수 학 강 의 록　　제 41 권

GROMOV-HAUSDORFF TOPOLOGY AND ITS APPLICATIONS TO RIEMANNIAN MANIFOLDS

DONG-PYO CHI
GABJIN YUN

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
held at the Seoul National University

Dong-Pyo Chi
Department of Mathematics,
Seoul National University,
Seoul, 151-742,
Korea

Gabjin Yun
Department of Mathematics,
Myong Ji University,
Yongin Kyunggi, 449-728,
Korea

펴낸날 : 1998년 6월 30일
지은이 : Dong-Pyo Chi and Gabjin Yun
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터 [TEL : 82-2-880-6562]

GROMOV-HAUSDORFF TOPOLOGY AND ITS APPLICATIONS TO RIEMANNIAN MANIFOLDS

Department of Mathematics, Seoul National University, Seoul, Korea 151-742
E-mail address: dpchi@math snu.ac.kr

Department of Mathematics, Myong Ji University, Yongin, Kyunggi, Korea 449-728
E-mail address: gabjin@wh.myongji.ac.kr

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

CONTENTS

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Preface

This is a survey article on the theory of Gromov-Hausdorff topology and its applications to Riemannian geometry. Classically, the Hausdorff distance is defined on the space of closed subsets of a fixed metric space.

In the beginning of $80'$s, M. Gromov ([G-L-P]) introduced generalized Hausdorff distance between abstract two metric spaces to what we shall call the Gromov-Hausdorff distance (For a precise definition, see §1.1).

Two metric spaces which are close in the Gromov-Hausdorff distance generally do not have to be topologically alike. Thus, it is interesting to find natural topological or geometric assumptions, which ensure that close spaces also are close topologically alike. In [G-L-P], M. Gromov had proved two important theorems, the precompactness theorem and the convergence theorem, and applied those to the study of the global Riemannian manifolds.

In this article, we describe compactness theorems and convergence theorems of Riemannian manifolds. There are good expositary survey articles about this topic (For examples, [Fu2], [Pet]). However, in the aspect of application to Riemannian geometry, those are focused on Riemannian manifolds with sectional curvature bounded below or pinched together with the diameter bounded above. Whereas we mainly are going to be concerned about Riemannian manifolds of Ricci curvature bounded below or pinched rather than the sectional curvature condition.

When replacing the condition of sectional curvature by Ricci curvature condition, one might face a difficulty in controlling Riemannian manifolds with Ricci curvature condition rather than those with sectional curvature condition. If one has a Riemannian manifold with sectional curvature bounded below or pinched, then one can apply the Toponogov triangle comparison theorem and Rauch comparison theorem ([C-E], [DoC], etc.) to this manifold. But one can't apply those theorems to Riemannian manifolds with Ricci curvature bounded below or pinched.

On the other hand, there is an effective and important theorem which is able to control Riemannian manifolds with Ricci curvature bounded below or pinched. Namely, the volume comparison theorem due to Bishop ([B-C]) and the relative volume comparison theorem due to Gromov ([G-L-P]) play an important role in Riemannian manifolds with Ricci curvature bounds (see § 1.4).

Also there is a tool to control those manifolds, so called "harmonic radius". The harmonic radius is the infimum of the radius of geodesic balls on a Riemannian manifold on which there exists a harmonic coordinates. The harmonic radius was introduced and studied in [An1] and [A-C].

After Gromov introduced the Gromov-Hausdorff convergence theory, many mathematician could proved several problems which are not clear so far. Now we have clearer image about the concept of Gromov-Hausdorff convergence.

This article consists of three chapters. In chapter 1, we shall discuss the Gromov-Hausdorff distance of metric spaces in a very general context and with a few geometric applications. In section 1.1, we introduce the definition of Gromov-Hausdorff distance and prove various properties about it. In section 1.2, we give an equivalent definition of the Gromov-Hausdorff distance. In section 1.3, we introduce the length space (inner metric space) and the Lipschitz distance which is stronger notion than the Gromov-Hausdorff distance.

In chapter 2, we are going to discuss precompactness theorems which is important and we shall also deal with Haudorff dimension of metric spaces. Precompactness theorems show that the Gromov-Hausdorff distance is really useful in the studying of global Riemannian geometry. Section 2.1 is devoted to prove the precompactness theorem for metric spaces and a class of Riemannian manifolds as an appliction. In section 2.2, we give a definition of the pointed Gromov-Hausdorff distance in noncompact case. In section 2.3, we deal with Hausdorff dimension and Hausdorff measure.

In chapter 3, we discuss applications of the Gromov-Hausdorff distance to some classes of Riemannian manifolds. In section 3.1, we introduce the harmonic coordinates and discuss regularity of the Riemannian metrics on a smooth manifold. In section 3.2, we define the harmonic radius and give some basic results. Section 3.3 is the main section in this article. We shall dicuss the convergence theorems of some classes of Riemaannian manifolds with Ricci curvature condition via hamonic radius.

We expect this article together with [Fu2] would be an exellent reference to those who want to study this kind of topic and research further in this direction. Finally, we would like to mention that we should omitted many important results. For these, one can refer the original papers quoted in the reference in this article.

CHAPTER I

GROMOV-HAUSDORFF TOPOLOGY

Global Riemnnian geometry has long history and many results are now known. For example, recall the classical uniformization theorem for surfaces. If M^2 is a closed oriented surface, then the uniformization theorem asserts that M^2 carries a smooth Riemannian metric of contstant Gaussian curvature equal to -1, 0 or +1. In 3-dimensional case, there is also a similar result due to W. Thurston ([Thu1-3]). But it is much more complicated and difficult . In general, let M^n be a compact smooth n-dimensional manifold. Consider the space of isometry classes of Riemannian metrics on M, which is the quotient space of Riemannian metrics by the group of diffeomorphisms of M. Now consider a subclass of it, e.g., the space of constant sectional curvature metrics, the space of Einstein metrics or the space of constant scalar curvature metrics, etc. Our main concern is to study the boundary of these moduli spaces for a suitable given topology. If the boundary is empty set, then any sequence in this space has a convergent subsequence such that the limit is also contained in the given space. However, if the boundary is not empty, then there is a sequence which degenerates and so we want to understand how this sequence does degenerate.

For example, let M be a smooth manifold and suppose $M \subset \mathbf{R}^n$ is embedded for some n. Consider metrics on M induced from *different embeddings* of M in $\mathbf{R}^n$. We could say two metrics on M are close if and only if the images of the embeddings are close in $\mathbf{R}^n$.

To develop results in Riemannian geometry in this direction, we need a topology as week as possible, but still able to describe what limits of sequences in the given space are.

§1.1 Gromov-Hausdorff distance

We start with the classical Hausdorff distance between subsets of a fixed metric space. And then we give generalized Hausdorff distance between abstract two metric spaces due to M. Gromov. Of course, we are mainly interested in Riemannian manifolds. But sometimes it is useful to work in wider classes of spaces, namely, metric spaces to analyze the limits.

Definition 1.1.1. Let Z be a metric space with metric d. For subsets A, B of Z, we define the *Hausdorff distance* between A and B in Z by

$$d_H^Z(A, B) = \inf\Big\{\epsilon : A \subset \overline{T}_\epsilon(B) \quad \text{and} \quad B \subset \overline{T}_\epsilon(A)\Big\} \tag{1.1.1}$$

where $T_\epsilon(B)$ is a tubular neighborhood of radius ϵ about B defined by $T_\epsilon(B) = \{z \in Z : d(z, B) < \epsilon\}$ and $\overline{T}_\epsilon(B)$ denotes the colsure of $T_\epsilon(B)$.

Remark 1.1.2. The Hausdorff distance is not actually a metric on space of subsets of Z. If we take the closed unit disk as $Z = A$ and open disk as B, then we have $d_H^Z(A, B) = 0$ but $A \neq B$. Of course, we consider only closed subsets and delete the closure in the definition, then it becomes a metic on the closed subsets. However even if the Hausdorff distance is not a metric, it does define a topology on the space of all subsets of Z.

M. Gromov generalized the concept of the Hausdorff distance to abstract two different metric spaces.

Definition 1.1.3. Let X, Y be arbitrary metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf_{Z}\left\{d_H^Z\left(f(X), g(Y)\right)\right\} \tag{1.1.2}$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $f : X \to Z$ and $g : Y \to Z$.

Remark 1.1.4. One can easily show that $d_{GH}(X, Y) < \infty$ for compact metric spaces X and Y. In fact, consider the disjoint union $Z = X \coprod Y$ endowed with the obvious metric on X, Y and $d(x, y) = \max\{diam(X), diam(Y)\}$ for $x \in X, y \in Y$, where $diam(X)$ denotes the diameter of X. It is easy to see that d is really a metric on $Z = X \coprod Y$. Thus, $d_{GH}(X, Y) < \infty$ by definition.

Example 1.1.5.

(1) $d_H([0, 1], \mathbf{Q}\cap[0, 1]) = 0$. Thus, two metric spaces of finite diameters whose Hausdorff distance is zero are not necessarily isometric.

(2) Let $A = \{a_1, a_2, a_3\}$ with metric $d(a_i, a_j) = 1$ for $i \neq j$ and let $B = \{b\}$ be a single set. For any isometric embeddings $f : A \to \mathbf{R}^n$, $g : B \to \mathbf{R}^n$, one can see from an equilateral triangle on $\mathbf{R}^n$ that

$$d_H^{\mathbf{R}^n}(f(A), g(B)) \geq \frac{1}{\sqrt{3}}.$$

However, we have

$$d_{GH}(A, B) = \frac{1}{2}.$$

by taking the disjoint union. In fact, defining $d(a_i, b) = 1/2$, we have $d(a_i, a_j) = d(a_i, b) + d(b, a_j)$. So d is a metric on $Z = A \coprod B$ and $d_{GH}(A, B) = 1/2$. □

Next we introduce point-set theoretic concepts which are needed to understand degeneration of a sequence of metric spaces and Riemannian manifolds.

Definition 1.1.6.

(1) For $\epsilon > 0$, a subset A of a metric space X is called an *ϵ-net* if

$$d(x, A) := \inf_{a \in A} d(x, a) \leq \epsilon \tag{1.1.3}$$

for any $x \in X$.

(2) For $\delta > 0$, a subset B of a metric space X is called *δ-separated* if

$$d(y_1, y_2) \geq \delta \quad \text{for all} \quad y_1, y_2 \in B, \; y_1 \neq y_2.$$

Obviously, for every $\epsilon > 0$ there exists a 2ϵ-separated ϵ-net. For example, every maximal 2ϵ-separated subset $Y \subset X$ is an ϵ-net.

Remark 1.1.7. From the definition, one can see immediately that if N_ϵ is an ϵ-net in X, then $d_{GH}(N_\epsilon, X) < \epsilon$.

From now on, we assume all nets consist of points.

Theorem 1.1.8 (Gromov). *Let X_i, X be metric spaces.*

(1) *Suppose $X_i \to X$ in the Gromov-Hausdorff topology, i.e., $d_{GH}(X_i, X) \to 0$. Then for any $\epsilon > 0$, any ϵ-net N of X is a quasi-isometric limit of 2ϵ-nets N_i of X_i for i sufficiently large, i.e., for $x_j, x_k \in N$ and $x^i_j, x^i_k \in N_i$,*

$$\left| \frac{d^{X_i}(x^i_j, x^i_k)}{d^X(x_j, x_k)} - 1 \right| \to 0 \quad \text{as} \quad i \to 0. \tag{1.1.4}$$

(2) *Conversely, if $\sup(diam(X_i), diam(X)) < \infty$ and for any $\epsilon > 0$, there exists an ϵ-nets $N_i \subset X_i$ as above, then $X_i \to X$ in the Gromov-Hausdorff topology.*

Proof. (1) Since $d_{GH}(X_i, X) \to 0$ as $i \to \infty$, there exist $\delta_i > 0, \delta_i \to 0$ and metric spaces Z_i, isometric embeddings $g_i : X_i \to Z_i$ and $f_i : X \to Z_i$ such that

$$d_H^{Z_i}(f_i(X), g_i(X_i)) < \delta_i. \tag{1.1.5}$$

Let $\{x_j\}_{j\in\Lambda}$ be an ϵ-net in X. Since $f_i(X) \subset T_{\delta_i}(g_i(X_i))$, there exists a set of points $\{x_j^i\} \subset X_i$ such that $d^{Z_i}(f_i(x_j), g_i(x_j^i)) < \delta_i$. Since we also have $g_i(X_i) \subset T_{\delta_i} f_i(X)$, it is easy to see that $\{x_j^i\}$ forms an $(\epsilon + 2\delta_i)$-net in X_i. In fact, if $y \in X_i$, there is a point $x \in X$ such that $d^{Z_i}(g_i(y), f_i(x)) < \delta_i$ since $g_i(X_i) \subset T_{\delta_i} f_i(X)$. Since $\{x_j\}_{j\in\Lambda}$ is an ϵ-net in X, there exists a point x_j such that $d(x, x_j) < \epsilon$. Thus

$$\begin{aligned} d(y, x_j^i) = d\left(g_i(y), g_i(x_j^i)\right) &\le d\left(g_i(y), f_i(x)\right) + d\left(f_i(x), f_i(x_j)\right) + d\left(f_i(x_j), g_i(x_j^i)\right) \\ &< \epsilon + 2\delta_i. \end{aligned}$$

Now we claim that $|d^X(x_j, x_k) - d^{X_i}(x_j^i, x_k^i)| < 2\delta_i$. One has

$$\begin{aligned} d^X(x_j, x_k) &= d^{Z_i}\left(f_i(x_j), f_i(x_k)\right) \\ &\le d^{Z_i}\left(f_i(x_j), g_i(x_j^i)\right) + d^{Z_i}\left(g_i(x_j^i), g_i(x_k^i)\right) + d^{Z_i}\left(g_i(x_k^i), f_i(x_k)\right) \\ &< 2\delta_i + d^{X_i}(x_j^i, x_k^i) \end{aligned}$$

Thus we have

$$d^X(x_j, x_k) - d^{X_i}(x_j^i, x_k^i) < 2\delta_i. \tag{1.1.6}$$

Similarly we can get

$$d^{X_i}(x_j^i, x_k^i) - d^X(x_j, x_k) < 2\delta_i. \tag{1.1.7}$$

Finally, dividing by $d^X(x_j, x_k)$, we obtain

$$\left|\frac{d^{X_i}(x_j^i, x_k^i)}{d^X(x_j, x_k)} - 1\right| < \frac{2\delta_i}{d^X(x_j, x_k)} \to 0 \quad \text{as} \quad i \to 0.$$

(2) Suppose $\{x_j\}$ is an ϵ-net in X and $\{x_j\}$ is a quasi-isometric limit of ϵ-nets $\{y^i_j\}$ in X_i. We have to find spaces Z_i and embeddings $X_i \subset Z_i, X \subset Z_i$ satisfying

$$d^{Z_i}(X_i, X) \to 0 \quad \text{as} \quad i \to \infty.$$

Let $Z_i = X \coprod X_i$ with obvious inclusoins of X_i, X in Z_i and define a metric d on Z_i as follows:

$$d^{Z_i}\Big|_{X\times X} = d^X \quad \text{and} \quad d^{Z_i}\Big|_{X_i\times X_i} = d^{X_i}$$

and for $x \in X, y \in X_i$, define

$$d(x,y) = \inf_j \left(d^X(x, x_j) + d^{X_i}(y^i_j, y) + \epsilon\right). \tag{1.1.8}$$

If d is a metric on Z_i, then we have $d^{Z_i}_H(X, X_i) < 3\epsilon$ and so $X_i \to X$ in the Gromov-Hausdorff topology. It remains to verify that d is really a metric on Z_i. We just check the triangle inequality, namely, for $x, x' \in X, y \in X_i$,

$$d(x,x') \le d(x,y) + d(y,x') \quad \text{and} \quad d(x,y) \le d(x,x') + d(x',y).$$

We have $d(x,x') \le d(x,x_j) + d(x_j,x_k) + d(x_k,x')$ by the triangle inequality and $d(x_j,x_k) \le (1+\delta_i)d(y^i_j,y^i_k)$ by assumption (quasi-isometric limit). So, choosing δ_i so that $\delta_i d(y^i_j, y^i_k) < 2\epsilon$, one has

$$\begin{aligned} d(x,x') &\le d(x,x_j) + d(y^i_j,y^i_k) + d(x_k,x') + \delta_i d(y^i_j,y^i_k) \\ &\le d(x,x_j) + d(y^i_j,y) + d(y,y^i_k) + d(x_k,x') + 2\epsilon. \end{aligned}$$

Taking the infimum over j, k, we get

$$d(x,x') \le d(x,y) + d(y,x').$$

Similarly, one can easily show that

$$d(x,y) \le d(x,x') + d(x',y). \qquad \square$$

Lemma 1.1.9. *Let X, Y be compact metric spaces and let $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^m$ be ϵ-nets in X and Y, respectively. Suppose they are related by*

$$|d(x_i, x_j) - d(y_i, y_j)| < \epsilon \tag{1.1.9}$$

Then $d_{GH}(X, Y) < 3\epsilon$.

Proof. As in the proof of Theorem 1.1.8, we define a metric on the disjoint union $X \coprod Y$ by

$$d(x, x') = d^X(x, x') \quad \text{and} \quad d(y, y') = d^Y(y, y')$$

and

$$d(x, y) = \min_{i=1,\ldots m} \left\{ d^X(x, x_i) + d^Y(y, y_i) + \epsilon \right\}$$

Then d satisfies the triangle inequality and so d is really a metric on $X \coprod Y$. Furthermore, it is easy to see $X \subset T_{3\epsilon}(Y, X \coprod Y)$ and $Y \subset T_{3\epsilon}(X, X \coprod Y)$. Hence $d_{GH}(X, Y) < 3\epsilon$. □

Example 1.1.10.

(1) Let X be a metric space with metric d. For each $\lambda > 0$, let λX denote the metric space $(X, \lambda d)$. If $diam(X, d) < \infty$, then the Gromov-Hausdorff limit of λX for $\lambda \to 0$ is a point.

(2) Let $X_n = Y \times Z_n$ and suppose $diam(Z_n) \to 0$ as $n \to \infty$. Then $d_{GH}(X_n, Y) \to 0$ as $n \to \infty$.

(3) Let $\pi : E \to M$ be a Riemannian submersion, i.e., E and M are Riemannian manifolds and π_* restricted to each normal space of the fibers $F = \pi^{-1}(x), x \in M$ is an isometry. Suppose that M and F are compact.

Let g_t be the Riemannian metric on E obtained from the original metric by multiplying the length of vertical vectors by $\sqrt{t}$. Then M is the Gromov-Hausdorff limit of $(E, g_t), t \to 0$.

(4) Any (path connected) compact set without interior in $\mathbf{R}^n$ is the Gromov-Hausdorff limit of smooth compact $(n-1)$-manifolds with extrinsic metrics. In fact, by Whitney theorem (see [Hir]), any closed set $Z \subset \mathbf{R}^n$ is the zero set of C^∞ function $f : \mathbf{R}^n \to \mathbf{R}$, i.e., $Z = f^{-1}(0)$. By Sard theorem. we can choose a sequence of regular values $\lambda_i \to 0$ so that $f^{-1}(\lambda_i)$ is a smooth $(n-1)$-manifold and one can see easily that $f^{-1}(\lambda_i) \to Z$ in the Gromov-Hausdorff topology.

Proposition 1.1.11. *Any compact Riemannian n-manifold M^n is a Gromov-Hausdorff limit of smooth compact Riemannian manifolds of dimension $\leq c \cdot n^2$ foe some constant $c > 0$.*

Proof. Applying the Nash's embedding theorem, we can embed M in $\mathbf{R}^{c \cdot n^2}$, i.e., there exists an embedding $i : M \to \mathbf{R}^{c \cdot n^2}$ such that $i^* g_o = g_M$, where g_o is the Euclidean Riemannian metric. Take the ϵ-tubular neighborhood and their boundary. Define

$$M_\epsilon = \partial \left(T_\epsilon(i(M))\right) = \left\{x \in \mathbf{R}^{c \cdot n^2} : dist(x, i(M)) = \epsilon\right\}.$$

For sufficiently small $\epsilon > 0$, M_ϵ is a smooth compact manifold with dimension $c \cdot n^2 - 1$. Give M_ϵ the extrinsic merics, i.e., $g_{M_\epsilon} = g_o|_{M_\epsilon}$. Then one can show that $M_\epsilon \to M$ as $\epsilon \to 0$. □

In all examples above, the dimension goes down under the Gromov-Hausdorff limit and we call such cases *collapsing*. However, this is of course not always true.

Example 1.1.12.

(1) Take any compact Riemannian manifold M and for $\epsilon_i \to 0$, choose ϵ_i-nets N_i in M. Then $N_i \to M$ as $i \to \infty$.

(2) Let

$$J_n = \left\{(x, y) \in \mathbf{R} \times \mathbf{R} : x = \frac{k}{n} + l \quad \text{or} \quad y = \frac{k}{n} + l, k = 0, 1, \ldots, n \quad \text{and} \quad l \in \mathbf{Z}\right\}.$$

If we give on J_n a metric $d(x, y)$, the length of shortest curve in J_n between x and y. Then as $n \to \infty$, $J_n \to (\mathbf{R}^2, ds)$, where $ds(x, y) = |x_2 - x_1| + |y_2 - y_1|$ which is a non-Riemannian metric on $\mathbf{R}^2$.

§1.2 An Equivalent Definition

In this section, we give another definition which is equivalent to the Gromov-Hausdorff topology.

Definition 1.2.1. A (not necessarily continuous) map $f : X \to Y$ between metric spaces is called an *ϵ-Hausdorff approximation* if

(1) $|d(f(x), f(y)) - d(x, y)| < \epsilon$ for all $x, y \in X$,

(2) The ϵ-neighborhood of $f(X)$ covers Y.

Then the *Hausdorff distance* $d'_{GH}(X, Y)$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations from X to Y and Y to X.

Lemma 1.2.2. *d_{GH} and d'_{GH} are equivalent on the set of all compact metric spaces (modulo isometry). In fact, we have $d_{GH} \le 9 \cdot d'_{GH}$ and $d_{GH} \ge c \cdot d'_{GH}$ for some constant $c > 0$.*

Proof. Let $f : X \to Y$ be an ϵ-Hausdorff approximation and choose an ϵ-net $\{x_i\}_{i=1}^m$ in X. Denoting $y_i = f(x_i)$, we claim $\{y_i\}_{i=1}^m$ is a 3ϵ-net in Y. In fact, if $y \in Y$ is any point, then one can choose $x \in X$ such that $d(f(x), y) < \epsilon$ since $Y = T_\epsilon(f(X))$. Since $\{x_i\}_{i=1}^m$ is an ϵ-net, there exists i such that $d(x, x_i) < \epsilon$. From the definition of Hausdorff approximation, one has

$$d(f(x), f(x_i)) < \epsilon + d(x, x_i) < 2\epsilon.$$

Thus,

$$d(f(x_i), y) \le d(f(x_i), f(x)) + d(f(x), y) < 3\epsilon.$$

Consequently, one has an ϵ-net $\{x_i\}_{i=1}^N$ in X and 3ϵ-net $\{y_i\}_{i=1}^N$ in Y. Moreover, by definition, one has also

$$|d(y_i, y_j) - d(x_i, x_j)| = |d(f(x_i), f(x_j)) - d(x_i, x_j)| < \epsilon.$$

Thus, applying Lemma 1.1.9, one has $d_{GH}(X, Y) \le 9\epsilon$.

Similarly, one can also prove $d_G H \ge c \cdot d'_{GH}$ for some constant $c > 0$. □

Lemma 1.1.9 also implies the following

Lemma 1.2.3. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then $d_{GH}(X, Y) \leq \epsilon$ if and only if there exists a metric on the disjoint union $X \coprod Y$ such that*

(1) $d\big|_X = d_X$ *and* $d\big|_Y = d_Y$

(2) $T_\epsilon(X) = X \coprod Y$ *and* $T_\epsilon(Y) = X \coprod Y$.

The notion of nets can be used defining another definition which is equivalent to the Gromov-Hausdorff distance. Note that if $\{x_i\}_{i=1}^N$ is a ϵ-net in a metric space X, then the map $f : X \to \mathbf{R}^N$ defined by

$$f(x) = (d(x, x_1), d(x, x_2), \cdots, d(x, x_N))$$

is an embedding, i.e., f is continuous and injective. It is obvious that f is continuous. To show f is one-one, let $f(x) = f(x')$ for $x, x' \in X$. Then

$$d(x, x_i) = d(x', x_i) \tag{1.2.1}$$

for all $i = 1, \cdots, N$. On the other hand, since $\{x_i\}_{i=1}^N$ is a ϵ-net, there exists j such that $x \in B(x_j, \epsilon)$. Then by (1.2.1) one has $x' \in B(x_j, \epsilon)$ and so $x = x'$.

Lemma 1.2.4. *Let X and Y be metric spaces. Then $d_{GH}(X, Y) \leq \epsilon$ if and only if there exist ϵ-nets $\{x_i\}_{i=1}^N$ in X and $\{y_i\}_{i=1}^N$ in Y such that the images of maps $f_1 : X \to \mathbf{R}^N$ and $f_2 : Y \to \mathbf{R}^N$ defined by*

$$f_1(x) = (d(x, x_1), d(x, x_2), \cdots, d(x, x_N))$$

and

$$f_2(y) = (d(y, x_1), d(y, x_2), \cdots, d(y, x_N))$$

lie in each other's ϵ-tubular neighborhoods in $\mathbf{R}^N$.

Proof. It follows from the original Definition 1.1.3 due to M. Gromov. □

Theorem 1.2.5. *Let M be a smooth compact manifold. Then there exists a positive number $\epsilon = \epsilon(M) > 0$, depending only on M, such that if $d_{GH}(M, N) < \epsilon$ and N is a smooth compact manifold, then there exists a canonical homotopy class of smooth map $f : N \to M$ which gives an ϵ-Hausdorff approximation. Furthermore, $f_* : \pi_1(N) \to \pi_1(M)$ is surjective.*

Proof. For a given smooth compact Riemannian manifold M, we can find an $\epsilon > 0$ such that all ϵ-nets $\{x_i\}_{i=1}^m$ in M have the property that the map $f_1 : M \to \mathbf{R}^m$ defined by $f_1(x) = (d(x, x_1), \cdots, d(x, x_m))$ is an embedding. (In fact, one can prove f_1 is almost an isometry with respect to the two metrics). If N is a smooth compact Riemannian manifold with $d_{GH}(N, M) \leq \epsilon$, then there exists a map $g : N \to \mathbf{R}^m$ whose image lies in an ϵ-tubular neighborhood of $f_1(M)$ by Lemma 1.2.4. But $T_\epsilon(f_1(M))$ retracts canonically onto $f_1(M)$ provided ϵ is sufficiently small. Let $r : T_\epsilon(f_1(M)) \to f_1(M)$ be the retraction. Define $F = r \circ g : N \to f_1(M)$ and $f = f_1^{-1} \circ F : N \to M$. Then it is easy to see that f is an ϵ-Hausdorff approximation. □

§1.3 Length space and Lipschitz distance

In this section, we will define Lipschitz distance which is stronger than the Gromov-Hausdorff topology in some sense. And then we also define special types of metric spaces, called length spaces which are lying between metric spaces and Riemannian manifolds. Length spaces play an important role in the Gromov-Hausdorff topology. Those spaces could be a limit of a sequence of Riemannian manifolds with curvautre bounded below.

Definition 1.3.1. Let X and Y be metric spaces. The *dilation* of a map $f : X \to Y$ is the (possibly infinite) number

$$dil(f) = \sup_{x_1 \neq x_2} \frac{d^Y(f(x_1), f(x_2))}{d^X(x_1, x_2)} \tag{1.3.1}$$

For a point $x \in X$, the *local dilation* of f at x is defined by

$$dil_x(f) = \lim_{\epsilon \to 0} dil\left(f\Big|_{B_\epsilon(x)}\right). \tag{1.3.2}$$

If $c = dil(f) < \infty$, then we say that f is a Lipschitz map with Lipschitz constant c. A Lipschitz homeomorphism is a homeomorphism f such that f and f^{-1} are both Lipschitz maps.

Lemma 1.3.2. *If $f : [a, b] \to X$ is a Lipschitz map of an interval $[a, b]$ into a metric space X, then the function $t \to dil_t(f)$ is measurable and bounded.*

Proof. It is easy to prove and we leave the proof to the reader. □

Definition 1.3.3. For a Lipschitz map $f : [a, b] \to X$ of an interval $[a, b]$ into a metric space X, define the *length* of f by

$$l(f) = \int_a^b dil_t(f)\, dt. \tag{1.3.3}$$

If f is only continuous, then we are able to define $l(f)$ as

$$\sup_{a=t_o<t_1<\cdots<t_{n+1}=b} \sum_{i=0}^{n} d(f(t_i), f(t_{i+1})) \tag{1.3.4}$$

for any subdivision on $[a, b]$.

Remark 1.3.4. If ϕ is a homeomorphism of a closed interval I' onto $[a, b]$, then we have $l(f \circ \phi) = l(f)$. In fact, it is true with weaker condition that ϕ is just stirctly monotone. This shows that $l(f)$ is invariant under the change of parameter.

Two definitions are equivalent when f is absolutely continuous (cf. [Rin], p. 106). This allows us to define $l(f)$ as the integral of the dilation when f is a Lipschitz map.

Definition 1.3.5. Let (X, d) be a metric space. We define the *distance of length on* X *by*

$$d_l(x, y) = \inf\Big\{l(\gamma) : \gamma(0) = x, \quad \gamma(1) = y\Big\}, \tag{1.3.5}$$

where the infimum is taken over all continuous curves in X *joining* x *and* y.

Remark 1.3.6. In general, there is no reason for that $d_l = d$ for a given metric space (X, d). In fact, the topologies induced by d and d_l may be different (cf. [G-L-P]).

Definition 1.3.7. A metric space (X, d) is called a *length space* (or *inner metric space*) if $d = d_l$.

Example 1.3.8.

(1) Any Riemannian manifold with a Riemannian metric is a length space.

(2) For the induced metric (extrinsic metric) in $\mathbf{R}^2$, $\mathbf{R}^2 - \{point\}$ is a length space but $\mathbf{R}^2 - \{segment\}$ is not.

(3) The sphere S^n is not a length space for the induced metric (extrinsic)in $\mathbf{R}^{n+1}$. Generally, if $M \subset \mathbf{R}^n$ is a smooth manifold, then M with the induced metric (extrinsic metric) in $\mathbf{R}^n$ is never Riemannian metric unless M is an affine space.

Theorem 1.3.9. *If (X, d) is a complete metric space satisfying the following condition: For any $x, y \in X$ and any $\epsilon > 0$, there exists an element $z \in X$ such that*

$$\max\Big\{d(x,z), \quad d(y,z)\Big\} \leq \frac{1}{2}d(x,y) + \epsilon. \tag{1.3.6}$$

Then X is a length space.

Conversely, any length space satisfies the condition (1.3.6).

Proof. Given $x, y \in X$ and any $\epsilon > 0$, choose $\epsilon_k > 0$ such that $\sum_k \epsilon_k = c < \infty$ and so that $\prod_{k=1}^{\infty}(1+\epsilon_k) < \infty$. Let $\delta = d(x,y)$. By (1.3.6), there exists $z_{1/2} \in X$ such that

$$\max\Big\{d(x, z_{1/2}), \quad d(y, z_{1/2})\Big\} \leq \frac{1}{2}\delta + \epsilon_1\frac{\delta}{2} = \frac{\delta}{2}(1+\epsilon_1).$$

Then there exist $z_{1/4}, z_{3/4} \in X$ such that

$$\max\Big\{d(x, z_{1/4}), \quad d(y, z_{3/4})\Big\} \leq \frac{1}{2}(\frac{\delta}{2}(1+\epsilon_1)) + \frac{\epsilon_2}{2}(\frac{\delta}{2}(1+\epsilon_1))$$

and

$$\max\Big\{d(z_{1/2}, z_{3/4}), \quad d(z_{3/4}, y)\Big\} \leq \frac{1}{2}(\frac{\delta}{2}(1+\epsilon_1)) + \frac{\epsilon_2}{2}(\frac{\delta}{2}(1+\epsilon_1))$$

by assumption. So, we have

$$\max\Big\{d(x, z_{1/4}), d(z_{1/4}, z_{1/2}), d(z_{1/2}, z_{3/4}), d(y, z_{3/4})\Big\}$$
$$\leq \frac{1}{2}(\frac{\delta}{2}(1+\epsilon_1)) + \frac{\epsilon_2}{2}(\frac{\delta}{2}(1+\epsilon_1)) = \frac{\delta}{2^2}(1+\epsilon_1)(1+\epsilon_2).$$

Continuing this process inductively, we get a map $f : \mathcal{R}[2] \subset [0,1] \to X$ such that

$$d\left(f(\frac{p}{2^n}), f(\frac{p+1}{2^n})\right) \leq \frac{\delta}{2^n}\prod_{k=1}^{\infty}(1+\epsilon_k) < \infty,$$

where $\mathcal{R}[2]$ is a set of rational numbers modulo 2. Since X is complete, f can be extended to a continuous map $\tilde{f} : [0,1] \to X$. Furthermore, we can choose $c = \sum_k \epsilon_k$ to be arbitrary close to 0 so that $\prod_{k=1}^{\infty}(1+\epsilon_k)$ is arbitrary close to 1. Hence $\tilde{f}$ is what we want and $l(\tilde{f})$ is arbitrary close to $d(x,y) = \delta$ and so X is a length space.

Conversely, suppose X is a length space and let $x, y \in X$ and $\epsilon > 0$ be given. Then since $d(x,y) = \inf_\gamma l(\gamma)$, it is easy to see that there exists $z \in X$ satisfying (1.3.6). □

Theorem 1.3.10. *Let X be a complete metric space and suppose $d_{GH}(X_i, X) \to 0$ as $i \to \infty$. If X_i's are complete length spaces, then X is a length space.*

Proof. It is enough to show that X satisfies the condition (1.3.6). Let $x, y \in X$ and $\epsilon > 0$ be given. By Definition 1.2.1, there exists an $\epsilon/8$-Hausdorff approximation[1] $f : X \to X_i$ such that

(i) $|d(f(x_1), f(x_2)) - d(x_1, x_2)| < \epsilon/8$,

(ii) $X_i = T_{\epsilon/8}(f(X))$.

[1] Rigorously speaking, it is not $\epsilon/8$-Hausdorff approximation. However since the original Gromov-Hausdorff topology is equivalent to it, we use the same constant without any modification

So, there exist points $x_i, y_i \in X_i$ such that

$$d(f(x), x_i) < \epsilon/8 \quad \text{and} \quad d(f(y), y_i) < \epsilon/8. \tag{1.3.7}$$

Since X_i is a complete length space, there is a point $z_i \in X_i$ such that

$$\max\Big\{d(x_i, z_i), d(y_i, z_i)\Big\} \leq \frac{1}{2}d(x_i, y_i) + \epsilon/8. \tag{1.3.8}$$

We also have

$$\begin{aligned} d(x_i, y_i) &\leq d(x_i, f(x)) + d(f(x), f(y)) + d(f(y), y_i) \\ &< \epsilon/4 + d(f(x), f(y)) < 3\epsilon/8 + d(x, y). \end{aligned}$$

Thus, from (1.3.8) we get

$$\max\Big\{d(x_i, z_i), \quad d(y_i, z_i)\Big\} < \frac{\epsilon}{2} + \frac{1}{2}d(x, y).$$

On the other hand, since $z_i \in X_i = T_{\epsilon/8}(X)$, there exists an element $z \in X$ such that $d(f(z), z_i) < \epsilon/8$. Thus, by triangle inequality and (1.3.7), we get

$$\begin{aligned} d(x, z) &< d(f(x), f(z)) + \epsilon/8 \\ &\leq d(f(x), x_i) + d(x_i, z_i) + d(z_i, f(z)) + \epsilon/8 \\ &< \epsilon + \frac{1}{2}d(x, y). \end{aligned}$$

Similarly, one can prove

$$d(z, y) < \epsilon + \frac{1}{2}d(x, y).$$

Hence

$$\max\Big\{d(x, z), \quad d(z, y)\Big\} \leq \frac{1}{2}d(x, y) + \epsilon$$

and so X is a length space. $\square$

Definition 1.3.11. Let (X, d) be a length space.

(1) A *minimizing geodesic* γ is a continuous curve $\gamma : [a, b] \to X$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for any $t, t' \in I = [a, b]$.

(2) γ is a *geodesic* if $\gamma\big|_{I'}$ is a minimizing geodesic for any $I' \subset I$ sufficiently small.

Definition 1.3.11 (2) does not look clear but the meaning is obvious, that is, it means for any point a in I there exists a sufficiently small interval I' of a in I such that $\gamma\big|_{I'}$ is a minimizing geodesic.

Recall the Hopf-Rinow theorem shows a Riemannian manifold is complete if and only if any two points can be joined by a minimizing geodesic. We have a theorem analogue to the Hopf-Rinow theorem in length spaces.

Lemma 1.3.12. *Let (X, d) be a complete length space and let $a \in X$ and $0 < \rho$. If $\overline{B}(a, r)$ is compact for all $r, 0 \leq r < \rho$, then $\overline{B}(a, \rho)$ is also compact.*

Proof. Recall for a metric space, compactness is equivalent to limit point compactness or sequential compactness.

Let (x_n) be any sequence in $\overline{B}(a, \rho)$. One has to show (x_n) has a convergent subsequence in $\overline{B}(a, \rho)$. Note that one may assume $d(a, x_n) \to \rho$ as $n \to \infty$. In fact, if there exists a $r_o, 0 < r_o < \rho$ such that $\overline{B}(a, r_o)$ contains infinitely many x_n, then (x_n) has a convergent subsequence since $\overline{B}(a, r_o)$ is compact.

Choose a sequence (ϵ_k) with $\epsilon_k > 0, \epsilon_k \to 0$ as $k \to \infty$ and assume $0 < \epsilon_k < \rho$. For each k, there exists $N = N(k) \in \mathbf{N}$ such that $|d(a, x_n) - \rho| < \epsilon_k$ for all $n \geq N$ since $d(a, x_n) \to \rho$. So, $\rho - \epsilon_k < d(a, x_n) < \rho$. Thus there is a point $y_n^k \in B(a, \rho - \epsilon_k)$ such that $d(x_n, y_n^k) < \epsilon_k$. Consequently for fixed k, $(y_n^k) \subset B(a, \rho - \epsilon_k)$, $n \geq$

$N(k)$. Since $\overline{B}(a, \rho - \epsilon_k)$ is compact, (y_n^k) has a convergent subsequence $(y_{n_i}^k)$. Thus

$$\begin{aligned} d(x_{n_i}, x_{n_j}) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}^k, y_{n_j}^k) + d(y_{n_j}^k, x_{n_j}) \\ &< \epsilon_{n_i} + \epsilon_{n_j} + d(y_{n_i}^k, y_{n_j}^k). \end{aligned}$$

That is, (x_{n_i}) is a Cauchy sequence. Since X is complete, (x_{n_i}) converges to a limit $x \in \overline{B}(a, \rho)$ and so $\overline{B}(a, \rho)$ is sequentially compact. □

Theorem 1.3.13. *Suppose that X is a complete, locally compact length space. Then we have*

(1) *Closed balls in X are compact, or equivalently, any closed and bounded set is compact in X.*
(2) *Given any $x, y \in X$, there exists a minimal geodesic $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proof.

(1) First note that since X is locally compact, for $a \in X$, closed ball $\overline{B}(a, r)$ is compact for $r > 0$ sufficiently small. We claim that

$$\sup\left\{ r \,\middle|\, \overline{B}(a, r) \text{ is compact} \right\}$$

is infinite. Suppose $\sup\left\{r \middle| \overline{B}(a,r) \text{ is compact}\right\} = \rho < \infty$. Then $\overline{B}(a, \rho)$ is compact by Lemma 1.3.12 and so is $S(a, \rho) = \{x \in X | d(x, a) = \rho\}$. The remainder is standard point-set topological argument. By choosing a finite covering of $S(a, \rho)$ by compact balls, we can find $\rho' > \rho$ such that $\overline{B}(a, \rho')$ is compact.

(2) Suppose first X is compact. Since X is a length space, given $x, y \in X$, there exist curves $\gamma_n : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ such that

$$l(\gamma_n) \leq d(x, y) + \frac{1}{n}. \tag{1.3.9}$$

We may assume γ_n is parametrized proportional to arc length. Thus, $\{\gamma_n\}$ are equicontinuous. Since X is compact, by Arzela-Ascoli theorem (see §3.3), there exists a subsequence of $\{\gamma_n\}$ which converges to a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since

$$d(x,y) \leq l(\gamma) \leq \liminf_{n\to\infty} l(\gamma_n) = d(x,y),$$

one has $d(x,y) = l(\gamma)$.

Now assume X is just locally compact which is not necessarily compact. Note that $\gamma_n \subset \overline{B}(a, 2d(x,y))$ from (1.3.9). Since $\overline{B}(a, 2d(x,y))$ is compact, by above, one can conclude $d(x,y) = l(\gamma), \quad \gamma \subset \overline{B}(a, 2d(x,y))$. □

Corollary 1.3.14. *Let X be a compact length space. Then for any free homotopy class in X, there exists a minimal geodesic of least length. Similarly, if one fix a base point $x_o \in X$, then there is a geodesic of least length based at x_o in given class in $\pi_1(X, x_o)$.*

Proof. It follows from the same argument as Theorem 1.3.13. □

Definition 1.3.15. The *Lipschitz distance* between metric spaces X and Y is defined by

$$d_L(X,Y) = \inf\{\, |\log dil(f)| + |\log dil(f^{-1})|\,\}, \tag{1.3.10}$$

where the infimum is taken over all Lipschitz homeomorphisms $f : X \to Y$. If no such homeomorphisms exist, we set $d_L(X,Y) = \infty$.

It is clear that d_L is symmetric, i.e., $d_L(X,Y) = d_L(Y,X)$.

Proposition 1.3.16. *If X and Y are compact metric spaces and $d_L(X,Y) = 0$, then X and Y are isometric.*

Proof. The assumption $d_L(X,Y) = 0$ implies that for each $n \in \mathbf{N}$, there exists a Lipschitz homeomorphism $f_n : X \to Y$ such that

$$1 - \frac{1}{n} < dil(f_n) < 1 + \frac{1}{n}. \tag{1.3.11}$$

Thus $\{f_n\}$ is equicontinuous. Since X and Y are compact, Arzela-Ascoli theorem implies that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a Lipschitz homeomorphism f such that $\{f_{n_k}\}$ converges to f. Obviously, $dil(f) = 1$ from (1.3.11) and hence f is an isometry. □

Theorem 1.3.17. *Suppose X, Y are compact metric spaces. If $d_{GH}(X,Y) = 0$, then X and Y are isometric.*

Proof. It suffices to show that $d_{GH}(X,Y) = 0$ implies $d_L(X,Y) = 0$ by Theorem 1.3.16. For each $n \in \mathbf{N}$ and for any $\frac{2}{n}$-net $N^{\frac{2}{n}}$ of Y, there exists a sequence of $\frac{1}{n}$-nets $N_i^{\frac{1}{n}}$ of X such that $N_i^{\frac{1}{n}} \to N^{\frac{2}{n}}$ in the Lipschitz distance by Throrem 1.1.8.

For each $\epsilon > 0$, define $f_\epsilon : Y \to X$ as follows:

For any $\epsilon > 0$, there exists an integer i_k and a function $f_{i_k} : N^{\frac{2}{n}} \to N_{i_k}^{\frac{1}{n}}$ such that

$$\max\{dil(f_{i_k}), dil(f_{i_k}^{-1})\} < 1 + \epsilon. \tag{1.3.12}$$

For each $y \in Y$ and for each n, there is an element $y_n \in N^{\frac{2}{n}}$ such that $d(y, y_n) < 2/n$. Then $f_{i_k}(y_n)$ is a sequence in $N_{i_k}^{\frac{1}{n}} \subset X$ by (1.3.12). Define

$$f_\epsilon(y) = \lim_{n \to \infty} f_{i_k}(y_n).$$

One can show that f_ϵ is an ϵ-isometry. In fact, since $\max\{dil(f_{i_k}), dil(f_{i_k}^{-1})\} < 1 + \epsilon$, one has $dil(f_\epsilon) \leq 1 + \epsilon$. Since ϵ is arbitrary, by diagonal method, we have $d_L(X,Y) = 0$. □

Corollary 1.3.18. *The Gromov-Hausdorff distance d_{GH} is a metric on the isometric classes of compact metric spaces.*

We remark, however, that without compactness the Gromov-Hausdorff distance d_{GH} is not a metric anymore.

Corollary 1.3.19. *Let (X_n) be a sequence of compact metric spaces and X be a metric space. If $d_L(X_n, X) \to 0$ as $n \to \infty$, then $d_{GH}(X_n, X) \to 0$ as $n \to \infty$.*

Proof. It follows from Theroem 1.1.8 (2). □

Remark 1.3.20. The converse of Corollary 1.3.19 is obviously wrong. If $d_L(X_n, X) \to 0$ as $n \to \infty$ and X_n is compact for all n, then obviously X must be compact. However, as mentioned above, the Gromov-Hausdorff distance d_{GH} just becomes a *pseudometric* on the set of all metric spaces and the limit of a sequence of compact metric spaces with respect to the Gromov-Hausdorff topology may be noncompact.

CHAPTER II

PRECOMPACTNESS THEOREMS AND HAUSDORFF DIMENSION

In this chapter, we shall discuss precompactness theorems, the pointed Gromov-Hausdorff distance between noncompact metric spaces and Hausdorff dimension of metric spaces. Also we are going to deal with applications of those to Riemannian manifolds. Precompactness theorem has two types. One is about metric spaces and another is about Riemannian manifolds as an application. These theorems are very important in studying global Riemannian geometry.

For unbounded metric spaces, the Gromov-Hausdorff distance is not useful and so we have to modify it appropriately. It is so called the *pointed* Gromov-Hausdorff distance.

In the last section of this chapter, we are going to discuss Hausdorff dimension for metric spaces. The Hausdorff dimension of a space need not be an integer. However this might be used to measure the structure of limit spaces of a sequence of metric spaces or Riemannian manifolds.

§2.1 Precompactness Theorem

In this section, we shall prove compactness theorems about metric spaces and some classes of Riemannian manifolds as application.

Definition 2.1.1. Let X be a compact metric space. For $\epsilon > 0$, define $Cov(X, \epsilon)$ as the minimal number of closed ϵ-balls needed to cover X and $Cap(X, \epsilon)$ as the maximal number of disjoint ϵ-balls in X. $Cov(X, \epsilon)$ is called ϵ-*covering* and $Cap(X, \epsilon)$ is called ϵ-*capacity* of X.

Lemma 2.1.2. *For a compact metric space X and $\epsilon > 0$,*

$$Cov(X, 2\epsilon) \leq Cap(X, \epsilon).$$

Proof. Let $N = Cap(X, \epsilon)$ and let $\{B_i = B(x_i, \epsilon)\}_{i=1}^N$ be a maximal disjoint ϵ-balls in X. Then it is easy to see that $\{B(x_i, 2\epsilon)\}_{i=1}^N$ cover X and so $Cov(X, 2\epsilon) \leq N$ by definition. □

Lemma 2.1.3. *Let X and Y be compact metric spaces. If $d_{GH}(X, Y) < \delta$, then for any $\epsilon > 0$, one has the estimates*

$$Cov(X, \epsilon) \geq Cov(Y, \epsilon + 2\delta)$$

and

$$Cap(X, \epsilon) \geq Cap(Y, \epsilon + 2\delta)$$

Proof. Suppose X is covered by N ϵ-balls, say, $\{B_i = B(x_i, \epsilon)\}_{i=1}^N$ so that $N = Cov(X, \epsilon)$. By assumption, we have a δ-Hausdorff approximation $f : X \to Y$. The same argument as Lemma 1.2.2 shows that the balls $\{B(f(x_i), \epsilon + 2\delta)\}$ of radius $\epsilon + 2\delta$ about $f(x_i)$ cover Y. Similarly, one can prove the second inequality. □

Let $(\mathcal{M}et, d_{GH})$ be the set of all isometric classes of compact metric spaces with Gromov-Hausdorff distance. One of the most important properties for Gromov-Hausdorff topology is convergence of a sequence of metric spaces or Riemannian manifolds in $\mathcal{M}et$. First, we shall prove the space $\mathcal{M}et$ is complete with respect to the Gromov-Hausdorff distance d_{GH}.

Theorem 2.1.4. *$(\mathcal{M}et, d_{GH})$ is complete, i.e., every Cauchy sequence in $(\mathcal{M}et, d_{GH})$ converges to a limit in $\mathcal{M}et$.*

Proof. Let $\{X_i\}$ be a Cauchy sequence. It suffices to show that some subsequence converges, so we can without loss of generality assume that $d_{GH}(X_i, X_{i+1}) < 2^{-i}$ for all $i = 1, 2, \ldots$. Then choose metrics $d^{i,i+1}$ on $X_i \coprod X_{i+1}$ such that $d_H^{i,i+1}(X_i, X_{i+1}) < 2^{-i}$. With these choices we can construct metrics $d^{i,j}$ on $X_i \coprod X_j$, where $i < j$ as follows

$$d^{i,j}(x, y) = \inf\Big\{\sum_{k=i}^{j-1} d^{k,k+1}(x_k, x_{k+1}) : x_k \in X_k \quad \text{and} \quad x_i = x, x_j = y\Big\}.$$

These metrics clearly satisfy

$$d^{i,k}(x_i, x_k) \le d^{i,j}(x_i, x_j) + d^{j,k}(x_j, x_k)$$

if $i \le j \le K$ and $x_i \in X_i, x_j \in X_j, x_k \in X_k$. Therefore

$$d_H^{i,j}(X_i, X_j) \le \sum_{k=i}^{j-1} d_H^{k,k+1}(X_k, X_{k+1}) \le 2^{-i+1} \quad \text{if} \quad i \le j.$$

Let $\hat{X} = \{(x_j) : x_j \in X_j \quad \text{and} \quad d^{i,j}_{(x_i, x_j)} \to 0 \quad \text{as} \quad i, j \to 0\}$. There is a pseudometric on $\hat{X}$ defined by $d((x_j), (y_j)) = \lim_{j\to\infty} d(x_j, y_j)$. We contend that the metric space X, obtained from $\hat{X}$ by identifying points which have zero distance, is the limit of $\{X_i\}$.

Construct a metric d^i on $X_i \coprod X$ by $d^i(y, (x_j)) = \limsup_{j\to\infty} d^{i,j}(y, x_j)$, where $y \in X_i$ and (x_i) represents an element in X. This is easily seen to give a well defined metric on $X \coprod X_i$. We claim that $d_H^i(X, X_i) < 2^{-i+2}$. Let (x_j) represent an element in X. Choose $n \geq i$ such that $d^n(x_n, (x_j)) < 2^{-i}$, and then $y \in X_i$ with $d^{i,n}(y, x_n) \leq 2^{-i+1}$. Thus

$$\begin{aligned} d^i(y, (x_j)) &= \limsup_{j\to\infty} d^{i,j}(y, (x_j)) \\ &\leq \limsup_{j\to\infty} d^{i,n}(y, x_n) + d^{n,j}(x_n, x_j) \\ &\leq d^{i,n}(y, x_n) + d^n(x_n, (x_j)) \leq 2^{-i} + 2^{-i+1} \leq 2^{-i+2}. \end{aligned}$$

Conversely suppose $y \in X_i$. We can then successively find $x_j \in X_j, j \geq i$ and $y = x_i$ and $d^{j,j+1}(x_j, x_{j+1}) < 2^{-j}$. The sequence (x_j) then defines an element in X and by construction

$$d^i(y, (x_i)) = \limsup_{j\to\infty} d^{i,j}(y, x_j) \leq \lim_{j\to\infty} \sum_{k=i}^{j-1} 2^{-k} = 2^{-i+1}.$$

□

Definition 2.1.5. A metrix space X is called *totally bounded* if for any $\epsilon > 0$, there exists a finite number $N(\epsilon)$ of ϵ-balls $\{B_i = B(x_i, \epsilon)\}_{i=1}^{N(\epsilon)}$ which cover X.

Note that a subspace of a metric space which is precompact is totally bounded. Furthermore, one has the following easy property.

Lemma 2.1.6. *A family $\mathcal{C} \subset \mathcal{M}et$ of metric spaces is totally bounded if and only if for any $\epsilon > 0$, there exists a finite ϵ-net in $\mathcal{C}$.*

Proof. Recall $\mathcal{M}et$ is the set of all isometric classes of compact metric spaces and the Gromov-Hausdorff distance d_{GH} becomes a metric on it by Corollary 1.3.18.

Suppose $\mathcal{C}$ is totally bounded. Then for any $\epsilon > 0$ there is a finite number $N = N(\epsilon)$ of ϵ-balls $\{\mathcal{B}_i = B^{GH}(X_i, \epsilon)\}_{i=1}^N$ which cover $\mathcal{C}$, where $X_i \in \mathcal{C}$ and $B^{GH}(X_i, \epsilon) = \{Y \in \mathcal{C} : d_{GH}(X_i, Y) < \epsilon\}$. Thus $\{X_i\}_{i=1}^N$ is an ϵ-net in $\mathcal{C}$. To prove the converse, let $\epsilon > 0$ be given. Then there exists a finite $\epsilon/2$-net $\{X_i\}$. It is then easy to see $\{B^{GH}(X_i, \epsilon)\}$ covers $\mathcal{C}$. □

Theorem 2.1.7 (Gromov Precompactness Theorem I). *Let $\mathcal{C} \subset \mathcal{M}et$ be a family. The followings are equivalent:*

(1) *$\mathcal{C}$ is precompact, i.e., any sequence in $\mathcal{C}$ has a subsequence which converges in $\mathcal{M}et$.*

(2) *There exists a function $N : (0, 1] \to (0, \infty)$ such that $Cap(X, \epsilon) \leq N(\epsilon)$ for any $\epsilon \in (0, 1], X \in \mathcal{C}$.*

(3) *There exists a function $N : (0, \frac{1}{2}] \to (0, \infty)$ such that $Cov(X, \epsilon) \leq N(\epsilon)$ for any $\epsilon \in (0, \frac{1}{2}], X \in \mathcal{C}$.*

Proof. (1) $\Rightarrow$ (2). Suppose $\mathcal{C}$ is precompact. Then it is totally bounded and so for any $\epsilon > 0$, there exists a finite set $X_1, \dots, X_{n(\epsilon)} \in \mathcal{C}$ such that for any $X \in \mathcal{C}$, there is an i such that $d_{GH}(X, X_i) \leq \epsilon/4$, i.e., $\{X_i\}_{i=1}^{n(\epsilon)}$ is a $\epsilon/4$-net in $\mathcal{C}$. Therefore, by Lemma 2.1.3, one has $Cap(X, \epsilon) \leq Cap(X_i, \epsilon - 2\epsilon/4) = Cap(X_i, \epsilon/2)$. So just defining $N(\epsilon) = \max_i Cap(X_i, \epsilon/2)$, we get (2).

(2) $\Rightarrow$ (3). This is immediate since $Cov(X, 2\epsilon) \leq Cap(X, \epsilon) \leq N(\epsilon)$.

(3) $\Rightarrow$ (1). It suffices to show that for every sequence $\{X_i\}$ in $\mathcal{C}$ and every $\epsilon > 0$ there is a subsequence $\{X'_k\}$ where $d_{GH}(X_k, X_l) < \epsilon$ for all elements in $\{X'_k\}$.

Every X_i is covered by at most $N = N(\epsilon)$ ϵ-balls. Thus, for fixed $N_1 \leq N$, there is a subseuqence $\{X'_k\}$ of $\{X_i\}$ such that X'_k is covered by exactly N_1 ϵ-balls. Let $\{x_k^\alpha\}_{\alpha=1}^{N_1}$ be the centers of these balls covering X'_k. For each k consider the matrix of numbers $\{d(x_k^\alpha, x_k^\beta)\}_{\alpha,\beta=1}^{N_1}$. All these number are bounded by $diam(X'_k) \leq \epsilon \cdot N_1$. So *Pigeon hole principle* implies there exists a subsequence X''_l of X'_k such that

$$|d(x_l^\alpha, x_l^\beta) - d(x_m^\alpha, x_m^\beta)| < \frac{\epsilon}{2}$$

for all l, m. By Lemma 1.1.9 one has

$$d_{GH}(X''_l, X''_m) \leq \epsilon.$$

for all l, m. □

The following theorem is the main result of this section.

Theorem 2.1.8 (Gromov Precompactness Theorem II). *For given $D > 0, k$ and $n \geq 2$, the space of compact Riemannian manifolds (M^n, g) satisfying*

$$Ric(M) \geq (n-1)k, \quad diam(M) \leq D \tag{2.1.1}$$

is precompact in the Gromov-Hausdorff topology.

This implies that if (M_i, g_i) is a sequence of Riemannian manifolds satisfying (2.1.1), then there exists a metric space X such that M_i converges to X in the Gromov-Hausdorff topology. We will see X must be compact in this case. In fact, X is a length space by Theorem 1.3.10 and one has $diam(X) \leq D$. Note that k might be also negative number.

To prove this, we need the following volume comparison theorems due to Bishop and Gromov.

Theorem 2.1.9 ([B-C], [G-H-L]). *Let (M, g) be a complete Riemannian manifold and $B_p(r)$ denotes the geodesic ball in M. If $Ric(M) \geq (n-1)k$, then $vol(B_p(r)) \leq V_k(r)$, where $V_k(r)$ denotes the volume of a ball of radius r in the space form of curvature k.*

Theorem 2.1.10 ([G-H-L], [G-L-P]). *If (M, g) is a complete Riemannian manifold with $Ric(M) \geq (n-1)k$, then for any point $p \in M$,*

$$\frac{vol(B_p(r))}{V_k(r)}$$

is nonincreasing, i.e., one has for $r' > r > 0$

$$\frac{vol(B_p(r'))}{V_k(r')} \leq \frac{vol(B_p(r))}{V_k(r)}. \tag{2.1.2}$$

In particular, if M is compact with the diameter $diam(M) = D$, then for $r \le D$ we have

$$\frac{vol(B_p(r))}{V_k(r)} \ge \frac{vol(M)}{V_k(D)}. \tag{2.1.3}$$

Proof of Theroem 2.1.8. For given $\epsilon > 0$ and a Riemannian n-manifold M satisfying (2.1.1), choose a maximal set of points $x_1, x_2, \ldots, x_N$ in M such that $d(x_i, x_j) \ge 2\epsilon$. Then $B(x_i, 2\epsilon)$ covers M and so $Cov(M, 2\epsilon) \le N$. On the other hand, since $B(x_i, \epsilon), i = 1, \ldots, N$, is mutually disjoint in M, one has $N \cdot volB(x_o, \epsilon) \le vol(M)$ where $volB(x_o, \epsilon) = \min\{volB(x_1, \epsilon), \ldots, volB(x_N, \epsilon)\}$. Thus Theorem 2.1.10 implies

$$N \le \frac{vol(M)}{volB(x_o, \epsilon)} \le \frac{V_\lambda(D)}{V_\lambda(\epsilon)}.$$

The proof follows from Theorem 2.1.7. □

We are going to close this section with the proof of the following theorem.

Theorem 2.1.11. *The diameter, diam, as a map from $\mathcal{M}et$ to $\mathbf{R}$ is continuous with respect to the Gromov-Hausdoff topology.*

Proof. It suffices to show that

$$|diam(X) - diam(Y)| \le 6 d_{GH}(X, Y) \tag{2.1.4}$$

for compact metric spaces X and Y. Given $\epsilon > d_{GH}(X, Y)$ there is by definition a metric d on $X \coprod Y$ extending the metrics on X and Y such that the Gromov-Hausdorff diastance between X and Y in $X \coprod Y$ is $\le 3\epsilon$ (see also Theorem 1.1.8). Then if $x_1, x_2 \in X$, there are $y_1, y_2 \in Y$ with $d(x_i, y_i) \le 3\epsilon$. Hence $d(y_1, y_2) \le 6\epsilon + d(x_1, x_2)$ by the triangle inequality. Consequently, $diam(Y) \le 6\epsilon + diam(X)$, and by symmetry this proves the inequality (2.1.4) □

§2.2 Pointed Gromov-Hausdorff convergence

In section 2.1, we have seen the Gromov-Hausdorff distance actually defines a metric on the set of all compact metric spaces. For unbounded spaces, it is not useful, but the notion of *pointed Gromov-Hausdorff distance* is effective. By a pointed metric space we mean a pair (X, p) of a metric space X and a point $p \in X$.

Definition 2.2.1. Let (X_i, p_i) and (X, p) be pointed metric spaces. We say (X_i, p_i) converges to (X, p) in the *pointed Gromov-Hausdorff distance* if for any $r > 0$ and any sequence of positive real numbers $\epsilon_i \to 0$, the closed balls $\overline{B}(p_i, r+\epsilon_i)$ in X_i converges to the closed ball $\overline{B}(p, r)$ in X in the Gromov-Hausdorff distance.

As in the section 1.2, we can define the notion of the pointed Gromov-Hausdorff distance by using Hausdorff approximation.

Definition 2.2.2. For pointed metric spaces (X, p) and (Y, q), the pointed Gromov-Hausdorff distance $d_{p.GH}((X, p), (Y, q))$ is defined as the infimum of $\epsilon > 0$ such that there exist ϵ-Hausdorff approximations $f : B^X(p, \frac{1}{\epsilon}) \to B^Y(q, \frac{1}{\epsilon}+\epsilon)$ and $g : B^Y(q, \frac{1}{\epsilon}) \to B^X(p, \frac{1}{\epsilon}+\epsilon)$ between metric balls with $f(p) = q$ and $g(q) = p$.

Example 2.2.3.

(1) Let $S^n(r)$ be the n-sphere with radius r and p be a point on it. Then one can easily see that $(S^n(r), p) \to (\mathbf{R}^n, 0)$ as $r \to \infty$.

(2) (Blowing up) Let (X, x, d) be a pointed compact metric space with the metric d. Then we have

$$\lim_{\lambda \to 0}(X, x, \lambda \cdot d) = \{point\}.$$

(3) One can easily prove the following fact: Let g_o be the Euclidean falt metric on $\mathbf{R}^n$. Then

$$\lim_{\lambda \to 0}(\mathbf{R}^n, 0, \lambda \cdot g_o) = (\mathbf{R}^n, 0, g_o).$$

In fact, since g_o is a flat metric, $\lambda \cdot g_o$ is isometric to g_o.

Theorem 2.2.4. *Let (M, g) be a Riemannian manifold. Then for any $x \in M$,*

$$\lim_{\lambda\to\infty}(M, x, \lambda \cdot g) = (T_xM, x, g|_{T_xM}),$$

where $(T_xM, x, g|_{T_xM})$ is isometric to $(\mathbf{R}^n, 0, g_o)$ with the Euclidean flat metric g_o.

Proof. Denote, for convenience, $(T_xM, x, g|_{T_xM}) = (T_xM, x, g_o)$. Consider balls $B(x, \frac{r}{\lambda}) \subset M$ and $B(0, \frac{r}{\lambda}) \subset (T_xM, g_o)$. For λ sufficiently large, these balls are almost isometric since the exponential map $exp_x : T_xM \to M$ is well-defined and $d(exp_x)(0) = I$, the identity map. Multiplying by λ, $B(x, \frac{r}{\lambda}) \subset M$ is equivalent to $B(x, r)$ on $(M, \lambda \cdot g)$ and $B(0, \frac{r}{\lambda}) \subset (T_xM, g_o)$ is equivalent to $B(0, r)$ on $(T_xM, \lambda \cdot g_o)$. Since g_o is a flat metric, $\lambda \cdot g_o$ is isometric to g_o as in Example 2.2.3 (3). So, $B(x, r) \subset (M, \lambda \cdot g)$ is almost isometric to $B(0, r)$ on (T_xM, g_o). Hence

$$\lim_{\lambda\to\infty}(M, x, \lambda \cdot g) = (T_xM, x, g|_{T_xM}).$$

□

For pointed metric spaces or complete metric spaces which are not necessarily compact, there is a similar tool to examine the convergence as in the section 2.1.

Definition 2.2.5. Given a complete length space X and two positive real numbers $\epsilon, R, 0 < \epsilon < R$, define $N(\epsilon, R, X)$ as the maximal number of disjoint ϵ-balls in R-ball $B(x, R)$ for all $x \in X$.

Remark 2.2.6. If $R = diam(X)$ or more precisely, $R = Rad(X)$, the *radius* of X defined by the smallest positive real number so that a single closed ball of such radius covers X, then $N(\epsilon, R, X)$ becomes $Cap(X, \epsilon)$, the ϵ-capacity of X.

If $\{x_i\}$ are the centers of the maximal disjoint ϵ-balls in $B(x, R)$, then $\{x_i\}$ is a 2ϵ-net in $B(x, R)$, i.e., the balls $\{B(x_i, 2\epsilon)\}$ cover $B(x, R)$.

Lemma 2.2.7. *For any fixed $\epsilon > 0$ and $R > \epsilon$, the function $X \to N(\epsilon, R, X)$ is almost continuous in the Gromov-Hausdorff distance.*

Proof. Let $B(x, R) \subset X$ and $B(y, R) \subset Y$ be R-balls in X and Y, respectively, such that $d_{GH}(B(x, R), B(y, R)) < \delta$, $\delta < \epsilon$. Let $N = N(\epsilon, R, X)$ and $\{B(x_i, \epsilon)\}$ be a maximal disjoint ϵ-balls in $B(x, R)$. Then $\{x_1, \ldots, x_N\}$ is a 2ϵ-net in $B(x, R)$ by Remark 2.2.6 and $d(x_i, x_j) \geq 2\epsilon$ if $i \neq j$. Since $d_{GH}(B(x, R), B(y, R)) < \delta$, there exists $y_i \in B(y, R)$ such that $d(x_i, y_i) < \delta$ for each $i = 1, \ldots, N$, where d is an extended metric on the disjoint union of X and Y, $X \coprod Y$. Thus,

$$2\epsilon \leq d(x_i, x_j) \leq d(x_i, y_i) + d(y_i, y_j) + d(y_j, x_j).$$

That is, $d(y_i, y_j) > 2(\epsilon - \delta)$. This means $\{B(y_i, \epsilon - \delta)\}$ are mutually disjoint in $B(y, R)$ and so $N(\epsilon - \delta, R, Y) \geq N(\epsilon, R, X)$. By interchanging X with Y, we also get $N(\epsilon - \delta, R, X) \geq N(\epsilon, R, Y)$. □

Corollary 2.2.8. *Let $\mathcal{F} = \{(X, d)\}$ be a set of complete locally compact length space. If $\mathcal{F}$ is precompact in the Gromov-Hausdorff topology, then $\{N(\epsilon, R, X) : X \in \mathcal{F}\}$ is uniformly bounded.*

Proof. It follows immediately from Lemma 2.2.7. □

One can also define the notion of *totally bounded* for a pseudometric space and it is easy to see Lemma 2.1.6 still holds for pseudometric spaces. Keeping these in mind, one has the following theorem which is similar to the precompactness theorem I.

Theorem 2.2.9. *A family $\mathcal{F}$ of complete, locally compact, pointed length spaces is precompact if and only if for any fixed $\epsilon, R > 0$, $\{N(\epsilon, R, X) : X \in \mathcal{F}\}$ is uniformly bounded*

Proof. One direction follows from Corollary 2.2.8. To show the converse, fix $\epsilon > 0$ and $R > 0$ and let $N = \sup\{N(\epsilon, R, X) : X \in \mathcal{F}\}$. By assumption and Theorem 2.1.7, $\{B^X(x, R) : X \in \mathcal{F}\}$ is precompact. A similar argument as in the proof of Theorem 2.1.7 and by using diagonal sequence, one can prove $\mathcal{F}$ is precompact. However the proof is quite long and so we omit it here. For reference, see [G-L-P]. □

Corollary 2.2.10. *Let (M_i, x_i, g_i) be a sequence of pointed n-dimensional Riemannian manifolds such that*

$$C_1 \leq \frac{vol(B^{M_i}(x_i, r))}{r^n} \leq C_2 \qquad \textit{for all} \quad r > 0,$$

where C_1, C_2 are fixed constants and $B^{M_i}(x_i, r)$ is a geodesic ball of radius r in M_i. Then $\{(M_i, x_i, g_i)\}$ is prcompact in the pointed Gromov-Hausdorff topology.

Proof. Given any $\epsilon > 0$ and $R > 0$, let $N(\epsilon, R, M_i) = N_i$ be the number of maximal disjoint ϵ-balls, $\{B(x_i^j, \epsilon)\}_{j=1}^{N_i}$, in $B(x_i, R) \subset M_i$. Then we have

$$C_1 \cdot N_i \cdot \epsilon^n \leq \sum_{j=1}^{N_i} volB(x_i^j, \epsilon) \leq C_2 \cdot R^n.$$

In other words, we get

$$N_i \leq \frac{C_2 \cdot R^n}{C_1 \cdot \epsilon^n},$$

i.e., N_i is uniformly bounded (independent of i). Hence $\{(M_i, x_i, g_i)\}$ is precompact by Theorem 2.2.9 □

Theorem 2.2.11 (Precompactness Theorem III). *For given $k \in \mathbf{R}$, the space of all pointed complete n-dimensional Riemannian manifolds (M, p) satisfying*

$$Ric(M) \geq (n-1)k$$

is precompact in the pointed Geomov-Hausdorff distance.

Proof. Let (M, p) be a pointed Riemannian n-manifold satisfying $Ric(M) \geq (n-1)k$. For $0 < \epsilon < R$, let $N = N(\epsilon, R, M)$ be the maximal number of disjoint ϵ-balls in $B(p, R)$ in M. Together with the relative volume comparison theorem, one has

$$N \cdot vol(B(\epsilon)) \leq vol(B(p, R)) \leq \frac{V_k(R)}{V_k(\epsilon)} vol(B(\epsilon)).$$

That is, N is uniformly bounded which is independent of (M, p). Then the proof follows from Theorem 2.2.9. □

We close this section with the notion of tangent cone of a length space or a Riemannain manifold. By Theorem 2.2.11, for any sequence, $r_i \to \infty$, there exists a subsequence, $r_j \to \infty$, such that the sequence of pointed Riemannian manifolds, $\{(M^n, x, r_j^{-1} g)\}$ or $\{(M^n, x, r_j g)\}$, converges to some length space, $(M_\infty, x_\infty, d_\infty)$ or (M_x, x, d) in the pointed Gromov-Hausdorff distance, respectively. Any such space, M_∞ or M_x, is called a *tangent cone at infinity* or *tangent cone at x*, respectively.

Definition 2.2.12. Let (X, d) be a length space and $x \in X$ be a point.

(1) A *tangent cone* to X at x is a space of the form

$$\overset{GH}{\lim_{\lambda_i \to \infty}} (X, x, \lambda_i \cdot d)$$

in the pointed Gromov-Hausdorff topology.

(2) A *tangent cone at infinity* to X is a space of the form

$$\overset{GH}{\lim_{\lambda_i \to 0}} (X, x, \lambda_i \cdot d)$$

in the pointed Gromov-Hausdorff topology.

Proposition 2.2.13. *For given postive real numbers k and D and a natural number $n \in \mathbf{N}$, let (M_i, g_i) be a sequence of Riemannian n-manifolds satisfying $Ric(M_i) \geq -(n-1)k$, $diam(M_i) \leq D$. Suppose X is a Gromov-Hausdorff limit of (M_i, g_i). Then every point $x \in X$ has a tangent cone*

$$\overset{GH}{\lim_{r_i \to \infty}} (X, x, r_i d) \overset{\text{def}}{\equiv} T_x X.$$

Proof. It suffices to show that there is a bound on $N = N(\epsilon, R, (X, r \cdot d))$ for any $r >> 1$ and $\epsilon, R > 0$ by Theorem 2.2.9. That is, one has to show N is independent of r. First note that

$$N(\epsilon, R, (X, r \cdot d)) = N(\frac{\epsilon}{r}, \frac{R}{r}, (X, d)) = N(\frac{\epsilon}{r}, \frac{R}{r}, X).$$

Since X is a limit of M_i and N is almost continuous,

$$N(\frac{\epsilon}{r}, \frac{R}{r}, X) \leq N(\frac{\epsilon}{r}, \frac{R}{r}, M_i) + 1$$

for i sufficiently large.

From the definition of N, one has

$$N(\frac{\epsilon}{r}, \frac{R}{r}, M_i) \cdot vol\left(B(x_{ij}, \frac{\epsilon}{r})\right) \leq vol\left(B(x_i, R)\right),$$

where $x_i \in M_i$ and $x_{ij} \in B(x_i, R, M_i)$. Thus

$$N(\frac{\epsilon}{r}, \frac{R}{r}, M_i) \leq \frac{vol\left(B(x_i, \frac{R}{r})\right)}{vol\left(B(x_{ij}, \frac{\epsilon}{r})\right)} \leq \frac{V_k(\frac{R}{r})}{V_k(\frac{\epsilon}{r})}.$$

The last inequality follows from the relative volume comparison theorem (Theorem 2.1.10) and the final term is independent of r.

In fact, for $r >> 1$ sufficiently large, one has

$$\frac{V_k(\frac{R}{r})}{V_k(\frac{\epsilon}{r})} \leq C \cdot \frac{\left(\frac{R}{r}\right)^n}{\left(\frac{\epsilon}{r}\right)^n} = C\left(\frac{R}{\epsilon}\right)^n$$

whatever the sign of λ is. □

For a complete locally compact length space, it is not true in general that limit is unique nor independent of $r_i's$. Also the limit $\lim^{GH}_{r_i \to \infty}(X, x, \lambda_i \cdot d) \equiv T_x X$ may not be a cone. Recall that a metric space is a cone over Y if X is homeomorphic to $Y \times [0,1]/Y \times \{0\}$, or $Y \times [0,\infty)/Y \times \{0\}$, where the metric $d\big|_{Y\times\{s\}} = s \cdot d\big|_Y$. However if we assume the volume condition, any tangent cone is a metric cone. More precisely, assume (X, d) is the Gromov-Hausdorff limit of a sequence, $\{ (M_i^n, p_i) \}$, of Riemannian n-manifolds satisfying $Ric(M_i) \geq -(n-1)$ and $vol(B_1(p_i)) \geq v > 0$. Then at any point $z \in X$,

$$\lim^{GH}_{r_i \to \infty} (X, z, r_i d)$$

is a metric cone ([C-C2]). For more details in this direction, refer to [B-G-P], [C-C1. C-C2], etc.

§2.3 Hausdorff dimension and measure

Let X be a separable metric space and p an arbitrary real number, $0 \leq p < \infty$. Given $\epsilon > 0$, let

$$H_{p,\epsilon}(X) = \frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} \inf\left\{ \sum_{i=1}^{\infty} r_i^p : X = \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq \epsilon \right\},$$

where $B(x_i, r_i)$ denotes the open ball in X with radius r_i centered at $x_i \in X$ and Γ is the Gamma function defined by

$$\Gamma(t) = \int_0^{\infty} e^{-s} s^{t-1}\, ds.$$

Note that if $p \geq 2$ is a positive integer, then the constant

$$\frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} = \frac{1}{p} vol(S^{p-1}) = \frac{\omega_{p-1}}{p},$$

which is exactly equal to $\frac{1}{p}$-multiple of the volume of round $(p-1)$-sphere. Recall that the volume of a ball of radius r in the Euclidean space $\mathbf{R}^p$ is given by $\frac{1}{p} vol(S^{p-1}) r^p$.

Definition 2.3.1. Let X be a separable metric space and p an arbitrary real number, $0 \leq p < \infty$. The p-dimensional *Hausdorff measure* of X is defined as

$$H_p(X) = \lim_{\epsilon \to 0} H_{p,\epsilon}(X).$$

The Hausdorff measure looks like the Lebesgue measure. In fact, if A is a Lebesgue measurable subset of $\mathbf{R}^n$, then the n-dimensional Hausdorff measure $H_n(A)$ is equal to Lebesgue maesure of A.

For smooth manifolds, the Hausdorff measure can be considered as a generalization of the Lebesgue measure. The basic properties are the followings

Proposition 2.3.2. *(A) For the zero dimensional Hausdorff measure, we have*

(1) $H_0(X) = 0$ *if* X *is empty*

(2) $H_0(X) = n$ *if* X *is a finite set of* n *points*

(3) $H_0(X) = \infty$ *if* X *is an infinite set*

(B) If $p < q$ *then* $H_p(X) \geq H_q(X)$; *in fact* $p < q$ *and* $H_p(X) < \infty$ *imply* $H_q(X) = 0$.

(C) An n*-dimensional polytope has finite* n*-dimensional Hausdorff measure. Consequently its* q*-dimensional Hausdorff measure is zero for all* $q > n$.

Theorem 2.3.3. *Let* X *be a compact separable metric space. Then* $H_p(X) = 0$ *if and only if for each* $\epsilon > 0$ *there exists a finite decomposition of* X:

$$X = B_1 \cup B_2 \cup \ldots \cup B_k, \quad B_i = B(x_i, r_i)$$

such that

$$r_1^p + \ldots + r_k^p < \epsilon.$$

Proof. Suppose $H_p(X) = 0$ and let $\epsilon > 0$ be given. By definition, there exists a countable number of balls $B(x_1, r_1'), B(x_2, r_2'), \ldots$ such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r_i') \quad \text{and} \quad \sum_{i=1}^{\infty} r'^p_i < \epsilon/2.$$

It is possible to enlarge each ball $B(x_i, r_i')$ slightly to an open ball $B(x_i, r_i)$ such that

$$r_i^p < r'^p_i + \frac{\epsilon}{2^{i+1}}.$$

Since X is compact, there is a finite cover $B_1, \ldots, B_k, \quad B_j = B(x_j, r_j)$ of X and so $r_1^p + \ldots + r_k^p < \epsilon$.

The converse is obvious from the definition. □

Definition 2.3.4. The *Hausdorff dimension* of an arbitrary separable metric space X is defined by

$$\begin{aligned} dim_H(X) &= \inf\{p \geq 0 : H_p(X) = 0\} \\ &= \sup\{p \geq 0 : H_p(X) > 0\}. \end{aligned}$$

Note that even if $dim_H(X) = p$, we may have $H_p(X) = 0, \infty$ or positive real number. In particular, for a smooth Riemannian n-manifold (M^n, g), we have $dim_H(M) = n$ and $H_n(M) = vol(M, g)$.

Theorem 2.3.5. *For given positive real numbers $k, D > 0$ and a natural number $n \in \mathbf{N}$, let (M_i, g_i) be a sequence of Riemannian n-manifolds satisfying $Ric(M_i) \geq -(n-1)k$, $diam(M_i) \leq D$. Suppose X is a Gromov-Hausdorff limit of (M_i, g_i). Then $dim_H X \leq n$.*

Proof. By proposition 2.3.2 (B), it suffices to show that $H_n(X) < \infty$. Recall that X is a compact length space with $diam(X) \leq D$. For any $\epsilon > 0$, recall also that $Cov(X, \epsilon)$ denotes the smallest number of closed ϵ-balls in X which cover X. Since $Cov(X, \epsilon)$ is continuous in the Gromov-Hausdorff distance(Lemma 2.1.3), one has

$$Cov(X, \epsilon) \leq Cov(M_i, \epsilon - \frac{3}{\delta_i}), \quad \delta_i \to 0 \text{ as } i \to \infty$$

for i sufficiently large. The volume comparison theorem together with $diam(M_i) \leq D$ implies that $Cov(M_i, \epsilon)$ depends only on ϵ, k and D. Thus one get

$$\begin{aligned} H_n(X) &= \lim_{\epsilon \to 0} \frac{\omega_{n-1}}{n} \epsilon^n Cov(X, \epsilon) \leq \lim_{\epsilon \to 0} \left(\limsup_{i \to \infty} \frac{\omega_{n-1}}{n} \epsilon^n Cov(M_i, \epsilon - \frac{3}{\delta_i}) \right) \\ &= \limsup_{i \to \infty} \left(\lim_{\epsilon \to 0} \frac{\omega_{n-1}}{n} \epsilon^n Cov(M_i, \epsilon - \frac{3}{\delta_i}) \right) \\ &= \limsup_{i \to \infty} H_n(M_i) \leq V_k(D) < \infty. \end{aligned}$$

The last inequality follows from the volume comparison theorem and the fact that $H_n(M_i) = vol(M_i)$ for a compact smooth Riemannian manifold. □

Remark 2.3.6. One can also show that, in the Theorem 2.3.5,

$$\liminf_{i \to \infty} H_n(M_i) \leq H_n(X).$$

Theorem 2.3.7. *For given positive real numbers $k, v, D > 0$ and a natural number $n \in \mathbf{N}$, let (M_i, g_i) be a sequence of Riemannian n-manifolds satisfying $Ric(M_i) \geq -(n-1)k$, $vol(M_i) \geq v$ and $diam(M_i) \leq D$. Suppose X is a Gromov-Hausdorff limit of (M_i, g_i). Then $dim_H X = n$ and the tangent cone $T_x X$ at any point $x \in X$ also has Hausdorff dimension n, i. e.,*

$$dim_H T_x X = dim_H \lim_{r \to \infty} (X, x, r \cdot d) = n$$

for any point $x \in X$.

Proof. Since $vol(M_i) \geq v$, one has $dim_H(X) \geq n$ and so it follows from the Theorem 2.3.5 that $dim_H(X) = n$. It remains to prove $dim_H T_x X = n$.

It is known from the volume condition that for each point $x \in X$, $T_x X$ is a metric cone, i.e., metrically cone on tangent space and so it becomes a length space. As in the proof of Proposition 2.2.13, one has $N(\epsilon, R, (X, rd)) \leq C\left(\frac{R}{\epsilon}\right)^n$ and so $dim_H T_x X \leq n$. To show the equality, it is enough to verify that for any $\alpha > 0, H_{n-\alpha}(T_x X \cap B(1)) > 0$, where $B(1)$ denotes the unit ball in $T_x X$ centered at x. The claim is that

$$N(\epsilon, R, (X, rd)) \geq C' \left(\frac{R}{\epsilon}\right)^n$$

for some positive constant $C' = C'(k, v, D, n) > 0$. Note that for $N = N(\epsilon, R, X)$,

$$volB(R) \leq \sum_{i=1}^{N} volB(2\epsilon) \leq N \cdot \max\{volB(2\epsilon)\}. \tag{2.3.1}$$

Since N is almost continuous by Lemma 2.2.7, one has

$$\begin{aligned} N(\epsilon, R, (X, rd)) &= N(\frac{\epsilon}{r}, \frac{R}{r}, (X, d)) \geq C_1 \cdot N(\frac{\epsilon}{r}, \frac{R}{r}, M_i) \\ &\geq \frac{volB(\frac{R}{r})}{\max\{volB(\frac{2\epsilon}{r})\}} \qquad \text{by (2.3.1)} \end{aligned}$$

Note that

$$volB(\frac{2\epsilon}{r}) \leq V_k(\frac{2\epsilon}{r}) \leq C \cdot \left(\frac{2\epsilon}{r}\right)^n \tag{2.3.2}$$

for $r >> 1$ sufficiently large. So

$$\begin{aligned}
\frac{volB(\frac{R}{r})}{\max\{volB(\frac{2\epsilon}{r})\}} &\geq C_2 \cdot \frac{volB(\frac{R}{r})}{(\frac{2\epsilon}{r})^n} && \text{by (2.3.2)} \\
&\geq C_2 \cdot \frac{vol(M)}{V_k(D)} \frac{V_k(\frac{R}{r})}{(\frac{2\epsilon}{r})^n} && \text{by relative volume comparison theorem} \\
&\geq C_2 \cdot \frac{v}{V_k(D)} \frac{(\frac{R}{r})^n}{(\frac{2\epsilon}{r})^n} = C' \left(\frac{R}{\epsilon}\right)^n, \quad C' = C'(k, v, D, n).
\end{aligned}$$

Finally, choose a δ-net in $B(1)$ so that balls of radius δ are disjoint and 2δ-balls cover $B(1)$. Then for any $\alpha > 0$,

$$\sum \frac{\omega_{n-1}}{n}\delta^{n-\alpha} = \frac{\omega_{n-1}}{n}\delta^{n-\alpha}N(\delta, 1, T_x X) = \frac{\omega_{n-1}}{n}\delta^{-\alpha}.$$

Hence $dim_H(T_x X) \geq n$ by definition. The proof is completed. □

CHAPTER III

HARMONIC RADIUS AND SMOOTH CONVERGENCE THEORY

In this chapter we will discuss smooth convergence theory of Riemannain manifolds as an application of the Gromov-Hausdorff topology. We have seen in the previous chapter that some classes of Riemannain manifolds satisfying geometric conditions are precompact in the metirc spaces with respect to the Gromov-Hausdorff topology. However the limit space is just a metric space not a smooth manifold in general. Thus, it is interesting to find conditions which the limit space becomes a smooth manifold. We will focus our attention on this problem throughout this chapter.

As is now well-known, the Cheeger-Gromov convergence theorem ([Che], [G-L-P]) implies that the space of compact Riemannian n-manifolds of sectioanal curvature $|K| \leq \Lambda$, volume $\geq v > 0$ and diameter $\leq D$, is precompact in the $C^{1,\alpha}$ topology (c.f. See also [G-W], [Kas], [Pet]). The key step is that the bounds above give a uniform lower bound for the injectivity radius of Riemannian manifolds in this class. Or in our sence, the bounds above give the existence of harmonic coordinates, i.e., charts for which the coordinate functions are harmonic functions, on balls of uniform size (depending only on the constants above), and uniform $C^{1,\alpha}$ estimates of the metric tensor g_{ij} in these coordinates ([Che], [J-K]). In section 3.1 and 3.2, we shall define the harmonic coordinate system and harmonic radius and prove some various properties about these concepts as preliminaries for convergence of a sequence of Riemannian manifolds. In section 3.3, we shall prove several types of convergence theorems related with the harmonic radius.

§3.1 Harmonic coordinates

Let (M^n, g) be an n-dimensional Riemannain manifold and let $\{x^i\}$ be a local coordinate system around a point in M. Then the Laplace operator in the coordinates $\{x^i\}$ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x^j}\right),$$

where $g = det(g_{ij})$, $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $(g^{ij}) = (g_{kl})^{-1}$. From now on, we shall follow the Einstein convention and so drop the summation notation. First we define the *harmonic coordinate system.*

Definition 3.1.1. A coordinate system $\{(h^1, \cdots, h^n)\}$ defined on an open subset of a Riemannian n-manifold (M, g) is called *harmonic coordinate system* if $\Delta h^i \equiv 0$ for all $i = 1, \cdots, n$.

For any C^2 function u on M, we can write down the Laplacian of u as

$$\Delta u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left((g^{ij}\sqrt{g})\right)\frac{\partial u}{\partial x^j}. \qquad (3.1.1)$$

So, for the coordinate function $u = x^k$, we have

$$\begin{aligned}\Delta x^k &= g^{ij}\frac{\partial^2 x^k}{\partial x^i \partial x^j} + \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial x^i}(g^{ij}\sqrt{g})\right)\frac{\partial x^k}{\partial x^j}\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(g^{ik}\sqrt{g}).\end{aligned}$$

We define

$$\Gamma^k = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(g^{ik}\sqrt{g}) \qquad (3.1.2)$$

so that x^k is harmonic if and only if $\Gamma^k \equiv 0$.

A strightforward computation shows the followings.

Excecise 3.1.2

(1) $\Gamma^k = \sum_{i,j=1}^n g^{ij}\Gamma_{ij}^k$, where Γ_{ij}^k is the Christoffel symbol of the metric g.

(2) If $\{x^i\}$ is a harmonic coordinate, i.e., $\Delta x^i = 0$ for all $i = 1, \cdots, n$, then for any function $u \in C^2(M)$,

$$\Delta u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j}. \tag{3.1.3}$$

(3) The Ricci curvature R_{ij} in the harmonic coordinate $\{x^i\}$ is given by

$$R_{ij} = -\frac{1}{2}\sum_{r,s=1}^n g^{rs}\frac{\partial^2 g_{ij}}{\partial x^r \partial x^s} + Q, \tag{3.1.4}$$

where Q is the lower order term.

Given a point $m \in M$ and $v \in T_m M$ with $|v| = 1$, we define

$$r(x) = d(m, x),$$

$$p(x) = exp_m(r(x)v),$$

$$q(x) = exp_m(-r(x)v),$$

and then set

$$l_v(x) = \frac{1}{4r(x)}\{d^2(x, q(x)) - d^2(x, p(x))\}.$$

In $\mathbf{R}^n$, l_v would be the linear functional determined by inner product with v, i.e., $l_v(x) = \langle v, x\rangle$. If $\{v_1 . \cdots, v_n\}$ is an orthonormal basis for $T_m M$ and let $l^i = l_{v_i}$, then $\{l^1, \cdots, l^n\}$ gives a local coordinate system on a ball of fixed radius ρ. On a geodesic ball $B(m, \rho)$, solving the following *Dirichlet problem* for each $i = 1, \cdots, n$:

$$\Delta h^i = 0 \quad \text{in} \quad B(m, \rho), \qquad h^i|_{\partial B} = l^i|_{\partial B},$$

we get n harmonic functions $\{h^1, \cdots, h^n\}$ which is a coordinate system on $B(m, \rho)$. In other words, since the Dirichlet problem is always solvable (see [G-T]), the harmonic coordinates alwyas exists on a small ball of given Riemannian manifold.

So far, we have discussed metric tensors g in smooth category. From now on, we will also consider metric tensors with weaker regularity, that is, in the category of $C^{k,\alpha}$ or the Sobolev space $L^{k,p}$. First of all, we consider the topology on the space of Riemannian metrics of a fixed manifold M. Let M be a compact smooth n-manifold and let $\{\phi_\alpha : B^n(1) \to M\}, B^n(1) \subset \mathbf{R}^n$ be a locally finite C^∞ atlas on M.

Definition 3.1.3. For $\epsilon > 0$, two covariant tensors T_1 and T_2 are called ϵ-close in the C^k topology on M if

$$\|\phi_\alpha^*(T_1 - T_2)\big|_{Im(\phi_\alpha)}\|_{C^k(B^n(1))} \leq \epsilon. \tag{3.1.5}$$

Note that expressing the tensor as local coordinate system, it becomes

$$\phi_\alpha^*(T_1 - T_2)\big|_{Im(\phi_\alpha)} = \sum T_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

Thus the C^k-norm means the summation of the C^k-norm of functions $T_{i_1 \cdots i_k}$. Recall that for a function f,

$$\|f\|_{C^k(B^n(1))} = \sup_{B^n(1)} |f| + \sup_{B^n(1)} |Df| + \cdots + \sup_{B^n(1)} |D^k f|.$$

If we change atlas to ψ_β, we get an equivalent norm so that ϵ is changed to $c \cdot \epsilon$ where c is a bound $\|(\psi_\beta)^{-1} \circ \phi_\beta\|_{C^k}$. Thus, this gives a well-defined metric topology on space of covariant k-tensors, independent of the choice of atlas. We also can do the same thing for $C^{k,\alpha}, \alpha \in (0,1)$, topology with $C^{k,\alpha}$-norm

$$\|f\|_{C^{k,\alpha}(B^n(1))} = \|f\|_{C^k(B^n(1))} + \sup_{x \neq y} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha}$$

and for the Sobolev space $L^{k,p}$ with the norm

$$\|f\|_{L^{k,p}(B^n(1))} = \left(\int_{B^n(1)} |f|^p \, dx + \int_{B^n(1)} |Df|^p \, dx + \cdots + \int_{B^n(1)} |D^k f|^p \, dx \right)^{1/p}.$$

Note that $C^\infty(B^n(1), \|\cdot\|_{L^{k,p}})$ is not complete and so making the completion of $C^\infty(B^n(1), \|\cdot\|_{L^{k,p}})$, we get a complete normed vector space, i.e., Banach space $L^{k,p}(B^n(1))$ which is called the Sobolev space.

Example 3.1.4. (C^o topology on metrics) Let g, h be two Riemannian metrics on a fiexd compact smooth n-manifold M and let $\{\phi_\alpha\}$ be a smooth atlas. Then the C^o-norm between g and h is given by

$$\|g - h\|_{C^o} = \|\phi_\alpha^*(g - h)\|_{C^o} = \sum_{i,j} \|g_{ij} - h_{ij}\|_{C^o},$$

where

$$\phi_\alpha^* g = g_{ij}, \quad g_{ij} = g((\phi_\alpha)_* \frac{\partial}{\partial x^i}, (\phi_\alpha)_* \frac{\partial}{\partial x^j})$$

and

$$\phi_\alpha^* h = h_{ij}, \quad h_{ij} = h((\phi_\alpha)_* \frac{\partial}{\partial x^i}, (\phi_\alpha)_* \frac{\partial}{\partial x^j})$$

in local atlas ϕ_α. Thus, $\|g-h\|_{C^0} \leq \epsilon$ is equivalent to $(1-\epsilon')h_{ij} \leq g_{ij} \leq (1+\epsilon')h_{ij}$ for some $\epsilon' = \epsilon(h_{ij})$. We call in this case h is ϵ' quasi-isometric to g. Hence the C^o topology gives the quasi-isometry of Riemannian metrics on M.

Now let $\mathcal{M}'$ denote the set of all Riemannian metrics on a compact smooth manifold M. In the compact open C^∞ topology, $\mathcal{M}'$ is an open convex cone. If $\mathcal{D}(M)$ denotes the diffeomorphism group of M, there is a natural right action on the space $\mathcal{M}'$ by pull-back, i.e.,

$$\mathcal{D}(M) \times \mathcal{M}' \to \mathcal{M}', \qquad (\phi, g) \to \phi^* g.$$

Clearly, two metrics in the same orbit have the same geometric properties since they are isometric. Thus from now on, we will consider only the isometry classes of metrics described by the quotient $\mathcal{M}'/\mathcal{D}(M)$, which is often called the space of Riemannian *structures* and we denote $\mathcal{M}'/\mathcal{D}(M)$ by $\mathcal{M}$. With the induced topology on $\mathcal{M}$, we can define the notion of ϵ-close between two metrics as follows.

Definition 3.1.5. Two metrics g_1 and g_2 in $\mathcal{M}$ are said to be ϵ-close in the $C^{k,\alpha}$ topology if there is a diffeomorphism $f : M \to M$ such that

$$\|f^* g_1 - g_2\|_{C^{k,\alpha}} \leq \epsilon$$

with resprct to the same atlas on M.

Lemma 3.1.6. *Let $\{x^i\}$ be a local coordinate system on an open subset $\Omega \subset M$ containing $m \in \Omega$. Suppose a metric tensor g on M is $C^{k,\alpha}$ in $\{x^i\}$ coordinates. Then there exist an open set $\Omega' \subset\subset \Omega$ and harmonic coordinates $\{h^1, \cdots, h^n\}$ on Ω' such that these harmonic functions, $\{h^i\}$, are $C^{k+1,\alpha}$ functions of $\{x^j\}$. In fact, we have*

$$\|h^i\|_{C^{k+1,\alpha}(\Omega',\{x^j\})} \leq C \cdot \|h^i\|_{C^\alpha}(\Omega'),$$

where $C = C(\|g\|_{C^{k,\alpha}(\Omega,\{x^j\})})$.

The proof of Lemma 3.1.6 follows from the existence of the Dirichlet problem and interior regularity theorem of second order elliptic partial differential operator due to Schauder. Let $\Omega \subset \mathbf{R}^n$ be a conneted bounded open subset which has smooth boundary $\partial\Omega$ and let

$$L = a^{ij} D_{ij} + b^i D_i + c, \qquad a^{ij} = a^{ji}$$

be a second order differential operator whose coefficient functions are defined in Ω. Moreover we assume that L is *elliptic*, that is,

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbf{R}^n,$$

for some positive constant λ.

Note that if g is a $C^{k,\alpha}$ metric tensor on M, then by eq. (3.1.1), the Laplace operator is an elliptic second order partial differential operator whose coefficient functions are $C^{k-1,\alpha}$. First we have

Theorem 3.1.7 ([G-T]). *Let L be an elliptic second order P.D.E and let f and the coefficients of L belong to $C^\alpha(\Omega)$. Assume the coefficient of zero order term c is nonpositive. Then, if ϕ is continuous on $\partial\Omega$, the Dirichlet problem*

$$Lu = f \quad in \quad \Omega, \quad u = \phi \quad \text{on} \quad \partial\Omega,$$

has a unique solution $u \in C^o(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$.

Lemma 3.1.8 (Local elliptic regularity). *Let $u \in L^1_{loc}(\Omega)$ be a solution of the equation $Lu = f$.*

(a) *(Schauder estimates) If $f \in C^{k,\alpha}$, then $u \in C^{k+2,\alpha}(K)$ for any compact subset $K \subset\subset \Omega$, and if $u \in C^\alpha(\Omega)$ then*

$$\|u\|_{C^{k+2,\alpha}(K)} \leq C\left(\|Lu\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^\alpha(\Omega)}\right), \tag{3.1.6}$$

where C depends only on the $C^{k,\alpha}(\Omega)$ norm of the coefficient functions a^{ij}, b^i and c.

(b) *If $f \in L^{k,p}(\Omega)$, then $u \in L^{k+2,p}(K)$ for any compact subset $K \subset\subset \Omega$, and if $u \in L^p(\Omega)$ then*

$$\|u\|_{L^{k+2,p}(K)} \leq C\left(\|Lu\|_{L^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right), \tag{3.1.7}$$

where C depends only on the $L^{k,p}(\Omega)$ norm of the coefficient functions a^{ij}, b^i and c.

By using local charts, these results can be transferred to a compact Riemannian manifold.

Theorem 3.1.9. *Let M be a compact smooth Riemannian manifold, and suppose $u \in L^1_{loc}(M)$ is a weak solution to $Lu = f$.*

(a) *If $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$, and*

$$\|u\|_{C^{k+2,\alpha}} \leq C\left(\|Lu\|_{C^{k,\alpha}} + \|u\|_{C^\alpha}\right). \tag{3.1.8}$$

(b) *If $f \in L^{k,p}(M)$, then $u \in L^{k+2,p}(M)$, and*

$$\|u\|_{L^{k+2,p}} \leq C\left(\|Lu\|_{L^{k,p}} + \|u\|_{L^p}\right). \tag{3.1.9}$$

Proof of Lemma 3.1.6. We have already seen that there exist harmonic coordinates $\{h^i\}$ on a subdomain $\Omega' \subset \Omega$. Recall that the Laplacian

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x^j}\right) = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^j\frac{\partial}{\partial x^j}$$

is an elliptic second order P.D.E with $C^{k-1,\alpha}$ coefficients. Applying Lemmas 3.1.7 and 3.1.8 to $\Delta h^i = 0$ in $\Omega' \subset\subset \Omega$, we get $h^i \in C^{k+1,\alpha}(\Omega')$ in $\{x^j\}$ coordinates and

$$\|h^i\|_{C^{k+1,\alpha}(\Omega')} \leq C\left(\|\Delta h^i\|_{C^{k,\alpha}(\Omega')} + \|h^i\|_{C^\alpha(\Omega')}\right) = C\|h^i\|_{C^\alpha(\Omega')},$$

where $C = C(\|g_{ij}\|_{C^{k,\alpha}(\Omega)}, \|\Gamma^j\|_{C^{k-1,\alpha}(\Omega)})$. □

Corollary 3.1.10. *Suppose g is a $C^{k,\alpha}$ metric in local coordinates $\{x^i\}$. Then g is also $C^{k,\alpha}$ in the harmonic coordiantes constructed above. In fact, g is $C^{k,\alpha}$ for any harmonic coordinates, i.e., harmonic coordinates give an optimal regularity for g. In geodesic normal coordinates, g is at best $C^{k-2,\alpha}$.*

Proof. Let $\{y^j\}$ be a harmonic coordinates so that $\{y^j\}$ are functions of $\{x^i\}$. Then $\frac{\partial}{\partial y^j}$ is $C^{k,\alpha}$ and so

$$g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^i} g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

is $C^{k,\alpha}$ functions. However, in a normal coordinates $\{z^k\}, z^k = tv^k$, the geodesic equation shows

$$\frac{\partial^2 z^k}{dt^2} + \Gamma^k_{ij}\frac{\partial z^i}{dt}\frac{\partial z^j}{dt} = 0.$$

Since the regularity of O.D.E shows $\{z^k\}$ is $C^{k+1,\alpha}$ functions of t and Γ^k_{ij} is $C^{k-1,\alpha}$, $\{z^k\}$ expect to at best $C^{k-1,\alpha}$ in other variables, i.e., S^{n-1} parameters, $\{v^k\}$. Thus, g is at best $C^{k-2,\alpha}$ in $\{z^k\}$. □

We close this section with the following theorem.

Theorem 3.1.11 ([J-K]). *Let M be a compact Riemannian n-manifold with $|K_M| \leq 1$ and let $\mu = inj(M)$. Then there exists $\epsilon = \epsilon(n, \mu)$ and $C = C(n, \mu)$ satisfying the following: For each point $p \in M$ there exists $h_i : B_p(\epsilon) \to \mathbf{R}$, $i = 1, \cdots, n$, such that $H = (h_1, \cdots, h_n) : B_p(\epsilon) \to U \subset \mathbf{R}^n$ is a diffeomorphism and that $\|g_{ij}\|_{C^{1,\alpha}} \leq C$, where the $g'_{ij}s$ are metric coefficients relative to H and the $C^{1,\alpha}$-norm is taken in the H-coordinates.*

Sketch of the Proof. We follow Green-Wu's method ([G-W]). Using eq. (3.1.5) and (3.1.6), one can get harmonic coordinates $H = (h_1, \cdots, h_n)$. Let $v_i(x) \in T_xM$ denote the parallel transport of v_i along the minimal geodesic. Using comparison theorems we have

$$\|\nabla_x l^i - v_i(x)\| < C \cdot r(x)^2, \quad \|D^2 l^i(x)\| < C \cdot r(x). \tag{3.1.10}$$

We can use these formulae to show that $(l^1, \cdots, l^n)$ is a diffoemorphism. Take $\epsilon << \rho << \mu$. Let h_i be the unique solution of

$$\Delta h_i = 0, \quad h_i|_{\partial B_p(\epsilon)} = l^i|_{\partial B_p(\epsilon)}. \tag{3.1.11}$$

Then, by formula (3.1.10), we have

$$\|\Delta(h_i - l^i)\| < C \cdot r(x)^2, \quad (h_i - l^i)|_{\partial B_p(\epsilon)} = 0.$$

Applying an elliptic regularity to the above inequality, we obtain

$$\|\nabla(h_i - l^i)\| < C \cdot \epsilon.$$

This formula implies that $H = (h_1, \cdots, h_n)$ is also a coordinate system. Now using Bochner technique and Nash-Moser type estimate, we obtain a uniform $C^{1,\alpha}$ bound of the metric tensor. $\square$

§3.2 Harmonic radius

As mentioned in introduction of this chapter, the harmonic radius gives one tool to handle the convergence of a sequence of Riemannian manifolds. Furthermore, we have seen that harmonic coordinates give, in a certain sense, optimal regularity for the coefficients of the metric tensor. We begin with definitions.

Definition 3.2.1. Let (M, g) be a Riemannian n-manifold and let a fixed positive constant $C > 0$ be given.

(1) For each point $x \in M$, the $C^{k,\alpha}$ *harmonic radius* at x, denoted by $r_h^{C^{k,\alpha}}(x) = r_h(x)$, is defined by the radius of the largest geodesic ball $B(x, r_h(x))$ on which there exist harmonic coordinates $u_i : B(x, r_h(x)) \to \mathbf{R}^n$ satisfying

$$e^{-C}\delta_{ij} \leq g_{ij} \leq e^{C}\delta_{ij}, \qquad \text{(as binear forms)} \tag{3.2.1}$$

$$[r_h(x)]^{k+\alpha} \|g_{ij}\|_{C^{k,\alpha}(B(x,r_h(x)))} \leq C. \tag{3.2.2}$$

(2) The $C^{k,\alpha}$ harmonic radius of (M, g) is defined by

$$r_h^{C^{k,\alpha}}(M) = \inf_{x \in M} r_h^{C^{k,\alpha}}(x).$$

(3) Similarly, the $L^{k,p}$ *harmonic radius* of a point x, $r_h^{L^{k,p}}(x) = r_h(x)$, is the radius of largest geodesic ball $B(x, r_h(x))$ on which there exist harmonic coordinates $u_i : B(x, r_h(x)) \to \mathbf{R}^n$ satisfying

$$e^{-C}\delta_{ij} \leq g_{ij} \leq e^{C}\delta_{ij}, \qquad \text{(as binear forms)}$$

$$[r_h(x)]^{k-\frac{n}{p}} \|g_{ij}\|_{L^{k,p}(B(x,r_h(x)))} \leq C. \tag{3.2.3}$$

(4) The $L^{k,p}$ harmonic radius of (M, g) is defined by

$$r_h^{L^{k,p}}(M) = \inf_{x \in M} r_h^{L^{k,p}}(x).$$

We remark that $g_{ij} = g(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$ and the left-hand side of the inequality (3.2.2) means, more precisely,

$$\sup_{B(x,r_h(x))} |g_{ij}| + r_h(x) \sup_{B(x,r_h(x))} |dg_{ij}| + r_h(x)^2 \sup_{B(x,r_h(x))} |d^2 g_{ij}| + \cdots$$
$$+ r_h(x)^k \sup_{B(x,r_h(x))} |d^k g_{ij}| + r_h(x)^{k+\alpha} \sup_{x \neq y} \frac{|d^k g_{ij}(x) - d^k g_{ij}(y)|}{|x-y|^\alpha}.$$

Similarly, denoting $B_x = B(x, r_h^{L^{k,p}}(x))$, the left-hand side of (2.2.4) means

$$r_h(x)^{-\frac{n}{p}} \left(\int_{B_x} |g_{ij}|^p \, dv_g \right)^{\frac{1}{p}} + r_h(x)^{1-\frac{n}{p}} \left(\int_{B_x} |dg_{ij}|^p \, dv_g \right)^{\frac{1}{p}} + \cdots$$
$$+ r_h(x)^{k-\frac{n}{p}} \left(\int_{B_x} |d^k g_{ij}|^p \, dv_g \right)^{\frac{1}{p}}.$$

It is easy to see that if (M, g) is a closed smooth (in fact, C^2) Riemannian n-manifold, then $r_h(M) > 0$. The harmonic radius, of course, may depend on the constant C. But we may omit the constant C and just say the harmonic radius unless one is confused. Also, the harmonic radius behaves like distance or raidius, i.e., if we rescale $\widetilde{g} = \lambda^2 g$, $\lambda > 0$, then $\widetilde{r}_h(x) = \lambda \cdot r_h(x)$, where $\widetilde{r}_h$ denotes the harmonic radius with respect to the metric $\widetilde{g}$. This is one reason that we put r factor in (3.2.2) and (3.2.3).

However, note that (3.2.1), (3.2.2) and (3.2.3) are scale invariant. In fact, if $\widetilde{g} = \lambda^2 g$, $\lambda > 0$ and $\{u_i\}$ are harmonic coordinates for the metric g, then setting $\{\widetilde{u}_i = \lambda \cdot u_i\}$, we have

$$\widetilde{g}_{ij} = \widetilde{g}(\frac{\partial}{\partial \widetilde{u}_i}, \frac{\partial}{\partial \widetilde{u}_j}) = \lambda^2 g(\frac{1}{\lambda}\frac{\partial}{\partial u_i}, \frac{1}{\lambda}\frac{\partial}{\partial u_j}) = g_{ij}. \tag{3.2.4}$$

On the other hand, for a smooth function $f \in C^\infty(M)$,

$$|df|_{\widetilde{g}} = \frac{1}{\lambda}|df|_g \quad \text{and} \quad |d^l f|_{\widetilde{g}} = \frac{1}{\lambda^l}|d^l f|_g. \tag{3.2.5}$$

Since $\widetilde{r}_h(x)^l = \lambda^l \cdot r_h(x)^l$, the $C^{k,\alpha}$ harmonic radius is scale invariant.

For the $L^{k,p}$ harmonic radius, recalling the volume form $dv_{\widetilde{g}} = \lambda^n \cdot dv_g$, we get for each $l = 0, 1, \cdots, k$,

$$\begin{aligned}\widetilde{r}_h(x)^{l-\frac{n}{p}} \left(\int_{B_x} |d^l f|^p_{\widetilde{g}}\, dv_{\widetilde{g}}\right)^{1/p} &= \lambda^{l-\frac{n}{p}} r_h(x)^{l-\frac{n}{p}} \left(\int_{B_x} \frac{1}{\lambda^{lp}} |d^l f|^p_g \lambda^n\, dv_g\right)^{1/p} \\ &= r_h(x)^{l-\frac{n}{p}} \left(\int_{B_x} |d^l f|^p_g\, dv_g\right)^{1/p}.\end{aligned}$$

Hence the $L^{k,p}$ harmonic radius is also scale invariant. We will consider, mainly, C^α, $C^{1,\alpha}$, $L^{1,p}$ and $L^{2,p}$ harmonic radius which are the most important cases in some sense. For instance, we have the following

Corollary 3.2.2. *For given $\Lambda > 0$ and $\mu > 0$, if (M, g) is a Riemannian n-manifold satisfying*

$$|K_M| \leq \Lambda, \quad inj(M) \geq \mu,$$

then

$$r_h^{C^{1,\alpha}}(M) \geq C = c(n, \Lambda, \mu).$$

Proof. It follows directly from Theorem 3.1.11. □

Furthermore, $L^{2,2}$ harmonic radius is used in [An3] to prove the geometrization conjecture in dimension 3 of nonpositive case.

Definition 3.2.3. A sequence of Riemannian n-manifolds (M_i, g_i) is said to converge in the $C^{k,\alpha}$ topology to a $C^{k,\alpha}$ Riemannian manifold (M, g) if M is a C^∞ manifold with a $C^{k,\alpha}$ metric tensor g, and there is a sequence of diffeomorphisms $F_i : M \to M_i$, for i sufficiently large, such that the metrics $F_i^* g_i$ converge to g in the $C^{k,\alpha}$ topology on M. Here the $C^{k,\alpha}$ structure is defined with respect to some fixed $C^{k,\alpha}$ atlas on M, compatible its C^∞ structure.

The next lemma shows that the harmonic radius is contiuous in the $C^{k,\alpha}$ topology or $L^{k,p}$ topology.

Lemma 3.2.4. *If (M_i, g_i) converges to (M, g) in the $C^{1,\alpha}$ topology, then*

$$r_h^{C^{1,\alpha}}(M_i) \to r_h^{C^{1,\alpha}}(M).$$

The same is true pointwise, i.e., for the harmonic radius at any sequence $\{z_i\} \to z \in M$.

Proof. First, note that on a $C^{1,\alpha}$ Riemannian manifold the Laplace operator is well defined as in (3.1.3), so that one may speak of harmonic functions on M, which are then at least in $C^{2,\alpha}$. Similarly, the concept of $C^{1,\alpha}$ harmonic radius is well defined on M. We first show that r_h is upper semi-continuous, namely,

$$r_h(M) \geq \limsup_{i\to\infty} r_h(M_i).$$

Let $r_i = r_h^{C^{1,\alpha}}(M_i) = r_h^{C^{1,\alpha}}(x_i, M_i)$ and let $\{u_i^j\}_{j=1}^n$ be harmonic coordinates on $B_i = B(x_i, r_i)$ such that with these coordinates

$$e^{-C}\delta_{kl} \leq (g_i)_{kl} \leq e^C\delta_{kl} \tag{3.2.6}$$

and

$$r_i^{1+\alpha}\|(g_i)_{kl}\|_{C^{1,\alpha}(B_i)} \leq C. \tag{3.2.7}$$

Since $M_i \to M$ in the $C^{1,\alpha}$ topology, that is, $g_i \to g$ in the $C^{1,\alpha}$ topology, the charts $\{u_i\}$ converge in the $C^{2,\alpha}$ topology to a limit map $u : B \to \mathbf{R}^n$, where $B = B(x, r), x_i \to x$ and $r = \limsup_{i\to\infty} r_i$. It is easy to see that the limit map u is also harmonic with respect to the metric g. Since the bounds (3.2.6) and (3.2.7) are clearly preserved under the $C^{1,\alpha}$ convergence, we have

$$r_h(M) \geq r = \limsup_{i\to\infty} r_i.$$

To obtain the converse

$$r_h(M) \le \liminf_{i\to\infty} r_i, \tag{3.2.8}$$

suppose $r \le r_h(M)$ is finite and let $\{x_k\}$ be harmonic coordinates on $B = B(r) \subset (M, g)$ satisfying (3.2.6) and (3.2.7). Via diffeomorphisms, we may view the metrics g_i on B, for i sufficiently large. Let Δ_i be the Laplace operator of g_i in the coordinates $\{x_k\}$ on B, i.e.,

$$\Delta_i = \frac{1}{\sqrt{g}_i} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} \left(g_i^{kl} \sqrt{g}_i \frac{\partial}{\partial x_l} \right).$$

Let $\{y_k^i\}$ be solutions to the Dirichlet problem for Δ_i on B with boundary values $\{x_k\}$, and set $w_k^i = x_k - y_k^i$. Thus,

$$\Delta_i w_k^i = \Delta_i x_k, \qquad w_k^i|_{\partial B} = 0.$$

By the Schauder estimates (Theorem 3.1.8), one has the estimates

$$\|w_k^i\|_{C^{1,\alpha}(B')} \le C(\|g_i\|_{C^\alpha}, B') \|\Delta_i w_k^i\|_{C^\alpha(B)}, \tag{3.2.9}$$

where $B' \subset\subset B$, since w_k^i has zero boundary values. By definition, we have $\Delta x_k = 0$. Furthermore,

$$\begin{aligned} \Delta_i w_k^i = \Delta_i x_k &= \frac{1}{\sqrt{g}_i} \frac{\partial}{\partial x_k} \left(g_i g_i^{kk} \right) \\ &\to \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_k} \left(g g^{kk} \right) = \Delta x_k = 0, \end{aligned}$$

in the C^α topology. In other words,

$$\|\Delta_i w_k^i\|_{C^\alpha} = \|\Delta_i x_k\|_{C^\alpha} \to 0 \quad \text{as} \quad i \to \infty.$$

Thus one has $w_k^i \to 0$ by (3.2.9) and so $y_k^i \to x_k$ in the $C^{2,\alpha}$ topology and uniformly on compact subsets of B. Since the bounds (3.2.6) and (3.2.7) are continuous in the $C^{1,\alpha}$ topology, they are satisfied for the charts $\{y_k^i\}$ on arbitary compact subsets $B' \subset\subset B$, with constants $C_i \to C$ as $i \to \infty$. Hence $r_h^{C^{1,\alpha}}(x_i, M_i) \geq r - \delta$ for any $\delta > 0$.This then establishes (3.2.8). $\square$

Remark 3.2.5.

(1) The same proof shows that Lemma 3.2.4 is true when $C^{1,\alpha}$ is replaced by $C^{k,\alpha}, k \geq 1$ by Theorem 3.1.8.

(2) The same proof with replacing the Schauder estimates by the L^p estimates

$$\|w_k^i\|_{L^{k,p}(B')} \leq C(B') \cdot \|\Delta_i x_k\|_{L^p(B)},$$

shows that the same is true for the $L^{k,p}, k \geq 1$ convergence.

(3) It is not clear that if the C^α harmonic radius is continuous in the C^α topology, since one does not have the estimates (3.2.8) in this case.

(4) The *injectivity radius* is not continuous in the $C^{1,\alpha}$ topology for any $\alpha < 1$. In fact, $inj_M(x)$ is upper semi-continuous in the C^o topology, but not lower semi-continuous in the $C^{1,\alpha}$ topology with $\alpha < 1$. However inj_M is lower semi-continuous in the C^2 topology (cf. [Sak]).

§3.3 Convergence theorems

In this section, we are going to prove some convergence thorems. Fisrt we start with the classical compactness and convergence theorems for functions which are called *Arzela-Ascoli* theorem and *Aloaglu theorem.*

Theorem 3.3.1 (Arzela-Ascoli). *Let Ω be a domain of $\mathbf{R}^n$ and let $\{f_i : \Omega \to \mathbf{R}\}$ be a sequence of smooth functions. Assume for $\alpha < 1$,*

$$\|f_i\|_{C^{k,\alpha}(\Omega)} \leq C$$

for some constant C which is independent of i. Then for any $\alpha' < \alpha$, there exists a subsequence $\{f_{i_k}\}$ which converges, in $C^{k,\alpha'}$ topology, to a $C^{k,\alpha}$ limit function f_∞.

Theorem 3.3.2 (Aloaglu). *Let B be a reflexive Banach space and let $\{f_i : B \to \mathbf{R}\}$be a sequence of functions satisfying*

$$\|f_i\|_B \leq C$$

for some constant C. Then a subsequence $\{f_{i_k}\}$ converges weakly to a function $f : B \to \mathbf{R}$.

Here *weakly* means f_i converges in the weak (*) topology to a function f, that is, for any element A in the dual space B^* of linear functionals, one has

$$A(f_i) \to A(f) \quad \text{as} \quad i \to \infty.$$

Corollary 3.3.3. *Let Ω be a domain of $\mathbf{R}^n$ and let $\{f_i : \Omega \to \mathbf{R}\}$ be a sequence of smooth functions satisfying*

$$\|f_i\|_{L^{k,p}(\Omega)} \leq C$$

for some constant C. Then a subsequence $\{f_{i_k}\}$ converges weakly to a function $f \in L^{k,p}(\Omega)$.

The next theorem gives one convergence criterion which is important.

Theorem 3.3.4. *Given a sequence (M_i, g_i) of closed Riemannian n-manifolds, suppose that there exist atlas $\mathcal{A}_i = \{F^i_k : U^i_k \to \mathbf{R}^n\}$ and constants $\delta_o > 0, C > 0$ such that*

(1) $r_h^{C^{k,\alpha}}(M_i, g_i) \geq \delta_o > 0$ *with bounds (3.2.1) and (3.2.2)*

(2) $(M_i, g_i, \mathcal{A}_i)$ *satisfies overlap condition, i.e., for some constant $\widetilde{C}$, independent of i, one has*

$$\|F^i_k \circ (F^i_l)^{-1}\|_{C^{k+1,\alpha}} \leq \widetilde{C}$$

on $U^i_k \cap U^i_l \neq \emptyset$.

(3) *There is a uniform bound N on the number of coordinate charts as well as on the multiplicity of their intersections.*

Then, there is a subsequence, say it also (M_i, g_i), and a smooth manifold M such that (M_i, g_i) converges, in the $C^{k,\alpha'}$ topology, for $\alpha' < \alpha$, to a limit $C^{k,\alpha}$ metric g on M.

Proof. The proof is consisted of two steps. The main idea is to embed M_i into an Euclidean space of large dimension so that the image looks like a graph in it.

Step 1 The first step is to prove M_i can be embedded in the Euclidean space of large dimension as the whitney embedding theorem. Without loss of generality, we may assume, by passing to a subsequence and adding new charts if necassary, that the cardinality of altas $\mathcal{A}_i$ are all equal to N and there is a covering $B^i_{x_j}(\delta_1)$ of M_i such that each ball is contained in a domain of some F^i_k, where $\delta_1 = \frac{1}{2}e^{-C}\delta_o$. Rearranging the subindex and normalizing so that $F^i_k(x_k) = 0 \in \mathbf{R}^n$ by overlap condition, we have

$$B_o(e^{-C}\delta_o) \subset F^i_k(B_{\delta_o}(x_k)) \subset B_o(e^C\delta_o)$$

by (3.2.1).

Choose a cut-off function $\xi : \mathbf{R}^+ \to \mathbf{R}$ satisfying

(i) $0 \le \xi \le 1, \quad supp(\xi) \subset [0, \delta_1]$

(ii) $\xi \equiv 1$ on $[0, \delta_2], 0 < \delta_2 < \delta_1$ and $\xi \equiv 0$ on $[\delta_o, \infty)$.

Define ξ_k on $B_{\delta_o}(x_k)$ by

$$\xi_k(x) = \xi(\|F^i_k(x)\|)$$

and extend it to M_i by 0. Then define a smooth map

$$\Phi : M_i \to \mathbf{R}^{nN+N}$$

by

$$\Phi(x) = (\xi_1(x)F^i_1(x), \cdots, \xi_N(x)F^i_N(x), \xi_1(x), \cdots, \xi_N(x))$$

We claim that Φ is an embedding. To show this, it is enough to prove Φ is one-one. Suppose $\Phi(x_1) = \Phi(x_2)$. Then for all $j = 1, \cdots, N$,

$$\xi_j(x_1)F^i_j(x_1) = \xi_j(x_2)F^i_j(x_2), \quad \text{and} \quad \xi_j(x_1) = \xi_j(x_2).$$

On the other hand, by definition of $\xi_j's$, there is an $j_o, 1 \leq j_o \leq N$ such that $\xi_{j_o}(x_1) = \xi_{j_o}(x_2) \neq 0$. Thus $F^i_{j_o}(x_1) = F^i_{j_o}(x_2)$ and so $x_1 = x_2$ since $F^i_{j_o}$ is a diffeomorphism.

Step 2 We claim that the image $\Phi(M_i)$ is locally a graph with uniformly bounded $C^{k+1,\alpha}$ norm. Consider for instance $\Phi(B'_1) \subset \mathbf{R}^{N(n+1)}$, where $B'_1 = B_{\delta'_1}(x_1) \subset\subset B_{\delta_o}(x_1) \subset U_1, \delta'_1 << \delta_o$. We will show that $\Phi(B'_1)$ is a graph over $F^i_1(B'_1) \subset \mathbf{R}^n$. Recall that

$$\Phi(B'_1) = \left\{ \left(\xi_1(x)F^i_1(x), \cdots, \xi_N(x)F^i_N(x), \xi_1(x), \cdots, \xi_N(x)\right) : x \in B'_1 \right\}.$$

Let $y \in B'_1$ so that $\xi_1(y) = 1$ and set $x = F^i_1(y) = \xi_1(y)F^i_1(y) \in \mathbf{R}^n$. The for any $j \geq 2$, we have

$$F^i_j \circ F^i_1(x) = F^i_j(y)$$

and letting $\phi_j = \xi(\|F^i_j \circ (F^i_1)^{-1}\|)$,

$$\varphi_j(x) = \xi(\|F^i_j \circ (F^i_1)^{-1}(x)\|) = \xi(\|F^i_j(y)\|) = \xi_j(y).$$

Thus,

$$\Phi(B'_1) = \{(x, \phi_2(x)F^i_2 \circ (F^i_1)^{-1}(x), \cdots, \phi_N(x)F^i_N \circ (F^i_1)^{-1}(x),$$
$$\phi_1(x), \cdots, \phi_N(x)) : x \in B'_1\}$$

which is a graph over $F^i_1(B'_1)$. Now letting $G^i_j = F^i_j \circ (F^i_1)^{-1}$, graphing functions have a ball of a fixed size contained in them, i.e.,

$$\|G^i_j\|_{C^{k+1,\alpha}} = \|F^i_j \circ (F^i_1)^{-1}\|_{C^{k+1,\alpha}} \leq \widetilde{C}$$

by the overlap condition and $\|\phi\|_{C^{k+1,\alpha}} \leq C'$. Applying the Arzela-Ascoli Theorem, we get a subsequence $\{(G^i_j, \phi_j)\}$ which converges in the $C^{k+1,\alpha'}, \alpha' < \alpha$ topology to limit functions $\{(G^\infty_j, \phi^\infty_j)\}$ as $i \to \infty$ and these are $C^{k+1,\alpha}$. Define

$$M_\infty = \bigcup graph(G^\infty_j, \phi^\infty_j)$$

which is smooth embedded manifold in $\mathbf{R}^N$. Moreover, note that the ϵ-tubular neighborhood of M_∞ satisfies $M_i \subset T_\epsilon(M_\infty)$ for i sufficiently large and so the retraction of M_i to M_∞ gives an immesion $\Psi_i : M_i \to M_\infty$ which is one to one. Thus, M_∞ is diffeomorphic to M_i for i sufficiently large. On the other hand, as proved in Cheeger's finiteness ([Che]), one can see that

$$g^i_{kl} = g^i(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) \to g_\infty$$

in the $C^{k,\alpha'}, \alpha' < \alpha$ topology. This completes the proof. □

Remark 3.3.5. (1) Although phrased in terms of compact manifolds, it is easily to see that Theorem 2.3.4 is also valid locally, for bounded domains in a Riemannian manifold, as well as for pointed complete Remannian manifolds, provided one works on compact subsets.

(2) We remark that the conditions (2) and (3) in Theorem 3.3.1 are less important than the condition (1) in some sense. For instance, suppose (M, g) be a compact Riemannian n-manifold with atlas $\{F_k : U_k \to \mathbf{R}^n\}$ of harmonic coordinates satisfying

$$e^{-C}\delta_{ij} \leq g_{ij} \leq e^C\delta_{ij}$$

and

$$\|g_{ij}\|_{C^{k,\alpha}(U_k)} \leq C \tag{3.3.1}$$

with respect to the harmonic coordinates F_k. Then the condition (2) holds automatically, that is, from (3.1.8)

$$\|F^i_k \circ (F^i_l)^{-1}\|_{C^{k+1,\alpha}(Im(F_k))} = \|F^i_k\|_{C^{k+1,\alpha}(U_k)} \leq \widetilde{C}$$

on $U_k \cap U_l \neq \emptyset$. On the other hand, the condition (3) follows from lower bound of Ricci curvature and upper bound of diameter of the given manifold. In fact, we have the following theorem.

Lemma 3.3.6 (packing lemma). *Let (M^n, g) be a complete Riemannian n-manifold with $Ric(M) \geq -(n-1)k$, $k > 0$. Then for any point $p \in M$ and for given positive real numbers $r, \epsilon > 0$, there exists a covering $\cup_1^N B(p_i, \epsilon) \supset B(p, r)$ with $p_i \in B(p, r)$ and $N \leq N_1(n, k, r, \epsilon)$. Moreover, the multiplicity of this covering is at most $N_2(n, k)$.*

Proof. The proof follows from the volume comparison theorem. For given $r, \epsilon > 0$, choose a maximal disjoint $\epsilon/2$ balls, $\{ B(p_i, \epsilon/2) \}$, in $B(p, r)$ so that $\{B(p_i, \epsilon)\}$ covers $B(p, r)$. If N denotes the number of $\epsilon/2$-balls, then we have

$$N \cdot vol(B(\epsilon/2)) \leq vol(B(p, r)).$$

Thus the volume comparison theorem implies that

$$N \leq \frac{vol(B(p, r))}{vol(B(\epsilon/2))} \leq \frac{V_k(r)}{V_k(\epsilon/2)}.$$

□

Hence if M is compact with $diam(M) = D$ and $Ric(M) \geq -(n-1)k^2$, then for any $\epsilon > 0$, we can choose a finite covering $M \subset \cup_1^N B(p_i, \epsilon)$, $N \leq N_1 = N_1(n, k, \epsilon)$.

The general scheme to prove a precompactness theorem for a sequence of Riemannian manifolds with geometric constraints by using the Gromov-Hausdorff topology is the following. First suppose the conclusion one wants to prove dose not hold. Then there is a sequence of Riemannian manifolds in the given class satisfying given geometric conditions. Rescale the sequence of Riemannian manifolds by some sequence of real numbers and then consider the limit space with respect to the Gromov-Hausdorff topology. After studying the structure of limit space by using given geometric conditions, one gets a contradiction. The following theorem which gives a structure of limit space is on line of this general scheme.

Theorem 3.3.7. *Assume M is a complete Riemannian n-manifold with non-negative Ricci curvature and infinite injectivity radius. Then M is isometric to the Euclidean flat manifold $\mathbf{R}^n$.*

Proof. Note that M must be noncompact. Fix a point $p \in M$ and choose a ray $\gamma(t)$ from p. Considering a function defined by $\phi_t(x) = d(x, \gamma(t)) - t$, we have

$$|\Delta \phi_t| \leq \frac{n-1}{d(\gamma(t), \cdot)} = \frac{n-1}{\phi_t + t} \to 0 \quad \text{as} \quad t \to \infty$$

since $Ric_M \geq 0$. Furthermore, by triangle inequality, the functions ϕ_t form a family of equicontinuous functions on M, bounded by $d(x,p)$ and non-increasing. Therefore, it we set $\beta(x) = \lim_{t\to\infty} d(x, \gamma(t)) - t$, it is well-known that

$$\Delta\beta \equiv 0, \qquad |\nabla\beta| \equiv 1.$$

To finish the proof, we use the Bochner formula

$$\frac{1}{2}\Delta|\nabla\beta|^2 = |D^2\beta|^2 + \langle\nabla\Delta\beta, \nabla\beta\rangle + Ric(\nabla\beta, \nabla\beta).$$

Thus we have

$$Ric(\nabla\beta, \nabla\beta) = 0 \quad \text{and} \quad |D^2\beta| = 0.$$

Since $\nabla\beta$ denotes an arbitary direction, it follows that M is Ricci flat and $D^2\beta = 0$ implies that $\nabla\beta$ is a parallel vector field. Thus the metric on M splits isometrically along $\nabla\beta$ and this shows M is isometric to $\mathbf{R}^n$. For more details, see [C-G], [S-Y] or [Bes]. □

As we have seen in Theorem 3.3.1, the main idea to prove a precompactness is the proof of the existence of harmonic coordinates on balls of uniform size (depending only on the given constants).

It is natural to seek geometric conditions which guarantee that a manifold has harmonic charts on balls of uniform size. We carry this out here in the next theorem.

Theorem 3.3.8. *For given positive constants Λ, D and i_o, let (M, g) be a Riemannian n-manifold (not necessarily complete) satisfying*

$$|Ric_M| \le \Lambda, \quad inj(M) \ge i_o, \quad diam(M) \le D. \tag{3.3.2}$$

Then for any $\alpha < 1$ and a constant $c > 0$ there exists a constant $C = C(n, \Lambda, i_o, c, \alpha) > 0$ such that the harmonic radius with constant c has the estimates

$$r_h^{C^{1,\alpha}}(x, M) \ge C \cdot \frac{d(x, \partial M)}{diam(M)} \cdot i_o, \tag{3.3.3}$$

where $\frac{d(x,\partial M)}{diam(M)} = 1$ if $\partial M = \emptyset$.

Proof. Suppose the theorem is false, i.e., the (3.3.3) dose not hold for any constant. Then there is a sequence of Riemannian n-manifolds (M_i, x_i, g_i) such that

$$|Ric_{M_i}| \le \Lambda, \quad inj(M_i) \ge i_o, \quad diam(M_i) \le D$$

but

$$\frac{r_h^{C^{1,\alpha}}(x_i, M_i)}{i_o} \cdot \frac{diam(M_i)}{d(x_i, \partial M_i)} \to 0 \tag{3.3.4}$$

as $i \to \infty$. Note that (3.3.4) is scale invariant. Assume without loss of generality that x_i is realized minimum value of the ratio (3.3.4). Denote $r_i = r_h^{C^{1,\alpha}}(x_i, M_i)$ so that $r_i \to 0$ as $i \to \infty$ since the diameter condition implies that the quotient of the second term in (3.3.4) does not go to 0. Consider the rescaled metrics $\widetilde{g}_i = r_i^{-2} g_i$ on M_i and let $\widetilde{r}_i = r_h^{C^{1,\alpha}}(x_i, M_i, \widetilde{g}_i)$. Then we have the followings

(i) $\widetilde{r}_i = r_h^{C^{1,\alpha}}(x_i, M_i, \widetilde{g}_i) = 1$

(ii) $|Ric_{M_i}(\widetilde{g}_i)| \le \Lambda \cdot r_i^2 \to 0$

(iii) $inj(M_i, \widetilde{g}_i) \ge r_i^{-1} \cdot i_o \to \infty$

(iv) $d_{\widetilde{g}_i}(x_i, \partial M_i) \to \infty$.

By rescaling the harmonic coordinates $\{u^i\}$ by setting $v^i = r_i^{-1} \cdot u^i$, together with (i), one has $h^i_{kl} = h^i(\nabla v_k, \nabla v_l)$ satisfying $h^i_{kl} = (g_i)_{kl}$, so that

$$e^{-c} \cdot I \le h^i(y) \le e^c \cdot I,$$

$$\|h^i(y)\|_{C^{1,\alpha}} \le c,$$

on all balls of radius ≈ 1 on (M_i, x_i, h^i), but no harmonic coordinate system satisfying inequalities above on $B_{1+\mu}(x_i), \mu > 0$.

In other words, there is a covering of (M_i, x_i, h^i) by geodesic balls of fixed (but not large) radius, on which one has harmonic coordinate system for which the metric tensor $\{h^i\}$ are uniformly bounded in the $C^{1,\alpha}$ norm. Thus by Theorem 3.3.4, it follows that a subsequence of $\{(M_i, x_i, h^i)\}$ converges, in the $C^{1,\alpha'}$ topology, $\alpha' < \alpha$, to a $C^{1,\alpha}$ Riemannian manifold (N, x, h), with $x = \lim x_i$. We note that, by (iv), (N, h) is a complete Riemannian manifold. Since $|Ric(M_i)(h^i)| \to 0$ in the C^o topology, by Exercise 3.1.2 (3), h is a weak solution to the Einstein equation

$$h^{ij} \frac{\partial^2 h_{rs}}{\partial x^i \partial x^j} + Q\left(\frac{\partial h_{kl}}{\partial x_m}\right) = 0.$$

Then the regularity theory implies that h is a smooth, in fact, real-analytic Ricci-flat metric on M. Furthermore, by (iii), one has $inj(N, h) = \infty$. Therefore by Theorem 3.3.9, (N, h) is isometri to the Euclidean $\mathbf{R}^n$ with the flat metric. Since the harmonic radius is continuous in the $C^{1,\alpha}$ topology, one has $r_h(x, \widetilde{g}) = 1$ but for $\mathbf{R}^n$, $r_h(x) = \infty$ which is a contradiction. Hence the harmonic radius is uniformly bounded below. □

Corollary 3.3.9. *The space $\mathcal{M}(\lambda, i_0, D)$ of compact Riemannia m-manifolds such that*

$$|Ric| \le \lambda, \quad inj \ge i_0 > 0, \quad diam \le D, \tag{3.3.5}$$

is precompact in the $C^{1,\alpha}$ topology. More precisely, given any sequence $\{(M_i, g_i)\} \in \mathcal{M}(\lambda, i_0, D)$, there are diffeomorphisms f_i of M_i such that $\{f_i^* g_i\}$ subconverges, in the $C^{1,\alpha''}$ topology, for $\alpha'' < \alpha$, to a $C^{1,\alpha}$ Riemannian manifold (M, g). In particular, there are only finitely many diffeomorphism types satisfying these bounds.

Proof. It follows from Theorem 3.3.8, Lemma 3.3.6 and Remark 3.3.5. □

As an application we will prove the well-known Cheeger-Gromov precompactness theorem.

Theorem 3.3.10 ([Che], [G-L-P], [G-W], [Kas]). *Let $\{(M_i, g_i)\}$ be a sequence of closed Riamannian m-manifolds satisfying*

$$|K_{M_i}| \le \Lambda, \quad vol(M_i) \ge v, \quad diam(M_i) \le D, \tag{+}$$

then there exists a subsequence which converges, in the $C^{1,\alpha}$ topology, for $\alpha'' < \alpha$, to a $C^{1,\alpha}$ manifold M for any $0 < \alpha < 1$.

Proof. By the well-known theorem due to Cheeger,the injectivity radius for the class of Riemannian m-manifolds satisfying the bounds (+) is uniformly bounded by below. Thus it follows from Corollary 3.3.9. □

Remark 3.3.11.

(1) Let (M, g) be a Riemannian m-manifold and assume

$$|K(M)| \le \Lambda, \quad diam(M) \le D.$$

Then by a Cheeger's theorem, the lower volume bound gives a lower bound for the inectivity radius and vice versa. However if one replaces $|K_M| \leq \Lambda$ by $|Ric| \leq \lambda$, then it is not true anymore (cf. [An2]). Thus it is interesting to ask what one can say if one replaces $inj(M) \geq i_o$ by $vol(M) \geq v$ in (3.3.5).

(2) Suppose a Riamannian n-manifold (M, g) satisfies

$$Ric \geq -\lambda, \quad inj \geq i_o. \tag{3.3.6}$$

Then an elementary packing argument, based on the volume comparison theorem (cf. [G-L-P]), shows that the bound $vol(M, g) \leq V$ is equivalent to a diameter bound $diam(M) \leq D$. It is obvoius that $diam(M) \leq D$ implies that $vol(M) \leq V$. To show the converse, first note that $vol(M) \leq V$ and $Ric(M) \geq -\lambda$ imply M is compact. Let N be the maximal number of disjoint i_o- balls in M. Then the volume comparison theorem and $inj(M) \geq i_o$ show that N is bounded above by a universal constant $C = C(V, i_o)$. Therefore, $diam(M) \leq 4i_o \cdot N \leq D = D(i_o, V)$.

Thus, we have the following.

Theorem 3.3.12. *The space of compact Riemannian n-manifolds (M, g) such that*

$$|Ric_M| \leq \Lambda, \quad inj_M \geq i_o, \quad vol(M) \leq V \tag{3.3.7}$$

is precompact in the $C^{1,\alpha}$ topology for any $\alpha < 1$. And so there are only finitely many diffeomorphism types of n-manifolds satisfying these bounds.

We will next consider the following class of Riemannian n-manifolds:

$$Ric \geq -\lambda, \quad inj \geq i_o, \quad vol \leq V. \tag{3.3.8}$$

As mentioned in Remark 3.3.11, the volume condition can be replaced by $diam(M) \leq D$. The remainder of this section is devoted to prove the following precompactness theorem.

Theorem 3.3.13 ([A-C]). *For given $\lambda > 0$, $i_o > 0$ and $V > 0$, the space of compact Riemannian n-manifolds (M, g) such that*

$$Ric(M) \geq -\lambda, \quad inj_M \geq i_o, \quad vol(M) \leq V \tag{3.3.9}$$

is precompact in the C^α topology for any $\alpha < 1$. More precisely, given any sequence of n-manifolds $\{(M_i, g_i)\}$ satisfying the bounds (3.3.9), and given any fixed $\alpha < 1$, there is a compact smooth manifold M, and diffeomorphisms $f_j : M \to M_j$, for a subsequence $\{j\}$ of $\{i\}$, such that the metrics $f_j^* g_j$ converge, in the $C^{\alpha'}$ topology for $\alpha' < \alpha$, to a Riemannian manifold (M, g) with C^α metric g.

To prove this theorem, one needs some lemmas.

Lemma 3.3.14. *Let M be a Riemannian n-manifold with $inj_M \geq i_o$ and $Ric_M \geq -\lambda^2, \lambda > 0$. Let $\rho = \rho_x \equiv dist(x, \cdot)$ be a distance function from $x \in M$. Then one has the estimate*

$$|\Delta\rho| \leq (n-1)\lambda \cdot \coth \lambda\rho, \tag{3.3.10}$$

provided $\rho < i_o/2$.

Proof. Recall $\Delta\rho$ is the mean curvature of geodesic spheres $\partial B_x(r)$ and it is given by

$$\Delta\rho(y) = \frac{V'(r)}{V(r)}, \qquad r = \rho(y) = d(x, y).$$

Thus, a well-known version of the Bishop volume comparison theorem implies that

$$\Delta\rho \leq (n-1)\lambda \cdot \coth \lambda\rho, \tag{3.3.11}$$

provided $\rho \leq i_o$. Given x fixed, let p be any point with $t = d(x, p) \leq i_o/2$; let γ be the geodesic with $\gamma(0) = x$ and $\gamma(t) = p$; then set $p_1 = \gamma(2t)$. Thus (3.3.10) holds for $\rho = \rho_x$ and for $\rho_1 = \rho_{p_1}$ on $B_p(i_o/2)$. On the other hand, the function $\sigma = \rho + \rho_1 - 2t : M \to \mathbf{R}$ is nonnegative by the triangle inequality, and achieves its minimum value, 0, along the line segment γ between p and p_1. Hence, we have

$$\Delta\sigma = \Delta(\rho + \rho_1)|_\gamma \geq 0,$$

i.e., $\Delta\rho \geq -\Delta\rho_1 \geq -(n-1)\lambda \cdot \coth \lambda\rho$, which establishes (3.3.10). □

Theorem 3.3.15. *Let (M_i, x_i, g_i) be a sequence of pointed Riemannian n-manifolds satisfying*

(i) $r_h^{L^{1,p}}(M_i, g_i) \geq 1$

(ii) $Ric(M_i, g_i) \geq -r_i^2 \cdot \lambda \to 0, \quad r_i > 0, \quad$ *as* $\quad i \to \infty$

(iii) $inj(M_i, g_i) \geq r_i^{-1} \cdot i_o \to \infty \quad$ *as* $\quad i \to \infty$

Then a subsequence converges in the strong $L^{1,p}$ topology for any $p < \infty$ to a limit $L^{1,p}$ Riemannian manifold (N, x_∞, g_o).

Proof. By the loval version of convergence criterion theorem 3.3.4, there is a subsequence which converges weakly to a $L^{1,p}$ manifold (N, x_∞, g_o) and uniformly on a compact subset of N. Now let us show this subsequene converges, in fact, in the strong $L^{1,p}$ topology to N. For any tangent vector $v_i \in T_{x_i}M_i$, let γ_i be the geodesic in M_i with $\gamma_i(0) = x_i$ and $\gamma'(0) = v_i$. Set $y_i = \gamma_i(-s_i)$, where $s_i = \frac{1}{2} i_o \cdot r_i^{-1} \to \infty$ as $i \to \infty$. Then the distance function $\rho_i = d(y_i, \cdot) - s_i$ is smooth on $B(x_i, s_i/2)$, and by Lemma 3.3.14, with conditions (i), (ii) and (iii), one obtains estimates

$$|\Delta_i \rho_i| \leq (n-1)\lambda_i \coth \lambda_i \rho_i \to 0 \quad \text{as} \quad i \to \infty \tag{3.3.12}$$

on $B(x_i, s_i/2)$, where $\lambda_i = \frac{\lambda \cdot r_i}{n-1}$.

On the other hand, by (i), on each ball $B = B_i \subset M_i$ of bounded distance to x_i but fixed radius, one has harmonic coordinates $\{u_k\} = \{u_k^i\}$ with $L^{1,p}$ bounds, and for which the Laplace operator has of the form

$$\Delta_i = \sum g_i^{kl} \frac{\partial^2}{\partial u_k \partial u_l}. \tag{3.3.13}$$

Then applying Theorem 3.1.8, one has

$$\|\rho_i\|_{L^{2,p}(B')} \leq C(q, B') \left(\|\Delta_i \rho_i\|_{L^q(B)} + \|\rho_i\|_{L^2(B)} \right) \tag{3.3.14}$$

on $B' \subset\subset B$ for any $q (q \geq p)$. In particular, $\{\rho_i\}$ is uniformly bounded in $L^{2,q}(B)$, and thus by compactness of the embedding $L^{2,q} \subset L^{1,q}$, $\{\rho_i\}$ has a subsequence converging strongly in $L^{1,q}$ and weakly in $L^{2,q}$ to a $L^{2,q}$ limit ρ, where ρ is a distance function (or more precisely a Busemann function) on (N, g_o).

We claim that in fact $\{\rho_i\}$ has a subsequence converging strongly in $L^{2,q}$. To see this, we apply L^q estimates Theorem 3.1.8 again to $\rho - \rho_i$. (Here we are abusing notation slightly, namely, ρ_i is actually $\rho_i \circ F_i^{-1} : \mathbf{R}^n \to \mathbf{R}$, where F_i is the harmonic coordinate chart $\{u_k\}$. Thus every function can be considered as a function defined on domains in $\mathbf{R}^n$. This is also implicit in what follows.) One has estimates

$$\|\rho - \rho_i\|_{L^{2,p}(B')} \leq C(q, B') \left(\|\Delta_i(\rho - \rho_i)\|_{L^q(B)} + \|\rho - \rho_i\|_{L^2(B)} \right) \tag{3.3.15}$$

Clearly, $\|\rho - \rho_i\|_{L^2} \to 0$. To show that $\|\Delta_i(\rho - \rho_i)\|_{L^q} \to 0$, we have $|\Delta_i \rho_i| \to 0$ by (3.3.12) and also $\Delta_i \rho \to \Delta \rho$ in L^q since $\rho \in L^{2,q}$, and the coefficients in (3.3.13) converge in $C^{\alpha'}$ topology. Thus we need to show that $\Delta \rho = 0$ in L^q. Letting $f \in C_o^\infty(B)$, we compute

$$\int f \cdot \Delta \rho \, dv = \int \Delta f \cdot \rho \, dv = \lim_{i \to \infty} \int \Delta_i f \cdot \rho_i \, dv_i = \lim_{i \to \infty} \int f \cdot \Delta_i \rho_i \, dv_i = 0,$$

which establishes the claim.

We are now in position to verify that $g_i \to g_o$ strongly in the $L^{1,q}$ topology for any q. Namely, fix i for the moment and consider the distance function $\rho = \rho_i)$ constructed above. Then we have

$$|\nabla \rho|^2 = \sum g^{kl} \rho_k \rho_l = 1,$$

where $\rho_k = \partial \rho / \partial u_k$, and the $\{u_k\}$ are harmonic coordinates on $B \subset (M_i, g_i)$. Choose for instance an orthonormal basis e_μ of $T_{z_i} M_i$, where z_i is the center point of B, and consider the $n(n+1)/2$ vectore $e_\mu, e_\mu + e_\nu, \mu, \nu = 1, \cdots, n$. We let $\rho^m, 1 \leq m \leq n(n+1)/2$ equations

$$\sum_{k,l} g^{kl} \rho_k^m \rho_l^m = 1 \tag{3.3.16}$$

on B. We view this as a system of linear equations with g^{kl} as unknowns and $\rho_k^m \rho_l^m$ as coefficients. Suppose we could solve this system for a moment. Then g^{kl} and so g_{kl} are rational functions of $\{\rho_l^m\}$. It has been shown that the $\{\rho_l^m\}$ converges strongly in the $L^{1,q}$ topology for any q to limit $L^{1,q}$ functions; hence the same is true for $\{g_{kl}\} = \{g_{kl}(i)\}$.

Finally we prove the solvability for the linear system (3.3.6). One may algebraically solve this system for g^{kl} provided the determinant of the coefficients is nonzero. Clearly, by choosing the constant c sufficiently close 0, where c is the constant asociated with harmonic radius, g^{kl} is arbitray close, in the $C^{\alpha'}$ topology, to the Euclidean flat metric δ^{kl}. By the L^q estimates (3.3.15) and the argument above, each $\rho^m = \rho^m(i)$ is close in the $C^{1,\alpha'}$ topology to a limit distance function ρ on $B \subset (N, g_o)$. By choosing sufficiently small ball $B' \subset B$ (depending on (N, g_o)), we see that all $\rho^m(i)$ are close, in the $C^{1,\alpha'}$ topology, to the correspondingly defined Euclidean distance functions on B'. One may easily check that the matrix $\rho_k^m \rho_l^m$ is nonsingular in $\mathbf{R}^n$ and thus, by continuity of (3.3.16), it is nonsingular on B' for i sufficiently large. This proves the final claim. □

Proof of Theorem 3.3.13. In view of Theorem 3.3.4, Remark 3.3.5 and Lemma 3.3.6, and the Sobolev embedding theorem $L^{1,p} \subset C^\alpha, n < p < \infty, \alpha = 1 - n/p$, it suffices to show that for any $p, n < p < \infty$

$$r_h^{L^{1,p}}(M, g) \geq C(\lambda, i_o, V, n). \tag{*}$$

This in fact gives one stronger result.

Suppose (*) does not hold. Then there is a sequence of Riemannian n-manifolds (M_i, g_i) satisfying bounds (3.3.9), but

$$r_h^{L^{1,p}}(M_i, g_i) =: r_i \to 0 \quad \text{as} \quad i \to \infty.$$

Let $x_i \in M_i$ be a point realizing $r_h^{L^{1,p}}(M_, g_)$, i.e.,

$$r_h^{L^{1,p}}(M_i, g_i) = r_h^{L^{1,p}}(M_i, x_i, g_i) = r_i.$$

Rescaling the metrics g_i by $\tilde{g}_i = r_i^{-2} \cdot g_i$, one obtains

$$r_h^{L^{1,p}}(M_i, x_i, \tilde{g}_i) = 1 \quad \text{and} \quad r_h^{L^{1,p}}(M_i, x, g_i) \geq 1, \forall x \in M_i.$$

And one has, as $i \to \infty$,

(i) $Ric(M_i, \tilde{g}_i) \geq -r_i^{-2} \cdot \lambda \to 0$

(ii) $inj(M_i, \tilde{g}_i) \geq r_i^{-1} i_o \to \infty$

(iii) $vol(M_i, \tilde{g}_i), diam(M_i, \tilde{g}_i) \to \infty$

Consider the sequence of pointed Riemannian manifolds $(M_i, x_i, \tilde{g}_i)$ which satisfies $r_h^{L^{1,p}}(M_i, \tilde{g}_i) = 1$. Since the convergence criterion theorem (Theorem 3.3.4) holds also locally, applying it to $(B(x_i, R), \tilde{g}_i)$, $R > 0$, $(B(x_i, R), \tilde{g}_i)$ subconverges weakly in the $L^{1,p}$ topology to a $L^{1,p}$ manifold $(B(x_\infty, R), \tilde{g}_o)$.

Now choose a sequence of real numbers R_j so that $R_j \to \infty$ as $i \to \infty$ and apply $(B(x_i, R_j), \tilde{g}_i)$ to the above argument. Taking the usual diagonal sequence procedure, one has a subsequence, say it also $(M_i, x_i, \tilde{g}_i)$, which converges weakly in the $L^{1,p}$ topology to a complete noncompact $L^{1,p}$ manifold $(N, x_\infty, \tilde{g}_o)$, and converges uniformly on a compact subset. But by Theorem 3.3.15, $(M_i, x_i, \tilde{g}_i)$converges in the strong $L^{1,p}$ topology to $(N, x_\infty, \tilde{g}_o)$. From (i) and (ii) one has $Ric(N) \geq 0$ and $inj(N) = \infty$, and so by Theorem 3.3.8, N is isometirc to $\mathbf{R}^n$, and hence

$$r_h^{L^{1,p}}(N, x_\infty, \tilde{g}_o) = r_h^{L^{1,p}}(\mathbf{R}^n) = \infty.$$

which is a contradiction since one has

$$r_h^{L^{1,p}}(N, \tilde{g}_o) = \lim_{i \to \infty} r_h^{L^{1,p}}(M_i, x_i, \tilde{g}_i) = 1$$

by Theorem 3.2.3 □

References

[An1] M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bonds*, Invent. Math. **102** (1990), 429-445.

[An2] ______, *Short geodesics and gravitational instantons*, J. Diff. Geom. **31** (1990), 265-275.

[An3] ______, *Scalar curvature and geometrization of 3-manifolds: The non-positive case*, preprint.

[A-C] M. Anderson and J. Cheeger, *C^α compactness for manifolds with Ricci curvature and injectivity radius bounded below*, Jour. Diff. Geom. **35** (1992), 265-275.

[Bes] A. Besse, *Einstein manifolds*, Berlin-Heidelberg: Springer-Verlag, 1978.

[B-C] R.L. Bishop and R.J Crittenden, *Geometry of Manifolds*, Academi press, New york, 1964.

[B-G-P] Y. Burado, M. Gromov and G. Perelman, *Alexandrov spaces with curvature bounded below*, Uspekhi Mat. Nauk. **47** (1992), 3-51.

[Che] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. of Math. **92** (1970), 61-74.

[C-C1] J. Cheeger and T. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. **144** (1996), 189-237.

[C-C2] ______, *On the structures of spaces with Ricci curvature bounded below I*, preprint.

[C-E] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[C-G] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geometry **6** (1971), 119-128.

[DoC] M. Do Carmo, *Riemannian Geometry*, Prentice-Hall, Inc., 1990.

[Fu1] K. Fukaya, *Theory of convergence for Riemannian orbifolds*, Japan J. Math. **12** (1986), 121-160.

[Fu2] ______, *Hausdorff convergence of Riemannian manifolds and its applications*, Advanced Studies in Pure Mathematics 18-I, Recent topics in differential and analytic geometry, 1990.

[G-H-L] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, 1987.

[G-T] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1977.

[G-L-P] M. Gromov, J. Lafontaine and P. Pansu, *Structures métriques pour les variétés riemannienness*, Paris:Cedic/Fernand Nathan, 1981.

[G-W] R. Greene and H. Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific journal of Math. **131** (1988), 119-141.

[Hir] M. W. Hirsch, *Differential Topology*, Springer-Verlag.

[H-W] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press.

[J-K] J. Jost and H. Karcher, *Geometrische Methoden zur Gewinnung von apriori Schranken für harmonische Abbildungen*, Manus. Math. **40** (1982), 27-71.

[Kas] A. Kasue, *A convergence theorem for Riemannian manifolds and some applications*, Nagoya Math. J. **114** (1989), 21-51.

[Pet] P. Petersen, *Gromov-Hausdorff convergence of metric spaces*, Proc. of Symp. in Pure Math. **54 , part 3** (1993), 489-504.

[Rin] W. Rinow, *Die Innere Geometrie der Metrischen Ŕaum*, Springer, 1961.

[Sak] T. Sakai, *Comparison and finiteness theorems in Riemannian geometry*, Advanced Studies in Pure Mathematics 3, Geometry of geodesics and related topics, 1984.

[S-Y] R. Schoen and S-T Yau, *Lectures on differential geometry*, International press, 1988.

[Thu1] W. Thurston, *Geometry and Topology of 3-manifolds*, preprint, Princeton University Press, 1980.

[Thu2] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. of A.M.S **6** (1982), 357-381.

[Thu3] ———, *Hyperbolic structures on 3-manifolds I, deformations of acylindrical manifolds*, preprint.

[Yam] T Yamaguchi, *manifolds of almost nonnegative curvature*, MPI (1993).

Lecture Notes Series

1. M.-H. Kim (ed.), Topics in algebra, algebraic geometry and number theory, 1992
2. J. Tomiyama, The interplay between topological dynamics and theory of C*-algebras, 1992 ; 2nd Printing, 1994
3. S. K. Kim, S. G. Lee and D. P. Chi (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part I, 1993
   H. Kim, C. Kang and C. S. Bae (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part II, 1993
4. T. P. Branson, The functional determinant, 1993
5. S. S.-T. Yau, Complex hyperface singularites with application in complex geometry, algebraic geometry and Lie algebra, 1993
6. P. Li, Lecture notes on geometric analysis, 1993
7. S.-H. Kye, Notes on operator algebras, 1993
8. K. Shiohama, An introduction to the geometry of Alexandrov spaces, 1993
9. J. M. Kim (ed.), Topics in algebra, algebraic geometry and number theory II, 1993
10. O. K. Yoon and H.-J. Kim, Introduction to differentiable manifolds, 1993
11. P. J. McKenna, Topological methods for asymmetric boundary value problems, 1993
12. P. B. Gilkey, Applications of spectral geometry to geometry and topology, 1993
13. K.-T. Kim, Geometry of bounded domains and the scaling techniques in several complex variables, 1993
14. L. Volevich, The Cauchy problem for convolution equations, 1994
15. L. Elden and H. S. Park, Numerical linear algebra algorithms on vector and parallel computers, 1993
16. H. J. Choe, Degenerate elliptic and parabolic equations and variational inequalities, 1993
17. S. K. Kim and H. J. Choe (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part I, The first Korea-Japan conference of partial differential equations, 1993
    J. S. Bae and S. G. Lee (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part II, 1993
    D. P. Chi, H. Kim and C.-H. Kang (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part III, 1993
18. H.-J. Kim (ed.), Proceedings of GARC Workshop on geometry and topology ' 93, 1993
19. S. Wassermann, Exact C*-algebras and related topics, 1994
20. S.-H. Kye, Notes on abstract harmonic analysis, 1994
21. K. T. Hahn, Bloch-Besov spaces and the boundary behavior of their functions, 1994
22. H. C. Myung, Non-unital composition algebras, 1994
23. P. B. Dubovskii, Mathematical theory of coagulation, 1994
24. J. C. Migliore, An introduction to deficiency modules and Liaison theory for subschemes of projective space, 1994
25. I. V. Dolgachev, Introduction to geometric invariant theory, 1994
26. D. McCullough, 3-Manifolds and their mappings, 1995
27. S. Matsumoto, Codimension one Anosov flows, 1995
28. J. Jaworowski, W. A. Kirk and S. Park, Antipodal points and fixed points, 1995
29. J. Oprea, Gottlieb groups, group actions, fixed points and rational homotopy, 1995
30. A. Vesnin, On volumes of some hyperbolic 3-manifolds, 1996
31. D. H. Lee, Complex Lie groups and observability, 1996
32. X. Xu, On vertex operator algebras, 1996
33. M. H. Kwack, Families of normal maps in several variables and classical theorems in complex analysis, 1996
34. A. V. Fursikov and O. Yu. Imanuvilov, Controllability of evolution equations, 1996
35. Y. W. Lee, Introduction to knot theory, 1996
36. H. Kitahara, Some topics on Carnot-Caratheodory metrics, 1996 ; 2nd Printing (revised), 1998
37. D. Auckly, Homotopy K3 surfaces and gluing results in seiberg-witten theory, 1996
38. D. H. Chae (ed.), Proceedings of Miniconference of Partial Differential Equations and Applications, 1997
39. H. J. Choe and H. O. Bae (ed.), Proceedings of Korea-Japan Partial Differential Equations Conference, 1997
40. P. B. Gilkey, J. V. Leahy and J. G. Park, Spinors, spectral geometry, and Riemannian submersions, 1998
41. D.-P. Chi and G. J. Yun, Gromov-Hausdorff topology and its applications to Riemannian manifolds, 1998