수 학 강 의 록 제 42 권

PROCEEDINGS OF INTERNATIONAL WORKSHOP ON MATHEMATICAL AND PHYSICAL ASPECTS OF NONLINEAR FIELD THEORIES

Edited by
DONGHO CHAE
SUNG-KI KIM

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
held at the Seoul National University

Dongho Chae
Department of Mathematics,
Seoul National University,
Seoul, 151-742,
Korea

Sung-Ki Kim
Department of Mathematics,
Seoul National University,
Seoul, 151-742,
Korea

펴낸날 : 1998년 7월 30일
편집인 : Dongho Chae and Sung-Ki Kim
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터 [TEL : 82-2-880-6562]

Preface

On Feb. 23–24, 1998 " International Workshop on Mathematical and Physical Aspects of Nonlinear Field Theories " was held at Seoul National University.

The main topic was mathematical and physical theories on vortices and monopole solutions in various self-dual gauge field theories.

The purpose was to make environment for fruitful exchange of ideas between theoretical physicists and mathematicians working in those areas.

We would like to thank for all the speakers for their inspiring lectures and contributions to this volume.

We would also like to thank Daewoo Foundation, Global Analysis Research Center and Center for Theoretical Physics for financial support.

Finally, we express our deep thanks to Hee-Seok Nam for helping us to edit this proceeding.

Dongho Chae and Sung-Ki Kim
Department of Mathematics
Seoul National University
Seoul 151-742, KOREA

참 가 자

건국대학교	:	윤종혁(P)
경희대학교	:	박규환(P), 정진모(P), 현승준(P)
고려대학교	:	안인경(M)
서울대학교	:	김남권(M), 김도한(M), 김상정(M), 김성기(M), 김찬주(P), 남희석(M), 신동우(M), 이준규(P), 이지훈(M), 조용민(P), 지동표(M), 채동호(M), 채명주(M), 최광석(M), 한승호(P), 한종민(M), 현정순(P)
서강대학교	:	이범훈(P)
성균관대학교	:	김기병(P), 김동현(P), 김성수(P), 김윤배(P), 오필렬(P), 이준형(P)
연세대학교	:	이재형(P)
인하대학교	:	최규홍(M)
제주대학교	:	고봉수(M)
한양대학교	:	김완세(M)
Columbia Univ.	:	Erick Weinberg(P)
Polytechnic Univ.	:	Yisong Yang(M)
KIAS	:	변재형(M)

* M : Mathematics , P : Physics

CONTENTS

Nonlinear Problems in Field Theories*

Yisong Yang
Department of Applied Mathematics and Physics
Polytechnic University
Brooklyn, New York 11201, USA

There are many interesting and challenging problems in the area of classical field theories. This area has attracted the attention of algebraists, geometers, and topologists in the past and has begun to attract more analysts. Analytically, the area offers all types of differential equation problems which come from the two basic sets of equations in physics describing fundamental interactions, namely, the Yang–Mills equations governing electromagnetic, weak, and strong forces, reflecting internal symmetry, and the Einstein equations governing gravity, reflecting external symmetry. Of course, a combination of these two sets of equations gives us a theory which couples both symmetries and unifies all forces. In these lectures, I will present some of the problems that I have been interested in which involve elliptic equations.

1 Self-dual cosmic strings

Cosmic strings arise as finite-energy static solutions of the coupled Einstein and Yang–Mills–Higgs equations in which the gravitational metric housed over a Riemann surface M and field configurations are independent of time and a third vertical variable. Such a structure leads to energy and curvature concentrations at the centers of strings, around which matter accretion takes place, which provides a mechanism for galaxy formation. Mathematically, Abelian self-dual strings are well understood.

Let us begin with the Abelian Higgs model for which the energy density and momentum tensor are given by

$$\mathcal{E} = \frac{1}{4}g^{jj'}g^{kk'}F_{jk}F_{j'k'} + \frac{1}{2}g^{jk}D_ju\overline{D_ku} + \frac{1}{8}(|u|^2 - \varepsilon^2)^2, \tag{1.1}$$

$$T_{jk} = g^{j'k'}F_{jj'}F_{kk'} + \frac{1}{2}(D_ju\overline{D_ku} + \overline{D_ju}D_ku) - g_{jk}\mathcal{E}, \tag{1.2}$$

where u is the complex Higgs field, A_j $(j = 1, 2)$ is a real vector gauge field, $F_{jk} = \partial_jA_k - \partial_kA_j$ is the magnetic field induced from A_j, $\varepsilon > 0$ is the Higgs vacuum

*Lectures at the International Workshop on Mathematical and Physical Aspects of Field Theories held at Seoul National University, Feb. 23 – 25, 1998.

expectation value which determines the energy breaking scale of the model (or how far the temperature is below the critical temperature), and $g = \{g_{jk}\}$ is the unknown gravitational metric of a two-surface M to be determined by coupling the Einstein equations and the gauge field model (1.1). In fact, if we use K_g to denote the Gauss curvature of (M, g), the coupled Einstein and gauge field equations are equivalent [43, 44] to the self-dual system derived by Linet [27] and Comtet–Gibbons [10]:

$$K_g = 8\pi G\mathcal{E}, \tag{1.3}$$

$$D_j u = -\mathrm{i}\varepsilon_j^k D_k u, \tag{1.4}$$

$$\varepsilon^{jk} F_{jk} = \varepsilon^2 - |u|^2, \tag{1.5}$$

where $G > 0$ is Newton's constant which is of the order of 10^{-30}. To find a solution with N strings located at the prescribed points

$$p_1, p_2, \cdots, p_N, \tag{1.6}$$

we need to obtain a finite-energy solution of (1.3)–(1.5) so that u vanishes exactly at these points. Eqs.(1.4) and (1.5) are simply the well-known vortex equations for the Abelian Higgs model [17]. It is the presence of the Einstein equation (1.3) that introduces many surprises.

Compact M: Let $\mathrm{d}\Omega_g$ be the canonical surface element of (M, g). Since the energy is quantized according to

$$\int_M \mathcal{E}\, \mathrm{d}\Omega_g = \pi\varepsilon^2 N, \tag{1.7}$$

integrating (1.3) and using the Gauss–Bonnet theorem, we have the constraint

$$\chi(M) = 4\pi\varepsilon^2 G N. \tag{1.8}$$

However, topologically M is a sphere with n handles and $\chi(M) = 2 - 2n$, thus the only possibility is $n = 0$ and $M = S^2$. Inserting $\chi(S^2) = 2$ into (1.8), we obtain the quantization of symmetry breaking scale,

$$\varepsilon = \varepsilon_N = \frac{1}{\sqrt{2\pi G N}}, \quad N = 1, 2, \cdots. \tag{1.9}$$

Another interesting thing is that the string number N now affects existence. More precisely, it can be shown that when $N \geq 3$, there are N string solutions with prescribed locations p's, whereas, when $N = 2$, a solution with 2 strings each sitting at one of the poles exists [22, 45], but when $N = 1$, there does not exist a solution which is symmetric about its string. In fact, it can be shown that there is no symmetric solution for any N if all the points p's listed in (1.6) coincide [45]. Hence I propose

Open Problem 1.1. Is a 1-string solution on S^2 symmetric with respect to its string? If the answer is yes, then it implies that there is no 1-string solution in any compact setting. More generally, is an N-string solution on S^2 symmetric about its point with N superimposed strings? If the answer is yes, it means in view of the above nonexistence result that multi-centered strings are necessary in our gravitational system to balance each other and avoid an energy blowup.

Noncompact M: The simplest case is when (M, g) is conformally Euclidean or $(M, g) = (\mathbb{R}^2, e^{\eta}\delta_{jk})$. There are also obstructions to existence.

For radially symmetric N-string solutions, it has been proved [8, 43, 44] that there exists a finite-energy solution if and only if the string number N satisfies

$$N < \frac{1}{2\pi\varepsilon^2 G}. \tag{1.10}$$

Therefore, weaker gravity (smaller G) allows more strings. In particular, when gravity is switched off by setting $G = 0$, there can be any number of strings, which coincide with the classical existence theorem obtained in [17] for superconducting vortices. The condition (1.10) is an energy constraint.

For multistrings, it has been shown [43, 44] that, in our category of field configurations, there exist solutions representing N prescribed strings which give rise to geodesically complete metrics if and only if the string number N satisfies

$$N \leq \frac{1}{4\pi\varepsilon^2 G}. \tag{1.11}$$

Thus, gravity also affects global topology in a noncompact setting. The condition (1.11) is therefore a topological constraint.

Open Problem 1.2. Prove the existence of multistring solutions for the string number N in the range

$$\frac{1}{4\pi\varepsilon^2 G} < N < \frac{1}{2\pi\varepsilon^2 G}. \tag{1.12}$$

Technically, the condition (1.11) comes from a suitable L_p-convergence requirement in the existence proof. We need $p \geq 1$ which is ensured by assuming (1.11).

Recall that in the compact case, multistrings with distinct location points p's in (1.6) exist for $N \geq 3$, in contrast to the nonexistence result for superimposed N strings with radial symmetry. This feature suggests that (1.10) may only be an obstruction to the existence of radially symmetric solutions. Hence I propose

Open Problem 1.3. Can it be shown that there are finite-energy multi-centered N-string solutions for N beyond the range stated in (1.10)?

Note that these strings are magnetic objects with quantized flux, energy, and total curvature, Φ, E, and $\mathcal{K}$, expressed by

$$\frac{1}{2}\int_M \varepsilon^{jk} F_{jk}\, \mathrm{d}\Omega_g = 2\pi N, \quad \int_M \mathcal{E}\, \mathrm{d}\Omega_g = \pi\varepsilon^2 N, \quad \int_M K_g\, \mathrm{d}\Omega_g = 8\pi^2\varepsilon^2 G N. \tag{1.13}$$

Let me next present a recent result on the coexistence of M strings and N anti-strings which satisfy the revised quantization formulas of the form,

$$\Phi = 2\pi(M - N), \quad E = 2\pi\varepsilon^2(M + N), \quad \mathcal{K} = 16\pi^2\varepsilon^2 G(M + N). \tag{1.14}$$

The energy density of our string-antistring model is rewritten as

$$\mathcal{E}=\frac{1}{4}g^{jj'}g^{kk'}F_{jk}F_{j'k'}+\frac{2}{(1+|u|^2)^2}g^{jk}(D_ju)\overline{(D_ku)}+\frac{1}{2}\Big(\frac{1-|u|^2}{1+|u|^2}\Big)^2, \tag{1.15}$$

where we have set $\varepsilon=1$ to simplify our discussion. The coupled Einstein and gauge field equations are

$$K_g = 8\pi G\mathcal{E}, \tag{1.16}$$

$$D_ju = -\mathrm{i}\varepsilon_j^k D_k u, \tag{1.17}$$

$$\frac{1}{2}\varepsilon^{jk}F_{jk} = \frac{1-|u|^2}{1+|u|^2}, \tag{1.18}$$

Given the sets of points

$$P=\{p_1,p_2,\cdots,p_N\},\quad Q=\{q_1,q_2,\cdots,q_M\}, \tag{1.19}$$

we are to look for a finite-energy solution (g,u,A_j) of the equations (1.16)–(1.18) for which the points p's and q's are (simple, for convenience) poles and zeros of u, respectively. We shall see that the points q's and p's indeed give rise to M strings and N antistrings.

Firstly, let $B=\varepsilon^{jk}F_{jk}/2$ represent the induced magnetic field. Then (1.18) leads to

$$B(q_j)=1,\quad j=1,2,\cdots,M;\quad B(p_j)=-1,\quad j=1,2,\cdots,N, \tag{1.20}$$

which says that the magnetic field at q's and p's are oriented along opposite directions.

Next, we specify the case that (M,g) is conformally Euclidean, $g_{jk}=e^{\eta}\delta_{jk}$. With the substitution $v=\ln|u|^2$, the equations (1.17)–(1.18) become a single scalar one,

$$\Delta v = 2e^{\eta}\Big(\frac{e^v-1}{e^v+1}\Big)-4\pi\sum_{s=1}^{N}\delta_{p_s}+4\pi\sum_{s=1}^{M}\delta_{q_s}, \tag{1.21}$$

where δ_p is the Dirac distribution concentrated at $p\in R^2$. In order to determine the conformal exponent η, we need to consider the Einstein equation, (1.16).

It is known that the Gauss curvature K_g now can be written in terms of η as

$$K_g=-\frac{1}{2}e^{-\eta}\Delta\eta. \tag{1.22}$$

Besides, in view of (1.17), (1.18) and $F_{12}=-\Delta v/2$, we have

$$\begin{aligned}\mathcal{E} &= e^{-\eta}F_{12}\Big(\frac{1-|u|^2}{1+|u|^2}\Big)+\frac{2e^{-\eta}}{(1+|u|^2)^2}(|D_1u|^2+|D_2u|^2)\\ &= e^{-\eta}\Big(\frac{\Delta v}{2}\Big[\frac{e^v-1}{e^v+1}\Big]+\frac{e^v|\nabla v|^2}{(e^v+1)^2}\Big)\end{aligned}$$

away from the points p_s's and q_s's. Since $\mathcal{E}$ is a smooth function, the above expression indicates that we can compensate the singular sources at p_s's and q_s's to arrive at the relation

$$e^\eta \mathcal{E} = \Delta\Big(\ln(1+e^v) - \frac{1}{2}v\Big) + 2\pi\sum_{s=1}^N \delta_{p_s} + 2\pi\sum_{s=1}^M \delta_{q_s}, \tag{1.23}$$

which is now valid in the full $\mathbb{R}^2$. Inserting (1.22), (1.23) into (1.16), we see that

$$\frac{\eta}{16\pi G} + \ln(1+e^v) - \frac{1}{2}v + \sum_{s=1}^N \ln|x-p_s| + \sum_{s=1}^M \ln|x-q_s|$$

is a harmonic function, which we assume to be a constant. Consequently the metric is determined by

$$e^\eta = g_0\Big(\frac{e^v}{(1+e^v)^2}\prod_{s=1}^N |x-p_s|^{-2}\prod_{s=1}^M |x-q_s|^{-2}\Big)^{8\pi G}. \tag{1.24}$$

Here g_0 is an arbitrary constant. Note that the metric (1.24) is everywhere regular. Thus, only infinity is to be concerned, and, at the vortex and anti-vortex points, q_s's and p_s's, respectively, we have opposite associated magnetic field as expected,

$$F_{12}(q_s) = e^{\eta(q_s)} > 0, \quad F_{12}(p_s) = -e^{\eta(p_s)} < 0. \tag{1.25}$$

Recall that we are interested in solutions in the broken symmetry category so that $v=0$ or $|u|^2=1$ at infinity. This fact and (1.24) imply the validity of the following global inequality in $\mathbb{R}^2$:

$$C_1(1+|x|)^{-16\pi G(M+N)} \le e^{\eta(x)} \le C_2(1+|x|)^{-16\pi G(M+N)}, \tag{1.26}$$

where $C_1, C_2 > 0$ are suitable constants. The inequality (1.26) enables us to draw the immediate conclusion that a solution leads to a geodesically complete metric if and only if the condition

$$M + N \le \frac{1}{8\pi G} \tag{1.27}$$

holds. Thus, in sense of a complete metric, the numbers of strings and antistrings play equal parts and a large number of strings (either type) or strong gravity (large Newton's constant G) will make the metric incomplete. It is interesting to compare (1.27) with (1.11).

An existence theorem can be proved exactly under the condition (1.27) by substituting (1.24) into (1.21) and solving the resulting governing equation using techniques from nonlinear functional analysis [47]. Besides, it can be shown that the solution approaches the asymmetric vacuum sufficiently fast and the magnetic flux is given by the formula

$$\Phi = \int_{R^2} F_{12} = \int_{\mathbb{R}^2} e^\eta \frac{1-e^v}{1+e^v} = 2\pi(M-N) \tag{1.28}$$

This result confirms that the two types of vortices counter-balance each other magnetically like charged particles.

We then calculate the energy. By completing quadratures and applying (1.17), (1.18), we can represent (1.15) as

$$\mathcal{E} = \frac{1}{2}\varepsilon^{jk}F_{jk} + \frac{1}{2}\varepsilon^{jk}J_{jk} = e^{-\eta}F_{12} + e^{-\eta}J_{12}. \tag{1.29}$$

where $J_{jk} = \partial_j J_k - \partial_k J_j$ and the new 2-current density J_k is defined by

$$J_k = \frac{\mathrm{i}}{1+|u|^2}(u\overline{D_k u} - \overline{u}D_k u). \tag{1.30}$$

Using (1.28) in (1.29), we have

$$\begin{aligned} E &= \int \mathcal{E}\,\mathrm{d}\Omega_\eta \quad (\mathrm{d}\Omega_\eta = e^\eta \mathrm{d}x) \quad = \int_{\mathbb{R}^2} F_{12} + \int_{\mathbb{R}^2} J_{12} \\ &= 2\pi(M-N) + \lim_{\rho\to\infty}\oint_{|x|=\rho} J_k \mathrm{d}x_k - \sum_{s=1}^{N}\lim_{\rho\to 0}\oint_{|x-p_s|=\rho} J_k \mathrm{d}x_k. \end{aligned} \tag{1.31}$$

Note that in (1.31) all path integrals (circulations of current) are taken counter-clockwise. The asymptotic estimates for a solution near infinity first imply that the second term on the right-hand side of (1.31) is zero. We now concentrate on the last term on the right-hand side of (1.31) which counts possible energy contributions from the antistrings at $p_1, p_2, \cdots, p_N$ but not from the strings at $q_1, q_2, \cdots, q_M$ because J_k is regular there.

To calculate the circulations near the antistrings, we may use $D_1 u = u\partial v$ and $D_2 u = \mathrm{i}u\partial v$ to get

$$\begin{aligned} I_s(\rho) = &\oint_{|x-p_s|=\rho} J_k\,\mathrm{d}x_k \\ = &\ \mathrm{i}\oint_{|x-p_s|=\rho} \frac{|u|^2}{1+|u|^2}([\overline{\partial}-\partial]v\mathrm{d}x_1 - \mathrm{i}[\overline{\partial}+\partial]v\mathrm{d}x_2) \\ = &\oint_{|x-p_s|=\rho} \frac{e^v}{1+e^v}(-\partial_2 v\mathrm{d}x_1 + \partial_1 v\mathrm{d}x_2). \end{aligned} \tag{1.32}$$

However, near p_s, v can be expressed as

$$v(x) = -\ln|x-p_s|^2 + w_s(x), \tag{1.33}$$

where w_s is a smooth function. Inserting (1.33) into (1.32) and letting $\rho \to 0$, we obtain

$$\lim_{\rho\to 0} I_s(\rho) = -4\pi,$$

which implies $\int J_{12} = 4\pi N$. Hence, we arrive at the quantized energy

$$E = 2\pi(M+N) \tag{1.34}$$

as desired. From this result and the Einstein equation (1.16), we derive also the quantized total curvature,

$$\mathcal{K} = \int K_g \, \mathrm{d}\Omega_g = 16\pi^2 G(M+N). \tag{1.35}$$

When gravity is switched off, $G = 0$, strings and antistrings are vortices and antivortices. An existence and uniqueness theorem may be proved for arbitrary numbers of vortices and antivortices and the same flux and energy formulas hold [46, 47].

The results (1.28), (1.33), and (1.34) imply that there is a symmetry between strings (vortices) and antistrings (antivortices). Here we observe that such a symmetry can be broken by an external field. To see this, we switch on a constant magnetic field along the x_3 axis, say $B = (0, 0, H)$. The energy density is now $\mathcal{E}' = \mathcal{E} - e^{-\eta} F_{12} H$. Hence (21) and (32) lead us to

$$E' = \int \mathcal{E}' \, \mathrm{d}\Omega_\eta = 2\pi M(1-H) + 2\pi N(1+H). \tag{1.36}$$

If $0 < H < 1$ (subcritical), the expression (1.36) says that strings are energetically preferred over antistrings; similarly, if $-1 < H < 0$, antistrings are preferred over strings. Consequently, in either case, the excited magnetic field F_{12} chooses to be aligned everywhere along the same direction of the applied field B. In other words, no matter how weak the external field is, its presence breaks the symmetry between strings and antistrings.

At the first sight, the obstructions (1.11) and (1.27) are rather different restrictions to the total numbers of strings in the two models. However, in terms of energy or total curvature, these obstructions are in fact *identical*,

$$E \le \frac{1}{4G} \quad \text{or} \quad \mathcal{K} \le 2\pi. \tag{1.37}$$

Open Problem 1.4. Find multistring solutions, which now define noncomplete metrics, when (1.27) is violated. Besides, how would the obstruction (1.10) look like for the model (1.15) with either M strings, or N antistrings, in terms of radially symmetric solutions? In particular, will (1.10) in terms of

$$E < \frac{1}{2G} \quad \text{or} \quad \mathcal{K} < 4\pi \tag{1.38}$$

continue to hold for either strings for antistrings with radial symmetry?

A Mathematical Application

Suppose that $\phi : \mathbb{R}^2 \to S^2$ is well behaved so that ϕ has a limiting image at the infinity of $\mathbb{R}^2$. Then ϕ may be viewed as a homotopy class in $\pi_2(S^2)$ so that it is characterized by an integer called the degree, $\deg(\phi)$, of ϕ which measures the number of times S^2 is being covered by itself ($S^2 = \mathbb{R}^2 \cup \{\infty\}$) under ϕ. Analytically, $\deg(\phi)$ can be represented by the integral

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \phi \cdot (\partial_1 \phi \wedge \partial_2 \phi). \tag{1.39}$$

It is therefore conceivable that (1.39) may not be an integer when ϕ does not have a definite limit at the infinity of $\mathbb{R}^2$. Here, we show that our string (vortex) and antistring (antivortex) solutions may be used to realize the topological integral (1.39) as any *half* integer of the form $n + 1/2$.

In fact, let (u, A_j) be a solution of our model representing M vortices and N antivortices. Using a stereographic transformation of the form

$$\phi_1 = \frac{2}{1+|u|^2}\,\mathrm{Re}(u), \quad \phi_2 = \frac{2}{1+|u|^2}\,\mathrm{Im}(u), \quad \phi_3 = \frac{1-|u|^2}{1+|u|^2}, \tag{1.40}$$

we get a map $\phi = (\phi_1, \phi_2, \phi_3) : \mathbb{R}^2 \to S^2$. From (1.40), it is seen that the zero and poles of u are mapped into the north and south poles of S^2, respectively, and that the vacuum space, $|u| = 1$, becomes the equator of S^2. Thus, as $|x| \to \infty$, $\phi(x)$ rotates around the equator of S^2, and (1.39) may fail to be an integer.

Indeed, we can show that the integral, I, has the explicit value

$$I = \frac{1}{2}(M + N), \tag{1.41}$$

which takes half-integer values unless $M = N \bmod(2)$.

Field theory origin

With (1.40) and the gauge-covariant derivative $D_j\phi = \partial_j\phi + A_j(\mathbf{n} \wedge \phi)$ where $\mathbf{n} = (0,0,1)$ denote the north pole of S^2, the model (1.15) comes from a version [46] of the gauged sigma model of Schroers [33], with broken symmetry, of the form

$$\mathcal{E} = \frac{1}{4} g^{jj'} g^{kk'} F_{jk} F_{j'k'} + \frac{1}{2} g^{jk} (D_j\phi) \cdot (D_k\phi) + \frac{1}{2}(\mathbf{n} \cdot \phi)^2. \tag{1.42}$$

2 Electroweak solitons and functional analysis

We are interested in soliton-like solutions in the Weinberg–Salam theory [23] which is a unified model for electromagnetic and weak interactions. We will describe some analytic work concerning the existence of vortices and dyons (monopoles). The main technique in the solution of these problems is the calculus of variations. More precisely, we shall look for critical points of a certain class of action functionals with integral type multiple constraints.

The gauge group is $SU(2) \times U(1)$. To generate solitons, it suffices to consider only the bosonic sector so that the Lagrangian density is of the form

$$\mathcal{L} = \frac{1}{4}(F^{\mu\nu} \cdot F_{\mu\nu} + G^{\mu\nu} G_{\mu\nu}) + (D^\mu\phi)^\dagger \cdot (D_\mu\phi) + \lambda(|\phi|^2 - \varepsilon^2)^2, \tag{2.1}$$

where ϕ is in the fundamental representation of the gauge group, hence is a complex doublet, and the field strength tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ lie in the Lie algebras of $SU(2)$ and of $U(1)$, which is $\mathbb{R}$, respectively, and D_μ is the gauge-covariant derivative. Here we choose not to discuss (2.1) in detail but only mention that unified electroweak forces are characterized by several structural parameters, including the weak coupling constant g, positron charge e, mixing angle θ, as well as the Higgs vacuum value $\varepsilon > 0$.

Vortices: In order to isolate physical properties, we need to specify the so-called unitary gauge. In this gauge the electroweak interactions through P photons, mediating electromagnetic forces, and W, Z bosons, mediating weak forces, are placed on a correct stage.

It was first found by Ambjorn and Olesen in [1, 2] that when the critical condition

$$\lambda = \frac{g^2}{8\cos^2\theta} \quad \text{(the Higgs particle mass = the } Z \text{ particle mass)} \tag{2.2}$$

is satisfied, the full equations of motion of the electroweak model in the unitary gauge has the following reduced form,

$$D_1 W = -\mathrm{i} D_2 W, \tag{2.3}$$

$$P_{12} = \frac{g}{2\sin\theta}\varepsilon^2 + 2g\sin\theta|W|^2, \tag{2.4}$$

$$Z_{12} = \frac{g}{2\cos\theta}(|\phi|^2 - \varepsilon^2) + 2g\cos\theta|W|^2, \tag{2.5}$$

$$Z_j = -\frac{\cos\theta}{g}\varepsilon_{jk}\partial_k \ln|\phi|^2, \tag{2.6}$$

over a two-dimensional periodic domain, Ω, where

$$D_j W = \partial_j W - \mathrm{i}g(P_j \sin\theta + Z_j\cos\theta)W, \quad P_{jk} = \partial_j P_k - \partial_k P_j.$$

Existence of N-vortex solutions. For any points $p_1, p_2, \cdots, p_N \in \Omega$, if the equations (2.3)–(2.6) have a solution (ϕ, W, P_j, Z_j) so that W vanishes exactly at these points, then

$$g^2\varepsilon^2 < \frac{4\pi N}{|\Omega|} < \frac{g^2\varepsilon^2}{\cos^2\theta}. \tag{2.7}$$

Furthermore, if in addition to (2.7), there holds

$$\frac{4\pi N}{|\Omega|} < \frac{8\pi\sin^2\theta}{|\Omega|} + g^2\varepsilon^2, \tag{2.8}$$

then (2.3)–(2.6) have a solution (ϕ, W, P_j, Z_j) so that W vanishes at $p_1, p_2, \cdots, p_N$, ϕ never becomes zero, and both the total energy and magnetic flux over Ω are proportional to N.

Note that, when $N = 1, 2$, (2.8) is contained in (2.7), and thus, (2.7) is a necessary and sufficient condition for existence of an N-vortex solution.

Another interesting fact concerning (2.7) is that, in order to have a solution, the number of vortices, N, should be neither small nor large.

Open Problem 2.1. Improve the sufficiency condition (2.8) for existence.

The proof [36] uses a multi-constrained variational principle and the Trudinger–Moser inequality of the form

$$\int_\Omega e^f \le C(\varepsilon) \exp\left(\left[\frac{1}{16\pi} + \varepsilon\right] \int_\Omega |\nabla f|^2\right), \quad \int_\Omega f = 0. \tag{2.9}$$

It is the optimal constant 16π in (2.9) that imposes on us the restrictive condition (2.8) in our associated minimization problem. Thus, to overcome (2.8), one might need to look for saddle points instead.

Using the substitution $f = \ln |W|^2, w = \ln |\phi|^2$, we have the governing equations

$$\Delta f = -4g^2 e^f - g^2 e^w + 4\pi \sum_{j=1}^{N} \delta_{p_j}, \tag{2.10}$$

$$\Delta w = 2g^2 e^f + \frac{g^2}{2\cos^2\theta}(e^w - \varepsilon^2) \tag{2.11}$$

on the torus Ω. To proceed, we introduce the functions u_0 and $U = e^{u_0}$ where u_0 satisfies

$$\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{N} \delta_{p_j}.$$

Now define $v = f - u_0$. Then (2.10) and (2.11) become

$$\Delta v = -4g^2 U e^v - g^2 e^w + \frac{4\pi N}{|\Omega|}, \tag{2.12}$$

$$\Delta w = 2g^2 U e^v + \frac{g^2}{2\cos^2\theta}(e^w - \varepsilon^2). \tag{2.13}$$

It is hard approach (2.12), (2.13) directly. Our trick is to use the transformation

$$v_1 = v + 2w, \quad v_2 = v,$$

to change the system into a "lower diagonal" form,

$$\Delta v_1 = -C_0 + g^2 \tan^2\theta e^{(v_1 - v_2)/2}, \tag{2.14}$$

$$\Delta v_2 = \frac{4\pi N}{|\Omega|} - g^2 e^{(v_1 - v_2)/2} - 4g^2 U e^{v_2}, \tag{2.15}$$

where the constant C_0 is define by

$$C_0 = \frac{g^2\varepsilon^2}{\cos^2\theta} - \frac{4\pi N}{|\Omega|}.$$

We can now integrate (2.14) and (2.15) to get the constraints

$$\int_\Omega e^{(v_1-v_2)/2} = C_0\frac{|\Omega|}{g^2}\cot^2\theta = C_1, \quad \int_\Omega Ue^{v_2} = \frac{|\Omega|}{4g^2\sin^2\theta}\left(\frac{4\pi N}{|\Omega|} - g^2\varepsilon^2\right) = C_2, \quad (2.16)$$

which lead to the necessary condition (2.7).

We can find a solution of (2.14) and (2.15) by minimizing the functional

$$I(v_1, v_2) = \int_\Omega \left\{\frac{\sigma}{2}|\nabla v_1|^2 + \frac{1}{2}|\nabla v_2|^2 - C_0\sigma v_1 + \frac{4\pi N}{|\Omega|}v_2\right\}, \quad \sigma = \cot^2\theta \qquad (2.17)$$

under the constraints in (2.16). We can show that this problem has a solution if (2.8) is fulfilled.

We use the function space $H_1(\Omega)$ (noting that Ω is a 2-torus). Define

$$M(f) = \frac{1}{|\Omega|}\int_\Omega f, \quad f \in H_1(\Omega).$$

We have the unique decomposition $f = M(f)+f'$ where $f' \in H_1(\Omega)$ satisfies $\int_\Omega f' = 0$. Hence I has the form

$$I(v_1, v_2) = \int_\Omega \left\{\frac{\sigma}{2}||\nabla v_1'|^2 + \frac{1}{2}|\nabla v_2'|^2\right\} - C_0\sigma|\Omega|M(v_1) + 4\pi N M(v_2). \qquad (2.18)$$

It will be crucial to estimate the tail terms in (2.18), i.e.,

$$T(v_1, v_2) \equiv -C_0\sigma|\Omega|M(v_1) + 4\pi N M(v_2),$$

in terms of v_1' and v_2'.

By (2.16), we have

$$M(v_1) = M(v_2) + 2\ln C_1 - 2\ln\left(\int_\Omega e^{(v_1'-v_2')/2}\right), \qquad (2.19)$$

$$M(v_2) = \ln C_2 - \ln\left(\int Ue^{v_2'}\right). \qquad (2.20)$$

As a consequence, we have

$$T(v_1, v_2) = (4\pi N - C_0\sigma|\Omega|)M(v_2) + 2C_0\sigma|\Omega|\ln\left(\int_\Omega e^{(v_1'-v_2')/2}\right) + C_3$$

where $C_3 = -2C_0\sigma|\Omega|\ln C_1$ is irrelevant. The term containing integral on the right-hand side of the above is bounded from below (with a lower bound independent of v_1', v_2') as may be seen by the Jensen inequality and the properties that $\int_\Omega v_j' = 0$ $(j = 1, 2)$. Moreover, we note that

$$C_4 \equiv 4\pi N - C_0\sigma|\Omega| = \frac{|\Omega|}{\sin^2\theta}\left(\frac{4\pi N}{|\Omega|} - g^2\varepsilon^2\right) > 0$$

in view of (2.7), and, that, by (2.9), there holds

$$\begin{aligned}\ln\Big(\int_\Omega Ue^{v_2'}\Big) &\le \frac{1}{p}\ln\Big(\int_\Omega U^p\Big)+\frac{1}{q}\ln\Big(\int_\Omega e^{qv_2'}\Big)\\ &\le \frac{1}{p}\ln\Big(\int_\Omega U^p\Big)+\frac{1}{q}\ln C(\varepsilon)+q\Big(\frac{1}{16\pi}+\varepsilon\Big)\|\nabla v_2'\|_2^2.\end{aligned}$$

Hence, inserting these results into (2.18), we obtain

$$\begin{aligned}I(v_1,v_2) &= \frac{\sigma}{2}\|\nabla v_1'\|_2^2+\frac{1}{2}\|\nabla v_2'\|_2^2+T(v_1,v_2)\\ &\ge \frac{\sigma}{2}\|\nabla v_1'\|_2^2+\gamma\|\nabla v_2'\|_2^2-C_5(\delta),\end{aligned}\qquad (2.21)$$

where the coefficient $\gamma=\gamma(q,\delta)$ is defined by

$$\begin{aligned}\gamma(q,\delta) &= \frac{1}{2}-C_4q\Big(\frac{1}{16\pi}+\delta\Big)\\ &= \frac{1}{2}\Big(1-\frac{2|\Omega|}{\sin^2\theta}\Big[\frac{4\pi N}{|\Omega|}-g^2\varepsilon^2\Big]\Big[\frac{1}{16\pi}+\delta\Big]q\Big).\end{aligned}\qquad (2.22)$$

Since $\gamma(1,0)>0$, we can make (2.22) positive for some $q>1$ and $\delta>0$. In view of (2.21) we see that I is bounded from below. In fact it also implies coerciveness of the problem. Hence a solution is obtained.

The structure of our problem above is similar to the prescribed Gauss curvature problem for 2-surfaces [3].

Dyons: Unlike monopoles [12], such solutions are to carry both electric and magnetic charges with localized field distributions [35, 54, 18, 31]. In 1996, Cho and Maison [9] published their results that dyons exist in the physically important Weinberg–Salam model [23]. Supported by rather convincing numerical stimulations, they made the following conclusions.

1. A new soliton (dyon) exists in the Weinberg–Salam theory which carries both electric and magnetic charges.
2. Dirac's monopole is contained as a component due reflecting the presence of electromagnetism.
3. Its magnetic charge q_m obeys also the Dirac condition,

$$q_m=\frac{4\pi}{e}.\qquad (2.23)$$

4. Its electric charge q_e is positive.
5. The Z boson stays neutral both electrically and magnetically, $q_e^Z=0, q_m^Z=0$.
6. The soliton is no more singular than the Dirac monopole.

Mathematically, the work of Cho and Maison leads to a difficult 4×4 system of nonlinear ordinary differential equations defined on the half line (radial variable) subject to a set of boundary conditions realizing regularity and a partially finite energy. Thus the existence of electroweak dyons becomes a differential equation problem. In [48], such a problem is thoroughly solved and we can state

Existence of electroweak dyons. The above described soliton by Cho and Maison exists in the Weinberg–Salam theory.

I now briefly describe how to solve this very interesting problem.

The system of equations is given below,

$$f'' = \frac{1}{r^2}(f^2-1)f + (\rho^2 - A^2)f, \tag{2.24}$$

$$(r^2\rho')' = \frac{1}{2}f^2\rho - \frac{1}{4}r^2(A-B)^2\rho + \frac{1}{2}r^2(\rho^2-1)\rho, \tag{2.25}$$

$$(r^2A')' = 2f^2A + r^2\rho^2(A-B), \tag{2.26}$$

$$(r^2B')' = r^2\rho^2(B-A), \tag{2.27}$$

supplemented with the boundary conditions

$$f(0)=1, \quad f(\infty)=0, \quad \rho(\infty)=1, \quad A(\infty)=B(\infty)=a_0, \tag{2.28}$$

$$\rho(0)=0, \quad A(0)=0, \quad B(0)=b_0. \tag{2.29}$$

We are to find a solution of (2.24)–(2.29) with a finite energy E where

$$\begin{aligned} E(f,\rho,A,B) &= \int_0^\infty \mathrm{d}r\Big\{(f')^2 + 2r^2(\rho')^2 + \frac{1}{2}r^2(A')^2 + \frac{1}{2}r^2(B')^2 + \frac{(f^2-1)^2}{2r^2} \\ &\quad + f^2\rho^2 + f^2A^2 + \frac{1}{2}r^2\rho^2(A-B)^2 + \frac{1}{2}r^2(\rho^2-1)^2\Big\}. \end{aligned} \tag{2.30}$$

The boundary condition (2.28) is an easy consequence of the finiteness of (2.30) but (2.29) is not which is one of the (minor) difficulties. In fact, the major difficulty is that (2.24)–(2.27) are *not* the Euler–Lagrange equations of the energy (2.30) but the following *indefinite* action functional

$$\begin{aligned} F(f,\rho,A,B) &= \int_0^\infty \mathrm{d}r\Big\{(f')^2 + 2r^2(\rho')^2 + \frac{(f^2-1)^2}{2r^2} + f^2\rho^2 + \frac{1}{2}r^2(\rho^2-1)^2 \\ &\quad - \frac{1}{2}r^2(A')^2 - \frac{1}{2}r^2(B')^2 - f^2A^2 - \frac{1}{2}r^2\rho^2(A-B)^2\Big\}. \end{aligned} \tag{2.31}$$

Problem. Find a critical point of F satisfying the boundary conditions (2.28) and (2.29) and $E < \infty$.

The negative terms in (2.31) make the problem a nonstandard one.

Our strategy is to find a suitable admissible space $\mathcal{C}$ which is large enough to contain a desired solution for us but is small enough so that, on which, the functional F becomes coercive in such a sense that a minimizing sequence weakly converges. Since the bad terms in (2.31) are the negative terms involving A and B, we are motivated to "freeze" them. However, we must also make sure that such a procedure does not jeopardizing losing solutions. Thus we may impose the following constraints to restrict A and B by

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}F(f,\rho,A+tA_1,B)\right)\Big|_{t=0}=0,\quad \left(\frac{\mathrm{d}}{\mathrm{d}t}F(f,\rho,A,B+tB_1)\right)\Big|_{t=0}=0, \tag{2.32}$$

where A_1, B_1 are test functions satisfying (2.28) and

$$E(f,\rho,A+A_1,B)<\infty,\quad E(f,\rho,A,B+B_1)<\infty<\infty.$$

In additional to the above, a less transparent constraint is the following diagonal one.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}F(f,\rho,A+tA_2,B+tB_2)\right)\Big|_{t=0}=0, \tag{2.33}$$

where A_2, B_2 satisfy (2.28) and $E(f,\rho,A+A_2,B+B_2)<\infty$.

This design of the admissible space allows us to obtain a minimizer of F in spite of the indefiniteness of F. Since the analysis is rather involved, I shall not discuss it here but, rather, indicate a few necessary steps. An interested reader may want to consult the original article [48].

Step 1. If a_0 is sufficiently small, then we have the partial coerciveness

$$F(f,\rho,A,B)\geq\int_0^\infty \mathrm{d}r\Big\{(f')^2+\frac{(f^2-1)^2}{2r^2}+2\varepsilon_1 r^2(\rho')^2+\varepsilon_2 f^2\rho^2+\frac{1}{2}r^2(\rho^2-1)^2\Big\}. \tag{2.34}$$

Here $\varepsilon_1,\varepsilon_2>0$ are small constants.

Step 2. Let $\{(f_n,\rho_n,A_n,B_n)\}$ be a minimizing sequence of F under the constraints (2.32) and (2.33), then (2.34) leads to weak convergence of $\{(f_n,\rho_n,A_n,B_n)\}$ in sense of subsequence.

Step 3. Show that in the weak limit, the constraints (2.32) and (2.33) are preserved and that a minimizer of F over the constrained class is obtained.

Step 4. Use (2.32) and (2.33) to show that the minimizer is a critical point of F satisfying the boundary condition (2.28).

Step 5. Recover the boundary condition (2.29) by the fact that the critical point is a minimizer of F over the constrained class.

Step 6. Establish asymptotic estimates and calculate the physical quantities stated earlier, and, hence, conclude the proof.

Electroweak strings: Consider a Minkowski manifold with metric $g_{\mu\nu}$. Denoting by R and $R_{\mu\nu}$ the scalar curvature and Ricci tensor, respectively, the Einstein tensor and the Einstein equations are then

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R,\quad G_{\mu\nu}=-8\pi G T_{\mu\nu}. \tag{2.35}$$

On the other hand, when cosmic string solutions are sought and a string metric is specified, $G_{\mu\nu}$ takes a special form. In fact, it has only two nontrivial components. $G_{00} = -G_{33}$, and all other components vanish if the metric is uniform along x_3 and independent of the time variable $t = x_0$. Such a property imposes a severe restriction on the energy-momentum tensor $T_{\mu\nu}$ through (2.35). In the Abelian gauge field models presented in Section, the associated energy-momentum tensors happen to enjoy the same property as $G_{\mu\nu}$ which made the Einstein equations (2.35) reduce into a single scalar curvature equation. However, when we try to extend such problems to non-Abelian gauge groups, such as $SU(2)$ and $SU(2)\times U(1)$, the energy-momentum tensor $T_{\mu\nu}$ no longer enjoys such a property due to the presence of nonvanishing commutator terms, and, in order to meet consistency, it is necessary to introduce the cosmological term into the Einstein equations, which now become

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2.36}$$

For example, for the electroweak strings in the framework discussed earlier, it is necessary for the cosmological constant Λ to take the unique value [49, 50, 51]

$$\Lambda = \pi G \frac{g^2 \varepsilon^4}{\sin^2\theta}. \tag{2.37}$$

The multistrings are governed by the following 2×2 system of nonlinear elliptic equations,

$$\Delta v_1 = C_{11} U(x)(e^{v_1} - 1) e^{a(2v_1+v_2) - b e^{v_1}} + C_{12} e^{v_2} + 4\pi \sum_{s=1}^{N} \delta_{p_s}, \tag{2.38}$$

$$\Delta v_2 = -C_{21} U(x) e^{(2a+1)v_1 + a v_2 - b e^{v_1}} - C_{21} e^{v_2}, \tag{2.39}$$

is a challenge to analysts [50, 51]. Here the constants C_{jk}'s are positive physical parameters and

$$a = \frac{4\pi G}{\sin^2\theta}, \quad b = 8\pi G, \quad U(x) = \left(\prod_{s=1}^{N} |x - p_s|\right)^{-16\pi G/\sin^2\theta} \tag{2.40}$$

Open Problem 2.2. Prove the existence of a finite energy solution of (2.38) and (2.39) under suitable conditions imposed on the coefficients and the string number N in these equations. In particular, a moderate problem would be an understanding of the radially symmetric solutions when all string points p's coincide.

3 Relativistic Chern–Simons equations

The Chern–Simons models arise in anyon physics and promise to give a theoretical framework for high-temperature superconductivity and the quantum Hall effect. In the context of field theories, the Chern–Simons vortices are both electrically and magnetically charged which are absent in the classical Yang–Mills theories. Mathematically, the general Chern–Simons equations are hard to solve even in their radially symmetric reductions [11, 22, 30]. It was the original work of Hong–Kim–Pac [14] and Jackiw–Weinberg [16] that brought light to a class of Chern–Simons equations which may be proved to have multivortex solutions by methods of nonlinear analysis. This class of equations are also self-dual equations. It may be interesting to bring to the attention of mathematicians that, away from self-duality, no solutions have been proved to exist even within radial symmetry ansatze, in contrast to the level of understanding on the classical Abelian Higgs model [17] where one can easily prove the existence of vortex solutions away from self-duality in the category of radially symmetric solutions.

Many of the non-relativistic Chern–Simons equations, Abelian or non-Abelian, are integrable (see Dunne [13] for a review and a comprehensive bibliography up to 1995). However, none of the relativistic equations is integrable. It is this class of equations that is interesting to people in the area of nonlinear partial differential equations. In the following lecture, I will discuss some recent progress in this direction.

Abelian case: The (2 + 1)-dimensional spacetime is equipped with the metric $g_{\mu\nu}$ =diag$(1,-1,-1)$. The Chern–Simons action density is

$$\mathcal{L} = \frac{1}{4}\kappa\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + (D_\alpha u)\overline{(D^\alpha u)} - \frac{1}{\kappa^2}|u|^2(1-|u|^2)^2, \tag{3.1}$$

where $A_\alpha = (A_0, A_1, A_2)$ is a gauge vector field, u a complex scalar Higgs field, $D_\alpha u = \partial_\alpha u - \mathrm{i}A_\alpha u$ the gauge-covariant derivative, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and $\kappa > 0$ the Chern–Simons coupling constant.

The equations of motion of (3.1) are

$$\frac{1}{2}\kappa\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma} = j^\alpha = \mathrm{i}(u\overline{D^\alpha u} - \overline{u}D^\alpha u), \tag{3.2}$$

$$D_\alpha D^\alpha u = -\frac{1}{\kappa^2}(2|u|^2[|u|^2-1] + [|u|^2-1]^2)u. \tag{3.3}$$

We are interested in static solutions of (3.2), (3.3). First it is seen from the temporal ($\alpha = 0$) component of (3.2) that

$$\kappa F_{12} = j^0 \quad \text{or} \quad \kappa B = \rho, \tag{3.4}$$

which says that there is an equivalence between magnetic and electric fields. Hence vortices are to carry both electric and magnetic charges. Such a feature is known to be impossible in any Yang–Mills model [18].

In [14, 16], it is found that (3.2),(3.3) can be reduced into the following self-dual system

$$D_1 u = \mp D_2 u, \tag{3.5}$$

$$F_{12} = = \pm\frac{2}{\kappa^2}|u|^2(1-|u|^2). \tag{3.6}$$

Namely, any solution of (3.5), (3.6) also satisfies (3.2), (3.3). However, none knows about the converse of this statement. Hence we propose

Open Problem 3.1. Are the systems of equations (3.2), (3.3) and (3.5), (3.6) equivalent for solutions of finite energy?

It is well known that, for the Abelian Higgs model, an equivalence theorem holds [17]. However, for the Abelian Higgs model, it is not as clear.

The solutions which satisfy the boundary condition $|u| \to 1$ as $|x| \to \infty$ are called topological and may be classified by winding numbers or $\pi_1(S^1)$. Existence and numerical approximation results have long been established [37, 41], but not uniqueness.

Open Problem 3.2. Given the locations and local winding charges of N vortices, is there a *unique* topological solution realizing such a prescription?

For radially symmetric solutions, the answer is yes [8], but it is unknown in the general case.

Non-topological solutions are those satisfying the boundary condition $|u| \to 0$ as $|x| \to \infty$ which are harder to obtain. A very interesting feature of such solutions is that they are not uniquely determined by their vortex locations and there is another free parameter to come into play: the fractional flux or charge. In fact, this fractional charge may assume any value in an explicitly given open interval. In literature, there are only rigorous proofs of such solutions with radial symmetry [8, 38]. Recently, Dongho Chae and Imanuvilov reported in [7] their progress on the construction of non-topological solutions with arbitrarily given vortex locations.

Vortices over a doubly periodic lattice domain Ω were first proved to exist in [6] (non-relativistic solutions were first constructed explicitly by Olesen [29]). More precisely, it is stated that there is a critical value κ_c satisfying

$$\kappa_c \le \frac{1}{2}\sqrt{\frac{|\Omega|}{\pi N}}, \tag{3.7}$$

so that, for $0 < \kappa < \kappa_c$, the self-dual Chern–Simons equations have an N-vortex solution (u, A) for which u vanishes exactly at the given vortex points $p_1, p_2, \cdots, p_N$, but for $\kappa > \kappa_c$, no solutions exist. In [39], Gabriella Tarantello sharpens this existence results in two directions. The first is that, at κ_c, solutions also exist. The second is the existence of another solution realizing the same vortex locations, which was quite unexpected.

With the substitution $v = \ln|u|^2$, the system (3.5), (3.6) governing vortices at the prescribed points $p_1, p_2, \cdots, p_N$ is seen to be equivalent to the elliptic equation [14, 16]

$$\Delta v = \lambda e^v(e^v - 1) + 4\pi \sum_{s=1}^{N} \delta_{p_s}. \tag{3.8}$$

Note that the non-relativistic version [15] of the equation is of the form

$$\Delta v = -\lambda e^v + 4\pi \sum_{s=1}^{N} \delta_{p_s}, \tag{3.9}$$

which is the classical Liouville equation and can be integrated explicitly (by Liouville's method, Backlünd transformation, or the inverse scattering theory). On the other hand, however, (3.8) is known to be non-integrable [32].

Non-Abelian case: Let G be a compact Lie group and $(\mathcal{G}, [\,,\,])$ the Lie algebra of G. The Chern–Simons action density is

$$\begin{aligned}\mathcal{L} &= \kappa\varepsilon^{\mu\nu\alpha}\mathrm{Tr}\Big(\partial_\mu A_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\Big) + \mathrm{Tr}([D_\mu\phi]^*[D^\mu\phi]) \\ &\quad -\frac{1}{4\kappa^2}\mathrm{Tr}\{([[\phi,\phi^*],\phi]-\phi)^*([[\phi,\phi^*],\phi]-\phi)\},\end{aligned} \tag{3.10}$$

where ϕ takes value in $\mathcal{G}$ (adjoint representation) and $D_\mu = \partial_\mu + [A_\mu, \cdot]$ is the gauge-covariant derivative.

As in the Abelian case, the original equations of motion for static solutions of (3.10) are hard to approach. However, again, there is a self-dual reduction to employ. We will use the notation

$$\begin{aligned} D_- &= D_1 - \mathrm{i}D_2, \quad \partial_\pm = \partial_1 \pm \mathrm{i}\partial_2, \\ A_\pm &= A_1 \pm \mathrm{i}A_2, \quad F_{+-} = \partial_- A_+ - \partial_+ A_- + [A_-, A_+]. \end{aligned}$$

Then the self-dual equations of the model (3.10) are now,

$$D_-\phi = 0, \tag{3.11}$$

$$F_{+-} = \frac{1}{\kappa^2}[\phi - [[\phi,\phi^*],\phi],\phi^*]. \tag{3.12}$$

Let r be the rank of $\mathcal{G}$. The reduced nonlinear elliptic equations governing multi-vortices, as the equation (3.8) for the Abelian case, are

$$\begin{aligned}\Delta v_a &= -\sum_{b=1}^{r} K_{ab}e^{v_b} + \sum_{b=1}^{r}\sum_{c=1}^{r} e^{v_b}K_{ab}e^{v_c}K_{bc} + 4\pi\sum_{s=1}^{N_a}\delta_{p_{as}}, \\ a &= 1,2,\cdots,r, \quad x\in\mathbb{R}^2,\end{aligned} \tag{3.13}$$

where $K = (K_{ab})_{r\times r}$ is the Cartan matrix of $\mathcal{G}$ which characterizes $\mathcal{G}$. Instead of reviewing the definition of K, here we only recall some of the useful properties of K for our purpose.

1. K is always invertible.

2. If K^{-1} denotes the inverse of K with entries $(K^{-1})_{ab}$, then $(K^{-1})_{ab} \geq 0$ for all a, b. In particular

$$\sum_{b=1}^{r}(K^{-1})_{ab} > 0, \quad a = 1, 2, \cdots, r. \tag{3.14}$$

3. K may or may not be symmetric.

Besides, we note that part of (3.13), namely,

$$\Delta v_a = -\sum_{b=1}^{r} K_{ab} e^{v_b}, \quad a = 1, 2, \cdots, r, \tag{3.15}$$

is the well-known Toda system which is integrable due to the elegant work of Kostant [21] and Leznov–Saveliev [25, 26].

We now establish a general existence theorem for the non-Abelian Chern–Simons equations (3.13).

For greater generality, we relax the assumption that K is the Cartan matrix of a semi-simple Lie algebra but assume that K has the decomposition

$$K = PS, \tag{3.16}$$

where P is an $r \times r$ diagonal matrix with positive diagonal entries and S is an $r \times r$ symmetric positive definite matrix.

Note. If K is a Cartan matrix, then

$$K_{ab} = 2\frac{(\vec{\alpha}_a, \vec{\alpha}_b)}{(\vec{\alpha}_a, \vec{\alpha}_a)}, \quad a, b = 1, 2, \cdots, r,$$

where $\vec{\alpha}_a$'s are the simple root vectors of $\mathcal{G}$. Thus, the matrix K in (3.13) of course satisfies (3.16).

We now state our results.

Suppose that

$$R_a = \sum_{b=1}^{r}(K^{-1})_{ab} > 0, \quad a = 1, 2, \cdots, r. \tag{3.17}$$

Then the Chern–Simons equations (3.13) have a solution satisfying

$$v_a \to \ln\left(\sum_{b=1}^{r}(K^{-1})_{ab}\right), \quad |\nabla v_a| \to 0, \quad \text{as } |x| \to \infty. \tag{3.18}$$

Besides, if $R = \text{diag}(R_1, R_2, \cdots, R_r)$ and

$$M = \frac{1}{2}(KRK + K^{\tau}RK^{\tau})$$

is positive definite (in particular, when K is symmetric), the above stated decays are exponentially fast and there hold the quantized integrals,

$$\begin{aligned}\int_{\mathbb{R}^2}\sum_{b=1}^{r} K_{ab} e^{v_b} - \int_{\mathbb{R}^2}\sum_{b=1}^{r}\sum_{c=1}^{r} e^{v_b} K_{ab} e^{v_c} K_{bc} &= 4\pi N_a, \\ a &= 1, 2, \cdots, r.\end{aligned} \tag{3.19}$$

Examples: We consider the physically most interesting group, $G = SU(N)$. Then

$$r = N - 1.$$

For $SU(2)$, $r = 1$, and the Chern–Simons system is a single scalar equation. In fact this equation is the same [24] as that in the Abelian case and is well understood.

The first "non-trivial" member is then $SU(3)$. So the Chern–Simons equations (due to Kao and Lee [19]) are

$$\Delta v_1 = -2e^{v_1} + e^{v_2} + 4e^{2v_1} - 2e^{2v_2} - e^{v_1+v_2} + 4\pi \sum_{s=1}^{N_1} \delta_{p_s}, \tag{3.20}$$

$$\Delta v_2 = e^{v_1} - 2e^{v_2} - 2e^{2v_1} + 4e^{2v_2} - e^{v_1+v_2} + 4\pi \sum_{s=1}^{N_2} \delta_{q_s}. \tag{3.21}$$

According to our results, the above system has a solution satisfying

$$v_1^2 + v_2^2 + |\nabla v_1|^2 + |\nabla v_2|^2 = O(e^{-\sqrt{2}|x|})$$

for $|x|$ large and there hold the quantized integrals

$$\int_{\mathbb{R}^2} (2e^{v_1} - e^{v_2} - 4e^{2v_1} + 2e^{2v_2} + e^{v_1+v_2}) = 4\pi N_1,$$

$$\int_{\mathbb{R}^2} (-e^{v_1} + 2e^{v_2} + 2e^{2v_1} - 4e^{2v_2} + e^{v_1+v_2}) = 4\pi N_2.$$

For the general gauge group $SU(N)$ ($N \geq 3$), the Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix}.$$

Our results apply well for this general case. We omit the details.

A variational method: We will use the Cholesky decomposition theorem to formulate a variational principle for the problem.

For the matrix S in the factoring (3.16), there is a unique lower triangular $r \times r$ matrix L with positive diagonal entries so that

$$S = LL^{\tau}. \tag{3.22}$$

We rewrite the Chern–Simons equations (3.13) in the matrix form

$$\Delta \mathbf{w} = -K\mathbf{U} + K\mathbf{U}K\mathbf{U} + \mathbf{g}, \tag{3.23}$$

where we have defined

$$
\begin{aligned}
v_a^0(x) &= -\sum_{s=1}^{N_a} \ln(1+|x-p_{as}|^{-2}), \\
g_a(x) &= 4\sum_{s=1}^{N_a} \frac{1}{(1+|x-p_{as}|^2)^2}, \\
v_a &= v_a^0 + w_a, \quad a=1,2,\cdots,r, \\
\mathbf{w} &= (w_1, w_2, \cdots, w_r)^\tau, \\
\mathbf{g} &= (g_1, g_2, \cdots, g_r)^\tau, \\
\mathbf{U} &= (e^{v_1^0+w_1}, e^{v_2^0+w_2}, \cdots, e^{v_r^0+w_r}).
\end{aligned}
$$

Setting

$$\mathbf{f} = (PL)^{-1}\mathbf{w}, \quad \mathbf{h} = (PL)^{-1}\mathbf{g},$$

we study the energy functional (crucial recognition)

$$I(\mathbf{f}) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\sum_{a=1}^{r} |\nabla f_a|^2 + \frac{1}{2}|L^\tau \mathbf{U} - (PL)^{-1}\mathbf{1}|^2 + \mathbf{h}\cdot\mathbf{w} \right\}, \tag{3.24}$$

where $\mathbf{1} = (1,1,\cdots,1)^\tau$ and $\mathbf{U}$ becomes

$$\mathbf{U} = (e^{v_1^0+\ell_{11}f_1}, \cdots, e^{v_r^0+\sum_{b=1}^r \ell_{rb}f_b})^\tau, \quad (\ell_{ab}) = PL.$$

We show that $I(\mathbf{f})$ gives us a correct variational principle for the equations (3.23). In fact define the following $r\times r$ matrix

$$U = \mathrm{diag}(e^{v_1^0+\ell_{11}f_1}, \cdots, e^{v_r^0+\sum_{b=1}^r \ell_{rb}f_b}).$$

Then, for any constant vector $\mathbf{c} = (c_1, c_2, \cdots, c_r)^\tau$, we have

$$D_\mathbf{f}(\mathbf{c}^\tau \cdot \mathbf{U}) = (PL)^\tau U\mathbf{c}.$$

¿From this we can see that a critical point of $I(\mathbf{f})$ satisfies

$$\Delta \mathbf{f} = -L^\tau \mathbf{U} + L^\tau PULL^\tau \mathbf{U} + \mathbf{h}.$$

Since $PU = UP$, we see that (3.23) follows.

To get a critical point for $I(\cdot)$, we minimize it. We notice that finite energy condition implies that

$$L^\tau \mathbf{U} - (PL)^{-1}\mathbf{1} = \mathbf{0}$$

at infinity. Namely, $\mathbf{U} = (K^{-1})\mathbf{1}$ at infinity. Hence

$$\lim_{|x|\to\infty} e^{v_a^0(x)+\sum_{b=1}^a \ell_{ab} f_b(x)} = \sum_{b=1}^r (K^{-1})_{ab}, \quad a = 1, 2, \cdots, r > 0. \tag{3.25}$$

For convenience, we may assume $\sum_{b=1}^r (K^{-1})_{ab} = 1$ for all a. Thus the boundary condition (3.25) simply becomes

$$\lim_{|x|\to\infty} f_a(x) = 0, \quad a = 1, 2, \cdots, r. \tag{3.26}$$

So, naturally, we can consider the optimization problem

$$\inf\{I(\mathbf{f}) \mid \mathbf{f} \in W^{1,2}(\mathbb{R}^2)\} \tag{3.27}$$

and topological solutions described earlier are therefore obtained.

Open Problem 3.3. Prove the existence of non-topological solutions of the system (3.13) for $r \geq 2$ which vanish at infinity.

Open Problem 3.4. Prove the existence of solutions of (3.13) over a doubly periodic domain Ω under some conditions similar to (3.7).

Other related problems: The method above may be used to study other systems of nonlinear equations. For example, we consider the system

$$\begin{aligned} \Delta v_j &= \sum_{k=1}^n a_{jk}(e^{v_k} - r_k) + 4\pi \sum_{s=1}^{N_j} \delta_{p_{js}}, \\ j &= 1, 2, \cdots, n, \quad x \in M, \end{aligned} \tag{3.28}$$

where M is a closed 2-surface or $\mathbb{R}^2$, $A = (a_{jk})$ is a symmetric positive definite matrix, and $r_j > 0$ are constants. This system has wide interest. When $n = 1$, it is simply the Abelian Higgs (Ginzburg–Landau) self-dual vortex equation thoroughly solved in [17]. It also arises as the two-dimensional Seiberg–Witten equation [42]. When $n = 2$, it arises in the two-Higgs extended electroweak theory of Bimonte and Lozano [5, 51]. The general case appears in the gauged sigma model of Schroers [34]. Hence it becomes highly desirable to conduct a unified treatment of (3.28). Indeed, a fairly complete study has recently been carried out [53]. Here we present the results obtained.

Case 1. M is closed. We use the notation

$$\mathbf{N} = (N_1, N_2, \cdots, N_n)^\tau, \quad \mathbf{r} = (r_1, r_2, \cdots, r_n)^\tau.$$

Then the system (3.28) has a solution if and only if (component-wise)

$$\frac{4\pi}{|M|} A^{-1}\mathbf{N} < \mathbf{r}. \tag{3.29}$$

Besides, if there is a solution, then solution must be unique.

Case 2. $M = \mathbb{R}^2$. In this case we need to supplement the equations (3.28) with the 'physical' boundary condition

$$\lim_{|x|\to\infty} e^{v_j(x)} = r_j, \quad j = 1, 2, \cdots, n. \tag{3.30}$$

Our existence theorem may be stated as follows: the system over $\mathbb{R}^2$ subject to the boundary condition (3.30) always has a unique solution. In fact, the solution approaches its asymptotic values given in (3.30) at infinity exponentially fast. Besides, there hold the quantized integrals

$$\int_{\mathbb{R}^2} \sum_{k=1}^{n} a_{jk}(e^{v_k} - r_k) = -4\pi N_j, \quad j = 1, 2, \cdots, n. \tag{3.31}$$

Case 3. $M = \mathbb{R}^2$ and some of the r_j's vanish. In this case we have some non-existence results.

(i) $r_j = 0$, $j = 1, 2, \cdots, n$. The boundary value problem (3.28), (3.30) has no solution.

(ii) $n = 2, r_1 > 0, r_2 = 0$ or $r_1 = 0, r_2 > 0$, and $a_{12} = a_{21} \leq 0$: No solution.

(iii) $n = 2, a_{12} = a_{21} > 0$, $r_1 > 0, r_2 = 0$, and

$$\frac{a_{12}}{a_{11}} N_1 \leq 1 + N_2. \tag{3.32}$$

In this case we can still prove that there is no solution.

The case $r_j > 0$ for all $j = 1, 2, \cdots$ represents a completely broken vacuum symmetry of the field theory model whereas the case that some of the r_j's vanish represents a situation when the vacuum symmetry is partially broken. The above non-existence results indicate that the existence problem in the partially broken symmetry case is rather subtle.

Open Problem 3.5. For $n = 2$, find suitable conditions for the case where $r_1 > 0$, $r_2 = 0$ and $a_{12} = a_{21} > 0$ for the boundary value problem (3.28), (3.30) such that it allows the existence of a solution. In particular, does there exist a solution when (3.32) is violated?

The author's research was supported in part by the National Science Foundation under grant DMS–9596041.

References

[1] J. Ambjorn and P. Olesen, Nucl. Phys. B **315**, 606 (1989).

[2] J. Ambjorn and P. Olesen, Nucl. Phys. B **330**, 193 (1990).

[3] T. Aubin, *Nonlinear Analysis on Manifolds: Monge–Ampére Equations*, Springer, Berlin and New York, 1982.

[4] A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).

[5] G. Bimonte and G. Lozano, Phys. Lett. B **326**, 270 (1994).

[6] L. Caffarelli and Y. Yang, Commun. Math. Phys. **168**, 321 (1995).

[7] D. Chae and O. Yu. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, Preprint, Seoul National Univ. 151-742, 1997.

[8] X. Chen, S. Hastings, J. B. McLeod, and Y. Yang, Proc. Roy. Soc. A **446**, 453 (1994).

[9] Y. M. Cho and D. Maison, Phys. Lett. B **391**, 360 (1996).

[10] A. Comtet and G. W. Gibbons, Nucl. Phys. B **299**, 719 (1988).

[11] H. J. de Vega and F. A. Schaposnik, Phys. Rev. Lett. **56**, 2564 (1986).

[12] P. A. M. Dirac, Proc. Roy. Soc. A **133**, 60 (1931).

[13] G. Dunne, *Self-dual Chern–Simons Theories*, Lect. Notes in Phys. **36**, Springer, Berlin 1995.

[14] J. Hong, Y. Kim, and P.-Y. Pac, Phys. Rev. Lett. **64**, 2330 (1990).

[15] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990).

[16] R. Jackiw and E. Weinberg, Phys. Rev. Lett. **64**, 2334 (1990).

[17] A. Jaffe and C. H. Taubes, *Vortices and Monopoles* (Birkhäuser, Boston, 1980).

[18] B. Julia and A. Zee, Phys. Rev. D. **11**, 2227 (1975).

[19] H.-C. Kao and K. Lee, Phys. Rev. D **50**, 6626 (1994).

[20] C. Kim and Y. Kim, Phys. Rev. D **50**, 1040 (1994).

[21] B. Kostant, Adv. Math. **34**, 195 (1979).

[22] C. N. Kumar and A. Khare, Phys. Lett. B **178**, 395 (1986).

[23] C. H. Lai (ed.), *Selected Papers on Gauge Theory of Weak and Electromagnetic Interactions*, World Scientific, Singapore.

[24] K. Lee, Phys. Rev. Lett. **66**, 553 (1991).

[25] A. Leznov and M. Saveliev, Lett. Math. Phys. **3**, 389 (1979).

[26] A. Leznov and M. Saveliev, *Group-theoretical Methods for Integration of Nonlinear Dynamical Systems*, Birkhäuser, Boston, 1992.

[27] B. Linet, Gen. Relat. Grav. **20**, 451 (1988).

[28] H. Nielsen and P. Olesen, Nucl. Phys. B **61**, 45 (1973).

[29] P. Olesen, Phys. Lett. B **265**, 361 (1991); (E) **267**, 541 (1991).

[30] S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986).

[31] M. Schechter and R. Weder, Ann. Phys. **132**, 293 (1981).

[32] J. Schiff, J. Math. Phys. **32**, 753 (1991).

[33] B. J. Schroers, Phys. Lett. B **356**, 291 (1995).

[34] B. J. Schroers, Nucl. Phys. B **475**, 440 (1996).

[35] J. Schwinger, Science **165**, 757 (1969).

[36] J. Spruck and Y. Yang, Commun. Math. Phys. **144**, 1 (1992).

[37] J. Spruck and Y. Yang, Ann. Inst. Henri Poincaré - Anal. Nonlin. **12**, 75 (1995).

[38] J. Spruck and Y. Yang, Commun. Math. Phys. **149**, 361 (1992).

[39] G. Tarantello, J. Math. Phys. **37**, 3769 (1996).

[40] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge Univ. Press, 1994).

[41] R. Wang, Commun. Math. Phys. **137**, 587 (1991).

[42] E. Witten, Math. Res. Lett. **1**, 769 (1994).

[43] Y. Yang, Phys. Rev. Lett. **73**, 10 (1994).

[44] Y. Yang, Commun. Math. Phys. **170**, 541 (1995).

[45] Y. Yang, Proc. Roy. Soc. A **453**, 581 (1997).

[46] Y. Yang, Phys. Rev. Lett. **80**, 26 (1998).

[47] Y. Yang, Strings of opposite magnetic charges in a gauge field theory, to be published.

[48] Y. Yang, Proc. Roy. Soc. A **454**, 155 (1998).

[49] Y. Yang, Commun. Math. Phys. **162**, 481 (1994).

[50] Y. Yang, Internat. J. Mod. Phys. A **11**, 203 (1996).

[51] Y. Yang, Physica D **101**, 55 (1997).

[52] Y. Yang, Commun. Math. Phys. **186**, 199 (1997).

[53] Y. Yang, On a system of nonlinear elliptic equations arising in theoretical physics, to be published.

[54] D. Zwanziger, Phys. Rev. **176**, 1489 (1968).

Massive Monopoles and Massless Monopole Clouds

Erick J. Weinberg

Department of Physics, Columbia University
New York, NY 10027, USA

Abstract

Magnetic monopole solutions naturally arise in the context of spontaneously broken gauge theories. When the unbroken symmetry includes a non-Abelian subgroup, investigation of the low-energy monopole dynamics by means of the moduli space approximation reveals degrees of freedom that can be attributed to massless monopoles. These do not correspond to distinct solitons, but instead are manifested as a cloud of non-Abelian field surrounding one or more massive monopoles. In these talks I explain how one is led to these solutions and then describe them in some detail.

1 Introduction

One-particle states arise in the spectra of weakly coupled quantum field theories in two rather different ways. By quantizing the small oscillations about the vacuum, one finds the states, with a characteristic mass m, that correspond to the quanta of the fundamental fields of the theory. It may also happen that the classical field equations of the theory possess localized solutions with energies of order m/λ, where λ is a typical small coupling of the theory; these soliton solutions also give rise to particles in the quantum theory. At first sight, these two classes of particles appear quite different: the former seem to be point particles with no internal structure, while the latter are extended objects described by a classical field profile $\phi(\mathbf{r})$.

However, these distinctions are not quite so clearcut. On the one hand, in an interacting theory even the fundamental point particles can be viewed as having a partonic substructure that evolves with momentum scale according to the DGLAP equations. On the other, one can analyze the behavior of the soliton states in terms of the normal modes of small fluctuations about the soliton. The modes in the continuum part of the spectra can be interpreted as scattering states of elementary quanta in the presence of the soliton; there may also be discrete eigenvalues corresponding to quanta bound to the soliton. This leaves only a small number of zero frequency modes whose quantization entails the introduction of the collective coordinates that may be viewed as the fundamental degrees of freedom of the soliton.

These considerations suggest that the particle states built from solitons and those based the elementary quanta do not differ in any essential way. Indeed, it can happen that states that appear as elementary quanta in one formulation of the theory correspond to solitons in another. The classic example of this is the correspondence between the sine-Gordon model and the Thirring model [1]. Of more immediate relevance to my talks is the conjecture by Montonen and Olive [2] that certain theories possess an exact electromagnetic duality relating magnetically charged solitons and electrically charged elementary quanta.

If there is such a duality, then one would expect the classical solutions to display particle-like properties. In particular, one would expect the classical solutions with higher charges to have a structure consistent with an interpretation in terms of component solitons of minimal charge. This is indeed found to be the case in many theories. However, in the course of studying magnetic monopoles in the context of larger gauge groups [3, 4], Kimyeong Lee, Piljin Yi, and I found [5] that in some theories there are classical solutions that do not quite fit this picture. As I will explain below, there is a sense in which these solutions can be understood as multimonopole solutions containing both massive and massless magnetic monopoles. While the massive components are quite evident when one examines the classical solutions, the massless monopoles appear to lose their individual identity and merge into a "cloud" of non-Abelian fields. Because these massless monopoles can be viewed as the duals to the massless gauge bosons of the theory, a better understanding of the nature of these unusual solutions may well provide deeper insight into the properties of non-Abelian gauge theories.

In these talks I will explain how one is led to these solutions and then describe them in some detail. I begin in the next section by reviewing some general properties of magnetic monopoles. In Sec. 3, I describe the Bogomolny-Prasad-Sommerfield, or BPS, limit [6] and its application to multimonopole solutions in an $SU(2)$ gauge theory. The extension of these results to larger gauge groups is discussed in Sec. 4. The treatment of low-energy monopole dynamics by means of the moduli space approximation is discussed in Sec. 5, while the actual determination of some moduli space metrics is described in Sec. 6. The theories where one actually encounters evidence of massless monopoles are those in which the unbroken gauge symmetry has a non-Abelian component. These are discussed in Sec. 7. Explicit examples of solutions in which the massless monopoles appear to condense into a non-Abelian cloud are described in Secs. 8 and 9. Section 10 contains some concluding remarks.

2 Magnetic monopoles

In the absence of sources, Maxwell's equations display a symmetry under the interchange of electric and magnetic fields. This suggests that there might also be a symmetry in sources, and that in addition to the familiar electric charges there might also be magnetically charged objects, usually termed magnetic monopoles, that act as sources for magnetic fields. A static monopole with magnetic charge Q_M would give rise to a Coulomb magnetic field

$$B_i = \frac{Q_M}{4\pi}\frac{\hat{r}_i}{r^2}\,. \tag{2.1}$$

In the canonical treatment of the behavior of charged particles in a magnetic field, either classically or quantum mechanically, it is most convenient to express the magnetic field as the curl of a vector potential $\mathcal{A}_i$. For the magnetic field of Eq. (2.1), a suitable choice is

$$\mathcal{A}_i = -\epsilon_{ij3}\frac{Q_M}{4\pi}\frac{\hat{r}_j}{r}\frac{(1-\cos\theta)}{\sin\theta}\,. \tag{2.2}$$

Note that this is singular along the negative z-axis. This "Dirac string" singularity is an inevitable consequence of trying to express a field with nonvanishing divergence as the curl of a potential; any potential leading to Eq. (2.1) will have a similar singularity along some curve running from the position of the monopole out to infinity. Physically, this singularity is a difficulty only if it actually observable. In classical physics, where only the magnetic field, and not the vector potential, is measurable, it causes no problem. However, there are quantum mechanical interference effects that are sensitive to the quantity

$$U = \exp\left[ie\oint_C \mathcal{A}_i dl_i\right] \tag{2.3}$$

where e is the electric charge of some particle and the integration is around any closed curve. If C is taken to be an infinitesimal closed curve in a region where $\mathcal{A}_i$ is nonsingular, U is clearly equal to unity. On the other hand, if the integral is taken around an infinitesimal closed curve encircling the Dirac string, U is not equal to unity, and the string is thus observable, unless the magnetic charge obeys the Dirac quantization condition[1]

$$Q_M = \frac{4\pi}{e}\left(\frac{n}{2}\right) \tag{2.4}$$

for some integer n. If we want the string to be unobservable, this condition must hold for all possible electric charges. This is only possible if all electric charges are integer multiples of some minimum charge for which Eq. (2.4) is satisfied. Thus, the existence of a single monopole in the universe would be sufficient to explain the observed quantization of electric charge.

There is an alternative approach that avoids the appearance of string singularities [7]. Instead, one introduces two gauge patches, one excluding the negative z-axis and one excluding the positive z-axis, and in each one chooses a vector potential that is nonsingular in that region. In the overlap of the two regions, the two vector potentials can differ only by the gauge transformation that relates the two patches. In order that this gauge transformation be single-valued in the overlap region, Eq. (2.4) must hold.

[1] I am assuming units in which $\hbar = 1$; otherwise there is a additional factor of $\hbar$ on the right hand side.

One can always incorporate magnetic monopoles into a theory with electrically charged particles simply by postulating a new species of fundamental particles. However, it turns out [8] that monopoles are already implicit in many theories with electrically charged fundamental fields. In these theories, the classical field equations have localized finite energy solutions with magnetic charge that correspond to one-particle states of the quantized theory. Although topological arguments are usually used to demonstrate the existence of these solutions, their existence and many of their features can in fact be understood on the basis of energetic arguments alone [9].

To begin, note that the Coulomb magnetic field Eq. (2.1) has a $1/r^2$ singularity at the origin. In contrast to the Dirac string, this a true physical singularity, as can be seen by noting that it leads to a $1/r^4$ divergence in the energy density $\mathcal{E} = \frac{1}{2}\mathbf{B}^2$. This singularity must somehow be tamed if finite energy classical solutions are to exist. One approach might be to replace the point magnetic charge by a charge distribution, but the Dirac quantization condition forbids such continuous charge distributions in theories with both electric and magnetic charges.

However, there is another possibility for removing the divergence. When placed in a magnetic field, a magnetic dipole $\mathbf{d}$ acquires an energy $-\mathbf{d}\cdot\mathbf{B}$. Thus, the singular energy density in the magnetic field might be cancelled by introducing a suitable (singular) distribution of magnetic dipoles. This idea can be implemented by introducing a complex vector field $\mathbf{W}$ with electric charge e and a magnetic dipole density $\mathbf{d} = ieg\mathbf{W}^* \times \mathbf{W}$, with g a real constant that for the moment can be taken to be arbitrary. Since we want there to be a lower bound on the energy, the energy density must also contain terms of higher order in $\mathbf{d}$. In particular, adding a term $\mathbf{d}^2/2$ allows the $1/r^4$ divergence of the energy density to be cancelled if $|\mathbf{W}| \sim 1/r$ near the origin.

Since we want the W field to be localized within a finite region, the energy density should contain a mass term of the form $M_W^2|\mathbf{W}|^2$. However, this would give a $1/r^2$ contribution to $\mathcal{E}$ near the origin. This singularity can be eliminated by allowing the W mass to be dependent on some spatially varying field ϕ. In particular, let us assume that $M_W = G\phi$, where G is a constant and the scalar field potential $V(\phi)$ is minimized by $\phi = v \neq 0$. Finiteness of the energy then implies that at large distances $\phi \approx v$ and $M_W \neq 0$, but at $r = 0$ both ϕ and M_W can be taken to vanish. Provided that the contribution from the gradients of the fields introduce no additional singularities (which can be arranged), the energy density will then be nonsingular everywhere.

An energy density of the sort described here can be obtained from a Lagrangian density of the form

$$\mathcal{L} = -\frac{1}{4}(\mathcal{F}_{\mu\nu} - iegW_\mu^* W_\nu)^2 - \frac{1}{2}|\mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu|^2 + G^2\phi^2|W_\mu|^2 - \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) \qquad (2.5)$$

where $\mathcal{D}_\mu = \partial_\mu + ie\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ are the electromagnetic covariant derivative and electromagnetic field strength. A solution of the resulting Euler-Lagrange equations that carries unit magnetic charge (i.e., $Q_M = 4\pi/e$) can be obtained by introducing the ansatz

$$\begin{aligned}
\mathcal{A}_i &= -\epsilon_{ij3}\frac{\hat{r}_j}{er}\frac{(1-\cos\theta)}{\sin\theta} \\
W_1 &= -\frac{i}{\sqrt{2}}\frac{u(r)}{er}\left[1 - e^{i\phi}\cos\phi(1-\cos\theta)\right] \\
W_2 &= \frac{1}{\sqrt{2}}\frac{u(r)}{er}\left[1 + e^{i\phi}\sin\phi(1-\cos\theta)\right] \\
W_3 &= \frac{i}{\sqrt{2}}\frac{u(r)}{er}e^{i\phi}\sin\theta \\
\phi &= h(r)\,.
\end{aligned} \qquad (2.6)$$

Substitution of this ansatz into the Euler-Lagrange equations leads to a pair of coupled second order ordinary differential equations that can be solved numerically to yield a finite energy solution. However, that the Dirac string singularity still remains.

A very important special case is obtained by setting $g = 2$ and $G = e$. With this choice of parameters, the theory described by Eq (2.5) is in fact an $SU(2)$ gauge theory in disguise. Let us define the components of an $SU(2)$ gauge field $A_\mu \equiv A_\mu^a T^a$ and a triplet Higgs field $\Phi \equiv \Phi^a T^a$ by

$$\begin{aligned}
A_\mu^1 + iA_\mu^2 &= W_\mu, \qquad A_\mu^3 = \mathcal{A}_\mu \\
\Phi^a &= \delta^{a3}\phi(r)
\end{aligned} \qquad (2.7)$$

where the T^a are the generators of $SU(2)$. The Lagrangian (2.5) can then be rewritten in the form

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\,F_{\mu\nu}^2 + \frac{1}{2}\mathrm{Tr}\,(D_\mu\Phi)^2 - V(\Phi)\,. \qquad (2.8)$$

Here

$$D_\mu = \partial_\mu + ieA_\mu \qquad (2.9)$$

is the non-Abelian covariant derivative and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \qquad (2.10)$$

is the field strength with magnetic components $B_i = (1/2)\epsilon_{ijk}F_{jk}$ and electric components $E_i = F_{oi}$. For definiteness, let us assume that the potential is of the form

$$V(\Phi) = -\frac{\mu^2}{2}\mathrm{Tr}\,\Phi^2 + \frac{\lambda}{4}(\mathrm{Tr}\,\Phi^2)^2 \qquad (2.11)$$

If $\mu^2 < 0$, the classical energy has a minimum at $\Phi = 0$ that preserves the $SU(2)$ symmetry. The spectrum of the quantum theory includes three massless gauge bosons and three massive scalars with equal masses. If instead $\mu^2 > 0$, there is a family of physically equivalent degenerate minima given by $\mathrm{Tr}\,\Phi_0^2 = \mu^2/\lambda \equiv v^2$; the "vacuum manifold" of such minima can be identified with the coset space $SU(2)/U(1) = S^2$. In each of these vacuum states the $SU(2)$ symmetry is spontaneously broken to the $U(1)$ subgroup that leaves Φ_0 invariant; this $U(1)$ subgroup may be identified with electromagnetism. After quantization of the theory, the small fluctuations about the vacuum lead to a spectrum of elementary particles that includes a massless photon, an electrically neutral Higgs scalar with mass $\sqrt{2}\mu$, and a pair of vector bosons with mass ev and electric charges $\pm e$.

In describing either the vacuum or configurations that are small perturbations about the vacuum, it is most natural to take the orientation of the Higgs field to be uniform in space; indeed, our ansatz for the monopole solution corresponds to a vacuum with $\Phi_0^a = v\delta^{a3}$. However, the orientation of the Higgs field is gauge-dependent quantity that need not be uniform. In particular, by applying a spatially varying $SU(2)$ gauge transformation (with a singularity along the negative z-axis), we can bring our monopole solution into the manifestly nonsingular "radial gauge" form

$$\begin{aligned} A_j^a &= \epsilon_{jak}\hat{r}_k \frac{1-u(r)}{er} \\ \Phi^a &= \hat{r}_a h(r)\,. \end{aligned} \qquad (2.12)$$

At large-distances the resulting magnetic field

$$B_i^a = \frac{1}{e}\frac{\hat{r}_i\hat{r}_a}{r^2} + O(1/r^3) \qquad (2.13)$$

is parallel to Φ in internal space, showing that it lies in the unbroken electromagnetic subgroup.

In any finite energy solution, the Higgs field must approach one of the minima of $V(\Phi)$ as $r \to \infty$ in any fixed direction. Hence, for any nonsingular solution the Higgs configuration gives a map from the S^2 at spatial infinity into the vacuum manifold $SU(2)/U(1)$. Any such map corresponds to an element of the homotopy group $\Pi_2(SU(2)/U(1)) = Z$ and can therefore be assigned an integer "topological charge" n. While the vacuum solutions correspond to the identity element with $n = 0$, the radial gauge monopole solution gives a topologically nontrivial map with $n = 1$. In fact, one can show that there is a one-to-one correspondence between the magnetic charge and the topological charge, with a nonsingular $SU(2)$ configuration of total magnetic charge $Q_M = 4\pi n/e$ having topological charge n. Although magnetic charges that are half-integer multiples of $4\pi n/e$ are allowed by the Dirac quantization condition, they cannot be obtained from nonsingular field configurations.

3 The BPS limit

An especially interesting special case, which I will assume for the remainder of these talks, is known as the Bogomolny-Prasad-Sommerfield, or BPS, limit [6]. It can be motivated by considering the expression for the energy of a static configuration with magnetic, but not electric, charge. Assuming for the moment that A_0 vanishes identically, we have

$$\begin{aligned} E &= \int d^3x \left[\frac{1}{2}\mathrm{Tr}\, B_i^2 + \frac{1}{2}\mathrm{Tr}\,(D_i\Phi)^2 + V(\Phi)\right] \\ &= \int d^3x \left[\frac{1}{2}\mathrm{Tr}\,(B_i \mp D_i\Phi)^2 + V(\Phi) \pm \mathrm{Tr}\, B_i D_i \Phi\right] . \end{aligned} \tag{3.1}$$

With the aid of the Bianchi identity $D_i B_i = 0$ the last term on the right hand side may be rewritten as a surface integral over the sphere at spatial infinity:

$$\int d^3x \mathrm{Tr}\, B_i D_i \Phi = \int d^3x \partial_i (\mathrm{Tr}\, B_i \Phi) = \int dS_i \mathrm{Tr}\, B_i \Phi \equiv Q_M v\,. \tag{3.2}$$

(The normalization of Q_M implied by the last equality agrees with that of Eq. (2.1).) Substituting this back into the previous equation yields the bound

$$\begin{aligned} E &= \pm Q_M v + \int d^3x \left[\frac{1}{2}\mathrm{Tr}\,(B_i \mp D_i\Phi)^2 + V(\Phi)\right] \\ &\geq |Q_M| v\,. \end{aligned} \tag{3.3}$$

The BPS limit is obtained by dropping the contribution of $V(\Phi)$ to the energy. This can be done most simply by letting $\mu^2 \to 0$, $\lambda \to 0$, with $v^2 = \mu^2/\lambda$ held fixed. It can also be obtained by considering the extension of this theory to a Yang-Mills theory with extended supersymmetry. The latter approach is particularly attractive from a physical point of view, and can be formulated in such a way that the BPS limit is preserved by higher order quantum corrections.

Now recall that any static configuration that is a local minimum of the energy is a stable solution of the classical equations of motion. Because the magnetic charge is quantized, any configuration that saturates the lower bound in Eq. (3.3) will be such a solution. With $V(\Phi)$ absent, the conditions for saturation of this bound are the BPS equations

$$B_i = D_i \Phi\,. \tag{3.4}$$

(I have assumed here, and henceforth, that $Q_M \geq 0$; the extension to the case $Q_M < 0$ is obvious.) One can easily verify by direct substitution that any solution of the first-order BPS equations is indeed a solution of the second-order Euler-Lagrange equations.

This result can easily be extended to the case of dyons, solutions carrying not only a magnetic charge Q_M but also a nonzero electric charge

$$Q_E = v^{-1} \int dS_i \mathrm{Tr}\, E_i \Phi \,. \tag{3.5}$$

The bound on the energy is generalized to

$$E \geq v\sqrt{Q_M^2 + Q_E^2} \tag{3.6}$$

with the minimum being achieved by configurations that satisfy

$$\begin{aligned} B_i &= \cos\beta D_i \Phi \\ E_i &= \sin\beta D_i \Phi \\ D_0 \Phi &= 0 \end{aligned} \tag{3.7}$$

with $\beta = \tan^{-1}(Q_E/Q_M)$.

An attractive feature, which was in fact one of the original motivations for the BPS approximations, is that it is possible to obtain a simple analytic expression for the singly charged monopole solution. By a rescaling of fields and distances the gauge coupling e can be set equal to unity; for the remainder of these talks I will assume that this has been done. The solutions can then be written as [6],

$$\begin{aligned} A_j^a &= \epsilon_{jak} \hat{r}_k \left[\frac{v}{\sinh(vr)} - \frac{1}{r} \right] \\ \Phi^a &= \hat{r}_a \left[v \coth(vr) - \frac{1}{r} \right] . \end{aligned} \tag{3.8}$$

Note that the Higgs field does not approach its asymptotic value exponentially fast, but instead has a $1/r$ tail. This is because the absence of a potential term makes the Higgs field massless. Since a massless scalar field carries a long-range force that is attractive between like objects, this raises the possibility that the magnetic repulsion between two BPS monopoles might be exactly cancelled by their mutual scalar attraction, thus allowing for the existence of static multimonopole solutions. In fact, it turns out that such solutions — indeed continuous families of solutions — exist for all values of Q_M.

The actual construction of these multimonopole solutions is a difficult, but fascinating, problem. For the moment, I will simply concentrate on the problem of counting the number of physically meaningful parameters, or "collective coordinates", needed to specify these solutions. Each of these corresponds to a zero frequency eigenmode (a "zero mode") in the spectrum of small fluctuations about a given solution. However, there are also an infinite number of zero modes, corresponding to local gauge transformations of the solution, that do not correspond to any physically meaningful parameter. To eliminate these, a gauge condition must be imposed on the fields.

I will start with the zero modes about the solution with unit magnetic charge. The elimination of the gauge modes is particularly transparent if we work in the singular "string gauge" of Eq. (2.7) where $\Phi^1 = \Phi^2 = 0$. This leaves only a $U(1)$ gauge freedom that can be fixed by imposing, e.g, the electromagnetic Coulomb gauge condition $\nabla \cdot \mathcal{A} = 0$. Explicit solution of the zero mode equations then shows that there are precisely four normalizable zero modes about the solution. Three of these correspond to infinitesimal spatial translations of the monopole; the corresponding parameters are most naturally chosen to be the spatial coordinates of the center of the monopole. The fourth zero mode corresponds to a spatially constant phase rotation of the massive vector field, $W_\mu(\mathbf{r}) \to e^{i\alpha} W_\mu(\mathbf{r})$. Since this mode is in fact a gauge mode that has no effect on gauge-invariant quantities, one might think that it should be discarded as unphysical. The justification for not doing so comes from considering the effect of allowing the collective coordinates to be time-dependent. In the case of the translation modes, this gives a solution with nonzero linear momentum. For the gauge mode, allowing the phase α to vary linearly in time produces a dyon solution that carries an electric charge proportional to $d\alpha/dt$.

Although explicit solution of the zero mode equations suffices for the case of unit magnetic charge, where the monopole solution is known explicitly, index theory methods are needed to count the zero modes about solutions with higher charges [10]. Each zero mode consists of perturbations δA_j and $\delta\Phi$ that can be viewed as three-component vectors transforming under the adjoint representation of $SU(2)$. Since these preserve the BPS equations, they must satisfy

$$\begin{aligned} 0 &= \delta(B_j - D_j\Phi) \\ &= D_j\delta\Phi - \phi\delta A_j - \epsilon_{jkl} D_k \delta A_l \,. \end{aligned} \tag{3.9}$$

(Here $D_j = \partial_j + A_j$ and A_j and Φ are 3×3 anti-Hermitian matrices in the adjoint representation of $SU(2)$.) These must be supplemented by a gauge condition that eliminates the unwanted gauge modes. A convenient choice is the background gauge condition

$$0 = D_j\delta A_j + \Phi\delta\Phi \tag{3.10}$$

which is equivalent to requiring that the perturbation be orthogonal, in the functional sense, to all normalizable gauge modes. The number of collective coordinates is just equal to the number of linearly independent normalizable solutions of Eqs. (3.9) and (3.10).

If we define [11]

$$\psi = I\delta\Phi + i\sigma_j \delta A_j \tag{3.11}$$

where I is the unit 2×2 matrix and the σ_j are the Pauli matrices, Eqs. (3.9) and (3.10) can be combined into the single Dirac-type equation

$$0 = (-i\sigma_j D_j + i\Phi)\psi \equiv \mathcal{D}\psi\,. \tag{3.12}$$

We must remember, however, that two solutions ψ and $i\psi$ that are linearly dependent as solutions of Eq. (3.12) actually correspond to linearly independent solutions of the original bosonic equations (3.9) and (3.10). The number of collective coordinates is thus actually twice the number of linearly independent normalizable zero eigenmodes of $\mathcal{D}$.

Note that if $\psi(\mathbf{r})$ is a solution of Eq. (3.12), then so is

$$\psi'(\mathbf{r}) = \psi(\mathbf{r})U \tag{3.13}$$

where U is any 2×2 unitary matrix. This fact, which be of importance later, implies that number of normalizable zero eigenmodes of the bosonic equation must be a multiple of four.

The next step is to define

$$\mathcal{I}(M^2) = \mathrm{Tr}\,\frac{M^2}{\mathcal{D}^\dagger\mathcal{D} + M^2} - \mathrm{Tr}\,\frac{M^2}{\mathcal{D}\mathcal{D}^\dagger + M^2} \tag{3.14}$$

where M is an arbitrary real number and

$$\mathcal{D}^\dagger = -i\sigma_j D_j - i\Phi \tag{3.15}$$

is the adjoint of $\mathcal{D}$. The quantity

$$\mathcal{I} = \lim_{M^2 \to 0} \mathcal{I}(M^2) \tag{3.16}$$

is then equal to the number of zero eigenvalues of $\mathcal{D}^\dagger\mathcal{D}$ minus the number of zero eigenvalues of $\mathcal{D}\mathcal{D}^\dagger$. Using the fact that the unperturbed solution obeys the BPS equations, one finds that

$$\begin{aligned} \mathcal{D}^\dagger\mathcal{D} &= -D_j^2 + 2\sigma_j B_j + \Phi^2 \\ \mathcal{D}\mathcal{D}^\dagger &= -D_j^2 + \Phi^2\,. \end{aligned} \tag{3.17}$$

The second equation shows that $\mathcal{D}\mathcal{D}^\dagger$ is a positive operator with no normalizable zero modes. Since every normalizable zero mode of $\mathcal{D}$ is also a normalizable zero mode of $\mathcal{D}^\dagger\mathcal{D}$, and conversely,

$\mathcal{I}$ would clearly give the desired counting of zero modes if it were not for the fact that these operators have continuous spectra extending down to zero.

The contribution from these continuous spectra can be written as

$$\mathcal{I}_{\text{continuum}} = \lim_{M^2 \to 0} \int \frac{d^3k}{(2\pi)^3} \frac{m^2}{k^2 + M^2} \left[\rho_{\mathcal{D}^\dagger\mathcal{D}}(k) - \rho_{\mathcal{D}\mathcal{D}^\dagger}(k)\right] \tag{3.18}$$

where $\rho_{\mathcal{O}}(k)$ is the density of continuum eigenstates of an operator $\mathcal{O}$. This contribution can be nonzero only if these density of states factors are singular near $k = 0$. For the case at hand, one can show that this is not the case. The essential idea is that such singularities are determined by the large r behavior of the potential terms in the operators. Since $\mathcal{D}^\dagger\mathcal{D} - \mathcal{D}\mathcal{D}^\dagger = 2\sigma_j B_j$, the potentially dangerous behavior is associated with the long-range behavior of the magnetic field. But, up to exponentially small corrections, the long-range part of the B_j lies in the unbroken $U(1)$ subgroup and so does not act on the massless components of the fields, which also lie in this $U(1)$. Since only these latter fields have spectra that extend down to zero, $\mathcal{I}_{\text{continuum}}$ vanishes.

Having eliminated the continuum contribution, let us now turn to the evaluation of $\mathcal{I}$. For this purpose it is convenient to adopt a pseudo-four-dimensional notation and define a four-vector V_μ with components $V_j = A_j$ for $j = 1, 2, 3$ and $V_4 = \Phi$. Because this is actually a three-dimensional space, $\partial_4 = 0$ and so $D_4 = \Phi$. Similarly, $G_{ij} = F_{ij}$ while $G_{i4} = -G_{4i} = D_i\Phi$. Finally, the Dirac matrices

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{3.19}$$

all anticommute with each other and all have square equal to unity.

With these definitions, we can write

$$\gamma_\mu D_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^\dagger & 0 \end{pmatrix} \tag{3.20}$$

and hence

$$\mathcal{I}(M^2) = -\mathrm{Tr}\,\gamma_5 \frac{M^2}{-(\gamma\cdot D)^2 + M^2} = -\int d^3x \langle x| \mathrm{tr}\,\gamma_5 \frac{M}{\gamma\cdot D + M} |x\rangle\,. \tag{3.21}$$

(Here Tr indicates a functional trace, while tr denotes a trace over Dirac and $SU(2)$ matrix indices.) The integrand in the last expression can be written as the three-dimensional divergence of the current

$$J_i = \frac{1}{2}\int d^3x \langle x| \mathrm{tr}\,\gamma_5\gamma_i \frac{1}{\gamma\cdot D + M}|x\rangle = -\frac{1}{2}\int d^3x \langle x|\mathrm{tr}\,\gamma_5\gamma_i(\gamma\cdot D)\frac{1}{-(\gamma\cdot D)^2 + M^2}|x\rangle \tag{3.22}$$

and so

$$\mathcal{I}(M^2) = \int d^3x \partial_i J_i(x) = \int dS_i J_i(x) \tag{3.23}$$

where the surface integral in the last term is over the sphere at spatial infinity.

We now write

$$\frac{1}{-(\gamma \cdot D)^2 + M^2} = \frac{1}{-\mathbf{D}^2 + \Phi^2 + M^2} + \frac{1}{-\mathbf{D}^2 + \Phi^2 + M^2}\left(\frac{i}{2}\gamma_\mu\gamma_\nu G_{\mu\nu}\right)\frac{1}{-\mathbf{D}^2 + \Phi^2 + M^2} + \cdots \tag{3.24}$$

where $\mathbf{D}^2 = D_j D_j$ and the dots represent terms of order G^2 or higher that vanish at least as fast as $1/r^4$ at spatial infinity. When this expansion is substituted into the expression for J_i, the contribution from the first term vanishes after the trace over Dirac indices is performed. The remaining terms give

$$\begin{aligned} \hat{x}_i J_i &= -\frac{i}{2}\epsilon_{i\lambda\mu\nu}\hat{x}_i \mathrm{tr}\,\langle x|D_\lambda \frac{1}{-\mathbf{D}^2 + \Phi^2 + M^2} G_{\mu\nu}\frac{1}{-\mathbf{D}^2 + \Phi^2 + M^2}|x\rangle + O(x^{-4}) \\ &= \frac{i}{2}\langle x|\Phi\frac{1}{-\nabla^2 + \Phi^2 + M^2}\hat{x}\cdot B\frac{1}{-\nabla^2 + \Phi^2 + M^2}|x\rangle + O(|x|^{-3}) \end{aligned} \tag{3.25}$$

where the trace is now only over $SU(2)$ indices.

The evaluation of this last expression is most transparent if we work in the singular "string gauge". If the magnetic charge is $Q_M = 4\pi n$, then asymptotically $\Phi \to vT^3$, and $\hat{x}\cdot B \to (n/x^2)T^3$ and one finds that

$$\hat{x}_i J_i = \frac{n}{2\pi x^2}\frac{v}{\sqrt{v^2 + M^2}} + O(|x|^{-3}). \tag{3.26}$$

It follows that

$$\mathcal{I}(M^2) = 2n\frac{v}{\sqrt{v^2 + M^2}} \tag{3.27}$$

and that the number of linearly independent normalizable zero modes of the original bosonic problem is

$$2\mathcal{I} = 4n. \tag{3.28}$$

A priori, one might have expected that classical solutions with higher charges could lead to new types of magnetically charged particles. Eq. (3.28), together with the fact that the BPS energy is strictly proportional to the magnetic charge, suggests that this is not the case. Instead, all higher charged solutions should be viewed as being multimonopole solutions composed of $n > 1$ unit monopoles, each with three translational and one $U(1)$ degree of freedom.[2] In the quantum theory, these solutions would thus correspond to multiparticle states.

[2]As was the case with the unit monopole, there is only a single gauge mode that is not eliminated by the gauge condition; this corresponds to a simultaneous $U(1)$ rotation of all the monopoles. The modes corresponding to relative $U(1)$ rotations are not simply gauge transformations of the underlying solution.

It is useful at this point to review the spectrum of particles in this theory. Quantization of the small fluctuations of the fundamental fields yielded two particles, the photon and the Higgs scalar, that have neither electric nor magnetic charge and that in the BPS limit are both massless. It also gave two massive vector particles, with electric charges $\pm e$, no magnetic charge, and mass ev. In addition to these we have the monopole and antimonopole, with no electric charge, magnetic charges $\pm 4\pi/e$, and mass $(4\pi/e)v$. A curious feature of this spectrum is that the pattern of masses and charges remains the same under the interchanges $e \leftrightarrow 4\pi/e$ and $Q_E \leftrightarrow Q_M$. There is a mismatch in spin, since the monopole and antimonopole are spinless, while the vector bosons have spin one, but this can be remedied by enlarging the theory so that it has $N = 4$ extended supersymmetry [12]; once this is done, the elementary electrically charged particles and the magnetically charged BPS soliton states form supermultiplets with corresponding spins. These facts suggest that this duality symmetry, which exchanges solitons and elementary particles, and weak and strong coupling, might in fact be an exact symmetry of the theory, as was first conjectured by Montonen and Olive [2].

4 Monopoles in theories with larger gauge groups

This analysis can be extended to the case of a Yang-Mills theory with an arbitrary simple gauge group G of rank r and dimension d and a Higgs field Φ transforming under the adjoint representation. To begin, recall that the generators of the Lie algebra of G can be chosen to be r commuting generators H_a that span the Cartan subalgebra, together with a number of generators $E_{\boldsymbol{\alpha}}$ associated with the $d - r$ root vectors $\boldsymbol{\alpha}$ that are defined by the commutation relations

$$[E_{\boldsymbol{\alpha}}, H_j] = \alpha_j E_{\boldsymbol{\alpha}} \,. \tag{4.1}$$

The asymptotic value of the Higgs field in some fixed reference direction can always be chosen to lie in the Cartan subalgebra. It thus defines an r-component vector $\mathbf{h}$ through the relation

$$\Phi_0 = \mathbf{h} \cdot \mathbf{H} \,. \tag{4.2}$$

The unbroken gauge symmetry is the subgroup G that leaves Φ_0 invariant. The maximal symmetry breaking occurs if $\mathbf{h}$ has nonzero inner products with all the root vectors, in which case the unbroken subgroup is the $U(1)^r$ generated by the Cartan subalgebra. If instead some of the roots are orthogonal to $\mathbf{h}$, then these form the root lattice for a non-Abelian group K of rank $k < r$ and the unbroken symmetry is $U(1)^{r-k} \times K$.

At large distances, $F_{\mu\nu}$ must commute with the Higgs field. Hence, along the same direction used to define $\mathbf{h}$, the asymptotic magnetic field may be chosen to also lie in the Cartan subalgebra and to be of the form

$$B_i = \mathbf{g} \cdot \mathbf{H} \frac{\hat{r}_i}{r^2} + O(r^{-3}) . \tag{4.3}$$

The generalized quantization condition on the magnetic charge then becomes [13]

$$e^{i\mathbf{g}\cdot\mathbf{H}} = I . \tag{4.4}$$

I will begin by considering the case of maximal symmetry breaking. Because $\Pi_2(G/U(1)^r) = \Pi_1(U(1)^r) = Z^r$, there are r topologically conserved charges. These can be identified in a particularly natural fashion by recalling that a basis for the root lattice can be chosen to be a set of r simple roots $\boldsymbol{\beta}_a$ with the property that all other roots are linear combinations of simple roots with coefficients that are either all positive or all negative. There are many possible choices for this basis. However, a unique set of simple roots can be specified by requiring that

$$\mathbf{h} \cdot \boldsymbol{\beta}_a \geq 0 \tag{4.5}$$

for all a. If all of the fields are in the adjoint representation, the quantization condition (4.4) then reduces to the requirement that

$$\mathbf{g} = 4\pi \sum_a n_a \boldsymbol{\beta}_a{}^* \tag{4.6}$$

where $\boldsymbol{\beta}_a{}^* = \boldsymbol{\beta}_a / \boldsymbol{\beta}_a^2$ and the integers n_a are are the topological charges.

The BPS mass formula is easily extended to this case. One finds that

$$M = \mathbf{g} \cdot \mathbf{h} = \sum_a n_a \left(\frac{4\pi}{e} \mathbf{h} \cdot \boldsymbol{\beta}_a \right) \equiv \sum_a n_a m_a . \tag{4.7}$$

The methods used to count the zero modes about $SU(2)$ solutions can also be applied here [14]. As before, there is no continuum contribution $\mathcal{I}_{\text{continuum}}$ because the long-range part of the magnetic field lies in the Cartan subalgebra and so does not act on the massless fields, which also lie in the Cartan subalgebra. The calculation of $\mathcal{I}$ proceeds very much as before until one gets to Eq. (3.27), which is replaced by

$$\mathcal{I}(M^2) = \frac{1}{4\pi} \sum_{\boldsymbol{\alpha}} \frac{(\boldsymbol{\alpha} \cdot \mathbf{h})(\boldsymbol{\alpha} \cdot \mathbf{g})}{[(\boldsymbol{\alpha} \cdot \mathbf{h})^2 + M^2]^{1/2}} = \frac{1}{2\pi} {\sum_{\boldsymbol{\alpha}}}' \frac{(\boldsymbol{\alpha} \cdot \mathbf{h})(\boldsymbol{\alpha} \cdot \mathbf{g})}{[(\boldsymbol{\alpha} \cdot \mathbf{h})^2 + M^2]^{1/2}} . \tag{4.8}$$

Here the first sum is over all roots $\boldsymbol{\alpha}$, while the prime on the second sum indicates that it is to be taken only over the positive roots (those that are positive linear combinations of simple roots). Taking the limit $M^2 \to 0$ gives

$$\mathcal{I} = \frac{1}{2\pi} {\sum_{\boldsymbol{\alpha}}}' \boldsymbol{\alpha} \cdot \mathbf{g} = 2 \sum_a n_a \left({\sum_{\boldsymbol{\alpha}}}' \boldsymbol{\alpha} \cdot \boldsymbol{\beta}_a{}^* \right) . \tag{4.9}$$

In the sum inside the parentheses, the contributions from the roots other than $\boldsymbol{\beta}_a$ cancel, so that the sum is just $\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_a{}^* = 1$. Hence, the number of normalizable zero modes is

$$2\mathcal{I} = 4\sum_a n_a\,. \tag{4.10}$$

It was argued above that the $SU(2)$ solutions with higher magnetic charge should be understood as being composed of a number of unit monopoles. The mass formula and the zero mode counting suggest that the higher charged solutions in the present case should also be understood as multimonopole solutions. Now, however, there are r different species of fundamental monopoles, with the ath fundamental monopole having mass m_a, topological charges $n_b = \delta_{ab}$ and four degrees of freedom. Classical solutions corresponding to these fundamental monopoles can be constructed by appropriate embeddings of the $SU(2)$ solution. Any root $\boldsymbol{\alpha}$ defines an $SU(2)$ subgroup of of G with generators

$$\begin{aligned} t^1(\boldsymbol{\alpha}) &= \frac{1}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}) \\ t^2(\boldsymbol{\alpha}) &= -\frac{i}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}) \\ t^3(\boldsymbol{\alpha}) &= \boldsymbol{\alpha}^* \cdot \mathbf{H}\,. \end{aligned} \tag{4.11}$$

If we denote by $A_i^s(\mathbf{r};v)$ and $\Phi^s(\mathbf{r};v)$ the unit $SU(2)$ monopole with Higgs expectation value v, then the embedded solution

$$\begin{aligned} A_i(\mathbf{r}) &= \sum_{s=1}^{3} A_i^s(\mathbf{r};\mathbf{h}\cdot\boldsymbol{\beta}_a)t^s(\boldsymbol{\beta}_a) \\ \Phi(\mathbf{r}) &= \sum_{s=1}^{3} \Phi^s(\mathbf{r};\mathbf{h}\cdot\boldsymbol{\beta}_a)t^s(\boldsymbol{\beta}_a) + (\mathbf{h} - \mathbf{h}\cdot\boldsymbol{\beta}_a^*\boldsymbol{\beta})\cdot\mathbf{H} \end{aligned} \tag{4.12}$$

gives the fundamental monopole corresponding to the root $\boldsymbol{\beta}_a$. It has the expected mass and topological charges and four zero modes, three corresponding to translational degrees of freedom and the fourth to a phase angle in the $U(1)$ generated by $\boldsymbol{\beta}_a \cdot \mathbf{H}$.

As an example, consider the case of $SU(3)$ broken to $U(1)\times U(1)$ by an adjoint representation Higgs field that can be represented by a traceless Hermitian 3×3 matrix. Let Φ_0 be diagonal, with its eigenvalues decreasing along the diagonal. With this convention, the $SU(2)$ subgroup defined by $\boldsymbol{\beta}_1$ lies in the upper left 2×2 block. Embedding the $SU(2)$ monopole in this block gives a solution with a mass m_1, topological charges $n_a = (1,0)$, and four zero modes. After quantization, there is a family of monopole and dyon one-particle states corresponding to this solution. Similarly, $\boldsymbol{\beta}_2$ defines an $SU(2)$ subgroup lying in the lower right 2×2 block. Using this subgroup for the embedding gives a solution with mass m_2, topological charges $(0,1)$, and again four zero modes. This, too, corresponds to a particle in the spectrum of the quantum theory.

There is a third $SU(2)$ subgroup, lying in the four corner matrix elements, defined by the composite root $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$. Using this subgroup to embed the $SU(2)$ monopole also gives a spherically symmetric BPS solution, with mass $m_1 + m_2$ and topological charges $(1, 1)$. However, Eq. (4.10) (as well as explicit solution of the zero mode equations) shows that there are not four, but eight zero modes. Hence, this embedding solution is just one out of a continuous family of two-monopole solutions; in contrast to the two fundamental solutions, it can be continuously deformed into a solution containing two widely separated fundamental monopoles. It does not lead to a new particle in the spectrum of the quantum theory, but instead corresponds to a two-particle state.

Let us now consider this result in the light of the Montonen-Olive duality conjecture. Although this conjecture was first motivated by the spectrum of the $SU(2)$ theory, it is natural to test it with larger gauge groups. The elementary particle sector of the theory contains a number of massless particles, carrying no $U(1)$ charges, that are presumably self-dual. There are also six massive vector bosons, one for each root of the root diagram, that carry electric-type charges in one or both of the unbroken $U(1)$'s. The duals of the $\pm\boldsymbol{\beta}_1$ and $\pm\boldsymbol{\beta}_2$ vector bosons are clearly the one-particle states corresponding to the $\boldsymbol{\beta}_1$- and $\boldsymbol{\beta}_2$-embeddings of the $SU(2)$ monopole and antimonopole solutions. One might have thought that the duals of the vector bosons corresponding to $\pm(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)$ would be obtained from the $(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)$-embedding solutions, but we have just seen that these do not correspond to single-particle states. Some other state must be found if the duality is to hold. The most likely candidate would be some kind of threshold bound state [15]. To explore this possibility, we need to understand the interactions of low-energy BPS monopoles. This can be done by making use of the moduli space approximation, to which I now turn.

5 The moduli space approximation

The essential idea of the moduli space approximation [16] is that, since the static multimonopole solutions are all BPS, the time-dependent solutions containing monopoles with sufficiently small velocities should in some sense also be approximately BPS.[3]

To make this more precise, let $\{A_i^{\rm BPS}(\mathbf{r}, z), \Phi^{\rm BPS}(\mathbf{r}, z)\}$ be a family of static, gauge-inequivalent

[3]Here velocities should be understood to include not only spatial velocities but also the time derivatives of the $U(1)$ phases. Thus, we are considering slowly moving dyons with small (and possibly zero) electric charges.

BPS solutions parameterized by a set of collective coordinates z_j. The moduli space approximation is obtained by assuming that the fields at any fixed time are gauge-equivalent to some configuration in this family, so that they can be written as

$$\begin{aligned} A_0(\mathbf{r},t) &= 0 \\ A_i(\mathbf{r},t) &= U^{-1}(\mathbf{r},t)A_i^{\rm BPS}(\mathbf{r},z(t))U((\mathbf{r},t) - iU^{-1}(\mathbf{r},t)\partial_i U((\mathbf{r},t) \\ \Phi(\mathbf{r},t) &= U^{-1}(\mathbf{r},t)\Phi^{\rm BPS}(\mathbf{r},z(t))U((\mathbf{r},t)\,. \end{aligned} \tag{5.1}$$

Their time derivatives are then of the form

$$\begin{aligned} \dot{A}_i &= \dot{z}_j\left[\frac{\partial A_i}{\partial z_j} + D_i\epsilon_j\right] \equiv \dot{z}_j\delta_j A_i \\ \dot{\Phi} &= \dot{z}_j\left[\frac{\partial \Phi}{\partial z_j} + [\Phi,\epsilon_j]\right] \equiv \dot{z}_j\delta_j\Phi \end{aligned} \tag{5.2}$$

where the gauge function $\epsilon_j(\mathbf{r},t)$ arises from the time derivative of $U(\mathbf{r},t)$. These are constrained by Gauss's law, which takes the form

$$\begin{aligned} 0 &= -D_\mu F^{\mu 0} + [\Phi,\partial_0\Phi] = D_i\dot{A}_i + [\Phi,\dot{\Phi}] \\ &= \dot{z}_j\left(D_i\delta_j A_i + [\Phi,\delta_j\Phi]\right)\,. \end{aligned} \tag{5.3}$$

Because they arise from variation of a collective coordinate, the quantities $\delta_j A_i$ and $\delta_j\Phi$ form a zero mode about the underlying solution BPS solution. The Gauss's law constraint shows that they obey the background gauge condition Eq. (3.10).

With A_0 identically zero, the Lagrangian of the theory can be written as

$$L = \frac{1}{2}\int d^3r \mathrm{Tr}\left[\dot{A}_i^2 + \dot{\Phi}^2 + B_i^2 + D_i\Phi^2\right]\,. \tag{5.4}$$

Since for fields obeying the ansatz (5.1) the configuration at any fixed time is BPS, the contribution of the last two terms to the integral is just the BPS energy determined by the topological charge. This is a time-independent constant that has no effect on the dynamics and so can be dropped. The remaining terms then give an effective Lagrangian

$$L_{\rm MS} = \frac{1}{2}g_{ij}(z)\dot{z}_i\dot{z}_j \tag{5.5}$$

where

$$g_{ij}(z) = \int d^3r\left[\delta_i A_k\delta_j A_k + \delta_i\Phi\delta_j\Phi\right]\,. \tag{5.6}$$

Thus, the full field theory dynamics for low energy monopoles has been reduced to a problem involving a finite number of degrees of freedom. If one views $g_{ij}(z)$ as a metric for the moduli space spanned by the collective coordinates, the dynamics described by $L_{\rm MS}$ is simply geodesic motion on the moduli space.

6 Determining the moduli space metric

Actually determining the moduli space metric is a nontrivial matter. To apply Eq. (5.6) directly one needs to know the zero modes, whereas we do not in general even know the underlying solution. However, some more indirect approaches can sometimes be brought to bear on the problem.

First, Gibbons and Manton [17] showed how one could obtain the metric for the region of moduli corresponding to widely separated monopoles. They pointed out that, since the moduli space metric determines the low energy dynamics, the metric can be inferred if this dynamics is known. The only long-range interactions between widely separated monopoles are those mediated by massless fields. These are the electromagnetic interactions and an interaction due to the massless Higgs field. The Lagrangian describing the interactions between moving point electric and magnetic forces is well known, while that for the scalar force is easily worked out. To obtain the metric, these must be expanded up to terms quadratic in the velocities and the electric charges. A Legendre transformation must then be used to replace the electric charges by the time derivatives of the corresponding phase angles. Apart from a constant term, the result is a Lagrangian, of the form of Eq. (5.5), from which the metric can be read off directly. For the case of many $SU(2)$ monopoles, each with mass m and magnetic charge g and with positions $\mathbf{x}_i$ and phase angles ξ_i, this gives

$$ds^2 = \frac{1}{2}M_{ij}d\mathbf{x}_i \cdot d\mathbf{x}_j + \frac{g^4}{2(4\pi)^2}(M^{-1})_{ij}(d\xi_i + \mathbf{W}_{ik}\cdot d\mathbf{x}_k)(d\xi_j + \mathbf{W}_{jl}\cdot d\mathbf{x}_l) \tag{6.1}$$

where

$$\begin{aligned} M_{ii} &= m - \sum_{k\neq i}\frac{g^2}{4\pi r_{ik}}, \\ M_{ij} &= \frac{g^2}{4\pi r_{ij}} \qquad \text{if } i \neq j, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} \mathbf{W}_{ii} &= -\sum_{k\neq i}\mathbf{w}_{ik}, \\ \mathbf{W}_{ij} &= \mathbf{w}_{ij} \qquad \text{if } i \neq j, \end{aligned} \tag{6.3}$$

with r_{ij} the distance between the ith and jth monopoles and $\mathbf{w}_{ij}$ the value at $\mathbf{x}_i$ of the Dirac vector potential due to the jth monopole, defined so that

$$\nabla_i \times \mathbf{w}_{ij}(\mathbf{x}_i - \mathbf{x}_j) = -\frac{\mathbf{x}_i - \mathbf{x}_j}{r_{ij}^3}. \tag{6.4}$$

The extension of this result to the case of maximal symmetry breaking of an arbitrary simple group G is quite simple. For a collection of fundamental monopoles, with the ith monopole corresponding to the simple root $\boldsymbol{\beta}_i$, we need only replace Eqs. (6.2) and (6.3) by

$$\begin{aligned} M_{ii} &= m_i - \sum_{k\neq i} \frac{g^2 \boldsymbol{\beta}_i^* \cdot \boldsymbol{\beta}_k^*}{4\pi r_{ik}}, \\ M_{ij} &= \frac{g^2 \boldsymbol{\beta}_i^* \cdot \boldsymbol{\beta}_j^*}{4\pi r_{ij}} \qquad\qquad \text{if } i \neq j, \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \mathbf{W}_{ii} &= -\sum_{k\neq i} \boldsymbol{\beta}_i^* \cdot \boldsymbol{\beta}_k^* \mathbf{w}_{ik}, \\ \mathbf{W}_{ij} &= \boldsymbol{\beta}_i^* \cdot \boldsymbol{\beta}_j^* \mathbf{w}_{ij} \qquad\qquad \text{if } i \neq j, \end{aligned} \tag{6.6}$$

with $m_i = g\,\boldsymbol{\beta}_i^* \cdot \mathbf{h}$.

Although the derivation of these expressions was only valid in the region of moduli space corresponding to widely separated monopoles, one might wonder whether the asymptotic metric could be exact. For the case of two $SU(2)$ monopoles, several considerations show that it cannot be. The matrix M of Eq. (6.2) reduces to

$$M = \begin{pmatrix} m - \frac{g^2}{4\pi r} & \frac{g^2}{4\pi r} \\ \frac{g^2}{4\pi r} & m - \frac{g^2}{4\pi r} \end{pmatrix} . \tag{6.7}$$

The determinant of this matrix vanishes at $r = g^2/2\pi m$, implying a singularity in the metric, despite the fact that there is no reason to expect any type of singular behavior near this value of the intermonopole distance. Furthermore, we know that there is a short-range force, carried by the massive vector bosons, that was ignored in the derivation of the metric. If one works in a singular gauge in which the Higgs field orientation is uniform in space, this interaction is proportional to the gauge-invariant quantity $\mathrm{Re}[\mathbf{W}^*_{(1)}\mathbf{W}_{(2)}]$ where $\mathbf{W}_{(1)}$ and $\mathbf{W}_{(2)}$ are the massive vector fields of the two monopoles. Because these fall exponentially with distance from the center of the monopole, their overlap, and hence the interaction, falls exponentially with the intermonopole separation.

Neither of these objections apply when the two monopoles are fundamental monopoles associated with different simple roots of a large gauge group. The simple roots have the property that their mutual inner products are always negative. The resulting sign changes in M eliminate the zero of the determinant and make the asymptotic metric everywhere nonsingular. In addition, the quantity characterizing the interactions carried by the massive vector fields is now $\mathrm{Re}[\mathrm{Tr}\mathbf{W}^\dagger_{(1)}\mathbf{W}_{(2)}]$, which vanishes when the two monopoles arise from different simple roots.

This, of course, is not sufficient to show that the asymptotic metric is exact. To do this, we first note that the coordinates for the moduli space can always be chosen so that three specify the position of the center-of-mass of the monopoles and a fourth is an overall $U(1)$ phase. The moduli space metric can then be written in the factorized form

$$\mathcal{M} = R^3 \times \left(\frac{R^1 \times \mathcal{M}_{\rm rel}}{D} \right) \tag{6.8}$$

where the factors of R^3 and R^1 are associated with the center-of-mass coordinates and the overall $U(1)$ phase, while $\mathcal{M}_{\rm rel}$ is the metric on the subspace spanned by the relative positions and phases. The factoring by the discrete group D arises from difficulties in globally factoring out an overall $U(1)$ phase.

$\mathcal{M}_{\rm rel}$ has several important properties. First, it must have a rotational isometry reflecting the fact that the interactions among an assembly of monopoles are unaffected by an overall spatial rotation of the entire assembly. Second, the $SU(2)$ relations among the zero modes shown in Eq. (3.13) imply that the moduli space metric must be hyper-Kahler[4] [18]. Finally, the relative moduli space for a collection of n monopoles is $4(n-1)$-dimensional. Hence, we are seeking a four-dimensional hyper-Kahler manifold with a rotational isometry. There are four such [18]:

1) Flat four-dimensional Euclidean space

2) The Eguchi-Hanson manifold [19]

3) The Atiyah-Hitchin manifold [18]

4) Taub-NUT space

The first of these would imply that there were no interactions at all between the monopoles, and so is clearly ruled out if $\boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2 \neq 0$. The Eguchi-Hanson metric has the wrong asymptotic behavior for large intermonopole separation, and so can be ruled out. At large r (but not at small r) the Atiyah-Hitchin metric approaches the two-monopole asymptotic metric with M given by Eq. (6.2). It thus describes the moduli space for two $SU(2)$ monopoles (or for two identical monopoles in a larger group), but not that for two distinct monopoles. The only remaining possibility is the Taub-NUT metric. This not only agrees at large r with the asymptotic metric, but is in fact equal to it everywhere. Thus, for the case of two distinct fundamental monopoles the asymptotic metric is in fact exact [3, 20, 21].

If a collection of more than two monopoles includes two corresponding to the same simple root, then the asymptotic metric develops a singularity when these approach each other.

[4]A metric is hyper-Kahler if it possess three covariantly constant complex structures that also form a quaternionic structure and if it is pointwise Hermitian with respect to each.

However, this metric is everywhere nonsingular if the monopoles are all distinct. It is therefore natural to conjecture that for this case also the asymptotic metric is exact [4]. Proofs of this conjecture have been given by Chalmers [22] and by Murray [23].

Let us now briefly return to the issue of duality in the theory with $SU(3)$ broken to $U(1) \times U(1)$. As noted above, duality is expected to hold only if the theory has an extended supersymmetry, which means that the low-energy fermion dynamics must be included. It turns out that these fermions will give rise to a supermultiplet of threshold bound states if and only if there is a normalizable harmonic form on the relative moduli space [24]. Having determined the metric for this moduli space metric, one can easily verify that such a harmonic form exists, and hence that the test of the duality conjecture is met [3, 20].

7 Nonmaximal symmetry breaking

Let us now turn to the case of non-maximal symmetry breaking, where the gauge symmetry G is spontaneously broken to $K \times U(1)^{r-k}$. As in the case of maximal symmetry breaking, we can require that inner products of the simple roots with $\mathbf{h}$ be all non-negative. It is useful to distinguish between those for which this inner product is greater than zero and those for which it vanishes. I will continue to denote the former by $\boldsymbol{\beta}_a$, and will label the latter, which form a set of simple roots for K, by $\boldsymbol{\gamma}_i$. In contrast with the previous case, the condition on the inner products with $\mathbf{h}$ does not uniquely determine the set of simple roots. Instead, there can be many acceptable sets, all related by Weyl reflections of the root diagram that result from global gauge transformations by elements of K.[5]

The quantization condition on the magnetic charge now takes the form

$$\mathbf{g} = 4\pi \left[\sum_a n_a \boldsymbol{\beta}_a{}^* + \sum_j q_j \boldsymbol{\gamma}_j{}^* \right] . \tag{7.1}$$

As in the case of maximal symmetry breaking, the integers n_a are the topological charges, one for each $U(1)$ factor of the unbroken group. They are gauge-independent, and thus independent of the choice of the set of simple roots. The q_j must also be integers, but they are neither topologically conserved nor gauge-invariant. We will see that there is an important distinction to be made between the case where

$$\mathbf{g} \cdot \boldsymbol{\gamma}_j = 0 , \qquad \text{all } j, \tag{7.2}$$

[5]Consider, for example, the case of $SU(3)$ broken to $SU(2) \times U(1)$. If one set of simple roots is denoted by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, with the latter being a root of the unbroken $SU(2)$, then another acceptable set is given by $\boldsymbol{\beta} + \boldsymbol{\gamma}$ and $-\boldsymbol{\gamma}$.

and that where some of the $\mathbf{g}\cdot\gamma_j$ are nonzero. (Note that these do not in general correspond to vanishing or nonvanishing q_j.) In the former case, the long-range magnetic fields are purely Abelian with only $U(1)$ components, whereas in the latter the configuration has a non-Abelian magnetic charge. We will see that there are a number of pathologies associated with the latter case.

The BPS mass formula takes the same form as before,

$$M = \sum_a n_a m_a \tag{7.3}$$

but with the sum running only over the indices corresponding to simple roots that are not orthogonal to $\mathbf{h}$.

The zero mode counting proceeds as before, but with some complications [25]. First, the continuum contribution cannot be immediately discarded, since the massless fields cannot all be brought into the Cartan subalgebra. Because of this, there can be a singularity in the density of states factor that is strong enough to give a nonzero $\mathcal{I}_{\rm continuum}$ if $\mathcal{D}^\dagger\mathcal{D} - \mathcal{D}\mathcal{D}^\dagger$ contains order $1/r^2$ terms that act on fields lying in the unbroken non-Abelian subgroup K. This will be the case whenever there is a net non-Abelian magnetic charge. Explicit solution of the zero mode equations in some simple cases shows that the number of zero modes is not equal to the expression for $2\mathcal{I}$ given below, implying that there is indeed a nonvanishing continuum contribution. This difficulty does not arise when Eq. (7.2) is satisfied.

Second, the expression for $\mathcal{I}$ is more complicated. The same procedures as used before again lead to

$$2\mathcal{I} = \lim_{M^2\to 0} \frac{1}{\pi} {\sum_{\boldsymbol{\alpha}}}' \frac{(\boldsymbol{\alpha}\cdot\mathbf{h})(\boldsymbol{\alpha}\cdot\mathbf{g})}{[(\boldsymbol{\alpha}\cdot\mathbf{h})^2 + M^2]^{1/2}} \tag{7.4}$$

with the prime indicating that the sum is only over positive roots. Now, however, the contribution from the roots orthogonal to $\mathbf{h}$ (i.e., those of the subgroup K) vanishes even for finite M^2 and so gives no contribution to the limit. As a result, the expression for $2\mathcal{I}$ is in general much less simple than before. But, again, matters simplify if the asymptotic magnetic field is purely Abelian. Because the roots of K are now all orthogonal to $\mathbf{g}$, they would not have contributed in any case, and the methods used for the case with maximal symmetry breaking yield

$$2\mathcal{I} = 4\left[\sum_a n_a + \sum_j q_j\right] . \tag{7.5}$$

As was noted earlier, the q_j are not gauge-invariant. However, when $\mathbf{g}$ is orthogonal to all of the γ_k, the sum appearing on the right-hand side of Eq. (7.5), and hence $2\mathcal{I}$, is gauge-invariant.

The difficulties with applying index theory when there is a non-Abelian magnetic charge are related to other known difficulties with such solutions. Since the unbroken gauge group acts nontrivially on these, one would expect to find gauge zero modes, analogous to the $U(1)$ modes of the maximally symmetric case, whose excitation would lead to "chromodyons", objects with non-Abelian electric-type charge. Instead, one finds that these modes are non-normalizable and that the expected chromodyon states are absent [26]. This can be traced to the fact that the existence of the non-Abelian magnetic charge creates an obstruction to the smooth definition of a set of generators for K over the sphere at spatial infinity; i.e., one cannot define "global color" [27].

It is instructive to return to the $SU(3)$ example considered in Sec. 4, but with the last two eigenvalues of Φ_0 taken to be equal so that the unbroken group is $SU(2)\times U(1)$. While n_1 remains a topological charge, n_2 must be replaced by the nontopological integer q_1. The first fundamental monopole solution of the maximally broken case, obtained by embedding in the upper left 2×2 block, is still present with a nonzero mass. As before, it has three translational zero modes and a $U(1)$ phase mode. There are no other normalizable zero modes, even though the solution is not invariant under the unbroken $SU(2)$, and even though Eq. (7.4) gives $2\mathcal{I}=6$. Embedding in the lower right 2×2 block, which previously gave a second fundamental monopole, is no longer possible. Indeed, if one starts with the maximally broken case, and follows the behavior of the second fundamental monopole as the last two eigenvalues of Φ_0 approach one another, one finds that its mass tends to zero, its core radius tends to infinity, and the fields at any fixed point approach their vacuum value. Finally, the embedding in the corner matrix elements, which previously gave a solution with eight zero modes that was naturally understood to be a two-monopole solution, now gives a solution that is gauge-equivalent to the first fundamental monopole and hence has only four zero modes. In all three of these cases the magnetic charge has a non-Abelian component.

Eqs. (7.3) and (7.5) are consistent with the idea that even for non-maximal symmetry breaking one should interpret all solutions — or at least those with purely Abelian magnetic charges — in terms of a number of component fundamental monopoles. However, there are clearly two quite different kinds of fundamental monopoles. The massive monopoles corresponding to the β_a carry $U(1)$ magnetic charges and appear to have four associated degrees of freedom. They can be realized as classical solitons, even though the latter may not be unique, as the $SU(3)$ example shows. The remaining fundamental monopoles, corresponding to the γ_j, would have to be massless. Indeed, the duality conjecture would lead us to expect to find massless magnetically charged states that would be the duals of the massless gauge bosons of the unbroken non-Abelian subgroup. The difficulty is that, precisely because they are massless, these monopoles cannot be associated with any localized classical solutions. To learn more about them, we must examine multimonopole solutions containing both massive and massless components.

The pathologies associated with non-Abelian magnetic charges suggest that this is best done by concentrating on configurations that obey Eq. (7.2). This should not impose any real physical restriction, since the additional monopoles needed to cancel any non-Abelian charge can be placed at an arbitrarily large distance. It also turns out to be useful to treat non-maximal symmetry breaking as a limit of maximal symmetry breaking in which one or more of the $\mathbf{h} \cdot \boldsymbol{\beta}_a^*$ tend to zero. As we will see, it appears that the moduli space for the maximally broken case behaves smoothly in this limit, with the limit of its metric being the metric for the non-maximally case. Although some of the fundamental monopoles become massless in this limit and no longer have corresponding soliton solutions, their degrees of freedom of these massless monopoles are still evident in the low-energy moduli space Lagrangian.

8 An $SO(5)$ example

A particularly simple example [5] for illustrating this arises with the gauge group $SO(5)$, whose root diagram is shown in Fig. 1 with the simple roots labeled $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Consider the solutions whose magnetic charge is such that

$$\mathbf{g} = 4\pi \left(\boldsymbol{\beta}^* + \boldsymbol{\gamma}^*\right) . \tag{8.1}$$

With $\mathbf{h}$ as in Fig. 1a, the symmetry breaking is maximal and there is an eight-parameter family of solutions composed of two monopoles, of masses $m_{\boldsymbol{\beta}}$ and $m_{\boldsymbol{\gamma}}$ respectively. Because the two monopoles correspond to different simple roots, the moduli space metric is known from the results of Sec. 6. If instead $\mathbf{h}$ is perpendicular to $\boldsymbol{\gamma}$, as in Fig. 1b, the unbroken gauge group is $SU(2) \times U(1)$, with the roots of the $SU(2)$ being $\pm\boldsymbol{\gamma}$. These are both orthogonal to $\mathbf{g}$, so Eq. (7.2) is satisfied and Eq. (7.5) tells us that there is again an eight-parameter family of solutions. It turns out that these solutions, which are spherically symmetric, can be found explicitly [28]. This makes it possible to determine the background gauge zero modes and then use Eq. (5.6) to obtain the moduli space metric directly. The result can then be compared with the $m_{\boldsymbol{\gamma}} \to 0$ limit of the first case.

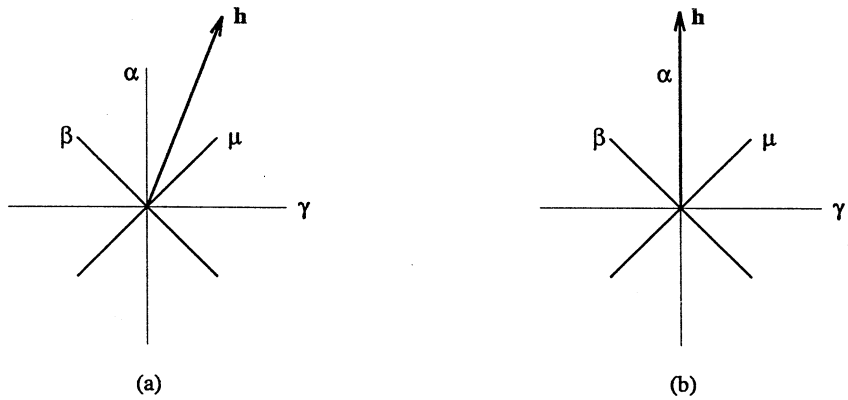

Figure 1: The root diagram of $SO(5)$. With the Higgs vector $\mathbf{h}$ oriented as in (a) the gauge symmetry is broken to $U(1) \times U(1)$, while with the orientation in (b) the breaking is to $SU(2) \times U(1)$.

I will begin by describing the $SU(2) \times U(1)$ solutions. Three of its eight parameters give the location of the center-of-mass. Four others are phase angles that specify the $SU(2) \times U(1)$ orientation. (All elements of the unbroken group act nontrivially on the solution.) This leaves only a single parameter, which I will denote by b, whose significance can be found by examining the detailed form of the solutions. To write these we need some notation. Let $\mathbf{t}(\alpha)$ and $\mathbf{t}(\gamma)$ be defined as in Eq. (4.11) and let

$$\mathcal{M} = \frac{i}{\sqrt{\beta^2}} \begin{pmatrix} E_\beta & -E_{-\mu} \\ E_\mu & E_{-\beta} \end{pmatrix} . \tag{8.2}$$

Any adjoint representation $SO(5)$ field P can then be decomposed into parts that are respectively singlets, triplets, and doublets under the unbroken $SU(2)$ by writing

$$P = \mathbf{P}_{(1)} \cdot \mathbf{t}(\alpha) + \mathbf{P}_{(2)} \cdot \mathbf{t}(\gamma) + \operatorname{tr} P_{(3)} \mathcal{M} . \tag{8.3}$$

With this notation, the solutions can be written as

$$\begin{aligned} A^a_{i(1)} &= \epsilon_{aim} \hat{r}_m A(r) & \phi^a_{(1)} &= \hat{r}_a H(r) \\ A^a_{i(2)} &= \epsilon_{aim} \hat{r}_m G(r,b) & \phi^a_{(2)} &= \hat{r}_a G(r,b) \\ A_{i(3)} &= \tau_i F(r,b) & \phi_{(3)} &= -iIF(r,b) \end{aligned} \tag{8.4}$$

where $A(r)$ and $H(r)$ are the same as the coefficient functions in the $SU(2)$ BPS monopole solution given in Eq. (3.8) and

$$\begin{aligned} F(r,b) &= \frac{v}{\sqrt{8}\cosh(vr/2)} L(r,b)^{1/2} \\ G(r,b) &= A(r) L(r,b) \end{aligned} \tag{8.5}$$

with

$$L(r,b) = [1 + (r/b)\coth(vr/2)]^{-1} \tag{8.6}$$

and $v = \mathbf{h}\cdot\boldsymbol{\alpha}$.

The parameter b, which has dimensions of length, can take on any positive real value. It only enters into the doublet and triplet components of the fields, and then only through the function $L(r,b)$. While the doublet fields decrease exponentially fast outside the monopole core, the triplet fields have long-range components whose character is determined by b. For $1/v \lesssim r \lesssim b$ these fall as $1/r$, resulting in a Coulomb magnetic field appropriate to a non-Abelian magnetic charge. At larger distances, however, the vector potential falls as $1/r^2$, implying a field strength falling as $1/r^3$ and thus showing that the magnetic charge is purely Abelian. Thus, one might view these solutions as being composed of a massive monopole, with a core of radius $\sim 1/v$, surrounded by a "non-Abelian cloud" of radius $\sim b$ that cancels the non-Abelian part of its charge.

In order to obtain the moduli space metric from Eq. (5.6), we need the background gauge zero modes about these solutions. An infinitesimal variation with respect to b gives one zero mode, which turns out to already be in background gauge. The three $SU(2)$ modes can then be obtained from this by a transformation of the type shown in Eq. (3.13). The translational and $U(1)$ modes could also be obtained in the usual fashion. However, we do not need to do so, since the corresponding parts of the metric can be inferred from the BPS mass formulas for monopoles and dyons. The result of all this is

$$ds^2_{SU(2)\times U(1)} = M d\mathbf{x}^2 + \frac{16\pi^2}{M} d\chi^2 + k\left[\frac{db^2}{b} + b\left(d\alpha^2 + \sin^2\alpha d\beta^2 + (d\gamma + \cos\alpha d\beta)^2\right)\right] \tag{8.7}$$

where M is the monopole mass, $\mathbf{x}$ is the location of the center of the monopole, χ is the $U(1)$ phase, and α, β, and γ are the three angles specifying the $SU(2)$ orientation of the solution. The coefficient k is a constant whose value is unimportant for our purposes.

This should be compared with the two-monopole moduli space metric when the symmetry is broken to $U(1)\times U(1)$. Let $M = m_\beta + m_\gamma$ and $\mu = m_\beta m_\gamma / M$ denote the total mass and reduced mass of the system. After transformation into center-of-mass and relative variables, the metric given by Eq. (6.1) takes the form

$$\begin{aligned} ds^2_{U(1)\times U(1)} &= M d\mathbf{x}^2_{\rm cm} + \frac{16\pi^2}{M} d\chi^2_{\rm tot} + \left(\mu + \frac{k}{r}\right)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &\quad + k^2\left(\mu + \frac{k}{r}\right)^{-1} (d\psi + d\cos\theta d\phi)^2 . \end{aligned} \tag{8.8}$$

Here $\mathbf{x}_{\rm cm}$ specifies the position of the center-of-mass, r, θ and ϕ are the spherical coordinates specifying the relative positions of the two monopoles, $\chi_{\rm tot}$ and ψ are overall and relative $U(1)$ phases, and k is the same constant as in Eq. (8.7). We are interested in the limit where $m_\gamma \to 0$ with M held fixed. In this limit the reduced mass μ vanishes, and the metric becomes

$$ds^2_{U(1)\times U(1)} = M d\mathbf{x}^2_{\rm cm} + \frac{16\pi^2}{M} d\chi^2_{\rm tot} + k\left[\frac{dr^2}{r} + r\left(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2\right)\right] . \quad (8.9)$$

This is exactly the same metric as in Eq. (8.7), but with a different notation: b replaced by r, and α, β, and γ replaced by θ, ϕ, and ψ, respectively. Thus, the moduli space metric behaves smoothly in the limit where the unbroken symmetry becomes non-Abelian, with the number of degrees of freedom being conserved. However, the interpretation of these coordinates undergoes a curious change. In particular, as one of the monopoles becomes massless, its position becomes somewhat ambiguous. While the separation r goes over into the cloud radius b, which has a definite gauge-invariant meaning, the directional angles θ and ϕ are replaced by two global $SU(2)$ gauge phases. Hence, two solutions with the same intermonopole separation but different values for θ and ϕ are physically distinct as long as the γ-monopole remains massive, but become gauge-equivalent when $m_\gamma = 0$.

9 More complex examples

Further insight into the nature of the massless monopoles and the non-Abelian cloud can be obtained by considering some more complex solutions that arise in $SU(N)$ gauge theories. The asymptotic value of the adjoint Higgs field in some fixed direction can be brought into the form

$$\Phi_0 = \text{diag }(t_N, t_{N-1}, \dots, t_1) \quad (9.1)$$

with $t_1 \le t_2 \le \dots \le t_N$. The set of simple roots picked out by Eq. (4.5) then generate the $SU(2)$ subgroups that lie in 2×2 blocks along the diagonal and the magnetic charge is given by

$$\mathbf{g}\cdot\mathbf{H} = 4\pi\,\text{diag }(n_{N-1}, n_{N-2} - n_{N-1}, \dots, n_1 - n_2, -n_1) . \quad (9.2)$$

If the t_j are all unequal, the symmetry breaking is maximal, to $U(1)^{N-1}$, and the n_j are the topological charges. Here I will be primarily interested instead in the case where the middle $N-2$ eigenvalues of Φ_0 are equal and the unbroken group is $U(1)\times SU(N-2)\times U(1)$. As explained previously, I will focus on configurations in which the asymptotic magnetic field is purely Abelian and commutes with all elements of the unbroken $SU(N-2)$; i.e., configurations for which the middle $N-2$ eigenvalues of $\mathbf{g}\cdot\mathbf{H}$ are all equal.

All choices for the $\{n_j\}$ that satisfy this condition can be written as combinations of three irreducible solutions[6]:

1) $n_j = j - 1$, so that

$$\mathbf{g} \cdot \mathbf{H} = 4\pi \,\mathrm{diag}\,((N-2), -1, -1, \ldots, -1, 0)\,. \tag{9.3}$$

2) $n_j = N - j - 1$, so that

$$\mathbf{g} \cdot \mathbf{H} = 4\pi \,\mathrm{diag}\,(0, 1, 1, \ldots, 1, -(N-2))\,. \tag{9.4}$$

3) $n_j = 1$ for all j, leading to

$$\mathbf{g} \cdot \mathbf{H} = 4\pi \,\mathrm{diag}\,(1, 0, 0, \ldots, 0, -1)\,. \tag{9.5}$$

(Note that the moduli space metric for this case can be obtained from the results of Sec. 6; the metrics for the first two cases are not known.)

Configurations of the first type can be viewed as containing $N-2$ massive and $(N-2)(N-3)/2$ massless monopoles, with the massive monopoles all corresponding to the last simple root. Eq. (7.5) shows that they depend on $2(N-1)(N-2)$ parameters. Of these, $4(N-2)$ presumably specify the positions and $U(1)$ phases of the massive monopoles. Specifying the $SU(N-2)$ orientation of the configuration requires another $\dim\,[SU(N-2)] = (N-2)^2 - 1$ parameters. Hence, the remaining $(N-3)^2$ parameters describe gauge-invariant aspects of the non-Abelian cloud, showing that it is possible for this cloud to have considerably more structure than it did in the $SO(5)$ example of the previous section.

Configurations of the second type also contain $(N-2)(N-3)/2$ massless and $N-2$ massive monopoles, but now with the latter corresponding to the first simple root.

Finally, configurations of the third type contain two massive monopoles (one of each massive species), together with $N-3$ massless monopoles. There are $4(N-1)$ parameters in all, 8 of which specify the positions and $U(1)$ phases of the massive monopoles. One might have expected to find an additional $(N-2)^2-1$ parameters associated with the unbroken $SU(N-2)$, as in the previous cases. Except for the simplest nontrivial case, $SU(4)$ broken to $U(1) \times SU(2) \times U(1)$, there are clearly not enough parameters. The explanation is that, as we will see more explicitly below, any configuration of this type for gauge group $SU(N)$ with $N > 4$ can be obtained by an embedding of an $SU(4)$ solution. As a result, there are only $\dim\,[SU(N-2)/U(N-4)] = 4N-13$

[6]The existence three types of irreducible solutions can be understood by noting that the states corresponding to the two species of massive monopoles in this theory transform under the $(N-2)$-dimensional fundamental and antifundamental representations of $SU(N-2)$. An $SU(N-2)$ singlet can be formed from $N-2$ fundamentals, $N-2$ antifundamentals, or from a fundamental and an antifundamental.

global gauge parameters. There is but a single remaining parameter, which is associated with the non-Abelian cloud.

Having explicit expressions for the solutions in these cases would clearly be quite helpful for understanding the nature and characteristics of the non-Abelian cloud. Such expressions are not known for the first two cases. However, solutions for the third case can be obtained explicitly, as I will now describe, by making use of Nahm's construction of the BPS monopole solutions [29].

The fundamental elements in Nahm's approach [30] are a triplet of matrices $T_a(t)$ that satisfy a set of nonlinear ordinary differential equations. These then define a set of linear differential equations for another set of matrices, $v(t, \mathbf{r})$, from which the spacetime fields $\mathbf{A}(\mathbf{r})$ and $\mathbf{\Phi}(\mathbf{r})$ can be readily obtained. I will now describe the details of this construction for the case of a gauge group $SU(N)$.

The eigenvalues t_j of Φ_0 divide the range $t_1 \le t \le t_N$ into $N-1$ intervals. On the jth interval, $t_j < t < t_{j+1}$, let $k(t) \equiv n_j$, where n_j is given by Eq. (9.2). The matrices $T_a(t)$ are required to have dimension $k(t) \times k(t)$. In addition, whenever two adjacent intervals have the same value for $k(t)$, there are three matrices α_j, of dimension $k(t_j) \times k(t_j)$, defined at the interval boundary t_j. These matrices are required to obey the Nahm equation,

$$\frac{dT_a}{dt} = \frac{i}{2}\epsilon_{abc}[T_b, T_c] + \sum_j (\alpha_j)_a \delta(t - t_j) . \tag{9.6}$$

where the sum in the last term is understood to only run over those values of j for which the α_j are defined. Having solved this equation, one must next find a $2k(t) \times N$ matrix function $v(t, \mathbf{r})$ and N-component row vectors $S_j(\mathbf{r})$ obeying the linear equation

$$0 = \left[-\frac{d}{dt} + (T_a + r_a) \otimes \sigma_a \right] v + \sum_j a_j^\dagger S_j \delta(t - t_j) \tag{9.7}$$

together with the orthogonality condition

$$I = \int dt\, v^\dagger v + \sum_j S_j^\dagger S_j . \tag{9.8}$$

In Eq. (9.7), a_j is a $2k(t_j)$-component row vector obeying

$$a_j^\dagger a_j = \boldsymbol{\alpha}_j \cdot \boldsymbol{\sigma} - i(\alpha_j)_0 I \tag{9.9}$$

with $(\alpha_j)_0$ chosen so that the above matrix has rank 1. Finally, spacetime fields obeying the BPS equations are given by

$$\Phi = \int dt\, t\, v^\dagger v + \sum_j t_j S_j^\dagger S_j \tag{9.10}$$

$$\mathbf{A} = -\frac{i}{2}\int dt\,\left[v^\dagger\nabla v - \nabla v^\dagger\, v\right] - \frac{i}{2}\sum_j \left[S_j^\dagger \nabla S_j - \nabla S_j^\dagger S_j\right] . \tag{9.11}$$

I will consider the case where the n_j are all equal to unity, so $k(t) = 1$ over the entire range and there are an α_j and an S_j for each value of j from 2 through $N-1$. To begin, I will assume that the t_j are all different, so that there are $N-1$ distinct massive monopoles, although I will soon turn to the case with unbroken $U(1) \times SU(N-2) \times U(1)$ symmetry. Eq. (9.6) is solved by the piecewise constant solution

$$\mathbf{T}(t) = -\mathbf{x}_j\,, \qquad t_j < t < t_{j+1}\,, \tag{9.12}$$

where the $\mathbf{x}_a$ have a natural interpretation as the positions of the individual monopoles. The a_j of Eq. (9.9) are simply two-component row vectors that may be taken to be

$$a_j = \sqrt{2|\mathbf{x}_j - \mathbf{x}_{j-1}|}\left(\cos(\theta/2)e^{-i\phi/2}, \sin(\theta/2)e^{i\phi/2}\right) \tag{9.13}$$

where θ and ϕ specify the direction of the vector $\boldsymbol{\alpha}_j = \mathbf{x}_{j-1} - \mathbf{x}_j$.

Next, we must find a $2 \times N$ matrix $v(t)$ and a set of N-component row vectors S_k ($k = 2, 3, \ldots, N-1$) that satisfy Eq. (9.7). To do this, let us first define a function $f_k(t)$ for each interval $t_k \le t \le t_{k+1}$, with

$$\begin{aligned} f_1(t) &= e^{(t-t_2)(\mathbf{r}-\mathbf{x}_1)\cdot\boldsymbol{\sigma}} \\ f_k(t) &= e^{(t-t_k)(\mathbf{r}-\mathbf{x}_k)\cdot\boldsymbol{\sigma}} f_{k-1}(t_k)\,, \qquad k > 1\,. \end{aligned} \tag{9.14}$$

These are defined so that at the boundaries between intervals $f_k(t_k) = f_{k-1}(t_k)$. An arbitrary solution of Eq. (9.7) can then be written as

$$v^a(t) = f_k(t)\eta_k^a\,, \qquad t_k < t < t_{k+1}\,, \tag{9.15}$$

where the η_k ($1 \le k \le N-1$) are a set of N-component row vectors. The discontinuities at the interval boundaries must be such that

$$\eta_k = \eta_{k-1} + [f_k(t_k)]_k^{-1} a_k^\dagger S_k\,. \tag{9.16}$$

The normalization condition Eq. (9.8), takes the form

$$I = \sum_{j=2}^{N-1} S_j^\dagger S_j + \sum_{k=1}^{N-1} {\eta_k}^\dagger N_k \eta_k \tag{9.17}$$

with

$$N_k = \int_{t_k}^{t_{k+1}} dt\, f_k^\dagger(t) f_k(t)\,. \tag{9.18}$$

These equations do not completely determine the η_k. This indeterminacy reflects the fact that Eq. (9.7) is preserved if v and the S_k are multiplied on the right by any $N\times N$ unitary matrix function of $\mathbf{r}$; this corresponds to an ordinary spacetime gauge transformation. A convenient choice is to take two columns of v, say v^1 and v^2, to be continuous. This can be done by setting $S_k^a = 0$ for $a = 1,2$ and choosing

$$\eta_k^a = N^{-1/2}\theta^a\,, \qquad a = 1,2\,, \tag{9.19}$$

with $N = \sum_k N_k$ and the θ^a being the two-component objects $\theta^1 = (1,0)^t$ and $\theta^2 = (0,1)^t$. Orthogonality of the other columns of v with the first two, as required by Eq. (9.8), then implies that

$$0 = \sum_{k=1}^{N-1} N_k \eta_k^\mu \tag{9.20}$$

where here and below Greek indices are assumed to run from 3 to N. Together with the discontinuity Eq. (9.16), this uniquely determines the η_k^μ. Substituting the result back into Eq. (9.17) then gives an equation for the S_k^μ,

$$\delta^{\mu\nu} = \sum_{i,j=2}^{N-1} S_i^{\mu\dagger}[\delta_{ij} + a_i M_{ij} a_j^\dagger] S_j^\nu \tag{9.21}$$

where the M_{ij} are matrices, constructed from the N_k and the $f_k(t_k)$, whose precise form is not important for our purpose. After solving this equation for the S_k^μ, one can then work back to obtain the η_k^μ and thus v, and then substitute into Eqs. (9.10) and (9.11) to obtain the spacetime fields.

Now let us specialize to the case of unbroken $U(1)\times SU(2)\times U(1)$ symmetry. The middle $N-2$ eigenvalues of Φ_0 are now degenerate, and so all but the first and last intervals in t vanish. Because the $f_k(t_k)$ are all equal to unity, the discontinuity equation for the η_k becomes

$$\eta_k = \eta_{k-1} + a_k^\dagger S_k\,. \tag{9.22}$$

In addition, the matrices M_{ij} in Eq. (9.21) no longer depend on i and j, but instead are all equal to a single matrix M.

When the symmetry breaking was maximal, the monopole positions entered both through the functions $f_k(t)$ and through the a_j. Now however, with the middle intervals having zero width, the positions associated with the massless monopoles enter only through the a_j. But these now appear in Eqs. (9.21) and (9.22) only in the combination $\sum_j a_j^\dagger S_j^\mu$. This fact has a striking consequence. Consider two sets of monopole positions $\mathbf{x}_k$ and $\tilde{\mathbf{x}}_k$ with identical positions for the massive monopoles, but with the massless monopoles constrained only by the requirement that

$\tilde{a}_j = W_{jk} a_k$, where W is any $(N-2) \times (N-2)$ unitary matrix. If S_j^μ is a solution of Eq. (9.21) for the first set of positions, then $\tilde{S}_j^\mu = W_{jk} S_k^\mu$ is a solution for the transformed set.

This implies that the positions of the massless monopoles are not all physically meaningful quantities. This result was anticipated by the parameter counting done earlier in this section, which indicated that there should be a single gauge-invariant quantity characterizing the non-Abelian cloud. This quantity can be identified by noting that these transformations leave invariant

$$\sum_j a_j^\dagger a_j = \sum_j \left[\alpha_j \cdot \boldsymbol{\sigma} - i\alpha_{j0} I\right] = (\mathbf{x}_1 - \mathbf{x}_{N-1}) \cdot \boldsymbol{\sigma} + \sum_{j=2}^{N-1} |\mathbf{x}_j - \mathbf{x}_{j-1}| \, . \tag{9.23}$$

The first term on the right hand side is determined by the positions of the massive monopoles, while the second is just the sum of the distances between successive massless monopoles. The latter can be used to define a cloud parameter b by

$$2b + R = \sum_{j=2}^{N-1} |\mathbf{x}_j - \mathbf{x}_{j-1}| \tag{9.24}$$

where R is the distance between the massive monopoles.

The subsequent analysis can be simplified by using a transformation of this type to choose a canonical set of massless monopole positions in which $\mathbf{x}_2$ is located on the straight line defined by $\mathbf{x}_1$ and $\mathbf{x}_{N-1}$ at a distance b from $\mathbf{x}_1$, while the remaining $N-3$ massless monopoles are located at $\mathbf{x}_{N-1}$. Once this choice is made, one is rather naturally led to choose a solution for the S_k, and hence for the η_k and v, such that the resulting expressions for the spacetime fields have nontrivial components only in a 4×4 block. Thus, as promised earlier, the solutions can all be obtained by embeddings of $SU(4) \to U(1) \times SU(2) \times U(1)$ solutions.

The fact that the solutions for arbitrary $SU(N)$ can be obtained from the $SU(4)$ solution underscores the difficulties in pinning down the massless monopoles. When viewed as an $SU(4)$ solution, the configuration contains a single massless monopole, but when it is interpreted as an $SU(N)$ solution there are $N-3$ massless monopoles. Thus, not only the positions, but even the number of massless components is ambiguous.

These $SU(4)$ solutions have some features that are reminiscent of the $SO(5)$ solutions discussed in the previous section. The fields can be decomposed into pieces that transform as triplets, doublets, and singlets under the unbroken $SU(2)$. Only the first two depend on b, and then only through a single function L, which is now a 2×2 matrix. Also, the triplet and doublet components of the Higgs field are given in terms of the same spacetime functions as the corresponding gauge field components, just as was the case with the $SO(5)$ solution.

The detailed form of these solutions [29] is rather complex. However, some insight into the nature of the non-Abelian cloud can be obtained by examining the asymptotic behavior of the fields well outside the cores of the massive monopoles. Consider first the case $b \gg R$. If the distances y_L and y_R from a point $\mathbf{r}$ to the two massive monopoles are both much less than b, the Higgs field and magnetic field can be written in the form

$$\Phi(\mathbf{r}) = U_1^{-1}(\mathbf{r}) \begin{pmatrix} t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & t_2 + \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & t_2 - \frac{1}{2y_L} & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y_L} \end{pmatrix} U_1(\mathbf{r}) + \cdots \tag{9.25}$$

$$\mathbf{B}(\mathbf{r}) = U_1^{-1}(\mathbf{r}) \begin{pmatrix} \frac{\hat{\mathbf{y}}_R}{2y_R^2} & 0 & 0 & 0 \\ 0 & -\frac{\hat{\mathbf{y}}_R}{2y_R^2} & 0 & 0 \\ 0 & 0 & \frac{\hat{\mathbf{y}}_L}{2y_L^2} & 0 \\ 0 & 0 & 0 & -\frac{\hat{\mathbf{y}}_L}{2y_L^2} \end{pmatrix} U_1(\mathbf{r}) + \cdots \tag{9.26}$$

where $U_1(\mathbf{r})$ is an element of $SU(4)$ and the dots represent terms that are suppressed by powers of R/b, y_L/b, or y_R/b. These are the fields that one would expect for two massive monopoles, each of whose magnetic charges has both a $U(1)$ component and a component in the unbroken $SU(2)$ that corresponds to the middle 2×2 block. If instead $y \equiv (y_L + y_R)/2 \gg b$,

$$\Phi(\mathbf{r}) = U_2^{-1}(\mathbf{r}) \begin{pmatrix} t_4 - \frac{1}{2y} & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y} \end{pmatrix} U_2(\mathbf{r}) + O(b/y^2) \tag{9.27}$$

$$\mathbf{B}(\mathbf{r}) = U_2^{-1}(\mathbf{r}) \begin{pmatrix} \frac{\hat{\mathbf{y}}}{2y^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hat{\mathbf{y}}}{2y^2} \end{pmatrix} U_2(\mathbf{r}) + O(b/y^3)\,. \tag{9.28}$$

Thus, at distances large compared to b the non-Abelian part of the Coulomb magnetic field is cancelled by the cloud, in a manner similar to that which we saw for the $SO(5)$ case.

In the opposite limit, $b = 0$, the solutions are essentially embeddings of $SU(3) \to U(1) \times U(1)$

solutions. At large distances, one finds that

$$\Phi(\mathbf{r}) = U_3^{-1}(\mathbf{r}) \begin{pmatrix} t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & t_2 - \frac{1}{2y_L} + \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y_L} \end{pmatrix} U_3(\mathbf{r}) \tag{9.29}$$

$$\mathbf{B}(\mathbf{r}) = U_3^{-1}(\mathbf{r}) \begin{pmatrix} \frac{\hat{\mathbf{y}}_R}{2y_R^2} & 0 & 0 & 0 \\ 0 & \frac{\hat{\mathbf{y}}_L}{2y_L^2} - \frac{\hat{\mathbf{y}}_R}{2y_R^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hat{\mathbf{y}}_L}{2y_L^2} \end{pmatrix} U_3(\mathbf{r}) \,. \tag{9.30}$$

Viewed as $SU(3)$ solutions, the long-range fields are purely Abelian. Viewed as $SU(4)$ solutions, the long-range part is non-Abelian in the sense that the unbroken $SU(2)$ acts nontrivially on the fields. However, because of the alignment of the fields of the two massive monopoles, the non-Abelian part of the field is a purely dipole field that falls as R/y^3 at large distances.

10 Concluding remarks

I have shown in these talks how one is naturally led to a class of multimonopole solutions that contain one or more massive monopoles, similar to those found in the $SU(2)$ gauge theory, surrounded by a cloud, of arbitrary size, in which there are nontrivial non-Abelian fields. Analysis of the moduli space Lagrangian that governs the low-energy monopole dynamics suggests that these clouds can be understood in term of the degrees of freedom of massless monopoles carrying purely non-Abelian magnetic charges.

There remain many open questions relating to these massless monopoles. First, it would clearly be desirable to obtain additional solutions containing non-Abelian clouds. Particularly useful would be solutions with charges such that the cloud depends on more than a single gauge-invariant parameter, and solutions containing more than a single cloud. Experience with the solutions described in Sec. 9 suggests that, as a first step, it might be feasible to attack the simplified problem of determining the cloud structure for a given set of massive monopole positions. From the more physical viewpoint, one would like to use these solutions to gain further insight in the properties of non-Abelian gauge theories. The massless monopoles clearly seem to be the duals of the massless gauge bosons. Hence, one should be able to find some kind of correspondence between the behavior of the non-Abelian clouds and that of the gauge bosons. Understanding this correspondence in detail remains an important challenge.

This work was supported in part by the U.S. Department of Energy.

References

[1] S. Coleman, Phys. Rev. D **11**, 2088 (1975).

[2] C. Montonen and D. Olive, Phys. Lett. **72B**, 117 (1977).

[3] K. Lee, E.J. Weinberg and P. Yi, Phys. Lett. B **376**, 97 (1996).

[4] K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D **54**, 1633 (1996).

[5] K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D **54**, 6351 (1996).

[6] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976); M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975); S. Coleman, S. Parke, A. Neveu and C.M. Sommerfield, Phys. Rev. D **15**, 544 (1977).

[7] T.T. Wu and C.N. Yang, Nucl. Phys. **B107**, 365 (1976).

[8] G 't Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, JEPT Lett. **20**, 194 (1974).

[9] K. Lee and E.J. Weinberg, Phys. Rev. Lett. **73**, 1203 (1994).

[10] E.J. Weinberg, Phys. Rev. D **20**, 936 (1979).

[11] L.S. Brown, R.D. Carlitz, and C. Lee, Phys. Rev. D **16**, 417 (1977).

[12] H. Osborn, Phys. Lett. **83B**, 321 (1979).

[13] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. **B125**, 1 (1977); F. Englert and P. Windey, Phys. Rev. D **14**, 2728 (1976).

[14] E.J. Weinberg, Nucl. Phys. **B167**, 500 (1980).

[15] A. Sen, Phys. Lett. **B329**, 217 (1994).

[16] N.S. Manton, Phys. Lett. **110B**, 54 (1982).

[17] G.W. Gibbons and N.S. Manton, Phys. Lett. **B356**, 32 (1995).

[18] M.F. Atiyah and N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton Univ. Press, Princeton (1988); Phys. Lett. **107A**, 21 (1985); Phil. Trans. R. Soc. Lon. **A315**, 459 (1985).

[19] T. Eguchi and A.J. Hanson, Ann. Phys. **120** (1979) 82.

[20] J.P. Gauntlett and D.A. Lowe, Nucl. Phys. **B472**, 194 (1996).

[21] S.A. Connell, *The dynamics of the $SU(3)$ charge (1,1) magnetic monopoles,* University of South Australia preprint.

[22] G. Chalmers, *Multi-monopole moduli spaces for $SU(N)$ gauge group*, ITP-SB-96-12, hep-th/9605182.

[23] M.K. Murray, J. Geom. Phys. **23**, 31 (1997).

[24] E. Witten, Nucl. Phys. **B202**, 253 (1982).

[25] E.J. Weinberg, Nucl. Phys. **B203**, 445 (1982).

[26] A. Abouelsaood, Phys. Lett. **125B**, 467 (1983); P. Nelson, Phys. Rev. Lett. **50**, 939 (1983).

[27] P. Nelson and A. Manohar, Phys. Rev. Lett. **50**, 943 (1983); A. Balachandran, G. Marmo, M. Mukunda, J. Nilsson, E. Sudarshan and F. Zaccaria, Phys. Rev. Lett. **50**, 1553 (1983); P. Nelson and S. Coleman, Nucl. Phys. **B237**, 1 (1984).

[28] E.J. Weinberg, Phys. Lett. **B119**, 151 (1982).

[29] E.J. Weinberg and P. Yi, *Explicit multimonopole solutions in $SU(N)$ gauge theory*, hep-th/9803164, to appear in Phys. Rev. D.

[30] W. Nahm, Phys. Lett. **90B**, 413 (1980); W. Nahm, in *Monopoles in quantum field theory*, N. Craigie et al. eds. (World Scientific, Singapore, 1982); W. Nahm, in *Group theoretical methods in physics*, G. Denardo et al. eds. (Springer-Verlag, 1984).

Painlevé test for extended nonlinear Schrödinger equation

Q-Han Park[1]

and

H.J. Shin[2]

Department of Physics
and
Research Institute of Basic Sciences
Kyunghee University
Seoul, 130-701, Korea

ABSTRACT

We briefly review the Painlevé test and the Painlevé property for ordinary and partial differential equations and describe the connection with integrability. The Kowalewski top, the Bullough-Dodd equation and the extended nonlinear Schrödinger equation are Painlevé tested and explained in the same vein.

[1]E-mail address; qpark@nms.kyunghee.ac.kr , Speaker
[2]E-mail address; hjshin@nms.kyunghee.ac.kr

Nonlinear ordinary and partial differential equations are essential in vast areas of physical sciences. Thus, the test of integrability of those equations is not only of mathematical interest but also an important problem for the practical applications. In this talk, We introduce one such test, the Painlevé test, which provides a criterion for integrability. Though, the Painlevé test neither prove or disprove the integrability of the equation at hand in an absolute manner, it provides us with quite a useful tool in testing integrability. The Painlevé test applies both to the ordinary and the partial differential equations and We will briefly discuss about the test for each cases, in particular, through explicit examples. That is, the Kowalewski top for the ODE test, the Bullough-Dodd equation and the extended nonlinear Schrödinger equation for the PDE test. Especially, the Painlevé test for the extended nonlinear Schrödinger equation is based on the recent work I have done with Prof. H.J. Shin which finds nice application in the field of nonlinear optics.

The history of the Painlevé test goes back to the work of Sonya Kovalevskaya (Kowalewski (1889, 1890)). She has applied the singular point analysis to the problem of spinning top and found a new case of integrable spinning top, now known as the Kowalewski top. The governing equation of spinning top is given by a set of first-order ODE as follows;

$$\begin{aligned} A\dot{p} &= (B-C)qr + Mg(\gamma y_0 - \beta z_0) \\ B\dot{q} &= (C-A)pr + Mg(\alpha z_0 - \gamma x_0) \\ C\dot{r} &= (A-B)pq + Mg(\beta x_0 - \alpha y_0) \\ \dot{\alpha} &= \beta r - \gamma q \\ \dot{\beta} &= \gamma p - \alpha r \\ \dot{\gamma} &= \alpha q - \beta p \end{aligned} \tag{0.1}$$

where $\vec{w} = (p, q, r)$ is the angular velocity vector, (α, β, γ) the directional cosines of the direction of gravity, (A, B, C) moments of inertia and (x_0, y_0, z_0) is the coordinate of the center of mass. In order to solve Eq. (0.1) completely, we need six functionally independent integrals among which five integrals are easily found. Kowalewski's idea in finding the sixth integral was to require the system to possess solutions without movable critical points. Movable singularities are singularities appearing only through solutions but not in the equation itself. Critical points are singularities which are not poles. She did so by assuming the solution to take the asymptotic form

$$\begin{aligned} p &= (t-t_0)^{-m_1} \sum_{j=0}^{\infty} p_j (t-t_0)^j \\ q &= (t-t_0)^{-m_2} \sum_{j=0}^{\infty} q_j (t-t_0)^j \\ r &= (t-t_0)^{-m_3} \sum_{j=0}^{\infty} r_j (t-t_0)^j \end{aligned}$$

$$
\begin{aligned}
\alpha &= (t-t_0)^{-n_1}\sum_{j=0}^{\infty}\alpha_j(t-t_0)^j \\
\beta &= (t-t_0)^{-n_2}\sum_{j=0}^{\infty}\beta_j(t-t_0)^j \\
\gamma &= (t-t_0)^{-n_3}\sum_{j=0}^{\infty}\gamma_j(t-t_0)^j
\end{aligned}
\tag{0.2}
$$

and looking for conditions which allow five arbitrary constants in order Eq. (0.2) to become a general solution. Then, she found that the conditions are $m_1 = m_2 = m_3 = 1, n_1 = n_2 = n_3 = 2$ and four distinct cases of constants;

$$
\begin{array}{ll}
1. & A = B = C,\ I_1 = px_0 + qy_0 + rz_0 \\
2. & x_0 = y_0 = z_0,\ I_2 = A^2p^2 + B^2q^2 + C^2r^2 \\
3. & A = B,\ \text{and } x_0 = y_0 = 0,\ I_3 = Cr \\
4. & A = B = 2C,\ z_0 = 0,\ I_4 = (p^2 - q^2 - \frac{Mgx_0\alpha}{C})^2 + (2pq - \frac{Mgx_0\beta}{C}
\end{array}
\tag{0.3}
$$

The first three cases have been known previously, the second was found by Euler and the third was by Lagrange. The fourth was a new finding by Kowalewski. Note the asymmetric relation among components of moments of inertia in the Kowalewski case. We will see this asymmetric behavior again when we consider PDE. The property that movable singularies of solutions are only poles is known as the Painlevé property. The application of the singularity analysis has also been made to other types of equations. In particular, in a series of papers, Painlevé and collaborators have made a systematic study on the second order ODE possessing the Painlevé property, and found that there exist fifty cannonical types of equations up to the transformation

$$
Y(X) = \frac{a(x)y + b(x)}{c(x)y + d(x)},\ X = \phi(x). \tag{0.4}
$$

Among those fifty equations, six equations are distinguished while all others either reduce to the six equation or become solved in terms of known functions. The six equations are solved in terms of new functions, called as Painlevé transcendents, and they are

$$
\begin{aligned}
PI\ \frac{d^2y}{dx^2} &= 6y^2 + x \\
PII\ \frac{d^2y}{dx^2} &= 2y^3 + xy + \alpha \\
PIII\ \frac{d^2y}{dx^2} &= \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
PIV\ \frac{d^2y}{dx^2} &= \frac{1}{2y}\left(\frac{dy}{dx}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y} \\
PV\ \frac{d^2y}{dx^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\left(\alpha + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}
\end{aligned}
$$

$$PVI \quad \frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\frac{\beta x}{y^2}+\frac{\gamma(x-1)}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right) \quad (0.5)$$

In the case of PDE, there are certain sets of equations which are known to be integrable in the sense that they allow solutions via the inverse scattering method. Though there does not exist a well defined notion of integrability, it is convetional to call a PDE integrable if it admits the inverse scattering method. The relation between the Painlevé property of integrability through the inverse scattering has been investigated and now it has been conjectured that every ODE which arises as a similarity reduction of an integrable PDE through the inverse scattering method is of Painlevé type up to a transformation of variables. (Ablowitz (1978,1980), Hastings (1980)). This is also known as the Painlevé ODE test for a given PDE. For example, the sine-Gordon equation

$$\phi_{xt} = sin\phi \quad (0.6)$$

under the similarty reduction

$$z = xt, \quad y = \exp(i\phi(z)) \quad (0.7)$$

becomes

$$z(yy^{''} - y^{'2}) + yy^{'} = y(y^2 - 1) \quad (0.8)$$

which is of Painlevé III type. Thus, it passes the Painlevé ODE test. When the same similarity reduction and the Painlevé ODE test is applied to the equation of the form $\phi_{xt} = f(\phi)$, following three distinct cases pass the test;

$$\begin{aligned} \text{sine-Gordon}\phi_{xt} &= \sin\phi \\ \text{Bullough-Dodd}\phi_{xt} &= e^{\phi} - e^{-2\phi} \\ \text{Liouville}\phi_{xt} &= e^{\phi} \end{aligned} \quad (0.9)$$

Note that the second case (Bullough-Dodd equation) also poses an asymmetric exponents similar to the Kowalewski top. Though, the Painlevé ODE test provides a useful test for a given PDE, in practice, it is not easy to test it with all possible similarity redutions. To circumvent this, Weiss et al (Weiss (1983)) has proposed the the Painlevé PDE test which directly tests the PDE without going through the ODE reduction procedure. Similar to the Kowalewski's test, one assumes an asymptotic form for the solution

$$\psi(z_1, ..., z_n) = f^{-p}\sum_{j=0}^{\infty}\psi_j f^j \quad (0.10)$$

where $f = 0$ defines a non-characteristic movable singularity manifold analytic in z_i. Inserting this ansatz to the equation, one obtains a recursive relation of the form

$$(j-\beta_1)(j-\beta_2)...(j-\beta_N)\psi_j = G_j(\psi_0, ..., \psi_{n-1}, f, z_i). \quad (0.11)$$

The coefficient function ψ_j is determined by Eq . (0.11) unless j is equal to one of the resonances $\beta_1, ..., \beta_N$. At the resonance, ψ_j is undetermined. If $G_j = 0$ at resonance values of j to saturate the consistency and if one finds enough numbers of undetermined functions required by the equation, then the given PDE passes the test. As for an explicit example of the PDE test, let me briefly mention our work. Recently, we have applied this test to the coupled nonlinear Schrödinger equation defined by (Park and Shin (1998a))

$$
\begin{aligned}
i\dot{q}_1 &= q_1'' + q_1(\gamma_1|q_1|^2 + \gamma_2|q_2|^2) + \gamma_3 q_1^* q_2^2 + \gamma_4 q_1^2 q_2^* \\
i\dot{q}_2 &= q_2'' + q_2(\gamma_2|q_1|^2 + \gamma_1|q_2|^2) + \gamma_3 q_2^* q_1^2 + \gamma_4 q_2^2 q_1^*.
\end{aligned} \tag{0.12}
$$

It turns out that resonances occur in various combination of parameters γ_1, γ_4 and after lengthy and painful anaysis, we found that it passes the Painlevé PDE test for the following cases;

$$
\begin{aligned}
&1. \quad \gamma_2 = \gamma_3 = \gamma_4 = 0 \\
&2. \quad \gamma_1 = \gamma_2, \text{ and } \gamma_3 = \gamma_4 = 0 \\
&3. \quad \gamma_2 = 2\gamma_1, \ \gamma_3 = -\gamma_1 \text{ and } \gamma_4 \text{ arbitrary} \\
&4. \quad \gamma_2 = 2\gamma_1, \ \gamma_3 = \gamma_1
\end{aligned} \tag{0.13}
$$

The first case is simply two sets of the usual nonlinear Schrödinger equation which has been solved by Zakharov and Shabat (Zakharov (1972)) using the inverse scattering method. The second case is the vector nonlinear Schrödinger equation which has been also solved (Manakov (1975), Zakharov and Manakov (1976)). The third the fourth cases are our new results. Note that they also possess asymmetric relations in coefficients likewise the Kowalewski top and the Bullough-Dodd equation. All of them share the ratio 2:1 to certain extent. The reason behind this conincidence is the sympletic group structure lurking these equations. Indeed, the integrability of the third and the fourh cases can be shown by embedding these two cases to the nonlinear Schrödinger equation generalized in association with the Hermitian symmetric space $Sp(2)/U(2)$. Details on the relationship of the nonlinear Schrödinger equation with the group strucutre will appear in a forthcoming paper (Park and Shin (1998b)).

ACKNOWLEDGEMENT

Q.P thanks Prof. Chae for his hospitality and effort in organizing this workshop.

- Kowalewski S (1889), *Sur une Problème de la Rotation d'un Corps Solide Autour d'un Point Fize*, Acta Mathematica **12** (1889), 177-232.
- Kowalewski S (1890), *Sur une Propriété du Système D'équations Differentielles qui Definit la Rotation d'un Corps Solide Autour d'un Point Fize* Acta Mathematica **14** (1890), 81-93.

- Ablowitz M.J., Ramani A. and Segur H. (1978), *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lett. Nuovo Cim. **23**, 333; *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I and II,* J. Math. Phys., **21**, 715, 1006.
- Hastings S.P. and McLeod J.B. (1980) *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vris equation*, Arch. Rat. Mech. Anal., **73**, 31.
- Weiss J., Tabor M. and Carnevale G. (1983), *The Painlevé property for partial differential equations*, J. Math. Phys. *24*, 522.
- Park Q-H. and Shin H.J. (1998a), *The Painlevé test of the coupled nonlinear Schrödinger equation with coherence terms* , to appear.
- Zakharov V.E. and Shabat A.B. (1972), *Exact theory of two-dimensional self-focusing and one-dimensional of waves in nonlinear media*, Sov. Phys. JETP, **34**, 62.
- Manakov S.V. (1975), *Complete integrability and stochastization of discrete dynamical systems*, Sov. Phys. JETP, **40**, 269.
- Zakharov V.E. and Manakov S.V. (1976), *The theory of resonance interaction of wave packets in nonlinear media*, Sov. Phys. JETP, **42**, 842.
- Park Q-H. and Shin H.J. (1998b), *Integrable coupling of optical waves in higher-order nonlinear Schrödinger equations*, to appear.

BOGOMOL'NYI SOLITONS AND HERMITIAN SYMMETRIC SPACES

Phillial Oh

Department of Physics, Sung Kyun Kwan University
Suwon 440-746, Republic of Korea

Abstract

We apply the coadjoint orbit method to construct relativistic nonlinear sigma models (NLSM) on the target space of coadjoint orbits coupled with the Chern-Simons (CS) gauge field and study self-dual solitons. When the target space is given by Hermitian symmetric space (HSS), we find that the system admits self-dual solitons whose energy is Bogomol'nyi bounded from below by a topological charge. The Bogomol'nyi potential on the Hermitian symmetric space is obtained in the case when the maximal torus subgroup is gauged, and the self-dual equation in the $CP(N-1)$ case is explored. We also discuss the self-dual solitons in the non-compact $SU(1,1)$ case and present a detailed analysis for the rotationally symmetric solutions.

1. INTRODUCTION

Recently, a coadjoint orbit method to formulate the nonlinear sigma model defined on the target space of homogeneous space G/H was proposed [1]. It was first applied to a non-relativistic spin system whose Poisson bracket between the dynamical variables defined on the coadjoint orbit satisfies the classical $\mathcal{G}$ algebra. The Euler-Lagrange equation of motion yields the generalized continuous Heisenberg ferromagnet. When the target space of coadjoint orbit is given by HSS which is a symmetric space equipped with complex structure [2], the generalized ferromagnet system becomes completely integrable in 1+1 dimension [1]. Later, this method was exploited to produce a class of integrable extension of relativistic NLSM in 1+1 dimension [3]. It was also discovered that incorporation the CS gauge field in 2+1 dimension on the same target space produces a class of self-dual field theories which admit Bogomol'nyi self-dual equations saturating the energy functional [4]. A detailed numerical investigation in $SU(2)$ [5] and the non-compact $SU(1,1)$ spin system [6] showed a rich structure of self-dual solitons, and the quantization of the system revealed that the symmetry algebra satisfies anomalous commutation relations, and the system describes anyons [6].

In this paper, we apply the coadjoint orbit method to construct relativistic NLSM on the target space of coadjoint orbits coupled with the CS gauge field and study self-dual solitons. When the target space is HSS, the Hamiltonian is bounded from below by a topological charge, and the resulting self-dual CS solitons satisfy a vortex-type equation, thus producing a class of new self-dual theories on HSS. This construction provides a unified framework for treating the previous gauged $O(3)$ model on S^2 and

$CP(N-1)$ models [7] which are well known examples of the coadjoint orbit G/H with $S^2 = SO(3)/SO(2) \approx SU(2)/U(1)$ and $CP(N-1) = SU(N)/SU(N-1) \times U(1)$. We also study the self-dual solitons in the non-compact HSS with $SU(1,1)$ group in which the target space is given by the upper sheeted hyperboloid and find various topological and nontopological solitons.

We first give a brief summary of NLSM on the target space of coadjoint orbit for completeness. Consider a group G, Lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^* : X \in \mathcal{G}; \quad u \in \mathcal{G}^*$. The adjoint action of G on the Lie algebra is defined by

$$\mathrm{Ad}(g)X = gXg^{-1}, \quad g \in G. \tag{1.1}$$

Denoting inner product between $\mathcal{G}$ and $\mathcal{G}^*$ by $< u, X >$, the coadjoint action of the group on $\mathcal{G}^*$ is defined in such a way to make the inner product invariant:

$$< \mathrm{Ad}^*(g)u, X > = < u, \mathrm{Ad}(g^{-1})X > . \tag{1.2}$$

The coadjoint orbit is given by the orbit of coadjoint action of the group G: Fix a point $u \in \mathcal{G}^*$, then the orbit is generated by

$$\mathcal{O}_u = \{x | x = \mathrm{Ad}^*(g)u, g \in G\}. \tag{1.3}$$

It can be shown that $\mathcal{O}_u \approx G/H$, where H is the stabilizer of the point u.

Let us assume that the inner product is given by the trace: $< u, X > = \mathrm{Tr}(Xu)$. Then, $\mathcal{G}$ and $\mathcal{G}^*$ are isomorphic and the coadjoint orbit can be parameterized by

$$Q = gKg^{-1} = Q^A t^B \eta_{AB}; \; t^A, K \in \mathcal{G} \; (A = 1, \cdots, \dim \mathcal{G}), \tag{1.4}$$

where η_{AB} is the G-invariant metric given by $\mathrm{Tr}(t^A t^B) = -\frac{1}{2}\eta^{AB}$ with t^A's being the generator of $\mathcal{G}$. The action for the NLSM on the target space of coadjoint orbit can be constructed as

$$S(g) = \epsilon \mathrm{Tr} \int d^3x \partial_\mu Q \partial^\mu Q. \tag{1.5}$$

$\epsilon = +1$ for the compact case -1 for the non-compact case [8]. Let us first choose the element K to be the central element of the Cartan subalgebra of $\mathcal{G}$ whose centralizer in $\mathcal{G}$ is H. Then, for the HSS, we have $J = \mathrm{Ad}(K)$ acting on the coset is a linear map satisfying the complex structure condition $J^2 = -1$, which gives the useful identity [1]:

$$[Q, [Q, \partial_\mu Q]] = -\partial_\mu Q. \tag{1.6}$$

This paper is organized as follows: In Section 2, starting from a CS gauged action of (1.5) on arbitrary HSS, we derive self-dual equations and Bogomol'nyi potential. We give explicit expressions in $CP(N-1)$ case. In Section 3, we deal with non-compact minimal $SU(1,1)$ model and discuss rotationally symmetric solutions in detail. In Section 4, we give the conclusion.

2. COMPACT MODEL

Let us consider the following CS gauged action of (1.5):

$$S_G = \int d^3x \left[-\epsilon \frac{1}{2} \left(D_\mu Q^A D^\mu Q^B \eta_{AB} \right) - W_G(Q^A) - \kappa \epsilon^{\mu\nu\rho} \mathrm{Tr} \left(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \right], \tag{2.1}$$

where the covariant derivative is defined by

$$D_\mu Q = \partial_\mu Q + [A_\mu, Q], \quad A_\mu = A_\mu^A t^B \eta_{AB}. \tag{2.2}$$

We assume that the potential is given by

$$W_G(Q^A) = \frac{1}{2} I^{AB} Q^A Q^B, \tag{2.3}$$

where I^{AB} is the symmetric tensor and its content will be determined by the self-duality condition. The equations of motion are given by

$$\begin{aligned} D_\mu[Q,\ D^\mu Q] + [\bar{Q}, Q] &= 0, \quad (\bar{Q} = I^{AB} Q^A t^B). \\ \frac{\kappa}{2}\epsilon^{\mu\nu\rho} F_{\nu\rho} &= [Q, D^\mu Q]. \end{aligned} \tag{2.4}$$

We first treat the compact case with $\eta_{AB} = \delta_{AB}$. To study self-dual solitons, we bring the energy functional into Bogomol'nyi expression:

$$\begin{aligned} E_G &= \int d^2x \left[\frac{1}{2}\left((D_0 Q^A)^2 + (D_i Q^A)^2\right) + W(Q^A)\right] \\ &= \int d^2x \left[\frac{1}{2}(D_0 Q^A)^2 + \frac{1}{4}(D_i Q^A \pm \epsilon_{ij}[Q, D_j Q]^A)^2 \right. \\ &\quad \left. + W(Q^A) \pm \frac{1}{2}\epsilon_{ij} F^A_{ij} Q^A\right] \pm 4\pi T_G, \end{aligned} \tag{2.5}$$

where the topological charge T_G is given by

$$T_G = \frac{1}{8\pi}\int d^2x [\epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A - 2\epsilon_{ij}\partial_i (Q^A A^A_j)]. \tag{2.6}$$

In deriving (2.5), we used the gauged version of (1.6) where ∂_μ is replaced by the covariant derivative D_μ [4]. Thus, the Hamiltonian is bounded from below by the topological charge T_G when the potential W_G is chosen such that

$$W_G \pm \frac{1}{2}\epsilon_{ij} F^A_{ij} Q^A = 0. \tag{2.7}$$

Here, F^A_{ij} is determined in terms of Q^A by the Gauss's law which is the time-component of (2.4).

The minimum energy arises when the self-duality equation is satisfied:

$$D_i Q = \mp \epsilon_{ij} [Q, D_j Q]. \tag{2.8}$$

Consistency with the static equations of motion (2.4) forces

$$F_{ij} = 0, \quad A_0 = \pm\frac{1}{\kappa} Q, \tag{2.9}$$

which in turn puts the potential $W_G = 0$ and $I^{AB} = 0$. Note that the gauge field can be chosen as a pure gauge in this case and the contents of the Bogomol'nyi solitons are precisely the two dimensional instantons which were completely classified on each HSS [9].

More interesting cases in which the system offers other solitons arise when we gauge the subgroup H. We consider gauging the maximal torus subgroup of G:

$$S_H = \int d^3x \left[-\frac{1}{2}\left(D_\mu Q^A D^\mu Q^A\right) - W_H(Q^A) + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\partial_\mu A^a_\nu A^a_\rho\right]. \tag{2.10}$$

Here, the index $a = 1, \cdots$, rank G denotes the maximal Abelian subgroup. Again, the content of the potential W_H will be determined from the self-duality condition.

Using the Gauss's law given by

$$\frac{\kappa}{2}\epsilon_{ij}F^a_{ij} = -[Q, D_0 Q]^a, \tag{2.11}$$

we find that the energy functional satisfies

$$\begin{aligned} E_H &= \frac{1}{2}\int d^2x \left[\left(D_0 Q^A \pm \frac{1}{\kappa}[Q, Q_H]^A\right)^2 + \frac{1}{2}(D_i Q^A \pm \epsilon_{ij}[Q, D_j Q]^A)^2\right] \pm 4\pi T_H, \\ T_H &= \frac{1}{8\pi}\int d^2x [\epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A - 2\epsilon_{ij}\partial_i (Q^a_H A^a_j)], \end{aligned} \tag{2.12}$$

when the Bogomol'nyi potential W_H is chosen as

$$W_H = \frac{1}{2\kappa^2}([Q_H, Q]^A)^2, \quad Q_H \equiv Q^a_H t^a = (Q^a - V^a)t^a. \tag{2.13}$$

Note that V^a's are free parameters associated with the vacuum symmetry breaking [10]. When the self-duality equations

$$D_i Q^A = \mp\epsilon_{ij}[Q, D_j Q]^A, \quad D_0 Q^A = \mp\frac{1}{\kappa}[Q, Q_H]^A, \tag{2.14}$$

are satisfied, we see that the energy is saturated by the topological charge:

$$E_H = 4\pi|T_H|. \tag{2.15}$$

The first order equation (2.14) in the static case fixes A^a_0 to be

$$A^a_0 = \pm\frac{1}{\kappa}Q^a_H, \tag{2.16}$$

which automatically solves the Euler-Lagrange equations of motion of the action (2.10) with the potential given by (2.13).

Let us examine (2.11) and (2.14) more closely in $CP(N-1)$ case. We use the expression of Q [1],

$$Q = i\Psi\Psi^\dagger - i\frac{I}{N} \tag{2.17}$$

where the column vector Ψ can be expressed by the Fubini-Study coordinate ψ_a's $(a = 1, 2, \cdots, N-1)$:

$$\Psi = \frac{1}{\sqrt{1+|\psi|^2}}\begin{pmatrix} 1 \\ \psi_1 \\ \vdots \\ \psi_{N-1}\end{pmatrix}, \tag{2.18}$$

with $|\psi|^2 = |\psi_1|^2 + \cdots + |\psi_{N-1}|^2$. Using the complex notation; $z = x+iy, \bar{z} = x - iy$, $A_z = \frac{1}{2}(A_1 - iA_2), A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$, and $D_z = \frac{1}{2}(D_1 - iD_2), D_{\bar{z}} = \frac{1}{2}(D_1 + iD_2)$, we obtain an alternative expression of the self-duality equation,

$$D_{\bar{z}} Q = \mp i[Q, D_{\bar{z}} Q]. \tag{2.19}$$

With the above parameterization of Q, the self-duality equation (2.19) for the plus sign becomes a set of $N-1$ equations [4],

$$D^a_- \equiv \partial_z + \frac{i}{2}(A^1_z + \frac{1}{\sqrt{3}}A^2_z + \cdots + \sqrt{\frac{2}{a(a-1)}}A^{a-1}_z + \sqrt{\frac{2(a+1)}{a}}A^a_z), \tag{2.20}$$

$$D^a_-\bar{\psi}_a = 0. \tag{2.21}$$

Similarly, for the minus sign, we have

$$D^a_+ \equiv \partial_z - \frac{i}{2}(A^1_z + \frac{1}{\sqrt{3}}A^2_z + \cdots + \sqrt{\frac{2}{a(a-1)}}A^{a-1}_z + \sqrt{\frac{2(a+1)}{a}}A^a_z), \tag{2.22}$$

$$D^a_+\psi_a = 0. \tag{2.23}$$

We concentrate on the plus sign from here on. With $\bar{\psi}_a = w_a \exp(i\phi_a)$, we find that (2.11), (2.14) and (2.21) produce the following new vortex-type equation:

$$\nabla^2 \log w_a + \epsilon_{ij}\partial_i\partial_j\phi_a = \sum_{i=1}^{a-1}\sqrt{\frac{1}{2i(i+1)}}\Gamma^i + \sqrt{\frac{a+1}{2a}}\Gamma^a, \tag{2.24}$$

where Γ^a is given by

$$\Gamma^a = \frac{V^b - Q^b}{\kappa^2}\left[\frac{2}{N}\delta^{ab} + d^{abc}Q^c - Q^aQ^b\right]. \tag{2.25}$$

We used the normalization: $\{\lambda^A, \lambda^B\} = (4/N)\delta^{AB}I + 2d^{ABC}\lambda^C$. Also the Bogomol'nyi potential (2.13) can be expressed by

$$W_H = \frac{1}{2\kappa^2}(V^a - Q^a)(V^b - Q^b)\left[\frac{2}{N}\delta^{ab} + d^{abc}Q^c - Q^aQ^b\right]. \tag{2.26}$$

Let us give an example in the case of $CP(1)$. With $w_1 = w, \phi_1 = \phi, V^1 = V$, the above potential becomes

$$W_H = \frac{1}{2\kappa^2}(V - Q^3)^2(1 - (Q^3)^2), \tag{2.27}$$

which is exactly the same as the potential in the $O(3)$ model [10]. Next, we find that (2.24) becomes

$$\nabla^2 \log w + \epsilon_{ij}\partial_i\partial_j\phi = \frac{1}{\kappa^2}\left[V - \frac{1-w^2}{1+w^2}\right]\left[1 - \left(\frac{1-w^2}{1+w^2}\right)^2\right] \tag{2.28}$$

A detailed numerical study of the above equation showed that the equation has various kinds of rotationally symmetric solitons solutions connected with symmetric and broken phases, and they are anyons carrying fractional angular momentum [10]. Similar results are expected in the more complicated higher $CP(N)$ case, but a detailed study will be addressed elsewhere.

3. NON-COMPACT $SU(1,1)$ SOLITON

In this section, we consider a non-compact HSS with $\epsilon = -1$. We restrict to the $SU(1,1)$ group with $\eta_{AB} = (-,-,+)$. The target space is given by the two-sheeted hyperboloid $H = SU(1,1)/U(1)$. Using the expression for the group element g of (1.4) given by

$$g = \frac{1}{\sqrt{1-|\psi|^2}}\begin{pmatrix} 1 & \psi^* \\ \psi & 1 \end{pmatrix} \tag{3.1}$$

which satisfies $gMg^\dagger = M$ with $M = \mathrm{diag}(1,-1)$, we have (with $K = i\sigma^3/2$)

$$Q = \frac{i}{2(1-|\psi|^2)}\begin{pmatrix} 1+|\psi|^2 & -2\psi^* \\ 2\psi & -(1+|\psi|^2) \end{pmatrix}. \tag{3.2}$$

We restrict to $|\psi| < 1$, which corresponds to the upper sheet of $\mathcal{M} = SU(1,1)/U(1)$. A couple of remarks at this point concerning the ungauged case are in order. First, some soliton solutions associated with non-compact NLSM were discussed in connection with the Ernst equation [11], which are not self-dual. Secondly, using the above expression, one can check that there actually exist self-dual soliton solutions which are analytic or anti-analytic as in the compact case [12], but the energy and topological charge diverge at the boundary $|\psi| = 1$. Coupling with CS gauge field greatly improves the situation, because the gauge field effectively provides a potential barrier to the boundary (see (3.7)) and prevents the system from diverging.

Again, with the parameterization $\bar{\psi} = w\exp(i\phi)$, we find the Bogomol'nyi potential (2.13) and the self-dual equation (2.24) are produced as follows:

$$W_H = \frac{1}{2\kappa^2}\left[V - \left(\frac{1+w^2}{1-w^2}\right)\right]^2 \left[\left(\frac{1+w^2}{1-w^2}\right)^2 - 1\right] \tag{3.3}$$

$$\nabla^2 \log w + \epsilon_{ij}\partial_i\partial_j\phi = \frac{1}{\kappa^2}\left[V - \frac{1+w^2}{1-w^2}\right]\left[\left(\frac{1+w^2}{1-w^2}\right)^2 - 1\right]. \tag{3.4}$$

Let us look for the rotationally symmetric solutions with the ansatz in the cylindrical coordinate (r,θ) given by

$$w = \tanh\frac{f(r)}{2},\ \phi = n\theta,\ A_i = \frac{\epsilon_{ji}x_j}{r^2}a(r). \tag{3.5}$$

Then, the Gauss's law and self-dual equation become $(' = d/dr)$

$$\begin{aligned} a'(r) &= (r/k^2)(-V + \cosh f(r))(1 - \cosh^2 f(r)), \\ rf'(r) &= (a(r) - n)\sinh f(r). \end{aligned} \tag{3.6}$$

Now, the combined equation of motion in (3.4) becomes an analogue of the one dimensional Newton's equation for $r > 0$, if we regard r as "time" and $u(r) \equiv \log\tanh\frac{f(r)}{2}$ as the "position" of the hypothetical particle with unit mass under a time-dependent friction, $(1/r)u'$, and an effective potential V_{eff}:

$$V_{eff}(u) = \frac{1}{2\kappa^2}\coth^2 u + \frac{V}{\kappa^2}\coth u. \tag{3.7}$$

The exerting force also includes an impact term at $r = 0$ due to $\epsilon^{ij}\partial_i\partial_j\phi = \frac{n}{r}\delta(r)$ in (3.4).

The inspection of the effective potential suggests that solitons are basically of two types; the non-topological vortices with $n \neq 0$ (negative integer) and the non-topological solitons with $n = 0$. In the former case, the "particle" starts from $u = -\infty$, reaches a turning point where it stops, changes the direction, and finally rolls down to $u = -\infty$. In the latter, the "particle" starting at some finite position, either rolls down to $u = -\infty$ directly, or moves to a turning point, changes the direction, and rolls down to $u = -\infty$. Let us look at the solutions more closely. Near $r = 0$, the condition for A_i to be non-singular forces $a(0) = 0$. First, when $n \neq 0$, we must have $f(0) = 0$. When $n = 0$, $\alpha \equiv f(0)$ can be arbitrary. The behavior of the solution near $r = \infty$ can be also read off from the conditions $f'(\infty) = a'(\infty) = 0$; $\beta \equiv f(\infty) = 0$ for arbitrary $\gamma \equiv a(\infty)$ and V. Putting $f(r) = f_\infty r^l$, $a(r) = \gamma + a_\infty r^s$ $(l, s < 0)$ near $r = \infty$, we find $l = \gamma - n, s = 2\gamma + 2 - 2n$ for $V \neq 1$. Since $l, s < 0$, we have consistency condition $\gamma < n - 1$. When $V = 1$, we have $l = \gamma - n, s = 4\gamma + 2 - 4n$ and $\gamma < n - \frac{1}{2}$. When

$\beta = \cosh^{-1} V$ for $V > 1$, γ must be equal to n ($\neq 0$). This solution which will show oscillatory behavior before it comes to rest does not exit. Near $r = \infty$, we assume an exponential approach $f(r) = \beta + f_\infty r^l e^{-ar}$, $a(r) = \gamma + a_\infty r^s e^{-br}$ $(a, b > 0)$. Then substitution leads to a contradictory output, $l = s$ and $l = s + 1$. Power law approach with $a, b = 0$ and $l, s < 0$ also leads to a contradiction. In view of the Bogomol'nyi potential (3.3), this excludes any solitons in the broken vacuum with $V = \cosh f(\infty)$, and all the solitons are in the symmetric phases.

Let us focus on the vicinity of $r = 0$. (a) $V \leq 1$, in which the effective potential (3.7) is a monotonically decreasing function. (a-i) $n \neq 0$; Trying power solutions of the form $f(r) = f_0 r^p$, $a(r) = a_0 r^q$ $(p, q > 0)$, we find $p = -n$, $q = 2 - 2n$ for $V < 1$. Hence n must be a negative integer. When $V = 1$, we have $p = -n$, $q = 2 - 4n$. (a-ii) $n = 0, \alpha \neq 0$; Let us try $f(r) = \alpha + f_0 r^p$, $a(r) = a_0 r^q$ $(p, q > 0)$. We find $p = q = 2$. We note that both a_0 and f_0 turns out to be negative, so that the solution rolls down to $r = \infty$. Climbing up at first and then rolling down the hill solution does not exist. When (b) $V > 1$, the effective potential (3.7) develops a pool with a local minimum at $f_m = \cosh^{-1} V$. (b-i) $n \neq 0$; The behavior is similar to (a-i) except the fact $a(r)$ passes the minimum at r_m twice in the process of climbing up, passing the turning point, and rolling down the hill to its original position. (b-ii) $n = 0, \alpha \neq 0$; There are two cases. When $\alpha < \cosh^{-1} V$, the solution, if exists, will behave similarly with (b-i) except that it starts at some finite point α. However, it cannot exist for the following reason; The initial "velocity" of the particle is given by $u'(0) \propto f'(0) = 0$ ($f'(r) \propto r$ from (a-ii)). Hence the particle does not carry enough kinetic energy to return to its starting point in this dissipative system with conservative potential. Note that when $n \neq 0$, even though the initial velocity is in general equal to 0 except $n = -1$, the solutions are possible because of the impact term at $r = 0$. In the opposite case $\alpha > \cosh^{-1} V$, it is similar to (a-ii) and only rolling down the hill is permitted. A detailed numerical study indeed confirms the existence of these solitons and is presented in Fig.1 and Fig.2.

Note that there does not exist any topological lump solutions, because $\pi_2(\mathcal{M}) = 0$. And the topological vortices does not exist, because there is no bump in the effective potential where the particle can stop at the top. In the solutions, the magnetic flux is given by $\Phi = 2\pi\gamma$, and the energy is saturated by the topological charge; $E = 4\pi|T| = 2\pi|\gamma(1 - V)|$. The system also carry non-vanishing angular momentum. Let us define

$$J = \int d^2x \epsilon_{ij} x_i D_0 Q^A D_j Q^B \eta_{AB}. \tag{3.8}$$

A simple calculation using the Gauss's Law (2.11) (with plus sign in the right hand side due to the ϵ factor), and self-dual equations (2.14) and (3.6), we find $J = \pi\kappa((\gamma - n)^2 - n^2)$. Thus the solitons in general carry a fractional angular momentum, representing anyons. For the non-topological solitons, it is simply $J = \pi\kappa\gamma^2$.

4. CONCLUSION

We showed that the coadjoint orbit approach for the relativistic NLSM coupled with CS gauge field leads to a class of new self-dual field theories on the target space of HSS which contain the previous $O(3)$ and $CP(N-1)$ models, and a new non-compact $SU(1,1)$ model. We also found an explicit expression of the Bogomol'nyi potential when the maximal torus subgroup is gauged, and showed that the non-compact NLSM admits self-dual soliton solutions which are saturated by the Bogomol'nyi bound, and gave a complete description of the rotationally symmetric solutions.

We also studied 1+1 dimensional NLSM on HSS coupled with Yang-Mills field, and found that this model also allows self-dual Bogomol'nyi soliton solutions [13]. In this case, the first order self-dual equation is given by the non-local integral equation as in the case of the self-dual Bogomol'nyi formulation of the nonlinear Schrödinger equation [14], and the soliton equations turn out to be the same as the dimensionally reduced vortex equations of Chern-Simons gauged NLSM on HSS [4].

There remains several further issues to be discussed. Firstly, note that the identity (1.6) and its gauged version on HSS is essential for the existence of self-duality. In this respect, it would be an intriguing problem to extend the above formalism to other non HSS coadjoint orbits, and also to higher non-compact group. Quantization of the model is another problem to be addressed. Secondly, it would be interesting to see whether there exists a well-defined procedure in which the non-relativistic NLSM of the generalized CS Heisenberg ferromagnet system defined on the coadjoint orbits [4] could emerge as a non-relativistic limit of the present relativistic NLSM. This will require in the course a revelation of the connection between the symplectic structure of HSS [4] for the non-relativistic NLSM and phase space structure of relativistic NLSM.

I thank D. Chae, Y. Kim, K. Kimm, K. Lee, Q-H. Park, and C. Rim for useful discussions, Sung-Soo Kim for his invaluable help, and especially Prof. D. Chae for his kind hospitality at "1st International Workshop on Nonlinear Field Theory". This work is supported in part by the Korea Science and Engineering Foundation through the project number (95-0702-04-01-3), and by the Ministry of Education through the Research Institute for Basic Science (BSRI/97-1419).

References

[1] P. Oh and Q-H. Park, *Phys. Lett. B* 383, 333 (1996).

[2] A. P. Fordy and P. P. Kulish, *Commum. Math. Phys.* 89, 427 (1983).

[3] P. Oh, *J. Phys. A: Math. Gen* 31, L325-L330 (1998).

[4] P. Oh and Q-H. Park, *Phys. Lett. B* 400, 157 (1997); (E) 416, 452 (1998).

[5] Y. Kim, P. Oh, and C. Rim, *Mod. Phys. Lett. A* 12, 3169 (1997).

[6] S.-S. Kim and P. Oh, hepth/9805010, *Int. J. Mod. Phys. A*, in press.

[7] G. Nardelli, *Phys. Rev. Lett.* 73, 2524 (1994); B. J. Schroers, *Phys. Lett. B* 356, 291 (1995); J. Gladikowski, B. M. A. G. Piette and B. J. Schroers, *Phys. Rev. D* 53, 844 (1996); K. Kimm, K. Lee and T. Lee, *Phys. Rev. D* 53, 4436 (1996); K. Arthur, D. H. Tchrakian and Y. Yang, *Phys. Rev. D* 54, 5245 (1996); P. K. Ghosh and S. K. Ghosh, *Phys. Lett. B* 366, 199 (1996); Y. M. Cho and K. Kimm, *Phys. Rev. D* 52, 7325 (1995); K. Kimm, K. Lee and T. Lee, *Phys. Lett. B* 380, 303 (1996); P. K. Ghosh, *Phys. Lett. B* 384, 185 (1996).

[8] We will only consider the simplest non-compact case $SU(1,1)$. Here, $\epsilon = -1$ is necessary to render the energy to be non-negative in the representation in which the element K is anti-Hermitian. Higher orbits of non-compact group needs more careful treatment.

[9] A. M. Perelomov, *Phys. Rep.* 146, 135 (1987).

[10] K. Kimm, K. Lee and T. Lee, *Phys. Lett. B* 380, 303 (1996).

[11] S. Takeno, *Progr. Theor. Phys.* 66, 1250 (1981); J. Gruszczak, *J. Phys. A: Math. Gen.* 14, 3247 (1981).

[12] A. A. Belavin and A. M. Polyakov, *JETP Lett.* 22, 245 (1975).

[13] P. Oh, unpublished.

[14] P. Oh and C. Rim, *Phys. Lett. B* 404, 89 (1997).

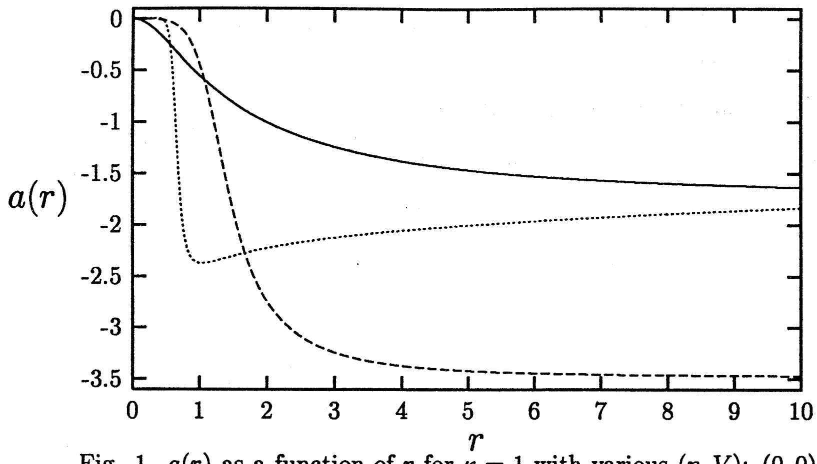

Fig. 1. $a(r)$ as a function of r for $\kappa = 1$ with various (n, V): $(0, 0)$ for solid line, $(-1, 0)$ for dashed line, and $(-1, 1.5)$ for dotted line.

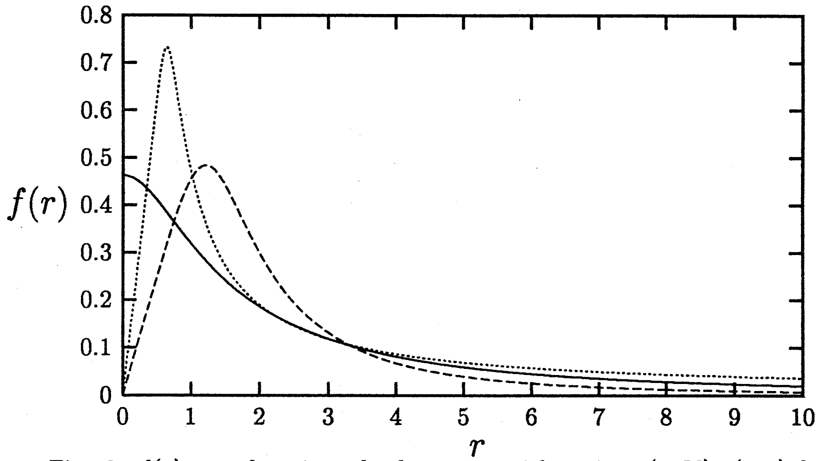

Fig. 2. $f(r)$ as a function of r for $\kappa = 1$ with various (n, V): $(0, 0)$ for solid line, $(-1, 0)$ for dashed line, and $(-1, 1.5)$ for dotted line.

Vortex Solutions of a Fermion Maxwell-Chern-Simons Theory*

Jae Hyung Yee
Department of Physics and Institute for Mathematical Sciences
Yonsei University
Seoul 120-749, Korea

July 3, 1998

Abstract

We explain how static multi-vortex solutions arise in non-linear field theories, by taking the non-linear Schrödinger equation coupled to Chern-Simons field (Jackiw-Pi model) and a fermion Chern-Simons theory as simple examples. We then construct a fermion Maxwell-Chern-Simons theory which has consistent static field equations, and show that it has the same vortex solutions as the Jackiw-Pi model, but gives rise to quite different vortex dynamics.

*To appear in the proceedings of "International Workshop on Mathematical and Physical Aspacts of Nonlinear Field Theories", 1998.

I. Introduction

Magnetic vortices are found to appear in high temperature superconductors as quantized vortex lines in the range of magnetic field between 10^{-2} tesla and 10^2 tesla [1]. It is known that the behavior of vortices dominates many physical properties of high temperature superconductors in this range of magnetic field. These vortices show very interesting dynamical properties:in some range of magnetic field and temperature they form vortex lattice or behave as liquid. This phenomenon poses an important theoretical challenge in understanding the dynamics of magnetic vortices.

One way to understand the dynamics of vortices is to consider non-linear field theories that possess classical vortex solutions and to study the dynamics of these solutions in the field theoretic framework. There exist many field theoretic models that support the static vortex solutions. Relativistic models of vortices include the Abelian Higgs model of Nielson and Olesen [2], the scalar Chern-Simons theory of Hong, Kim and Pac, and Jackiw and Weinberg [3], and the fermion Chern-Simons thoery of Lie and Bhaduri [4]. There also exist many non-relativistic models including the Ginzburg-Landau model [5] which is the non-relativistic limit of the Abelian Higgs model, Jackiw-Pi model [6] which is the non-relativistic limit of the scalar Chern-Simons theory, and non-relativistic spinor Chern-Simons theories [7].

The most of these models are not soluble in closed form except for the Jackiw-Pi model and some of its generalizations. For the study of vortex dynamics in the field theoretic framework, however, it would be more convenient to have completely soluble models. We therefore consider the field theoretic models which are completely soluble.

In the next section we give a brief review of the Jackiw-Pi model to show how to find the static self-dual solutions. In section III, the simplest fermion Chern-Simons theory which supports the static vortex solutions as solutions of the Liouville equations is presented. In section IV, we present a fermion Maxwell-Chern-Simons theory which supports static vortex solutions. We show that although the static solutions are the same as those of Jackiw-Pi model and the simplest fermion Chern-Simons theory, this model gives rise to quite different moduli space dynamics from those simple models. We conclude with some discussions in the last section.

II. Non-linear Schrödinger Equation coupled to Chern-Simons Gauge Field

As a simplest model field theory that possesses vortex solutions we consider the non-linear Schrödinger field theory coupled to a Chern-Simons gauge field described by the Lagrangian [6],

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_\gamma + i\phi^* D_0\phi - \frac{1}{2}|\vec{D}\phi|^2 + \frac{1}{2\kappa}(\phi^*\phi)^2, \tag{1}$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ A_\mu &= (A^0, -\vec{A}) \\ D_0 &= \partial_0 + iA^0, \qquad \vec{D} = \vec{\nabla} - i\vec{A}. \end{aligned}$$

One way to see the existence of static solutions is to consider the energy functional and to find the field configurations that minimize the energy functional [5]. To this end it is convenient to write the Lagrangian (1) as

$$\mathcal{L} = -\frac{\kappa}{2}\epsilon_{ij}A^i\dot{A}^j + i\phi^*\dot{\phi} - A^0(\phi^*\phi - \kappa F_{12}) - \frac{1}{2}|\vec{D}\phi|^2 + \frac{1}{2\kappa}(\phi^*\phi)^2, \tag{2}$$

where indices i and j run for the spatial components 1 and 2. The momentum conjugates to the field variables A^i and ϕ are defined by

$$\begin{aligned} \pi_i &= \frac{\delta\mathcal{L}}{\delta\dot{A}^i} = \frac{\kappa}{2}\epsilon_{ij}A^j \\ P &= \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = i\phi^*, \end{aligned} \tag{3}$$

respectively. The Lagrangian (2) can then be written as

$$\mathcal{L} = \pi_i\dot{A}^i + P\dot{\phi} - \mathcal{H} - A^0(\phi^*\phi - \kappa F_{12}), \tag{4}$$

where $\mathcal{H}$ is the Hamiltonian density and the last term represents the Gauss' law constraint,

$$B = \vec{\nabla}\times\vec{A} = -F_{12} = -\frac{1}{\kappa}\phi^*\phi. \tag{5}$$

The Gauss' law constraint (5) shows that the magnetic field is proportional to the charge density of the scalar field,

$$\rho = \phi^*\phi. \tag{6}$$

¿From Eqs. (2) and (4) one finds that the Hamiltonian density of the system is given by

$$\mathcal{H} = \frac{1}{2}|\vec{D}\phi|^2 - \frac{1}{2\kappa}(\phi^*\phi)^2. \tag{7}$$

Using the identity,

$$|\vec{D}\phi|^2 = |(D_1 - iD_2)\phi|^2 - (B\rho + \vec{\nabla}\times\vec{J}), \tag{8}$$

where $\vec{J} = Im\phi^* \vec{D}\phi$ is the current density, one can write the Hamiltonian density as

$$\mathcal{H} = \frac{1}{2}|(D_1 - iD_2)\phi|^2 - \frac{1}{2}\vec{\nabla} \times \vec{J}. \tag{9}$$

The energy of the system then becomes

$$E = \int d^2x \mathcal{H} = \frac{1}{2}\int d^2x \big[|(D_1 - iD_2)\phi|^2 - \vec{\nabla} \times \vec{J}\big]. \tag{10}$$

If the fields are well-behavecd at infinity, then the last term does not contribute to the energy since it is a surface term. Then the energy of the system is positive definite:

$$E = \frac{1}{2}\int d^2x |(D_1 - iD_2)\phi|^2 \geq 0, \tag{11}$$

where the equality is called the Bogomol'nyi bound. Note that the coupling constant of the quartic interaction of the scalar field is so chosen that the energy functional is positive definite.

Eq.(11) shows that the minimum energy configurations of the system are determined by the static first-order differential equation,

$$(D_1 - iD_2)\phi = 0, \tag{12}$$

which is called the self-dual equation.

To find the static soliton solutions one has to solve the self-dual equation (12) together with the Gauss' law constraint (5). To do this we write the scalar field as

$$\phi = \rho^{\frac{1}{2}} e^{i\omega}. \tag{13}$$

Substituting (13) into (12) one finds

$$A_- = -i\phi^{-1}\partial_-\phi = -\frac{i}{2}\partial_- \ln\rho + \partial_-\omega, \tag{14}$$

or

$$\vec{A} = \vec{\nabla}\omega - \frac{1}{2}\vec{\nabla} \times \ln\rho, \tag{15}$$

where $A_\pm = A_1 \pm iA_2$ and $\partial_\pm = \partial_1 \pm i\partial_2$. This shows that the gauge field $\vec{A}$ is completely determined by the scalar field. ¿From (14) we obtain

$$\partial_- A_+ - \partial_+ A_- = 2i\vec{\nabla} \times \vec{A} = 2iB = i\vec{\nabla}^2 \ln\rho, \tag{16}$$

which, upon substituting into the Gauss' law constraint, reduces to the Liouville equation,

$$\nabla^2 \ln\rho = -\frac{2}{\kappa}\rho, \quad \kappa > 0, \tag{17}$$

which is completely integrable.

The simplest solution of the Liouville equation (17) is the spherically symmetric solution,

$$\rho(\vec{r}) = \frac{4\kappa N^2}{r^2}\Big[\big(\frac{r_0}{r}\big)^N + \big(\frac{r}{r_0}\big)^N\Big]^{-2}, \tag{18}$$

where r_0 and N are the constants representing the scale and the flux number of the soliton, respectively. To find the restriction on the number N, we observe that the regularity of the solution at the origin and at infinity requires $N \geq 1$. And the single-valuedness of the scalar field ϕ requires N to be an integer. Note that the regularity of the gauge field at the origin,

$$A_i(\vec{r}) \stackrel{r\to 0}{\longrightarrow} \partial_i\omega - \frac{1}{2}\epsilon_{ij}\frac{r^j}{r^2}(N-1), \tag{19}$$

determines the function ω:

$$\omega = (1-N)\theta. \tag{20}$$

We thus find the static solution,

$$\phi(\vec{r}) = \frac{2\sqrt{\kappa}N}{r}\Big[\big(\frac{r_0}{r}\big)^N + \big(\frac{r}{r_0}\big)^N\Big]^{-1} e^{i(1-N)\theta}, \tag{21}$$

with $\vec{A}$ determined by Eq.(15)

¿From the explicit solution (21) one finds the magnetic flux of the soliton solution,

$$\Phi = \int d^2xB = -\frac{1}{\kappa}\int d^2x\rho = -2\pi(2N), \tag{22}$$

the electric charge,

$$Q = -\kappa\Phi, \tag{23}$$

and the angular momentum,

$$J = 2\pi\kappa(2N). \tag{24}$$

This shows that the solution (18) and (21) represents the charged vortex solution with quantized magnetic flux.

The solution (18) represents the case of N solitons superimposed at the origin. The general solution of the Liouville equation is also known:

$$\rho(\vec{r}) = \frac{4\kappa|f'(z)|^2}{(1+|f(z)|^2)^2}, \tag{25}$$

where $z = re^{i\theta}$ and $f(z)$ is an arbitrary function of z such that ρ is non-singular. $f(z)$ for the most general solution can be written as

$$f(z) = \sum_{i=1}^{N}\frac{c_i}{z-z_i}, \tag{26}$$

where c_i and z_i are arbitrary constants. This solution contains $4N$ arbitrary parameters which represent $2N$ position parameters, N scale parameters and N phase variables of the solitons. In fact, index theorem confirms that $(4N)$ is the maximal number of parameters contained in the general solution [8] .

The Jackiw-Pi model described above is invariant under the Galilean transformation, and is the non-relativistic limit of the relativistic scalar Chern-Simons theory [3] ,

$$\mathcal{L}_{rel} = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_\gamma + D_\mu\phi D^\mu\phi^* - \frac{1}{\kappa^2}|\phi|^2(|\phi|^2 - \kappa)^2. \tag{27}$$

This theory possesses two types of soliton solutions: the topological solitons in the symmetry broken sector, and the non-topological solitons in the symmetric sector. The magnetic flux of the topological solitons is quantized as

$$\Phi = 2\pi(N-1), \tag{28}$$

while the magnetic flux of the non-topological solitons is not quantized:

$$\Phi = 2\pi(N+\alpha), \tag{29}$$

where α is an arbitrary parameter such that $\alpha \geq N$.

The non-relativistic limit of the model (27) in the symmetric sector reduces to the Jackiw-Pi model. Why then the non-relativistic solitons have quantized flux while magnetic flux of the relativistic case is not quantized? The answer lies in the fact that the Jackiw-Pi model has inversion symmetry which is not respected by the relativistic model [9]. To see this note that the charge density satisfies the equation,

$$\nabla^2 \ln\rho = \begin{cases} \frac{4m^2}{\kappa}\rho(1-\frac{\rho}{\kappa}), & \text{relativistic case} \\ -\frac{2}{\kappa}\rho, & \text{non-relativistic case.} \end{cases} \tag{30}$$

One can easily show that the Liouville equation is invariant under the inversion transformation:

$$\begin{aligned} r \to \frac{1}{r}, \qquad \theta \to \theta \\ \rho(r) \to \rho(\frac{1}{r}) = r^4\rho(r), \end{aligned} \tag{31}$$

while the relativistic equation is not. The charge density for both models behaves as

$$\rho(r) \longrightarrow \begin{cases} r^{2(N-1)}, & \text{as} \quad r \to 0 \\ r^{-2(\alpha+1)}, & \text{as} \quad r \to \infty, \end{cases} \tag{32}$$

and thus the magnetic flux is given by

$$-\Phi = \int_{r\to\infty} \vec{A}\cdot d\vec{l}$$

$$
\begin{aligned}
&= \int_{r\to\infty} (-\vec{\nabla}\omega + \frac{1}{2}\vec{\nabla} \times \ln \rho) \cdot dl \\
&= 2\pi(n-1) + 2\pi(\alpha+1).
\end{aligned} \tag{33}
$$

Due to the inversion symmetry (31) of the Jackiw-Pi model, the behavior of $\rho(\vec{r})$ at the origin and at infinity (32) must be related. This give rises to the relation $\alpha = N$, and explains the quantization of flux numbers in the case of the non-relativistic theory.

III. Vortex Solutions in a Fermionic Chern-Simons Theory

Vortex phenomena in realistic systems are known to be dominated by the electronic structure of the systems. Thus in attempting to understand the vortex dynamics it would be preferrable to use the fermionic field theories that support the static vortex solutions. The simplest fermionic Chern-Simons theory that supports the vortex solutions is the one proposed by Li and Bhaduri [4], described by the Lagrangian,

$$
\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + eA_\mu j^\mu, \tag{34}
$$

where ψ is a two-component spinor field, the Dirac matrices are chosen to be

$$
\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2 \tag{35}
$$

in terms of the Pauli matrices σ^i , and

$$
j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \bar{\psi} = \psi^\dagger\gamma^0. \tag{36}
$$

We will denote the charge density as $j^0 = \psi^\dagger\psi = \rho$.

To find the static vortex solutions in this system, it is simpler to start from the field equations,

$$
\begin{aligned}
\gamma^\mu(i\partial_\mu + eA_\mu)\psi - m\psi = 0 \ , \\
\frac{\kappa}{2}\epsilon^{\mu\nu\alpha}F_{\nu\alpha} = -ej^\mu \ .
\end{aligned} \tag{37}
$$

The second equation of (37) can be decomposed into the Gauss' law constraint,

$$
B = F_{12} = \frac{e}{\kappa}\rho, \tag{38}
$$

and

$$
E^i = F_{0i} = \frac{e}{\kappa}\epsilon^{ij}j^j. \tag{39}
$$

By writing

$$\psi = \begin{pmatrix} \psi_+(\vec{x},t) \\ \psi_-(\vec{x},t) \end{pmatrix}, \tag{40}$$

the fermion field equation, the first of (37), reduces to the two coupled equations for $\psi_+(\vec{x},t)$ and $\psi_-(\vec{x},t)$:

$$\begin{aligned} (i\partial_0 - m)\psi_+(\vec{x},t) &= (D_1 - iD_2)\psi_-(\vec{x},t) \\ -(i\partial_0 + m)\psi_-(\vec{x},t) &= (D_1 + iD_2)\psi_+(\vec{x},t), \end{aligned} \tag{41}$$

where $D_i = \partial_i - ieA^i$.

We now seek the stationary solution of the form,

$$\psi = \psi_+(\vec{x}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iE_f t}, \tag{42}$$

where E_f is a constant. Then the fermion field equation reduces to

$$(E_f - m)\psi_+(\vec{x}) = 0, \tag{43}$$

which determines the constant $E_f = m$, and the self-dual equation,

$$(D_1 + iD_2)\psi_+(\vec{x}) = 0. \tag{44}$$

Note that this is the same type of self-dual equation as that of the Jackiw-Pi model.

If we take ,

$$\psi = \psi_-(\vec{x}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iE_f t}, \tag{45}$$

instead, the fermion field equation reduces to

$$(E_f + m)\psi_-(\vec{x}) = 0 \tag{46}$$

which determines $E_f = -m$ and the self-dual equation,

$$(D_1 - iD_2)\psi_-(\vec{x}) = 0. \tag{47}$$

To find the static solutions we choose the gauge, $A^0 = 0$, and take $\vec{A}$ to be static. Then the both sides of Eq.(39) consistently vanish, and the spatial components of matter current vanish for spinor fields (42) and (45), since γ^i's are off-diagonal. Thus for the upper component spinor field (42), the field equations reduce to the self-dual equation(44) and the Gauss' law constraint (38). If we write the spinor field as

$$\psi_+(\vec{x}) = \rho^{\frac{1}{2}} e^{i\omega}, \tag{48}$$

Eq.(44) reduces to

$$eA_+ = -\frac{i}{2}\partial_+ \ln\rho + \partial_+\omega, \tag{49}$$

which determines the gauge field in terms of the matter fields:

$$\vec{A} = \vec{\nabla}\omega - \frac{1}{2}\vec{\nabla} \times \ln \rho. \tag{50}$$

¿From Eq.(49) together with the Gauss' law constraint (38), one finds that the matter charge density satisfies the Liouville equation:

$$\nabla^2 \ln \rho = -\frac{2e^2}{\kappa}\rho. \tag{51}$$

Thus if we take the upper-component for spinor field (42), $\kappa > 0$ is required in order to have non-singular positive charge density ρ.

If we take the lower-component for the matter field, Eq(45), on the other hand, the field equations reduce to

$$\nabla^2 \ln \rho = \frac{2e^2}{\kappa}\rho, \tag{52}$$

where κ must be negative for the regularity of the charge density ρ.

Since the matter charge density ρ satisfies the same Liouville equation (51) and (52) as that of Jackiw-Pi model, we find the same structure of the static vortex solutions except that the energy of the solution is now given by

$$E = \pm m \int d^2x \rho_{\pm}, \tag{53}$$

where $\rho_{\pm} = {\psi_{\pm}}^{\dagger}(\vec{x})\psi_{\pm}(\vec{x})$ for the upper and lower component matter fields, respectively. We thus find that the fermion Chern-Simons theory (34) supports the finite energy static vortex solutions with quantized magnetic flux and charge.

In general field theories with Chern-Simons term, parity is known to be violated by the Chern-Simons term. One can restore the parity invariance by introducing appropriate parity partners for each field in the theory. A parity invariant fermion Chern-Simons theory is proposed by Hagen [10]. This parity invariant fermion Chern-Simons theory also supports the static vortex solutions as solutions of Liouville equation [11].

IV. Vortex Solutions of a Fermion Maxwell-Chern-Simons Theory

To find a field theoretic model which supports the vortex solutions with physically more interesting properties, we introduce a Maxwell term to the gauge field part of the Lagrangian and a couple of new interaction terms [12]:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_{\gamma} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi + eA_{\mu}(J^{\mu} + lG^{\mu}) + \frac{1}{2}g(\bar{\psi}\psi)^2, \tag{54}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, J^\mu = \bar{\psi}\gamma^\mu\psi$ and the new current G^μ is defined by

$$G^\mu = \epsilon^{\mu\nu\rho}\partial_\nu J_\rho. \tag{55}$$

The new gauge coupling term can be written as

$$\begin{aligned} A_\mu G^\mu &= \partial_\mu A_\nu \epsilon^{\mu\nu\rho} J_\rho + \text{surface term} \\ &= F_{\mu\nu}\epsilon^{\mu\nu\rho} J_\rho + \text{surface term}, \end{aligned} \tag{56}$$

which is the magnetic moment coupling in 3-dimensional space-time. This is the reason why this new coupling is called an anomalous magnetic moment coupling.

If the gauge field part of the Lagrangian (54) is eliminated, then the theory reduces to the 3-dimensional Gross-Neveu model. If we let $l \to 0, g \to 0$ and $e^2 \to \infty$ with $\frac{\kappa}{e^2}$ fixed, the theory becomes the fermion Chern-Simons theory discussed in the last section.

Field equations of the new theory are

$$\begin{aligned} \partial_\nu F^{\mu\nu} + \tfrac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} &= -e(J^\mu + lG^\mu) \\ \gamma^\mu(i\partial_\mu + eA_\mu)\psi - m\psi + g(\bar{\psi}\psi)\psi - el\epsilon^{\mu\nu\rho}(\partial_\nu A_\mu)\gamma_\rho\psi &= 0, \end{aligned} \tag{57}$$

the first of which is the field equation for a massive gauge field in 3-dimensions.

To find static solutions we choose the gauge, $A^0 = 0$, and take $\vec{A}$ to be static. As in the last section we write the spinor field in component form,

$$\psi = \begin{pmatrix} \psi_+(\vec{x}) \\ \psi_-(\vec{x}) \end{pmatrix} e^{-iE_f t} \tag{58}$$

where E_f is a constant. Then the spinor field equation reduces to the coupled equations,

$$\begin{aligned} [E_f - m + g(|\psi_+|^2 - |\psi_-|^2) - el\epsilon^{ij}\partial_j A_i]\psi_+ &= D_-\psi_- \\ [-E_f - m + g(|\psi_+|^2 - |\psi_-|^2) + el\epsilon^{ij}\partial_j A_i]\psi_- &= D_+\psi_+, \end{aligned} \tag{59}$$

where $D_\pm = D_1 \pm iD_2$, and $D_i = \partial_i - ieA^i$.

If we take the upper-component spinor field,

$$\psi = \psi_+(\vec{x}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iE_f t}, \tag{60}$$

then Eq.(59) reduces to

$$\begin{aligned} [E_f - m + g\rho_+ - el\epsilon^{ij}\partial_j A_i]\psi_+ &= 0 \\ D_+\psi_+ &= 0, \end{aligned} \tag{61}$$

where $\rho_+ = J_+{}^0 = |\psi_+|^2$ is the charge density of the matter field. For this choice of the spinor field, we find,

$$\begin{aligned} J_+{}^i &= \bar{\psi}_+\gamma^i\psi_+ = 0 \\ G^0 &= 0 \\ G^i &= \epsilon^{ij}\partial_j\rho_+, \end{aligned} \tag{62}$$

since γ^i's are off-diagonal matrices. Then the gauge field equation reduces to

$$\begin{aligned} \epsilon^{ij} F_{ij} &= -\frac{2e}{\kappa}\rho_+ \\ \partial_j F^{ij} &= elG^i = el\epsilon^{ij}\partial_j \rho_+. \end{aligned} \tag{63}$$

For these two equations to be consistent, the coupling constants l and κ must be related by

$$l = -\frac{1}{\kappa}. \tag{64}$$

This shows that, for the fermion Maxwell-Chern-Simons theory (54) to have consistent static field equations, one need to introduce the anomalous magnetic moment coupling term.

To find static solutions, therefore, one has to solve Eqs. (61) and (63), which now reduce to

$$\begin{aligned} (E_f - m + g\rho_+ + \frac{e}{\kappa}\epsilon^{ij}\partial_j A_i]\psi_+ &= 0 \\ D_+\psi_+ &= 0 \\ B = -F_{12} &= \frac{e}{\kappa}\rho. \end{aligned} \tag{65}$$

Note that, for the first equation of Eq.(66) to be consistent with the other two equations, one need to require,

$$\begin{aligned} E_f &= m \\ g &= -\frac{e^2}{\kappa^2}. \end{aligned} \tag{66}$$

In other words, in order for the theory (54) to have consistent static self-dual solutions, the quartic coupling constant of the matter field must be determined by the Chern-Simons coupling by (66) as in the case of the Jackiw-Pi model.

To solve the second equation of (66), we write

$$\psi_+ = \rho_+{}^{\frac{1}{2}} e^{i\omega}. \tag{67}$$

Then this equation determines the gauge field in terms of the matter field:

$$\begin{aligned} eA_+ &= -\frac{i}{2}\rho_+^{-1}\partial_+\rho_+ + \partial_+\omega \\ eA_- &= \frac{i}{2}\rho_+^{-1}\partial_-\rho_+ + \partial_-\omega, \end{aligned} \tag{68}$$

from which one finds,

$$\partial_- A_+ - \partial_+ A_- = 2iB = -\frac{i}{e}\partial_+\partial_- \ln \rho_+. \tag{69}$$

By substituting this equation to the Gauss' law constraint, the third equation of (66), one finds the Liouville equation for the matter charge density:

$$\nabla^2 \ln \rho_+ = -\frac{2e^2}{\kappa}\rho_+, \tag{70}$$

where κ must be positive for the regularity of the matter density. This shows that the fermion Maxwell-Chern-Simons theory (54) also supports the static vortex solutions as solutions of Liouville equation.

If we choose the lower component spinor field,

$$\psi = \psi_-(\vec{x}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iE_f t}, \tag{71}$$

the field equations reduce to

$$\begin{aligned} (E_f + m + g\rho_- + \frac{e}{\kappa}\epsilon^{ij}\partial_j A_i)\psi_- &= 0 \\ D_-\psi_- &= 0 \\ B &= \frac{e}{\kappa}\rho_-, \end{aligned} \tag{72}$$

where $\rho_- = \psi^\dagger{}_-\psi_-$. For these equations to be consistent, we also need to require,

$$\begin{aligned} E_f &= -m \\ g &= -\frac{e^2}{\kappa^2}. \end{aligned} \tag{73}$$

By writing $\psi_- = \rho_-{}^{\frac{1}{2}} e^{i\omega}$, we again obtain the Liouville equation for the matter density,

$$\nabla^2 \ln \rho_- = -\frac{2e^2}{\kappa}\rho_-, \tag{74}$$

where $\kappa < 0$ is required for the regularity of the density ρ_-.

We now consider the energy of the static vortex solutions. The Hamiltonian density is given by

$$\mathcal{H} = \pm m\rho_\pm + \frac{1}{2}F_{12}{}^2 - \frac{1}{2}g\rho_\pm{}^2 + \frac{e^2 l}{\kappa}\rho_\pm{}^2. \tag{75}$$

Due to the consistency requirements (64), (66), and (73), the last three terms of (75) cancel out, and the energy of the static solutions reduces to

$$E = \pm m \int d^2\rho_\pm = \frac{2\kappa}{e^2} m N_\pm, \tag{76}$$

where $2N_\pm$ denotes the flux number of the solutions. The cancellation of the quartic terms in the Hamiltonian density is also reminiscent of that in the Jackiw-Pi model.

One may wonder why one studies such a complicated model with an unusual interaction term if it gives the same static vortex solutions as those of the Jackiw-Pi model or the simple fermion Chern-Simons theory. Although the theory (54) gives the same static solutions as those of the Jackiw-Pi model, the moduli space dynamics of this model is quite different from the Jackiw-Pi model because of the Maxwell term in the Lagrangian. This may give more interesting, and hopefully more realistic vortex dynamics.

The theory we have described is a $U(1)$ gauge theory. This theory can be generalized to $SU(N)$ gauge theories as has been done by Jackiw and Pi [6] for the Jackiw-Pi model. There exist many possibilities to formulate corresponding $SU(N)$ gauge theories as a generalization of the fermion Maxwell-Chern-Simons theory. The simplest example is to take adjoint representation for the matter field and use the ansatz that the matter density and the gauge field A_μ lie in the Cartan subalgebra of the grhoup [13]. One then finds the equations satisfied by the components of matter density,

$$\nabla^2 \ln \rho_\alpha = \mp \frac{2e}{\kappa} \sum_{\beta=1}^{r} K_{\alpha\beta} \rho_\beta, \tag{77}$$

where $K_{\alpha\beta}$ is the Cartan matrix. Eq.(77) is not completely integrable in general, but can be integrated numerically [6]. For the $SU(2)$ case, Eq.(77) reduces to the Liouville equation, and it corresponds to an embeding of $U(1)$ in $SU(2)$.

V. Discussions

We have constructed a fermion Maxwell-Chern-Simons theory that possesses completely integrable multi-vortex solutions, and have shown that the solutions are the same as those of Jackiw-Pi model. Although the model gives the same static solutions as the Jackiw-Pi model, the moduli space dynamics of the static solutions will be quite different due to the Maxwell term in the Lagrangian. The Maxwell term is quadratic in the time-derivatives of gauge field, i.e., $\frac{1}{2}\dot{A}_i\dot{A}_i$, which gives rise to a non-trivial contribution to the moduli space metric, while the Chern-Simons term is linear in the time-derivatives of gauge field. This may give an interesting vortex dynamics and may be more relevant in understanding the vortex phenomena arising in the high temperature superconductors.

Similar construction can be done for more relatistic, non-relativistic fermion gauge theories. Some models have already been constructed and their static solutions have been studied [7]. These models provide simple starting point in studying more realistic vortex dynamics in the field theoretic framework.

The most well-known method of studying vortex dynamics is that of moduli space dynamics for slowly moving vortices proposed by Manton [14]. This method, however, has a difficulty in obtaining the moduli space metric for the theories with kinetic terms linear in time-derivatives, such as Chern-Simons theory, relativistic fermion theories and general non-relativistic field theories. This difficulty can be avoided by using the method proposed by Bak and Lee [15], where one integrates out the momentum variables to make the kinetic terms quadratic in time-derivatives of field variables. One can then use the Manton's method to obtain the correct moduli space metric.

Another way to study the vortex dynamics is the so-called string formulation of vortex dynamics [16]. The authors of ref.[16] note that at the centers of vortices the field function(scalar field ϕ in the Jackiw-Pi model and spinor field ψ in the fermionic models) vanishes, and treat the points of zeros in 3-dimensional space-time (or string of zeros in 4-dimensional space-time) as fundamental objects. They then write exact equations of motion for points(or strings) in terms of fields that surround them. These equations of motions are rather complicated and need approximation for practical applications.

As mentioned above, it is not a simple matter to use these methods to study vortex dynamics in the field theoretic framework. The field theoretic models discussed here have static vortex solutions in closed form, and it is relatively simple to apply these methods to understand the dynamics of magnetic vortices.

Acknowledgements

The author would like to thank Professor Dongho Chae, the organizer of the Workshop. This work was supported in part by the Korea Science and Engineering Foundation, under Grant No. 97-07-02-02-01-3 and 065-0200-001-2, the Center for Theoretical Physics(SNU), and the Basic Science Research Institute Program, Ministry of Education, under project No. BSRI-97-2425.

References

[1] G. W. Crabtree and D. R. Nelson, Physics Today, April 1997, pp38.

[2] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**,451(1973)

[3] J. Hong, Y. Kim and P. Y. Pac, Phys. Rev. Lett. **64**, 2230(1990)
R. Jackiw and E. Weinberg, Phys. Rev. Lett. **64**, 2234(1990)

[4] S. Li and R. K. Bhaduri, Phys. Rev. **D43**, 3573(1991)

[5] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449(1976)
H. de Vega and F. Schaposnik, Phys. Rev. **D14**, 1100(1976)
L. Jacobs and C. Rebbi, Phys. Rev. **B19**, 4486(1978)

[6] R. Jackiw and S. -Y. Pi, Phys. Rev. Lett. **64**, 2969(1990);
Phys. Rev. **D42**, 3500(1990);Prog. Theor. Phys. Suppl. **107**, 1(1992)

[7] C. Duval, P. A. Horváthy, and L. palla. Phys. Rev. **D52**,4700(1995)
Z. Németh, Phys. Rev. **D56**, 5066(1997)

[8] S. K. Kim, K, S, Soh and J. H. Yee, Phys. Rev. **D42**, 4139(1990)

[9] S. K. Kim, W. Namgung, K, S, Soh and J. H. Yee, Phys. Rev. **D46**, 1882(1992)

[10] C. R. Hagen, Phys. Rev. Lett. **68**, 3821(1992)

[11] J. Shin and J. H. Yee, Phys. Rev. **D50**, 4223(1994)

[12] S. Hyun, J. Shin, H,-j. Lee, Phys. Rev. **D55**, 3900(1997).
The computation of energy in this paper is in error. The total energy of static solutions should consist only of the mass term of the fermion field.

[13] H. -j. Lee, J. Y. Lee and J. H. Yee, preprint, hep-th/9802062

[14] N. S. Manton, Phys. Lett. **B110**, 54(1982); ibid **B154**, 397(1985)

[15] D. Bak and H. -j. Lee, preprint, hep-th/9706102.

[16] E. Schöder and O. Törnkvist, preprint, hep-th/9711195

APCTP-97-12, SNUTP-97-065

Monopoles in Electroweak Theory

Y. M. Cho

Asia Pacific Center for Theoretical Physics

and

Department of Physics, Seoul National University, Seoul 151-742, Korea

Kyoungtae Kimm

Department of Physics, Sung Kyun Kwan University, Suwon 440-746, Korea

Abstract

We show that the Weinberg-Salam model, with a simple modification of the 4-point coupling constant of the W-boson, can be made to allow finite energy monopole and dyon solutions. The existence of the new solutions is based on the fact that the Weinberg-Salam model can be interpreted as a gauge theory of $SU(2)_{\rm em}$ which is spontaneously broken by a Higgs triplet. Our result suggests that genuine electroweak monopole and dyon could exist whose mass scale is much smaller than the grand unification scale.

PACS Number(s): 14.80.Hv, 11.15.Tk, 12.15.-y

I. INTRODUCTION

Ever since Dirac [1] has introduced the concept of the magnetic monopole, the monopoles have remained a fascinating subject in theoretical physics. The Abelian monopole has been generalized to the non-Abelian gauge theory by Wu and Yang [2] who showed that the pure $SU(2)$ gauge theory allows a point-like monopole, and by 't Hooft and Polyakov [3] who have constructed a finite energy monopole solution in the Georgi-Glashow model as a topological soliton. In the interesting case of the electroweak theory of Weinberg and Salam, however, it has generally been believed that there exists no topological monopole of physical interest. The basis for this "non-existence theorem" is, of course, that with the spontaneous symmetry breaking the quotient space $SU(2) \times U(1)/U(1)_{\rm em}$ allows no non-trivial second homotopy. This has led many people to conclude that there is no topological structure in the Weinberg-Salam model which can accommodate a finite energy magnetic monopole.

This, however, is shown to be not true. Indeed recently Cho and Maison [4] have established that the Weinberg-Salam model has exactly the same topological structure as the Georgi-Glashow model, and demonstrated the existence of a new type of monopole and dyon solutions in the standard Weinberg-Salam model. *This was based on the observation that the Weinberg-Salam model, with the hypercharge $U(1)$, could be viewed as a gauged CP^1 model in which the (normalized) Higgs doublet plays the role of the CP^1 field.* So the Weinberg-Salam model does have exactly the same nontrivial second homotopy as the Georgi-Glashow model. Once this is understood one could proceed to construct the desired monopole and dyon solutions in the Weinberg-Salam model. Originally the solutions of Cho and Maison were obtained by a numerical integration, but now a mathematically rigorous existence proof has been established which supports the numerical results [5].

The monopole of Cho and Maison may be viewed as a hybrid between the Dirac monopole and the 't Hooft-Polyakov monopole, because it has a $U(1)$ point singularity at the center even though the $SU(2)$ part is completely regular. Consequently it carries an infinite energy so that at the classical level the mass of the monopole remains arbitrary. Of course there is nothing wrong with this as far as one wants to interpret the monopole as an elementary particle. Nevertheless one may wonder whether one can have an analytic electroweak monopole which has a finite energy. This is indeed shown to be possible with a minor modification of the theory [6], and the purpose of this paper is to discuss the finite energy electroweak monopole and dyon solutions in detail. We show that *the existence of the finite energy solu-*

tions are based on the fact that the Weinberg-Salam model not only has the same topological structure as the Georgi-Glashow model but in fact can be interpreted as a Georgi-Glashow model which has an extra interaction between the W boson and the Higgs triplet. The new monopoles could have important physical implications in the phenomenology of electroweak interaction.

II. MONOPOLES IN WEINBERG-SALAM MODEL

Before we construct the finite energy monopole solutions we must understand how one could obtain the infinite energy solutions first. So we will briefly review the singular solutions in the Weinberg-Salam model. Let us start with the Lagrangian which describes (the bosonic sector of) the standard Weinberg-Salam model

$$\begin{aligned}\mathcal{L}_0 &= -|\hat{D}_\mu\boldsymbol{\phi}|^2 - \frac{\lambda}{2}\Big(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\Big)^2 - \frac{1}{4}(\boldsymbol{F}_{\mu\nu})^2 - \frac{1}{4}(G_{\mu\nu})^2, \qquad (2.1)\\ \hat{D}_\mu\boldsymbol{\phi} &= \Big(\partial_\mu - i\frac{g}{2}\boldsymbol{\tau}\cdot\boldsymbol{A}_\mu - i\frac{g'}{2}B_\mu\Big)\boldsymbol{\phi} = \Big(D_\mu - i\frac{g'}{2}B_\mu\Big)\boldsymbol{\phi},\end{aligned}$$

where $\boldsymbol{\phi}$ is the Higgs doublet, $\boldsymbol{F}_{\mu\nu}$ and $G_{\mu\nu}$ are the gauge field strengths of $SU(2)$ and $U(1)$ with the potentials $\boldsymbol{A}_\mu$ and B_μ, and g and g' are the corresponding coupling constants. Notice that D_μ describes the covariant derivative of the $SU(2)$ subgroup only. From (2.1) one has the following equations of motion

$$\begin{aligned}\hat{D}_\mu(\hat{D}_\mu\boldsymbol{\phi}) &= \lambda\Big(\boldsymbol{\phi}^\dagger\boldsymbol{\phi} - \frac{\mu^2}{\lambda}\Big)\boldsymbol{\phi},\\ D_\mu\boldsymbol{F}_{\mu\nu} &= -\boldsymbol{j}_\nu = i\frac{g}{2}\Big[\boldsymbol{\phi}^\dagger\boldsymbol{\tau}(\hat{D}_\nu\boldsymbol{\phi}) - (\hat{D}_\nu\boldsymbol{\phi})^\dagger\boldsymbol{\tau}\boldsymbol{\phi}\Big], \qquad (2.2)\\ \partial_\mu G_{\mu\nu} &= -k_\nu = i\frac{g'}{2}\Big[\boldsymbol{\phi}^\dagger(\hat{D}_\nu\boldsymbol{\phi}) - (\hat{D}_\nu\boldsymbol{\phi})^\dagger\boldsymbol{\phi}\Big].\end{aligned}$$

Now we choose the following static spherically symmetric ansatz

$$\boldsymbol{\phi} = \frac{1}{\sqrt{2}}\rho(r)\xi(\theta,\varphi),$$

$$\xi = i\begin{pmatrix}\sin(\theta/2)\ e^{-i\varphi}\\ -\cos(\theta/2)\end{pmatrix}, \qquad \hat{\boldsymbol{\phi}} = \xi^\dagger\boldsymbol{\tau}\xi = -\hat{r},$$

$$
\begin{aligned}
\mathbf{A}_\mu &= \frac{1}{g}A(r)\partial_\mu t\hat{\phi} + \frac{1}{g}(f(r)-1)\hat{\phi}\times\partial_\mu\hat{\phi}, \\
B_\mu &= -\frac{1}{g'}B(r)\partial_\mu t - \frac{1}{g'}(1-\cos\theta)\partial_\mu\varphi,
\end{aligned} \tag{2.3}
$$

where (t, r, θ, φ) are the spherical coordinates. Notice that the apparent string singularity along the negative z-axis in ξ and B_μ is a pure gauge artifact which can easily be removed with a hypercharge $U(1)$ gauge transformation. Indeed with the $U(1)$ gauge transformation $\xi \longrightarrow e^{i\varphi}\xi$, $B_\mu \longrightarrow B_\mu + (2/g')\partial_\mu\varphi$, one can move the string to the positive z-axis. This proves that the above ansatz describes a genuine spherically symmetric ansatz of a $SU(2)\times U(1)$ dyon. Here we emphasize the importance of the non-trivial $U(1)$ degrees of freedom to make the ansatz spherically symmetric. Without the extra $U(1)$ the Higgs doublet does not allow a spherically symmetric ansatz. This is because the spherical symmetry for the gauge field involves the embedding of the radial isotropy group $SO(2)$ into the gauge group that requires the Higgs field to be invariant under the $U(1)$ subgroup of $SU(2)$. This is possible with a Higgs triplet, but not with a Higgs doublet [7]. In fact, in the absence of the hypercharge $U(1)$ degrees of freedom, the above ansatz describes the $SU(2)$ sphaleron which is not spherically symmetric [8]. The situation changes with the inclusion of the extra hypercharge $U(1)$, which allows us to circumbent this difficulty.

The spherically symmetric ansatz (2.3) reduces the equations of motion to

$$
\begin{aligned}
&\ddot{f} - \frac{f^2-1}{r^2}f = \Big(\frac{g^2}{4}\rho^2 - A^2\Big)f, \\
&\ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{f^2}{2r^2}\rho = -\frac{1}{4}(B-A)^2\rho + \lambda\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)\rho, \\
&\ddot{A} + \frac{2}{r}\dot{A} - \frac{2f^2}{r^2}A = \frac{g^2}{4}\rho^2(A-B), \\
&\ddot{B} + \frac{2}{r}\dot{B} = \frac{g'^2}{4}\rho^2(B-A).
\end{aligned} \tag{2.4}
$$

The smoothness of the solution requires the following boundary conditions near the origin,

$$
\begin{aligned}
f &\simeq 1 + \alpha_1 r^2, \\
\rho &\simeq \beta_1 r^\delta, \\
A &\simeq a_1 r, \\
B &\simeq b_0 + b_1 r,
\end{aligned} \tag{2.5}
$$

where $\delta = (-1+\sqrt{3})/2$. On the other hand asymptotically the finiteness of energy requires the following condition,

$$
\begin{aligned}
f &\simeq f_1 \exp(-\kappa r), \\
\rho &\simeq \rho_0 + \rho_1 \frac{\exp(-\sqrt{2}\mu r)}{r}, \\
A &\simeq A_0 + \frac{A_1}{r}, \\
B &\simeq A + B_1 \frac{\exp(-\nu r)}{r},
\end{aligned}
\tag{2.6}
$$

where $\rho_0 = \sqrt{2\mu^2/\lambda}$, $\kappa = \sqrt{(g\rho_0)^2/4 - A_0^2}$, and $\nu = \sqrt{(g^2+g'^2)}\rho_0/2$. Notice that asymptotically $B(r)$ must approaches to $A(r)$ with an exponential damping.

To determine the electric and magnetic charge of the dyon we now perform the following gauge transformation on (2.3)

$$
\xi \longrightarrow \xi' = U\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.7}
$$

$$
U = -i \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{-i\varphi} \\ \sin(\theta/2)e^{i\varphi} \\ -\cos(\theta/2) \end{pmatrix},
$$

and find that in this unitary gauge

$$
\boldsymbol{A}_\mu \longrightarrow \boldsymbol{A}'_\mu = \frac{1}{g} \begin{pmatrix} f(r)(\sin\varphi\partial_\mu\theta + \sin\theta\cos\varphi\partial_\mu\varphi) \\ -f(r)(\cos\varphi\partial_\mu\theta - \sin\theta\sin\varphi\partial_\mu\varphi) \\ -A(r)\partial_\mu t - (1 - \cos\theta)\partial_\mu\varphi \end{pmatrix}. \tag{2.8}
$$

So expressing the electromagnetic potential $\mathcal{A}_\mu$ and the neutral potential Z_μ with the Weinberg angle $\theta_{\rm w}$

$$
\begin{aligned}
\begin{pmatrix} \mathcal{A}_\mu \\ Z_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_{\rm w} \\ \sin\theta_{\rm w} \\ -\sin\theta_{\rm w} \\ \cos\theta_{\rm w} \end{pmatrix} \begin{pmatrix} \mathrm{B}_\mu \\ \mathrm{A}^3_\mu \end{pmatrix} \\
&= \frac{1}{\sqrt{g^2+g'^2}} \begin{pmatrix} \mathrm{g} \\ \mathrm{g}' \\ -\mathrm{g}' \\ \mathrm{g} \end{pmatrix} \begin{pmatrix} \mathrm{B}_\mu \\ \mathrm{A}^3_\mu \end{pmatrix},
\end{aligned}
\tag{2.9}
$$

we have

$$\begin{aligned}
\mathcal{A}_\mu &= -e\Big(\frac{1}{g^2}A + \frac{1}{g'^2}B\Big)\partial_\mu t - \frac{1}{e}(1-\cos\theta)\partial_\mu\varphi, \\
Z_\mu &= \frac{e}{gg'}(B-A)\partial_\mu t,
\end{aligned} \tag{2.10}$$

where e is the electric charge

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = g\sin\theta_{\rm w}.$$

From this one has the following electromagnetic charges of the dyon

$$\begin{aligned}
q_e &= 4\pi e\Big[r^2\Big(\frac{1}{g^2}\dot{A} + \frac{1}{g'^2}\dot{B}\Big)\Big]\Big|_{r=\infty} = \frac{4\pi}{e}A_1 \\
&= \frac{8\pi}{e}\sin^2\theta_{\rm w}\int_0^\infty f^2 A dr, \\
q_m &= \frac{4\pi}{e}.
\end{aligned} \tag{2.11}$$

Also, from the asymptotic condition (2.6) we conclude that our dyon does not carry any neutral charge,

$$\begin{aligned}
Z_e &= -\frac{4\pi e}{gg'}\big[r^2(\dot{B}-\dot{A})\big]\big|_{r=\infty} = 0, \\
Z_m &= 0,
\end{aligned} \tag{2.12}$$

which is what one should have expected.

With the boundary conditions one can integrate (2.4) and find the dyon solution of Cho and Maison shown in Fig.1 [4]. The regular part of the solution looks very much like the well-known Prasad-Sommerfield solution of the Julia-Zee dyon [9]. But there is a crucial difference. The above dyon has a non-trivial $B-A$, which represents the non-vanishing neutral Z boson content of the dyon as shown by (2.10). To understand the behavior of the solutions, remember that the mass of the W and Z bosons are given by $M_W = g\rho_0/2$ and $M_Z = \sqrt{g^2+g'^2}\rho_0/2$, and the mass of Higgs boson is given by $M_H = \sqrt{2}\mu$. This confirms that $\sqrt{(M_W)^2-(A_0)^2}$ and M_H determines the exponential damping of f and ρ, and M_Z determines the exponential damping of $B-A$, to their vacuum expectation values asymptotically. These are exactly what one would have expected.

With the ansatz (2.3) the energy of the dyon is given by

$$E = E_0 + E_1, \tag{2.13}$$

$$E_0 = \frac{2\pi}{g^2}\int_0^\infty \frac{dr}{r^2}\Big\{\frac{g^2}{g'^2} + (1-f^2)^2\Big\},$$

$$E_1 = \frac{4\pi}{g^2}\int_0^\infty dr\Big\{\frac{g^2}{2}(r\dot\rho)^2 + \frac{g^2}{4}f^2\rho^2 + \frac{g^2r^2}{8}(B-A)^2\rho^2 + \frac{\lambda g^2 r^2}{2}\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)^2$$
$$+(\dot f)^2 + \frac{1}{2}(r\dot A)^2 + \frac{g^2}{2g'^2}(r\dot B)^2 + f^2A^2\Big\}.$$

Now with the boundary conditions (2.5) and (2.6) one could easily find that E_1 is finite. As for E_0 we can minimize it with the boundary condition $f(0) = 1$, but even with this E_0 becomes infinite. Of course the origin of this infinite energy is obvious, which is precisely due to the hypercharge $U(1)$ singularity at the origin. This means that one can not predict the mass of dyon. It remains arbitrary at the classical level.

Recently a rigorous existence proof of the above solutions has been provided by Yang [5], who also established a very interesting constraint on the boundary conditions (2.6) and (2.7),

$$0 \le b_0 \le \frac{A_0}{e^2},$$
$$0 \le A_0 \le e\rho_0. \tag{2.14}$$

Notice that since $e\rho_0 = 2M_W \sin\theta_W$, $\kappa = \sqrt{(M_W)^2 - (A_0)^2}$ always remains positive with the experimental value of $\sin\theta_W$=0.4822. From the mathematical point of view the existence proof was a nontrivial task because the action functional of our dyon is not positive definite. With an indefinite action the standard minimization method in the variational calculus does not guarantee the existence of a solution.

Notice that (2.4) has another monopole/dyon solution which is much simpler. Clearly $f = 0$, $\rho = \rho_0$, $A = B = const/r$ also becomes a solution of the system. This, of course, is nothing but the "pure" magnetic monopole of the Weinberg-Salam model which has no $SU(2)$ tails. Unlike the original Dirac monopole, however, this one carries a magnetic charge $4\pi/e$, not $2\pi/e$. This is due to the fact that $U(1)_{em}$ of the Weinberg-Salam model is a composite subgroup of $SU(2)\times U(1)$. But this does not mean that the embedding of the original Dirac monopole with $q_m = 2\pi/e$ in the Weinberg-Salam model is impossible. In fact the Dirac monopole may be described by

$$\mathbf{A}_\mu = \frac{1}{g}\Big(A\partial_\mu t - \frac{1-\cos\theta}{2}\partial_\mu\varphi\Big)\hat\phi - \frac{1}{g}\hat\phi\times\partial_\mu\hat\phi \quad (\hat\phi = -\hat r),$$

$$B_\mu = -\frac{1}{g'}B\partial_\mu t - \frac{1-\cos\theta}{2g'}\partial_\mu\varphi. \tag{2.15}$$

Notice that under the $SU(2) \times U(1)$ gauge transformation

$$U = \exp\left(i\frac{1}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\phi}}\ \varphi\right) \times \exp\left(i\frac{\varphi}{2}\right), \tag{2.16}$$

one has

$$\begin{aligned}
\xi &\longrightarrow e^{i\varphi}\xi, \\
\boldsymbol{A}_\mu &\longrightarrow \frac{1}{g}\Big(A\partial_\mu t + \frac{1+\cos\theta}{2}\partial_\mu\varphi\Big)\hat{\boldsymbol{\phi}} - \frac{1}{g}\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}}, \\
B_\mu &\longrightarrow -\frac{1}{g'}B\partial_\mu t + \frac{1+\cos\theta}{2g'}\partial_\mu\varphi.
\end{aligned} \tag{2.17}$$

This proves that the gauge transformation (2.16) moves the string along the negative z-axis in (2.15) to the positive z-axis, and confirms the spherical symmetry of the ansatz (2.15). Now, in the unitary gauge (2.7), (2.15) becomes

$$\boldsymbol{A}_\mu \longrightarrow \frac{1}{g}\begin{pmatrix} 0 \\ 0 \\ \text{-A(r)}\partial_\mu t - \frac{1-\cos\theta}{2}\partial_\mu\varphi \end{pmatrix}, \tag{2.18}$$

so that one has

$$\begin{aligned}
\mathcal{A}_\mu &= -e\Big(\frac{1}{g^2}A + \frac{1}{g'^2}B\Big)\partial_\mu t - \frac{1}{2e}(1-\cos\theta)\partial_\mu\varphi, \\
Z_\mu &= \frac{e}{gg'}(B-A)\partial_\mu t.
\end{aligned} \tag{2.19}$$

This confirms that (2.15) indeed describes the Dirac monopole with $q_m = 2\pi/e$.

From the above analysis one may wonder whether the Dirac monopole could admit a nontrivial $SU(2)$ tail in the Weinberg-Salam model. Unfortunately this does not seem to be possible. The reason is that the W-boson does not allow a spherically symmetric configuration which is consistent with (2.15). This means that *only the monopole of Cho and Maison, not the Dirac monopole, allows a spherically symmetric W-boson tail in the Weinberg-Salam model.*

III. ANALYTIC SOLUTIONS

At this stage one may ask whether there is any way to make the energy of the above solutions finite. A simple way to make the energy finite is to introduce the gravitational interaction [10]. But the gravitational interaction is not likely remove the singularity at the origin, and one may still wonder if there is any way to regularize the singular solutions. In this section we will discuss how one can construct the monopole solutions explicitly which have not only a finite energy but also analytic everywhere.

To do this we first notice that a non-Abelian gauge theory in general is nothing but a special type of an Abelian gauge theory which has a well-defined set of charged vector fields as its source. This must be obvious, but this trivial observation reminds us the fact that *the finite energy non-Abelian monopoles are really nothing but the Abelian monopoles whose singularity is regularized by the charged vector fields* [11]. From this perspective one can try to make the energy of the above solutions finite by introducing additional interactions and/or charged vector fields. In the followings we will present two ways which allow us to achieve this goal along this line, and construct analytic electroweak monopole and dyon solutions with finite energy.

A. Electromagnetic Regularization

Remember that the origin of the infinite energy of the above solutions is the magnetic singularity of $U(1)_{\rm em}$ at the origin. We could try to regularize this singularity with a judicious choice of an extra electromagnetic interaction of the charged vector field with the Abelian monopole. This regularization would provide a most economic way to make the energy of the singular solution finite, because here we could use the already existing W boson without introducing a new source.

To show that this is indeed possible we first notice that in the unitary gauge the Lagrangian (2.1) can be written as

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{4}(G_{\mu\nu})^2 - \frac{1}{2}|D_\mu W_\nu - D_\nu W_\mu|^2 \\ &+ igF_{\mu\nu}W_\mu^* W_\nu + \frac{1}{4}g^2(W_\mu^* W_\nu - W_\nu^* W_\mu)^2 \\ &- \frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{4}\rho^2\big(g^2 W_\mu^* W_\mu + \frac{1}{2}(g' B_\mu - gA_\mu)^2\big) - \frac{\lambda}{2}\big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\big)^2, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} W_\mu &= \frac{1}{\sqrt{2}}(A_\mu^1 + iA_\mu^2), \\ A_\mu &= A_\mu^3, \\ D_\mu W_\nu &= (\partial_\mu + igA_\mu)W_\nu. \end{aligned}$$

This Lagrangian describes the dynamics of two $U(1)$ gauge fields A_μ and B_μ interacting with a charged vector field W_μ and a real scalar field ρ. Notice that in the unitary gauge the spherically symmetric ansatz (2.3) is written as

$$\begin{aligned}
\rho &= \rho(r) \\
W_\mu &= -\frac{i}{g}\frac{f(r)}{\sqrt{2}}e^{i\varphi}(\partial_\mu\theta + i\sin\theta\partial_\mu\varphi), \\
A_\mu &= -\frac{1}{g}A(r)\partial_\mu t - \frac{1}{g}(1-\cos\theta)\partial_\mu\varphi, \\
B_\mu &= -\frac{1}{g'}B(r)\partial_\mu t - \frac{1}{g'}(1-\cos\theta)\partial_\mu\varphi,
\end{aligned} \tag{3.2}$$

which must be clear from (2.8).

To regularize the singular dyon, we now introduce an extra interaction $\delta\mathcal{L}_1$ to (3.1) which modifies the coupling strength of the 4-point interaction of the W boson,

$$\delta\mathcal{L}_1 = \frac{\alpha}{4}g^2(W_\mu^* W_\nu - W_\nu^* W_\mu)^2, \tag{3.3}$$

where α is an arbitrary constant. With this modification the energy of system is given by

$$E = E_0' + E_1, \tag{3.4}$$

where now E_0' is given by

$$E_0' = \frac{2\pi}{g^2}\int_0^\infty \frac{dr}{r^2}\Big\{\frac{g^2}{g'^2} + 1 - 2f^2 + (1+\alpha)f^4\Big\}. \tag{3.5}$$

Notice that with $\alpha = 0$, E_0' reduces to E_0 and becomes infinite. Clearly, for the energy (3.4) to be finite, E_0' must be free not only from the $O(1/r^2)$ singularity but also $O(1/r)$ singularity at the origin. This requires us to have

$$1 + \frac{g^2}{g'^2} - 2f^2(0) + (1+\alpha)f^4(0) = 0,$$

$$f(0) - (1+\alpha)f^3(0) = 0. \tag{3.6}$$

Thus we arrive at the following condition for a finite energy solution

$$\frac{1}{1+\alpha} = 1 + \frac{g^2}{g'^2} = \frac{1}{\sin^2\theta_{\rm w}}, \tag{3.7}$$

from which we have

$$f(0) = \frac{1}{\sin\theta_{\rm w}}. \tag{3.8}$$

At the first glance one might think that this is unacceptable because, according to the ansatz (2.3), the above boundary condition creates a singularity in $\boldsymbol{A}_\mu$. But we will see that this is precisely what one need to counter the magnetic singularity of B_μ.

Actually, for the purpose of a finite energy solution one could have tried a more general $\delta\mathcal{L}_1$,

$$\delta\mathcal{L}_1 = \frac{\alpha}{4}g^2(W_\mu^* W_\nu - W_\nu^* W_\mu)^2 + i\beta g F_{\mu\nu} W_\mu^* W_\nu \tag{3.9}$$

The additional β-term will introduce an extra coupling constant to the theory, and leave $f(0)$ (and α) arbitrary. But clearly the β-term will create an "anomalous" magnetic moment for the W-boson, which one may wish to avoid. For this reason we will keep $\beta = 0$ in this paper.

To understand the finite energy solutions it is important to realize that the Weinberg-Salam model can be interpreted as a Georgi-Glashow model. To see this notice that with (3.3) the modified Lagrangian is given by

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}(\mathcal{F}_{\mu\nu})^2 - \frac{1}{2}|\mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu|^2 + ie\mathcal{F}_{\mu\nu} W_\mu^* W_\nu - \frac{1}{4}e^2(W_\mu^* W_\nu - W_\nu^* W_\mu)^2 \\
& -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{4}\frac{e^2}{\sin^2\theta_{\rm w}}\rho^2 W_\mu^* W_\mu - \frac{\lambda}{2}\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)^2 - \frac{1}{4}(Z_{\mu\nu})^2 \\
& -\frac{1}{2}\frac{e^2}{\sin^2 2\theta_{\rm w}}\rho^2 Z_\mu^2 + ie\cot\theta_{\rm w} Z_{\mu\nu} W_\mu^* W_\nu + e^2\cot^2\theta_{\rm w}(Z_\mu^2|W_\nu|^2 - |Z_\mu W_\mu|^2) \\
& +ie\cot\theta_{\rm w}\mathcal{D}_\mu W_\nu^*(Z_\mu W_\nu - Z_\nu W_\mu) - ie\cot\theta_{\rm w}\mathcal{D}_\mu W_\nu(Z_\mu W_\nu^* - Z_\nu W_\mu^*),
\end{aligned} \tag{3.10}$$

where

$$\mathcal{D}_\mu W_\nu = (\partial_\mu + ie\mathcal{A}_\mu)W_\nu, \quad \mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu, \quad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu.$$

Notice that the only change made by (3.3) is the coupling constant of the 4-point interaction of the W-boson from $g^2/4$ to $e^2/4$. Now it must become clear that the above Lagrangian has a hidden $SU(2)_{\rm em}$ symmetry which is different from the old $SU(2)$ symmetry of Weinberg and Salam. *In fact, when $Z_\mu = 0$, the Lagrangian (3.10) describes an $SU(2)_{\rm em}$ gauge theory with the physical coupling constant e (not g) which is spontaneously broken by a Higgs triplet whose gauge potentials consist of the real electromagnetic $\mathcal{A}_\mu$ (not A_μ) and $W_\mu^\pm$, except that here Higgs triplet has an extra interaction with the W boson given by*

$$\delta\mathcal{L}_2 = e^2\Big(1 - \frac{1}{4\sin^2\theta_{\rm w}}\Big)\rho^2 W_\mu^* W_\mu = \Big(e^2 - \frac{g^2}{4}\Big)\rho^2 W_\mu^* W_\mu. \tag{3.11}$$

To demonstrate this we introduce a Higgs triplet

$$\boldsymbol{\Phi} = \rho\hat{\boldsymbol{\phi}},$$

and write

$$\boldsymbol{A}_\mu^{\rm em} = \hat{\boldsymbol{A}}_\mu^{\rm em} + \boldsymbol{W}_\mu \quad (\hat{\boldsymbol{\phi}}\cdot\boldsymbol{W}_\mu = 0), \tag{3.12}$$

$$\hat{\boldsymbol{A}}_\mu^{\rm em} = \mathcal{A}_\mu^{\rm em}\hat{\boldsymbol{\phi}} - \frac{1}{e}\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}} \qquad (\mathcal{A}_\mu^{\rm em} = \hat{\boldsymbol{\phi}}\cdot\boldsymbol{A}_\mu^{\rm em}).$$

Notice that in this decomposition $\hat{\boldsymbol{A}}_\mu^{\rm em}$ is the gauge potential which parallelizes $\hat{\boldsymbol{\phi}}$ [12],

$$\hat{\mathcal{D}}_\mu\hat{\boldsymbol{\phi}} = \partial_\mu\hat{\boldsymbol{\phi}} + e\hat{\boldsymbol{A}}_\mu^{\rm em}\times\hat{\boldsymbol{\phi}} = 0. \tag{3.13}$$

Also notice that under an infinitesimal gauge transformation

$$\delta\boldsymbol{A}_\mu^{\rm em} = -\frac{1}{e}\mathcal{D}_\mu\boldsymbol{\theta}, \qquad \delta\hat{\boldsymbol{\phi}} = \boldsymbol{\theta}\times\hat{\boldsymbol{\phi}}, \tag{3.14}$$

one has

$$\delta\hat{\boldsymbol{A}}_\mu^{\rm em} = -\frac{1}{e}\hat{\mathcal{D}}_\mu\boldsymbol{\theta}, \qquad \delta\boldsymbol{W}_\mu = \boldsymbol{\theta}\times\boldsymbol{W}_\mu, \tag{3.15}$$

so that $\boldsymbol{W}_\mu$ becomes a gauge covariant vector field orthogonal to $\hat{\boldsymbol{\phi}}$. With this the Lagrangian (3.10) now can be written in a gauge invariant form

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}(\boldsymbol{F}_{\mu\nu}^{\rm em})^2 - \frac{1}{2}(\mathcal{D}_\mu\boldsymbol{\Phi})^2 + e^2\Big(1-\frac{1}{4\sin^2\theta_{\rm w}}\Big)(\boldsymbol{W}_\mu\times\boldsymbol{\Phi})^2 - \frac{\lambda}{2}\Big(\frac{1}{2}\boldsymbol{\Phi}^2 - \frac{\mu^2}{\lambda}\Big)^2 \\ &-\frac{1}{4}(Z_{\mu\nu})^2 - \frac{1}{2}\frac{e^2}{\sin^2 2\theta_{\rm w}}\boldsymbol{\Phi}^2 Z_\mu^2 + \frac{e}{2}\cot\theta_{\rm w}\hat{\boldsymbol{\phi}}\cdot\big(\hat{\mathcal{D}}_\mu\boldsymbol{W}_\nu\times(\boldsymbol{W}_\nu Z_\mu - \boldsymbol{W}_\mu Z_\nu)\big) \\ &-\frac{e}{2}\cot\theta_{\rm w} Z_{\mu\nu}(\hat{\boldsymbol{\phi}}\cdot\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu) + \frac{e^2}{2}\cot^2\theta_{\rm w}\big(Z_\mu^2(\boldsymbol{W}_\nu)^2 - Z_\mu Z_\nu\boldsymbol{W}_\mu\cdot\boldsymbol{W}_\nu\big),\end{aligned} \tag{3.16}$$

where $\boldsymbol{F}_{\mu\nu}^{\rm em}$ is the field strength of $\boldsymbol{A}_\mu^{\rm em}$ and

$$\mathcal{D}_\mu\boldsymbol{\Phi} = \partial_\mu\boldsymbol{\Phi} + e\boldsymbol{A}_\mu^{\rm em}\times\boldsymbol{\Phi}.$$

In this form the Lagrangian describes an $SU(2)_{\rm em}\times U(1)_{\rm z}$ gauge theory where $SU(2)_{\rm em}$ is spontaneously broken by a Higgs triplet but $U(1)_{\rm z}$ is explicitly broken.

The above result immediately tells that the Weinberg-Salam model itself can be interpreted as a Georgi-Glashow model of the new $SU(2)_{\rm em}$ in the absence of the Z boson. This must be clear because (3.16) is expressed by

$$\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}_1, \tag{3.17}$$

where now

$$\delta\mathcal{L}_1 = -\frac{e^2}{4}\cot^2\theta_{\rm w}(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu)^2. \tag{3.18}$$

It is really remarkable that one could reinterpret the Weinberg-Salam model as a Georgi-Glashow model with such a minor modification. The origin for this, of course, is the realization of the fact that the original Weinberg-Salam model is really nothing but a gauged CP^1 model which has exactly the same topological structure as the Georgi-Glashow model. Without this understanding one could not possibly have arrived at this interpretation.

Clearly the above interpretation guarantees the existence of the finite energy monopole and dyon solutions. To obtain the desired solutions notice that in the regular $SU(2)_{\rm em}$ gauge the ansatz (3.2) is written as

$$\begin{aligned}
\boldsymbol{\Phi} &= \rho\hat{\boldsymbol{\phi}} \qquad (\hat{\boldsymbol{\phi}} = -\hat{r}),\\
\boldsymbol{A}_\mu^{\rm em} &= -e\Big(\frac{1}{g^2}A + \frac{1}{g'^2}B\Big)\partial_\mu t\hat{\boldsymbol{\phi}} + \frac{1}{e}(f\sin\theta_{\rm w} - 1)\hat{\boldsymbol{\phi}}\times\partial_\mu\hat{\boldsymbol{\phi}},\\
Z_\mu &= \frac{e}{gg'}(B-A)\partial_\mu t.
\end{aligned} \tag{3.19}$$

This tells that the boundary condition (3.7) is precisely what one need to remove the magnetic singularity at the origin. Notice that the old boundary condition $f(0) = 1$ cannot remove the singularity of the hypercharge $U(1)$, although it does make the original $SU(2)$ part regular. Now, the equations of motion of the Lagrangian (3.16) with the above ansatz can be written as

$$\begin{aligned}
&\ddot{f} - \frac{f^2\sin^2\theta_{\rm w} - 1}{r^2}f = \Big(\frac{g^2}{4}\rho^2 - A^2\Big)f,\\
&\ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{f^2}{2r^2}\rho = -\frac{1}{4}(B-A)^2\rho + \lambda\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)\rho,\\
&\ddot{A} + \frac{2}{r}\dot{A} - \frac{2f^2}{r^2}A = \frac{g^2}{4}(A-B)\rho^2,\\
&\ddot{B} + \frac{2}{r}\dot{B} = \frac{g'^2}{4}(B-A)\rho^2.
\end{aligned} \tag{3.20}$$

One could integrate this with the boundary conditions

$$\begin{aligned}
&f(0) = 1/\sin\theta_{\rm w}, \quad A(0) = a_0, \quad B(0) = b_0, \quad \rho(0) = 0,\\
&f(\infty) = 0, \quad A(\infty) = B(\infty) = A_0, \quad \rho(\infty) = \rho_0.
\end{aligned} \tag{3.21}$$

For the monopole this boundary condition is enough to guarantee an analytic solution. For a dyon $A(\infty) = B(\infty)$ guarantees the finiteness of the energy. One could further try to impose the condition $a_0/g^2 + b_0/g'^2 = 0$ to make the electric part of the solution smooth at the origin in the regular gauge (3.19). Notice, however, that (3.20) is invariant under $(A, B) \rightarrow (-A, -B)$. From this symmetry and the last two equations of (3.20) one can show that $B(r) \geq A(r) \geq 0$ everywhere [5], so that one must have $b_0 \geq a_0 \geq 0$. This tells that in the regular gauge the dyon solution develops a cusp at the origin, which is harmless. The results of the numerical integration for the monopole and dyon solution are shown in Fig.2 and Fig.3. *It is really remarkable that the finite energy solutions look almost identical to the solutions of Cho and Maison, even though they no longer have the magnetic singularity at the origin.*

Clearly the energy of the above solutions must be of the order of M_W. Indeed for the monopole the energy can be expressed as

$$E = \frac{4\pi}{e^2} C(\sin^2\theta_w, \lambda/g^2) M_W \tag{3.22}$$

where C the dimensionless function of $\sin^2\theta_w$, and λ/g^2. With experimental value $\sin^2\theta_w$, C becomes slowly varying function of λ/g^2 with $C = 1.407$ for $\lambda/g^2 = 0.5$. This demonstrates that the finite energy solutions can indeed be interpreted as the electroweak monopole and dyon, and are really nothing but the original solutions of Cho and Maison which have been regularized to have a mass of the electroweak scale.

The similarity between the Weinberg-Salam model and the Georgi-Glashow model can be made more precise just by making the interaction of the Higgs triplet with the W boson "normal". This can be easily done by removing the extra interaction (3.11)

$$\delta\mathcal{L}_2 = \Big(1 - \frac{1}{4\sin^2\theta_w}\Big) e^2 (\boldsymbol{W}_\mu \times \boldsymbol{\Phi})^2 \tag{3.23}$$

from (3.16). With this the Lagrangian (3.16) reduces to nothing but a Georgi-Glashow model in the absence of Z boson. This observation allows us to have a Bogomol'nyi-Prasad-Sommerfield monopole solution. Indeed in the absence of (3.23) the monopole energy functional in the Prasad-Sommerfield limit $\lambda = 0$ becomes

$$\begin{aligned} E &= \int d^3x \Big\{ \frac{1}{4} (\boldsymbol{F}_{ij})^2 + \frac{1}{2} (\mathcal{D}_i \boldsymbol{\Phi})^2 \Big\} \\ &= \frac{1}{4} \int d^3x \Big(\boldsymbol{F}_{ij} \mp \epsilon_{ijk} \mathcal{D}_k \boldsymbol{\Phi} \Big)^2 \pm \frac{1}{2} \int d^3x \epsilon_{ijk} \boldsymbol{F}_{ij} \cdot \mathcal{D}_k \boldsymbol{\Phi}, \end{aligned} \tag{3.24}$$

which, with the Bianchi identity $\epsilon_{ijk}\mathcal{D}_i\boldsymbol{F}_{jk}=0$, gives the desired energy bound

$$E \geq \left|\frac{1}{2}\int d^3x \epsilon_{ijk}\partial_k(\boldsymbol{F}_{ij}\cdot\boldsymbol{\Phi})\right|. \tag{3.25}$$

The bound is saturated by the well-known Bogomol'nyi equation

$$\boldsymbol{F}_{ij} = \pm\epsilon_{ijk}\mathcal{D}_k\boldsymbol{\Phi}, \tag{3.26}$$

which automatically satisfies the equations of motion in the absence of Z boson. The equation with the ansatz (3.19) (with $A = B = 0$) reduces to

$$\begin{aligned}
&\dot{f} \pm e\rho f = 0,\\
&\dot{\rho} \mp \frac{1}{er^2}(1 - f^2\sin^2\theta_{\rm w}) = 0,
\end{aligned} \tag{3.27}$$

which has the following solution

$$\begin{aligned}
&f\sin\theta_{\rm w} = \frac{e\rho_0 r}{\sinh(e\rho_0 r)},\\
&\rho = \rho_0\coth(e\rho_0 r) - \frac{1}{er}.
\end{aligned} \tag{3.28}$$

Clearly the energy of the Bogomol'nyi-Prasad-Sommerfield solution is given by

$$E = \frac{4\pi}{e}\rho(\infty) = \frac{4\pi}{e^2}M_W, \tag{3.29}$$

where now M_W is given by $e\rho_0$.

B. Embedding $SU(2)\times U(1)$ to $SU(2)\times SU(2)$

As we have noticed the origin of the infinite energy of the solutions obtained by Cho and Maison was the magnetic singularity of the hypercharge $U(1)$ field B_μ. So one could try to to obtain a finite energy monopole solution by regularizing this hypercharge singularity. This could be done by introducing a hypercharged vector field to the theory. A simplest way to do this is, of course, to enlarge the hypercharge $U(1)$ and embed it to another $SU(2)$.

To construct the desired solutions we generalize the Lagrangian (3.1) by adding the following Lagrangian

$$\begin{aligned}
\mathcal{L}' \;=\; &-\frac{1}{2}|\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu|^2 + ig'G_{\mu\nu}X_\mu^* X_\nu + \frac{1}{4}g'^2(X_\mu^* X_\nu - X_\nu^* X_\mu)^2\\
&-\frac{1}{2}(\partial_\mu\sigma)^2 - g'^2\sigma^2 X_\mu^* X_\mu - \frac{\kappa}{4}\Big(\sigma^2 - \frac{m^2}{\kappa}\Big)^2,
\end{aligned} \tag{3.30}$$

where X_μ is a hypercharged vector field, σ is a Higgs field, and $\tilde{D}_\mu X_\nu = (\partial_\mu + ig'B_\mu)X_\nu$. Notice that, if we introduce a hypercharge $SU(2)$ gauge field $\boldsymbol{B}_\mu$ and a scalar triplet $\boldsymbol{\Phi}$ and identify

$$\begin{aligned} X_\mu &= \frac{1}{\sqrt{2}}(B_\mu^1 + iB_\mu^2), \\ B_\mu &= B_\mu^3, \\ \boldsymbol{\Phi} &= (0,0,\sigma), \end{aligned} \tag{3.31}$$

the above Lagrangian in an arbitrary gauge can be written as

$$\mathcal{L}' = -\frac{1}{2}(\tilde{D}_\mu \boldsymbol{\Phi})^2 - \frac{\kappa}{4}\Big(\boldsymbol{\Phi}^2 - \frac{m^2}{\kappa}\Big)^2 - \frac{1}{4}(\boldsymbol{G}_{\mu\nu})^2. \tag{3.32}$$

This clearly shows that Lagrangian (3.30) is nothing but the embedding of the hypercharge $U(1)$ to an $SU(2)$ Georgi-Glashow model.

From (3.1) and (3.30) one has the following equations of motion

$$\partial_\mu(\partial_\mu \rho) = \frac{1}{2}g^2 W_\mu^* W_\mu \rho + \frac{1}{4}(g'B_\mu - gA_\mu)^2 \rho + \lambda\Big(\frac{\rho^2}{2} + \frac{\mu^2}{\lambda}\Big)\rho,$$

$$D_\mu(D_\mu W_\nu - D_\nu W_\mu) = igF_{\mu\nu}W_\mu - g^2 W_\mu(W_\nu W_\mu^* - W_\nu^* W_\mu) + \frac{1}{4}g^2\rho^2 W_\nu,$$

$$\begin{aligned} \partial_\mu F_{\mu\nu} &= \frac{1}{4}g\rho^2(gA_\nu - g'B_\nu) + ig\Big(W_\mu^*(D_\mu W_\nu - D_\nu W_\mu) - (D_\mu W_\nu^* - D_\nu W_\mu^*)W_\mu\Big) \\ &\quad + ig\partial_\mu(W_\mu^* W_\nu - W_\nu^* W_\mu), \end{aligned}$$

$$\begin{aligned} \partial_\mu G_{\mu\nu} &= \frac{1}{4}g'\rho^2(g'B_\nu - gA_\nu) + ig'\Big(X_\mu^*(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu) - (\tilde{D}_\mu X_\nu^* - \tilde{D}_\nu X_\mu^*)X_\mu\Big) \\ &\quad + ig'\partial_\mu(X_\mu^* X_\nu - X_\nu^* X_\mu), \end{aligned}$$

$$\partial_\mu(\partial_\mu \sigma) = 2g'^2 X_\mu^* X_\mu \sigma + \kappa\Big(\sigma^2 - \frac{m^2}{\kappa}\Big)\sigma,$$

$$\tilde{D}_\mu(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu) = ig'G_{\mu\nu}X_\mu - g'^2 X_\mu(X_\mu^* X_\nu - X_\nu^* X_\mu) + (g')^2\sigma^2 X_\nu \tag{3.33}$$

Now for a static spherically symmetric ansatz we choose (3.2) and assume

$$\begin{aligned} \sigma &= \sigma(r), \\ X_\mu &= -\frac{i}{g'}\frac{h(r)}{\sqrt{2}}e^{i\varphi}(\partial_\mu\theta + i\sin\theta\partial_\mu\varphi). \end{aligned} \tag{3.34}$$

With the spherically symmetric ansatz (3.33) is reduced to

$$\ddot{f} - \frac{f^2-1}{r^2}f = \Big(\frac{g^2}{4}\rho^2 - A^2\Big)f,$$

$$\ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{f^2}{2r^2}\rho = -\frac{1}{4}(B-A)^2\rho + \lambda\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)\rho,$$

$$\ddot{A} + \frac{2}{r}\dot{A} - \frac{2f^2}{r^2}A = \frac{g^2}{4}\rho^2(A-B), \tag{3.35}$$

$$\ddot{h} - \frac{h^2-1}{r^2}h = (g'^2\sigma^2 - B^2)h, \tag{3.36}$$

$$\ddot{\sigma} + \frac{2}{r}\dot{\sigma} - \frac{2h^2}{r^2}\sigma = \kappa\Big(\sigma^2 - \frac{m^2}{\kappa}\Big)\sigma,$$

$$\ddot{B} + \frac{2}{r}\dot{B} - \frac{2h^2}{r^2}B = \frac{g'^2}{4}\rho^2(B-A).$$

Notice that the energy of the above configuration is given by

$$E = E_W + E_X, \tag{3.37}$$

$$\begin{aligned} E_W &= \frac{4\pi}{g^2}\int_0^\infty dr\Big\{(\dot{f})^2 + \frac{(1-f^2)^2}{2r^2} + \frac{1}{2}(r\dot{A})^2 + f^2A^2 \\ &\quad + \frac{g^2}{2}(r\dot{\rho})^2 + \frac{g^2}{4}f^2\rho^2 + \frac{g^2r^2}{8}(B-A)^2\rho^2 + \frac{\lambda g^2r^2}{2}\Big(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\Big)^2\Big\} \\ &= \frac{4\pi}{g^2}C_1(\lambda/g^2)M_W, \end{aligned}$$

$$\begin{aligned} E_X &= \frac{4\pi}{g'^2}\int_0^\infty dr\Big\{(\dot{h})^2 + \frac{(1-h^2)^2}{2r^2} + \frac{1}{2}(r\dot{B})^2 + h^2B^2 \\ &\quad + \frac{g'^2}{2}(r\dot{\sigma})^2 + g'^2h^2\sigma^2 + \frac{\kappa g'^2r^2}{4}(\sigma^2 - \sigma_0^2)^2\Big\} \\ &= \frac{4\pi}{g'^2}C_2(\kappa/g'^2)M_X, \end{aligned}$$

where $M_W = g\rho_0/2$,and $M_X = g'\sigma_0 = g'\sqrt{m^2/\kappa}$. The boundary conditions for a regular field configuration can be chosen as

$$f(0) = h(0) = 1, \quad A(0) = B(0) = \rho(0) = \sigma(0) = 0,$$

$$f(\infty) = h(\infty) = 0, \quad A(\infty) = B(\infty) = A_0, \quad \rho(\infty) = \rho_0, \quad \sigma(\infty) = \sigma_0. \tag{3.38}$$

Notice that the origin of the condition $B(0) = 0$ is the same as $A(0) = 0$. With the boundary condition (3.38) one may try to find the desired solution. From the physical point of view one could assume $M_X \gg M_W$, where M_X is an intermediate scale which lies somewhere between the grand unification scale and the electroweak scale. Now, let $A = B = 0$ for simplicity. Then (3.37) decouples to describes two independent systems so that the monopole solution has two cores, the one with the size $O(1/M_W)$ and the other with the size $O(1/M_X)$. With $M_X = 10M_W$ we obtain the solution shown in Fig.4 in the limit $\lambda = \kappa = 0$. In this limit we find $C_1 = 1.946$ and $C_2 = 1$ so that the energy of the solution is given by

$$E = \frac{4\pi}{e^2}\Big(\cos^2\theta_{\rm w} + 0.195\sin^2\theta_{\rm w}\Big)M_X. \tag{3.39}$$

Clearly the solution describes the monopole of Cho and Maison whose singularity is regularized by a Prasad-Sommerfield monopole of the size $O(1/M_X)$.

Notice that even if the energy of the monopole is fixed by the intermediate scale, the monopole could be interpreted as an electroweak monopole. To see this remember that the size of the monopole is fixed by the electroweak scale. Furthermore from the outside the monopole looks exactly the same as the monopole of Cho and Maison. Only the inner core is regularized by the hypercharged vector field. This justifies it as an electroweak monopole.

IV. CONCLUSIONS

In this paper we have discussed two ways to regularize the singular monopole and dyon solutions of the Weinberg-Salam model, and explicitly constructed genuine finite energy electroweak monopole solutions which are analytic everywhere including the origin. The finite energy solutions are obtained with a simple modification of the interaction of the W boson or with the embedding of the hypercharge $U(1)$ to a compact $SU(2)$. It has generally been believed that the finite energy monopole must exist only at the grand unification scale [13]. But our result tells that this belief is unfounded, and suggests the existence of a new class of electroweak monopole whose mass is much smaller than the monopoles of the grand unification. Obviously the electroweak monopoles are topological solitons which must be stable.

Strictly speaking the finite energy solutions are not the solutions of the Weinberg-Salam model, because their existence requires a modification of the model. But from the physical point of view there is no doubt that they could be interpreted as the electroweak monopole and dyon, because they are really nothing but the regularized solutions of Cho and Maison whose size is fixed at the electroweak scale. In spite of the fact that the singular solutions are obviously the solutions of the Weinberg-Salam model one could try to object them as the electroweak dyons under the presumption that the singular solutions could be regularized only at the grand unification scale. Our work shows that this objection is groundless, and assures that it is not necessary for us to go to the grand unification scale to make the energy of the singular solutions finite. This really reinforces the dyons of Cho and Maison as the electroweak dyons which must be taken seriously.

We close with the following remarks:

1) A most remarkable aspect of our result is that, unlike the original Dirac monopole, the magnetic charge of the above electroweak monopoles satisfy the Schwinger quantization condition $q_m = 4\pi n/e$. On the other hand, since the Weinberg-Salam model has an unbroken $U(1)_{\rm em}$, one should be able to embed the original Dirac monopole with the charge $q_m = 2\pi/e$ and identify it as a classical solution of the Weinberg-Salam model. However, we emphasize that the electroweak unification forbids such an embedding, as far as one wants to add a non-trivial structure to the monopole. The existence of a finite size monopole with $q_m = 2\pi/e$ is simply not compatible with the Weinberg-Salam model. Whether this conclusion applies only to the electroweak theory or to a more general type of theories is not clear at this moment.

2) Another important result of our work is that the Weinberg-Salam model, with a minor modification, can actually be interpreted as a Georgi-Glashow model. At the first thought this may come as a surprise. But we emphasize that this is precisely what one should have expected, if one realizes that the Weinberg-Salam model has a hidden $SU(2)_{\rm em}$ which is spontaneously broken down to $U(1)_{\rm em}$ with the massless photon. There is only one gauge theory, the Georgi-Glashow model, which can describe such a spontaneous symmetry breaking.

3) The electromagnetic regularization of the Dirac monopole with the charged vector fields is nothing new. In fact it was this regularization which made the energy of the 't Hooft-Polyakov monopole finite. Furthermore it has been known that the 't Hooft-Polyakov monopole is the only analytic solution which one could obtain with this technique [11]. What we have shown here is that the same technique also works to regularize the solutions

of Cho and Maison, but only with (3.3).

4) The additional interactions (3.3) and (3.11) could spoil the renormalizability of the theory. How serious would this offense be, however, is not clear at this moment. If one views the Weinberg-Salam model as a low energy effective theory of a renormalizable theory, it need not necessarily be renormalizable by itself. Here we simply notice that the introduction of a non-renormalizable interaction (like a gravitational interaction) has been an acceptable practice to study finite energy classical solutions.

5) The embedding of the electroweak $SU(2) \times U(1)$ to a larger $SU(2) \times SU(2)$ or $SU(2) \times SU(2) \times U(1)$ could naturally arise in the left-right symmetric grand unification models, in particular in the $SO(10)$ grand unification, although the embedding of the hypercharge $U(1)$ to a compact $SU(2)$ may turn out to be too simple to be realistic. Independent of the details, however, our discussion suggests that the electroweak monopoles at an intermediate scale M_X could be possible in a realistic grand unification.

The existence of the finite energy electroweak monopoles could have important physical implications [14]. We will discuss on the physical implications of the electroweak monopoles separately.

ACKNOWLEDGMENTS

It is a pleasure to thank C.K. Lee and Y. Yang for discussions. The work is supported in part by the Ministry of Education through the Basic Science Research Program (BSRI 97-2418) and by the Korean Science and Engineering Foundation through the Center for Theoretical Physics (SNU).

REFERENCES

[1] P.A.M. Dirac, Phys. Rev. **74**, 817 (1948).

[2] T.T. Wu and C.N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York) 1969; Nucl. Phys. **B107**, 365 (1976); Phys. Rev. **D16**, 1018 (1977).

[3] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974);
A.M. Polyakov, JETP Lett. **20**, 194 (1974).

[4] Y.M. Cho and D. Maison, Phys. Lett. **B391**, 360 (1997).

[5] Yisong Yang, Proc. Roy. Soc. Lond. **A454**, 155 (1998).

[6] Y.M. Cho and K. Kimm, APCTP-97-10, Phys. Lett. **B**, submitted.

[7] P. Forgács and N.S. Manton, Commun. Math. Phys. **72**, 15 (1980).

[8] R.F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. **D10**, 4138 (1974).

[9] B. Julia and A. Zee, Phys. Rev. **D11**, 2227 (1975);
M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).

[10] F.A. Bais and R.J. Russell, Phys. Rev. **D11**, 2692 (1975);
Y.M. Cho and P.G.O. Freund, Phys. Rev. **D12**, 1711 (1975);
P. Breitenlohner, P.Forgacs, and D. Maison, Nucl. Phys. **B383**, 357 (1992).

[11] K. Lee and E. Weinberg, Phys. Rev. Lett, **73**, 1203 (1994);
C. Lee and P. Yi, Phys. Lett. **B348**, 100 (1995).

[12] Y.M. Cho, Phys. Rev. **D21**, 1080 (1980); **D23**, 2415 (1981).

[13] C.P. Dokos and T.N. Tomaras, Phys. Rev. **D21**, 2940 (1980);
Y.M. Cho, Phys. Rev. Lett. **44**, 1115 (1980).

[14] J. Preskill, Phys. Rev. Lett. **43**, 1365 (1979);
C.G. Callan, Phys. Rev. **D25**, 2141 (1982);
V.A. Rubakov, Nucl. Phys. **B203**, 311 (1982).

FIGURES

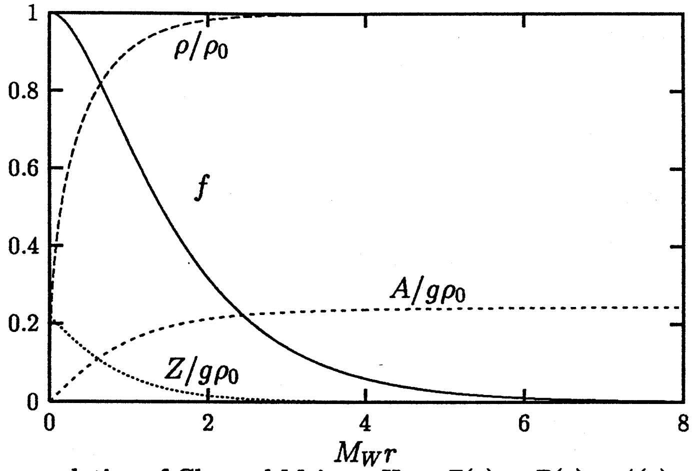

FIG. 1. The dyon solution of Cho and Maison. Here $Z(r) = B(r) - A(r)$ and we have chosen $\sin^2\theta_w = 0.2325$, $\lambda/g^2 = M_H^2/2M_W^2 = 0.5$, and $A_0 = M_W/2$.

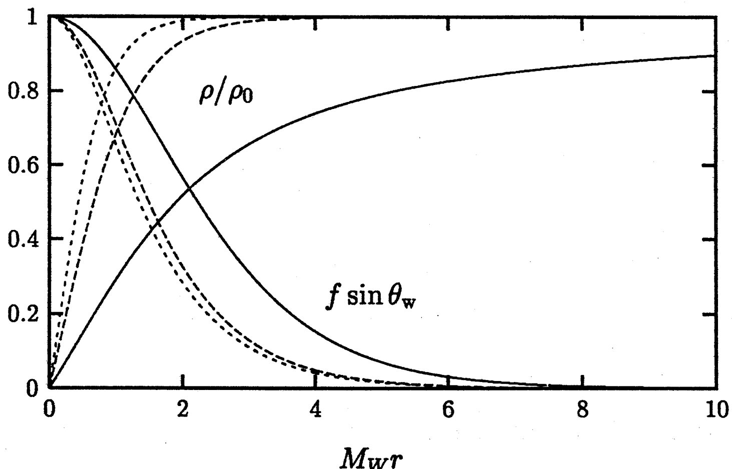

FIG. 2. The finite energy electroweak monopole solution obtained with different values of $\lambda/g^2 = 0$ (solid line), 0.5(dashed line), and 2.0(dotted line).

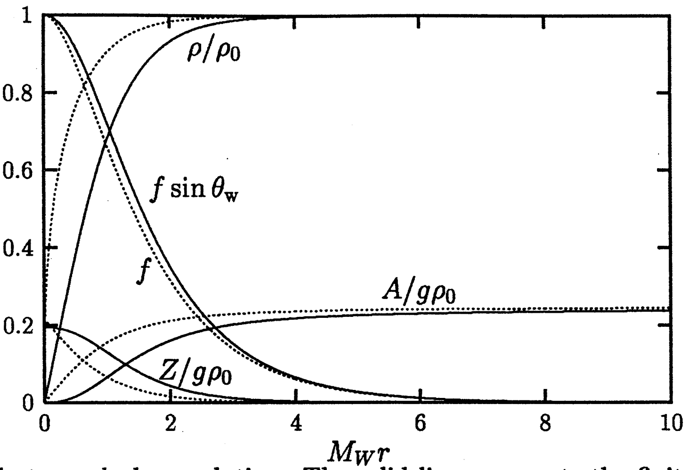

FIG. 3. The electroweak dyon solution. The solid line represents the finite energy dyon and dotted line represents the dyon of Cho and Maison, where we have chosen $\lambda/g^2 = 0.5$, $a_0 = 0$, $b_0 = 0.45 M_W$, and $A_0 = M_W/2$.

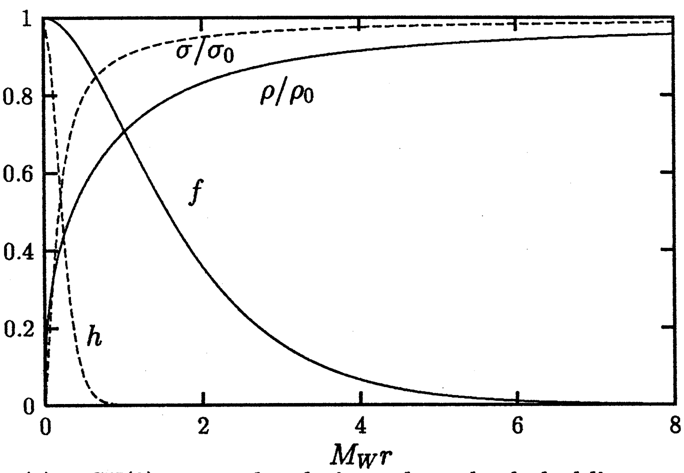

FIG. 4. The $SU(2) \times SU(2)$ monopole solution, where the dashed line represents hypercharge part which describes a Bogomol'nyi-Prasad-Sommerfield solution.

ON THE EXISTENCE OF SOLUTIONS OF THE HEAT EQUATION FOR HARMONIC MAP

DONG PYO CHI, HYUN JUNG KIM*, AND WON KUK KIM

ABSTRACT. In this paper, we prove the existence of solutions of the heat equation for harmonic map on a compact manifold with a boundary when the target manifold is allowed to have positively curved parts.

§1. Introduction

Let (M, f) and (N, γ) be Riemannian manifolds of dimension m and n respectively. Let $\{x^\alpha\}_{\alpha=1}^m$ and $\{y^i\}_{i=1}^n$ be the local coordinates of M and N, respectively, and let f defined by $f = \sum_{\alpha\beta} f_{\alpha\beta} dx^\alpha dx^\beta$ in this local expression.

Let $u : M \times [0, \infty) \to N$ be a map which is represented as $u = (u^1, \ldots, u^n)$ in terms of the above local coordinates. We say u satisfies the heat equation for harmonic maps if it is a solution of the following nonlinear parabolic systems:

$$(\Delta - \frac{\partial}{\partial t}) u^i(x,t) = f^{\alpha\beta}(x) \Gamma^i_{jk}(u(x,t)) \frac{\partial u^j}{\partial x^\alpha}(x,t) \frac{\partial u^k}{\partial x^\beta}(x,t),$$

for $i = 1, \ldots, n$, where $(f^{\alpha\beta}) = (f_{\alpha\beta})^{-1}$ and $\Gamma^i_{jk}(y)$ is the Christoffel symbol at y in N.

Let $\Lambda = (M \times \{0\}) \cup (\partial M \times [0, \infty))$, and $\Lambda_T = (M \times \{0\}) \cup (\partial M \times [0, T))$, for all $T > 0$. Let $\psi : \Lambda \to N$ be a given map. The boundary value problem of heat equation for harmonic map is to find a map $u : M \times [0, \infty) \to N$ which satisfies

$$\begin{aligned} (1.1) \qquad (\Delta - \frac{\partial}{\partial t}) u^i(x,t) &= \gamma^{\alpha\beta}(x) \Gamma^i_{jk}(u(x,t)) \frac{\partial u^j}{\partial x^\alpha}(x,t) \frac{\partial u^k}{\partial x^\beta}(x,t) \\ u|_\Lambda(x,t) &= \psi(x,t), \end{aligned}$$

for $i = 1, \ldots, n$.

Heat equation for harmonic map has been investigated by many mathematicians for many years. Eells and Sampson proved the existence of the unique solution of (1.1) when the domain manifold is a compact manifold without boundary [E-S]. R. Hamilton proved it for the case when the domain manifold is a compact manifold with a boundary, but he dealt with only the case when target manifold is negatively curved [Ha]. W. Kendall showed the the existence problem of solutions of the heat equation for harmonic map when the target

*This work was supported partially by Research Fund at Heseo University.

manifold has positive curvature parts and the domain manifols is a compact manifold with boundary in a similar fashion as R. Hamilton's [Ke]. He proved it using not analytic method, which was used in [E-S] and [Ha], but probabilistic method. The goal of this paper is to give a anlytic proof of the same results of W. Kendall. Our domain manifold is the same as Hamilton's but the target manifold is different so our method of proof is different from it. We get the solution of (1.1) by applying the Leray-Schauder degree theory to the nonlinear parabolic system. The present idea of proof is from the proof in [H-K-W], but we get the gradient estimate of solution of (1.1), which is the important part of the proof in using the Leray Schauder degree theory, in a different way. Hilderbrandt et al. [H-K-W] used the distance function on N from a fixed point as a convex function on N, because the target manifold N is only nonpositively curved. Since our target manifold N is allowed to have positive curvature parts as well we need to define a new convex function instead of the distance function.

We would like to thank Professor Hyeong In Choi, who helped us to use Leray-Scauder degree theory.

§2. Preliminaries

Suppose that (N, γ) is a Riemannian manifold with the sectional curvature bounded above by a positive constant $K > 0$. Without loss of generality, we set $K = 1$. Let $q \in N$ be given and $B_r(q)$ is the geodesic ball with raius $r \leq \{\frac{\pi}{2}, \tau\}$ and center q and where τ is the injectivity radius at q. Then $B_r(q)$ is diffeomorphic to a Euclidean ball in $\mathbb{R}^n$ with center $0 = (0, \ldots, 0)$ and radius r, the diffeomorphism being given by any normal coordinate system at q. Hence using the normal coordinates, any map $u : M \times [0, \infty) \to B_r(q)$ can be represented by vector valued functions $u = (u^1, \ldots, u^n) : M \times [0, \infty) \to \mathbb{R}^n$.

Now the notations which will be used through the present paper are introduced. Choose an orthonormal frame $\{e_\alpha, \frac{\partial}{\partial t}\}$ in a neighborhood of $(x, t) \in M \times [0, \infty)$ and an local orthonormal frame $\{f_i\}$ in a neighborhood of $u(x, t) \in N$. Let $\{\theta_\alpha, dt\}$ and $\{\omega_i\}$ be the dual coframes of $\{e_\alpha, \frac{\partial}{\partial t}\}$ and $\{f_i\}$, respectively.

Denote $d = d_M + \frac{\partial}{\partial t} dt$ is a cannonical differential on $M \times [0, \infty)$ where d_M is a differential on M. Let us define $u_{i\alpha}$ by

$$u^*(\omega_i) = \sum_\alpha u_{i\alpha}\theta_\alpha + u_{it}dt.$$

By taking thc covariant derivative of the above equation, we get $u_{i\alpha\beta}$ by

$$\begin{aligned}&\sum_\beta u_{i\alpha\beta}\theta_\beta + u_{i\alpha t}dt \\ &= du_{i\alpha} + \sum_j u_{j\alpha}u^*\omega_{ji} + \sum_\beta u_{i\beta}\theta_\beta\alpha.\end{aligned}$$

Since $du_{i\alpha} = d_M u_{i\alpha} + u_{i\alpha t}dt$,

$$\sum_\beta u_{i\alpha\beta}\theta_\beta = d_M u_{i\alpha} + \sum_j u_{j\alpha}u^*\omega_{ji} + \sum_\beta u_{i\beta}\theta_{\beta\alpha}.$$

It is well known that the heat equation for harmonic map (1.1) is equivalent to

$$u_{it} = u_{i\alpha\alpha}$$

for $i = 1, \ldots, n$.

We define the energy function $e(u)$ of u by $e(u)(x,t) = \sum_{i\alpha} u_{i\alpha}^2(x,t)$.

For $p \in \overline{B_r(q)}$, let us define a function $\phi_p : N \to \mathbb{R}$ by

$$\begin{aligned}\phi_p(y) &= \frac{1 - \cos\rho(y,p)}{\cos\rho(y,q)} \\ &= \frac{1-\cos\rho_p(y)}{\cos\rho_q(y)} =: \frac{g(y)}{h(y)}\end{aligned}$$

where $p \in \overline{B_r(q)}$ and ρ_q, ρ_p are the distance functions from q, p on N, respectively [J-Ka].

Lemma 2.1. *ϕ_p is convex for all $p \in \overline{B_r(q)}$. Furthermore,Suppose that $u : M \times [0,\infty) \to N$ is a heat equation for harmonic map with $u(M \times [0,\infty)) \subset B_r(q)$. Then*

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})(\phi_p \circ u)(x,t) &\geq \frac{1}{2}\phi_p(u(x,t))e(u)(x,t) \\ &\geq \frac{1}{2}(1-\cos\rho_p(u))e(u)(x,t) \geq 0,\end{aligned}$$

for all $(x,t) \in M \times [0,\infty)$.

Proof. We have

$$\begin{aligned}(\phi_p)_{ij} &= (\frac{g}{h})_{ij} \\ &= \frac{h^2(g_{ij}h - g_ih_j - g_jh_i - gh_{ij}) + 2hgh_ih_j}{h^4}.\end{aligned}$$

We can get $g_{ij} \geq \cos\rho_p\delta_{ij}$ and $h_{ij} \leq -\cos\rho_q\delta_{ij}$, on $B_r(q)$. Inserting these into $(\phi_p)_{ij}$, one can obtain

$$\begin{aligned}&(\phi_p)_{ij} \\ &\geq \frac{h\{\cos\rho_q\delta_{ij} + \sin\rho_p\sin\rho_q((\rho_p)_i(\rho_q)_j + (\rho_q)_i(\rho_p)_j)\} + 2(1-\cos\rho_p)\sin^2\rho_q(\rho_q)_i(\rho_q)_j}{h^3}.\end{aligned}$$

This proves $(\phi_p)_{ij}\psi^i\psi^j \geq 0$ for all functions $\psi = (\psi^i) : N \to \mathbb{R}^n$, which is the proof of convexity of ϕ_p.

Now for the convenience of notation, let $g = g \circ u$ and $h = h \circ u$. Then since

$$\begin{aligned}&(\Delta - \frac{\partial}{\partial t})(g \circ u)(x,t) \geq \cos\rho_p(u(x,t))e(u(x,t)) \\ \text{and } &(\Delta - \frac{\partial}{\partial t})(h \circ u)(x,t) \leq -\cos\rho_q(u(x,t))e(u(x,t)),\end{aligned}$$

we can get

$$\begin{aligned}
&(\Delta - \frac{\partial}{\partial t})(\psi_p \circ u) \\
&= \frac{1}{h^2}\{h(u)(\Delta - \frac{\partial}{\partial t})g + g(\Delta - \frac{\partial}{\partial t})h\} - \frac{g_\alpha^2}{2gh} + \frac{(hg_\alpha - 2gh_\alpha)^2}{2h^3 g} \\
&\geq \frac{1}{h^2}\{h(u)(\Delta - \frac{\partial}{\partial t})g + g(\Delta - \frac{\partial}{\partial t})h\} - \frac{\sin^2 \rho_p(u)}{2hg} e(u) \\
&\geq \frac{2g - \sin^2 \rho_p(u)}{2hg} e(u) \\
&= \frac{g}{2h} e(u) \geq \frac{1}{2} he(u).
\end{aligned}$$

□

Before we state the main theorems, let us introduce the following notations. Let $y = (y^1, \ldots, y^n)$ be normal coordinates at q of $B_r(q)$. If $u : M \times [0, \infty) \to B_r(q) \subset N$, u can be wrutten as $u = (u^1, \ldots, u^n)$ with respect to this normal coordnates. Then the norm $|u(x,t)|$ in $\mathbb{R}^n$ is the same as the distance $\rho(u(x,t), q)$ from q in N. We shall use the following two kinds of norms

$$\begin{aligned}
||u||_{C_T^1} &= \sup_{M \times [0,T]} |u(x,t)| + \sup_{M \times [0,T]} |D_x u(x,t)|, \\
||u||_{C_T^{1+\alpha}} &= ||u||_{C_T^1} + \sup_{M \times [0,T]} \frac{|u(x,t) - u(x',t)|}{\gamma(x,x')^\alpha} + \sup_{M \times [0,T]} \frac{|u(x,t) - u(x,t')|}{|t - t'|^\alpha},
\end{aligned}$$

where $\gamma(x, x')$ is the distance between x and x' on M, for $0 < \alpha < 1$ and $0 < T < \infty$. These are defined in the usual manner, using an arbitrary, but fixed, finite atlas of M. These two different atlases yield equivalent norms.

§3. Gradient Estimate and Existence Theorem

For any given $C^{1+\alpha}$ function $\psi : \Lambda \to B_r(q) \subset N$, we consider the following system:

$$\begin{aligned}
(\Delta - \frac{\partial}{\partial t}) u^i(x,t) &= f^{\alpha\beta}(x) \Gamma^i_{jk}(u(x,t)) \frac{\partial u^j}{\partial x^\alpha}(x,t) \frac{\partial u^k}{\partial x^\beta}(x,t), \\
u|_\Lambda(x,t) &= \psi(x,t),
\end{aligned}$$

for $i = 1, \ldots, n$.

First, we have to get the C^1- estimate of the solution of (1.1) in order to use the Leray-Schauder degree theory. C^0- estimate of the solution of (1.1) and Interior estimate of energy of the solution of (1.1) can be easily obtained by the same method as that in [Ci-C-K]. The boundary estimate of energy is obtained by a modification of the proof in [H-K-W].

Theorem 3.1. *Suppose $\psi : \Lambda \to B_r(q)$ is of class C^{1+c} and u is a solution of (1.1), where $c > 0$. Then for all $T > 0$,*

$$||u||_{C_T^{1+c}} \le C,$$

where C depends only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N.

Proof. First, we have to claim that $u(M \times [0,\infty) \subset B_\tau(p)$, that is a C^0−estimate of u.

As the same in Lemma 2.1, define $g(x,t) : M \times [0,\infty) \to \mathbb{R}$ by

$$g(x,t) = 1 - \cos(\rho(u(x,t),p)).$$

Then

$$(\Delta - \frac{\partial}{\partial t})g(x,t) \ge \cos\rho_p(u(x,t))e(u(x,t)) \ge 0.$$

Since $u|_\Lambda(x,t) = \psi(x,t) \subset B_\tau(p)$ for all $(x,t) \in \Lambda$, we can get $g|_\Lambda(x,t) < 1 - \cos\tau$. Then by the maximum principle, we have $\Gamma(x,t) \le 1 - \cos\tau$. Therefore $\cos(\rho(u(x,t),p)) \ge \cos\tau$, which implies that $u(x,t) \in B_\tau(p)$ for all $(x,t) \in M \times [0,\infty)$.

Let $x_0 \in M$ be any point and $a > 0$. Let γ be the distance function from x_0 in M and let $B_a(x_0)$ be the closed geodesic ball of radius a and center x_0 in M.Take any $T > 0$. Let $\sup_\Lambda \rho_q(\psi(x,t)) = b_1$. And we can choose a constant $b > 0$ such that $\sup_{M\times[0,\infty)} g(x,t) < b_1 < b$.

Let us consider the function

$$\Phi = \left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2} \right\},$$

which is defined on $(B_a(x_0) \cap M) \times [0,T]$.

Since $\Phi|\partial B_a(x_0) = 0$, Φ attains its maximum on $(B_a(x_0) \cap M) \times [0,T]$. Let

$$\Phi(x_1,t_1) = \max_{(B_a(x_0)\cap M)\times[0,T]} \left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g^2)^2} \right\}.$$

Then we can have the three cases : $(x_1,t_1) \in B_a(x_0) \times \{0\}$, $(B_a(x_0) - \partial M) \times (0,T]$ or $(B_a(x_0) \cap \partial M) \times (0,T]$.

In the first case i.e.$(x_1,t_1) \in B_a(x_0) \times \{0\}$,

$$\begin{aligned} \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}(x,t) &\le \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2}(x_1,t_1) \\ &\le \frac{a^4}{(b-g)^2} \sup_{\Lambda_T} e(\psi), \end{aligned}$$

for $(x,t) \in B_a(x_0) \times [0,T]$. Then we have, for $(x,t) \in B_{\frac{a}{2}}(x_0) \times [0,T]$,

$$e(u)(x,t) < \frac{16}{9}\frac{b^4}{(b-b_1)^2} \sup_{\Lambda_T} e(\psi). \tag{3.1}$$

In the second case, i.e. when $(x_1, t_1) \in B_a(x_0) \times (0, T]$, by a similar computations as in [C], we have

$$e(u)(x_1, t_1) \le 4 \max \left\{ \frac{128\gamma^2}{(a^2 - \gamma^2)^2}, \quad (b-g) + \frac{C_1(1+\gamma)(b-g)}{(a^2-\gamma^2)} + \frac{8\gamma^2(b-g)}{(a^2-\gamma^2)^2} \right\}.$$

For any $(x, t) \in B_{\frac{a}{2}}(x_0) \times (0, T]$,

$$\left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2} \right\}(x,t) \le \left\{ \frac{(a^2-\gamma^2)^2 e(u)}{(b-g)^2} \right\}(x_1, t_1) \le 4 \max \left\{ \frac{16a^2}{(b-g(x_1,t_1))^2}, \frac{a^2}{(b-g(x_1,t_1))} + \frac{C_1(1+a)a^2}{(b-g(x_1,t_1))} + \frac{8a^2}{(b-g(x_1,t_1))^2} \right\}.$$

Therefore

$$e(u)(x,t) \le 4 \max \left\{ \frac{256a^2b^2}{9(b-b_1)^2 a^4}, \frac{16Ka^4b^2}{9(b-b_1)a^4} + \frac{16C_1(1+\sqrt{K}a)a^2b^4}{9a^4(b-b_1)} + \frac{128a^2b^2}{9a^4(b-b_1)} \right\}, \tag{3.2}$$

for $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0.T]$.

We consider the last case $(x_1, t_1) \in \partial M \times [0, T]$. Let n be the outer normal vector of ∂M at (x_1, t_1), and $p = u(x_1, t_1)$. Since $u(x,t) = \psi(x,t)$ for all $(x,t) \in \Lambda_T$, $e(u)(x_1,t_1) \le C_1(||\psi||^2_{C^1_T} + ||\frac{\partial u}{\partial n}||^2_{(x_1,t_1)})$, for the same constant C_1 depending only on the geometries of M and N. Hence it suffices to get the estimate of $||\frac{\partial u}{\partial n}||_{(x_1,t_1)}$.

One can choose a sufficiently small $\delta > 0$ such that $1 - 2\sin\frac{\delta}{2} > 0$, and let $p_0 \in N$ be a point on the geodesic in the direction $\frac{\partial u}{\partial n}|_{(x_1,t_1)}$ with $\rho_p(p_0) = \delta$. Define

$$w(x,t) = f_p(u(x,t)) + \frac{1}{2}\{1 - \cos\rho(u(x,t), p_0)\} - \eta,$$

where f_p is as defined in Section 2 and η is the solution of the following equation:

$$(\Delta - \frac{\partial}{\partial t})\eta = 0$$

$$\eta|_{\Lambda_T} = \frac{1-\cos\rho_p(\psi)}{\cos\rho_q(\psi)} + \frac{1}{2}\{1 - \cos\rho(\psi, p_0)\}.$$

The well-known Schauder estimate for the partial differential equations of parabolic type implies $||\eta||_{C_T^1} \leq C(||\psi||_{C_T^{1+c}})$. Applying Lemma 2.1, we can easily get

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})w &\geq \frac{1}{2}\{1 - \cos\rho_p(u) + \cos\rho(u, p_0)\}e(u) \\ &= \frac{1}{2}\left\{1 - 2\sin\frac{\rho_p(u) + \rho(u, p_0)}{2}\sin\frac{\rho_p(u) - \rho(u, p_0)}{2}\right\}e(u) \\ &\geq \frac{1}{2}(1 - 2\sin\frac{\delta}{2})e(u) > 0,\end{aligned}$$

where the last inequality comes from the fact that $\rho_p(u) - \rho(u, p_0) \leq \rho_p(p_0) \leq \delta$ i.e. w is a subsoluiton of linear heat equation. By the easy computation, $\frac{\partial}{\partial n}|_{(x_1,t_1)}\phi_p(u(x,t)) = 0$.

Since $w|_{\Lambda_T} = 0$, and

$$\frac{\partial}{\partial n}|_{(x_1,t_1)}(1 - \cos\rho(u(x,t), p_0)) = -\sin\delta||\frac{\partial u}{\partial n}||_{(x_1,t_1)},$$

we get

$$\begin{aligned}0 \leq \frac{\partial w}{\partial n}|_{(x_1,t_1)} &= -\sin\delta||\frac{\partial u}{\partial n}||_{(x_1,t_1)} - \frac{\partial \eta}{\partial n}|_{(x_1,t_1)} \\ ||\frac{\partial u}{\partial n}||_{(x_1,t_1)} &\leq \frac{1}{\sin\delta}||\frac{\partial \eta}{\partial n}||_{(x_1,t_1)} \leq \frac{1}{\sin\delta}||\eta||_{C^1} \leq C_2||\psi||_{C_T^{1+\alpha}},\end{aligned}$$

for some constant C_2 depending only on δ and the geometries of M and N.

We have by the above computation (3.1) and (3.2),

$$\begin{aligned}e(u)(x,t) \leq 4\max\Bigg\{ &\frac{b^2}{(b^2 - b_1^2)^2}\sup_{\Lambda_T} e(\psi), \frac{256a^2b^4}{9(b^2 - b_1^2)^2a^4}, \\ &\frac{16a^4b^4}{9(b^2 - b_1^2)a^4} + \frac{16C_1(1+a)a^2b^4}{9a^4(b^2 - b_1^2)} \\ &+ \frac{128a^2b^4}{9a^4(b^2 - b_1^2)}, C_2||\psi||_{C_T^{1+c}}\Bigg\},\end{aligned}$$

for $(x,t) \in B_{\frac{a}{2}}(x_0) \times (0.T]$.

Since a is arbitrary, as a goes to infinity, we have

$$\sup_{M\times[0,T)} e(u) \leq C_3,$$

for C_3 depends only on $||\psi||_{C_T^{1+c}}$ and the geometries of M and N. □

Let $y = (y^1, \ldots, y^n)$ be normal coordinates at q of $B_r(q)$, and let h_{ij} be the metric of N with respect to this normal coordinates. For $0 \leq s \leq 1$, let us define a new metric ${}^sh_{ij}(y) = h_{ij}(sy)$. Note that ${}^1\gamma_{ij} = \gamma_{ij}$ and ${}^0\gamma_{ij}$ is a flat metric on $B_r(q)$. Furthermore, since there is no change of metric in the radial direction, y is still a normal coordinates for the metric ${}^sh_{ij}$. The Christoffel symbol ${}^s\Gamma^i_{ik}(y)$ with respect to ${}^s\gamma$ is $s\Gamma^i_{ik}(sy)$, and the heat equation for harmonic map with ${}^sh_{ij}$ on the target becomes

$$(H_s)\qquad \begin{aligned}&(\Delta - \frac{\partial}{\partial t})u^i_s(x,t) + f^{\alpha\beta}(x,t)s\Gamma^i_{jk}(su_s(x,t))\frac{\partial u^j_s}{\partial x^\alpha}(x,t)\frac{\partial u^k_s}{\partial x^\beta}(x,t) = 0\\ &u_s(x,t) = \psi(x,t), \qquad (x,t) \in \Lambda,\end{aligned}$$

Note that the solution of (H_1) is the solution of (1.1). To prove the main theorem, it is important to get the energy estimate of u_s independent of s. It is easy to check that the upper bound of the sectional curvaturre with repect to ${}^sh$ is the same as that with repect to h. And we can get the following theorem.

Theorem 3.2. *Suppose for all $0 \le s \le 1$, $\psi : \Lambda \to B_r(q)$ is of class C^{1+c} and u_s is a solution of (H_s). Then for all $T > 0$,*

$$||u||_{C^{1+c}_T} \le C,$$

where C depends only on $||\psi||_{C^{1+c}_T}$ and the geometries of M and N.

Now we prove the existence of solutions of the heat equations for harmonic using Leray-Schauder degree theory.

Theorem 3.2. *Let (M, f) be a Riemannian manifold with boundary ∂M and (N, γ) a Riemannian manifold with the sectional curvature bounded above by $K > 0$. Let $\Lambda = (M \times \{0\}) \cup (\partial M \times [0,\infty))$. To a given $C^{1+\alpha}$ function $\psi : \Lambda \to B_r(q)$, there exists the solution $u : M \times [0,\infty) \to B_r(q)$ of (1.1) in class C^3 on $M \times [0,\infty)$ and C^1 on Λ.*

Proof. Without loss generality, we may assume $K = 1$. We prove this theorem by using Leray-Schauder degree theory technique. To apply the Leray-Schauder degree theory, we need an appropriate Banach space, a bounded domain of the Banach space and a homotopy of maps. Let $T > 0$ be fixed.

First let us define the space B by the set of all C^1 maps from $M \times [0,T)$ to $\mathbb{R}^n$. Then clearly $(B, ||\cdot||)$ becomes a Banach space, where $||\cdot||$ is the C^1_T-norm.

Now, we define a homotopy of maps. Let $0 \le s \le 1$. For $u = (u^1, \ldots, u^n) \in B$, define

$${}^sF^i(u) = \sum_{i,j,\alpha,\beta} s\Gamma^i_{jk}(su) f^{\alpha\beta}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta},$$

for all $i = 1, \ldots, n$. Define $\Psi_s : B \to B$ by $\Psi_s(u) := v = (v^1, \ldots, v^n)$, where v is the solution of the following problem:

$$\begin{aligned}&(\Delta - \frac{\partial}{\partial t})v^i(x,t) = {}^sF^i(u)(x,t) \qquad \text{on } M \times [0,\infty),\\ &v|_\Lambda = 0,\end{aligned}$$

for all $i = 1, \ldots, n$.

For $u \in B$, $\Psi(u)$ is of class $C^{1+\beta}$ for some $0 < \beta < 1$ (see [F]), and Arzela-Ascoli theorem implies that Ψ is a compact mapping from B into B. Now let $h = h(\phi)$ be the uniquely determined solution of boundary value problem of the linear equation

$$\begin{aligned} &(\Delta - \frac{\partial}{\partial t})h(x,t) = 0 \qquad\qquad \text{on } M \times [0,\infty), \\ &h|_\Lambda = \psi. \end{aligned}$$

Let us define a homotopy $H_s : B \to B$ as follows,

$$H_s(u) = u - \Psi_s(u) - h.$$

By Theorem 3.2, there is a constant C_4 depending only on $||\psi||_{C_T^{1+\alpha}}$ and the geometries of M and N such that $||u_s||_{C_T^1} \le C_4$, for all the solution u of (H_s), where C_4 is independent of s. Let $D = \{u \in B \mid ||u|| \le 2C_4\}$. Here the degree of H_s is calculated with respect to the set D and the element $0 \in B$. Note that for all $0 \le s \le 1$, any solution u_s of $H_s(u) = 0$ is the solution of (H_s) on $M \times [0,T]$, which is in D and

$$\sup_{M\times[0,T)} e(u_s) \le C_4,$$

as avove. And for all $0 \le s \le 1$, $u_s \notin \partial D$, from which $\deg(H_s, D, 0)$ is well-defined and is finite. Since $\Psi_0 = 0$, a solution of $H_0(u) = 0$ is a the solution of linear heat equation with ψ on the bondary Λ, $\deg(H_0, D, 0) \ne 0$. Then the homotopy invariance of degree implies that $\deg(H_1, D, 0) \ne 0$. Since $H_1^{-1}(0)$ is not nonempty set, $H_1(u) = 0$ has the solution $u \in D$ that is the harmonic map for heat equation on $M \times [0,T)$.

Since $T > 0$ is arbitrary and the solution of (1.1) on $M \times [0,T]$ is unique (see Section 4 of IV in [Ha]), we can obtain a unique solution of (1.1) on $M \times [0,\infty)$. □

References

[C] H. Choi, *On the Liouville theorem for harmonic maps*, Proc. Amer. Math. Soc. **85** (1982), 91–94.

[Ci-C-K] D. Chi, H. Choi, and H. Kim, *Heat equation for harmonic maps of the compactification of complete manifolds*, to appear in J. Geo. Anal..

[Ce-E] J. Cheeger and D. G. Ebin, *Comparison Theorems in Geometry*, North-Holland, Amsterdam, 1975.

[Ch] S. Cheng, *Liouville theorem for harmonic maps*, Proc. Sym. Pure Math. **36** (1980), 147–151.

[E-S] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[F] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[Ha] R. Hamilton, *On homotopic harmonic maps*, Canadian J. Math. **19** (1967), 673–687.

[H-K-W] S. Hilderbrandt, H. Kaul, and K. Widman, *Harmonic mappings into Riemannian manifold with nonpositive sectional curvature*, Math. Scand. **37** (1975), 257–263.

[J-Ka] W. Jager and H. Kaul, *Uniqueness and stability of harmonic maps and their Jacobi fields*, Manuscripta Math. **28** (1979), 269–291.

[Ke] W. Kendall, *Probability, convexity, and harmonic maps with small image I: uniqueness and fine existence*, Proc. London Math. Soc. **61** (1990), 371–406.

[Li-T] P. Li and L. F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105** (1991), 1–46.

DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

DEPARTMENT OF MATHEMATICS, HOSEO UNIVERSITY, BAEBANG MYUN, ASAN 337-795, KOREA

DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, STONY BROOK, NY 11794-3651, U.S.A.

Non-topological Multivortex Solutions in the Self-Dual Chern-Simons Theories

Dongho Chae
Department of Mathematics
Seoul National University
Seoul 151-742, Korea
e-mail address: *dhchae@math.snu.ac.kr*

and

Oleg Yu. Imanuvilov
Korea Institute for Advanced Study
207-43 chungrangri-dong, Dongdaemun-ku, Seoul, Korea
e-mail address: *oleg@kais.kaist.ac.kr*

Abstract

In this lecture we present our recent results on the constructions of non-topological multivortex solutions of verious self-dual Chern-Simons-Higgs systems in $\mathbf{R}^2$ which makes the energy functional finite. Our method of proof is basically the Newton-Kantorovich iteration. Thus, not only we prove existence of solutions, but also we present a method of approximaion scheme. We also study the "Chern-Simons limit problem" for a nonrelativistic Maxwell-Chern-Simons model, and show that there is a sequence of our solutions that converges to a solutions of the non-relativistic Chern-Simons equation.

1 Relativistic Chern-Simons theory

The Lagrangian density(Hong-Kim-Pac and Jackiw-Weinberg, 1990, Phys. Rev. Lett.):

$$\mathcal{L} = \frac{\kappa}{4}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + (D_\mu\phi)\overline{(D^\mu\phi)} - \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2$$

Here

- $A_\mu (\mu = 0,1,2)$; the gauge field on $\mathbf{R}^2 \times [0,\infty)$
- $F_{\mu\nu} = \frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu$; field strength (curvature) tensor
- $\phi = \phi_1 + i\phi_2 (i = \sqrt{-1})$; complex field, called the Higgs field
- $D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$; the gauge covariant derivative
- $\varepsilon_{\mu\nu\rho}$; totally skewsymmetric tensor with $\varepsilon_{012} = 1$
- $\kappa > 0$; the Chern-Simons coupling constant
- Our metric; $(g_{\mu\nu})$ =diag$(1,-1,-1)$.
- Physical motivations of the model:
  (i) Study of vortex solutions of the Abelian Higgs model which carry both electric and magnetic charges(cf. Ginzburg-Landau model).
  (ii) A possible candidate of models for **high** T_c **superconductivity**, explanaion of the **quantum Hall effect.**
- The static energy functional:

$$\begin{aligned}\mathcal{E}(\phi, A) &= \int_{\mathbf{R}^2}\left\{\frac{\kappa^2}{4}\frac{F_{12}^2}{|\phi|^2} + \sum_{j=1}^{2}|D_j\phi|^2 + \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2\right\}dx \\ &= \int_{\mathbf{R}^2}\left\{|(D_1 \pm iD_2)\phi|^2 + |\frac{F_{12}}{\phi} \mp \frac{2}{\kappa^2}\bar{\phi}(|\phi|^2-1)|^2\right\}dx \\ &\qquad \pm \int_{\mathbf{R}^2} F_{12}\,dx\end{aligned}$$

- Lower bound of the energy:

$$\mathcal{E} \geq \left|\int_{\mathbf{R}^2} F_{12}\,dx\right|,$$

- The minimum of energy is saturated if and only if (ϕ, A), $A = (A_1, A_2)$ satisfies the **self-duality equations**, or the **Bogomol'nyi equations**:

$$\begin{cases} (D_1 + iD_2)\phi = 0 \\ F_{12} + \frac{2}{\kappa^2}|\phi|^2(|\phi|^2 - 1) = 0 \end{cases}.$$

- Natural boundary conditions:

  (i) Topological ($\int_{\mathbf{R}^2} F_{12}dx = \text{integer} \times \Phi_0$):

$$|\phi(x)| \to \ 1 \ \ as \ |x| \to \ \infty$$

  (ii) Non-Topological($\int_{\mathbf{R}^2} F_{12}dx = \text{integer} \times \Phi_0 + \alpha$):

$$|\phi(x)| \to \ 0 \ \ as \ |x| \to \ \infty$$

Following Jaffe-Taubes we introduce new variable (u, θ) by

$$\phi = e^{\frac{1}{2}(u+i\theta)}, \ \ \theta = 2\sum_{j=1}^{m} n_j arg(z - z_j)\,, \ \ z = x_1 + ix_2 \in \mathbf{C^1},$$

where z_j and $n_j (j = 1, 2, ..., m)$ are the **prescribed** zeros, called the centres of the vorticities, and their multiplicities respectively of $\phi(z)$.

$$\Delta u = \frac{4}{\kappa^2} e^u(e^u - 1) + 4\pi\sum_{j=1}^{m} n_j\delta(z - z_j),$$

with B.C. as $|x| \to \infty$:

$$u(x) \ \to \ 0 \qquad \text{(topological solutions)}$$

or

$$u(x) \ \to \ -\infty \qquad \text{(non-topological solutions).}$$

Brief history of study of the equation

(i) Topological solutions:

- R. Wang(1991, CMP), existence by variational method
- Spruck-Yang(1995, Ann. Inst. H. Poincaré), construction of maximal solution by an iteration

(ii) Periodic solutions(Vortex condesate):

- Caffarelli-Yang(1995, CMP), existence by super-sub solution method
- G.Tarantello(1996, JMP), multiple existence
- Ding-Jost-Li-Wang(1997, preprint), analysis of two vortices solution

(iii) Non-Topological Solutions:

- Spruck-Yang(1992, CMP), existence of radially symmetric solution
- Chen-Hastings-McLeod-Yang(1994, Pro. R. Soc. Lond. A.), analysis of radially symmetric solution

Our results

1. Existence of general type of non-topological solution by an iteration
2. Asymptotic decay estimates of solutions
3. Extensions of our method to more complicated Chern-Simons systems (Maxwell-Chern-Simons, Non-Abelian Chern-Simons)

Theorem 1 *Let* $\{z_j\}_{j=1}^m \subset \mathbf{C}^1$, $\{n_j\}_{j=1}^m \subset \mathbf{Z}^+$ *be arbitrarily given, and* $\beta \in (0, 2N+2)$, *where* $N = \sum_{j=1}^m n_j$. *Then, there exists a non-topological multivortex solution* (ϕ, A) *such that the function* $\phi(z)$ *has the zeros* $\{z_j\}_{j=1}^m$ *with multiplicities* $\{n_j\}_{j=1}^m$, *and the pair* (ϕ, A) *make the energy functional finite; moreover, those solutions satisfy the decay estimates*

$$|\phi|^2 + |F_{12}| + |D_1\phi|^2 + |D_2\phi|^2 = O\left(\frac{1}{|x|^{2N+4-\beta}}\right)$$

$$\text{as } |x| \to \infty.$$

2 Idea of Proof

Step 1 : The Newton- Kantorovich Theorem

Theorem 2 *Let* B_1 *and* B_2 *be Banach spaces, and* $\Omega \subset B_1$ *be an arbitrary domain,* $P : \Omega \to B_2$ *be a given mapping which has a continous second derivative in* Ω_0, *where* $\Omega_0 = \{v \in \Omega | \, \|v - v_0\|_{B_1} \leq r\}$. *Suppose, in addition, that*

(i) $\Gamma_0 = [P'(v_0)]^{-1}$ *exists and continous linear operator;*
(ii) $\|\Gamma_0(P(v_0))\| \leq \eta$ *;*
(iii) $\|\Gamma_0 P''(v)\| \leq K \quad \forall v \in \Omega_0$*;*
Then, provided

$$h = K\eta \leq \frac{1}{2}, \quad and \ r \geq r_0 = \frac{1 - \sqrt{1-2h}}{h}\eta.$$

Then the sequence $\{v_n\}_{n=0}^{\infty}$ *defined by*

$$v_{n+1} = v_n - \Gamma_0(P(v_n))$$

converges in B_1 *to a solution* v^* *to the functional equation*

$$P(v) = 0 \quad in\ \Omega.$$

Moreover,

$$\|v^* - v_0\|_{B_1} \leq r_0.$$

Step 2 : Introduction of Function Spaces

Introduce Hilbert spaces

$$X_\alpha = \left\{ u(x) \in L^2_{loc}(\mathbf{R}^2) | \int_{\mathbf{R}^2} (1 + |x|^{2+\alpha}) u^2 dx < \infty \right\},$$

equipped with the inner product

$$(u, v)_{X_\alpha} = \int_{\mathbf{R}^2} (1 + |x|^{2+\alpha}) uv dx,$$

and

$$Y_\alpha = \left\{ u \in W^{2,2}_{loc}(\mathbf{R}^2) | \ \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u}{1 + |x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbf{R}^2)} < \infty \right\}$$

equipped with the inner product

$$(u, v)_{Y_\alpha} = (\Delta u, \Delta v)_{X_\alpha} + \int_{\mathbf{R}^2} \frac{uv}{1 + |x|^{2+\alpha}} dx.$$

Lemma 1 *Let* $\alpha \in (0, 1)$*, then there exists* $C_1 > 0$ *such that for all* $v \in Y_\alpha$

$$|v(x)| \leq C_1 \|v\|_{Y_\alpha} (\ell n^+ |x| + 1) \qquad \forall x \in \mathbf{R}^2.$$

Step 3 : Liouville's equation

Let us consider the Liouville equation:

$$\Delta\rho = -\frac{4}{\kappa^2}e^{\rho}, \quad x \in \mathbf{R}^2.$$

Let $f(z)$ be an analytic function on whole of $\mathbf{C}^1$, and $\mu > 0$. Then, the function

$$\rho(x_1, x_2) = \ell n\left\{\frac{2\kappa^2\mu|f'(z)|^2}{(1+\mu|f(z)|^2)^2}\right\}, \quad z = x_1 + ix_2 \in \mathbf{C}^1,$$

where $f'(z) = \frac{\partial}{\partial z}f(z)$, is a solution of the Liouville equation beside the zeros of $f'(z)$. We define $f(\cdot)$ by

$$f(z) = \int_0^z f'(t)dt, \ \mathit{with}\ f'(z) = \prod_{j=1}^{m}(z-z_j)^{n_j}, \sum_{j=1}^{m} n_j = N, n_j \in \mathbf{Z}^+$$

Then, the function $\rho_{0,\mu}(z)$ defined by

$$\rho_{0,\mu}(z) = \ell n\frac{2\kappa^2\mu|f'(z)|^2}{(1+\mu|f(z)|^2)^2}$$

is a solution of

$$\Delta\rho_{0,\mu} = -\frac{4}{\kappa^2}e^{\rho_{0,\mu}} + 4\pi\sum_{j=1}^{m} n_j\delta(z-z_j) \qquad \text{in } \mathbf{R}^2,$$

$$\rho_{0,\mu}(z) \to -\infty \qquad as \qquad |z| \to \infty.$$

Step 4 : Functional setting of the problem

First we remove the Dirac delta functions singularities in the equation.

Defining $v = u - \rho_{0,\mu}$, we obtain:

$$\Delta v = \frac{4}{\kappa^2}e^{v+\rho_{0,\mu}}(e^{v+\rho_{0,\mu}} - 1) + \frac{4}{\kappa^2}e^{\rho_{0,\mu}}.$$

Next we introduce the mapping $P : \Omega \to X_\alpha$ defined by

$$P(v) = \Delta v - \frac{4}{\kappa^2}e^{v+\rho_{0,\mu}}(e^{v+\rho_{0,\mu}} - 1) - \frac{4}{\kappa^2}e^{\rho_{0,\mu}}.$$

Here Ω is a domain in $\tilde{Y}_\alpha \subset Y_\alpha$ to be defined later.
Then the problem of construction of a solution reduces to that of finding a solution(root) v of

$$P(v) = 0.$$

Step 5 : Existence of $\Gamma_0 = P'(0)^{-1}$ and its norm estimate

Write

$$P'(0)^{-1} = \Delta + \frac{4}{\kappa^2} e^{\rho_{0,\mu}} Id - \frac{8}{\kappa^2} e^{2\rho_{0,\mu}} Id$$

"formally" as

$$\Gamma_0 = K_\mu (Id + K_\mu^{-1} A_\mu),$$

where

$$K_\mu = \Delta + \frac{4}{\kappa^2} e^{\rho_{0,\mu}} Id,$$

$$A_\mu = -\frac{8}{\kappa^2} e^{2\rho_{0,\mu}} Id$$

If K_μ^{-1} exists, and A_μ is "small", then Γ_0 exists.

We first show that K_μ^{-1} exists on suitable $\mathcal{D}(\mathcal{K}_\mu)$.
For a given $\varepsilon > 0$ and $z_0 \in \mathbf{C}^1$ we set

$$f_\varepsilon(z, z_0) = \int_0^z f'(t)(t - z_0)^\varepsilon dt, \quad z^\varepsilon = e^{\varepsilon \ln z} = e^{\varepsilon (\ln r + i\theta)}.$$

Then, the function

$$\rho_{\varepsilon,\mu}(z) = \ell n \frac{2\kappa^2 \mu |f'_\varepsilon(z, z_0)|^2}{(1 + \mu |f_\varepsilon(z, z_0)|^2)^2}$$

satisfies the equation

$$\Delta \rho_{\varepsilon,\mu} = -\frac{4}{\kappa^2} e^{\rho_{\varepsilon,\mu}} + 4\pi \sum_{j=1}^m n_j \delta(z - z_j) + 4\pi\varepsilon\delta(z - z_0)$$

$$\rho_{\varepsilon,\mu}(z) \to -\infty \quad as \quad |z| \to +\infty.$$

We set

$$\hat{\rho}_\mu(z, z_0) = \frac{\partial \rho_{\varepsilon,\mu}(z)}{\partial \varepsilon}|_{\varepsilon=0}.$$

Then, $\frac{1}{4\pi}\hat{\rho}_\mu(z, z_0)$ is a Green's function of the linear differential operator:

$$K_\mu = \Delta + \frac{4}{\kappa^2} e^{\rho_{0,\mu}} Id.$$

Lemma 2 *Let $\mu \in (0,1)$. Then there exists a constant C independent of $z_0 \in \mathbf{C}^1$ and μ such that*

$$|\hat{\rho}_\mu(z, z_0)| \le C(|\ell n|z - z_0|| + 1) \quad z, z_0 \in \mathbf{C}^1.$$

Now we consider the operator equation

$$K_\mu v = (\Delta + \frac{4}{\kappa^2} e^{\rho_{0,\mu}})v = g \quad \text{in } \mathbf{R}^2.$$

Lemma 3 *Let $\alpha \in (0,1)$, $g \in X_\alpha$. Then there exists $\mu_0 \in (0,1)$ such that the following inequality holds*

$$\|v\|_{Y_\alpha} \le \frac{C}{\varepsilon} \mu^{-\frac{\alpha+\varepsilon}{4N+4}} \|g\|_{X_\alpha} \quad \forall(\mu, \varepsilon) \in (0, \mu_0) \times (0,1),$$

where constant C is independent of ε and μ.

We decompose Y_α as

$$Y_\alpha = Ker\, K_\mu \oplus (Ker\, K_\mu)^\perp$$

and denote $\tilde{Y}_\alpha = (Ker\, K_\mu)^\perp$.
Then the above Lemma implies that K_μ is an **isomorphism** beetwen $\tilde{Y}_\alpha$ and X_α for $\alpha \in (0,1)$. Moreover the inverse operator $K_\mu^{-1} : X_\alpha \to \tilde{Y}_\alpha$ exists and its norm satisfies the inequality

$$\boxed{\|K_\mu^{-1}\|_{\mathcal{L}(X_\alpha, \tilde{Y}_\alpha)} \le \frac{C}{\varepsilon} \mu^{-\frac{\alpha+\varepsilon}{4N+4}} \quad \forall \mu \in (0, \mu_0), \varepsilon \in (0,1).}$$

We consider another operator $A_\mu : \tilde{Y}_\alpha \to X_\alpha$ defined by

$$A_\mu v = -\frac{8}{\kappa^2} e^{2\rho_{0,\mu}} v.$$

Lemma 4 *Let $\mu \in (0, \mu_0)$, then A_μ satisfies*

$$\|A_\mu v\|_{X_\alpha} \le \frac{C}{\varepsilon}\mu^{\frac{4-\alpha-\varepsilon}{4N+4}}\|v\|_{Y_\alpha} \quad \forall v \in \tilde{Y}_\alpha,\ \forall \mu \in (0, \mu_0)$$

for all $\alpha, \varepsilon \in (0,1)$, where C is independent of μ and ε.

Step 6 : Estimate of norm of $P''(v)$

We denote

$$B_r = \{v \in \tilde{Y}_\alpha \,|\, \|v\|_{Y_\alpha} \le r\}.$$

Lemma 5 *Let $\alpha \in (0,1)$, and C_1 be the constant defined in Lemma 1. Then for all $r < r_1 = \frac{1}{C_1}(N+1)$ the mapping $P : B_r \to X_\alpha$ is well defined.*
Moreover, $P \in C^2(B_r, X_\alpha)$ and its second derivative, $P''(v) \in \mathcal{L}(\tilde{Y}_\alpha \times \tilde{Y}_\alpha, X_\alpha)$ satisfies the inequality

$$\|P''(v)\| \le \frac{C}{\varepsilon^2}\mu^{-\frac{\alpha+2\varepsilon+2C_1 r}{4N+4}} \quad \forall \mu \in (0, \mu_1),\ \forall v \in B_r,$$

where C is independent of ε, r and μ.

Step 7 : Proof of the existence

We apply the Kanorovich Theorem to our functional equation.

$$P(v) = \Delta v - \frac{4}{\kappa^2}e^{v+\rho_{0,\mu}}(e^{v+\rho_{0,\mu}} - 1) - \frac{4}{\kappa^2}e^{\rho_{0,\mu}}.$$

Set $B_1 = \tilde{Y}_{\frac{1}{8}}$, $B_2 = X_{\frac{1}{8}}$, and $v_0 \equiv 0$.
Also, we set $\alpha = \varepsilon = \frac{1}{8}$, $r = \frac{5}{16C_1}$, $\mu_1 = \mu_1(\frac{1}{8}, \frac{1}{8})$.

Then, there exists the operator $\Gamma_0 = [P'(0)]^{-1} = L_\mu^{-1}$ for all $\mu \in (0, \mu_1)$, and its norm can be estimated as

$$\|\Gamma_0\|_{\mathcal{L}(X_{\frac{1}{8}}, \tilde{Y}_{\frac{1}{8}})} \le C\mu^{-\frac{1}{16N+16}} \quad \forall \mu \in (0, \mu_1).$$

Since $P(0) = -\frac{4}{\kappa^2}e^{2\rho_{0,\mu}}$, we obtain

$$\|\Gamma_0 P(0)\|_{Y_{\frac{1}{8}}} \le \|\Gamma_0\|_{\mathcal{L}(X_{\frac{1}{8}}, \tilde{Y}_{\frac{1}{8}})}\|P(0)\|_{X_{\frac{1}{8}}}$$

$$\leq C\|\Gamma_0\|_{\mathcal{L}(X_{\frac{1}{8}},\tilde{Y}_{\frac{1}{8}})}\|A_\mu\|_{\mathcal{L}(\tilde{Y}_{\frac{1}{8}},X_{\frac{1}{8}})} \leq C\mu^{\frac{13}{16N+16}} = \eta(\mu)$$

for all $\mu \in (0, \mu_1)$.

On the other hand, we have

$$\|\Gamma_0 P''(v)\|_{Y_\alpha} \leq \|\Gamma_0\|_{\mathcal{L}(X_{\frac{1}{8}},\tilde{Y}_{\frac{1}{8}})}\|P''(v)\|$$

$$\leq C\mu^{-\frac{1}{16N+16}}\mu^{-\frac{1}{4N+4}} \leq C\mu^{-\frac{5}{16N+16}} = K(\mu)$$

for all $v \in \Omega_0 = \{v \in \tilde{Y}_{\frac{1}{8}} \mid \|v\|_{Y_{\frac{1}{8}}} < \frac{5}{16C_1}\}$.

Hence

$$h(\mu) = K(\mu)\eta(\mu) \leq C\mu^{\frac{1}{2N+2}} \quad \forall \mu \in (0, \mu_1).$$

Taking the parameter μ sufficiently small, we have

$$h(\mu) < \frac{1}{2}.$$

Thus all conditions of the Kantorovich Theorem are satisfied, and there exists a solution, $v^* \in \Omega_0$ to $P(v) = 0$ such that

$$\|v^*\|_{Y_{\frac{1}{8}}} \leq \frac{1-\sqrt{1-2h}}{h}\eta.$$

Since $\frac{1-\sqrt{1-2h}}{h}\eta \to 0$ as $\mu \to 0$, one can choose $\mu \in (0, \mu_1)$ so that

$$C_1\frac{1-\sqrt{1-2h}}{h}\eta \leq \beta,$$

where we fix $\beta \in (0,1)$.

Then, $v^*(x)$ satisfies

$$|v^*(x)| \leq \beta(\ell n^+|x| + 1),$$

and, since

$$\rho_{0,\mu}(x) = -(2N+4)\ell n|x| + o(\ell n|x|)$$

as $|x| \to \infty$, we obtain that

$$u(x) = v^*(x) + \rho_{0,\mu}(x) \leq -(2N+4-\beta)\ell n^+|x| + C$$

for all sufficiently large $|x|$.

3 Nonrelativistic Maxwell-Chern-Simons theory

- The Lagrangian density(Dunne-Trugenberger, 1991, Phys. Rev.):

$$\begin{aligned}\mathcal{L}(A,\psi,\mathcal{N}) &= -\frac{1}{4q^2}F_{\mu\nu}F^{\mu\nu} + \frac{\gamma}{4q^2}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \\ &\quad + i\overline{\psi}D_0\psi - \frac{1}{2m}D_\mu\psi\overline{D^\mu\psi} \\ &\quad - \frac{1}{2q^2}\partial_\mu\mathcal{N}\partial^\mu\mathcal{N} - |\psi|^2\mathcal{N} \\ &\quad + \frac{1}{2}\left(\frac{q}{2m}|\psi|^2 - \frac{\gamma}{q}\mathcal{N}\right)^2,\end{aligned}$$

where $\mathcal{N}$ is a new neutral(real) scalar field (self-duality $\leftrightarrow$ supersymmetry).

- The self-duality equations:

$$\begin{cases} (D_1 + iD_2)\psi = 0 \\ \partial_1 A_2 - \partial_2 A_1 + \frac{q^2}{4\gamma}|\psi|^2 - \gamma\mathcal{N} = 0 \\ (\Delta - \gamma^2)\mathcal{N} + \frac{q^2}{2}\left(1 + \frac{\gamma}{2m}\right)|\psi|^2 = 0 \end{cases}$$

with the natural B.C.(finite energy functional):

$$A(x), \psi(x), \mathcal{N}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Following Jaffe-Taubes' reduction procedure, we obtain:

$$\begin{cases} \Delta u = \frac{q^2}{2m}e^u - 2\gamma\mathcal{N} + 4\pi\sum_{j=1}^k n_j\delta(z - z_j) \\ \Delta\mathcal{N} = \gamma^2\mathcal{N} - \frac{q^2}{2}\left(1 + \frac{\gamma}{2m}\right)e^u \end{cases}$$

with B.C. as $|x| \to \infty$.

$$u(x) \to -\infty, \quad \mathcal{N}(x) \to 0 \quad \text{(non-topological)}.$$

For this system our result is:

Theorem 3 *Let* $\{z_j\}_{j=1}^k \subset \mathbf{C}^1$, $\{n_j\}_{j=1}^k \subset \mathbf{Z}^+$ *and* $\beta \in (0, 2N + 2)$, *where* $N = \sum_{j=1}^k n_j$. *Then there exists a non-topological solution* $(\psi, A, \mathcal{N})$ *of finite energy such that the function* $\psi(z)$ *has the zeros*

$\{z_j\}_{j=1}^k$ with multiplicites $\{n_j\}_{j=1}^k$.
Moreover, these solutions satisfy the decay estimate,

$$|\psi(x)|^2 + |D_1\psi(x)|^2 + |D_2\psi(x)|^2 = O\left(\frac{1}{|x|^{2N+4-\beta}}\right)$$

as $|x| \to \infty$, and the relation

$$\mathcal{N}(x) \geq \frac{q^2}{4\gamma m}|\psi(x)|^2 \qquad \forall x \in \mathbf{R}^2.$$

As $q, \gamma \to \infty$ with γ/q^2 kept fixed **(the Chern-Simons limit),** *there exists a sequence $\{\mathcal{N}_{\gamma,q}, u_{\gamma,q}\}$ of our constructed solutions, denoted by the same notation, $\{\mathcal{N}_{\gamma,q}, u_{\gamma,q}\}$ and its limit $\{\tilde{\mathcal{N}}, \tilde{u}\}$ such that for all $\varepsilon \in (0,1)$*

$$\mathcal{N}_{\gamma,q} \to \tilde{\mathcal{N}}, \quad in\ C^{0,\varepsilon}(\mathbf{R}^2), \quad u_{\gamma,q} \to \tilde{u}$$

almost everywhere in $\mathbf{R}^2$, and

$$\tilde{\mathcal{N}} = \frac{1}{2m\kappa}e^{\tilde{u}}, \quad \Delta\tilde{u} = -\frac{2}{\kappa}e^{\tilde{u}} + 4\pi\sum_{j=1}^{k} n_j\delta(z - z_j).$$

cf. Existence results with the **periodic B.C.**:

- Spruck-Yang; for restricted range of parameter γ, q, m (1997, to appear in JDE)
- G. Tarantello; for all range of the parameters (1997, preprint)

Idea of Proof

Step 1 : Transform of the system

First transform $(\mathcal{N}, u) \to (S, u)$ by

$$\mathcal{N} = \frac{q^2}{2\gamma^2}\left(1 + \frac{\gamma}{2m}\right)e^u + S,$$

then we obtain the equation:

$$\Delta u = -\frac{q^2}{\gamma} e^u - 2\gamma S + 4\pi \sum_{j=1}^{k} n_j \delta(z - z_j),$$

and then, using this, the second part of the self-duality equations is transformed into:

$$\Delta S = \gamma^2 S - \frac{q^2}{2\gamma^2}\left(1 + \frac{\gamma}{2m}\right)|\nabla u|^2 e^u + \frac{q^4}{2\gamma^3}\left(1 + \frac{\gamma}{2m}\right)e^{2u}$$
$$+\frac{q^2}{\gamma}\left(1 + \frac{\gamma}{2m}\right)Se^u.$$

Next we transform $(S, u) \to (S, v)$ by

$$u = v + \rho_{0,\mu},$$

where $\rho_{0,\mu}$ is a solution of the Liouville equation.

Then the self-duality equations become:

$$\Delta v = -ae^{v+\rho_{0,\mu}} + ae^{\rho_{0,\mu}} - 2\gamma S,$$

$$\Delta S = \gamma^2 S - b|\nabla(v + \rho_{0,\mu})|^2 e^{v+\rho_{0,\mu}} + abe^{2v+2\rho_{0,\mu}} + 2b\gamma e^{v+\rho_{0,\mu}} S,$$

where we set

$$a = \frac{q^2}{\gamma}, \quad b = \frac{q^2}{2\gamma^2}\left(1 + \frac{\gamma}{2m}\right).$$

Step 2 : Functional setting of the problem

Introduce the mapping $P(\cdot, \cdot) : B_1 \times B_2 \mapsto \tilde{B}_1 \times \tilde{B}_2$ defined by

$$P(v, S) = (\Delta v + ae^{v+\rho_{0,\mu}} - ae^{\rho_{0,\mu}} + 2\gamma S,$$
$$\Delta S - r^2 S + b|\nabla(v + \rho_{0,\mu})|^2 e^{v+\rho_{0,\mu}}$$
$$-abe^{2v+2\rho_{0,\mu}} - 2b\gamma e^{v+\rho_{0,\mu}} S).$$

We note that

$$P(0,0) = (0, b|\nabla \rho_{0,\mu}|^2 e^{\rho_{0,\mu}} - abe^{2\rho_{0,\mu}}).$$

For all $(w,R) \in B_1 \times B_2$ we have

$$P'(0,0)(w,R) = (\quad \Delta w + ae^{\rho_{0,\mu}} w + 2\gamma R, \quad \Delta R - \gamma^2 R + b|\nabla \rho_{0,\mu}|^2 e^{\rho_{0,\mu}} w + 2be^{\rho_{0,\mu}} \nabla \rho_{0,\mu} \cdot \nabla w - 2abe^{2\rho_{0,\mu}} w - 2b\gamma e^{\rho_{0,\mu}} R).$$

For all $((w_1, R_1); (w_2, R_2)) \in (B_1 \times B_2)^2$ we have

$$\begin{aligned} P''(v,S)((w_1,R_1);(w_2,R_2)) &= (ae^{v+\rho_{0,\mu}} w_1 w_2, \\ & b|\nabla(v+\rho_{0,\mu})|^2 e^{v+\rho_{0,\mu}} w_1 w_2 + 2be^{v+\rho_{0,\mu}} \nabla w_1 \cdot \nabla w_2 \\ & +2be^{v+\rho_{0,\mu}} \nabla v \cdot \nabla w_2 w_1 + 2be^{v+\rho_{0,\mu}} \nabla \rho_{0,\mu} \cdot \nabla w_2 w_1 \\ & +2be^{v+\rho_{0,\mu}} \nabla v \cdot \nabla w_1 w_2 + 2be^{v+\rho_{0,\mu}} \nabla \rho_{0,\mu} \cdot \nabla w_1 w_2 \\ & -4abe^{2v+2\rho_{0,\mu}} w_1 w_2 - 2b\gamma e^{v+\rho_{0,\mu}} S w_1 w_2 \\ & -2b\gamma e^{v+\rho_{0,\mu}} R_1 w_2 - 2b\gamma e^{v+\rho_{0,\mu}} w_1 R_2). \end{aligned}$$

Step 3 : Follow previous argument

Need to find Green's function of the linear equation:

$$(\Delta w + ae^{\rho_{0,\mu}} w + 2\gamma R, \Delta R - \gamma^2 R) = (g_1, g_2)$$

4 Relativistic Maxwell-Chern-Simons theory

- The Lagrangian density(C. Lee-K. Lee-Min, 1991, Phys. Lett.):

$$\mathcal{L}(A,\phi,\mathcal{N}) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\gamma}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho - D_\mu \phi \overline{D^\mu \phi} -\frac{1}{2}\partial_\mu \mathcal{N} \partial^\mu \mathcal{N} - q^2 \mathcal{N}^2 |\phi|^2 - \frac{1}{2}(q|\phi|^2 + \gamma \mathcal{N} - q)^2,$$

- The self-duality equations:

$$\begin{cases} A_0 = -\mathcal{N} \\ (D_1 + iD_2)\phi = 0 \\ \partial_1 A_2 - \partial_2 A_1 + q|\phi|^2 + \gamma \mathcal{N} - q = 0 \\ \Delta \mathcal{N} = \gamma q(|\phi|^2 - 1) + (\gamma^2 + 2q^2|\phi|^2)\mathcal{N} \end{cases}$$

- B.C. as $|x| \to \infty$: either

$$|\phi|^2 \to 1 \quad \text{and} \quad \mathcal{N} \to 0, \quad \text{(topological)}$$

or

$$\phi \to 0 \quad \text{and} \quad \mathcal{N} \to \frac{q}{\gamma}, \quad \text{(non-topological).}$$

By the similar Jaffe-Taubes' reduction procedure we obtain:

$$\begin{cases} \Delta u = 2q^2(e^u - 1) + 2\gamma q \mathcal{N} + 4\pi \sum_{j=1}^k n_j \delta(z - z_j) \\ \Delta \mathcal{N} = \gamma q (e^u - 1) + (\gamma^2 + 2q^2 e^u)\mathcal{N} \end{cases}$$

with B.C. as $|x| \to \infty$: either

$$u(x), \quad \mathcal{N}(x) \to 0 \quad \text{(topological),}$$

or

$$u(x) \to -\infty, \quad \mathcal{N}(x) \to \frac{q}{\gamma} \quad \text{(non-topological).}$$

For **topological** solutions existence and analysis of the Chern-Simons limit are done by C. - Kim(1996, JDE).

For **non-topological solutions** our main result is:

Theorem 4 *Let $\{z_j\}_{j=1}^k \subset \mathbf{C}^1$, $\{n_j\}_{j=1}^k \subset \mathbf{Z}^+$ and $\beta \in (0, 2N+2)$, where $N = \sum_{j=1}^k n_j$. Then there exists a solution $(\phi, A, \mathcal{N})$ of finite energy such that the function $\phi(z)$ has the zeros $\{z_j\}_{j=1}^m$ with multiplicites $\{n_j\}_{j=1}^k$. Moreover, these solutions satisfy the decay estimate*

$$|\phi(x)|^2 + |D_1\phi(x)|^2 + |D_2\phi(x)|^2 = O\left(\frac{1}{|x|^{2N+4-\beta}}\right)$$

as $|x| \to \infty$.

Idea of Proof

Step 1: Transform of the system

Transform $(u, \mathcal{N}) \to (u, S)$ by

$$\mathcal{N} = -S - \left(\frac{q}{\gamma} + \frac{2q^3}{\gamma^3}\right) e^u + \frac{q}{\gamma}$$

then the self-duality equations reduce to:

$$\Delta u = -2\gamma q S - \frac{4q^4}{\gamma^2}e^u + 4\pi \sum_{j=1}^{k} n_j \delta(z - z_j)$$

$$\Delta S = \gamma^2 S + \left(\frac{2q^3}{\gamma} + \frac{8q^5}{\gamma^3} + \frac{8q^7}{\gamma^5}\right) e^{2u} + \left(4q^2 + \frac{4q^4}{\gamma^2}\right) Se^u - \left(\frac{q}{\gamma} + \frac{2q^3}{\gamma^3}\right) |\nabla u|^2 e^u.$$

$$u(x) \to -\infty, \quad S(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Note the similarity with the nonrelativistic case.

Step 2 : Follow the argument for nonrelativistic case

5 Relativistic $SU(3)$ Chern-Simons theory

- The Lagrangian density(K. Lee, 1991, Phys. Rev. Lett.):

$$\mathcal{L} = -\frac{\kappa}{2}\varepsilon^{\mu\nu\rho} tr\left(A_\mu \partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - tr\left((D_\mu\phi)^\dagger D^\mu \phi\right) - \frac{1}{\kappa^2}\left(([[\phi,\phi^\dagger],\phi] - \phi)^\dagger([[\phi,\phi^\dagger],\phi] - \phi)\right),$$

  where ϕ and A_μ are the Lie algebra valued fields given by

$$\phi = \phi^a T^a, \qquad A_\mu = A_\mu^a T^a$$

  with $\{T^a\}_{a=1}^8$, the antihermitian generators of the Lie algebra of $SU(3)$, which consist of 3×3 martices satisfying

$$[T^a, T^b] = i f^{abc} T^c, \qquad tr(T^a T^b) = -\delta^{ab},$$

  where f^{abc} as the structure constants of $SU(3)$.

- The self-duality equations:

$$\begin{cases} D_1\phi - iD_2\phi = 0 \\ F_{12} = -\frac{1}{\kappa^2}([\phi^\dagger, [\phi, [\phi^\dagger, \phi]]] - [\phi^\dagger, \phi]) \end{cases}$$

- Dunne's algebraic ansatz(choice of other "good" basis): $\Rightarrow$

$$4\partial_{\bar{z}}\partial_z ln|\phi^a|^2 = -\frac{1}{\kappa^2}\sum_{b=1}^{2} K_{ab}|\phi^b|^2 + \frac{1}{\kappa^2}\sum_{b=1}^{2}\sum_{c=1}^{2} K_{bc}K_{ac}|\phi^b|^2|\phi^c|^2, \qquad a = 1,2, \qquad x \in$$

where

$$(K_{ab}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

is the Cartan matrix.

- Follow Jaffe-Taubes' reduction procedure: $\Rightarrow$

$$\begin{aligned} \Delta u_a &= -\frac{1}{\kappa^2}\sum_{b=1}^{2} K_{ab}e^{u_b} + \frac{1}{\kappa^2}\sum_{b=1}^{2}\sum_{c=1}^{2} K_{bc}K_{ac}e^{u_b}e^{u_c} \\ &\quad +4\pi\sum_{j=1}^{k_a} n_{aj}\delta(z - z_{aj}), \\ &\quad a = 1,2, \qquad z \in \mathbf{R}^2 \end{aligned}$$

- cf. Toda system:

$$\Delta u_a = -\frac{1}{\kappa^2}\sum_{b=1}^{2} K_{ab}e^{u_b}, \qquad a = 1,2.$$

- For **topological** solutions existence is done by Y. Yang (1998, to appear in CMP).
- Our main result:

Theorem 5 *Let* $\{z_{a,j}\}_{j=1}^{k_a}, \subset \mathbf{C}^1$, $\{n_{a,j}\}_{j=1}^{k_a} \subset \mathbf{Z}^+$, $a = 1,2$ *be arbitrarily given, and* $\beta \in (0, 2M+2)$, *where* $M = \min\{N_1, N_2\}$, $N_a = \sum_{j=1}^{k_a} n_{a,j}$, $a = 1,2$. *Then, there exists a* **non-topological** *multi-vortex solution* (ϕ^a, A^a) *of finite energy such that the function* $\phi^a(z)$ *has the zeros* $\{z_{a,j}\}_{j=1}^{k_a}$ *with multiplicities* $\{n_{a,j}\}_{j=1}^{k_a}$. *Moreover, those solutions satisfy the decay estimates*

$$tr(\phi^\dagger\phi) + tr(F_{12}^\dagger F_{12}) + tr\left((D_j\phi)^\dagger D_j\phi\right) = O\left(\frac{1}{|x|^{2M+4-\beta}}\right)$$

as $|x| \to \infty$.

Proof Perform Newton-Kantorovich iteration starting from an explicit solution of the Toda system.

References

[1] L. Caffarelli and Y. Yang, *Vortex condensation in the Chern-Simons-Higgs model:an existence theory*, Comm. Math. Phys. **168**, (1995) pp. 321-336.

[2] D. Chae and N. Kim, *Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs System*, J. Diff. Eqns, **134** , No.1, (1997), pp. 154-182.

[3] D. Chae and N. Kim, *Vortex condensates in the relativistic self-dual Maxwell-Chern-Simons-Higgs system*, submitted.

[4] D. Chae and O. Yu Imanuvilov, *The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory*, submitted.

[5] D. Chae and O. Yu Imanuvilov, *Non-Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs systems*, RIM-GARC preprint no. 97-76.

[6] D. Chae and O. Yu Imanuvilov, *Non-Topological multivortex solutions of the relativistic self-dual $SU(3)$ Chern-Simons-Higgs theory*, manuscript in preparation.

[7] X. Chen, S. Hastings, J. B. McLeod and Y. Yang, *Nonlinear elliptic equations arising from gauge field theory and cosmology*, Proc. R. Soc. Lond. A. **446**, (1994), pp. 453-478

[8] G. Dunne, *"Self-Eual Chern-Simons Theories"*, Lecture Notes in physics, **M36**, Springer -Verlag, Berlin , New York, (1995).

[9] G. Dunne and C. Trugenberger, *Self-Duality and Non-Relativistic Maxwell-Chern-Simons Solitons* Phys. Rev. D, (1991), pp. 1323

[10] J. Hong, Y. Kim, and P.Y. Pac, *Multivortex Solutions of the Abelian Chern-Simons-Higgs Theory*, Phys. Rev. Lett., **64**, (1990), pp. 2230.

[11] R. Jackiw and S. Y. Pi, *Soliton Solutions to the Gauged Nonlinear Schrödinger Equation on the Plane*, Phys. Rev. Lett., **64**, (1990), pp. 2969.

[12] R. Jackiw and E. J. Weinberg, *Self-Dual Chern-Simons Vortices*, Phys. Rev. Lett., **63**, (1990), pp. 2234.

[13] A. Jaffe and C. Taubes, *" Vortices and Monopoles"*, Birkhäuser, Boston, (1980).

[14] L. V. Kantorovich and G. P. Akilov, *Functional Analysis* 2nd ed.,Pergamon Press, (1982).

[15] C. Lee, K. Lee and H. Min, *Self-dual Maxwell-Chern-Simons solitons*, Phys. Lett. B, **252**, (1990), pp. 79-83.

[16] K. Lee, *Self-dual nonabelian Chern-Simons solitons*, Phys. Rev. Lett. **66**, (1991), pp. 553-355.

[17] J. Spruck and Y. Yang, *Topological solutions in the self-dual Chern-Simons theory: Existence and approximation*, Ann. Inst. Henri Poincaré **1**, (1995), pp. 75-97.

[18] J. Spruck and Y. Yang, *The existence of nontopological solitons in the self-dual Chern-Simons theory*, Comm. Math. Phys., **149**, (1992), pp. 361-376.

[19] J. Spruck and Y. Yang, *Existence Theorems for Periodic Non-Relativistic Maxwell-Chern-Simons Solitons*, to appear in J. Diff. Eqns.

[20] G. Tarantello, *Vortex-Condensations for a non-relativistic Maxwell-Chern-Simons theory*, preprint.

[21] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs Theory*, J. Math. Phys. **37**, (1996), pp. 3769-3796.

[22] R. Wang, *The existence of Chern-Simons Vortices*, Comm. Math. Phys., **137**, (1991), pp. 587-597.

Lecture Notes Series

1. M.-H. Kim (ed.), Topics in algebra, algebraic geometry and number theory, 1992
2. J. Tomiyama, The interplay between topological dynamics and theory of C*-algebras, 1992 ; 2nd Printing, 1994
3. S. K. Kim, S. G. Lee and D. P. Chi (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part I, 1993
   H. Kim, C. Kang and C. S. Bae (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part II, 1993
4. T. P. Branson, The functional determinant, 1993
5. S. S.-T. Yau, Complex hyperface singularites with application in complex geometry, algebraic geometry and Lie algebra, 1993
6. P. Li, Lecture notes on geometric analysis, 1993
7. S.-H. Kye, Notes on operator algebras, 1993
8. K. Shiohama, An introduction to the geometry of Alexandrov spaces, 1993
9. J. M. Kim (ed.), Topics in algebra, algebraic geometry and number theory II, 1993
10. O. K. Yoon and H.-J. Kim, Introduction to differentiable manifolds, 1993
11. P. J. McKenna, Topological methods for asymmetric boundary value problems, 1993
12. P. B. Gilkey, Applications of spectral geometry to geometry and topology, 1993
13. K.-T. Kim, Geometry of bounded domains and the scaling techniques in several complex variables, 1993
14. L. Volevich, The Cauchy problem for convolution equations, 1994
15. L. Elden and H. S. Park, Numerical linear algebra algorithms on vector and parallel computers, 1993
16. H. J. Choe, Degenerate elliptic and parabolic equations and variational inequalities, 1993
17. S. K. Kim and H. J. Choe (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part I, The first Korea-Japan conference of partial differential equations, 1993
    J. S. Bae and S. G. Lee (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part II, 1993
    D. P. Chi, H. Kim and C.-H. Kang (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part III, 1993
18. H.-J. Kim (ed.), Proceedings of GARC Workshop on geometry and topology ' 93, 1993
19. S. Wassermann, Exact C*-algebras and related topics, 1994
20. S.-H. Kye, Notes on abstract harmonic analysis, 1994
21. K. T. Hahn, Bloch-Besov spaces and the boundary behavior of their functions, 1994
22. H. C. Myung, Non-unital composition algebras, 1994
23. P. B. Dubovskii, Mathematical theory of coagulation, 1994
24. J. C. Migliore, An introduction to deficiency modules and Liaison theory for subschemes of projective space, 1994
25. I. V. Dolgachev, Introduction to geometric invariant theory, 1994
26. D. McCullough, 3-Manifolds and their mappings, 1995
27. S. Matsumoto, Codimension one Anosov flows, 1995
28. J. Jaworowski, W. A. Kirk and S. Park, Antipodal points and fixed points, 1995
29. J. Oprea, Gottlieb groups, group actions, fixed points and rational homotopy, 1995
30. A. Vesnin, On volumes of some hyperbolic 3-manifolds, 1996
31. D. H. Lee, Complex Lie groups and observability, 1996
32. X. Xu, On vertex operator algebras, 1996
33. M. H. Kwack, Families of normal maps in several variables and classical theorems in complex analysis, 1996
34. A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, 1996
35. Y. W. Lee, Introduction to knot theory, 1996
36. H. Kitahara, Some topics on Carnot-Caratheodory metrics, 1996 ; 2nd Printing (revised), 1998
37. D. Auckly, Homotopy K3 surfaces and gluing results in seiberg-witten theory, 1996
38. D. H. Chae (ed.), Proceedings of Miniconference of Partial Differential Equations and Applications, 1997
39. H. J. Choe and H. O. Bae (ed.), Proceedings of Korea-Japan Partial Differential Equations Conference, 1997
40. P. B. Gilkey, J. V. Leahy and J. G. Park, Spinors, spectral geometry, and Riemannian submersions, 1998
41. D.-P. Chi and G. J. Yun, Gromov-Hausdorff topology and its applications to Riemannian manifolds, 1998
42. D. H. Chae and S.-K. Kim (ed.), Proceedings of international workshop on mathematical and physical aspects of nonlinear field theories, 1998