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Proceedings of The 1st GARC SYMPOSIUM on Pure and Applied Mathematics

PART I

Edited by

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PREFACE

The first GARC Symposium on Pure and Applied Mathematics was held at the Global Analysis Research Center, the Seoul National University, from Thursday, February 13 to Friday, February 21, 1992. All the meetings were held at the Department of Mathematics Building.

The Global Analysis Research Center was inaugurated on March 1, 1991 under the Science Research Center Program of the Korea Science and Engineering Foundation to promote research ability in the field of mathematics in Korea. The central aim of the Global Analysis Research Center is the cooperative study of various analytic problems defined on manifolds such as partial differential equations, nonlinear analysis, operator algebra, dynamical systems and other related problems. The approach is a comprehensive one that also requires basic understandings of topological, geometric and algebraic properties of manifolds.

In order to maximize the efficiency of research, the Global Analysis Research Center has 6 Research Sections adapted to the natural division of research activities of the participating members. In accordance with the 6 Research Sections of the Global Analysis Research Center, the first GARC Symposium was carried out in 6 sessions; Partial Differential Equations, Nonlinear Analysis, Operator Algebra, Differential Geometry and Dynamical System, Topology and Geometry of Manifolds, and Complex Analytic Manifolds and Varieties.

The aim of the GARC Symposium was intended to set up mutual understandings on the interest of each research member and to explore current problems in the area of Global Analysis. Accordingly, almost all the research members of the Research Center including post doctors participated at the symposium. In addition, the organization committee invited several mathematicians from abroad. A few speakers were asked to survey their fields but the majority of speakers presented their recent research works.

In this proceedings of two issues, we collect all the lecture materials which were presented at the symposium. We would like to thank all the speakers, especially those professors from abroad, for their enthusiastic participation and their cooperation in writing up their talks. We would also like to thank the Korea Science and Engineering Foundation for their support to the Global Analysis Research Center and the Department of Mathematics of the Seoul National University for its hospitality.

We hope that in publishing this proceedings we will allow much wider audience to share in some of the work and enthusiasm of the participants at the symposium.

1992.10.

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REPRESENTATION OF QUASIANALYTIC ULTRADISTRIBUTIONS

SOON-YEONG CHUNG AND DOHAN KIM

ABSTRACT. We give the following representation theorem for a class containing quasianalytic ultradistributions and all the non-quasianaly- tic ultradistributions: Every ultradistribution u in this class can be written as

$$u = P(\Delta)g(x) + h(x)$$

where $g(x)$ is a bounded continuous function, $h(x)$ is a bounded real analytic function and $P(d/dt)$ is an ultradifferential operator. Also, we show that the boundary value of every heat function with some exponential growth condition determines an ultradistribution in this class. Our interest lies in the quasianalytic case, although the theorems do not exclude non-quasianalytic classes. The proofs of the theorems in this paper will appear in [C-K].

1. A class of quasianalytic ultradistributions

Let M_p, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers. An infinitely differentiable function ϕ on Ω is called an ultradifferentiable function of class (M_p) (of class $\{M_p\}$ resp.) if for any compact set K of Ω for each $h > 0$ (there exist constants $h > 0$ resp.) such that

$$|\phi|_{M_p,K,h} = \sup_{\substack{x\in K \\ \alpha\in \mathbf{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}M_{|\alpha|}} \tag{1.1}$$

is finite. We impose the following conditions on M_p:

(M.0) For any $A > 0$ there exists a constant $C > 0$ such that

$$p! \le CA^p M_p, \; p = 0, 1, 2, \cdots.$$

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(M.1) $M_p^2 \le M_{p-1}M_{p+1},\ p = 1, 2, \cdots$.
(M.2) There are constants C and H such that

$$M_{p+q} \le CH^{p+q}M_pM_q, \quad p, q = 0, 1, 2, \cdots.$$

We call the above sequence M_p the defining sequence and denote by $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$ resp.) the space of all ultradifferentiable functions of class (M_p) (of class $\{M_p\}$ resp.) on Ω

As usual, we denote by $\mathcal{E}'_{(M_p)}(\Omega)$ ($\mathcal{E}'_{\{M_p\}}(\Omega)$ resp.) the strong dual space of $\mathcal{E}_{(M_p)}(\Omega)$ (of $\mathcal{E}_{\{M_p\}}(\Omega)$ resp.) and we call its elements ultradistributions of Beurling type (of Roumieu type resp.) with compact support in Ω. Let $K \subset \mathbf{R}^n$ be a compact set. We denote by $\mathcal{E}'_{(M_p)}(K)$ ($\mathcal{E}'_{\{M_p\}}(K)$ resp.) the set of ultradistributions of class (M_p) (of class $\{M_p\}$ resp.) with support in K. For each defining sequence M_p we define for $t > 0$

$$(1.3)\qquad \begin{aligned} M(t) &= \sup_p \log \frac{t^p M_0}{M_p} \\ M^*(t) &= \sup_p \log \frac{p!t^p M_0}{M_p} \\ \overline{M}(t) &= \sup_p \log \frac{p!t^p M_0^2}{M_p^2} \end{aligned}$$

An operator of the form

$$(1.4)\qquad P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha, \quad a_\alpha \in \mathbf{C}$$

is called an ultradifferential operator of class (M_p) (of class $\{M_p\}$ resp.) if there are constants L and C (for every $L > 0$ there is a constant $C > 0$ resp.) such that

$$(1.5)\qquad |a_\alpha| \le CL^{|\alpha|}/M_{|\alpha|}, \quad \alpha \in \mathbf{N}_0^n.$$

2. Structure Theorems

In this section it will be shown that every $u \in \mathcal{E}'_{(M_p)}(K)$ can be written as an infinite sum of derivatives of a continuous function modulo a bounded real analytic function and that every $u \in \mathcal{E}'_{(M_p)}(K)$ can be represented by the boundary value of a heat function satisfying some exponential growth condition.

We denote by $E(x,t)$ the n-dimensional heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Lemma 2.1([M]). *$E(\cdot,t)$ is an entire function of order 2 for every $t > 0$. We have the following properties on E:*

(i) $\int_{\mathbf{R}^n} E(x,t)dx = 1, \quad t > 0$

(ii) *There are positive constants C and a such that*

$$|\partial_x^\alpha E(x,t)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{\frac{1}{2}} \exp[-a|x|^2/4t], \quad t > 0$$

where a can be chosen as close as desired to 1 and $0 < a < 1$.

Lemma 2.2. *Let K be a compact subset of $\mathbf{R}^n$ and Ω be a bounded open set containing K. For every $\phi \in \mathcal{E}_{(M_p)}$ let*

$$\phi_t(x) = \int_\Omega E(x-y,t)\phi(y)dy.$$

Then ϕ_t converges to ϕ in $\mathcal{E}_{(M_p)}(\Omega)$ as $t \to 0+$.

For each defining sequence M_p we impose the following condition:

(C) There exists a positive integer k such that

$$\liminf_{p\to\infty} \left(\frac{m_{kp}}{m_p}\right)^2 > k$$

where $m_p = M_p/M_{p-1}, \quad p = 1, 2, \cdots$.

Remark 2.3. *(i) Let $m_p = p(\log p)^\alpha$, $\alpha > 0$. Then $M_p = m_2 \cdots m_p$ satisfies (C). Thus the defining sequence for this standard quasianalytic class satisfies (C).*

(ii) The Gevrey sequence $M_p = p!^s$, $s > 1$, satisfies (C).

(iii) Furthermore, if M_p satisfies the strong nonquasianalytic condition (M.3) in Komatsu [K1] then it satisfies (C). In fact, (M.3) is equivalent to the fact that for some integer $k > 0$

$$\text{(M.3)}'' \qquad \liminf_{p\to\infty} \frac{m_{kp}}{m_p} > k.$$

Thus the condition (C) is equivalent to the fact that $N_p = M_p^2$ satisfies (M.3) (see [P], p.300)

Lemma 2.4. *Let L be an arbitrary positive number and let*

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} (1 + \frac{L\zeta}{m_p}), \quad \zeta \in \mathbf{C}^n. \tag{2.1}$$

(i) *If M_p satisfies (M.1), (M.2) and (M.3) then $P(\partial)$ is an ultradifferential operator of class (M_p).*

(ii) *If M_p satisfies (M.1) and $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$ then for any $\varepsilon > 0$ there exist functions $v, w \in C_0^\infty(\mathbf{R})$ such that*

$$\operatorname{supp} v \subset [0, \varepsilon], \ \operatorname{supp} w \subset [\varepsilon/2, \ \varepsilon] \tag{2.2}$$

$$|v(t)| \le C \exp[-M^*(L/t)], \quad t > 0 \tag{2.3}$$

$$P(d/dt)v(t) = \delta(t) + w(t), \tag{2.4}$$

where δ is a Dirac measure.

Now we are in a position to state the main theorem of this paper.

Theorem 2.5. *Let M_p be a defining sequence satisfying (C) and $u \in \mathcal{E}'_{(M_p)}(K)$. Then there exists an ultradifferential operator $P(d/dt)$ such that for some $C > 0$ and $L > 0$*

$$P(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k, \quad |a_k| \le CL^k/M_k^2 \tag{2.5}$$

and there exist a bounded continuous function $g(x)$ and a bounded real analytic function $h(x)$ such that

$$u = P(\Delta)g(x) + h(x) \tag{2.6}$$

where $g(x) \in C^\infty(\mathbf{R}^n \backslash K)$, $P(\Delta)g(x) + h(x) = 0$ in $\mathbf{R}^n \backslash K$, and Δ is the Laplacian.

Every distribution and hyperfunction with compact support can be represented by the boundary value of holomorphic functions. Here we will give a similar result for $\mathcal{E}'_{(M_p)}(K)$.

Theorem 2.6. *Let M_p be a defining sequence satisfying (C) and $U(x,t)$ be an infinitely differentiable function in $\mathbf{R}^{n+1}_+$ satisfying the following conditions:*

(i) $(\partial_t - \Delta)U(x,t) = 0$ *in* $\mathbf{R}^{n+1}_+$

(ii) *For any $\delta > 0$ there exist $C > 0$ and $\varepsilon > 0$ such that*

$$|U(x,t)| \leq C \exp\left[\overline{M}(\varepsilon/t) - d(x, K_\delta)^2/8t\right] \quad \text{in } \mathbf{R}^{n+1}_+. \tag{2.7}$$

Then there exists a unique element $u \in \mathcal{E}'_{(M_p)}(K)$ such that

$$U(x,t) = u_y(E(x-y,t)), \quad t > 0 \tag{2.8}$$

and

$$\lim_{t \to 0+} U(x,t) = u \tag{2.9}$$

in the following sense:

$$u(\phi) = \lim_{t \to 0+} \int_\Omega U(x,t)\phi(x)dx, \quad \phi \in \mathcal{E}_{(M_p)}(\mathbf{R}^n) \tag{2.10}$$

where Ω is an arbitrary bounded neighborhood of K.

For a compact set K of $\mathbf{R}^n$ we denote by $\mathcal{M}_K^{\text{tame}}$ the totality of C^∞ solutions $U(x,t)$ of the heat equation $(\partial_t - \Delta)U(x,t) = 0$ in $\mathbf{R}^{n+1}_+$ which satisfy the following condition:

For any $\delta > 0$ there exist C and $\varepsilon > 0$ such that

$$|U(x,t)| \leq C \exp\left[\overline{M}(\varepsilon/t) - d(x, K_\delta)^2/8t\right] \text{ on } \mathbf{R}^{n+1}_+. \tag{2.11}$$

Note that $\mathcal{M}_K^{\text{tame}}$ is a DF-space with the best constants C as semi-norms. Then we have the following theorem in view of Theorem 2.5 and 2.6:

Theorem 2.7. *Let M_p be a defining sequence satisfying (C). Then there exists an isomorphism:*

$$\mathcal{M}_K^{\text{tame}} \cong \mathcal{E}'_{(M_p)}(K).$$

Matsuzawa [M] has proved similar theorems for the case of hyperfunctions and ultradistributions of Gevrey class. Thus the above theorem is an extension of Matsuzawa's result for a class of quasianalytic ultradistributions.

Finally, we conjecture that our assertions should also remain valid without the condition (C), i.e., for the general ultradistributions, both quasianalytic and nonquasianalytic.

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UNIQUENESS CLASSES FOR THE CAUCHY PROBLEM WITHOUT UNIFORM CONDITION ON TIME

SOON-YEONG CHUNG AND DOHAN KIM

0. Introduction

In the theory of heat conduction it is well known that the temperature of an infinite rod is not uniquely determined by its initial temperature. Consider the following famous example

$$u(x,t) = \sum_{n=0}^{\infty} f^{(n)}(t) x^{2n}/(2n)! \tag{0.1}$$

where

$$f(t) = \begin{cases} e^{-1/t^2} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

for the nonuniqueness theorems for the Cauchy problem of the heat equation. The function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $t > 0$, and $u(x, 0+) = 0$ for $-\infty < x < \infty$.

In general there are several uniqueness theorems of solutions of the heat equation as follow :

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Theorem A([7]). *Let $u(x,t)$ be a continuous function on $\mathbf{R}^n \times [0,T]$ satisfying that*

$$(\partial_t - \Delta)u(x,t) = 0 \quad on \quad \mathbf{R}^n \times (0,T)$$

and for some $C > 0$

$$|u(x,t)| \leq Ce^{a|x|^2} \quad on \quad \mathbf{R}^n \times [0,T] \tag{0.2}$$

for some $a > 0$. Then $u(x,0) = 0$ on $\mathbf{R}^n$ implies that $u(x,t) \equiv 0$ on $\mathbf{R}^n \times [0,T]$.

Theorem B([1]). *Let $u(x,t)$ be a continuous function on $\mathbf{R}^n \times [0,T]$ satisfying that*

$$(\partial_t - \Delta)u(x,t) = 0 \quad on \quad \mathbf{R}^n \times (0,T)$$

and

$$\int_0^T \int_{\mathbf{R}^n} |u(x,t)|e^{-a|x|^2}\, dx\, dt \tag{0.3}$$

is finite for some $a > 0$. Then $u(x,0) = 0$ on $\mathbf{R}^n$ implies that $u(x,t) \equiv 0$ on $\mathbf{R}^n \times [0,T]$.

It is clear that Theorem B is much stronger than Theorem A. Note that the growth condition (0.2) or (0.3) is quite unrestricted with respect to x variable, but too restricted with respect to t variable to apply this theorem in many cases (see [4], for example).

In this paper, we give more generalized uniqueness theorems of Cauchy problem under the following much weaker growth condition instead of (0.2) and (0.3)

$$|u(x,t)| \leq C \exp k(|x|^2 + 1/t), \quad t > 0$$

for some $C > 0$ and $k > 0$. Moreover, this growth condition does not require the continuity of $u(x,t)$ on $t = 0$. Also, we give an example of nontrivial solutions for the Cauchy problem of the heat equation which is uniformly bounded with respect to x variable.

The proofs of the theorems in this paper will appear elsewhere.

1. Example of nonuniqueness

We give here an exmaple of nonuniqueness for the Cauchy problem of heat equation. In fact, this example is motivated by Morimoto and Yoshino [5] as an example of an analytic functional with noncompact carrier.

Theorem 1.1. *There exists a continuous function $u(x,t)$ on $\mathbf{R} \times [0,\infty)$ which is a C^∞-function on $\mathbf{R} \times (0,\infty)$ satisfying*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on} \quad \mathbf{R} \times (0,\infty) \tag{1.1}$$

$$u(x,0) = 0 \quad \text{on} \quad \mathbf{R} \tag{1.2}$$

and for any $\varepsilon > 0$

$$|u(x,t)| \le C \exp(\varepsilon/t), \quad \text{on} \quad \mathbf{R} \times (0,\infty) \tag{1.3}$$

for some $C = C(\varepsilon)$, but which is not identically zero.

Remark. We note that the above $u(x,t)$ is uniformly bounded with respect to space variables. Considering the growth condition (0.2) or (0.3), the above theorem implies that the uniqueness for the Cauchy problem of heat equation depends strongly on the growth along the time.

2. Uniqueness Theorems

In this section we will give the more generalized uniqueness theorems than Theorem A and Theorem B.

Definition 2.1. We denote by $\mathrm{Exp}(k)$ the set of all infinitely differentiable functions ϕ in $\mathbf{R}^n$ such that for any $h > 0$

$$|\phi|_{\mathrm{Exp}(k),h} = \sup_{\substack{x \in \mathbf{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi(x)| \exp k|x|^2}{h^{|\alpha|}\alpha!} \tag{2.1}$$

is finite. The topology in $\mathrm{Exp}(k)$, defined by the above seminorms makes $\mathrm{Exp}(k)$ a FS-space. In fact, it is the projective limit topology over all $h > 0$. We denote by $\mathrm{Exp}'(k)$ the strong dual of the space $\mathrm{Exp}(k)$.

Lemma 2.2. *Let $P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha$ be a differential operator of infinite order with constant coefficients satisfying that there exist $L > 0$ and $C > 0$ such that*

$$|a_\alpha| \leq CL^{|\alpha|}/\alpha! \tag{2.2}$$

for all α. Then the operators

$$P(\partial) : \mathrm{Exp}(k) \to \mathrm{Exp}(k) \tag{2.3}$$

and

$$P(\partial) : \mathrm{Exp}'(k) \to \mathrm{Exp}'(k) \tag{2.4}$$

are continuous.

Proposition 2.3. *Let $g(x)$ be a continuous function satisfying that for some $C > 0$ and $k > 0$*

$$|g(x)| \leq C \exp k|x|^2, \quad x \in \mathbf{R}^n, \tag{2.5}$$

and $G(x,t) = g(x) * E(x,t)$ where $*$ denotes the convolution with respect to x variable. Then $G(x,t)$ is a well defined C^∞-function in $\mathbf{R}^n \times (0, 1/4k)$ and satisfies that

$$\begin{aligned} &\text{(i) } (\partial_t - \Delta)G(x,t) = 0, \quad 0 < t < 1/4k \\ &\text{(ii) } |G(x,t)| \leq C\exp(2k|x|^2), \quad 0 < t < 1/16k \\ &\text{(iii) } G(x,t) \to g(x) \text{ locally uniformly on } \mathbf{R}^n \text{ as } t \to 0+. \end{aligned} \tag{2.6}$$

The following lemma is very useful later. For the details of the proof we refer to Komatsu [3] :

Lemma 2.4. *For any $L > 0$ and $\varepsilon > 0$ there exist a function $v(t) \in C_0^\infty(\mathbf{R})$ and a differential operator $P(d/dt)$ of infinite order such that*

$$\begin{aligned} &\text{supp } v \subset [0,\varepsilon], \ |v(t)| \le C\exp(-L/t), \quad 0 < t < \infty; \\ &P(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k, \ |a_k| \le C_1 L_1^k / k!^2, \quad 0 < L_1 < L; \\ &P(d/dt)v = \delta + w(t) \end{aligned} \tag{2.7}$$

where $w \in C_0^\infty(\mathbf{R})$ with $\text{supp}\, w \subset [\varepsilon/2, \varepsilon]$ and δ is a Dirac measure.

Now we are in a position to state the main theorem in this paper.

Theorem 2.5. *Let $u(x,t)$ be a function on $\mathbf{R}^n \times (0,T)$ satisfying that*

$$\begin{aligned} &\text{(i) } (\partial_t - \Delta) u(x,t) = 0, \quad 0 < t < T, \\ &\text{(ii) For some } k > 0, \text{ there exists } C > 0 \text{ such that} \\ &\qquad |u(x,t)| \le C \exp k(|x|^2 + 1/t), \quad 0 < t < T, \\ &\text{(iii) } \lim_{t \to 0+} \int u(x,t)\phi(x)\, dx = 0 \text{ for every } \phi \in \text{Exp}(2k). \end{aligned} \tag{2.8}$$

Then $u(x,t)$ is identically zero on $\mathbf{R}^n \times [0,T]$. Here T may be ∞.

Remark. (i) In the condition of the above theorem, the continuity of $u(x,t)$ on $\mathbf{R}^n \times [0,T]$ is not required. Thus this uniqueness theorem is much stronger than any other one already known.

(ii) It can be easily seen that this Theorem generalizes the Theroem A and B.

(iii) The initial condition (iii) of this theorem is, more or less, unsatisfactory. But in view of the example as seen in the Section 2, it can be regard as optimal one. The space $\text{Exp}(2k)$ can be replaced by $\text{Exp}(k')$ for some $k' > k$. Also, it can be weakened as follows:

$$\lim_{t \to 0+} \int u(x,t) e^{-k|x|^2}\, dx = 0.$$

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ORTHOGONAL POLYNOMIALS SATISFYING DIFFERENTIAL EQUATIONS

K. H. KWON, J. K. LEE AND B. H. YOO

1. Introduction.

All polynomials in this work are assumed to be real polynomials in one variable.

Let P be the set of all polynomials. For any $\phi(x)$ in P, we let $\deg \phi$ denote the degree of $\phi(x)$ with convention $\deg 0 = -1$. A sequence of polynomials $\{\phi_n(x)\}_0^\infty$ is called on orthogonal polynomial set (OPS in short) if there is a linear functional σ on P (which we shall call a moment functional) such that

$$\langle \sigma, \phi_m \phi_n \rangle = K_n \delta_{mn}, \qquad m \text{ and } n = 0, 1, 2, \ldots \tag{1.1}$$

where $K_n \neq 0$, $n = 0, 1, 2, \ldots$, are real constants.

In 1929, Bochner [3] showed that there are essentially (that is, up to a linear change of variable) only four OPS's arising as eigenfunctions of a second order Sturm-Liouville differential equation of the form

$$\alpha(x)y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = \lambda y(x) \tag{1.2}$$

where $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are polynomials of degree 2, 1, and 0 respectively independent of the parameter n and $\lambda = \lambda_n$ is a constant depending on n.

They are now known as classical OPS's of Jacobi, Laguerre, Hermite, and Bessel polynomials. There are many other characterizations of classical orthogonal polynomials (see Al-Salam [1]) among which we are particularly interested in the Hahn's theorem saying that the classical OPS's are the only OPS's whose derivatives also form OPS's.

On the other hand, Krall [5] found a necessary and sufficient condition for differential equations of the form

$$L_N y = \sum_0^N \ell_i(x) y^{(i)} = \lambda_n y \tag{1.3}$$

to have an OPS as solutions, where $\ell_i(x)$ are polynomials independent of the parameter $n = 0, 1, 2, \ldots$ and λ_n is a real constant. In [6], he also classified all differential equations of the form (1.3) for $N = 4$ which have an OPS as solutions.

In this work, we shall generalize Hahn's theorem to the characterization theorem of all OPS's that arise as eigenfunctions of the differential equation (1.3).

We are grateful to Professor W. N. Everitt who suggested the problem for $N = 4$ at the conference on orthogonal polynomials and applications, Granada, Spain, 1991 (see also [12]). This work is supported by KOSEF (Grant No. 90-08-00-02) and GARC of Seoul National University.

2. Main results.

By a polynomial set, we mean a sequence of polynomials $\{P_n(x)\}_0^\infty$ such that $\deg P_n = n$, $n = 0, 1, 2, \ldots$. Any polynomial set $\{P_n(x)\}_0^\infty$ determines a unique moment functional σ satisfying

$$\langle \sigma, 1 \rangle = 1 \quad \text{and} \quad \langle \sigma, P_n \rangle = 0, \qquad n = 1, 2, 3, \ldots, \tag{2.1}$$

We call σ the canonical moment functional of $\{P_n(x)\}_0^\infty$ and the moments $\{\sigma_n\}_0^\infty$ of σ given by

$$\sigma_n := \langle \sigma, x^n \rangle, \qquad n = 0, 1, 2, \ldots$$

the moments of $\{P_n(x)\}_0^\infty$.

Note that if a polynomial set is an OPS, then it must be orthogonal relative to its canonical moment functional.

Lemma 2.1 (H. L. Krall [5]). *A polynomial set $\{P_n(x)\}_0^\infty$ is an OPS satisfying the differential equation* (1.3) *if and only if $\ell_i(x)$ are polynomials of degree $\leq i$, say,*

$$\ell_i(x) = \sum_{j=0}^{i} \ell_{ij} x^j, \qquad i = 0, 1, \ldots, N$$

and

$$\lambda_n = \ell_{00} + n\ell_{11} + \cdots + n(n-1)\cdots(n-N+1)\ell_{NN} \tag{2.2}$$

and the moments $\{\sigma_n\}_0^\infty$ of $\{P_n(x)\}_0^\infty$ satisfy

$$\Delta_n = \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0, \qquad n = 0, 1, 2, \ldots \tag{2.3}$$

$$(2.4)\quad S_k(m) := \sum_{i=2k+1}^{N} \sum_{j=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\ell_{i,i-j}\sigma_{m-j} = 0$$

for $1 \leq 2k+1 \leq N$, $m = 2k+1, 2k+2, \ldots$, where $P(n,k) = n(n-1)\cdots(n-k+1)$.

Furthermore, if then, N must be an even integer, say, $N = 2r$.

Following Littlejohn [11], we call an OPS $\{P_n(x)\}_0^\infty$ a Bochner-Krall OPS if $P_n(x)$ satisfies

$$(2.5)\qquad L_{2r}(y) = \sum_{i=0}^{2r} \ell_i(x) y^{(i)} = \sum_{i=0}^{2r}\sum_{j=0}^{i} \ell_{ij} x^j y^{(i)} = \lambda_n y$$

for $n = 0, 1, 2, \ldots$ with $\ell_{2r}(x) \not\equiv 0$ and λ_n given by the equation (2.2).

We call then the r recurrence relations $S_k(m) = 0$, $k = 0, 1, \ldots, r-1$, the moment equations for $\{P_n(x)\}_0^\infty$. Our main theorems are

Theorem 2.2 ([8]). *For any OPS $\{P_n(x)\}_0^\infty$ relative to a moment functional σ, the following statements are all equivalent.*

(a) *$\{P_n(x)\}_0^\infty$ is a Bochner-Krall OPS of order $2r$ satisfying (2.5).*

(b) *σ satisfies an overdetermined system of r homogeneous differential equations*

$$(2.6)\qquad R_k\sigma := \sum_{2k+1}^{2r} (-1)^i \binom{i-k-1}{k} (\ell_i\sigma)^{(i-2k-1)} = 0$$

for $k = 0, 1, \ldots, r-1$.

(c) *σL_{2r} is formally symmetric on polynomials in the sense that*

$$(2.7)\qquad \langle L_{2r}(\phi)\sigma, \psi\rangle = \langle L_{2r}(\psi)\sigma, \phi\rangle$$

for every polynomial $\phi(x)$ and $\psi(x)$.

(d) *There are $r+1$ moment functionals $\{\tau_i\}_0^r$ such that $\tau_r \not\equiv 0$ and*

$$(2.8)\qquad L_{2r}(\phi)\sigma = \sum_{0}^{r} (-1)^i [\phi^{(i)}\tau_i]^{(i)}$$

for every polynomial $\phi(x)$.

Moreover, the moment functionals σ and $\{\tau_i\}_0^r$ are related by the equations

$$\ell_k(x)\sigma = \sum_{[\frac{k+1}{2}]}^{\min(r,k)} (-1)^i \binom{i}{k-i} \tau_i^{(2i-k)}, \qquad k = 0, 1, \ldots, 2r \tag{2.9}$$

where $[x]$ is the largest integer $\leq x$.

The proof of Theorem 2.2 is based on the characterization of symmetry factor (cf. [10, 13]) of linear differential operators. It is interesting to note that every differential operator L_{2r} having an OPS $\{P_n(x)\}_0^\infty$ as polynomial solutions can be made symmetric on polynomials in the sense of Theorem 2.2 (c) and each $P_n(x)$ is an eigenfunction corresponding to an eigenvalue λ_n. However, in general, not every differential operator is symmetrizable if it is of order ≥ 4.

Theorem 2.3 ([8]). *An OPS $\{P_n(x)\}_0^\infty$ is a Bochner-Krall OPS of order $2r$ if and only if there are $r+1$ moment functionals $\{\tau_i\}_0^r$ such that $\tau_r \not\equiv 0$ and*

$$\sum_{i=0}^{r} \langle \tau_i, P_m^{(i)} P_n^{(i)} \rangle = 0 \qquad \text{for} \quad m \neq n \geq 0. \tag{2.10}$$

Moreover, $\{P_n(x)\}_0^\infty$ is a symmetric Bochner-Krall OPS of order $2r$ if and only if $\{\tau_i\}_0^r$ can be taken to be symmetric.

As a special case of Theorem 2.3 for $r = 1$, we obtain an extension of Hahn's characterization theorem of classical orthogonal polynomials.

Corollary 2.4 ([7, 8]). *An OPS $\{P_n(x)\}_0^\infty$ is a classical orthogonal polynomials (i.e., a Bochner-Krall OPS of order 2) if and only if there is a nontrivial moment functional τ such that*

$$\langle \tau, P_m' P_n' \rangle = 0 \tag{2.8}$$

for $m \neq n \geq 1$.

3. Remarks and open problems.

Bochner-Krall OPS's of order 2 and 4 are completely classified by S. Bochner [3] and H. L. Krall [6] and several Bochner-Krall OPS's of order ≥ 6 are now known (cf. [4, 9, 11]). However, the problem of classifying all Bochner-Krall OPS's is far from being complete. In this work, we have used moment functionals to introduce the orthogonality, but due to the classical theorem of R. P. Boas [2] on Stieltjes moment problem, for any OPS $\{P_n(x)\}_0^\infty$, there must exists (in fact infinitely many) a function of bounded variation $\mu(x)$ on $[0,\infty)$ such that

$$\int_0^\infty P_m(x)P_n(x)\,d\mu(x) = K_n\delta_{mn}, \qquad K_n \neq 0 \tag{3.1}$$

for m and $n \geq 0$.

When the Stieltjes measure $d\mu(x)$ is of the form $w(x)\,dx$, we call $w(x)$ an orthogonalizing weight of $\{P_n(x)\}_0^\infty$. For all known Bochner-Krall OPS's we have representations of their orthogonalizing weights as generalized functions of distributions or hyperfunctions. However, in order to develop the spectral theory of differential operators having OPS's as solutions, it is important to construct functions $\mu(x)$ as in (3.1) explicitly, which are not known yet except some special cases.

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RELATION BETWEEN SOLVABILITY AND A REGULARITY OF CONVOLUTION OPERATORS IN $\mathcal{K}'_p$, $p > 1$

DAE HYEON PAHK AND BYUNG KEUN SOHN

Let $\mathcal{K}'_p$ be the space of distributions on R^n which grow no faster than $e^{k|x|^p}$ for some $k > 0$; $\mathcal{K}'_p$ is the dual of space $\mathcal{K}_p$, which we deribe later. We denote by $O'_c(\mathcal{K}'_p, \mathcal{K}'_p)$ the space of convolution operators in $\mathcal{K}'_p$.

In [6] S. Sznajder and Z. Zielezny proved that, if S is a distributions in $O'_c(\mathcal{K}'_p, \mathcal{K}'_p)$ and $\hat{S}$ is its Fourier transform, the following conditions are equivalent :

(a) There exist positive constants A, C and a positive integer N such that

$$\sup_{\substack{x \in R^n \\ |x-\xi| \leq A(\log(2+|\xi|))^{\frac{1}{q}}}} |\hat{S}(x)| \geq \frac{c}{(1+|\xi|)^N}, \xi \in R^n \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

(b) $S * \mathcal{K}'_p = \mathcal{K}'_p$.

In this paper we prove that, for $S \in \mathcal{E} \subset O'_c(\mathcal{K}'_p, \mathcal{K}'_p)$, the statements (a) and (b) are equivalent to the following : every distribution $u \in O'_c(\mathcal{K}'_p, \mathcal{K}'_p)$ satisfying $S * u \in \mathcal{K}_p$ is in $\mathcal{K}_p$.

The motivation for this problem comes from the paper [7]. Here S. Sznajder and Z. Zielezny proved that, if S is a distribution in $O'_c(\mathcal{K}'_1, \mathcal{K}'_1)$ and $\hat{S}$ is its Fourier transform, the following statements are equivalent :

(i) There exist positive constants N, r, c such that

$$\sup_{z\in C^n,\,|z|\leq r} |\hat{S}(\xi+z)| \geq \frac{c}{(1+|\xi|^N)}, \xi \in R^n,$$

(ii) $S * \mathcal{K}_1' = \mathcal{K}_1'$

(iii) If $u \in O_c'(\mathcal{K}_1', \mathcal{K}_1')$ and $S * u \in \mathcal{K}_1$, then $u \in \mathcal{K}_1$

In fact the equivalence of the first two statements is proved in [5] and the last one in [7] by them.

Moreover, they considered the solvability condition of convolution operators in the tempered distribution space which is still open. In [7] they proved some necessary properrties for the solvability in the tempered distribution space, which is related to the result in $\mathcal{K}_1'$. We naturally ask the some complemented condition in the space $\mathcal{K}_p'$, $p > 1$. But we only succeed for convolution operators with compact support in $O_c'(\mathcal{K}_p', \mathcal{K}_p')$.

We have also studied the complemented condition for the solvability of convolution operators in the generalized distribution spaces of Beurling type which include the classical distribution space. The report of our progress in this direction will be avaible soon.

We now state our theorem, which complements partly the result in [6] :

Theorem. *Let S be a distribution with compact support and $\hat{S}$ be its Fourier transform, then the following conditions are equivalent:*

(*a*) *There exist positive constants A, C and a positive integer N such that*

$$\sup_{\substack{x\in R^n \\ |x-\xi|\leq A(\log(2+|\xi|))^{\frac{1}{q}}}} |\hat{S}(x)| \geq \frac{C}{(1+|\xi|)^N}, \quad \xi \in R^n \text{ and } \frac{1}{p}+\frac{1}{q}=1$$

(*b*) $S * \mathcal{K}_p' = \mathcal{K}_p'$

(*c*) *If* $u \in O_c'(\mathcal{K}_p', \mathcal{K}_p')$ *and* $S * u \in \mathcal{K}_p$, *then* $u \in \mathcal{K}_p$.

However, the conditons (a) and (b) are equivalent for every distribution in $O_c'(\mathcal{K}_p', \mathcal{K}_p')$ as was shown in [6].

Before presenting the proof of our theorem we recall briefly the basic facts about the space $\mathcal{K}_p'$; for further details we refer to [4].

The space $\mathcal{K}_p'$. Let $\mathcal{K}_p$, $p > 1$, be the space of all C^∞-function ϕ in R^n such that

$$v_k(\phi) = \sup_{|\alpha|\leq k,\; x\in R^n} e^{k|x|^p} |D^\alpha \phi(x)| < \infty, \quad k = 0, 1, 2, \ldots$$

where $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$ and $D_j = i^{-1}(\frac{\partial}{\partial x_j})$. Provided with the topology defined by the seminorms v_k, $\mathcal{K}_p$ is a Frechet space. The dual $\mathcal{K}_p'$ of $\mathcal{K}_p$ is the space of continuous linear functionals on $\mathcal{K}_p$. Then a distribution u is in $\mathcal{K}_p'$ if and only if there exist positive integers m, k and a bounded continuous function f(x) on R^n such that

$$u = \frac{\partial^{mn}}{\partial x_1^m ... \partial x_n^m}[e^{k|x|^p} f(x)].$$

$\mathcal{K}_p'$ is endowed with the topology of uniform convergence on all bounded sets in $\mathcal{K}_p$.

If $u \in \mathcal{K}_p'$ and $\phi \in \mathcal{K}_p$, then the convolution $u * \phi$ is a C^∞-function defined by

$$u * \phi(x) = < u_y, \phi(x-y) > .$$

The space $O_c'(\mathcal{K}_p', \mathcal{K}_p')$. The space $O_c'(\mathcal{K}_p', \mathcal{K}_p')$ of convolution operators in $\mathcal{K}_p'$, consists of distributions $S \in \mathcal{K}_p'$ satisfying one of the following equivalent conditions:

(i) The distributions $S_k = e^{k|x|^p} S, k = 1, 2, ...,$ are in tempered distribution spaces.

(ii) For every integer $k \geq 0$, there exists an integer $m \geq 0$ such that

$$S = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

where f_α, $|\alpha| \leq m$, are continuous functions in R^n whose products with$e^{k|x|^p}$ are bounded.

(iii) For every $\phi \in \mathcal{K}_p$, the convolution $S * \phi$ is in $\mathcal{K}_p$.

If $u \in O_c'(\mathcal{K}_p', \mathcal{K}_p')$ and $\check{u}$ is obtained from u by symmetry with respect to the origin, i.e., $< \check{u}, \phi > = < u_x, \ \phi(-x) >$ for $\phi \in \mathcal{K}_p$, then $\check{u}$ is also in $O_c'(\mathcal{K}_p', \mathcal{K}_p')$. The convolution of u with $v \in \mathcal{K}_p'$ is defined by

(1) $$< u * v, \ \phi > = < v, \ \check{u} * \phi >, \quad \phi \in \mathcal{K}_p.$$

Proof of Theorem. Since the space of distributions with compact support is a subspace of $O_c'(\mathcal{K}_p', \mathcal{K}_p')$, it suffices to show that $(b) \Rightarrow (c) \Rightarrow (a)$.

$(b) \Rightarrow (c)$. The proof goes along exactly the same lines as proof of Theorem 1 in [6]. For the completeness we give the proof. If S is a distribution with compact support, then so is $T = \check{S}$ and, by (1), the mapping $S^* : u \to S * u$ of $\mathcal{K}_p'$ into $\mathcal{K}_p'$ is the transpose of the mapping $T^* : \phi \to T * \phi$ of $\mathcal{K}_p$ into $\mathcal{K}_p$. Condition (b) is satisfied if and only if T^* is an isomorphism of $\mathcal{K}_p$ onto $T * \mathcal{K}_p$(see e.g.,[1, Corollary on p92]). In particular the inverse $T * \phi \to \phi$ must be continuous.

Suppose now that $S * u = \phi$ where $u \in O_c'(\mathcal{K}_p', \mathcal{K}_p')$ and $\phi \in \mathcal{K}_p$. Then

$$T * \check{u} = (-1)^n \check{\phi} \tag{2}$$

and for the proof it suffices to show that $\check{u} \in \mathcal{K}_p$. If ψ is a C^∞-function with $\mathrm{supp}(\psi) \subset B(0,1) = \{x \in R^n : |x| \le 1\}$ and $\hat{\psi}(0) = 1$, we define $\psi_k(x) = k^n \psi(kx)$, $k = 1, 2, \ldots$. From (2) it follows that

$$T * (\check{u} * \psi_k) = (-1)^n \check{\phi} * \psi_k,$$

and the convolutions $\check{u} * \psi_k$ and $(-1)^n \check{\phi} * \psi_k$ are in $\mathcal{K}_p$. Moreover, the sequence $\{\psi_k\}$ converges in $O_c'(\mathcal{K}_p', \mathcal{K}_p')$ to δ, the Dirac measure at the origin. Hence $(-1)^n \check{\phi} * \psi_k \to (-1)^n \check{\phi}$ in $\mathcal{K}_p$ and $\check{u} * \psi_k \to \check{u}$ in $O_c'(\mathcal{K}_p', \mathcal{K}_p')$. On the other hand, the sequence $\{\check{u} * \gamma_k\}$ converges in $\mathcal{K}_p$, by the assumption that the inverse of T^* is continuous. The limit must be again $\check{u}$, and so $\check{u}$ is a function in $\mathcal{K}_p$.

$(c) \Rightarrow (a)$. Let $\mathcal{F}$ be the space of all functions $u \in C(R^n)$ such that

$$\sup_{x \in R^n} e^{k|x|^p} |u(x)| < \infty, \quad \text{for all } k$$

and $S * u \in \mathcal{K}_p$. We provide $\mathcal{F}$ with the topology defined by the seminorms

$$\|u\|_k = \sup_{x \in R^n} e^{k|x|^p} |u(x)| + v_k(S * u), \quad k = 0, 1, 2, \ldots.$$

Then $\mathcal{F}$ becomes a Frechet space. Further, let $\mathcal{G}$ be the space of all functions $u \in C^1(R^n)$ such that

$$\|u\| = \sup_{x \in R^n \; |\alpha| \le 1} |D^\alpha u(x)| < \infty\ ;$$

with the norm $\| \quad \|$, $\mathcal{G}$ is a Banach space.

By the fact $\mathcal{F} \subset O_c'(\mathcal{K}_p', \mathcal{K}_p')$ and the assumption (c), each function $u \in \mathcal{F}$ is in $\mathcal{G}$. Also, the natural mapping $\mathcal{F} \to \mathcal{G}$ is closed and therefore continuous. Consequently there exist an integer $\mu > 0$ and a constant C such that

$$\|u\| \le C\|u\|_\mu = C\{\sup_{x \in R^n} e^{\mu|x|^p} |u(x)| + v_\mu(S * u)\}$$

for all $u \in \mathcal{F}$, which gives

$$\|u\| - C \sup_{x \in R^n} e^{\mu|x|^p}|u(x)| \leq Cv_\mu(S * u). \tag{3}$$

Suppose now the condition (a) is not satisfied. Then there exist a sequence $\{\xi_j\}$ such that $|\xi_j| \to \infty$ and

$$\sup_{\substack{x \in R^n \\ |x-\xi_j| \leq A(\log(2+|\xi_j|)^{\frac{1}{q}}}} |\hat{S}(x)| < \frac{1}{(1+|\xi_j|)^N} \tag{4}$$

for given $N > 2^p\mu^{(1-p)}+(\mu+n+1)$ and $A > \{e^{(2^p\mu^{1-p}+2(N_0+\mu+n+1)+d)}-1\}\mu$. Here N_0 is a positive integer satisfying $|\hat{S}(\xi)| \leq C_0(1+|\xi|)^{N_0}, \xi \in R^n$ and $d = \log C_1$ satisfying $|\hat{\psi}(\xi)| \leq C_1(1+|\xi|)^{-1}, \xi \in R^n$. Both inequalities are given by the Paley-Wiener theorem for the given distribution S and the given test function ψ in the above proof.

Now let $k_j = [\log(2+|\xi_j|)]$ and $\alpha_j = \mu k_j^{\frac{1}{q}}$, where [ ] is the Gaussian integer. We define the function ϕ_j by

$$\phi_j(x) = e^{i<x,\xi_j>} \underbrace{\psi_{\alpha_j} * \dots * \psi_{\alpha_j}}_{k_j\text{-times}}$$

where $\psi_{\alpha_j}(x) = \alpha_j^n\psi(\alpha_j x)$. Then ϕ_j is a C^∞-function with $\int_{R^n}|\phi_j(x)|dx = 1$ and $\text{supp}\phi_j \subset B(0, \mu^{-1}k_j^{\frac{1}{p}})$.

Substituting ϕ_j's into the inequality (3), we will show that the left-side of (3) goes to ∞ and the right to 0, as $j \to \infty$, which gives the desired contradiction.

To show this, we first estimate

$$\|\phi_j\| = \sup_{x \in R^n} \sup_{|\alpha| \leq 1} |D^\alpha\phi_j(x)| \geq \frac{|\xi_j|}{n} \sup_{x \in R^n} |\phi_j(x)| \tag{5}$$

and

$$\begin{aligned} \sup_{R^n} e^{\mu|x|^p}|\phi_j(x)| &= \sup_{|x| \leq \mu^{-1}k_j^{\frac{1}{p}}} e^{\mu|x|^p}|\phi_j(x)| \\ &\leq (2+|\xi_j|)^{\mu^{1-p}}\|\phi_j\|_\infty. \end{aligned} \tag{6}$$

Viewing

$$1 = \int_{R^n} |\phi_j(x)|dx \le C_2(\mu^{-1}k_j^{\frac{1}{p}})^n \|\phi_j\|_\infty$$

where $C_2(\mu^{-1}k_j^{\frac{1}{p}})^n$ is the volumn of the ball $B(0, \mu^{-1}k_j^{\frac{1}{p}})$ in R^n, we have

$$\|\phi_j\|_\infty \ge \frac{1}{C_2}(\mu^{-1}k_j^{\frac{1}{p}})^{-n}. \tag{7}$$

Substituting (5), (6) and (7) into (3), the left-hand side of (3) behaves, as $j \to \infty$,

$$\lim_{j\to\infty} \{\|\phi_j\| - C \sup_{R^n} e^{\mu|x|^p} |\phi_j(x)|\}$$

$$\ge \lim_{j\to\infty} \{\frac{1}{n} - C\frac{(2+|\xi_j|)^{\mu^{1-p}}}{|\xi_j|}\}|\xi_j|\frac{1}{C_2}(\mu^{-1}k_j^{\frac{1}{p}})^{-n}$$

$$= \frac{1}{C_2} \lim_{j\to\infty} \frac{\mu^n}{n}|\xi_j|[\log(2+|\xi_j|)]^{-\frac{n}{p}} = \infty.$$

On the other hand, we say supp(S) $\subset B(0,r)$ for some $r > 0$ and then $\operatorname{supp}(S * \phi_j) \subset B(0, r+\mu^{-1}k_j^{\frac{1}{p}})$. Hence

$$v_\mu(S*\phi_j) \le e^{\mu|r+\mu^{-1}k_j^{\frac{1}{p}}|^p} \sup_{x\in R^n, |\alpha|\le\mu} |D^\alpha(S*\phi_j)|$$

$$\le e^{2^p\mu(r^p+\mu^{-p}k_j)} \sup_{|\alpha|\le\mu} |\frac{1}{(2\pi)^n}\int_{R^n} e^{i<x,\eta>}\eta^\alpha \widehat{S*\phi_j}(\eta)d\eta|$$

$$\le C_3(2+|\xi_j|)^{2^p\mu^{1-p}} \sup_{\eta\in R^n} [|\widehat{S*\phi_j}(\eta)|(1+|\eta|)^{\mu+n+1}]$$

(8)

$$\cdot \sup_{|\alpha|\le\mu} [\frac{1}{(2\pi)^n}\int_{R^n} \frac{|\eta|^{|\alpha|}}{(1+|\eta|)^{\mu+n+1}}d\eta]$$

$$\le C_4(2+|\xi|)^{2^p\mu^{1-p}}\{ \sup_{|\eta-\xi_j|\le A(\log(2+|\xi_j|))^{\frac{1}{q}}} [|\widehat{S*\phi_j}(\eta)|(1+|\eta|)^{\mu+n+1}]$$

$$+ \sup_{|\eta-\xi_j|> A(\log(2+|\xi_j|))^{\frac{1}{q}}} |\widehat{S*\phi_j}(\eta)|(1+|\eta|)^{\mu+n+1}]\}$$

It now suffices to prove that both terms in (8) go to 0, as $j \to \infty$. We first observe

$$\hat{\phi}_j(\eta) = [\hat{\psi}_{\alpha_j}(\eta - \xi_j)]^{k_j} = [\hat{\psi}(\frac{\eta - \xi_j}{\alpha_j})]^{k_j}$$

and, by the Paley-Wiener theorem for ψ,

$$|\hat{\phi}_j(\eta)| \le C_1^{k_j}(1 + \frac{|\eta - \xi_j|}{\alpha_j})^{-k_j}$$

$$\le (C_1\alpha_j^{-1})^{k_j}(1 + |\eta - \xi_j|)^{-k_j}.$$

From these observations, (4) and Peetre's inequality,

the first term of the last estimate in (8) is bounded by

$$C_4(2 + |\xi_j|)^{2^p\mu^{1-p}}(C_1\alpha_j^{-1})^{k_j}(1 + |\eta - \xi_j|)^{-k_j}(1 + |\eta|)^{\mu+n+1}(1 + |\xi_j|)^{-N}$$

$$\le C_5(1 + |\xi_j|)^{2^p\mu^{1-p}-N}(C_1\alpha_j^{-1})^{k_j}(1 + |\eta - \xi_j|)^{-k_j+\mu+n+1}2^{\mu+n+1} \cdot (1 + |\xi_j|)^{\mu+n+1}.$$

Therefore the first term of the last part in (8) approaches to 0 as $j \to \infty$ because of $N > 2^p\mu^{1-p} + (\mu + n + 1)$ and $\alpha_j, k_j \to \infty$ as $j \to \infty$.

From Peetre's inequality and $C_1^{k_j} \le (2 + |\xi_j|)^d$,

the second term of the last estimate in (8) is bounded by

$$C_6(1 + |\xi_j|)^{2^p\mu^{1-p}} \sup_{|\eta-\xi_j|>A(\log(2+|\xi_j|))^{\frac{1}{q}}} \{(1 + |\eta|)^{N_0}(C_1(1 + \frac{|\eta - \xi_j|}{\alpha_j})^{-1})^{k_j} \cdot (1 + |\eta|)^{\mu+n+1}\}$$

$$\le C_6 \sup_{|\eta-\xi_j|>A(\log(2+|\xi_j|))^{\frac{1}{q}}} \{2^{N_0+\mu+n+1}(1 + |\xi_j|)^{N_0+\mu+n+1+2^p\mu^{1-p}} \cdot (1 + |\eta - \xi_j|)^{N_0+\mu+n+1}(2 + |\xi_j|)^d(1 + \frac{|\eta - \xi_j|}{\alpha_j})^{-k_j}\}$$

$$\leq C_7(1+|\xi_j|)^{N_0+\mu+n+1+d+2^p\mu^{1-p}} \sup_{|\eta-\xi_j|>A(\log(2+|\xi_j|))^{\frac{1}{q}}} \{\alpha_j{}^{N_0+\mu+n+1+d}$$

$$\cdot (1+\frac{|\eta-\xi_j|}{\alpha_j})^{N_0+\mu+n+1-k_j}\}$$

$$\leq C_7(1+|\xi_j|)^{N_0+\mu+n+1+d+2^p\mu^{1-p}}\{(\mu[\log(2+|\xi_j|)]^{\frac{1}{q}})^{N_0+\mu+n+1}$$

$$\cdot (1+\frac{A(\log(2+|\xi_j|))^{\frac{1}{q}}}{\mu[\log(2+|\xi_j|)]^{\frac{1}{q}}})^{N_0+\mu+n+1-k_j}\}$$

$$\leq C_8(1+|\xi_j|)^{2(N_0+\mu+n+1)+d+2^p\mu^{1-p}}(1+\frac{A}{\mu})^{N_0+\mu+n+1-k_j}$$

$$\leq C_9(1+|\xi_j|)^{2(N_0+\mu+n+1)+d+2^p\mu^{1-p}}(1+\frac{A}{\mu})^{-k_j}$$

$$\leq C_9(1+|\xi_j|)^{2(N_0+\mu+n+1)+d+2^p\mu^{1-p}}e^{-a[\log(2+|\xi_j|)]}$$

$$\leq C_9(1+|\xi_j|)^{2(N_0+\mu+n+1)+d+2^p\mu^{1-p}}e^{-a(\log(2+|\xi_j|)-1)}$$

$$\leq C_{10}(1+|\xi_j|)^{2(N_0+\mu+n+1)+d+2^p\mu^{1-p}-a}$$

where $a=\log(1+\frac{A}{\mu})$ and C_i, $1\leq i\leq 10$, are positive constants which are independent of j. Hence the second term of the last part in (8) approaches to 0 as $j\to\infty$ because of

$$A>\{e^{2^p\mu^{1-p}+2(N_0+\mu+n+1)}-1\}\mu.$$

Combining both estimates we have

$$\lim_{j\to\infty}\nu(S*\phi_j)=0$$

which gives the desired contradiction.

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THE STUDY OF A NONLINEAR SUSPENSION BRIDGE EQUATION BY A VARIATIONAL REDUCTION METHOD

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Abstract Let $Lu = u_{tt} + u_{xxxx}$ and H be the complete normed space spanned by the eigenfunctions of L. A nonlinear suspension bridge equation $(3 < b < 15)$

$$Lu + bu^+ = 1 + \epsilon h(x,t) \text{ in } H$$

has at least three solutions. It is shown by a variational reduction method.

KEYWORDS: **Eigenvalue, critical points, variational reduction method.**

Introduction

In this paper we inverstigate solutions of the nonlinear suspension bridge equation

$$u_{tt} + u_{xxxx} + bu^+ = 1 + \epsilon h(x,t) \ in \ (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R},$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{0.1}$$

$$u \ is \ \pi - periodic \ in \ t \ and \ even \ in \ x.$$

McKenna and Walter [6] prove that if $3 < b < 15$ then there exist at least two solutions of (0.1) by the degree theory, with replacing the condition for $u(t, x)$ in (0.1) by

$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t.$$

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We improve this earlier result of [6] in two ways; first we reduce the symmetry requirements and second we show that there are at least three (as opposed at least two) solutions, two of which are large amplitude. For $3 < b < 15$, one solution is positive and the existence of the other solutions can be proved by the dual variational method.

Our method shall be to reduce the problem in an infinite dimensional Hibert space to an equivalent finite-dimensional one via a variational reduction method. These methods were first used in [3], [4] and were afterwards extended in [1], to the case we wish to use.

1. Main results

Let L be the differential operator

$$Lu = u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u(x,t)$

$$\begin{gathered} Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbf{R}, \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0. \\ u(x,t) = u(-x,t) = u(x, t+\pi) \end{gathered} \tag{1.1}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x \quad \text{for} \quad n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cos(2n+1)x \quad \text{for} \quad m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cos(2n+1)x \quad \text{for} \quad m > 0, n \geq 0. \end{aligned}$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal base in H_0.

We define a subspace H of H_0 as follows

$$H = \{u \in H_0 : u = \sum(h_{mn}\phi_{mn} + \tilde{h}_{mn}\psi_{mn}),\ \sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2) < \infty\}$$

with a norm

$$|||u||| = [\sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2)]^{\frac{1}{2}}.$$

Then this normed space is complete and we have the following simple properties.

Proposition 1.1 (i) $Lu \in H$ implies $u \in H$,

(ii) $|||u||| \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u.

(iii) $\|u\| = 0$ iff $|||u||| = 0$.

Proof (i) Let

$$Lu = \sum \lambda_{mn}h_{mn}\phi_{mn} + \sum \lambda_{mn}\tilde{h}_{mn}\psi_{mn}.$$

Then

$$\begin{aligned} \infty &> |||Lu|||^2 = \sum |\lambda_{mn}|(\lambda_{mn}^2 h_{mn}^2 + \lambda_{mn}^2 \tilde{h}_{mn}^2) \\ &\geq \sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2) = |||u|||^2, \end{aligned}$$

because $|\lambda_{mn}| \geq 1$ for all m, n. (ii) and (iii) are trivial. ■

We note that even if $1 \in H_0$, $1 \notin H$. In fact, since 1 is expressed by

$$1 = \sum_{n=0}^{\infty}(-1)^n \frac{2\sqrt{2}}{2n+1}\phi_{0n},$$

$$|||1||| = \left(\sum_{n=0}^{\infty}(2n+1)^4 \frac{8}{(2n+1)^2}\right)^{\frac{1}{2}} = \infty.$$

Hence 1 does not belong to H.

Lemma 1.1 Let δ be not an eigenvalue of L. Let $u \in H_0$. Then $(L+\delta)^{-1}u \in H$.

Proof Suppose that δ be not an eigenvalue of L and finite. We recall that

$$\lambda_{mn} = (2n+1)^4 - 4m^2 = (4n^2+4n+1)^2 - (2m)^2.$$

If n is fixed, we define

$$\begin{aligned} \lambda_n^+ &= \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 8n^2 + 8n + 1, \\ \lambda_n^- &= \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3. \end{aligned}$$

When $n \longrightarrow \infty, \lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$. Hence we know that the number of $\{\lambda_{mn} : |\lambda_{mn}| < |\delta|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum (h_{mn}\phi_{mn} + \tilde{h}_{mn}\psi_{mn}).$$

Then

$$(L+\delta)^{-1}u = \sum \left(\frac{1}{\lambda_{mn}+\delta} h_{mn}\phi_{mn} + \frac{1}{\lambda_{mn}+\delta}\tilde{h}_{mn}\psi_{mn}\right).$$

Hence we have

$$\begin{aligned} |||(L+\delta)^{-1}u|||^2 &= \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}+\delta)^2}(h_{mn}^2 + \tilde{h}_{mn}^2) \\ &\leq C\sum (h_{mn}^2 + \tilde{h}_{mn}^2) \end{aligned}$$

for some C, which means that

$$|||(L+\delta)^{-1}u||| \leq C_1\|u\|, \quad C_1 = \sqrt{C}. \qquad \blacksquare$$

With the above Lemma 1.1, we can obtain the following lemma.

Lemma 1.2 Let $w(x,t) \in H$ and δ not an eigenvalue of L. Then all solution in H_0 of

$$Lu + \delta u^+ = w(x,t) \ \text{ in } \ H_0$$

belong to H.

Let V be the 2 dimensional subspace of H which is the closure of the span of the functions ϕ_{10} and ψ_{10}, both of which have the same eigenvalue $\lambda_{10} = -3$. Then $|||v||| = \sqrt{3}\|v\|$ for $v \in V$. Let W be the orthogonal complement of V in H.

We first consider the uniqueness theorem when $-1 < b < 3$.

Theorem 1.1 *Let $w(x,t) \in H_0$ and $-1 < b < 3$. Then the equation*

$$Lu + bu^+ = w(x,t) \tag{1.2}$$

has a unique solution in H_0. Furthermore if $w(x,t) \in H$, then the equation (1.2) has a unique solution in H.

Proof Let $w(x,t) \in H_0$ and $-1 < b < 3$. Let $\delta = 1$. The equation (1.2) is equivalent to

$$u = (L+\delta)^{-1}[-(b-\delta)u^+ - \delta u^- + w(x,t)], \tag{1.3}$$

where $(L+\delta)^{-1}$ is a compact, self-adjoint, linear map from H_0 into H_0 with norm $\frac{1}{2}$. We note that

$$\begin{aligned} &\|(b-\delta)(u_2^+ - u_1^+) + \delta(u_2^- - u_1^-)\| \\ &\le \max\{|b-\delta|, \delta\}\|u_2 - u_1\| < 2\|u_2 - u_1\|. \end{aligned}$$

It follows that the right hand side of (1.3) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < \frac{1}{2}2 = 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$ of (1.3). On the other hand, by Lemma 1.1, if $w(x,t) \in H$, then we know that the solution of (1.3) belongs to H. ■

Our main result in this paper is the following :

Theorem 1.2 *Let $h \in W, |||h||| = 1$, be given. Let $3 < b < 15$. Then there exists $\epsilon_0 > 0$ (depending on h and b) such that if $|\epsilon| < \epsilon_0$ the equation*

$$Lu + bu^+ = 1 + \epsilon h(x,t) \text{ in } H \tag{1.4}$$

has at least three solutions.

2. Proof of Theorem 1.2

Let us define the functional on H

$$I_b(u) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) + \frac{b}{2}|u^+|^2 - u - \epsilon h(x,t)u]dtdx. \tag{2.1}$$

The solutions of (1.4) coincide with the critical points of I_b.

Proposition 2.1 I_b is continuous in H and Fréchet differentiable at each u in H.

Proof. Let u be in H. To prove the continuity of I_b, we consider

$$\begin{aligned} &I_b(u+v) - I_b(u) \\ = &\int [u(v_{tt} + v_{xxxx}) + \frac{1}{2}v(v_{tt} + v_{xxxx}) + \frac{b}{2}(|(u+v)^+|^2 - |u^+|^2) \\ &-v - \epsilon hv]dxdt. \end{aligned}$$

Let $u = \sum(h_{mn}\varphi_{mn} + \tilde{h}_{mn}\psi_{mn})$, $v = \sum(k_{mn}\varphi_{mn} + \tilde{k}_{mn}\psi_{mn})$. Then

$$|\int u(v_{tt} + v_{xxxx})dxdt| = |\sum(\lambda_{mn}h_{mn}k_{mn} + \lambda_{mn}\tilde{h}_{mn}\tilde{k}_{mn})| \leq |||u||| \cdot |||v|||,$$

$$|\int \frac{1}{2}v(v_{tt} + v_{xxxx})dxdt| = |\sum \lambda_{mn}(k^2_{mn} + \tilde{k}^2_{mn})| \leq |||v|||^2.$$

On the other hand,

$$||(u+v)^+|^2 - |u^+|^2| \leq 2u^+|v| + |v|^2,$$

and hence

$$|\int(|(u+v)^+|^2 - |u^+|^2)dxdt| \leq 2\|u^+\|\|v\| + \|v\|^2 \leq 2|||u||||||v||| + |||v|||^2.$$

With the above results, we see that I_b is continuous at u.

Now let us prove that I_b is Fréchet differentiabl at u in H, with

$$I_b'(u)v = \int(Lu + bu^+ - 1 - \epsilon h)vdxdt.$$

To prove the above equation, it is enough to compute the following :

$$\begin{aligned}
&|I_b(u+v) - I_b(u) - I_b'(u)v| \\
= \; & |\int \left[\frac{1}{2}v(Lv) + \frac{b}{2}(|(u+v)^+|^2 - |u^+|^2 - 2u^+v)\right] dxdt| \\
\leq \; & \frac{1}{2}|||v|||^2 + \frac{|b|}{2}|\int(|(u+v)^+|^2 - |u^+|^2 - 2u^+v)dxdt| \\
\leq \; & \frac{1}{2}|||v|||^2 + \frac{|b|}{2}\int v^2 dxdt \\
\leq \; & \frac{1}{2}(1+|b|)|||v|||^2,
\end{aligned}$$

since $0 \leq |(u+v)^+|^2 - |u^+|^2 - 2u^+v \leq |v|^2$. ■

Lemma 2.1 *For $b > -1$, the boundary value problem*

$$y^{(4)} + by^+ = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \tag{2.2}$$

has a unique solution y, which is even and positive and satisfies

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0.$$

For the proof see [6]. From Lemma 2.1 we can obtain the following theorem :

Theorem 2.1 *Let $-1 < b$, with b not an eigenvalue of L. Let $h \in H$, with $|||h||| = 1$, be given. Then there exists $\epsilon_0 > 0$ (depending on b and h) such that if $|\epsilon| < \epsilon_0$ the boundary value problem*

$$Lu + bu^+ = 1 + \epsilon h(x,t) \text{ in } H$$

has a positive solution u.

Proof From Lemma 2.1 the problem

$$y^{(4)} + by^+ = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0$$

has a unique positive solution y_0. We note that if b is not an eigenvalue of L, then the following linear partial differential equation

$$Lu + bu = \epsilon h(x,t) \text{ in } H \tag{2.3}$$

has a unique solution u_ϵ. We can choose sufficiently small $\epsilon_0 > 0$ (depending on b and h) such that if $|\epsilon| < \epsilon_0$ then $u_\epsilon + y_0 > 0$, which is a solution of (1.4).∎

Next we shall use a variational reduction method to apply the mountain pass theorem.

Let $P : H \longrightarrow V$ denote the orthogonal projection of H onto V and $I - P : H \longrightarrow W$ denote that of H onto W, where V and W are defined in Section 1.

Lemma 2.2 *Let $3 < b < 15$, $h \in W$ with $|||h||| = 1$, and let $v \in V$ be given. Then for small $\epsilon > 0$, there exists a unique solution $z \in W$ of the equation*

$$Lz + (I-P)[b(v+z)^+ - 1 - \epsilon h(x,t)] = 0 \text{ in } W. \tag{2.4}$$

If $z = \theta(v)$, then θ is continuous on V and we have $DI_b(v + \theta(v))(w) = 0$ for all $w \in W$. If $\tilde{I}_b : V \longrightarrow R$ is defined by $\tilde{I}_b(v) = I_b(v + \theta(v))$, then $\tilde{I}_b$ has a continuous Fréchet derivative $D\tilde{I}_b$ with respect to v and

$$D\tilde{I}_b(v)(h) = DI_b(v + \theta(v))(h) \quad \text{for all} \quad h \in V.$$

If v_0 is a critical point of $\tilde{I}_b$, then $v_0 + \theta(v_0)$ is a solution of (1.4) and conversely every solution of (1.4) is of this form. In particular $\theta(v)$ satisfies a uniform Lipschitz condition in v with respect to the $L^2(Q)$ norm (also the norm $|||\cdot|||$).

Proof. Let $3 < b < 15, \delta = 7$, and $g(\xi) = b\xi^+$. If $g_1(\xi) = g(\xi) - \delta\xi$, the equation (2.4) is equivalent to

$$z = (L+\delta)^{-1}(I-P)[-g_1(v+z) + 1 + \epsilon h(x,t)]. \tag{2.5}$$

Since $(L+\delta)^{-1}(I-P)$ is a self-adjoint, compact, linear map from $(I-P)H$ into itself, the eigenvalues of $(L+\delta)^{-1}(I-P)$ in W are $(\lambda_{mn}+\delta)^{-1}$, where $\lambda_{mn} \geq 1$ or $\lambda_{mn} \leq -15$. Therefore its L_2 norm is $\frac{1}{8}$. Since

$$|g_1(\xi_2) - g_1(\xi_1)| \leq \max\{|b-\delta|, \delta\}|\xi_2 - \xi_1| < 8|\xi_2 - \xi_1|,$$

it follows that the right hand side of (2.5) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)H_0$ into itself with Lipschitz constant $\gamma < \frac{1}{8}8 = 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I-P)H_0$ (also $z \in (I-P)H$) which satisfies (2.5). Since the constant δ does not depend on v, it follows from standard arguments that if $\theta(v)$ denotes the unique $z \in (I-P)H$ which solves (2.5) then θ is continuous. In fact, if $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$ then we have

$$\begin{aligned}
&\|z_1 - z_2\| \\
= \ & \|(L+\delta)^{-1}(I-P)(-g_1(v_1+z_1) + g_1(v_2+z_2))\| \\
= \ & \|(L+\delta)^{-1}(I-P)([(\delta-b)(v_1+z_1)^+ - \delta(v_1+z_1)^-] \\
& \qquad -[(\delta-b)(v_2+z_2)^+ - \delta(v_2+z_2)^-])\| \\
\leq \ & \gamma\|(v_1+z_1) - (v_2+z_2)\| \\
\leq \ & \gamma(\|v_1 - v_2\| - \|z_1 - z_2\|).
\end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq c\|v_1 - v_2\|, \qquad c = \frac{\gamma}{1-\gamma}.$$

With this inequality we have

$$\begin{aligned}
&|||z_1 - z_2||| \\
= \ & |||(L+\delta)^{-1}(I-P)[(-b(z_1+v_1)^+ + \delta(z_1+v_1)) \\
& \qquad -(-b_1(z_2+v_2)^+ + \delta(z_2+v_2))]|||| \\
\leq \ & \frac{1}{2}\|(I-P)[(-b(z_1+v_1)^+ + \delta(z_1+v_1)) \\
& \qquad -(-b(z_2+v_2)^+ + \delta(z_2+v_2))]\| \\
\leq \ & 4(\|z_1 - z_2\| + \|v_1 - v_2\|) \\
\leq \ & \frac{4}{\sqrt{3}}(c+1)|||v_1 - v_2|||.
\end{aligned}$$

Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (2.4) we see that

$$\int_Q (-z_t w_t + z_{xx} w_{xx} + b(v+z)^+ w - w - \epsilon h(x,t) w) dt dx = 0.$$

Since

$$\int_Q v_t w_t = 0 \quad \text{and} \quad \int_Q v_{xx} w_{xx} = 0,$$

we have

$$DI_b(v + \theta(v))(w) = 0 \quad \text{for} \quad w \in W. \tag{2.6}$$

Let W_1 be the subspace of H which is the closure of the span of functions ϕ_{mn} and ψ_{mn} whose eigenvalues are $\lambda_{mn} \leq -15$ and let W_2 be the subspace of H which is the closure of the span of functions ϕ_{mn} and ψ_{mn} whose eigenvalues are $\lambda_{mn} \geq 1$. Let $v \in V$ and consider the function $h : W_1 \times W_2 \longrightarrow \mathbf{R}$ defined by

$$h(w_1, w_2) = I_b(v + w_1 + w_2).$$

The function h has continuous partial Fréchet derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_i h(w_1, w_2)(y_i) = DI_b(v + w_1 + w_2)(y_i)$$

for $y_i \in W_i$, $i = 1, 2$. Therefore, if we set $\theta(v) = \theta_1(v) + \theta_2(v)$ with $\theta_i(v) \in W_i$ for $i = 1, 2$, it follows from (2.6) that

$$D_i h(\theta_1(v), \theta_2(v)) = 0, \quad i = 1, 2. \tag{2.7}$$

If w_2 and y_2 are in W_2 and $w_1 \in W_1$, then

$$\begin{aligned} & [D_2 h(w_1, w_2) - D_2 h(w_1, y_2)](w_2 - y_2) \\ = \; & (DI_b(v + w_1 + w_2) - DI_b(v + w_1 + y_2))(w_2 - y_2) \\ = \; & \int_Q [-|(w_2 - y_2)_t|^2 + |(w_2 - y_2)_{xx}^2 + b((v + w_1 + w_2)^+ \\ & -(v + w_1 + y_2)^+)(w_2 - y_2)]dtdx. \end{aligned}$$

Since $(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) \geq 0$ for arbitrary ξ_1 and ξ_2, and

$$\int_Q [-|(w_2 - y_2)_t|^2 + (w_2 - y_2)_{xx}^2]dtdx = |||w_2 - y_2|||,$$

it follows that

$$(D_2 h(w_1, w_2) - D_2 h(w_1, y_2))(w_2 - y_2) \geq |||w_2 - y_2|||.$$

Therefore, h is strictly convex with respect to the second variable. Similarly, using the fact that $(g(\xi_2) - g(\xi_1))(\xi_2 - \xi_1) \leq b(\xi_2 - \xi_1)^2$, we see that if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned} & (D_1 h(w_1, w_2) - D_1 h(y_1, w_2))(w_1 - y_1) \\ & \leq -|||w_1 - y_1|||^2 + b\|w_1 - y_1\|^2 \leq (-1 + \frac{b}{15})|||w_1 - y_1|||^2, \end{aligned}$$

where $-15+b<0$. Therefore, h is strictly concave with respect to the first variable. From (2.7) it follows that

$$I_b(v+\theta_1(v)+\theta_2(v)) \le I_b(v+\theta_1(v)+y_2) \tag{2.8}$$

for $y_2 \in W_2$ with equality if and only if $y_2=\theta_2(v)$ and

$$I_b(v+\theta_1(v)+\theta_2(v)) \ge I_b(v+y_1+\theta_2(v)) \tag{2.9}$$

for $y_1 \in W_1$ with equality iff $y_1=\theta_1(v)$.

Since h is strictly concave (convex) with respect to its first (second) variable, Theorem 2.3 of [1] implies that $\tilde{I}_b$ is C^1 with respect to v and

$$D\tilde{I}_b(v)(h)=DI_b(v+\theta(v))(h), \quad h\in V. \tag{2.10}$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{I}_b(v_0)=0$. From (2.10) it follows that $DI_b(v_0+\theta(v_0))(v)=0$ for all $v\in V$. Since (2.6) holds for all $w\in W$ and H is the direct sum of V and W, it follows that $DI_b(v_0+\theta(v_0))=0$ in H. Therefore, $u=v_0+\theta(v_0)$ is a solution of (1.4).

Conversely our reasoning shows that if u is a solution of (1.4) and $v=Pu$, then $D\tilde{I}_b(v)=0$ in V.

We will see that $\theta(v)$ satisfies a uniform Lipschitz condition in v. Let $v_1, v_2 \in V$ and let $z_k=\theta(v_k)$, $\quad k=1,2$. From (2.5) it follows that

$$\begin{aligned} z_1-z_2 &= (L+\delta)^{-1}(I-P)[-g_1(v_1+z_1)+g_1(v_1+z_2)] \\ &\quad +(L+\delta)^{-1}(I-P)[-g_1(v_1+z_2)+g_1(v_2+z_2)], \end{aligned}$$

where $\delta=7$. Since $|g_1(\xi_1)-g_1(\xi_2)|\le \max\{|b-\delta|,\delta\}|\xi_1-\xi_2|$,

$$\begin{aligned} r &= \max\{(\lambda_{mn}+\delta)^{-1} : \lambda_{mn}\ge 1 \text{ or } \lambda_{mn}\le -15\} \\ &= \|(L+\delta)^{-1}(I-P)\| = \frac{1}{8}, \end{aligned}$$

and $\gamma = r\max\{|b-\delta|,\delta\} < \frac{1}{8}8=1$, it follows that

$$\|z_1-z_2\| \le \gamma\|z_1-z_2\|+\gamma\|v_1-v_2\|.$$

Hence

$$\|\theta(v_1)-\theta(v_2)\| \le k\|v_1-v_2\|,$$

where $k=\gamma(1-\gamma)^{-1}$ and the claim is established. ∎

Let $h\in W$ with $|||h|||=1$ and $3<b<15$. From theorem 2.1 there exists small $\epsilon_0>0$ (depending on h and b) such that for all ϵ with $|\epsilon|<\epsilon_0$, the equation (1.4) has a positive solution u_0, which belongs to W. By the above

Lemma 2.2 u_0 can be written by $u_0 = v_0 + \theta(v_0), \quad v_0 \in V$. Since the positive solution u_0 belongs to W, $v_0 = 0$. Therefore we have $u_0 = 0 + \theta(0)$.

Lemma 2.3 *Let $3 < b < 15$ and $h \in W$ with $|||h||| = 1$. Then there exists $\epsilon_0 > 0$ depending on h and b, and a small open neighborhood B of 0 in V such that for all ϵ with $|\epsilon| < \epsilon_0$, in B, $v = 0$ is a strict local point of minimum of $\tilde{I}_b$.*

Proof Since for $|\epsilon| < \epsilon_0$ the equation (1.4) has the positive solution u_0 which is of the form $u_0 = 0 + \theta(0)$ and $I + \theta$, where I is an identity map on V, is continuous, it follows that there exists a small open neighborhood B of 0 in V such that if $v \in B$ then $v + \theta(v) > 0$. We note that $\theta(v) = \theta(0)$ in B. Therefore, if $v \in B$, then for $z = \theta(v)$ we have

$$
\begin{aligned}
\tilde{I}_b(v) &= I_b(v+z) \\
&= \int_Q [\frac{1}{2}(-|(v+z)_t|^2 + |(v+z)_{xx}|^2) + \frac{b}{2}|(v+z)^+|^2 \\
&\quad -(v+z) - \epsilon h(x,t)(v+z)]dtdx \\
&= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) + \frac{b}{2}v^2]dtdx \\
&\quad + \int_Q [-v_t z_t + v_{xx} z_{xx} + bvz - v - \epsilon h(x,t)v]dtdx \\
&\quad + \int_Q [\frac{1}{2}(-|z_t|^2 + |z_{xx}|^2) + \frac{b}{2}z^2 - z - \epsilon h(x,t)z]dtdx \\
&= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) + \frac{b}{2}v^2]dtdx + C,
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \int_Q [\frac{1}{2}(-|z_t|^2 + |z_{xx}|^2) + \frac{b}{2}z^2 - z - \epsilon h(x,t)z]dtdx \\
&= I_b(z) = \tilde{I}_b(0).
\end{aligned}
$$

If $v \in V$, then $v = c_{10}\phi_{10} + c'_{10}\psi_{10}$, where the eigenvalue of ϕ_{10} and ψ_{10} is the same integer $\lambda_{10} = -3$. Therefore we have, in B,

$$
\begin{aligned}
\tilde{I}_b(v) - \tilde{I}_b(0) &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) + \frac{b}{2}v^2]dtdx \\
&= \frac{1}{2}(-3+b)\int_Q v^2 dtdx.
\end{aligned}
$$

Since $3 < b < 15$, it follows that $v = 0$ is a strict local point of minimum of $\tilde{I}_b$. ■

Lemma 2.4 *Let $h \in H$ with $|||h||| = 1$. For $-1 < b < 15$ and all $\epsilon \in [-1,1]$ the functional $\tilde{I}_b$, defined on V, satisfies the Palais-Smale condition : Any sequence $\{v_n\} \subset V$ for which $\tilde{I}_b(v_n)$ is bounded and $D\tilde{I}_b(v_n) \longrightarrow 0$ possesses a convergent subsequence.*

Proof. If $\tilde{I}_b(v_n)$ is bounded and $D\tilde{I}_b(v_n) \longrightarrow 0$ in V for any sequence $\{v_n\} \subset V$, then since V is 2 dimensional and spanned by smooth functions we have with $u_n = v_n + \theta(v_n)$

$$Lu_n + bu_n^+ = DI_b(u_n) + 1 + \epsilon h(x,t) \text{ in } H. \tag{2.11}$$

Assuming [P.S.] condition does not hold, that is $\|v_n\| \longrightarrow +\infty$, we see that $\|u_n\| \longrightarrow +\infty$. Dividing by $\|u_n\|$ and taking $w_n = \|u_n\|^{-1}u_n$ we have

$$Lw_n + bw_n^+ = \|u_n\|^{-1}(DI_b(u_n) + 1 + \epsilon h(x,t)). \tag{2.12}$$

Since $DI_b(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|u_n\| \longrightarrow +\infty$, the right hand side of (2.12) converges to 0 in $L^2(Q)$ as $n \longrightarrow \infty$. Moreover (2.12) shows that $\|Lw_n\|$ is bounded. Since L^{-1} is a compact operator, passing to a subsequence we get : $w_n \longrightarrow w_0$ in H_0. Since $\|w_n\| = 1$, it follows that $\|w_0\| = 1$. Taking the limit of both sides of (2.12), we find

$$Lw_0 + bw_0^+ = 0$$

with $\|w_0\| \neq 0$. This contradicts to the fact that for $-1 < b < 15$ the following equation

$$Lu + bu^+ = 0 \quad \text{in} \quad H_0$$

has only the trivial solution (cf. [6]). ∎

Let us define the functional on H :

$$I_b^*(u) = \int_Q \frac{1}{2}[(-|u_t|^2 + |u_{xx}|^2) + \frac{b}{2}|u^+|^2]dtdx.$$

Critical points of $I_b^*(u)$ coincide with solutions of the equation

$$Lu + bu^+ = 0 \quad \text{in} \quad H.$$

In fact, since, for $-1 < b < 15$, the equation $Lu + bu^+ = 0$, in H, has only the trivial solution $u = 0$ (cf. [6]), $I_b^*(u)$ $(-1 < b < 15)$ has only one critical point $u = 0$. Let $-1 < b < 15$. Given $v \in V$, let $\theta^*(v) \in W$ be a unique solution of the equation

$$Lz + (I - P)(b(v + z)^+) = 0 \quad \text{in} \quad W.$$

Let us define the reduced functional $\tilde{I}_b^*(v)$, on V, by $I_b^*(v+\theta(v))$. We note that we can obtain the same results as Lemma 2.2 when we replace $\theta(v)$ and $\tilde{I}_b(v)$ by $\theta^*(v)$ and $\tilde{I}_b^*(v)$. We also note that, for $-1 < b < 15$, $\tilde{I}_b^*(v)$ has only one critical point $v = 0$.

Lemma 2.5 *For $c > 0$, $\tilde{I}_b^*(cv) = c^2\tilde{I}_b^*(v)$.*

Proof If for $v \in V, z \in W$, $Lz + (I-P)(b(v+z)^+) = 0$ in W, then $L(cz) + (I-P)(b(cv+cz)^+) = 0$ for $c > 0$. Therefore $\theta^*(cv) = c\theta^*(v)$. From the definition of I_b^* we see that

$$I_b^*(cu) = c^2 I_b^*(u) \text{ for } u \in H \textit{ and } c > 0.$$

Hence, for $v \in V$ and $c > 0$,

$$\tilde{I}_b^*(cv) = I_b^*(cv + \theta^*(cv)) = c^2 I_b^*(v + \theta^*(v)) = c^2 \tilde{I}_b^*(v). \blacksquare$$

Lemma 2.6 *Let $3 < b < 15$. Then we have $\tilde{I}_b^*(v) < 0$ for all $v \in V$ with $v \neq 0$.*

Proof To prove this lemma it suffices to show that $\tilde{I}_b^*(v)$ does not satisfy the following four cases;

(i) $\tilde{I}_b^*(v) \geq 0$ and $\tilde{I}_b^*(v_0) = 0$ for some $v_0 \in V$ with $v_0 \neq 0$,

(ii) $\tilde{I}_b^*(v) \leq 0$ and $\tilde{I}_b^*(v_1) = 0$ for some $v_1 \in V$ with $v_1 \neq 0$,

(iii) $\tilde{I}_b^*(v) > 0$ for all $v \in V$ with $v \neq 0$,

(iv) There exists v_1 and v_2 in V such that $\tilde{I}_b^*(v_1) < 0$ and $\tilde{I}_b^*(v_2) > 0$.

Suppose that (i) holds. It follows that $\tilde{I}_b^*$ has an absolute minimum at v_0 and hence, $D\tilde{I}_b^*(v_0) = 0$. Therefore, by Lemma 2.2, $u_0 = v_0 + \theta^*(v_0)$ is a nontrivial solution of the equation $Lu + bu^+ = 0$ in H, which is a contradiction. A similar argument shows that it is impossible that (ii) holds.

Suppose that (iii) holds. Then there exists $t_0 \in (0,1)$ such that for all $t \leq t_0$

$$t\tilde{I}_b^*(v) + (1-t)\tilde{I}_0^*(v) \leq 0 \text{ for all } v \neq 0.$$

We note that there exists $v_0 \neq 0$ and $t(\leq t_0)$ such that $t\tilde{I}_b^*(v_0) + (1-t)\tilde{I}_0^*(v_0) = 0$. Let t_1 be the greatest number such that

$$t\tilde{I}_b^*(v_0) + (1-t)\tilde{I}_0^*(v_0) = 0$$

for some v_0 and t. Then $0 < t_1 \leq t_0$. Since $t_1\tilde{I}_b^*(v) + (1-t_1)\tilde{I}_0^*(v) \leq 0$ for all $v \neq 0$ and hence v_0 is a point of maximum of $t_1\tilde{I}_b^*(v) + (1-t_1)\tilde{I}_0^*(v)$, we have

$$D[t_1\tilde{I}_b^*(v_0) + (1-t_1)\tilde{I}_0^*(v_0)] = 0.$$

Let $v \in V$ be given and $0 < t_1 < 1$. Let $\theta^*_{t_1}(v)$ be the unique solution of the equation

$$Lz + (I-P)(t_1 b(v+z)^+) = 0 \text{ in } W.$$

We note that we can obtain the same results as Lemma 2.2 if we replace $\theta(v)$ and $\tilde{I}^*_b(v)$ by $\theta^*_{t_1}(v)$ and $t_1\tilde{I}^*_b(v) + (1-t_1)\tilde{I}^*_0(v)$. Therefore, it follows that $v_0 + \theta^*_{t_1}(v_0)$ is a nontrivial solution of the equation

$$t_1(Lu + bu^+) + (1-t_1)Lu = 0 \text{ in } H,$$

that is,

$$Lu + t_1 bu^+ = 0 \text{ in } H,$$

which contradicts to the fact that the above equation has only the trival solution because $0 < t_1 b < 15$.

A similar argument shows that it is impossible that (iv) holds. This proves our lemma. ∎

Lemma 2.7 *Let $3 < b < 15$ and $h \in W$ with $\|h\| = 1$. Then we have $\tilde{I}_b(v) \longrightarrow -\infty$ as $\|v\| \longrightarrow \infty$ ($|||v||| = \sqrt{3}\|v\|$).*

Proof We showed in Lemma 2.6 that $\tilde{I}^*_b(v) < 0$ for all $v \neq 0$. Suppose that it is not true that $\tilde{I}_b(v) \longrightarrow -\infty$ as $\|v\| \longrightarrow \infty$. This means that there exists a sequence $\{v_n\}_1^\infty$ in V and a number $M < 0$ such that $\|v_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\tilde{I}_b(v_n) \geq M$.

For given $v_n \in V$ let $w_n = \theta(v_n)$ be the unique solution of the equation

$$Lw + (I-P)(b(v_n+w)^+ - 1 - \epsilon h(x,t)) = 0 \text{ in } W.$$

According to Lemma 2.2 we have that for some constant k

$$\|\theta(v_n) - \theta(0)\| \leq k\|v_n\|, \quad |||\theta(v_n) - \theta(0)||| \leq k|||v_n|||.$$

From this we see that the sequence $\left\{\frac{w_n + v_n}{\|v_n\|}\right\}$ is bounded in H. Let $z_n = v_n + w_n$, $v^*_n = \frac{v_n}{\|v_n\|}$, $w^*_n = \frac{w_n}{\|v_n\|}$ and $z^*_n = v^*_n + w^*_n$ for $n \geq 1$. For $w_n = \theta(v_n)$ and $w^*_n = \frac{w_n}{\|v_n\|}$, we have

$$w^*_n = L^{-1}(I-P)(-b\left(\frac{v_n + w_n}{\|v_n\|}\right)^+ + \frac{1}{\|v_n\|} + \frac{\epsilon h(x,t)}{\|v_n\|}) \text{ in } W.$$

Since $\left\{\frac{w_n + v_n}{\|v_n\|}\right\}$ is bounded and $\frac{1}{\|v_n\|} + \frac{\epsilon h(x,t)}{\|v_n\|} \longrightarrow 0$ as $\|v_n\| \longrightarrow \infty$, it follows that $-b\left(\frac{v_n + w_n}{\|v_n\|}\right)^+ + \frac{1}{\|v_n\|} + \frac{\epsilon h(x,t)}{\|v_n\|}$ is bounded in H. Since L^{-1} is a compact operator, passing to a subsequence we get that w_n^* converge to w^* in W. Since V is 2 dimensional space, we may assume that $\{v_n^*\}_1^\infty$ converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, we can assume that $\{z_n^*\}_1^\infty$ converges to an element z^* in H.

On the other hand, since $\tilde{I}_b(v_n) \geq M$ for all n, we have, for all n,

$$\int_Q (\frac{1}{2}Lz_n \cdot z_n + \frac{b}{2}|z_n^+|^2 + z_n - \epsilon h(x,t)z_n)dtdx \geq M.$$

Dividing the above inequality by $\|v_n\|^2$, we obtain

$$\int_Q [\frac{1}{2}(-|(z_n^*)_t|^2 + |(z_n^*)_{xx}|^2) + \frac{b}{2}|(z_n^*)^+|^2 - \frac{z_n^*}{\|v_n\|} \tag{2.13}$$

$$-\epsilon h(x,t)\frac{z_n^*}{\|v_n\|}]dtdx \geq \frac{M}{\|v_n\|^2}.$$

From the definition of $w_n = \theta(v_n)$, it follows that for any $y \in W$ and $n \geq 1$

$$\int_Q [-(z_n)_t y_t + (z_n)_{xx} y_{xx} + bz_n^+ y - y - \epsilon h(x,t)y]dtdx = 0. \tag{2.14}$$

If we set $y = w_n$ in (2.14) and divide by $\|v_n\|^2$, then we obtain

$$\int_Q [-|(w_n^*)_t|^2 + |(w_n^*)_{xx}|^2 + (b(z_n^*)^+ - \frac{1}{\|v_n\|} - \frac{\epsilon h(x,t)}{\|v_n\|})w_n^*]dtdx = 0 \tag{2.15}$$

for all $n \geq 1$.

Let $y \in W$ be arbitrary. Dividing (2.14) by $\|v_n\|$ and letting $n \longrightarrow \infty$, we obtain

$$\int_Q [-(z^*)_t y_t + (z^*)_{xx} y_{xx} + b(z^*)^+ y]dtdx = 0. \tag{2.16}$$

we see that (2.16) can be written in the form $DI_b^*(v^* + w^*)(y) = 0$ for all $y \in W$. Hence by Lemma 2.2 $w^* = \theta^*(v^*)$. Letting $n \longrightarrow \infty$ in (2.15), We obtain

$$\begin{aligned}
& \lim_{n\to\infty} \int_Q (-|(w_n^*)_t|^2 + |(w_n^*)_{xx}|^2)dtdx \\
= & -\lim_{n\to\infty} \int_Q b(z_n^*)^+ w_n^* dtdx \\
= & -\int_Q b(z^*)^+ w^* dtdz \\
= & \int_Q (-(z^*)_t (w^*)_t + (z^*)_{xx}(w^*)_{xx})dtdx \\
= & \int_Q (-|(w^*)^t|^2 + |(w^*)_{xx}|^2)dtdx,
\end{aligned}$$

where we have used (2.16). Hence

$$\lim_{n\to\infty}\int_Q[-|(z_n^*)_t|^2+|(z_n^*)_{xx}|^2]dtdx=\int_Q[-|(z^*)_t|^2+|(z^*)_{xx}|^2]dtdx.$$

Letting $n\longrightarrow\infty$ in (2.13), we obtain

$$\tilde{I}_b^*(v^*)=\int_Q[\frac{1}{2}(-|(z^*)_t|^2+|(z^*)_{xx}|^2)+\frac{b}{2}|(z^*)^+|^2]dtdx\geq 0.$$

Since $\|v^*\|=1$, this contradicts to the fact that $\tilde{I}_b^*(v)<0$ for all $v\neq 0$. This proves that $\tilde{I}_b(v)\longrightarrow-\infty$ as $\|v\|\longrightarrow\infty$. ∎

We now use the familiar deformation lemma.

Lemma 2.8 *Let E be a real Banach space and $I\in C^1(E,R)$. Suppose I satisfies Palais-Smale condition. Let N be a given neighborhood of the set K_c of the critical points of I at a given level c. Then there exists $\epsilon>0$, as small as we want, and a deformation $\eta:[0,1]\times E\longrightarrow E$ such that, denoting by A_b the set $\{x\in E: I(x)\leq b\}$:*

$$\begin{array}{ll}(i) & \eta(0,x)=x\quad\forall x\in E,\\ (ii) & \eta(t,x)=x\quad\forall x\in A_{c-2\epsilon}\cup(E\backslash A_{c+2c}),\ \forall t\in[0,1],\\ (iii) & \eta(1,\cdot)(A_{c+\epsilon}\backslash N)\subset A_{c-\epsilon}.\end{array}$$

The proof of Lemma 2.8 can be found in [2].

Now let us prove our main results stated in the end of Section 1.

Proof of Theorem 1.2. By Lemma 2.3, there exists $\epsilon_0>0$ and a small open neighborhood B of 0 in V such that for all ϵ with $|\epsilon|<\epsilon_0$, in B, $v=0$ is a strict local point of minimum of $\tilde{I}_b$. Since $\tilde{I}_b(v)\longrightarrow-\infty$ as $|||v|||\longrightarrow\infty$ (Lemma 2.7) and $\tilde{I}_b\in C^1(V,R)$ satisfies Palais-Smale condition, $\max_{v\in V}\tilde{I}_b(v)$ exists and is a critical value of $\tilde{I}_b$. Hence there exists a critical point v_0 of $\tilde{I}_b$ such that

$$\tilde{I}_b(v_0)=\max_{v\in V}\tilde{I}_b(v).$$

Let C be an open neighborhood of v_0 in V such that $B\cap C=\phi$. Since $\tilde{I}_b(v)\longrightarrow-\infty$ as $|||v|||\longrightarrow\infty$, we can choose $v_1\in V\backslash(B\cup C)$ such that $\tilde{I}_b(v_1)<\tilde{I}_b(0)$. Let Γ be the set of all paths in V joining 0 and v_1. We write

$$c=\inf_{\gamma\in\Gamma}\sup_{\gamma}\tilde{I}_b(v).$$

Let $\Gamma'=\{\gamma\in\Gamma:\gamma\cap C=\phi\}$ and

$$c'=\inf_{\gamma\in\Gamma'}\sup_{\gamma}\tilde{I}_b(v).$$

The fact that in B, $v = 0$ is a strict local point of minimum of $\tilde{I}_b$ when $|\epsilon| < \epsilon_0$, the fact that $\tilde{I}_b(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$, the fact that $\tilde{I}_b$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem (cf. [2]) imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{I}_b(v).$$

is a critical value of $\tilde{I}_b$.

First we prove that if $\tilde{I}_b(v_0) = c$, then there exists a critical point v of $\tilde{I}_b$ at level c such that $v \neq v_0$ (of course $v \neq 0$ since $c \neq \tilde{I}_b(0)$).

We claim that if $\tilde{I}_b(v_0) = c$, then $c = c'$. In fact, since $\Gamma' \subset \Gamma$, $c \leq c'$. On the other hand, $c' \leq c$ since c is the maximum value of $\tilde{I}_b$. Hence $c = c'$. Suppose by contradiction $K_c = \{v_0\}$. By the above claim $c = c'$. Let us fix ϵ, η as in Lemma 2.8 with $E = V$, $I = \tilde{I}_b, c = c$, $N = C$ and taking $\epsilon < \frac{1}{2}(c - \tilde{I}_b(0))$. Taking $\gamma \in \Gamma'$ such that $\sup_\gamma \tilde{I}_b \leq c$. From Lemma 2.8 $\eta(1, \cdot) \circ \gamma \in \Gamma$ and

$$\sup \tilde{I}_b(\eta(1, \cdot) \circ \gamma) \leq c - \epsilon < c,$$

which is a contradition. Therefore, there exists a critical point v of $\tilde{I}_b$ at level c such that $v \neq v_0$, 0, which means that the equation (1.4) has at least 3 solutions when $3 < b < 15$.

Finally, if $\tilde{I}_b(v_0) \neq c$, then there exists a critical point v of $\tilde{I}_b$ at level c such that $v \neq v_0$, 0 (since $c \neq \tilde{I}_b(v_0)$ and $c > \tilde{I}_b(0)$). Therefore, in case $\tilde{I}_b(v_0) \neq c$, the equation (1.4) has also at least 3 solutions. ∎

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REFLECTED BROWNIAN MOTION AND HARNACK PRINCIPLE

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1. Introduction

The boundary Harnack principle without boundary condition may be stated as follows.

Theorem 1.1 *Let D be a domain and V an open set. Then for any compact $K \subset V$, there exists a constant c_0 such that for all nonnegative harmonic functions u and v in D that vanish continuously on $(\partial D) \cap D$ with $u(x) = v(x)$ for some $x \in K \cap D$,*

$$c_0^{-1} u(y) \leq v(y) \leq c_0 u(y)$$

for all $y \in K \cap D$.

When D is Lipschitz, Bass and Burdzy gave a probabilistic proof of Theorem 1.1 using elementary properties of Brownian motion. Also by a similar way, Theorem 1.1 is proved when D is Hölder domain of order α, $1/2 < \alpha < 1$ and Theorem 1.1 is not true if $\alpha < 1/2$ in [BB].

The main purpose of this paper is to give a probabilistic proof of boundary Harnack principle with boundary condition, that is the following theorem.

Theorem 1.2 *Let D be a Lipschitz domain of constant $\gamma > 1$. There exists $c > 0$, depending only on γ, such that if $z \in D$, $r > 0$, h is nonnegative and harmonic in $B(z, 6r) \cap D$ and h has zero normal derivative on $B(z, 6r) \cap \partial D$, then*

$$c^{-1} h(y) \leq h(x) \leq c h(y)$$

for $x, y \in B(z, r) \cap D$.

Theorem 1.2 is proved in [BP] by properties of reflected Brownian motion (abbreviated RBM). Also a generalization is given by Y.Kwon in [Kw] when the reflection on the boundary may not be normal, moreover not continuous but satisfies some certain condition.

In section 2, we establish Theorem 1.2 by the key estimate of exit distribution of RBM and the following Proposition 1.3.

Define

$$Osc_C f = \sup_C f - \inf_C f.$$

Proposition 1.3 *There exists $\rho \in (0,1)$, depending only on γ(the Lipschitz constant) such that if $x \in D$, $r > 0$, h is harmonic in $B(x,r) \cap D$, continuous on $\overline{B(x,r)} \cap D$ and h has zero normal derivative on $B(x,r) \cap \partial D$, then*

$$Osc_{B(x,r/2)\cap D} h \leq Osc_{B(x,r)\cap D} h.$$

2. RBM

Let us consider $(\Omega, \mathcal{F}, P)$ -a complete probability space with an increasing family of sub σ-field $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$. We suppose we are given a d-dimensional $\mathcal{F}_t$-Brownian motion $(B_t)_{t\geq 0}$ and domain $D \subset R^d$. Then in C^1 domains, Lions and Sznitman [LS] proved the existence and uniqueness of continuous $\mathcal{F}_t$-semimartingale $(X_t)_{t\geq 0}$ satisfying;

there exists the continuous bounded variation process L_t such that $X_t \in \overline{D}$ for all $t \geq 0$ a.s.

$$\begin{aligned} X_t &= x + B_t + \textstyle\int_0^t n(X_s) dL_s \\ L_t &= \textstyle\int_0^t 1_{(X_s \in \partial D)} dL_s \end{aligned}$$

where n is the unit inward normal. We call above X_t as RBM. Let P^x denote the probability measure on Ω such that $P^x(X_0 = x) = 1$ and E^x be the integral with respect to P^x. In case of bounded Lipschitz domain D, Bass and Hsu proved the existence of RBM and L_t. More precisely; let σ be the surface measure of boundary and $p(t,x,y)$ be the density of X_t, that is, $P^x(X_t = y) = p(t,x,y)$. Then for any $\lambda > 0$ and $x \in \overline{D}$,

$$\int_0^\infty e^{-\lambda t} p(t,x,y)\sigma(y) dy = E^x[\int_0^\infty e^{-\lambda t} dL_t].$$

Another way to see L_t is following;

we consider the Newmann boundary value problem on D;

$$\text{(2.1)} \qquad \Delta u = 0 \quad \text{on} \quad D$$

$$\frac{\partial u}{\partial n} = -f \quad \text{on} \quad \partial D$$

where f is a bounded measurable and $\int_{\partial D} f(x)\sigma(dx) = 0$. Then there exists a unique solution of (2.1) satisfying $\int_D u(x)dx = 0$ and $u(x)$ is represented as

$$u(x) = \lim_{t\to\infty} \frac{1}{2} E^x[\int_0^t f(X_s)dL_s].$$

We introduce the notations.

(2.2) $D_\varepsilon = \{x \in D : dist(x, \partial D) < \varepsilon\}$

(2.3) $A(x,r) = B(x,r) \cap D$

and if X_t is RBM on D, then define

(2.4) $\tau_r = \inf\{t : |X_t - X_0| > r\}$

and

(2.5) $T_B = \inf\{t : X_t \in D\}$ for Borel set B.

3. Estimates

In this section, we assume the domain D is bounded and smooth, but estimates will only depend on γ, the Lipschitz constant of D not on any further smoothness of D.

Proposition 3.1 *Let $x \in D$. Given $\eta > 0$, there exists $\delta > 0$ depending on η but not x, such that if $C \subset A(x,1)$ and $|C| > \eta$, then*

$$P^x[T_C < \tau_2] > \delta.$$

Here $|C|$ is the Lebeque measure of C and T_A and τ_2 are definded in section 2.
Proof. By the fact that D is Lipschitz,

$$|D_\varepsilon \cap B(x,1)| \leq c\varepsilon$$

for some $c > 0$. So if ε is taken small enough, $C' = C - D_\varepsilon$ will be a positive distance from ∂D and $|C'| > \eta/2$. We can find a large integer N depending only on ε, η and γ such that we can cover $B(x,2) - D_\varepsilon$ by at most N balls of radius $\varepsilon/4$ with center in $B(x,2) - D_\varepsilon$. For at least one of these balls, say $B(y, \varepsilon/4)$,

$$|C' \cap B(y, \varepsilon/4)| > \eta/2N.$$

We then taken $C'' = C' \cap B(y, \varepsilon/4)$ and we show there exists $\delta > 0$ such that

$$P^x[T_{C''} < \tau_2] > \delta.$$

By theorem 3.2 of [BP], we can find t_0 such that

$$P^x[\tau_1 < t_0] \leq 1/4.$$

Let $G(x, y)$ be the Green function for P. Then

$$\begin{aligned} P^x[X_s \in D_\varepsilon \cap B(x,\lambda) \quad \text{for all } s \leq 1] &\leq E^x[\textstyle\int_0^1 1_{D_\varepsilon \cap B(x,\lambda)}(X_s)ds] \\ &\leq \textstyle\int_{D_\varepsilon \cap B(x,\lambda)} G(x,w)dw \\ &\leq 1/4 \end{aligned}$$

if ε, depending on γ, is sufficiently small. So we can take $\varepsilon' \in (0, \varepsilon/4)$ sufficiently small so that

$$P^x[X_s \in D_{\varepsilon'} \quad \text{for all } s \leq t_0] \leq 1/4.$$

So

$$P^x[X_s \in A(x,1) - D_{\varepsilon'} \quad \text{for some } s < \tau_1] \geq 1/2.$$

By the strong Markov property, the support theorem of Brownian motion ([SV] pp 168-169) and geometrical consideration, there exists $\delta' > 0$ such that

$$P^x[X_s \in B(y, \varepsilon/4) \quad \text{for some } s < \tau_2] \geq \delta'.$$

And if $z \in B(y, \varepsilon/4)$,

$$P^z[T_{C''} < \tau_{\varepsilon/2} \geq \int_{C''} p^0(1, z, w)dw \geq c|C''|,$$

where p^0 is the transition density for Brownian motion killed on exiting $B(y, \varepsilon/2)$. Therefore with the strong Markov property, we prove the result with

$$\delta = c\eta\delta'/2N.$$

Note that if a is a constant, aX_{t/a^2} is again a Brownian motion in the interior of aD. Also aD is the region above a Lipschitz function with the same Lipschitz constant γ as F. We refer this property as scaling.

Proposition 3.2 *There exists $\rho \in (0,1)$ depending only on γ, such that if $x \in D$, $r > 0$, h is harmonic in $A(x,r)$, continuous on $\overline{B(x,r)} \cap D$, and h has zero normal derivative on $B(x,r) \cap \partial D$, then*

$$Osc_{A(x,r/2)} h \leq \rho Osc_{A(x,r)} h.$$

Proof. By considering $ah+b$ for suitable a and b, we may assume $\sup_{A(x,r)} h = 1$, $\inf_{A(x,r)} h = 0$. Moreover, by considering $1-h$ if necessary, we may assume

$$|\{x \in A(x,r) : h(x) \geq 1/2\}| \geq 1/2|A(x,r)|.$$

Let $C = \{x \in A(x,r) : h(x) \geq 1/2\}$. Then by Proposition 3.1, scaling and the fact that ∂D is smooth,

$$h(y) = E^y[h(X_{\tau_r \wedge T_C})] \geq 1/2P^y[T_C \leq \tau_r] \geq \delta > 0$$

for $y \in A(x,r/2)$. Since $h \leq 1$ in $A(x,r/2)$ by the maximum principle,

$$Osc_{A(x,r/2)}h \leq 1-\delta = (1-\delta)Osc_{A(x,r)}h.$$

Now take $\rho = 1-\delta$.

Now we can prove a Harnack principle valid up to the boundary of D for harmonic function with zero argument as Theorem 3.9 of [BP].

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THE HYPERBOLIC CAUCHY PROBLEM

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1. Review on basic facts

1.1 Hyperbolicity

Let P be a differential operator of order m defined on an open set Ω in $\mathbb{R}^{d+1}$ and let H be a hypersurface in Ω. The Cauchy problem for P with respect to the hypersurface H is:

Find a solution u to the equation $Pu = 0$ of which the first m terms in the Taylor expansion on H coincide with given functions on H?

This is not always possible and hence our main concern is:

For which operators P and hypersurfaces H this problem could be solved?

One almost necessary condition to this problem is that P is non-characteristic with respect to H. That is

DEFINITION 1.1.1: P is said to be *non-characteristic* with respect to H at $\bar{x} \in H$ if

$$\lim_{\lambda\to\infty} \lambda^{-m} e^{-\lambda h(x)} P e^{\lambda h(x)} \neq 0 \text{ at } \bar{x}, \tag{1.1.1}$$

where $h(x)$ is a defining function of H, in the sense that $H = \{h(x) = 0\}$, $dh(x) \neq 0$ on H.

In the analytic category, (1.1.1) is sufficient to assure the solvability of the Cauchy problem (Cauchy-Kowalevsky Theorem). On the other hand, (1.1.1) is far from sufficient to garantee the solvability of the Cauchy problem for general C^∞ data.

REMARK: If P is of constant coefficients and H is a hyperplane, this is really necessary ([10]). In the case of variable coefficients, if we assume the existence of a finite dependence domain, this is also necessary ([30], [14], [11]).

Taking the remark in mind, we assume, in what follows, that P is non-characteristic with respect to H.

With a system of local coordinates $x = (x_0, x_1, ..., x_d)$ in Ω, P is expressed as follows:

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha = \sum_{j=0}^{m} P_j(x, D), \tag{1.1.2}$$

where $a_\alpha(x)$ are C^∞ functions on Ω and D is the differential monomial

$$D^\alpha = D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_d^{\alpha_d}, D_j = -i\frac{\partial}{\partial x_j}$$

and $\alpha = (\alpha_0, \alpha_1, ..., \alpha_d) \in \mathbb{N}^{d+1}$, $|\alpha| = \sum_{j=0}^{d} \alpha_j$,

$$P_j(x, D) = \sum_{|\alpha|=j} a_\alpha(x) D^\alpha.$$

We choose the local coordinates so that

$$h(x) = x_0, \bar{x} = 0,$$

near $\bar{x}$ and we write P_m in the following form,

$$P_m = \sum_{k=0}^{m} Q_{m-k}(x, D') D_0^k, \tag{1.1.3}$$

where Q_j is a differential operator of order j with respect to $x' = (x_1, ..., x_d)$. In this situation, the condition (1.1.1) yields that

$$Q_0(\bar{x}) \neq 0.$$

Hence, dividing P by $Q_0(x)$, we can assume that the coefficient of D_0^m in (1.1.2) is equal to one near $\bar{x}$.

Here we give an elegant formulation of the Cauchy problem due to [14], DEFINITION 1.1.2: Let P be a partial differential operator of order m with coefficients in $C^\infty(\Omega)$. Let $t = t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued function. Then the Cauchy problem for P is C^∞ *well posed* at $\overline{x}$ with respect to $t(x)$ if there exist a neighborhood $\omega \subset \Omega$ of $\overline{x}$ and a number $\varepsilon > 0$ such that

$$P : E_\tau = \{v \in C^\infty(\omega) | v = 0 \text{ in } t(x) < t(\overline{x}) + \tau\} \to E_\tau \tag{1.1.4}$$

is an isomorphism if $|\tau| < \varepsilon$.

Our main concern is to characterize differential operators for which the Cauchy problem is C^∞ well posed, that is to characterize *hyperbolic operators*. Another very closely related problem is to characterize strongly hyperbolic operators:

DEFINITION 1.1.3: Let P be a differential operator of order m with $C^\infty(\Omega)$ coefficients and $t(x) \in C^\infty(\Omega)$, be real valued with $dt(x) \neq 0$ in Ω. Then P (or the principal part P_m of P) said to be *strongly hyperbolc* at $\overline{x} \in \Omega$ with respect to $t(x)$ if, for any differential operator Q of order at most $m-1$ with $C^\infty(\Omega)$ coefficients, the Cauchy problem for $P + Q$ is C^∞ well posed at $\overline{x}$ with respect to $t(x)$.

1.2 Operators with constant coefficients

We take $t(x) =< \theta, x >, \theta \in \mathbb{R}^{d+1}$ as a linear function in x so that $dt(x) = \theta$. In this case, the hyperbolicity is completely characterized. Let

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \tag{1.2.1}$$

be a polynomial in $D_0, ..., D_d$. We introduce the following condition; there exists $T > 0$ such that

$$\xi \in \mathbb{R}^{d+1}, \tau \in \mathbb{C}, P(\xi + \tau\theta) = 0 \Longrightarrow |\text{Im}\tau| \leq \text{T}. \tag{1.2.2}$$

Theorem 1.2.1: *Let P have constant coefficients. In order that P to be hyperbolic at $\overline{x}$ w.r.t. θ, it is necessary and sufficient that (1.1.1) and (1.2.2) hold ([8]).*

Here we remark that the hyperbolicity is independent of $\overline{x}$ if $t(x)$ is linear in x. Recall that the principal part of P is given by

$$P_m = P_m(D) = \sum_{|\alpha| = m} a_\alpha D^\alpha. \tag{1.2.3}$$

If P is hyperbolic w.r.t. θ, then P_m is also hyperbolic w.r.t. θ. On the other hand if P is a homogeneous polynomial satisfying (1.1.1) then, for this P, the condition (1.2.2) is equivalent to that

$$\xi \in \mathbb{R}^{d+1}, P(\xi + \tau\theta) = 0 \Longrightarrow \tau \text{ is real}, \tag{1.2.4}$$

The following is also an important characterization of the hyperbolicity.

Theorem 1.2.2: *Suppose that $P(D)$ satisfies (1.1.1) and $P_m(D)$ is hyperbolic w.r.t. θ. In order that P is hyperbolic w.r.t. θ it is necessary and sufficient that we have*

$$|P(\xi)| \le C \sum_{\alpha} |D^{\alpha} P_m(\xi)| \text{ for any } \xi \in \mathbb{R}^{d+1},$$

with some $C > 0$, where the sum is taken over all order derivatives w.r.t. ξ ([48]).

DEFINITION 1.2.1: Let $P(D)$ be given by (1.2.1) , $P(D)$ is said to be *strictly hyperbolic* w.r.t. θ if the roots of the equation $P_m(\xi + \tau\theta) = 0$ in τ are all real and distinct for any $\xi \in \mathbb{R}^{d+1} \setminus \mathbb{R}\theta$.

Theorem 1.2.3: *Let P have constant coefficients. For P to be strongly hyperbolic w.r.t. θ, it is necessary and sufficient that P is strictly hyperbolic w.r.t. θ.*

Let $P(D)$ be hyperbolic w.r.t. $t(x) = x_0$. Then a fundamental solution E of the Cauchy problem for $P(D)$ is a distribution satisfying

$$P(D)E = \delta(x), E = 0 \text{ in } x_0 < 0, \tag{1.2.5}$$

where $\delta(x)$ is the Dirac measure at the origin.

We define $\Gamma(P_m, \theta)$ by

$$\Gamma(P_m, \theta) = \text{the component of } \theta \text{ in } \{\xi | P_m(\xi) \neq 0\},$$

which is a cone with vertex at the origin. Then one can prove that the support of E is contained in $\Gamma^{\circ}(P_m, \theta)$, which is the dual cone of $\Gamma(P_m, \theta)$:

$$\Gamma^{\circ}(P_m, \theta) = \{x | < x, y > \ge 0, y \in \Gamma(P_m, \theta)\}.$$

For more detailed studies on the hyperbolicity of operators with constant coefficients, we refer to [8], [1] and [10].

1.3 Strict hyperbolicity

With a system of local coordinates $x = (x_0, ..., x_d)$ in Ω, P is given by

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$

Recall that the *principal part* of P is defined by

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha. \tag{1.3.1}$$

$P_m(x, \xi)$ is invariantly defined as a function on the cotangent bundle $T^*\Omega$.

A first basic result in the characterization of hyperbolicity, in the variable coefficients case, is

Theorem 1.3.1: *Suppose that P is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$. Then there is a neighborhood U of $\overline{x}$ such that $P_m(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ for every $x \in U$, that is $P_m(x, \cdot)$ satisfies (1.2.4) ([25], [31]).*

DEFINITION 1.3.1: We say that a point $z = (x, \xi) \in T^*\Omega \backslash 0$ is a *characteristic of order k* of P_m if

$$d^j P_m(z) = 0, j \leq k-1, d^k P_m(z) \neq 0. \tag{1.3.2}$$

where $d^j P_m$ is the j-th differential of P_m.

DEFINITION 1.3.2: P is said to be *strictly hyperbolic* at $\overline{x} \in \Omega$ w.r.t. $t(x) \in C^\infty(\Omega)$ if there exists a neighborhood $\omega \subset \Omega$ of $\overline{x}$ such that for any $x \in \omega$, $P_m(x, \cdot)$ is strictly hyperbolic w.r.t. $dt(x)$ in the sense of the definition 1.2.1.

We note that $P_m(x, \cdot)$ is a polynomial on $T_x^*\Omega$ and $dt(x) \in T_x^*\Omega$.

Lemma 1.3.2: *Assume that $P_m(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\overline{x}$. Then P is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ if and only if there is a neighborhood $\omega \subset \Omega$ of $\overline{x}$ such that every characteristic on $T^*\omega \setminus 0$ of P_m is simple.*

Theorem 1.3.3: *If P is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ then P is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([46], [23], [9]).*

We assume that P is strictly hyperbolic in Ω w.r.t. $t(x)$ and we define $\Gamma^-(x)$ as follows

$$\cup_{x(s)} \{y \in \mathbb{R}^n | y \in x(s)\}$$

where $x(s)$ varies over all Lipschitz curves such that $(d/ds)x(s)$ belongs to $\Gamma^\circ(p(x(s), \cdot), dt(x(s)))$, $x(0) = x$, x_0 is decreasing along on $x(s)$.

We take $\omega \subset \Omega$, a neighborhood of $\overline{x}$, so that

$$\Gamma^-(x) \cap \{t(x) \geq t(\overline{x})\} \subset\subset \omega \text{ if } x \in \omega^+ = \omega \cap \{t(x) \geq t(\overline{x})\}. \tag{1.3.3}$$

Then we have

Theorem 1.3.4: *Assume that*

$$Pu = f \text{ in } \omega^+,\ u = 0 \text{ in } t(x) < t(\bar{x}) \text{ and } f = 0 \text{ on } \Gamma^-(x).$$

Then it follows that

$$u = 0 \text{ on } \Gamma^-(x)$$

([23]).

The same conclusion holds for the singularities of the solution of (1.3.4), i.e. if f is C^∞ in a neighborhood of $\Gamma^-(x)$ then so is u. A more refined version of this is the celebrated theorem in the propagation of singularities. For this we need to microlocalize the notion that u is singular at $\bar{x}$, i.e. that u is not C^∞ in some neighborhood of $\bar{x}$ to that of wave front set.

Now we introduce the bicharacteristic of P_m which carries the wave front set of solutions. In the following we assume that P_m is real valued and set

$$P_m(x, \xi) = p(x, \xi)$$

for simplicity. The Hamilton vector field H_p of p is given by

$$H_p = \sum_{j=0}^{d} \frac{\partial p(x,\xi)}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p(x,\xi)}{\partial x_j}\frac{\partial}{\partial \xi_j} \tag{1.3.4}$$

which is a vector field on $T^*\Omega$.

DEFINITION 1.3.3: A *bicharacteristic* of p is an integral curve of H_p on $\{p = 0\}$.

Let γ be a bicharacteristic of p issuing from $z = (\bar{x}, \bar{\xi})$ with $p(z) = 0$, on which x_0 is decreasing.

Theorem 1.3.5: *Assume that P is strictly hyperbolic at $\bar{x}$. If $u \in \mathcal{D}'(\omega)$ satisfies*

$$Pu = f \text{ near } \bar{x} \text{ and } z \notin WF(f),$$

then

$$z \notin WF(u),$$

if $\gamma(-\epsilon) \notin WF(u)$ with a sufficiently small $\epsilon > 0$, where $WF(u)$ denotes the wave front set of u ([11]).

1.4 Operators with constant multiple characteristics

We begin with the following definition.

DEFINITION 1.4.1: Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set. P is said to be of *constant multiple characteristics* if $P_m(x,\xi)$ can be factorized as

$$P_m(x,\xi) = \prod_{j=1}^{k} q_j(x,\xi)^{r_j},$$

where each $q_j(x,\xi)$ is of simple characteristics in Ω and the sets $q_j^{-1}(0)$ are mutually disjoint.

Next, in order to introduce the Levi condition, we define the characteristic function of q_j.

DEFINITION 1.4.2: $\phi(x)$ is a *characteristic function* of q at $\overline{x} \in \Omega$ if there is a neighborhood U of $\overline{x}$ such that

$$q(x, d\phi(x)) = 0, x \in U, d\phi(\overline{x}) \neq 0.$$

DEFINITION 1.4.3: Let P be of constant multiple characteristics. We say that P satisfies the *Leivi condition* at $\overline{x} \in \Omega$ if we have

$$e^{-i\lambda\phi}P(ae^{i\lambda\phi}) = O(\lambda^{m-r_j}), (\lambda \to \infty),$$

for any characteristic function ϕ of q_j and any $a \in C^\infty(\Omega)$ on whose support $d\phi \neq 0$, $j = 1, 2, ..., k$.

Theorem 1.4.1: *Let P be of constant multiple characteristics. If P is hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$, then each q_j is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ and P satisfies the Levi condition at $\overline{x}$. Conversely, if each q_j is strictly hyperbolic at $\overline{x}$ w.r.t. $t(x)$ and P satisfies the Levi condition near $\overline{x}$, then P is hyperbolic at $\overline{x}$ w.r.t. $t(x)$ ([26], [24], [32], [33], [7], [6]).*

EXAMPLE 1.4.1: We give the simplest example in $\mathbb{R}^2$. Let

$$P(x,D) = D_0^2 + a(x)D_0 + b(x)D_1 + c(x),$$

where $x = (x_0, x_1) \in \mathbb{R}^2$ and $a(x), b(x), c(x)$ are C^∞ functions defined near the origin. Then in order that the Cauchy problem for this P is C^∞ well posed at $x = 0$ it is necessary and sufficient that $b(x) = 0$ near the origin.

2. Effective hyperbolicity

2.1 Effective hyperbolicity

There was a surprising discorvery around 1970, that is there are operators of second order with double characteristics which are strongly hyperbolic. Of course this phenomenon never occur in constant coefficient case.

EXAMPLE 2.1.1: Let

$$P(x,D) = D_0^2 - x_0^2 D_1^2 + a(x)D_0 + b(x)D_1 + c(x)$$

where $x = (x_0, x_1) \in \mathbb{R}^2$. The Cauchy problem for this P is C^∞ well posed at the origin with respect to $t(x) = x_0$ for any $a(x)$, $b(x)$, $c(x) \in C^\infty$ near the origin. On the other hand it is obvious that $z = (0,0,0,1)$ is a double characteristic of P_2. The main feature of this Cauchy problem is that the solution of the Cauchy problem loses the regularity compared with initial data and the loss of derivatives depends on $b(x)$.

Lemma 2.1.1: *Assume that P_m is strongly hyperbolic at $\bar{x} \in \Omega$ w.r.t. $t(x)$. Then there is a neighborhood U of $\bar{x}$ such that all characteristics of P_m in $T^*U \setminus 0$ are at most double ([14]).*

Let (x,ξ) be a system of symplectic coordinates in $T^*\Omega$. Then the natural symplectic 2-form σ in $T^*\Omega$ is given by

$$\sigma = \sum_{j=0}^{d} d\xi_j \wedge dx_j.$$

Let $h(x,\xi)$ be a smooth function on $T^*\Omega \setminus 0$ and $z = (x,\xi) \in T^*\Omega \setminus 0$ be a double characteristic so that $h(z) = dh(z) = 0$.

DEFINITION 2.1.1: The *Hamilton map* $F_h(z)$ of h at z is defined by

$$\sigma(X, F_h(z)Y) = Q(X,Y), \text{ for any } X, Y \in T_z(T^*\Omega),$$

where Q is the quadratic form corresponding to the Hessian of $h/2$ at z.

Lemma 2.1.2: *Suppose that $P_m(x,\cdot)$ is hyperbolic near $\bar{x}$ w.r.t. $dt(x)$. Let $z \in T^*_{\bar{x}}\Omega \setminus 0$ be a double characteristic of P_m. Then all eigenvalues of $F_{P_m}(z)$ are on the pure imaginary axis possibly with an exception of a pair of $\pm e$, $e \in \mathbb{R}$, $e \neq 0$.*

DEFINITION 2.1.2: Suppose that $P_m(x,\cdot)$ is hyperbolic near $\bar{x}$ w.r.t. $dt(x)$. We shall say that P_m is *effectively hyperbolic* at a double characteristic $z \in T^*_{\bar{x}}\Omega \setminus 0$ if $F_{P_m}(z)$ has non-zero real eigenvalue.

Theorem 2.1.3: *In order that P_m is strongly hyperbolic at $\bar{x} \in \Omega$ w.r.t. $t(x)$ it is necessary and sufficient that $P_m(x,\cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\bar{x}$ and P_m is effectively hyperbolic at every double characteristic on $T^*_{\bar{x}}\Omega \setminus 0$ ([14], [15], [29], [17], [34]).*

Let $z \in T^*_{\bar{x}}\Omega \setminus 0$ be a double characteristic of p and assume that p is effectively hyperbolic at z.

DEFINITION 2.1.3: Let

$$\gamma : s \mapsto \gamma(s) = (x(s), \xi(s))$$

be a bicharacteristic of p defined in $[s_0, +\infty)$,(resp. $(-\infty, s_0]$) with some s_0. We say that γ is *incoming* (resp. *outgoing*) with respect to z if

$$\gamma(s) \to z \text{ as } s \uparrow +\infty \text{ (resp. as } s \downarrow -\infty).$$

Proposition 2.1.4: *There are exactly two incoming (resp. outgoing) bicharacteristics of p with respect to z. Furthermore one of the incoming (resp. outgoing) bicharacteristics is naturally continued to the other one, and the resulting two curves are C^∞ regular near z as submanifolds of $T^*\Omega$. These two curves are (real) analytic near z whenever p is assumed to be analytic there ([18], [20]).*

2.2 A geometric characterization

We start by the following definition:

DEFINITION 2.2.1: Let z be a multiple characteristic of p. *The localization* $p_z(X)$ of p at z is defined by

$$p_z(X) = d^r p(z; X, ..., X)/r!, X \in T_z(T^*\Omega)$$

which is a homogeneous polynomial of degree r in $X \in T_z(T^*\Omega)$, the tangent space of $T^*\Omega$ at z.

Note that the hyperbolicity of $p_z(X)$ with respect to $\Theta = -H_{x_0}$ follows from the hyperbolicity of $p(x, \cdot)$ with respect to $dt(x) = dx_0$ near $\bar{x}$. Recall that H_ϕ denotes the Hamilton vector field of ϕ defined by

$$\sigma(X, H_\phi(z)) = d\phi(X), X \in T_z(T^*\Omega).$$

Naturally we are led to consider the hyperbolicity cone $\Gamma(p_z, \Theta)$ of p_z. We recall the definition:

$$\Gamma(p_z, \Theta) = \text{ the component of } \Theta \text{ in } \{X \in T_z(T^*\Omega) | p_z(X) \neq 0\}.$$

DEFINITION 2.2.2: *The propagation cone* $\Gamma^\sigma(p_z, \Theta)$ of p_z is defined by

$$\Gamma^\sigma(p_z, \Theta) = \{X \in T_z(T^*\Omega) | \sigma(X, Y) \leq 0, \forall Y \in \Gamma(p_z, \Theta)\}.$$

DEFINITION 2.2.3: Let $t(x, \xi)$ be homogeneous of degree 0 in ξ, C^1 in a conic neighborhood of z. We say that $t(x, \xi)$ is a *time function* at z w.r.t. $\Gamma(p_z, \Theta)$ if $t(z) = 0$ and

$$-H_t(z) \in \Gamma(p_z, \Theta).$$

Note that $t(x, \xi)$ is a time function at z w.r.t. $\Gamma(p_z, \Theta)$ if and only if

$$\Gamma^\sigma(p_z, \Theta) \cap T_z(\{t(x, \xi) = 0\}) = \{0\}.$$

The propagation cone is the *minimal* cone containing the tangents of bicharacterisitcs of p with limit point z. More precisely:

Lemma 2.2.1: *Let $z \in T^*\Omega \setminus 0$ be a characteristic of order r of p. Assume that there are simple characterisitcs z_j and positive numbers λ_j such that*

$$z_j \to z \text{and } \lambda_j p_{z_j}(\Theta) H_p(z_j) \to X(\neq 0) \text{ as } j \to \infty.$$

Then $X \in \Gamma^\sigma(p_z, \Theta)$ ([51]).

Let $q(X)$ be a homogeneous hyperbolic polynomial on $T_z(T^*\Omega)$ with respect to $\Theta \in T_z(T^*\Omega)$. Denote by $\Lambda(q)$ the linearity space of q:

$$\Lambda(q) = \{X \in T_z(T^*\Omega) | q(tX + Y) = q(Y), \forall t \in \mathbb{R}, \forall Y \in T_z(T^*\Omega)\}.$$

Note that $\Lambda(p_z) = \mathrm{Ker} F_p(z)$ if $d^2 p(z) \neq 0$. We now state a geometric characterization of the effective hyperbolicity.

Proposition 2.2.2: *Notations as above. Let $z \in T^*\Omega \setminus 0$ be a double characteristic of p. Then the following conditions are equivalent:*
(a) $\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) = \{0\}$,
(b) $F_p(z)$ *has a non-zero real eigenvalue.*

Let $\theta = (1, 0, ..., 0)$ and assume that the coefficient of D_0^m is equal to 1. Factorizing $p(x, \xi)$ as

$$p(x,\xi) = \prod_{j=1}^{m} q_j(x,\xi)$$

where $q_j(x,\xi) = \xi_0 - \lambda_j(x,\xi')$, we define $h_j(x,\xi)$ as

$$|p(x,\xi - is\theta)|^2 = \sum_{j=0}^{m} s^{2(m-j)} h_j(x,\xi).$$

It is clear that

$$h_k(x,\xi) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} |q_{j_1}(x,\xi)|^2 \cdots |q_{j_k}(x,\xi)|^2, k = 1, 2, ..., m,$$

and $h_0(x,\xi) = 1$, $h_m(x,\xi) = |p(x,\xi)|^2$. We now characterize the effective hyperbolicity in terms of time functions.

Proposition 2.2.3: *Let $z \in T^*\Omega \setminus 0$ be a double characteristic. Assume that p is effectively hyperbolic at z. Then there is a time function $t(x,\xi)$ at z with respect to $\Gamma(p_z, \Theta)$ satisfying*

$$h_{m-1}(x,\xi) \ge c t(x,\xi)^2 |\xi'|^{2(m-1)}$$

near z with a positive constant c. Conversely if the conclusion holds then p is effectively hyperbolic at z.

DEFINITION 2.2.4: $\gamma^+(z)$ (resp. $\gamma^-(z)$) denotes the union of two bicharacteristics of p with the limit point z along which a time function with respect to $\Gamma(p_z, \Theta)$ is increasing (resp. decreasing).

Theorem 2.2.4: *Let $t(x,\xi)$ be a time function with respect to $\Gamma(p_z, \Theta)$. Assume that*

$$WF(u) \cap \{t(x,\xi) = -\epsilon\} \cap \gamma^+(z) = \phi$$

and $z \notin WF(Pu)$ with a sufficiently small $\epsilon > 0$. Then $z \notin WF(u)$ ([29], [35]).

2.3 A generalization of effective hyperbolicity

Here we generalize the notion of effective hyperbolicity at characteristics of order exceeding two employing the geometric characterization. We introduce the following assumption:

$(A.i)_z$: there are a conic neighborhood U of z and finite number of time functions $t_l(x,\xi)$, $l=1,2,...,n$ such that

$$h_{m-1}(x,\xi)\geq ct(x,\xi)^2h_{m-2}(x,\xi)|\xi'|^2, \forall(x,\xi)\in U$$

where $t(x,\xi)=\min_{1\leq l\leq n}|t_l(x,\xi)|$.

Lemma 2.3.1: *Assume that $(A.i)_z$ holds. Then we have*

$$\Gamma^\sigma(p_z,\Theta)\cap\Lambda(p_z)=\{0\}.$$

QUESTION : Let z be a characteristic of order greater than two. When the conclusion of Lemma 2.3.1 implies $(A.i)_z$?

This is motivated by the geometric characterization given in Proposition 2.2.3. Let us denote by P_j the homogeneous part of degree j so that P is the sum of P_j, $j=0,1,...,m-1$ and $p=P_m$. A general necessary condition for hyperbolicity at a multiple characterisitc is:

Theorem 2.3.2: *Let $z=(\bar{x},\bar{\xi})\in T^*\Omega\setminus 0$ be a characteristic of order r of p. Suppose that P is hyperbolic at $\bar{x}$, that is the Cauchy problem for P is C^∞ well posed at $\bar{x}$ w.r.t. $t(x)=x_0$. Then P_j vanishes at least of order $r-2(m-j)$ at z whenever $r-2(m-j)>0$ ([14]).*

We assume the following:

$(A.ii)_z$: there are a conic neighborhood U of z and $C>0$ such that

$$|P_j(x,\xi)|\leq C|h_{2j-m}(x,\xi)|^{1/2}|\xi'|^{m/2}, \forall(x,\xi)\in U$$

for $[m/2]+1\leq j\leq m-1$.

It should be noted that $h_{2j-m}(x,\xi)\neq 0$ near z if $2j-m\leq m-r$, i.e. $j\leq(2m-r)/2$ when z is a characteristic of order r because there are $m-r$ of q_j which do not vanish at z and hence $(A.ii)_z$ gives no restriction on P_j near z if $j\leq(2m-r)/2$.

Now we have

Theorem 2.3.3: *Assume that the conditions* $(A.i)_z$ *and* $(A.ii)_z$ *are satisfied for every multiple characteristic* $z \in T^*_{\overline{x}}\Omega \setminus 0$. *Then the Cauchy problem for* P *is* C^∞ *well posed at* $\overline{x}$ *w.r.t.* $t(x) = x_0$ *([21]).*

In the next theorem we follow the notations in subsection 1.3 and assume the condition (1.3.3).

Theorem 2.3.4: *Assume that the conditions* $(A.i)_z$ *and* $(A.ii)_z$ *are fulfilled at every multiple characteristic* $z \in T^*\Omega \setminus 0$. *Suppose that*

$$Pu = f \text{ in } \omega^+,\ u = 0 \text{ in } t(x) < t(\overline{x}) \text{ and } f = 0 \text{ on } \Gamma^-(x).$$

Then it follows that

$$u = 0 \text{ on } \Gamma^-(x)$$

([21]).

EXAMPLE 2.3.1: Here we give a simple example to elucidate the geometric meanings of the conditions $(A.i)_z$ and $(A.ii)_z$. Let $p(x,\xi)$ be factorized as

$$p(x,\xi) = e(x,\xi)\prod_{j=1}^{r} q_j(x,\xi), q_j(z) = 0$$

in a conic neighborhood U of z where $e(x,\xi)$, $q_j(x,\xi)$ are smooth near z, homogeneous of degree $m-r$ and 1 respectively and $e(z) \neq 0$, $dq_j(z) \neq 0$ and $q_j(\overline{x},\theta) > 0$. Assume, for simplicity, that dq_j are linearly independent at z. Recall that the cone generated by the Hamilton vector fields $H_{q_j}(z)$ of q_j forms the propagation cone $\Gamma^\sigma(p_z,\Theta)$ of the localization $p_z(X)$. Then the condition $(A.i)_z$ is fulfilled if and only if $\Gamma^\sigma(p_z,\Theta)$ is transversal to the tangent space at z of each intersection of any two hypersurfaces $\{q_k = 0\}$, $\{q_l = 0\}$:

$$\Gamma^\sigma(p_z,\Theta) \cap T_z\{q_k = 0, q_l = 0\} = \{0\}, \forall k \neq l.$$

On the other hand, the condition $(A.ii)_z$ is satisfied if and only if $P_j(x,\xi)$ vanishes of order $r - 2(m-j)$ on each intersection of any two hypersurfaces $\{q_k = 0\}$, $\{q_l = 0\}$ near z whenever $r - 2(m-j) > 0$.

EXAMPLE 2.3.2: Here we give an example verifying the conditions $(A.i)_z$ and $(A.ii)_z$ which is not necessarily factorized smoothly. Denote by Σ the set of characteristics of order r of p:

$$\Sigma = \{(x,\xi) \in T^*\Omega \setminus 0 | p(x,\xi) = dp(x,\xi) = \cdots = d^{r-1}p(x,\xi) = 0\}.$$

We assume that

(i) Σ is a C^∞ manifold near $z = (\bar{x}, \bar{\xi})$.

It then follows that

$$p_z(X + tY) = p_z(X) \; \forall t \in \mathbb{R}, \forall Y \in T_z\Sigma, \forall X \in T_z(T^*\Omega)$$

so that $T_z\Sigma = \Lambda(p_z)$ and we may regard $p_z(X)$ as a polynomial on $N_\Sigma(T^*\Omega)_z$ which is defined by $T_z(T^*\Omega)/T_z\Sigma$. Denoting by $[X]$ the equivalence class of $X \in T_z(T^*\Omega)$ we assume that

(ii) $p_z([X])$ is strictly hyperbolic with respect to $[\Theta] \in N_\Sigma(T^*\Omega)_z$

and that $\Gamma^\sigma(p_z, \Theta)$ is transversal to Σ at z:

$$\Gamma^\sigma(p_z, \Theta) \cap T_z\Sigma = \{0\}.$$

We also assume that

(iii) $P_j(x, \xi)$ vanishes of order $r - 2(m - j)$ on Σ near z when $r - 2(m - j) > 0$.

Then the conditions $(A.i)_z$ and $(A.ii)_z$ are fulfilled for p.

2.4 Non effective hyperbolicity

The necessity of effective hyperbolicity in Theorem 2.1.3 is a special case of a more general condition for hyperbolicity. At any double characteristic $z \in T^*\Omega \setminus 0$ of p, the subprincipal symbol of P is well defined by reference to any local coordinates x:

$$P^s(x, \xi) = P_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=0}^{d} \frac{\partial^2}{\partial x_j \partial \xi_j} p(x, \xi).$$

DEFINITION 2.4.1: We define *the positive trace* $Tr^+ F_p$ of p at z as

$$Tr^+ F_p(z) = \sum i\mu_j$$

where $i\mu_j$ are the eigenvalues of $F_p(z)$ on the imaginary axis, repeated according to their multiplicities.

Theorem 2.4.1: *Let $z = (\bar{x}, \bar{\xi}) \in T^*_{\bar{x}}\Omega \backslash 0$ be a double characteristic. Assume that P is hyperbolic at $\bar{x}$ w.r.t. $t(x) = x_0$. Then we have*

$$\mathrm{Im}\ P^s(z) = 0, \ |\mathrm{Re}\ P^s(z)| \leq Tr^+ F_p(z)$$

([14], [12]).

For the converse of Theorem 2.4.1 we refer to [16], [12]. When the multiplicity of z exceeds 2, according to Proposition 2.2.2, it would be natural to call that P is not of effective type at z if

$$\Gamma^\sigma(p_z, \Theta) \cap \Lambda(p_z) \neq \{0\}.$$

In what follows, in this subsection, we study operators of non effective type. We recall that the localization $p_z(X)$ is a well defined hyperbolic polynomial on $T^*(\Omega)/\Lambda(p_z)$ w.r.t. $[\Theta]$, where $[X]$ denotes the equivalence class of X as in EXAMPLE 2.3.2. It is easy to check that if z is a double characteristic then $p_z(X)$ is strictly hyperbolic on $T^*_z(\Omega)/\Lambda(p_z)$ w.r.t. $[\Theta]$. It is then natural to assume, as an ideal case, that $p_z(X)$ is strictly hyperbolic w.r.t. $[\Theta]$ even when z is a characteristic of order greater than 2. When z is a triple characteristic with

$$\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z)$$

we refer to a recent work [2].

As for the case

$$\Gamma^\sigma(p_z, \Theta) \not\subset \Lambda(p_z)$$

we state a typical necessary condition in order that P is hyperbolic at $\bar{x}$ w.r.t. $t(x) = x_0$ when p has a triple characteristic $z \in T^*_{\bar{x}}\Omega \setminus 0$ (see also [3]). We list up the assumptions we make:

The localization $p_z(X)$ of p at z satisfies the following conditions:

(i) $p_z(X) = L(X)Q(X)$ where $L(X)$ is a linear form and $Q(X)$ is a real quadratic form such that

$$\mathrm{Ker}\ F_Q^2 \cap \mathrm{Im}\ F_Q^2 = \{0\}.$$

(ii) $H_L \in \Lambda(p_z)$.

Theorem 2.4.2: *In order that P is hyperbolic at $\bar{x}$ w.r.t. $t(x) = x_0$ the followings are necessary:*

(L1) $P^s(z) = 0$
(L2) $\mathrm{Im}\ H_{P^s}(z) = 0, \ Tr^+ F_Q\ H_L \pm \mathrm{Re}\ H_{P^s}(z) \in \Gamma^\sigma(p_z, \Theta)$

([4]).

QUESTION : What conditions are necessary when we drop the assumption (ii) in Theorem 2.4.2?

REMARK: Assuming that P is not effective type at a multiple characteristic z, we could expect, in general, neither $\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z)$ nor p_z is factorized. In such general cases, few facts are known concerning with both necessity and sufficiensy of C^∞ well posedness of the Cauchy problem. However see [40]. An interesting approach to this problem is found in [41].

3. First order systems

3.1 Preliminaries

Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$. Let (x, ξ) be a system of local coordinates on $T^*\Omega$ and $e_1, ..., e_N$ be a frame in $\mathbb{C}^N$. With these coordinates and frame, the principal symbol of L is given by

$$L_1(x, \xi) = \sum_{j=0}^{d} L_j(x)\xi_j. \tag{3.1.1}$$

We set

$$h(x, \xi) = \det L_1(x, \xi),$$

which is invariantly defined as a function on $T^*\Omega$.

DEFINITION 3.1.1: Let $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. We say that L is *non-characteristic* w.r.t. $H = \{t(x) = 0\}$ at $\bar{x} \in H$ if

$$\lim_{\lambda \to \infty} \lambda^{-1} e^{-\lambda t(x)} L(e^{\lambda t(x)}),$$

is a surjection on $\mathbb{C}^N$ at $\bar{x}$.

As in subsection 1.1 we are assuming that L is non-characteristic w.r.t. H at the reference point $\bar{x}$.

DEFINITION 3.1.2: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. Then L is said to be *hyperbolic* w.r.t. $t(x)$ at $\bar{x} \in \Omega$ if there are a neighborhood $\omega \subset \Omega$ of $\bar{x}$ and $\epsilon > 0$ such that

$$L : E_\tau = \{U \in C^\infty(\omega, \mathbb{C}^N) | U = 0 \text{ on } t(x) < t(\bar{x}) + \tau\} \to E_\tau$$

is an isomorphism if $|\tau| < \epsilon$.

DEFINITION 3.1.3: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$ be real valued. Then L_1 is said to be *strongly hyperbolic* at $\bar{x}$ w.r.t. $t(x)$ if, for any $Q \in C^\infty(\Omega, M(N, \mathbb{C}))$, $L + Q$ is hyperbolic at $\bar{x}$ w.r.t. $t(x)$.

3.2 Systems with constant coefficients

Let

$$L(D) = \sum_{j=1}^{d} A_j D_j + B,$$

where A_j, B are constant square matrices of order N. We take $t(x) =< \theta, x >$ as a linear function in x.

Theorem 3.2.1: *Assume that L is of constant coefficients. For that L to be hyperbolic at $\overline{x}$ w.r.t. θ it is necessary and sufficient that* det $L(D)$ *is hyperbolic at $\overline{x}$ w.r.t. θ ([1]).*

For $L(D)$ to be strongly hyperbolic the strict hyperbolicity of det $L(D)$ is sufficient but not necessary:

Theorem 3.2.2: *Assume that L is of constant coefficients. In order that L is strongly hyperbolic w.r.t. θ it is necessary and sufficient that the following condition holds for every $\xi \in \mathbb{R}^{d+1} \setminus \mathbb{R}\theta$,*

$$|L_1(\xi + \tau\theta)^{-1}| \le C(\text{Re } \tau)^{-1} \textit{ for } \text{Re } \tau > 0,$$

([22], [47]).

Theorem 3.2.3: *If h is strictly hyperbolic at $\overline{x} \in \Omega$ w.r.t. $t(x)$ then L is strongly hyperbolic at $\overline{x}$ w.r.t. $t(x)$. This statement also holds in the variable coefficient case.*

Recall that symmetric or symmetrizable systems are always strongly hyperbolic. It happens that the converse is also true. We first recall that L_1 is said to be symmetrizable if there is a positive definite Hermitian symmetric matrix $S \in M(N, \mathbb{C})$ such that $SL_1(\xi)$ becomes to be Hermitian symmetric for every $\xi \in \mathbb{R}^{d+1}$.

Proposition 3.2.4: *Every 2×2 strongly hyperbolic system is symmetrizable ([47]).*

Without restrictions we may assume that $A_0 = I$, the identity matrix of order N. Let us set

$$d(L_1) = \dim \text{span}\{I, A_1, ..., A_d\}$$

which is called *the reduced dimension* of L_1.

Proposition 3.2.5: *Assume that A_j are real and*

$$d(L_1) \geq \frac{(d+1)(d+2)}{2} - 1.$$

If L_1 is strongly hyperbolic then L_1 is symmetrizable ([36]).

For another related results we refer to [49].

QUESTION : Let $N > 2$. For what k can one find a strongly hyperbolic system L_1 with $d(L_1) = k$ which is not symmetrizable? Recently a complete classification of 3×3 strongly hyperbolic systems with real constant coefficients is given in [42], [43].

3.3 Systems with constant multiple characterisitcs

DEFINITION 3.3.1: L is said to be of *constant multiple characteristics* if $h(x,\xi)$
$= \det L_1(x,\xi)$ satisfies the conditions in the definition 1.4.1.

If L is of constant multiple characteristics then $h(x,\xi)$ can be factorized as

$$h(x,\xi) = \prod_{j=1}^{k} q_j(x,\xi)^{r_j}.$$

We introduce the following hypothesis.

For every characteristic function ϕ of q_j at $\bar{x} \in \Omega$, we have

$$\text{rank } L_1(\bar{x}, d\phi(\bar{x})) = N - 1 \text{ for any } j. \tag{3.3.1}$$

If we assume that (3.3.1) holds near $\bar{x} \in \Omega$ then we can find $N_j \in C^\infty(\omega, \mathbb{C}^N)$ such that

$$L_1(x, d\phi(x))N_j(x, d\phi(x)) = 0, \ 1 \leq j \leq k,$$

where ω is a neighborhood of $\bar{x}$. Using N_j we introduce the Levi condition.

DEFINITION 3.3.2: We shall say that L satisfies the Levi condition at $\bar{x}$ if, for every characteristic function ϕ of q_j at $\bar{x}$ and for every $a \in C_0^\infty(\Omega)$ on whose support $d\phi \neq 0$, there exists $V_i^{(j)}(x;\phi,a)$ belonging to $C^\infty(\Omega, \mathbb{C}^N)$ such that

$$e^{-i\lambda\phi}L\{e^{i\lambda\phi}(aN_j + \sum_{i=1}^{r_j-1} \lambda^{-i}V_i^{(j)})\} = O(\lambda^{1-r_j}),$$

for $j = 1, 2, ..., k$.

Theorem 3.3.1: *Assume that L is of constant multiple characteristics and the hypothesis (3.3.1) is realized near $\bar{x}$. If L is hyperbolic at $\bar{x} \in \Omega$ w.r.t. $t(x)$ then each q_j is strictly hyperbolic at $\bar{x}$ w.r.t. $t(x)$ and L satisfies the Levi condition at $\bar{x}$. Conversely if each q_j is strictly hyperbolic at $\bar{x}$ w.r.t. $t(x)$ and L satisfies the Levi condition near $\bar{x}$, then L is hyperbolic at $\bar{x}$ w.r.t. $t(x)$ ([44], [52]).*

Theorem 3.3.2: *Assume that L is of constant multiple characteristics. In order that L is strongly hyperbolic at $\bar{x}$ w.r.t. $t(x)$ it is necessary and sufficient that*

$$\dim \operatorname{Ker} L_1(x,\xi) = r_j,\ \forall (x,\xi) \text{ with } q_j(x,\xi) = 0,\ x \text{ near } \bar{x}$$

for $j = 1, 2, ..., k$ ([19]).

For studies on hyperbolicity of systems with constant multiple characteristics without the condition (3.3.1), we refer to recent works [50] and [27].

3.4 Systems with double characteristics

With a system of local coordinates (x, ξ) in $T^*\Omega$ and a frame in $\mathbb{C}^N$, the full symbol of $L(x, D)$ is expressed as follows

$$L(x,\xi) = L_1(x,\xi) + L_0(x).$$

We define $\mathcal{L}(x,\xi)$ by

$$\mathcal{L}(x,\xi) = L^s(x,\xi)L_1^{co}(x,\xi) - \frac{i}{2}\{L_1, L_1^{co}\}(x,\xi),$$

where

$$L^s(x,\xi) = L_0(x) + \frac{i}{2}\sum_{j=0}^{d} \frac{\partial^2}{\partial x_j \partial \xi_j} L_1(x,\xi),$$

$$\{L_1, L_1^{co}\} = \sum_{j=0}^{d} \frac{\partial L_1}{\partial \xi_j}\frac{\partial L_1^{co}}{\partial x_j} - \frac{\partial L_1}{\partial x_j}\frac{\partial L_1^{co}}{\partial \xi_j}$$

and $L_1^{co}(x,\xi)$ is the *cofactor matrix* of $L_1(x,\xi)$. Note that $\mathcal{L}(x,\xi)$ is invariantly defined at a multiple characteristic z in

$$\operatorname{Hom}(\mathbb{C}^N, \mathbb{C}^N)/L_1(z)\operatorname{Hom}(\mathbb{C}^N, \mathbb{C}^N)$$

Theorem 3.4.1: *Assume that L is hyperbolic at $\bar{x} \in \Omega$ w.r.t. $t(x)$ and $h = \det L_1$ is not effectively hyperbolic and the rank of L_1 is $N-1$ at the multiple characteristic $z \in T^*_{\bar{x}}\Omega \setminus 0$. Then there is a real number $\alpha, |\alpha| \le 1$ such that*

$$\mathcal{L}(z) + \alpha Tr^+ h(z) I = O,$$

in Hom $(\mathbb{C}^N, \mathbb{C}^N)/L_1(z)$Hom $(\mathbb{C}^N, \mathbb{C}^N)$ *([37]).*

Corollary 3.4.2: *Assume that L_1 is strongly hyperbolic at $\bar{x} \in \Omega$ w.r.t. $t(x)$ and $z \in T^*_{\bar{x}}\Omega \setminus 0$. Then h is effectively hyperbolic at z or the rank of $L_1(z)$ is less than or equal to $N-2$.*

Theorem 3.4.3: *Suppose that $h(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\bar{x} \in \Omega$ and h is effectively hyperbolic at every multiple characteristic in $T^*_{\bar{x}}\Omega \setminus 0$. Then L_1 is strongly hyperbolic at $\bar{x}$ w.r.t. $t(x)$ ([38]).*

In the following we assume that all characteristics are at most double and we denote by Σ the doubly characteristic set:

$$\Sigma = \{z | h(z) = dh(z) = 0\}.$$

We introduce the following hypotheses concerning the doubly characteristic set.

(i) Σ is a C^∞ manifold,
(ii) rank Hess h = codim Σ.

Theorem 3.4.4: *Assume that (i) and (ii) hold and $h(x, \cdot)$ is hyperbolic w.r.t. $dt(x)$ near $\bar{x}$ and one of the following conditions is verified at every point $z \in T^*_{\bar{x}} \cap \Sigma$,*

(a) h is effectively hyperbolic at z,
(b) rank $L_1 \le N-2$, near z on Σ.

Then L is strongly hyperbolic ([38], [37], [5]).

We present some interesting facts which are valid at double characteristics. Let us denote by Hess $h(z)$ the Hessian of h at z.

Lemma 3.4.5: *Let $z \in T^*_{\bar{x}}\Omega \setminus 0$ be a double characteristic. Then we have*

$$\text{rank Hess} h(z) \le 4.$$

If all $L_j(x)$ are real valued then we have

$$\text{rank Hess} h(z) \le 3.$$

DEFINITION 3.4.1: We say that a double characteristic z is *non degenerate* if

$$\text{rank Hess} h(z) = 4$$

(resp. rank Hess$h(z) = 3$ if all $L_j(x)$ are real valued).

Proposition 3.4.6: *Let $\overline{z}$ be a non degenerate double characteristic.*

(i) The doubly characteristic set $\Sigma = \{z|h(z) = dh(z) = 0\}$ is a smooth manifold near $\overline{z}$ of codimension 4 (resp. 3 if $L_j(x)$ are real).
(ii) There is a smooth symmetrizer of $L_1(x,\xi)$ near $\overline{z}$, that is there is a positive definite Hermitian symmetric matrix $S(x,\xi')$, smoothly depending on (x,ξ') satisfying

$$S(x,\xi')L_1(x,\xi) = L_1(x,\xi)^* S(x,\xi')$$

([37], [5]).

We now turn to the stability of non degenerate double characteristics. Let

$$\tilde{L}_1(x,\xi) = \sum_{j=0}^{d} \tilde{L}_j(x)\xi_j$$

be another system and set $\tilde{h}(x,\xi) = \det \tilde{L}_1(x,\xi)$. We assume that $\tilde{h}(x,\cdot)$ is hyperbolic w.r.t. x_0.

Proposition 3.4.7: *Suppose that $\tilde{L}_j(x)$ are sufficiently close to $L_j(x)$ in C^3 near $\overline{x}$. Then $\tilde{h}$ has a non degenerate double characteristic near $\overline{z} = (\overline{x},\overline{\xi})$ ([13]).*

This shows that non degenerate double characteristics are very stable and we can not remove them by small perturbations.

We now introduce the notion of localization of L_1 at a multiple characteristic z following [49].

DEFINITION 3.4.2: Let z be a characteristic of order r with

$$\dim \operatorname{Ker} L_1(z) = r.$$

Let Ker $L_1(z) = \text{span }\{u_1,...,u_r\}$ and Ker ${}^tL_1(z) = \text{span }\{v_1,...,v_r\}$. We set

$U = (u_1,...,u_r)$ and $V = (v_1,...,v_r)$ and define *the localization* of L_1 at z as

$$L_{loc}(U,V)(X) = dL(U,V)(z;X)$$

where $L(U,V) ={}^t VL_1(x,\xi)U$.

Lemma 3.4.8: *Let $\tilde{U}, \tilde{V}$ be another pair of basis for* Ker $L_1(z)$ *and* Ker ${}^tL_1(z)$. *Then with some non singular M_i we have*

$$L_{loc}(\tilde{U}, \tilde{V})(X) = M_1 L_{loc}(U, V)(X) M_2.$$

DEFINITION 3.4.3: Let z be a characteristic of order r with dim Ker$L_1(z) = r$. We say that z is *non degenerate* if

$$d(L_{loc}(U, V)) \geq r(r+1)/2.$$

QUESTION : Assume that dim Ker$L_1(z) = r(z)$, the multiplicity of z, for every multiple characteristic near $\bar{z}$. Suppose that $\bar{z}$ is non degenerate. Let $\tilde{L}_1$ be sufficiently close to $\tilde{L}_1$ in C^∞. Is there a characteristic of order $r = r(\bar{z})$ of $\tilde{h}$ near z? More moderately is there a multiple characteristic of $\tilde{h}$ near $\bar{z}$?

QUESTION : Assume that dim Ker$L_1(z) = r(z)$, the multiplicity of z, for every multiple characteristic near $\bar{z}$. Suppose that $\bar{z}$ is non degenerate. Is there a smmoth symmetrizer of $L_1(x, \xi)$ near $\bar{z}$?

3.5 Systems with multiple characteristics

In this subsection we state some recent necessary conditions for strong hyperbolicity of first order systems at characteristics of order exceeding two. We adopt the following definitions.

DEFINITION 3.5.1: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$ in Ω, be real valued. Then L is said to be *hyperbolic* w.r.t. $t(x)$ both *future and past* at $\bar{x} \in \Omega$ if there are a neighborhood $\omega \subset \Omega$ of $\bar{x}$ and $\epsilon > 0$ such that both

$$L : E_\tau^\pm = \{U \in C^\infty(\omega, \mathbb{C}^N) | U = 0 \text{ on } \pm(t(x) - t(\bar{x})) < \tau\} \rightarrow E_\tau^\pm$$

are isomorphisms if $|\tau| < \epsilon$.

DEFINITION 3.5.2: Let L be a differential operator of first order on $C^\infty(\Omega, \mathbb{C}^N)$ and $t(x) \in C^\infty(\Omega)$ be real valued. Then L_1 is said to be *strongly hyperbolic* at $\bar{x}$ w.r.t. $t(x)$ if, for any $Q \in C^\infty(\Omega, M(N, \mathbb{C}))$, $L + Q$ is hyperbolic at $\bar{x}$ both future and past w.r.t. $t(x)$.

Let us denote by M the cofactor matrix $L_1^{co}(x, \xi)$ of $L_1(x, \xi)$. As before we set $h(x, \xi) = \det L_1(x, \xi)$. Recall that

$$L_1(x, \xi) = \sum_{j=0}^{d} L_j(x)\xi_j.$$

Theorem 3.5.1: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a characteristic of order r of $h(x,\xi)$. Then if L is strongly hyperbolic at the origin w.r.t. $t(x) = x_0$, it follows that*

$$d^j M(z) = O, j < r-2 \text{ i.e. } \partial_\xi^\alpha \partial_x^\beta M(z) = O, |\alpha + \beta| < r-2.$$

Moreover every element of $d^{r-2}M(z;X) = d^{r-2}M(z;X,...,X)/(r-2)!$ is divisible by $\prod g_j(X)^{r_j-1}$ where $\prod g_j(X)^{r_j}$ is an irreducible factorization of $p_z(X)$ ([39]).

Corollary 3.5.2: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a multiple characteristic of $h(x,\xi)$ and V_0 be the generalized eigenspace for $L_1(z)$ associated to the zero eigenvalue. Then if L is strongly hyperbolic at the origin w.r.t. x_0 we have*

$$(L_1(z)|_{V_0})^2 = O,$$

where $L_1|_{V_0}$ is the restriction of $L_1(z)$ to V_0.

This corollary clearly corresponds to Lemma 2.1.1.

Theorem 3.5.3: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a characteristic of order r of $h(x,\xi)$. Suppose that*

$$\Gamma^\sigma(p_z, \Theta) \subset \Lambda(p_z).$$

Then if L is strongly hyperbolic at the origin w.r.t. $t(x) = x_0$ we have

$$d^j M(z) = O, j < r-1,$$

([39]).

Corollary 3.5.4: *Assume that $L_j(x)$ are real analytic in Ω and $0 \in \Omega$. Let $z \in T_0^*\Omega \setminus 0$ be a characteristic of order r of $h(x,\xi)$ with $\Gamma^\sigma(p_z,\Theta) \subset \Lambda(p_z)$. If L is strongly hyperbolic at the origin w.r.t. x_0 then we have*

$$\dim \operatorname{Ker} L_1(z) = r.$$

For another approach to systems with multiple characteristics, we refer to [28], [53].

QUESTION : In Theorems 3.5.1 and 3.5.3 can we drop the assumption of analyticity?

Finally we state two basic questions.

QUESTION: Let z be a characteristic of order r. Assume that L_1 is strongly hyperbolic and $\Gamma^{\sigma}(p_z, \Theta) \cap \Lambda(p_z) \neq \{0\}$. Then dim Ker$L_1(z) = r$ is necessary?

If this is affirmative, combining Theorem 3.5.3, we could conclude that; if L is strongly hyperbolic and z is a characteristic of order r then we have either

$$\Gamma^{\sigma}(p_z, \Theta) \cap \Lambda(p_z) = \{0\}$$

or

$$\dim \operatorname{Ker} L_1(z) = r.$$

Clearly the first case corresponds to a generalization of effective hyperbolicity and the second case means the symmetrizability of L_1 at z.

QUESTION : Let z be a characteristic of order r with $\Gamma^{\sigma}(p_z, \Theta) \subset \Lambda(p_z)$. Assume that L_1 is strongly hyperbolic . Then the localization $L_{loc}(U, V)(X)$ is strongly hyperbolic? More moderately the localization is diagonalizable?

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NONLINEAR ANALYSIS

INVARIANCE OF DOMAIN THEOREM FOR DEMICONTINUOUS MAPPINGS OF TYPE (S_+)

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1. Introduction

Wellknown invariance of domain theorems are Brower's invariance of domain theorem for continuous mappings defined on a finite dimensional space and Schauder-Leray's invariance of domain theorem for the class of mappings $I+C$ defined on a infinite dimensional Banach space with I the identity and C compact. The two classical invariance of domain theorems were proved by applying the homotopy invariance of Brower's degree and Leray-Schauder's degree respectively.

Degree theory for some class of mappings is a useful tool for mapping theorems. And mapping theorems (or surjectivity theorems of mappings) are closely related with invariance of domain theorems for mappings.

In [4,5], Browder and Petryshyn constructed a multi-valued degree theory for A-proper mappings. From this degree Pertyshyn [9] obtained some invariance of domain theorems for locally A-proper mappings.

Recently Browder [6] has developed a degree theory for demicontinuous mappings of type (S_+) from a reflexive Banach space X to its dual X^*. By applying this degree we obtain some invariance of domain theorems for a demicontinuous mappings of type (S_+).

2. Preliminaries

In what follows it will always be assumed that X is a reflexive Banach space with norm $\| \ \|$ and its dual space X^*. We use $B(x_0, r)$ and $\overline{B}(x_0, r)$

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to denote respectively the open ball and the closed ball in X or X^* with the center x and radius $r > 0$ while $\partial B(x_0, r)$ will denote its strong boundary.

In the followings 'locally' means that a mapping satisfies some properties on a neighborhood of any point in its domain. Notations $\longrightarrow$ and $\rightharpoonup$ denote the strong and weak convergence respectively. A map $T : D(T) \subset X \longrightarrow X^*$ is continuous if for any sequence $\{x_n\}$ in $D(T)$ with $x_n \longrightarrow x \in D(T)$, we have $Tx_n \longrightarrow Tx$. We need the following definitions of mappings of various monotone types.

[M] A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be monotone if for any x, $y \in D(T)$, we have

$$(Tx - Ty \ , \ x - y) \geq 0.$$

[SM] A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be strongly monotone if for any x, $y \in D(T)$, we have

$$(Tx - Ty \ , \ x - y) \geq c\|x - y\|^2,$$

where c is a positive constant.

[$S\phi E$] A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be strongly ϕ-expansive if for any x,$y \in D(T)$, we have

$$(Tx - Ty \ , \ x - y) \geq \phi(\|x - y\|),$$

where $\phi : R^+ \longrightarrow R^+$ is strictly increasing, continuous in a neighborhood of 0 and $\phi(0) = 0$.

[S] A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be of type (S) if for any sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x \in X$, such that $\lim(Tx_n \ , \ x_n - x) = 0$, we have $x_n \longrightarrow x$.

[S_+] A mapping $T : D(T) \subset X \longrightarrow X^*$ is said to be of type (S_t) if for any sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x \in X$ and $\limsup(Tx_n \ , \ x_n - x) \leq 0$, we have $x_n \longrightarrow x$.

The duality mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* | (x^*, x) = \|x\|^2 = \|x^*\|^2\}.$$

Let X be a reflexive Banach space which is normed so that both X and X^* are locally uniformly convex. Then the duality mapping J is single valued, bicontinuous, strictly monotone and of type (S_+) (see Browder [6]). Browder [6] obtained the degree theory for demicontinuous mappings of type (S_+) via Galerkin approximation processes. In this degree theory the normalized mapping is the duality mapping and the homotopies are of type (S_+). Futhermore Browder [6] showed that linear homotopy is a homotopy of type (S_+).

3. Invariance of domain theorem

By applying Browder's degree we have the following invariance of domain theorem.

Theorem 1. *Let G be an open subset of a reflexive Banach space and $T : G \longrightarrow X^*$ be demicontinuous and locally strongly ϕ-expansive. Then $T(G)$ is open in X^*.*

Proof. We choose $r > 0$ such that T is strongly ϕ-expansive on $\overline{B}(x_0, r) \subset G$. Let $y_0 = Tx_0$. Since T is strongly ϕ-expansive, T is one-to-one and $y_0 \notin T(\partial B(x_0, r))$. And $T(\partial B(x_0, r))$ is closed. Indeed, for any sequence $\{y_n\}$ in $T(\partial B(x_0, r))$ with $y_n \longrightarrow y$, $Tx_n = y_n$, $x_n \in \partial B(x_0, r)$, we have

$$(Tx_m - Tx_n \ , \ x_m - x_n) \geq \phi(\|x_m - x_n\|).$$

Hence $\|Tx_m - Tx_n\| \ \|x_m - x_n\| \geq \phi(\|x_m - x_n\|)$. Since $\{x_n\}$ is bounded and $\{Tx_n = y_n\}$ is a Cauchy sequence. Hence $x_n \longrightarrow x \in \partial B(x_0, r)$. Since T is demicontinuous, $y_n = Tx_n \rightharpoonup Tx$. Therefore $y = Tx \in T(\partial B(x_0, r))$. Since $T(\partial B(x_0, r))$ is closed, we choose $\rho > 0$ such that $\overline{B}(t_0, \rho) \bigcap T(\partial B(x_0, r)) = \phi$. Since T is demicontinuous and strongly ϕ-expansive on $\overline{B}(x_0, r)$, T is demicontinuous and of type (S_+). We have a homotopy of (S_+)

$$H(t, x) = tTx + (1 - t)J(x - x_0), \ \ y(t) = ty_0.$$

Then $y(t) \notin H(t, \partial B(x_0, r))$ for any t in [0,1]. Indeed, on the contrary we have, for some t in [0,1], for some $x \in \partial B(x_0, r)$,

$$\begin{aligned} ty_0 &= tTx + (1 - t)J(x - x_0) \\ &\Longrightarrow t(Tx_0 - Tx) = (1 - t)J(x - x_0) \\ &\Longrightarrow t(Tx_0 - Tx \ , \ x - x_0) = (1 - t)\|x - x_0\|^2 \ \ \cdots (1) \end{aligned}$$

From (1) and ϕ-expansiveness of T we have a contradiction. Therefore $d(H(t, \bullet)$, $B(x_0, r)$, $y_t)$ is constant. That is,

$$d(T(\bullet), \ B(x_0, r), \ y_0) = d(J(\bullet - x_0), \ B(x_0, r), \ 0) \ \ \cdots (2)$$

On the other hand, from a homotopy of (S_+)

$$G(t, x) = tJ(x - x_0) + (1 - t)Jx, \ \ y(t) = tJx_0$$

we have

$$d(J(\bullet - x_0), \ B(x_0, r), \ 0) = d(J\bullet, \ B(x_0, r), \ J(x_0)) = 1 \ \ \cdots (3)$$

By (2) and (3), $d(T,\ B(x_0,r),\ y_0) = 1$. Since $\overline{B}(u_0,\rho)\bigcap T(\partial B(x_0,r)) = \phi$, for any $y \in B(t_0,\rho)$ the path $y(t) = ty_0 + (1-t)y \notin T(\partial B(x_0,r))$. Hence

$$d(T,\ B(x_0,r),\ y_0) = d(T,\ B(x_0,r),\ y) = 1.$$

Therefore $y \in T(\overline{B}(x_0,r)) \subset T(G)$. Hence $B(y_0,\rho) \subset T(\overline{B}(x_0,r)) \subset T(G)$. The proof is completed.

Corollary 1. *Let X be a reflexive Banach space. If $T : D(T) = X \longrightarrow X^*$ is demicontinuous and strongly monotone, then T is a homeomorphism from X to X^*.*

Proof. Since T is strongly monotone, T is one to one and $T(X)$ is closed. By Theorem 1 $T(X)$ is open. Therefore T is onto and T is a homeomorphism.

In Hilbert space we have the following result of Minty [8] and Browder [2].

Corollary 2. *[2,8] Let H be a Hilbert space, $G \subset H$ be open and let $T : G \longrightarrow H$ be demicontinuous and locally strongly monotone. Then $T(G)$ is open in H.*

Proof. The proof of Corollary 2 is obvious from Theorem 1.

By applying Corollary 2 and Kirszbraun's theorem. Schönberg [10] obtained the following theorem.

Schönberg's Theorem[10, Theoreml]: Let H be a Hilbert space, $G \subset H$ be open and let $T : \overline{G} \longrightarrow H$ be demicontinuous and strongly monotone. If $K \subset H$ is connected such that $K\bigcap T(G) \neq \phi$ and $K\bigcap T(\partial G) = \phi$, then $K \subset T(G)$.

Similar results are obtained by Z.Guan[7] for demicontinuous monotone mappings defined on a closure of open bounded convex subset of a reflexive Banach space. On the other hand Browder[1] has the similar results for demicontinuous monotone mapping defined on all of X. But Browder's Theorem is for bounded closed convex subsets of a reflexive Banach space.

Now we have another following similar result in Hilbert spaces.

Theorem 2. *Let G be a bounded open subset of a Hilbetr space X and $T : \overline{G} \longrightarrow X$ be demicontinuous and monotone. If $K \subset X$ is path-connected such that $K\bigcap T(G) \neq \phi$ and $K\bigcap \overline{T(\partial G)} = \phi$, then $K \subset T(G)$.*

Proof. Without loss of generality we may assume $T(0) = 0 \in K$ and $0 \in G$. For any fixed $y \in K$ we have a path $y(t)$ $(y(0) = 0\ ,\ y(1) = y)$ in K. Let $T_n(x) = T(x) + \frac{1}{n}x$. Since $K \bigcap \overline{T(\partial G)} = \phi$ for all sufficiently large n, we have

$$y(t) \notin T_n(\partial G) \quad \cdots (4)$$

For such n, let $s = \{t \in [0,1] \mid y(t) \in T_n(G)\}$. Since T_n is strongly monotone, $T_n(G)$ is open by Theorem 1. Hence S is open. Since $0 \in S$, S is nonempty. S is closed. Indeed, if $t_m \in S$, $t_m \longrightarrow t$, then we have $y(t_m) = T_n(x_m)$, $x_m \in G$, $y(t_m) \longrightarrow y(t)$. Since T_n is strongly monotone, $\{x_m\}$ is a Cauchy sequence and $x_m \longrightarrow x \in \overline{G}$. Since T_n is demicontinuous, $T_n(x_m) = y(t_m) \rightharpoonup T_n(x)$. and $y(t) = T_n(x)$. From (4) $y(t) \in T_n(G)$. Hence S is closed. We conclude that $S = [0,1]$ and $y \in T_n(G)$ for all sufficiently large n. That is, for some z_n in G

$$y = T_n(z_n) = T(z_n) + \frac{1}{n}z_n \quad \cdots (5)$$

Since T is monotone,

$$(\frac{1}{n}z_n - \frac{1}{m}z_m\ ,\ z_n - z_m) \leq 0 \quad \cdots (6)$$

Due to Crandall and Pazy [3, Lemma2.4] and (6), $z_n \longrightarrow x \in \overline{G}$. By (5) and boundedness of G we have $Tx = y$, $x \in G$. Hence $y \in T(G)$. Therefore $K \subset T(G)$.

In the following theorem we generalize the results of Petryshyn's invariance of domain theorem [9,Theorem5].

Theorem 3. *If T is a demicontinuous, of type (S), locally one to one mapping of an open subset G of a reflexive Banach space X into X^*, then $T(G)$ is open in X^*.*

Proof. For any x_0 in X we choose $r > 0$ such that T is monotone and one to one on $\overline{B}(x_0, r) \subset G$. Since T is one to one, $y_0 \notin T(\partial B(x_0, r))$. Since T is demicontinuous and of type (S), it is easy to show that T is demicontinuous and of type (S_+) (see[7]). So $d(T,\ B(x_0,\ r), y_0)$ is well-defined. Moreover the image of closed subset under T is closed. Indeed, let $y_n \in T(C)$, (C is a closed subset of $\overline{B}(x_0, r)$) $y_n = Tx_n$, $x_n \in C \subset \overline{B}(x_0, r)$, $y_n \longrightarrow y$. Because X is reflexive, we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$ for some x in $\overline{B}(x_0, r)$. Since $y_{n_i} = Tx_{n_i} \longrightarrow y$ and $x_{n_i} \rightharpoonup x$,

$$\lim(Tx_{n_i} - y\ ,\ x_{n_i} - x) = 0$$

$$\Longrightarrow \lim(Tx_{n_i}\ ,\ x_{n_i} - x) = 0$$

Since T is of type (S), $x_{n_i} \longrightarrow x \in C$. Since T is demicontinuous, $Tx_{n_i} \rightharpoonup Tx = y$. Therefore $y \in T(C)$. Hence $T(\partial B(x_0, r))$ is closed and we choose $\rho > 0$ such that $\overline{B}(Tx_0, \rho) \bigcap T(\partial B(x_0, r)) = \phi$. By similar methods of proof in Theorem 1 $T(G)$ is open in X^*.

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THE WEAK ATTOUCH-WETS TOPOLOGY AND THE METRIC ATTOUCH-WETS TOPOLOGY

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ABSTRACT. The purpose of this paper is to find some relations between the weak Attouch- Wets topology and the metric Attouch-Wets topology for the nonempty closed convex subsets of a metrizable locally convex space X. We verify that the former is coaser than the latter. Moreover, we show that X is normable if and only if the two uniformities determining the two topologies for the closed convex subsets of $X \times R$ respectively are equivalent. Our results strengthen and sharpen those of Holà in terms of uniformity itself rather than the topology determined by the uniformity.

1. Introduction

As a successful generalization of the classical Kuratowski convergences of closed convex sets in finite dimensions [8], Attouch-Wets topology [1] in a general normed space X has lately attracted considerable attention. The reason why this topology receives a good deal of attention is that it is stable with respect to duality without reflexivity or even completeness. This Attouch-Wets topology is the topology of uniform convergence of distance functionals on bounded subsets of X, and is well suited for approximation and convex optimization. Its rich developements can be found in the literature [2] [4] [5].

Recently, Beer [3] defined, in the context of a locally convex space, the weak Attouch-Wets topology and the strong Attouch-Wets topology for the nonempty closed convex subsets. These topologies are, in general, different. In fact, it is essentially only in the normed setting that we get the same topology (see [3, Theorem 4.13]). One [3, Theorem 4.9] of his main theorems tells us that the strong convergence of a net of continuous linear functionals

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on a Hausdorff locally convex space X can be explained in terms of the convergence of the corresponding net of its graphs in $X \times R$ with respect to the weak Attouch-Wets topology for the closed convex subsets $C(X \times R)$ of $X \times R$.

On the other hand, Holà [6] considered a "metric" Attouch-Wets topology for the closed convex subsets of a metrizable locally convex space, equipped with a translation invariant metric d. By an elementary method in functional analysis, he has shown that the metric Attouch-Wets convergence of graphs of linear functionals is stronger than convergence of the functionals in the strong topology, and that two notions coincide if and only if X is normable.

When X is a metrizable locally convex space with a translation invariant metric d, there are two topologies, namely, the weak Attouch-Wets topology and the metric Attouch-Wets topology for the nonempty closed convex subsets of X. In that case, it is natural to ask what the relation between the two topologies is. In the present paper, we will show that the latter is stronger than the former [Theorem 1]. Moreover, X is normable if and only if the two topologies for the nonempty closed convex subsets $C(X \times R)$ of $X \times R$ coincide. In fact, X is normable if and only if two uniformities determining the two topologies for $C(X \times R)$ respectively are equivalent [Theorem 2]. Our results strengthen and sharpen those of Holà [6, Theorems 3 and 4] in terms of uniformity itself rather than the topology determined by the uniformity.

2. Preliminaries

We mainly refer to Beer [3]. As mentioned in the introduction, if X is a normed space, then the Attouch-Wets topology τ_{AW} on the nonempty closed convex subsets $C(X)$ is the topology of uniform convergence of distance functionals on bounded subsets of X. As is well-known, the Attouch-Wets topology τ_{AW} can be presented as a uniform space. There are two standard uniformities representing τ_{AW}. A weaker uniformity determining τ_{AW} has a base consisting of all sets of the form

$$\{(A, C) \mid A \cap B \subset C + \epsilon U \text{ and } C \cap B \subset A + \epsilon U\}$$

where U is the solid unit ball of X, B is a bounded subset of X, and $\epsilon > 0$. Motivated by this, Beer [3, Definition, p.7] gave the following definition in the locally convex setting.

Let X be a locally convex space. *The weak Attouch-Wets topology* τ_{AW}^{W} on $C(X)$ is the topology determined by the uniformity with typical basic entourages of the form

$$\Omega(B, U) = \{(A, C) \mid A \cap B \subset C + U \quad \text{and} \quad C \cap B \subset A + U\}$$

where B is a closed bounded balanced convex subset and U is a convex balanced neighborhood of the origin.

Now we turn our attention to the metric space setting. Let (X, d) be a metrizable space with a compatible metric d. For $x_0 \in X$ and $\epsilon > 0$, $S_d[x_0, \epsilon]$ denotes the open d-ball with center x_0 and radius $\epsilon > 0$, and $S_d[A, \epsilon] = \cup_{a \in A} S_d[a, \epsilon]$ does the ϵ-parallel body for a subset A of X. Let $CL(X)$ be the nonempty closed subsets of X. *The Attouch-Wets topology* $\tau_{AW}(d)$ on $CL(X)$ is presented by a uniformity $\sum_d$ which has a countable base consisting of all sets of the form

$$U_d[x_0, n] = \{(A, C) \mid A \cap S_d[x_0, n] \subset S_d[C, \frac{1}{n}] \text{ and } C \cap S_d[x_0, n] \subset S_d[A, \frac{1}{n}]\}$$

where x_0 is a fixed but arbitrary point of X and $n \in Z^+$. In particular, if X is a metrizable locally convex space with a translation invariant (in short, invariant) metric d, the relativized Attouch-Wets topology $\tau_{AW}(d)$ on $C(X)$ the nonempty closed convex subsets is called *the "metric" Attouch-Wets topology* in this paper.

In the sequel, X will be a metrizable locally convex space with an invariant metric d, X^* its continuous dual, and $\mathcal{U}$ will be the family of convex balanced neighborhoods of the origin θ. The product $X \times R$ will be understood to be equipped with the box metric, denoted by $d \times |\cdot|$. Also we denote by $C(X)$ the nonempty closed convex subsets of X. Let us write $BC(X)$ for the family of all closed, bounded, balanced convex subsets of X.

3. Main Results

A set E in X is *bounded* if, for every neighborhood V of θ, we have $E \subset tV$ for all sufficiently large t. A set $E \subset X$ is said to be *d-bounded* if there is a number $M < \infty$ such that $d(x, y) \leq M$ for all x and y in E. In general, the bounded sets and the d-bounded ones need not be the same, even if d is invariant. If X is a normed space and d is the metric induced by the norm, then the two notions of boundedness coincide; but if d is replaced by $d_1 = d/(1 + d)$, (an invariant metric which induces the same topology) they do not. However, we always assert the following.

Lemma. *Let X be a metrizable locally convex space with an invariant metric d. Then the family of d-bounded subsets contains the family of bounded ones.*

Proof. Let E be bounded but not d-bounded. We may choose a sequence $\{x_n\}$ in E satisfying $d(\theta, x_n) \geq n^2$. Since d is invariant, we have

$$d(\theta, nx) \leq nd(\theta, x)$$

for every $x \in X$ and for $n = 1, 2, 3, \cdots$. Taking $x = x_n/n$, we obtain

$$\frac{1}{n}d(\theta, x_n) \leq d(\theta, \frac{x_n}{n}).$$

Hence $d(\theta, x_n/n)(\geq n)$ does not tend to zero. Since d is a compatible metric, this implies x_n/n is not convergent to the origin θ. This contradicts the boundedness of E ([9, Theorem 1.30, p.22]).

This simple lemma plays the crucial role in our results.

Theorem 1. *Let X be a metrizable locally convex space with an invariant metric d. Then the uniformity $\sum_d$ determining τ_{AW}^W for $C(X)$ is stronger than the one doing τ_{AW}^W for $C(X)$. Therefore, τ_{AW}^W is coaser than $\tau_{AW}(d)$.*

Proof. It is sufficient to verify that every basic entourage $\Omega(B, U)$ contains some $U_d[\theta, n]$ in $\sum_d$, where $B \in BC(X)$ and $U \in \mathcal{U}$. Since B is bounded, by Lemma there is an $n_0 \in Z^+$ such that $B \subset S_d[\theta, n_0]$. The family $\{S_d[\theta, 1/n]\}_{n=1}^{\infty}$ is a local base of the origin θ, so we may assume that $S_d[\theta, 1/n_0] \subset U$. Observe that for a subset $E \subset X$ and $r > 0$, we have $S_d[E, r] = E + S_d[\theta, r]$ because d is invariant. Then for $A, C \in C(X)$ we have

$$A \cap S_d[\theta, n_0] \subset S_d[C, \frac{1}{n_0}] = C + S_d[\theta, \frac{1}{n_0}] \Longrightarrow A \cap B \subset C + U$$

$$C \cap S_d[\theta, n_0] \subset S_d[A, \frac{1}{n_0}] = A + S_d[\theta, \frac{1}{n_0}] \Longrightarrow C \cap B \subset A + U.$$

Thus $U_d[\theta, n_0] \subset \Omega(B, U)$ as desired. Therefore, τ_{AW}^W is weaker than $\tau_{AW}(d)$.

As a direct consequence, we obtain the following;

Corollary. Holà ([6, Theorem 3]) *Let* $\{f_n\}$ *be a net in* X^* *and let* $f \in X^*$. *The* $\tau_{AW}(d\times \mid \cdot \mid)$ *- convergence of* Grf_n *to* Grf *implies that* f_n *is convergent to* f *in the strong topology. Here* Grf *denotes the graph of* f *in* $X \times R$.

Proof. By Theorem 1, $Gr\ f_n$ converges to $Gr\ f$ in the weak Attouch-Wets topology τ_{AW}^W for $C(X \times R)$. Moreover, τ_{AW}^W-convergence is equivalent to the strong convergence of f_n to f in virtue of Beer's result [3, Theorem 4.9]. This forces us to get the result.

Remark. In the meantime, we provided a simple proof for Holà's result [6, Theorem 3].

Theorem 2. *X is normable if and only if the two uniformities* $\{\Omega(B,U)\}$ *and* $\sum_{d\times|\cdot|}$ *determining* τ_{AW}^W *and* $\tau_{AW}(d\times \mid \cdot \mid)$ *for* $C(X \times R)$ *respectively are equivalent (If X is a normed space, we take* $d = \|\cdot\|$ *the norm).*

Proof. If X is a normed space and d is the metric induced by the norm $\|\cdot\|$, the box metric $d\times \mid \cdot \mid$ is a norm (easily checked). Hence the boundedness and the $d\times \mid \cdot \mid$-boundedness on the normed space $(X \times R, d\times \mid \cdot \mid)$ coincide. Recall that the ball $S_{d\times|\cdot|}[\theta, n]$ is convex balanced in this case. It is direct from these and Theorem 1 that the two uniformities $\{\Omega(B,U)\}$ and $\sum_{d\times|\cdot|}$ for $C(X \times R)$ are equivalent. Conversely, if the two uniformities are equivalent, then τ_{AW}^W and $\tau_{AW}(d\times \mid \cdot \mid)$ for $C(X \times R)$ are the same. Thus, the strong convergence of a net $\{f_n\}$ to f in X^* coincides with the $\tau_{AW}(d\times \mid \cdot \mid)$-convergence of its graphs by means of Beer's result [3, Theorem 4.9]. By Holà's result [6, Theorem 4], X is normable. This completes our proof.

Remark. Theorem 2, in fact, is a strengthened form of Holà's theorem [6, Theorem 4].

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THE GENERALIZED KKM THEOREMS ON SPACES HAVING CERTAIN CONTRACTIBLE SUBSETS

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ABSTRACT. The concept of a convex space is extended to an H-space; that is, a space having certain contractible subsets. Recently, Chang and Zhang introduced the concept of the generalized KKM multifuntion. Applying this concept to H-spaces, we obtain some general versions of the KKM theorem, Ky Fan's minimax inequality, and systems of inequalities.

1. Introduction

Applications of the classical Knaster-Kuratowski-Mazurkiewicz theorem and the fixed point theory of funtions defined on convex subsets of topological vector spaces have been greatly improved by adopting the concept of convex spaces due to Lassonde [8]. This concept has been extended by Horvath [4-7] to pseudo-convex spaces, contractible spaces, or spaces having certain families of contractible subsets (simply, H-spaces [1]) .

Recently, Chang and Zhang [2] introduced the concept of the generalized KKM multifuntion. Applying this concept to H-spaces, we obtain some general versions of the KKM theorem, Ky Fan's minimax inequality, and systems of inequalities.

2. Preliminaries

Let X and Y be two sets. A *multifunction* $F : X \to 2^Y$ is a function from X into the power set 2^Y of Y. Let $F(X) = \bigcup\{Fx : x \in X\}$ and

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$F^{-1}y = \{x \in X : y \in Fx\}$ for $y \in Y$. Let $\langle X \rangle$ be the family of all nonempty finite subsets of X.

A *convex space* Y is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. For details, see Lassonde [8]. A multifunction $F : Y \to 2^Y$ is said to be *KKM* if $\mathrm{co}\, A \subset F(A)$ for each $A \in \langle Y \rangle$ where co denotes the convex hull.

A subset C of a topological space Y is said to be *compactly closed* [resp. *open*] in Y if for every compact set $K \subset Y$ the set $C \cap K$ is closed [resp. open] in K. A topological space Y is said to be *contractible* if the identity map 1_Y of Y is homotopic to a constant map.

According to Pietsch [11], a collection $\mathcal{G}$ of real-valued funtions defined on a set X is *concave* if, given any finite subset $\{g_1, \cdots g_n\}$ of $\mathcal{G}$ and $\alpha_1, \cdots \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, there exists a $g \in \mathcal{G}$ such that $g(x) \geq \sum_{i=1}^n \alpha_i g_i(x)$ for all $x \in X$.

Given any two collections $\mathcal{G}$ and $\mathcal{G}'$ of real-valued functions on a set X, we shall write $\mathcal{G} \leq \mathcal{G}'$ if for any $f \in \mathcal{G}$, there exists a $g \in \mathcal{G}'$ such that $f(x) \leq g(x)$ for all $x \in X$.

Park [9,10] introduced the following notions. A triple $(Y, D; \Gamma)$ is called an *H-space* if Y is a topological space, D a nonempty subset of Y, and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of Y indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = Y$, we denote $(Y; \Gamma)$ instead of $(Y, Y; \Gamma)$. For an H-space $(Y; \Gamma)$ and any nonempty subset X of Y, we have an H-space $(Y, X; \Gamma)$. A subset L of Y is called an *H-subspace* of $(Y, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D \rangle$, $\Gamma_A \cap L$ is contractible. This is equivalent to saying that the triple $(L, L \cap D; \{\Gamma_A \cap L\})$ is an H-space.

Any convex space Y is an H-space $(Y; \Gamma)$ by putting $\Gamma_A = \mathrm{co}\, A$. Other examples of $(Y; \Gamma)$ are any pseudo-convex space [4], any homeomorphic image of a convex space, any contractible space, and so on. For other examples, see [1]. Every n-simplex Δ_n is an H-space $(\Delta_n, D; \Gamma)$, where D is the set of vertices and $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle D \rangle$.

Motivated by Chang and Zhang [2], we introduce the following notions. Let X be a nonempty set and $(Y; \Gamma)$ an H-space. A multifuntion $G : X \to 2^Y$ is said to be *generalized H-KKM* if for any $A \in \langle X \rangle$, there exists a function $\alpha_A : A \to Y$ such that for any $J \subset A$, we have $\Gamma_{\alpha_A(J)} \subset G(J)$. In particular, if $X = Y$ and $\alpha_A = 1_A$ for each $A \in \langle X \rangle$, the multifunction G is called an *H-KKM* multifunction. Let $\phi : X \times Y \to \overline{\mathbf{R}}$ and $\gamma \in \overline{\mathbf{R}}$. ϕ is called *γ-generalized H-quasi-convex* [*γ- generalized H-quasi-concave*, resp.] in x, if for any $A \in \langle X \rangle$, there exists a function $\alpha_A : A \to Y$ such that for any $J \subset A$ and any $y_0 \in \Gamma_{\alpha_A(J)}$, we have

$$\gamma \leq \max_{x \in J} \phi(x, y_0)$$

$$[\gamma \geq \min_{x \in J} \phi(x, y_0), \text{resp.}].$$

The classical KKM theorem can be stated as followes.

LEMMA A. *Let $R_0, \cdots, R_n$ be closed subsets of the standard n-simplex and let $\{b_0, \cdots, b_n\}$ be the set of its vertices. If for any $\{i_1, \cdots, i_k\} \subset \{i : i = 0, \cdots, n\}, co\{e_j : j = i_1, \cdots, i_k\}$ is contained in $\bigcup\{R_j : j = i_1, \cdots, i_k\}$. Then $\bigcap\{R_i : i = 0, \cdots, n\} \neq \emptyset$.*

LEMMA B. *Let $(Y;\Gamma)$ be an H-space where $D = \{y_0, \cdots, y_n\} \in \langle Y \rangle$. Then there exists a continuous function $f : \Delta_n \to Y$ such that $f(\Delta_J) \subset \Gamma_J$ for each $J \subset D$, where Δ_J is the face of Δ_n corresponding to J.*

Lemma B is given by Horvath [6, Theorem 1.1].

3. Main results

We begin with the following generalized KKM theorem for H-spaces.

THEOREM 1. *Let X be a nonempty set, $(Y;\Gamma)$ an H-space and $G : X \to 2^Y$ a generalized H-KKM multifunction with compactly closed values. Then $\{Gx : x \in X\}$ has the finite intersection property.*

Further if there exists a nonempty compact subset K of Y such that either

(i) *$\bigcap\{Gx : x \in M\} \subset K$ for some $M \in \langle X \rangle$;or*

(ii) *for each $N \in \langle X \rangle$, there exists compact H-subspace L_N of Y containing $\alpha_N(N)$ such that $L_N \cap \bigcap\{Gx : x \in N\} \subset K$.*

Then $K \cap \bigcap\{Gx : x \in X\} \neq \emptyset$.

Proof. For each $A \in \langle X \rangle$ there exists a function $\alpha_A : A \to Y$ such that for any $J \subset A$, we have $\Gamma_{\alpha_A(J)} \subset G(J)$. Let $\alpha_A(A) = \{y_0, \cdots, y_n\}$ and $\Delta_n = b_0 \cdots b_n$. Then by Lemma B, there exists a continuous function $f : \Delta_n \to Y$ such that $f(\Delta_J) \subset \Gamma_{\alpha_A(J)}$ for each $J \subset A$, where Δ_J is the face of Δ_n corresponding to $\alpha_A(J)$. For each $x_i \in A, f^{-1}Gx_i$ is a closed subset of Δ_n. Moreover, the function $H : \{b_0, \cdots, b_n\} \to 2^{\Delta_n}$ given by $Hb_i = f^{-1}Gx_i$ is a KKM function since $f(\Delta_J) \subset \Gamma_{\alpha_A(J)} \subset G(J)$. Therefore, by the KKM theorem, $\bigcap_{i=0}^{n} f^{-1}Gx_i \neq \emptyset$, that is $\bigcap_{i=0}^{n} Gx_i \neq \emptyset$. This completes our proof of the first part.

Case (i). Clear from the above proof.

Note that, from Case (i), if Y itself is compact, then Theorem 1 holds without assuming (i) or (ii). From this fact, we can deduce Case (ii) as follows:

Case (ii). Suppose that $K \cap \bigcap\{Gx : x \in X\} = \emptyset$; that is, $K \subset \bigcup\{Y \backslash Gx : x \in X\}$. Since the compact subset K is covered by compactly open sets $Y\backslash Gx$, $x \in X$, there exists an $N \in \langle X \rangle$ such that $K \subset \bigcup\{Y\backslash Gx : x \in N\}$. Let L_N be set in (ii) and $G' : N \to 2^{L_N}$ a multifunction defined by $G'x = Gx \cap L_N$ for $x \in N$. Then each $G'x$ is closed in L_N. There exists a function $\alpha_N : N \to Y$ such that for any $J \subset N$, we have $\Gamma_{\alpha_N(J)} \subset G(J)$, so $\Gamma_{\alpha_N(J)} \cap L_N \subset G(J) \cap L_N = G'(J)$. Hence G' is a generalized H-KKM multifunction. Therefore, we have

$$\bigcap_{x \in N} G'x = L_N \cap \bigcap_{x \in N} Gx \neq \emptyset.$$

Let $z \in L_N \cap \bigcap\{Gx : x \in N\}$. If $z \in L_N \cap K$, then

$$z \in K \subset \bigcup_{x \in N} (X \backslash Gx)$$

and hence $z \notin Gx$ for some $x \in N$, which is a contradiction. Therefore, we have $z \in L_N \backslash K$. This implies $z \notin \bigcap\{Gx : x \in N\}$ by (ii), which leads another contradiction. Therefore, we must have $K \cap \bigcap\{Gx : x \in X\} \neq \emptyset$. This completes our proof.

A converse of Theorem 1 also holds under some additional condition.

THEOREM 2. *Let X be a nonempty set and $(Y;\Gamma)$ an H-space such that for each $y \in Y$, there is a $z \in Y$ such that $\Gamma_{\{z\}} = \{y\}$. Let $G : X \to 2^Y$ be a multifunction. If the family of sets $\{Gx : x \in X\}$ has the finite intersection property, then G is a generalized H-KKM multifuntion.*

Proof. For each $A \in \langle X \rangle$, $\bigcap\{Gx : x \in A\} \neq \emptyset$. Take $y_* \in \bigcap\{Gx : x \in A\}$ and define $\alpha_A : A \to Y$ by $\alpha_A(x) = z_*$ for each $x \in A$ such that $\Gamma_{\{z_*\}} =$

y_*. For any $J \subset A$, we have $\Gamma_{\alpha_A(J)} = \{y_*\} \subset \bigcap\{Gx : x \in A\} \subset G(J)$. This implies that $G : X \to 2^Y$ is a generalized H-KKM multifuntion. This completes our proof.

REMARK. If X is a convex subset of a Hausdorff topological vector space $E = Y$, then Theorem 1 and 2 generalizes Chang and Zhang [2,Theorem 3.1].

Next we establish the relations between the concept of the generalized H-KKM multifuntion and the γ-generalized H-quasi-convexity [γ-generalized H-quasi-concavity, resp.].

PROPOSITION 3. *Let X be a nonempty set, $(Y;\Gamma)$ an H-space, and $\phi : X \times Y \to \overline{\mathbf{R}}$. Then the followings are equivalent:*

(i) *The multifuntion $G : X \to 2^Y$ given by*

$$Gx = \{y \in Y : \phi(x,y) \le \gamma\}$$

$$[Gx = \{y \in Y : \phi(x,y) \ge \gamma\}, \text{ for } x \in X resp.]$$

is generalized H-KKM.

(ii) *ϕ is γ-generalized H-quasi-concave [γ-generalized H-quasi-convex, resp.] in x.*

Proof. For the sake of the simplicity, we prove the conclusion only for the first case in (i) and (ii). The other case can be proved similarly.

(i) $\Longrightarrow$ (ii). Since $G : X \to 2^Y$ is a generalized H-KKM multifunction, for any $A \in \langle X \rangle$, there exists a function $\alpha_A : A \to Y$ such that for any $J \subset A$ and $y_0 \in \Gamma_{\alpha_A(J)}$, we have $y_0 \in G(J)$. Hence there exists an $x \in J$ such that $y_0 \in Gx$, so we have $\phi(x, y_0) \le \gamma$. Therefore we have $\min_{x \in J} \phi(x, y_0) \le \gamma$; that is, ϕ is γ-generalized H- quasi-concave in x.

(ii) $\Longrightarrow$ (i). Since ϕ is γ-generalized H- quasi-concave in x, for any $A \in \langle X \rangle$, there exists a function $\alpha_A : A \to Y$ such that for any $J \subset A$ and $y_0 \in \Gamma_{\alpha_A(J)}$, we have $\gamma \ge \min_{x \in J} \phi(x, y_0)$. Hence there exists an $x \in J$ such that $\phi(x, y_0) \le \gamma$. This implies $y_0 \in Gx$. By the arbitrariness of $y_0 \in \Gamma_{\alpha_A(J)}$, we have $\Gamma_{\alpha_A(J)} \subset G(J)$. Therefore $G : X \to 2^Y$ is a generalized H-KKM multifunction. This completes our proof.

REMARK. Proposition 3 generalizes Chang and Zhang [2, Proposition 2.1].

From Theorem 1, we have

THEOREM 4. *Let X be a nonempty set, $(Y;\Gamma)$ an H-space, and $\alpha \geq \beta$. Suppose that $\phi, \psi : X \times Y \to \overline{\mathbf{R}}$ satisfy the following conditions:*

(1) *$\phi(x,y) \leq \psi(x,y)$ for all $(x,y) \in X \times Y$;*
(2) *ψ is β-generalized H-quasi-concave in x;*
(3) *for each $x \in X$, $\{y \in Y : \phi(x,y) > \alpha\}$ is compactly open.*

Suppose that there exists a nonempty compact subset K of Y such that either

(i) *for some $M \in \langle X \rangle$, $\bigcap_{x \in M}\{y \in Y : \phi(x,y) \leq \alpha\} \subset K$; or*
(ii) *for each $N \in \langle X \rangle$, there exists a compact H-subspace L_N of Y containing $\alpha_N(N)$ such that for each $y \in L_N \backslash K$, there exists an $x \in N$ satisfying $\phi(x,y) > \alpha$.*

Then there exists a $y_0 \in K$ such that $\sup_{x \in X} \phi(x,y_0) \leq \alpha$.

Proof. Define multifuntions $F, G : X \to 2^Y$ by

$$Fx = \{y \in Y : \psi(x,y) \leq \beta\},$$
$$Gx = \{y \in Y : \phi(x,y) \leq \alpha\} \text{ for } x \in X.$$

Then for each $x \in X$, $Fx \subset Gx$ and Gx is compactly closed in Y. By (2) and Proposition 3, F is a generalized H-KKM multifunction, and so is G. Since the coercivity condition (i) or (ii) of Theorem 1 holds, so $K \cap \bigcap\{Gx : x \in X\} \neq \emptyset$. Taking $y_0 \in \bigcap\{Gx : x \in X\} \cap K$, we have $\phi(x,y_0) \leq \alpha$ for all $x \in X$. Hence $\sup_{x \in X} \phi(x,y_0) \leq \alpha$. This completes our proof.

As an immediate consequence of Theorem 4 we obtain the following general version of Ky Fan's minimax inequality.

COROLLARY 5. *Under the hypothesis of Theorem 4, suppose that* $X = (Y;\Gamma) = K$, $\phi = \psi$ *and* $\alpha = \beta = \sup_{x\in X}\phi(x,x)$. *Then there exists an* $y_0 \in X$ *such that*

$$\sup_{x\in X}\phi(x,y_0) \le \sup_{x\in X}\phi(x,x).$$

From Theorem 1 we also have the following slight generalization of Ding, Kim, and Tan [3, Theorem 8].

THEOREM 6. *Let* $(X;\Gamma)$ *be a normal* H*-space,* K *a nonempty compact subset of* X*, and* $\rho \in \mathbf{R}$. *Let* $\mathcal{G}$, $\mathcal{G}'$ *and* $\mathcal{G}''$ *be three collections of real-valued functions on* X *such that*

(a) $\mathcal{G} \le \mathcal{G}' \le \mathcal{G}''$;
(b) *for each* $f \in \mathcal{G}$ *and* $\alpha \in \mathbf{R}$, $\{x \in X : f(x) > \alpha\}$ *is compactly open;*
(c) *the collection* $\mathcal{G}''$ *is concave.*

Suppose that for any finite subset $\{g_1,\cdots,g_n\}$ *of* $\mathcal{G}'$ *and nonzero real numbers* $\beta_1,\cdots,\beta_n$ *with* $\sum_{i=1}^n \beta_i = 1$*, the following holds:*

(d) *for each* $A \in \langle X\rangle$ *there is a function* $\alpha_A : A \to X$ *such that for any* $J \subset A$ *and* $y \in \Gamma_{\alpha_A(J)}$*, we have* $\sum_{i=1}^n \beta_i g_i(y) \le \sum_{i=1}^n \beta_i g_i(x) + \rho$ *for some* $x \in J$*; and*
(e) *there exists a nonempty compact subset* K *of* X *such that either*
(i) *for some* $M \in \langle X\rangle, \sum_{i=1}^n \beta_i g_i(y) \le \sum_{i=1}^n \beta_i g_i(x) + \rho$ *for all* $x \in M$ *implies* $y \in K$*; or*
(ii) *for each* $N \in \langle X\rangle$*, there exists a compact* H*-subspace* L_N *of* X *containing* $\alpha_N(N)$ *such that for each* $y \in L_N\backslash K$*, there exists an* $x \in N$ *satisfying* $\sum_{i=1}^n \beta_i g_i(y) > \sum_{i=1}^n \beta_i g_i(x) + \rho$*, where* $\alpha_N : N \to X$ *is the function in (d).*

Then one of the following properties holds:

(1) *there exists an* $h \in \mathcal{G}''$ *such that* $\inf_{x\in X} h(x) > \rho$*;*
(2) *there exists a point* $y_0 \in K$ *such that* $f(y_0) \le \rho$ *for all* $f \in \mathcal{G}$.

Proof. Without loss of generality, we may assume that $\rho = 0$. For each $f \in \mathcal{G}$, let $Q(f) = \{x \in K : f(x) \le 0\}$; then $Q(f)$ is closed in K by (b). If the family $\{Q(f) : f \in \mathcal{G}\}$ has the finite intersection property, then by the compactness of K we obtain the alternative (2). Suppose $\{Q(f) : f \in \mathcal{G}\}$ does not have the finite intersection property. Then there are $f_1,\cdots,f_n \in \mathcal{G}$

such that $\bigcap_{i=1}^n Q(f_i) = \emptyset$. For each $i = 1, \cdots, n$, let $V_i = X \backslash Q(f_i)$. Then each V_i is open in X and $\{V_1, \cdots, V_n\}$ is an open cover of the normal space X. Let $\{\beta_1, \cdots, \beta_n\}$ be a continuous partition of unity surbordinate to this open cover. Thus for each $i = 1, \cdots, n$, $\beta_i : X \to [0,1]$ is continuous and Supp $\beta_i \subset V_i$ such that $\sum_{i=1}^n \beta_i(y) = 1$ for each $y \in X$. Choose $g_1, \cdots, g_n \in \mathcal{G}'$ and $h_1, \cdots, h_n \in \mathcal{G}''$ such that $f_i \leq g_i \leq h_i$ on X for each $i = 1, \cdots, n$. Define multifunctions $F, G : X \to 2^X$ as follows:

$$Fx = \{y \in X : \sum_{i=1}^n \beta_i(y) g_i(y) - \sum_{i=1}^n \beta_i(y) g_i(x) \leq \rho\},$$

$$Gx = \{y \in X : \sum_{i=1}^n \beta_i(y) f_i(y) - \sum_{i=1}^n \beta_i(y) g_i(x) \leq \rho\} \text{ for } x \in X.$$

Then $\overline{F}$ satisfies all of the hypotheses of Theorem 1 and so does G, since $Fx \subset Gx$ and Gx is closed. Hence there exists a $y_0 \in K$ such that

$$\sum_{i=1}^n \beta_i(y_0) f_i(y_0) \leq \sum_{i=1}^n \beta_i(y_0) g_i(x)$$

for all $x \in X$. By (c), there is an $h \in \mathcal{G}''$ satisfying $h(x) \geq \sum_{i=1}^n \beta_i(y_0) h_i(x)$ for all $x \in X$. Therefore for all $x \in X$,

$$0 < \sum_{i=1}^n \beta_i(y_0) f_i(y_0) \leq \sum_{i=1}^n \beta_i(y_0) g_i(x) \leq \sum_{i=1}^n \beta_i(y_0) h_i(x) \leq h(x).$$

This proves the alternative (1).

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SOME COINCIDENCE THEOREMS ON ACYCLIC MULTIFUNCTIONS AND APPLICATIONS TO KKM THEORY, II

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§1. Introduction

In this talk, we discuss several topics on which studies were initiated from our previous work [**P6**], which will be called Part I.

An upper semicontinuous (u.s.c.) multifunction with nonempty compact convex values will be called a *Kakutani map*. Recently, Lassonde [**L2**] extended the well-known fixed point theorems due to Kakutani [**Kk**] and Himmelberg [**Hi**] to multifunctions factorizable by Kakutani maps through convex sets in topological vector sapces. Such multifunctions arise in a natural way in minimax and coincidence theories. For the literature, see [**L2**], [**GL2**], [**Gr1,2**].

On the other hand, Ben-El-Mechaiekh [**Bn1**] obtained an elementary proof of a fairly general fixed point theorem for composites of Kakutani maps defined on a class of general extension spaces containing locally convex and some not necessarily locally convex topological vector spaces. He also deduced some general coincidence theorems for composities of multifunctions. The aim in [**L2**], [**Bn1**] lies to give elementary approach to the convex-valued multifunctions not using homological methods.

An u.s.c. multifunction with compact acyclic values will be called an *acyclic map*. In Part I, from a Lefschetz type fixed point theorem for composites of acyclic maps, we obtained a general Fan-Browder type coincidence theorem, which was shown to be equivalent to a matching theorem and a KKM type theorem. From the coincidence theorem, we deduced the Himmelberg type fixed point theorem for acyclic compact multifunctions, acyclic versions of general geometric properties of convex sets, abstract variational

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inequality theorems, new minimax theorems, and non-continuous versions of the Brouwer and Kakutani type fixed point theorems with very generous boundary conditions. Further applications to acyclic maps were also given in the author's works [**P9,10**].

Our main purpose in the present talk is to give improvements of results in [**P6,9,10**] and further applications of results in Part I. The detailed versions of each section will be published separately.

In Section 3, we show that all of the key results of Sehgal *et al.* [**SSW**], Lassonde [**L3**], Shioji [**So**], Liu [**L**], Chang and Zhang [**CZ**], and Guillerme [**Gu**] are simple consequences of Part I [**P6**, Theorem 3].

Section 4 deals with a generalization of the main coincidence theorem of Part I [**P6**, Theorem 1] to the class of composites of "admissible" maps which properly includes that of multifunctions factorizable by Kakutani or acyclic maps. Our new results generalize those of [**L2,4**], [**Bn1**], [**BD2**], [**P6**]. Moreover, fundamental theorems in the KKM theory can be obtained in far-reaching generalized forms related to admissible maps.

Finally, in Section 5, we obtain mainly sufficient conditions for the existence of fixed points of compact composites of admissible maps defined on a convex subset of a topological vector space E on which its topological dual E^* separates points. Our arguments are based on fundamental theorems of the KKM theory. Our main consequences are admissible map versions of well-known fixed point theorems due to Fan [**F3,5**], Halpern and Bergman [**HB**], Himmelberg [**Hi**], Reich [**R5**], Ha [**H**], Granas and Liu [**GL2**], and many others.

Consequently our new results extend, improve, and unify main theorems in more than one hundred published works.

§2. Preliminaries

For terminology and notations, we follow mainly [**P6**].

Let X and Y be sets. A *multifunction* $S : X \to 2^Y$ is a function from X into the power set 2^Y of Y; that is, $Sx \subset Y$ for each $x \in X$.

For $A \subset X$, let $S(A) = \bigcup\{Sx : x \in A\}$. For $y \in Y$, let $S^- y = \{x \in X : y \in Sx\}$. For any $B \subset Y$, the *(lower) inverse* of B under S is defined by $S^-(B) = \{x \in X : Sx \cap B \neq \emptyset\}$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Let X be a set (in a vector space) and D a nonempty subset of X. Then (X, D) is called a *convex space* if convex hulls of any $N \in \langle D \rangle$ is contained in X and X has a topology that induces the Euclidean topology on such convex hulls. Such conve hull will be called a *polytope*. A subset A of (X, D) is said to be *D-convex* if, for any $N \in \langle D \rangle$, $N \subset A$ implies $\operatorname{co} N \subset A$, where co denotes the convex hull. If $X = D$, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [**L1**]. Note that for a convex space (X, D), X itself is not necessarily convex. For example, let X be any space containing an n-simplex Δ_n as a subspace and D the set of vertices of Δ_n. Then (X, D) is a convex space, X is not convex, but D-convex.

For a convex space (X, D), a multifunction $G : D \to 2^X$ is called a *KKM map* if $\operatorname{co} N \subset G(N)$ for each $N \in \langle D \rangle$. The *KKM theory* is the study of KKM maps and their applications. For the literature, see [**A**], [**AE**], [**Gr1,2**], [**P6**], [**Z**].

A subset B of a topological space Y is said to be *compactly closed* [resp. *open*] in Y if for every compact set $K \subset Y$ the set $B \cap K$ is closed [resp. open] in K.

For topological spaces X and Y, a multifunction $F : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) if $F^-(C)$ is closed for each closed set $C \subset Y$. F is said to be *compact* if $F(X)$ is contained in a compact subset of Y.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic. For a topological space Y, $ka(Y)$ denotes the set of all compact acyclic subsets of Y. For a convex space Y, $kc(Y)$ denotes the set of all nonempty compact convex subsets of Y.

A family of sets is said to have the *finite intersection property* if the intersection of each finite subfamily is not empty.

Given a class $\mathbb{L}$ of multifunctions, we define

$$\mathbb{L}(X, Y) := \{T : X \to 2^T \mid T \in \mathbb{A}\};$$
$$\mathbb{L}_c := \{T = T_m T_{m-1} \cdots T_1 \mid T_i \in \mathbb{A}\}.$$

We also define

$T \in \mathbb{K}(X, Y) \iff T$ is a Kakutani map; that is, Y is a convex space and T is u.s.c. with $Tx \in kc(Y)$ for $x \in X$.

$T \in \mathbb{V}(X, Y) \iff T$ is an acyclic map; that is, T is u.s.c. with $Tx \in ka(Y)$ for $x \in X$.

We now introduce an abstract class $\mathbb{A}$ of multifunctions motivated by [**BD2**].

A class $\mathbb{A}$ of multifunctions is said to be *admissible* if

(i) $\mathbb{A}$ contains the class $\mathcal{C}$ of continuous functions;

(ii) each $F \in \mathbb{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathbb{A}_c(P, P)$ has a fixed point.

We list a few examples of admissible classes:

(1) $\mathbb{A} = \mathcal{C}$.

(2) $\mathbb{A} = \mathbb{K}$. [**L2**], [**Bn1**].

(3) $\mathbb{A} = \mathbf{V}$. See Górniewicz and Granas [**GG**].

(4) $\mathbb{A}$ is the class of approachable maps in a topological vector space. See Ben-El-Mechaiekh and Deguire [**BD2**].

For related examples, see [**BD1,2**], [**L4**].

§3. A unified approach to generalizations of the KKM type theorems

In Part I, we established several KKM type theorems which subsume and strengthen many of known generalizations of the KKM theorem. Actually, one of our results [**P6**, Theorem 3] is a consequence of the Lefschetz fixed point theory on acyclic maps and includes important KKM type theorems of Fan [**F2,7,9**], Lassonde [**L1**], Chang [**C**], Park [**P2,5**], and others as particular cases. However, after the author completed Part I, he came to know that there have appeared other generalizations or equivalent forms of the KKM theorem; e.g., Sehgal, Singh, and Whitfield [**SSW**], Lassonde [**L3**], Shioji [**So**], Liu [**L**], Chang and Zhang [**CZ**], and Guillerme [**Gu**].

In this section, we show that all of the key results of those papers are simple consequences of [**P6**, Theorem 3]. Consequently, those recent results can be stated in more general forms and unified in a single theorem. We also discuss the most general forms of the other KKM type theorems which can not be covered by [**P6**, Theorem 3].

We begin with the following:

Theorem 3.1. (Park [**P6**, Theorem 3]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $F \in \mathbb{V}(X,Y)$, and K a nonempty compact subset of Y. Let $G : D \to 2^Y$ be a multifunction such that*

(1) *for each $x \in D$, Gx is compactly closed;*

(2) *for each $N \in \langle D \rangle$, $F(\operatorname{co} N) \subset G(N)$; and*

(3) *for each $N \in \langle D \rangle$, there exists an $L_N \in kc(X)$ containing N such that $F(L_N) \cap \bigcap\{Gz : z \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

From Theorem 3.1, we can deduce all of the main results of Shioji [**So**]. In fact, we have

Theorem 3.2. (Shioji [**So**, Theorem 1]) *Let X be a nonempty subset of a vector space E, Y a Hausdorff space, $G : X \to 2^Y$, and $T : \operatorname{co} X \to ka(Y)$ such that*

(1) *for each $N \in \langle X \rangle$, $T(\operatorname{co} N) \subset G(N)$;*

(2) *for each $N \in \langle X \rangle$, $T \,|\, \operatorname{co} N$ is u.s.c., where $\operatorname{co} N$ is endowed with the Euclidean simplex topology; and*

(3) *for each $N \in \langle X \rangle$ and each $x \in N$, $Gx \cap T(\operatorname{co} N)$ is relatively closed in $T(\operatorname{co} N)$.*

Then, for each $N \in \langle X \rangle$,

$$\bigcap\{Gx \,|\, x \in N\} \cap T(\operatorname{co} N) \neq \emptyset.$$

Note that other results of [**So**] are consequences of Theorem 3.2.

In [**CZ**], Chang and Zhang extended the concept of KKM multifunctions to generalized KKM multifunctions, and obtained general versions of the KKM theorem, Fan's minimax inequality, and the Browder-Hartman-Stampacchia variational inequality.

Let D be a nonempty set and Y a convex space. A multifunction $G : D \to 2^Y$ is said to be *generalized KKM* if, for any $N \in \langle D \rangle$, there is a function $\sigma : N \to Y$ such that $M \subset N$ implies $\operatorname{co} \sigma(M) \subset G(M)$ [**CZ**].

From Theorem 3.1, we can obtain

Theorem 3.3. *Let D be a nonempty subset of a convex space X, Y a convex space, and $G : D \to 2^Y$ a multifunction with compactly closed values. Then $\{Gx : x \in D\}$ has the finite intersection property if and only if G is generalized KKM.*

If $X = D$ is a convex subset of a Hausdorff topological vector space $E = Y$, then Theorem 3.3 reduces to Chang and Zhang [**CZ**, Theorem 3.1].

Moreover, from Theorem 3.3, we have the following:

Theorem 3.4. *Let D be a nonempty subset of a convex space X, and $G : D \to 2^X$ with compactly closed values. Suppose that there exists a nonempty compact subset K of X such that, for each $N \in \langle D \rangle$, there exists an $L_N \in kc(X)$ containing N such that*

$$L_N \cap \bigcap\{Gz : z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$ if and only if G is a generalized KKM multifunction.

Note that Theorem 3.4 generalizes Chang and Zhang [**CZ**, Theorem 3.2]. From Theorem 3.4, we have

Corollary. (Lassonde [**L2**, Principe KKM]) *Let D be a subset of a convex set X (in a vector space) and $F : D \to 2^X$ a multifunction with closed values in X_f (that is; X with the finite topology). If, for every $A \in \langle D \rangle$, we have $\operatorname{co} A \subset F(A)$, then, for every $N \in \langle D \rangle$, we have*

$$\operatorname{co} N \cap \bigcap \{Fx : x \in N\} \neq \emptyset.$$

In [**L**], Liu proposed a form of the KKM principle and obtained some applications. His main result can be extended as follows:

Theorem 3.5. *Let X be a convex space, D a nonempty subset of X, K a nonempty compact subset of X, Y a topological space, $S : X \to 2^Y$, and $A : D \to 2^Y$ such that*

(1) *S is u.s.c.;*
(2) *Ax is closed for each $x \in D$; and*
(3) *for each $N \in \langle D \rangle$, there exists an $L_N \in kc(X)$ containing N such that*

$$L_N \cap \{z \in X : Sz \cap Ax \neq \emptyset \text{ for all } x \in L_N \cap D\} \subset K.$$

Then either

(I) *there exists an $x_0 \in K$ such that $Sx_0 \cap Ax \neq \emptyset$ for all $x \in D$; or*
(II) *there exist an $N \in \langle D \rangle$ and an $x_0 \in \operatorname{co} N$ such that $Sx_0 \cap A(N) = \emptyset$ (that is, S is trappable by A^c [**L**]).*

For $D = X$ and under a slightly weaker condition than (3), Theorem 3.5 reduces to the main result of Liu [**L**, Theorem 2.1].

In certain cases, the KKM theorem holds for open-valued multifunctions. Such results were first obtained by W. K. Kim [**Ki1**]. However, the main idea was given in the earlier work of Ky Fan [**F9**, Theorem 2] as a matching theorem for closed coverings. Later, results of Kim [**Ki1,2**] were generalized by the present author [**P2,4**]. Recently, Lassonde [**L2**] refined Kim's idea and gave some applications.

Note that the following encompasses all of the open-valued KKM theorems due to Kim [**Ki1,2**], Park [**P4**], and Lassonde [**L2**]:

Theorem 3.6. (Park [**P2**, Theorem 8]) *Let D be a nonempty subset of a convex space X, Y a topological space, $G : D \to 2^Y$ a multifunction and $s : X \to Y$ a continuous function. Suppose that*

(1) *for each $x \in D$, Gx is compactly open in Y; and*
(2) *for each $N \in \langle D \rangle$, $s(\operatorname{co} N) \subset G(N)$.*

Then the family $\{Gx : x \in D\}$ has the finite intersection property.

Theorem 3.1 is equivalent to another form of whole intersection property as follows:

Theorem 3.7. *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $F \in \mathbb{V}(X,Y)$, and K a nonempty compact subset of Y. Let $G : D \to 2^Y$ and $H : X \to 2^Y$ be multifunctions such that*

(1) *for each $x \in D$, $Hx \subset Gx$ and Gx is compactly closed;*

(2) *for each $x \in X$, $Fx \subset Hx$;*

(3) *for each $y \in F(X)$, $X \backslash H^- y$ is convex; and*

(4) *for each $N \in \langle D \rangle$, there exists an $L_N \in kc(X)$ containing N such that $F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

A particular form of Theorem 3.7 appeared first in Tarafdar [**T1**]. Some other variations, see Horvath [**H3,4**], Bardaro and Ceppitelli [**BC1**], and Ding and Tan [**DT**].

From Theorem 3.7 we have the following:

Corollary. (Guillerme [**Gu**, Theorem V.1]) *Let X and Y be convex spaces and $F \subset S \subset T \subset G \subset X \times Y$ relations such that*

(1) *for each $y \in Y$ and each polytope P of X, $F^- y \cap P$ is open in P;*

(2) *for each $x \in X$, Sx is convex in Y;*

(3) *for each $y \in Y$, $X \backslash T^- y$ is convex*

(4) *for each $x \in X$, Gx is closed.*

If Gx_0 is compact for some $x_0 \in X$ and $Fx \neq \emptyset$ for each $x \in X$, then $\bigcap\{Gx : x \in X\} \neq \emptyset$.

Note that Guillerme [**Gu**, Theorem IV.1] is a particular form of Corollary with $S = T$. He used his results to obtain some minimax inequalities.

Now, we discuss other types of generalizations of the KKM theorem which can not be covered by Theorems 3.1 and 3.6.

(1) The key results of Lassonde [**L1**, Theorems I, II, and III] were generalized, unified, and strengthened by Jiang [**J3**, Theorem 2.2], which is further extended by Park [**P6**, Theorem 4]. This result is not comparable to Theroems 3.1 or 3.6. Note that Lassonde [**L1**, Theorems I and III] are included in Theorem 3.1.

(2) There is another generalization of the KKM theorem due to Shapley [**Sh**]. Its generalizations or applications appear in Ichiishi [**I1,2**], Fan [**F8,9**], Simons [**Si**], Shih and Tan [**ST1,2**], Ichiishi and Idzik [**II**], and Park [**P3**]. Among them the most general forms seem to be [**Si**, Theorem 7.1], [**ST1**, Theorems 2 and 3], [**II**, Theorem 2.2], and [**P3**, Theorem 3].

(3) The concept of a convex space is extended to an H-space or spaces having certain contractible subsets in the sense of Horvath [**H1-5**]. The KKM theorem can be extended to such spaces. See [**BC1-3**], [**DT**], [**DKT1,2**], and

[**P7,8**]. The H-space versions of Theorems 3.1, 3.6 and [**P6**, Theroem 4] for the case F is single-valued are obtained as [**P7**, Theorems 4, 14, and 3], respectively.

(4) The concept of convexity is generalized by many authors. In one of such directions, Bielawski [**B**, Proposition 4.7 and Corollary 4.8] obtained the KKM theorem for a space having "a finitely local convexity."

§4. Fundamental theorems in the KKM theory via coincidences of composites of admissible u.s.c. multifunctions

As we have seen in Part I, there exist mutually equivalent fundamental theorems in the KKM theory from which most of important results in the theory can be deduced. Recently, the author has found that the acyclic map in the most of KKM type theorems in Part I can be replaced by any composite of admissible u.s.c. multifunctions. This remark also applies to the results in Section 3.

In this section, we give a generalization of the main coincidence theorem of Part I to the class of composites of admissible maps. Our new result generalizes that of [**L2,4**], [**Bn1**], [**BD2**], [**P6**]. Moreover, fundamental theorems in the KKM theory can be obtained in far-reaching generalized forms related to admissible maps. Those are the KKM theorem, matching theorems, the Fan-Browder fixed point theorem, the Ky Fan minimax inequality, analytic alternatives, geometric properties of convex sets, and others.

The following is the main result of this section:

Theorem 4.1. *Let* (X, D) *be a convex space,* Y *a Hausdorff space,* $S : D \to 2^Y$, $T : X \to 2^Y$ *multifunctions, and* $F \in \mathbb{A}_c(X, Y)$. *Suppose that*

(1) *for each* $x \in D$, $Sx \subset Tx$ *and* Sx *is compactly open;*
(2) *for each* $y \in F(X)$, $T^{-}y$ *is* D*-convex;*
(3) *there exists a nonempty compact subset* K *of* Y *such that* $\overline{F(X)} \cap K \subset S(D)$*; and*
(4) *for each* $N \in \langle D \rangle$, *there exists a compact* D*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \backslash K \subset S(L_N \cap D)$.

Then F *and* T *have a coincidence point.*

The origin of Theorem 4.1 is due to Browder [**B2,3**] and Fan [**F2,6**] for $X = D = Y = K$ and $F = 1_X$. Note that numerous applications of the Fan-Browder fixed point theorem have appeared in various fields as fixed point theory, minimax theory, variational inequalities, and so on.

For $\mathbb{K}$ or $\mathbb{V}$ instead of $\mathbb{A}_c$, Theorem 4.1 reduces to Park [**P6**, Theorem 1], which includes earlier works of Browder [**B2-4**], Tarafdar [**T1-4**], Tarafdar and Husain [**TH**], Ben-El-Mechaiekh *et al.* [**BDG1,2**], Yannelis and Prabhakar [**YP**], Lassonde [**L1,2**], Ko and Tan [**KT**], Simons [**Si**], Takahashi [**T**], Komiya [**Ko**], Mehta [**Me**], Mehta and Tarafdar [**MT**], Sessa [**Ss**], Jiang [**J1-3**], McLinden [**Mc**], Granas and Liu [**GL1,2**], Park [**P1,2,5**], and Chang [**C**].

As in Part I, from Theorem 4.1, we obtain the following KKM theorem:

Theorem 4.2. *Let (X, D) be a convex space, Y a Hausdorff space, and $F \in \mathbb{A}_c(X, Y)$. Let $G : D \to 2^Y$ be a multifunction such that*

(1) *for each $x \in D$, Gx is compactly closed in Y;*
(2) *for any $N \in \langle D \rangle$, $F(\operatorname{co} N) \subset G(N)$; and*
(3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

The origin of Theorem 4.2 goes back to Sperner [**Sp**] and Knaster, Kuratowski, and Mazurkiewicz [**KKM**] for $X = Y = K = \Delta^n$ an n-simplex, D its set of vertices, and $F = 1_X$. Note that Theorem 4.2 properly generalizes Theorem 3.1.

As we have done in Part I, Theorems 4.1 and 4.2 can be reformulated other form of fundamental theorems in the KKM theory. The following is one of them and useful in the next section of this paper:

Theorem 4.3. *Let (X, D) be a convex space, Y a Hausdorff space, $F \in \mathbb{A}_c(X, Y)$, $A, B \subset Z$ sets, $f, g : X \times Y \to Z$ functions, and K a nonempty compact subset of Y. Suppose that*

(1) *for each $x \in D$, $\{y \in Y : g(x, y) \in A\}$ is compactly open and contained in $\{y \in Y : f(x, y) \in B\}$;*

(2) *for each $y \in F(X)$, $\{x \in X : f(x, y) \in B\}$ is D-convex; and*

(3) *for each $N \in \langle D \rangle$, there exists a compact D-convex subset L_N of X containing N such that for each $y \in F(L_N) \backslash K$, there exists an $x \in L_N \cap D$ satisfying $g(x, y) \in A$.*

Then either

(i) *there exists a $\hat{y} \in \overline{F(X)} \cap K$ such that $g(x, \hat{y}) \notin A$ for all $x \in D$, or*
(ii) *there exists an $(\hat{x}, \hat{y}) \in F$ such that $f(\hat{x}, \hat{y}) \in B$.*

The first form of Theorem 4.3 seems to be Lassonde [**L1**, Theorem 1.1′]. Note also that if F is single-valued, then Y is not necessarily Hausdorff. Lassonde used his result to generalize earlier works of Iohvidov [**I**], Fan [**F4**], and Browder [**B2**]. Applications of this kind results to the Tychonoff fixed point theorem and the study of invariant subspaces of certain linear operators were given in [**I**], [**F4**].

For $X = D$ and $\mathbb{V}$ instead of $\mathbb{A}_c$, Theorem 4.3 reduces to [**P6**, Theorem 5].

§5. Fixed points of admissible multifunctions on topological vector spaces

As an application of fundamental theorems of the KKM theory, in this section, we obtain mainly sufficient conditions for the existence of fixed points of compact composites of admissible maps defined on a convex subset of a topological vector space E on which its topological dual E^* separates points. Such class of spaces properly contains locally convex Hausdorff topological vector spaces. Our arguments are based on a geometric property of a convex set and a variational inequality related to acyclic maps. Our main consequences are admissible map versions of well-known fixed point theorems due to Fan [**F3,5**], Halpern and Bergman [**HB**], Himmelberg [**Hi**], Reich [**R5**], Ha [**H**], Granas and Liu [**GL2**], and many others.

From Theorem 4.3, we obtain the following variational inequality with a lopsided saddle point:

Theorem 5.1. *Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathbb{A}_c(X, Y)$. Let $g : X \times Y \to \mathbb{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto g(x, y)$ is quasi-convex on X. Then there exists an $(x_0, y_0) \in T$ such that*

$$g(x_0, y_0) \leq g(x, y_0) \quad \text{for all} \quad x \in X.$$

For $\mathbb{V}$ instead of $\mathbb{A}_c$, Theorem 5.1 reduces to Park [**P9**, Theorem 2], which extends Ha [**H**, Theorem 2] for $\mathbb{K}$ instead of $\mathbb{A}_c$.

For a subset X of a topological vector space E, the *inward* and *outward sets* of X at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) := \{x + r(u - x) \in E \mid u \in X,\ r > 0\},$$
$$O_X(x) := \{x - r(u - x) \in E \mid u \in X,\ r > 0\},$$

The closures of $I_X(x)$ and $O_X(x)$ are denoted by $\overline{I}_X(x)$ and $\overline{O}_X(x)$, resp. In the sequel, $W(x)$ denotes either $\overline{I}_X(x)$ or $\overline{O}_X(x)$.

Let $\mathcal{P}$ denote the family of all continuous seminorms on a topological vector space E.

As an application of Theorem 5.1, we obtain the following Ky Fan type fixed point theorem for admissible maps as in [**P9**]:

Theorem 5.2. *Let X be a compact convex subset of a topological vector space E on which E^* separates points and $T \in \mathbb{A}_c(X, E)$. Then either T has a fixed point or there exist an $(x_0, y_0) \in T$ and a $p \in \mathcal{P}$ such that*

$$0 < p(x_0 - y_0) \leq p(x - y_0) \quad \text{for all} \quad x \in \overline{I}_X(x_0).$$

For a locally convex Hausdorff topological vector space E, Theorem 5.2 for $\mathbb{V}$ instead of $\mathbb{A}_c$ reduces to [**P9**, Theorem 3], which extends earlier works of Fan [**F5**], Reich [**R3**], and Ha [**H**].

As a direct consequence of Theorem 5.2, we have the following as in [**P9,10**]:

Theorem 5.3. *Let X be a compact convex subset of a topological vector space E on which E^* separates points and $T \in \mathbb{A}_c(X, E)$. If T satisfies one of the following conditions, then T has a fixed point.*

For each $x \in \operatorname{Bd} X$,

(0) *for each $y \in Tx$ and each $p \in \mathcal{P}$, $p(y - x) > 0$ implies $p(y - x) > p(y - z)$ for some $z \in \overline{I}_X(x)$.*

(i) *for each $y \in Tx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1 - \lambda) y \in \overline{I}_X(x).$$

(ii) $Tx \subset \overline{I}_X(x)$.

(iii) *for each $y \in Tx$, there exists a number λ (as in (i)) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1 - \lambda) y \in X.$$

(iv) $Tx \subset IF_X(x) = \{x + c(u - x) \mid u \in X, \ \operatorname{Re}(c) > 1/2\}$.

(v) $Tx \subset X$.

(vi) $T(X) \subset X$.

For locally convex spaces, Theorem 5.3 for $\mathbb{V}$ instead of $\mathbb{A}_c$ reduces to [**P9**, Theorem 4], which extends well-known earlier theorems of Brouwer [**B**], Schauder [**S**], Tychonoff [**Ty**], Rothe [**R**], Kakutani [**Kk**], Fan [**F1**], Glicksberg [**G**], Halpern [**Hl**], Browder [**B1,4**], Reich [**R1,4,6**], Fitzpatrick and Petryshyn [**FP**], Ha [**H**], and others. For details, see [**P9**]. For topological vector spaces E on which E^* separates points, Theorem 5.3 for $\mathbb{V}$ instead of $\mathbb{A}_c$ reduces to [**P10**, Theorem 5], which includes Fan [**F3**, Corollaire 2], Halpern and Bergman [**HB**, Theorems 4.1 and 4.3], Kaczynski [**Ka**, Théorèmes 1-4], Granas and Liu [**GL2**, Theorem 10.5], and Park [**P1**, Theorems 6 and 8].

From Theorem 4.3, we obtain the following extension of the Himmelberg fixed point theorem:

Theorem 5.4. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E. If $T \in \mathbb{A}_c(X,X)$ is compact, then T has a fixed point.*

If $T \in \mathbb{K}(X,X)$, then Theorem 5.4 reduces to Himmelberg [**Hi**, Theorem 2], which extends earlier works of Schauder [**S**], Mazur [**M**], Bohnenblust and Karlin [**BK**], Hukuhara [**Hu**], Singbal [**Sn**], Tychonoff [**Ty**], Kakutani [**Kk**], Fan [**F1**], and Glicksberg [**G**]. An extended version of Theorem 5.4 for $\mathbb{V}$ instead of $\mathbb{A}_c$ was given in [**P6**, Theorem 7]. Some particular forms can be seen also in [**Bn1**], [**BD2**], [**L2**].

As an application of Theorem 5.4, we obtain the following acyclic version of Reich's theorem on condensing maps with the Leray-Schauder boundary condition. For the definition of condensing multifunctions, see Su and Sehgal [**SS**].

Theorem 5.5. *Let C be a nonempty closed subset of a locally convex Hausdorff topological vector space E and $T : C \to 2^E$ an u.s.c. multifunction with nonempty closed acyclic values. Suppose that T has a bounded range, and that there is a point $w \in \operatorname{Int} C$ such that*

(L-S) *for every $x \in \operatorname{Bd} C$ and $y \in Tx$,*

$$y - w \neq m(x - w) \quad \text{for all} \quad m > 1.$$

Then F has a fixed point if one of the following holds:

(i) *T is compact.*
(ii) *T is condensing and E is quasi-complete.*
(iii) *T is condensing with compact values and C is quasi-complete.*

If T has convex values, then Theorem 5.5 reduces to Reich [**R5**, Theorem]. Reich [**R2**] noted that his earlier version of Theorem 5.5 holds for a metrizable E. However, we do not assume the metrizability of E in Theorem 5.5.

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ORDERING PRINCIPLES AND DROP THEOREMS

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ABSTRACT. We show that generalized drop theorems can be derived from the author's ordring principle. Drop properties of norms of Banach spaces are discussed. We also consider the drop property with respect to another topology in a normed space.

I. Introduction

Let E be a real Hausdorff topological vector space (TVS), $B \subset E$ a convex set and $x \in E \backslash B$. The convex hull of $\{x\} \cup B$ will be called a ***drop*** determined by x and denoted by $D(x, B)$. In 1972, Daneš [3]proved the following "Drop theorem":

Theorem. *Let $(E, \|\cdot\|)$ be a real Banach space, $\overline{B}$ the closed unit ball in E, and A a nonempty closed subset of E such that $inf\{\|a\| \mid a \in A\} > 1$. Then there exists a $v \in A$ such that $D(v, \overline{B}) \cap A = \{v\}$.*

Some generalizations of drop theorem are obtained by Daneš [4], Georgiev [6]and Mizoguchi [11]. Penot [h]obtained a flower petal theorem in complete metric spaces and he showed that drop they are equivalent to Ekeland's variational principle [5].

On the other hand, Kutzarova and Rolewicz [10], Giles, Sims and Yorke [7], and Rolewicz [13]studied the drop property of the norm of a Banach space. Giles, Sims and Yorke [7]and Rolewicz [13]proved that in a Banach space E, the norm has (weak) drop property if and only if every stream in $E \backslash \overline{B}$ contains a (weakly) convergent subsequence if and only if E is reflexive.

Brézis-Browder [1]and Brøndsted [2]observed that Daneš' drop theorem can be derived from an ordering principle. Also, they proved Ekeland's variational principle [5]using their ordering principles. Mizoguchi [11]proved a drop theorem in locally convex spaces using a generalization of Brøndstead's principle. In [8], [9], the author proved a general ordering principle which contains the results of Brézis-Browder [1]and Brøndsted [2]. In this paper, we apply the author's ordering principle in [8], [9]to prove some generalized drop theorems. We also consider the drop property with respect to another topology in a normed space.

II. Main Results

Let X be a nonempty set and $\preccurlyeq$ a quasi-order (that is, a reflexive and transitive relation) on X. We say that $(X, \preccurlyeq)$ is an ***ordered set.*** For $x \in X$, we denote $S(x) = \{y \in X \mid x \preccurlyeq y\}$. X is said to be ***countably inductive (a CIO set)*** if every nondecreasing sequence in X has an upper bound. If A is a well ordered subset of X, A can be indexed by a well ordered set Λ by $A = \{a_\lambda\}_{\lambda \in \Lambda}$ such that $\lambda \leq \omega$ implies $a_\lambda \preccurlyeq a_\omega$. Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers and $\leq$ is the usual order in the set of real numbers, cardinal numbers or ordinal numbers. For any set A, $|A|$ denotes the cardinality of A.

First, we begin with the following basic ordering principle :

Theorem A [8, Theorem I. 2. 1]. *Let $(X, \preccurlyeq)$ be an ordered set and Ω be any nonempty set. Let $\{A_\omega\}_{\omega \in \Omega}$ be a collection of subsets of X. Suppose that*

(A.1) if A is a nonempty well ordered subset of X such that $|A| \leq |\Omega|$, then A has an upper bound, and

(A.2) for any $x \in X$ and $\omega \in \Omega$, there is a $y \in S(x)$ satisfying $S(y) \subset A_\omega$.

Then for any $x \in X$, there is a $v \in S(x)$ such that $S(v) \subset \bigcap_{\omega \in \Omega} A_\omega$.

From Theorem A, we can deduce the following :

Theorem B [9, Theorem 2]. *Let $(X, \preccurlyeq)$ be a CIO set and $d : X \times X \to \mathbb{R}^+$ a function satisfying*

(B. 1) *for any $x \in X$ and $\varepsilon > 0$, there is a $y \in S(x)$ such that $d(z, w) < \varepsilon$ if $y \preccurlyeq z \preccurlyeq w$.*

Then for any $x \in X$, there is a $v \in S(x)$ such that $d(v, w) = 0$ for all $w \in S(v)$.

Let E be a real Hausdorff topological vector space (TVS), $B \neq E$ a convex subset of E and $x \in E \setminus B$. As mentioned above, the convex hull $D(x, B)$ of $\{x\} \cup B$ will be called a *drop* determined by x. We say that B has the *drop property* if for any nonempty closed set A disjoint with B, there exists a point $a \in A$ such that $D(a, B) \cap A = \{a\}$. If E is a normed space and the closed unit ball has the drop property, then we say that the norm of E has the *drop property*. For a given B, if a sequence $\{x_n\}$ satisfies $x_{n+1} \in D(x_n, B) \setminus B$, then we call $\{x_n\}$ a *stream*.

In the same situation, define a relation $\preccurlyeq$ on A by

$$x \preccurlyeq y \Longleftrightarrow y \in D(x, B).$$

Then it is easy to show that $\preccurlyeq$ is a partial order on A and $S(x) = D(x, B) \cap A$ for all $x \in A$. So in the following, we consider A as an ordered set. Note also that a stream in A is merely a nondecreasing sequence in A.

Let p be a seminorm on E, $x \in E$ and $A,\ B \subset E$. We denote

$$p(x, B) = inf\{p(x - b) \mid b \in B\}$$

and

$$p(A, B) = inf\{p(a - b) \mid a \in A \text{ and } b \in B\}.$$

We say that a net $\{x_\lambda\}_{\lambda \in \Lambda}$ is *p-Cauchy* if for every $\varepsilon > 0$, there is a $\lambda_0 \in \Lambda$ such that $p(x_\lambda - x_\omega) < \varepsilon$ for all $\lambda, \omega \geq \lambda_0$. If $p = \|\cdot\|$ is a norm, then we use $d(x, B)$ and $d(A, B)$ instead of $p(x, B)$ and $p(A, B)$.

Theorem 1. *Let E be a real Hausdorff TVS and p a seminorm on E. Let A, B be nonempty subsets of E such that B is convex, p is bounded on B and $p(A,B) > 0$. Then every well ordered subset of A is a p-Cauchy net. Furthermore, if every nondecreasing p-Cauchy sequence in $S(x) = D(x,B) \cap A$ converges in $S(x)$ for all $x \in A$, then for all $x \in A$, there exists a $v \in D(x,B) \cap A$ such that $p(v-w) = 0$ for all $w \in D(v,B) \cap A$.*

Proof. Let $x_0 \in A$, $A_0 = D(x_0,B) \cap A$ and $x \in A_0$. Itis easy to show that p is bounded on $A_0 - B$. So we can choose an $M > 0$ so that $p(x-b) \le M$ for all $x \in A_0$ and $b \in B$. We claim that for all $x \in A_0$ and $y \in S(x) = D(x,B) \cap A$,

$$p(x,B) \ge p(y,B) \text{ and } p(y-x) \le \frac{M}{p(A,B)}(p(x,B) - p(y,B)).$$

Since $y \in D(x,B) \cap A$, there exist $t \in [0,1]$ and $b \in B$ such that $y = (1-t)x + tb$. By the convexity of p,

$$\begin{aligned} p(y,B) &\le (1-t)p(x,B) + tp(b,B). \\ &= (1-t)p(x,B) \\ &\le p(x,B) \end{aligned}$$

And so

$$t \le \frac{p(x,B) - p(y,B)}{p(x,B)}.$$

Since $y - x = t(b-x)$,

$$\begin{aligned} p(y-x) &= tp(b-x) \\ &\le \frac{M}{p(x,B)}(p(x,B) - p(y,B)) \\ &\le \frac{M}{p(A,B)}(p(x,B) - p(y,B)) \end{aligned}$$

Let $\{x_\omega\}_{\omega \in \Omega}$ be a well-ordered subset of A indexed by a well ordered set Ω. Then $\{p(x_\omega, B)\}_{\omega \in \Omega}$ is a bounded decreasing net in the set of real numbers. Since it is a Cauchy net, $\{x_\omega\}_{\omega \in \Omega}$ is also a Cauchy net.

Assume that every nondecreasing p-Cauchy sequence in $S(x)$ converges in $S(x)$ for all $x \in A$. Then every nondecreasing sequence $\{x_n\}$ in A, being Cauchy, converges to a point x_0 in $S(x_1)$. But $\{x_n\}_{n \ge m}$ is contained in $S(x_m)$ and so converges to a point in $S(x_m)$ for each $m \in \mathbb{N}$. Since E is Hausdorff, $x_0 = \lim_{n \to \infty} x_n \in S(x_m)$ for all $m \in \mathbb{N}$. This shows that A is a CIO set. Moreover, by defining $d(x,y) = p(x-y)$, $x, y \in A_0$, the above inequality

$$p(y-x) \leq \frac{M}{p(A,B)}(p(x,B) - p(y,B))$$

for $x \in A_0$, $y \in S(x)$ implies (B. 1). So the conclusion follows from Theorem B.

Theorem A and Theorem 1 can be used to prove the following Mizoguchi's generalization [11]of drop theorem to locally convex spaces.

Theorem 2 [11, Theorem 3]. *Let E be a real locally convex space whose topology is generated by a family $\{p_\lambda\}_{\lambda \in \Lambda}$ of seminorms on E. If A is a complete subset of E and B is a closed, convex and bounded subset of E with $p_\lambda(A,B) > 0$ for any $\lambda \in \Lambda$. Then for all $x \in A$, there is a $v \in D(x,B) \cap A$ such that $D(x,B) \cap A = \{v\}$.*

Proof. For $\lambda \in \Lambda$, let $A_\lambda = \{v \mid p_\lambda(v-w) = 0$ for all $w \in D(v,B) \cap A\}$. By Theorem 1, every well-ordered subset of A is p_λ-Cauchy for all $\lambda \in \Lambda$. So every nondecreasing sequence in $S(x) = D(x,B) \cap A$ is a Cauchy sequence and hence converges in the complete set $S(x)$.

Theorem 1 shows that for any $x \in A$ and $\lambda \in \Lambda$, there is a $y \in S(x)$ such that $y \in A_\lambda$. Note that if $y \in A_\lambda$, then $S(y) \subset A_\lambda$. By Theorem A, for any $x \in A$, there is a $v \in S(x)$ such that $S(v) \subset \bigcap_{\lambda \in \Lambda} A_\lambda$. That is, $D(v,B) \cap A = \{v\}$.

Theorem 3. *Let $(E, \|\cdot\|)$ be a real normed space and τ be any Hausdorff vector topology weaker than norm topology. Let A be a sequentially τ-closed subset of E and B is a norm bounded, convex and sequentially τ-complete subset of E with $d(A,B) > 0$. Then for all $x \in A$, there is a $v \in D(x,B) \cap A$ such that $D(x,B) \cap A = \{v\}$.*

Proof. For each $x \in A$, $S(x) = D(x,B) \cap A$ is sequentially τ-complete. Since τ is weaker than the norm topology, every (norm)-Cauchy sequence in $S(x)$ converges in $S(x)$ for all $x \in A$. The conclusion follows from Theorem 1.

If τ is the norm topology, then Theorem 3 reduces to the generalized drop theorem due to Daneš [4]. In case τ is the weak topology, the following holds :

Corollary. *Let $(E, \|\cdot\|)$ be a real normed space. Let A be a weakly sequentially closed subset of E and B is a norm bounded, convex and weakly sequentially complete subset of E with $d(A, B) > 0$. Then for all $x \in A$, there is a $v \in D(x, B) \cap A$ such that $D(x, B) \cap A = \{v\}$.*

In the situation of Theorem 3, we say that B has the ***τ-drop property*** if for any sequentially τ-closed subset A of E disjoint from B, there is a $v \in D(x, B) \cap A$ such that $D(x, B) \cap A = \{v\}$. And if the closed unit ball has the τ-drop property, then we say that the norm has the ***τ-drop property.***

Theorem 4. *Let $(E, \|\cdot\|)$ be a real normed space and τ be any Hausdorff vector topology weaker than norm topology. Suppose that $\|\cdot\|$ is lower semi-continuous with respect to τ. Then $\|\cdot\|$ has the drop property if and only if every stream in $E \setminus \overline{B}$ has a τ-convergent subsequence, where $\overline{B}$ is the closed unit ball in E.*

Proof. Suppose that there is a stream $\{x_n\}$ in $E \setminus \overline{B}$ which does not have any τ-convergent subsequence. Then $A = \{x_n\}$ is a sequentially τ-closed subset of E. We may assume that A is an infinite set. Since $\{x_n\}_{n \geq m} \subset D(x_m, B) \cap A$, there is no $x_m \in A$ such that $D(x_m, B) \cap A = \{x_m\}$.

Conversely, suppose that every stream in $E \setminus \overline{B}$ has a τ-convergent subsequence and the norm $\| \; \|$ does not have the τ-drop property. Then there is a sequentially τ-closed set A disjoint from $\overline{B}$ such that for each $x \in A$, $d(D(x, \overline{B}) \cap A, \overline{B}) = 0$. (Otherwise, A has a maximal element by Theorem 3.) That is,

$$inf\{\|y\| \mid y \in D(x, \overline{B}) \cap A\} = 1.$$

So we can choose a sequence $\{x_n\}$ in A such that $x_{n+1} \in D(x_n, \overline{B}) \cap A$ and $\lim_{n \to \infty} \|x_n\| = 1$. By assumption, $\{x_n\}$ contains a τ-convergent subsequence $\{x_{n_k}\}$ which converges to a point $x_0 \in A$. Since $\|\cdot\|$ is lower semi-continuous with respect to τ,

$$\|x_0\| \leq \lim_{n \to \infty} \|x_{n_k}\| = 1.$$

So $x_0 \in \overline{B}$, which contradicts the fact that $A \cap \overline{B} = \emptyset$.

If τ is the norm topology, then Theorem 4 reduces to [**12, Proposition 2**]. And if τ is the weak topology, it is a part of [**7, Theorem 5**].

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POSITIVE LINEAR MAPS IN THE THREE-DIMENSIONAL MATRIX ALGEBRA

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1. Introduction.

Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field. The structure of the positive cone $\mathcal{P}(M_n)$ of all positive linear maps between M_n is very complicated even in lower dimensions. In this note, we will discuss various examples of positive linear maps between the 3-dimensional matrix algebra. For interesting examples for the 4-dimensional case, we refer to [**Ro83, Ro85, Ta86**].

We denote by $M_k(M_n)$ the matrix algebra of order k over M_n. For a linear map $\phi : M_n \to M_n$, we define two linear maps ϕ_k and ϕ^k between $M_k(M_n)$ by

$$\phi_k([a_{ij}]_{i,j=1}^k) = [\phi(a_{ij})]_{i,j=1}^k$$
$$\phi^k([a_{ij}]_{i,j=1}^k) = [\phi(a_{ji})]_{i,j=1}^k,$$

for $[a_{ij}] \in M_k(M_n)$. The linear map ϕ is said to be *k-positive* (respectively *k-copositive*) if ϕ_k (respectively ϕ^k) is positive, and ϕ is *completely positive* (*completely copositive*) if ϕ is k-positive (respectively k-copositive) for each positive integer $k = 1, 2, \ldots$. It is well known that $\phi : M_n \to M_n$ is completely positive if and only if ϕ is n-positive, and this is equivalent to the positivity of the matrix $\phi_n([E_{ij}]_{i,j=1}^n)$ in M_{n^2}, where $\{E_{ij} : i, j = 1, 2, \ldots, n\}$ is the usual matrix unit [**Ch75a**]. Similarly $\phi : M_n \to M_n$ is completely copositive if and only if $\phi^n([E_{ij}]_{i,j=1}^n)$ is a positive matrix. It is also known that every completely positive linear map is of the form

$$A \mapsto \sum_i V_i^* A V_i, \qquad A \in M_n,$$

with $V_i \in M_n$, and similarly for completely copositive linear maps.

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Recall that a linear map ϕ is said to be *decomposable* if ϕ is the sum of a completely positive map and a completely copositive map. The notion of decomposability is closely related with the question whether a real biquadratic form can be written as the sum of squares of linear forms [**Ch75b, CL77**]. Although it is known that every positive linear map between M_2 is decomposable [**St63, Wo76**], this is not the case for M_n with $n \geq 3$. We will see that there are even positive linear maps which are not the sums of 2-positive linear maps and 2-copositive linear maps.

2. Variants of Choi's example

One of interesting examples may be obtained by adjusting diagonal entries and attaching minus signs to other entries, which was initiated by Choi [**Ch72**]. For nonnegative real numbers a, b and c, we define the linear map $\Psi[a, b, c]$ (denoted by just Ψ if there is no confusion) by

$$\Psi[a, b, c](x) = \Psi_1[a, b, c](x) - x,$$

where

$$\Psi_1[a, b, c]((x_{ij})) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & 0 & 0 \\ 0 & ax_{22} + bx_{33} + cx_{11} & 0 \\ 0 & 0 & ax_{33} + bx_{11} + cx_{22} \end{pmatrix},$$

for each $(x_{ij}) \in M_3$. $\Psi[2, 2, 2]$ is just the Choi's example mentioned above. This was the first example of 2-positive linear map which is not completely positive. Choi also showed that $\Psi[2, 0, \mu]$ with $\mu \geq 1$ is not decomposable [**Ch80**]. From the following result [**CKL**], we have plenty of examples of indecomposable maps, positive maps which are not 2-positive, 2-positive maps which are not completely positive.

Theorem 2.1.

(1) *The linear map $\Psi[a, b, c]$ is positive if and only if the following three conditions are satisfied;*

$$a \geq 1,$$
$$a + b + c \geq 3,$$
$$bc \geq (2 - a)^2 \quad if \quad 1 \leq a \leq 2,$$

if and only if the inequality

$$\frac{\alpha}{a\alpha + b\beta + c\gamma} + \frac{\beta}{a\beta + b\gamma + c\alpha} + \frac{\gamma}{a\gamma + b\alpha + c\beta} \le 1$$

holds for every positive real numbers α, β, γ.

(2) *The linear map* $\Psi[a,b,c]$ *is completely positive if and only if the following condition is satisfied:*

$$a \ge 3.$$

(3) *The linear map* $\Psi[a,b,c]$ *is completely copositive if and only if it is 2-copositive if and only if the following conditions are satisfied:*

$$a \ge 1, \quad bc \ge 1.$$

(4) *The linear map* $\Psi[a,b,c]$ *is decomposable if and only if the following two conditions are satisfied:*

$$a \ge 1,$$
$$bc \ge \left(\frac{3-a}{2}\right)^2 \quad \text{if} \quad 1 \le a \le 3.$$

(5) *The linear map* $\Psi[a,b,c]$ *is 2-positive if and only if*

$$a \ge 3 \qquad \text{or}$$
$$2 \le a < 3, \quad bc = (3-a)(b+c) > 0.$$

The second condition of (5) follows from the inequality

$$\frac{b\alpha + c\gamma}{ab\alpha + bc\beta + ac\gamma} + \frac{b\beta + c\alpha}{ab\beta + bc\gamma + ca\alpha} + \frac{b\gamma + c\beta}{ab\gamma + bc\alpha + ca\beta} \le 1,$$

which holds for every nonnegative real numbers α, β and γ under the condition, whenever not all of them are zero.

Remark. It would be an interesting result if we know the conditions for Schwarz inequalities. When $a = b = c$, Ψ satisfies the Schwarz inequality if and only if $a \ge \frac{4}{3}$, and $\Psi \otimes I_2$ satisfies the Schwarz inequality if and only if $a \ge \frac{7}{3}$.

Because $\Psi[2,0,1]$ is extremal but not 2-positive, it follows that it can not be even expressed as the sum of 2-positive and 2-copositive maps. Tomiyama **[TT88]** showed this directly and called such a map as an *atom*.

In order to find another examples of atoms, we consider **[Ky]** the following maps motivated by **[Os1]**: For nonnegative real numbers a, c_1, c_2 and c_3, we define the linear map $\Theta[a; c_1, c_2, c_3]$ from M_3 into M_3 by

$$\Theta[a; c_1, c_2, c_3](x_{ij}) = \begin{pmatrix} ax_{11} + c_1x_{33} & 0 & 0 \\ 0 & ax_{22} + c_2x_{11} & 0 \\ 0 & 0 & ax_{33} + c_3x_{22} \end{pmatrix} - (x_{ij})$$

for each $(x_{ij}) \in M_3$. Note that $\Theta[a; c, c, c] = \Psi[a, 0, c]$.

Theorem 2.2.

(1) *The linear map $\Theta[a; c_1, c_2, c_3]$ is positive if and only if the following two conditions are satisfied:*

$$a \geq 2,$$
$$c_1c_2c_3 \geq (3-a)^3$$

(2) *The linear map $\Theta[a; c_1, c_2, c_3]$ is completely positive if and only if it is 2-positive if and only if the following condition is satisfied:*

$$a \geq 3.$$

(3) *For the positive real numbers a, c_1, c_2 and c_3 satisfying the conditions:*

$$2 \leq a < 3 \qquad \text{and} \qquad c_1c_2c_3 \geq (3-a)^3,$$

the maps $\Theta[a; c_1, c_2, c_3]$ are atomic positive linear maps between M_3.

Remark. It is easy to see that the map $\mathbb{R}^4 \to \mathcal{P}(M_3)$ given by

$$(a, c_1, c_2, c_3) \mapsto \Theta[a; c_1, c_2, c_3]$$

is an affine map. Therefore, the map $\Theta[a; c_1, c_2, c_3]$ is not extremal if $a > 2$ or $c_1c_2c_3 > (3-a)^3$. Recently, Osaka **[Os2]** showed that $\Theta[2; c_1, c_2, c_3]$, with $c_1c_2c_3 = 1$ is extremal using the theory of biquadratic forms.

3. Another examples.

In this section, we consider positive linear maps in 3-dimensional matrix algebras which fix diagonal elements. The identity map and the tranpose map are, of course, typical examples of this type. It is easy to see that such maps are of the forms;

$$\Phi_P : (x_{ij}) \mapsto \begin{pmatrix} x_{11} & \alpha_1 x_{12} + \alpha_2 x_{21} & \bar{\beta}_1 x_{13} + \bar{\beta}_2 x_{31} \\ \bar{\alpha}_1 x_{21} + \bar{\alpha}_2 x_{12} & x_{22} & \gamma_1 x_{23} + \gamma_2 x_{32} \\ \beta_1 x_{31} + \beta_2 x_{13} & \bar{\gamma}_1 x_{32} + \bar{\gamma}_2 x_{23} & x_{33} \end{pmatrix},$$

for $(x_{ij}) \in M_3$. Note that the map Φ_P is determined by a point $P = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$ in $\mathbb{C}^6$. Because the correspondence $P \mapsto \Phi_P$ is an affine isomorphism, we will confuse them in some cases.

For these linear maps, they are completely positive (respectively completely copositive) if and only if they are 2-positive (respectively 2-copositive). Also, they are decomposable if and only if they are the sums of 2-positive and 2-copositive linear maps. In order to state more explicitly, we introduce the following two linear maps

$$\Phi_i(x_{ij}) = \begin{pmatrix} a_i x_{11} & \alpha_i x_{12} & \bar{\beta}_i x_{13} \\ \bar{\alpha}_i x_{21} & b_i x_{22} & \gamma_i x_{23} \\ \beta_i x_{31} & \bar{\gamma}_i x_{32} & c_i x_{33} \end{pmatrix}, \quad i = 1, 2,$$

where a_i, b_i and c_i are nonnegative real numbers. We also denote by

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Proposition 3.1.

(1) *The linear map Φ is completely positive if and only if it is 2-positive if and only if $\alpha_2 = \beta_2 = \gamma_2 = 0$ and the matrix $\Phi_1(J)$, with $a_1 = b_1 = c_1 = 1$, is semi-definite positive.*

(2) *The linear map Φ is completely copositive if and only if it is 2-copositive if and only if $\alpha_1 = \beta_1 = \gamma_1 = 0$ and the matrix $\Phi_2(J)$, with $a_2 = b_2 = c_2 = 1$, is semi-definite positive.*

Proposition 3.2. *For the linear map Φ, the following are equivalent:*

(i) *Φ is decomposable.*

(ii) *Φ is the sum of a 2-positive linear map and a 2-copositive linear map.*

(iii) *There exist real numbers a_1, a_2, b_1, b_2, c_1 and c_2 such that*

$$a_1 + a_2 = 1, \quad b_1 + b_2 = 1, \quad c_1 + c_2 = 1,$$
$$\Phi_1(J) \geq 0, \qquad \Phi_2(J) \geq 0$$

In order to characterize the positivity of Φ, we denote by R the projection onto the one-dimensional subspace spanned by the vector $(x, y, z) \in \mathbb{C}^3$. Then Φ is positive if and only if the matrix

$$\Phi(R) = \begin{pmatrix} |x|^2 & \alpha_1\bar{x}y + \alpha_2\bar{y}x & \bar{\beta_1}\bar{x}z + \bar{\beta_2}\bar{z}x \\ \bar{\alpha_1}\bar{y}x + \bar{\alpha_2}\bar{x}y & |y|^2 & \gamma_1\bar{y}z + \gamma_2\bar{z}y \\ \beta_1\bar{z}x + \beta_2\bar{x}z & \bar{\gamma_1}\bar{z}y + \bar{\gamma_2}\bar{y}z & |z|^2 \end{pmatrix} \tag{3.1}$$

is positive for every $(x, y, z) \in \mathbb{C}^3$. Considering 2×2-submatrices, we have

$$|x|^2|y|^2 - |\alpha_1\bar{x}y + \alpha_2\bar{y}x|^2 \geq 0, \qquad x, y \in \mathbb{C}.$$

This is equivalent to the condition

$$|\alpha_1| + |\alpha_2| \leq 1. \tag{3.2}$$

Similarly, we have

$$|\beta_1| + |\beta_2| \leq 1, \qquad |\gamma_1| + |\gamma_2| \leq 1. \tag{3.3}$$

Also, if we consider the determinant of the matrix (3.1), it follows that the inequality

$$\begin{aligned} &|z|^2|\alpha_1\bar{x}y + \alpha_2\bar{y}x|^2 + |y|^2|\beta_1\bar{z}x + \beta_2\bar{x}z|^2 + |x|^2|\gamma_1\bar{y}z + \gamma_2\bar{z}y|^2 \\ &\qquad \leq |x|^2|y|^2|z|^2 + 2\operatorname{Re}(\alpha_1\bar{x}y + \alpha_2\bar{y}x)(\beta_1\bar{z}x + \beta_2\bar{x}z)(\gamma_1\bar{y}z + \gamma_2\bar{z}y) \end{aligned}$$

holds for every $x, y, z \in \mathbb{C}$. Because this inequality is trivial if one of x, y, z is zero, we divide by $|x|^2|y|^2|z|^2$, and may assume that $|x| = |y| = |z| = 1$. Hence, the above inequality is equivalent to the following condition;

$$\begin{aligned} &|\alpha_1 + \alpha_2 e^{i\theta}|^2 + |\beta_1 + \beta_2 e^{i\sigma}|^2 + |\gamma_1 + \gamma_2 e^{i\tau}|^2 \\ &\qquad \leq 1 + 2\operatorname{Re}(\alpha_1 + \alpha_2 e^{i\theta})(\beta_1 + \beta_2 e^{i\sigma})(\gamma_1 + \gamma_2 e^{i\tau}), \end{aligned} \tag{3.4}$$

with $\theta + \sigma + \tau = 0$. Summing up, we have;

Proposition 3.3. *The linear map Φ is positive if and only if the conditions (3.2), (3.3) and (3.4) hold for any θ, σ, τ with $\theta + \sigma + \tau = 0$.*

Because the condition (3.4) is not so easy to apply, we will give an intrinsic characterization of positivity in the case of

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}. \tag{3.5}$$

This condition is not so restrictive because if Φ_P is positive then $\Phi_{\tilde{P}}$ is also positive, where $\tilde{P} \in \mathbb{R}^6$ is obtained by taking the real parts of each entry of $P \in \mathbb{C}^6$. Under this condition (3.5), our problem becomes to find the maximum value of

$$A\cos\theta + B\cos\sigma + C\cos\tau$$

under the constraint

$$\theta + \sigma + \tau = 0,$$

where A, B and C are real numbers. Now, we put

$$\begin{aligned}
A &= \alpha_1\alpha_2 - \alpha_1\beta_2\gamma_2 - \alpha_2\beta_1\gamma_1, \\
B &= \beta_1\beta_2 - \alpha_2\beta_1\gamma_2 - \alpha_1\beta_2\gamma_1, \\
C &= \gamma_1\gamma_2 - \alpha_2\beta_2\gamma_1 - \alpha_1\beta_1\gamma_2, \\
D &= 1 + 2(\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2) - (\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + \gamma_1^2 + \gamma_2^2).
\end{aligned}$$

and denote by Δ the set of all triplet (p, q, r) of nonnegative real numbers satisfying

$$p \le q + r, \qquad q \le r + p, \qquad r \le p + q.$$

Theorem 3.4. *Under the condition (3.5), the linear map Φ is positive if and only if the conditions (3.2) and (3.3) together with the following conditions are satisfied;*

$$\begin{aligned}
&ABC \ge 0 \\
&\quad \Longrightarrow 2(|A| + |B| + |C|) \le D, \\
&ABC < 0, \ (|AB|, |BC|, |CA|) \in \Delta \\
&\quad \Longrightarrow -\frac{A^2B^2 + B^2C^2 + C^2A^2}{ABC} \le D. \\
&ABC < 0, \ (|AB|, |BC|, |CA|) \notin \Delta \\
&\quad \Longrightarrow 2\max\{|A| + |B| - |C|, |A| - |B| + |C|, -|A| + |B| + |C|\} \le D.
\end{aligned}$$

Because the conditions for decomposability envolves cubic or biquadratic curves on the plane, it seems to be very difficult to find an intrinsic conditions for decomposability. Especially, the author was unable to determine whether every positive linear map Φ is decomposable or not. Instead, we give a geometric characterization for unique decomposability.

We say that a decomposable linear map Φ is *uniquely decomposable* if any two decompositions $\Phi = \Phi_1 + \Phi_2$ and $\Phi = \Phi_1' + \Phi_2'$ by completely positive and completely copositive maps lead to $\Phi_1 = \Phi_1'$ and $\Phi_2 = \Phi_2'$. We denote by Δ_Φ the set of all $(a, b, c) \in \mathbb{R}^3$ such that the numbers

$$a_1 = a,\ a_2 = 1 - a,\ b_1 = b,\ b_2 = 1 - b,\ c_1 = c,\ c_2 = 1 - c$$

satisfies the conditions in (iii) of Proposition 3.2. It is clear that Φ is uniquely decomposable if and only if Δ_Φ consists of just one point. It is also clear that Δ_Φ is a convex body in $\mathbb{R}^3$. By examining the shape of Δ_Φ, We characterize the unique decomposability. Now, we denote by $\mathcal{D}$ the conex set of all $P \in \mathbb{C}^6$ such that Φ_P is decomposable, and say that Φ_P is on the *boundary* of $\mathcal{D}$ if

$$\sup\{k : \Phi_{kP} \in \mathcal{D}\} = 1.$$

While Δ_Φ is a one point or a straight line in the case that the equalities hold in (3.2) or (3.3), it may be shown that the set Δ_Φ consists of one point or is a three dimensional body if the strict inequality hold in (3.2) and (3.3). From this, we have

Theorem 3.5. *Assume that the strict inequalities hold in (3.2) and (3.3). Then Φ_P is on the boundary of $\mathcal{D}$ if and only if Φ_P is uniquely decomposable.*

We close this note by examining several examples to exhibit how Theorems 3.4 and 3.5 work.

Example 3.6. First, we consider the case

$$\alpha_1 = \beta_1 = \gamma_1 = s \in \mathbb{R}, \qquad \alpha_2 = \beta_2 = \gamma_2 = t \in \mathbb{R}.$$

In this case, we have $A = B = C = st(1 - s - t)$, and if $st \geq 0$ then the first condition of Theorem 3.4 becomes $2s + 2t + 1 \geq 0$. If $st < 0$ then the second condition becomes $(2s - t + 1)(-s + 2t + 1) \geq 0$. Considering the condition (3.2), we see that the region for positivity of Φ is given by the convex body on st-plane determined by the four points $(1, 0)$, $(0, 1)$, $(-\frac{1}{2}, 0)$ and $(0, -\frac{1}{2})$. In this case, it is also clear that every positive linear map is decomposable. Note that the point $(1, 0)$ and $(0, 1)$ represent the identity map and the transpose map, respectively. Also, the point $(-\frac{1}{2}, 0)$ represents the linear $\frac{1}{2}\Psi[3, 0, 0]$ in Section 2.

Example 3.7. Now, we consider the case

$$\alpha_1 = \alpha_2 = \alpha \in \mathbb{R}, \quad \beta_1 = \beta_2 = \beta \in \mathbb{R}, \quad \gamma_1 = \gamma_2 = \gamma \in \mathbb{R}.$$

In this case, we have $A = \alpha(\alpha - 2\beta\gamma)$, $B = \beta(\beta - 2\gamma\alpha)$ and $C = \gamma(\gamma - 2\alpha\beta)$. First, assume that $ABC \geq 0$. If two of A, B and C are negative and one is nonnegative, it is easy to see that the condition (3.2) or (3.3) is violated by the case-by-case investigation. So, we have $A, B, C \geq 0$, and the first condition becomes

$$\alpha^2 + \beta^2 + \gamma^2 \leq \frac{1}{4} + 4\alpha\beta\gamma. \tag{3.6}$$

Now, we consider the case $ABC < 0$. Because the second condition becomes

$$\frac{(4\alpha^2\beta^2\gamma^2 + \alpha\beta\gamma - \alpha^2\gamma^2 - \beta^2\gamma^2 - \gamma^2\alpha^2)^2}{ABC} \geq 0,$$

we need not consider the case $(|AB|, |BC|, |CA) \in \Delta$. Finally, the numbers in the left side of the third condition are among

$$A + B + C, \quad A - B - C, \quad -A + B - C, \quad -A - B + C.$$

By a calulation, we see that the third condition becomes (3.6) again, or vacuous. Hence, the linear map Φ is positive if and only if the condition (3.6) is satisfied. In this case, every positive linear map is decomposable with $a_i = b_i = c_i = \frac{1}{2}$ in Proposition 3.2, and Φ is uniquely decomposable if and only if the equality holds in (3.6).

Example 3.8. We also consider the case

$$\alpha_1 = \beta_2 = \gamma_1 = r \in \mathbb{R}, \qquad \alpha_2 = \beta_1 = \gamma_2 = s \in \mathbb{R},$$

in which the third condition of Theorem 3.4 is should be considered. In this case, we have

$$A = C = rs(1 - r - s), \quad B = rs - r^3 - s^3 := f(r, s).$$

First, note that

$$D + \frac{A^2B^2 + B^2C^2 + C^2A^2}{ABC} = \frac{g(r, s)g(s, r)}{f(r, s)},$$

where

$$g(r,s) = 2r^3 + s^3 - rs^2 + r^2 + s^2 - rs - r.$$

Case 1. $rs \geq 0$, $f(r,s) \geq 0$. It follows that $A, B, C \geq 0$ and the first condition becomes $2r + 2s \geq 1$.

Case 2. $rs = 0$, $f(r,s) \leq 0$. The first condition is $1 - 3s^2 - 2s^3 \geq 0$ or $1 - 3r^2 - 2s^3 \geq 0$ depending on $r = 0$ or $s = 0$. So, we have $r \leq \frac{1}{2}$ and $s \leq \frac{1}{2}$.

Case 3. $rs < 0$, $f(r,s) > 0$. We have $A, C < 0$ and $B > 0$. Then the first condition is

$$h(r,s) = 2(r-s)^2 - (r+s) - 1 \leq 0.$$

Case 4. $rs > 0$, $f(r,s) < 0$. $(|AB|, |BC|, |CA|) \in \Delta$ if and only if

$$k(r,s) = 2r^3 + 2s^3 + r^2 s + rs^2 - 3rs \geq 0.$$

In this case, the second condition is $g(r,s)g(s,r) \leq 0$. If $k(r,s) < 0$ then the third condition is

$$\ell(r,s) = 2(r-s)^2 + (r+s) - 1 \leq 0.$$

Case 5. $rs < 0$, $f(r,s) < 0$. $(|AB|, |BC|, |CA|) \in \Delta$ if and only if

$$m(r,s) = 2r^3 + 2s^3 - r^2 s - rs^2 - rs \geq 0,$$

then we have $g(r,s)g(s,r) \leq 0$ as above. If $m(r,s) < 0$ then we have

$$h(r,s) \leq 0, \qquad \ell(r,s) \leq 0.$$

Summing up, we have the picture of the region for positivity on the rs-plane. Note that the three curves $g(r,s) = 0$, $h(r,s) = 0$ and $m(r,s) = 0$ intersect at the common point, at which g and h are tangent to each other.

Now, we show that every positive map of this type is decomposable. Note that this is clear when $r \leq 0, s \leq 0$. So, we may assume that

$$s > 0, \qquad s \geq r,$$

by the symmetry between r and s. In this case, the boundary consists of $g(r,s) = 0$ and $h(r,s) = 0$ and the region of positivity is divided by the following two cases:

Case 1. $g(r,s) = 0$. In this case, we have

$$\Delta_\Phi = \{(s, \frac{r^2}{s}, s)\}.$$

Case 2. $h(r,s) = 0$, $m(r,s) < 0$. In this case, we parametrize the curve $h(r,s) = 0$ as follows;

$$r = \frac{2x^2 + x - 1}{2}, \qquad s = \frac{2x^2 - x - 1}{2}.$$

Then we have

$$-1 \leq x \leq \frac{1}{2}, \quad 2x^3 - x^2 + 1 \geq 0, \quad 2x^3 + x^2 - 1 \leq 0.$$

We put

$$a = c = \frac{4x^3 - x + 1}{2}, \qquad b = \frac{2x^2 + x - 1}{2x}.$$

Then, we have $\Delta_\Phi = \{(a,b,c)\}$.

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ON THE DISTANCE BETWEEN THE UNITARY ORBITS OF TWO HERMITIAN FUNCTIONALS ON A SEMIFINITE FACTOR

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§1. Introduction

Let M be a von Neumann algebra, M_* its predual, $M_*^h = \{\varphi \in M_* : \varphi^* = \varphi\}$, $M_*^+ = \{\varphi \in M_*^h : \varphi \geq 0\}$, and $U(M)$ = the group of all unitary elements in M. For $u \in U(M)$, $\varphi \in M_*$, we define $u\varphi u^* \in M_*$ by

$$(u\varphi u^*)(x) = \varphi(u^* x u), \quad \text{for all} \quad x \in M.$$

The *unitary orbit* of $\varphi \in M_*$ means the set $\{u\varphi u^* : u \in U(M)\}$. As an extension of the notion in (§2[5]), we say that $\varphi, \psi \in M_*$ are *equivalent*, $\varphi \sim \psi$, if ψ is in the norm closure of the unitary orbit of φ. It is immediate to see that this relation $\sim$ is a true equivalence relation in M_*.

Let $[\varphi]$ denote the equivalence class containing $\varphi \in M_*$. The quotient set $M_*/\sim$ is a metric space with the metric

$$d([\varphi], [\psi]) = \inf\{\|\varphi' - \psi'\| : \varphi' \sim \varphi,\ \psi' \sim \psi\}.$$

Because M_*^h and M_*^+ are norm closed subsets of M_* invariant under the action $(u, \varphi) \in U(M) \times M_* \to u\varphi u^* \in M_*$, we see that both $M_*^h/\sim$ and $M_*^+/\sim$ inherit the metric d.

Lemma 2.1 [5] and its proof is translated verbatim to $M_*/\sim$ and $M_*^h/\sim$.

Note: After finishing this article, some of results (e.g. Propositions 6,20) have been generalized. See [13].

Lemma 1. *Under the metric d, both $M_*/\sim$ and $M_*^h/\sim$ are complete metric spaces.*

We regard M_*^h as a real vector space also equipped with the natural order structure determined by its positive cone M_*^+.

§2. A lower estimate for the distance between two elements in $M_*^h/\sim$

Throughout the section, let M be a semifinite factor on a Hilbert space H, with separable predual space M_*, and τ_0 a fixed normal faithful semifinite tracial weight on M^+. When M is of type I with a minimal projection e, or type II_1, we assume that $\tau_0(e) = 1$ or $\tau_0(1) = 1$ respectively.

Let P_f denote the set of all finite projections in M and put

$$J = \tau_0(P_f).$$

Let $\mathcal{M}$ denote the $*$-algebra of all τ_0-measurable operators affiliated with M ([11] p.117. Theorem 5.3). It is known that τ_0 is extended on $\mathcal{M}^+$ by

$$\tau_0(T) = \lim_{\epsilon \to 0+} \tau_0(T(1 + \epsilon T)^{-1})$$

for every $T \in \mathcal{M}_+$, where $\mathcal{M}_+$ denotes the set of all positive operators in $\mathcal{M}$ ([11] p.121 (5.11)).

Consider

$$L^1(M, \tau_0) = \{T \in \mathcal{M} : \|T\|_1 < \infty\},$$

where $\|T\|_1 = \tau(|T|)$ $(T \in \mathcal{M})$. By Theorem 5.10 ([11] p.121), we see that $L^1(M, \tau_0)$ is a Banach space with the norm $\|\cdot\|_1$, in which $M \cap L^1(M, \tau_0)$ is dense, that τ_0 is extended as a bounded linear functional on $L^1(M, \tau_0)$, and that $(x, T) \in M \times L^1(M, \tau_0) \to \tau_0(xT) \in \mathbb{C}$ gives rise to an isometric linear $*$-isomorphism of M_* onto $L^1(M, \tau_0)$. The restriction of this canonical identification to M_*^h is an order-isomorphism of M_*^h onto $L^1(M, \tau_0)^h$, where $L^1(M, \tau_0)^h$ denotes the set of all hermitian elements in $L^1(M, \tau_0)$ ([6] p.63 Proposition 4.2, p.71 Theorem 5.12. Or see [9] p.65 Proposition 4.5, p.67 Theorem 4.10 (V)).

For every $\varphi \in M_*$, we define the *Radon-Nikodym derivative* $\frac{d\varphi}{d\tau_0} \in L^1(M, \tau_0)$ of φ, as the element in $L^1(M, \tau_0)$ to which φ is mapped under the canonical isomorphism $M_* \to L^1(M, \tau_0)$. Thus $\frac{d\varphi}{d\tau_0}$ is a closed (densely defined) operator affiliated with M such that $\tau_0(\chi_{(a,\infty)}(|\frac{d\varphi}{d\tau_0}|)) < \infty$ for all $a \in (0, \infty)$, or equivalently $\lim_{a\to\infty} \tau_0(\chi_{(a,\infty)}(|\frac{d\varphi}{d\tau_0}|)) = 0$ ([11] p.120 Corollary, [12] Chap.I, Proposition 21). Note that $\frac{d\varphi}{d\tau_0} \in L^1(M,\tau_0)^h$, if $\varphi \in M_*^h$.

For any selfadjoint (densely defined) operator T in a Hilbert space, if we put

$$T_+ = T\chi_{(0,\infty)}(T),$$
$$T_- = -T\chi_{(-\infty,0)}(T)$$

then T_+, T_- are also selfadjoint (densely defined) operator such that $\mathcal{D}(T) = \mathcal{D}(T_+) \cap \mathcal{D}(T_-)$, where $\mathcal{D}(\cdot)$ denotes the domain. When, in particular, T is affiliated with M, so are T_+ and T_-. Anyway, we can write

$$T = T_+ - T_-, \quad \text{and} \quad s(T_+)\perp s(T_-),$$

where $s(\cdot)$ denotes the support. If, in addition, $T \in \mathcal{M}$, then this expression of T is unique under such construints, where the substraction in $T_+ - T_-$ is carried out in the sense of the substraction in $\mathcal{M}$.

Lemma 2. *For any (densely defined) selfadjoint operator T on a Hilbert space, we have that, for every $a \in (0,\infty)$,*

$$\chi_{(a,\infty)}(T) = \chi_{(a,\infty)}(T_+),$$
$$\chi_{(-\infty,-a)}(T) = \chi_{(a,\infty)}(T_-)$$

Proof. To get the first equality, we apply the Borel function calculus to the equation $g \circ h = \chi_{(a,\infty)}$, where $h(t) = t\chi_{(0,\infty)}(t)$, $g(s) = \chi_{(a,\infty)}(s)$ $(t, s \in \mathbb{R})$. The second one in the lemma is proven similarly.

Let $\varphi \in M_*^h$. As an extension of the notion for the case M_*^+ ([5] Lemma 4.2), we define $f_\varphi : (0,\infty) \to J - J$ by

$$f_\varphi(a) = \tau_0(\chi_{(a,\infty)}(\frac{d\varphi}{d\tau_0}) - \chi_{(-\infty,-a)}(\frac{d\varphi}{d\tau_0})) \quad (a \in (0,\infty)),$$

which is right continuous.

Here if we consider the Jordan decomposition of φ, $\varphi = \varphi_+ - \varphi_-$, then $\frac{d\varphi}{d\tau_0} = \frac{d\varphi_+}{d\tau_0} - \frac{d\varphi_-}{d\tau_0}$ is regarded as the "Jordan decomposition of $\frac{d\varphi}{d\tau_0}$" in $L^1(M,\tau_0)^h$, since $s(\frac{d\varphi_+}{d\tau_0}) = s(\varphi_+)$, $s(\frac{d\varphi_-}{d\tau_0}) = s(\varphi_-)$ ([9] p.65 (1)).

Thus

$$\left(\frac{d\varphi}{d\tau_0}\right)_+ = \frac{d\varphi_+}{d\tau_0},$$
$$\left(\frac{d\varphi}{d\tau_0}\right)_- = \frac{d\varphi_-}{d\tau_0}.$$

Consequently, by Lemma 2,

$$\chi_{(a,\infty)}\left(\frac{d\varphi}{d\tau_0}\right) = \chi_{(a,\infty)}\left(\frac{d\varphi_+}{d\tau_0}\right),$$
$$\chi_{(-\infty,-a)}\left(\frac{d\varphi}{d\tau_0}\right) = \chi_{(a,\infty)}\left(\frac{d\varphi_-}{d\tau_0}\right).$$

Consequently, for every $a \in (0,\infty)$, $f_\varphi(a) = f_{\varphi_+}(a) - f_{\varphi_-}(a)$.

We have to prepare some additional facts taken mostly from §3 and §4 in [5].

Since $\sigma_t^{\tau_0} = \iota_M$ for all $t \in \mathbb{R}$, where ι_M is the identity automorphism of M, the *crossed product* $M \underset{\sigma^{\tau_0}}{\times} \mathbb{R}$ that will be denoted by N, is the von Neumann algebra on $L^2(\mathbb{R}, H)$ generated by $\{\pi(x), \lambda(t); x \in M,\ t \in \mathbb{R}\}$, where $(\pi(x)\xi)(s) = x(\xi(s))$, $(\lambda(t)\xi)(s) = \xi(s-t)$, where $\xi \in L^2(\mathbb{R}, H)$, so that

$$\lambda(t)\pi(x)\lambda(-t) = \pi(x).$$

The *dual action* $\{\theta_s : s \in \mathbb{R}\}$ of σ^{τ_0} ([10] p.257 Definition 4.1) is determined by following conditions;

$$\theta_s(\pi(x)) = \pi(x)$$
$$\theta_s(\lambda(s)) = e^{-ist}\lambda(t)$$

$(x \in M,\ s, t, \in \mathbb{R})$.

By [4], there is a faithful normal semifinite operator volued weight W from the extended positive part $\widehat{N_+}$ onto that $\widehat{\pi(M)}_+$ of $\pi(M)$ determined by

$$W(y) = \int_{-\infty}^{\infty} \theta_s(y)ds \quad (y \in N^+),$$

where ds denotes the Lebesgue measure on $\mathbb{R}$.

For any normal semifinite weight φ on M^+, its dual weight $\tilde{\varphi}$ on N^+ ([3] p.112 Definition 3.1) is given by

$$\tilde{\varphi} = \varphi \circ \pi^{-1} \circ W$$

([4] p.131 Theorem 3.1 (d) or p.138 Corollary 3.6)

On the other hand, there exists a positive selfadjoint operator h affiliated with N such that

$$\lambda(t) = h^{it} \quad (t \in \mathbb{R}),$$

and the weight τ on N^+ defined by

$$\tau(y) = \tilde{\tau}_0(h^{-1}y) \quad (y \in N^+)$$

is a faithful normal semifinite trace on N satisfying

$$\tau \circ \theta_s = e^{-s}\tau \quad (s \in \mathbb{R})$$

([10] p.282 Lemma 8.2). τ is called the *canonical trace* on N.

Now let $\varphi \in M_*$. By considering the Cartesian decomposition first, $\varphi = \varphi_1 + i\varphi_2$, $\varphi_1, \varphi_2 \in M_*^h$, and then by considering $((\varphi_i)_+)^{\sim}$, $((\varphi_i)_-)^{\sim}$ $(i = 1, 2)$, we can define $\frac{d\tilde{\varphi}}{d\tau}$ as a τ-measurable operator affiliated with N ([12] Chap. II, Corollary 6) by

$$\frac{d\tilde{\varphi}}{d\tau} = \frac{d((\varphi_1)_+)^{\sim}}{d\tau} - \frac{d((\varphi_1)_-)^{\sim}}{d\tau} + i\Big(\frac{d((\varphi_2)_+)^{\sim}}{d\tau} - \frac{d((\varphi_2)_-)^{\sim}}{d\tau}\Big),$$

since N is also a semifinite von Neumann algebra with τ ([11] §5. pp.114–128). Here the operations in the right hand side of the above expression is carried out in the $*$-algebra of all τ-measurable operators.

On the other hand, let $L^\infty(\mathbb{R}, M)$ denote the Banach space of all M-valued essentially bounded Lebesgue measurable functions with respect to the σ-weak topology on M ([1] p.183 Proposition 11). Then $L^\infty(\mathbb{R}, M)$ becomes a von Neumann algebra by making it act on $L^2(\mathbb{R}, H)$, as

$$(a\xi)(t) = a(t)(\xi(t))$$

$(a \in L^\infty(\mathbb{R}, M),\ \xi \in L^2(\mathbb{R}, H),\ t \in \mathbb{R})$.

For $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, let $\tilde{g}$ denote the inverse Fourier transform of g,

$$\tilde{g}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)e^{its}ds \quad (t \in \mathbb{R}).$$

There is a Hilbert space isomorphism of $L^2(\mathbb{R}, H)$ onto itself determined by sending every $g(\cdot)\xi_0$ to $\tilde{g}(\cdot)\xi_0$ $(g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}),\ \xi_0 \in H)$, under which $\pi(x)$, $\lambda(t) \in N$ are transformed to $(s \in \mathbb{R} \to x)$, $e^{it(\cdot)}1 \in L^\infty(\mathbb{R}, M)$ respectively, where $x \in M$, $t \in \mathbb{R}$ and 1 is the identity operator in M. Also the center $Z(N)$ of N is transformed to $L^\infty(\mathbb{R})$ $(= L^\infty(\mathbb{R})1)$.

The next Proposition generalizes Lemma 4.2 of [5].

Proposition 3. *Let $\varphi, \psi \in M_*^h$. Then,*

(i) $\varphi(1) = \int_0^\infty f_\varphi(a)da$.
(ii) *If* $\varphi \leq \psi$ *then* $f_\varphi \leq f_\psi$.
(iii) $d([\varphi], [\psi]) \geq \int_0^\infty |f_\varphi(a) - f_\psi(a)|da$

Proof. For any $t \in \mathbb{R}$, we see that

$$t = \int_{a \in (0,\infty)} [\chi_{(a,\infty)}(t) - \chi_{(-\infty,-a)}(t)]da,$$

so that

$$\frac{d\varphi}{d\tau_0} = \int_0^\infty \left[\chi_{(a,\infty)}(\frac{d\varphi}{d\tau_0}) - \chi_{(-\infty,-a)}(\frac{d\varphi}{d\tau_0})\right] da.$$

Hence,

$$\begin{aligned}\varphi(1) &= \tau_0(\frac{d\varphi}{d\tau_0}1) \\ &= \tau_0(\frac{d\varphi}{d\tau_0}) \\ &= \int_0^\infty \tau_0\left[\chi_{(a,\infty)}(\frac{d\varphi}{d\tau_0}) - \chi_{(-\infty,-a)}(\frac{d\varphi}{d\tau_0})\right] da \\ &= \int_0^\infty f_\varphi(a)da, \quad \text{proving (i).}\end{aligned}$$

Now let $\varphi \leq \psi$. Then $\varphi_+ \leq \psi_+$ and $\psi_- \leq \varphi_-$. By (ii) of Lemma 4.2 [5], we see that $f_{\varphi_+} \leq f_{\psi_+}$ and $f_{\psi_-} \leq f_{\varphi_-}$. Consequently,

$$f_\varphi = f_{\varphi_+} - f_{\varphi_-} \leq f_{\psi_+} - f_{\psi_-} = f_\psi,$$

which proves (ii).

To show (iii)

$$\begin{aligned}\|\varphi - \psi\| &= 2(\varphi \vee \psi)(1) - \varphi(1) - \psi(1) \\ &= \int_0^\infty [2f_{\varphi\vee\psi}(a) - f_\varphi(a) - f_\psi(a)]da \\ &\geq \int_0^\infty [2(f_\varphi \vee f_\psi)(a) - f_\varphi(a) - f_\psi(a)]da \\ &\qquad (\,, \quad \text{by (ii).}) \\ &= \int_0^\infty |f_\varphi(a) - f_\psi(a)|da.\end{aligned}$$

Now, for every $u \in U(M)$, we note that

$$\begin{aligned} f_{u\varphi u^*} &= f_{(u\varphi u^*)_+} - f_{(u\varphi u^*)_-} \\ &= f_{u\varphi_+ u^*} - f_{u\varphi_- u^*} \\ &= f_{\varphi_+} - f_{\varphi_-} = f_\varphi, \end{aligned}$$

for any $\varphi \in M_*^h$. This together with the previous paragraph imply (iii).

Lemma 4. *Let $\varphi \in M_*$ and $\varphi = v|\varphi|$ be the polar decomposition. Define $f_\varphi : (0, \infty) \to \mathbb{C}$ by*

$$f_\varphi(a) = \tau_0(v\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})),$$

$a \in (0, \infty)$, and $e_\varphi \in L^\infty(\mathbb{R}, M)$ by

$$e_\varphi = {}^\oplus\int_{-\infty}^{\infty} v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma.$$

Then, for all $z \in Z(N)$,

$$\tau(e_\varphi z) = \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma})e^{-\gamma}d\gamma.$$

Proof. If $z \in Z(N) = L^\infty(\mathbb{R})$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma})e^{-\gamma}d\gamma \\ &= \int_{-\infty}^{\infty} z(\gamma)\tau_0(v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0}))e^{-\gamma}d\gamma \\ &= \int_{-\infty}^{\infty} \tau_0(z(\gamma)v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0}))e^{-\gamma}d\gamma \\ &= \tau({}^\oplus\int_{-\infty}^{\infty} z(\gamma)v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma) \\ &= \tau(z{}^\oplus\int_{-\infty}^{\infty} v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma) \\ &= \tau(ze_\varphi). \end{aligned}$$

Lemma 5. *Let $\varphi \in M_*$. Then*

$$\frac{d\tilde{\varphi}}{d\tau} = \oplus \int_{-\infty}^{\infty} (ve^{\gamma} \frac{d|\varphi|}{d\tau_0}) d\gamma.$$

Proof.

$$\begin{aligned}
\oplus \int_{-\infty}^{\infty} (ve^{\gamma} \frac{d|\varphi|}{d\tau_0}) d\gamma &= v \frac{d|\varphi|}{d\tau_0} \otimes m(e^{\gamma}) \\
&= \frac{d(v|\varphi|)}{d\tau_0} \otimes m(e^{\gamma}) \\
&= \frac{d\varphi}{d\tau_0} \otimes m(e^{\gamma}) \\
&= \frac{d(\varphi \otimes d\gamma)}{d(\tau_0 \otimes e^{-\gamma} d\gamma)} = \frac{d\tilde{\varphi}}{d\tau}.
\end{aligned}$$

Lemma 6. *Let $\varphi \in M_*$. Then,*

$$|\varphi\widehat{|}(1) = \|\varphi\|.$$

Proof.

$$\begin{aligned}
|\varphi\widehat{|}(1) &= \int_{-\infty}^{\infty} f_{|\varphi|}(e^{-\gamma}) e^{-\gamma} d\gamma \\
&= \int_{-\infty}^{\infty} \tau_0(\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})) e^{-\gamma} d\gamma \\
&= \int_{0}^{\infty} \tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}) da \\
&= \tau_0(\int_{0}^{\infty} \chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}) da) \\
&= \tau_0(\frac{d|\varphi|}{d\tau_0}) = \left\| \frac{d|\varphi|}{d\tau_0} \right\|_1 = \| \, |\varphi| \, \| = \|\varphi\|.
\end{aligned}$$

Corollary 7. *Let $\varphi \in M_*$. Then,*

$$\|\varphi\| = \left\| f_{|\varphi|} \right\|_1.$$

Lemma 8. *Let $\varphi \in M_*$. Then,*

$$\|\hat{\varphi}\| = \|f_\varphi\|_1.$$

Proof. For every $z \in Z(N)$,

$$\hat{\varphi}(z) = \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} d\gamma.$$

We can show that

$$|\hat{\varphi}|(z) = \int_{-\infty}^{\infty} z(\gamma) |f_\varphi(e^{-\gamma})| e^{-\gamma} d\gamma$$

for all $z \in Z(N)$. To see this, let us put

$$\rho(z) = \int_{-\infty}^{\infty} z(\gamma) |f_\varphi(e^{-\gamma})| e^{-\gamma} d\gamma.$$

Define $U \in N$ by $U = {}^{\oplus}\int_{-\infty}^{\infty} u(\gamma) 1_M d\gamma$, where

$$u(\gamma) = \begin{cases} 0, & \text{when } f_\varphi(e^{-\gamma}) = 0 \\ \frac{f_\varphi(e^{-\gamma})}{|f_\varphi(e^{-\gamma})|}, & \text{when } f_\varphi(e^{-\gamma}) \neq 0 \end{cases},$$

so that $f_\varphi(e^{-\gamma}) = u(\gamma)|f_\varphi(e^{-\gamma})|$ for all $\gamma \in \mathbb{R}$.

We claim that $U\rho = \hat{\varphi}$ and $U^*U = \operatorname{supp}(\rho)$, so that $\rho = |\hat{\varphi}|$, as desired. Now for all $z \in Z(N)$,

$$\begin{aligned}
(U\rho)(z) &= \rho(Uz) \\
&= \rho\left({}^{\oplus}\int_{-\infty}^{\infty} u(\gamma) z(\gamma) 1 d\gamma\right) \\
&= \int_{-\infty}^{\infty} u(\gamma) z(\gamma) |f_\varphi(e^{-\gamma})| e^{-\gamma} d\gamma \\
&= \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} d\gamma = \hat{\varphi}(z).
\end{aligned}$$

Put $V = \{\gamma \in \mathbb{R} : f_\varphi(e^{-\gamma}) \neq 0\}$. Then

$$\begin{aligned}
U^*U &= {}^{\oplus}\int_{-\infty}^{\infty} \overline{u(\gamma)} u(\gamma) 1_M d\gamma \\
&= \int_{-\infty}^{\infty} \chi_V(\gamma) 1_M d\gamma
\end{aligned}$$

and

$$1_N - U^*U = \int_{-\infty}^{\infty} \chi_{\mathbb{R}\sim V}(\gamma)1_M d\gamma$$

In general, for an element $z \in N^+$, we have that

$$\begin{aligned} z \in \ker\rho &\iff \int_{-\infty}^{\infty} z(\gamma)|f_\varphi(e^{-\gamma})|e^{-\gamma}d\gamma = 0 \\ &\iff \int_V z(\gamma)|f_\varphi(e^{-\gamma})|e^{-\gamma}d\gamma = 0 \\ &\iff z(\gamma) = 0 \quad \text{a.e.} \quad \gamma \in V. \end{aligned}$$

In particular when z is a projection in N^+, z is of the form

$$z = {}^{\oplus}\int_{-\infty}^{\infty} \chi_E(\gamma)d\gamma,$$

so that

$$\chi_E(\gamma) = 0 \quad \text{a.e.} \quad \gamma \in V.$$

Then ${}^{\oplus}\int_{-\infty}^{\infty} \chi_{\mathbb{R}\sim V}(\gamma)d\gamma$ is one of such projection in N^+ lying in the kernel of ρ. Actually this is the langest among such kind. Hence $1_N \sim {}^{\oplus}\int_{-\infty}^{\infty} \chi_{\mathbb{R}\sim V}(\gamma)d\gamma$, i.e., ${}^{\oplus}\int_{-\infty}^{\infty} \chi_V(\gamma)d\gamma$ is the support of ρ. That is, U^*U is the support of ρ, as desired. We thus have shown that

$$|\hat{\varphi}|(z) = \int_{-\infty}^{\infty} z(\gamma)|f_\varphi(e^{-\gamma})|e^{-\gamma}d\gamma$$

for all $z \in Z(N)$.

Thus,

$$\begin{aligned} \|\hat{\varphi}\| &= \| \, |\hat{\varphi}| \, \| = |\hat{\varphi}|(1) \\ &= \int_{-\infty}^{\infty} |f_\varphi(e^{-\gamma})|e^{-\gamma}d\gamma \\ &= \int_0^{\infty} |f_\varphi(a)|da \\ &= \|f_\varphi\|_1. \end{aligned}$$

Lemma 9. *Let $\varphi, \psi \in M_*$. Then*

$$\|\hat{\varphi} - \hat{\psi}\| = \|f_\varphi - f_\psi\|_1.$$

Proof.

$$\begin{aligned}
\|\hat{\varphi} - \hat{\psi}\| &= \sup\{|\hat{\varphi}(z) - \hat{\varphi}(z)| : z \in Z(N),\ \|z\| = 1\} \\
&= \sup_{\substack{z\in Z(N)\\ \|z\|=1}} |\int_{-\infty}^{\infty} z(\gamma)(f_\varphi(e^{-\gamma}) - f_\psi(e^{-\gamma}))e^{-\gamma}d\gamma| \\
&= \sup_{\substack{z\in Z(N)\\ \|z\|=1}} |\int_{0}^{\infty} z(-\log a)(f_\varphi(a) - f_\psi(a))da| \\
&= \int_0^\infty |f_\varphi(a) - f_\psi(a)|da \\
&= \|f_\varphi - f_\psi\|_1,
\end{aligned}$$

as desired.

Lemma 10. *Let $\varphi \in M_*$, $\varphi = v|\varphi|$ be the (left) polar decomposition of φ. Define*

$$V = {}^{\oplus}\int_{-\infty}^{\infty} v d\gamma \in L^\infty(\mathbb{R}, M) \cong N.$$

Then $e_\varphi = Ve_{|\varphi|}$ and this expression is the (left) polar decomposition of e_φ.

Proof. Since

$$e_{|\varphi|} = {}^{\oplus}\int_{-\infty}^{\infty} \chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma,$$

we see that

$$\begin{aligned}
Ve_{|\varphi|} &= {}^{\oplus}\int_{-\infty}^{\infty} v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma \\
&= e_\varphi.
\end{aligned}$$

Clearly $e_{|\varphi|}$ is a positive (unbounded, in general) operator affiliated with N. Now

$$\operatorname{supp}(e_{|\varphi|}) = {}^{\oplus}\int_{-\infty}^{\infty} \operatorname{supp}(\frac{d|\varphi|}{d\tau_0})d\gamma.$$

On the other hand $\frac{d\varphi}{d\tau_0} = v\left|\frac{d\varphi}{d\tau_0}\right|$ is the polar decomposition of $\frac{d\varphi}{d\tau_0}$. Consequently,

$$\mathrm{supp}(\frac{d|\varphi|}{d\tau_0}) = v^*v.$$

Thus

$$\mathrm{supp}(e_{|\varphi|}) = {}^{\oplus}\int_{-\infty}^{\infty} v^*v d\gamma = V^*V.$$

This implies that $e_\varphi = Ve_{|\varphi|}$ is the polar decomposition of e_φ.

Corollary 11. $|e_\varphi| = e_{|\varphi|}$.

Corollary 12. $e_\varphi = V\chi_{(1,\infty)}(\left|\frac{d\tilde{\varphi}}{d\gamma}\right|)$ *is the left polar decomposition of* e_φ.

Proof. By 3) of Theorem 7 in Chapter II [12], we see that

$$|\tilde{\varphi}| = |\varphi|\tilde{}, \quad \text{for every } \varphi \in M_*.$$

But

$$\chi_{(1,\infty)}(\frac{d(|\varphi|)\tilde{}}{d\gamma}) = {}^{\oplus}\int_{-\infty}^{\infty} \chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0})d\gamma$$
$$= e_{|\varphi|}.$$

Hence $e_\varphi = Ve_{|\varphi|} = V\chi_{(1,\infty)}(\frac{d|\tilde{\varphi}|}{d\gamma}) = V\chi_{(1,\infty)}(\left|\frac{d\tilde{\varphi}}{d\gamma}\right|)$ is the left polar decomposition of e_φ.

Corollary 13. *For every* $\varphi \in M_*$, $\hat{\varphi}(z) = \tau(e_\varphi z) = \tau(Ve_{|\varphi|}z)$, *for all* $z \in Z(N) \cong L^\infty(\mathbb{R})$.

Lemma 14. *Let* $\varphi \in M_*$. *Then*

$$|\hat{\varphi}| \leq |\varphi|\hat{}, \quad \|\hat{\varphi}\| \leq \|\varphi\|.$$

Proof. Note that

$$|\hat{\varphi}|(z) = \int_{-\infty}^{\infty} z(\gamma)\left|\tau_0(v\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0}))\right|e^{-\gamma}d\gamma$$

for all $z \in Z(N)$. But, for $a \in (0,\infty)$, we have that

$$\begin{aligned}
&|\tau_0(v\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))|^2 \\
&= |\tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}) \cdot v\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))|^2 \\
&\leq \tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \cdot \tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})v^*v\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&\leq \left\{\tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})\right\}^2.
\end{aligned}$$

Hence,

$$|\tau_0(v\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))| \leq \tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))$$

Hence for every $z \in Z(N)^+$,

$$\begin{aligned}
|\hat{\varphi}|(z) &\leq \int_{-\infty}^{\infty} \tau_0(\chi_{(e^{-\gamma},\infty)}(\frac{d|\varphi|}{d\tau_0}))e^{-\gamma}d\gamma \\
&= (|\varphi|\hat{)}(z).
\end{aligned}$$

Consequently, $|\hat{\varphi}| \leq |\varphi\hat{|}$.

Thus, by Lemma 7 and Lemma 6, we have

$$\|f_\varphi\|_1 - \|\hat{\varphi}\| = \|\,|\hat{\varphi}|\,\| = |\hat{\varphi}|(1) \leq (|\varphi\hat{|})(1) = \|\varphi\|,$$

as desired.

The following lemma implies that our old definition of f_φ, for $\varphi \in M_*^h$ agrees with the new definition of f_φ in Lemma 4 when $\varphi \in M_*^h$.

Lemma 15. *Let $\varphi \in M_*^h$, and $\varphi = (e^+ - e^-)|\varphi|$ be the unique left polar decomposition of φ, where e^+, e^- are orthogonal projections in M such that $e^+ + e^- = s(|\varphi|)$. Also let $\varphi = \varphi^+ - \varphi^-$ be the Jordan decomposition of φ so that $\varphi^+ = e^+|\varphi|$, $\varphi^- = e^-|\varphi|$. Then,*

$$f_{\varphi^+}(a) = \tau_0(e^+\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})),$$

$$f_{\varphi^-}(a) = \tau_0(e^-\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))$$

for all $a \in (0,\infty)$.

Proof. Note that $s(|\varphi|) = s(\frac{d|\varphi|}{d\tau_0})$ [p.65 [9] Stratila (1)), so that $e^+ + e^- = s(\frac{d|\varphi|}{d\tau_0})$. Consequently, both e^+ and e^- commutes with $s(\frac{d|\varphi|}{d\tau_0})$.

We first prove that for any $a, b \in (0,\infty)$ with $a < b$,

$$\chi_{(a,b)}(e_+ \frac{d|\varphi|}{d\tau_0} e_+) = e_+ \chi_{(a,b)}(\frac{d|\varphi|}{d\tau_0})e_+.$$

(Here all operations are considered in the setting of the $*$-algebra of all τ_0-measurable operators.)

Let $K \in (0,\infty)$ be a fixed number such that $K > b$. For all sufficiently large positive integer n, $a < a + \frac{1}{n} < b - \frac{1}{n} < b$. For all such large positive integer n, we find a real valued continuous function $f_n : [0,K] \to [0,1]$ such that $f_n(x) = 0$ for all $x \in [0,a] \cup [b,K]$ and $f_n(y) = 1$ for all $y \in [a+\frac{1}{n},\ b-\frac{1}{n}]$, by the Urysohn's lemma for a normal space.

The Stone-Weierstrass theorem now implies that there are real polynomials p_n on $[0,K]$ such that

$$\|p_n - f_n\|_K \to 0 \quad \text{as} \quad n \to \infty,$$

where $\|\cdot\|_K$ is the supremum norm on K. Then $\{\|p_n\|_K\}_{n=1,2,\ldots}$ is a bounded sequence. Also it is clear that $f_n(x) \to \chi_{(a,b)}(x)$ for every $x \in [0,K]$. Consequently $p_n(x) \to \chi_{(a,b)}(x)$ for every $x \in [0,K]$. Consequently

$$p_n(\frac{d|\varphi|}{d\tau_0}) \to \chi_{(a,b)}(\frac{d|\varphi|}{d\tau_0})$$

in the strong operator topology, which can be checked by the Lebesgue dominated convergence theorem. Then

$$e_+ p_n(\frac{d|\varphi|}{d\tau_0})e_+ \to e_+ \chi_{(a,b)} \frac{d|\varphi|}{d\tau_0})e_+$$

in the strong operator topology. But

$$e_+ p_n(\frac{d|\varphi|}{d\tau_0})e_+ = p_n(e_+ \frac{d|\varphi|}{d\tau_0} e_+)$$

and these converge to $\chi_{(a,b)}(e_+ \frac{d|\varphi|}{d\tau_0} e_+)$ in the strong operator topology.

It follows that

$$e_+ \chi_{(a,b)}(\frac{d|\varphi|}{d\tau_0})e_+ = \chi_{(a,b)}(e_+ \frac{d|\varphi|}{d\tau_0}(e_+)$$

Now letting $b \to \infty$, we get that

$$\begin{aligned} e_+ \chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})e_+ &= \chi_{(a,\infty)}(e_+ \frac{d|\varphi|}{d\tau_0}e_+) \\ &= \chi_{(a,\infty)}(\frac{d(e_+|\varphi|)}{d\tau_0}) \end{aligned}$$

i.e., $e^+\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})e^+ = \chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0})$. Similarly,

$$e^-\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})e^- = \chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0}),$$

from which our desired equalities follow.

As we already noted, if $\varphi \in M_*^h$. Then $f_\varphi = f_{\varphi^+} - f_{\varphi^-}$.
Indeed, for every $a \in (0, \infty)$,

$$\begin{aligned} f_\varphi(a) &= \tau_0((e^+ - e^-)\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\ &= \tau_0(e^+\chi_{(a,\infty)}(\frac{d|\phi|}{d\tau_0})) - \tau_0(e^-\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\ &= f_{\varphi^+}(a) - f_{\varphi^-}(a), \end{aligned}$$

as desired.

Corollary 16. *Let* $\varphi \in M_*^h$. *Then*

$$f_{|\varphi|} = f_{\varphi^+} + f_{\varphi^-}.$$

Proof. For every $a \in (0,\infty)$,

$$\begin{aligned}
(f_{\varphi^+} + f_{\varphi^-})(a) &= f_{\varphi^+}(a) + f_{\varphi^-}(a) \\
&= \tau_0(\chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0})) + \tau_0(\chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0})) \\
&= \tau_0(e^+\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&\quad + \tau_0(e^-\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&\qquad (\text{ , see the proof of Lemma 18}) \\
&= \tau_0((e^+ + e^-)\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&= \tau_0(\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&\qquad (\text{ , since } e^+ + e^- \text{ is the support of } \frac{d|\varphi|}{d\tau_0}.) \\
&= f_{|\varphi|}(a), \quad \text{as desired.}
\end{aligned}$$

Let $\mathcal{S}$ denote, the set of all complex valued L^1-functions f on $(0,\infty)$ into $(J-J)+i(J-J)$, which are continuous from the right and the absolute value functions $|f|$ are decreasing. Put

$$\begin{aligned}
\mathcal{S}^h &= \{f \in \mathcal{S} : f \text{ is real valued.}\}, \\
\mathcal{S}^+ &= \{f \in \mathcal{S}^h : f \text{ is nonnegative real valued}\}.
\end{aligned}$$

The next theorem is our first main result related with Theorem 4.4 of [5].

Proposition 17. *The map* $[\varphi] \to f_\varphi$ $(\varphi \in M_*^h)$ *is a map of* $M_*^h/\sim$ *onto* $\mathcal{S}^h$. *The image of* $M_*^+/\sim$ *under this map is* $\mathcal{S}^+$.

Proof. The well-definedness of the map $[\varphi] \to f_\varphi(\varphi \in M_*^h)$ follows from (iii) of Lemma 6.

For the case when $f \in \mathcal{S}^+$, the existence of $\varphi \in M_*^+$ such that $f_\varphi = f$ is known in the earlier part of the proof of Theroem 4.4 [5]. Now let $f \in \mathcal{S}^h$, and consider $f = f_+ - f_-$. We thus can find $\varphi_+, \varphi_- \in M_*^+$ such that $f_{\varphi_+} = f_+$, $f_{\varphi_-} = f_-$. Since $f_{\varphi_+}f_{\varphi_-} = f_+f_- = 0$, one can easily show that

$$s(\frac{d\varphi_+}{d\tau_0}) \perp s(\frac{d\varphi_-}{d\tau_0}),$$

so that

$$s(\varphi_+) \perp s(\varphi_-).$$

Then, by putting

$$\varphi = \varphi_+ - \varphi_-,$$

we get the Jordan decomposition of φ. Consequently,

$$f_\varphi = (f_\varphi)_+ - (f_\varphi)_- = f_{\varphi_+} - f_{\varphi_-} = f_+ - f_- = f,$$

proving the surjectivity of $[\varphi] \in M_*/ \sim \to f_\varphi \in \mathcal{S}^h$.

Let M be a semifinite factor with a fixed semifinite normal tracial weight τ_0.

Theorem 18. *Let $\varphi \in M_*^h$. Then there are $\rho_1, \rho_2 \in M_*^+$ such that $[\rho_1] = [\varphi^+]$, $[\rho_2] = [\varphi^-]$, $\|\rho_1 - \rho_2\| = \int_0^\infty |f_\varphi(a)| da$, where $\varphi = \varphi^+ - \varphi^-$ is the Jordan decomposition of φ.*

Proof. Let e^+, e^- be mutually orthogonal projections in the abelian von Neumann algebra generated by the spectral projections of $\frac{d\varphi}{d\tau_0}$ such that

$$\frac{d\varphi}{d\tau_0} = (e^+ - e^-)\frac{d|\varphi|}{d\tau_0}$$

is the left polar decomposition of $\frac{d\varphi}{d\tau_0}$.

Let h, k denote the positive selfadjoint operator affiliated with M such that

$$\begin{aligned}
\chi_{(a,\infty)}(h) &= p(\tau_0(e^+\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})) \\
&= p\tau_0(\chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0})), \\
\chi_{(a,\infty)}(k) &= p(\tau_0(e^-\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))) \\
&= p\tau_0(\chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0}))
\end{aligned}$$

for all $a \in (0, \infty)$.

We put

$$X_1 := \{a \in (0,\infty) : \chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0}) \succsim\!\!\!\!/\ \chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0})\},$$
$$X_0 := \{a \in (0,\infty) : \chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0}) \sim \chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0})\},$$
$$X_2 := \{a \in (0,\infty) : \chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0}) \precsim\!\!\!\!/\ \chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0})\}.$$

Define $\rho_1 = \tau_0(h(\cdot))$, $\rho_2 = \tau_0(k(\cdot))$. Then one can show that $\rho_1, \rho_2 \in M_*^+$. We note that $f_{\rho_1} = f_{\varphi^+}$, $f_{\rho_2} = f_{\varphi^-}$. By [5] Lemma 4.3, we see that $[\rho_1] = [\varphi^+]$ and $[\rho_2] = [\varphi^-]$. Also we note that $\|\rho_1 - \rho_2\| = \int_{(0,\infty)} |f_\varphi(a)|da$.

Theorem 19. *We keep the notation in Theorem 21. If, in particular, $\rho_1 - \rho_2$ is the Jordan decomposition of $\rho = \rho_1 - \rho_2 \in M_*^h$ satisfying the conditions in Theorem 21, then either $\varphi \in M_*^+$ or $-\varphi \in M_*^+$.*

Proof. Assume that $\rho = \rho_1 - \rho_2$ is the Jordan decomposition of $\rho \in M_*^+$. Then

$$\|\rho_1 - \rho_2\| = \|\rho_1\| + \|\rho_2\|.$$

Thus, from the condition in Theorem 1,

$$\begin{aligned}
\int_0^\infty |f_\varphi(a)|da &= \|\rho_1\| + \|\rho_2\| \\
&\geq d([\rho_1],[0]) + d([\rho_2],[0]) \\
&\geq \int_{(0,\infty)} f_{\rho_1}(a)da + \int_{0,\infty)} f_{\rho_2}(a)da \\
&= \int_{(0,\infty)} f_{\varphi^+}(a)da + \int_{(0,\infty)} f_{\varphi^-}(a)da.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{X_1} |f_\varphi(a)|da + \int_{X_0} |f_\varphi(a)|da + \int_{X_2} |f_\varphi(a)|da \\
\geq &\int_{(0,\infty)} f_{\varphi^+}(a)da + \int_{(0,\infty)} f_{\varphi^-}(a)da\ .
\end{aligned}$$

$$\int_{X_1} |f_{\varphi^+}(a) - f_{\varphi^-}(a)|da + \int_{X_0} |f_{\varphi^+}(a) - f_{\varphi^-}(a)|da$$

$$+\int_{X_2} |f_{\varphi^+}(a) - f_{\varphi^-}(a)| da$$

$$\geq \int_{(0,\infty)} f_{\varphi^+}(a)da + \int_{(0,\infty)} f_{\varphi^-}(a)da$$

$$\int_{X_1} (f_{\varphi^+}(a) - f_{\varphi^-}(a))da + \int_{X_2} (f_{\varphi_-}(a) - f_{\varphi^+}(a))da$$

$$\geq \int_{(0,\infty)} f_{\varphi^+}(a)da + \int_{(0,\infty)} f_{\varphi^-}(a)da \; .$$

Because

$$0 \leq \int_{X_1} (f_{\varphi^+}(a) - f_{\varphi^-}(a)da \leq \int_{X_1} f_{\varphi^+}(a)da \leq \int_{(0,\infty)} f_{\varphi^+}(a)da,$$

$$0 \leq \int_{X_2} (f_{\varphi^-}(a) - f_{\varphi^+}(a))da \leq \int_{X_2} f_{\varphi^-}(a)da \leq \int_{(0,\infty)} f_{\varphi^-}(a)da,$$

we see that

$$\int_{X_1} (f_{\varphi^+}(a) - f_{\varphi^-}(a))da + \int_{X_2} (f_{\varphi^-}(a) - f_{\varphi^+}(a))da$$

$$= \int_{(0,\infty)} f_{\varphi^+}(a)da + \int_{(0,\infty)} f_{\varphi^-}(a)da$$

Put

$$\alpha' = \int_{X_1} (f_{\varphi^+}(a) - f_{\varphi^-}(a))da,$$

$$\beta' = \int_{X_2} (f_{\varphi^-}(a) - f_{\varphi^+}(a))da,$$

$$\alpha = \int_{(0,\infty)} f_{\varphi^+}(a)da,$$

$$\beta = \int_{(0,\infty)} f_{\varphi^-}(a)da.$$

Thus we have the following situation:

$$0 \leq \alpha', \; \beta', \; \alpha, \; \beta < \infty$$

$$\alpha' \leq \alpha, \; \beta' \leq \beta, \quad \text{and}$$

$$\alpha' + \beta' = \alpha + \beta.$$

Consequently, $(\alpha-\alpha')+(\beta-\beta')=0$ with $0\le\alpha-\alpha'<\infty$, $0\le\beta-\beta'<\infty$. We thus can condude that $\alpha=\alpha'$ and $\beta=\beta'$.

$$\int_{X_1}(f_{\varphi^+}(a)-f_{\varphi^-}(a)da=\int_{(0,\infty)}f_{\varphi^+}(a)da$$
$$=\int_{X_1}f_{\varphi^+}(a)da+\int_{X_0}f_{\varphi^+}(a)da+\int_{X_2}f_{\varphi^+}(a)da,$$
$$\int_{X_2}(f_{\varphi^-}(a)-f_{\varphi^+}(a))da=\int_{(0,\infty)}f_{\varphi^-}(a)da$$
$$=\int_{X_1}f_{\varphi^-}(a)da+\int_{X_0}f_{\varphi^-}(a)da+\int_{X_2}f_{\varphi^-}(a)da.$$

Hence

$$0\ge-\int_{X_1}f_{\varphi^-}(a)da=\int_{X_0}f_{\varphi^+}(a)da+\int_{X_2}f_{\varphi^+}(a)da\ge 0$$
$$0\ge-\int_{X_2}f_{\varphi^+}(a)da=\int_{X_1}f_{\varphi^-}(a)da+\int_{X_0}f_{\varphi^-}(a)da\ge 0.$$

Consequently,

$$\int_{X_0}f_{\varphi^+}(a)da=0,\quad \int_{X_2}f_{\varphi^+}(a)da=0$$
$$\int_{X_1}f_{\varphi^-}(a)da=0,\quad \int_{X_0}f_{\varphi^-}(a)da=0.$$

Thus, f_{φ^+}, f_{φ^-} are respectively equal to zero almost everywhere on X_0.

f_{φ^+} is equal to zero almost everywhere on X_2, f_{φ^-} is equal to zero almost everywhere on X_1.

But $(0,\infty)=X_1\dot{\cup}X_0\dot{\cup}X_2$ (disjoint union).

Consequently

$$f_{\varphi^+}(a)f_{\varphi_-}(a)=0$$

almost everywhere on $(0,\infty)$. Because both f_{φ^+}, f_{φ^-} are right continuous on $(0,\infty)$, we see that $f_{\varphi^+}f_{\varphi^-}$ is also right continuous on $(0,\infty)$. Hence $f_{\varphi^+}(a)f_{\varphi^-}(a)=0$ for every $a\in(0,\infty)$.

It follows that at every $a\in(0,\infty)$, either

$$\chi_{(0,\infty)}(\frac{d\varphi^+}{d\tau_0})=0 \quad \text{or}$$
$$\chi_{(0,\infty)}(\frac{d\varphi^-}{d\tau_0})=0$$

This means that the origin 0 of the real line is either an accumulation point of $\{a \in (0,\infty) : \chi_{(a,\infty)}(\frac{d\varphi^+}{d\tau_0}) = 0\}$, or it is an accumulation point of $\{a \in (0,\infty) : \chi_{(a,\infty)}(\frac{d(\varphi^-}{d\tau_0}) = 0\}$.

For the former case, we see that

$$\chi_{(a,\infty)}(\frac{d(\varphi^+}{d\tau_0}) = 0$$

for all $a \in (0,\infty)$, so that $\varphi^+ = 0$.

For the latter case, we see that

$$\chi_{(a,\infty)}(\frac{d\varphi^-}{d\tau_0}) = 0$$

for all $a \in (0,\infty)$, so that $\varphi^- = 0$.

Thus either $\varphi = -\varphi^-$ or $\varphi = \varphi^+$ i.e., $-\varphi \in M_*$ or $\varphi \in M_*^+$.

Corollary 20. *Let $\varphi \in M_*^h$. If $\|\varphi\| = \int_0^\infty |f_\varphi(a)|da$, then either $\varphi \in M_*^+$ or $-\varphi \in M_*^+$.*

Theorem 21. *Let M be a semifinite factor with a fixed normal semifinite trace τ_0. Assume that $\varphi, \psi \in M_*^h$ be given as satisfying either φ or $-\varphi \in M_*^+$, and also eithe ψ or $-\psi \in M_*^+$. Then*

$$d([\varphi],[\psi]) = \int_0^1 |f_\varphi(a) - f_\psi(a)|da$$

Proof. Case (i). Both $\varphi, \psi \in M_*^+$: This is just Theorem 4.4 [5]. Case (ii) Both $-\varphi, -\psi \in M_*^+$: Let e be the support of $|\varphi|$. Then

$$\varphi = (-e)|\varphi|,$$
$$-\varphi = e|\varphi|$$

are the left polar decompositions of φ and $-\varphi$ respectively. Thus, for every $a \in (0,\infty)$,

$$f_\varphi(a) = \tau_0(-e\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0}))$$
$$f_{-\varphi}(a) = \tau_0(e\chi_{(a,\infty)}(\frac{d|\varphi|}{d\tau_0})),$$

so that $f_{-\varphi}(a) = -f_\varphi(a)$.

Similarly,

$$f_{-\psi}(a) = -f_\psi(a), \quad a \in (0,\infty)$$

Hence

$$\begin{aligned}\int_0^\infty |f_\varphi(a) - f_\psi(a)|da &= \int_0^\infty |-f_{-\varphi}(a) + f_{-\psi}(a)|da \\ &= \int_0^\infty |f_{-\varphi}(a) - f_{-\psi}(a)|da \\ &= d([-\varphi],[-\psi]),\end{aligned}$$

by case (i).

On the while,

$$\begin{aligned} d([-\varphi],[-\psi]) &= \inf_{U\in U(M)} \|U(-\varphi)U^* - (-\psi)\| \\ &= \inf_{U\in U(M)} \|U\varphi U^* - \psi\| \\ &= d([\varphi],[\psi]).\end{aligned}$$

Consequently,

$$\int_0^\infty |f_\varphi(a) - f_\psi(a)|da = d([\varphi],[\psi])$$

Case (iii) $\varphi \in M_*^+ - \psi \in M_*^+$:

We have to show that

$$d([\varphi],[\psi]) = \int_{(0,\infty)} |f_\varphi(a) - f_\psi(a)|da$$

i.e.,

$$d([\varphi],[\psi]) = \int_{(0,\infty)} |f_\varphi(a) + f_{-\psi}(a)|da$$

i.e.,

$$\inf_{U\in U(M)} \|U\varphi U^* - \psi\| = \int_{(0,\infty)} (f_\varphi(a) + f_{-\psi}(a))da$$

But

$$\begin{aligned}\|U\varphi U^* - \psi\| &= \|U\varphi U^* + (-\psi)\| \\ &= [U\varphi U^* + (-\psi)](1) \\ &= (U\varphi U^*)(1) + (-\psi)(1) \\ &= \int_0^\infty f_{U\varphi U^*}(a)da + \int_0^\infty f_{-\psi}(a)da \\ &\qquad (\text{ ,by (i) of Lemma 4.2 [5]}) \\ &= \int_0^\infty (f_\varphi(a)da + f_{-\psi}(a))da\end{aligned}$$

Consequently,

$$\begin{aligned}d([\varphi],[\psi]) &= \inf_{U\in U(M)} \|U\varphi U^* - \psi\| \\ &= \int_0^\infty (f_\varphi(a) + f_{-\psi}(a))da,\end{aligned}$$

as desied.

Remark. In this case we actually have shown that

$$\begin{aligned}\|\varphi - \psi\| &= d([\varphi],[\psi]) \\ &= \int_0^\infty |f_\varphi(a) - f_\psi(a)|da.\end{aligned}$$

Case (iv). $-\varphi \in M_*^+$, $\psi \in M_*^+$:
The proof is symmetric to Case (iii) and omitted.

For every $\varphi \in M_*^h$, we recall that $\frac{d\tilde{\varphi}}{d\tau}$ is a τ-measurable operator, so that, when we put $e_\varphi = (\chi_{(1,\infty)} + \chi_{(-\infty,-1)}(\frac{d\tilde{\varphi}}{d\tau})$, we have $e_\varphi \in L^1(N,\tau) \cap N$. Indeed, by Lemma 5, the right hand side is $\chi_{(1,\infty)}((\frac{d\tilde{\varphi}}{d\tau})_+) - \chi_{(1,\infty)}((\frac{d\tilde{\varphi}}{d\tau})_-)$, while by Proposition 4 in Chap. II of [12] $\frac{d\tilde{\varphi}}{d\tau} = \frac{d\widetilde{\varphi_+}}{d\tau} - \frac{d\widetilde{\varphi_-}}{d\tau}$ is the "Jordan decomposition" of $\frac{d\tilde{\varphi}}{d\tau}$ in the sense that $s(\frac{d\widetilde{\varphi_+}}{d\tau}) \perp s(\frac{d\widetilde{\varphi_-}}{d\tau})$. Hence

$$e_\varphi = e_{\varphi_+} - e_{\varphi_-} \in L^1(N,\tau) \cap N.$$

When $\varphi \in M_*$, we consider the Cartesian decomposition $\varphi = \varphi_1 + i\varphi_2$ ($\varphi_1, \varphi_2 \in M_*^h$), and define $e_\varphi = e_{\varphi_1} + ie_{\varphi_2} \in L^1(N,\tau) \cap N$.

As an extension of Definition 3.2 [5], we define $\hat{\varphi} \in Z(N)_*$, by

$$\hat{\varphi}(z) = \tau(e_\varphi z) \quad (z \in Z(N)),$$

where $Z(N)$ denotes the center of N. Then, by Lemma 4.5 [5], we see that

$$\hat{\varphi}(z) = \int_{-\infty}^{\infty} z(\gamma) f_\varphi(e^{-\gamma}) e^{-\gamma} d\gamma,$$

$z \in Z(N) \cong L^\infty(\mathbb{R})$, where $\cong$ denotes the canonical identification.

The next lemma extends (i) of Theore 4.7 [5]. Its proof is similar to that for (i) of Theorem 4.7 [5] and hence omitted.

Lemma 22. *Let $\varphi, \psi \in M_*$. Then,*

$$d([\varphi, [\psi]) = \|\hat{\varphi} - \hat{\psi}\|.$$

Let $\mathcal{Y}^h$ denote the set of all $\rho \in Z(N)_*^h$ satisfying the following conditions.

(i) There is an everywhere defined real valued L^1-function on $(0, \infty)$, say f, depending on ρ, with values in $J - J$ such that

$$\rho(z) = \int_{-\infty}^{\infty} z(\gamma) f(e^{-\gamma}) e^{-\gamma} d\gamma$$

for all $z \in Z(N) \cong L^\infty(\mathbb{R})$.

(ii) For the Jordan decomposition $\rho = \rho_+ - \rho_-$,

$$\rho_+ \circ \theta_s \geq e^{-s} \rho_+ \quad \text{and}$$

$$\rho_- \circ \theta_s \geq e^{-} \rho_- \quad \text{for all} \quad s \in (0, \infty).$$

We also put $\mathcal{Y}^+ = \{\rho \in \mathcal{Y}^h : \rho \geq 0\}$.

The following lemma extends (ii) of Theorem 4.7 [5] that dealt with the case of $\mathcal{Y}^+$. Its proof can be done by aid of Lemma 4.6 [5] and also omitted.

Lemma 23. $\{\hat{\varphi} : \varphi \in M_*^h\} = \mathcal{Y}^h$.

We are now ready to state the extension of Corollary 4.8 [5].

Theorem 24. *Let M be a type II factor with separable predual. Then the map $[\varphi] \to \hat{\varphi}$ is an isometry of $M_*^h / \sim$ into $Z(N)_*$. When M is of II_∞, then the range of this map is the set*

$$\{\rho \in Z(N)_*^h : \rho_+ \circ \theta_s \geq e^{-s} \rho_+ \quad \text{and}$$
$$\rho_- \circ \theta_s \geq e^{-s} \rho_- \quad \text{for all} \quad s \in (0, \infty)\}.$$

When M is of II_1, then the range of this map is the set

$$\{\rho \in Z(N)_*^h : \rho_+ \circ \theta_s \geq e^{-s} \rho_+,$$
$$\rho_- \circ \theta_s \geq e^{-s} \rho_- \quad \text{for all}$$
$$s \in (0, \infty) \quad \text{and}$$
$$\rho_+, \rho_- \leq \tau|_{Z(N)}\}.$$

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G-SIMPLICITY OF C^*-DYNAMICAL SYSTEMS AND SIMPLICITY OF C^*-CROSSED PRODUCTS*

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1. Introduction

Let (A, G, α) be a C^*-dynamical system. Our aim is to continue the investigation of the relationship between the property of the C^*-dynamical system (A, G, α) and the ideal structure of the corresponding C^*-crossed product $A \times_\alpha G$. This problem first appeared in [**5**] and has been studied in [**2, 3, 4, etc**]. Olesen and Pedersen [**6**] gave the necessary and sufficient condition of simplicity of C^*-crossed products by locally compact abelian groups. When G is a discrete group and A is an AF-algebra, Elliott [**2**] showed that if (A, G, α) is properly outer and A is G-simple, the reduced crossed product $A \times_{\alpha r} G$ is simple. Later Kawamura and Tomiyama [**3**] obtained the same result when A is an abelian C^*-algebra. In this paper we study simplicity of the C^*-crossed product $A \times_\alpha G$ for a general C^*-algebra when G is a discrete group.

Let (A, G, α) be a C^*-dynamical system and G be a discrete group. Let A'' be the universal enveloping von Neumann algebra of a C^*-algebra A. Then the action $\alpha : g \to \alpha_g$ induces the action $\alpha'' : g \to \alpha''_g$ on A''. Then (A'', G, α'') becomes a W^*-dynamical system. It is said that G is a central shift in (A, G, α) if there exists a family $\{p_j\}$ of mutually orthogonal central projections in A'' such that $\sum p_j = 1$ and $\alpha''_g(p_j)p_j = 0$ for every $g \in G - \{e\}$ where e is the identity of G.

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2. G- SIMPLICITY

Let (A, G, α) be a C^*-dynamical system. We say that A is *G-simple* if it has no non-trivial α-invariant closed two-sided ideal of A. And A is *G-prime* if any two non-zero α-invariant closed two-sided ideals of A have a non-zero intersection.

Let H be a subgroup of G. We denote the fixed point algebra under α_H by A^H, i.e.

$$A^H = \{x \in A |\ \alpha_g(x) = x, \quad g \in H\}.$$

If H is a normal subgroup of G, then we can consider the C^*-dynamical system $(A^H, G/H, [\alpha]^H)$ obtained naturally from (A, G, α) on the quotient group G/H.

Let G be a locally compact abelian group and K be a subset of G. We denote the annihilator of K by $A(K)$.

Lemma 2.1. *Let (A, G, α) be a C^*-dynamical system and G be a locally compact abelian group. Let $\{G_i\}_{i\in I}$ be the directed system of compact subgroups of G. Let $A(G_i)$ be the annihilator of G_i for each $i \in I$. If* $\mathrm{Sp}(\alpha)$ *is contained in $\bigcup_i A(G_i)$, then $\bigcup_i A^{G_i}$ is a dense $*$-subalgebra of A.*

Proof. Since G_i is a compact group, $A(G_i)$ is equal to

$$\{\gamma \in \widehat{G} \mid |\gamma(g) - 1| < \sqrt{3}, \quad g \in G_i\}.$$

Therefore $A(G_i)$ is open. Since the Arveson spectrum $\mathrm{Sp}(\alpha)$ is closed in $\widehat{G}$, $\{A(G_i) \mid i \in I\} \cup \{\widehat{G} - \mathrm{Sp}(\alpha)\}$ is an open covering of $\widehat{G}$. Let K be a compact subset of $\widehat{G}$. There exists a finite subset $\{i_k\}_{k=1}^n$ of I such that $K \cap \mathrm{Sp}(\alpha)$ is contained in $\bigcup_{i_k} A(G_{i_k})$. Hence there exists an index $i_0 \in I$ such that $K \cap \mathrm{Sp}(\alpha)$ is contained in $A(G_{i_0})$. Since $A^\alpha(K \cap \mathrm{Sp}(\alpha)) = A^\alpha(K)$, $A^\alpha(K)$ is contained in $A^{G_{i_0}}$. From the definition of A_F^α, A_F^α is contained in $\bigcup_i A^{G_i}$. Since A_F^α is dense in A, $\bigcup_i A^{G_i}$ is dense in A.

Theorem 2.2. *Let (A, G, α) be a C^*-dynamical system and G be a locally compact abelian group. Let $\{G_i\}_{i\in I}$ be the directed system of compact subgroups of G such that $\bigcup_i A(G_i)$ contains* $\mathrm{Sp}(\alpha)$. *Let J be an α-invariant closed two-sided ideal of A. Then for each $i \in I$ there exists $[\alpha]^i$-invariant* *closed two-sided ideal J_i of the C^*-dynamical system $(A^{G_i}, G/G_i, [\alpha]^i)$ such that J is the norm closure of $\bigcup J_i$.*

Proof. Let J be an α-invariant closed two-sided ideal of A. Then $J \cap A^{G_i}$ is an $[\alpha]^i$-invariant closed two-sided ideal of A^{G_i} for each $i \in I$. Since $\bigcup_i A^{G_i}$ is dense in A by Lemma 2.1, for each self-adjoint element a in J and $\epsilon > 0$, there exists a self-adjoint element $b \in A^{G_{i_0}}$ such that $\|a - b\| < \epsilon/3$ for some $i_0 \in I$. Choose a continuous function $f(t)$ on $\mathbf{R}$ such that

$$|f(t) - t| < \frac{\epsilon}{2}, \quad t \in \mathbf{R},$$

$$f(t) = 0, \qquad |t| < \frac{\epsilon}{3}.$$

Then by the functional calculus, $f(b) \in A^{G_{i_0}}$. We consider the homomorphism $\pi_J : A \to A/J$. Since $\|\pi_J(b)\| < \frac{\epsilon}{3}$, we have

$$\pi_J(f(b)) = f(\pi_J(b)) = 0.$$

Therefore $f(b)$ is contained in $A^{G_{i_0}} \cap J$. Since $\|a - f(b)\| < \epsilon$, $\bigcup_i (J \cap A^{G_i})$ is dense in J.

Corollary 2.3. *Let A be a unital C^*-algebra. Under the same hypothesis of Theorem 2.2, if the C^*-dynamical system $(A^{G_i}, G/G_i, [\alpha]^i)$ is G/G_i-simple for each $i \in I$, then (A, G, α) is also G-simple.*

Proof. Let J be a non-zero α-invariant closed two-sided ideal of A. By Theorem 2.2, there exists an element $i \in I$ such that

$$J \cap A^{G_i} \neq \{0\}.$$

Furthermore $J \cap A^{G_i}$ is an $[\alpha]^i$-invariant closed two-sided ideal of A^{G_i}. Since A^{G_i} is G/G_i-simple, $J \cap A^{G_i} = A^{G_i}$. Since A^{G_i} has the unit of A, we have $J = A$.

Corollary 2.4. *Under the same hypothesis of Theorem 2.2 if the C^*- dynamical system $(A_{G_i}, G/G_i, [\alpha]^i)$ is G/G_i-prime for each $i \in I$, then the C^*-dynamical system (A, G, α) is G-prime.*

Proof. Let J_1, J_2 be non-zero α-invariant closed two-sided ideals of A. Then there exist elements i_1 and $i_2 \in I$ such that

$$A^{G_{i_1}} \cap J_1 \neq \{0\}, \qquad A^{G_{i_2}} \cap J_2 \neq \{0\}.$$

There exists an element $i_0 \in I$ such that $A^{G_{i_1}}$ and $A^{G_{i_2}}$ are contained in $A^{G_{i_0}}$. Then $A^{G_{i_0}} \cap J_1$ and $A^{G_{i_0}} \cap J_2$ are non-zero $[\alpha]^{i_0}$ -invariant closed two-sided ideals of $A^{G_{i_0}}$. Since $A^{G_{i_0}}$ is G/G_{i_0}-prime, $A^{G_{i_0}} \cap J_1$ and $A^{G_{i_0}} \cap J_2$ have non-zero intersection. So we have $J_1 \cap J_2 \neq \{0\}$.

3. Main Result

Let (M, G, α) be a W^*-dynamical system and $M \subset B(H)$ for a Hilbert space H. The W^*-crossed product $M \times_\alpha G$ is the von Neumann algebra on $L^2(G, H)$ generated by $\{\pi_\alpha(x), \lambda_g |\, x \in M,\ g \in G\}$, where

$$(\pi_\alpha(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s), \quad (\lambda_g\xi)(s) = \xi(g^{-1}s)$$

for $x \in M, s, g \in G$ and $\xi \in L^2(G, H)$.

Lemma 3.1. *Let (A, G, α) be a C^*-dynamical system. Let I be an α-invariant ideal of A. Then $I \times_\alpha G$ is an ideal of $A \times_\alpha G$.*

Proof. Let $(\pi \times \lambda)$ be the universal representation of $A \times_\alpha G$ induced by a some covariant representation (π, λ, H) of A and $(A \times_\alpha G)''$ be the enveloping von Neumann algebra of $A \times_\alpha G$. Let I be an α-invariant ideal of A. Let p be a projection in the center of $\overline{\pi(A)}^{\sigma w}$ such that

$$\overline{\pi(I)}^{\sigma w} = \overline{\pi(A)}^{\sigma w} p.$$

It is clear that p is contained in $(A \times_\alpha G)''$. Since I is α-invariant, we have that $\lambda_g p \lambda_g = p$ for all $g \in G$. We put

$$\lambda_f = \int_G f(g)\lambda_g dg$$

for all f in $L^1(G)$. For $x \in A$ and $f \in L^1(G)$ we obtain that

$$\pi(x)\lambda_f p = \pi(x)p\lambda_f = p\pi(x)\lambda_f.$$

Since $(A \times_\alpha G)''$ is generated by $\{\pi(x)\lambda_f |\ x \in A,\ f \in L^1(G)\}$, p is contained in the center of $(A \times_\alpha G)''$. Also, since $(I \times_\alpha G)''$ is generated by $\{\pi(x)\lambda_f |\ x \in I,\ f \in L^1(G)\}$,

$$\overline{(\pi \times \lambda)(A \times_\alpha G)}^{\sigma w} p = \overline{(\pi \times \lambda)(I \times_\alpha G)}^{\sigma w}.$$

Hence $I \times_\alpha G$ is a norm closed two-sided ideal of $A \times_\alpha G$.

Theorem 3.2. *Let (A, G, α) be a C^*-dynamical system and G be a discrete group. Let G be a central shift in (A, G, α). Then A is a G-simple if and only if $A \times_\alpha G$ is simple.*

Proof. Let A'' be the enveloping von Neumann algebra of A and (A'', G, α'') be the W^*-dynamical system induced by the C^*-dynamical system (A, G, α). Since G is a central shift in (A, G, α), by [1] there exists an *-isomorphism ϕ from the enveloping von Neumann algebra $(A \times_\alpha G)''$ of $A \times_\alpha G$ onto the W^*-crossed product $A'' \times_{\alpha''} G$. Let J be a norm closed two-sided ideal of $A \times_\alpha G$. There exists a projection p_0 in the center of $(A \times_\alpha G)''$ such that

$$\overline{J}^{\sigma w} = (A \times_\alpha G)'' p_0$$

where $\overline{J}^{\sigma w}$ denotes the σ-weak closure of J. Then we have

$$\phi(\overline{J}^{\sigma w}) = (A'' \times_{\alpha''} G)\phi(p).$$

Since $\phi(p_0)$ is contained in the center of $A'' \times_{\alpha''} G$ and (A'', G, α'') acts centrally freely, there exists a projection q_0 in the center of A'' such that

$$\phi(p_0) = \pi_\alpha(q_0).$$

Let W_s be an operator on $l^2(G, H)$ to H such that

$$W_s\xi = \xi(s^{-1})$$

for each $s \in G$ and $\xi \in l^2(G, H)$. Put

$$E(x) = W_e x W_e^*$$

for $x \in B(l^2(G,H))$. We denote also the restriction of E to $A'' \times_{\alpha''} G$ by E. Then $E : A'' \times_{\alpha''} G \to A''$ is a faithful normal positive linear map. Let $\{p_k\}$ be an approximate unit of J. Then p_0 is the least upper bound of $\{p_k\}$. Since p_i exists in J for all $i \in I$, $E(\phi(p_i))$ is contained in A. Since $\phi(p_i) \le \phi(p_0)$, we get

$$E(\phi(p_i))q_0 = E(\phi(p_i)\pi_\alpha(q_0)) = E(\phi(p_i)\phi(p_0)) = E(\phi(p_i)).$$

Thus we have

$$\pi_\alpha(E(\phi(p_i))) = \pi_\alpha(E(\phi(p_i)))\phi(p_0).$$

Therefore $\pi_\alpha(E(\phi(p_i)))$ is contained in $\phi(J) \cap \pi_\alpha(A)$ for each $i \in I$. Since A is G-simple, $\phi(J) \cap \pi_\alpha(A)$ is $\{o\}$ or $\pi_\alpha(A)$. Since J is nonzero, $\phi(J) \cap \pi_\alpha(A) = \pi_\alpha(A)$. So we have

$$J = A \times_\alpha G.$$

The converse is an immediate consequence of Lemma 3.1.

Corollary 3.3. *Let (A, G, α) be a C^*-dynamical system and G be a discrete group. Assume that G is a central shift in (A, G, α). Then A is G-simple if and only if the reduced crossed product $A \times_{\alpha r} G$ is simple.*

Proof. Let J be a nonzero norm closed two-sided ideal of the reduced crossed product $A \times_{\alpha r} G$. There exists a projection p in the center of the W^*-crossed product $A'' \times_{\alpha''} G$ such that

$$\overline{J}^{\sigma w} = (A'' \times_{\alpha''} G)p$$

where $\overline{J}^{\sigma w}$ denotes the σ-weak closure of J. We then proceed the remaining part of this proof in the similar manner as the proof of Theorem 3.2.

Corollary 3.4. *Let (A, G, α) be a C^*-dynamical system and G be a discrete group. Assume that G is a central shift in (A, G, α) Then A is G-prime if and only if $A \times_\alpha G$ is prime*

Under the same hypothesis of Corollary 3.4 we can also show that A is G-prime if and only if the reduced crossed product $A \times_{\alpha r} G$ is prime.

Let (A, G, α) be a C^*-dynamical system and G be a discrete group. If G is a central shift in (A, G, α) then α is properly outer. It was shown that properly outerness and G-simplicity is the sufficient condition for simplicity of the reduced crossed product in the case of abelian C^*-algebras and AF-algebras. The above statement that properly outerness and G-simplicity is the sufficient condition for simplicity for the C^*-crossed product may be conjectured valuable for more general C^*-algebras.

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ON POSITIVE MULTILINEAR MAPS

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1. Definitions and Preliminaries

Let $\mathcal{E}$ be a vector space over **C**. Throughout this paper let $M_{m,n}(\mathcal{E})$ denote the vector space of $m \times n$ matrices with entries from $\mathcal{E}$, let $M_{m,n}$ denote the $m \times n$ complex matrices with C^*-norm. We set $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$ and $M_n = M_{n,n}$.

If $\mathcal{B}$ be a C^*-algebra and $\mathcal{E}$ be a subspace, then we call $\mathcal{E}$ an operator space. If $\mathcal{E}$ is a subset of a C^*-algebra $\mathcal{B}$, then we set

$$\mathcal{E}^* = \{a : \ a^* \in \mathcal{E}\},$$

and we call $\mathcal{E}$ self–adjoint when $\mathcal{E} = \mathcal{E}^*$. If $\mathcal{B}$ has a unit I and $\mathcal{E}$ is a self adjoint subspace of $\mathcal{B}$ containing I, then we call $\mathcal{E}$ an operator system.

Throught the paper $\mathcal{B}$ and $\mathcal{C}$ will denote unital C^*-algebras, $\mathcal{S}$ will denote operator system, and $\bar{\mathcal{S}}$ will denote the norm closure of $\mathcal{S}$.

Definition 1. A multilinear map $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{B}$ is selfadjoint if $\phi(x_1^*, \cdots, x_n^*) = \phi(x_1, \cdots, x_n)^*$ for $x_k \in \mathcal{S}_k$ ($1 \le k \le n$). A multilinear map $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{A}$ is positive if $\phi(x_1, \cdots, x_n)$ is positive whenever x_k is positive in $\mathcal{S}_k$ for $1 \le k \le n$, and bounded if $\|\phi\| = sup\{\|\phi(x_1, \cdots, x_n)\| : x_k \in \mathcal{S}_k, \|x_k\| \le 1\}$ is finite.

Definition 2. Let $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{B}$ be multilinrar positive. we define $\phi_k : M_k(\mathcal{S}_1) \times \cdots \times M_k(\mathcal{S}_n) \to M_k(\mathcal{B})$ by $\phi_k([x_{ij}^1], [x_{ij}^2], \cdots, [x_{ij}^n]) = [\phi(x_{ij}^1, x_{ij}^2, \cdots, x_{ij}^n]$. We say ϕ is k-positive if and only if ϕ_k is positive, ϕ is completely positive if and only if ϕ is positive for $k \in N$ and ϕ is completely bounded if and only if $\|\phi\|_{cb} = sup\{\|\phi_k\| : k \in N\}$ is finite.

2. Some Results

Proposition 1. *If a multilinear map $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{B}$ is positive, then ϕ is bounded and* $\|\phi\| \leq 2^n \|\phi(1, \cdots, 1)\|$ (cf. [12, Proposition 2.1]).

Proposition 2. *Let X_k be compact Hausdroff spaces for $1 \leq k \leq n, C(X_k)$ the continuous functions on X_k, and let $\phi : C(X_1) \times \cdots \times C(X_n) \to \mathcal{B}$ be positive multilinear. Then* $\|\phi\| = \|\phi(1, \cdots, 1)\|$ (cf. [12, Theorem 2.4]).

Proposition 3. *Let $\mathcal{B}_1, \cdots, \mathcal{B}_n, \mathcal{C}$ be unital C^*-algebras, let $\mathcal{A}_k$ be a subalgebra of $\mathcal{B}_k$ with $1_k \in A_k$, and let $\mathcal{S}_k = \mathcal{A}_k + \mathcal{A}_k^*$. If $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{C}$ is multilinear positive, then $\|\phi(a_1, \cdots, a_n)\| \leq \|\phi(1, \cdots, 1)\| \|a_1\| \cdots \|a_n\|$ for all a_k in $\mathcal{A}_k$* (cf. [12, Corollary 2.8]).

Proposition 4. *Let $\phi : \mathcal{S}_1 \times \cdots \mathcal{S}_n \to B$ be a unital 2-positive multilinear map. Then ϕ is contractive* (cf. [12, Proposition 3.2]).

Proposition 5. *Let $\phi : \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \to \mathcal{B}$ be a completely positive multilinear map. Then ϕ is completely bounded and* $\|\phi(1, \cdots, 1)\| = \|\phi\| = \|\phi\|_{cb}$ (cf. [12, Proposition 3.5]).

Proposition 6. *Let X_k be a compact Hausdorff Space for $1 \leq k \leq n$, $C(X_k)$ the continuous functions on X_k, and let $\phi : C(X_1) \times \cdots \times C(X_n) \to \mathcal{B}$ be a positive multilinear map. Then ϕ is completely positive* (cf. [12, Theorem 3.8]).

For a matrix A, let $A_k = [A_{ij}]$ denote the $k \times k$ matrix with $A_{ij} = A$ for $1 \leq i, j \leq k$.

Proposition 7. *Let $\phi : M_{k_1} \times \cdots \times M_{k_n} \to \mathcal{B}$ be a multilinear map, let $\{E_{ij}^{k_l}\}$ denote the standard matrix units for M_{k_l} and let $a = k_1 \cdots k_n$, $t_l = k_1 \cdots k_{l-1}$, $s_l = k_{l+1} \cdots k_n$, $t_1 = 1$, $s_n = 1$, $E_l = [E_{ijs_l}^{k_l}]_{t_l}$. Then the following are equivalent.*

(1) *ϕ is completely positive.*
(2) *ϕ is a-positive*
(3) *$\phi_a(E_1, \cdots, E_n)$ is positive.*

Proposition 8. *Let $\mathcal{H}$ be a Hilbert space and let $\phi : \mathcal{B}_1 \times \cdots \times \mathcal{B}_r \to \mathcal{B}(\mathcal{H})$ be a n-positive multilinear map. Then if $a_{ki} \in \mathcal{B}_k$, $i = 1, 2, \cdots, n-1$ $k = 1, \cdots, r$, we have*

$$[\phi(a_{1i}^*, \cdots, a_{ri}^*)\phi(a_{1j}, \cdots, a_{rj})] \leq \|\phi\|[\phi(a_{1i}^* a_{1j}, \cdots, a_{ri}^* a_{rj})]$$

in $M_{n-1}(\mathcal{B}(\mathcal{H}))$ (cf. [5, Theorem 2].

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PIMSNER POPA BASIS FOR A PAIR OF FINITE VON NEUMANN ALGEBRAS

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P.Jolissant [Jol] extends the Jones' index to the case of a pair $N \subset M$ of finite σ-finite von Neumann algebras using the completely positive maps between $Z(M)$ and $Z(N)$, also J. Bion-Nadal [Bi] extends the Jones index to the case of a pair $N \subset M$ of a II_1 factor M and a von Neumann subalgebra N, using the correspondence associated to the inclusion $N \subset M$. First we review the work of P. Jolissant ([Jol]).

Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras, and let tr_M be a faithful normal normalized finite trace on M and $L^2(M, tr_M)$ be the standard representation of M. Then N is of finite index in M if $L^2(M, tr_M)$ is a finite representation of the pair $N \subset M$, i.e. M, M', N, N' are finite on $L^2(M, tr_M)$ and the coupling operators c_N and $c_{N'}$ are bounded. In this case define $S_N^M : Z(M) \to Z(N)$ and $T_N^M : Z(N) \to Z(M)$ by $S_N^M(z) = c_N z^{\natural_{N'}}$, $T_N^M(w) = w^{\natural_M}$ and let $C_N^M = S_N^M T_N^M$, $D_N^M = T_N^M S_N^M$. Then define the index $[M : N]$ of N in M as the spectral radius of C_N^M or D_N^M ([Jol]).

Let $E_N : M \to N$ be the tr_M preserving conditional expectation i.e. $tr_M(E_N(x)y) = tr_M(xy)$ for $x \in M, y \in N$. Let ξ_0 be the canonical cyclic trace vector in $L^2(M, tr_M)$ and identify M with the algebra of left multiplication operators on $L^2(M, tr_M)$ then the conditional expectation E_N extends to a projection e_N onto $L^2(N, tr_M)$ via $e_N(x\xi_0) = E_N(x)\xi_0$. Let J be the involution $x\xi_0 \to x^*\xi_0$. Then ([Jon])

(1) $e_N x e_N = E_N(x) e_N$ for $x \in M$,
(2) If $x \in M$ then $x \in N \Leftrightarrow e_N x = x e_N$,
(3) $N' = \{M' \cup \{e_N\}\}''$,
(4) J commutes with e_N.

Let $\langle M, e_N \rangle$ be the von Neumann algebra generated by M and e_N and say the basic construction of $N \subset M$. The followings are given by V. Jones [Jon];

(1) $\langle M, e_N \rangle$ is a factor iff N is a factor,
(2) $\langle M, e_N \rangle$ is finite iff N' is finite,
(3) $\langle M, e_N \rangle = JN'J$.

LEMMA 1. *Let $N \subset M \subset \langle M, e_N \rangle \subset B(L^2(M))$ be a basic construction of a pair $N \subset M$ of finite σ finite von Neumann algebras with finite index. Then*

(1) $(JNJ)' = JN'J, \quad Z(JNJ) = JZ(N)J;$
(2) $x^{\natural_{JNJ}} = J(JxJ)^{\natural_N} J, \quad y^{\natural_{JN'J}} = J(JyJ)^{\natural_{N'}} J, \quad x \in JNJ,\ y \in JN'J;$
(3) $x^{\natural_{M'}} = (Jx^*J)^{\natural_M}, \quad x \in M';$
(4) $c_{JNJ} = Jc_N J, \quad c_{JN'J} = Jc_{N'} J;$
(5) $e_N{}^{\natural_{N'}} = c_N^{-1}, e_N{}^{\natural_{\langle M, e_N \rangle}} = Jc_N^{-1} J;$
(6) $[\langle M, e_N \rangle : M] = [M : N]$.

Proof. Using $J^2 = 1$ and the uniqueness of $\natural_{JNJ}$ and $\natural_{JN'J}$ (1),(2) can be easily varified. For each ξ in $L^2(M)$, we get

$$\begin{aligned}(e_\xi^{JN'J})^{\natural_{JNJ}} &= J(Je_\xi^{JN'J}J)^{\natural_N} J = J(e_{J\xi}^{N'})^{\natural_N} J \\ &= Jc_N(e_{J\xi}^{N})^{\natural_{N'}} J = Jc_N JJ(Je_\xi^{JNJ}J)^{\natural_{N'}} J \\ &= Jc_N J(e_\xi^{JNJ})^{\natural_{JN'J}}.\end{aligned}$$

So that $c_{JNJ} = Jc_N J$ and similarly we have $c_{JN'J} = Jc_{N'} J$. Let ξ_0 be the separating cyclic vector for M in $L^2(M)$ then $e_{\xi_0}^{N'} = 1$ and $e_N = e_{\xi_0}^{N}$. So $1 = (e_{\xi_0}^{N'})^{\natural_N} = c_N(e_N)^{\natural_{N'}}$. Thus $e_N{}^{\natural_{N'}} = c_N^{-1}$ and $e_N{}^{\natural_{\langle M, e_N \rangle}} = e_N{}^{\natural_{JN'J}} = J(Je_N J)^{\natural_{N'}} J = J(e_N)^{\natural_{N'}} J = Jc_N^{-1} J.$ Q.E.D.

DEFINITION (1) A faithful normal finite trace ϕ on M is a *Markov trace* of modulus α for the pair $N \subset M$ if it extends to a faithful normal finite trace $\widetilde{\phi}$ on $\langle M, e_N \rangle$ such that $\widetilde{\phi}(e_N x) = \alpha^{-1}\phi(x)$ for each x in M.

(2) The inclusion $N \subset M$ is *connected* if $Z(M) \cap Z(N) = \mathbf{C}$.

(3) If F is a bounded normal linear map from $Z(M)$ to $Z(N)$ define a map F_* from $N_{*,c}$ to $M_{*,c}$ by $F_*(\phi) = \phi \circ F \circ \natural_M$ for every ϕ in $N_{*,c}$, where $M_{*,c} = \{ \phi \in M_* \mid \phi(xy) = \phi(yx)$ for every x, y in $M \}$.

REMARK Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras with finite index .

(1) If M is a II_1 factor then the canonical trace tr_M is the Markov trace for $N \subset M$ of modulus $[M : N]$ and $[M : N] = tr_M(c_N)$.

(2) If N is a II_1 factor then $\phi(x) = tr_{N'}(x^{\natural_M})$, $x \in M$, is the Markov trace for $N \subset M$ of modulus $[M : N]$ and $[M : N] = c_N$.

(3) Let $N \subset M \subset \langle M, e_N \rangle$ be a basic construction. If tr be a normalized Markov trace for $M \subset \langle M, e_N \rangle$ of modulus $[\langle M, e_N \rangle : M]$ then the restriction $tr|M$ is a normalized Markov trace for $N \subset M$ of modulus $[M : N]$

(4) If $Z(M)$ or $Z(N)$is of finite dimensional then they are both finite dimensional, ([Jol] Lemma 4.1).

(5) If M is a factor then there exists minimal central projections $q_1, \cdots, q_n$ in N such that $N = N_{q_1} \oplus \cdots \oplus N_{q_n}$, and by ([Jol],Example 2.7) $[M : N] = \sum_{i=1}^{n} [M_{q_i} : N_{q_i}]$.

Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras of finite index. If M and N have atomic centers then the sets Min(M) and Min(N) , sets of minimal central projections of M and N respectively , are countable sets, and by([Jol], Lemma 4.1) Min(M) is a finite set if and only if Min(N) is a finite set. Let Min(M)=$\{ p_i \mid i \in I \}$ and Min(N)=$\{ q_j \mid j \in J \}$, then $Z(M) = \{ \sum_{i \in I} \lambda_i p_i | \sup_{i \in I} |\lambda_i| < \infty \}$, $Z(N) = \{ \sum_{j \in J} \lambda_j q_j \mid \sup_{j \in J} |\lambda_j| < \infty \}$ and $\mathrm{Min}(\langle M, e_N \rangle) = \{ \tilde{q_j} \mid j \in J \}$, where $\tilde{q_j} = J q_j J$.

Proposition 2. *Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras of finite index. Suppose that there is a normalized Markov trace tr_M for $N \subset M$ of modulus $[M : N]$, and let tr be the extension of tr_M to $\langle M, e_N \rangle$, then*

(1) $tr_M(x^{\natural_M}) = tr_M(x), \quad x \in M;$
(2) $tr_M(c_N) = [M : N] = tr(e_N)^{-1};$
(3) $tr(x) = [M : N]^{-1} tr(c_N J x^{*\natural_{M_1}} J) \quad x \in \langle M, e_N \rangle.$

If M and N have atomic centers then

(1) $tr_M(x) = \sum_{i\in I} tr_i(xp_i)tr_M(p_i), \quad x \in M;$

(2) $tr(x) = [M : N]^{-1} \sum_{j\in J} tr_M(c_N q_j)\widetilde{tr}_j(x\widetilde{q}_j), \quad x \in \langle M, e_N\rangle.$

where $\widetilde{tr}_j$ and tr_i are canonical traces on $\langle M, e_N\rangle_{q_j}$ and M_{p_i} respectively.

Proof. Let $\alpha = [M : N]$. Since tr_M is the Markov trace for $N \subset M$ of modulus α , for each x in M
$tr_M(x) = \alpha^{-1}D_*(tr_M)(x) = \alpha^{-1}tr_M(D(x^{\natural_M})) = \alpha^{-1}tr_M((c_N x^{\natural_M \natural_{N'}})^{\natural_M})$,
so $tr_M(c_N) = \alpha$ and $tr_M(x^{\natural_M}) = tr_M(x)$. By Lemma 1.3 the extension tr of tr_M is given by $\alpha^{-1}\widetilde{S}_*(tr_M)$, where $\widetilde{S} = S_M^{\langle M, e_N\rangle}$. So that for each x in $M_1 = \langle M, e_N\rangle$ we get

$$\begin{aligned} tr(x) &= \alpha^{-1}\widetilde{S}_*(tr_M)(x) = \alpha^{-1}tr_M((c_{M_1'} x^{\natural_{M_1}})^{\natural_{M'}}) \\ &= \alpha^{-1}tr_M((Jc_{M_1'} x^{*\natural_{M_1}} J)^{\natural_M}) \\ &= \alpha^{-1}tr_M((Jc_{M_1'} x^{*\natural_{M_1}} J)) \end{aligned}$$

Suppose that M and N have atomic centers. Then for each p_i in $\text{Min}(M)$, $tr_M(p_i)^{-1}tr_M$ is the canonical trace on M_{p_i}, so $tr_M(x) = \sum_{i\in I} tr_M(xp_i) = \sum_{i\in I} tr_i(xp_i)tr_M(p_i)$, for each x in M. And for each $x \in M_1$, we get $x^{\natural_{M_1}} = \sum_{j\in J} \widetilde{tr}_j(x\widetilde{q}_j)$. So that

$$\begin{aligned} tr(x) &= \alpha^{-1}tr_M((Jc_{M_1'} x^{*\natural_{M_1}} J)) = \alpha^{-1}tr_M(c_N J \sum_{j\in J} \widetilde{tr}_j(x^*\widetilde{q}_j)\widetilde{q}_j J) \\ &= \alpha^{-1} \sum_{j\in J} tr_M(c_N q_j)\widetilde{tr}_j(x\widetilde{q}_j) \end{aligned}$$

Q.E.D.

LEMMA 3. ([Jol] Lemma 4.2) *Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras of finite index. Suppose that M and N have atomic centers. Then*

(1) *If $x \in \langle M, e_N\rangle$, there is a unique $y \in M$ for which $xe_N = ye_N$;*

(2) $\langle M, e_N\rangle = Me_N M = \{ \sum_{i=1}^n a_i e_N b_i \mid n \geq 1\ a_i, b_i \in M \}$.

LEMMA 4. *Let M be a finite σ-finite von Neumann algebra.*

(1) *If p is a central projection in M then*

$$(x_p)^{\natural_{M_p}} = (x^{\natural_M})_p \quad \text{for any } x \text{ in } M.$$

(2) *If M has atomic center, then*

$$x^{\natural_M} = \sum_{i \in I} tr_i(xp_i)p_i \quad \text{for any } x \text{ in } M,$$

where tr_i is the canonical trace on M_{p_i}.

Proof. Since p is a central projection in M , $Z(M_p) = Z(M)p$. Then by the uniqueness of the center valued trace (1) can be easily proved. Since p_i's are minimal central projections in M, $M_{p_i} = Mp_i$ is a factor. And for each $x \in M$, $(xp_i)^{\natural_M} = x^{\natural_M}p_i$. So $\natural_M$ is the normalized centervalued trace on M_{p_i}. Therefore $x^{\natural_M} = \sum (xp_i)^{\natural_M} = \sum tr_i(xp_i)p_i$ for any x in M. Q.E.D.

Definition Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras of finite index, and let $[M : N] = l + \varepsilon$ for some integer l and $0 \le \varepsilon < 1$. *Pimsner-Popa basis* for $N \subset M$ is a family $\{m_j\}_{1 \le j \le l+1}$ of elements in M satisfying the followings:

(1) $E_N(m_j^* m_k) = 0$ if $j \neq k$;
(2) $E_N(m_j^* m_j) = 1$ if $j = 1, \cdots, l$;
(3) $E_N(m_{l+1}{}^* m_{l+1})$ is a projection of trace ε.

REMARK (a) If there is a Markov trace for $N \subset M$ of modulus $[M : N]$ then any Pimsner-Popa basis $\{m_j\}_{1 \le j \le l+1}$ for a pair $N \subset M$ satisfies the followings :

(1) $m_i e_N$ are partial isometries , $1 \le i \le l + 1$;

(2) $\sum_{i=1}^{l+1} m_i e_N m_i^* = 1$;

(3) $\sum_{i=1}^{l+1} m_i m_i^* = [M : N]$;

(4) Every element $m \in M$ has a unique decomposition ;

$$m = \sum_{i=1}^{l+1} m_i y_i$$

where $y_i \in N,\ 1 \le i \le l,\ \ y_{l+1} \in E_N(m_{l+1}{}^* m_{l+1})N$.

(5) If $\{m_i'\}_{1 \le i \le l+1}$ is another Pimsner-Popa basis for $N \subset M$ then the matrix $(E_N(m_i^* m_j'))_{1 \le i,j \le l+1}$ is a unitary element in N_α such that

$$m_k' = \sum_{j=1}^{l+1} m_j E_N(m_j^* m_k')$$

where $\alpha = [M : N]$ and we shall identify the elements in the amplification N_α of N with $(l+1)\times(l+1)$ matrices $(a_{ij})_{i,j}$ such that the entries a_{ij} satisfy $a_{ij} \in N,\ \ a_{i,l+1} \in Np,\ \ a_{l+1,j} \in pN,\ \ a_{l+1,l+1} \in pNp$.

(b) If N and M are II_1 factors M. Pimsner and S. Popa ([PP]) shows that there exists a Pimsner-Popa basis for $N \subset M$. But in our case the Pimsner-Popa basis does not exist in general. In next theorem we consider the necessary and sufficient condition under which the Pimsner-Popa basis exists. The theorem generalize the Pimsner and Popa's result.

THEOREM 5. ([Boo] Theorem 2.1.) *Let $N \subset M$ be a pair of finite σ-finite von Neumann algebras with atomic centers, N being of finite index in M. Suppose that there is a normalized Markov trace for $N \subset M$ of modulus $[M : N]$. Then there is a Pimsner-Popa basis for the pair $N \subset M$ if and only if $c_N(L^2(M)) \ge l$.*

EXAMPLE Let M be a II_1 factor and K be a subfactor of M such that $[M : K] = k + \varepsilon$, where k is an integer and $0 < \varepsilon < \frac{1}{n}$ for some integer $n > 1$. Choose projections $p_1, \cdots, p_n$ in K such that $\sum p_i = 1, tr(p_1) = \frac{\alpha}{nk}$ and $tr(p_i) = tr(p_j)$ for each $i, j = 2, \cdots, n$, where $\alpha = k + \varepsilon$. Let $N = K_{p_1} \oplus \cdots \oplus K_{p_n}$ and let $c_N(L^2(M)) = \sum \lambda_i p_i$, then we have

$$\lambda_i = c_{K_{p_i}}(p_i L^2(M)) = tr(p_i)^{-1} c_K(L^2(M)) = tr(p_i)^{-1}\alpha.$$

So by Proposition 1.7, $[M : N] = n\alpha$ and $nk = \lambda_1 < \lambda_2 = \cdots = \lambda_n$. Thus there is a Pimsner-Popa basis for $N \subset M$ but c_N is not a constant and $\lambda_n - \lambda_1 > n\varepsilon$.

COROLLARY 6. *The elements of* $\langle M, e_N\rangle$ *are of the form* $\sum a_i e_N b_i$, *for some finite sets* $\{a_i\}, \{b_i\}$ *of elements in* M.

Proof. For any x in $\langle M, e_N\rangle$

$$x = \sum x m_i e_N m_i^* = \sum y_i e_N m_i^*$$

for some y_i in M. Q.E.D.

Let $\{m_i\}$ be the Pimsner-Popa basis for the pair $N \subset M$ as given in theorem 2.1 and let $K = \oplus_{i=1}^{l} L^2(N) \oplus pL^2(N)$, where $p = E_N(m_{l+1}{}^* m_{l+1})$. Then $N_\alpha \subset B(K)$. Define for each $(y_{i,j}) \in N_\alpha$, $\phi((y_{i,j})) = \sum_{i,j} m_i y_{i,j} e_N m_j{}^*$.

COROLLARY 7. N_α *is spatially isomorphic with* $\langle M, e_N\rangle$.

Proof. For $x \in M$ with $x = \sum m_i x_i$, the decomposition of x using the Pimsner-Popa basis, define $u(x\xi_0) = (x_1\xi_0, \cdots, x_{n+1}\xi_0) \in K$. Then u is a unitary operator from $L^2(M)$ onto K, since for $x = \sum m_i x_i$, $y = \sum m_i y_i$,

$$\begin{aligned} < x\xi_0, y\xi_0 > &= tr_M(y^* x) = tr_M(\sum_{i,j} y_i{}^* m_i{}^* m_j x_j) \\ &= \sum_{i,j} tr_M(E_N(y_i{}^* m_i{}^* m_j x_j)) = \sum_{i,j} tr_M(y_i{}^* E_N(m_i{}^* m_j) x_j) \\ &= \sum tr_M(y_i x_i) = \sum < x_i\xi_0, y_i\xi_0 > . \end{aligned}$$

And for $\xi = x\xi_0 \in L^2(M)$, $x = \sum m_i x_i$, and $(y_{i,j}) \in N_\alpha$,

$$\begin{aligned} u^*(y_{i,j})u(\xi) &= u^*(y_{i,j})(x_1\xi_0, \cdots, x_{n+1}\xi_0) \\ &= u^*(\sum_j y_{1,j} x_j \xi_0, \cdots, \sum_j y_{n+1,j} x_j \xi_0) = \sum_i m_i \sum_j y_{i,j} x_j \xi_0 \\ &= \sum_{i,j} m_i y_{i,j} E_N(m_j{}^* x)\xi_0 = \sum_{i,j} m_i y_{i,j} e_N m_j{}^* x \xi_0 \\ &= \sum_{i,j} m_i y_{i,j} e_N m_j{}^*(\xi) = \phi((y_{i,j}))(\xi) \end{aligned}$$

Thus $\phi = Ad(u)$ and the surjectivity is clear by $\sum m_i e_N m_i{}^* = 1$. Q.E.D.

REMARK (1) For each $x \in \langle M, e_N \rangle$ the element $(y_{i,j})$ of N_α which corresponds with x via ϕ is given by $y_{i,j} = \alpha E_N E_M(m_i{}^* x m_j e_N)$.
(2) If q is a self-adjoint element of $Z(N)$ then $JqJ \in \langle M, e_N \rangle$ corresponds to the diagonal matrix in N_α with diagonal $(q, \cdots, q, pq)$.
(3) Each element x of M corresponds to $(E_N(m_i{}^* x m_j))$ and the projection e_N corresponds to $(E_N(m_i{}^*)E_N(m_j))$.

EXAMPLE Let $p_1, \cdots, p_n$ be projections in a II_1 factor M such that $\sum p_i = 1$ and let $N = M_{p_1} \oplus \cdots \oplus M_{p_n} \subset M$. Then $c_N = \sum tr_M(p_i)^{-1} p_i$ and $[M : N] = n$. So that there is a Pimsner-Popa basis for $N \subset M$ if and only if $tr_M(p_i) = \frac{1}{n}$ for all i. In this case there are partial isometries $w_{i,j}, i, j = 1, \cdots, n$ such that $w_{i,,j} w_{k,l} = \delta_{j,k} w_{i,l}$ and $w_{i,i} = p_i$ and the extension tr_{M_1} of tr_M to $M_1 = \langle M, e_N \rangle$ is given by $tr_{M_1}(x) = \frac{1}{n} \sum tr_i(x\widetilde{p_i})$ for each x in M_1, where $\widetilde{p_i} = Jp_iJ$ and tr_i is the canonical trace on $\langle M, e_N \rangle_{\widetilde{p_i}}$. Now $\langle M, e_N \rangle_{\widetilde{p_i}} = M_{\widetilde{p_i}}$ and $\langle M, e_N \rangle = \oplus_{i=1}^n M_{\widetilde{p_i}}$. So $E_N(x) = \sum_{i=1}^n p_i x p_i$, for each x in M and $E_M(x) = \frac{1}{n} \sum_{i=1}^n x_i$ for each x in M_1, where $x = \sum x_i \widetilde{p_i}$ with $x_i \in M$ and also $e_N = \sum p_i \widetilde{p_i}$. Let $m_k = \sum_{i=1}^n w_{[i+k-1],i}$ for $k = 1, \cdots, n$ where $1 \leq [i+k-1] \leq n$ with $[i+k-1] = i+k-1 \pmod n$, then $\{m_k\}_{1 \leq k \leq n}$ is a Pimsner-Popa basis for $N \subset M$. In correspondence between $\langle M, e_N \rangle$ and N_α, e_N corresponds to the diagonal operator $\mathrm{diag}(1, 0, \cdots, 0)$, $p_i, i = 1, \cdots, n$, corresponds to the diagonal operator $\mathrm{diag}(p_i, p_{i-1}, \cdots, p_1, p_n, \cdots, p_{i+1})$ and $\widetilde{p_i}$ corresponds to $\mathrm{diag}(p_i, \cdots, p_i)$.

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