n, \mathcal{Z}_n)$ is independent of $n \in \mathbf{N}$. By Corollary 1.2, the δ-outer part $\mathcal{O}_n$ of G_n is controlled in the following sense:

(2.3): $\lim_{\delta\to 0} \sup_n\{\text{Vol}(\mathcal{O}_n)\} = 0$, and

(2.4): There exists a δ-microchip decomposition $\mathcal{C}_n$ on $\mathcal{O}_n$ with

$$\lim_{\delta\to 0} \sup_n \{\text{Area}(\partial \mathcal{C}_n)\} = 0.$$

Again by passing to a subsequence if necessary, one can modify $\mathcal{I}$, G_n, f'_n, h_n, $\mathcal{I}_n$, $\mathcal{O}_n$ by surgery along $\partial\mathcal{I}_{(n)} = \partial\mathcal{O}_n - \partial\mathcal{Z}_n$ in order to construct new manifolds $\widehat{\mathcal{I}}$, $\widehat{G}_n$; maps $\widehat{f}_n : \widehat{G}_n \longrightarrow N_n$, $\widehat{h}_n : \widehat{\mathcal{I}} \longrightarrow \widehat{\mathcal{I}}_n$; and a decomposition $\widehat{\mathcal{I}}_n$, $\widehat{\mathcal{O}}_n$, $\widehat{\mathcal{Z}}_n\,(= \mathcal{Z}_n)$ on $\widehat{G}_n$ which satisfy the same properties as (2.1)–(2.4) and moreover

(2.5): $\rho_\Sigma(\pi_1(\Sigma))$ is a non-trivial parabolic group for each component Σ of $\partial\widehat{\mathcal{I}}$, where $\rho_\Sigma : \pi_1(\Sigma) \longrightarrow \text{Isom}^+(\mathbf{H}^3)$ is the restriction of the holonomy of $\widehat{\mathcal{I}}$.

Here, let us first fix a constant $\lambda_0 > 0$ so that $\widehat{f}_n(\mathcal{L}_n^-) \cap N_{n,\text{thin}(\lambda_0)} = \emptyset$, where $\mathcal{L}_n^-$ is some part added to $\widehat{\mathcal{O}}_n$ under our modification. By using the parabolicity (2.5), one can next choose $\delta > 0$ so that $\widehat{f}_n(\partial\widehat{\mathcal{I}}_n) \subset \text{int}N_{n,\text{thin}(\lambda_0)}$ for all $n \in \mathbf{N}$.

We finally retake $\varepsilon > 0$ with $\varepsilon \ll \lambda_0$ so that $\widehat{f}_n(\widehat{\mathcal{I}}_n) \cap N_{n,\text{thick}(\varepsilon)} = \emptyset$ for all $n \in \mathbf{N}$. By using (ii), $\widehat{f}_n$ can be modified again so that the resulting map $\widehat{\psi}_n : \widehat{G}_n \longrightarrow N_n$ satisfies that $\widehat{\psi}_n|\partial\widehat{\mathcal{I}}_n$ is a non-zero degree map onto $\partial N_{n,\text{thin}(\varepsilon)}$ and $\widehat{\psi}_n(\widehat{\mathcal{I}}_n \cup \mathcal{Z}_n)$ is contained in the union of $N_{n,\text{thin}(\varepsilon)}$ and a star Γ_n in $N_{n,\text{thick}(\varepsilon)}$. We note that the property (2.4) is crucial in our argument. In fact, without the (2.4), one would only show that $\widehat{\psi}_n(\widehat{\mathcal{I}}_n \cup \mathcal{Z}_n)$ would lie in the union of $N_{n,\text{thin}(\varepsilon)}$ and a 2-complex in $N_{n,\text{thick}(\varepsilon)}$, and hence one can not invoke Lemma 2.2. Since the sequence $\{g_n \circ \widehat{f}_n \circ \widehat{h}_n : \widehat{\mathcal{I}} \longrightarrow N_{\text{thick}(\varepsilon)}\}$ converges uniformly to a map $\widehat{f} : \widehat{\mathcal{I}} \longrightarrow N_{\text{thick}(\varepsilon)}$, one can assume that $g_n \circ \widehat{\psi}_n \circ \widehat{h}_n|\partial\widehat{\mathcal{I}} = g_{n'} \circ \widehat{\psi}_{n'} \circ \widehat{h}_{n'}|\partial\widehat{\mathcal{I}}$. Then, by Lemma 2.2, $g_{n'}^{-1} \circ g_n : N_{n,\text{thick}(\varepsilon)} \longrightarrow N_{n',\text{thick}(\varepsilon)}$ would be extended to a homeomorphism $N_n \longrightarrow N_{n'}$, a contradiction. This completes our reduction to absurdity and hence the proof of Theorem 2.1. □

§3. Non-zero degree maps to Haken manifolds

A compact, connected, irreducible 3-manifold is called *Haken* if the manifold contains an incompressible surface. According to Thurston's Uniformization Theorem [17] and the Torus Decomposition Theorem (Jaco-Shalen [7], Johannson [8]), for any closed, Haken manifold N, there exists a union $\mathcal{T}$ of mutually disjoint, incompressible tori (possibly empty) such that each component P of $N - \text{int}\mathcal{N}(\mathcal{T})$ has either an interior hyperbolic structure of finite volume or a Seifert fibered structure, where $\mathcal{N}(\mathcal{T})$ is a regular neighborhood of $\mathcal{T}$ in N. In the former case, P is called a *hyperbolic piece*, and in the latter, a *Seifert piece.* The the union $\mathcal{H}(N)$ of hyperbolic pieces is determined uniquely up to ambient isotopy on N. As was defined in Introduction, a compact, connected 3-manifold H is said to be *dominated by* a closed, connected 3-manifold M *as a hyperbolic piece* if H is homeomorphic to a component of the hyperbolic piece union $\mathcal{H}(N)$ of a Haken manifold N dominated by M.

The following theorem is the hyperbolic piece version of Theorem 2.1.

Theorem 3.1 ([15]). *Suppose that M is any closed, connected, orientable 3-manifold. Then, there are only finitely many, mutually non-homeomorphic 3-manifolds dominated by M as hyperbolic pieces.*

A complete, simplicial complex G is said to *simplicially dominate* a hyperbolic 3-manifold W of finite volume if there exists a proper, non-zero degree map $\varphi : G \longrightarrow W$ such that, for each 3-simplex Δ_i in G, a lift of the restriction $\varphi|\Delta_i : \Delta_i \longrightarrow W$ to the universal covering is a smooth (but not necessarily isometric) embedding onto a non-degenerate, straight 3-simplex in $\mathbf{H}^3$ all vertices of which are contained in S^2_∞.

For the proof of Theorem 3.1, we need the following lemma.

Lemma 3.2 ([15]). *Any closed, connected, irreducible 3-manifold M admits a finite set $\mathcal{G} = \{G_1, \dots, G_n\}$ of complete, simplicial complexes such that, for any 3-manifold H dominated by M as a hyperbolic piece, a hyperbolic 3-manifold W homeomorphic to* int*H is simplicially dominated by at least one G_i of $\mathcal{G}$.*

Sketch of Proof. Fix a simplicial decomposition $\mathcal{D}_M$ on M, and let $\mathcal{D}_M^{(3)}$ be the set of 3-simplices in $\mathcal{D}_M$. Take a non-zero degree map $f : M \longrightarrow N$ to a Haken manifold N such that $\mathcal{H}(N)$ has a component homeomorphic to H. By [5, Lemma 6.5], for the union $\mathcal{T}$ of tori determining a torus decomposition on N, we may assume that each component of $\mathcal{F} = f^{-1}(\mathcal{T})$ is an incompressible surface in M. We deform $\mathcal{F}$ by an ambient isotopy on M in the same manner as in [5, Lemmas 13.2, 3.14]. Let L be the components of $f^{-1}(H)$ such that the degree of $f|L : L \longrightarrow H$ is non-zero. The set of components $\Delta \cap L$ for all $\Delta \in \mathcal{D}_M^{(3)}$ defines a polyhedral decomposition $\mathcal{E}_L$ on L. Consider a certain subdivision $\mathcal{D}_L$ of $\mathcal{E}_L$ which gives a simplicial decomposition on L, where any vertex in $\mathcal{D}_L^{(0)} - \mathcal{E}_L^{(0)}$ is contained in the interior of some $\nabla \in \mathcal{E}_L^{(3)}$, and this correspondence $\mathcal{D}_L^{(0)} - \mathcal{E}_L^{(0)} \longrightarrow \mathcal{E}_L^{(3)}$ is bijective. Here, note that we cannot estimate the number of 3-simplices in $\mathcal{D}_L$. To avoid the difficulty, straighten these simplices by using the map $f|L$ and the hyperbolic structures on W. Then, the number of 3-simplices in $\mathcal{D}_L$ which are non-degenerate after straightening is not greater than $4\#(\mathcal{D}_M^{(3)})$. We note that the number $4\#(\mathcal{D}_M^{(3)})$ depends only on $\mathcal{D}_M$, but is independent of the dominating map f. Let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a maximal set of complete, simplicial complexes consisting of at most $4\#(\mathcal{D}_M^{(3)})$ 3-simplices and such that any two elements of $\mathcal{G}$ have distinct combinatorial types. Then, W is simplicially dominated by at least one G_i of $\mathcal{G}$. $\square$

Proof of Theorem 3.1. We may assume that M is irreducible, and hence one can apply Lemma 3.2. Let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a set of complete, simplicial complexes given in Lemma 1. This means that, for any compact 3-manifold H dominated by M as a hyperbolic piece, a hyperbolic 3-manifold W homeomorphic to int$H$ is simplicially dominated by some $G_i \in \mathcal{G}$. On the other hand, the argument quite similar to that in Theorem 2.1

implies that, for each G_i, the number m_i of hyperbolic 3-manifolds W admitting simplicially dominating maps $\varphi : G_i \longrightarrow W$ is finite. In fact, the induced hyperbolic structure on each ideal 3-simplex Δ_β of G_i via φ is parametrized by a complex number $z_W(e_\beta)$ as in §1, where e_β is a fixed edge of Δ_β. Then, the proof of Theorem 2.1 works also in the present case without any essential changes. This implies that the number of 3-manifolds dominated by M as hyperbolic pieces is at most $m_1 + \cdots + m_n$. This completes the proof. □

Corollary 3.3 ([15]). *Let M be any closed, connected, orientable 3-manifold. Then, there exists an integer $n_1(M)$ depending only on M such that, for any family of Haken manifolds N_i $(i = 1, \dots, n)$ with $n \geq n_1(M)$ and dominated by M, at least two of the hyperbolic unions $\mathcal{H}(N_i)$ have the same topological type.*

Proof. Let $f : M \longrightarrow N$ be a non-zero degree map from a closed, connected, 3-manifold M to a closed, Haken manifold N. Let $\mathcal{H}(N) = H_1 \cup \cdots \cup H_m$ be the union of hyperbolic pieces in a torus decomposition on N. According to [16, Chapter 6], there exists a constant $\mathbf{v}_0 > 0$ such that the volume of any hyperbolic 3-manifold is not less than $\mathbf{v}_0$. By Gromov's cutting-off theorem [2, §4.2], we have

$$||H_1|| + \cdots + ||H_m|| \leq ||N|| \leq \frac{||M||}{|\deg(f)|} \leq ||M||.$$

Since $||H_i|| = \mathrm{Vol}(W_i)\mathbf{v}_3^{-1} \geq \mathbf{v}_0\mathbf{v}_3^{-1}$, we have $m \leq ||M||\mathbf{v}_0^{-1}\mathbf{v}_3$, where W_i is a hyperbolic 3-manifold homeomorphic to $\mathrm{int}H_i$. This fact together with Theorem 3.1 completes the proof. □

A closed, irreducible 3-manifold M is said to be *geometric* if M is either hyperbolic or Seifert-fibered or Haken. The corollary above together with some results in Rong [10], [11], Hayat-Legrand, Wang and Zieschang [3], [4] and Soma [14] implies Theorem 2 below concerning a sequence of degree-one maps between geometric 3-manifolds. This gives a complete answer to Problem 3.100 (B) in [9] if the Geometrization Conjecture by Thurston [17] holds. In fact, Rong considered the case where geometric 3-manifolds M_j in the theorem were irreducible. On the other hand, Thurston conjectured that any closed, irreducible 3-manifold is geometric.

Theorem 3.4 ([15]). *For any closed, orientable 3-manifold M, there exists an integer $n_2(M)$ depending only on M and satisfying the following (*).*
(*) *Consider any sequence*

$$M \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$$

such that $\deg(f_i) = 1$ for $i = 0, 1, \dots, n-1$ and M_j is geometric for $j = 1, \dots, n$. If the length n of the sequence is not less than $n_2(M)$, then at least one of f_i's is a homotopy equivalence. □

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DEPARTMENT OF MATHEMATICAL SCIENCES, COLLEGE OF SCIENCE AND ENGINEERING, TOKYO DENKI UNIVERSITY, HATOYAMA-MACHI, SAITAMA-KEN 350-0394, JAPAN

E-mail address: soma@r.dendai.ac.jp

KÄHLER SURFACES OF POSITIVE SCALAR CURVATURE

MYONG-HEE SUNG

ABSTRACT. We construct Kähler metrics of positive scalar curvature on almost all blown-up ruled surfaces of arbitrary genus. The metrics have an explicit form on ruled surfaces blown up at most twice successively from a minimal model. Our surfaces are generic in the sense that they make up a dense set in the deformations of a given ruled surface.

1. INTRODUCTION

Here we consider the existence of Kähler metrics with positive scalar curvature on a compact complex surface of Kähler type. By the surface classification theory, we are left with rational surfaces and ruled surfaces for candidates. As a matter of fact, even if we loosen the condition by merely requiring the existence of *Riemannian* metrics with positive scalar curvature, the same is true by recent works of LeBrun [16](for minimal surfaces) and Friedman-Morgan [1] using Seiberg-Witten theory. Note that this metric doesn't have to be parallel or hermitian with respect to the complex structure as a Kähler metric is.

On the other hand, Hitchin [5] constructed Kähler metrics of positive scalar curvature on rational surfaces obtained by blowing up a minimal model at finitely many distinct points. These rational surfaces are generic in the sense of deformation of the complex structure. In this paper, we will extend Hitchin's work to ruled surfaces and prove

Theorem 1. *There exist Kähler metrics of positive scalar curvature on almost all rational or ruled surfaces.*

More precisely, we construct such metrics on the ruled surfaces obtained by blowing up distinct points on a minimal model. We see that a generic ruled surface falls into this category by using deformation theory. This means that as a *smooth* manifold any blown-up ruled surface admits such a metric. In fact, the construction of metrics goes through for ruled surfaces which can be obtained by blowing up at most twice successively from a minimal model with arbitrarily many distinct blown-up points. This is a weak version of the following which was conjectured in [16] based on overwhelming evidence [3, 20, 23, 1, 16, 5, 24].

Conjecture 1. *Let X be a compact complex surface of Kähler type. Then the following are equivalent*:
(a) *X admits a Riemannian metric of positive scalar curvature*;

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(b) *X admits a Kähler metric of positive scalar curvature;*
(c) *X is either rational or ruled.*

The missing implication is in (c) $\Rightarrow$ (b) about the other blown-up ruled surfaces. A previous result in this direction is Theorem B in [6] which states that if a ruled surface is blown up sufficiently many times Hyperbolic Ansatz leads us to the existence of such a metric.

The rest of the paper is structured as follows:
In Section 2 we set up our curvature convention which will be used later. In Section 3 we introduce Hitchin's construction of metrics on a space blown up at a point. Then in Section 4, the metrics are constructed on ruled surfaces with distinct blow-ups and we consider genericity of those. In Section 5 it is shown that there exist Kähler metrics of positive scalar curvature on a ruled surface which was successively blown up twice from a minimal model.

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2. Notations and curvature conventions

Let's assume $\{z^\alpha\}$ is in use for local coordinates around a point p. When ω is a Kähler metric, we can write

$$\omega = i\sum g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta.$$

$g^{\bar\beta\alpha}$ is defined to be the inverse of $g_{\alpha\bar\beta}$: $g_{\alpha\bar\beta}g^{\bar\beta\gamma} = \delta^\gamma_\alpha$. The curvature tensor R of g is

$$R_{\alpha\bar\beta\gamma\bar\delta} = g^{\bar\epsilon\tau}\partial_\gamma g_{\alpha\bar\epsilon}\bar\partial_\delta g_{\tau\bar\beta} - \partial_\gamma\bar\partial_\delta g_{\alpha\bar\beta}$$

and in particular, $R_{\alpha\bar\beta\gamma\bar\delta} = -\partial_\gamma\bar\partial_\delta g_{\alpha\bar\beta}$ when all the first derivatives of the metric vanish. The *Ricci curvature* is defined by $ric_{\alpha\bar\beta} = g^{\bar\delta\gamma}R_{\gamma\bar\delta\alpha\bar\beta}$; the *scalar curvature* is defined by $s = g^{\bar\beta\alpha}ric_{\alpha\bar\beta}$. It is useful to know the efficient way to compute the scalar curvature when you don't need the full curvature tensor.

$$\begin{aligned} s &= -g^{\bar\beta\alpha}\partial_\alpha\bar\partial_\beta \log\det g\ , \text{ or} \\ s\,\mathrm{dvol}_g &= -\omega\wedge i\partial\bar\partial\log\left(\frac{\omega\wedge\omega}{\eta}\right) \end{aligned}$$

where $d\mathrm{vol}_g$ is the volume form of g and η is the coordinate volume form. The *holomorphic sectional curvature* and the Ricci curvature at p in the direction of z^α are also defined:

$$\begin{aligned} K(p)(z) &= R_{\alpha\bar\beta\gamma\bar\delta}(p)z^\alpha\bar z^\beta z^\gamma\bar z^\delta/r^4; \\ ric(p)(z) &= ric_{\alpha\bar\beta}(p)z^\alpha\bar z^\beta/r^2 \end{aligned}$$

where $r^2 = g_{\alpha\bar\beta}(p)z^\alpha\bar z^\beta$.

3. Metrics on a blown-up space

Here we introduce the metrics Hitchin used on the blown-up space to have the scalar curvature positive [5]. These metrics are also the ones Kodaira used for his famous Embedding Theorem. For the reader's convenience we summarize Hitchin's setup using our conventions wherein the scalar curvature of the space blown up at a point will be computed in terms of the various curvatures of the original space.

Let $\widetilde{X} \xrightarrow{\beta} X$ be the blow-up of X at a point p. Also, let $\varphi : X \to [0,1]$ be a smooth cut-off function such that $\varphi \equiv 1$ on U'' and $\varphi \equiv 0$ outside U' where $p \in U'' \subset U' \subset U \subset X$ and U has local geodesic coordinates $\{z^\alpha\}$ around p. Then, if ω is the given Kähler metric on X, consider the following metric on $\widetilde{X}$:

$$\widetilde{\omega} = \beta^*\omega + t\, i\partial\bar{\partial}\left[(\beta^*\varphi)\log \|z\|^2\right]. \tag{1}$$

If $t\,(>0)$ is sufficiently small, $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$.

We'd like to show that if ω has positive scalar curvature s, then $\widetilde{\omega}$ also has positive scalar curvature $\widetilde{s}$ for small t under some extra conditions on the curvatures of ω. If we can show that $\widetilde{s} > \lambda > 0$ on $\beta^{-1}(U'') \setminus E$ for some λ and small t, then $\widetilde{s} > 0$ on $\beta^{-1}(U'')$ by continuity and we are done. Using the isomorphism $\beta^{-1}(U'') \setminus E \overset{\beta}{\cong} U'' \setminus \{p\}$, we need to prove that $\widetilde{s} > \lambda > 0$ on $U'' \setminus \{p\}$ with all computations thought of as being done on the deleted neighborhood of p on X.

On $U'' \setminus \{p\}$ with geodesic coordinates $\{z^\alpha\}$, we can write $\widetilde{\omega}$ as

$$\widetilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{t}{r^2}\left(\delta_{\alpha\beta} - \frac{\bar{z}^\alpha z^\beta}{r^2}\right)$$

where we can expand the metric g on X as a Taylor series about p relative to the geodesic coordinates:

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} - R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) z^\gamma \bar{z}^\delta + \cdots . \tag{2}$$

Then $(g_{\alpha\bar{\beta}} - \delta_{\alpha\beta})/r^2 =: C_{\alpha\beta}$ is bounded on U''. If we introduce the matrix $P_{\alpha\beta} := \bar{z}^\alpha z^\beta / r^2$, then P is the projection onto the vector z^α and in particular, $P^2 = P$ and trace $P = 1$. Using matrix notation, we rewrite the metric $\widetilde{g}$ as:

$$\widetilde{g} = \frac{r^2+t}{r^2}\left(1 - \frac{tP}{r^2+t}\right)\left(1 + \frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right). \tag{3}$$

Now we will compute $\widetilde{s} = -\widetilde{g}^{\bar{\beta}\alpha}\partial_\alpha\bar{\partial}_\beta \log\det\widetilde{g}$. If n is the complex dimension of the space, we have

$$\begin{aligned}\partial_\alpha\bar{\partial}_\beta \log\det\widetilde{g} &= (n-1)\left(\frac{\delta_{\alpha\beta}}{r^2+t} - \frac{\bar{z}^\alpha z^\beta}{(r^2+t)^2} - \frac{\delta_{\alpha\beta}}{r^2} + \frac{\bar{z}^\alpha z^\beta}{r^4}\right)\\ &- \partial_\alpha\bar{\partial}_\beta\left(\frac{1}{r^2(r^2+t)}(tR_{\rho\bar{\sigma}\gamma\bar{\delta}}(0) z^\rho \bar{z}^\sigma z^\gamma \bar{z}^\delta + r^4 ric_{\gamma\bar{\delta}}(0) z^\gamma \bar{z}^\delta)\right)\\ &+ O(r^2)\end{aligned} \tag{4}$$

where $f \in O(r^n)$ means $|f| < Ar^n$ as $r \to 0$ where A is independent of t. We also have

$$\begin{aligned}\widetilde{g}^{-1} &= \left(\frac{r^2}{r^2+t} + \frac{tP}{r^2+t}\right) - \left(\frac{r^6C}{(r^2+t)^2} + \frac{tr^4PC}{(r^2+t)^2} + \frac{tr^4CP}{(r^2+t)^2} + \frac{t^2r^2PCP}{(r^2+t)^2}\right)\\ &+ O(r^4).\end{aligned} \tag{5}$$

Also, notice that we can safely replace $C_{\alpha\beta}$ in 5 by $B_{\alpha\beta} = -R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0)z^\gamma\bar{z}^\delta/r^2$. If we multiply 5 and 4 and take the trace according to 3, then

$$\widetilde{s} = s_1 + O(r^2)$$

for $n = 2$ where s_1 is a linear combination of various curvatures of g at p.

Lemma 2.

$$\begin{aligned} s_1 &= \frac{1}{(r^2+t)^3}\left[r^4(r^2+t)s(0) + 2tr^2\left((n+2)r^2+4t\right)ric(0)(z)\right. \\ &+ \left. t\left(-4nr^4-(2n+3)r^2t+t^2\right)K(0)(z)\right]. \end{aligned}$$

Proof. Curvature expressions in 5 contribute one term a_1 to s_1 and the two curvature terms in 4 contribute a_2 and a_3:

$$\begin{aligned} a_1 &= \frac{n-1}{(r^2+t)^2}\text{ trace }\left(r^6B+tr^4PB+tr^4BP+t^2r^2PBP\right)\left(\frac{1}{r^2+t}\right. \\ &- \left.\frac{r^2P}{(r^2+t)^2}-\frac{1}{r^2}+\frac{P}{r^2}\right) \\ &= \frac{n-1}{(r^2+t)^3}\left(tr^4ric(0)(z)-tr^2(2r^2+t)K(0)(z)\right). \end{aligned}$$

Now,

$$a_2 = \left(\frac{r^2\delta_{\alpha\beta}}{r^2+t}+\frac{tz^\alpha\bar{z}^\beta}{r^2(r^2+t)}\right)\partial_\alpha\bar{\partial}_\beta\left(\frac{tu}{r^2(r^2+t)}\right)$$

where $u = R_{\rho\bar{\sigma}\gamma\bar{\delta}}(0)z^\rho\bar{z}^\sigma z^\gamma\bar{z}^\delta = r^4K(0)(z)$. After some computation we get

$$a_2 = \frac{4tr^2}{(r^2+t)^2}ric(0)(z)+\frac{tK(0)(z)}{(r^2+t)^3}\left(-nr^2(2r^2+t)-2r^4-4r^2t+t^2\right).$$

Next,

$$\begin{aligned} a_3 &= \left(\frac{r^2\delta_{\alpha\beta}}{r^2+t}+\frac{tz^\alpha\bar{z}^\beta}{r^2(r^2+t)}\right)\partial_\alpha\bar{\partial}_\beta(fv) \\ &= \frac{tr^2ric(0)(z)}{(r^2+t)^3}\left((n+1)r^2+4t\right)+\frac{r^4s(0)}{(r^2+t)^2}. \end{aligned}$$

where $v = r^4ric_{\gamma\bar{\delta}}(0)z^\gamma\bar{z}^\delta = r^6ric(0)(z)$. Finally,

$$\begin{aligned} s_1 &= a_1+a_2+a_3 \\ &= \frac{1}{(r^2+t)^3}\left[r^4(r^2+t)s(0)+2tr^2\left((n+2)r^2+4t\right)ric(0)(z)\right. \\ &+ \left. t\left(-4nr^4-(2n+3)r^2t+t^2\right)K(0)(z)\right] \end{aligned}$$

as stated.

Corollary 3. *If X is a complex surface, then*

$$s_1 = \frac{1}{(r^2+t)^2}\left[r^4 s(0) + 8tr^2 ric(0)(z) + t(-8r^2+t)K(0)(z)\right].$$

Now that we have $\widetilde{s} = s_1 + O(r^2)$ for surfaces, if $s_1 > \lambda_1 > 0$ for some λ_1, there is a δ which is independent of t such that $r^2 < \delta$ implies $\widetilde{s} > \lambda > 0$ for some λ. Taking $U'' = \{z \in U | r^2 < \delta\}$ in the construction of $\widetilde{g}$, we have $\widetilde{s} > \lambda > 0$ on $U'' \setminus \{p\}$, which is exactly what we aimed for in the beginning. So we need to show $s_1 > \lambda_1 > 0$ to prove that $\widetilde{s} > 0$ on the entire $\widetilde{X}$. We will use this method to obtain the main results in the following.

4. Curvature of ruled surfaces and genericity

If $\widetilde{X}$ is a ruled surface obtained from a minimal model $X = \mathbf{P}(V) \xrightarrow{\pi} C$ where V is a rank 2 vector bundle on the Riemann surface C by blowing up finitely many distinct points, then we will consider the metric $\widetilde{\omega}$ constructed in Section 3 with a cut-off function centered around each blown-up point. Here, we use Yau's metric for the minimal surface. In [24], Yau constructed Kähler metrics with positive scalar curvature on any (minimal) ruled manifold as follows: The metric is written as

$$\omega = \pi^*\omega_C + \epsilon\, i\partial\bar\partial \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle$$

where ω_C is a Kähler metric on C and $\langle\cdot,\cdot\rangle$ is a hermitian metric on $V\setminus\{0\}$. When ϵ (> 0) is small, ω is Kähler with positive scalar curvature.

Since the construction of $\widetilde{g}$ is local we might as well consider X blown up at one point, p. Then with the same notation as before,

$$\widetilde{\omega} = \beta^*\omega + t\, i\partial\bar\partial\left[(\beta^*\varphi)\log r^2\right].$$

We are going to make ω more specific by requiring the following:
Choose ω_C such that regardless of the genus of C the metric around p locally looks like the standard Fubini-Study metric on $\mathbf{C}P^1$. We can achieve this for example by deforming the uniform metric on C conformally around p until it is the Fubini-Study metric on a neighborhood U of p. We also take a local trivialization of V on a neighborhood of $\pi(p)$ and let $\langle,\rangle$ be the standard inner product on the fiber. If we call the local inhomogeneous coordinates for C and the fiber z^1 and z^2, respectively, then by making U smaller if necessary, we can express this metric as

$$\omega = i\partial\bar\partial\left[\log(1+|z^1|^2) + \epsilon\log(1+|z^2|^2)\right]$$

on U. Notice that this is just the product metric on $\mathbf{C}P^1 \times \mathbf{C}P^1$ with the fiber-shrinking parameter ϵ. We can use this simple local expression for ω in the computation of $\widetilde{s}$ from the last section if the support of the cut-off function is chosen to lie in U. So our computational problem is reduced to Hitchin's and we have positivity of the scalar curvature of $\widetilde{\omega}$.

Theorem 4. *There exist Kähler metrics of positive scalar curvature on a ruled surface which is obtained from a minimal model by blowing up at distinct points.*

Now we study how the blow-up structure of a ruled surface behaves under small deformations of the complex structure. For a rational surface, Hitchin [5] explicitly constructed a semi-universal family from that of a minimal model of the given surface. We extend his result [5, Proposition 6.1] to ruled surfaces.

Proposition 5. *Let X be a ruled surface. Then there exists a semi-universal (relative to local deformations) family $Z \to B$ for X such that there is an open dense set $U \subset B$ so that the fibers over U are ruled surfaces obtained by blowing up distinct points on a minimal model.*

The proof is identical to Hitchin's except the following additions: by the celebrated works of Kodaira-Spencer [11, 12] and Kuranishi [13, 14], for every compact complex manifold X there exists a semi-universal (relative to local deformations) family of deformations of X.[1] We apply this to our situation and take W to be a trivial deformation for the rigid minimal models $\mathbf{C}P^2$ and $\mathbf{C}P^1 \times \mathbf{C}P^1$, and a semi-universal family for other minimal ruled surfaces. If X is obtained by blowing up k times from a minimal model, take $Z = V^k(W)$ in Hitchin's notation and $B = V^{k-1}(W)$. Since the ruled structure is preserved under small deformations, fibers of $Z \to B$ are all ruled surfaces.

So roughly speaking, the constructed semi-universal family for X is just the semi-universal family for the minimal model of X plus the deformations given by the configuration of blow-ups. We conclude that a generic ruled surface admits Kähler metrics of positive scalar curvature.

5. More ruled surfaces: successive blow-ups

An *essentially-twice-successively-blown-up* ruled surface happens when the fiber point on an exceptional curve is chosen to be the center of the second blow-up. Otherwise, we can always blow down in a different way to consider the given surface as obtained by blowing up distinct points. We extend our previous construction over to an essentially-twice-successively-blown-up ruled surface.

We start with $\widetilde{X}$ from the previous section and take local coordinates $w^1 = z^1/z^2, w^2 = z^2$ around the fiber point on the exceptional curve. Then we blow up $\widetilde{X}$ at 0 and consider the metric

$$\tilde{\omega} + \tilde{t}\, i\partial\bar{\partial}(\tilde{\varphi}\log r^2)$$

on the blown-up space as constructed in Section 3 . So $\widetilde{X}$ is equipped with the metric expressed as

$$\begin{aligned} \tilde{\omega} &= i\partial\bar{\partial}\left[\log(1+|z^1|^2) + \epsilon\log(1+|z^2|^2) + t\log(|z^1|^2 + \epsilon|z^2|^2)\right] \\ &= i\partial\bar{\partial}\left[\log(1+|w^1w^2|^2) + \epsilon\log(1+|w^2|^2) + t\log(|w^1|^2 + \epsilon)\right] \end{aligned}$$

on $\beta^{-1}(U'')$.

From the last paragraph of Section 3, all we need to show is $s_1 > \lambda_1 > 0$ for some constant λ_1. Let's compute the full curvature of $\tilde{\omega}$ at the point that is blown up ($w^1 =$

[1] When $H^2(X, \Theta) \neq 0$ where Θ is the sheaf of holomorphic vector fields on X, the parameter space could have a singularity at the point that corresponds to X.

$w^2 = 0$) to see the various curvatures that appear in s_1. First we compute the components of the metric $\widetilde{\omega} = i \sum g_{\alpha\bar{\beta}}\, dw^\alpha \wedge d\bar{w}^\beta$:

$$\begin{aligned} g_{1\bar{1}} &= \frac{|w^2|^2}{(1+|w^1w^2|^2)^2} + \frac{t\epsilon}{(|w^1|^2+\epsilon)^2}, \\ g_{1\bar{2}} &= \frac{\bar{w}^1 w^2}{(1+|w^1w^2|^2)^2} = \overline{g_{2\bar{1}}}, \\ g_{2\bar{2}} &= \frac{|w^1|^2}{(1+|w^1w^2|^2)^2} + \frac{\epsilon}{(1+|w^2|^2)^2}. \end{aligned}$$

We have

$$\begin{aligned} R_{1\bar{1}1\bar{1}}(0) &= -\partial_1\bar{\partial}_1 g_{1\bar{1}}(0) = 2t/\epsilon^2, \\ R_{1\bar{2}1\bar{1}}(0) &= -\partial_1\bar{\partial}_2 g_{1\bar{1}}(0) = 0, \\ R_{1\bar{2}1\bar{2}}(0) &= -\partial_1\bar{\partial}_2 g_{1\bar{2}}(0) = 0, \\ R_{1\bar{1}2\bar{2}}(0) &= -\partial_1\bar{\partial}_1 g_{2\bar{2}}(0) = -1, \\ R_{1\bar{2}2\bar{2}}(0) &= -\partial_1\bar{\partial}_2 g_{2\bar{2}}(0) = 0, \\ R_{2\bar{2}2\bar{2}}(0) &= -\partial_2\bar{\partial}_2 g_{2\bar{2}}(0) = 2\epsilon, \end{aligned}$$

and the rest of the components can be obtained by symmetry of the tensor and the metric. Now we compute the various curvatures:

$$\begin{aligned} K(w)(0) &= \left(\frac{2t}{\epsilon^2}|w^1|^4 - 4|w^1w^2|^2 + 2\epsilon|w^2|^4\right)/r^4 \\ &\quad \text{where } r^2 = \frac{t}{\epsilon}|w^1|^2 + \epsilon|w^2|^2 =: \frac{t}{\epsilon}x + \epsilon y, \\ ric_{1\bar{1}}(0) &= g^{\bar{1}1}R_{1\bar{1}1\bar{1}}(0) + g^{2\bar{2}}R_{2\bar{2}1\bar{1}}(0) = 2/\epsilon - 1/\epsilon = 1/\epsilon, \\ ric_{1\bar{2}}(0) &= g^{\bar{1}2}R_{2\bar{1}1\bar{2}}(0) = 0, \\ ric_{2\bar{2}}(0) &= g^{\bar{1}1}R_{1\bar{1}2\bar{2}}(0) + g^{\bar{2}2}R_{2\bar{2}2\bar{2}}(0) = -\epsilon/t + 2, \\ ric(w)(0) &= \left(\frac{1}{\epsilon}|w^1|^2 + \left(2 - \frac{\epsilon}{t}\right)|w^2|^2\right)/r^2, \\ s(0) &= 2/\epsilon. \end{aligned}$$

Finally from Corollary 3,

$$\begin{aligned} s_1 &= \frac{1}{(r^2+\tilde{t})^2}\left\{2r^4/\epsilon + 8\tilde{t}\left(\frac{1}{\epsilon}|w^1|^2 + (2-\epsilon/t)|w^2|^2\right)\right. \\ &\quad \left. + \frac{\tilde{t}}{r^4}(-8r^2+\tilde{t})\left(\frac{2t}{\epsilon^2}|w^1|^4 - 4|w^1w^2|^2 + 2\epsilon|w^2|^4\right)\right\}. \end{aligned}$$

We consider the positivity of the following:

$$\begin{aligned} & r^4(r^2+\tilde{t})^2(s_1-\lambda_1) \\ = \;& x^2\left\{\frac{t^2}{\epsilon^2}(2/\epsilon-\lambda_1)r^4-\frac{2t}{\epsilon^2}(4+\lambda_1 t)r^2\tilde{t}+\frac{t}{\epsilon^2}(2-\lambda_1 t)\tilde{t}^2\right\} \\ + \;& 2xy\left\{t(2/\epsilon-\lambda_1)r^4+2(8+4t/\epsilon-\lambda_1 t)r^2\tilde{t}-(2+\lambda_1 t)\tilde{t}^2\right\} \\ + \;& y^2\left\{\epsilon^2(2/\epsilon-\lambda_1)r^4-\frac{2\epsilon^2}{t}(4+t\lambda_1)r^2\tilde{t}+\epsilon^2(2/\epsilon-\lambda_1)\tilde{t}^2\right\} \\ =: \;& Ex^2+2Fxy+Gy^2. \end{aligned}$$

We will show $E, G > 0$ and $F^2 - EG < 0$ for suitable values of t/ϵ. First, we make the first coefficients positive by making $\lambda_1 < 2/\epsilon$. E has the discriminant which is negative if $\lambda_1 < \frac{2(t/\epsilon-4)}{t^2/\epsilon+5t}$ where we need to impose the condition $t/\epsilon > 4$. So $E > 0$. G is also positive since its discriminant is negative if λ_1 is small enough in terms of ϵ and t. F has the discriminant which is positive if λ_1 is small in terms of ϵ and t. Without loss of generality we assume $F < 0$. We will use this inequality in the following computation. We compute $F^2 - EG$ directly:

$$\begin{aligned} F^2-EG &= 2(t/\epsilon+3)\tilde{t}\Big[8t(2/\epsilon-\lambda_1)r^6+\left(2(15t/\epsilon+16)-15t\lambda_1\right)r^4\tilde{t} \\ &\quad -2(4+3t\lambda_1)r^2\tilde{t}^2\Big]+\tilde{t}^4\left(4(1-t/\epsilon)+2t(t/\epsilon+3)\lambda_1\right) \\ &< 2\tilde{t}^2\Big[(t/\epsilon+3)(-96-34t/\epsilon+t\lambda_1)r^4+2(t/\epsilon+3)(4+t\lambda_1)r^2\tilde{t} \\ &\quad +\left(2(1-t/\epsilon)+t(t/\epsilon+3)\lambda_1\right)\tilde{t}^2\Big] \end{aligned}$$

This is again a quadratic expression in r^2 and $\tilde{t}$ with negative first coefficient for small λ_1. Also for λ_1 small enough in terms of t and ϵ, the discriminant $2t\lambda_1(17t^2/\epsilon^2+104t/\epsilon+155)-4(17t^2/\epsilon^2+27t/\epsilon-60)$ is negative under our condition $t/\epsilon > 4$ that was imposed to ensure the positivity of E and G. Therefore, $F^2 - EG < 0$ for sufficiently large t/ϵ and sufficiently small λ_1 . Thus, $s_1 > \lambda_1 > 0$ and we have the following results.

Proposition 6. *There exists Kähler metrics of positive scalar curvature on a ruled surface that is essentially-successively-blown-up twice from a minimal surface.*

Removing the technical terms, we have

Theorem 7. *There exists Kähler metrics of positive scalar curvature on a ruled surface that is blown up twice from a minimal surface.*

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INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD, USA

E-mail address: myonghee@ipst.umd.edu

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