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Proceedings of The 1st GARC SYMPOSIUM on Pure and Applied Mathematics

PART II

Edited by

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PREFACE

The first GARC Symposium on Pure and Applied Mathematics was held at the Global Analysis Research Center, the Seoul National University, from Thursday, February 13 to Friday, February 21, 1992. All the meetings were held at the Department of Mathematics Building.

The Global Analysis Research Center was inaugurated on March 1, 1991 under the Science Research Center Program of the Korea Science and Engineering Foundation to promote research ability in the field of mathematics in Korea. The central aim of the Global Analysis Research Center is the cooperative study of various analytic problems defined on manifolds such as partial differential equations, nonlinear analysis, operator algebra, dynamical systems and other related problems. The approach is a comprehensive one that also requires basic understandings of topological, geometric and algebraic properties of manifolds.

In order to maximize the efficiency of research, the Global Analysis Research Center has 6 Research Sections adapted to the natural division of research activities of the participating members. In accordance with the 6 Research Sections of the Global Analysis Research Center, the first GARC Symposium was carried out in 6 sessions; Partial Differential Equations, Nonlinear Analysis, Operator Algebra, Differential Geometry and Dynamical System, Topology and Geometry of Manifolds, and Complex Analytic Manifolds and Varieties.

The aim of the GARC Symposium was intended to set up mutual understandings on the interest of each research member and to explore current problems in the area of Global Analysis. Accordingly, almost all the research members of the Research Center including post doctors participated at the

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symposium. In addition, the organization committee invited several mathematicians from abroad. A few speakers were asked to survey their fields but the majority of speakers presented their recent research works.

In this proceedings of two issues, we collect all the lecture materials which were presented at the symposium. We would like to thank all the speakers, especially those professors from abroad, for their enthusiastic participation and their cooperation in writing up their talks. We would also like to thank the Korea Science and Engineering Foundation for their support to the Global Analysis Research Center and the Department of Mathematics of the Seoul National University for its hospitality.

We hope that in publishing this proceedings we will allow much wider audience to share in some of the work and enthusiasm of the participants at the symposium.

1992.10.

Jongsik Kim
Director
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TOPOLOGY AND GEOMETRY OF MANIFOLDS

THE GENERALIZED MYERS THEOREMS ON SPACE-TIMES

SEON-BU KIM

1. INTRODUCTION

Let M^n be a Riemannian manifold and γ a geodesic joining two points of M^n. Recall that Myers[11] actually showed that if along γ the Ricci curvature, Ric, satisfies

$$Ric(T,T) \geq a > 0$$

and the length of γ exceeds $\pi\sqrt{n-1}/\sqrt{a}$ where T is the unit tangent to γ, then γ is not minimal.

Moreover, there have been several applications of Myers method to general relativity. T. Frankel[6] has used Myers theorem to obtain a bound on the size of a fluid mass in stationary space-time universe. In [7], G. Galloway made use of Frankel's method to obtain a closure theorem(which has as its conclusion the "finiteness" of the "spatial part" of a space-time obeying certain cosmological assumptions for cosmological models more general than the classical Friedmann models. S. Markvosen[9] obtained another extension similar to G. Galloway's work.

On the other hand, J. K. Beem and P. E. Ehrlich[1,2] proved that if (M,g) is a globally hyperbolic space-time with all Ricci curvature positive and bounded away from zero, then (M,g) has finite timelike diameter.

In this paper, we used generalized Myers theorem on Riemannian manifolds given by G. Galloway[7] to extend the Lorentzian version of Myers theorem given by J. K. Beem and P. E. Ehrlich. Moreover, we compute the upper bound of $diam_K(M,g)$ with respect to the spacelike submanifold K for the suitable curvature tensor and second fundamental tensor conditions.

2. PRELIMINARIES

Let (M, g) be an arbitrary space-time. Given $p, q \in M$ with $p \leq q$ i.e., $p = q$ or there is a smooth future directed nonspacelike curve from p to q, let $\Omega_{p,q}$ denote the path space of all future directed nonspacelike curves $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. The *Lorentzian arc lenth* $L : \Omega_{p,q} \to \mathbf{R}$ is then defined as follows. Given a piecewise smooth curve $\gamma \in \Omega_{p,q}$, choose a partition $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ such that $\gamma|(t_i, t_{i+1})$ is smooth for each $i = 0, 1, 2, ..., n-1$. Then we define

$$L(\gamma) = \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))}dt.$$

Thus it is natural to make the *Lorentzian distance* $d : M \times M \to \mathbf{R} \cup \{\infty\}$ given by, if $q \notin J^+(p) = \{q \in M | p \leq q\}$ set $d(p.q) = 0$, and if $q \in J^+(p)$ set $d(p,q) = sup\{L(\gamma) : \gamma \in \Omega_{p,q}\}$. Now, we define the *timelike diameter*, $diam(M, g)$, of the space-time by

$$diam(M, g) = sup\{d(p,q) | p, q \in M\}.$$

If a complete Riemannian manifold has finite diameter, it is compact by the Hopf-Rinow theorem. But for a space-time (M, g), since

$$L(\gamma) \leq d(p,q)$$

for all future-directed nonspacelike curve γ from p to q, every timelike geodesic must satisfy

$$L(\gamma) \leq diam(M, g).$$

A space-time (M, g) is *timelike complete* if all timelike geodesics may be defined for all values $-\infty < t < \infty$ of an affine parameter t. Thus if a space-time (M, g) has finite timelike diameter, all timelike geodesics have finite length and hence are incomplete, cf.[2, p.329]. Physically, the timelike diameter represents the supremum of possible proper times that any particle could possibly experience in the given space-time.

A space-time (M, g) is *strongly causal* if (M, g) does not contain any point p of M such that there are future-directed nonspacelike curves leaving arbitrarily small neighborhood of p and then returning. Thus a strong causal space-time (M, g) is said to be *globally hyperbolic* if $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ where $J^-(p) = \{q \in M | q \leq p\}$. It should be noted that global hyperbolicity guarantees the existance of a maximal geodesic segment $\gamma \in \Omega_{p,q}$, i.e., a future directed nonspacelike geodesic γ from p to q with $L(\gamma) = d(p,q)$. cf.[2, Theorem 5.1]. But global hyperbolicity does not imply any of these forms of geodesic completeness. This may be seen by fixing points p and q in Minkowski space with $p << q$, i.e., there is a future-directed piecewise smooth timelike curve in M from p to q, and equipping $M = I^+(p) \cap I^-(q)$ (here $I^+(p) = \{q \in M | p << q\}$ and $I^-(q) = \{p \in M | p << q\}$ with the Lorentzian metric it inherits as an open subset of Minkowski space. Moreover, the global hyperbolicity does not imply the existance of the timelike maximal geodesic segment joining all pairs of causally related points. With respect to the conjugate points it is well known that a timelike geodesic is not maximal beyond the first conjugate point.

For a complete Riemannian manifold, the diameter is finite if the manifold is compact. In this case we may always find two points whose distance realizes the diameter. On the other hand, on an arbitrary space-time (M,g), suppose $d(p,q) = diam(M,g) < \infty$ and let $q' \in I^+(q)$ be arbitrary. Then

$$d(p,q') \geq d(p,q) + d(q,q') > d(p,q) = diam(M,g),$$

in contradiction. So $d(p,q) = \infty$. Thus the timelike diameter is never realized by any pair of points in (M,g) of finite timelike diameter.

Let $\gamma : [0,b] \longrightarrow (M,g)$ be a unit timelike geodesic segment. One considers an $\mathbf{R}$-vector space $V^\perp(\gamma)$ of continuous piecewise smooth vector fields Y along γ perpendicular to γ' and let $V_0^\perp(\gamma) = \{Y \in V^\perp(\gamma) | Y(0) = Y(b) = 0\}$. Then we may define the *Lorentzian index form* $I : V^\perp(\gamma) \times V^\perp(\gamma) \longrightarrow \mathbf{R}$ given by, for $X, Y \in V^\perp(\gamma)$

$$I(X,Y) = -\int_0^b [g(X',Y') - g(R(X,\gamma')\gamma',Y)]dt$$

where R is the curvature tensor with respect to the Levi-Civita connection ∇ on (M,g). Moreover, $t_1, t_2 \in [0,b]$ with $t_1 \neq t_2$ are *conjugate* with respect to the timelike geodesic γ if there is a nontrivial Jacobi field J (i.e., $J'' + R(J,\gamma')\gamma' = 0$) along γ with $J(t_1) = J(t_2) = 0$. Then we have the following maximality property of Jacobi fields with respect to the index form, cf. [1,2].

Proposition 2.1. *Let $\gamma : [0,b] \longrightarrow (M,g)$ be a unit speed timelike geodesic with no conjugate points and let $J \in V^\perp(\gamma)$ be any Jacobi field. Then, for any $Y \in V^\perp(\gamma)$ with $Y \neq J$ and $Y(0) = J(0)$, $Y(b) = J(b)$, we have $I(J,J) > I(Y,Y)$.*

Corollary 2.2. *Let $\gamma : [0,b] \to M$ haqve no conjugate points. Then the index form I is negative definite on $V_0^\perp(\gamma) \times V_0^\perp(\gamma)$.*

In [1.2], J. K. Beem and P. E. Ehrlich proved the Lorentzian analogue of Myers theorem for complete Riemannian manifolds given in [3,8] as follows.

Theorem 2.3. *Let (M,g) be a globally hyperbolic space-time of dimension $n \geq 2$ satisfying*

$$Ric(\gamma', \gamma') \geq (n-1)k > 0$$

for any unit timelike unit geodesic γ. Then

$$diam(M,g) \leq \pi/\sqrt{k}.$$

In fact, if $(n-1)k = a$, we may check that this theorem reduces to Myers result on complete Riemannian manifolds.

On a space-time (M,g), a future-directed nonspacelike curve γ from p to q is *maximal* if $L(\gamma) = d(p,q)$. Now we may generalize Myers theorem for the Lorentzian version given in [1,2] as follows.

Proposition 2.4. *Let (M,g) be an arbitrary space-time of dimension $n \geq 2$ and let $\gamma : [0,b] \longrightarrow (M,g)$ be any unit timelike geodesic joining two points of M with length L. Suppose that*

$$Ric(\gamma', \gamma') \geq a + \frac{df}{ds}$$

where $a > 0$, f is a differentiable function of arc length s with $|f(s)| \leq c$ along γ, and $L > \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right)$. Then γ can not be maximal.

Note that if $f = c = 0$ then Proposition 3.1 reduces to Myers theorem. In this Proposition 3.1 the Ricci curvature does not require positiveness along γ. Finally, we are ready to give the Lorentzian analogue of Myers diameter theorem for complete Riemannian manifolds.

Theorem 2.5. *Let (M,g) be a globally hyperbolic space-time of dimension $n \geq 2$ and let $\gamma : [0,b] \longrightarrow (M,g)$ be any unit timelike geodesic joinnig two points of M with length L. Suppose that*

$$Ric(\gamma', \gamma') \geq a + \frac{df}{ds}$$

where $a > 0$, f is a differentiable function of arc length s such that $|f(s)| \leq c$ along γ. Then

$$diam(M,g) \leq \frac{\pi}{a}\left(c + \sqrt{c^2 + a(n-1)}\right).$$

3. Existance of Maximal Geodesics for the Spacelike Submanifolds

Let K be a spacelike submanifold of dimension $k \geq 0$ and let for $q \in M$, $K << q$ if there exists $p \in K$ such that $p << q$. $K \leq q$ if there exists $p \in K$ with $p \leq q$. And let $I^+(K) = \{q \in M | K << q\}$ chronological future of K, $I^-(K) = \{q \in M | q << K\}$ chronological past of K, $J^+(K) = \{q \in M | K \leq q\}$ causal future of K, $J^-(K) = \{q \in M | q \leq K\}$ causal past of K.

Clearly, $I^+(K) = \sum_{p \in K} I^+(p)$.

Now, let $\Omega_{K,q}$ be the path space of all future directed nonspacelike curves $\gamma : [0,b] \to (M,g)$ with $\gamma(0) \in K$ and $\gamma(b) = q$. The *Lorentzian arc length* $L : \Omega_{K,q} \longrightarrow \mathbf{R}$ for a partition $0 = t_0 < t_1 < \ldots\ldots < t_n = b$ such that $\gamma|_{(t_{i-1},t_i)}$ is smooth for $i = 1, 2, \ldots.., n$ given by

$$L = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sqrt{-g(\gamma'(t), \gamma'(t))}.$$

Given a timelike curve γ from K to q, we have a variation α of $\gamma(t)$ and define the variation vector filed V of α along γ by

$V(t) = \frac{\partial}{\partial s}\alpha(t,s)|_{s=0}$, $V(b) = 0$, $V(0) \in T_{\gamma(0)}K$.

Then we have some facts:

if $\gamma : [0,b] \to (M,g)$ is a unit speed timelike geodesic segment, then $L'(0) = g(V(0), \gamma'(0))$. Thus, γ is maximal iff γ is orthogonal at $\gamma(0)$ to K.

Moreover, if $\gamma : [0,b] \to (M,g)$ is a unit timelike geodesic which is orthogonal at $\gamma(0)$ to the spacelike submanifold K and assume that V is a piecewise smooth vector field along γ orthogonal to γ', then we have

$L''(0) = g(S_{\gamma'}V(0), V(0)) + I(V,V)$,

where $I(V,V) = -\int_0^b [g(V',V') - g(R(V,\gamma')\gamma', V)]dt$ and where $S_{\gamma'}$ is the second fundamental tensor given by $S_{\gamma'}x = -(\nabla_x \gamma')^T$ for $x \in T_pK$ where T means "tangential part".

Hence we may define the *Lorentzian submanifold index form*

$$I_{(b,K)} : V^{\perp}(\gamma, K) \times V^{\perp}(\gamma, K) \to \mathbf{R}$$

on $V^{\perp}(\gamma, K)$ the vector space of piecewise smooth vector fields Y with $Y \perp \gamma', Y(0) \in T_pK$ or $V_0^{\perp}(\gamma, K)$ the subspace of $V^{\perp}(\gamma, K)$ with $Y(b) = 0$ as follows; for $X, Y \in V^{\perp}(\gamma, K)$,

$$I_{(b,K)} = g(S_{\gamma'(0)}X(0), Y(0)) + I(X,Y)$$

where I is the index form on $V^{\perp}(\gamma)$.

Now a smooth vector field $J \in V^{\perp}(\gamma, K)$ is called a K-Jacobi field along γ if J satisfies

(1)$J'(0) + S_{\gamma'(0)}J(0) \in (T_pK)^{\perp}$,

(2)$J'' + R(J, \gamma')\gamma' = 0$.

Hence we may define a K-focal point $\gamma(t_0), t_0 \in (0, b]$ if there is a nontrivial K-Jacobi field with $J(t_0) = 0$.

Theorem 3.1. *(Maximaity of K-Jacobi fields) Let $\gamma : [0, b] \to M$ be a timelike geodesic segment with no K-focal points and let $X \in V^{\perp}(\gamma, K)$. If $J \in V^{\perp}(\gamma, K)$ is a K-Jacobi field along γ with $J(b) = X(b)$, then*

$$I_{(b,K)}(X, X) \leq I_{(b,K)}(J, J),$$

and equality holds if and only if $X = J$.

Corollary 3.2. *Let $\gamma : [0, b] \to M$ have no K-focal points. Then the index form $I_{(b,K)}$ is negative definite on $V_0^{\perp}(\gamma, K) \times V_0^{\perp}(\gamma, K)$.*

Recently in [4,5], P. E. Ehrlich and S. B. Kim extended Proposition 2.1. to the focal points for null geodesics.

Now we define the ***Lorentzian distance*** from K to q by

$$d(K, q) = \begin{cases} 0, & \text{if } q \notin J^{+}(K); \\ sup\{L(\gamma) | \gamma \in \Omega_{K,q}\}, & \text{if } q \in J^{+}(K). \end{cases}$$

Clearly, $d(K, q) > 0$ iff $q \in I^{+}(K)$. $q \in J^{+}(K) - I^{+}(K)$ implies that $d(K, q) = 0$, But the converse does not hold since $d(K, q) = 0$ for $q \notin J^{+}(K)$.

Using the index form $I_{(b,K)}$ it is well known that a timelike geodesic orthogonal to a spacelike submanifold K fails to maximize arc length to K after the first focal point.[2,9]

Theorem 3.3. *Let γ be a unit speed timelike geodesic segment orthogonal to a spacelike submanifold K at $\gamma(0) = p \in K$. If there exists $t_0 \in (0, b)$ such that $\gamma(t_0)$ is a K-focal point along γ, then there exists a variation vector field $Z \in V^{\perp}(\gamma, K)$ such that $I_{(b,K)}(Z, Z) > 0$, i.e., there exists a timelike curve from K to q longer than γ.*

4. The focal Myers-Galloway theorem on space-times

Now, we generalize Proposition 2.4 to the K-focal sense.

Theorem 4.1. *Let (M, b) be a space-time of dimension ≥ 2 and $\gamma : [0, b] \to (M, g)$ be any unit speed timelike geodesic segment in $\Omega_{K,q}$ for any spacelike submanifold K and any point $q \in M$. Suppose $g(R(u, \gamma'(t))\gamma'(t), u) > \frac{1}{n-1}(a + \frac{df}{dt})$ for all $u \in (\gamma'(t))^{\perp}$ with $g(u, u) = 1$ along γ, and suppose $g(S_{\gamma'(0)}w, w) > \frac{f(0)}{n-1}$ for all $w \in T_{\gamma(0)}K$ with $g(w, w) = 1$, where $a > 0, c \geq 0$ and f is a differentiable function with $|f(t)| \leq c$. Assume*

$$L(\gamma) > \frac{\pi}{a}\left((1 - \frac{k}{2(n-1)})c + \sqrt{(1 - \frac{k}{2(n-1)})^2 c^2 + a((n - 1 - \frac{3k}{4})}\right).$$

Then γ can not be maximal.

PROOF. Suppose that $\gamma : [0, L] \to M$ is parametrized as a unit speed timelike geodesic with length L. Set $E_n(t) = \gamma'(t)$ and let $\{E_1, E_2, ..., E_{n-1}\}$ be $n-1$ spacelike spacelike parallel fields such that $\{E_1(t), E_2(t), ..., E_k(t)\}$ forms an orthonormal basis of $T_{\gamma(t)}K$ and $\{E_1(t), E_2(t), ..., E_n(t)\}$ the orthonormal basis of $T_{\gamma(t)}M$. Set

$$W_i = \begin{cases} \cos(\frac{\pi(t)}{2L})E_i, & i = 1, 2, ..., k \\ \sin(\frac{\pi(t)}{L})E_i, & i = k+1, ..., n-1. \end{cases}$$

Then

$$W_i(0) = \begin{cases} E_i(0) \in T_{\gamma(0)}K, & i = 1, 2, ..., k \\ 0 \in T_{\gamma(0)}K, & i = k+1, ..., n-1. \end{cases}$$

Since $W_i(L) = 0,\ i = 1, 2, ..., n-1,\ W_i \in V_0^{\perp}(\gamma, K)$. Now,

$$I_{(b,K)}(W_i, W_i) = g(S_{\gamma'(0)}W_i(0), W_i(0)) + \int_0^L [g(R(W_i, \gamma')\gamma', W_i) - g(W_i', W_i')]dt.$$

For i=1,2,.....,k,

$$\begin{aligned}
I_{(b,K)}(W_i, W_i) =& g(S_{\gamma'(0)}E_i(0), E_i(0)) + \int_0^L [\cos^2(\frac{\pi t}{2L}) g(R(E_i, \gamma')\gamma', E_i) \\
&- (\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L}) g(E_i, E_i)] dt, \\
>& \frac{f(0)}{n-1} + \int_0^L [\cos^2(\frac{\pi t}{2L}) \frac{1}{n-1}(a + \frac{df}{dt}) - (\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L})] dt \\
=& \frac{f(0)}{n-1} + \frac{a}{n-1} \int_0^L \cos^2(\frac{\pi t}{2L}) dt \frac{1}{n-1} \int_0^L \cos^2(\frac{\pi t}{2L}) \frac{df}{dt} dt \\
&- \int_0^L (\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L}) dt \\
=& \frac{f(0)}{n-1} + \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}[\cos^2(\frac{\pi t}{2L}) f(t)|_0^L \\
&+ \int_0^L (\frac{\pi}{2L}) \sin(\frac{\pi t}{L}) f(t) dt] - (\frac{\pi}{2L})^2 \frac{L}{2} \\
>& \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}\frac{\pi}{2L}(-Lc) - (\frac{\pi}{2L})^2 \frac{L}{2}.
\end{aligned}$$

For i=k+1,......,n-1,

$$\begin{aligned}
I_{(b,K)}(W_i, W_i) =& \int_0^L [\sin^2(\frac{\pi t}{L}) g(R(E_i, \gamma')\gamma', E_i) - (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L}) g(E_i, E_i)] dt \\
>& \int_0^L [\sin^2(\frac{\pi t}{L}) \frac{1}{n-1}(a + \frac{df}{dt}) - (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L})] dt \\
=& \frac{a}{n-1} \int_0^L \sin^2(\frac{\pi t}{L}) dt + \frac{1}{n-1} \int_0^L \sin^2(\frac{\pi t}{L}) \frac{df}{dt} dt \\
&- \int_0^L (\frac{\pi}{L})^2 \cos^2(\frac{\pi t}{L}) dt \\
=& \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}[\sin^2(\frac{\pi t}{L}) f(t)|_0^L \\
&- \int_0^L (\frac{\pi}{L}) \sin(\frac{2\pi t}{L}) f(t) dt] - (\frac{\pi}{L})^2 \frac{L}{2} \\
>& \frac{a}{n-1}\frac{L}{2} + \frac{1}{n-1}(-(\frac{\pi}{L})Lc) - (\frac{\pi}{L})^2 \frac{L}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n-1} I_{(b,K)}(W_i, W_i) &> \frac{aL}{2} - \pi c + \frac{k\pi c}{2(n-1)} - \frac{(n-1)\pi^2}{2L} + \frac{3k\pi^2}{8L} \\ &= \frac{1}{2L}[aL^2 - 2(1 - \frac{k}{2(n-1)})\pi c L - (n-1-\frac{3k}{4})\pi^2] \\ &> 0.\end{aligned}$$

The last inequality is given by our hypothesis:

$$L > \frac{\pi}{a}\left((1 - \frac{k}{2(n-1)})c + \sqrt{(1 - \frac{k}{2(n-1)})^2 c^2 + a(n-1-\frac{3k}{4})}\right).$$

By Corollary 3.2, γ has a K-focal point. By Theorem 3.3, γ can not be maximal.

Set $diam_K(M,g) = sup\{d(K,q) | q \in M\}$. Then the following corollay may be shown similar to Theorem 3.2.

Corollary 4.2. *Let (M,g) be a globally hyperbolic space-time of dimension $n \geq 2$ and suppose $g(R(u,\gamma'(t))\gamma'(t),u) > \frac{1}{n-1}(a + \frac{df}{dt})$ for all $u \in (\gamma'(t))^\perp$ with $g(u,u) = 1$ along any unit speed timelike geodesic segment $\gamma : [0.L] \to (M,g)$ in $\Omega_{K,q}$ with length L for any spacelike submanifold K and for any $q \in M$, and suppose $g(S_{\gamma'(0)}w.w) > \frac{f(0)}{n-1}$ for all $w \in T_{\gamma(0)}K$ with $g(w,w) = 1$, where $a > 0, c \geq 0$ and f is a differentiable function with $|f(t)| \leq c$ along γ. Then*

$$\begin{aligned}diam_K(M,g) \leq &\frac{\pi}{a}\left((1 - \frac{k}{2(n-1)})c\right. \\ &\left.+\sqrt{(1 - \frac{k}{2(n-1)})^2 c^2 + a(n-1-\frac{3k}{4})}\right).\end{aligned}$$

Note that if $k = 0$ we have the same result of Theorem 3.2 and that if K is any spacelike hypersurface of M we have

$$diam_K(M,g) \leq \frac{\pi}{2a}\left(c + \sqrt{c^2 + (n-1)a}\right),$$

which is exactly a half of the upper bound of $diam(M,g)$ given in Theorem 2.5.

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ANALYTIC TORSION

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ABSTRACT. The torsion of a CW complex was first introduced by Reidemeister [Rei], Franz [Fra] and de Rham [deR] and developed by Milnor [Mil2], Ray-Singer [RS], Cheeger [Che] and Müller [Mül1]. This article is an elementary introduction to the analytic torsion, which reappeared recently in Quillen's metric [Qui], Jones-Witten theory of knots [Bra, Mül2, FG], Quantum Gauge Theory [Wit] and Casson's invariant [Joh].

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1. NOTATION

For a finite dimensional real vector space E, let

$$\det E := \wedge^{\dim E} E,$$

be the highest exterior power of E. When $E = \{0\}$, $\det E = \mathbb{R}$ by definition. The dual space of E will be denoted by E^{-1}. When $\dim E = 1$ and $e \in E - \{0\}$, $e^{-1} \in E^{-1}$ is the unique element with $e^{-1}(e) = 1$.

If $f : E \to F$ is an isomorphism between finite dimensional vector spaces, then f induces a map from $\det E$ to $\det F$, or an element of $(\det E)^{-1} \otimes$

(det F). Note that when $E = F$ this element "is" the ordinary determinant of f. The torsion is a generalization of "determinant" to arbitrary chain complex.

For any finite sequence $C^\bullet = \{C^0, C^1, \ldots, C^n\}$ of finite dimensional vector spaces, let[1]

$$\det C^\bullet := (\det C^0) \otimes (\det C^1)^{-1} \otimes \cdots \otimes (\det C^n)^{(-1)^n}.$$

2. LINEAR ALGEBRA

First, we will review the *torsion* for a chain complex of finite dimensional vector spaces over $\mathbb{R}$ [RS, Che, Mül2, BGS].

Fundamental Theorem of Torsion. *For any chain complex*

$$(2.1) \qquad (C^\bullet, d^\bullet) : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} C^n \to 0$$

of finite dimensional vector spaces, there exists a canonical isomorphism

$$\det C^\bullet \simeq \det H^\bullet,$$

or a nonzero element $\tau \in (\det C^\bullet)^{-1} \otimes (\det H^\bullet)$, *where* $H^\bullet := \{H^0, H^1, \ldots, H^n\}$ *denotes the "cohomology spaces" of the chain complex* (2.1).

Proof. As a special case of the proposition, we will first see that for any short exact sequence

$$(E^\bullet, f^\bullet) : 0 \to E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \to 0$$

of finite dimensional vector spaces, there exists a canonical isomorphism

$$\det E^0 \otimes \det E^2 \simeq \det E^1$$

or equivalently,

$$\det E^\bullet \simeq \mathbb{R}.$$

To show this it suffices to find a nonzero element ξ of $\det E^\bullet$. Let $\dim E^k = n_k$. Then $n_1 = n_0 + n_2$. Choose a basis $e_1^0, \ldots, e_{n_0}^0$ of E^0. Since f^0 is injective, one can extend $f^0(e_1^0), \ldots, f^0(e_{n_0}^0)$ to a basis

$$f^0(e_1^0), \ldots, f^0(e_{n_0}^0), e_1^1, \ldots, e_{n_2}^1$$

[1] Some authors have a different convention of the sign.

for E^1. Then

$$\begin{aligned}\xi :=&(e_1^0 \wedge \cdots \wedge e_{n_0}^0) \otimes (f^0(e_1^0) \wedge \cdots \wedge f^0(e_{n_0}^0) \wedge e_1^1 \wedge \cdots \wedge e_{n_2}^1)^{-1} \otimes \\ &\otimes (f^1(e_1^1) \wedge \cdots \wedge f^1(e_{n_2}^1))\end{aligned}$$

is the desired element of $\det E^\bullet$. This construction is independent of the choice of basis.

Now for the general case, we split the chain complex (2.1) into the short exact sequences

$$0 \to Z^{k-1} \to C^{k-1} \to B^k \to 0 \tag{2.2}$$

$$0 \to B^k \to Z^k \to H^k \to 0 \tag{2.3}$$

where $Z^k = \ker d^k$ and $B^k = \operatorname{im} d^{k-1}$. Then we have

$$\begin{aligned}\det C^\bullet &= \otimes_k(\det C^{k-1})^{(-1)^{k-1}} \simeq \otimes_k(\det Z^{k-1} \otimes \det B^k)^{(-1)^{k-1}} \\ &\simeq (\otimes_k(\det Z^k)^{(-1)^k}) \otimes (\otimes_k(\det B^k)^{(-1)^{k-1}}) \\ &\simeq \otimes_k(\det Z^k \otimes (\det B^k)^{-1})^{(-1)^k} \\ &\simeq \otimes_k(\det H^k)^{(-1)^k} = \det H^\bullet\end{aligned}$$

This completes the proof.

3. Torsion τ

Definition. The element $\tau = \tau(C^\bullet, d^\bullet) \in (\det C^\bullet)^{-1} \otimes (\det H^\bullet)$ in the above Fundamental Theorem is called the *torsion* of the chain complex $(C^\bullet, d^\bullet)$.

Explicitly, the torsion is given as follows: Let

$$\dim C^k = c^k, \ \dim Z^k = z^k, \ \dim B^k = b^k, \ \dim H^k = h^k$$

so that

$$z^{k-1} - c^{k-1} + b^k = 0, \quad b^k - z^k + h^k = 0.$$

Pick a basis

$$\eta_1^k, \eta_2^k, \ldots, \eta_{h^k}^k$$

for H^k, and let

$$\eta^k := \eta_1^k \wedge \eta_2^k \wedge \cdots \wedge \eta_{h^k}^k \in \det H^k - \{0\}.$$

Then

$$\eta := \eta^0 \otimes (\eta^1)^{-1} \otimes \cdots \otimes (\eta^n)^{(-1)^n} \in \det H^\bullet - \{0\}.$$

Now choose elements

$$e_1^k, \ldots, e_{h^k}^k \in Z^k$$

which represent the cohomology classes $\eta_1^k, \eta_2^k, \ldots, \eta_{h^k}^k$. Now extend the basis $e_1^0, \ldots, e_{h^0}^0$ for Z^0 to a basis

$$e_1^0, \ldots, e_{h^0}^0, v_1^0, \ldots, v_{b^1}^0$$

for C^0. Then

$$d(v_1^0), \ldots, d(v_{b^1}^0), e_1^1, \ldots, e_{h^1}^1$$

is a basis for Z^1. Extend this to a basis

$$d(v_1^0), \ldots, d(v_{b^1}^0), e_1^1, \ldots, e_{h^1}^1, v_1^1, \ldots, v_{b^2}^1$$

for C^1. This way, we find, inductively on k,

$$v_1^k, \ldots, v_{b^{k+1}}^k \in C^k$$

so that

$$d(v_1^{k-1}), \ldots, d(v_{b^k}^{k-1}), e_1^k, \ldots, e_{h^k}^k, v_1^k, \ldots, v_{b^{k+1}}^k$$

is a basis for C^k. Put

$$\xi^k := d(v_1^{k-1}) \wedge \cdots \wedge d(v_{b^k}^{k-1}) \wedge e_1^k \wedge \cdots \wedge e_{h^k}^k \wedge v_1^k \wedge \cdots \wedge v_{b^{k+1}}^k$$

and

$$\xi = \xi^0 \otimes (\xi^1)^{-1} \otimes \cdots \otimes (\xi^n)^{(-1)^n} \in \det C^\bullet - \{0\}.$$

Then

$$\tau = \xi^{-1} \otimes \eta.$$

3.1. If $0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$ consists of trivial maps, then $\tau \in (\det C^\bullet)^{-1} \otimes (\det C^\bullet) \simeq \mathbb{R}$ is equal to 1.

3.2. Let $(C_1^\bullet, d_1^\bullet)$ and $(C_2^\bullet, d_2^\bullet)$ be finite chain complexes with cohomology spaces $H_1^\bullet$ and $H_2^\bullet$, respectively. Then the cohomology spaces of

$$(C^\bullet, d^\bullet) := (C_1^\bullet, d_1^\bullet) \oplus (C_2^\bullet, d_2^\bullet)$$

is $H^\bullet = H_1^\bullet \oplus H_2^\bullet$ and

$$\det C^\bullet \simeq \det C_1^\bullet \otimes \det C_2^\bullet$$
$$\det H^\bullet \simeq \det H_1^\bullet \otimes \det H_2^\bullet.$$

Now

$$(\det C^\bullet)^{-1} \otimes (\det H^\bullet) \simeq ((\det C_1^\bullet)^{-1} \otimes (\det H_1^\bullet)) \otimes ((\det C_2^\bullet)^{-1} \otimes (\det H_2^\bullet))$$

and

$$\tau(C^\bullet, d^\bullet) = \pm \tau(C_1^\bullet, d_1^\bullet) \otimes \tau(C_2^\bullet, d_2^\bullet).$$

3.3. This time, suppose

$$(C^\bullet, d^\bullet) = (C_1^\bullet, d_1^\bullet) \otimes (C_2^\bullet, d_2^\bullet),$$

i.e., $C^k = \sum_{p+q=k} C_1^p \otimes C_2^q$ and $d(v_1^p \otimes v_2^q) := d_1 v_1^p \otimes v_2^q + (-1)^p v_1^p \otimes d_2 v_2^q$, for $v_1^p \in C_1^p$, $v_2^q \in C_2^q$ so that $H^\bullet \simeq H_1^\bullet \otimes H_2^\bullet$.

Now

$$\begin{aligned}
\det C^\bullet &= \otimes_k (\det C^k)^{(-1)^k} \\
&\simeq \otimes_k \left(\otimes_{p+q=k} (\det C_1^p)^{c_2^q} \otimes (\det C_2^q)^{c_1^p} \right)^{(-1)^k} \\
&\simeq \otimes_p \otimes_q \left((\det C_1^p)^{c_2^q} \otimes (\det C_2^q)^{c_1^p} \right)^{(-1)^{p+q}} \\
&\simeq \left[\otimes_q \left(\otimes_p (\det C_1^p)^{(-1)^p} \right)^{(-1)^q c_2^q} \right] \otimes \left[\otimes_p \left(\otimes_q (\det C_2^q)^{(-1)^q} \right)^{(-1)^p c_1^p} \right] \\
&\simeq (\det C_1^\bullet)^{\chi(C_2^\bullet)} \otimes (\det C_2^\bullet)^{\chi(C_1^\bullet)}
\end{aligned}$$

where $c_1^p = \dim C_1^p$, $c_2^q = \dim C_2^q$, and χ denotes the Euler-Poincaré characteristic of the complex. Similarly,

$$\det H^\bullet \simeq (\det H_1^\bullet)^{\chi(C_2^\bullet)} \otimes (\det H_2^\bullet)^{\chi(C_1^\bullet)}.$$

Thus

$$\begin{aligned}
&(\det C^\bullet)^{-1} \otimes (\det H^\bullet) \\
&\qquad \simeq \left((\det C_1^\bullet)^{-1} \otimes \det H_1^\bullet \right)^{\chi(C_2^\bullet)} \otimes \left((\det C_2^\bullet)^{-1} \otimes \det H_2^\bullet \right)^{\chi(C_1^\bullet)}.
\end{aligned}$$

Now we have

$$\tau(C_1^\bullet \otimes C_2^\bullet) \simeq \tau(C_1^\bullet)^{\chi(C_2^\bullet)} \otimes \tau(C_2^\bullet)^{\chi(C_1^\bullet)}.$$

4. INNER PRODUCTS

Now we define analytic torsion for a chain complex of finite dimensional *inner product spaces.*

We will use the following notation. If f is an endomorphism of a finite dimensional vector space E, then

$$\det{}^{\times} f \in \mathbb{R} - \{0\}$$

denotes the product of all nonzero (complex) eigenvalues of f counted with multiplicity, which may be called the "renormalized determinant" of f. Thus $\det{}^{\times} f$ is a finite nonzero real number equal to

$$\lim_{\lambda\to 0} \frac{(-1)^k \det(f-\lambda)}{\lambda^k} = \lim_{\lambda\to 0} \frac{\det(f+\lambda)}{\lambda^k}$$

for some nonnegative integer $k = \lim_{l\to\infty} \dim\ker f^l \le \dim E$. The renormalized determinant has the following properties

(i) If f is nilpotent, i.e., $f^k = 0$ for some positive integer k, then $\det{}^{\times} f = 1$.

(ii) If $f_1 : E_1 \to E_1$ and $f_2 : E_2 \to E_2$, then $\det{}^{\times}(f_1 \oplus f_2) = \det{}^{\times} f_1 \cdot \det{}^{\times} f_2$.

(iii) If f is an isomorphism, then $\det{}^{\times} f$ is equal to the ordinary determinant.

(iv) For any nonzero real number c, $\det{}^{\times}(cf) = c^l \det{}^{\times} f$, where $l = \lim_{k\to\infty} f^k(E)$.

(v) In general, $\det{}^{\times}(f \circ g) \neq \det{}^{\times} f \cdot \det{}^{\times} g$. For instance, if $f = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $f \circ g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\det{}^{\times}(f \circ g) = 1 \neq 2 \cdot 1 = \det{}^{\times} f \cdot \det{}^{\times} g$. But $\det{}^{\times}$ is invariant under the conjugation, i.e., if g is an automorphism, then $\det{}^{\times}(g \circ f \circ g^{-1}) = \det{}^{\times} g$.

(vi) Suppose that E is equipped with an inner product and let f^* be the adjoint of $f : E \to E$. Then

$$\det{}^{\times} f^* = \det{}^{\times} f.$$

If we consider hermitian inner products over complex vector spaces, then we have

$$\det{}^{\times} f^* = \overline{\det{}^{\times} f}.$$

Suppose we are given a chain complex

$$(C^\bullet, d^\bullet) : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^n \to 0 \tag{4.1}$$

of finite dimensional inner product spaces. Then we have the adjoint $(d^k)^* : C^{k+1} \to C^k$ of $d^k : C^k \to C^{k+1}$, and the operator

$$\Delta^k := (d^k)^* \circ d^k + d^{k-1} \circ (d^{k-1})^*$$

is a self-adjoint positive semi-definite endomorphism of C^k.

The *analytic torsion* $T = T(C^\bullet, d^\bullet)$ of the chain complex (4.1) is a positive real number defined by

$$T := \prod_{k=0}^{n} (\det{}^\times \Delta^k)^{(-1)^{k+1} k/2}.$$

4.2. Lemma. *Let* $(C_1^\bullet, d_1^\bullet)$ *and* $(C_2^\bullet, d_2^\bullet)$ *be two chain complexes of finite dimensional inner product spaces and let* $(C^\bullet, d^\bullet) = (C_1^\bullet, d_1^\bullet) \oplus (C_2^\bullet, d_2^\bullet)$. *Then*

$$T(C^\bullet, d^\bullet) = T(C_1^\bullet, d_1^\bullet) \cdot T(C_2^\bullet, d_2^\bullet).$$

Proof. Let Δ^k, Δ_1^k and Δ_2^k be the Laplacians of $(C^\bullet, d^\bullet)$, $(C_1^\bullet, d_1^\bullet)$ and $(C_2^\bullet, d_2^\bullet)$, respectively. Then

$$\Delta^k = \Delta_1^k \oplus \Delta_2^k$$

and hence

$$\det{}^\times \Delta^k = \det{}^\times \Delta_1^k \cdot \det{}^\times \Delta_2^k.$$

Thus

$$\begin{aligned} T(C^\bullet, d^\bullet) &= \prod (\det{}^\times \Delta^k)^{(-1)^{k+1} k/2} = \prod (\det{}^\times \Delta_1^k \cdot \det{}^\times \Delta_2^k)^{(-1)^{k+1} k/2} \\ &= T(C_1^\bullet, d_1^\bullet) \cdot T(C_2^\bullet, d_2^\bullet). \end{aligned}$$

This completes the proof.

4.3. Example. Let

$$f = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad ff^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\tau(f) = \det{}^\times (ff^*)^{1/2} = \sqrt{2} \neq 1 = |\det{}^\times f|$$

5. $|\tau| = T$

We now see the relation between the torsion τ and the analytic torsion T of a chain complex (4.1) of finite dimensional inner product spaces. In this case, the cohomology spaces H^k are canonically isomorphic to the kernel of Δ^k, by Hodge theory, and hence it has an inner product induced from C^k. Thus one can measure the length $|\tau|$ of the torsion

$$\tau \in (\det C^\bullet)^{-1} \otimes (\det H^\bullet).$$

5.1. Proposition. $|\tau| = T$.

Proof. Let

$$C^k := \oplus_{\lambda \geq 0} C^k_\lambda$$

be the eigenspace decomposition of C^k with respect to the Laplacian Δ^k. Since Δ commutes with d, we have

$$d(C^k_\lambda) \subset C^{k+1}_\lambda.$$

Similarly, we have

$$d^*(C^k_\lambda) \subset C^{k-1}_\lambda.$$

The chain complex (4.1) splits into the direct sum of

$$(5.2) \qquad (C^\bullet_\lambda, d^\bullet_\lambda) : 0 \to C^0_\lambda \to C^1_\lambda \to \cdots \to C^n_\lambda \to 0.$$

Note that $\Delta^k_\lambda := \Delta^k | C^k_\lambda = \lambda\,\mathrm{id}$ and hence

$$\det{}^\times \Delta^k_\lambda = \begin{cases} 1 & \lambda = 0 \\ \lambda^{c^k_\lambda} & \lambda > 0 \end{cases}$$

where $c^k_\lambda = \dim C^k_\lambda$.

If $\lambda = 0$, the maps in (5.2) is "trivial" and the cohomology space is

$$C^k_0 = \ker \Delta^k =: H^k_\Delta$$

which is isomorphic to the cohomology space H^k_d of (4.1). Then

$$(5.3) \qquad T(C^\bullet_0, d^\bullet_0) = 1 = \tau(C^\bullet_0, d^\bullet_0).$$

Now assume $\lambda > 0$. Then the sequence (5.2) is exact and the associated cohomology spaces are trivial. We have

$$T(C^{\bullet}_{\lambda}, d^{\bullet}_{\lambda}) = \prod_{k=0}^{n} \lambda^{(-1)^{k+1} k c^{k}_{\lambda}/2}, \quad (\lambda \neq 0).$$

Put $z^k_\lambda := \dim \ker d^k_\lambda$. Then

$$c^k_\lambda = z^k_\lambda + z^{k+1}_\lambda$$

and hence

$$\sum_k (-1)^{k+1} k c^k_\lambda = \sum_k (-1)^{k+1} z^k_\lambda.$$

We have, therefore,

$$T(C^{\bullet}_{\lambda}, d^{\bullet}_{\lambda}) = \prod_{k} \lambda^{(-1)^{k+1} z^{k}_{\lambda}/2}.$$

Note that

$$C^k_\lambda = d(C^{k-1}_\lambda) \oplus d^*(C^{k+1}_\lambda), \quad \lambda \neq 0$$

and hence

$$\dim d(C^{k-1}_\lambda) = z^k_\lambda, \quad \dim d^*(C^{k+1}_\lambda) = z^{k+1}_\lambda$$

for $\lambda > 0$. Thus the torsion τ is obtained as follows. First pick a basis $e^0_1, e^0_2, \dots$ for C^0_λ and let $e^0 = e^1_1 \wedge e^0_2 \wedge \dots$. Then extend $d(e^0_1), d(e^0_2), \dots$ to a basis $d(e^0_1), d(e^0_2), \dots, e^1_1, e^1_2, \dots$ for C^1_λ, where $e^1_1, e^1_2, \dots$ are now elements of $d^*(C^2_\lambda)$. Put $d(e^0) = d(e^0_1) \wedge d(e^0_2) \wedge \dots$ and $e^1 = e^1_1 \wedge e^1_2 \wedge \dots$. Now extend $d(e^1_1), d(e^1_2), \dots \in C^2_\lambda$ to a basis for C^2_λ by adding elements $e^2_1, e^2_2, \dots$ of $d^*(C^3_\lambda)$. Put $d(e^1) = d(e^1_1) \wedge d(e^1_2) \wedge \dots$ and $e^2 = e^2_1 \wedge e^2_2 \wedge \dots$. This way we obtain

$$\tau = \Big((e^0) \otimes (de^0 \wedge e^1)^{-1} \otimes (de^1 \wedge e^2) \otimes \dots \otimes (de^{n-1})^{(-1)^n} \Big)^{-1}.$$

Thus

$$\begin{aligned} |\tau| &= |e^0|^{-1} \cdot (|de^0| \cdot |e^1|) \cdot (|de^1| \cdot |e^2|)^{-1} \dots (|de^{n-1}|)^{(-1)^{n-1}} \\ &= (|e^0|^{-1} \cdot |de^0|)(|e^1|^{-1} |de^1|)^{-1} \dots (|e^{n-1}|^{-1} |de^{n-1}|)^{(-1)^{n-1}}. \end{aligned}$$

Now since $e^k \in \det(d^*(C_\lambda^{k+1}))$, we have $\Delta^k e^k = \lambda^{z_\lambda^{k+1}} e^k$, and

$$\frac{|de^k|^2}{|e^k|^2} = \frac{\langle d^*de^k, e^k\rangle}{|e^k|^2} = \frac{\langle (d^*d + dd^*)e^k, e^k\rangle}{|e^k|^2} = \frac{\langle \Delta^k e^k, e^k\rangle}{|e^k|^2} = \lambda^{z_\lambda^{k+1}}.$$

Thus

$$|\tau|^2 = \prod_{k=0}^{n-1} (\lambda^{z_\lambda^{k+1}})^{(-1)^k} = \prod_{k=0}^{n} \lambda^{(-1)^{k+1} z_\lambda^k}$$

and hence $T(C_\lambda^\bullet, d_\lambda^\bullet) = |\tau(C_\lambda^\bullet, d_\lambda^\bullet)|$. Now

$$T(C^\bullet, d^\bullet) = \prod_{\lambda \geq 0} T(C_\lambda^\bullet, d_\lambda^\bullet) = \prod_{\lambda \geq 0} |\tau(C_\lambda^\bullet, d_\lambda^\bullet)| = |\tau(C^\bullet, d^\bullet)|.$$

This completes the proof.

5.2. Corollary. $T(C_1^\bullet \otimes C_2^\bullet) = T(C_1^\bullet)^{\chi(C_2^\bullet)} \cdot T(C_2^\bullet)^{\chi(C_1^\bullet)}$

Proof. This follows from 3.3. □

6. Zeta Function and the Determinant of an Elliptic Operator

We now explain the "renormalized (zeta) determinant" of a self-adjoint positive semi-definite elliptic (pseudo differential) operator P of order $m > 0$ defined on the space $\mathcal{C}^\infty(M, E)$ of sections of a Riemannian vector bundle E over an n-dimensional compact Riemannian manifold (M, g).

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

be the eigenvalues of P, each nonzero eigenvalue repeated according to its multiplicity.

6.1. Theorem [See, Bro]. *As $k \to \infty$, λ_k is asymptotic to $c\, k^{m/n}$ for some constant c.*

We will prove rather easy statement.

Proposition. *There exist a constant $c > 0$ and $\delta = \delta(n, m) > 0$ such that*

$$\lambda_k \geq c\, k^\delta$$

for all $k = 1, 2, \ldots$.

Proof. Take an integer l large enough so that

$$lm - n/2 > 0.$$

Then the Sobolev L^2_{lm} sections of E are continuous, and there is a constant $c_1 > 0$ such that

$$\|s\|_\infty \le c_1 \|s\|_{2,lm}, \quad s \in L^2_{lm}(M, E) \tag{6.2}$$

where $\|\cdot\|_p$ denotes the L^p-norm and $\|\cdot\|_{2,lm}$ is the Sobolev L^2_{lm}-norm. Now P^l is a self-adjoint elliptic operator of order lm and hence there is a constant $c_2 > 0$ such that

$$\|s\|_{2,lm} \le c_2 \|P^l s\|_2 \tag{6.3}$$

for sections s of E orthogonal to $\ker P = \ker P^l$.

Let $\phi_1, \phi_2, \ldots, \phi_k$ be orthonormal sections of E with

$$P(\phi_j) = \lambda_j \phi_j, \quad j = 1, 2, \ldots, k.$$

Then for any scalars $a_1, a_2, \ldots, a_k$, the section

$$s(x) := \sum_{j=1}^{k} a_j \phi_j(x), \quad x \in M$$

of E satisfies the inequality

$$\|P^l s\|_2 = \|\sum_j a_j \lambda_j^l \phi_j\|_2 = \left(\sum_j (a_j \lambda_j^l)^2\right)^{\frac{1}{2}} \le \left(\sum_j a_j^2\right)^{\frac{1}{2}} \lambda_k^l$$

and hence

$$|s(x)| \le \|s\|_\infty \le c_1 \|s\|_{2,lm} \le c_1 c_2 \|P^l s\|_2 \le c_1 c_2 \left(\sum_j a_j^2\right)^{\frac{1}{2}} \lambda_k^l.$$

We fix a point $x \in M$ and choose a local orthonormal frame $e_1, \ldots, e_k$ for E at x. Then

$$\phi_j(x) = \sum_{\mu=1}^{r} \phi_{j\mu} e_\mu(x)$$

for some constant $\phi_{j\mu}$ and

$$|s(x)| = |\sum_{j,\mu} a_j \phi_{j\mu} e_\mu(x)| = \left(\sum_\mu (\sum_j a_j \phi_{j\mu})^2\right)^{\frac{1}{2}}.$$

In particular, for each μ, the inequality

$$|\sum_j a_j \phi_{j\mu}| \leq |s(x)| \leq c_1 c_2 \left(\sum_j a_j^2\right)^{\frac{1}{2}} \lambda_k^l$$

is true for any scalars $a_1, a_2, \ldots, a_k$. With $a_j = \phi_{j\mu}$ we get

$$\sum_{j=1}^k \phi_{j\mu}^2 \leq c_1^2 c_2^2 \lambda_k^{2l}$$

for each $\mu = 1, \ldots, r$ and hence

$$\sum_{j=1}^k |\phi_j(x)|^2 = \sum_j (\sum_\mu \phi_{j\mu}^2) = \sum_\mu (\sum_j \phi_{j\mu}^2) \leq r c_1^2 c_2^2 \lambda_k^{2l}.$$

Now we integrate this term over M and get

$$k \leq r c_1^2 c_2^2 \operatorname{vol}(M, g) \lambda_k^{2l}.$$

Take $l = [\frac{n}{2m}] + 1 \leq (n + 2m)/2m$. Then

$$\lambda_k \geq c\, k^\delta, \quad k = 1, 2, \ldots$$

for $\delta = \delta(n, m) = \frac{1}{2([(\frac{n}{2m}] + 1)} \geq \frac{m}{n+2m}$ and $c = (r c_1^2 c_2^2 \operatorname{vol}(M, g))^{-\delta}$. This completes the proof.

Now from the Proposition, the series [Gil]

$$\zeta_P(s) := \sum_{k=1}^\infty \frac{1}{\lambda_k^s}, \quad \operatorname{Re}(s) > n/m$$

converges and the series[2]

$$h_P(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}$$

converges for any $t > 0$, as

$$\int_1^{\infty} e^{-ax^b}\, dx < \infty$$

for any $a, b > 0$.

6.4. Proposition [Gil]. *As $t \searrow 0$, there is an asymptotic expansion*

$$h_P(t) \sim \sum_{k=0}^{\infty} a_k(P) t^{(k-n)/m}$$

for some real number $a_k(P)$.

Recall that Euler's *Gamma function* is defined by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt = \lim_{n\to\infty} \frac{n!n^s}{s(s+1)\dots(s+n)}, \quad \mathrm{Re}(s) > 0.$$

Note that $\Gamma(1) = 1$, $\Gamma(n+1) = n!$, for positive integers n, and $\Gamma(1/2) = \sqrt{\pi}$. The functional equation $\Gamma(s+1) = s\Gamma(s)$ extends Γ to a meromorphic

---

[2] For each $t > 0$, let

$$e^{-tP} : C^{\infty}(M, E) \to C^{\infty}(M, E)$$

be the *heat operator* of P [LM]. The heat operator is defined by the *heat kernel*

$$K_t(x, y) : E_y \to E_x, \quad (x, y) \in M \times M,\ t > 0$$

and

$$(e^{-tP}u)(x) := \int_M K_t(x, y) u(y)\, dg(y)$$

for any section u of E, and

$$h_P(t) = \sum_k e^{-\lambda_k t} = (e^{-tP}) - \dim\ker P.$$

function on $\mathbb{C}$ with simple poles at $s = -k$ of residue $(-1)^k/k!$, for $k = 0, -1, -2, \ldots$. The Gamma function satisfies the relation [Lan, Ahl]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and has no zeros. Thus $1/\Gamma$ is an entire function with simple zeros at $s = 0, -1, -2, \ldots$.

From the Mellin transform[3]

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t}\, dt, \quad \lambda > 0, \operatorname{Re} s > 0$$

we have

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} h_P(t)\, dt, \quad \operatorname{Re}(s) > n/m.$$

6.5. Proposition. *The "zeta function" $\zeta_P(s)$ of P extends to a meromorphic function on $\mathbb{C}$, which is holomorphic at $s = 0$. Moreover, $\zeta_P(s)$ is real for real s.*

Proof. For $\operatorname{Re} s > n/m$, we have

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 t^{s-1} h_P(t)\, dt + \int_1^\infty t^{s-1} h_P(t)\, dt \right).$$

From (6.1) it is easy to see that

$$s \mapsto \int_1^\infty t^{s-1} h_P(t)\, dt$$

can be analytically continued to all of $\mathbb{C}$.

Now given any positive integer k_0, we have

$$h_P(t) = \left(\sum_{k=0}^{k_0 - 1} a_k(P) t^{(k-n)/m} + O(t^{(k_0-n)/m}) \right)$$

[3]The Mellin transform is *scale invariant*, i.e., for any absolutely integrable function $f(t)$ on $0 < t < \infty$,

$$\int_0^\infty f(t) \frac{dt}{t} = \int_0^\infty f(\lambda t) \frac{dt}{t}$$

for any $\lambda > 0$.

for $0 < t < 1$. Thus for $\mathrm{Re}\, s > n/m$

$$\int_0^1 t^{s-1} h_P(t)\, dt = \int_0^1 t^{s-1} \left(\sum_{k=0}^{k_0-1} a_k(P) t^{(k-n)/m} + O(t^{(k_0-n)/m}) \right) dt$$
$$= \sum_{k=0}^{k_0-1} \frac{a_k(P)}{s - \frac{n-k}{m}} + \int_0^1 O(t^{s-\frac{n-k_0}{m}-1})\, dt$$

where the last integral is holomorphic for $\mathrm{Re}\, s > \frac{n-k_0}{m}$. This shows that

$$\int_0^1 t^{s-1} h_P(t)\, dt$$

is a meromorphic function on $\mathbb{C}$ having, at worst, simple poles at $s = \frac{n-k}{m}$, with residue $a_k(P)$, $k = 0, 1, 2, \ldots$. Thus $\zeta_P(s)$ is a meromorphic function on $\mathbb{C}$. Since $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, $\zeta_P(s)$ is regular at $s = 0$ and $\zeta_P(0) = a_n(P)$. The remainder of the proof is obvious. This completes the proof.

Now we define

$$\det{}^{\times} P := \exp(- \left.\frac{d}{ds}\right|_0 \zeta_P(s)),$$

which is a positive real number, and call it the *renormalized determinant* of P. Formally,

$$\det{}^{\times} P \approx \prod_{k=1}^{\infty} \lambda_k.$$

6.6. Riemann zeta function. Recall that the Riemann zeta function is defined by, for $\mathrm{Re}\, s > 1$,

$$\zeta(s) = \prod_{p:\text{prime}} (1 - \frac{1}{p^s})^{-1} = \sum_{k=1}^{\infty} \frac{1}{k^s}$$
$$= \sum_{k=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-kt}\, dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{k=1}^{\infty} e^{-tk} \right) dt$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt.$$

The integral

$$s \mapsto \int_1^\infty \frac{t^{s-1}}{e^t - 1}\,dt$$

extends easily to an entire function. We show that

$$s \mapsto \int_0^1 \frac{t^{s-1}}{e^t - 1}\,dt \tag{6.7}$$

extends to a meromorphic function on $\mathbb{C}$. Note that

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}t^{2k-1}$$

where B_k's are Bernoulli numbers. Thus

$$\int_0^1 \frac{t^{s-1}}{e^t - 1}\,dt = \frac{1}{s-1} - \frac{1}{2}\frac{1}{s} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}\frac{1}{s-(1-2k)}$$

and, as before, (6.7) is a meromorphic function on $\mathbb{C}$ with simple poles at $s = -k$ of residue $(-1)^k/k!$ for $k = 0, 1, 2, \ldots$. We conclude that $\zeta(s)$ is a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with $\operatorname{Res}_{s=1}\zeta(s) = 1$, vanishes at $s = -2, -4, -6, \ldots$, and $\zeta(0) = -\frac{1}{2}$, $\zeta(1-2k) = (-1)^k B_k/2k$, for $k = 1, 2, 3, \ldots$.

It is well known [Tit] that

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)\zeta(1-s)$$

and

$$\zeta'(0) = -\log\sqrt{2\pi} \tag{6.8}$$

Formally, we have

$$\infty! \approx \exp(-\zeta'(0)) = \sqrt{2\pi}.$$

6.9. Example. On the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, let $P = -d^2/dt^2$ be defined on the space of smooth functions. Then

$$0, 1^2, 1^2, 2^2, 2^2, 3^2, 3^2, \ldots$$

are the eigenvalues of P and hence

$$\zeta_P(s) = \sum_{k=1}^{\infty}\frac{2}{k^{2s}} = 2\zeta(2s),$$

where ζ is the Riemann zeta function. From (6.8) we have $\zeta_P'(0) = -4\log\sqrt{2\pi}$ and hence $\prod_{k=1}^{\infty} k^4 \approx \exp(-\zeta_P'(0)) = 4\pi^2$, or

$$(\infty!)^4 \approx (\sqrt{2\pi})^4.$$

7. FLAT BUNDLES AND ANALYTIC TORSION

Now suppose we have a flat connection d_A on a Riemannian vector bundle E over a compact Riemannian manifold M. Then we have a chain complex

$$0 \to \Omega^0(M,E) \xrightarrow{d_A} \Omega^1(M,E) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Omega^n(M,E) \to 0 \tag{7.1}$$

of infinite dimensional pre-Hilbert spaces. The "Laplacians"

$$\Delta^k := (d_A^k)^* \circ d_A^k + d_A^{k-1} \circ (d_A^{k-1})^*$$

are self-adjoint positive semi-definite elliptic operators and have the "renormalized zeta determinant" $\det{}^\times \Delta^k$. Then the *analytic torsion* $T(d_A)$ of the complex (7.1) is defined by

$$T(d_A) := \prod_{k=0}^{n} (\det{}^\times \Delta^k)^{(-1)^{k+1}k/2}$$

or

$$+\log T(d_A) = \tfrac{1}{2}\sum_{k=0}^{n}(-1)^k k \zeta'_{\Delta^k}(0).$$

Note that a flat connection on a Riemannian vector bundle is equivalent to an orthogonal representation of the fundamental group of M, see e.g., [Kob]. Recently Müller has generalized the notion of analytic torsion for unimodular representations [Mül2].

7.2. Proposition. *If M is even dimensional and orientable, then $T(d_A) = 1$.*

Proof. Fix an orientation of M and let $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ be the Hodge star. Then $\star$ extends to an isometry

$$\star : \Omega^k(M,E) \to \Omega^{n-k}(M,E)$$

and

$$d_A^* = \star^{-1} d_A \star .$$

Let $\zeta_k(s)$ be the zeta function of the Laplacian Δ^k. Then it suffices to show that

$$\sum_{k=0}^{n}(-1)^k k \zeta_k(s) = 0.$$

Let C_λ^k be the λ-eigenspace of $\Omega^k(M,E)$ with respect to Δ^k. Then for $\lambda \neq 0$,

$$C_\lambda^k = d(C_\lambda^{k-1}) \oplus d^*(C_\lambda^{k+1}).$$

Since the Hodge star commutes with the Laplacian, we have isomorphisms

$$\star : C_\lambda^k \simeq C_\lambda^{n-k}$$

and

$$\star : d(C_\lambda^{k-1}) \simeq d^*(C_\lambda^{n-k+1}).$$

Let $c_\lambda^k := \dim C_\lambda^k$, $z_\lambda^k = \dim d(C_\lambda^{k-1})$. Then

$$c_\lambda^k = z_\lambda^k + z_\lambda^{k+1}, \quad z_\lambda^k = z_\lambda^{n-k+1}.$$

Thus

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k k\zeta_k(s) &= \sum_{k=0}^{n}(-1)^k k(\sum_\lambda \frac{c_\lambda^k}{\lambda^s}) \\
&= \sum_\lambda \sum_k (-1)^k k(z_\lambda^k + z_\lambda^{k+1})/\lambda^s \\
&= \sum_\lambda \sum_k (-1)^k z_\lambda^k/\lambda^s \\
&= \sum_\lambda \sum_k (-1)^k z_\lambda^{n-k+1}/\lambda^s \\
&= -(-1)^n \sum_\lambda \sum_k (-1)^k z_\lambda^k/\lambda^s \\
&= -(-1)^n \sum_{k=0}^{n}(-1)^k k\zeta_k(s).
\end{aligned}$$

This completes the proof.

7.3. Theorem. *Let M be orientable. Then the analytic torsion T is independent of the choice of a Riemannian metric on M.*

The proof for the "acyclic case" is given in [RS]. Once we notice that the "heat trace" $h_{\Delta^k}(t)$, $t > 0$, is

$$h_{\Delta^k}(t) = (e^{-t\Delta^k}) - h^k$$

where $h^k = \dim H_{d_A}^k(M,E)$ is independent of the choice of the Riemannian metric on M, then the proof is exactly the same as in [RS].

As a consequence, if ρ is an orthogonal representation of the fundamental group of a compact oriented smooth manifold M, then the analytic torsion $T(\rho)$ is an *invariant* of the smooth structure of M. We will see, however, that $T(\rho)$ is *not* a homeomorphism invariant.

7.4. Example. The lens space $L_p^3(q)$ for a prime p and $q \in \{1, \ldots, p-1\}$ is defined by the quotient of the 3-sphere $\mathbf{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ by the $\mathbb{Z}_p$ action induced by

$$(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2), \quad (z_1, z_2) \in \mathbf{S}^3.$$

The fundamental group of $L_p^3(q)$ is isomorphic to $\mathbb{Z}_p$, which acts on $\mathbb{C}$ as the p-th roots of unity. This representation ρ of the fundamental group of $L_p^3(q)$ induces a flat bundle E_ρ.

It is known that $L_p^3(q_1)$ and $L_p^3(q_2)$ are of the same homotopy type preverving the orientation (respectively reversing the orientation) if and only if $q_1 {q_2}^{-1}$ is a square (respectively a negative of a square) in $\mathbb{Z}_p$. For instance, $L_5^3(1)$ and $L_5^3(2)$ are not homotopically equivalent. On the other hand, $L_7^3(q)$, $q = 1, 2, \ldots, 6$, are all homotopic to each other. But one can show that [Ray]

$$T(L_p^3(q), \rho) = -2 \log |(e^{2\pi i/p} - 1)(e^{2\pi i r/p} - 1)|, \quad r = q^{-1} \in \mathbb{Z}_p.$$

In particular, $T(L_7^3(1), \rho) \neq T(L_7^3(2), \rho)$ and hence $L_7^3(1)$ and $L_7^3(2)$ are not homeomorphic. On the other hand, by a theorem of Mazur [Mil2, Maz, DFN][4], $M_1 = L_7^3(1) \times \mathbf{S}^4$ and $M_2 = L_7^3(2) \times \mathbf{S}^4$ are homeomorphic. Since

$$T(M_1, \rho) = T(L_7^3(1), \rho)^{\chi(\mathbf{S}^4)} = T(L_7^3(1), \rho)^2 \neq T(L_7^3(2), \rho)^2 = T(M_2, \rho),$$

we conclude that M_1 and M_2 are not diffeomorphic, a counter-example to the Hauptvermutung [Mil1].

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ON THE MOD 2 COHOMOLOGY OF THE LOOP SPACE OF *Spin(n)*

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1. Introduction

In this note we compute $H^*(\Omega Spin(n); \mathbf{Z}/(2))$ by induction on n by studying the spectral sequence for the fibration.

$$\Omega Spin(n) \longrightarrow \Omega Spin(n+1) \longrightarrow \Omega S^n$$

After this we will try other methods to compute $H^*(\Omega Spin(n); \mathbf{Z}/(2))$ using the Eilenberg–Moore spectral sequence with the path fibration:
$\Omega Spin(n) \to P(Spin(n)) \to Spin(n)$ and exploiting the Verschiebung map V with $H_*(\Omega Spin(n); \mathbf{Z}/(2))$.
The computation is not hard but it is the background for the computation of $H_*(\Omega^2 Spin(n); \mathbf{Z}/(2))$ and $H_*(\Omega^3 Spin(n); \mathbf{Z}/(2))$[2].

2. Priliminaries

Let $E(x)$ be the exterior algebra on x and $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x which is free over $\gamma_i(x)$ with coproduct

$$\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$$

and the product

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x) .$$

First recall the following standard fact. Let $V(x_{i_1}, \ldots, x_{i_t})$ be the commutative associative algebra over $\mathbf{Z}/(2)$ such that

1. $\{(x_{i_1})^{\epsilon_i}, \ldots, (x_{i_t})^{\epsilon_i} : \epsilon_i = 0, 1\}$ is a basis.

2. $(x_{i_q})^2 = x_{i_s}$ if $2i_q = i_s$ for some $1 \le s \le t$
   $(x_{i_q})^2 = 0$ otherwise.
   where $|x_i| = i$.

For $Spin(n)$ choose s where $2^s < n \le 2^{s+1}$.

$$H^*(Spin(n); \mathbf{Z}/(2)) = V(x_i | 3 \le i \le n-1 \text{ and } i \ne 2^j) \otimes E(z),$$
$$Sq^r(x_i) = \binom{i}{r} x_{i+r} \text{ where } |z| = 2^{s+1} - 1. \tag{0.1}$$

In fact, we have the steenrod operation on z [4]. But we do not need it here. For example

$$H^*(Spin(10); \mathbf{Z}/(2)) = V(x_3, x_5, x_6, x_7, x_9) \otimes E(z), where |z| = 15$$

So $(x_3)^2 = x_6, (x_5)^2 = 0, (x_6)^2 = 0, (x_7)^2 = 0, (x_9)^2 = 0$. Hence

$$H^*(Spin(10); \mathbf{Z}/(2)) = P(x_3)/(x_3)^4 \otimes E(x_5, x_7, x_9) \otimes E(z)$$
$$Sq^2 x_3 = x_5, Sq^4 x_5 = x_9, Sq^2 x_7 = x_9$$

3. The cohomology of $\Omega Spin(n)$

Lemma 3.1 $H^*(\Omega Spin(8n); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le (n-1))$$
$$\otimes \Gamma(c_{8n-2+2k} : 0 \le k \le (4n-2),\ k \not\equiv 3 \bmod 4)$$
$$\textit{where } \nu_i \textit{ is the power of } 2 \textit{ such that } 8n \le \nu_i(4i-2) \le 16n-8.$$

$H^*(\Omega Spin(8n+1); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le (n-1))$$
$$\otimes \Gamma(c_{8n+2k} : 0 \le k \le (4n-1),\ k \not\equiv 2 \bmod 4)$$
$$\textit{where } \nu_i \textit{ is the power of } 2 \textit{ such that } 8n \le \nu_i(4i-2) \le 16n-8.$$

$H^*(\Omega Spin(8n+2); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le (n-1))$$
$$\otimes \Gamma(c_{8n+2+2k} : 0 \le k \le (4n-2),\ k \not\equiv 1 \bmod 4)$$
$$\bigotimes_{i \ge 0} P(\gamma_{2^i}(d_{8n}))/(\gamma_{2^i}(d_{8n}))^4$$
$$\textit{where } \nu_i \textit{ is the power of } 2 \textit{ such that } 8n+8 \le \nu_i(4i-2) \le 16n$$

$H^*(\Omega Spin(8n+3); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le n-1)$$
$$\otimes \Gamma(c_{8n+2+2k} : 0 \le k \le 4n,\ k \not\equiv 1 \bmod 4)$$
where ν_i *is the power of* 2 *such that* $8n+8 \le \nu_i(4i-2) \le 16n$.

$H^*(\Omega Spin(8n+4); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le n)$$
$$\otimes \Gamma(c_{8n+2+2k} : 0 \le k \le 4n,\ k \not\equiv 1 \bmod 4)$$
where ν_i *is the power of* 2 *such that* $8n+8 \le \nu_i(4i-2) \le 16n$.

$H^*(\Omega Spin(8n+5); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2+4k} : 0 \le k \le n)$$
$$\otimes \Gamma(c_{8n+6+2k} : 0 \le k \le 4n,\ k \not\equiv 3 \bmod 4)$$
where ν_i *is the power of* 2 *such that* $8n+8 \le \nu_i(4i-2) \le 16n$.

$H^*(\Omega Spin(8n+6); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n+1)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+6+4k} : 0 \le k \le n-1)$$
$$\otimes \Gamma(c_{8n+6+2k} : 0 \le k \le 4n,\ k \not\equiv 3 \bmod 4)$$
$$\otimes_{i \ge 0} P(\gamma_{2^i}(b_{8n+4})/(\gamma_{2^i}(b_{8n+4}))^4$$
where ν_i *is the power of* 2 *such that* $8n+8 \le \nu_i(4i-2) \le 16n+8$.

$H^*(\Omega Spin(8n+7); \mathbf{Z}/(2))$ *is*

$$P(a_{4i-2} : 1 \le i \le n+1)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+6+4k} : 0 \le k \le n-1)$$
$$\otimes \Gamma(c_{8n+6+2k} : 0 \le k \le 4n+2,\ k \not\equiv 3 \bmod 4)$$
where ν_i *is the power of* 2 *such that* $8n+8 \le \nu_i(4i-2) \le 16n+8$.

Note that a_i *become the stable element.*

Proof. Let $H^*(\Omega S^n; \mathbf{Z}/(2)) = \Gamma(a_{n-1})$. We will prove this lemma by induction. Remind that $\Omega Spin(3) \simeq \Omega S^3$. For $H^*(\Omega Spin(8n+4))$, we have

$$\Omega Spin(8n+3) \longrightarrow \Omega Spin(8n+4) \longrightarrow \Omega S^{8n+3}$$

Since $H^*(\Omega G; \mathbf{Z}/(2))$ is concentrated in the even dimensions for any finite H–space G and $H^*(\Omega S^{8n+3}; \mathbf{Z}/(2))$ is even dimensional, the spectral sequence collapses. And there is no extension problem by the dimension reason.

For next step consider the following fibration.

$$\Omega Spin(8n+4) \longrightarrow \Omega Spin(8n+5) \longrightarrow \Omega S^{8n+4}$$

Since $H^*(\Omega Spin(8n+5); \mathbf{Z}/(2))$ is concentrated in the even dimensions and $H^*(\Omega S^{8n+4}; \mathbf{Z}/(2))$ contains an $(8n+3)$ dimensonal element, we have the first non–zero differential which comes from a $(8n+2)$–dimensional generater and goes to a_{8n+3}. But in $H^*(\Omega Spin(8n+4); \mathbf{Z}/(2))$ we have two generator a_{8n+2}, c_{8n+2} of that dimension. Consider

$$\begin{array}{ccccc}
\Omega Spin(8n+3) & \longrightarrow & \Omega Spin(8n+5) & \longrightarrow & \Omega Spin(8n+5)/Spin(8n+3) \\
f \downarrow & & \downarrow & & \downarrow \\
\Omega Spin(8n+4) & \longrightarrow & \Omega Spin(8n+5) & \longrightarrow & \Omega S^{8n+4} \\
g \downarrow & & \downarrow & & h \downarrow \\
\Omega S^{8n+3} & \longrightarrow & * & \longrightarrow & S^{8n+3}.
\end{array}$$

From the naturality of the differential we have

$$\begin{aligned}
\tau(g^*(a_{8n+2})) &= h^*(\tau(a_{8n+2})) \\
&= h^*(x_{8n+3}) \\
&= 0.
\end{aligned}$$

Hence we have the differential with the source c_{8n+2} and $\gamma_2(c_{8n+2})$ hits $c_{8n+2} \cdot a_{8n+3}$ and so on. Hence $\Gamma(c_{8n+2})$ are the sourse of the differentials. $\gamma_{2^i}(a_{8n+3})$ survives permanently for $i \geq 1$. Put $\gamma_2(a_{8n+3}) = c_{16n+6}$.

For $H^*(\Omega Spin(8n+6))$ consider the following.

$$\Omega Spin(8n+5) \longrightarrow \Omega Spin(8n+6) \longrightarrow \Omega S^{8n+5}$$

By the same reason as $H^*(\Omega Spin(8n+4); \mathbf{Z}/(2))$, the spectral sequence collapses. So we get E_∞–term for $H^*(\Omega Spin(8n+6); \mathbf{Z}/(2))$ is

$$\begin{gathered}
P(a_{4i-2} : 1 \leq i \leq n)/(a_{4i-2}^{\nu_i}) \otimes \Gamma(a_{4n+2}, a_{4n+2}, \ldots, a_{8n+2}) \otimes \Gamma(a_{8n+4}) \\
\otimes \Gamma(c_{8n+6+2k} : 0 \leq k \leq 4n,\ k \not\equiv 3 mod 4) \\
where\, \alpha_i = 2^{\ell+2}, \ell \geq 0\, and\, 8n+8 \leq \nu_i(4i-2) \leq 16n.
\end{gathered}$$

But in this case there are extension problems. We claim that $(a_{4n+2})^2 = a_{8n+4}$. From $H^*(Spin(8n+6) : \mathbf{Z}/(2))$ we can compute $\mathrm{Tor}_{H^*(Spin(8n+6))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. Since $Sq^{4n+2}x_{4n+3} = \binom{4n+3}{4n+2} x_{8n+5} = x_{8n+5}$ in

$H^*(Spin(8n+6))$ by (2.1), $(a_{4n+2})^2 = Sq^{4n+2}\sigma(x_{4n+3}) = \sigma(x_{8n+5}) = a_{8n+4}$. So $\Gamma(a_{4n+2}) \otimes \Gamma(a_{8n+4}) = P(\gamma_{2^i}(a_{4n+2}))/(\gamma_{2^i}(a_{4n+2}))^4$, $i \geq 0$. Let

$$P(\gamma_i(a_{4n+2}))/(\gamma_i(a_{4n+2}))^4 = P(a_{4n+2})/(a_{4n+2})^4 \otimes P(\gamma_{2^{i+1}}(a_{4n+2}))/(\gamma_{2^{i+1}}(a_{4n+2}))^4,$$

$i \geq 0$ and let $\gamma_2(a_{4n+2}) = b_{8n+4}$. Hence we extend the conditions: $1 \leq i \leq n+1$, $\nu_i(4i-2) \leq 16n+8$.

Consider the next fibration.

$$\Omega Spin(8n+6) \longrightarrow \Omega Spin(8n+7) \longrightarrow \Omega S^{8n+6}$$

Since $H^*(\Omega S^{8n+6})$ contains a_{8n+5}, we have a nonzero first differential from b_{8n+4} to a_{8n+5} and the next differentials from $\gamma_2(b_{8n+4})$ to $a_{8n+5} \cdot b_{8n+4}$, from $b_{8n+4} \cdot \gamma_2(b_{8n+4})$ to $a_{8n+5} \cdot \gamma_2(b_{8n+4})$ and so on. Then $(\gamma_{2^i}(b_{8n+4}))^2$ is a permanent cycle for each $i \geq 0$. But $(\gamma_{2^i}(b_{8n+4}))^2 = (\gamma_{2^{i+1}}(a_{4n+2}))^2 = \gamma_{2^{i+1}}(a_{8n+4})$ for $i \geq 0$ in the previous step and $\gamma_{2^i}(a_{8n+5})$ is also a permanent cycle for each $i \geq 1$. Let $(\gamma_1(b_{8n+4}))^2 = c_{16n+8}$ and $\gamma_2(a_{8n+5}) = c_{16n+10}$.

We can prove the other four cases in similiar ways, however, compared with $H^*(\Omega Spin(8n+6); \mathbf{Z}/(2))$, we have a little different extension problems for $H^*(\Omega Spin(8n+2); \mathbf{Z}/(2))$. Note that in $H^*(Spin(8n+2); \mathbf{Z}/(2))$ $Sq^{4n}x_{4n+1} = x_{8n+1}$, $Sq^{2n}x_{2n+1} = x_{4n+1}$. So $a_{8n} = \sigma(x_{8n+1}) = \sigma(Sq^{4n}x_{4n+1}) = Sq^{4n}\sigma(x_{4n+1}) = Sq^{4n}(a_{4n}) = (a_{4n})^2 = (\sigma(x_{4n+1}))^2 = (\sigma(Sq^{2n}x_{2n+1}))^2 = (Sq^{2n}a_{2n})^2 = a_{2n}^4$. In fact, the difference come from the property of the number: $8n = 2^2 2n, 8n+4 = 2(4n+2)$. ∎

Remark 3.2 *In fact, using the Eilenberg–Moore spectral sequence with* $E_2 = \mathrm{Tor}_{H^*(Spin(n);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$, *we can choose the primitive generators* a_i, b_i, c_i *such that* $\sigma(x_i) = a_j^{2^k}$ *where* $2^k j = i-1$ *or* b_{i-1} *according to the dimension and* $\sigma(z_i) = c_{i-1}$ *and* $\rho(x_i^{2^k}) = c_{2^k i-2}$ *where* $\rho(x_i^{2^k})$ *is the transpotence of* $x_i^{2^k}$. *Here we can solve the extension problems using the Steenrod operation.*

For example, for $H^*(\Omega Spin(10); \mathbf{Z}/(2))$

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(Spin(10);\mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\ &= \mathrm{Tor}_{P(x_3)/(x_3)^4 \otimes E(x_5,x_7,x_9) \otimes E(z_{15})}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \\ &= E(a_2) \otimes \Gamma(a_4, a_6, a_8) \otimes \Gamma(c_{10}, c_{14}) \end{aligned}$$

Since E_2–term concentrates in even dimension, the spectral sequence collapses from E_2, i.e. , $E_2 = E_\infty$. But we have the extension problems. Since the Eilenberg–Moore spectral sequence is the spectral sequence of the steenrod module,

$$\begin{aligned}(a_2)^2 &= Sq^2 a_2 = a_4, \\ (a_4)^2 &= Sq^4 a_4 = a_8, \\ (\gamma_{2^i}(a_4))^2 &= \gamma_{2^i}(a_8), i \geq 1\end{aligned}$$

Hence $H^*(\Omega Spin(10); \mathbf{Z}/(2))$ is

$$\begin{gathered} P(a_2)/(a_2^8) \otimes \Gamma(a_6) \\ \otimes \Gamma(c_{10}, c_{14}) \\ \bigotimes_{i \geq 0} P(\gamma_{2^i}(d_8))/(\gamma_{2^i}(d_8))^4 \end{gathered}$$

,where $d_8 = \gamma_2(a_4)$.

Now we will try another method to compute $H^*(\Omega Spin(n); \mathbf{Z}/(2))$ using the Verschiebung map V with the information of $H_*(\Omega Spin(n); \mathbf{Z}/(2))$.
The generating variety for the homology of $\Omega Spin(n+2)$ is

$$G'_{2,n} = SO(n+2)/SO(2) \times SO(n)$$

and $\lim_{n\to\infty} G'_{2,n} = CP^\infty$ is a generating variety for the homology of $\Omega Spin = SO/U$, that is, we have a map from $G'_{2,n}$ to $\Omega Spin(n+2)$ such that the image of $H_*(G'_{2,n})$ under the induced map generates $H_*(\Omega Spin(n+2))$. Note that $H_*(\Omega Spin; \mathbf{Z}/(2)) = E(\alpha_2, \alpha_4, \alpha_6, \ldots)$. From [1] or [5],we know that
$H_*(G'_{2,2n}; \mathbf{Z}/(2))$ is free on

$$\{\alpha_2, \alpha_4, \ldots, \alpha_{2n}\} \cup \{\beta_{2n}, \beta_{2n+2}, \ldots, \beta_{4n}\}$$

as a module and
$H_*(G'_{2,2n+1}; \mathbf{Z}/(2))$ is free on

$$\{\alpha_2, \alpha_4, \ldots, \alpha_{2n}\} \cup \{\beta_{2n+2}, \beta_{2n+4}, \ldots, \beta_{4n+2}\}$$

as a module. The following lemma just comes from the Eilenberg–Moore spectral sequence with $E_2 = \text{Ext}_{H^*(Spin(n); \mathbf{Z}/(2))}(\mathbf{Z}/(2), \mathbf{Z}/(2))$.
Note that $H_*(\Omega Spin; \mathbf{Z}/(2)) = E(\alpha_{2i} : i \geq 1)$.

Lemma 3.3 $H_*(\Omega Spin(4n);\mathbf{Z}/(2))$ *is*

$$E(\alpha_{2i}:1\le i\le n-1)\otimes P(\alpha_{2i}:n\le i\le 2n-1)\otimes P(\beta_{4i-2}:n\le i\le 2n-1).$$

$H_*(\Omega Spin(4n+1);\mathbf{Z}/(2))$ *is*

$$E(\alpha_{2i}:1\le i\le n-1)\otimes P(\alpha_{2i}:n\le i\le 2n-1)\otimes P(\beta_{4i+2}:n\le i\le 2n-1)$$

$H_*(\Omega Spin(4n+2);\mathbf{Z}/(2))$ *is*

$$E(\alpha_{2i}:1\le i\le n-1)\otimes P(\alpha_{2i}:n\le i\le 2n)\otimes P(\beta_{4i+2}:n\le i\le 2n-1)$$

$H_*(\Omega Spin(4n+3);\mathbf{Z}/(2))$ *is*

$$E(\alpha_{2i}:1\le i\le n)\otimes P(\alpha_{2i}:n+1\le i\le 2n)\otimes P(\beta_{4i+2}:n\le i\le 2n)$$

Note that α_{2i} in $H_*(G'_{2,n};\mathbf{Z}/(2))$ corresponds to

$$\alpha_{2i} \text{ in } H_*(\Omega Spin(n+2);\mathbf{Z}/(2))$$

and β_{2i} corresponds to β_{2i} or $(\alpha_i)^2$ according to the dimension. In order to compute the cohomology we recall by [1] or [5]

$$\begin{aligned} H^*(G'_{2,2n};\mathbf{Z}/(2)) &= x_2/x_2^{n+1}\otimes E(y_{2n}) \\ H^*(G'_{2,2n+1};\mathbf{Z}/(2)) &= x_2/x_2^{n+1}\otimes E(y_{2n+2}) \end{aligned}$$

From here we can consider the Verschiebung map V. The good explanation for the Verschiebung map are in ([6] or [3],p234-235).

$$\begin{aligned} V(\alpha_{4i}) &= \alpha_{2i} \\ V(\alpha_{4i+2}) &= 0 \\ V(\beta_{2i}) &= 0 \end{aligned}$$

Let A be a Hopf algebra and $Q(A)$ be the module of indecomposable elements and $P(A)$ be the module of primitive elements. We have the useful relations between primitive elements and indecomposable elements (generators);

$$Q(A)^* = P(A)$$

, where$Q(A)^*$ is the dual of $Q(A)$. So if we pass from $H_*(X;\mathbf{Z}/(2))$ to $H^*(X;\mathbf{Z}/(2))$, each primitive element in $H_*(X;\mathbf{Z}/(2))$ corresponds to a generator in $H^*(X;\mathbf{Z}/(2))$ and each generator in $H_*(X;\mathbf{Z}/(2))$ corresponds to a primitive in $H^*(X;\mathbf{Z}/(2))$. We also have the same situations if we pass from $H^*(X;\mathbf{Z}/(2))$ to $H_*(X;\mathbf{Z}/(2))$.
For example

$$\begin{aligned} H_*(\Omega Spin(8);\mathbf{Z}/(2)) &= E(\alpha_2)\otimes P(\alpha_4,\alpha_6)\otimes P(\beta_6,\beta_{10}) \\ V(\alpha_4) &= \alpha_2 \end{aligned}$$

So we have the following primitives:

$$\alpha_2,\ \alpha_4^{2^{i+1}},\ \alpha_6^{2^i},\ \beta_6^{2^i},\ \beta_{10}^{2^i} \text{ for } i\geq 0\,.$$

Hence we get

$$H^*(\Omega Spin(8);\mathbf{Z}/(2)) \ = \ P(a_2)/a_2^4\otimes\Gamma(a_6)\otimes\Gamma(c_6,c_8,c_{10})\,.$$

We will give two more examples. First

$$\begin{aligned} H_*(\Omega Spin(10);\mathbf{Z}/(2)) &= E(\alpha_2)\otimes P(\alpha_4,\alpha_6,\alpha_8) \\ &\quad \otimes P(\beta_{10},\beta_{14}) \end{aligned}$$

with

$$\begin{aligned} V(\alpha_4) &= \alpha_2 \\ V(\alpha_8^{2^i}) &= \alpha_4^{2^i}, i\geq 0\,. \end{aligned}$$

Hence we get the following primitives:

$$\alpha_2,\ \alpha_4^{2^{i+1}},\alpha_6^{2^i},\ \beta_{10}^{2^i},\ \beta_{14}^{2^i} \text{ for } i\geq 0\,.$$

From the above

$$\begin{aligned} H^*(\Omega Spin(10);\mathbf{Z}/(2)) &= P(a_2)/(a_2^8)\otimes\Gamma(a_6)\otimes \\ &\quad \Gamma(c_{10},c_{14})\otimes \\ &\quad \textstyle\bigotimes_{i\geq 0} P(\gamma_{2^i}(d_8)/(\gamma_{2^i}(d_8))^4\,. \end{aligned}$$

For the next

$$\begin{aligned} H_*(\Omega Spin(14);\mathbf{Z}/(2)) &= E(\alpha_2,\alpha_4)\otimes P(\alpha_6,\alpha_8,\alpha_{10},\alpha_{12}) \\ &\quad \otimes P(\beta_{14},\beta_{18},\beta_{22}) \end{aligned}$$

with

$$\begin{aligned} V(\alpha_4) &= \alpha_2, V(\alpha_8)=\alpha_4 \\ V(\alpha_{12}^{2^i}) &= \alpha_6^{2^i}, i\geq 0\,. \end{aligned}$$

Hence we get the following primitives:

$$\alpha_2,\ \alpha_6^{2^i},\ \alpha_8^{2^{i+1}},\ \alpha_{10}^{2^i},\ \beta_{14}^{2^i},\ \beta_{18}^{2^i},\ \beta_{22}^{2^i} \text{ for } i \geq 0\,.$$

Therefore

$$\begin{aligned} H^*(\Omega Spin(14); \mathbf{Z}/(2)) &= P(a_2)/a_2^8 \otimes P(a_6)/a_6^4 \otimes \Gamma(a_{10}) \otimes \\ &\quad \Gamma(c_{14}, c_{16}, c_{18}, c_{22}) \otimes \\ &\quad \bigotimes_{i \geq 0} P(\gamma_{2^i}(b_{12}))/(\gamma_{2^i}(b_{12}))^4\,. \end{aligned}$$

,where $b_{12} = \gamma_2(a_6)$. The above two examples illustrate the difference between $H^*(\Omega Spin(8n+2); \mathbf{Z}/(2))$ and $H^*(\Omega Spin(8n+6); \mathbf{Z}/(2))$. In this way we can also compute $H^*(\Omega Spin(n); \mathbf{Z}/(2))$.

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INVOLUTION ON THE MODULI SPACE OF ANTI-SELF-DUAL CONNECTIONS

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§1. Introduction

Let σ be an involution on a smooth simply connected closed 4-manifold M. Suppose that the fixed point set B of σ on M is a 2-dimensional submanifold. Let $M' = M/\sigma$ be the orbit space and $p : M \to M'$ the projection map. There is a smooth struction on the orbit space M' such that the projection map p is smooth. The involution map σ induces an isomorphism σ^* on $H^n(M)$ to itself. The projection map p induces an isomorphism $p^* : H^n(M') \to H^n(M)^\sigma$ where $H^n(M)^\sigma$ is the invariant subspace of $H^n(M)$ under the involution σ^*. Using this isomorphism p^* we have the relations $\chi(M) = 2\chi(M') - \chi(B)$ and $\tau(M) = 2\tau(M') - B^2$ where $\tau(M)$ is the signature of M and B^2 is the intersection number of B. In case M is Kähler manifold there are such relations as $b_1(M) = 2b_1(M')$, $b_2(M) = 2b_2(M') - \chi(B) + 2$ and $b_2^+(M) = 1 + 2b_2^+(M')$ where $b_2^+(M)$ is the rank of the maximal positive subspace of $H^2(M)$ under the intersection pairing.

Let E be a vector bundle over M with the structure group $SU(2)$. Suppose that E has a lifting endomorphism of the involution σ. This lifting acts on the connections on E by pull-back, and the induced action on the orbit space of the gauge equivalent classes of the connections is independent of the choice of the liftings. There is a lifting of σ to $\tilde{\sigma} : E \to E$ which restricts trivially to the sub-bundle E/B. We can form the quotient bundle $E' = E/\tilde{\sigma}$ over the orbit space $M' = M/\sigma$. We have the moduli space of the $\tilde{\sigma}$-invariant anti-self-dual connections. We define polynomial invariants on this moduli space. We also have the moduli space of anto-self-dual connections on the quotient bundle $E' \to M'$ and we can construct polynomial invariants on the moduli space.

Our goal is to understand the orbit space M' and the invariant part of the moduli space of anti-self-dual connections, and polynomial invariants on the moduli spaces.

§2. The Equivarient Geometry on Bundles

Let G be a finite Abelian group. Let G acts on a smooth, closed, simply connected 4-manifold M and lift to an $SU(2)$ vector bundle $\pi : E \to M$ on which π is a G-map. Choose Riemannian metrics on M and E on which G acts as isometries. A Riemannian connection is a linear map. $\nabla : \Omega^0(E) \to \Omega^1(E)$ which satisfies

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$
$$d\langle\sigma, \tau\rangle = \langle\nabla\sigma, \tau\rangle + \langle\sigma, \nabla\tau\rangle$$

where $f \in C^\infty(M)$ and $\sigma, \tau \in \Omega^0(E)$ and $\langle\ ,\ \rangle$ is the given Riemannian metric on E. There are Riemannian connections on E. Let $\mathcal{C}$ be the space of all Riemannian connections on E. In fact $\mathcal{C}$ is an affine space as a model space $\Omega^1(adP)$ where adP is the associated Lie algebra bundle of E. For $\nabla \in \mathcal{C}$ let $F^\nabla \in \Omega^2(adP)$ be the curvature of ∇. Let $\mathcal{G}$ be the group of gauge transformations of E which are the sections of the associated Lie group bundle by the adjoint action of $SU(2)$ on itself. The group $\mathcal{G}$ acts on $\mathcal{C}$ by $g(\nabla) = g \circ \nabla \circ g^{-1}$.

Let $B = \mathcal{C}/\mathcal{G}$ be the orbit space and $\pi : \mathcal{C} \to B$ be the projection. Let $*$ be the Hodge star operator on the oriented Riemannian 4-manifold M. A connection $\nabla \in \mathcal{C}$ is said anti-self-dual if $*F^\nabla + F^\nabla = 0$.

Let $\mathcal{A}$ be the subspace of $\mathcal{C}$ consisting of all anti-self-dual connections on E. The action of $\mathcal{G}$ on $\mathcal{C}$ can restrict on $\mathcal{A}$. The orbit space $\mathcal{A}/\mathcal{G} \equiv \mathcal{M}$ is called the moduli space of the gauge equivalence classes of anti-self-dual connections on E. the action of G on the bundle $E \to M$ induces on action of G on $\mathcal{C}$. If $\sigma \in \Omega^0(E)$ and $h \in G$, then we define

$h(\sigma) = h \circ \sigma \circ h^{-1}$ where h^{-1} is an action on M and h is an action on E as a bundle map.

If $v \in TM$, and $\nabla \in \mathcal{C}$, then we define $h(\nabla)v(\sigma) = h(\nabla_{h_*^{-1}}h^{-1} \circ \sigma \circ h)$.

Finally we define an action of G on $\Omega^K(adP)$ by

$$(h\Phi)_{v_1,\dots,v_k} = h \circ \Phi_{h_*^{-1}(v_1),\dots,h_*^{-1}(v_k)} \circ h^{-1}.$$

Lemma 2.1. *For any $\nabla \in \mathcal{C}$ and any $h \in G$, we have $h(\nabla) \in \mathcal{C}$.*

Proof. Let $v \in TM$, $\sigma, \sigma_1, \sigma_2 \in \Omega^0(E)$ and $f \in C^\infty(M)$

$$\begin{aligned} h(\nabla)(f\sigma) &= h[\nabla h^{-1} \circ (f\sigma) \circ h] \\ &= h[\nabla f(h^{-1} \circ \sigma \circ h)] \\ &= h(df \otimes h^{-1}\sigma h) + h(f \circ \nabla h^{-1} \circ \sigma \circ h) \\ &= df \otimes \sigma + f[h(\nabla)\sigma] \end{aligned}$$

Compatibility of the connection $h(\nabla)$ with the Riemannian Structure:

$$\begin{aligned} &\langle h(\nabla)v\sigma_1, \sigma_2\rangle Ex + \langle \sigma_1, h(\nabla)v\sigma_2\rangle Ex \\ &= \langle h(\nabla_{h_*^{-1}v} h^{-1} \circ \sigma \circ h), \sigma_2\rangle Ex + \langle \sigma_1, h(\nabla_{h_*^{-1}v} h^{-1} \circ \sigma_2 \circ h)\rangle Ex \\ &= \langle \nabla_{h_*^{-1}v} h^{-1}\sigma h, h^{-1}\sigma_2\rangle E_{h^{-1}(x)} + \langle h^{-1}\sigma_1, \nabla_{h_*^{-1}v}(h^{-1}\sigma_2 h)\rangle E_{h^{-1}(x)} \\ &\qquad \text{by } h \text{ is an isometry on } E \\ &= \langle \nabla_v h^{-1}\sigma_1 h, h^{-1}\sigma_2 h\rangle Ex + \langle h^{-1}\sigma_1 h, \nabla_v h^{-1}\sigma_2 h\rangle Ex \\ &\qquad \text{by } h \text{ is an isometry on } M \\ &= v\langle h^{-1}\sigma_1 h, h^{-1}\sigma_2 h\rangle Ex \\ &= v\langle \sigma_1, \sigma_2\rangle Ex \quad \text{by isometries on } M \text{ and } E. \end{aligned}$$

Since G acts on M as isometries, the action of G on $\mathcal{C}$ preserves $\mathcal{A}$. The set of invariant connections of E denotes $\mathcal{C}^G = \{\nabla \in \mathcal{C} \mid h(\nabla) = \nabla\}$ and the invariant anti-self-dual connections $\mathcal{A}^G = \mathcal{C}^G \cap \mathcal{A}$. The G-equivariant gauge group denotes $\mathcal{G}^G = \{g \in \mathcal{G} \mid hg = gh \text{ for all } h \in G\}$. Then $\mathcal{G}^G$ acts on $\mathcal{C}^G$ and $\mathcal{A}^G$. Let and $B^G = \mathcal{C}^G/\mathcal{G}^G$ and $\mathcal{M}^G = \mathcal{A}^G/\mathcal{G}^G$ and $\Omega^k(adP)^G$ be the G-invariant subspace of $\Omega^k(adP)$. Connection on E induces a connection on adP by for $\Phi \in \Omega^0(adP)$ and for $\sigma \in \Omega^0(E)$.

$$(\nabla\Phi)(\sigma) = (\Phi\sigma) - \Phi(\nabla\sigma).$$

Then we have the following immediate consequences.

Lemma 2.2. *If ∇ is a G-invariant anti-self-dual connection.*

1. $0 \to \Omega^0(adP)^G \xrightarrow{d^0} \Omega^1(adP)^G \xrightarrow{d_+} \Omega^2_+(adP) \to 0$ *is an elliptic complex.*
2. $F^\nabla \in \Omega^2(adP)^G$.
3. $\mathcal{G}^G_{k+1}$ *acts on* $\mathcal{C}^G_k$ *and* $\mathcal{A}^G_k$ *where* $\mathcal{C}^G_k$ *is the Sobolev-Completion* $\{\nabla + A \mid A \in L^2_k(\Omega^1(adP)^G)\}$ *by the* L^2_k*-norm.*

Remark 2.3. If the bundle $E \to M$ is an $SO(3)$-bundle. Let $\pi : \mathcal{C} \to \mathcal{B}$ be the projection. Then

$$\mathcal{B}^{G*} = \mathcal{C}^{G*}/\mathcal{G}^G = \pi(\mathcal{C}^{G*})$$
$$\mathcal{M}^{G*} = \mathcal{A}^{G*}/\mathcal{G}^G = \pi(\mathcal{A}^{G*})$$

Proof. Suppose ∇ is a G-invariant irreducible connection and $\nabla' = g(\nabla)$. For each $h \in G$, $h(g(\nabla)) = h(\nabla) = g(\nabla) = g(h(\nabla))$ and $g^{-1}h^{-1}gh(\nabla) = \nabla$, $g^{-1}h^{-1}gh$ is an element of the isotropy subgroup Γ^∇ of the gauge transformation group $\mathcal{G}$. Since the connection ∇ is irreducible, the gauge transformation g is G-invariant.

§3. Involution on $K3$ surface

Let an anti-holomorphic involution σ on a $K3$-surface M lift to an $SU(2)$ vector bundle $E \xrightarrow{\pi} M$ on which π is σ-equivariant.

Remark 3.1. Let $E_2 \to S^4$ be the canonical quaternion line bundle. Let $f : M' \to S^4$ be a map with degree k and let $E_1 = f^*E_2$ and $E = \pi^*E_1$. The involution $\sigma : M \to M$ can be lifted to $E_1 \to E_1$ by choosing an isomorphism $\tau : E \to \sigma^*E$. Since f has degree $k\langle c_2(E_1), M'\rangle = \langle c_2(f^*E_2), M'\rangle$, $\langle f^*c_2(E_2), M'\rangle = \langle c_2(E_2), f * M'\rangle = \langle c_2(E_2), kS^4\rangle = k$, and since π is a branched double cover with codimension 2 branch set $\langle c_2(E), M\rangle = \langle \pi^*c_2(E_1), M\rangle = \langle c_2(E_1), \pi * M\rangle = \langle c_2(E_1), 2M'\rangle = 2k$. The involution σ on the $K3$ surface M is an orientation preserving isometry $\tilde\sigma$ acts on the set $\mathcal{C}$ of connections for $E \to M$ and acts on the group $\mathcal{G}$ of gauge transformations, via, $\tilde\sigma(\nabla) = \tilde\sigma \nabla \tilde\sigma^{-1}$ and $\tilde\sigma(g) = \tilde\sigma\, g\, \tilde\sigma^{-1}$ for $\nabla \in \mathcal{C}$ and $g \in \mathcal{G}$. Since $\tilde\sigma$ is an isometry $\tilde\sigma$ acts on the set of anti-self-dual connections.

Lemma 3.2. (1) *$\tilde{\sigma}$ acts $\mathcal{B}$ and $\mathcal{M}$.*

(2) *Let $\mathcal{M}_{k,M}$ be the moduli space of equivalence classes of anti-self-dual connections with $c_2 = k$ on $E \to M$. Let $M' = M/\sigma$ be the orbit space of σ. Then $\dim \mathcal{M}_{2k,M} = 2 \cdot \dim \mathcal{M}_{k,M'}$.*

Proof. (2)

$$\begin{aligned}\dim \mathcal{M}_{2k,M} &= 8(2k) - 3(1 + b^+ M) && \text{since} \quad b^+(M) = 3 \\ &= 2(8k) - 3(4) && \text{since} \quad b^+(M') = 1 \\ &= 2[8k - 3(1 + b^+(M')] \\ &= 2 \dim \mathcal{M}_{k,M'}.\end{aligned}$$

Theorem 3.3. *The moduli space $\mathcal{M}_{k,M'}$ on the orbit space M' can be identified one of the components of the σ-invariant moduli space $\mathcal{M}^{\sigma}_{2k,M}$ on M.*

Proof. Consider the fundamental elliptic complex

$$0 \to \Omega^0(\mathcal{G}_E)^{\sigma} \underset{\delta}{\overset{\nabla}{\rightleftarrows}} \Omega^1(\mathcal{G}_E)^{\sigma} \xrightarrow{d_+^{\nabla}} \Omega^2_+(\mathcal{G}_E)^{\sigma} \to 0$$

where $\nabla \in \mathcal{M}^{\sigma}_{2k,M}$. By Lefschetz fixed point theorem $\dim \mathcal{M}^{\sigma}_{2k,M} = \mathrm{Ind}(\delta^{\nabla} + d_+^{\nabla}) = 1/2 \sum_{\sigma \in G} L(\sigma, D)$ where $D : T(V_+ \otimes V_- \otimes \mathcal{G}_{\mathbf{C}})^{\tilde{\sigma}} \to T(V_+ \otimes V_+ \otimes \mathcal{G}_{\mathbf{C}})^{\tilde{\sigma}}$ is the twisted Dirac operator and $\mathcal{G}_{\mathbf{C}} = \mathcal{G}_E \otimes_R \mathbf{C}$. Let $\Delta : T(V_+ \otimes V_-)^{\sigma} \to T(V_+ \otimes V_+)^{\sigma}$. Let

$$\begin{aligned}L(1, D) &= P_1(\mathcal{G}_E \otimes \mathbf{C})[X] + 3 \ \mathrm{Ind}\,\Delta \\ &= 2k \cdot 8 - 3/2 \cdot 2[(\chi(M') - \sigma(M')) - (d\chi - d\sigma)] \\ &= 2 \cdot [8k - 3/2(\chi(M') - \sigma(M')) - 3/2(d\chi - d\sigma)] \\ &= 2 \cdot \dim \mathcal{M}_{k,M'} - 3(d\chi - d\sigma)\end{aligned}$$

$L(\tilde{\sigma}, D) = 3\,L(\sigma, \Delta)|_{M^{\sigma}} = 3(d\chi - d\sigma)$ since σ acts trivially on $\mathcal{G}_E|_{M^{\sigma}}$. Thus $\dim\ \mathcal{M}^{\sigma}_{2k,M} = 1/2[L(1, D) + L(\tilde{\sigma}, D)] = \dim \mathcal{M}_{k,M'}$.

Remark 3.4. When we compute

$$\begin{aligned}&\mathrm{Ind}_\sigma(D)\\&=\frac{ch_g(V_- - V_+)ch(V_+)ch_g(\mathcal{G}_E\otimes\mathbb{C})(T\otimes\mathbb{C})td(T^\sigma\otimes\mathbb{C})}{ch_g(\Lambda_{-1}N)}[M^\sigma],\end{aligned}$$

if M^σ non-orientable, twisted coefficients are used. Let $\pi : M \to M' = M/\sigma$ be the projection and let $\mathcal{A}(E)$ be the space of the connection with $c_2(E) = 2k$. And let $\mathcal{A}(E_1)$ be the space of the connection with $c_2(E_1) = k$ and $E = \pi^* E_1$. As before choose a σ-invariant metric g on M, then π induces a singular metric g_1 on M' such that $g = \pi^*(g_1)$. We would like to pull back the connections $\pi^* : \mathcal{A}(E_1) \to \mathcal{A}(E)$ by setting for $\nabla \in \mathcal{A}(E_1)$. $\phi \in \Omega^0(E)$ and $v \in TM$, $\pi^*(\nabla)_v\phi = \nabla_{\pi^* v}\phi(\pi)$. Since $E = \pi^* E_1, \phi(\pi)$ is a section of $E_1 \to M'$. For any function $f \in C^\infty(M)$, we extend the definition of $\pi^*(\nabla)$ to be a connection on $E \to M$;

$$\pi^*(\nabla)_v(f\phi) = v(f)\cdot\phi + f\cdot\pi^*(\nabla)_v\phi$$

where $\pi^*(\nabla)_v\phi = \nabla_{\pi^* v}\phi(\pi)$. By construction $\pi^*\sigma^* = \pi^*$ and $\pi * \sigma * = \pi * \sigma(\pi^*(\nabla))_v\phi = \sigma((\pi^*\nabla)_{\sigma * v}\sigma\phi\sigma) = \sigma(\nabla_{\pi * \sigma * v}\phi(\nabla)) = \nabla_{\pi^* v}\phi(\pi) = \pi^*(\nabla)_v\phi$.

Hence the image $\pi^* : \mathcal{A}(E_1) \to \mathcal{A}(E)$ is contained in the invariant subspace $\mathcal{A}(E)^\sigma$. Thus we complete the proof of theorem 3.3.

§4. Polynomial Invariant on $K3$-surface

For any $\alpha \in H_2(M : \mathbb{Z})$, we choose an embedded oriented surface Σ representing α. For $\pi * (\alpha) = \beta \in H_2(M' : \mathbb{Z})$, we choose an embedded oriented surface Σ_1 representing β.

Let N and N_1 be small tubular nbds of Σ and Σ_1 respectively. We may assume that $\pi(\Sigma) = \Sigma_1$. Let $\gamma_\Sigma : B(M) \to B(N)$ be the restriction map and $\gamma_{\Sigma_1} : M(B') \to B(N_1)$. For $A \in \mathcal{M}_{k,M'}$, consider the twisted Dirac Operators over N and N_1:

$$\begin{array}{ccc}\Gamma(V_\Sigma^-\otimes E) & \xrightarrow{\partial_{\pi * A}} & \Gamma(V_\Sigma^+\otimes E)\\ {\scriptstyle\pi^*}\uparrow & & \uparrow{\scriptstyle\pi^*}\\ \Gamma(V_{\Sigma_1}^-\otimes\Sigma_1) & \xrightarrow{\partial_A} & \Gamma(V_{\Sigma_1}^+\otimes\Sigma_1)\end{array}$$

For any $(\sigma \otimes s) \in \Gamma(V_{\Sigma_1}^{+} \otimes \Sigma_1)$,

$$\begin{aligned}
\not{D}_{\pi^* A}\pi^*(\sigma \otimes s) &= \not{D}_A(\pi^*\sigma \otimes \pi^* s) \\
&= \not{D}_\Sigma(\pi^*\sigma \otimes \pi^* s + \pi^*\sigma \otimes \pi^*(A)(\pi^* s)) \\
&= \pi^*(\not{D}_{\Sigma_1}\sigma \otimes s + \sigma \otimes A(s)) \\
&= \pi^* \not{D}_A(\sigma \otimes s), \quad \text{since } \pi^*(\not{D}_{\Sigma_1}\sigma) = \not{D}_\Sigma(\pi^*\sigma)
\end{aligned}$$

Thus the inclusion map $\pi^* : \mathcal{M}_{k,M'} \to \mathcal{M}_{2k,M}$ from $\mathcal{M}_{k,M'}$ into the invariant anti-self-dual connections of $\mathcal{M}_{2k,M}$ induces a bundle map on he determinant line bundles.

$$\begin{array}{ccc}
L_{\Sigma_1} & \longrightarrow & L_\Sigma \\
\downarrow & & \downarrow \\
\mathcal{M}_{k,M'} & \xrightarrow{\pi^*} & \mathcal{M}_{2k,M}
\end{array}$$

There is a universal bundle $\mathbb{E}$ over $M \times \mathcal{C}^*/\mathcal{G}$ with its Chern class $c_2(\mathbb{E}) \in H^4(M \times \mathcal{C}^*/\mathcal{G})$. For any class Σ in $H_2(M)$, we have a map $\mu M : H_2(M) \to H^2(\mathcal{C}^*/\mathcal{G})$ which is defined by the slant product $\mu(\Sigma) = c_2(\mathbb{E})/\Sigma$.

Lemma 4.1. *Given our two fold branched cover $\pi : M \to M'$, we have the following commutative diagram*

$$\begin{array}{ccc}
H_2(M)^\sigma & \xrightarrow{2\cdot\mu M} & H^2(\mathcal{M}^\sigma_{2k,M}) \\
\pi^*\downarrow & & \downarrow\pi^* \\
H_2(M') & \xrightarrow{\mu M'} & H^2(\mathcal{M}_{k,M'})
\end{array}$$

Proof. For any $\alpha \in H_2(M)^\sigma$, let Σ be an embedded oriented surface representing α. Since the indusion map $\pi^* : \mathcal{M}_{k,M'} \to \mathcal{M}_{2k,M}$ induces a bundle map and $\mu M(\alpha) = c_1(L_\Sigma)$ and $\pi * (\Sigma) = \Sigma_1$, we have

$$\begin{aligned}
\pi^* 2\mu M(\alpha) &= \pi^*(2c_1(L_\Sigma)) = c_1(\pi^*(L_\Sigma)) \\
&= c_1(L_{\Sigma_1}) = c_1(L_{\pi * \Sigma}) \\
&= \mu M'(\pi * (\alpha)).
\end{aligned}$$

Given homology classes $\alpha_1 \cdots \alpha_d \in H_2(M : \mathbb{Z})$, we represent them by embedded surfaces $\Sigma_1 \cdots \Sigma_d$ in M in general position such that any triple intersections $N_i \cap N_j \cap N_k$ of small tubular neighborhoods of Σ_i, Σ_j and Σ_k respectively are empty. There is a determinant line bundle L_Σ over B_M with a section whose zero set V_Σ is a codimension 2 submanifold of B_M and meets all of the moduli spaces $\mathcal{M}_i$ for $i \leq k$ transversaly. If $4K > 3(1 + b_2^+)$, then the intersection $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_k$ is compact.

Definition 4.2. For any homology class $\alpha_1 \cdots \alpha_d \in H_2(M : \mathbb{Z})$, the polynomial invariant is defined to be

$$q_k \times (\alpha_1, \ldots, \alpha_d)\langle \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_d), [\mathcal{M}_{k,M}]\rangle$$
$$= \#\langle V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_{k,M}\rangle$$

where # denotes a count with signs.

Remark 4.3. Suppose that g_1, g_2 are two different metrics on the base 4-manifold M. The moduli spaces $\mathcal{M}(g_1)$ and $\mathcal{M}(g_2)$ of gauge equivalence classes of anti-self-dual connections with respect to the metrics g_1 and g_2 respectively differ by a boundary in B^*. The pairing is independent of the choice of metric if $b_2^+ > 1$. Thus Donaldson invariants are smooth invariants.

Let the manifold M be Kählerian.

Let ω we the Kähler class in $H^2(M : \mathbb{C})$. By the Hodge index

$$H_+^2 = \omega \cdot \mathbb{R} \oplus H^{2,0} \quad \text{and} \quad b_2^+ = 1 + 2Pg.$$

We fix the orientation of H_+^2 by $-\omega\wedge$ (complex orientation of $H^{2,0}(M; \mathbb{C})$ which specifies the orientation of the moduli space. Let H be the hyperplane class in $H_2(M; \mathbb{Z})$ which is the Poincaré dual to ω. As a real manifold H is a compact Riemann surface. Over a small tubular neighborhood of H in M, we have, for each connection A, a coupled Dirac operator $\partial\!\!\!/ H : \Gamma(V^- \otimes E) \to \Gamma(V^+ \otimes E)$. We may construct the determinant line bundle L_H by using the index of the Dirac operators $\partial\!\!\!/ H$ over B_Σ:

$$L_H = \wedge \max(\ker \partial\!\!\!/_H)^* \otimes \wedge \max(\ker \partial\!\!\!/_H^*).$$

The determinant line bundle L_H descends to a bundle over $\mathcal{M}_H$. In fact, $\mathcal{M}_H$ is a complex manifold and each connection A defines the associated $\overline{\partial}$-operator $\overline{\partial}_A : \Omega^0(E) \to \Omega^{0,1}(E)$ and defines a holomorphic structure on E. Thus we have the holomorphic line bundle $L_H \to \mathcal{M}_H$.

Theorem 4.4. (Donaldson) (1) *The determinant line bundle* $L_H \to \mathcal{M}_H$ *is an ample line bundle.*

(2) $\mu(\alpha) = c_1(L_\Sigma) \in H^2(\mathcal{M}_M)$, *where* L_Σ *is the pull back of* L_H *on* $\mathcal{M}_M$ *and* Σ *is a representative of* $[\alpha] \in H_2(M : \mathbb{Z})$.

In the $SU(2)$ universal bundle $\mathbb{E} \to M \times \mathcal{C}/\mathcal{G}$, the orthogonal complement to orbits of $SU(2)$ gives the connection $\mathbb{A}$. The curvature $F_\mathbb{A}$ of A is a horizontal 2-form with values in the Lie algebra $SU(2)$. The tangent vectors are of type $(2,0)$, $(1,1)$ and $(0,2)$ in the tangent space $T_p(M) \times T_A(\mathcal{C}/\mathcal{G})$. Let $\rho \in H^2_{deR}(M : \mathbb{R})$ be the poincaré dual to a homology class $\Sigma \in H_2(M : \mathbb{Z})$. The slant product $c_2(\mathbb{E})/\Sigma \in H^2(B^* : \mathbb{Z})$ and

$$\begin{aligned}\mu(\Sigma) = c_2(\mathbb{E})/\Sigma &= \int_x c_2(\mathbb{E}) \wedge \rho \\ &= \frac{1}{8\pi^2} \int_x \mathrm{Tr}(F_\mathbb{A}^2) \wedge \rho\end{aligned}$$

is the de-Rham representation of $\mu(\Sigma)$.

If $a, b \in T_A B^* \simeq \operatorname{Ker} \delta^A \subset \Omega^1(\mathcal{G}_E)$, then $\mu(\Sigma)(a \wedge b) = \frac{1}{8\pi^2} \int_x \mathrm{tr}(a \wedge b) \wedge \rho$. Let M be a complex Kähler surface with Kähler from ω. Let H be the Poincaré dual to ω. $\mu(H)(a \wedge b) = \frac{1}{8\pi^2} \int_x \mathrm{tr}(a \wedge b) \wedge \rho$ is the Kähler form on the moduli space $\mathcal{M}$ for the usual L^2-metric on $\mathcal{M}$. For the generic metric on M, the tangent bundle of $\mathcal{M}$ is the index bundle for $\not{D}_A : \Gamma(V^- \otimes V^+ \otimes \mathcal{G}_\mathbb{C}) \to \Gamma(V^+ \otimes V^+ \otimes \mathcal{G}_\mathbb{C})$.

If A_1 is a connection on $L_\Sigma \to \mathcal{M}_k$, then

$$c_1(L) = \frac{i}{2\pi} F_{A_1} \quad \text{and} \quad F_{A_1}(a, b) = -\frac{i}{4\pi} \int_M \mathrm{tr}(a \wedge b) \wedge (\rho \cdot d\Sigma)$$

where $a, b \in \Omega^1(\mathcal{G}_E)$.

Theorem 4.5. *Suppose that* $L_\Sigma \to \mathcal{M}_k$ *is the complex line bundle induced by a homology class* $\Sigma \in H_2(M : \mathbb{Z})$ *and* A *is a connection of this bundle. Then the curvature is given by*

$$F_A(a, b) = \frac{1}{4\pi i} \int_M \mathrm{tr}(a \wedge b) \wedge \theta,$$

where $a, b \in \Omega^1(\mathcal{G}_E)$ and θ = Poincaré dual of Σ.

§5. Involution on polynomial invariants

Recall that a $K3$-surface is a compact, simply connected complex surface with trivial canonical bundle. All $K3$-surface with trivial canonical bundle. All $K3$-surfaces are diffeomorphic and Kahlerain, but not necessarily biholomorphically equivalent. Some $K3$-surface are elliptic surfaces, that is they admit a holomorphic map $\pi : M \to \mathbb{C}P'$ whose generic fiber is an elliptic curve. $b^+(M) = +3$, the stable range $4k > 3(1+b)$, i.e., $k > 4$, the invariant $\mathcal{Q}_{k,M}$ is a multilinear function of degree $d = 1/2 \dim \mathcal{M}_{k,M} = 4k - 6$.

Let Q be the quadratic form of the intersection form on M, and let K be the linear function $K : H_2(M) \to Z$ defined by the pairing $K(\alpha) = \langle c_1(M), \alpha\rangle$ for any α in $H_2(M)$.

Let $D_1 \simeq CP^2\#9\overline{CP}^2$ be the complex surface formed by 9 points blow-up on CP^2. Let $D_1(2, 2q+1)$ be obtained from D_1 by logarithmic transformations of multiplicities 2 and 2^{q+1}. Let $D_2(p,q)$ be obtained from $K3$-surface D_2 by logarithmic transfortions of multiplicities p and q.

The polynomial invariants can be expressed as polynomials in Q and $K : H_2(D_2(p,q)) \to Z$. And have the form <u>5.1</u> : $\mathcal{Q}_k,\ D_2(p,q) = pqQ^{[\ell]} + \Sigma_{a_i} Q^{[\ell-i]}K^{2i}$, where $\ell = d/2 = 1/4 \dim \mathcal{M}_k$, and $Q^{[\ell]} = 1/\ell! Q^\ell$ and $Q^\ell(\Sigma_1, \cdots, \Sigma_{2\ell}) = (1/2!)^\ell \cdot \Sigma_\sigma Q(\Sigma_{\sigma_1}, \Sigma_{\sigma_2}) \cdots Q(\Sigma_{\sigma_2\ell-1}, \Sigma_{\sigma_2\ell})$.

We introduce the results of Friedman and Morgan.

Theorem 5.2. (Friedman, Morgan)

1. *No two of the manifolds $D_1(2, 2q+1)(q = 0, 1, \cdots)$ are diffeomorphic, but all homeomorphic.*
2. *The product pq is a smooth invariant.*

In particular, no two of $D_2(1, 2k+1)$ are diffeomorphic.

Example 5.3. Let M be a $K3$-surface. Let $E \to M$ be an $SU(2)$ vector bundle with $c_2(E) = 4$. The moduli space $\mathcal{M}_{4,M}$ of anti-self-dual connections has dimension 20.

Let Q be the quardratic form of the intersection on M. By (5.1) we have, for $z_1, \ldots, z_{10} \in H_2(M, Z)$,

$$\begin{aligned} \mathcal{Q}_{4,M}(z_1, \ldots, z_{10}) &= Q^{(5)}(z_1, \ldots, z_{10}) \\ &= 1/5!\,1/2^5 \sum_{\sigma \in S_{10}} Q(z_{\sigma_1}, z_{\sigma_2}), \cdots, Q(z_{\sigma_1}, z_{\sigma_{10}}) = 1. \end{aligned}$$

This result also proved by Fintushel and Stern for homology $K3$-surface.

Suppose that $\alpha_1, \cdots, \alpha_d \in H_2(M : Z)^\sigma$. We use the Lemma 4.1 to compute polynomial invariant.

$$\begin{aligned} &\langle 2\mu M(\alpha_1) \cdots 2\mu M(\alpha_d), \pi^* \mathcal{M}_{k,M'} \rangle \\ &= \langle \pi^*(2\mu M(\alpha)1) \cdots 2\mu M(\alpha_d)), \mathcal{M}_{k,M'} \rangle \\ &= \langle \pi^* 2\mu M(\alpha_1) \cdots \pi^* 2\mu M(\alpha_d), \mathcal{M}_{k,M'} \rangle \\ &= \langle \mu M'[\pi_*(\alpha_1)] \cdots \mu M[\pi_*(\alpha_d)], \mathcal{M}_{k,M'} \rangle \\ &= \mathcal{Q}_{k,M'}(\pi_*\alpha_1, \cdots, \pi_*\alpha_d) \end{aligned}$$

Thus we have a theorem.

Theorem 5.4. *Let σ be an anti-holomorphic involution on a $K3$-surface M. If $M' = M/\sigma$ is the orbit space, then the polynomial invariant $\mathcal{Q}_{k,M'}$ can be computed by the pairing invariant moduli space $\mathcal{M}^{\delta}_{2k,M}$ and the cohomology classes in $\mathcal{M}_{2k,M}$.*

For a $K3$-surface M, let O_Q be the isometry group of the intersection form Q on the integral homology $H_2(M)$ and the homomorphism $h : \mathrm{Diff}(M) \to O_Q$.

The isometry group O_Q contains an index Q subgroup O_Q^+ consisiting of transformations which preserve the orientation of the positive part $H_2^+(M) \simeq Z^3$ of H_2. Since -1 does not lie in O_Q^+ there is a splitting $O_Q = O_Q^+ \oplus (-1) O_Q^+$.

Theorem 5.5. (Donaldson and Matumoto)

The image of $h : \mathrm{Diff}(M) \to O_Q$ is the subgroup O_Q^+.

Theorem 5.6. *Let σ be an anti-holomorphic involution on a $K3$-surface M. Then $\sigma^* \mathcal{Q}_{2k,M} = \mathcal{Q}_{2k,M} : S^{2d}(H^2(M, Z)) \to Z$.*

Proof. If σ is an anti-holomorphic involution on a $K3$-surface M then σ is an orientation preserving diffeomorphism. By Donaldson and Matumoto Theorem, σ^* is an isometry on $H^2(M : Z)$ and preserves the orientation of the positive part of the second cohomology group $H^2(M : Z)^+ = Z \oplus Z \oplus Z$. Thus σ^* preserves the polynomial invariants on $S^{2d}(H_2(M; Z))$, where $S^{2d}(H_2(M; \mathbb{Z}))$ is the symmetric product of $2d$ copies of $H_2(M; \mathbb{Z})$.

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DIFFERENTIAL GEOMETRY AND DYNAMICAL SYSTEM

THE SHARP ISOPERIMETRIC INEQUALITY FOR MINIMAL SURFACES WITH RADIALLY CONNECTED BOUNDARY IN HYPERBOLIC SPACE

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Given a plane domain D bounded by a curve C, it has long been known that the area A of D and the length L of C are related by the classical isoperimetric inequality

$$4\pi A \leq L^2,$$

where equality holds if and only if C is a circle. Many mathematicians have also sought isoperimetric inequalities for a domain in a curved space. An interesting one for a domain in the sphere was obtained by F. Bernstein in 1905 [B] :

$$4\pi A \leq L^2 + A^2.$$

Then Schmidt [S] proved in 1940 the analogue for the hyperbolic plane :

$$4\pi A \leq L^2 - A^2.$$

In each case, equality holds if and only if the domain is a geodesic disk. In fact, these three isoperimetric inequalities can all be expressed in one inequality as follows :

$$4\pi A \leq L^2 + KA^2,$$

where K is the Gauss curvature of the simply connected space form in which D lies.

On the other hand, it has been a long-standing conjecture that the classical isoperimetric inequality $4\pi A \leq L^2$ should hold for an arbitrary domain in a minimal surface in $\mathbf{R}^n$. Until now this inequality has been proved only for minimal surfaces with one or two boundary components, or more generally, with weakly or radially connected boundary ([C, OS, LSY, Ch]). In view of this conjecture and the works of Bernstein and Schmidt, one may ask whether their inequalities hold for domains on a minimal surface in S^n or H^n. In

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this paper we show that any two-dimensional minimal surface Σ^2 in H^n such that $\partial\Sigma$ is *radially connected* from some point p of Σ, *i.e.* such that $\{ r = dist(p,q),\ q \in \partial\Sigma\}$ is a connected interval, satisfies the sharp isoperimetric inequality

$$4\pi A \le L^2 - A^2.$$

But the isoperimetric inequality $4\pi A \le L^2 + A^2$ for a minimal surface in S^n still remains open.

In our companion paper [CG] we obtain two different types of isoperimetric inequalities: First, we introduce a modified area $M(D)$ of a domain D, and show that

$$4\pi M(D) \le L(\partial D)^2,$$

where D is a domain on a minimal surface in S^n_+ or H^n, whose boundary is radially connected or weakly connected in analogy with [LSY]. Second, weaker isoperimetric inequalities

$$2\pi A \le L^2 + KA^2$$

are obtained for *any* minimal surface Σ in S^n_+ or in H^n, where $K = 1$ or -1 depending on whether Σ is in S^n_+ or in H^n. Surprisingly, while the modified-area inequality is valid for S^n_+ or for $\mathbf{R}^n$, the result of this paper is valid for H^n or for $\mathbf{R}^n$; compare *Remark 1* below.

We would like to thank Henry Wente for suggesting a shorter proof of *Lemma 3.*

1. Estimates for the volume and angle of a cone

Every minimal surface considered in this paper is assumed to be differentiable up to its boundary.

Blaschke, earlier than [Ch], pointed out the value of comparing a minimal surface Σ in $\mathbf{R}^n$ with the cone over its boundary [Bl, p. 247]. Estimates for the volume of the cone $p \divideontimes \partial\Sigma$ and for the angle of $\partial\Sigma$ viewed from an interior point of Σ play crucial roles in the proof of the sharp isoperimetric inequality for Σ with radially connected boundary in [Ch]. In this section we obtain the analogous estimates for minimal surfaces in H^n. In fact, this will require a more exacting choice of test function: compare *Proposition 1* and *Proposition 2* of [Ch] with *Proposition 2* and *Proposition 1* below.

Lemma 1 *Suppose $h'(r) = r\varphi(r)$ for some smooth $\varphi : [0,\infty) \longrightarrow \mathbf{R}$, and write $h(q) = h(r_p(q))$ where $r = r_p(q) = dist(p,q)$ for a fixed $p \in H^n$. If $\Sigma^k \subset H^n$ is either minimal or a cone over p, then*

$$\Delta h = r\varphi' + \varphi Q + (1 - |\nabla r|^2)(r\varphi \coth r - \varphi - r\varphi')$$

where $Q(r) = 1 + (k-1)r\coth r$.

Proof. One shows that the Hessian in H^n,

$$\overline{\nabla}^2 \cosh r = (\cosh r)g,$$

from which it follows that the Laplacian on Σ^k,

$$\Delta r = \coth r(k - |\nabla r|^2)$$

when Σ is either minimal or a cone over p. See *Lemma 5(b)* of [CG]. *Lemma 1* follows by direct computation.

The following lemma addresses the case where $h(r)$ is the solution of $\Delta h \equiv 1$ on the totally geodesic submanifold $\Sigma = H^k \subset H^n$. The conclusions may also be found on p.483 of [A].

Lemma 2 *Let $\varphi(r) = \alpha(r)/(r\alpha'(r))$, where $\alpha(r)$ is the volume of the geodesic ball of radius r in k-dimensional hyperbolic space H^k; thus $\alpha(0) = 0$ and $\alpha'(r) = k\omega_k \sinh^{k-1} r$. Define $Q(r)$ as in Lemma 1. Then*

(*a*) *for all $r > 0$, $\varphi'(r) < 0$ and $0 < \varphi(r) < \varphi(0) = 1/k$;*

and

(*b*) $r\varphi'(r) + \varphi(r)Q(r) \equiv 1$.

Proof. Differentiation of $r\varphi(r) = \alpha/\alpha'$ yields

$$r\varphi' + \varphi = 1 + r\varphi\alpha''/\alpha' = 1 - (k-1)\varphi r \coth r,$$

from which (*b*) follows. Elementary asymptotic analysis shows that $\varphi(0) = 1/k$ and $\varphi'(0) = 0$. Since $\sinh r \cosh r > r$, we find $Q'(r) > 0$, so that $Q(r) > Q(0) = k$, for all $r > 0$. The derivative of (*b*) now yields $r\varphi'' + (1+Q)\varphi' < 0$, or $(\varphi'(r)\exp P(r))' < 0$ where $P'(r) = (1+Q)/r$. Since $\varphi'(0) = 0$, we conclude that $\varphi'(r) < 0$ for positive r.

Definition Let $C \subset H^n$ be a $(k-1)$-dimensional rectifiable set and p a point in H^n. The $(k-1)$-dimensional *angle* $A^{k-1}(C,p)$ of C viewed from p is defined by setting

$$A^{k-1}(C,p) = \sin^{1-k} t \cdot \mathit{Volume}[(p \divideontimes C) \cap S(p,t)],$$

where $S(p,t)$ is the geodesic sphere of radius $t < \mathit{dist}(p,C)$ centered at p, and the volume is measured counting multiplicity. Clearly, the angle does not depend on t.

Note that

$$A^{k-1}(C,p) = k\omega_k \Theta^k(p \divideontimes C, p),$$

where $\Theta^k(p \divideontimes C, p)$ is the k-dimensional density of $p \divideontimes C$ at p.

Proposition 1 *Let Σ be a k-dimensional compact minimal submanifold with boundary in H^n, and let p be an interior point of Σ. Then*

$$A^{k-1}(\partial\Sigma, p) \geq k\omega_k.$$

Equality holds if and only if Σ is a domain on a totally geodesic H^k that is star-shaped with respect to p.

Proof. We use the Green's function $G(r)$ of H^k: $G'(r) = \sinh^{1-k} r$. Writing $G'(r) = r\varphi(r)$, we see that $r\varphi' + \varphi Q \equiv 0$ and

$$r\varphi \coth r - \varphi - r\varphi' = k \sinh^{-k} r \cosh r > 0$$

for $r > 0$, where $Q = 1 + (k-1) r \coth r$. Thus by *Lemma 1*, G is subharmonic on Σ, and harmonic on the cone $p \divideontimes \partial\Sigma$. Let ν be the exterior unit normal vector to Σ and η the exterior unit normal vector to the cone along $\partial\Sigma$. Then

$$\frac{\partial r}{\partial \nu} \leq \frac{\partial r}{\partial \eta},$$

implying

$$\begin{aligned} k\omega_k &= k\omega_k + \lim_{t\to 0} \int_{\Sigma - B(p,t)} \Delta G = \int_{\partial\Sigma} G'(r) \frac{\partial r}{\partial \nu} \\ &\leq \int_{\partial\Sigma} \sinh^{1-k} r \cdot \frac{\partial r}{\partial \eta} = A^{k-1}(\partial\Sigma, p). \end{aligned}$$

Equality holds if and only if $\Delta G(r) = 0$, $\Theta^k(\Sigma, p) = 1$, and $\nu = \eta$ if and only if Σ is a star-shaped minimal cone with density at the center equal to 1. Since

S^{k-1} is the only $(k-1)$-dimensional minimal submanifold in S^{n-1} with volume $k\omega_k$, we conclude that Σ lies in a totally geodesic H^k.

The next proposition will allow us to replace a minimal submanifold Σ^k in H^n by the cone over its boundary, relying on the monotone dependence of the isoperimetric inequality on the volume of Σ. This proposition and *Lemma 2* are closely related to the monotonicity formula of M. Anderson[A, p. 481].

Proposition 2 *Let Σ be a k-dimensional immersed compact minimal submanifold with boundary in hyperbolic space H^n, and let p be any point of H^n. Then*

$$\textit{Volume}(\Sigma) \leq \textit{Volume}(p \divideontimes \partial\Sigma);$$

if equality holds, then $p \in \Sigma$, and Σ must be totally geodesic and star-shaped with respect to p.

Proof. Let $h(q) = h(r_p(q))$, where $h'(r) = \alpha(r)/\alpha'(r)$ as in *Lemma 8*. Let ν be the outward unit normal vector to $\partial\Sigma$, which is tangent to Σ, and η the unit vector tangent to $p \divideontimes \partial\Sigma$; as in the proof of *Proposition 1*, we have $\partial r/\partial\nu \leq \partial r/\partial\eta$. This implies

$$\int_\Sigma \Delta h = \int_{\partial\Sigma} \frac{\partial h}{\partial \nu} \leq \int_{\partial\Sigma} \frac{\partial h}{\partial \eta} = \int_{p \divideontimes \partial\Sigma} \Delta h,$$

since $h'(r) > 0$ for all $r > 0$. But according to *Lemmas 1* and *2*,

$$\Delta h = 1 + (1 - |\nabla r|^2)[(r \coth r - 1)\varphi - r\varphi']$$

either on Σ or on $p \divideontimes \partial\Sigma$, where $\varphi(r) > 0$ and $\varphi'(r) < 0$ for $r > 0$. In particular, $\Delta h \geq 1$; and further, $\Delta h > 1$ unless $|\nabla r| = 1$ or $r = 0$. On the cone $p \divideontimes \partial\Sigma$, we have $|\nabla r| = 1$. Therefore,

$$\textit{Volume}(\Sigma) \leq \int_\Sigma \Delta h \leq \int_{p \divideontimes \partial\Sigma} \Delta h = \textit{Volume}(p \divideontimes \partial\Sigma).$$

Equality would imply $|\nabla r| = 1$ a.e. on Σ, which is to say that Σ coincides with a subset of the cone $p \divideontimes \partial\Sigma$. Equality also requires $\partial h/\partial\nu = \partial h/\partial\eta$, hence for every $q \in \partial\Sigma$ the entire geodesic segment from p to q lies in Σ. At p, each such segment is tangent to the tangent plane to Σ. This implies that Σ is totally geodesic.

Remark 1 *Proposition 2* is false when H^n is replaced by the hemisphere S^n_+, even for $n = 3$ and $k = 2$. For example, let Σ be half of the *Clifford torus*:

$$\Sigma = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| = |y| = 1/\sqrt{2},\, x_1 > 0\},$$

and $p = (1,0,0,0)$. Then $Area(\Sigma) = \pi^2$, which is greater than $Area\,(p⋇\partial\Sigma) = 2\sqrt{2}\,\pi$. Nonetheless, for domains $\Omega \subset \Sigma$ we have an isoperimetric inequality $L^2 \geq min\{4\pi A, 8\pi^2\}$ which implies the sharp S^2-isoperimetric inequality

$$4\pi\, A \leq L^2 + A^2.$$

It is an interesting question whether this last inequality is valid for every two-dimensional minimal surface in the hemisphere S^n_+.

2. Approximation lemma

In light of *Proposition 2* we would like to prove that certain hyperbolic cones satisfy the isoperimetric inequality $4\pi\, A \leq L^2 - A^2$. This inequality was proved in great generality by Bol, namely, for any smooth, simply connected, two-dimensional manifold with Gauss curvature $K \leq -1$. The following approximation lemma may be interpreted as stating in a precise way that a hyperbolic cone has generalized Gauss curvature ≤ -1 if the angle at its vertex is at least 2π. It is well known that a two- dimensional hyperbolic cone has Gauss curvature $\equiv -1$ away from its vertex.

Lemma 3 *Let $\Sigma_0 = (\mathbf{R}^2, ds^2)$ be the singular Riemannian 2-manifold (a hyperbolic cone) with metric given in geodesic polar coordinates (r,θ) by*

$$ds^2 = dr^2 + (a_0/2\pi)^2 \sinh^2 r\, d\theta^2.$$

If $a_0 \geq 2\pi$, then ds^2 may be approximated in $C^1_{loc}(\mathbf{R}^2\backslash\{0\})$ by smooth metrics ds^2_δ having Gauss curvature $K_\delta \leq -1$.

Proof. If $a_0 = 2\pi$, then $ds^2_\delta = ds^2$ suffices. For any angle $a_0 > 2\pi$, we shall construct ds^2_δ in the form

$$ds^2_\delta = dr^2 + g_\delta(r)^2 d\theta^2$$

for an appropriate function $g_\delta : [0,\infty) \longrightarrow [0,\infty)$. Similarly, write $g(r) = (a_0/2\pi)\sinh r$. The Gauss curvature K_δ of $(\mathbf{R}^2, ds^2_\delta)$ is determined by the Jacobi equation

$$(J) \qquad g''_\delta(r) + K_\delta(r) g_\delta(r) = 0.$$

The C^∞ function g_δ will be a smooth approximation to a $C^{1,1}$ function g_0 defined by

$$\begin{aligned} g_0(r) &= \beta^{-1}\sinh\beta r,\ 0 \le r \le r_1; \\ g_0(r) &= g(r-\epsilon),\ r \ge r_1; \end{aligned}$$

where $\epsilon > 0$, $r_1 > \epsilon$, and $\beta > 1$ are appropriately chosen parameters. Continuity of g_0'/g_0 at r_1 is equivalent to

$$(*) \qquad \beta \coth \beta r_1 = \coth(r_1 - \epsilon).$$

This plus the continuity of g_0 at r_1 imply that

$$(a_0/2\pi)^2 = 1 + (1-\beta^{-2})\sinh^2 \beta r_1,$$

which determines r_1 uniquely as a function of $\beta \in (1,\infty)$ since $a_0 > 2\pi$. Now let $\epsilon = \epsilon(\beta) < r_1(\beta)$ be defined by equation $(*)$. Then the $C^{1,1}$ metric

$$ds_0^2 = dr^2 + g_0(r)^2 d\theta^2$$

has Gauss curvature $K_0 \equiv -\beta^2$ on the disk $B_{r_1}(0)$ and $K_0 \equiv -1$ on $\mathbf{R}^2\backslash B_1(0)$. Note also that the mapping given in polar coordinates by $(r,\theta) \mapsto (r-\epsilon,\theta)$ is an isometry from $\mathbf{R}^2\backslash \overline{B}_{r_1}(0)$ with the metric ds_0^2 to $\Sigma_0\backslash \overline{B}_{r_1-\epsilon}(0)$. Since $\coth \beta r_1 > 1$, it follows from $(*)$ that $r_1(\beta) - \epsilon(\beta) \to 0$ as $\beta \to \infty$, so that the complement of an arbitrarily small neighborhood of the singularity in Σ_0 is isometric to a subset of $(\mathbf{R}^2, ds_0^2)$. Further, it may be seen from the definition of $r_1(\beta)$ that $r_1(\beta) \to 0$ as $\beta \to +\infty$, and hence also $\epsilon(\beta) \to 0$.

We may now construct the smooth approximation g_δ by smoothing the Gauss curvature K_δ of ds_0^2: we choose $K_\delta \in C_0^\infty([0,\infty))$ with $K_\delta(r) \equiv -\beta^2$ $(0 \le r \le r_1 - \delta)$, $K_\delta(r) \equiv -1$ $(r \ge r_1+\delta)$ and $K_\delta'(r) \ge 0$ for all r. We then solve the Jacobi equation (J) with $g_\delta(0) = 0$, $g_\delta'(0) = 1$. Since $-\beta^2 \le K_\delta(r) \le -1$, this initial- value problem has a unique solution $g_\delta : [0,\infty) \longrightarrow [0,\infty)$ which is moreover positive on $(0,\infty)$. For any exponent $1 < p < \infty$, we have $K_\delta \to K_0$ in $L^p([0,\infty))$. This implies that $g_\delta \to g_0$ in $W^{2,p}$ on any bounded interval, and hence also in $C^{1,\alpha}$ for any $\alpha < 1$ on any bounded interval. By choosing β sufficiently large, we make $r_1(\beta)$ and $\epsilon(\beta)$ as small as desired; choosing also δ sufficiently close to 0 results in a metric ds_δ^2 arbitrarily close to ds^2 in $C^{1,\alpha}_{loc}(\mathbf{R}^2\backslash\{0\})$.

3. The sharp isoperimetric inequality

As was hinted in the preceding section, we shall prove the sharp isoperimetric inequality for cones in H^n by combining Bol's theorem and the approximation lemma. The analogous result for cones in $\mathbf{R}^n$ was proved in [Ch, *Lemma 1*] by a substantially different method of developing the cone into a planar domain.

Lemma 4 *Choose $p \in H^n$, and let C be a compact 1-dimensional submanifold of H^n such that C is radially connected from p and $A^1(C,p) \geq 2\pi$. Then the length L of C and the area A of the cone $p \divideontimes C$ satisfy the sharp isoperimetric inequality of domains in H^2 :*

$$4\pi A \leq L^2 - A^2.$$

Proof. Write $r(q) = dist(p,q)$, as usual, for the distance in H^n. We shall first show that on any radially connected 1-manifold C, there are a finite number of points $q_1, \ldots, q_m, p_1, \ldots, p_m = p_0$ such that

(i) $r(q_i) = r(p_i)$ for all $1 \leq i \leq m$;

(ii) p_i and q_{i+1} lie in the same component of C for all $0 \leq i \leq m-1$; and

(iii) C may be oriented so that the union of the m closed arcs of C from p_i to q_{i+1} in the positive sense, $0 \leq i \leq m-1$, covers C exactly once.

The proof is by induction on the number J of connected components of C. If $J = 1$, the assertion is obvious with $m = 1$. Now suppose the assertion holds for 1-manifolds in H^n with $(J-1)$ connected components. Write the connected components of C as $\Gamma_1, \ldots, \Gamma_J$, where $min\{r(q) : q \in \Gamma_1\} \geq min\{r(q) : q \in \Gamma_j\}$ for all $2 \leq j \leq J$. Then $\Gamma_2 \cup \cdots \cup \Gamma_J$ is radially connected from p. Applying the induction hypothesis, we may write $\{Q_1, \ldots, Q_M, P_1, \ldots, P_M = P_0\}$ for a set of points satisfying $(i), (ii)$ and (iii) with $\Gamma_2 \cup \cdots \cup \Gamma_J$ in place of C. Since C is radially connected, there are points $P \in \Gamma_1$ and $Q \in \Gamma_2 \cup \cdots \cup \Gamma_J$ with $r(P) = r(Q)$ (for example, $r(P) = min\{r(q) : q \in \Gamma_1\}$). Let P_k and Q_{k+1} be the endpoints of the interval in which Q falls, according to (iii). Define $p_l = P_l$ and $q_l = Q_l$ for $1 \leq l \leq k$; $q_{k+1} = Q = p_{k+2}$; $p_{k+1} = P = q_{k+2}$; and $q_l = Q_{l-2}, p_l = P_{l-2}$ for $k+3 \leq l \leq m = M+2$. Then $\{q_1, \ldots, q_m, p_1, \ldots, p_m = p_0\}$ satisfy $(i), (ii)$ and (iii) as claimed. (Incidentally, one may note that $m + 1 = 2J$.)

Write $a_0 = A^1(C,p)$. We may now show that $p⋇C$ may be mapped discontinuously, but locally isometrically, into an abstract hyperbolic cone $\Sigma_0 = (\mathbf{R}^2, ds^2)$ with the singular Riemannian metric

$$ds^2 = dr^2 + (a_0/2\pi)^2 \sinh^2 r d\theta^2,$$

so that $r = dist(p,\cdot)$ is preserved. Namely, let $\{q_1, \ldots, q_m, p_1, \ldots, p_m = p_0\}$ be a set of points in C such that properties $(i), (ii)$ and (iii) are valid. For $0 \leq i \leq m-1$, write $C(p_i, q_{i+1})$ for the closed oriented arc of C from p_i to q_{i+1}. Then $p⋇C(p_0, q_1)$ may be mapped isometrically into Σ_0 so that for all $q \in C(p_0, q_1)$ the H^n-geodesic from p to q is mapped onto a geodesic segment $\theta = const.$ starting at the vertex $0 \in \Sigma_0$. The next sector $p⋇C(p_1, q_2)$ of $p⋇C$ is then mapped isometrically onto an adjacent sector of Σ_0, so that the geodesics from p to q_1 and from p to p_1 are mapped to the same radial geodesic segment. This process continues until $p⋇C(p_{m-1}, q_m)$ is mapped isometrically into Σ_0, so that the geodesics from p to q_{m-1} and from p to p_{m-1} are identified, and the geodesics from p to q_m and from p to $p_m = p_0$ are identified. This process closes up exactly since the angle at the vertex of Σ_0 is $a_0 = A^1(C,p) = \sum_{i=0}^{m-1} A^1(C(p_i, q_{i+1}), p)$. Observe that $p⋇C$ is mapped, almost everywhere one-to-one, onto a star-shaped domain $\Omega \subset \Sigma_0$ of area A, such that $\partial\Omega$ has length L. We may assume that $p \notin C$, since $Area(p⋇C)$ varies continuously with p, and since $A^1(C,p)$ is lower semi-continuous. Then Ω is a star-shaped neighborhood of 0 in Σ_0. Applying *Lemma 3* we see that for each δ near 0 there is a smooth Riemannian surface $(\mathbf{R}^2, ds_\delta^2)$, with Gaussian curvature $K_\delta \leq -1$, which converges locally uniformly to Σ_0, and which converges $C^{1,\alpha}$ to Σ_0 on compact sets in $\mathbf{R}^2\backslash\{0\}$. Then with respect to ds_δ^2, $\partial\Omega$ has length $L(\delta) \to L$ and Ω has area $A(\delta) \to A$ as $\delta \to 0$. By Bol's theorem [Bol, p.230] the isoperimetric inequality

$$4\pi A(\delta) \leq L(\delta)^2 - A(\delta)^2$$

holds, and the conclusion of *Lemma 4* follows.

Remark 2 *Lemma 4* is false for submanifolds of dimension $k \geq 3$ in H^n or even in $\mathbf{R}^n$. In $\mathbf{R}^n$, we may choose the reference point p near $p_0 = 0$. Given $R > 1$, $0 < \epsilon \ll 1$ and a point $q_1 \in \mathbf{R}^n$ with $|q_1|^2 = R^2 - 1$, let the $(k-1)$-submanifold C be formed from the two unit $(k-1)$-spheres $S_R^{n-1}(0) \cap S_1^{n-1}(\pm q_1) \cap \mathbf{R}^{k+1}$ plus a thin "bridge" of the form $[-R, R] \times S_\epsilon^{k-2}$ connecting points q_2 and $-q_2$ on the unit spheres, and smoothed. Then for sufficiently small ϵ, there is an immersed minimal k-submanifold Σ with boundary C, which is uniformly close to the union of the two flat unit k-dimensional balls with a thin "bridge" of the form $[-R, R] \times B_\epsilon^{k-1}$, by a theorem of N. Smale [Sm]. Choose $p \in \Sigma$ with $dist(p, p_0) < \epsilon$. Then the angle

$$A^{k-1}(C, p) \geq k\omega_k$$

by *Proposition 1.* Thus C satisfies conditions analogous to all hypotheses of *Lemma 4.* But

$$Volume(C) = 2k\omega_k + O(R\epsilon^{k-2}),$$

while a longer computation shows that

$$Volume(p \divideontimes C) = 2R\omega_k + O(R\epsilon^{k-1}),$$

so that for large R the k-dimensional Euclidean isoperimetric inequality

$$(Volume(C))^k \geq k^k\omega_k(Volume(p \divideontimes C))^{k-1}$$

is certainly false. Thus there is no hope of extending *Lemma 4* to submanifolds of dimension greater than two. On the other hand, the minimal submanifold Σ has

$$Volume(\Sigma) \leq 2\omega_k + 2R\omega_{k-1}\epsilon^{k-1},$$

as follows from the proof of Smale's theorem. For small ϵ, Σ itself therefore satisfies the k-dimensional Euclidean isoperimetric inequality

$$(Volume(\partial\Sigma))^k \geq k^k\omega_k(Volume(\Sigma))^{k-1}.$$

That this inequality be valid for every k-dimensional minimal submanifold Σ of $\mathbf{R}^n$ remains a challenging conjecture; an eventual proof cannot be found through the straightforward intermediation of a cone $p \divideontimes \partial\Sigma$.

Using *Proposition 1*, *Proposition 2*, and *Lemma 4*, and the monotonicity of the quadratic function $4\pi A + A^2$ for positive area A, we may now prove our main result.

Theorem 1 *Let Σ^2 be an immersed compact minimal surface with boundary in hyperbolic space H^n. Assume there exists $p \in \Sigma$ such that $\partial\Sigma$ is radially connected from p. Then $Area(\Sigma)$ and $Length(\partial\Sigma)$ satisfy the isoperimetric inequality*

$$4\pi A \leq L^2 - A^2,$$

with equality if and only if Σ is a geodesic ball in a totally geodesic $H^2 \subset H^n$.

Remark 3 If $\partial\Sigma$ has two components, choose two points p_1 and p_2, one from each component. Then there exists a point q on Σ with $dist(q, p_1) = dist(q, p_2)$, which implies that $\partial\Sigma$ is radially connected from q. Consequently Σ satisfies the above isoperimetric inequality.

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AN ISOPERIMETRIC INEQUALITY FOR A COMPACT N-MANIFOLD IN N-SPACE

Y. D. CHAI

§I. Introduction

For a planer compact set K with area A and perimeter L the classical isoperimetric inequality states:

$$L^2 - 4\pi A \geq 0.$$

Geometric inequality involving integral of absolute mean curvature $|M|$ and surface area S in 3-space has been founded by I.A.Danelich [4,5].

He found the following facts:

(1) $|M|(W_2) \geq |M|(W_1)$ if W_1 and W_2 are compact sets and W_1 is a convex set contained in W_2 [4].

(2) If W is a compact set with bounded integral of absolute mean curvature, then

$$|M|^2 \geq (\pi^2/2)S(W) \quad [5].$$

In this paper we first characterize compact n-manifold in n-space by means of the topological behavior of their intersections with regular balls. From the characterization we get analogous results of (1) and (2) for compact n-manifold in n-space.

§II. Intersection property of $P^*(t)$-convex sets

This section is devoted to list lemmas proved in [1] and to study a topological structure of the intersection of $p^*(t)$-convex set and regular balls in almost all positions in R^n.

Now we proceed with some definitions. The tangent space of C^2-manifold M at a point p will be denoted by TM_p. If $g : M \to N$ is a C^2-function with $g(p) = q$, then the induced linear map of tangent spaces will be denoted by $g_* : TM_p \to TM_q$. If f is a C^2-real valued function on a manifold M

and the induced map $f_* : TM_p \to TM_{f(p)}$ is zero, then the point p is called critical point of f and $f(p)$ is called critical value of f. Let $(x_1, \ldots, x_n)$ be a local coordinate system in a neighborhood of p. A critical point p of f is called nondegenerate if the Hessian matrix $[(\partial^2 f/\partial x_i \partial x_j)(p)]$ is non-singular and then index of the critical point is defined to be the index of the Hessian matrix.

For a fixed point a in R^n, distance function d_a on a k-dimensional manifold M in R^n is defined by

(3) $d_a(x(u_1, \ldots, u_k)) = \|x(u_1, \ldots, u_k) - a\|^2$, where $(u_1, \ldots, u_k)$ is a local coordinate system on M. It is well known that for almost all a, the function d_a is a Morse function, that is, the function d_a has only nondegenerated critical points for almost all a in R^n. From (3), we have

(4) $\partial d_a/\partial u_i = 2 \cdot \partial x/\partial u_i (x - a)$. Thus d_a has a critical point q if and only if vector $q - a$ is normal to M at q. So if W is an n-manifold with C^2-boundary ∂W in R^n and q is a critical point of a distance function $d_a|_{\partial W}$ defined on ∂W from a point a in R^n different from q, then the outward normal vector at q to W is either pointing towards the origin a or escaping from the origin a. The following Lemma 1 and Lemma 2 show the only critical points of a distance function whose outward normal vectors are pointing towards the origin of the distance function play the role to decide topological behavior of the distance function near the critical points.

Lemma 1. *If W is an n-manifold with C^2-boundary ∂W in R^n, the distance function $d_a|_{\partial W}$ defined on ∂W from a point a in R^n is a Morse function and r_1 is a positive critical value of $d_a|_{\partial W}$ such that all the critical points in the level r_1 are the critical points whose outward normal vectors are escaping from the origin a and $(d_a|_{\partial W})^{-1}[r, r_1)$ contains no critical points, then $H_i(d_a|_W{}^{-1}[0, r_1]) \cong H_i((d_a|_W)^{-1}[0, r])$ for all i, $i \geq 0$, where H_i is the i-dimensional homology group with integer coefficients.*

Proof. see [1].

Lemma 2. *Let W be a compact n-manifold with C^2-boundary ∂W in R^n and $H_i(W) = 0$ for all i, $i \geq p$. Let d be a distance function such that $d|_{\partial W}$ is a Morse function. If $H_i(d^{-1}[0, {r_0}^2]) \neq 0$ for some $r_0 > 0$, and for some $i \geq p$ and if we set $r_1 = \inf\{r \mid H_i(d^{-1}[0, r^2]) = 0$ for all $i \geq p$ and for all $r \geq r_0\}$, then ${r_1}^2$ is a critical value of $d|_{\partial W}$ and $H_i(d^{-1}[0, {r_1}^2]) = 0$ for all i, $i \geq p$.*

Proof. see [1].

Now we generalize the concepts of convexity in the following:

Definition 3. An n-manifold W with C^2-boundary ∂W in R^n is called $p^*(t)$-convex if each point on ∂W has at most p principal normal curvatures less than $-\frac{1}{t}$ (with respect to the inward normal vector) and homology groups $H_i(W)$ of W of dimension i greater than $p-1$ vanish.

Theorem 4. *Let W be a compact n-manifold with C^2-boundary ∂W in R^n. Then W is $p^*(t)$-convex only if for almost all x in R^n, $H_i(d_x|_W{}^{-1}[0,r^2]) = 0$ for all $i, i \geq p$, and for all r less than (t-diameter of W).*

Proof. To show the sufficiency let A be the set of all focal points of ∂W in R^n. Then it is easy result of Sard theory that A has measure zero. Now suppose that there is a point x_0 in A such that $(H_{i_0}(d_{x_0}|_W{}^{-1}\ [0,{r_0}^2]) \neq 0$ for some integer i_0 greater than $p-1$, and for some r_0 less than (t-diameter of W). Then the distance function $d_{x_0}|_{\partial W}$ is a Morse function. If we set $r_1 = \inf\{r|H_i(d_{x_0}|_W{}^{-1}[0,r^2]) = 0$ for all $i \geq p$ and for all $r \geq r_0\}$, then r_1 is less than (r_0+ diameter of W). By Lemma 2, ${r_1}^2$ is a critical value of $d_{x_0}|_{\partial W}$ and $H_i(d_{x_0}|_W{}^{-1}[0,{r_1}^2]) = 0$ for all $i \geq p$. Let $p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_m$ be the critical points in $d_{x_0}|_{\partial W}{}^{-1}\{{r_1}^2\}$ such that $p_1, p_2, \ldots, p_k$ are critical points whose outward normal vectors are pointing towards x_0 and $p_{k+1}, \ldots, p_m$ are critical points whose outward normal vectors are escaping from x_0. The construct new n-manifold W^* with C^2-boundary ∂W^* on which $d_{x_0}|_{\partial W^*}$ is a Morse function and $d_{x_0}|_{\partial W^*}{}^{-1}[{r_1}^2 - \varepsilon, {r_1}^2]$ contains no critical points for all sufficiently small ε. Let $U_{k+1}, \ldots, U_m$ be the sufficiently small disjoint neighborhoods of the critical points $p_{k+1}, \ldots, p_m$ in ∂W, respectively. Then we will construct W^* from W by perturbing W in those neighborhoods in the outward normal directions. Assume that x_0 is the origin of R^n and define W^* as follows:

$$W^* = W \cup \{y \in R^n \,|\, y = x + t\cdot\varepsilon_i\cdot\lambda_i(x)\cdot\frac{x}{\|x\|},\ x \in \partial W,\\ 0 \leq t \leq 1,\ \max\varepsilon_i \to 0\},$$

where the C^2-map λ_i defined on ∂W has been chosen to have support in U_i and equals 1 on some neighborhood V_i of p_i contained in U_i and $\varepsilon_i > 0$ is constant on U_i. Then W^* is an n-manifold with C^2-boundary ∂W^* and then the distance function $d_{x_0}|_{\partial W^*}$ is a C^2-function defined by $d_{x_0}|_{\partial W^*}(y) = d_{x_0}|_{\partial W}(x) + {\varepsilon_i}^2\cdot\lambda_i^2(x) + 2\cdot\varepsilon_i\cdot\lambda_i(x)\cdot[d_{x_0}|_{\partial W}(x)]^{1/2}$. Since $d_{x_0}|_{\partial W}$ is a Morse function, $d_{x_0}|_{\partial W^*}$ is also Morse function. Also critical points $p_1, p_2, \ldots, p_k$ are nondegenerated critical points since $d_{x_0}|_{\partial W^*} = d_{x_0}|_{\partial W}$ around the points. Since by the construction of W^*, set $d_{x_0}|_{\partial W^*}{}^{-1}[0,{r_1}^2]$ is deformation retraction of $d_{x_0}|_{\partial W}{}^{-1}[0,{r_1}^2]$ for each nonnegative real number r,

$$r_1 = \inf\{r \,|\, H_i(d_{x_0}|_{W^*}{}^{-1}[0,r^2]) = 0 \text{ for all } i \geq p \text{ and for all } r \geq r_0\}.$$

This implies that for some integer i greater than $p-1$ and for all sufficiently small positive real number ε, $H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \neq 0$.

On the other hand, the set $d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2]$ has the homotopy type of $d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon]$ with cells $e^{\lambda(1)},\dots,e^{\lambda(k)}$ attached, where $\lambda(i)$ is the dimension of the cell which corresponds to the critical point p_i. The critical points are critical points whose outward normal vectors are escaping from x_0 and those are the only critical points of $d_{x_0}|_{\partial \overline{W^{*c}}}$ in the level $r_1{}^2$ where $\overline{W^{*c}}_{is}$ the closure of the complement of W^*. Since $H_i(d_{x_0}|_{\partial\overline{W^{*c}}}{}^{-1}[0,r_1{}^2-\varepsilon]) \cong H_i(d_{x_0}|_{\partial\overline{W^{*c}}}{}^{-1}[0,r_1{}^2])$ for all sufficiently small positive number ε and for all $i, i \geq 0$ by Lemma 1, using the Mayer-Vietoris exact homology sequence, we have the following relations:

$$\begin{aligned}(5)\qquad H_i(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) &\cong H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \\ &\oplus H_i(d_{x_0}|_{\overline{W^{*c}}}{}^{-1}[0,r_1{}^2-\varepsilon]) \quad \text{and} \\ H_i(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2+\varepsilon]) &\cong H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2+\varepsilon]) \\ &\oplus H_i(d_{x_0}|_{\overline{W^{*c}}}{}^{-1}[0,r_1{}^2+\varepsilon]) \quad \text{for all } i,\ i \geq 0.\end{aligned}$$

Now consider the following exact homology sequence with the sufficiently small positive number ε such that $H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \neq 0$ for some i, $i \geq p$:

(6)
$$\begin{aligned}\cdots &\to H_{i+1}(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2+\varepsilon], d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \\ &\to H_i(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \to H_i(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2+\varepsilon]) \to \cdots\end{aligned}$$

Since $H_i(d_{x_0}|_{\overline{\partial W^*}}{}^{-1}[0,r_1{}^2+\varepsilon]) \cong H_i(d_{x_0}|_{\overline{W^{*c}}}{}^{-1}[0,r_1{}^2+\varepsilon])$, $H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \neq 0$ and $H_i(d_{x_0}|_{W^*}{}^{-1}[0,r_1{}^2+\varepsilon]) = 0$, the ker of h is nontrivial. Exactness of the homology sequence (6) shows that the image of f is nontrivial, that is, $H_{l+1}(d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon] \cup e^{\lambda(1)} \cup \cdots \cup e^{\lambda(k)}, d_{x_0}|_{\partial W^*}{}^{-1}[0,r_1{}^2-\varepsilon]) \neq 0$ for some l, $l \geq p$. This implies that at least one of the $\{\lambda(j)\,|\,j=1,\dots k\}$ is $l+1$ for some l, $l \geq p$. Therefore we have a critical point q of $d_{x_0}|_{\partial W}$ with index greater than l, $l \geq p$, and on which outward normal vector is pointing towards x_0. Morse index theorem [6] for the distance function tells that there are $l+1$ principal normal curvatures $\kappa_{i_1},\dots,\kappa_{i_{l+1}}$ such that $\vec{q}+\kappa_{i_1}{}^{-1}\vec{v} = t_1(\vec{x_0}-\vec{q})+\vec{q},\cdots,\vec{q}+\kappa_{i_{l+1}}{}^{-1}\vec{v} = t_{l+1}(\vec{x_0}-\vec{q})+\vec{q}$ where $0 < t_1,\dots,t_{l+1} < 1$ and $\vec{v}$ is the unit inward normal vector at the point q. So $\kappa_{i_1} = \frac{1}{t_1 r_1} < -\frac{1}{r_1} < -\frac{1}{r_0+\text{diameter of } W} < -\frac{1}{t},\cdots,\kappa_{i_{l+1}} = -\frac{1}{t_{l+1}r_1} < -\frac{1}{r_1} < -\frac{1}{r_0+\text{diameter of } W} < -\frac{1}{t}$. Therefore we have a point q at which at least $p+1$ principal normal curvatures which are less than $-\frac{1}{t}$. This contradicts that W is a $p^*(t)$-convex set in R^n.

§III. An isoperimetric inequality for $1^*(\infty)$-convex set

In this section, we use the characterization studied in section II to obtain a geometric inequality for $1^*(\infty)$-convex set in n-space.

Theorem 5. *If W is $1^*(\infty)$-convex set in n-space, then*

$$V(W_r) \leq V(W) + (n \cdot O_{n-1})^{-1} \sum_{i=0}^{n-2} {}_nC_{i+1} \cdot M_i{}^*(W) \cdot r^{i+1} + M_{n-2}{}^*(W) \cdot r^{n-1} + V(U) \cdot r^n$$

for all nonnegative real number r, where O_{n-1} is the surface area of the $(n-1)$-dimensional unit ball, ${}_nC_{i+1} = n!/[(i+1)!(n-i-1)!]$, $M_i{}^*(W)$ is the i-th integral of mean curvature of boundary of W and U is the n-dimensional unit ball.

Proof. By Theorem 4, if $(d_x|_W{}^{-1}[0, r^2])$ is nonempty, then the Euler characteristic $\chi(d_x|_W{}^{-1}[0, r^2])$ of $(d_x|_W{}^{-1}[0, r^2])$ is greater than or equal to 1 for almost all position x and all nonnegative real number r. Note that $(d_x|_W{}^{-1}[0, r^2]) = W \cap (x + D_r)$. So by the "kinematic fundamental formula in n-space," [2], we have

$$\begin{aligned} O_1 \cdot O_2 \cdots O_{n-1} V(W_r) &= \int_{W \cap (x+D_r) \neq \emptyset} dk \\ &\leq \int_{W \cap (x+D_r) \neq \emptyset} \chi(W \cap (x + D_r)) dk \\ &= O_1 \cdot O_2 \cdots O_{n-2}[O_{n-1} V(D_r) + O_{n-1} V(W) \\ &\quad + (1/n) \sum_{i=0}^{n-2} {}_nC_{i+1} \cdot M_i{}^*(W) \cdot M_{n-i-2}{}^*(D_r)], \end{aligned}$$

where O_i denotes the surface area of the i-dimensional unit ball. So we have

$$V(W_r) \leq V(W) + (n \cdot O_{n-1})^{-1} \sum_{i=0}^{n-2} {}_nC_{i+1} \cdot M_i{}^*(W) \cdot r^{i+1} + M_{n-2}{}^*(W) \cdot r^{n-1} + V(U) \cdot r^n \tag{7}$$

for all nonnegative real number.

Theorem 5. *If W is a $1^*(\infty)$-convex set in n-space, then*

$$M_{n-2}^*(W) \geq n \cdot [V(U)]^{(n-1)/n}[V(W)]^{1/n}.$$

Proof. The Brun-Minkowski inequality [7] states

$$(8) \qquad V(W_r) \geq V(W) + [\sum_{i=1}^{n-2} {}_nC_i \cdot V(W)^{(n-1)/n} V(D_r)^{1/n})] + n \cdot V(W)^{1/n} V(U)^{(n-1)/n} r^{n-1} + V(D_r).$$

From (7) and (8), if r tends to ∞, then we have the desired result.

Theorem 6. *If W is a $1^*(\infty)$-convex set and K is a convex subset of W, then*

$$M_{n-2}^*(W) \geq M_{n-2}^*(K).$$

Proof. This follows from the fact $V(K_r) \geq V(W_r)$ and Theorem 5 by letting r to ∞.

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THE STABILITY OF COMPLETE NONCOMPACT SURFACES WITH CONSTANT MEAN CURVATURE

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Let M be a complete, noncompact, orientable surface and N be a simply connected three dimensional Riemannian manifold whose sectional curvature is nonpositive and lower bounded by a negative constant $-a^2$. It is shown that if the immersion of M in N has constant mean curvature H with $|H| > |a|$, it cannot be a stable immersion.

1. Introduction

Let M be an orientable surface and N be a simply connected three dimensional Riemannian manifold. For a compact domain D of M, let F_D be the set of all piecewise smooth functions $f : M \to R$ with compact support in D satisfying the constraint $\int f dM = 0$. We say that an immersion $\phi : M \to N$ with constant mean curvature H is *stable* if the inequality

$$(*) \qquad \int |\nabla f|^2 dM \geq \int (Ric(\eta) + |B|^2) f^2 dM$$

holds for all $f \in F_D$ where η is the unit normal vector of M in N, $Ric(\eta)$ is the Ricci curvature of N in the direction of η and $|B|^2$ is the square of the norm of the second fundamental form B of M in N. Several years ago, da Silveira showed that, in $\mathbf{R}^3$, there is no stable immersion of complete, noncompact orientable surface with nonzero constant mean curvature [1]. An observation of his proof gives the following result.

Theorem. *Let M be a complete, noncompact, orientable surface. Assume that the sectional curvature K_N of N satisfies $-a^2 \leq K_N \leq 0$. If the immersion $\phi : M \to N$ with constant mean curvature H is stable, then $|H| \leq |a|$.*

In the same paper, he also proved that for $N = \mathbf{H}^3$, the hyperbolic space, we must have $|H| \leq 1$ and a horosphere is stable with $|H| = 1$. So, our result is sharp.

2. Proof

We begin with the following Proposition. The proof is known (cf. (3.1) of [1]), but we think our proof is more straightforward.

Proposition. *Under the same hypothesis, the index of L on M is at most one.*

Proof. Let f be the second (Dirichlet) eigenfunction of L on a compact domain D of M. Since L is an elliptic operator, the number of nodal domains of f is exactly two by the Courant's nodal domain theorem. Set $D_1 = \{x \in D; f(x) > 0\}$, $D_2 = \{x \in D; f(x) < 0\}$ and set $f_1 = max\{f, 0\}$, $f_2 = min\{f, 0\}$ and define $g = af_1 - bf_2$ where $a = \int_D f_2 dM$, $b = \int_D f_1 dM$. Then g satisfies the constraint condition $\int g dM = 0$ and (*) implies

$$a^2 \int_D (|\nabla f_1|^2 - (Ric(\eta)+|B|^2)f_1^2)dM$$

$$+ b^2 \int_D (|\nabla f_2|^2 - (Ric(\eta) + |B|^2)f_2^2)dM \geq 0. \tag{1}$$

Since $\lambda_1(D_i) = \lambda_2(D)$ and f_i is the first eigenfunction of D_i, respectively, $i = 1, 2$, we have

$$\int_D (|\nabla f_i|^2 - (Ric(\eta) + |B|^2)f_i^2)dM = \lambda_2(D) \int_D f_i^2 dM \tag{2}$$

for $i = 1, 2$. Now (1) implies $\lambda_2(D) \geq 0$. □

Let K be the Gaussian curvature of M. Then the Gauss curvature equation gives

$$K_{12} - K = h_{12}^2 - h_{11}h_{22} \tag{3}$$

$$|B|^2 = 4H^2 + 2(K_{12} - K) \tag{4}$$

where K_{12} is the sectional curvature of N for the section determined by the orthonormal basis e_1, and e_2 of the tangent plane of M and h_{ij} is defined by the equation $B(e_i, e_j) = h_{ij}\eta, i, j = 1, 2$. Then the elliptic operator $L = \Delta + Ric(\eta) + |B|^2$ associated with the stability inequality (*) can be written as

$$L = \Delta + Ric(\eta) + 3H^2 + K_{12} + (H^2 + K_{12} - K) - K.$$

Since

$$H^2 + K_{12} - K = (\frac{h_{11} + h_{22}}{2})^2 + h_{12}^2 - h_{11}h_{22} = (\frac{h_{11} - h_{22}}{2})^2 + h_{12}^2 \geq 0,$$

if we assume $|H| > a$, (3),(4) and the curvature assumption on N give

$$Ric(\eta) + 3H^2 + K_{12} + (H^2 + K_{12} - K) \geq 3(H^2 - a^2) > 0.$$

By Theorem 1.5 of [1], this inequality and Proposition imply that

$$3(H^2 - a^2) \int_M dM \leq \int_M (Ric(\eta) + |B|^2 + K)dM < \infty.$$

Since $3(H^2 - a^2)$ is a positive constant, this implies that the area of M is finite, which is a contradiction to Theorem 1.7 of [1]. This completes the proof of our Theorem.

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ON THE ELLIPTIC EQUATION $\frac{4(n-1)}{n-2}\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ AND THE CONFORMAL DEFORMATION OF RIEMANNIAN METRICS

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§1. Introduction

In this paper, we prove some existence theorem for positive solutions of the elliptic nonlinear partial differential equation arising from conformal deformation of Riemannian metrics.

On compact manifolds of dimension $n(\geq 3)$ and with metric g_0, the problem of conformal deformation of metric is to find conditions on the function $K(x)$ so that $K(x)$ is the scalar curvature of a conformally related metric $g_1 = u^{4/(n-2)}g_0$, where u is a positive function on M.

If M admits $k \equiv 0$ as the scalar curvature of g_0, then this is equivalent to the problem of solving the elliptic equation

$$(1) \qquad \frac{4(n-1)}{n-2}\Delta u + Ku^{\frac{n+2}{n-2}} = 0, \quad u > 0,$$

where Δ is the Laplacian in the g_0 metric (See [K.W1,2], [A] or [N]).

Although throughout this paper we will assume that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs we need only assume that the curvature candidate $K(x)$ is Hölder continuous. In this case, the resulting metric with curvature $K(x)$ has Hölder continuous second derivatives.

J. L. Kazdan and F. W. Warner ([K.W.1]) have shown that there are topological obstructions to zero scalar curvature. In fact, if M is a compact spin manifold with A genus not zero and with first Betti number not zero, then M does not admit a metric of zero scalar curvature. But they have studied the necessary conditions of the solvability of (1), that is, K changes sign and $\overline{K} < 0$ (See [K.W.1]) and they conjectured that these two necessary conditions on $K(x)$ would be sufficient, much as in Theorem 5.3 of [K.W.3].

In this paper, we shall have another new necessary condition (In fact, because of the Remark in Theorem 6, this necessary condition may be omitted, so we can see that their conjecture is right) and prove that if K satisfies our necessary conditions, then there exists a solution of (1). For basic existence theorems, we use the method of upper and lower solutions (See [K.W.1] or [C.H.], pp.370–371).

It turns out that (1) is easier to analyze if we free it from geometry and consider instead

$$\Delta u + Hu^a = 0, \quad u > 0, \tag{2}$$

where H is an arbitrary function and $a > 1$ is a constant.

§2. Preliminaries on $\Delta u + Hu^a = 0$

Let M be a compact connected n-dimensional manifold, which is not necessarily orientable and possesses a given Riemannian structure g. We denote the volume element of this metric by dV, the gradient by ∇, and the associated Laplacian by Δ. The mean value of a function f on M is written $\overline{f}$, that is,

$$\overline{f} = \frac{1}{\mathrm{vol}(M)} \int_M f\, dV.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in $L_p(M)$. The norm on $H_{s,p}(M)$ will be denoted by $\| \ \|_{s,p}$. The usual norm $L_2(M)$ inner product will be written $\| \ \|$.

Lemma 1. *Let (M, g) be a compact Riemannian manifold. There exists a weak solution $w \in H_{1,2}(M)$ of $\Delta w = f$ if and only if $\overline{f} = 0$. The solution w is unique up to a constant. Moreover, if f is smooth, then w is also smooth.*

Proof. See Theorem 4.7 in [A].

Lemma 2. *Let $H \in L_p(M)$ for some $p > n = \dim M$. If there exist function $u_+, u_- \in H_{2,p}(M)$ such that*

$$\Delta u_+ + Hu_+^a \le 0, \quad \Delta u_- + Hu_-^a \ge 0,$$

with $0 < u_- \le u_+$, then there is a $u \in H_{2,p}(M)$ satisfying (2) and $u_- \le u \le u_+$. Moreover, u is smooth in any open set in which H is smooth.

Proof. For detail, see Lemma 9.3 in [K.W.3] or Lemma 2.6 in [K. W.1] or a standard argument in pp. 370–371 in [C.H].

Here u_+ and u_- are called upper and lower (or super and sub) solutions of (2), respectively.

Lemma 3. *If a positive solution u of (2) exists and $H \not\equiv 0$, then H must change sign and $\overline{H} < 0$.*

Proof. See Lemma 2.5 and Proposition 5.3 in [K.W.1].

Lemma 4. *If (2) has a positive solution for given H and if $H_1 = mH$ for some constant $m > 0$, then (2) has a positive solution for H_1.*

Proof. If u is a solution of (2) for H, then $m^{-1/(a-1)}u$ is a solution of (2) for $H_1 = mH$.

Theorem 5. [Existence of an upper (weak) solution] *Let $H(\not\equiv 0)$ belong to $C^\infty(M)$ such that H changes sign and $\overline{H} < 0$. Then there exists an upper solution $u_+ > 0$ of (2), that is,*

$$\Delta u_+ + Hu_+^a \leq 0.$$

Proof. Taking the change of variable $u_+ = e^v$,

$$\Delta u_+ + Hu_+^a = e^v(\Delta v + |\nabla v|^2 + He^{av} \leq 0.$$

Hence it is sufficient to find v satisfying

$$\Delta v + |\nabla v|^2 + He^{cv} \leq 0, \tag{3}$$

where $c = a - 1 > 0$ is a constant.

But Lemma 1 implies that there exists a solution w of $\Delta w = \overline{H} - H$. We can pick $b > 0$ so small that $|e^{cbw} - 1| \leq -\overline{H}/(4\|H\|_\infty)$ and $b|\nabla w|^2 < -\overline{H}/4$. Let $e^{cr} = b$. Put $v = bw + r$. Then

$$\begin{aligned}
\Delta v + |\nabla v|^2 + He^{cv} &= \Delta(bw + r) + |\nabla(bw + r)|^2 + He^{cbw+cr} \\
&= b\Delta w + b^2|\nabla w|^2 + bHe^{cbw} \\
&= b\overline{H} + b^2|\nabla w|^2 + bH(e^{cbw} - 1) \\
&\leq b\overline{H} + b^2|\nabla w|^2 + b\|H\|_\infty |e^{cbw} - 1| \\
&\leq b\overline{H} - b\overline{H}/4 - b\overline{H}/4 = b\overline{H}/2 < 0.
\end{aligned}$$

Thus $u_+ = e^v = e^{bw+r}$ is an upper (weak) solution of (2).

From the above theorem, if $\overline{H} < 0$, then we can always have an upper solution of (2). Hence in order to show that (2) has a solution, it suffices to find a lower (weak) solution u_- such that $0 < u_- < u_+$ and

$$\Delta u_- + Hu_-^a \geq 0.$$

§3. Scalar curvatures on compact manifolds

In this section, we assume that M is a compact connected $n(\geq 3)$- dimensional manifold which is not necessarily orientable and has a given Riemannian structure g.

We consider the first eigenvalue of $Lu = -\Delta u - Hu$, that is,

$$\lambda_1 = \inf_{v\not\equiv 0, v\in H_{1,2(M)}} (\|\nabla v\|^2 - \int Hv^2 dV)/\|v\|^2$$
$$= \inf(\|\nabla v\|^2 - \int Hv^2 dV) \quad \text{on } \{v \in H_{1,2}(M),\ \|v\|^2 = 1\}.$$

Note that the eigenfunction is never zero and smooth. In fact, since $|\nabla v| = |\nabla |v||$ almost everywhere (See Proposition 3.69 in [A]), the variational characterization of λ_1 shows that one can take $v \geq 0$, while the strong maximum principle shows that $v > 0$. Thus the eigenspace has dimension 1 and we can assume that the eigenfunction is positive.

Now we prove another new necessary condition and prove that these necessary conditions are also sufficient conditions for the sovability of (2).

Theroem 6. *If a solution u of (2) exists, then the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for some $m > 0$.*

Proof. Let u_1 be a solution of (2) for H. By Lemma 4 there exists a large number $M > 0$ such that $0 < u = M^{-\frac{1}{a-1}} u_1 < 1$ and

$$\Delta u + MHu^a = 0.$$

Choose $1 < q < a$ and put $p = (q-a)/(a-1)$. Then $-1 < p < 0$. By the change of variable $v = u^{1/(p+1)}$, i.e., $u = v^{(p+1)}$,

$$\Delta v + (p|\nabla v|^2)/v + M(p+1)^{-1} Hv^q = 0.$$

By multiflying v^{2-q} and integrating this equation,

$$-\int (2-q) v^{1-q|\nabla v|^2} dV + p\int |\nabla v|^2 v^{1-q} dV$$
$$+ M(p+1)^{-1} \int Hv^2\, dV = 0,$$
$$\int (p-2+q)|\nabla v|^2 v^{1-q}\, dV + M(p+1)^{-1} \int Hv^2\, dV = 0.$$

Since $0 < u < 1$, so $0 < v < 1$. Thus

$$\begin{aligned} 0 &= -\int (p-2+q)|\nabla v|^2 v^{1-q}\, dV - M(p+1)^{-1}\int Hv^2\, dV \\ &\geq \int |\nabla v|^2\, dV - m\int Hv^2\, dV, \end{aligned}$$

where $m = M(p+1)^{-1} > 0$ and q is so close to 1 that $2-p-q$ is greater than 1. Hence the variational characterization of λ_1 shows that $\lambda_1 < 0$, that is,

$$\begin{aligned} -\Delta f - mHf &= \lambda_1 f, \quad f > 0, \\ \Delta f + mHf &= -\lambda_1 f, \quad f > 0, \end{aligned}$$

where f is an eigenfunction.

Remark. We may consider the following fact instead of Theorem 6, that is, if H changes sign, then the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for some large $m > 0$. In fact, since H changes sign and M is a compact manifold, there exists a smooth nonnegative function u on M such that u is positive on some open ball in $\{x \in M \mid H(x) > 0\}$ and $u = 0$ otherwise. Then, for sufficiently large $m > 0$, $\|\nabla u\|^2 - m\int Hu^2 dV < 0$. Thus the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for sufficiently large $m > 0$. Therefore, by the following Theorem 7, we can see that Kazdan and Warner's conjecture is right.

Theorem 7. *If the first eigenvalue of $Lu = -\Delta u - mHu$ is negative for some $m > 0$, then there exists a solution of (2) for mH, so by Lemma 4, there exists a solution of (2) for H.*

Proof. Step 1. Since $m\overline{H} < 0$, Theorem 5 implies that there exists an upper solution u_+ of (2) for mH.

Step 2. Now we have only to show that there exists a lower solution $0 < u_- < u_+$ of (2) for mH. Like the case of the existence of an upper solution of (2), we consider the equation (3) for mH instead of (2), that is, we show that there exists v such that $e^v \leq u_+$ and

$$\Delta v + |\nabla v|^2 + mHe^{cv} \geq 0.$$

Let $f > 0$ be a corresponding eigenfunction of L, that is,

$$\Delta f + mHf = -\lambda_1 f, \quad f > 0. \tag{4}$$

Since cf is also an eigenfunction of (4), we can assume that $f > 1$. Now put $v = b(f^r - r^r)^{1+1/\sqrt{r}} + t$, where r is a sufficiently small positive real number and b and t are chosen suitably so that our conditions are satisfied. Then

$$\nabla v = b(r+\sqrt{r})(f^r - r^r)^{1/\sqrt{r}} f^{r-1}\nabla f$$

$$\Delta v = b(r+\sqrt{r})(f^r - r^r)^{\frac{1}{\sqrt{r}}} f^{r-1}[\Delta f + \{\frac{r-1}{f} + \frac{\sqrt{r}\, f^r}{(f^r - r^r)f}\}|\nabla f|^2].$$

Since $\frac{f^r - r^r}{\sqrt{r}} = \frac{f^r \log f - r^r(\log r + 1)}{1/(2\sqrt{r})} = 2\sqrt{r}[f^r \log f - r^r(\log r + 1)] \to +0$ as $r \to +0$, $\frac{r-1}{f} + \frac{\sqrt{r}\, f^r}{(f^r - r^r)f} > 0$ for sufficiently small positive real number r. Hence for $e^{ct} = b(r+\sqrt{r})(1-r^r)^{1/\sqrt{r}}$

$$\begin{aligned}
&\Delta v + |\nabla v|^2 + mHe^{cv}\\
&\geq \Delta v + mHe^{cv}\\
&= b(r+\sqrt{r})(f^r - r^r)^{\frac{1}{\sqrt{r}}} f^{r-1}[\Delta f + \{\frac{r-1}{f} + \frac{\sqrt{r}\, f^r}{(f^r - r^r)f}\}|\nabla f|^2\\
&\quad + mHe^{cb(f^r - r^r)^{1+\frac{1}{\sqrt{r}}}} f^{1-r}\{\frac{1-r^r}{f^r - r^r}\}^{1/\sqrt{r}}]\\
&\geq b(r+\sqrt{r})(f^r - r^r)^{\frac{1}{\sqrt{r}}} f^{r-1}[-\lambda_1 f\\
&\quad + mHf\{e^{cb(f^r - r^r)^{1+\frac{1}{\sqrt{r}}}} f^{-r}(\frac{1-r^r}{f^r - r^r})^{1/\sqrt{r}} - 1\}].
\end{aligned}$$

For sufficiently small $r > 0$, $(\frac{1-r^r}{f^r - r^r})^{1/\sqrt{r}} \to 1$ as $r \to +0$ (Note the Remark). Therefore, pick $b > 0$ so small that $0 < e^v < u_+$ and

$$|e^{cb(f^r - r^r)^{1+\frac{1}{\sqrt{r}}}} f^{-r}(\frac{1-r^r}{f^r - r^r})^{1/\sqrt{r}} - 1| < \frac{-\lambda_1}{2m\|H\|_\infty}.$$

Then $u_- = e^v$ is our desired lower solution of (2).

Remark. Put $y = \{(1-r^r)/(f^r - f^r)\}^{1/\sqrt{r}}$. Since $r^r \to 1$ as $r \to +0$, L'Hospital's theorem implies that

$$\frac{1-r^r}{f^r - r^r} = \frac{-r_1{}^{r_1}(\log r_1 + 1)}{f^{r_1}\log f - r_1{}^{r_1}(\log r_1 + 1)} \quad \text{for some } 0 < r_1 < r.$$

Applying the theorem once more,

$$\frac{1-r^r}{f^r - r^r} = \frac{-r_2{}^{r_2} r_2(\log r_2 + 1)^2 - r_2{}^{r_2}}{f^{r_2} r_2(\log f)^2 - r_2{}^{r_2} r_2(\log r_2 + 1)^2 - r_2{}^{r_2}} \to 1$$

for some $0 < r_2 < r_1 < r$. Then

$$\begin{aligned}
&\lim |\log y| \\
&= \lim_{r\to 0}\left|\frac{1}{\sqrt{r}}\log\frac{-r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}{f^{r_2}r_2(\log f)^2 - r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}\right| \\
&\le \lim_{r_2\to 0}\left|\frac{1}{\sqrt{r_2}}\log\frac{-r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}{f^{r_2}r_2(\log f)^2 - r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}\right| \\
&= \lim_{r_2\to 0} 2\sqrt{r_2}\left|\frac{-r_2{}^{r_2}r_2(\log r_2+1)^3 - r_2{}^{r_2}(\log r_2+1)^2 - 3r_2{}^{r_2}(\log r_2+1)}{-r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}\right. \\
&\quad - \frac{f^{r_2}r_2(\log f)^3 + f^{r_2}(\log f)^2 - r_2{}^{r_2}r_2(\log r_2+1)^3}{f^{r_2}r_2(\log f)^2 - r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}} \\
&\quad \left. + \frac{r_2{}^{r_2}(\log r_2+1)^2 + 3r_2{}^{r_2}(\log r_2+1)}{f^{r_2}r_2(\log f)^2 - r_2{}^{r_2}r_2(\log r_2+1)^2 - r_2{}^{r_2}}\right| \\
&= 0 \quad \text{because} \quad \sqrt{r_2}(\log r_2+1)^2 \to 0 \text{ as } r_2 \to 0.
\end{aligned}$$

Added in proof. [1]. In [K.W.3], J.L.Kazdan and F.W.Warner conjectured that there exists a solution of

$$\Delta v + He^v = 0 \tag{5}$$

if $H(\neq 0)$ changes sign and $\overline{H} < 0$ on a compact manifold with dimension $n(\geq 3)$. The equation (5) is related to the problem of pointwise conformal deformation of metrics on two dimensional compact connected manifolds with zero curvature (For details, see [K.W.3]). They have studied the necessary and sufficient conditions of the solvability of (5) on the two dimensional compact connected manifolds, that is, a solution of (4) exists if and only if both $\overline{H} < 0$ and H changes sign (Here their proofs for sufficient conditions depend on the dimension of the given manifold). These necessary conditions must still be satisfied in the $n(\geq 3)$-dimensional case, too. They conjectured that these two necessary conditions on H for the solvability of (5) on M of dimension $n(\geq 3)$ would be sufficient, much as in Theroem 5.3 of [K.W.3] (Also see some open problems in [K], p.47). But like the proof of Theorem 7, we can prove the existence of a solution of (5) by the method of upper and lower solutions, regardless of the dimension of the given manifold.

[2]. When the given manifold admits zero scalar curvature, by the proofs of Theorem 5 and Theorem 7, we can see that there exist many conformal metrics with K as the scalar curvature if K changes sign and $\overline{K} < 0$.

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COMPLEX ANALYTIC MANIFOLDS AND VARIETIES

BOUNDARIES FOR ALGEBRAS OF ANALYTIC FUNCTIONS ON DUAL BANACH SPACES

YUN SUNG CHOI AND SUNG GUEN KIM

Introduction

Let E be a Banach space with the open unit ball $B(E)$, the closed unit ball $\bar{B}(E)$ and the unit sphere $S(E)$. We denote by $A^\infty(E)$ the Banach algebra of all bounded continuous complex-valued functions on $\bar{B}(E)$, holomorphic on $B(E)$ with sup norm, and by $A_U(E)$ the closed subalgebra of $A^\infty(E)$ which are uniformly continuous on $\bar{B}(E)$. For the dual Banach space E^* of E we denote by $A_{w^*}(E^*)$ the Banach algebra of all w^*-continuous complex-valued functions on $\bar{B}(E^*)$, holomorphic on $B(E^*)$. Each of these algebras is an infinite dimensional analogue of the classical disc algebra $A(D)$ where D is replaced by $B(E)$. In [3] it was shown that $A_U(E)$ is a proper subspace of $A^\infty(E)$ as long as E is a infinite dimensional Banach space. For general background on holomorphic functions, we refer to [4] and [8].

Let K be a Hausdorff topological space and A a function algebra on K. A subset F of K is called a boundary for A if $\sup_{x\in K}|f(x)| = \sup_{x\in F}|f(x)|$ for all $f \in A$. If the intersection of all closed boundaries for A is again a boundary for A, then it is called the *Shilov boundary* for A and denoted by ∂A. Since A is not a uniform algebra in general, the existence of the Shilov boundary for A is not guaranteed. In fact, Globevnik in [6] showed that the Shilov boundary for $A_U(C_0)$ does not exist. See also [3] for a more complete discussion of this situation.

A point $x \in K$ is called a *peak point* for A if there exists some $f \in A$ such that $f(x) = 1$ and $|f(y)| < 1$, $y \neq x$ in K.

We recall some definitions and results about a uniform algebra (see, e.g., [7]). If A is a uniform algebra on a compact Hausdorff space K, it is well known that the Shilov boundary for A exists. The set of all peak points for A is called the *Bishop boundary* for A and denoted by ρA. The set of all real extreme points of M is denoted by $Ext_{\mathbf{R}}(M)$. A point e of M is called a *complex extreme* *point* of M if there is no nonzero x in E with $\{e+\lambda x : |\lambda| \leq 1, \lambda \in \mathbf{C}\} \subset M$. The set of all complex extreme points of M is denoted by $Ext_{\mathbf{C}}(M)$.

Let A^* be the dual Banach space of A and let S_1^* be the intersection of the unit sphere $S(A^*)$ of A^* with the hyperplane $\{x^* \in A^* : x^*(1) = 1\}$. To each $x \in K$ corresponds the element $\delta_x \in S_1^*$, where $\delta_x(f) = f(x)$ for all $f \in A$. The set $\chi A = \{x \in K : \delta_x \in Ext_{\mathbf{R}}(S_1^*)\}$ is called the *Choquet boundary* for A.

It is a well-known result that if A is a uniform algebra on a metrizable compact Hausdorff space K, then $\rho A = \chi A$ and the closure of ρA is ∂A.

Aron, Choi, Lourenço and Paques [2] showed that for $1 \leq p < \infty$, $\rho A_{w^*}(\ell_p) = \chi A_{w^*}(\ell_p) = S(\ell_p)$ and $\partial A_{w^*}(\ell_p) = \bar{B}(\ell_p)$. In this article, generalizing their result on a dual Banach space E^* of a separable Banach space E, we will show that $\rho A_{w^*}(E^*) = \chi A_{w^*}(E^*) = Ext_{\mathbf{C}}(\bar{B}(E^*))$ and that $\partial A_{w^*}(E^*)$ is the weak-star closure of $Ext_{\mathbf{C}}(\bar{B}(E^*))$. Applying this result to ℓ_p $(1 \leq p < \infty)$, we will prove that $\partial A_U(\ell_p) = \partial A^\infty(\ell_p) = S(\ell_p)$, which is a different proof from that given in [2].

Main Results

Lemma 1. Let E be a separable Banach space. Then E^* is separable with respect to the weak-star topology.

Proof. Let $\{x_n\}$ be a countable dense subset of E. For each positive integer n choose $x_n^* \in E^*$ so that $\|x_n^*\| \leq 1$ and $|x_n^*(x_n)| \geq \|x_n\|/2$. Let L be the set of all finite linear combinations of the elements x_n^* with rational coefficients. Then L is countable. If the weak-star closure of L is not E^*, then there is nonzero x in E such that $x^*(x) = 0$ for all $x^* \in L$. Since $\{x_n\}$ is a dense subset of E, we can choose a sequence (x_{n_j}) from $\{x_n\}$ converging x in norm topology. From this it follows that

$$\|x_{n_j} - x\| \geq |x_{n_j}^*(x_{n_j} - x)| \geq |x_{n_j}^*(x_{n_j})| \geq \|x_{n_j}\|/2.$$

Hence $\|x_{n_j}\|$ converges to 0 and $x = 0$, which contradicts that x is nonzero. Q.E.D.

Theorem 2. Let E be a separable Banach space. Then

$$\rho A_{w^*}(E^*) = \chi A_{w^*}(E^*) = Ext_{\mathbf{C}}(\bar{B}(E^*))$$

and $\partial A_{w^*}(E^*)$ is the weak-star closure of $Ext_{\mathbf{C}}(\bar{B}(E^*))$.

Proof. Let $P(E^*)$ be the uniform algebra generated by the constants and restrictions to $\bar{B}(E^*)$ of all continuous linear functionals on (E^*, w^*). Since (E^*, w^*) is separable by Lemma 1, we obtain $\chi P(E^*) = Ext_{\mathbf{C}}(\bar{B}(E^*))$ from Arenson's result in [1]. Since $\bar{B}(E^*)$ is metrizable and compact with respect to the weak-star topology, $\rho P(E^*) = \chi P(E^*) = Ext_{\mathbf{C}}(\bar{B}(E^*)$ and $\partial P(E^*)$ is the weak star closure of $Ext_{\mathbf{C}}(\bar{B}(E^*)$. Since $\rho P(E^*) \subset \rho A_{w^*}(E^*)$, it is clear that $Ext_{\mathbf{C}}(\bar{B}(E^*)$ is contained in $\rho A_{w^*}(E^*)$. By Globevnik [5] it is easy to see that $\rho A_{w^*}(E^*) \subset Ext_{\mathbf{C}}(\bar{B}(E^*)$. Hence $\rho A_{w^*}(E^*) = \chi A_{w^*}(E^*) = Ext_{\mathbf{C}}(\bar{B}(E^*))$ and $\partial A_{w^*}(E^*)$ is the weak-star closure of $Ext_{\mathbf{C}}(\bar{B}(E^*))$. Q.E.D.

From Theorem 2 it follows that for $1 \le p < \infty$,

$$\rho A_{w^*}(\ell_p) = Ext_{\mathbf{C}}(\bar{B}(\ell_p)) = S(\ell_p),$$

which will be used in proving $\partial A_U(\ell_p) = \partial A^\infty(\ell_p) = S(\ell_p)$.

Theorem 3. Let $1 \le p < \infty$. Let P be a finite dimensional coordinate projection such that

$$P(x) = \sum_{j=1}^{\ell} x_{n_j} e_{n_j} \qquad (x = (x_j) \in \ell_p).$$

Let

$$S_P(\ell_p) = \{(x_i) \in \ell_p \ : \ \sum_{j=1}^{\ell} |x_{n_j}|^p = 1\} \ (1 \le p < \infty)$$

and

$$S_P(\ell_\infty) = \{(x_i) \in \ell_\infty \ : \ |x_{n_j}| = 1, \ j = 1, \dots, \ell\}.$$

If $D \subset \overline{B_1}(\ell_p)$ is a boundary for $A_U(\ell_p)$, then $\overline{P(D)}$ contains $S_P(\ell_p)$.

Proof. Let us consider first the case $1 \le p < \infty$. Assume that $\overline{P(D)}$ does not contain $S_P(\ell_p)$. Then there is a point $x_0 \in S_P(\ell_p) \setminus \overline{P(D)}$. Since $A_{w^*}(\ell_p) \subset A_U(\ell_p)$, there is $f \in A_U(\ell_p)$ such that $f(x_0) = 1$ and $|f(x)| < 1$ for every $x \in \overline{B}(\ell_p)$, $x \neq x_0$.

Define $F : \overline{B}(\ell_p) \to \mathbf{C}$ by

$$F(x) = f \circ P(x) \qquad (x \in \overline{B}(\ell_p)).$$

Then $F \in A_U(\ell_p)$. Since $\overline{P(D)}$ is compact and $x_0 \notin \overline{P(D)}$, there is some $\delta > 0$ such that $|f(x)| < 1 - \delta$ for all $x \in \overline{P(D)}$. Thus

$$\begin{aligned} \sup_{x\in D} |F(x)| &= \sup_{x\in D} |f \circ P(x)| \\ &\le \sup_{t\in \overline{P(D)}} |f(t)| \le 1 - \delta. \end{aligned}$$

But $F(x_0) = f(P(x_0)) = f(x_0) = 1$. Hence $\sup_{x\in D} |F(x)| < \|F\| = 1$, which implies that D is not a boundary for $A_U(\ell_p)$.

In case $p = \infty$, the proof is similar to that of Theorem 1.1 in [3]. Q.E.D.

Theorem 4. Let $1 \le p < \infty$. Then $\partial A_U(\ell_p) = \partial A^\infty(\ell_p) = S(\ell_p)$.

Proof. Suppose that D is a boundary for $A_U(\ell_p)$. Let $x = (x_j) \in S(\ell_p)$ and $\epsilon > 0$ be given. Choose a sufficiently large positive integer N satisfying

$$\left(1 - \left(1 - \frac{1}{N}\epsilon\right)^p\right)^{1/p} < \frac{N-3}{N}\epsilon,$$

and choose also a positive integer k so that

$$\left(\sum_{j=k+1}^{\infty} |x_j|^p\right)^{1/p} < \frac{1}{N}\epsilon.$$

Put $\alpha = (x^1, \ldots, x_k, (1 - \sum_{j=1}^{k} |x_j|^p)^{1/p}, 0, 0, \ldots)$. Then α is a finite vector in $S(\ell_p)$. Let P be a finite dimensional coordinate projection with support $\{1, 2, \ldots, \ell\}$, where ℓ is the smallest positive integer such that $\alpha \in S_P(\ell_p)$. Clearly $1 \le \ell \le k+1$. By Theorem 3, there is $y = (y_j) \in D$ such that $\|P(y)\|_p > 1 - \frac{1}{N}\epsilon$. It follows that

$$\begin{aligned} \|x - y\|_p &\le \|x - \alpha\|_p + \|\alpha - y\|_p \\ &\le \|\sum_{j=1}^{k} x_j e_j - \alpha\|_p + \|\sum_{j=k+1}^{\infty} x_j e_j\|_p \\ &\quad + \|P(y) - \alpha\|_p + \|\sum_{j=\ell+1}^{\infty} y_j e_j\|_p \\ &\le 2\left(\sum_{j=k+1}^{\infty} |x_j|^p\right)^{1/p} + \|P(y) - \alpha\|_p + (1 - \|P(y)\|_p^p)^{1/p} \\ &< \frac{2}{N}\epsilon + \frac{1}{N}\epsilon + \left(1 - \left(1 - \frac{\epsilon}{N}\right)^p\right)^{1/p} \\ &< \frac{3}{N}\epsilon + \frac{N-3}{N}\epsilon = \epsilon. \end{aligned}$$

Hence $x \in \overline{D}$ and we proved that $S(\ell_p) \subset \overline{D}$ if D is a boundary for $A_U(\ell_p)$. By the Maximum Modulus Theorem, $S(\ell_p)$ is also a closed boundary for both $A_U(\ell_p)$ and $A^\infty(\ell_p)$. Therefore $\partial A_U(\ell_p) = S(\ell_p)$. Since every boundary for $A^\infty(\ell_p)$ is also a boundary for $A_U(\ell_p)$, $\partial A^\infty(\ell_p) = S(\ell_p)$. Q.E.D.

It follows immediately from Theorem 4 that the converse of Theorem 3 holds for $1 \leq p < \infty$ and that D is a boundary for $A_U(\ell_p)$ if and only if it is a boundary for $A^\infty(\ell_p)$ for $1 \leq p < \infty$.

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ON THE ADJOINT LINEAR SYSTEM

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§1. Introduction

Throughout this paper, we are working on the complex number field $\mathbf{C}$.

The aim of this paper is to explain the applications of theorem A. In the surface theory, the adjoint linear system has played important roles and many tools have been developed to understand it. In a threefold, we don't have any useful tools so far. Theorem A implies that it is enough to compute the dimension of the adjoint linear system to check the birationality. We can compute, somehow, the dimension of the adjoint linear system. For example, we can get an information about $h^0(X, \mathcal{O}_X(K_X + D))$ from Euler characteristic of $|K_X + D|$ and some vanishing theorems.

We are going to show the applications of theorem A in cases of smooth threefold of general type, smooth Fano variety, and Calabi-Yau threefold.

§2. Main

Let X be a smooth projective threefold.

We denote a linear equivalence by $\sim$. Denote by $\mathrm{Div}(X)$ a free abelian group generated by the divisors on X. Denote the canonical divisor of X by K_X.

Then we say that $D \in \mathrm{Div}(X)$ is *nef* if $D \cdot C \geqq 0$ for any curve C on X, and *big* if $\kappa(D, X) = dim\ X$, where $\kappa(D, X)$ is the Kodaira dimension of D on X.

For $D \in \mathrm{Div}(X)$, $\Phi_{|D|}$ denotes the rational map associated with the complete linear system $|D|$ if $h^0(X, \mathcal{O}_X(D)) \neq 0$. Let's denote $h^0(X, \mathcal{O}_X(nD))$ by $p_n(D)$.

Theorem 1. *(Kawamata-Viehweg vanishing theorem) Let X be a nonsingular projective variety and $D \in \mathrm{Div}(X)$. If D is nef and big, then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

For a proof, see Kawamata [2].

Lemma 1. *Let X be a smooth projective threefold, and $D \in \mathrm{Div}(X)$. Then we have the following:*
(i) $\chi(\mathcal{O}_X(D)) = D^3/6 - K_X \cdot D^2/4 + D \cdot (K_X^2 + c_2)/12 + \chi(\mathcal{O}_X)$, where c_2 is the second Chern class of X. Moreover, $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24$.
(ii) $K_X \cdot D^2$ is even.

Proof. (i) is the Riemann-Roch theorem.
(ii) comes from the following:

$$\chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X(-D)) = -K_X \cdot D^2/2 + 2\chi(\mathcal{O}_X) \in \mathbf{Z}. \quad \square$$

Lemma 2. *Let X be a smooth threefold with a canonical divisor K_X. Let $D \in \mathrm{Div}(X)$.*
(i) When K_X is nef and big, $p_n(K_X) = \dfrac{n(n-1)(2n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X)$ for $n \geqq 2$.
(ii) When $-K_X$ is ample, $p_n(-K_X) = \dfrac{n(n+1)(2n+1)}{12}(-K_X^3) + (2n+1)$ for $n \geqq 1$.
(iii) When $K_X \sim 0$, and D is nef and big, $p_n(D) = \dfrac{n^3 D^3}{6} + \dfrac{nD \cdot c_2}{12}$ for $n \geqq 1$.

Proof. Suppose that $L \in \mathrm{Div}(X)$ is nef and big.

$$\begin{aligned}\chi(\mathcal{O}_X(K_X + L)) = \ & h^0(X, \mathcal{O}_X(K_X + L)) - h^1(X, \mathcal{O}_X(K_X + L)) \\ & + h^2(X, \mathcal{O}_X(K_X + L)) - h^3(X, \mathcal{O}_X(K_X + L)).\end{aligned}$$

Since L is nef and big, $h^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$ by theorem 1. Thus $\chi(\mathcal{O}_X(K_X + L)) = h^0(X, \mathcal{O}_X(K_X + L))$.
For (i), take $L = (n-1)K_X$.

$$\begin{aligned}p_n(K_X) &= h^0(X, \mathcal{O}_X(K_X + (n-1)K_X)) \\ &= \chi(\mathcal{O}_X(K_X + (n-1)K_X)).\end{aligned}$$

Then our claim follows from (i) of lemma 1.

For (ii), take $L = (n+1)(-K_X)$.

$$\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) - h^3(X, \mathcal{O}_X).$$

For $i > 0$, $h^i(X, \mathcal{O}_X) = h^{3-i}(X, \mathcal{O}_X(K_X)) = 0$ since $-K_X$ is ample. So $\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$.

$$\begin{aligned} p_n(K_X) &= h^0(K_X + (n+1)(-K_X))) \\ &= \chi(\mathcal{O}_X(K_X + (n+1)(-K_X))). \end{aligned}$$

And apply (i) of lemma 1.

For (iii), take $L = nD$. Since $K_X \sim 0$, $\chi(\mathcal{O}_X) = -c_2 \cdot K_X/24 = 0$. We will get our claim from (1) of lemma 1. □

Lemma 3. *Let X be a smooth threefold with a canonical divisor K_X. Let $D \in Div(X)$.*

(i) *When K_X is nef and big, $p_n(K_X) \geqq 4$ for $n \geqq 2$.*

(ii) *When $-K_X$ is ample, $p_n(-K_X) \geqq 4$ for $n \geqq 1$.*

(iii) *When $K_X \sim 0$, and D is nef and big, $p_n(D) \geqq 2$ for $n \geqq 2$.*

Proof. When K_X is nef, and L is nef, then $L \cdot (3c_2 - c_1^2) \geqq 0$ by the pseudo-effectivity of $3c_2 - c_1^2$ (See Miyaoka [4].)

For (i), take $L = K_X$. It follows that $\chi(\mathcal{O}_X) < 0$ from (i) of lemma 1. Since K_X^3 is a positive even integer, and $\chi(\mathcal{O}_X)) < 1$,

$$p_n(K_X) \geqq \frac{n(n-1)(2n-1)}{6} + (2n-1) \geqq 4 \qquad \text{for } n \geqq 2.$$

For (ii), $-K_X^3$ is a positive even integer since $-K_X$ is ample.

$$p_n(-K_X) \geqq \frac{n(n+1)(2n+1)}{6} + (2n+1) \geqq 4 \qquad \text{for } n \geqq 1.$$

For (iii), take $L = nD$. Since $K_X \sim 0$, and D is nef, we have $D \cdot c_2 \geqq 0$. Thus,

$$p_n(D) \geqq \frac{n^3 D^3}{6} \geqq \frac{4}{3} \qquad \text{for } n \geqq 2.$$

Hence $p_n(D) \geqq 2$ for $n \geqq 2$. □

Theorem A. *Let X be a smooth projective threefold and let D be a nef and big divisor on X. Assume that $h^0(X, \mathcal{O}_X(mD)) \geqq 2$ for some positive integer m. Then $\Phi_{|K_X+nD|}$ is birational for a positive integer $n \geqq m+4$ such that $h^0(X, \mathcal{O}_X((n-m)D)) \geqq 1$.*

For a proof, see Shin [5].

Theorem B. *Let X be a smooth projective threefold with a canonical divisor K_X and let D be a nef and big divisor on X.*

(i) *When K_X is nef and big, $\Phi_{|nK_X|}$ is birational for $n \geqq 7$. (cf. See Matsuki [3].)*

(ii) *When $-K_X$ is ample, $\Phi_{|-nK_X|}$ is birational for $n \geqq 4$.*

(iii) *When $K_X \sim 0$, $\Phi_{|nD|}$ is birational for $n \geqq 6$.*

Proof. We are going to apply theorem A to each case. So, first of all, we have to choose the number "m" in the theorem A as small as possible.

For (i), take $m = 2$. For an integer $n \geqq 7$, $\Phi_{|nK_X|} = \Phi_{|K_X+(n-1)K_X|}$ and $n-1 \geqq m+4$. $h^0(X, \mathcal{O}_X((n-1)-m)K_X)) \geqq 4$ since $n-1-m \geqq 2$. Hence theorem A implies that $\Phi_{|nK_X|}$ is birational for $n \geqq 7$.

For (ii), take $m = 1$. For an integer $n \geqq 4$, by similar way, we can show that n satisfies all the conditions in theorem A. Hence $\Phi_{|nK_X|}$ is birational for $n \geqq 4$.

For (iii), take $m = 1$. Since $K_X \sim 0$, $\Phi_{|K_X+nD|} = \Phi_{|nD|}$. For an integer $n \geqq 6$, n satisfies all the conditions in theorem A. Hence $\Phi_{|nK_X|}$ is birational for $n \geqq 6$. □

Remark. (i) Smooth threefold of general type is of the case (i) in the theorem B.

(ii) Smooth Fano threefold is of the case (ii).

(iii) Calabi-Yau threefold is of the case (iii).

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실 정함수를 차례로 미분함에 따른 근의 이동에 관한 Pólya 의 원리

김 영 원

1. 머리말

1942 년에 G. Pólya는 미국 수학회에서 행한 초청강연에서 "실 정함수 $f(z)$의 증가지수가 2보다 작으면 $f^{(n)}(z) = 0$의 근은 n이 증가함에 따라 실축으로 끌려오는 것 같으며, 증가지수가 2보다 크면 그 반대의 현상이 일어나는 것 같다"는 말을 하였다. 이 논문의 목적은 증가지수가 2보다 작은 경우에 대하여 Pólya의 직관적 원리를 뒷받침하는 몇몇 구체적 결과를 제시하는 데 있다.

우리의 결과 및 그에 대한 배경을 소개하기에 앞서 필요한 개념 및 용어를 도입하자. 우선 정함수(entire function) $f(z)$의 '증가지수(order)' $\rho = \rho(f)$를 다음과 같이 정의 한다.

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r;f)}{\log r},$$

여기서 $M(r;f) = \max_{|z|=r} |f(z)|$이다. $0 < \rho < \infty$일 때, 증가지수가 ρ인 정함수 $f(z)$의 '증가계수(type)'는

$$\tau = \overline{\lim_{r\to\infty}} \frac{\log M(r;f)}{r^\rho}$$

으로 정의한다. 정함수 $f(z)$의 증가지수 및 증가계수가 각각 ρ, τ일 때, '$f(z)$는 증가율이 (ρ, τ)인 함수'라고 말하며, $f(z)$의 증가지수가 ρ보다 작거나, 또는 증가지수가 ρ이고 증가계수가 τ를 넘지 못하는 경우 '$f(z)$의 증가율이 (ρ, τ)를 넘지 못한다'고 말한다. 따라서 $\rho > 0$, $\tau < \infty$일 때, $f(z)$의 증가율이 (ρ, τ)를 넘지 못할 필요충분조건은 모든 양수 ϵ에 대하여 $r \to \infty$에 따라

$$M(r;f) = o\left(e^{(\tau+\epsilon)r^\rho}\right)$$

이 성립하는 것이다. 정함수 $f(z)$의 증가지수가 1보다 작으면, $f(z)$는 다음과 같이 1차 인수의 곱으로 분해된다.

$$f(z) = cz^n \prod_j \left(1 - \frac{z}{a_j}\right),$$

여기서 c는 상수이고, n은 0 이상인 정수이며, $\sum_j |a_j|^{-1}$이 수렴해야 된다. 이와 같은 꼴로 표시될 수 있는 정함수를 '0 종(genus 0) 정함수'라고 부른다. 0 종 정함수 가운데는 증가율이 (1,0)인 것도 있으며, 증가율이 (1,0)을 넘는 정함수는 0 종이 될 수 없다. 0 종이 아닌 정함수 $f(z)$가 '1 종 정함수'라는 말은 $f(z)$가 다음과 같이 표현될 수 있음을 뜻한다.

$$f(z) = cz^n e^{\gamma z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j}}.$$

물론 우변의 무한곱이 절대수렴하기 위하여 $\sum_j |a_j|^{-2}$이 수렴한다는 조건이 필요하다. 증가율이 2보다 작은 모든 정함수는 0 또는 1 종이며, 1 종 정함수의 증가율은 (2,0)을 넘지 않고, (1,0)보다 작지 않다. 종(genus)의 일반적인 정의 및 종과 증가율 사이의 관계에 대하여는 [A, Chapter 5]를 참고하기 바란다.

한편 $f(z)$가 '실 정함수(real entire function)'라는 말은 $f(z)$를 원점에서 Taylor 급수로 전개했을 때 모든 Taylor 계수가 실수로 주어짐을 뜻한다. 따라서 $f(z)$가 실 정함수이면, 방정식 $f(z) = 0$의 근은 실축에 대하여 대칭으로 분포되어 있다. 앞으로는 '방정식 $f(z) = 0$의 근'을 '함수 $f(z)$의 근'이라고 줄여 부르기로 하자.

$f(z)$가 상수함수가 아닌 실 정함수라고 하자. ξ가 $f^{(n)}(z)$의 중복도 m인 실근이면서 $f^{(n-1)}(z)$의 근은 아닐 때, 즉

$$\begin{aligned} &f^{(n-1)}(\xi) \neq 0, \\ &\quad f^{(n)}(\xi) = f^{(n+1)}(\xi) = \cdots = f^{(n+m-1)}(\xi) = 0, \\ &f^{(n+m)}(\xi) \neq 0 \end{aligned}$$

이 성립할 때, k를 다음과 같이 두자.

$$k = \begin{cases} \dfrac{m}{2}, & m\text{이 짝수일 때,} \\ \dfrac{m+1}{2}, & m\text{이 홀수이고}\quad f^{(n-1)}(\xi)f^{(n+m)}(\xi) > 0\text{일 때,} \\ \dfrac{m-1}{2}, & m\text{이 홀수이고}\quad f^{(n-1)}(\xi)f^{(n+m)}(\xi) < 0\text{일 때.} \end{cases}$$

$k > 0$일 때 ξ를 '$f^{(n)}(z)$의 중복도 k인 Fourier 임계근(Fourier critical zero)' 또는 간단히 임계근이라고 부른다. $n = 1, 2, \cdots$에 대하여 중복도를 포함하여 센

$f^{(n)}(z)$의 임계근의 개수를 $K(f^{(n)})$으로 나타내며, 모든 n에 대한 $K(f^{(n)})$의 합 $\sum_{n=1}^{\infty} K(f^{(n)})$을 $K_T(f)$로 나타낸다. 한편 중복도를 고려하여 센 $f(z)$의 허근의 개수는 $Z_C(f)$로 나타낸다. ($f(z)$가 상수함수이면, $K_T(f)$ 및 $Z_C(f)$를 0으로 둔다.)

$f(z)$가 실계수 다항식이면 다음이 성립함을 쉽게 알 수 있다.

$$\text{(A)} \qquad Z_C(f) - Z_C(f') = 2K(f'),$$

$$\text{(B)} \qquad \lim_{n\to\infty} Z_C(f) = 0.$$

따라서 모든 실계수 다항식 $f(z)$에 대하여 다음 등식이 성립한다.

$$\text{(C)} \qquad Z_C(f) = 2K_T(f).$$

증가지수가 2보다 작은 실 정함수 가운데 가장 간단한 것은 실계수 다항식이므로, (A)와 (B)는 실계수 다항식에 대하여 Pólya의 원리가 성립함을 구체적으로 나타낸 것이라고 볼 수 있다. 실계수 다항식을 미분하면, 허근의 개수는 늘어나지 않고, 만약 미분함에 따라 허근의 개수가 줄어들면 줄어든 개수의 반만큼 임계근이 생기며, 충분히 미분하면 허근이 모두 없어진다. 또한 다음 절에서는, 미분함에 따라 허근이 없어지면서 생긴 임계근은 없어진 허근에 가까이 위치함을 알게 될 것이다.

우리는, 이 논문을 통하여, 실 정함수 $f(z)$의 증가지수 및 근의 범위를 적당히 제한하면, (A), (B), (C), 또는 그와 유사한 결과를 얻을 수 있음을 보이려 한다.

Pólya는 적당한 0 또는 1 종 실 정함수 $g(z)$와, 음 아닌 실수 α에 대하여, $f(z) = e^{-\alpha z^2} g(z)$의 꼴로 표시될 수 있는 함수 $f(z)$를 'function of genus 1*'라고 불렀다. 편의상 그러한 함수를 g1* 함수라고 부르자. g1* 함수로서 실근만 가지는 것을 'Laguerre–Pólya 함수'라고 부른다. 따라서 정함수 $f(z)$가 Laguerre–Pólya 함수이면, $f(z)$는 다음과 같이 표현된다.

$$\text{(1)} \qquad f(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j}},$$

여기서 n은 자연수이며, α는 0보다 작지 않은 실수이고, c, β 및 모든 a_j는 실수로서 $\sum_j |a_j|^{-2}$이 수렴해야 된다. 모든 Laguerre–Pólya 함수들로 이루어진 집합을 $\mathcal{LP}$로 나타내자. 이러한 함수를 Laguerre–Pólya 함수라고 부르는 까닭은 정함수 $f(z)$가 (1)과 같은 꼴로 표현된다는 말과 "실근만을 가진 실계수 다항식들로 이루어진 함수열 $\{P_n(z)\}$가 존재하여 모든 유계집합에서 $\{P_n(z)\}$가 $f(z)$로 평등수렴한다"는 말이 서로 동치라는 사실을 Laguerre와 Pólya가 증명하였기 때문이다. 이제 적당한 실계수 다항식 $P(z)$와 적당한 Laguerre–Pólya 함수 $f(z)$의 곱으로 표현할 수 있는 정함수들로 이루어진 집합을 $\mathcal{LP}^*$로 나타내자. 당연히 $\mathcal{LP}^*$는 g1* 함수 가운데 허근을 유한 개만 가지는 것들로 이루어진 집합이며, $\mathcal{LP}$는 $\mathcal{LP}^*$에 속하는 함수 가운데 실근만 가지는 것들로 이루어진 집합이다. 또한 Laguerre와 Pólya의 결과 및 Rolle의 정리로

부터 $f \in \mathcal{LP}^*$일 때 $f' \in \mathcal{LP}^*$ 및 $Z_C(f') \le Z_C(f)$가 성립함을 알 수 있다. 특별히 $\mathcal{LP}$ 및 $\mathcal{LP}^*$는 각각 미분에 대하여 닫혀있다.

1930 년에 Pólya는 모든 $f \in \mathcal{LP}^*$에 대하여 (A)가 성립함을 증명하였다. 한편 증가지수가 (2,0)을 넘지 않는 모든 $f \in \mathcal{LP}^*$에 대하여 (B)가 성립한다는 사실이 1987 년에 Craven, Csordas, Smith에 의하여 증명되었으며, 모든 $f \in \mathcal{LP}^*$에 대하여 (B)가 성립한다는 사실은 1990 년에 저자에 의하여 증명되었다. 따라서 다음 정리를 얻는다.

정 리. 모든 $f \in \mathcal{LP}^*$에 대하여 다음 등식이 성립한다.

$$Z_C(f) = 2K_T(f).$$

한편 g1* 함수가 아닌 모든 정함수 $f(z)$에 대하여 (B)가 성립하지 않음이 알려져 있고, [LO], [HSW], [HW1], [HW2], [S], 또한 g1* 함수가 아닌 실 정함수로서 (A) 또는 (C)를 만족하지 않는 것의 보기는 쉽게 들 수 있다. e^{z^2}, e^{z^3}, $e^{e^z} + e^z e^{e^z}$ 등이 그러한 함수이다. 그러므로, $\mathcal{LP}^*$에 속하는 함수 외에, (A), (B), (C) 또는 그와 유사한 성질을 가지는 실 정함수를 더 찾기 위해서는 g1* 함수들 가운데 허근을 무수히 가지는 함수를 살펴보아야 될 것이다. 그러나, 그러한 g1* 함수 가운데도 (A), (B) 또는 (C)를 만족하지 않는 함수가 있는데 $e^z + 1$ 및 $e^z + e^{2z}$ 등이 그러한 함수이다. 이 두 함수의 증가지수는 1이고, 근은 허축 위에 일정한 간격으로 분포되어 있다. 따라서 증가지수가 1보다 작은 실 정함수, 또는 g1* 함수로서 허근이 실축에 충분히 가까이 있는 정함수 가운데 허근을 무수히 많이 가지는 함수를 고려하는 것은 자연스러운 일이라 하겠다. 실제로 1930 년에 Pólya는 다음과 같은 정리를 예상하였다.

Pólya의 예상(1930). 모든 0 종 실 정함수 $f(z)$에 대하여, $Z_C(f) = \infty$인 경우까지 포함하여, 다음 등식이 성립한다.

$$Z_C(f) = 2K_T(f).$$

이 예상에 대하여는 현재까지 증명도 반례도 발표되지 않은 것으로 보인다. 그러므로, 문제를 더 쉽게 하기 위해서, 허근이 실축에 충분히 가까이 있는 g1* 함수들로 대상을 제한하는 일이 필요하다고 본다. 이 논문에서는 적당한 양수 A에 대하여 $|\mathrm{Im}\, z| \le A$에서만 근을 가지는 g1* 함수를 대상으로 하였다.

이제 이 논문의 내용에 대하여 알아 보자. 우선 모든 g1* 함수 $f(z)$에 대하여 (A)가 국소적으로 성립함을 보였다.

정리 A. $f(z)$가 g1* 함수라고 하자. 두 실수 a와 b가, $a < b$, $f(z)$의 모든 허근 c_j에 대하여

$$|a - \mathrm{Re}\, c_j| > |\mathrm{Im}\, c_j| \quad \text{및} \quad |b - \mathrm{Re}\, c_j| > |\mathrm{Im}\, c_j|$$

를 만족하면, $f(z)$와 $f'(z)$는 영역 $a \le \mathrm{Re}\, z \le b$에서 허근을 유한 개 가진다. 또한 이 영역에서 $f(z)$와 $f'(z)$가 가지는 허근의 개수를 각각 $2J$, $2J'$이라고 하면, $f'(z)$는 구간 $[a,b]$에서 정확히 $J - J'$ 개의 임계근을 가진다.

또한 적당한 양수 A에 대하여 $|\mathrm{Im}\, z| \le A$에서만 근을 가지는 g1* 함수에 대하여 (B)가 국소적으로 성립함을 보였다.

정리 B. $f(z)$가 적당한 양수 A에 대하여 $|\mathrm{Im}\, z| \le A$에서만 근을 가지는 g1* 함수라고 하자. 2를 넘지 않는 양수 ρ에 대하여 $f(z)$의 증가율이 $(\rho, 0)$을 넘지 않으면, 모든 양수 C에 대하여 자연수 n_1이 존재하여 모든 $n \ge n_1$에 대하여 $f^{(n)}(z)$는 $|\mathrm{Re}\, z| \le Cn^{\frac{1}{\rho}}$에서 실근만 가진다.

끝으로, 위에 있는 두 정리를 이용하여, Pólya의 예상을 부분적으로 증명하였다.

정리 C. 증가지수가 2/3보다 작고, 적당한 양수 A에 대하여 $|\mathrm{Im}\, z| \le A$에서만 근을 가지는 모든 실 정함수 $f(z)$에 대하여, $Z_C(f) = \infty$인 경우까지 포함하여, 다음 등식이 성립한다.

$$Z_C(f) = 2K_T(f).$$

2 절에서는 정리의 증명에 필요한 예비지식을 다루었으며, 3 절은 이 정리들의 증명으로 이루어져 있다. 또한 4 절에서는 정리 B의 간단한 응용 및 정리 C를 증가지수가 1보다 작은 함수로까지 확장하는 데 필요하다고 생각하는 명제를 제시하였다.

2. 정리의 증명에 필요한 예비지식

우선 모든 g1* 함수에 대하여 공통적으로 성립하는 성질에 대하여 알아 보자. $f(z)$가 g1* 함수이면 $f(z)$를 다음과 같이 나타낼 수 있다.

$$f(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_k (1 - \frac{z}{a_k}) e^{\frac{z}{a_k}} \prod_j (1 - \frac{z}{c_j})(1 - \frac{z}{\bar{c}_j}) e^{(\frac{1}{c_j} + \frac{1}{\bar{c}_j})z},$$

여기서 n은 0 이상인 정수이며, $\alpha \ge 0$, c, β 및 모든 a_k는 실수이고, $\sum |a_k|^{-2}$ 및 $\sum |c_j|^{-2}$이 수렴해야 된다. 따라서 $f(z)$의 대수 도함수(logarithmic derivative)인 $f'(z)/f(z)$는 다음과 같이 주어진다.

$$\text{(2)} \qquad \frac{f'(z)}{f(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_k \left(\frac{1}{z - a_k} + \frac{1}{a_k} \right) + \sum_j \left(\frac{1}{z - c_j} + \frac{1}{z - \bar{c}_j} + \frac{2\mathrm{Re}\, c_j}{|c_j|^2} \right).$$

이러한 함수 $f(z)$에 대하여 영역 $\mathcal{J}(f)$를

$$\mathcal{J}(f) = \bigcup_j \{z \mid |z - \operatorname{Re} c_j| \leq |\operatorname{Im} c_j|\}$$

로 정의하면, $\mathbb{R} \cup \mathcal{J}(f)$에 속하지 않은 모든 z에 대하여 (2)의 각 항의 허수부는 z의 허수부와 부호가 반대임을 쉽게 알 수 있다. 이 사실은 정리 A의 증명에서 중요한 역할을 한다. 특별히, 모든 $z \notin \mathbb{R} \cup \mathcal{J}(f)$에 대하여 $f'(z) \neq 0$이 성립한다. 또한 $\xi \in \mathbb{R} \setminus \mathcal{J}(f)$에 대하여 $f'(\xi) = 0$이면, $f(\xi)f'(\xi) < 0$이 성립함을 간단한 계산을 통하여 알 수 있다. 이 사실로부터 다음 정리를 얻는다.

<u>Jensen 정리</u>. $f(z)$가 g1* 함수이면, $f'(z)$의 모든 허근과 임계근은 $\mathcal{J}(f)$에 속해 있다.

c_j가 g1* 함수 $f(z)$의 허근일 때, c_j와 $\overline{c}_j$를 잇는 선분을 지름으로 하는 닫힌 원판을 'Jensen 원판'이라고 부르며, 단축이 c_j와 $\overline{c}_j$를 잇는 선분이고 장축의 길이가 $2\sqrt{n}|\operatorname{Im} c_j|$인 타원을 n 차 Jensen 타원이라고 한다. $\mathcal{J}(f)$는 다름 아닌 $f(z)$의 모든 Jensen 원판들의 합집합이다. 또한 Jensen 정리를 이용하면, $f^{(n)}(z)$의 모든 허근과 임계근은 $f(z)$의 n 차 Jensen 타원과 그 내부로 이루어진 영역의 합집합에 속해 있음을 증명할 수 있다. 따라서 다음과 같은 따름정리를 얻는다.

<u>따름정리</u>. $f(z)$가 적당한 양수 A에 대하여 영역 $|\operatorname{Im} z| \leq A$에서만 근을 가지는 g1* 함수라고 하자. $\alpha_1, \alpha_2, \cdots$을 $f(z)$의 모든 허근의 실수부를 늘어 놓은 것이라고 하면, 모든 $n = 0, 1, 2, \cdots$에 대하여 다음 포함관계가 성립한다.

$$\mathcal{J}(f^{(n)}) \subset \bigcup_j \{z \mid \alpha_j - A\sqrt{n+1} \leq \operatorname{Re} z \leq \alpha_j + A\sqrt{n+1}\}.$$

$A \geq 0$일 때 영역 $|\operatorname{Im} z| \leq A$에서만 근을 가지는 g1* 함수들로 이루어진 집합을 $\mathcal{LP}^A$로 나타내자. 정함수 $f(z)$가 $\mathcal{LP}^A$의 구성원이 될 필요충분조건은 $|\operatorname{Im} z| \leq A$에서만 근을 가지는 실계수 다항식들로 이루어진 함수열 $\{P_n(z)\}$가 존재하여 $\{P_n(z)\}$가 $f(z)$로 모든 유계집합에서 평등수렴한다는 것을, [L]의 8 장에 있는 정리 8을 이용하여, 쉽게 증명할 수 있다. 이 사실 및 Gauss–Lucas 정리에 따라 $\mathcal{LP}^A$는 미분에 대하여 닫혀있다.

$\mathcal{LP}^A$가 미분에 대하여 닫혀있으므로, $f \in \mathcal{LP}^A$일 때, z_n이 반평면 $\operatorname{Im} z > 0$에 놓여 있는 $f^{(n)}(z)$의 허근이면, Jensen 정리에 따라 복소수열 $\{z_0, \cdots, z_{n-1}\}$이 존재하여 모든 $j = 0, 1, \cdots, n-1$에 대하여 $f^{(j)}(z_j) = 0$ 및

(3) $$|z_{j+1} - \operatorname{Re} z_j| \leq \operatorname{Im} z_j$$

를 만족한다. 이러한 성질을 가지는 복소수열에 관한 다음 보조정리가 정리 B의 증명에 쓰인다.

보조정리 1. $\operatorname{Im} z > 0$에 놓여 있는 점들로 이루어진 복소수열 $\{z_0, \cdots, z_n\}$이 모든 $j = 0, 1, \cdots, n-1$에 대하여 (3)을 만족하면, $\operatorname{Im} z_j = \beta_j$로 둘 때, 다음 부등식이 성립한다.

$$|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| \le \beta_0 - \beta_n + \sqrt{n({\beta_0}^2 - {\beta_n}^2)}.$$

이 보조정리는 n에 대한 귀납법으로 쉽게 증명할 수 있다.

정리 B의 증명에는 앞에 열거한 결과 말고도, Gontcharoff가 발견한 두 개의 부등식이 더 쓰인다.

Gontcharoff의 첫째 부등식. $f(z)$가 볼록한 영역(convex domain) $\mathcal{D}$에 정의된 해석적 함수라고 하자. $f(z)$가 $\mathcal{D}$에서 유계이면, $\sup_{z\in\mathcal{D}} |f(z)| = M$으로 둘 때, 모든 $z, z_0, z_1, \cdots, z_{n-1} \in \mathcal{D}$에 대하여 다음과 같은 부등식이 성립한다.

$$\left| \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f(\zeta_n) d\zeta_n \cdots d\zeta_2 d\zeta_1 \right|$$
$$\le \frac{M}{n!} \left(|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|\right)^n.$$

Gontcharoff의 둘째 부등식. $f(z)$가 정함수이고, 0보다 큰 실수 A, B, ρ, λ가 다음 방정식을 만족한다는 가정을 하자.

$$A\rho\lambda^{\rho-1}(\lambda - B) = 1.$$

충분히 큰 모든 양수 r에 대하여 부등식 $M(r; f) < e^{Ar^\rho}$가 성립하면, 다음이 성립한다.

$$\overline{\lim_{n\to\infty}}\, n^{\frac{1}{\rho}} \left(\frac{M(Bn^{\frac{1}{\rho}}; f^{(n)})}{n!} \right)^{\frac{1}{n}} \le A\rho\lambda^{\rho-1} e^{A\lambda^\rho}.$$

이 두 부등식의 증명은 [G]에서 찾아볼 수 있다.

끝으로, 정리 C를 증명하는 데는 다음 보조정리가 필요하다.

보조정리 2. $\{\alpha_n\}$이 0보다 큰 실수들로 이루어진 수열이고, ρ는 0보다 크며 1 이하인 상수라고 하자. $\sum \alpha_n^{-\rho}$이 수렴하면 다음이 성립한다.

$$\overline{\lim_{n\to\infty}} \frac{\alpha_{n+1} - \alpha_n}{\alpha_n^{1-\rho}} = \infty.$$

<u>증명</u>. 결론이 거짓이라고 가정하자. 그러면 적당한 양수 C가 존재하여, 다음 부등식이 모든 n에 대하여 성립한다.

$$\alpha_{n+1} \le \alpha_n + C\alpha_n^{1-\rho}.$$

$n = 1, 2, \cdots$에 대하여 β_n을

$$\beta_n = (\alpha_1^\rho + C\rho n)^{\frac{1}{\rho}}$$

으로 정의하면, 모든 자연수 n에 대하여 부등식

$$\beta_{n+1} - \beta_n \ge C(\alpha_1^\rho + C\rho n)^{\frac{1-\rho}{\rho}} = C\beta_n^{1-\rho}$$

이 성립하는데, 이것은 다음과 동치이다.

$$\beta_n + C\beta_n^{1-\rho} \le \beta_{n+1}, \quad n = 1, 2, \cdots .$$

따라서

$$\begin{aligned}
&\alpha_1 \le \beta_1, \\
&\alpha_2 \le \alpha_1 + C\alpha_1^{1-\rho} \le \beta_1 + C\beta_1^{1-\rho} \le \beta_2, \\
&\cdots \\
&\alpha_{n+1} \le \alpha_n + C\alpha_n^{1-\rho} \le \beta_n + C\beta_n^{1-\rho} \le \beta_{n+1}, \\
&\cdots
\end{aligned}$$

을 얻고, 이는 다음과 같이 가정에 모순이다.

$$\sum_{n=1}^{\infty} \alpha_n^{-\rho} \ge \sum_{n=1}^{\infty} \beta_n^{-\rho} = \infty. \quad \square$$

3. 정리의 증명

<u>정리 A의 증명</u>. $f(z)$가 g1* 함수라고 하자. 정리 A에 있는 두 실수 a와 b에 대한 조건은, 2 절의 표기법을 사용하면, 다름 아닌 $a, b \notin \mathcal{J}(f)$라는 것이다. $a, b \notin \mathcal{J}(f)$이므로, $\mathcal{J}(f)$의 정의 및 Jensen 정리에 따라, $f(z)$와 $f'(z)$가 영역 $a \le \operatorname{Re} z \le b$에서 가지는 모든 근의 허수부의 절대값은 $(b-a)/2$보다 작다. 특별히 $f(z)$와 $f'(z)$는 영역 $a \le \operatorname{Re} z \le b$에서 근을 유한 개만 가진다. 따라서 a를 왼쪽으로 조금 움직이거나 b를 오른쪽으로 조금 움직여도 영역 $a \le \operatorname{Re} z \le b$에서 $f(z)$와 $f'(z)$가 가지는

근의 개수는 변하지 않는다. 이 사실 때문에, 우리는 $\mathrm{Re}\, z = a, b$에서 $f(z)f'(z) \neq 0$이라는 가정을 덧붙일 수 있다.

$f(z)$와 $f'(z)$가 영역 $a \leq \mathrm{Re}\, z \leq b$에서 각각 $2J$, $2J'$ 개의 허근과 N, N' 개의 실근을 가진다고 하자. 앞에서 한 논의에 따라, 이들 근은 모두 영역

$$R = \left\{ z \mid a \leq \mathrm{Re}\, z \leq b,\ |\mathrm{Im}\, z| \leq \frac{b-a}{2} \right\}$$

의 내부에 존재한다. R의 경계를 이루는 직사각형을 C라고 한 다음, C에서 $\mathrm{Im}\, z \geq 0$에 있는 부분을 C_1, $\mathrm{Im}\, z \leq 0$에 있는 부분을 C_2라고 하자. C와 $\mathcal{J}(f)$는 공통부분이 없으므로, z가 C_1을 따라 움직일 때는 $w = f'(z)/f(z)$가 $\mathrm{Im}\, w \leq 0$에 머물러 있으며, z가 C_2를 따라 움직일 때는 $w = f'(z)/f(z)$가 $\mathrm{Im}\, w \geq 0$에 머물러 있다. 따라서 z가 C를 따라 왼쪽으로 한 바퀴 도는 데 따른 $\arg[f'(z)/f(z)]$의 증가량은 -2π 또는 0 또는 2π가 될 것이다. 또한 이 증가량이 -2π, 0, 2π 가운데 어느 것이 되는가는 $f'(a)/f(a)$와 $f'(b)/f(b)$의 부호에 따라 결정된다. 실제로 다음과 같음을 알 수 있다.

$$\frac{1}{2\pi} \Delta_C \arg \frac{f'(z)}{f(z)} = \frac{1}{2}[\mathrm{sgn} f(a) f'(a) - \mathrm{sgn} f(b) f'(b)], \tag{4}$$

여기서 $\mathrm{sgn}\, r$은 0 아닌 실수 r에 대하여 $r > 0$이면 $+1$, $r < 0$이면 -1을 나타낸다.

[P1]에서, Pólya는 $f'(z)$가 구간 $[a, b]$에서 가지는 임계근의 개수를 K라고 할 때 다음 등식이 성립함을 증명하였다.

$$2K = N' - N - \frac{1}{2}[\mathrm{sgn} f(a) f'(a) - \mathrm{sgn} f(b) f'(b)].$$

(실제로 이 등식은 모든 실해석적(real analytic) 함수 $f(z)$에 대하여 성립한다.) (4)는

$$N' + 2J' = N + 2J + \frac{1}{2}[\mathrm{sgn} f(a) f'(a) - \mathrm{sgn} f(b) f'(b)]$$

을 뜻하므로, 원하던 대로 다음을 얻는다.

$$2K = N' - N - \frac{1}{2}[\mathrm{sgn} f(a) f'(a) - \mathrm{sgn} f(b) f'(b)] = 2(J - J').$$

이렇게 하여 정리 A를 증명하였다. □

<u>정리 B의 증명</u>. $0 < \rho \leq 2$이고, $f(z)$는 $\mathcal{LP}^A$의 구성원이며, $f(z)$의 증가율이 $(\rho, 0)$을 넘지 못한다고 가정하자.

z_n이 $\operatorname{Im} z > 0$에 있는 $f^{(n)}(z)$의 허근이면, Jensen 정리에 따라 복소수열 $\{z_0, z_1, \cdots, z_{n-1}\}$이 존재하여 모든 $j = 0, 1, \cdots, n-1$에 대하여 $f^{(j)}(z_j) = 0$ 및

(5) $$|z_{j+1} - \operatorname{Re} z_j| \leq \operatorname{Im} z_j$$

을 만족한다. 모든 $j = 0, 1, \cdots, n-1$에 대하여 $f^{(j)}(z_j) = 0$이고 $f(z)$는 정함수이므로, 임의로 주어진 복소수 z에 대하여 $f(z)$를

$$f(z) = \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f^{(n)}(\zeta_n) d\zeta_n \cdots d\zeta_2 d\zeta_1$$

으로 나타낼 수 있고, Gontcharoff의 첫째 부등식에 따라 다음을 얻는다.

(6) $$|f(z)| \leq \frac{M}{n!} \left(|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|\right)^n,$$

여기서 M은 $\{z, z_0, z_1, \cdots, z_{n-1}\}$을 품는 최소 볼록집합(convex hull)에서 $|f^{(n)}(\zeta)|$가 취하는 최대값이다.

한편 (5)와 보조정리 1에서

$$|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| \leq \operatorname{Im} z_0 (1 + \sqrt{n}) \leq A(1 + \sqrt{n})$$

을 얻는데, 이로부터 다음 두 부등식이 성립함을 알 수 있다.

(7) $$|z_j| \leq |z_n| + |z_j - z_n| \leq |z_n| + A(1 + \sqrt{n}), \quad j = 0, 1, \cdots, n-1,$$

(8)
$$\begin{aligned} |z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}| &\leq |z| + |z_0| + A(1 + \sqrt{n}) \\ &\leq |z| + |z_n| + 2A(1 + \sqrt{n}). \end{aligned}$$

만일 결론이 거짓이라면, 적당한 양수 C가 존재하여 무수히 많은 n에 대하여 $f^{(n)}(z)$가 영역 $|\operatorname{Re} z| \leq Cn^{\frac{1}{\rho}}$에서 허근을 가질 것이며, (6), (7), (8)에 따라, 적당한 양수 B가 존재하여 $|z| \leq 1$ 및 무수히 많은 n에 대하여 다음 부등식이 성립할 것이다.

(9) $$|f(z)| \leq \frac{M(Bn^{\frac{1}{\rho}}; f^{(n)})}{n!} \left(Bn^{\frac{1}{\rho}}\right)^n.$$

$\epsilon > 0$일 때 λ_ϵ이 다음 방정식을 만족하는 양수라고 하자.

$$\epsilon \rho \lambda_\epsilon^{\rho - 1} (\lambda_\epsilon - B) = 1.$$

모든 양수 ϵ에 대하여 이러한 양수 λ_ϵ이 정확히 하나 존재하며,

$$\lim_{\epsilon\to 0} \epsilon\lambda_\epsilon^{\rho-1} e^{\epsilon\lambda_\epsilon^\rho} = 0 \tag{10}$$

이 성립함을 쉽게 보일 수 있다. 한편 모든 양수 ϵ에 대하여 $M(r;f) = o(e^{\epsilon r^\rho})$가 성립하므로, Gontcharoff의 둘째 부등식에 따라 다음이 성립한다.

$$\overline{\lim_{n\to\infty}}\, Bn^{\frac{1}{\rho}} \left(\frac{M(Bn^{\frac{1}{\rho}}; f^{(n)})}{n!} \right)^{\frac{1}{n}} < B\epsilon\rho\lambda_\epsilon^{\rho-1} e^{\epsilon\lambda_\epsilon^\rho}, \quad \epsilon > 0. \tag{11}$$

따라서 우리의 결론이 거짓이면, (9), (10), (11)에 의하여 다음이 성립할 수 밖에 없다.

$$f(z) = 0, \quad |z| \le 1.$$

이것은 모순이므로, 원하는 결론을 얻는다. □

<u>정리 C의 증명</u>. $f(z)$가 적당한 양수 A에 대하여 $|\mathrm{Im}\, z| \le A$에서만 근을 가지는 실 정함수이며, $f(z)$의 증가지수가 2/3보다 작다고 가정하자. $f(z)$의 증가지수가 2/3보다 작기 때문에 $f(z)$는 0 종 실 정함수이며, 따라서 당연히 $\mathcal{LP}^A$의 구성원이다. 또한 모든 정함수의 증가율은 미분에 대하여 불변이고 [L, p. 6], $\mathcal{LP}^A$는 미분에 대하여 닫혀 있으므로, $f'(z)$도 $f(z)$와 같은 조건을 만족한다.

만일 적당한 자연수 n_0에 대하여 $f^{(n_0)}(z)$가 허근을 유한 개만 가지면, 함수 $f^{(n_0)}(z)$는 $\mathcal{LP}^*$의 구성원이므로, $\lim_{n\to\infty} Z_C(f^{(n)}) = 0$이 성립한다. 이러한 함수 $f(z)$에 대하여는, $Z_C(f) = \infty$인 경우를 포함하여, $Z_C(f) = 2K_T(f)$가 성립함을 최근에 발표된 논문 [KK]를 통하여 김상문 교수와 저자가 증명하였다. 따라서 우리는 모든 자연수 n에 대하여 $Z_C(f^{(n)}) = \infty$라고 가정해도 된다. 이러한 가정을 하면 모든 $f^{(n)}(z)$가 $f(z)$와 같은 조건을 만족하므로, $K_T(f)$의 정의에 따라, $K_T(f) > 0$만 증명해도 충분하다. 또한 그렇게 하기 위해서는, 정리 A에 따라, 다음 세 조건을 만족하는 자연수 J, N 및 양수 B가 존재함을 보이면 되겠다.

(a) $f(z)$는 영역 $|\mathrm{Re}\, z| \le B$에서 허근을 $2J > 0$ 개 가진다.

(b) $f^{(N)}(z)$는 영역 $|\mathrm{Re}\, z| \le B$에서 실근만 가진다.

(c) $\pm B \notin \mathcal{J}(f) \cup \mathcal{J}(f') \cup \cdots \mathcal{J}(f^{(N-1)})$.

이제 $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2, \cdots$ 을 $f(z)$의 모든 허근을 늘어놓은 것이라고 하자. 변수를 적당히 변환하여, $|\beta_j| \le 1$, $j = 1, 2, \cdots$, 즉 $f \in \mathcal{LP}^1$ 및 $0 < |\alpha_1| \le |\alpha_2| \le \cdots$ 을 가정할 수 있다. $f(z)$의 증가지수가 2/3보다 작고, $f \in \mathcal{LP}^1$이므로, $\sum |\alpha_j|^{-\frac{2}{3}}$은

수렴한다 [L, Chapter 1, Theorem 6]. 보조정리 2에 따라, 모든 양수 C에 대하여 부등식

$$(12) \qquad |\alpha_j| + C|\alpha_j|^{\frac{1}{3}} < |\alpha_{j+1}|$$

을 만족하는 자연수 j가 무수히 많이 존재한다. 한편 정리 B에 따라 다음과 같은 자연수 n_1이 존재한다.

(13) 모든 $n \geq n_1$에 대하여 $f^{(n)}(z)$는 영역 $|\mathrm{Re}\, z| \leq 2n^{\frac{3}{2}}$에서 실근만 가진다.

(12)에 따라 다음 두 조건을 만족하는 자연수 J를 찾을 수 있다.

$$(14) \qquad n_1^{\frac{3}{2}} < |\alpha_J| \text{ 및 } |\alpha_J| + 4|\alpha_J|^{\frac{1}{3}} < |\alpha_{J+1}|.$$

이제 N을 $|\alpha_J|^{\frac{2}{3}}$보다 작지 않은 최소의 자연수라고 하자.

$$(15) \qquad |\alpha_J|^{\frac{2}{3}} \leq N < |\alpha_J|^{\frac{2}{3}} + 1.$$

(14)와 (15)로부터, 다음과 같은 부등식을 얻는다:

$$(16) \qquad n_1 < |\alpha_J|^{\frac{2}{3}} \leq N,$$

$$(17) \qquad |\alpha_J| + \sqrt{N} \leq N^{\frac{3}{2}} + \sqrt{N} < 2N^{\frac{3}{2}},$$

$$\sqrt{N} < (|\alpha_J|^{\frac{2}{3}} + 1)^{\frac{1}{2}} < |\alpha_J|^{\frac{1}{3}} + \frac{1}{2}|\alpha_J|^{-\frac{1}{3}} < 2|\alpha_J|^{\frac{1}{3}},$$

$$(18) \qquad |\alpha_J| + \sqrt{N} < |\alpha_J| + 2|\alpha_J|^{\frac{1}{3}} < |\alpha_{J+1}| - 2|\alpha_J|^{\frac{1}{3}} < |\alpha_{J+1}| - \sqrt{N}.$$

(17), (18)에서 다음 두 조건을 만족하는 양수 B가 존재함을 알 수 있다.

$$(19) \qquad |\alpha_J| + \sqrt{N} < B < |\alpha_{J+1}| - \sqrt{N},$$

$$(20) \qquad B \leq 2N^{\frac{3}{2}}.$$

이제 (a)와 (b)는 (13), (16), (19), (20)의 결과이다. 끝으로, 모든 β_j의 절대값이 1을 넘지 않는다는 가정과 Jensen 정리의 따름정리로부터, 다음 포함관계를 얻는다.

$$\mathcal{J}(f^{(n)}) \subset \bigcup_j \{z \mid \alpha_j - \sqrt{n+1} \leq \mathrm{Re}\, z \leq \alpha_j + \sqrt{n+1}\}, \; n = 0, 1, 2, \cdots.$$

(c)는 이 포함관계 및 (19)의 결과이므로, 정리 C의 증명이 끝났다. □

4. 맺음말

이 절에서는 정리 B의 간단한 응용과, 정리 C에서 정함수 $f(z)$의 증가지수 ρ가 2/3보다 작다는 제한을 ρ가 1보다 작다는 제한으로 완화하는 데 필요하다고 생각하는 명제를 하나 제시함으로써 맺음말을 대신하려 한다.

$m, n = 0, 1, 2, \cdots, N$에 대하여 $c_{m,n}$이 모두 실수일 때,

$$f(z) = \sum_{m,n=0}^{N} c_{m,n} \cos^m z \sin^n z$$

로 정의된 함수 $f(z)$를 생각해 보자. $f(z)$는 e^{iz}와 e^{-iz}의 다항식으로 나타낼 수 있으므로, 영역 $0 \le \operatorname{Re} z < 2\pi$에서 근을 유한 개 가지며 이 영역에서 가지는 근의 개수는 미분에 대하여 불변이다.

이 영역에서 $f(z)$와 $f'(z)$가 각각 $2J$, $2J'$ 개의 허근과 N, N' 개의 실근을 가진다고 하면, 등식 $2J + N = 2J' + N'$ 및 $f(z)$와 $f'(z)$의 주기가 2π라는 사실을 이용하여, 정리 A의 증명과 같은 방법으로, 구간 $[0, 2\pi)$에 $f'(z)$의 임계근이 정확히 $J - J'$ 개 있음을 보일 수 있다.

또한 $f(z)$는 $0 \le \operatorname{Re} z < 2\pi$에서 근을 유한 개만 가지며 주기가 2π인 함수이므로, 적당한 양수 A가 존재하여 $f(z)$는 $|\operatorname{Im} z| \le A$에서만 근을 가진다. $f(z)$의 증가율은 1이므로, 정리 B의 결과로서, 충분히 큰 모든 n에 대하여 $f^{(n)}(z)$는 $0 \le \operatorname{Re} z < 2\pi$에서 실근만 가진다. $f(z)$의 모든 도함수의 주기도 2π이므로, 충분히 큰 모든 n에 대하여 $f^{(n)}(z)$는 실근만 가진다.

따라서 $f(z)$에 대한 결론은 다음과 같다: $f(z)$가 영역 $0 \le \operatorname{Re} z < 2\pi$에서 가지는 허근의 개수는 $f(z)$를 계속 미분함에 따라 구간 $[0, 2\pi)$에서 나타나는 임계근의 개수를 모두 합한 것의 두 배이다.

이제는 정리 C의 확장에 대하여 알아보자. $f(z)$가 g1* 함수라고 하자. 정리 A는 ($f(z)$의 모든 Jensen 원판의 합집합인) $\mathcal{J}(f)$의 모든 연결단위(connected component)에서 등식 (A)가 개별적으로 성립함을 뜻한다. 이제 $f(z)$의 증가지수가 1보다 작으면, $f(z)$의 근의 역수의 절대값의 합이 수렴한다. 따라서, "적당한 양수 A에 대하여 $f(z)$가 $|\operatorname{Im} z| \le A$에서만 근을 가진다"는 가정까지 하면, $f(z)$의 모든 Jensen 원판의 반지름이 A를 넘지 못하기 때문에, $\mathcal{J}(f)$는 무수히 많은 연결단위로 이루어져있다. 또한 정리 B에 따라 $f^{(n)}(z)$가 허근을 가지지 않는 범위는 n이 증가하면서 임의로 넓어지므로, 이러한 함수에 대하여 (C)가 성립할 것처럼 보인다. 그러나 사실은 그렇게 간단하지 않은데, 그 까닭은 미분함에 따라 Jensen 원판이 움직이기 때문이다. 실제로 우리는 "$\mathcal{LP}^A$의 구성원인 $f(z)$의 증가지수가 2/3보다 작으면, n이 증가함에 따라 $f^{(n)}(z)$가 실근만 가지는 범위가 충분히 빠른 속도로 넓어지며 $\mathcal{J}(f)$의 연결단위들은 서로가 충분히 멀리 떨어져 있기 때문에, 그들이 움직여서 서로 붙기 전에 $f^{(n)}(z)$가 실근만 가지는 범위에 포함됨"을 보임으로써 정리 C를 증명한 셈이다. 그러나 증가지수가 2/3 이상인 함수에 대하여는 이 방법이 적용되지 않는다.

이러한 배경에서, 우리는 증가지수가 1보다 작은 모든 $f \in \mathcal{LP}^A$에 대하여 (C)가 성립함을 (Pólya는 이보다 훨씬 더 일반적인 명제를 예상하였다.) 증명하는 데 필요하다고 여겨지는 다음 명제가 증명되기를 희망한다.

<u>명제</u>. $\mathcal{LP}^A$의 구성원인 함수 $f(z)$가 영역 $a \leq \operatorname{Re} z \leq b$에서 k 개의 근을 가지면 $f'(z)$는 $a - A \leq \operatorname{Re} z \leq b + A$에서 적어도 $k-1$ 개의 근을 가진다.

이 명제가 정리 A와 모순되지 않음을 눈여겨보기 바란다. 또한 모든 $f \in \mathcal{LP}^A$는 $|\operatorname{Im} z| \leq A$에서만 근을 가지는 실계수 다항식들로 이루어진 함수열의 극한이므로, 이 명제를 증명하기 위해서는 다음 명제만 증명하여도 충분하다.

<u>명제</u>. 실계수 다항식 $P(z)$가 영역 $|\operatorname{Im} z| \leq A$에서만 근을 가진다고 하자. $P(z)$가 영역 $a \leq \operatorname{Re} z \leq b$에서 k 개의 근을 가지면 $P'(z)$는 $a - A \leq \operatorname{Re} z \leq b + A$에서 적어도 $k-1$ 개의 근을 가진다.

ABSTRACT

Let $f(z)$ be a real entire function of genus 1^*, that is $f(z)$ can be expressed in the form $f(z) = e^{-\alpha z^2} g(z)$, where $\alpha \geq 0$ and $g(z)$ is a real entire function of genus at most 1. In this paper, we obtain the following theorems.

Theorem A. *Assume that a, b, $a < b$, do not lie in the union of the Jensen disks of $f(z)$. If $f(z)$ and $f'(z)$ have $2J$ and $2J'$ nonreal zeros in the region $a \leq \operatorname{Re} z \leq b$, then $f'(z)$ has exactly $J - J'$ Fourier critical zeros in the interval $[a, b]$.*

Theorem B. *If there is a positive constant A such that $f(z) \neq 0$ for $|\operatorname{Im} z| > A$, and if $f(z)$ is at most of order ρ, $\rho \leq 2$, and minimal type, then for each $C > 0$ there is a positive integer n_1 such that for all $n \geq n_1$ $f^{(n)}(z)$ has only real zeros in $|\operatorname{Re} z| \leq Cn^{\frac{1}{\rho}}$.*

Theorem C. *If there is a positive constant A such that $f(z) \neq 0$ for $|\operatorname{Im} z| > A$, and if $f(z)$ is of order less than 2/3, then $f(z)$ has just as many Fourier critical points as couples of nonreal zeros, including the case that $f(z)$ has infinitely many nonreal zeros.*

The definitions of the Jensen disks of $f(z)$ and the Fourier critical zeros of $f'(z)$ can be found in [CCS1] and [P1].

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A METHOD OF PROLONGATION AND HOLOMORPHIC EXTENSION OF CR FUNCTIONS

CHONG-KYU HAN AND JAE-NYUN YOO

Introduction

In this paper we survey some jet theoretic aspects of tangential Cauchy-Riemann equations and propose several problems on holomorphic extension of CR functions.

Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u_\alpha : |\alpha| \leq m) = 0, \quad \lambda = 1, \cdots, l, \tag{1}$$

for a system of unknown functions $u = (u^1, \cdots, u^q)$ of independent variables $x = (x^1, \cdots, x^n)$, where $\alpha = (\alpha^1, \cdots, \alpha^n)$ are multi-indices, $|\alpha| = \alpha^1 + \cdots + \alpha^n$, u_α denotes the q-vector $(\frac{\partial}{\partial x^1})^{\alpha_1} \cdots (\frac{\partial}{\partial x^n})^{\alpha_n} u$ and $u_\alpha = u$ if $|\alpha| = 0$.

A compatibility equation of (1) is a differential equation obtained by prolongation, that is, a process of differentiation and algebraic operations on (1). To be precise, let X be an open set containing the origin of $\mathbf{R}^n$ and U the space of q real variables $u = (u^1, \cdots, u^q)$. For each nonnegative integer r, the r-th jet space, denoted by $J_r(X, U)$, is the space of the partial derivatives of unknown function $u = (u^1, \cdots, u^q)$ up to order r, namely,

$$J_r(X, U) \equiv \{(x, u^{(r)}) : x \in X\},$$

where $u^{(r)} = \{(u_\alpha) : |\alpha| \leq r\}$.

A differential function of order r is a smooth (C^∞) function $a(x, u^{(r)})$ defined on $J_r(X, U)$. By $A^{(r)}$ we denote the algebra of differential functions

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of order r. For integers r, s with $s > r$, we define a projection map $\pi_r^s : A^{(s)} \longrightarrow A^{(r)}$ by

$$\pi_r^s(a(x,u^{(s)})) = \begin{cases} a(x,u^{(s)}), & \text{if the highest order of the arguments of } a \leq r \\ 0, & \text{otherwise .} \end{cases}$$

The jet approach to differential equations is to study the properties of the subvariety $\mathcal{S}_\Delta$ of $J_m(X,U)$ defined by (1), or equivalently, an ideal Δ of $A^{(m)}$ generated by Δ_λ, $\lambda = 1, \cdots, l$.
If $a(x,u^{(r)})$ is a differential function of order r, for each $i = 1, \cdots, n$, i-th total derivative of a is

$$D_i a \equiv \frac{\partial a}{\partial x^i} + \sum_{|\alpha| \leq r} \sum_{k=1}^{q} \frac{\partial a}{\partial u_\alpha^k} u_{\alpha,i}^k,$$

where α, i is the multi-index $(\alpha^1, \cdots, \alpha^{i-1}, \alpha^i + 1, \alpha^{i+1}, \cdots, \alpha^n)$. The first prolongation of (1), which we denote by $\Delta^{(1)}$, is an ideal of $A^{(m+1)}$ generated by (1) and its total derivatives. r-th prolongation of (1), denoted by $\Delta^{(r)}$, is an ideal of $A^{(m+r)}$ generated by (1) and its total derivatives of order up to r.

A compatibility equation of (1) is an equation of the form $a(x,u^{(r)}) = 0$, where a is an element of a prolongation of (1) of any order. In general, one loses information by differentiating a given system of partial differential equations, however, if the highest order terms are eliminated in the process of algebraic operations, the resulting equation, which will be called a compatibility equation of Finzi type (cf. [CH],[OLV]), reveals those properties of solutions that are due to lower order terms of (1). More precisely, a compatibility equation of (1) of Finzi type is an equation $b(x,u^{(m+s)}) = 0$, where $b \in \pi_{m+s}^{m+r}(\Delta^{(r)}) \setminus \Delta^{(s)}$, for some $r > s$.

Example 1. Consider a system of first order equations for two real unknown functions u, v of two variables (x, y)

$$\text{(2)} \qquad \Delta_1 \equiv u_x + yv^2 = 0$$
$$\text{and} \quad \Delta_2 \equiv u_y + u + x^2 = 0.$$

Let $\Delta = (\Delta_1, \Delta_2)$. Then the first prolongation $\Delta^{(1)}$ is the ideal in $A^{(2)}$ generated by Δ_1, Δ_2, $D_x\Delta_1 = u_{xx} + 2yvv_x$, $D_y\Delta_1 = u_{xy} + v^2 +$

$2yvv_y$, $D_x\Delta_2 = u_{xy} + u_x + 2x$ and $D_y\Delta_2 = u_{yy} + u_y$. We see that $D_y\Delta_1 - D_x\Delta_2$ yields a compatibility equation of Finzi type

$$v^2 + 2yvv_y - u_x - 2x = 0.$$

We refer to [CH] for general method of construction of compatibility equations of Finzi type.

In §1 of this paper we introduce the notion of complete system. In §2, we show that CR equivalences between certain pseudoconvex real hypersurfaces of a complex space satisfy a complete system of order 3. In §3, we present two basic theorems on the problem of holomorphic extension of CR functions. In §4, we discuss the rigidity of CR embeddings, and finally propose several problems on holomorphic extension from the viewpoint of the rigidity of embeddings.

§1. Prolongation to complete systems

In certain overdetermined cases, (1) has a level of prolongation in which one can solve for all the $(m+r)$-th order partial derivatives of $(u^1, \cdots, u^q)$ in terms of lower order derivatives :

$$u^i_\alpha = H^i_\alpha(x, u_\beta : |\beta| \leq m + r - 1), \quad i = 1, \cdots, q, \ |\alpha| = m + r, \tag{1.1}$$

where each H^i_α is smooth in its arguments. (1.1) is called a complete system of order $m+r$. In this case (1) is said to admit a prolongation to a complete system of order $m+r$, which means that a prolongation $\Delta^{\rho(r)}$ of sufficiently high order $\rho(r)$ contains differential functions of the form

$$u^i_\alpha - H^i_\alpha(x, u_\beta : |\beta| \leq m + r - 1), \quad i = 1, \cdots, q, \text{ all } \alpha \text{ with } \ |\alpha| = m + r.$$

Example 1.1 Consider the following overdetermined system of 1st order equations for two unknown functions u, v of independent variables (x, y).

$$\begin{cases} \Delta_1 \equiv u_x + xy = 0 \\ \Delta_2 \equiv u_y + v_x = 0 \\ \Delta_3 \equiv v_y + u^2 + y^2 + 1 = 0. \end{cases} \tag{1.2}$$

Total differentiate Δ_1, Δ_2 and Δ_3 to get

$$\begin{aligned}
D_x\Delta_1 &= u_{xx} + y = 0\\
D_y\Delta_1 &= u_{xy} + x = 0\\
D_x\Delta_2 &= u_{yx} + v_{xx} = 0\\
D_y\Delta_2 &= u_{yy} + v_{xy} = 0\\
D_x\Delta_3 &= v_{yx} + 2uu_x = 0\\
D_y\Delta_3 &= v_{yy} + 2uu_y + 2y = 0.
\end{aligned}$$

The above six equations can be solved for six variables $u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}$ and v_{yy}, thus (1.2) admits a prolongation to a complete system of order 2.

If a system Δ of order m admits a complete system of order $m + r$ then the $\Delta^{\rho(r)}$ is the final level of prolongation we want ; further prolongation provides essentially no more information. If the subvariety $\mathcal{S}_\Delta$ defined by (1) is a manifold and satisfies a mild condition called regular (cf. [P1]) the existence of a complete system of order $m + r$ implies that for any integer t with $t > \rho(r)$ the submanifold of $J_{m+t}(X, U)$ defined by $\Delta^{(m+t)}$ is diffeomorphic to the submanifold of $J_{m+\rho(r)}(X, U)$ defined by $\Delta^{\rho(r)}$, via the natural projection map $J_{m+t}(X, U) \longrightarrow J_{m+\rho(r)}(X, U)$. If (1) admits a complete system then the problem of the existence and properties of solutions reduces to that of ordinary differential equations as we see in the following

Proposition 1.2 Suppose that a system of functions $u = (u^1, \cdots, u^q)$ defined on an open set X of $\mathbf{R}^n$ satisfies a complete system of order m

$$u^i_\alpha = H^i_\alpha(x, u^{(m-1)}), \qquad i = 1, \cdots, q, \quad |\alpha| = m, \tag{1.3}$$

where each H^i_α is real analytic in its arguments. Then u is real analytic provided that u is m times continuously differentiable.

Proof. On $J_{m-1}(X, U)$ we define a real analytic n-dimensional distribution $\mathcal{D}$ defined by the system of 1-forms

$$\omega^j = du^j - \sum_{k=1}^n u^j_k dx^k,$$

$$\omega^j_\beta = du^j_\beta - \sum_{k=1}^n u^j_{\beta,k} dx^k,$$

$$\text{and} \quad \omega^j_\alpha = du^j_\alpha - \sum_{k=1}^n H^j_{\alpha,k}(x, u^{(m-1)}) dx^k,$$

where $j = 1, \cdots, q$, $|\beta| \le m-2$, $|\alpha| = m-1$, and β, k is the multi-index $(\beta^1, \cdots, \beta^{k-1}, \beta^k + 1, \beta^{k+1}, \cdots, \beta^n)$ if $\beta = (\beta^1, \cdots, \beta^n)$. Let $f = (f^1, \cdots, f^q)$ is a system of functions that satisfies (1.3) we claim that the n-dimensional submanifold $\mathcal{I}$ of $J_{m-1}(X, U)$ defined by

$$x \longmapsto (x, \partial^\gamma f^j \quad : \quad j = 1, \cdots, q, \quad 0 \le |\gamma| \le m-1)$$

is an n-dimensional integral manifold of $\mathcal{D}$. The tangent space to the above submanifold is generated by n linearly independent vectors

$$E_k = \partial_{x^k} + \sum_{j=1}^{q} \partial^k f^j(x) \partial_{u^j} + \sum_{j=1}^{q} \sum_{|\beta|=1}^{m-1} \partial^{\beta,k} f^j(x) \partial_{u^j_\beta}, \quad k = 1, \cdots, n,$$

where β, k is the multi-index defined above. It is easy to see that $\omega^j(E_k) = \omega^j_\gamma(E_k) = 0$ for all $j = 1, \cdots, q$, $|\gamma| \le m-1$, $k = 1, \cdots, n$. Therefore, the submanifold $\mathcal{I}$ of $J_{m-1}(X, U)$ is an integral submanifold of $\mathcal{D}$, so it is real analytic and hence, in particular, f is real analytic. □

Example 1.3 Let M^n be a smooth manifold with smooth Riemann metric g. A vector field X is an infinitesimal isometry(Killing field) if $L_X g = 0$, where L is the Lie derivative. If we write $X = \sum_{i=1}^n \xi^i \partial/\partial x^i$ in terms of local coordinates, then ξ^i, $i = 1, \cdots, n$, satisfies a complete system of order 2 [H3].

Example 1.4 If $F = (f^1, \cdots, f^{n+1})$ is an isometric embedding of a n-dimensional Riemannian manifold M into $\mathbf{R}^{n+1}$, F satisfies a non-linear system of first order partial differential equations

$$\sum_{\alpha=1}^{n+1} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} = g_{ij}.$$

This system can be prolonged to a complete system of order 3 if the embedding is rigid [CHO].

§2. Complete system for CR equivalences

Let M^{2n+1} be a smooth(C^∞) real hypersurface in $\mathbf{C}^{n+1}$, $n \geq 1$, defined by $\rho(z) = 0$, where $z = (z^1, \cdots, z^{n+1})$ is the standard coordinates of $\mathbf{C}^{n+1}$ and $\rho(z) = 0$ is a real valued function defined on a neighborhood of M such that $d\rho(z) \neq 0$ on M. A nonzero complex vector field of the form

$$\mathbf{L} = \sum_{j=1}^{n+1} a_j \frac{\partial}{\partial \bar{z}_j}$$

is said to be tangent to M if $\mathbf{L}\rho = 0$ on M. $\mathbf{L}$ is called a tangential Cauchy-Riemann operator. If we assume that $\partial\rho/\partial\bar{z}_{n+1} \neq 0$,

$$\mathbf{L}_j \equiv \frac{\partial \rho}{\partial \bar{z}_{n+1}} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_{n+1}}, \qquad j = 1, \cdots, n,$$

are linearly independent tangential Cauchy-Riemann operators. Let $\mathcal{V}$ be the subbundle of the complexfied tangent bundle $T_{\mathbf{C}}M$ over M. It is easy to see that

$$\mathcal{V} \cap \bar{\mathcal{V}} = \{0\} \tag{2.1}$$

and

$$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V} \qquad \text{(integrability condition)}, \tag{2.2}$$

where $\bar{\mathcal{V}}$ is the complex conjugate of $\mathcal{V}$ and $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$ means that the bracket of any two section of $\mathcal{V}$ is again a section of $\mathcal{V}$. $\mathcal{V}$ is the CR structure bundle of M, induced by the complex structure of $\mathbf{C}^{n+1}$.

An abstract CR manifold of hypersurface type is a smooth real manifold M of dimension $2n + 1, n \geq 1$, which admits a smooth subbundle $\mathcal{V}$ of complex dimension n of the complexified tangent bundle of M satisfying (2.1) and (2.2). Let $(M^{2n+1}, \mathcal{V})$ be an abstract CR manifold of hypersurface type. A complex valued function f on M is a CR function if f is annihilated by $\mathcal{V}$. A system of CR functions $F = (f^1, \cdots, f^{n+1})$ is a CR embedding if $df^1 \wedge \cdots \wedge df^{n+1} \neq 0$. If F is a CR embedding the pull back by F of the induced CR structure bundle of $F(M)$ coincides with $\mathcal{V}$. At each point $p \in M$ let $\mathcal{W}_p = (T_{\mathbf{C}}M)_p/(\mathcal{V}_p + \bar{\mathcal{V}}_p)$. The Levi form is a hermitian form $\mathcal{L}_p : \mathcal{V}_p \times \mathcal{V}_p \to \mathcal{W}_p$ defined by $\mathcal{L}_p(u, v) = \sqrt{-1}[u, \bar{v}]$, where u, v are sections of $\mathcal{V}$. Levi form is nondegenerate at $p \in M$ if $\mathcal{L}_p(u, v) = 0$ for all $v \in \mathcal{V}_p$ implies that $u = 0$.

Now we will construct a complete system for CR equivalence between CR manifolds with non-degenerate Levi form. We work in analytic category even though the whole argument is valid in the C^∞ category also.

Theorem 2.1. Let $(M_i, \mathcal{V}_i)$, $i = 1, 2$, be real analytic CR manifolds of hypersurface type of dimension $2n + 1, n \geq 1$, with nondegenerate Levi form. Suppose that a diffeomorphism F of M_1 onto M_2 is a CR equivalence, namely, $F_* \mathcal{V}_1 = \mathcal{V}_2$. Then F satisfies a complete system of order 3.

Proof. For simplicity, we will give proof for the case $n = 1$ and M_2 is a hypersurface in $\mathbf{C}^2$. Let M_1 be a 3-dimensional real analytic CR manifold with the structure bundle $\mathcal{V}$. Let $\bar{Z}$ (instead of using the notation $\mathbf{L}$) be a C^ω generator of $\mathcal{V}$. By a holomorphic change of coordinates M_2 is given by

$$-2u = |z|^2 + \sum_{\substack{j+k\geq 6\\ k\geq 2\\ j\geq 2}} N_{jk}(u) z^j \bar{z}^k,$$

where $\mathbf{C}^2 = \{(z, w)\}, w = u + \imath v$ (cf. [J]).
Let

$$\rho(z, w) = w + \bar{w} + |z|^2 + \sum N_{jk}(u) z^j \bar{z}^k.$$

Writing $F = (f, g)$ coordinatewise we have

$$\rho \circ F = g + \bar{g} + f \cdot \bar{f} + \sum_{\substack{j+k\geq 6\\ k\geq 2\\ j\geq 2}} (N_{jk} \circ F) f^j \cdot \bar{f}^k = 0, \tag{2.3}$$

$$\bar{Z} f = 0, \tag{2.4}$$

and

$$\bar{Z} g = 0 \qquad \text{(tangential Cauchy-Riemann equations)}. \tag{2.5}$$

We will get all the third order derivatives of f and g from (2.3) - (2.5) by prolongation. Let $T = \sqrt{(-1)}[Z, \bar{Z}]$. Then $\bar{T} = -\sqrt{(-1)}[\bar{Z}, Z] = \sqrt{(-1)}[Z, \bar{Z}] = T$, thus T is a C^ω real vector field. The non-degeneracy of the Levi form implies that T is transversal to $H(M) \equiv Re\mathcal{V} + Im\mathcal{V}$. We use $\{Z, \bar{Z}, T\}$ as a basis of $T_{\mathbf{C}}M$. All the derivatives of the form $T^i Z^j \bar{Z}^k f$ and $T^i Z^j \bar{Z}^k g$ with $i + j + k = 3, k \neq 0$, are zero from (2.4) and (2.5), respectively. If one changes the order of applications of T, Z and $\bar{Z}$ to f and g, the results differ from the former by lower order derivatives that arise from the commutations. So, it is enough to find the derivatives of the form T^iZ^jf and T^iZ^jg with $i+j=3$. First, apply Z to (2.3), then by (2.4) and (2.5) we get

$$Zg + Zf\cdot\bar{f} + Z\Big(\sum_{\substack{j+k\geq 6\\ k\geq 2\\ j\geq 2}} N_{jk}(F)f^j\cdot\bar{f}^k\Big) = 0. \tag{2.6}$$

We assume at the reference point $0\in M$, $f(0)=g(0)=0$. Furthermore, $Zf(0)\neq 0$ for the following reason: Since $F_*\bar{Z} = (\bar{Z}\bar{f})\partial/\partial\bar{z} + (\bar{Z}\bar{g})\partial/\partial\bar{w}$, evaluation at $0\in M$ gives

$$(\bar{Z}\bar{f})(0)\frac{\partial}{\partial\bar{z}} + (\bar{Z}\bar{g})(0)\frac{\partial}{\partial\bar{w}} = \lambda\cdot\frac{\partial}{\partial\bar{z}} \qquad \text{for some } \lambda\neq 0,$$

therefore, $\bar{Z}\bar{f}(0)\neq 0, \bar{Z}\bar{g}(0)=0$. Solving (2.6) for $\bar{f}$, we have

$$\bar{f} = h(x,f,g,Zf,Zg,\bar{g}),$$

where $\partial h/\partial\bar{g}(0)=0$ and $h\in C^\omega$. Apply $\bar{Z}^3$ to the above, to get

$$\bar{Z}^3\bar{f} = a(x,f,g,Zf,Zg,Tf,Tg,\bar{g},\bar{Z}\bar{g},\bar{Z}^2\bar{g},\bar{Z}^3\bar{g}), \tag{2.7}$$

where $\partial a/\partial(\bar{Z}^3\bar{g})(0)=0$. Now apply Z^2 to (2.6), to get

$$Z^3g = b(x,\bar{f},\bar{g},f,g,Zf,Z^2f,Z^3f,Zg,Z^2g), \tag{2.8}$$

where $\partial b/\partial(Z^3f)(0)=0, b\in C^\omega$. Solving (2.7) and the complex conjugate of (2.8) we get

$$\bar{Z}^3\bar{f} = c(x,f,g,Zf,Zg,Tf,Tg,\bar{f},\bar{Z}\bar{f},\bar{Z}^2\bar{f},\bar{g},\bar{Z}\bar{g},\bar{Z}^2\bar{g}) \tag{2.9}$$

and

$$\bar{Z}^3\bar{g} = d(x,f,g,Zf,Zg,Tf,Tg,\bar{f},\bar{Z}\bar{f},\bar{Z}^2\bar{f},\bar{g},\bar{Z}\bar{g},\bar{Z}^2\bar{g}), \tag{2.10}$$

where $c,d\in C^\omega$. Apply Z repeatedly to (2.9) and (2.10) and reduce the orders of the arguments in the right hand side by the complex conjugate of already obtained ones, we get $T\bar{Z}^2\bar{f}$, $T^2\bar{Z}\bar{f}$, $T^3\bar{f}$, $T\bar{Z}^2\bar{g}$, $T^2\bar{Z}\bar{g}$ and $T^3\bar{g}$ as C^ω functions of the derivatives of order less than and equal to 2. □

Observe that in the process of prolongation from (2.3) - (2.5) to the complete system, differentiations are applied six times. Hence, we see that the complete system consists of compatibility equations of Finzi type in addition to twice differentiations of (2.4) and (2.5). A generalization of Theorem 2.1 to higher dimensions and degenerate Levi form is in [H1]. [H2] deals with the construction of complete systems for rigid immersions.

§3. Holomorphic extension of CR functions

Let M be a real hypersurface in $\mathbf{C}^{n+1}, n \geq 1$, and f be a holomorphic function on a neighborhood of M. Then f satisfies the Cauchy-Riemann equations and therefore, the restriction of f on M satisfies the tangential Cauchy-Riemann equations. However, not every CR function extends to a holomorphic function of the ambient space as the following example shows :

Example 3.1 Let $M = \{(z,w) \in \mathbf{C}^2 : Im\, w = 0\}$. Observe that M is Levi flat, namely the Levi form is degenerate everywhere. The tangential Cauchy-Riemann operator on M is $\partial/\partial\bar{z}$. Let $f(z,w) = \sum_{finite} a_j(u)h_j(z)$, where each $h_j(z)$ is a holomorphic function and each $a_j(u)$ is a differentiable (C^1) function which is not C^ω. Then f is a CR function that can not be extended to a holomorphic function on a neighborhood of M, for a restriction of a holomorphic function to M must be C^ω on M.

The holomorphic extension problem has been extensive study in several different approaches (cf. [BR], [B2]). In this section we present two basic theorems on holomorphic extension which are well known but are not found in the literature. One is the analytic disk method due to S. Bochner, which seems to be the most function theoretic and simple minded. For simplicity, we stay in $\mathbf{C}^2$ with coordinates (z,w). Let $\Omega \subset\subset \mathbf{C}^2$ be an open set with smooth boundary. Let $\pi : \mathbf{C}^2 \to \mathbf{C}^1$ be the projection $\pi(z,w) = z$. Let $A = \pi(\Omega)$. Let $\Omega_z = \{w \in \mathbf{C} : (z,w) \in \Omega\}$ and $\partial\Omega_z = \{w \in \mathbf{C} : (z,w) \in \partial\Omega\}$ for a fixed $z \in A$.

Proposition 3.1 Let $\Omega \subset\subset \mathbf{C}^2$, Ω_z and A be the same as above. Suppose that there exists a global smooth parameterization $\gamma(t,z) : [0,2\pi] \times A \to \partial\Omega$, such that for each fixed z, $\gamma(t,z) : [0,2\pi] \to \partial\Omega_z$ is a smooth parameterization of $\partial\Omega_z$. Let u be a smooth CR function on $\partial\Omega$. Define

$$U(z,w) = \frac{1}{2\pi i} \int_{\partial\Omega_z} \frac{u(z,\zeta)}{\zeta - w}\, d\zeta.$$

Then $U(z,w)$ is holomorphic on Ω.

Proof. Let $M = \{\rho = 0\}$, $\bar\partial\rho \neq 0$ on M. First, we find a smooth extension $\tilde u$ of u to a neighborhood of M such that $\bar\partial\tilde u = 0$ on M. Let u_1 be any ambient extension of u. Since $\bar\partial u_1 \wedge \bar\partial\rho = 0$ on M (tangential Cauchy-Riemann equations) $\bar\partial u_1 = \psi\bar\partial\rho$ on M for some smooth function ψ to a neighborhood of M. Let $\tilde\psi$ be a smooth extension of ψ to a neighborhood of M. Now let $\tilde u = u_1 - \rho\tilde\psi$. Then

$$\begin{aligned}\bar\partial\tilde u &= \bar\partial u_1 - \tilde\psi\bar\partial\rho - \rho\bar\partial\tilde\psi\\ &= 0 \qquad \text{on } M.\end{aligned}$$

Now we show that

$$\begin{aligned}U(z,w) =& \frac{1}{2\pi i}\int_{\partial\Omega_z}\frac{\tilde u(z,\zeta)}{\zeta - w}\,d\zeta\\ =& \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\tilde u\circ\gamma)\partial\gamma_2/\partial t}{\gamma_2(t,z) - w}\,dt\end{aligned}$$

is holomorphic on Ω, where we write $\gamma = (\gamma_1, \gamma_2)$, coordinatewise. Note that $\gamma_1(t,z) = z$. $U(z,w)$ is clearly holomorphic in w. To show that $\frac{\partial U}{\partial\bar z} = 0$, first we compute $\frac{\partial}{\partial\bar z}$ applied to the integrand :

$$\frac{\partial}{\partial\bar z}\frac{(\tilde u\circ\gamma)\partial\gamma_2/\partial t}{\gamma_2(t,z) - w} = \frac{\partial}{\partial\bar z}\left(\frac{(\tilde u\circ\gamma)}{\gamma_2 - w}\right)\frac{\partial\gamma_2}{\partial t} + \frac{(\tilde u\circ\gamma)}{\gamma_2 - w}\frac{\partial^2\gamma_2}{\partial\bar z\partial t}. \tag{3.1}$$

By chain rule,

$$\begin{aligned}\frac{\partial}{\partial\bar z}(\tilde u\circ\gamma) &= \frac{\partial\tilde u}{\partial z}\frac{\partial\gamma_1}{\partial\bar z} + \frac{\partial\tilde u}{\partial\bar z}\frac{\partial\bar\gamma_1}{\partial\bar z} + \frac{\partial\tilde u}{\partial w}\frac{\partial\gamma_2}{\partial\bar z} + \frac{\partial\tilde u}{\partial\bar w}\frac{\partial\bar\gamma_2}{\partial\bar z}\\ &= \frac{\partial\tilde u}{\partial w}\frac{\partial\gamma_2}{\partial\bar z} \qquad \text{on } \partial\Omega,\end{aligned}$$

since $\frac{\partial\gamma_1}{\partial\bar{z}} = \frac{\partial\tilde{u}}{\partial\bar{z}} = \frac{\partial\tilde{u}}{\partial\bar{w}} = 0$ on $\partial\Omega$. Thus on $\partial\Omega$ the RHS of (3.1) is equal to

$$\left[\frac{\partial\tilde{u}}{\partial w}\frac{\partial\gamma_2}{\partial\bar{z}}/(\gamma_2 - w) - (\tilde{u}\circ\gamma)\frac{\partial\gamma_2}{\partial\bar{z}}/(\gamma_2 - w)^2\right]\frac{\partial\gamma_2}{\partial t} + \frac{\tilde{u}\circ\gamma}{\gamma_2 - w}\frac{\partial^2\gamma_2}{\partial\bar{z}\partial t},$$

and therefore, we have

$$\text{(3.2)}\quad \frac{\partial U}{\partial\bar{z}} = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\frac{\partial\tilde{u}}{\partial w}\frac{\partial\gamma_2}{\partial\bar{z}}\frac{\partial\gamma_2}{\partial t}}{\gamma_2 - w}dt - \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\tilde{u}\circ\gamma)\frac{\partial\gamma_2}{\partial\bar{z}}\frac{\partial\gamma_2}{\partial t}}{(\gamma_2 - w)^2}dt + \frac{1}{2\pi i}\int_0^{2\pi}\frac{\tilde{u}\circ\gamma}{\gamma_2 - w}\frac{\partial^2\gamma_2}{\partial\bar{z}\partial t}dt.$$

By integration by parts the last term of (3.2) becomes

$$-\frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial}{\partial t}\left(\frac{\tilde{u}\circ\gamma}{\gamma_2 - w}\right)\frac{\partial\gamma_2}{\partial\bar{z}}dt$$

$$= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{\frac{\partial\tilde{u}}{\partial w}\frac{\partial\gamma_2}{\partial t}\frac{\partial\gamma_2}{\partial\bar{z}}}{\gamma_2 - w}dt + \frac{1}{2\pi i}\int_0^{2\pi}\frac{(\tilde{u}\circ\gamma)\frac{\partial\gamma_2}{\partial t}\frac{\partial\gamma_2}{\partial\bar{z}}}{(\gamma_2 - w)^2}dt,$$

so that the RHS all cancel out and we have $\frac{\partial U}{\partial\bar{z}} = 0$. □

A proof by $\bar{\partial}$ method for the existence of holomorphic extension is found in p. 31 of [HOE]. Next we show the following

Theorem 3.2. Suppose that M is a real analytic (C^ω) hypersurface of $\mathbf{C}^{n+1}$, $n \geq 0$, and f is a CR function on M then f extends to a holomorphic function of a neighborhood of M if and only if f is C^ω.

Proof. "Only if " part is trivial. Let ρ be an C^ω local defining function of M with $\frac{\partial\rho}{\partial\bar{z}_{n+1}} \neq 0$. For each $j = 1, \cdots, n$, let $\bar{Z}_j = \frac{\partial\rho}{\partial\bar{z}_{n+1}}\frac{\partial}{\partial\bar{z}_j} - \frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial}{\partial\bar{z}_{n+1}}$. Then the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{z}_{n+1}} = 0 \\ \qquad u = f \qquad \text{on } M, \end{cases}$$

has a unique C^ω solution F on a neighborhood $\Omega \subseteq \mathbf{C}^{n+1}$ of $0 \in M$ by the Cauchy-Kowalevsky theorem (M is non-characteristic with respect to $\frac{\partial}{\partial \bar{z}_{n+1}}$). We will show that F is holomorphic on a smaller neighborhood of $0 \in M$. Since $\frac{\partial F}{\partial \bar{z}_{n+1}} = 0$ on Ω, it suffices to show that $\frac{\partial F}{\partial \bar{z}_j} = 0, \quad j = 1, \cdots, n$, on a smaller neighborhood of $0 \in M$. We have for each $j = 1, \cdots, n$

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_{n+1}}\left(\frac{\partial F}{\partial \bar{z}_j}\right) &= \frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial F}{\partial \bar{z}_{n+1}}\right) \\ &= 0. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} \frac{\partial \rho}{\partial \bar{z}_{n+1}}\left(\frac{\partial F}{\partial \bar{z}_j}\right) &= \left(\frac{\partial \rho}{\partial \bar{z}_{n+1}}\frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j}\frac{\partial}{\partial \bar{z}_{n+1}}\right) F \\ &= \bar{Z}_j F \\ &= \bar{Z}_j f \quad \text{on } M, \text{ for } \bar{Z}_j \text{ is tangential to } M. \\ &= 0, \quad \text{for } f \text{ is a CR function.} \end{aligned}$$

Since $\frac{\partial \rho}{\partial \bar{z}_{n+1}} \neq 0$, we have

$$\frac{\partial F}{\partial \bar{z}_j} = 0 \qquad \text{on } M. \tag{3.4}$$

(3.3) and (3.4) implies that $\frac{\partial F}{\partial \bar{z}_j}$ is a C^ω solution to the Cauchy problem.

$$\begin{cases} \dfrac{\partial u}{\partial \bar{z}_{n+1}} = 0 \\ \qquad u = 0 \qquad \text{on } M. \end{cases} \tag{3.5}$$

The uniqueness of C^ω solution to (3.5) implies that $\frac{\partial F}{\partial \bar{z}_j} = 0$ on a smaller neighborhood $\tilde{\Omega} \subseteq \Omega$ of $0 \in M$. □

Theorem 3.2 can also be proved by complexification argument, see [HA]. Combining Proposition 1.2, Theorem 2.1 and Theorem 3.2, we have

Corollary 3.3 Suppose that M_i, $i = 1, 2$, are C^ω real hypersurfaces in $\mathbf{C}^{n+1}$ and $F : M_1 \to M_2$ is a CR equivalence of class C^3. Then F extends to a biholomorphic mapping between neighborhoods of M_1 and M_2.

[H1] and [BJT] are generalizations of corollary 3.3 to the cases of degenerate Levi form.

§4. Rigidity of CR embeddings and open problems

Let M^{2n+1} is a smooth CR manifold of hypersurface type with the structure bundle $\mathcal{V}$. A CR embedding $f = (f^1, \cdots, f^{n+1})$ of M into $\mathbf{C}^{n+1}$ is said to be rigid if for any CR embedding $g : M \to \mathbf{C}^{n+1}$, there exists a biholomorphic mapping Φ of neighborhood of $f(M)$ onto a neighborhood of $g(M)$ such that $g = \Phi \circ f$. We have

Proposition 4.1 A smooth real hypersurface M is rigid if and only if every CR function extends locally to a holomorphic function.

Proof. Suppose that M is rigid and f is a CR function on M. If $df \neq 0$, $df \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_n} \neq 0$, for some $i_1, \cdots, i_n$. We may assume that $df \wedge dz_1 \wedge \cdots \wedge dz_n \neq 0$. Then $\phi = (z_1, \cdots, z_n, f)$ is a CR mapping of M onto $\phi(M)$. Since M is rigid there exists a biholomorphic mapping $\Phi = (\phi_1, \cdots, \phi_{n+1})$ of a neighborhood of M onto that of $\phi(M)$. Then ϕ_{n+1} is a holomorphic extension of f. If $df = 0$ at a reference point, $f + z_{n+1}$ extends holomorphically by the same argument and therefore, f extends holomorphically. The converse is trivial. □

Returning to the jet theory, we now define the notions of automorphic system. By $J_r(X, U)$, we denote the space of r-jets of a map from an open subset $X \subseteq \mathbf{R}^n$ of a system of functions $u = (u^1, \cdots, u^q), q \geq n$. Let $\Pi_r(X, U)$ be the open set of $J_r(X, U)$ with the condition rank(u^i_j) = n. Let $\mathcal{R}_r \subseteq \Pi_r(U, U)$ be a system of partial differential equations of order r (or a subvariety given by the system of partial differential equations, which we will identify with the system itself) defining a Lie pseudogroup (cf. [P1], [P2]).

We define an action $\Pi_r(U,U)$ on $\Pi_r(X,U)$ by the composition of jets at the target :

$$\Pi_r(X,U) \times \Pi_r(U,U) \to \Pi_r(X,U)$$

action defined by $(f_r, g_r) \longmapsto g_r \circ f_r$, where $g_r \circ f_r \equiv (g \circ f)_r$, and g (f, resp.) is any function whose r-jet is g_r (f_r, resp.).

A system of partial differential equations $\mathcal{A}_r \subseteq \Pi_r(X,U)$ is said to be formally invariant under the action of $\mathcal{R}_r$ if each $g_r \in \mathcal{R}_r$ sends $\mathcal{A}_r$ to $\mathcal{A}_r$.

Definition 4.2 A system $\mathcal{A}_r \subseteq \Pi_r(X,U)$ is an automorphic system for $\mathcal{R}_r \subseteq \Pi_r(U,U)$ if the action $\mathcal{A}_{r+s} \times \mathcal{R}_{r+s} \to \mathcal{A}_{r+s}$ is free and transitive, for each integer $s \geq 0$, where $\mathcal{A}_{r+s}$ and $\mathcal{R}_{r+s}$ denote the s-th prolongation of $\mathcal{A}_r$ and $\mathcal{R}_r$, respectively.

Finally we propose the following

Problem 4.3

Let $M^{2n+1}, n \geq 1$, be a smooth real hypersurface in $\mathbf{C}^{n+1}$. Find conditions on the Levi form which imply the rigidity of M.

Problem 4.4

Let $M^{2n+1}, n \geq 1$, be a CR manifold of hypersurface type. Let $\mathcal{A}_1 \subseteq \Pi_1(M, \mathbf{C}^{n+1})$, be the system of tangential Cauchy-Riemann equations,

$$\bar{Z}_j u = 0, \qquad j = 1, \cdots, n, \tag{4.1}$$

where $\bar{Z}_j$ is a basis of CR structure bundle over M and let $\mathcal{R}_1 \subseteq \Pi_1(\mathbf{C}^{n+1}, \mathbf{C}^{n+1})$ be the Cauchy-Riemann equations. Then $\mathcal{A}_r$ is formally invariant under group action of $\mathcal{R}_r$, for each integer $r = 1, 2, \cdots$. Under what conditions on Levi form of M is $\mathcal{A}_1$ an automorphic system for $\mathcal{R}_1$? Observe that a solution of an automorphic system is automatically rigid. So, this is a problem of finding sufficient conditions for any CR function to be holomorphically extendable.

Problem 4.5

Let $M^{2n+1}, n \geq 1$, be a CR manifold of hypersurface type. Find compatibility equations of Finzi type for CR embeddings of M into $\mathbf{C}^{n+1}$. Under what conditons on the Levi form of M does the CR embedding equations (4.1) admit a prolongation to an elliptic system ? If a real analytic hypersurface M satisfies these conditions on the Levi form then any CR function on M is holomorphically extendable.

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ANALYTIC CLASSIFICATION OF PLANE CURVE SINGULARITIES DEFINED BY SOME HOMOGENEOUS POLYNOMIALS.

CHUNGHYUK KANG

0. Introduction

Let $V = \{(z, y) : f(z, y) = 0\}$ be an analytic subvariety of a polydisc near the origin in $\mathbf{C}^2$ where f is a homogeneous polynomical and square-free. We know that any homogeneous polynomial with two variables which is square-free can be written as $z^n + a_{n-1}yz^{n-1} + \cdots + a_1y^{n-1}z + y^n$ where $a_0, a_1, \cdots, a_{n-1}$ are constant by a suitable nonsingular linear change of coordinates in $\mathbf{C}^2$. Here we assume that f has the following form : (1) $f = z^n + a_iy^{n-i}z^i + \cdots + a_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2i + 3)$. (2) either $f = z^3 + ay^2z + y^3$ or $f = z^4 + ay^3z + y^4$. If $g = z^n + b_jy^{n-j}z^j + \cdots + b_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2j + 3)$, then in section 1 we show by the elementary method that f is analytically equivalent to g if and only if there is a unit ω with $\omega^n = 1$ such that $b_k = a_k\omega^k$ for each $k = 1, 2, \cdots, i = j$. In section 2 we prove that all homogeneous polynomials of degree three each of which is square-free are analytically equivalent and that if $f = z^4 + ay^3z + y^4$ and $g = z^4 + by^3z + y^4$ where f and g are square-free, then f and g are analytically equivalent if and only if $a^4 = b^4$. Moreover, we give examples with which we understand the condition that $n \geq 5$ and $n \geq 2i + 3$.

1. Analytic classification of plane curve singularities defined by $f = z^n + a_iy^{n-i}z^i + \cdots + a_1y^{n-1}z + y^n$ $(n \geq 5,\ n \geq 2i + 3)$.

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Definition 1.1. *Let $V = \{(z,y) : f(z,y) = 0\}$ and $W = \{(z,y) : g(z,y) = 0\}$ be germs of analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f, g are holomorphic and square-free near the origin in $\mathbf{C}^2$. V and W are said to be analytically equivalent if there exists a germ at the origin of biholomorphisms $\psi : (U_1, 0) \to (U_2, 0)$ such that $\psi(V) = W$ and $\psi(O) = O$ where U_1 and U_2 are open subsets containing the origin in $\mathbf{C}^2$. In this case we call $f(z,y)$ and $g(z,y)$ analytically equivalent near the origin and denote this relation by $f \approx g$. Note by [3] that $f \approx g$ if and only if $f(Az+By, Cz+Dy) = ug(z,y)$ for $u \neq 0$ and $AD - BC \neq 0$ whenever f and g are homogeneous.*

Before proving the main result, we need the following Lemma.

Lemma 1.2. *Recall the notation ${}_nC_k = \binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$. Then*

$$D = \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-1}C_1 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-1}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_k & {}_{n+1}C_k & \cdots & {}_{n+k-1}C_k \end{vmatrix}$$

$$= \begin{vmatrix} {}_nC_1 & {}_nC_0 & 0 & \cdots & 0 \\ {}_nC_2 & {}_nC_1 & {}_nC_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_nC_{k-2} & {}_nC_{k-3} & \cdots & {}_nC_0 \\ {}_nC_k & {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 \end{vmatrix}$$

$$= (-1)^{k(k-1)/2} \begin{vmatrix} 0 & \cdots & 0 & {}_{n+k-2}C_0 & {}_{n+k-1}C_1 \\ 0 & \cdots & {}_{n+k-3}C_0 & {}_{n+k-2}C_1 & {}_{n+k-1}C_2 \\ \vdots & & \vdots & \vdots & \vdots \\ {}_nC_0 & \cdots & {}_{n+k-3}C_{k-3} & {}_{n+k-2}C_{k-2} & {}_{n+k-1}C_{k-1} \\ {}_nC_1 & \cdots & {}_{n+k-3}C_{k-2} & {}_{n+k-2}C_{k-1} & {}_{n+k-1}C_k \end{vmatrix}$$

$$= {}_{n+k-1}C_k.$$

Proof. To compute D, subtracting $(k-1)$-column from k-column, we have

$$D = \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-2}C_1 & {}_{n+k-2}C_0 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-2}C_2 & {}_{n+k-2}C_1 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_nC_k & {}_{n+1}C_k & \cdots & {}_{n+k-2}C_{k-1} & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

Applying the same technique to $(k-1)-$column, $(k-2)-$column, $\cdots$, the second column in order, we get

$$D = \begin{vmatrix} {}_nC_1 & {}_nC_0 & \cdots & {}_{n+k-3}C_0 & {}_{n+k-2}C_0 \\ {}_nC_2 & {}_nC_1 & \cdots & {}_{n+k-3}C_1 & {}_{n+k-2}C_1 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_nC_k & {}_nC_{k-1} & \cdots & {}_{n+k-3}C_{k-1} & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

Using the same technique, by induction we get

$$D = \begin{vmatrix} {}_nC_1 & {}_nC_0 & \cdots & 0 & 0 \\ {}_nC_2 & {}_nC_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 & {}_nC_0 \\ {}_nC_k & {}_nC_{k-1} & \cdots & {}_nC_2 & {}_nC_1 \end{vmatrix}$$

This is the first form which we want.
With respect to the k-th column only, D is linear and so, D can be represented in the following:

$$D = \begin{vmatrix} {}_nC_1 & {}_nC_0 & \cdots & 0 & 0 \\ {}_nC_2 & {}_nC_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 & {}_{n-1}C_0 \\ {}_nC_k & {}_nC_{k-1} & \cdots & {}_nC_2 & {}_{n-1}C_1 \end{vmatrix} + \begin{vmatrix} {}_nC_1 & {}_nC_0 & \cdots & 0 \\ {}_nC_2 & {}_nC_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ {}_nC_{k-2} & {}_nC_{k-3} & \cdots & {}_nC_0 \\ {}_nC_{k-1} & {}_nC_{k-2} & \cdots & {}_nC_1 \end{vmatrix}$$

$= D_1 + D_2$ where D_1 is the $k \times k$ matrix and D_2 is the $(k-1) \times (k-1)$ matrix. Applying the same technique to D_1 as in the beginning of the proof, then we have

$$D_1 = \begin{vmatrix} {}_{n-1}C_1 & {}_{n-1}C_0 & \cdots & 0 & 0 \\ {}_{n-1}C_2 & {}_{n-1}C_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n-1}C_{k-1} & {}_{n-1}C_{k-2} & \cdots & {}_{n-1}C_1 & {}_{n-1}C_0 \\ {}_{n-1}C_k & {}_{n-1}C_{k-1} & \cdots & {}_{n-1}C_2 & {}_{n-1}C_1 \end{vmatrix}$$

Next, applying the same method to D_1 and D_2 reversely as in the beginning of the proof, then we get

$$D = D_1 + D_2 = \begin{vmatrix} {}_{n-1}C_1 & {}_nC_1 & \cdots & {}_{n+k-2}C_1 \\ {}_{n-1}C_2 & {}_nC_2 & \cdots & {}_{n+k-2}C_2 \\ \vdots & \vdots & & \vdots \\ {}_{n-1}C_k & {}_nC_k & \cdots & {}_{n+k-2}C_k \end{vmatrix} + \begin{vmatrix} {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-2}C_1 \\ {}_nC_2 & {}_{n+1}C_2 & \cdots & {}_{n+k-2}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_{n+1}C_{k-1} & \cdots & {}_{n+k-2}C_{k-1} \end{vmatrix}$$

By induction on $n+k$, then $D_1 = {}_{n+k-2}C_k$ and $D_2 = {}_{n+k-2}C_{k-1}$. Therefore $D = {}_{n+k-1}C_k$. Now to express D in another way, subtracting the second column from the first column, the third column from the second column, $\cdots$, the k-column from $(k-1)$-column in D in order, then we have

$$D = (-1)^k \begin{vmatrix} {}_nC_0 & {}_{n+1}C_0 & \cdots & {}_{n+k-1}C_1 \\ {}_nC_1 & {}_{n+1}C_1 & \cdots & {}_{n+k-1}C_2 \\ \vdots & \vdots & & \vdots \\ {}_nC_{k-1} & {}_{n+1}C_{k-1} & \cdots & {}_{n+k-1}C_k \end{vmatrix}.$$

Using the same process by induction on k, we have the desired result.

Theorem 1.3. *Let $V = \{(z,y) : f = z^n + a_i y^{n-i} z^i + \cdots + a_1 y^{n-1} z + y^n = 0\}$ and $W = \{(z,y) : g = z^n + b_j y^{n-j} z^j + \cdots + b_1 y^{n-1} z + y^n = 0\}$ be analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f and g are homogeneous polynomials and square-free, and $n \geq 2i+3$, $n \geq 2j+3$ and $n \geq 5$. Then $f \approx g$ if and only if there is a unit ω with $\omega^n = 1$ such that $b_k = a_k \omega^k$ for $k = 1, 2, \cdots, i = j$.*

Proof. Assume that $f \approx g$. Then we know by [3] that $f(Az+By, Cz+Dy) = (Az+By)^n + a_i(Cz+Dy)^{n-i}(Az+By)^i + a_{i-1}(Cz+Dy)^{n-i+1}(Az+By)^{i-1} + \cdots + a_1(Cz+Dy)^{n-1}(Az+By) + (Cz+Dy)^n = ug(z,y)$ for a nonzero constant u where $AD - BC \neq 0$. Because $n-(i+2) \geq i+1$ and i and j may be viewed as same integers, coefficients of the following monomials

$yz^{n-1}, y^2z^{n-2}, \cdots, y^{i+2}z^{n-(i+2)}$ in the polynomial $f(Az+By, Cz+Dy)$ are zero. Let us write down these coefficients in detail as follows :

([1])

$$yz^{n-1} : \binom{n}{1}A^{n-1}B + a_i \sum_{k+l=1} \binom{n-i}{k}\binom{i}{l} C^{n-i-k}D^kA^{i-l}B^l$$
$$+ \cdots + a_1 \sum_{k+l=1} \binom{n-1}{k}\binom{1}{l} C^{n-1-k}D^kA^{1-l}B^l + \binom{n}{1}C^{n-1}D = 0.$$

([2])

$$y^2z^{n-2} : \binom{n}{2}A^{n-2}B^2 + a_i \sum_{k+l=2} \binom{n-i}{k}\binom{i}{l} C^{n-i-k}D^kA^{i-l}B^l$$
$$+ \cdots + a_1 \sum_{k+l=2} \binom{n-1}{k}\binom{1}{l} C^{n-i-k}D^kA^{1-l}B^l + \binom{n}{2}C^{n-2}D^2 = 0.$$

([$i+2$])

$$y^{i+2}z^{n-(i+2)} : \binom{n}{i+2}A^{n-(i+2)}B^{i+2} + a_i \sum_{k+l=i+2} \binom{n-i}{k}\binom{i}{l} \times$$
$$C^{n-i-k}D^kA^{i-l}B^l + \cdots + a_1 \sum_{k+l=i+2} \binom{n-1}{k}\binom{1}{l} C^{n-1-k}D^kA^{1-l}B^l$$
$$+ \binom{n}{i+2}C^{n-(i+2)}D^{i+2} = 0.$$

Considering $1, a_i, a_{i-1}, \cdots, a_1, 1$ as a nontrivial solution of the above [$i+2$]-homogeneous equations, then we get an $(i+2) \times (i+2)$ square matrix Δ consisting of coefficients of $1, a_i, a_{i-1}, \cdots, a_1, 1$ in these equations whose determinant $|\Delta|$ must be zero. Now write down the determinant $|\Delta|$:

$$0 = |\Delta| =$$
$$\begin{vmatrix} \binom{n}{1}A^{n-1}B & \sum_{k+l=1} \binom{n-i}{k}\binom{i}{l}C^{n-i-k}D^kA^{i-l}B^l & \ldots & \binom{n}{1}C^{n-1}D \\ \binom{n}{2}A^{n-2}B^2 & \sum_{k+l=2} \binom{n-i}{k}\binom{i}{l}C^{n-i-k}D^kA^{i-l}B^l & \ldots & \binom{n}{2}C^{n-2}D^2 \\ \vdots & \vdots & & \vdots \\ \binom{n}{i+2}A^{n-(i+2)}B^{i+2} & \sum_{k+l=i+2} \binom{n-i}{k}\binom{i}{l}C^{n-i-k}D^kA^{i-l}B^l & \ldots & \binom{n}{i+2}C^{n-(i+2)}D^{i+2} \end{vmatrix}$$

Then we claim that $|\Delta| = kA^{n-(i+2)}B[C^{n-(i+2)}D]^{i+1}(AD - BC)^{{}_{i+2}C_2}$ for some nonzero constant k. Note that $\sum_{k+l=j}\binom{n-i}{k}\binom{i}{l} = \binom{n}{j}$ for a given nonnegative integer j. We know that each element in the first column of Δ has $A^{n-(i+2)}B$ as common factor. Now we are going to prove that any elementary signed product from Δ has $[C^{n-(i+2)}D]^{i+1}$ as common divisor. Consider the degree of C of each element in Δ as follows :

$$\begin{pmatrix} 0 & n-(i+1) & \dots & n-2 & n-1 \\ 0 & n-(i+2) & \dots & n-3 & n-2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n-(i+i+2) & \dots & n-(i+3) & n-(i+2) \end{pmatrix}$$

So the degree of C for each elementary signed product from Δ is greater than or equal to the following number : $n-(2i+2)+n-(2i+1)+\cdots+n-(i+2)+0+1+\cdots+i = (i+1)(n-i-2)$. Similarly, we can prove that the degree of D for each elementary signed product from Δ is greater than or equal to the integer $(i+1)$.

Now it is enough to prove that $|\Delta|/\{A^{n-i-2}B(C^{n-i-2}D)^{i+1}\} = k(AD-BC)^{{}_{i+2}C_2}$ for a nonzero constant k. To show this, divide each element in the first column of Δ by $A^{n-i-2}B$ and each element in the remaining columns of Δ by $C^{n-i-2}D$. Let the matrix got in this way from Δ be $P = P(A,B,C,D)$. Then it suffices to prove that the determinant $|P| = k(AD-BC)^{{}_{i+2}C_2}$. Note that each elementary signed product from $|P|$ has the same degree $(i+2)(i+1)$ since $n(i+2)-(n-i-1)-(n-i-1)(i+1) = (i+2)(i+1)$. Let $Q(A,B) = P(A,B,A,B)$. Note that $|Q(A,A)| = |P(A,A,A,A)| = 0$ and $|Q(A,-A)| = |P(A,-A,-A,A)| = 0$ because any two column vectors in $Q(A,A)$ or $\Delta(A,A,A,A)$ are same and any two column vectors in $Q(A,-A)$ or $\Delta(A,-A,-A,A)$ are same up to a sign. Therefore $|Q(A,B)| = (A^2-B^2)^{{}_{i+2}C_2}Q_1(A,B)$. But since the degree of each elementary signed product from $Q(A,B)$ is always $(i+2)(i+1)$, $Q_1(A,B)$ must be a constant, say k. Therefore $|\Delta| = A^{n-(i+2)}B(C^{n-(i+2)}D)^{i+1}[P_1(A,B,C,D)]^{{}_{i+2}C_2}$ where $P_1(A,B,C,D)$ is a homogeneous polynomial of degree 2.

We claim that $P_1(A,B,C,D) = m(AD-BC)$ for a nonzero constant m. Let $P_1(A,B,C,D) = s_1A^2+s_2AB+s_3AC+s_4AD+s_5B^2+s_6BC+s_7BD+s_8C^2+s_9CD+s_{10}D^2$.

First we want to prove that $s_1 = s_5 = s_8 = s_{10} = 0$. To prove $s_1 = 0$, let $\alpha_0 = \mathrm{Max}\{\alpha : cA^\alpha B^\beta C^\gamma D^\delta$ is an any nonzero elementary signed product from $\Delta\}$. Considering each elementary signed product from Δ, we see easily that $\alpha_0 = n-1+i+(i-1)+\cdots+1+0 = n-1+i(i+1)/2$. But if $P_1(A,B,C,D)$ contains a nonzero term s_1A^2 in its expansion, the highest degree of A among all elementary signed products from Δ would be an integer $n-(i+2)+(i+2)(i+1)$. Note that $n-(i+2)+(i+2)(i+1)-\alpha_0 = n-(i+2)+(i+2)(i+1)-[n-1+i(i+1)/2] = (i+1)(i+2)/2 > 0$ for $i \geq 0$. So $s_1 = 0$. Similarly, we can get $s_5 = s_8 = s_{10} = 0$.

Next, we prove that $s_2 = s_3 = s_7 = s_9 = 0$. To prove $s_3 = 0$, let $a_0 = \text{Max}\{\alpha+\gamma : cA^\alpha B^\beta C^\gamma D^\delta$ is an any nonzero elementary signed product from $\Delta\}$. If $P_1(A,B,C,D)$ contains a nonzero term s_3AC in its expansion, $a_0 = n-(i+2)+(n-(i+2))(i+1)+(i+2)(i+1) = (n-1)(i+2)$. In fact, a_0 is equal to an integer $(n-1)+(n-2)+\cdots+(n-i-2) = (2n-i-3)(i+2)/2$, looking at all elements in Δ. Note that $(n-1)(i+2)-(2n-i-3)(i+2)/2 = (i+1)(i+2)/2 > 0$ for $i \geq 0$. So $s_3 = 0$. Similarly, we can get $s_2 = s_7 = s_9 = 0$.

Therefore $P_1(A,B,C,D) = s_4AD + s_6BC$. Recalling that $|Q(A,B)| = |P(A,B,$
$A,B)| = k(A^2-B^2)^{{}_{i+2}C_2}$ and $|Q(A,A)| = |Q(A,-A)| = 0$, $s_4 = -s_6$. Thus we proved that $|\Delta| = kA^{n-i-2}B(C^{n-i-2}D)^{i+1}(AD-BC)^{{}_{i+2}C_2}$ for some constant k.

To prove $k \neq 0$, consider the term rB^d for a nonzero coefficient r in $|\Delta(1,B,1,1)|$ where $\Delta = \Delta(A,B,C,D)$ and d is the degree of $|\Delta(1,B,1,1)|$ as a polynomial of B. To find rB^d, write down elements of Δ only whose degree of B is the maximum on each column as follows :

$$\begin{pmatrix} * & * & \cdots & {}_{n-1}C_0B^1 & {}_nC_1B^0 \\ \vdots & \vdots & & \vdots & \vdots \\ * & {}_{n-i}C_0B^i & \cdots & {}_{n-1}C_{i-1}B^1 & {}_nC_iB^0 \\ * & {}_{n-i}C_1B^i & \cdots & {}_{n-1}C_iB^1 & {}_nC_{i+1}B^0 \\ {}_nC_{i+2}B^{i+2} & {}_{n-i}C_2B^i & \cdots & {}_{n-1}C_{i+1}B^1 & {}_nC_{i+2}B^0 \end{pmatrix}$$

Then we see that rB^d is equal to

$$(-1)^{i+3}{}_nC_{i+2}B^{i+2+i(i+1)/2}\begin{vmatrix} 0 & 0 & \cdots & {}_nC_1 \\ \vdots & \vdots & & \vdots \\ 0 & {}_{n-i+1}C_0 & \cdots & {}_nC_{i-1} \\ {}_{n-i}C_0 & {}_{n-i+1}C_1 & \cdots & {}_nC_i \\ {}_{n-i}C_i & {}_{n-i+1}C_2 & \cdots & {}_nC_{i+1} \end{vmatrix}$$

$= (-1)^{i+3}(-1)^{(i+1)i/2}{}_nC_{i+2}\cdot{}_nC_{i+1}\cdot B^{i+2+i(i+1)/2}$ by Lemma 1.2. But from $|\Delta(A,B,$
$C,D)| = kA^{n-i-2}B(C^{n-i-2}D)^{i+1}(AD-BC)^{{}_{i+2}C_2}|\Delta(1,B,1,1)|$ is $kB(1-$

$B)^{{}_{i+2}C_2}$. Thus $kB(-B)^{{}_{i+2}C_2} = (-1)^{i+3}(-1)^{(i+1)i/2}{}_nC_{i+2}\cdot{}_nC_{i+1}B^{i+2+i(i+1)/2}$, and so $k = {}_nC_{i+2}\cdot{}_nC_{i+1}$. Therefore we get $|\Delta(A,B,C,D)| = 0$ if and only if $ABCD = 0$.

Claim that $|\Delta(A,B,C,D)| = 0$ if and only of $B = C = 0$ whenever $a_i \neq 0$ for some i $(2i+3 \leq n)$. It is enough to consider the following cases separately:

(a) $C = 0$: It suffices to check the coefficient of yz^{n-1} in the expansion of $f(Az+By, Cz+Dy)$. Then $AB = 0$ implies $B = 0$ since $AD - BC \neq 0$.

(b) $D = 0$: Check the coefficient of $y^{i+2}z^{n-(i+2)}$ in $f(Az+By, Cz+Dy)$. Since $l \leq i$, $AB = 0$ implies $A = 0$ since $AD - BC \neq 0$. Looking at the coefficient of yz^{n-1}, then $A = D = 0$ implies $C^{n-1}Ba_1 = 0$. Since $AD - BC = -BC \neq 0$, $a_1 = 0$. Next, apply the result $A = D = a_1 = 0$ to the coefficient of y^2z^{n-2}. Trivially $a_2 = 0$. Apply this technique in order to $a_3, \cdots, a_i$. Then we get easily that $a_1 = a_2 = \cdots = a_i = 0$. So $f(Az+By, Cz+Dy) = f(By, Cz) = (By)^n + (Cz)^n = ug(z,y) = u(z^n + b_iy^iz^{n-i} + \cdots + b_1y^{n-1}z + y^n)$ for a nonzero constant u implies that $B^n = C^n = u$ and $a_k = b_k = 0$ for $1 \leq k \leq i$ $(2i+3 \leq n)$.

(c) $A = 0$: Since each element of the first column in the matrix is zero if $A = 0$, as in the beginning of the proof, consider $a_i, a_{i-1}, \cdots, a_1, 1$ as a nontrivial solution of the homogeneous equations $[1], [2], \cdots, [i+1]$ assuming that $A = 0$. Then we get an $(i+1) \times (i+1)$ square matrix $A_{i+2,1}(A,B,C,D)$ consisting of coefficients of $a_i, a_{i-1}, \cdots, a_1, 1$ from the equations $[1], [2], \cdots, [i+1]$. In fact, $A_{i+2,1}(A,B,C,D)$ is called a minor matrix of Δ by deleting the first column and the last row of Δ. Then $A_{i+2,1}(0,B,C,D) =$

$$\begin{pmatrix} 0 & 0 & \cdots & \binom{n}{1}C^{n-1}D \\ \vdots & \vdots & & \vdots \\ 0 & \binom{n-i+1}{0}C^{n-i+1}B^{i-1} & \cdots & \binom{n}{i-1}C^{n-i+1}D^{i-1} \\ \binom{n-i}{0}C^{n-i}B^i & \binom{n-i+1}{1}C^{n-i}DB^{i-1} & \cdots & \binom{n}{i}C^{n-i}D^i \\ \binom{n-i}{1}C^{n-i-1}DB^i & \binom{n-i+1}{2}C^{n-i-1}D^2B^{i-1} & \cdots & \binom{n}{i+1}C^{n-i-1}D^{i+1} \end{pmatrix}$$

Then $|A_{i+2,1}(0,B,C,D)| = B^{i(i+1)/2}C^{n(i+1)-(i+2)(i+1)/2}D^{i+1} \times$

$$\begin{vmatrix} 0 & 0 & \cdots & {}_nC_1 \\ \vdots & \vdots & & \vdots \\ 0 & {}_{n-i+1}C_0 & \cdots & {}_nC_{i-1} \\ {}_{n-i}C_0 & {}_{n-i+1}C_1 & \cdots & {}_nC_i \\ {}_{n-i}C_1 & {}_{n-i+1}C_2 & \cdots & {}_nC_{i+1} \end{vmatrix}$$

$= B^{i(i+1)/2}C^{n(i+1)-(i+2)(i+1)/2}D^{i+1}\cdot(-1)^{(i+1)i/2}{}_nC_{i+1}$ by Lemma 1.2.
Since $|A_{i+2,1}(0,B,C,D)| = 0$, $A = 0$ and $AD - BC \neq 0$, D must be zero. From coefficients of $yz^{n-1}, y^2z^{n-2}, \cdots, y^iz^{n-i}$ in the homogeneous equations $[1], [2], \cdots, [i]$, then we have $BCa_1 = BCa_2 = \cdots = BCa_i = 0$ because $A = D = 0$. So we get $a_1 = a_2 = \cdots = a_i = 0$. Thus we have the same result as in the case (b).

(d) $B = 0$: Since $AD - BC \neq 0$, $AD \neq 0$. Just as in the case (c), note that each element of the first column of Δ is zero if $B = 0$. So by the similar method as in the case (c) it is enough to consider the minor matrix $A_{i+2,1}(A,B,C,D)$. Let us compute $A_{i+2,1}(A,O,C,D)$. Then $A_{i+2,1}(A,O,C,D) =$

$$\begin{pmatrix} \binom{n-i}{1}C^{n-i-1}DA^i & \binom{n-i+1}{1}C^{n-i}DA^{i-1} & \cdots & \binom{n}{1}C^{n-1}D \\ \binom{n-i}{2}C^{n-i-2}D^2A^i & \binom{n-i+1}{2}C^{n-i-1}D^2A^{i-1} & \cdots & \binom{n}{2}C^{n-2}D^2 \\ \vdots & \vdots & & \vdots \\ \binom{n-i}{i+1}C^{n-i-i-1}D^{i+1}A^i & \binom{n-i+1}{i+1}C^{n-i-i}D^{i+1}A^{i-1} & \cdots & \binom{n}{i+1}C^{n-(i+1)}D^{i+1} \end{pmatrix}$$

Then $|A_{i+2,1}(A,O,C,D)| = A^{i(i+1)/2}D^{(i+1)(i+2)/2}C^{(i+1)(n-i-1)}\times$

$$\begin{vmatrix} {}_{n-i}C_1 & {}_{n-i+1}C_1 & \cdots {}_nC_1 \\ {}_{n-i}C_2 & {}_{n-i+1}C_2 & \cdots {}_nC_2 \\ \vdots & & \\ {}_{n-i}C_{i+1} & {}_{n-i+1}C_{i+1} & \cdots {}_nC_{i+1} \end{vmatrix}$$

$= {}_nC_{i+1}A^{i(i+1)/2}D^{(i+1)(i+2)/2}C^{(i+1)(n-i-1)}$ by Lemma 1.2.
Since $|A_{i+2,1}(A,O,C,D)| = 0$, $B = 0$ and $AD - BC \neq 0$, $C = 0$.

In the case of (a) and (d), that is, $B = C = 0$, by [3] $f(Az + By, Cz + Dy) = f(Az, Dy) = (Az)^n + a_i(Dy)^{n-i}(Az)^i + \cdots + a_1(Dy)^{n-1}Az + (Dy)^n = ug(z,y) = u(z^n + b_iy^{n-i}z^i + \cdots + b_1y^{n-1}z + y^n)$ for some nonzero constant u. Thus we get : $A^n = u$, $D^n = u$, $D^{n-k}A^ka_k = ub_k (2k + 3 \leq n)$. Since $(A/D)^n = 1$, put $\omega = A/D$. Also $D^{n-k}A^ka_k = ub_k$ implies that $(A/D)^ka_k = b_k$. Thus we get $b_k = a_k\omega^k$ for $k = 1, 2, \cdots, i$ $(2i + 3 \leq n)$.
In the case of (b) and (c), that is, $A = D = 0$, there is nothing to prove.

Conversely, suppose that there exists a unit ω with $\omega^n = 1$ such that $b_k = a_k\omega^k$ for $k = 1, 2, \cdots, i = j$. Define the map ψ by $\psi(z,y) = (\omega z, y)$. Then $f \circ \psi(z,y) = z^n + a_iy^{n-i}(\omega z)^i + \cdots + a_1y^{n-1}(\omega z) + y^n = z^n + b_iy^{n-i}z^i + \cdots + b_1y^{n-1}z + y^n = g(z,y)$. Thus the theorem is proved.

Corollary 1.4. *Let f and g be defined as in the Theorem 1.3. If $f \approx g$ and $f(Az+By, Cz+Dy) = ug(z,y)$ for some nonzero constant u, then either $B = C = 0$, or $A = D = 0$ and $a_k = b_k = 0$ for $1 \le k \le i = j$ with $n \ge 2i+3$.*

2. Analytic classification of plane curve singularities defined by $z^3 + ay^2z + y^3$ or $z^4 + ay^3z + y^4$.

Theorem 2.1. *Let $V = \{(z,y) : f = z^3 + ay^2z + y^3 = 0\}$ be an analytic subvariety of a polydisc near the origin in $\mathbf{C}^2$ where f is square-free. Then any f is analytically equivalent each other for any number a.*

Proof. We know that any homogeneous polynomial with two variables of degree three which is square-free can be written into $f = z^3 + ay^2z + y^3$ by a nonsingular linear change of coordinate at the origin, and also this f can be transformed into $u(z^3 + \alpha yz^2 + \beta y^2 z) = uz(z^2 + \alpha yz + \beta y^2)$ for a nonzero constant u by another linear change of coordinates. Note that $uz(z^2 + \alpha yz + \beta y^2) = uz(z^2 + 2\alpha_1 y_1 z + y_1^2)$ by a linear change of coordinates and that this polynomial becomes $uz((1-\alpha_1^2)z^2 + (y_1 - \alpha_1 z)^2) = u_1 z_1 (z_1^2 + y_2^2)$ for a nonzero constant u_1 where $z_1 = (1-\alpha_1^2)^{1/2} z$, $y_2 = y_1 - \alpha_1 z$ and $\alpha_1 \neq 1$. Thus $f \approx z(z^2 + y^2)$.

Theorem 2.2. *Let $V = \{(z,y) : f = z^4 + \alpha y^3 z + y^4 = 0\}$ and $W = \{(z,y) : g = z^4 + \beta y^3 z + y^4 = 0\}$ be analytic subvarieties of a polydisc near the origin in $\mathbf{C}^2$ where f and g are square-free. Then $f \approx g$ if and only if $\alpha^4 = \beta^4$.*

Proof. Assume that $f \approx g$. Then $f(Az+By, Cz+Dy) = (Az + By)^4 + \alpha(Cz+Dy)^3(Az+By) + (Cz+Dy)^4 = (A^4 + \alpha AC^3 + C^4)z^4 + (4A^3B + (3C^2DA + C^3B)\alpha + 4C^3D)yz^3 + (6A^2B^2 + (3CD^2A + 3C^2DB)\alpha + 6C^2D^2)y^2z^2 + (4AB^3 + (D^3A + 3CD^2B)\alpha + 4CD^3)y^3z + (B^4 + \alpha BD^3 + D^4)y^4 = ug = u(z^4 + \beta y^3 z + y^4)$ for a nonzero constant u by [3]. So we have

$$A^4 + AC^3\alpha + C^4 = u \tag{1}$$

$$4A^3B + (3C^2DA + C^3B)\alpha + 4C^3D = 0 \tag{2}$$

$$6A^2B^2 + (3CD^2A + 3C^2DB)\alpha + 6C^2D^2 = 0 \tag{3}$$

$$4AB^3 + (D^3A + 3CD^2B)\alpha + 4CD^3 = u\beta \tag{4}$$

$$B^4 + BD^3\alpha + D^4 = u \tag{5}$$

Subtracting the equation (5) from the equation (1), we get

$$(A^4 - B^4) + (AC^3 - BD^3)\alpha + C^4 - D^4 = 0 \tag{6}$$

Now consider the following two cases : (i) $ABCD\alpha = 0$ and (ii) $ABCD\alpha \neq 0$.

(i) Let $ABCD\alpha = 0$. Then $A = 0$, $B = 0$, $C = 0$, $D = 0$ or $\alpha = 0$.

(a) $A = 0$: From equations (2) and (3), we get

$$C^3 B\alpha + 4C^3 D = 0,$$
$$3C^2 DB\alpha + 6C^2 D^2 = 0.$$

These two equations give $6C^3D^2 = 0$ and so $D = 0$ since $AD - BC \neq 0$. From (2), $C^3B\alpha = 0$ implies $\alpha = 0$. From (1) and (5), $C^4 = u = B^4$ and (4) implies $\beta = 0$. Thus $\alpha^4 = \beta^4 = 0$.

(b) $D = 0$: By (3), $AB = 0$ and so $A = 0$. Then we get the same result as in the case (a).

(c) $C = 0$: By (2), $AB = 0$ and so $B = 0$. By (1), (4) and (5), $A^4 = D^4 = u$ and $D^3A\alpha = u\beta$. Thus $A\alpha = D\beta$ and so $\alpha^4 = \beta^4$.

(d) $B = 0$: From (2) and (3), we get

$$3C^2 DA\alpha + 4C^3 D = 0,$$
$$3CD^2 A\alpha + 6C^2 D^2 = 0.$$

These two equations give $2C^3D^2 = 0$ and so $C = 0$. Then we get the same result as in the case (c).

(e) $\alpha = 0$: From (2) and (3), we get

$$0 = \begin{vmatrix} 4A^3B & 4C^3D \\ 6A^2B^2 & 6C^2D^2 \end{vmatrix} = 24A^2BC^2D \begin{vmatrix} A & C \\ B & D \end{vmatrix}.$$

So $ABCD = 0$. Then we get the same result as in the case (a), (b), (c) or (d).

(ii) Hereafter we assume that $ABCD\alpha \neq 0$.

Then from (2), (3) and (6) which are considered homogeneous equations, we get

$$0 = \begin{vmatrix} 4A^3B & 3C^2DA + C^3B & 4C^3D \\ 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ A^4 - B^4 & AC^3 - BD^3 & C^4 - D^4 \end{vmatrix}$$
$$= 6C(AD - BC)^3(A^2C^3 - BD^2(2AD + BC)).$$

Since $ABCD \neq 0$, $2AD + BC \neq 0$. So

$$BC + 2AD = A^2C^3/(BD^2) \tag{7}$$

From (2) and (3), we get

$$C^2(3AD + BC)\alpha = -4(A^3B + C^3D) \quad \text{and}$$
$$CD(AD + BC)\alpha = -2(A^2B^2 + C^2D^2),$$

which by eliminating α, give $0 = 2D(AD + BC)(A^3B + C^3D) - C(3AD + BC)(A^2B^2 + C^2D^2) = (A^2B(2AD + BC) - C^3D^2)(AD - BC)$. Thus

$$2AD + BC = C^3D^2/(A^2B) \tag{8}$$

From (7) and (8), $A^2C^3/(BD)^2 = C^3D^2/(A^2B)$ and so we get

$$A^4 = D^4 \tag{9}$$

From (2), (3) and (4) we are going to compute α as follows : Let

$$\Delta = \begin{pmatrix} 4A^3B & 3C^2DA + C^3B & 4C^3D \\ 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ 4AB^3 & D^3A + 3CD^2B & 4CD^3 \end{pmatrix}, \text{ and then}$$

$$|\Delta| = 24ABC^2D^2(AD - BC)^3.$$

$$\alpha = \frac{1}{|\Delta|} \begin{vmatrix} 4A^3B & 0 & 4C^3D \\ 6A^2B^2 & 0 & 6C^2D^2 \\ 4AB^3 & u\beta & 4CD^3 \end{vmatrix} = \frac{-u\beta A}{D(AD - BC)^2} \tag{10}$$

Again, from (3), (4) and (5), we want to compute α as follows. Let

$$\Delta' = \begin{pmatrix} 6A^2B^2 & 3CD^2A + 3C^2DB & 6C^2D^2 \\ 4AB^3 & D^3A + 3CD^2B & 4CD^3 \\ B^4 & BD^3 & D^4 \end{pmatrix}$$

Then $|\Delta'| = 6B^2D^4(AD - BC)^3$ and so

$$\alpha = \frac{1}{|\Delta'|} \begin{vmatrix} 6A^2B^2 & 0 & 6C^2D^2 \\ 4AB^3 & u\beta & 4CD^3 \\ B^4 & u & D^4 \end{vmatrix} \tag{11}$$
$$= \frac{u}{D^2(AD - BC)^3} \cdot ((AD + BC)\beta - 4AC)$$

From (10) and (11), we get

$$\frac{-u\beta A}{D(AD-BC)^2} = \frac{u}{D^2(AD-BC)^3}\cdot((AD+BC)\beta - 4AC)$$

Thus

$$\beta = \frac{4AC}{2AD+BC} \tag{12}$$

Eliminating the first terms from the equations (2) and (3), we get $[6B(3C^2DA + C^3B) - 4A(3CD^2A + 3C^2DB)]\alpha + 24BC^3D - 24AC^2D^2 = 0$. Simplifying the above, we have

$$\alpha = -\frac{4CD}{2AD+BC} \tag{13}$$

From (12) and (13), $\beta/A = -\alpha/D$. Since $A^4 = D^4$ by (9), we get $\beta^4 = \alpha^4$.

Now, conversely, if $\alpha^4 = \beta^4 \neq 0$, then $f(\beta z, \alpha y) = \beta^4 z^4 + \alpha(\alpha^3 y^3)\beta z + \alpha^4 y^4 = \beta^4(z^4 + \beta y^3 z + y^4) = \beta^4 g(z,y)$. If $\alpha = \beta = 0$, there is nothing to prove.

Corollary 2.3. *Let f and g be defined as in Theorem 2.2. If $f \approx g$ and $f(Az+By, Cz+Dy) = ug(z,y)$ for a nonzero constant u, then $ABCD$ may not be zero.*

Proof. It is enough to show that there is such an example with $ABCD \neq 0$. Let $f(z,y) = z^4 - 4e^{\pi i/4}y^3z + y^4$ and $A = 1$, $B = e^{3\pi i/4}$, $C = e^{\pi i/4}$ and $D = 1$. Then $AD - BC = 2 \neq 0$ and $f(Az+By, Cz+Dy) = 4(z^4 + 4e^{\pi i/4}y^3z + y^4) = 4g(z,y)$ by tedius computations. Note that $ABCD \neq 0$.

Finally we are going to give an example which is a help to understand the condition for restriction on the degree of homgeneous polynomials in Theorem 1.3 as follows :

Let $V = \{(z,y) : f(z,y) = z^5 + 10y^3z^2 + 5y^4z + y^5 = 0\}$. By a linear transformation $T : (z,y) \mapsto (y, z-y)$,

$$\begin{aligned}(f\circ T)(z,y) &= f(y, z-y)\\ &= y^5 + 10(z-y)^3y^2 + 5(z-y)^4y + (z-y)^5\\ &= z^5 - 10y^3z^2 + 15y^4z - 5y^5.\end{aligned}$$

By another linear transformation $S : (z, y) \mapsto (z, -5^{-1/5}y)$, $f \circ T$ will be $g(z, y) = z^5 + 10 \cdot 5^{-3/5}y^3z^2 + 15 \cdot 5^{-4/5}y^4z + y^5$. Note that without the condition in Theorem 1.3, $f \approx g$.

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REDUCIBLE HILBERT SCHEME OF SMOOTH CURVES WITH POSITIVE BRILL-NOETHER NUMBER*

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0. Introduction

In [S], Severi has asserted with an incomplete proof that the subscheme $\mathcal{I}'_{d,g,r}$ which is the union of the irreducible components of the Hilbert scheme $\mathcal{H}_{d,g,r}$ whose general points correspond to smooth, irreducible and non-degenerate curves of degree d and genus g in $\mathbb{P}^r$ is irreducible if $d \geq g+r$. Also in [H], it has been conjectured that $\mathcal{I}'_{d,g,r}$ is irreducible if the Brill-Noether number $\rho(d,g,r) := g-(r+1)(g-d+r)$ is positive.

In this paper we demonstrate various reducible examples of the subscheme $\mathcal{I}'_{d,g,r}$ with positive Brill–Noether number. Indeed an example of a reducible $\mathcal{I}'_{d,g,r}$ with positive $\rho(d,g,r)$, namely the example $\mathcal{I}'_{2g-8,g,g-8}$ (or other variations of it), has been known to some people (including the author), but it seems to have first appeared in the literature in [EH]. The purpose of this paper is to add a wider class of examples to the list of such reducible examples by using general k-gonal curves. We also show that $\mathcal{I}'_{d,g,r}$ is irreducible for the range of $d \geq 2g-7$ and $g-d+r \leq 0$. Throughout we will be working over the field of complex numbers.

1. Terminologies, notations and some preliminary results

We first recall that, given non-negative integers r, d, for every point p of the moduli space $\mathcal{M}_g$ of smooth curves of genus g and any sufficiently small connected neighborhood U of p, there are a smooth connected variety $\mathcal{M}$, a finite ramified covering:

$$h : \mathcal{M} \to U$$

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and two varieties, proper over $\mathcal{M}$:

$$\xi : \mathcal{C} \to \mathcal{M}, \quad \pi : \mathcal{G}^r_d \to \mathcal{M}$$

with the following properties:

(1) $\mathcal{C}$ is a universal curve over $\mathcal{M}$, i.e. for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$.

(2) $\mathcal{G}^r_d$ parametrizes pairs $(p, \mathcal{D})$, where $p \in \mathcal{M}$ and $\mathcal{D}$ is a linear system (possibly incomplete) of degree d and of dimension r, which is denoted by g^r_d, on $C = \xi^{-1}(p)$.

Let $\mathcal{G}$ be the union of irreducible components of $\mathcal{G}^r_d$ whose general element corresponds to pairs $(p, \mathcal{D})$ such that $\mathcal{D}$ is a very ample linear system on $\xi^{-1}(p) = C$, i.e. $\mathcal{D}$ induces an embedding of C into $\mathbb{P}^r$.

In order to show the irreducibility of $\mathcal{I}'_{d,g,r}$, it is sufficient to demonstrate the irreducibility of $\mathcal{G}$ since the open subset of $\mathcal{I}'_{d,g,r}$ consisting of points corresponding to smooth curves is a $\mathbb{P}GL_{r+1}$ bundle over an open subset of $\mathcal{G}$. Also we will utilize the following fact which is basic in the theory for our purposes; see [AC1] or [H] for detailed discussion and proof.

Proposition 1.1. *There exists a unique component $\mathcal{G}_0$ of $\mathcal{G}$ which dominates $\mathcal{M}$ (or $\mathcal{M}_g$) if the Brill-Noether number $\rho(d, g, r)$ is positive. Furthermore in this case, for any possible component $\mathcal{G}'$ of $\mathcal{G}$ other than $\mathcal{G}_0$, a general element $(p, \mathcal{D})$ of $\mathcal{G}'$ is such that $\mathcal{D}$ is a special linear system on $C = \xi^{-1}(p)$.*

Remark 1.2. In the Brill-Noether range, i.e. in the range $\rho(d, g, r) > 0$, we call the unique component $\mathcal{G}_0$ of $\mathcal{G}$ which dominates $\mathcal{M}$, the principal component. We call other possible components exceptional components.

The following facts will also turn out to be useful for our purposes; see [AC2] for the proof.

Proposition 1.3. (i) *Any component of $\mathcal{G}^r_d$ has dimension at least $3g - 3 + \rho(d, g, r)$.*

(ii) *Suppose $g > 0$ and let X be a component of $\mathcal{G}^2_d$ whose general element $(p, \mathcal{D})$ is such that $\mathcal{D}$ is a linear system on $C = \xi^{-1}(p)$ which is not composed with an involution. Then*

$$\dim X = 3g - 3 + \rho(d, g, 2) = 3d + g - 9.$$

(iii) *The variety* $\mathcal{G}^1_d$ *is smooth of dimension:*

$$\rho(d, g, 1) + \dim \mathcal{M}_g.$$

By using (1.3)-(ii), one can prove the following fact regarding a subvariety of $\mathcal{G}^r_d$ consisting of birationally very ample linear series; see [KK].

Proposition 1.4. *Let $\mathcal{W}$ be an irreducible closed subvariety of $\mathcal{G}^r_d$, $r \geq 2$, whose general element $(p, \mathcal{D})$ is such that $\mathcal{D}$ is complete, special and birationally very ample on $C = \xi^{-1}(p)$. Then*

$$\dim \mathcal{W} \leq 3d + g - 4r - 1.$$

Corollary 1.5. *Whenever*

$$2g + 1 + \frac{3 - 3g}{r} < d \leq 2g - 2, \quad (r \geq 3)$$

$\mathcal{G}$ (and hence $\mathcal{I}'_{d,g,r}$) is irreducible with the expected dimension $3g - 3 + \rho(d, g, r)$.

Proof. Suppose there exists an exceptional component $\mathcal{G}'$ of $\mathcal{G}$. Sine we are in the Brill-Noether range, by Proposition (1.1) there is an open set $\mathcal{V}$ of $\mathcal{G}'$ whose elements consist of pairs $(p, \mathcal{D})$ such that $\mathcal{D}$ is a special very ample linear system on $C = \xi^{-1}(p)$. Consider the map

$$\psi : \mathcal{V} \to \mathcal{G}^\alpha_d$$

defined by $\psi(p, \mathcal{D}) = (p, |D|)$ where $D \in \mathcal{D}$, $\alpha = \dim |D|$. Then by Proposition (1.4) and by noting the fact that the dimension of a fiber of ψ over a point in $\psi(\mathcal{V})$ is $\dim \mathbb{G}(r, \alpha)$, we have $\dim \mathcal{G}' = \dim \mathcal{V} \leq 3d + g - 4\alpha - 1 + (r + 1)(\alpha - r) = 3d + g - 1 - r^2 - r + (r - 3)\alpha$.

On the other hand, by Castelnuovo theory the largest possible α in case $d \geq g$ is $\frac{2d-g+1}{3}$. Thus the above inequality implies

$$\dim \mathcal{G}' \leq 3d + g - 1 - r^2 - r + (r - 3)\frac{2d - g + 1}{3} < 3g - 3 + \rho(d, g, r),$$

which is contradictory to (i) of Proposition (1.3).

Remark 1.6. (i) (1.5) was also known to L. Ein; see [E-1]. He later gave a wider range of d, g, r for which $\mathcal{I}'_{d,g,r}$ is irreducible when $r \geq 5$; see [E-2].

(ii) It is quite easy to show that in case $d \geq 2g-1$, $\mathcal{I}'_{d,g,r}$ is empty if $r > d-g$, and is irreducible if $r \leq d-g$: see [H], page 61.

2. Irreducibility of $\mathcal{I}'_{d,g,r}$ with large d

Theorem 2.1. *$\mathcal{I}'_{d,g,r}$ is irreducible for $d \geq 2g-7$ and $g+r \leq d$, $r \geq 3$.*

Proof. For the case $d \geq 2g-2$, it is a consequence of the Corollary (1.5) and Remark (1.6)-(ii). For the case $2g-7 \leq d \leq 2g-3$, we proceed as follows. Let $d = 2g-2-k$ where $1 \leq k \leq 5$. Suppose there exists an exceptional component $\mathcal{G}'$ of $\mathcal{G}$. Then by the Proposition (1.1), a general element $(p, \mathcal{D}) \in \mathcal{G}'$ is such that $\mathcal{D}$ is a special linear system on $C = \xi^{-1}(p)$, i.e. $\dim|\mathcal{D}| > d-g$. Let $\mathcal{V}$ be an open subset of $\mathcal{G}'$ consisting of elements $(p, \mathcal{D})$ with $\dim|\mathcal{D}| = \alpha > d-g$. Consider the map

$$\Psi : \mathcal{V} \to \mathcal{G}_k^{k+\alpha+1-g}$$

defined by $\Psi(p, \mathcal{D}) = (p, |K-D|)$ where $D \in \mathcal{D}$ and K is a canonical divisor on $C = \xi^{-1}(p)$. Then by noting the fact that the dimension of a fiber of Ψ over a point in $\mathcal{G}_k^{k+\alpha+1-g}$ is at most $\dim \mathbb{G}(r, \alpha)$, we have

$$\dim \mathcal{G}' = \dim \mathcal{V} \leq \dim \mathcal{G}_k^{k+\alpha+1-g} + (r+1)(\alpha - r). \tag{2.1.1}$$

By Clifford theorem and the inequality $\alpha > d-g = g-2-k$, we have $g-1-k \leq \alpha \leq g-1-\frac{k}{2}$. Thus the following pairs for (k, α)'s are possible: (i) $\{(k, g-k-1); 1 \leq k \leq 5\}$ (ii) $\{(k, g-k); 2 \leq k \leq 5\}$ (iii) $\{(k, g-k+1); k = 4 \text{ or } 5\}$.

For the case (i), by the inequality (2.1.1) and the hypothesis $g+r \leq d$,

$$\dim \mathcal{G}' \leq \dim \mathcal{G}_k^0 + (r+1)(g-k-1-r) < 3g-3+\rho(2g-2-k, g, r)$$

which is a contradiction.

For the case (ii), again by (2.1.1) and the hypothesis $g+r \leq d$, we have

$$\begin{aligned}\dim \mathcal{G}' &\leq \dim \mathcal{G}_k^1 + (r+1)(g-k-r) = 2g-5+2k+(r+1)(g-k-r) \\ &< 3g-3+\rho(d, g, r)\end{aligned}$$

which is a contradiction.

(iii) Suppose $(k, \alpha) = (4, g-3)$. Because $\alpha = \frac{d}{2}$, any $(p, \mathcal{D}) \in \mathcal{V}$ is such that $C = \xi^{-1}(p)$ is a hyperelliptic curve by Clifford theorem. But this is a contradiction since a hyperelliptic curve cannot have a birationally very ample special linear system.

Suppose $(k, \alpha) = (5, g-4)$. Consider $\Psi(p, \mathcal{D}) = (p, |K - D|) \in \mathcal{G}^2_5$. If the complete $|K - D|$ has no base point, $|K - D|$ induces a birational map on $C = \xi^{-1}(p)$ and $g(C) \leq 6$, contrary to the hypothesis $g + r \leq d$ and $r \geq 3$. Thus $|K - D|$ has a base point and there exists a g^2_4 on C whence C is a hyperelliptic curve by Clifford theorem. Again this is a contradiction because there cannot exist a birationally very ample special linear system on a hyperelliptic curve.

To demonstrate the reducibility of $\mathcal{I}'_{2g-8,g,g-8}$, we do need the following lemma whose elementary proof we omit here.

Lemma 2.2. *Let C be a trigonal curve of genus $g \geq 8$ with the trigonal pencil g^1_3. Then $|K - 2g^1_3|$ is very ample and any g^2_6 is equal to $2g^1_3$.*

Theorem 2.3. (i) *For $r < \frac{2g-7}{3}$, $r \leq g-8$ and $r \geq 3$, $\mathcal{I}'_{2g-8,g,r}$ is irreducible.*

(ii) *For $\frac{2g-7}{3} \leq r \leq g-8$ and $r \geq 3$, $\mathcal{I}'_{2g-8,g,r}$ is reducible with two components. Furthermore, a general element of the exceptional component is trigonal.*

Proof. We use all the notations used in the proof of Theorem (2.1). Let $\mathcal{G}'$ be an exceptional component of $\mathcal{G}$ and $\alpha = \dim |\mathcal{D}|$ for general $(p, \mathcal{D}) \in \mathcal{G}'$. By Clifford theorem, we have $\alpha = g-7, g-6$ or $g-5$.

(i) If $\alpha = g-7$ or $g-6$, one can use the inequality (2.1.1) and proceed exactly as in the previous theorem to show that these cases do not occur.

(ii) If $\alpha = g-5$, $|K - D| = g^2_6$ where $D \in \mathcal{D}$ for a general $(p, \mathcal{D}) \in \mathcal{G}'$. By the hypothesis $3 \leq r \leq g-8$, the map induced by $|K - D|$ on $C = \xi^{-1}(p)$ is not birational. Instead, C may be either hyperelliptic, trigonal or elliptic-hyperelliptic. But C cannot be hyperelliptic because an hyperelliptic curve cannot have a very ample special linear system. If C is elliptic-hyperelliptic, $|K - D| = g^2_6 = \phi^*(g^2_3)$ where ϕ is the map of degree 2 onto an elliptic curve E. Then $|\mathcal{D}| = |K - g^2_6| = g^{g-5}_{2g-8}$ is not even birationally very ample because $|K - g^2_6 - P - Q| = g^{g-6}_{2g-10}$ where $P + Q = \phi^*(R)$, $R \in E$. Thus C cannot be elliptic-hyperelliptic.

If C is a trigonal curve, $|K-D| = g^2_6 = 2g^1_3$ and $|\mathcal{D}| = |D|$ is very ample by Lemma (2.2). Thus the only possible exceptional component of $\mathcal{G}$ may arise in this way; in other words, $\mathcal{V}$ surjects onto an open set of $\mathcal{M}^1_{g,3}$ if such $\mathcal{G}'$ exists. Hence

$$\dim \mathcal{G}' = \dim \mathcal{V} = \dim \mathcal{G}^1_3 + (r+1)(g-5-r) \geq 3g-3+\rho(d,g,r),$$

which proves the first half of the theorem.

On the other hand, suppose the above inequality holds and let $\mathcal{W}$ be the closed subvariety of $\mathcal{G}^r_d$ whose general element $(p,\mathcal{D})$ is such that p corresponds to a trigonal curve and $\mathcal{D}$ is a general r-dimensional subspace of $|K-2g^1_3|$ on $C = \xi^{-1}(p)$, i.e. $\mathcal{W}$ is just the locus in $\mathcal{G}^r_{2g-8}$ over trigonal curves. By the preceding discussion, $\mathcal{W}$ is indeed a component of $\mathcal{G}$ other than $\mathcal{G}_0$ because $\mathcal{D}$ is very ample. Furthermore, the uniqueness of such an exceptional component $\mathcal{G}' = \mathcal{W}$ is also obvious from the preceding discussion.

3. Exceptional components over general k-gonal curves

We now construct more examples of reducible $\mathcal{I}'_{d,g,r}$ with positive Brill-Noether number by using general k-gonal curves. We do need the following lemma due to Ballico ; [B], Proposition 1.

Lemma 3.1. *Fix positive integers g,k,ℓ with $k \geq 2$, $g \geq 2k-2$ and $1 \leq \ell \leq [\frac{g}{k-1}]$. Let $|E| = g^1_k$ be the unique pencil of degree k on a general k-gonal curve of genus g. Then $\dim|\ell E| = \ell$.*

Corollary 3.2. *Fix positive integers g,k,ℓ with $k \geq 3$, $g \geq 2k-2$ and $1 \leq \ell \leq [\frac{g}{k-1}]-2$. Let $|E| = g^1_k$ be the unique pencil of degree k on a general k-gonal curve C of genus g. Then for any $P,Q \in C$, $\dim|\ell E+P+Q| = \ell$.*

Proof. We first claim that $\dim|\ell E+P| = \ell$ for any $P \in C$: Suppose $\dim|\ell E+P| = \ell+1$ for some $P \in C$. By Lemma (3.1), $\dim|(\ell+1)E| = \dim|\ell E+P+E'-P| = \dim|\ell E+P| = \ell+1$, $E' \in |E|$. Then $E'-P \succ 0$ is the base locus of $|(\ell+1)E|$ which is in fact base-point-free.

Suppose that $\dim|\ell E+P+Q| = \ell+1$ for some $P,Q \in C$. By the first claim and Lemma (3.1), we have $\ell+1 = \dim|\ell E+P+Q| = \dim|(\ell+1)E| = \dim|(\ell+1)E+P| = \dim|\ell E+P+Q+E''-Q|$, where $E'' \in |E|$ and $E''-Q \succ 0$.

Then $E'' - Q$ is the base locus of the linear system $|(\ell+1)E+P|$. But this is a contradiction because the actual base locus of $|(\ell+1)E+P|$ is P.

Lemma (3.2) implies the following immediate corollary.

Corollary 3.3. *Let g, k, ℓ be positive integers such that $k \geq 3$, $g \geq 2k-2$ and $1 \leq \ell \leq [g/k-1]-2$. Let C be a general k-gonal curve with the unique pencil $|E|$ of degree k. Then $|K-\ell E|$ is very ample.*

Theorem 3.4. *Let g, k, ℓ, r be integers such that $k \geq 3$, $r \geq 3$, $2 \leq \ell \leq [\frac{g}{k-1}]-2$, $\frac{2g+2-2k}{\ell+1}-1 < r \leq g-2-\ell k$ and $d = 2g-2-\ell k$. Then $\mathcal{I}'_{d,g,r}$ is reducible with at least one exceptional component containing the family of general k-gonal curves.*

Proof. By Corollary (3.3), there exists a family $\mathcal{A}$ of k-gonal curves in $\mathbb{P}^r$ of degree $2g-2-\ell k$ embedded by a general r-dimensional sub-system of $|K-\ell g^1_k|$. Furthermore

$$\begin{aligned}\dim \mathcal{A} &\geq \dim \mathcal{M}^1_{g,k} + \dim \mathbb{G}(r, g-\ell k+\ell-1) + \dim(\operatorname{Aut}\mathbb{P}^r)\\ &> 3g-3+\rho(d,g,r)+(r+1)^2-1\end{aligned}$$

in the given range of g, k, ℓ and r. Thus there must be an exceptional component containing the family of general k-gonal curves and hence $\mathcal{I}'_{d,g,r}$ is reducible.

Remark 3.5. (i) In all the examples we demonstrated so far, we deliberately chose the numbers d, g and r so that the Brill-Noether number was positive, in particular $\rho(d,g,r) \geq g$. On the other hand, one can come up with a bunch of examples e.g. $\mathcal{I}'_{2g-2-\ell k, g, g-\ell k+\ell-1}$ which violate the so called Brill-Noether-Petri Principle (see [EH], § 2) for those g, k and ℓ in the same range as in Theorem (3.4) and in these cases the Brill-Noether number becomes negative.

(ii) If $\ell = 2$ in Theorem (3.4), one can show that the family of general k-gonal curves contained in an exceptional component of $\mathcal{I}'_{2g-2-2k,g,r}$ is indeed dense in the component.

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