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PROCEEDINGS OF KOREA-JAPAN PARTIAL DIFFERENTIAL EQUATIONS CONFERENCE

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HI JUN CHOE
HYEONG-OHK BAE

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

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Hi Jun Choe and Hyeong-Ohk Bae
Department of Mathematics,
KAIST, Taejon 305-701,
Korea

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Preface

From Dec. 16 – Dec. 18, 1996, THE 4TH KOREA–JAPAN PARTIAL DIFFERENTIAL EQUATIONS CONFERENCE was held at KAIST in Taejon, Republic of Korea. Many different topics, including fluid mechanics, solid mechanics, semilinear equations, Schrödinger equations and etc., were discussed and presented. Most of the talks were in the highest quality and the audience were very much involved in the discussions. There have been good communications among all the participants. We would like to express good gratitude for the help of CAM at KAIST and GARC at SNU.

Dec. 1, 1997

Hi Jun Choe and Hyeong–Ohk Bae,
KAIST, Taejon, Korea

Participants:

Bae, Hyeong–Ohk	KAIST, Taejon, Korea
Baek, Jeong–Seon	Chonnam National University, Kwangju, Korea
Byeon, Jae Young	GARC, Seoul National University, Seoul, Korea
Chae, Dongho	Seoul National University, Seoul, Korea
Choe, Hi Jun	KAIST, Taejon, Korea
Choi, Q–Heung	Inha University, Incheon, Korea
Ishii, Katsuyuki	Kobe University of Mercantile Marine, Kobe, Japan
Jeon, Youngmok	Ajou University, Suwon, Korea
Jimbo, Shuichi	Hokkaido University, Sapporo, Japan
Kang, Hyeongbae	Seoul National University, Seoul, Korea
Kawashima, S.	Kyushu University, Fukuoka, Japan
Kim, Namkwon	Seoul National University, Seoul, Korea
Kim, Wan Se	Hanyang University, Seoul, Korea
Koike, Shigeaki	Saitama University, Saitama, Japan
Kwak, Minkyu	Chonnam National University, Kwangju, Korea
Lee, Choon–Ho	Hoseo University, Cheonan, Korea
Lee, Hyung–Chun	Ajou University, Suwon, Korea
Lee, Yong–Hoon	Pusan National University, Pusan, Korea
Misawa, Masashi	University of Electro–Communications, Japan
Miyakawa, Tetsuro	Kyushu University, Fukuoka, Japan
Nakao, Mitsuhiro	Kyushu University, Fukuoka, Japan
Nam, Heeseok	Seoul National University, Seoul, Korea
Nii, Shunsaku	Saitama University, Saitama, Japan
Ogawa, Takayoshi	Nagoya University, Nagoya, Japan
Sakaguchi, Shigeru	Ehime University, Ehime, Japan
Seo, Jin Keun	Yonsei University, Seoul, Korea
Sheen, Dongwoo	Seoul National University, Seoul, Korea
Suzuki, Takashi	Osaka University, Toyonakashi, Japan
Takagi, Izumi	Tôhoku University, Sendai, Japan
Tsutsumi, M.	Waseda University, Japan
Yajima, Kenji	University of Tokyo, Tokyo, Japan
Yanagida, Eiji	University of Tokyo, Tokyo, Japan

Contents

L^∞-BOUND OF WEAK SOLUTIONS TO NAVIER-STOKES EQUATIONS

HYEONG-OHK BAE AND HI JUN CHOE

ABSTRACT. We prove that a solution $u = (u_1, u_2, u_3)$ to Navier-Stokes equations is locally bounded, therefore, u is locally smooth if any two components of u are locally bounded.

1. INTRODUCTION AND STATEMENT OF THE RESULT.

In this paper we study the regularity of the weak solutions of the incompressible Navier-Stokes equations with viscosity ν

$$\begin{cases} \dfrac{du_i}{dt} - \nu\Delta u_i + (u\cdot\nabla)u_i + \nabla_i p = f_i, \\ \nabla\cdot u = 0 \end{cases} \tag{1.1}$$

in $Q = \mathbf{R}^3 \times (0,\infty)$. We assume that any weak solution

$$u \in L^2(0,\infty; H^1(\mathbf{R}^3)) \cap L^\infty(0,\infty; L^2(\mathbf{R}^3))$$

satisfies

$$\int u\cdot\phi_t - \nu\nabla u\cdot\nabla\phi - (u\cdot\nabla)u\cdot\phi + p\nabla\cdot\phi + f\cdot\phi \, dx\, dt = 0$$

for all $\phi \in C_0^\infty(Q)$. The existence of weak solutions was proved by {Leray, [5]} and {Hopf, [4]}. Since the viscosity can be treated by scaling, we simply assume that $\nu = 1$. Also for the simplicity we assume that f is a smooth function in Q.

It is well known that if the viscosity is large, or initial data are small, then the solution lies in $L^\infty(0,\infty; H^1(\mathbf{R}^3)) \cap L^2(0,\infty; H^2(\mathbf{R}^3))$. We know that boundedness of u implies higher regularity of u in the interior and hence we can bound various higher norms in terms of L^∞-norm of u. From Sobolev's embedding theorem we know that the solution space of weak solutions $L^2(0,\infty; H^1(\mathbf{R}^3)) \cap L^\infty(0,\infty; L^2(\mathbf{R}^3))$ is continuously embedded in $L^{\frac{10}{3}}_{loc}(Q)$. But we do not know yet how to bound L^∞-norm of u in terms of $L^{\frac{10}{3}}$-norm of u. On the other hand as far as interior is concerned, it was proved by {Serrin, [6]} that any weak solution u of (1) on a cylinder

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$B \times (a,b)$ satisfying

$$\int_a^b \left(\int_B |u|^r \, dx \right)^{\frac{r'}{r}} dt < \infty \quad \text{with} \quad \frac{3}{r} + \frac{2}{r'} < 1, \quad r \geq 3$$

is necessarily L^∞ function on any compact subsets of the cylinder. Observe that when $r = r' = 5$, u is in L^5 and 5 is the critical number for the homogeneous Lebesgue space. The limiting case $3/r + 2/r' = 1, r > 3$ for the initial value problem was considered by {Fabes, Jones & Riviere, [2]} and their method seems not applicable to local problems. See also {Giga, [3]} and {Sohr, von Wahl, [7]}. Also {Struwe, [8]} improved Serrin's method and proved the boundedness of weak solutions in the interior for the critical cases, that is, $\frac{3}{r} + \frac{2}{r'} = 1, r > 3$. Since all the above results uses the vorticity equations, they are not applicable to boundary L^∞ estimate. {Takahashi, [9]} found some criterion for L^∞ regularity near boundary for the weak solution satisfying $u \in L^{r,r'}, \frac{3}{r} + \frac{2}{r'} \leq 1$. He imposed some integrability conditions on the velocity gradient and pressure in domain Q, that is,

$$\nabla u, p \in L^{r_0, r_0'}(Q) \quad \text{for all} \quad 1 < r_0, r_0' < \infty \quad \text{with} \quad \frac{3}{r_0} + \frac{2}{r_0'} = 3.$$

{Choe, [1]} showed the L^∞ regularity of u up to boundary for the limiting case that $u \in L^{r,r'}(Q), \frac{3}{r} + \frac{2}{r'} \leq 1$ with $r \geq 3$ or $u \in L^{3,\infty}$ with $\|u\|_{L^{3,\infty}} \leq \varepsilon_0$ for some small $\varepsilon_0 > 0$ under the assumption that the boundary data of the pressure is bounded. He also showed that the weak solution is as regular as the boundary data of the pressure.

We show that if any two components, for example u_1, u_2 of a weak solution $u = (u_1, u_2, u_3)$ lie in $L^\infty(Q)$, then u is in $L^\infty_{loc}(0,\infty; H^1_{loc}(\mathbf{R}^3)) \cap L^2_{loc}(0,\infty : H^2_{loc}(\mathbf{R}^3))$. To show this, we use the result of {Struwe, [8]} that if u lies in $L^5(Q)$ then u is locally bounded and hence regular in Q. Here $L^{r,r'}(Q)$ is the set of u that

$$\int \left(\int |u|^r \, dx \right)^{r'/r} dt < \infty \qquad \text{with } \frac{3}{r} + \frac{2}{r'} \leq 1.$$

The Sobolev space $W^{k,m}(Q)$ is the space of functions in $L^m(Q)$ with derivatives of order less than or equal to k in $L^m(Q)$ (k an integer). This is a Banach space with the norm $\|f\|_{W^{k,m}(Q)} = \left(\sum_{|j| \leq k} \|\nabla^j f\|^m_{L^m(Q)} \right)^{\frac{1}{m}}$. In particular when $m = 2$, $W^{k,2}(Q) = H^k(Q)$ is Hilbert space. We denote C a constant.

Now we state our main result.

Theorem. *Suppose that (u,p) is a weak solution. Let f is locally smooth, that is, $f \in L^2(0,T;W^{1,12/11}_{loc}(\mathbf{R}^3))\cap L^2_{loc}(Q)$, where $W^{1,12/11}_{loc}(\mathbf{R}^3)$ is a Sobolev space. Let $u \overset{\text{def}}{=} (u_1,u_2,u_3)$ and $Q \overset{\text{def}}{=} \mathbf{R}^3 \times (0,\infty)$. If any two of the three components u_1,u_2,u_3 of u belong to $L^\infty_{loc}(Q)$, for example $u_1,u_2 \in L^\infty_{loc}(Q)$, then*

$$u \in L^\infty_{loc}(Q).$$

Therefore, u is locally smooth.

2. Proof of Main Theorem

We let $u \overset{\text{def}}{=} (u_1,u_2,u_3)$ be a weak solution to Navier-Stokes equations

$$\begin{cases} \dfrac{du}{dt} - \Delta u + u\cdot\nabla u + \nabla p = f, \\ \nabla\cdot u = 0 \end{cases} \tag{2.1}$$

in $Q = \mathbf{R}^3 \times (0,\infty)$ and the initial condition $u(x,0) = u_0$ for $x \in \mathbf{R}^3$.

To estimate that $\|u_3\|_\infty \le C$, we use the the result of {Struwe, [8]}:

> Suppose that $u \in L^{2,\infty}(\Omega\times(0,T);\mathbf{R}^3)$ with $|\nabla u| \in L^2(\Omega\times(0,T))$ is a weak solution to the Navier-Stokes equations for an open domain $\Omega \subset \mathbf{R}^3$, and that $u \in L^{r,r'}(\Omega\times(0,T))$, where $1 < r, r' < \infty$, $3/r + 2/r' \le 1$. Then u is locally bounded in $\Omega\times(0,\infty)$.

We use the above theorem for $r = r' = 5$. In other words, we want to show that $u_3 \in L^{5,5}_{loc}(Q)$. We notice that, for a solution v of Navier-Stokes equations,

$$\begin{aligned}\iint |v|^5\eta^5\,dx\,dt &= \iint |v|^3\eta^3|v|^2\eta^2\,dx\,dt \\ &\le C\int\Big(\int|v\eta|^9\,dx\Big)^{1/3}\Big(\int|v\eta|^3\,dx\Big)^{2/3}dt \\ &\le C\int\Big(\int|\nabla|v\eta|^{3/2}|^2\,dx\Big)\Big(\int|v\eta|^3\,dx\Big)^{2/3}dt \\ &\le C\sup_t\|v\eta\|^2_{L^3}\iint|\nabla|v\eta|^{3/2}|^2\,dx\,dt,\end{aligned}$$

where η is a smooth cut-off function. Thus, if

$$\sup_t\int|v\eta|^3dx \le C \tag{2.2}$$

and

$$\iint_Q|\nabla|v\eta|^{3/2}|^2\,dx\,dt \le C, \tag{2.3}$$

then we have $v \in L^5_{loc}(Q)$. From now on, we show that for $T>0$ and for a smooth cut-off function η, (2.2) and (2.3) hold.

To get our result, we consider the third component u_3. Let $w \overset{\text{def}}{=} u_3$ for short.

$$\frac{dw}{dt} - \Delta w + u\cdot\nabla w + \frac{\partial p}{\partial x_3} = f_3, \qquad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{2.4}$$

where f_3 is the third component of f. Denote $D_i \overset{\text{def}}{=} \frac{\partial}{\partial x_i}$, $\nabla \overset{\text{def}}{=} (D_1, D_2, D_3)$ and $D_t \overset{\text{def}}{=} \frac{d}{dt}$ for short. Let $0<\rho<\delta$ be real numbers and $\eta \overset{\text{def}}{=} \eta(x,t)$ a standard cut-off function such that $\eta = 1$ in a cylinder Q_ρ and $\eta = 0$ on the parabolic boundary of Q_δ, where $Q_\rho = B_\rho(x_0)\times(t_0-\rho^2, t_0)$ for some generic point (x_0, t_0). Let $k>0$ be an integer which will be determined later.

Before going further, note that for any T with $0<T<\infty$,

$$\begin{aligned}\int_0^T\int_{\mathbf{R}^3} |u|^3\,dx\,dt &= \int_0^T\int_{\mathbf{R}^3} |u|^{3/2}|u|^{3/2}\,dx\,dt\\ &\le \int_0^T \Big(\int_{\mathbf{R}^3}|u|^6\,dx\Big)^{1/4}\Big(\int_{\mathbf{R}^3}|u|^2\,dx\Big)^{3/4}\,dt\\ &\le \sup_t \|u\|_{L^2}^{3/2}\int_0^T\Big(\int_{\mathbf{R}^3}|\nabla u|^2\,dx\Big)^{3/4}dt\\ &\le C\Big(\int_0^T\int_{\mathbf{R}^3}|\nabla u|^2\,dx\,dt\Big)^{3/4} \le C.\end{aligned} \tag{2.5}$$

We consider the inner product of (2.4) with $|w|w\eta^k$. Notice that

$$\langle D_t w, |w|w\eta^k\rangle = \frac{1}{3}D_t\int_{\mathbf{R}^3}|w|^3\eta^k\,dx - \frac{k}{3}\int_{\mathbf{R}^3}|w|^3\eta^{k-1}D_t\eta\,dx, \tag{2.6}$$

$$\langle -\Delta w, |w|w\eta^k\rangle = 2\int |w|\,|\nabla w|^2\eta^k\,dx + k\int w|w|D_i w\,\eta^{k-1}D_i\eta\,dx, \tag{2.7}$$

$$\langle u\cdot\nabla w, |w|w\eta^k\rangle = -\frac{k}{3}\int u_i|w|^3\eta^{k-1}D_i\eta\,dx, \tag{2.8}$$

$$\langle D_3 p, w|w|\eta^k\rangle = -2\int p|w|D_3 w\eta^k - k\int pw|w|\eta^{k-1}D_3\eta\,dx. \tag{2.9}$$

By inner product of (2.4) with $|w|w\eta^k$ and integrating with respect to time variable over $(0,T)$, we have from (2.6) – (2.9), and (2.5) that

$$\begin{aligned}&\frac{1}{3}\sup_t\int|w|^3\eta^k\,dx + 2\iint|w|\,|\nabla w|^2\eta^k\,dx\,dt\\ &\le C\iint|w|^2|D_iw|\eta^{k-1}|D_i\eta|\,dx\,dt + C\iint|w|^4\eta^{k-1}|D_3\eta|\,dx\,dt\\ &\quad + C\iint|p|\,|w|\,|D_3w|\eta^k\,dx\,dt + C\iint|p|\,|w|^2\eta^{k-1}|D_3\eta|\,dx\,dt\\ &\quad + \iint|f_3|\,|w|^2\eta^k\,dx\,dt + C.\end{aligned} \tag{2.10}$$

We now estimate the right terms. By Hölder's, Young's and Sobolev's inequalities, we have

$$\begin{aligned}
&\iint |w|^4\eta^{k-1}|D_3\eta|\ dx\,dt\\
&\le \|\nabla\eta\|_\infty \iint \eta^{k-1}|w|^{18/7}|w|^{10/7}\ dx\,dt\\
&\le C\int \Big(\int \eta^{7(k-1)/2}|w|^9\ dx\Big)^{2/7}\Big(\int |w|^2\ dx\Big)^{5/7}\ dt\\
&\le C\sup_t \|w\|_{L^2}^{10/7}\int\Big(\int |\nabla\big(|w|^{3/2}\eta^{7(k-1)/12}\big)|^2\ dx\Big)^{6/7}\ dt\\
&\le \frac{C}{\epsilon}+\epsilon C\iint \Big|\nabla\big(|w|^{3/2}\eta^{7(k-1)/12}\big)\Big|^2\ dx\,dt\\
&\le \frac{C}{\epsilon}+\epsilon C\iint \big(|w|\ |\nabla w|^2\eta^{7(k-1)/6}+|w|^3\eta^{(7k-19)/6}|\nabla\eta|^2\big)\ dxdt\\
&\le \frac{C}{\epsilon}+\epsilon C\iint \eta^k|w|\,|\nabla w|^2\ dx\,dt
\end{aligned}$$

for $k \ge 7$, because of (2.5). Thus,

$$\iint |w|^4\eta^{k-1}|D_3\eta|\ dx\,dt \le \frac{C}{\epsilon}+\epsilon C\iint \eta^k|w|\,|\nabla w|^2\ dx\,dt. \tag{2.11}$$

Notice that

$$\begin{aligned}
\iint |f_3|\,|w|^2\eta^k\ dx\,dt &\le \iint |f_3|^2\eta^k\ dx\,dt+\iint |w|^4\eta^k\ dx\,dt\\
&\le C+\iint |w|^4\eta^k\ dx\,dt
\end{aligned}\tag{2.12}$$

if $f \in L^2_{loc}(Q)$. Look at the first term of (2.10);

$$\begin{aligned}
&\iint |w|^2|D_iw|\,\eta^{k-1}|D_i\eta|\ dx\,dt\\
&\le \epsilon C\iint \eta^{2(k-2)}|w|\,|D_iw|^2\ dx\,dt+\frac{C}{\epsilon}\iint |w|^3\eta^2\ dx\,dt\\
&\le \epsilon C\iint \eta^k|w|\,|D_iw|^2\ dx\,dt+C
\end{aligned}\tag{2.13}$$

for $4 \le k$. Notice that, by Young's inequality,

$$\begin{aligned}
\iint |p|\,|w|\,|D_3w|\eta^k\ dx\,dt \le{}& \frac{C}{\epsilon}\iint |p|^2\,|w|\eta^k\ dx\,dt\\
&+\epsilon C\iint |w|\,|\nabla w|^2\eta^k\ dx\,dt.
\end{aligned}\tag{2.14}$$

From (2.11) – (2.14), we have

$$\begin{aligned}
&\sup_t\int |w|^3\eta^k\ dx+6\iint |w|\,|\nabla w|^2\eta^k\ dx\,dt\\
&\le C+\epsilon C\iint |w|\,|\nabla w|^2\eta^k\ dx\,dt\\
&+C\iint |p|^2|w|\eta^k\ dx\,dt+C\iint |p|\,|w|^2\eta^{k-1}\ dx\,dt.
\end{aligned}$$

Take $\epsilon > 0$ such that $\epsilon C \le 1$. Then, for $k \ge 7$,

$$\begin{aligned} &\sup_t \int |w|^3 \eta^k \, dx + 5 \iint |w| \, |\nabla w|^2 \eta^k \, dx \, dt \\ &\quad \le C + C \iint |p|^2 |w| \eta^k \, dx \, dt + C \iint |p| \, |w|^2 \eta^{k-1} \, dx \, dt. \end{aligned} \tag{2.15}$$

We now estimate $\iint |p|^2 \, |w| \eta^k \, dx \, dt$. To do this, we need to represent p locally. We observe that

$$\Delta p = -D_i D_j (u_i u_j) + \nabla \cdot f$$

in $\mathbf{R}^3 \times (0, \infty)$. That is, p can be represented as Newtonian potential

$$\begin{aligned} p(x,t) &= \lim_{\varepsilon \to 0} \int_{\mathbf{R}^3 \setminus B_\varepsilon(x)} G(x-y) \Delta_y p \, dy \\ &= -\frac{1}{3} |u|^2 (x,t) - \int D_{y_i} D_{y_j} G(x-y) u_i u_j \, dy + \int G(x-y) \nabla \cdot f dy \end{aligned}$$

where $G(x) = -3/(4\pi |x|)$ is the fundamental solution of Laplace equation, $\Delta_y = \frac{\partial^2}{\partial y_i^2}$, and $D_{y_i} = \frac{\partial}{\partial y_i}$. Let $\ell > 0$ be an integer with $\ell < k$. Since

$$\Delta \eta^\ell = \ell(\ell-1) \eta^{\ell-2} |\nabla \eta|^2 + \ell \eta^{\ell-1} \Delta \eta,$$

one has

$$\Delta(p\eta^\ell) = \eta^\ell \Delta p + 2\ell \eta^{\ell-1} \nabla p \cdot \nabla \eta + \ell(\ell-1) p \eta^{\ell-2} |\nabla \eta|^2 + \ell p \eta^{\ell-1} \Delta \eta.$$

Thus, by Newtonian potential, we have

$$p\eta^\ell = \lim_{\varepsilon \to 0} \int_{\mathbf{R}^3 \setminus B_\varepsilon(x)} G(x-y) \Delta_y (p\eta^\ell) \, dy. \tag{2.16}$$

Integrating by parts, we have

$$\begin{aligned} \int G(x-y) \eta^\ell \Delta p \, dy &= -\frac{1}{3} \eta^l |u|^2 (x,t) - \int G(x-y) \eta^\ell D_i D_j (u_i u_j) \, dy \\ &\qquad + \int G(x-y) \eta^\ell \nabla \cdot f \, dy \\ &= -\int \big(D_j D_i G(x-y) \eta^\ell + 2\ell D_i G(x-y) \eta^{\ell-1} D_j \eta \\ &\qquad + \ell(\ell-1) G(x-y) \eta^{\ell-2} D_i \eta D_j \eta \\ &\qquad + \ell G(x-y) \eta^{\ell-1} D_j D_i \eta \big) u_i u_j \, dy \\ &\quad + \int G(x-y) \eta^\ell \nabla \cdot f \, dy, \end{aligned}$$

and

$$\begin{aligned}
&\ell \int G(x-y)\eta^{\ell-1}\nabla p\cdot\nabla\eta\,dy = \int G(x-y)\nabla p\cdot\nabla(\eta^\ell)\,dy\\
&= -\int D_iG(x-y)D_ip\,\eta^\ell\,dy - \int G(x-y)\eta^\ell\Delta p\,dy\\
&= \int \Delta G(x-y)p\eta^\ell\,dy + \int D_iG(x-y)D_i(\eta^\ell)\,p\,dy - \int G(x-y)\eta^\ell\Delta p\,dy\\
&= p(x)\eta^\ell(x,t) + \int D_iG(x-y)D_i(\eta^\ell)\,p\,dy - \int G(x-y)\eta^\ell\Delta p\,dy.
\end{aligned}$$

Thus, from (2.16), we have

$$\begin{aligned}
p\eta^\ell(x,t) = {}& -\frac{1}{3}\delta_{ij}u_iu_j\eta^{2\ell} - \int D_iD_jG(x-y)\eta^\ell u_iu_j\,dy\\
&- 2\ell\int D_iG(x-y)\eta^{\ell-1}D_j\eta\,u_iu_j\,dy\\
&- \ell(\ell-1)\int G(x-y)\eta^{\ell-2}D_i\eta D_j\eta\,u_iu_j\,dy\\
&- \ell\int G(x-y)\eta^{\ell-1}D_iD_j\eta\,u_iu_j\,dy - 2\int D_iG(x-y)pD_i(\eta^\ell)\,dy\\
&- \ell(\ell-1)\int G(x-y)p\eta^{\ell-2}|\nabla\eta|^2\,dy - \ell\int G(x-y)p\eta^{\ell-1}\Delta\eta\,dy\\
&+ \int G(x-y)\nabla\cdot f\eta^\ell\,dy\\
&\stackrel{\text{def}}{=} \sum_{i,j=1}^{3}\big(-\frac{1}{3}\delta_{ij}u^iu^j\eta^{2\ell} + P_{1;ij} + P_{2;ij} + P_{3;ij} + P_{4;ij}\big)\\
&\qquad + \sum_{i=1}^{3}P_{5;i} + P_6 + P_7 + P_8.
\end{aligned}$$

We now return to estimate $\iint |p|^2|w|\,\eta^k dx\,dt$. Using Sobolev's inequality for $b = 3a/(3-2a)$, where $a = 12/11$ and $b = 4$, we have

$$\begin{aligned}
\iint |P_8|^2|w|\eta^{k-2\ell}\,dx\,dt &\le C\int\Big(\int|P_8|^4\,dx\Big)^{1/2}\Big(\int|w|^2\eta^{2(k-2\ell)}\,dx\Big)^{1/2}dt\\
&\le C\int\Big(\int|\nabla^2P_8|^{12/11}\,dx\Big)^{22/12}dt\\
&\le C\int\Big(\int|\nabla\cdot f|^{12/11}\eta^{12\ell/11}\,dx\Big)^{22/12}dt \le C.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\iint |p|^2|w|\eta^k\,dx\,dt\\
&\quad\le C\sum\iint\big(|-\frac{1}{3}\delta_{ij}u_iu_j\eta^{2\ell} + P_{1;ij}|^2 + |P_{2;ij}|^2 + |P_{3;ij}|^2 + |P_{4;ij}|^2\big)|w|\,\eta^{k-2\ell}\,dx\,dt\\
&\quad + C\sum\iint|P_{5;i}|^2|w|\eta^{k-2\ell}\,dx\,dt + C\iint\big(|P_6|^2+|P_7|^2\big)|w|\eta^{k-2\ell}\,dx\,dt + C.
\end{aligned}$$

First we consider $-\frac{1}{3}\delta_{ij}u_iu_j\eta^{2\ell} + P_{1;ij}$. For $i = 1,2$ and $j = 1,2,3$, we obtain, by Young's and Calderon-Zygmund type inequalities,

$$\begin{aligned}\iint &| -\frac{1}{3}\delta_{ij}u_iu_j\eta^{2\ell} + P_{1;ij}|^2|w|\eta^{k-2\ell}\,dx\,dt\\ &\le C\iint | -\frac{1}{3}\delta_{ij}u_iu_j\eta^{2\ell} + P_{1;ij}|^3\,dx\,dt + C\iint |w|^3\eta^{3(k-2\ell)}\,dx\,dt\\ &\le C\iint |u_j|^3\,dx\,dt + C \le C.\end{aligned}$$

Notice that

$$\begin{aligned}-\frac{1}{3}(u_3(x,t))^2 + P_{1;33} &= 2\int D_{y_3}G(x-y)\eta^\ell u_3 D_{y_3}u_3\,dy + \ell\int D_{y_3}G(x-y)u_3^2\eta^{\ell-1}D_{y_3}\eta\,dy\\ &\stackrel{\text{def}}{=} P_{9;33} + P_{10;33}.\end{aligned}$$

By the divergence free condition of (2.1), we have

$$\begin{aligned}P_{9;33} &= -2\int D_{y_3}G(x-y)u_3(D_{y_1}u_1 + D_{y_2}u_2)\eta^\ell\,dy\\ &\stackrel{\text{def}}{=} P_{11;33} + P_{12;33}.\end{aligned}$$

Clearly, $P_{12;33}$ can be treated similarly to $P_{11;33}$. Integrating by parts, we have

$$\begin{aligned}P_{11;33} &= 2\int D_{y_1}D_{y_3}G(x-y)u_3u_1\eta^\ell\,dy + 2\int D_{y_3}G(x-y)D_{y_1}u_3\,u_1\eta^\ell\,dy\\ &+ 2\ell\int D_{y_3}G(x-y)u_3u_1\eta^{\ell-1}D_{y_1}\eta\,dy.\end{aligned}$$

The first term of $P_{11;33}$ can be treated as done for $P_{1;i3}$, and the last term can be treated as for $P_{2;ij}$, which will be done later. Let

$$P_{13;33} \stackrel{\text{def}}{=} \int D_{y_3}G(x-y)D_{y_1}u_3\,u_1\eta^\ell\,dy.$$

Then, for $i \ne 3$

$$\begin{aligned}D_iP_{13;33} &= \int D_{x_i}D_{y_3}G(x-y)D_{y_1}u_3\;u_1\eta^\ell\,dy\\ &= -\int D_{y_i}D_{y_3}G(x-y)D_{y_1}u_3\;u_1\eta^\ell\,dy,\end{aligned}$$

and for $i = 3$

$$\begin{aligned}D_iP_{13;33} &= -\frac{1}{3}D_{x_1}u_3(x,t)u_1(x,t)\eta^\ell(x) + \int D_{x_i}D_{y_3}G(x-y)D_{y_1}u_3\;u_1\eta^\ell\,dy\\ &= -\frac{1}{3}D_{x_1}u_3(x,t)u_1(x,t)\eta^\ell(x) - \int D_{y_i}D_{y_3}G(x-y)D_{y_1}u_3\;u_1\eta^\ell\,dy.\end{aligned}$$

Therefore, by Calderon-Zygmund type inequality,

$$\begin{aligned}\int\int |\nabla P_{13;33}|^2\,dx\,dt &\le C\int\int |\nabla u_3|^2\,|u_1|^2\eta^{2\ell}\,dx\,dt\\ &\le C\int\int |\nabla u_3|^2\eta^{2\ell}\,dx\,dt \le C.\end{aligned}$$

Thus,

$$\begin{aligned}
&\iint |P_{13;33}|^2 |w| \eta^{k-2\ell}\, dx\, dt \\
&\le C \int \Big(\int |P_{13;33}|^6 dx \Big)^{1/3} \Big(\int |w|^{3/2} \eta^{3(k-2\ell)/2}\, dx \Big)^{2/3} dt \\
&\le C \int \Big(\int |\nabla P_{13;33}|^2 dx \Big) \Big(\int |w|^2 \eta^{2(k-2\ell)}\, dx \Big)^{1/2} dt \\
&\le C \sup_t ||u||_{L^2(\mathbf{R}^3)} \iint |\nabla P_{13;33}|^2\, dx\, dt \le C.
\end{aligned}$$

Now return to $P_{10;33}$. By Sobolev's and Calderon-Zygmund type inequalities, we have

$$\begin{aligned}
&\iint |P_{10;33}|^2 |w| \eta^{k-2\ell}\, dx\, dt \\
&\le C \int \Big(\int |\nabla P_{10;33}|^2 dx \Big) \Big(\int |w|^{3/2} \eta^{3(k-2\ell)/2}\, dx \Big)^{2/3} dt \\
&\le C \int \Big(\int |\nabla P_{10;33}|^2 dx \Big) \Big(\int |w|^2 \eta^{2(k-2\ell)}\, dx \Big)^{1/2} dt \\
&\le C \int \int |w|^4 \eta^{2(\ell-1)}\, dx\, dt \\
&\le \frac{C}{\epsilon} + \epsilon C \int \int |w|\, |D_3 w|^2 \eta^k\, dx\, dt,
\end{aligned}$$

for $k > \ell \ge 3k/7 + 1$. Thus we have finished estimating the terms containing $P_{1;ij}$;

$$\sum \iint |P_{1;ij}|^2 |w| \eta^{k-2\ell}\, dx\, dt \le C + \epsilon C \iint |w|\, |\nabla w|^2 \eta^k\, dx\, dt. \tag{2.17}$$

We now estimate $P_{2;ij}$. For $i, j = 1, 2$ and for $i = 1, 2,\ j = 3$, $P_{2;ij}$ can be treated as done for $P_{13;33}$. For $P_{2;33}$, consider

$$\begin{aligned}
&\int \int |P_{2;33}|^2 |w| \eta^{k-2\ell}\, dx\, dt \\
&\le C \int \Big(\int |\nabla P_{2;33}|^2 dx \Big) \Big(\int |w|^{3/2} \eta^{3(k-2\ell)/2}\, dx \Big)^{2/3} dt \\
&\le C \int \int |\nabla P_{2;33}|^2\, dx\, dt \le C \int \eta^{2(\ell-1)}\, w^4\, dx\, dt,
\end{aligned} \tag{2.18}$$

which can be estimated in the same way to (2.11).

We now estimate $P_{3;ij}$. We use Sobolev's inequality for $b = 3a/(3-2a)$, where $a = 5/4$ and $b = 15/2$. By Sobolev's and Hölder's and Calderon-Zygmund type inequalities,

$$\begin{aligned}
&\int\int |P_{3;ij}|^2 |w| \eta^{k-2\ell}\, dx\, dt \\
&\quad \le C \int \Big(\int |P_{3;ij}|^{15/2} dx \Big)^{4/15} \Big(\int |w|^{15/11} \eta^{15(k-2\ell)/11}\, dx \Big)^{11/15} dt \\
&\quad \le C \int \Big(\int |\nabla^2 P_{3;ij}|^{5/4} dx \Big)^{8/5} \Big(\int |w|^2 \eta^{2(k-2\ell)}\, dx \Big)^{1/2} dt \\
&\quad \le C \int \Big(\int \eta^{5(\ell-2)/4} |u_i u_j|^{5/4} dx \Big)^{8/5} dt.
\end{aligned}$$

For $i = 1, 2$ and $j = 1, 2, 3$,

$$\begin{aligned}
\int\int |P_{3;ij}|^2 |w| \eta^{k-2\ell}\, dx\, dt &\le C \int \Big(\int \eta^{5(\ell-2)/4} |u_j|^{5/4} dx \Big)^{8/5} dt \\
&\le C \int\int \eta^{2(\ell-2)} |u_j|^2\, dt \le C.
\end{aligned}$$

For $i, j = 3$,

$$\begin{aligned}
\int\int |P_{3;33}|^2 |w| \eta^{k-2\ell}\, dx\, dt &\le C \int \Big(\int \eta^{5(\ell-2)/4} |w|^{5/2} dx \Big)^{8/5} dt \\
&\le C \int\int \eta^{2(\ell-2)} |w|^4\, dx\, dt,
\end{aligned}$$

which can be treated similarly to (2.11) for $k > \ell \ge 3k/7 + 2$. Thus, we have

$$(2.19) \qquad \sum \iint |P_{3;ij}|^2 |w| \eta^{k-2\ell}\, dx\, dt \le C + \epsilon C \iint |w|\, |\nabla w|^2 \eta^k\, dx\, dt.$$

We can treat with $P_{4;ij}$ in the way that we done for $P_{3;ij}$.

To estimate $P_{5;i}$, we use Sobolev's inequality for $b = 3a/(3-a)$, where $a = 12/7$, $b = 4$;

$$\begin{aligned}
&\iint |P_{5;i}|^2 |w| \eta^{k-2\ell}\, dx\, dt \\
&\quad \le C \int \Big(\int |P_{5;i}|^4 dx \Big)^{1/2} \Big(\int w^2 \eta^{2(k-2\ell)}\, dx \Big)^{1/2} dt \\
&\quad \le C \sup_t \|w\|_{L^2} \int \Big(\int |\nabla P_{5;i}|^{12/7} dx \Big)^{7/6} dt \\
&\quad \le C \int \Big(\int |p|^{12/7} \eta^{12(\ell-1)/7} dx \Big)^{7/6} dt.
\end{aligned}$$

Here, notice that

$$p = \frac{1}{3} \delta_{ij} u_i u_j - \int D_i D_j G(x-y) u_i u_j\, dy + \int G(x-y) \nabla \cdot f\, dy \stackrel{\text{def}}{=} \sum p_{ij} + \nabla \cdot f,$$

which $\nabla\cdot f$ term is bounded as we mentioned before. For $i,j=1,2$ it has no problem. For $i=1,2$ and $j=3$,

$$\int\Big(\int |p_{i3}|^{12/7}\eta^{12(\ell-1)/7}\,dx\Big)^{7/6}dt \le C\iint |p_{i3}|^2\eta^{2(\ell-1)}\,dx\ dt$$
$$\le C\iint |w|^2\,dx\,dt + C \le C.$$

For $i,j=3$, using the divergence free condition of (2.1), we have

$$\begin{aligned}
p_{33} &= -2\int D_3G(x-y)wD_3u_3\,dy\\
&= 2\int D_3G(x-y)w(D_1u_1+D_2u_2)\,dy\\
&= -2\int D_1D_3G(x-y)wu_1\,dy - 2\int D_3G(x-y)u_1D_1w\,dy\\
&\quad -2\int D_2D_3G(x-y)wu_2\,dy - 2\int D_3G(x-y)u_2D_2w\,dy\\
&\overset{\text{def}}{=} P_{14}+P_{15}+P_{16}+P_{17}.
\end{aligned}$$

For P_{14} and P_{16},

$$\int\Big(\int |P_{14}|^{12/7}\eta^{12(\ell-1)/7}\,dx\Big)^{7/6}dt \le C\iint |w|^2\,dx\,dt \le C.$$

For P_{15} and P_{17},

$$\begin{aligned}
&\int\Big(\int |P_{15}|^{12/7}\eta^{12(k-\ell)/7}\,dx\Big)^{7/6}dt\\
&\le C\int\Big(\int |P_{15}|^6\,dx\Big)^{1/3}dt \le C\iint |\nabla P_{15}|^2\,dx\,dt\\
&\le C\iint |u_1D_1w|^2\,dx\,dt \le C\iint |D_1w|^2\,dx\,dt \le C.
\end{aligned}$$

Thus, we have

$$\iint |P_{5;i}|^2|w|\eta^{k-2\ell}\,dx\,dt \le C. \tag{2.20}$$

Terms P_6 and P_7 can be treated in the same way. To estimate P_6, use Sobolev's inequality for $a=\frac{11}{10}$, $b=\frac{33}{8}$;

$$\begin{aligned}
&\int\int |P_6|^2|w|\eta^{k-2\ell}\,dx\,dt\\
&\le C\int\Big(\int |P_6|^{33/8}dx\Big)^{16/33}\Big(\int w^{33/17}\eta^{33(k-2\ell)/17}\,dx\Big)^{17/33}dt\\
&\le C\int\Big(\int |\nabla^2P_6|^{11/10}dx\Big)^{20/11}\Big(\int w^{33/17}\eta^{33(k-2\ell)/17}\,dx\Big)^{17/33}dt\\
&\le C\int\Big(\int |p|^{11/10}\eta^{11(\ell-2)/10}dx\Big)^{20/11}\Big(\int w^2\eta^{2(k-2\ell)}\,dx\Big)^{1/2}dt\\
&\le C\int\Big(\int |p|^{11/10}\eta^{11(\ell-2)/10}\,dx\Big)^{20/11}dt.
\end{aligned}$$

For $i = 1, 2$ and $j = 1, 2, 3$,

$$\int \Big(\int |p_{ij}|^{11/10} \eta^{11(\ell-2)/10} \, dx \Big)^{20/11} dt \le C \iint |p_{ij}|^2 \eta^{2(\ell-2)} \, dx \, dt$$
$$\le C \iint |u_j|^2 \, dx \, dt \le C.$$

For P_{14} and P_{16} of p_{33},

$$\int \Big(\int |P_{14}|^{11/10} \eta^{11(\ell-2)/10} \, dx \Big)^{20/11} dt \le C.$$

For P_{15} and P_{17} of p_{33},

$$\int \Big(\int |P_{15}|^{11/10} \eta^{11(k-2\ell)/10} \, dx \Big)^{20/11} dt$$
$$\le C \int \Big(\int |P_{15}|^6 \eta^{6(k-2\ell)} \, dx \Big)^{1/3} dt \le C \iint |\nabla P_{15}|^2 \, dx \, dt$$
$$\le C \iint |u_1 D_1 w|^2 \, dx \, dt \le C \iint |D_1 w|^2 \, dx \, dt \le C.$$

Thus, we have

$$\iint |P_6|^2 |w| \eta^{k-2\ell} \, dx \, dt \le C. \tag{2.21}$$

From (2.17), (2.18), (2.19), (2.20) and (2.21), we have

$$\iint |p|^2 |w| \eta^k \, dx \, dt \le C + \epsilon C \iint |w| \, |\nabla w|^2 \eta^k \, dx \, dt. \tag{2.22}$$

We finally consider $\iint |p| \, |w|^2 \eta^{k-1} \, dx \, dt$. Notice that, for $i = 1, 2$, $j = 1, 2, 3$,

$$\iint |p_{ij}| \, |w|^2 \eta^{k-1} \, dx \, dt \le C \int \Big(\int |p_{ij}|^3 \, dx \Big)^{1/3} \Big(\int |w|^3 \eta^{k-1} \, dx \Big)^{2/3} dt$$
$$\le C \iint |w|^3 \, dx \, dt \le C.$$

In fact P_{14} and P_{16} has no problem. For P_{15} and P_{17}, use Sobolev's and Calderon-Zygmund inequalities,

$$\begin{aligned}
&\iint |P_{15}|\,|w|^2\eta^{k-1}\,dx\,dt\\
&\le C\int\Big(\int |P_{15}|^2\eta^2\,dx\Big)^{1/2}\Big(\int |w|^4\eta^{2(k-2)}\,dx\Big)^{1/2}dt\\
&\le C\int\Big(\int |P_{15}|^6\eta^6\,dx\Big)^{1/6}\Big(\int |w|^4\eta^{2(k-2)}\,dx\Big)^{1/2}dt\\
&\le C\int\Big(\int |\nabla P_{15}|^2\,dx\Big)^{1/2}\Big(\int |w|^4\eta^{2(k-2)}\,dx\Big)^{1/2}dt\\
&\le C\int\Big(\int |\nabla w|^2\,dx\Big)^{1/2}\Big(\int |w|^4\eta^{2(k-2)}\,dx\Big)^{1/2}dt\\
&\le C\iint |\nabla w|^2\,dx\,dt + C\iint |w|^4\eta^{2(k-2)}\,dx\,dt\\
&\le \frac{C}{\epsilon} + \epsilon C\iint |w|\,|\nabla w|^2\eta^k\,dx\,dt.
\end{aligned}$$

Thus, we have

$$\iint |p|\,|w|^2\eta^{k-1}\,dx\,dt \le C + \epsilon C\iint |w|\,|\nabla w|^2\eta^k\,dx\,dt. \tag{2.23}$$

By taking $\epsilon > 0$ such that $\epsilon C \le 1$ and $k \ge 7$, we have from (2.15), (2.22) and (2.23) that

$$\sup_t \int |w|^3\eta^k\,dx + \iint |w|\,|\nabla w|^2\eta^k\,dx\,dt \le C.$$

This completes the proof.

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Department of Mathematics, KAIST, Taejon, Republic of Korea
E-mail address: `hobae@math.kaist.ac.kr ch@math.kaist.ac.kr`

A DEGENERATE PARABOLIC EQUATION WITH APSORPTION: II. UNIQUENESS OF THE VERY SINGULAR SOLUTION

JEONG-SEON BAEK AND MINKYU KWAK

Department of Mathematics
Chonnam National University
Kwangju, 500-757, KOREA
jsbaek@chonnam.chonnam.ac.kr
mkkwak@chonnam.chonnam.ac.kr

ABSTRACT. We prove the uniqueness of the very singular solution for an equation of the form

$$u_t = \Delta_p u - |u|^{q-1}u \quad \text{in} \quad Q = \mathbf{R}^N \times (0, \infty), \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $2N/(N+1) < p < 2$, $N \geq 2$, and $1 < q < p-1+p/N$.

The solution we find is of the form

$$u(x,t) = t^{-1/(q-1)} f(\eta), \quad \eta = |x| t^{-\frac{q-p+1}{p(q-1)}},$$

where f is the unique solution of an ordinary differential equation

$$(|f'|^{p-2}f')' + \frac{N-1}{\eta}|f'|^{p-2}f' + \frac{q-p+1}{p(q-1)}f' + \frac{1}{q-1}f - |f|^{q-1}f = 0, \quad \eta > 0 \tag{2}$$

with conditions:
$f > 0$ on $[0, \infty)$, $f'(0) = 0$ and $\lim_{\eta\to\infty} \eta^{p/(q-p+1)} f(\eta) = 0$.

1. INTRODUCTION

In this paper we consider a quasilinear degenerate diffusion equation - involving the p-Laplacian - with absorption

$$u_t = \Delta_p u - |u|^{q-1}u \quad \text{in} \quad Q = \mathbf{R}^N \times (0, \infty) \tag{1.1}$$

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where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, with $p > 2N/(N+1)$, $N \geq 1$, and $\max\{1, p-1\} < q < p - 1 + p/N$. A very singular solution W of (1.1) is a nonnegative continuous function in $\bar{Q} - \{(0,0)\}$ such that

(i) $W(x,0) = 0$ for $x \neq 0$;

(ii) $\nabla w \in L^1_{loc}(0,\infty : W^{1,p-1}_{loc}(\mathbf{R}^N))$ and (1.1) is satisfied in the sense of distribution in Q;

(iii) $\int_{\mathbf{R}^N} W(x,t)dx \to \infty$ as $t \to 0$.

Such a solution arises natually when we study the long time behaviour of solutions of (1.1) with an initial data $u(x,0) = u_0(x)$ satisfying

$$\lim_{|x|\to\infty} |x|^{p/(q-p+1)} u_0(x)dx = 0, \tag{1.2}$$

see [4], [5], and [8]. Brezis, Peletier and Terman ([1]) found in 1986 that the heat equation with absorption admits a unique very singular solution for the range corresponding to $p = 2$. For $p > 2$, Peletier and Wang ([6]) have proved an existence of a very singular solution of (1.1) and Kamin and Vazquez ([3]) has proved its uniqueness for more general absorptive terms. Later the existence proof was extended for the case $2N/(N+1) < p < 2$ in [8]. The main purpose of this paper is to show its uniqueness.

Theorem A. *Let $2N/(N+1) < p < 2$, $N \geq 2$ and $1 < q < p - 1 + p/N$, then there exists a unique very singular solution $W_0(x,t)$ for (1.1).*

For the proof, we borrow some ideas from [3] and construct a minimal and a maximal very singular solution. These solutions are invariant under a scaling transformation and become self-similar solutions. Hence these solutions will be of the form

$$W(x,t) = t^{-1/(q-1)} f(\eta), \quad \eta = |x| t^{-\frac{q-p+1}{p(q-1)}},$$

where f is the unique solution of an ordinary differential equation

$$(|f'|^{p-2}f')' + \frac{N-1}{\eta}|f'|^{p-2}f' + \frac{q-p+1}{p(q-1)}f' + \frac{1}{q-1}f - |f|^{q-1}f = 0, \quad \eta > 0 \tag{1.3}$$

with conditions:

$$f > 0 \quad \text{on} \quad [0,\infty), f'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \eta^{p/(q-p+1)} f(\eta) = 0. \tag{1.4}$$

The uniqueness proof is then reduced to showing that the above O.D.E problem has a unique solution. When $p > 2$, f has a compact support and the support of the scaled function $f_\lambda(\eta) = \lambda f(\lambda^{-\delta}\eta)$, $\delta = (p-2)/p$, covers the support of f for $\lambda > 1$. These facts were essential in the proof of the uniqueness, see [3] for details. On the other hand when $2N/(N+1) < p \le 2$, the support of f becomes the whole $\mathbf{R}^N$ and the argument for the case $p > 2$ can not be applied directly. The case $p = 2$ has been treated in a different way in [1] (see [1], p. 206) and the proof can not be applied to the case $p < 2$, either. We here investigate the exact asymptotic decay rate of solutions of (1.4) and prove Theorem A in a rather simple way by adapting ideas from [3].

Concerning the asymptotic behaviour of solutions, we in particular prove the following.

Theorem B. *Let $u(x,t)$ be a solution of (1.1) with a nonnegative initial data $u(x,0) = u_0(x)$ satisfying (1.2), then*

$$\lim_{|x|\to\infty} |x|^{\frac{p}{2-p}} u(x,t) = K_* t^{\frac{1}{\beta}(\frac{p}{2-p}-\alpha)}$$

for every $t > 0$, where

$$K_* = \left(\frac{\beta p_*^{p-1}}{1-(\alpha-N)/(p_*-N)} \right)^{\frac{1}{2-p}}$$

and $p_* = p/(2-p)$.

2. A PRIORI ESTIMATES

Thoughout this paper we assume

$$\text{(2.1)} \qquad N \ge 2, \quad \frac{2N}{N+1} < p < 2, \quad \text{and} \quad 1 < q < p - 1 + \frac{p}{N}$$

and we denote for notational simplicity

$$\alpha = \frac{p}{q-p+1}, \qquad \beta = \frac{p(q-1)}{q-p+1}.$$

Then the assumption implies that

$$N < \alpha < \frac{p}{2-p}.$$

We now consider an ordinary differential equation

$$\text{(2.2)} \qquad (|u'|^{p-2}u')' + \frac{N-1}{x}|u'|^{p-2}u' + \frac{x}{\beta}u' + \frac{1}{q-1}u - |u|^{q-1}u = 0, \quad x > 0$$

with conditions:

$$\text{(2.3)} \qquad u \ge 0 \quad \text{on} \quad [0,\infty), \quad u'(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} x^\alpha u(x) = 0.$$

Let u be a nontrivial solution of (2.2), (2.3). Then the main purpose of this section is to derive the exact decay rate of solution $u(x)$. We first have a lower bound.

Lemma 2.1.

$$\liminf_{x\to\infty} x^{2/(2-p)}|u'| \geq \gamma^{1/(p-2)}, \tag{2.4}$$

$$\liminf_{x\to\infty} x^{p/(2-p)}u(x) \geq \frac{2-p}{p}\gamma^{1/(p-2)}, \tag{2.5}$$

where γ is a constant given in (2.8).

Proof. We see from Lemma 1, [6] that $u(x) \leq c^* = (q-1)^{-1/(q-1)}$ for all $x \geq 0$ and $u(x)$ is nonincreasing on $[0,\infty)$. Thus $1/(q-1)u - u^q \geq 0$ and $u' \leq 0$. The equation (2.2) is rewritten as

$$(x^{N-1}|u'|^{p-2}u')' + \frac{1}{\beta}x^N u' + \frac{1}{q-1}x^{N-1}u - x^{N-1}u^q = 0. \tag{2.6}$$

Let $w = x^{N-1}|u'|^{p-2}u'$, then $u' = -(|w|/x^{N-1})^{1/(p-1)}$ and we have from (2.6) that

$$w' - \frac{1}{\beta}x^{N-\frac{N-1}{p-1}}|w|^{1/(p-1)} \leq 0, \quad x > 0, \tag{2.7}$$

or

$$|w|^{-1/(p-1)}w' \leq \frac{1}{\beta}x^{N-\frac{N-1}{p-1}}.$$

An integration of this inequality over (ϵ, x), $\epsilon > 0$, yields

$$|w(x)| \geq [|w(\epsilon)|^{(p-2)/(p-1)} + \gamma(x^{N+1-\frac{N-1}{p-1}} - \epsilon^{N+1-\frac{N-1}{p-1}})]^{(p-1)/(p-2)}, x > \epsilon,$$

where

$$\gamma = \frac{2-p}{p-1}\cdot\frac{1}{\beta}\cdot\frac{1}{N+1-\frac{N-1}{p-1}} = \frac{1}{N\beta}\cdot\frac{2-p}{p-2+\frac{p}{N}} > 0. \tag{2.8}$$

In a limit, one gets (2.4).

Moreover, given any $\epsilon > 0$, there exists $R > 0$ such that

$$u'(x) \leq -(1+\epsilon)\gamma^{1/(p-2)}x^{2/(p-2)} \quad \text{for all} \quad x > R. \tag{2.9}$$

If we integrate over (x, ∞), $x > R$, we obtain

$$u(x) \geq (1+\epsilon)\gamma^{1/(p-2)}\frac{2-p}{p}x^{p/(p-2)}, \quad x > R.$$

Since ϵ is arbitarily chosen, we have (2.5).

Lemma 2.1 and Lemma 1, [6] imply that $u(x) > 0$ and $u'(x) < 0$ for all $x > 0$. Moreover Lemma 2.1 suggests that $\lim_{x\to\infty} x^{p/(2-p)}u(x)$ might exist and nontrivial. In fact, this turns out to be true as we will see below.

Lemma 2.2.
$$\lim_{x\to\infty} x^{N-1}|u'|^{p-2}u' = 0.$$

Proof. An itegration of (2.6) over $(0, x)$ yields

$$x^{N-1}|u'|^{p-2}u'(x) + \frac{x^N}{\beta}u(x) = -\frac{\alpha - N}{\beta}\int_0^x s^{N-1}u(s)ds + \int_0^x s^{N-1}u^q(s)ds. \tag{2.10}$$

The assumption (2.1) implies $\alpha > N$ and the decay condition in (2.3) implies that integrals on the right side of (2.10) has a limit as taking $x \to \infty$. Hence a limit $\lim_{x\to\infty} x^{N-1}|u'|^{p-2}u' = -l$, $l \geq 0$, exists. Suppose $l > 0$. There exists $R_0 > 0$ such that

$$x^{N-1}|u'|^{p-2}u'(x) < -\frac{l}{2} \quad \text{for} \quad x > R_0,$$

or

$$u'(x) < -(l/2)^{1/(p-1)}x^{-(N-1)/(p-1)} \quad \text{for} \quad x > R_0. \tag{2.11}$$

Choose $R > x > R_0$, and integrate (2.11) over (x, R) to obtain

$$u(R) - u(x) < -(l/2)^{1/(p-1)}\frac{p-1}{p-N}(R^{\frac{p-N}{p-1}} - x^{\frac{p-N}{p-1}}).$$

Since $p < 2 \leq N$, one has (by taking a limit as $R \to \infty$)

$$u(x) \geq (l/2)^{1/(p-1)}\frac{p-1}{N-p}x^{(p-N)/(p-1)} \quad \text{for} \quad x > R_0.$$

This is incompatible with the decay condition $\lim_{x\to\infty} x^\alpha u(x) = 0$ since $\alpha > N$ and $N > (N-p)/(p-1)$. This completes the proof.

We now integrate (2.6) over (x, ∞) to obtain

$$x^{N-1}|u'|^{p-1} + \frac{\alpha - N}{\beta}\int_x^\infty s^{N-1}u(s)ds = \frac{x^N}{\beta}u + \int_x^\infty s^{N-1}u^q(s)ds. \tag{2.12}$$

Dividing by $x^N u$, we have

$$\frac{|u'|^{p-1}}{xu} + \frac{\alpha - N}{\beta}\frac{\int_x^\infty s^{N-1}u(s)ds}{x^N u} = \frac{1}{\beta} + \frac{\int_x^\infty s^{N-1}u^q(s)ds}{x^N u}. \tag{2.13}$$

Then we prove

Proposition 2.3.

(1)
$$\lim_{x\to\infty}\frac{x|u'|}{u}=p_*,$$

(2)
$$\lim_{x\to\infty}\frac{|u'|^{p-1}}{xu}=\frac{1}{\beta}(1-\frac{\alpha-N}{p_*-N}),$$

(3)
$$\lim_{x\to\infty}x^{p/(2-p)}u(x)=K_*.$$

Here

$$K_*=(\frac{\beta p_*^{p-1}}{1-\frac{\alpha-N}{p_*-N}})^{1/(2-p)}\quad and\quad p_*=\frac{p}{2-p}.$$

Proof. Assume that $\lim_{x\to\infty}\frac{x|u'|}{u}=l<\infty$ exists. By l'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\int_x^\infty s^{N-1}u(s)}{x^Nu}=\lim_{x\to\infty}\frac{-x^{N-1}u(x)}{x^Nu'+Nx^{N-1}u}=\lim_{x\to\infty}\frac{1}{x|u'|/u-N}=\frac{1}{l-N}\tag{2.14}$$

and

$$\lim_{x\to\infty}\frac{\int_x^\infty s^{N-1}u^q(s)}{x^Nu}=\lim_{x\to\infty}\frac{u^{q-1}(x)}{x|u'|/u-N}=0.\tag{2.15}$$

Using these in (2.13) we obtain

$$\lim_{x\to\infty}\frac{|u'|^{p-1}}{xu}=\frac{1}{\beta}-\frac{1}{\beta}\cdot\frac{\alpha-N}{l-N}.$$

Thus $l\geq\alpha$.

Since we assume that $\lim_{x\to\infty}x^\alpha u=0$, we have

$$\lim_{x\to\infty}x^{\alpha+1}|u'|=0\tag{2.16}$$

and thus $l>\alpha$. In fact if we define $\phi(x)=\int_x^\infty s^{N-1}u(s)ds$, then (2.12) is rewritten as

$$\phi'(x)+\frac{\alpha-N}{x}\phi(x)=-\beta x^{N-2}|u'|^{p-1}+\frac{\beta}{x}\int_x^\infty s^{N-1}u^q(s)ds.\tag{2.17}$$

Mutiplying (2.17) by an integrating factor $x^{\alpha-N}$, we get

$$(x^{\alpha-N}\phi(x))' = -\beta x^{\alpha-2}|u'|^{p-1} + \beta x^{\alpha-N-1}\int_x^\infty s^{N-1}u^q(s)ds. \tag{2.18}$$

Now, $x^{\alpha q-N}\phi(x) = x^{\alpha-N}\phi(x)/x^{\alpha(1-q)}$ and by l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to\infty} x^{\alpha q-N}\phi(x) &= \lim_{x\to\infty}\frac{(x^{\alpha-N}\phi(x))'}{\alpha(1-q)x^{\alpha-\alpha q-1}}\\ &= \frac{\beta}{\alpha(1-q)}(-\lim_{x\to\infty} x^{\alpha q-1}|u'|^{p-1} + \lim_{x\to\infty} x^{\alpha q-N}\int_x^\infty s^{N-1}u^q(s)ds),\end{aligned}$$

which becomes 0 from (2.16) and another application of l'Hôpital's rule. Note that $\alpha q - 1 = (\alpha+1)(p-1)$. This and (2.14) imply that $\lim_{x\to\infty} x^{\alpha q}u(x) = 0$, which in turn implies $l > \alpha$.

Writing

$$\frac{|u'|^{p-1}}{xu} = \left(\frac{x|u'|}{u}\right)^{p-1}\cdot\frac{1}{x^p u^{2-p}},$$

we have $\lim_{x\to\infty} x^p u^{2-p} = \beta l^{p-1}/(1-\frac{\alpha-N}{l-N})$ and $\lim_{x\to\infty} x^{p/(2-p)}u(x) \neq 0$. Hence one must have $l = p/(2-p)$ and it suffices to prove that $\lim_{x\to\infty} x|u'|/u$ exists for the completion of proof. This is done in the next lemma.

lemma 2.4. $\lim_{x\to\infty}\frac{x|u'|}{u}$ *exists and is finite.*

Proof. We first observe that for every $R > 0$ there exists $x > R$ such that $x|u'|/u > \alpha$. Otherwise, there exists $R > 0$ such that $x|u'|/u \le \alpha$ for all $x \ge R$. Upon integrating one sees that $x^\alpha u(x) \ge R^\alpha u(R)$, $x \ge R$, which contradicts to the assumption (2.3).

A differentiation gives

$$\left(\frac{x|u'|}{u}\right)' = \frac{|u'|}{u} - \frac{xu''}{u} + \frac{x|u'|^2}{u^2}. \tag{2.19}$$

We multiply (2.19) by $(p-1)|u'|^{p-2}/x$ and use (2.2) and the fact that $(|u'|^{p-2}u')' = (p-1)|u'|^{p-2}u''$ to obtain

$$\frac{(p-1)|u'|^{p-2}}{x}\left(\frac{x|u'|}{u}\right)' = \frac{|u'|^{p-1}}{xu}((p-1)\frac{x|u'|}{u} + p - N) - \frac{1}{\beta}\frac{x|u'|}{u} + \frac{\alpha}{\beta} - u^{q-1}. \tag{2.20}$$

At each point x where $\frac{x|u'|}{u} = \frac{N-p}{p-1}$, the right side of (2.20) becomes $1/\beta(\alpha-(N-p)/(p-1)) - u^{q-1}$, which is positive for large x and $x|u'|/u$ increases. Hence $\liminf_{x\to\infty} x|u'|/u$ is not less than $(N-p)/(p-1)$. When $\frac{x|u'|}{u} = \alpha$,

$$\frac{(p-1)|u'|^{p-2}}{x}\left(\frac{x|u'|}{u}\right)' = \frac{|u'|^{p-1}}{xu}\left((p-1)\alpha + p - N - \frac{xu^q}{|u'|^{p-1}}\right)$$

and $x|u'|/u$ also increases for large x since

$$\frac{xu^q}{|u'|^{p-1}} = \left(\frac{u}{x|u'|}\right)^{p-1} (x^\alpha u)^{q-p+1},$$

which tends to 0 as $x \to \infty$. This reveals that $\liminf_{x\to\infty} x|u'|/u \geq \alpha$.

If $\limsup_{x\to\infty} x|u'|/u = \alpha$, then $\lim_{x\to\infty} x|u'|/u = \alpha$. Hence we may assume that $\limsup_{x\to\infty} x|u'|/u > \alpha$. By the above reasoning, $x|u'|/u > \alpha$ for large x.

(2.20) is rewritten as

(2.21)

$$\begin{aligned}(p-1)x\left(\frac{x|u'|}{u}\right)' &= \frac{x|u'|}{u}((p-1)\frac{x|u'|}{u} + p - N) \\ &+ \frac{x|u'|/u}{|u'|^{p-1}/(xu)}\left(-\frac{1}{\beta}\frac{x|u'|}{u} + \frac{\alpha}{\beta} - u^{q-1}\right).\end{aligned}$$

Let us assume that $(x|u'|/u)' = 0$ for some $x = x_0 > 0$. Differentiating (2.21), we obtain, at $x = x_0$,

(2.22)

$$\begin{aligned}(p-1)x\left(\frac{x|u'|}{u}\right)'' =& -(q-1)\frac{x|u'|/u}{|u'|^{p-1}/(xu)}u^{q-2}u' \\ &- \frac{x|u'|/u}{(|u'|^{p-1}/(xu))^2}\left(-\frac{1}{\beta}\frac{x|u'|}{u} + \frac{\alpha}{\beta} - u^{q-1}\right)\left(\frac{|u'|^{p-1}}{xu}\right)'.\end{aligned}$$

With the use of (2.6), one has

$$\begin{aligned}x\left(\frac{|x'|^{p-1}}{xu}\right)' &= x\left(\frac{x^{N-1}|u'|^{p-1}}{x^N u}\right)' \\ &= \frac{(x^{N-1}|u'|^{p-1})'}{x^{N-1}u} - x^N|u'|^{p-1}\frac{x^N u' + Nx^{N-1}u}{x^{2N}u^2} \\ &= -\frac{1}{\beta}\frac{x|u'|}{u} + \frac{\alpha}{\beta} - u^{q-1} + \frac{|u'|^{p-1}}{xu}\left(\frac{x|u'|}{u} - N\right).\end{aligned} \tag{2.23}$$

When $(x|u'|/u)' = 0$, one has by (2.20)

$$\frac{|u'|^{p-1}}{xu}((p-1)\frac{x|u'|}{u} + p - N) = \frac{1}{\beta}\frac{x|u'|}{u} - \frac{\alpha}{\beta} + u^{q-1}.$$

Using this in (2.23), one gets

$$x\left(\frac{|x'|^{p-1}}{xu}\right)' = \frac{|u'|^{p-1}}{xu}((2-p)\frac{x|u'|}{u} - p)$$

at $x = x_0$. Now equation (2.22) is rewritten as

(2.24)
$$\begin{aligned}(p-1)x^2\left(\frac{x|u'|}{u}\right)'' &= \frac{x|u'|/u}{|u'|^{p-1}/(xu)}\{(q-1)u^{q-1}\frac{x|u'|}{u} \\ &+ \left(\frac{1}{\beta}\frac{x|u'|}{u} - \frac{\alpha}{\beta} + u^{q-1}\right)\left((2-p)\frac{x|u'|}{u} - p\right)\} \\ &= \frac{x|u'|/u}{|u'|^{p-1}/(xu)}\{u^{q-1}\left((q-p+1)\frac{x|u'|}{u} - p\right) \\ &+ \frac{1}{\beta}\left(\frac{x|u'|}{u} - \alpha\right)\left((2-p)\frac{x|u'|}{u} - p\right)\} \\ &= \frac{x|u'|/u}{|u'|^{p-1}/(xu)}\left(\frac{x|u'|}{u} - \alpha\right)\{(q-p+1)u^{q-1} \\ &+ \frac{1}{\beta}\left((2-p)\frac{x|u'|}{u} - p\right)\}.\end{aligned}$$

Suppose that $\lim_{t\to\infty} x|u'|/u$ does not exist. At the critical point $x = x_0$, if $x|u'|/u \geq p/(2-p)$, then $x|u'|/u$ is convex and thus $x|u'|/u$ does not have any local maximum larger than $p/(2-p)$. Hence $\limsup_{x\to\infty} x|u'|/u \leq p/(2-p)$. On the other hand if $\alpha < x|u'|/u < (p - \beta(q-p+1)u^{q-1})/(2-p)$, then $x|u'|/u$ is concave. Since $\lim_{x\to\infty} u^{q-1} = 0$, for large x $x|u'|/u$ does not have any local minimum smaller than $1/2(\limsup_{x\to\infty} x|u'|/u + \liminf_{x\to\infty} x|u'|/u)$, which is strictly less than $p/(2-p)$. Therefore we may conclude that $\lim_{x\to\infty} x|u'|/u$ exists.

Moreover $\lim_{x\to\infty} x|u'|/u \leq p/(2-p)$. Otherwise, there exist $\epsilon > 0$ and $R > 0$ such that

$$\frac{x|u'|}{u} \geq \frac{p}{2-p} + \epsilon \quad \text{for all} \quad x \geq R.$$

An integration yields that $x^{p/(2-p)+\epsilon}u(x) \leq R^{p/(2-p)+\epsilon}u(R)$ for $x \geq R$. This implies that $\lim_{x\to\infty} x^{p/(2-p)}u(x) = 0$, which is incompatible with Lemma 2.1. This completes the proof.

3. Uniqueness

We now turn to the proof of the uniqueness of the very singular solution. A very singular solution may be found as a monotone limit of singular solutions and we see that a monotone limit of $W_A(x,t)$ yields also a very sigular solution, where $W_A(x,t)$ is a unique solution of (1.1) with an initial data $u(x,0) = A|x|^{-\alpha}$, see Proposition 4.1, [5].

As we have seen in [3] and [5], we can show that there exist a minimal and a maximal very singular solution. Remarked earlier, such a solution has to be invariant under a scaling transformation T_λ which associates to any solution of (1.1) another solution $T_\lambda u$ defined by

$$(T_\lambda u)(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t) \tag{3.1}$$

and becomes spherical symmetric self-similar solution. Hence such a solution must be of the form

$$u(x,t) = t^{-1/(q-1)} f(r), \quad r = |x| t^{-1/\beta}.$$

Moreover f satisfies an ordinary differential equation

$$(|f'|^{p-2}f')' + \frac{N-1}{r}|f'|^{p-2}f' + \frac{q-p+1}{p(q-1)}f' + \frac{1}{q-1}f - |f|^{q-1}f = 0, \quad r > 0 \tag{3.2}$$

and additional conditions:

$$f > 0 \quad \text{on} \quad [0,\infty), \quad f'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{p/(q-p+1)} f(r) = 0. \tag{3.3}$$

The uniqueness proof is then reduced to showing that the above O.D.E problem has a unique solution.

Let F and f be solutions of (3.2) and (3.3). Without loss of generality, we may assume that $F \geq f$. Following [3], we define

$$f_k(r) = k f(k^\delta r), \quad \delta = \frac{2-p}{p} \tag{3.4}$$

and then f_k will be larger that F on $[0,\infty)$ for sufficiently large k. We first observe that when $u(x,t) = t^{-1/(q-1)} f(r)$, $r = |x|t^{-1/\beta}$, is a solution of (1.1), $u_k(x,t) = t^{-1/(q-1)} f_k(r)$ satisfies

$$(u_k)_t - \Delta_p u_k + u_k^q = k(k^{q-1} - 1)u^q(k^\delta x, t) \geq 0$$

and is a super-solution of (1.1). By Proposition 2.3,

$$\lim_{r\to\infty} r^{p/(2-p)} f(r) = \lim_{r\to\infty} r^{2/(2-p)} F(r) = l$$

for some $l > 0$ and

(3.5)

$$\begin{aligned}
\lim_{|x|\to\infty} |x|^{p/(2-p)} u_k(x,2) &= \lim_{|x|\to\infty} |x|^{p/(2-p)} 2^{-1/(q-1)} k f\left(k^\delta \frac{|x|}{2^{1/\beta}}\right) \\
&= \lim_{|x|\to\infty} 2^{1/\beta(p/(2-p)-\alpha)} \left(\frac{k^\delta |x|}{2^{1/\beta}}\right)^{p/(2-p)} f\left(k^\delta \frac{|x|}{2^{1/\beta}}\right) \\
&= 2^{1/\beta(p/(2-p)-\alpha)} l > l
\end{aligned}$$

uniformly for $k \geq 1$. Thus we may find k large enough so that $u_k(x,2) \geq F(|x|)$ for all $x \in \mathbf{R}^N$. Using the Maximum Principle ([2] and [7]) we obtain that

$$u_k(x,t+1) \geq t^{-1/(q-1)}F(|x|t^{-1/\beta}) \tag{3.6}$$

for every $x \in \mathbf{R}^N$ and $t \geq 1$. Put $r = |x|t^{-1/\beta}$. Then

$$k\left(\frac{t}{t+1}\right)^{1/(q-1)} f\left(\left(\frac{t}{t+1}\right)^{1/\beta} k^\delta r\right) \geq F(r) \quad \text{for} \quad t \geq 1.$$

We now let $t \to \infty$ to get $f_k(r) \geq F(r)$ for all $r \geq 0$ as we claimed.

We now define

$$m = \min\{k \geq 1 : f_k(r) \geq F(r), \quad 0 \leq r < \infty\}. \tag{3.7}$$

The uniqueness proof is now reduced to showing that m is not greater than 1. Suppose $m > 1$ to the contrary.

$$\begin{aligned}\tau^{-1/(q-1)}F\Big(\frac{r}{\tau^{1/\beta}}\Big) - f(r) &= \int_1^\tau \frac{d}{ds}\Big(s^{-1/(q-1)}f\Big(\frac{r}{s^{1/\beta}}\Big)\Big)ds \\ &= \int_1^\tau s^{-1/(q-1)-1}f\Big(\frac{r}{s^{1/\beta}}\Big)\Big(-\frac{1}{q-1}+\frac{1}{\beta}\frac{r/s^{1/\beta}|f'(r/s^{1/\beta})|}{f(r/s^{1/\beta})}\Big)ds.\end{aligned}$$

We see from Proposition 2.3 that the right side is positive for large r and there exists $R > 0$ such that

$$\tau^{-1/(q-1)}f\Big(\frac{r}{\tau^{1/\beta}}\Big) \geq f(r) \tag{3.8}$$

for all $r \geq R$ and $1 \leq \tau \leq 2$. Thus

$$u_m(x,\tau) = \tau^{-1/(q-1)}mf(m^\delta|x|\tau^{-1/\beta}) \geq mf(m^\delta|x|) \geq F(|x|) \tag{3.9}$$

for all $|x| \geq R$ and $1 \leq \tau \leq 2$.

We also note that $f_m(r)$ does not touch $F(r)$ in a compact subset of $[0,\infty)$. In fact $f_m(r)$ solves

$$(|f_m'|^{p-2}f_m')' + \frac{N-1}{r}|f_m'|^{p-2}f_m' + \frac{r}{\beta}f_m' + \frac{1}{q-1}f_m - f_m^q = (m-m^q)f^q. \tag{3.10}$$

If f_m touches F at $r_0 > 0$, then $f_m'(r_0) = F'(r_0) \neq 0$ and

$$(|f_m'|^{p-2}f_m')'(r_0) < (|F'|^{p-2}F')'(r_0).$$

But $f_m(r) \geq F(r)$ near $r = r_0$, which obviously violates the Strong Maximum Principle ([2] and [7]). The other possibility to be checked is the case when f_m and F touch at the origin. We have from (2.15), [5] that

$$(|f_m'|^{p-2}f_m')'(0) = m(|f'|^{p-2}f')'(0) = \frac{m}{N}(f^q(0) - \frac{1}{q-1}f(0))$$
$$< \frac{m}{N}(m^q f^q(0) - \frac{l}{q-1}f(0)) = (|F'|^{p-2}F')'(0),$$

which leads to a contradiction since $f_m(r) \geq F(r)$ near the origin. Hence we may find $\epsilon > 0$ and $\tau > 1$ so that

$$u_{m-\epsilon}(x,\tau) \geq F(|x|) \tag{3.11}$$

for $|x| \leq R$. By (3.10), (3.11), and the comparison argument as above, we obtain that

$$f_{m-\epsilon}(r) \geq F(r) \quad \text{for} \quad r \geq 0,$$

which means that we can slightly reduce the factor m. Hence we may conclude that $m = 1$ and $f(r) = F(r)$, which proves the uniqueness of the very singular solution.

4. Asymptotic behaviour

Let $u(x,t)$ be a solution of (1.1) with a nonnegative and nontrivial initial data $u_0(x)$ satisfying (1.2). We have already seen that the asymptotic behavior as $t \to \infty$ (and $|x| \to \infty$) are deduced from the limiting behavior of a family of scaled functions $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ and determined by the very singular solution $W_0(x,t)$, see Theorem C, [k]. In fact we see that

$$\lim_{t\to\infty} |u(x,t) - W_0(x,t)| = 0$$

uniformly on the set $\{x \in \mathbf{R}^N : |x| \leq \gamma t^{1/\beta}\}$ for every $\gamma > 0$. The main purpose of this section is to derive the exact decay property in Theorem B.

For fixed $t > 0$, we put $\lambda = |x|$ and $s = t/|x|^\beta$, then $\lambda \to \infty$ and $s \to 0$ as $|x| \to \infty$. Thus

$$\lim_{|x|\to\infty} |x|^{p/(2-p)}u(x,t) = \lim_{s\to 0}\lim_{\lambda\to\infty}(\frac{t}{s})^{1/\beta(p/(2-p)-\alpha)}\lambda^\alpha u(\lambda\frac{x}{|x|},\lambda^\beta s)$$
$$= \lim_{s\to 0}(\frac{t}{s})^{1/\beta(p/(2-p)-\alpha)}W_0(\frac{x}{|x|},s)$$
$$= t^{1/\beta(p/(2-p)-\alpha)}\lim_{s\to 0}(s^{-1/\beta})^{p/(2-p)}f(s^{-1/\beta})$$
$$= K_* t^{1/\beta(p/(2-p)-\alpha)} \quad \text{(by Proposition 2.3).}$$

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Jumping Problem in a Wave Equation

Q-Heung Choi[1], Sungki Chun and Tacksun Jung[2]

0. Introduction

In this paper we investigate multiplicity of solutions $u(x,t)$ for a piecewise linear perturbation $-(bu^+ - au^-)$ of the one-dimensional wave operator $u_{tt} - u_{xx}$ under Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the variable t,

$$u_{tt} - u_{xx} + bu^+ - au^- = f(x,t) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{0.1}$$

$$u(\pm\frac{\pi}{2}, t) = 0, \tag{0.2}$$

$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t. \tag{0.3}$$

When a string with nonuniform density vibrates up and down, the upward restoring coefficient and the downward restoring coefficient of it are different. Hence it happens a nonlinear perturbation in a wave equation. Here we assumed that the upward restoring coefficient and the downward one in the vibrating of the string are constant and they are different.

We let L the wave operator, $Lu = u_{tt} - u_{xx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

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with (0.2) and (0.3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by

$$\begin{aligned}\phi_{0n} &= \frac{\sqrt{2}}{\pi}\cos(2n+1)x \; for \; n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi}\cos 2mt\cos(2n+1)x \; for \; m > 0, n \geq 0.\end{aligned}$$

We note that all eigenvalues in the interval $(-9, 9)$ are given by

$$\lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \left\{ u \in L^2(Q) : \quad u \text{ is even in } x \text{ and } t \quad \right\}.$$

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H. Hence equation (0.1) with (0.2) and (0.3) is equivalent to

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H. \tag{0.4}$$

Our concern is to investigate multiplicity of solutions of (0.4) when the nonlinearity $-(bu^+ - au^-)$ crosses finite eigenvalues and the source term f is generated by two eigenfunctions ϕ_{00}, ϕ_{10}.

Let V be the two dimensional subspace of H spanned by ϕ_{00} and ϕ_{10}. Let $\Phi : V \to V$ be a map (cf. equation (1.6)) defined by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V.$$

In Section 1, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues $\lambda_{00}, \lambda_{10}$ and the source term f is generated by ϕ_{00} and ϕ_{10}. In subsection 1.1, we investigate the properties of the map Φ and we reveal a relation between multiplicity of solutions and source terms in equation (0.4) when f belongs to the two-dimensional space V (cf. Theorem 1.2). In subsection 1.2, we determine the region of source terms in which (0.4) has no solution. The main result of this section is the following.

THEOREM A. *(cf. Theorem 1.1, 1.3) Suppose* $-5 < a < -1$ *and* $3 < b < 7$. *Let* $f = s_1\phi_{00} + s_2\phi_{10}$. *Then we have :*

(i) If f *belongs to the interior* $\text{Int}R_1$ *of* R_1, (0.4) has a positive solution, a negative solution, and at least two sign changing solutions.

(ii) If f belongs to the boundary ∂R_1 of R_1, (0.4) has a positive solution, a negative solution, and at least one sign changing solution.

(iii) If f belongs to $\text{Int}(R_3 \backslash R_1)$, (0.4) has a negative solution and at least one sign changing solution.

(iv) If f belongs to ∂R_3, (0.4) has a negative solution.

(v) If f does not belong to the cone R_3, (0.4) has no solution.

In Section 2, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses an eigenvalue λ_{00} and the source term f is generated by ϕ_{00} and ϕ_{10}. In subsection 2.1, we investigate the properties of the map Φ (cf. Lemma 1.3, Theorem 2.2). In subsection 2.2, we reveal a relation between multiplicity of solutions and source terms in equation (0.4) when f belongs to the two dimensional space V. That is, the main theorem of this section is the following.

THEOREM B. *(cf. Theorem 2.3) Let* $-1 < a < 3 < b < 7$ *satisfy* $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1$. *Then there are cones* R_1, R_2', R_3, R_4' *in* V *such that the followings hold.*

(i) If $f \in \text{Int } R_1$, *then equation (0.4) has a positive solution and at least two*

sign changing solutions.

(ii) If $f \in \partial R_1$, then equation (0.4) has a positive solution and at least one sign changing solution.

(iii) If $f \in \text{Int } R_i'(i = 2, 4)$, then equation (0.4) has at least one sign changing solution.

(iv) If $f \in \text{Int } R_3$, then equation (0.4) has only the negative solution.

(v) If $f \in \partial R_3$, then equation (0.4) has a negative solution.

1. The Nonlinearity Crosses Two Eigenvalues

In this section, we investigate multiplicity of solutions $u(x, t)$ for a piecewise linear perturbation $-(bu^+ - au^-)$ of the one-dimensional wave operator $u_{tt} - u_{xx}$ with the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues. We suppose that $-5 < a < -1$ and $3 < b < 7$. Under this assumption, we have a concern with a relation between multiplicity of solutions and source terms of a nonlinear wave equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H. \tag{1.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}, that is, $f = s_1\phi_{00} + s_2\phi_{10}(s_1, s_2 \in \mathbb{R})$.

To study equation (1.1), we use the contraction mapping theorem to reduce the problem from an infinite dimensional one in H to a finite dimensional one.

1.1. A Variational Reduction Method

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection

H onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$. Hence equation (1.1) is equivalent to

$$Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \tag{1.2}$$

$$Lv + P(b(v + w)^+ - a(v + w)^-) = s_1\phi_{00} + s_2\phi_{10}. \tag{1.3}$$

We look on (1.2) and (1.3) as a system of two equations in the two unknowns v and w.

LEMMA 1.1. *For fixed $v \in V$, (1.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous in terms of v.*

Proof. We use the contraction mapping theorem. Let $\delta = \frac{1}{2}(a+b)$. Rewrite (1.2) as

$$(-L - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w)),$$

or equivalently,

$$w = (-L - \delta)^{-1}(I - P)g_v(w), \tag{1.4}$$

where

$$g_v(w) = b(v + w)^+ - a(v + w)^- - \delta(v + w).$$

Since

$$|g_v(w_1) - g_v(w_2)| \le |b - \delta||w_1 - w_2|,$$

we have

$$\|g_v(w_1) - g_v(w_2)\| \le |b - \delta|\|w_1 - w_2\|,$$

where $\| \ \|$ is the L^2 norm in H. The operator $(-L - \delta)^{-1}(I - P)$ is a self adjoint compact linear map from $(I - P)H$ into itself. The eigenvalues of

$(-L-\delta)^{-1}(I-P)$ in W are $(\lambda_{mn}-\delta)^{-1}$, where $\lambda_{mn} \geq 7$ or $\lambda_{mn} \leq -5$. Therefore its L^2 norm is $\max\{\frac{1}{7-\delta}, \frac{1}{5+\delta}\}$. Since $|b-\delta| < \min\{7-\delta, 5+\delta\}$, it follows that for fixed $v \in V$, the right hand side of (1.4) defines a Lipschitz mapping W into itself with Lipschitz constant $\gamma < 1$. Hence, by the contraction mapping principle, for given $v \in V$, there is a unique $w \in W$ which satisfies (1.2).

Also, it follows, by the standard argument principle, that $\theta(v)$ is Lipschitz continuous in terms of v. ∎

By Lemma 1.1, the study of the multiplicity of solutions of (1.1) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = s_1\phi_{00} + s_2\phi_{10} \tag{1.5}$$

defined on the two dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}\}$.

While one feels instinctively that (1.5) ought to be easier to solve, there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some very important v's.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (1.2) reduces to

$$L0 + (I-P)(bv^+ - av^-) = 0$$

which is satisfied because $v^+ = v$, $v^- = 0$ and $(I-P)v = 0$, since $v \in V$.

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$ and $\phi_{00}(x,t) > 0$ in Q, there exists a cone C_1 defined by

$$C_1 = \left\{ v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq \frac{c_1}{\sqrt{2}} \right\}$$

so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \left\{ v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq \frac{|c_1|}{\sqrt{2}} \right\}$$

so that $v \le 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now, we define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V. \tag{1.6}$$

Then Φ is continuous on V and we have the following lemma.

LEMMA 1.2. *$\Phi(cv) = c\Phi(v)$ for $c \ge 0$.*

Proof. Let $c \ge 0$. If v satisfies

$$L\theta(v) + (I-P)(b(v+\theta(v))^+ - a(v+\theta(v))^-) = 0,$$

then

$$L(c\theta(v)) + (I-P)(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned}\Phi(cv) &= L(cv) + P(b(cv+\theta(cv))^+ - a(cv+\theta(cv))^-) \\ &= L(cv) + P(b(cv+c\theta(v))^+ - a(cv+c\theta(v))^-) \\ &= c\Phi(v).\end{aligned}$$

∎

We investigate the images of the cones C_1 and C_3 under Φ. First we consider the image of the cone C_1. If $v = c_1\phi_{00} + c_2\phi_{10} \ge 0$, we have

$$\begin{aligned}\Phi(v) &= L(v) + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= c_1\lambda_{00}\phi_{00} + c_2\lambda_{10}\phi_{10} + b(c_1\phi_{00} + c_2\phi_{10}) \\ &= c_1(b+\lambda_{00})\phi_{00} + c_2(b+\lambda_{10})\phi_{10}.\end{aligned}$$

Thus the images of the rays $c_1\phi_{00} \pm \frac{c_1}{\sqrt{2}}\phi_{10} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b + \lambda_{00})\phi_{00} \pm \frac{c_1}{\sqrt{2}}(b + \lambda_{10})\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone R_1 in the right half-plane of V, where

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \frac{1}{\sqrt{2}} \left(\frac{b + \lambda_{10}}{b + \lambda_{00}} \right) d_1 \right\}.$$

Second we consider the image of C_3. If $v = -c_1\phi_{00} + c_2\phi_{10} \leq 0$, we have

$$\begin{aligned} \Phi(v) &= L(v) + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= Lv + P(av) \\ &= -c_1\lambda_{00}\phi_{00} + c_2\lambda_{10}\phi_{10} - ac_1\phi_{00} + ac_2\phi_{10} \\ &= -c_1(\lambda_{00} + a)\phi_{00} + c_2(\lambda_{10} + a)\phi_{10}. \end{aligned}$$

Thus the images of the rays $-c_1\phi_{00} \pm \frac{c_1}{\sqrt{2}}\phi_{10}$ $(c_1 \geq 0)$ can be explicitly calculated and they are

$$-c_1(\lambda_{00} + a)\phi_{00} \pm \frac{c_1}{\sqrt{2}}(\lambda_{10} + a)\phi_{10} \qquad (c_1 \geq 0).$$

Thus Φ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, \ |d_2| \leq \frac{1}{\sqrt{2}} \left(\frac{\lambda_{10} + a}{\lambda_{00} + a} \right) d_1 \right\},$$

which is in the right half-plane of V and $R_1 \subset R_3$, since $-5 < a < -1$ and $3 < b < 7$.

Last we investigate the images of the cones C_2 and C_4 under Φ, where

$$C_2 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0, \ |\frac{c_1}{\sqrt{2}}| \leq c_2\},$$

$$C_4 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \le 0,\ |\frac{c_1}{\sqrt{2}}| \le |c_2|\}.$$

To investigate the images of the cones C_2, C_4, we need the following lemma.

LEMMA 1.3. *For every $v = c_1\phi_{00} + c_2\phi_{10}$ in V, there exists a constant $d > 0$ such that*

$$(\Phi(v), \phi_{00}) \ge d|c_2|.$$

Proof. Let us write $h(u) = bu^+ - au^-$. Let $u = c_1\phi_{00} + c_2\phi_{10} + \theta(c_1, c_2)$. Then we have

$$\Phi(v) = L(c_1\phi_{00} + c_2\phi_{10}) + P(h(c_1\phi_{00} + c_2\phi_{10} + \theta(c_1, c_2))).$$

Hence we have

$$(\Phi(v), \phi_{00}) = ((L - \lambda_{00})(c_1\phi_{00} + c_2\phi_{10}), \phi_{00}) + (h(u) + \lambda_{00}u, \phi_{00}).$$

The first term is zero because $(L - \lambda_{00})\phi_{00} = 0$ and L is self-adjoint. The second term satisfies

$$\begin{aligned} h(u) + \lambda_{00}u &= bu^+ - au^- + \lambda_{00}u^+ + \lambda_{00}u^- \\ &= (b + \lambda_{00})u^+ + (\lambda_{00} + a)u^- \ge \gamma|u|, \end{aligned}$$

where $\gamma = \min\{b + \lambda_{00}, \lambda_{00} + a\} > 0$. Therefore

$$(\Phi(v), \phi_{00}) \ge \gamma \int |u|\phi_{00}.$$

Now there exists $d > 0$ so that $\gamma\phi_{00} \ge d|\phi_{10}|$ and therefore

$$\gamma \int |u|\phi_{00} \ge d \int |u||\phi_{10}| \ge d\left|\int u\phi_{10}\right| = d|c_2|.$$

This proves the lemma. ∎

Lemma 1.3 means that the image of Φ is contained in the right half-plane of V. That is, $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the right half-plane of V. The image of C_2 under Φ is the cone containing

$$R_2 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, -\frac{1}{\sqrt{2}}\left(\frac{\lambda_{10}+a}{\lambda_{00}+a}\right) d_1 \leq d_2 \leq \frac{1}{\sqrt{2}}\left(\frac{\lambda_{10}+b}{\lambda_{00}+b}\right) d_1 \right\}$$

and the image of C_4 under Φ is the cone containing

$$R_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, -\frac{1}{\sqrt{2}}\left(\frac{\lambda_{10}+b}{\lambda_{00}+b}\right) d_1 \leq d_2 \leq \frac{1}{\sqrt{2}}\left(\frac{\lambda_{10}+a}{\lambda_{00}+a}\right) d_1 \right\}$$

We note that all the cones R_2, R_3, R_4 contain the cone R_1. Also R_3, R_2 contain the cone $R_2 \backslash R_1$, and R_3, R_4 contain the cone $R_4 \backslash R_1$. Hence we have the theorem.

THEOREM 1.1. *Suppose* $-5 < a < -1$ *and* $3 < b < 7$. *Let* $f = s_1\phi_{00} + s_2\phi_{10}$. *Then we have :*

(i) If f *belongs to the interior* $\text{Int}R_1$ *of* R_1, (1.1) has a positive solution, a negative solution, and at least two sign changing solutions.

(ii) If f belongs to the boundary ∂R_1 of R_1, (1.1) has a positive solution, a negative solution, and at least one sign changing solution.

(iii) If f belongs to $\text{Int}(R_3 \backslash R_1)$, (1.1) has a negative solution and at least one sign changing solution.

(iv) If f belongs to ∂R_3, (1.1) has a negative solution.

REMARK. *If* $f = s_1\phi_{00} + s_2\phi_{10}$ *and* $s_1 < 0$, *then* (1.1) has no solution. Also, if $f = s_1\phi_{00} + s_2\phi_{10}$ and $s_1 = 0$, $s_2 \neq 0$, then (1.1) has no solution.

Proof. If we consider the inner product

$$(Lu + bu^+ - au^-, \phi_{00}) = (f, \phi_{00}),$$

then the right hand side is s_1 and the left hand side is nonnegative. Hence we have the conclusion. ∎

REMARK. *With Theorem 1.1, we can not claim that if f does not belong to R_3 then* (1.1) has no solution.

1.2. A Region without Solution

In this subsection, we conclude the region of source terms without solution. We assume that $-5 < a < -1$, $3 < b < 7$ and f is generated by $\{\phi_{00}, \phi_{10}\}$, that is, $f = s_1\phi_{00} + s_2\phi_{10}(s_1, s_2 \in \mathbb{R})$. We consider a semilinear beam equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H, \tag{1.7}$$

The study of the map $\Phi : V \to V$ defined in (1.6) will give a powerful theorem to conclude the region of source terms without solution. We consider the restriction $\Phi|_{C_i}(1 \le i \le 4)$ of Φ to the cone C_i. Let $\Phi_i = \Phi|_{C_i}$, i.e.,

$$\Phi_i : C_i \to V.$$

First, we consider the restriction Φ_1. The restriction Φ_1 maps C_1 onto R_1. Let l_1 be the segment in R_1, defined by

$$l_1 = \left\{\phi_{00} + d_2\phi_{10} \mid |d_2| \le \frac{1}{\sqrt{2}}\left(\frac{b+\lambda_{10}}{b+\lambda_{00}}\right)\right\}.$$

Then the inverse image $\Phi_1^{-1}(l_1)$ of l_1 is a segment, in C_1

$$\mathcal{L}_1 = \left\{\frac{1}{b+1}(\phi_{00} + c_2\phi_{10}) \mid |c_2| \le \frac{1}{\sqrt{2}}\right\}.$$

It follows from Lemma 1.2 that $\Phi_1 : C_1 \to R_1$ is a bijection.

Second, we consider the restriction $\Phi_3 : C_3 \to V$. It maps C_3 onto R_3. If we let l_3 the segment in R_3, defined by

$$l_3 = \left\{ \phi_{00} + d_2\phi_{106} \mid |d_2| \leq \frac{1}{\sqrt{2}} \left(\frac{a+\lambda_{10}}{a+\lambda_{00}} \right) \right\},$$

then the inverse image $\Phi_3^{-1}(l_3)$ is a segment

$$\mathcal{L}_3 = \left\{ \frac{1}{a+1}(\phi_{00} + c_2\phi_{10}) \mid |c_2| \leq \frac{1}{\sqrt{2}} \right\}.$$

It follows from Lemma 1.2 that $\Phi_3 : C_3 \to R_3$ is a bijection.

Now we study the restrictions Φ_2 and Φ_4. Let $i = 2, 4$. Let γ be a simple path in R_i. We investigate the inverse image $\Phi_i^{-1}(\gamma)$ of γ and conclude the region of source terms that (1.1) has no solution. We note that $\Phi_i(C_i)$ contains R_i.

LEMMA 1.4. *Let $i = 2, 4$. Let γ be any simple path in R_i with end points on ∂R_i, where each ray (starting from the origin) in R_i intersects only one point of γ. Then the inverse image $\Phi_i^{-1}(\gamma)$ of γ is a simple path in C_i with end points on ∂C_i, where any ray (starting from the origin) in C_i intersects only one point of this path.*

Proof. We note that $\Phi_i^{-1}(\gamma)$ is closed since Φ is continuous and γ is closed in V. Suppose that there is a ray (starting from the origin) in C_i which intersects two points of $\Phi_i^{-1}(\gamma)$, say, p, $\alpha p\,(\alpha > 1)$. Then by Lemma 1.2,

$$\Phi_i(\alpha p) = \alpha\Phi_i(p),$$

which implies that $\Phi_i(p) \in \gamma$ and $\Phi_i(\alpha p) \in \gamma$. This contradicts that each ray (starting from the origin) C_i intersects only one point of γ.

We regard a point p in the plane V as a radius vector. For a point v in V, we define the argument $\arg p$ of p by the angle from the positive ϕ_{00}-axis to p.

We claim that $\Phi_i^{-1}(\gamma)$ meets all ray in C_i, starting from the origin. In fact, if not, $\Phi_i^{-1}(\gamma)$ is disconnected in C_i. Since $\Phi_i^{-1}(\gamma)$ is closed and meets at most one point of any ray in C_i, there are two points p_1 and p_2 in C_i such that $\Phi_{-1}(\gamma)$ does not contain any point p with

$$\arg p_1 < \arg p < \arg p_2.$$

On the other hand, if we let l the segment in C_i with end points p_1 and p_2, then $\Phi_i(l)$ is a path in R_i, where $\Phi_i(p_1)$ and $\Phi_i(p_2)$ belong to γ. Choose a point q in $\Phi_i(l)$ such that $\arg q$ is between $\arg \Phi_i(p_1)$ and $\arg \Phi_i(p_2)$. Then there exist a point q' in γ such that $q' = \beta q$ for some $\beta > 0$ since γ is a simple path. But $\Phi_i^{-1}(q')$ satisfies

$$\arg p_1 < \arg \Phi_i^{-1}(q') < \arg p_2,$$

which is a contradiction. This completes the lemma. ∎

With Lemma 1.4, we have the following theorem, which is very important to investigate the multiplicity of solutions of a semilinear wave equation when the source term varies.

THEOREM 1.2. *For $1 \le i \le 4$, the restriction Φ_i maps C_i onto R_i. Therefore, Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.*

The above theorem implies Theorem 1.1. Furthermore, we conclude the region of source terms in which (1.1) has no solution.

THEOREM 1.3. *Under the same condition as in Theorem 1.1, if f does not belong to the cone R_3, then equation* (1.1) has no solution.

2. The Nonlinearity Crosses An Eigenvalue

In this section, we investigate multiplicity of solutions $u(x,t)$ for a piecewise linear perturbation $-(bu^+ - au^-)$ of the one-dimensional wave operator $u_{tt} - u_{xx}$ with the nonlinearity $-(bu^+ - au^-)$ crossing the eigenvalue λ_{10}. We suppose that $-1 < a < 3$ and $3 < b < 7$. Under this assumption, we have a concern with a relation between multiplicity of solutions and source terms of a nonlinear wave equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H. \tag{2.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}.

We shall use the contraction mapping theorem to reduce the problem from an infinite dimensional one in H to a finite dimensional one and investigate multiplicity of solutions and source terms of equation (2.1).

2.1. A Variational Reduction Method

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I-P)u$. Hence equation (2.1) is equivalent to a system

$$Lw + (I-P)(b(v+w)^+ - a(v+w)^-) = 0, \tag{2.2}$$

$$Lv + P(b(v+w)^+ - a(v+w)^-) = s_1\phi_{00} + s_2\phi_{10}. \tag{2.3}$$

LEMMA 2.1. *For fixed* $v \in V$, (2.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to L^2 norm) in terms of v.

The proof of the lemma is similar to that of Lemma 1.1 in Section 1.

By Lemma 2.1, the study of multiplicity of solutions of (2.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = s_1\phi_{00} + s_2\phi_{10} \tag{2.4}$$

defined on the two dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}\}$.

Let $C_i (1 \leq i \leq 4)$ be the same cones of V as in Section 1. We define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V. \tag{2.6}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma.

LEMMA 2.2. $\Phi(cv) = c\Phi(v)$ *for* $c \geq 0$.

Lemma 2.2 implies that Φ maps a cone with vertex 0 onto a cone with vertex 0. Let $C_i (1 \leq i \leq 4)$ be the same cones of V as in Section 1. We investigate the images of the cones C_1 and C_3 under Φ. First we consider the image of the cone C_1. If $v = c_1\phi_{00} + c_2\phi_{10} \geq 0$, we have

$$\begin{aligned}
\Phi(v) &= L(v) + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\
&= c_1\lambda_{00}\phi_{00} + c_2\lambda_{10}\phi_{10} + b(c_1\phi_{00} + c_2\phi_{10}) \\
&= c_1(b+\lambda_{00})\phi_{00} + c_2(b+\lambda_{10})\phi_{10}.
\end{aligned}$$

Thus the images of the rays $c_1\phi_{00} \pm \frac{1}{\sqrt{2}}c_1\phi_{10} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b+\lambda_{00})\phi_{00} \pm \frac{1}{\sqrt{2}}c_1(b+\lambda_{10})\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \frac{1}{\sqrt{2}}\left(\frac{b+\lambda_{10}}{b+\lambda_{00}}\right) d_1 \right\}.$$

The cone R_1 is in the right half-plane of V and the restriction $\Phi|_{C_1} : C_1 \rightarrow R_1$ is bijective.

We determine the image of the cone C_3. If $v = -c_1\phi_{00} + c_2\phi_{10} \leq 0$, we have

$$\begin{aligned}
\Phi(v) &= L(v) + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\
&= Lv + P(av) \\
&= -c_1(\lambda_{00}+a)\phi_{00} + c_2(\lambda_{10}+a)\phi_{10}.
\end{aligned}$$

Thus the images of the rays $-c_1\phi_{00} \pm \frac{1}{\sqrt{2}}c_1\phi_{10}$ $(c_1 \geq 0)$ can be explicitly calculated and they are

$$-c_1(\lambda_{00}+a)\phi_{00} \pm \frac{1}{\sqrt{2}}c_1(\lambda_{10}+a)\phi_{10} \qquad (c_1 \geq 0).$$

Thus Φ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \leq 0,\ d_2 \leq \frac{1}{\sqrt{2}}\left|\frac{\lambda_{10}+a}{\lambda_{00}+a}\right||d_1| \right\}.$$

The cone R_3 is in the left half-plane of V and the restriction $\Phi|_{C_3} : C_3 \rightarrow R_3$ is bijective. We note that R_1 is in the right half plane and R_3 is in the left half plane.

THEOREM 2.1. *(i) If f belongs to R_1, then equation (2.1) has a positive solution and no negative solution. (ii) If f belongs to R_3, then equation (2.1) has a negative solution and no positive solution.*

The cones C_2, C_4 are as follows

$$C_2 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0,\ c_2 \geq \frac{1}{\sqrt{2}}|c_1|\},$$

$$C_4 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0,\ c_2 \leq -\frac{1}{\sqrt{2}}|c_1|\}.$$

Then the union of four cones C_i $(1 \leq i \leq 4)$ is the space V.

Lemma 2.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R_2' = \left\{d_1\phi_{00} + d_2\phi_{10} \mid d_2 \geq 0, -\sqrt{2}|\frac{\lambda_{00}+a}{\lambda_{10}+a}|d_2 \leq d_1 \leq \sqrt{2}\left|\frac{b+\lambda_{00}}{b+\lambda_{10}}\right| d_2\right\},$$

$$R_4' = \left\{d_1\phi_{00} + d_2\phi_{10} \mid d_2 \leq 0, \sqrt{2}\left(\frac{\lambda_{00}+a}{\lambda_{10}+a}\right) d_2 \leq d_1 \leq \sqrt{2}\left(\frac{b+\lambda_{00}}{b+\lambda_{10}}\right)|d_2|\right\}.$$

Then the union of four cones R_1, R_2', R_3, R_4' is also the space V.

To investigate a relation between multiplicity of solutions and source terms in the nonlinear beam equation

$$Lu + bu^+ - au^- = f \quad \text{in} \quad H, \tag{2.6}$$

we consider the restrictions $\Phi|_{C_i}(1 \leq i \leq 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, i.e.,

$$\Phi_i : C_i \to V.$$

For $i = 1, 3$, the image of Φ_i is R_i and $\Phi_i : C_i \to R_i$ is bijective.

From now on, our goal is to find the image of C_i under Φ_i for $i = 2, 4$. Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ is on the ray $c_1(b+\lambda_{00})\phi_{00}+\frac{1}{\sqrt{2}}c_1(b+\lambda_{10})\phi_{10}, c_1 \geq 0$ (a boundary ray of R_1) and the image of the other end point of γ under Φ is on the ray $-c_1(\lambda_{00}+a)\phi_{00}+\frac{1}{\sqrt{2}}c_1(\lambda_{10}+a)\phi_{10}, c_1 \geq 0$ (a boundary ray of R_3). Since Φ is continuous, $\Phi(\gamma)$ is a path in V. By Lemma 1.2, $\Phi(\gamma)$ does not meet the origin. Hence the path $\Phi(\gamma)$ meets all rays (starting from the origin) in $R_1 \cup R_4'$ or all rays (starting from the origin) in $R_2' \cup R_3$.

Therefore it follows from Lemma 1.2 that the image $\Phi(C_2)$ of C_2 contains one of sets $R_1 \cup R_4'$ and $R_2' \cup R_3$.

Similarly, we have that the image $\Phi(C_4)$ of C_4 contains one of sets $R_1 \cup R_2'$ and $R_4' \cup R_3$.

LEMMA 2.3. *Let A be one of the sets $R_1 \cup R_4'$ and $R_2' \cup R_3$ such that it is contained in $\Phi(C_2)$. Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_2^{-1}(\gamma)$ of γ is a simple path in C_2 with end points on ∂A, where any ray (starting from the origin) in C_2 intersects only one point of this path.*

Proof. We note that $\Phi_2^{-1}(\gamma)$ is closed since Φ is continuous and γ is closed in V. Suppose that there is a ray (starting from the origin) in C_2 which intersects two points of $\Phi_2^{-1}(\gamma)$, say, p, $\alpha p\,(\alpha > 1)$. Then by Lemma 1.2,

$$\Phi_2(\alpha p) = \alpha \Phi_2(p),$$

which implies that $\Phi_2(p) \in \gamma$ and $\Phi_2(\alpha p) \in \gamma$. This contradicts that each ray (starting from the origin) in A intersect only one point of γ.

We regard a point p as a radius vector in the plane V. Then for a point p in V, we define the argument $\arg p$ of p by the angle from the positive ϕ_{00}-axis to p.

We claim that $\Phi_2^{-1}(\gamma)$ meets all ray (starting from the origin) in A. In fact, if not, $\Phi_2^{-1}(\gamma)$ is disconnected in A. Since $\Phi_2^{-1}(\gamma)$ is closed and meets at most one point of any ray in A, there are two points p_1 and p_2 in C_2 such that $\Phi_2^{-1}(\gamma)$ does not contain any point p with

$$\arg p_1 < \arg p < \arg p_2.$$

On the other hand, if we let l the segment with end points p_1 and p_2, then $\Phi_2(l)$ is a path in A, where $\Phi_2(p_1)$ and $\Phi_2(p_2)$ belong to γ. Choose a point q in $\Phi_2(l)$ that $\arg q$ is between $\arg \Phi_2(p_1)$ and $\arg \Phi_2(p_2)$. Then there exist a point q' such that $q' = \beta q$ for some $\beta > 0$. But $\Phi_2^{-1}(q')$ meets l and

$$\arg p_1 < \arg \Phi_2^{-1}(q') < \arg p_2,$$

which is a contradiction. This completes the lemma. ∎

Similarly, we have the following lemma.

LEMMA 2.3′. *Let A be one of the sets $R_1 \cup R_2'$ and $R_4' \cup R_3$ such that it is contained in $\Phi(C_4)$. Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_4^{-1}(\gamma)$ of γ is a simple path in C_4 with end points on ∂A, where any ray (starting from the origin) in C_4 intersects only one point of this path.*

With Lemma 2.3 and Lemma 2.3′, we have the following theorem, which is very important to investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation.

THEOREM 2.2. *For $i = 2, 4$, if we let $\Phi_i(C_i) = R_i$, then R_2 is one of sets $R_1 \cup R_4'$, $R_2' \cup R_3$ and R_4 is one of sets $R_3 \cup R_4'$, $R_1 \cup R_2'$. Furthermore, for each $1 \le i \le 4$, the restriction Φ_i maps C_i onto R_i. In particular, Φ_1 and Φ_3 are bijective.*

To determine the images $R_2 = \Phi(C_2)$ and $R_4 = \Phi(C_4)$, we shall investigate the nonlinear beam equation

$$Lu + bu^+ - au^- = s\phi_{00} \quad \text{in} \quad H,$$

where we $-1 < a < 3 < b < 7$ and s is real.

2.2. Multiplicity of Solutions and Source Terms

In this subsection we reveal the relation between multiplicity of solutions and source terms in the nonlinear beam equation (2.1). Now we remember the map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V,$$

where $-1 < a < 3 < b < 7$, $\theta(v)$ is a solution of (2.2), and V is the two-dimensional subspace of H_0 spanned by two eigenfunctions $\lambda_{00}, \lambda_{10}$. The map Φ is continuous on V, since θ is continuous on V. For $1 \le i \le 4$, let C_i be the same cone, in V, as in subsection 2.1.

For $f \in V$, we establish an *a priori* bound for solutions of

$$Lv + P(b(v + \theta(v))^+ - a\theta(v))^-) = f \quad \text{in} \quad V. \tag{2.7}$$

LEMMA 2.4. *Let $C = \{(a, b) : \frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} = 1\}$. Let $k(\ge 16)$ be fixed and $f \in V$ with $\|f\| = k$. Let $\alpha, \beta, \epsilon > 0$ be given. Let $3 + \alpha < b < 7 - \alpha$, $-1 + \beta < a < 3 - \beta$ satisfy the condition $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} \ne 1$ and $dist((a, b), C) \ge$*

ϵ. Then there exists $R_0 > 0$ (depending only on k and α, β, ϵ) such that the solutions of (2.7) satisfy $\|v\| < R_0$.

Proof. Let $-1 < a < 3 < b < 7$, $f \in V$. Let $v \in V$ be given. Then there exists a unique solution $z \in W$ of the equation

$$Lz + (I - P)[b(v+z)^+ - a(v+z)^- - f] = 0 \quad \text{in} \quad W.$$

If $z = \theta(v)$, then θ is continuous on V and we have $DI_{b,a}(v + \theta(v))(w) = 0$ for all $w \in W$. In particular $\theta(v)$ satisfies a uniform Lipschitz in v with respect to th L^2 norm (cf. [7]).

Suppose the lemma does not hold. Then there is a sequence (b_n, a_n, v_n) such that $b_n \in [-1+\alpha, 7-\alpha]$, $a_n \in [-1+\beta, 3-\beta]$ satisfy $dist((a,b), C) \geq \epsilon$, $\|v_n\| \to +\infty$, and

$$v_n = L^{-1}(f - P(b(v_n + \theta(v_n))^+ - a(v_n + \theta(v_n))^-) \quad \text{in} \quad V.$$

Let $u_n = v_n + \theta(v_n)$. Then the sequence (b_n, a_n, u_n) with $b_n \in [-1+\alpha, 7-\alpha]$, $a_n \in [-1+\beta, 3-\beta]$ satisfies $\|u_n\| \to +\infty$ and

$$u_n = L^{-1}(f - bu_n^+ + au^-) \quad \text{in} \quad H.$$

Put $w_n = \frac{u_n}{\|u_n\|}$. Then we have

$$w_n = L^{-1}(\frac{f}{\|u_n\|} - bw_n^+ + aw^-).$$

The operator L^{-1} is compact. Therefore we may assume that $w_n \to w_0$, $b_n \to b_0 \in (-1, 7)$, $a_n \to a_0 \in (-1, 3)$ with $(a_0, b_0) \notin C$. Since $\|w_n\| = 1$ for all n, $\|w_0\| = 1$ and w_0 satisfies

$$w_0 = L^{-1}(-b_0 w_0^+ + aw_0) \quad \text{in} \quad H_0.$$

This contradicts the fact that for $-1 < a, b < 7$ with the condition $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} \neq 1$ $Lu + bu^+ - au^- = 0$ has only the trivial solution. ■

LEMMA 2.5. *Let $-1 < a < 3$, $-1 < b < 7$ satisfy*

$$\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1. \tag{2.8}$$

Let $k(\geq b+1)$ be fixed and $f \in V$ with $\|f\| = k$. Then we have

$$d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R, 0) = 1$$

for all $R \geq R_0$.

Proof. Let $b = a = 0$. Then we have

$$d(v - L^{-1}(f), B_R, 0) = 1,$$

since the map is simply a translation of the identity and since $\|L^{-1}(f)\| < R_0$ by Lemma 2.12.

In case $b, a \neq 0(-1 < a < 3, -1 < b < 7)$ with $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{a+1}} < 1$, the result follows in the usual way by invariance under homotopy, since all solutions are in the open ball B_{R_0} (cf. [15]). ■

LEMMA 2.6. *Let $-1 < a < 3 < b < 7$ satisfy the condition (2.8) and $f = (b+1)\phi_{00}$. Then equation (2.7) has a positive solution in* $\text{Int}C_1$*, at least one sign changing solution in* $\text{Int}C_2$*, and at least one sign changing solution in* $\text{Int}C_4$*.*

Proof. *First we compute the degree ($R > R_0$)*

$$\begin{aligned} &d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap C_1, 0) \\ &= d(v - L^{-1}(f - bv), B_R \cap C_1, 0) = -1, \end{aligned}$$

since $v - L^{-1}(f - bv) = 0$ has a unique solution in $\mathrm{Int}C_1$ and $1 + \frac{b}{\lambda_{00}} > 0$, $1 + \frac{b}{\lambda_{10}} < 0$. Since, for $f = (b+1)\phi_{00}$, equation (2.7) has no negative solution in $\mathrm{Int}C_3$,

$$d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap C_3, 0) = 0.$$

By the domain decomposition lemma,

$$d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap (C_2 \cup C_4), 0) = 2.$$

Hence equation (2.7) has at least one sign changing solution in $\mathrm{Int}(C_2 \cup C_4)$.

Suppose that (2.7) has a solution in Int C_2. Then $\Phi(C_2) \cap R_1 \neq \phi$ and hence $R_2 = \Phi(C_2) = R_1 \cup R_4'$ by Theorem 1.2. Let $B : V \to V$ be a linear map, where the matrix B is given by

$$\begin{pmatrix} \frac{b+a+2\lambda_{00}}{2} & \frac{b-a}{\sqrt{2}} \\ \frac{b-a}{2\sqrt{2}} & \frac{b+a+2\lambda_{10}}{2} \end{pmatrix}.$$

Then $B(C_2) = R_2 = \Phi(C_2)$ and $Bv = \Phi(v)$ for all $v \in \partial C_2$. Now we may assume that the solution of $Bv = f$ is in B_{R_0}. Hence if $0 \leq t \leq 1$ and $R \geq R_0$, then we have

$$tBv + (1-t)\Phi(v) \neq f, \quad v \in \partial(B_R \cap C_2).$$

So we have

$$\begin{aligned} &d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap C_2, 0) \\ &= d(v - L^{-1}(f - Bv + Lv), B_R \cap C_2, 0) = 1, \end{aligned}$$

since $Bv = f$ has a unique solution in $\mathrm{Int}C_2$ and $\det(L^{-1}B) > 0$. Since $d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R, 0) = 1$ and $d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap C_3, 0) = 0$,

$$d(v - L^{-1}(f - P(b(v + \theta(v))^+ - a(v + \theta(v))^-)), B_R \cap C_4, 0) = 1.$$

Therefore (2.7) has at least one solution in $\text{Int}C_4$.

Similarly, if we assume that (2.7) has a solution in $\text{Int}C_4$, *then* $d(v - L^{-1}(f - P(b(v+\theta(v))^+ - a(v+\theta(v))^-)), B_R \cap C_4, 0) = 1$ *and hence we get*

$$d(v - L^{-1}(f - P(b(v+\theta(v))^+ - a(v+\theta(v))^-)), B_R \cap C_2, 0) = 1.$$

Therefore (2.7) has at least one solution in $\text{Int}C_2$. ■

With Theorem 2.2, Lemma 2.6, we get the following.

LEMMA 2.7. *Let* $-1 < a < 3 < b < 7$ *satisfy the condition (2.8). For* $1 \leq i \leq 4$, *let* $\Phi(C_i) = R_i$. *Then* $R_2 = R_1 \cup R_4'$ *and* $R_4 = R_1 \cup R_2'$, *where* R_2', R_4' *are the same cones as in subsection 2.1.*

Proof. It follows from Lemma 2.6 that $R_2 \cap R_1 \neq \phi$. Since R_2 is one of sets $R_1 \cup R_4'$, $R_3 \cup R_2'$ (Theorem 2.2), the image R_2 of C_2 under Φ must be $R_1 \cup R_4'$.

On the other hand, it follows from Lemma 2.6 that $R_4 \cap R_1 \neq \phi$. Since R_4 is one of sets $R_1 \cup R_2'$, $R_3 \cup R_4'$ (Theorem 1.2), the image R_4 of C_4 under Φ must be $R_1 \cup R_2'$. ■

If a solution of (2.4) is in C_1, then it is positive. If a solution of (2.4) is in C_3, then it is negative. If a solution of (2.4) is in $\text{Int}(C_2 \cup C_4)$, then it has both signs. Therefore we have the main theorem of this paper with aid of Theorem 2.1, Theorem 2.2, and Lemma 2.7.

THEOREM 2.3. *Let* $-1 < a < 3 < b < 7$ *satisfies the condition (2.7). Then we have the followings.*

(i) If $f \in \text{Int } R_1$, *then equation (2.1) has a positive solution and at least two*

sign changing solutions.

(ii) If $f \in \partial R_1$, then equation (2.1) has a positive solution and at least one sign changing solution.

(iii) If $f \in \text{Int } R_i'(i=2,4)$, then equation (2.1) has at least one sign changing solution.

(iv) If $f \in \text{Int } R_3$, then equation (2.1) has only the negative solution.

(v) If $f \in \partial R_3$, then equation (2.1) has a negative solution.

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Q-Heung Choi and Sungki Chun
Department of Mathematics
Inha University
Incheon 402-751, Korea
Tacksun Jung
Department of Mathematics
Kunsan National University
Kunsan 573-360, Korea

Multiplicity and stability result for semilinear parabolic equations

Norimichi Hirano
Department of Mathematics
Faculty of Engineering
Yokohama National University
Yokohama, Japan

Wan Se Kim
Department of Mathematics
College of Sciences
Hanyang University
Seoul 133-791, Korea

1. Introduction. Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$. We denote by $\lambda_1 < \lambda_2 < \cdots$ the eigenvalues of the problem

$$(L) \qquad -\Delta_x u = \lambda u, \quad u \in H_0^1(\Omega).$$

In the last decade, the existence, multiplicity and stability of periodic solutions of semilinear parablic equations of the form

$$(P) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u = g(x,t,u) & \text{in} \quad R \times \Omega \\ u(t,x) = 0 & \text{on} \quad R \times \partial\Omega \\ u(0,x) = u(2\pi, x) & \text{on} \quad \Omega \end{cases}$$

has been studied by many authors(cf. [1],[2],[9],[13],[15], [18]). The main tools to attack this kind of problems are Leray Schauder degree theory and sub- and supersolution method. We restrict ourselves to the case that g is given by the form $g(t,x,u) = g(u) + h(t,x)$, where g is a continuous function on R and $h \in C(R, L^2(\Omega))$ is a periodic function with period 2π. In this case, the existence, multiplicity and stability of periodic solutions depends on the growth condition imposed on g. It is known that if

$$\lim_{v \to \pm\infty} g(v)/v < \lambda_1,$$

then problem (P) possesses at least one periodic solution(cf. Amann[2]). Moreover if $h \in C(R, C_0^1(\overline{\Omega}))$, then problem (P) has at least one stable solution (See Dancer and Hess[9] and Hess[13]). In case that

$$\lambda_k < \lim_{v \to \pm\infty} g(v)/v < \lambda_{k+1},$$

for some $k \geq 1$, problem (P) has a periodic solution(cf. Hirano & Mioguchi[15]) . On the other hand, In case that g satisfies the jumping nonlinearity condition:

$$\lim_{v \to -\infty} g(v)/v < \lambda_1 < \lim_{v \to +\infty} g(v)/v,$$

the existence of stable and unstable periodic solutions of (P) was studied by the authors[14]. In the present paper, we consider the case that g and h satisfies the Ambrosetti - Prodi type condition. That is we consider the problem

$$(AP) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u - \lambda_1 u + g(u) = s\varphi + h & \text{in} \quad R \times \Omega \\ u(t,x) = 0 & \text{on} \quad R \times \partial\Omega \\ u(0,x) = u(2\pi, x) & \text{on} \quad \Omega \end{cases}$$

where ϕ denotes the positive normalized eigenfunction corresponding to the first eigenvaule λ_1 of problem (L), $s \in R$, and $h \in C([0, 2\pi], C_0^1(\overline{\Omega}))$ with

$$(H) \qquad \int_Q h(t,x)\phi(x)dtdx = 0.$$

This type of result, so called an Ambrosetti-Prodi type result has been initiated by Ambrosetti-Prodi [3] in 1972 in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. A notable discussion for AP type results for periodic and Dirichlet boundary value problem has been done by Fabry, Mawhin and Nkashama [11] and Chiappinelli, Mawhin and Nugari [7], respectively, for second order ordinary differential equations. For AP type results for periodic solutions of higher order ordinary differential equations, we refer the results of Ding and Mawhin in [10]. AP type results for Lienard systems have been done by Hirano and Kim [13], AP type results for dissipative hyperbolic equations have been done by Kim [17]. Lazer and Mckenna treated AP type multiplicity result for elliptic and parabolic equations in [18]. In our result, we assume the coercive growth condition on g and consider the multiple existence of solutions of (AP) and stabilility and instability of the solutions.

We assume that $g \in C^2(R)$ and satisfies

$$(G1) \qquad \lim_{|v|\to\infty} \inf g(v) = \infty,$$

$$(G2) \qquad \lim_{v\to-\infty} \sup g(v)/v < \lambda_2 - \lambda_1.$$

Then we have that

Theorem. For each $h \in C([0, 2\pi], C_0^1(\overline{\Omega})) \cap C^1((0, 2\pi), C(\overline{\Omega}))$ satisfying $h(0) = h(2\pi)$ and (H), there exist real numbers $s_0 \leq s_1$ such that
(i) (AP) has no solution for $s < s_0$;
(ii) (AP) has at least one solution for $s = s_1$;
(iii) (AP) has at least one stable solution and one unstable solution for $s > s_1$.

2. Preliminaries. Throughout the rest of this paper, we assume that (G1) and (G2) hold, and fix $h \in C([0, 2\pi], C_0^1(\overline{\Omega})) \cap C^1((0, 2\pi), C(\overline{\Omega}))$ satisfying (H). We put $Q = (0, 2\pi) \times \Omega$. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner pr oduct of $L^2(\Omega)$. $\|\cdot\|_Q$ and $\langle\langle\cdot,\cdot\rangle\rangle$ stand for the norm and the inner product of

$L^2(Q)$, respectively. $\|\cdot\|_C$ and $\|\cdot\|_{C^1}$ stand for the norm of $C_0(\overline{\Omega})$ and $C^1(\overline{\Omega})$, respectively. We define a linear operator $L : Dom(L) \subset L^2(Q) \to L^2(Q)$ by

$$Lu = \frac{\partial u}{\partial t} - \Delta_x u - \lambda_1 u$$

on

$$Dom(L) = \{u \in L^2((0,2\pi), H^2(\Omega) \cap H_0^1(\Omega)) : \\ \frac{\partial u}{\partial t} \in L^2(Q), u(0,x) = u(2\pi,x), x \in \Omega\}$$

Then it is easy to see that $kerL = \{c\varphi : c \in R\}$ and L is a surjective operator from $(DomL) \cap E \to E$, where E is a subspace of $L^2(Q)$ such that $L^2(Q) = kerL \oplus E$. We denote by P_1 and P_2 the projections from $L^2(\Omega)$ onto $kerL$ and E, respectively. We set $X_+ = \{u \in C_0^1(\overline{\Omega}) : u \geq 0 \text{ on } \Omega\}$. Then X_+ is a closed cone in $C_0^1(\overline{\Omega})$. We employ the standard order in $C_0^1(\overline{\Omega})$

$$x \geq y \leftrightarrow x - y \in X_+$$
$$x > y \leftrightarrow x \geq y, x \neq y$$
$$x >> y \leftrightarrow x - y \in intX_+.$$

¿From (G1), we have that

$$d = \inf\{g(s) : s \in R\} > -\infty. \tag{2.1}$$

On the other hand, it follows from (G2) that there exists $a \in (0, \lambda_2 - \lambda_1)$ and $C > 0$ satisfying that

$$| g(s) | \leq a | s | + C \qquad \text{for all } s \leq 0. \tag{2.2}$$

Here we consider the initial boundary value problem

$$(IP) \qquad \begin{cases} Lu + g(u) = s\varphi + h & \text{in} \quad R^+ \times \Omega \\ u(t,x) = 0 & \text{on} \quad R^+ \times \partial\Omega \\ u(0,x) = u_0 & \text{on} \quad \Omega \end{cases}$$

where $u_0 \in C_0(\overline{\Omega})$. It is known for each $u_0 \in C_0(\overline{\Omega})$, problem (IP) has a local solution. Let $[0, t_{u_0})$ be the maximal interval on which the solution u of (IP) exists. Then u can be represented by the integral form

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(g(x,u(s)) + h(s,x))ds \tag{2.3}$$

for $0 < t < t_{u_0}$. Here $\{S(t)\}$ is the semigroup of linear operators generated by $-\Delta$. It is known that for each $q \geq 2$, there exists $c(q) > 0$ satisfying

$$\| S(t)f \|_{W^{1,q}(\Omega)} \leq c(q)t^{-1/2} \| f \|_{L^q(\Omega)} \qquad \text{for all } f \in L^q(\Omega) \text{ and } t > 0. \quad (2.4)$$

(cf. Tanabe[19], Amann[2]).

Lemma 2.1. For each $u_0 \in C_0(\overline{\Omega})$, problem (IP) has a global solution $u \in C([0,\infty), C_0(\overline{\Omega}))$.

Proof. Let $u_0 \in C_0(\overline{\Omega})$ and u be the solution of (IP) on $[0, t_{u_0})$. To show that $t_{u_0} = \infty$, it is suficient to show that

$$\sup_{t\in[0,t_{u_0})} \| u(t) \|_C < \infty. \quad (2.5)$$

For each $n \geq 1$, we define a trancation $\overline{g}_n$ of g by

$$\overline{g}_n(s) = \min\{g(s), n\} \qquad \text{for } s \in R,$$

and consider a initial value problem of the form

$$(IP_n) \qquad \begin{cases} Lu + \overline{g}_n(u) = s\varphi + h & \text{in} \quad R^+ \times \Omega \\ u(t,x) = 0 & \text{on} \quad R^+ \times \partial\Omega \\ u(0,x) = u_0 & \text{on} \quad \Omega. \end{cases}$$

Since $\overline{g}_n$ satisfies that there exists $C_n > 0$ and

$$| \overline{g}_n(s) | \leq a \mid s \mid + C_n \qquad \text{for all } s \in R, \quad (2.6)$$

we have by a standard argument(cf. theorem 1 of Amann[2]) that problem (IP_n) has a global solution u_n for each $n \geq 1$ and $\sup_{t\in[0,\infty)} \| u_n(t) \| < \infty$. Here we fix $n_0 \geq 1$ such that $\| u_0 \|_C < n_0$. ¿From the definition of $\overline{g}_{n_0}$, we have

$$Lu + \overline{g}_{n_0}(u) \leq s\varphi + h \quad \text{in} \quad R^+ \times \Omega.$$

Then

$$L(u - u_{n_0}) + (\overline{g}_{n_0}(u) - \overline{g}_{n_0}(u_{n_0})) \leq 0.$$

Then by the parabolic maximum principle, we have that $u(t) \leq u_{n_0}(t)$ for all $t \in [0, t_{u_0})$. Then from the definition of $\overline{g}_n$, we have that there exists $n_1 \geq 1$ such that $u(t) < n_1$ on $[0, t_{u_0})$. This implies that $u_{n_1}(t) = u(t)$ on $[0, t_{u_0})$ and therefore

$$\sup_{t\in[0,u_0)} \| u(t) \|_C < \sup_{t\in[0,u_0)} \| u_{n_1}(t) \|_C < \infty.$$

This completes the proof. ∎

It is known that if $u_0 \in D = H^2(\Omega) \cap C_0(\overline{\Omega})$ and u is a solution of (IP), then $u \in C^{1,2}((0, 2\pi) \times \overline{\Omega})$(cf. Amann[2]). We define a mapping T(Poincaré mapping) by $Tu_0 = u(2\pi)$. It is known that T is a compact mapping on $C_0^1(\overline{\Omega})$ and T is strongly order preserving(cf. [1], [2], [16]), i.e., $u > v$ implies that $Tu >> Tv$. ¿From the definition of T, the solution u of (IP) is a solution of (AP) if and only of u_0 is a fixed point of T. We denote by $F(T)$ the set of fixed point of T. The function $u \in C^{1,2}((0, 2\pi) \times \Omega) \cap C^{0,1}((0, 2\pi) \times \overline{\Omega})$ is called supersolution for (AP) provided

$$\begin{cases} Lu + g(u) \geq s\varphi + h & \text{in } [0, 2\pi] \times \Omega \\ u(t, x) \geq 0 & \text{on } [0, 2\pi] \times \partial\Omega \\ u(0, x) \geq u(2\pi, x) & \text{on } \Omega. \end{cases}$$

A supersolution is said to be a strict supersolution if it is not a solution. Correspondingly, subsolution and strict subsolution are defined by reversing the inequality signs. Let $u_0 \in C_0^1(\overline{\Omega})$ and u be the solution of (IP). If u is a supersolution of (AP), then from the parabolic maximum principle, $Tu_0 \leq u_0$. If u is a subsolution of (AP), the converse relation holds. We note that from the maximum principle, the eigenfunction φ satisfies

$$\frac{\partial \varphi}{\partial \nu} > 0 \quad \text{on } \partial\Omega. \tag{2.7}$$

Here we define a homotopy $\{g_c : c \in [0, 1]\}$ of mappings defined by

$$g_c(s) = (1 - c)g(s) + cm \qquad \text{for } s \in R \tag{2.8}$$

where $m > 0$. For each $c \in [0, 1]$, we denote by (AP_c) the boundary value problem (AP) and (IP_c) the initial value problem (IP_c) with g replaced by g_c, respectively. We denote by T_c the Poincaré mapping associate with (IP_c) , and $F(T_c)$ stands for the set of fixed points of T_c. If the constant m in (2.8) is sufficiently large, we can see that the set $\cup_c T_c$ is bounded from below. That is we have

Lemma 2.2. Let $v_0 \in C_0^1(\overline{\Omega})$. Let s be a real number. Then there exists $\overline{m} > 0$ such that for each $m > \overline{m}$,

$$\sup\{\| u \|_{C^1} : u \in F(T_c) \cap (-\infty, v_0] : c \in [0, 1]\} < \infty,$$

where $(-\infty, v_0] = \{u \in C_0^1(\overline{\Omega}) : u \leq v_0\}$.

Proof. Let $v_0 \in C_0^1(\overline{\Omega})$ and $s \in R$. We choose $\epsilon > 0$ and $\delta > 0$ so small that $1 - 7\epsilon > 0$, $1 > \delta$ and that

$$\int_A |\varphi|^2 \, dx < \epsilon \| \varphi \|^2, \tag{2.9}$$

for any measurable set $A \subset \overline{\Omega}$ with $|A| < \delta$. We next choose $\Omega_0 \subset \Omega$ such that $\overline{\Omega}_0 \subset \Omega$ and $|\Omega \backslash \Omega_0| < \delta/4$. Let $\beta > 0$ such that

$$\varphi(x) \geq \beta \quad \text{on } \Omega_0. \tag{2.10}$$

We fix a positive number M_0 such that

$$(\delta/4)M_0 + d \int_Q \varphi dx > 2\pi s. \tag{2.11}$$

We set $\overline{m} = \max\{2M_0/\beta, 4s\pi\}$ and fix $m \geq \overline{m}$. Let $n_0 \geq 1$ such that $\sup_{t\in[0,2\pi]} \| v(t) \|_c < n_0$, where $v(t)$ is the solution of (IP) with $u_0 = v_0$. Then since

$$g_c(s) = (1-c)g_{n_0}(s) + cm \qquad \text{for } s \in (-\infty, n_0]$$

we have that

$$|g_c(s)| \leq a|s| + \widetilde{C} \qquad \text{for all } s \in (-\infty, n_0] \tag{2.12}$$

where $\widetilde{C}$ is a positive constant independent of s. This implies that

$$\| g_c(u) \|_{L^q(\Omega)} \leq a \| u \|_{L^q(\Omega)} + C_q \tag{2.13}$$

for all $q \geq 2$ and $u \in L^q(\Omega)$, where C_q is a positive constant independent of c and u. We denote by S_c the set of solutions u of (AP_c) with $u(0) \leq v_0$ for $c \in [0,1]$. Let $u \in S_c$, $c \in [0,1]$. Then u satisfies $u(t) < n_0$ on Q and

$$\frac{\partial u}{\partial t} - \Delta_x u - \lambda_1 u + g_c(u) = s\varphi + h \quad \text{in } Q \tag{2.14}$$

We will see that there exists $C_0 > 0$ such that

$$\| u(t) \| < C_0 \qquad \text{for all } t \in [0, 2\pi] \text{ and } u \in S_c. \tag{2.15}$$

Multiplying (2.14) by $\partial u/\partial t$ and integrating over Q, we find from the periodicity of u that

$$\| \frac{\partial u}{\partial t} \|_Q^2 = \langle\langle h, \frac{\partial u}{\partial t} \rangle\rangle \leq \| h \|_Q \| \frac{\partial u}{\partial t} \|_Q .$$

Then we find that

$$\| \partial u/\partial t \|_Q \leq \| h \|_Q . \tag{2.16}$$

Then to show that (2.15) holds, it is sufficient to show that there exists $M > 0$ such that

$$\| u \|_Q < M \qquad \text{for all } u \in S_c, c \in [0,1]. \tag{2.17}$$

Suppose contrary that there exists $\{u_n\} \subset \cup_{c\in[0,1]} S_c$ such that $u_n \in S_{c_n}$ and $\lim_{n\to\infty} \| u_n \|_Q = \infty$. For each $n \geq 1$, we put $u_n = u_n^1 + u_n^2$, where $u_n^1 = P_1 u_n$ and $u_n^2 = P_2 u_n$. By extracting subsequences, we may assume that $\alpha = \lim_{n\to\infty} \| u_n^1 \|_Q / \| u_n^2 \|_Q$ exists. We first assume that $\alpha = \infty$. Then we have from the positivity of φ that

$$\lim_{n\to\infty} | u_n(t,x) | = \infty \qquad \text{a.e. on } Q.$$

Since g is bounded from bellows and φ is positive , we find

$$\lim_{n\to\infty} \langle\langle g_{c_n}(u_n), \varphi \rangle\rangle \geq m. \tag{2.18}$$

Here we multiply (2.14) by φ and integrate over Q. Then we have

$$4s\pi \leq \langle\langle g_{c_n}(u_n), \varphi \rangle\rangle = \langle\langle s\varphi, \varphi \rangle\rangle = 2s\pi. \tag{2.19}$$

This is a contradiction . We next assume that $0 < \alpha < \infty$. Then there exists $n_0 \geq 1$ such that

$$(\alpha/2) \| u_n^2 \|_Q \leq \| u_n^1 \|_Q \leq (3\alpha/2) \| u_n^2 \|_Q \qquad \text{for all } n \geq n_0. \tag{2.20}$$

On the other hand, we have from the definition of g_c that

$$m_0 = \sup\{| s |: \beta g_c(s) < M_0 \qquad \text{for some } c \in [0,1]\} < \infty.$$

We put

$$Q_n = \{(t,x) \in [0,T] \times \Omega : | u_n(t,x) | \geq m_0\} \qquad \text{for } n \geq 1.$$

Then from the definition of m_0, we have that $| Q_n | \leq \delta/2$. In fact, if $| Q_n | > \delta/2$, then $| Q_n \cap \Omega_0 | > \delta/4$ and we have

$$\begin{aligned} 2\pi s = \langle\langle g_{c_n}(u_n), \varphi \rangle\rangle &= \int_{Q_n} g_{c_n}(u_n)\varphi dx dt + \int_{Q\setminus Q_n} g_{c_n}(u_n)\varphi dx dt \\ &> (\delta/4)M_0 + d\int_Q \varphi dx dt > 2\pi s. \end{aligned}$$

This is a contradiction. Therefore we find by (2.9) that

$$\int_{Q\setminus Q_n} | u_n^1 |^2 \, dxdt \geq (1-\epsilon) \| u_n^1 \|^2 \qquad \text{for } n \geq 1. \tag{2.21}$$

On the other hand, we have

$$\begin{aligned} 0 &= \langle\langle u_n^1, u_n^2 \rangle\rangle \\ &= \int_{Q\setminus Q_n} u_n^1 \cdot u_n^2 dxdt + \int_{Q_n} u_n^1 \cdot u_n^2 dxdt \\ &= (1/2) \int_{Q\setminus Q_n} (| u_n^1 + u_n^2 |^2 - | u_n^1 |^2 - | u_n^2 |^2) dxdt \\ &\qquad\qquad + \int_{Q_n} | u_n^1 || u_n^2 | \, dxdt. \end{aligned}$$

¿From the definition of M_0, (2.21) , (2.20) and (2.9),

$$\begin{aligned} 0 &\leq (1/2 - \delta/4) m_0^2 - (1/2)(1-\epsilon)(\alpha/2) \| u_n^2 \|^2 + \epsilon(3\alpha/2) \| u_n^2 \|^2 \\ &= (1/2 - \delta/4)) m_0^2 - (\alpha/4)(1-7\epsilon) \| u_n^2 \|^2 \end{aligned}$$

for all $n \geq n_0$. That is

$$(\alpha/4)(1-7\epsilon) \| u_n^2 \|^2 \leq (2-\delta) m_0^2/4 \qquad \text{for all } n \geq n_0.$$

Then since $\| u_n^2 \| \to \infty$, this is a contradiction. We lastly assume that $\alpha = 0$. We multiply (2.14) by u_n and integrate over Q. Then we find

$$(\lambda_2 - \lambda_1) \| u_n^2 \|_Q^2 + \langle\langle g_{c_n}(u_n), u_n \rangle\rangle \leq s \| u_n^1 \|_Q + \| h \|_Q \| u_n^2 \|_Q \, .$$

¿From (2.12) , we have

$$\begin{aligned} &(\lambda_2 - \lambda_1) \| u_n^2 \|_Q^2 - a \| u_n \|^2 \\ &\qquad\qquad \leq \widetilde{C} \, | Q |^{1/2} \| u_n \|_Q + s\alpha \| u_n^1 \|_Q + \| h \|_Q \| u_n^2 \|_Q \, . \end{aligned}$$

Since $\alpha = 0$, we have $\lim \| u_n^2 \| / \| u_n \| = 1$. Then we have

$$\limsup_{n\to\infty} (\lambda_2 - \lambda_1 - \alpha) \| u_n^2 \|_Q \leq \widetilde{C} \, | Q |^{1/2} + \| h \|_Q \, .$$

Then we obtain that $\{\| u_n^2 \|_Q\}$ is bounded and this contradicts to the assumption. Thus we obtain that (2.15) holds. Then applying (2.4) and (2.13) to the equation (2.3) repeatedly, we have that

$$\sup\{\| u(t) \|_{C([0,2\pi],L^q(\Omega))} : u \in S_c\} < \infty \qquad \text{for each } q \geq 2.$$

Then we obtain the assertion by using the Sobolev's embedding theorem. ∎

3. Proof of Theorem.

Lemma 3.1. There exists $s_0 \in R$ such that there exists no solution of (AP) for any $s < s_0$.

Proof. We can choose $s_0 < 0$ such that

$$| \Omega |^{1/2} d > s_0.$$

Suppose that $s < s_0$ and u be a solution of (AP). That is u satisfies (2.14) with $g_s = g$. Then by multiplying (2.14) with φ and integrating over Q, we find that

$$2\pi | \Omega |^{1/2} d < \langle\langle g(u), \varphi \rangle\rangle = 2\pi s \| \varphi \|^2 + \langle h, \varphi \rangle = 2\pi s.$$

This is a contradiction. ∎

Lemma 3.2. There exists $s_1 > 0$ such that for each $s > s_1$, there exists a strict subsolution $\underline{v}$ of (AP).

Proof. We put $V = E \cap H_0^1(\Omega)$. Then from the definition of L and E, we have that $L : V = E \cap H_0^1(\Omega) \to V^*$ is a maximal monotone operator (cf. [5]) . Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, $P_2 g : V \to L^2(\Omega) \subset V^*$ is a compact mapping. Then we have the sum $L + P_2 g$ is a pseudo-monotone operator from V to V^*(cf. Browder[6]). On the other hand, from (2.3), we have that for each $v \in E \cap H_0^1(\Omega)$,

$$\begin{aligned}\langle\langle Lv + P_2 g(v), v \rangle\rangle &= \langle\langle Lv + g(v), v \rangle\rangle \\ &\geq (\lambda_2 - \lambda_1) \| v \|_Q^2 - a \| v \|_Q^2 - (C - d) | Q |^{1/2} \| v \| .\end{aligned}$$

Then we have $L + P_2 g$ is coercive. That is

$$\lim_{v \in E, \|v\|_Q \to \infty} \langle\langle Lv + P_2 g(v), v \rangle\rangle / \| v \|_Q = \infty.$$

Then we obtain that there exists a solution v such that $Lv + P_2 g(v) = 0$(cf. Browder[6]). Therefore we find

$$Lv + g(v) = Lv + P_2 g(v) + P_1 g(v) = P_1 g(u). \tag{3.1}$$

¿From (2.7), we can choose $s_1 > 0$ so large that

$$P_1 g(v) \leq s_1 \varphi + h.$$

Then from (3.1) , we obtain that for $s > s_1$, v is a strict subsolution of (AP). ∎

Here we fix a C^1 mapping g_0 such that g_0 is monotone increasing and

$$\begin{aligned} g_0(s) &= g(s) && \text{for } s \in (-\infty, -1] \\ g_0(s) &= s && \text{for } s \in [0,1], \\ g_0(s) &= 1 && \text{for } s \in [2, \infty) \end{aligned}$$

We next set

$$g_1(v) = \max\{g_0(v), g(v)\} \qquad \text{for } v \in R.$$

Then from the definition of g_1, we find that

$$M = \sup\{| \, g(s) - g_1(s) \, |: s \geq 0\} < \infty.$$

Lemma 3.3. For each real number $s > s_1$, there exists a strict subsolution $\underline{v}$ and a strict supersolution $\overline{v}$ of (AP) satisfying

$$\underline{v} << \overline{v}. \tag{3.2}$$

Proof. Fix a real number $s > s_1$. By Lemma 3.2, there exists a subsolution $\underline{v}$ of (AP). We put $c = \langle M, \varphi \rangle$. Then $P_1 M = c\varphi$. Since $P_2 L^2(Q) \subset Range(L)$, there exists a solution $v_0 \in E$ of the problem

$$Lv_0 = P_2 M.$$

Since $v_0 \in C([0, 2\pi], C_0^1(\overline{\Omega}))$ and $g_1(s)/s \geq 1$ on $[0,1]$, we can see that there exists $b_0 > 0$ such that for each $b > b_0$,

$$g_1(b\varphi + v_0) > (s + c)\varphi + h \qquad \text{on } \Omega.$$

We now choose $b > b_0$ so large that

$$b\varphi + v_0 >> \underline{v}.$$

Then putting $\overline{v} = b\varphi + v_0 > 0$ on Ω, we have

$$\begin{aligned} L\overline{v} + g(\overline{v}) &\geq L\overline{v} + g_1(\overline{v}) - M \\ &= Lv_0 + g_1(\overline{v}) - M \\ &> -P_1 M + (s+c)\varphi + h \\ &= s\varphi + h. \end{aligned}$$

Thus we have seen that $\overline{v}$ is a supersolution of (AP) satisfying $\underline{v} << \overline{v}$. ∎

Proof of Theorem. (i) follows from Lemma 3.1. We will see that (iii) holds. Fix $s > s_1$. Then by Lemma 3.2 and Lemma 3.3, there exist a subsolution $\underline{v}$ of (AP) and a supersolution $\overline{v}$ of (AP) satisfying $\underline{v} << \overline{v}$. Then by theorem 1 of [9], we have that there exists a stable solution u_1 of (AP) with $\underline{v} \leq u_1 \leq \overline{v}$. On the other hand, by Lemma 2.2 and Zorn's lemma, we can find a a minimal element u_0 of $F(T)$ in $(-\infty, u_1]$ with respect to the order defined in $C_0^1(\overline{\Omega})$. Let u be a solution of (AP) with $u(0) = u_0$. We will see that u_0 is an unstable solution of (AP). We fix $m > 0$ so large that the assertion of Lemma 2.2 holds with $v = u_0$ and

$$m \int_Q \varphi dxdt > s \text{ and } m > g(u) \quad \text{on } Q. \tag{3.3}$$

Let $\{g_c : c \in [0,1]\}$ be the homotopy of mappings define in section 2. Then since

$$g_c(u) \geq g(u) \qquad \text{for all } c \in [0,1],$$

we have by the maximum principle that u is a supersolution of (AP_c) for all $c \in [0,1]$. This implies that

$$T_c u_0 \leq u_0 = T u_0 \qquad \text{for all } c \in [0,1].$$

We put $X = (-\infty, u_0] = \{u \in C_0^1(\overline{\Omega}) : u \leq u_0\}$. Then from the observation above, we have that $T_c(X) \subset X$ for all $c \in [0,1]$. Then by proposition 1 of Dancer[8], to show that u_0 is unstable, it is sufficient to show that $\text{index}_X(T, (-\infty, u_0]) \neq 1$. ¿From the homotopy invariance of indices, we have that

$$\text{index}_X(T, (-\infty, u_0]) = \text{index}_X(T_0, (-\infty, u_0]) = \text{index}_X(T_1, (-\infty, u_0]).$$

Let $v_0 \in F(T_1)$ and v be the solution of (AP_1) corresponding to v_0. Then v satisfies

$$Lv + m = s\varphi + h.$$

Multiplying the equality above by φ and integrating over Q, we have

$$m \int \varphi dxdt = \langle\langle m, \varphi \rangle\rangle = \langle\langle s\varphi, \varphi \rangle\rangle = s.$$

This contradicts to (3.3) . Thus we obtain that $F(T_1) = \phi$. This implies that $\text{index}_X(T_1, (-\infty, u_0]) = 0$. Therefore we obtain that u_0 is unstable. This complete the proof of (iii). We lastly show that (ii) holds. Suppose that $s = s_1$. Then from the arguement in the proof of Lemma 3.2, there exists a subsolution $\underline{v}$ of (AP). We note that $\underline{v}$ is not neccesarily a strict subsolution. On the other hand, we have by the argument in the proof of Lemma 3.3 that there exists a supersolution $\overline{v}$ of (AP) such that $\underline{v} << \overline{v}$. Then again by theorem 1 of [9], we have that there exists a solution u of (AP). We note that u is not necessarily stable because $\underline{v}$ is not necessarily a strict subsolution. ∎

Remark. The argument to find an unstable solution of (AP) was suggested by Prof. E. N. Dancer in personal communications.

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Regularity and convergence of crystalline motion in the plane

Katsuyuki Ishii
Department of Mathematics, Kobe University of Mercantile Marine
Higashinada, Kobe 658 JAPAN

1 Introduction

This is a brief report of my joint work with Prof. H. M. Soner [14].

Several models in phase transition give rise to geometric equation relating the normal velocity of the interface to its curvature. The curvature term is related to surface tension and the surface energy is a function of the normal direction, indicating the prefered direction of the underlying crystal structure.

Let $\Gamma \subset \mathcal{R}^n$ be a smooth hypersurface. When the surface tension H is smooth and convex, the evolution of the hypersurface is governed by

$$V = -\nabla \cdot (DH(\vec{n})), \tag{1}$$

where $\vec{n}$, V are, respectively, the outward unit normal vector and the normal velocity and the curvature of the solution $\Gamma(t)$. The mean curvature flow corresponds to $H(p) = |p|$. The equations of this type have been treated by many authors. We notice that, in two dimensional case, we may rewrite this equation as

$$V = -(f(\theta) + f''(\theta))\kappa, \tag{2}$$

where $\vec{n} = (\cos\theta, \sin\theta)$, κ is the curvature of the curve $\Gamma(t)$ and

$$f(\theta) = H(\cos\theta, \sin\theta).$$

In this report we consider the case f is a nonsmooth energy. It arises in models for crystal growth, as it is well known that solid crystals can exist in polygonal shapes. Especially we treat the *crystalline* energy, whose Frank diagram $\{x = r(\cos\theta, \sin\theta) \in \mathcal{R}^2 \ : \ rf(\theta) = 1\}$ is a polygon. Let $\Theta := \{\theta_1, \cdots, \theta_n\}$ be the angles corresponding to the corner points of the Frank diagram of f. We note that we only consider polygonal solutions with normal angles taking values in Θ.

For simplicity, we restrict our considerations to the case where the Frank diagram of the enregy f is the regular n-polygons circumscribing the unit circle. Then

$$\Theta = \Theta_n := \left\{ \frac{2\pi k}{n} \ : \ k = 0, 1, \cdots, n-1 \right\}.$$

In this case the evolution of the side i is governed by

$$V_i(t) = -\frac{2\tan(\pi/n)}{l_i(t)}\chi_i, \tag{3}$$

where $V_i(t)$, $l_i(t)$ and χ_i, are, respectively, the normal velocity, the length and the discrete curvature of the side i. The discrete curvature χ_i is equal to $+1$ if both edges of the side i are locally convex, it is equal to -1 if they are locally concave, and it is equal to 0 if otherwise. The evolution rule for the length of $l_i(t)$'s, the sides of a solution of (3), consists of a system of ordinary differential equations:

$$\frac{d}{dt}l_i(t) = \frac{1}{\cos(\pi/n)}\left(\frac{2\cos(2\pi/n)\chi_i^2}{l_i(t)} - \frac{\chi_{i-1}^2}{l_{i-1}(t)} - \frac{\chi_{i+1}^2}{l_{i+1}(t)}\right). \tag{4}$$

We call a solution $\{\Gamma(t)\}$ of (3) *crystalline motion* or *crystalline flow.*

For the discussions to more general case, see Angenent - Gurtin [2], Gurtin [8].

In the sequal we treat a two dimensional problem with a crystalline energy whose Frank diagram is a regular n-polygon and consider the following problems

- Existence of "smooth" solutions of (3) globally in time and its behavior,
- Behavior of "smooth" solutions as $n \to +\infty$.

2 n-smooth solutions of (3)

In this section we discuss the n-smooth solutions of (3). At first we define the notion of the "smoothness" of polygons. For a polygon $\Gamma \subset \mathcal{R}^2$, let $N(\Gamma)$ be the total number of sides of Γ.

Definition 2. 1 *We say that a closed polygon Γ is n-smooth, if $N(\Gamma)$ is finite and*

(1) Γ *encloses a simply connected, bounded, open subset of* $\mathcal{R}^2$,

(2) *for every* $i = 1, \cdots, N(\Gamma)$, *the normal angle* θ_i *belongs to* Θ_n,

(3) $|\theta_i - \theta_{i-1}| = 2\pi/n$ *for every* $i = 1, \cdots, N(\Gamma)$.

We say a family of polygons $\{\Gamma(t)\}_{t>0}$ is an n-smooth solution of (3) or n-smooth crystalline flow if, for each $t > 0$, $\Gamma(t)$ is an n-smooth polygon, each side of $\Gamma(t)$ moves by the law (3) and continuously in t.

As to the existence and behavior of n-smooth solutions of (3), we have the following theorem.

Theorem 2. 2 *Let Γ_0 be an n-smooth polygon enclosing an open set Ω_0. Then there exist n-smooth polygons $\{\Gamma(t)\}_{0\le t<T}$ solving (3) with the initial condition $\Gamma(0)=\Gamma_0$. Moreover $\Gamma(t)$ shrinks to a point as $t\uparrow T$, where T is given by*

$$T=\frac{|\Omega_0|}{2n\tan(\pi/n)}.$$

Remark 2. 3 (1) This theorem is an discrete analogue of a theorem of Gage - Hamilton [8] and Grayson [12].
(2) In [16] Taylor showed the existence of n-smooth solutions of (3) globally in time in more general situations. But she did not obtain the behavior of solutions.
(3) Uniqueness follows from Gurtin [13], Giga - Gurtin [9] and Taylor [16].

Outline of the proof. The proof is very similar to [16]. Using the system (4), we can easily prove the local existence of n-smooth solutions $\{\Gamma(t)\}_{t\ge 0}$ of (3) satisfying the initial data. Let $t_1>0$ be the first time this solution is no longer n-smooth. Then there are two possibilities:

- the length of onr or more sides tend to 0,
- the solution self-intersects.

However, the latter does not happen. Assume the side i vanishes at t_1. Then we show $\chi_i=0$ by contradiction arguments. This is the crucial point in the proof. Hence we observe $\Gamma(t_1)$ is still n-smooth and $N(\Gamma(t_1))\le N(\Gamma(0))-2$. We repeat this procedure starting from $\Gamma(t_1)$ and have only to do so finitely many times since $N(\Gamma(0))$ is finite.

Let $\bar{t}>0$ be the time when all sides such that $\chi_i=-1$ or 0 vanish. Then $\Gamma(t)$ is convex for all $t\ge\bar{t}$. By the above arguments we can see $\Gamma(t)$ shrinks to a point at finite time.

The extinction time T of $\Gamma(t)$ is determined by the equality:

$$\frac{d}{dt}|\Omega(t)|=-2n\tan\frac{\pi}{n}.$$

□

3 Weak Viscosity Limits

Let $\{\Gamma_n(t)\}_{0\le t<T_n}$ be a sequence of n-smooth solutions of (3) and let $\Omega_n(t)$ be the open set enclosed by $\Gamma_n(t)$. For $t\in[0,T)$, we define the upper limit $\{\widehat{\Omega}(t)\}_{0\le t<T}$ and the lower limit $\{\underline{\Omega}(t)\}_{0\le t<T}$ as follows:

$$\widehat{\Omega}(t):=\bigcap_{\substack{r>0\\ N\ge 1}}\mathrm{cl}\left(\bigcup_{\substack{|s-t|\le r,0\le s<T\\ n\ge N}}\Omega_n(t)\right)$$

$$\underline{\Omega}(t) := \bigcup_{\substack{r>0 \\ N\geq 1}} \mathrm{int} \left(\bigcap_{\substack{|s-t|\leq r, 0\leq s<T \\ n\geq N}} \Omega_n(t) \right)$$

Assume that there is a constant $R > 0$ for which $\Omega_n(t) \subset B(0,R)$ for all $n \in \mathcal{N}$ and $t \in (0,T)$. Then we have

Lemma 3. 1 (1) $\{\widehat{\Omega}(t)\}_{0\leq t<T}$ *is a weak subsolution of the mean curvature flow* $V = -\kappa$ *in the following sense: for any family of smooth compact subsets* $\{O(t)\}_{0<t<T}$,

$$V_O(x_0,t_0) \leq -\kappa_O(x_0,t_0),$$

at each $t_0 \in (0,T)$ *and* $x_0 \subset \partial O(t_0)$ *satisfying*

$$\widehat{\Omega}(t) \subset\subset O(t) \qquad \forall t \neq t_0,$$
$$\widehat{\Omega}(t_0) \subset O(t_0), \quad \partial\widehat{\Omega}(t_0) \cap \partial O(t_0) = \{x_0\}.$$

(2) $\{\underline{\Omega}(t)\}_{0\leq t<T}$ *is a weak supersolution of the mean curvature flow* $V = -\kappa$ *in the following sense: for any family of smooth compact subsets* $\{O(t)\}_{0<t<T}$,

$$V_O(x_0,t_0) \geq -\kappa_O(x_0,t_0),$$

at each $t_0 \in (0,T)$ *and* $x_0 \subset \partial O(t_0)$ *satisfying*

$$O(t) \subset\subset \underline{\Omega}(t) \qquad \forall t \neq t_0,$$
$$O(t_0) \subset \underline{\Omega}(t_0), \quad \partial O(t_0) \cap \partial\underline{\Omega}(t_0) = \{x_0\}.$$

This lemma is a set-theoretic analogue of the stability result for viscosity solutions of nonlinear PDEs by Barles - Perthame [3, 4]. Also, see Crandall - Ishii - Lions [5]. We note that, in general, this type of the stability is a simple consequence of the maximum principle. However, the crystalline flow is not defined for smooth curve and this fact is the major difficulty in proving this lemma.

4 Convergence

Let $\Gamma_0 = \partial\Omega_0$ be a smooth Jordan curve and $\Gamma_{n0} = \partial\Omega_{n0}$ be an n-smooth approximation of Γ_0 satisfying

(5) $$\lim_{n\to+\infty} d_H(\Omega_{n0}, \Omega_0) = 0,$$

where d_H is the Hausdorff distance. We have already known that there is a unique n-smooth solution of (3) satisfying the initial condition $\Gamma_n(0) = \Gamma_{n0}$ by Theorem 2. 2 . Moreover, the extinction time T_n of $\Gamma_n(t)$ satsifies

$$T_n \to T_0 := \frac{|\Omega_0|}{2\pi} \qquad (n \to +\infty).$$

Let $\{\widehat{\Omega}(t)\}_{0\leq t<T}$ and $\{\underline{\Omega}(t)\}_{0\leq t<T}$ be as in the previous section. Then our convergence resuls is stated as follows

Theorem 4. 1 *Let $\Gamma_n(t) = \partial\Omega_n(t)$ be the n-smooth solution of (3) with initial data Γ_{n0} and let $\Gamma(t) = \partial\Omega(t)$ be the smooth mean curvature flow with initial data Ω_0. Assume (5), then*

$$\lim_{n\to+\infty} d_H(\Omega_n(t), \Omega_0(t)) = 0,$$

locally uniformly in $t \in [0, T_0)$.

Remark 4. 2 (1) By the results of Gage - Hamilton [8] and Grayson [12], we know the existence and uniqueness of smooth mean curvature flow in $\mathcal{R}^2$.
(2) This convergence result has already been proved by Girao [10] for convex solutions and Girao - Kohn [11] for graph-like solutions. They also obtained the rate of convergence. We generalize their convergence result to general curves which are not necessarily convex.

Outline of the proof. We devide our consideration into some steps.

Step 1. $\widehat{\Omega}(0) = \mathrm{cl}\,\Omega(0) = \mathrm{cl}\,\underline{\Omega}(0)$.

The proof of this equality is based on the comtainment principle for crystalline motions (cf. Gurtin [13] and Giga - Gurtin [9]).

Step 2. Let $\widehat{d}(x,t)$ (resp., $\underline{d}(x,t)$) be the sigened distance function for $\{\widehat{\Omega}(t)\}_{0\le t<T}$ (resp., $\{\underline{\Omega}(t)\}_{0\le t<T}$) and let $(\widehat{d} \wedge 0)(x,t) = \max\{\widehat{d}(x,t), 0\}$ (resp., $(\underline{d} \vee 0)(x,t) = \min\{\underline{d}(x,t), 0\}$). By Lemma 3. 1 we observe that $(\widehat{d} \wedge 0)(x,t)$ (resp., $(\underline{d} \vee 0)(x,t)$) is a viscosity subsolution (resp., a viscosity subsolution) of the mean curvature flow equation:

$$(6) \qquad u_t - |Du|\mathrm{div}\frac{Du}{|Du|} = 0 \qquad \mathcal{R}^2 \times (0,T).$$

See Soner [15] and Ambrosio - Soner [1] for the main part of this proof and the notion of the distance solution for the mean curvature flow.

Step 3. Let $d(x,t)$ be the signed distance function for $\{\Omega(t)\}_{0\le t<T_0}$. Since $\{\Omega(t)\}_{0\le t<T_0}$ is a smooth mean curvature flow, we have the following properties: for any $\delta > 0$, there are positive constants σ, K such that

- $u(x,t) = e^{-Kt}[(d \vee 0)(x,t) \wedge \sigma]$ is a viscosity subsolution of (6) with $T = T_0 - \delta$.
- $v(x,t) = e^{Kt}[(d \wedge 0)(x,t) \vee \sigma]$ is a viscosity supersolution of (6) with $T = T_0 - \delta$.

Step 4. Let $\widehat{T}$ (resp., $\underline{T}$) be the extinction time of $\widehat{\Omega}(t)$ (resp., $\underline{\Omega}(t)$) and $\tilde{T} := \min\{\underline{T}, T_0, \widehat{T}\}$. Then, by the above steps and the comparison principle for viscosity solutions (cf. [5, 6, 7], we observe

$$\widehat{\Omega}(t) \subset \mathrm{cl}\,\Omega(t) \subset \mathrm{cl}\,\underline{\Omega}(t) \quad \forall t \in [0, \tilde{T} - \delta).$$

Since $\mathrm{cl}\,\underline{\Omega}(t) \subset \widehat{\Omega}(t)$ by construction, we have, by letting $\delta \to 0$,

$$\widehat{\Omega}(t) = \mathrm{cl}\,\Omega(t) = \mathrm{cl}\,\underline{\Omega}(t) \quad \forall t \in [0, \tilde{T}).$$

By a lengthy elememtary argument we have our desired result. The uniform convergence implies that $\tilde{T} = T_0$. □

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Ginzburg-Landau equation with variable coefficients -an approach to prescribing zeros

Shuichi JIMBO

Department of Mathematics, Faculty of Science

Hokkaido University, Sapporo 060 Japan

§1. Introduction

In this note we consider certain kinds of stable solutions (with zeros) of the Ginzburg - Landau (type) equations. The Ginzburg-Landau (GL) equation was first introduced as a model of the low-temperature superconductivity and the idea of that formulation has been used in many fields of physics and so the similar that type equations come into mathematical physics in many aspects. The following semilinear elliptic equation can be regarded as one simplified version of the GL equations by neglecting the magnetic effect. However it is still regarded as a good model of those phenomena. We are interested in the stable solutions of the Ginzburg-Landau equation (Neumann B.C.)

$$\Delta\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \quad \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega \tag{1.1}$$

where the unknown variable Φ is a $\mathbb{C}$−valued function in Ω and ν is the outward unit normal vector on $\partial\Omega$. Note that in this case a stable solution corresponds to a local minimizer of the following (GL) functional

$$\mathcal{H}_\Omega(\Phi) = \int_\Omega \left(\frac{1}{2}|\nabla\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2\right) dx \tag{1.2}$$

First we note a certain property of solutions of (1.1).

Proposition 1.1. If Ω is bounded and simply-connected, any non-constant solution Φ of (1.1) has zero in $\overline{\Omega}$.

(Proof) If a non-constant solution Φ does not have any zeros and Ω is simply-connected, it can be expressed as

$$\Phi(x) = w(x)\exp(\phi(x)), \quad w(x) > 0, \quad \phi : \Omega \longrightarrow \mathbb{R}.$$

ϕ, w are as smooth as Φ. Note that ϕ satisfies

$$\operatorname{div}(w^2\nabla\phi) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega.$$

From the maximum principle, ϕ must be a constant function and we have $\phi(x) \equiv c$. Denote Φe^{-ic} by Φ. Φ is a positive valued function and satisfies the following equation,

$$\Delta\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega.$$

From $\Phi > 0$ in Ω, we have $\Phi \equiv 1$. Consequently the solution Φ is constant. □

The property of zeros of solutions arise as a mathematical subject if we consider nontrivial solutions.

Remark. There are several related works on the zeros of solutions of GL equations (see References). Among them, the results in Baumann-Carlson-Phillips [1], Brezis-Bethuel-Helein [2] are excellent. They are dealing with characterization of the locations of zeros of global minimizers of the GL functional under the first kind boundary condition. The problem for local minimizers seem to be more interesting. In the proof of our main result, we apply the idea in [1].

Notation.

$$Z[\Phi] = \{x \mid \Phi(x) = 0\}.$$

Basically we are interested in the existence of non-constant stable solution. In Jimbo and Morita [7], it was proved:
(∗) If Ω is convex, there are no non-constant stable soutions to (1.1).

We conjectured that the conclusion will be still true even if the assumption in (∗) is weakened. That is the "convex" may be replaced by "non-simply-conneced". However it turned out that this is not true in general. That is, there exists a contractible domain with non-constant stable solutions in Dancer [4] and Jimbo and Morita [8]. Note that in this example zeros necessarily arise. See Figure 1.

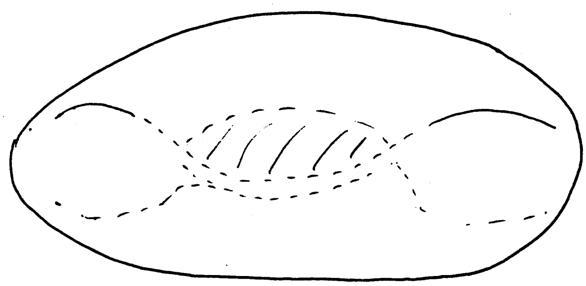

Fig. 1 : Ω domain with thin part

It should be mentioned that the domains of these examples are 3−dimensional or higher and so the conjecture is still open for 2−dimensional domain. From this reason, we consider a case of a variable coefficient equation in place of (1.1) to construct a stable non-constant solution.

$$\text{div}(a(x)\nabla\Phi) + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

$$\frac{1}{a(x)}\text{div}(a(x)\nabla\Phi) + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

§2. Physical background

As we mentioned in §1, the equations (1.1) are related with the superconductivity (or superfluid) phenomena. One of the important features those phenomena is the vortex pinning. The vortex is trapped by some defect or some special part of the material. This is a pattern made by a non-uniform environment of non-uniform shape of the material. Note that the vortex corresponds to zero of Φ. Let us consider a thin 3–dimensional domain,

$$\widetilde{\Omega}(\eta) = \{(x', x_3) \in \mathbb{R}^3 \mid 0 < x_3 < \eta\, a(x'),\, x' \in \Omega\}, \quad x' = (x_1, x_2)$$

where $\Omega \subset \mathbb{R}^2$.

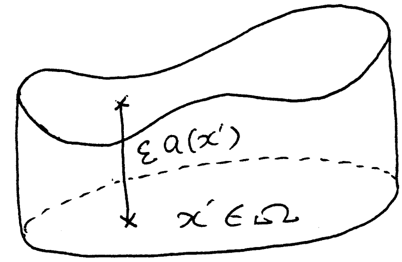

Fig. 2

The vortex is expected to stay around a narrow part of the domain. Because solutions corresponding to realizable phenomena are local minimizers (stable solutions) of GL functional and a stable solution want to make the value of the GL energy smaller. Note that the neighborhood of the zero gives large contribution in the GL energy and we naturally think that the solution will place its zero point in the narrow region of the domain. It suggests that the non-uniform a may bring a stable pattern featuring the shape of the domain. When $\eta \downarrow 0$, the Laplacian $\Delta = \sum_{j=1}^{3} \partial^2/\partial x_j^2$ with the Neumann boundary condition on $\partial\widetilde{\Omega}(\eta)$ approaches the operator

$$\frac{1}{a(x')}\mathrm{div}(a(x')\nabla\,\cdot)$$

and so the GL equation in $\widetilde{\Omega}(\eta)$ approaches the equation in (1.4). We want to construct a stable solution due to a non-uniform shape of domain to explain the pattern formation of vortex pinning. (1.3) and (1.4) can be dealt with quite similarly. Hereafter (1.3) is mainly considered.

§3. Main result

We consider the following equation.

$$\text{(3.1)} \qquad \mathrm{div}(a(x)\nabla\Phi) + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary and ν is the unit outward normal vector on $\partial\Omega$.

Problem. Can we construct a non-constant stable solution to (3.1) (with zeros)? Can we control the location of zeros by giving $a = a(x)$?
Note that solutions of (3.1) are critical points of the functional:

$$\mathcal{H}_{a,\Omega}(\Phi) = \int_\Omega \left(\frac{a(x)}{2}|\nabla\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2 \right) dx \tag{3.2}$$

We can give a partial answer to the problem.

Theorem (X.Y.Chen, S.Jimbo, Y.Morita). Let $p_1, p_2, \cdots, p_N$ be any N distinct points in Ω. For any $\epsilon > 0$, there exists a variable coefficient $a(x) > 0$ and a stable solution Φ to (3.1) for $\lambda = 1$ such that

$$Z[\Phi] \subset \cup_{k=1}^N B(p_k;\epsilon), \quad \#(Z[\Phi] \cap B(p_k;\epsilon)) = 1$$

where $B(p_k;\epsilon)$ is the ball of radius ϵ centered at p_k and $\#(C)$ denotes the number of the set C.

Remark. This result is a sharper than that of Chen, Jimbo, Morita [3].

§4. Sketch of the proof

Step 1: Punctured Domain $D(\rho)$

The main idea is to consider a punctured domain

$$D(\rho) = \Omega \setminus \cup_{k=1}^N \overline{B(p_k;\rho)}.$$

In such a domain we can construct a stable solution (without zero). From this stable solution we aim to get a true solution by approximately extending that solution to the whole Ω. We review a previous result concerning existence of stable solution in $D(\rho)$ from Jimbo, Morita, Zhai [1].

Proposition 4.1. Let γ be any continuous map from $\overline{D(\rho)}$ into S^1. There exists a $\lambda_0 > 0$ such that the equation

$$\Delta\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad D(\rho), \quad \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial D(\rho) \tag{4.1}$$

has a stable solution Φ_λ for $\lambda > \lambda_0$ such that

(i) $\Phi_\lambda(x) \neq 0$ in $\overline{D(\rho)}$,

(ii) the map $\overline{D(\rho)} \ni x \longmapsto \Phi_\lambda(x)/|\Phi_\lambda(x)| \in S^1 \subset \mathbb{C}$ is homotopic to γ.

The above stable solution Φ_λ is of course a local minimizer of the functional:

$$\mathcal{H}_{D(\rho)}(\Phi) = \int_{D(\rho)} \left(\frac{1}{2}|\nabla\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2 \right) dx \tag{4.2}$$

We can also claim a more comprehensive stability property for this solution.

Proposition 4.2 (Stability inequality). There exist $\delta > 0$, $\delta' > 0$,

$$\mathcal{H}_{D(\rho)}(\Phi_\lambda + \Psi) - \mathcal{H}_{D(\rho)}(\Phi_\lambda) \geqq \delta \|\Psi\|^2_{L^2(D(\rho))} \tag{4.3}$$

for $\Psi \in N(\Phi_\lambda)$ such that $\|\Psi\|_{L^2(D(\rho))} \leqq \delta'$ where

$$N(\Phi_\lambda) = \{\Psi \in H^1(D(\rho); \mathbb{C}) \mid \int_{D(\rho)} \mathrm{Im}(\Psi \overline{\Phi}_\lambda) dx = 0\}.$$

Remark. Prop. 4.1 holds also for higher dimensional case (cf. [9]).

Hereafter in this proof, we choose $\gamma \in C^0(\overline{D(\rho)}; S^1)$ as a special one, as follows, (**): For any anticlockwise cycle ι_k around $\partial B(p_k; \rho)$ (which is a generator of $\pi_1(D(\rho))$), $\gamma \circ \iota_k$ is winding number 1 in S^1.

Step 2: Construction by Variational method

We prepare a subset of $H^1(\Omega; \mathbb{C})$ in which we seek for a local minimizer. First we construct an approximate solution $\widetilde{\Phi}_{\lambda,\epsilon} \in W^{1,\infty}(\overline{\Omega})$, such that $\widetilde{\Phi}_{\lambda,\epsilon}(x) = \Phi_\lambda(x)$ for $x \in D(\rho)$.

$$\widetilde{\Phi}_{\lambda,\epsilon}(x) = \Phi_\lambda(x) \quad \text{for} \quad x \in \overline{D(\rho)},$$

$$\widetilde{\Phi}_{\lambda,\epsilon}(x) = \frac{|x-p_k|^2}{\epsilon^2} \frac{\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})}{|\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})|} \quad \text{for} \quad x \in B(p_k; \epsilon),$$

$$\widetilde{\Phi}_{\lambda,\epsilon}(x) = \frac{\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})}{|\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})|} \quad \text{for} \quad x \in B(p_k; \rho/2) \setminus B(p_k; \epsilon)$$

$$\widetilde{\Phi}_{\lambda,\epsilon}(x) = (1 - 2\frac{|x-p_k|}{\rho}) \Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|}) + 2\frac{|x-p_k|}{\rho} \frac{\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})}{|\Phi_\lambda(p_k + \frac{x-p_k}{|x-p_k|})|}$$
$$\text{for} \quad x \in B(p_k; \rho) \setminus B(p_k; \rho/2),$$

for $1 \leqq k \leqq N$.

We define the diffusion coefficient $a = a_\epsilon \in W^{1,\infty}(\Omega)$ (uniform in $\epsilon > 0$).

$$a_\epsilon(x) = \begin{cases} 1 & \text{for} \quad x \in D(\rho) \\ q_\epsilon(|x-p_k|) & \text{for} \quad x \in B(p_k; \rho) \setminus B(p_k; \rho/2) \\ \epsilon^3 & \text{for} \quad x \in B(p_k; \rho/2) \text{ for } 1 \leqq k \leqq N, \end{cases} \tag{4.4}$$

where

$$q_\epsilon(r) = \epsilon^3(\rho - r)/(\rho/2) + (r - \rho/2)/(\rho/2) \quad \text{for} \quad \rho/2 \leqq r \leqq \rho.$$

Let us define a set

$$E(\xi,\eta,\epsilon) = \{\Phi \in H^1(\Omega;\mathbb{C}) \cap C^0(\overline{\Omega};\mathbb{C}) \mid \mathcal{H}_{\Omega,a_\epsilon}(\Phi) - \mathcal{H}_{\Omega,a_\epsilon}(\widetilde{\Phi}_{\lambda,\epsilon}) \leqq \eta,$$
$$\inf_{0 \leqq \theta < 2\pi} \|\Phi - e^{i\theta}\widetilde{\Phi}_{\lambda,\epsilon}\|_{L^2(D(\rho))} \leqq \xi\} \quad (\xi,\eta,\epsilon > 0 : \text{ positive parameters}).$$

Now we will consider the minimizing problem of the functional $\mathcal{H}_{\Omega,a_\epsilon}$ in $E(\xi,\eta,\epsilon)$ and seek for a global minimizer as an interior point. We can assert by the aid of the following lemma, that the minimizing sequence can not approach $\partial E(\delta,\eta,\epsilon)$ for small $\epsilon > 0$.

For that purpose we prepare the following main lemma.

Lemma 4.3 There exists a constant $\delta'' > 0$ and $c' > 0$ such that

$$\mathcal{H}_{\Omega,a_\epsilon}(\Phi) - \mathcal{H}_{\Omega,a_\epsilon}(\widetilde{\Phi}_{\lambda,\epsilon}) \geqq (\delta/2)\|\Phi - \widetilde{\Phi}_{\lambda,\epsilon}\|^2_{L^2(D(\rho))} - c'\epsilon$$

for $\Phi \in E(\xi,\eta,\epsilon)$, $(\Phi - \widetilde{\Phi}_{\lambda,\epsilon})_{|D(\rho)} \in N(\Phi_\lambda)$, $0 < \xi < \delta''$.

(Proof) This is a straightforward calculation from the construction of $\widetilde{\Phi}_{\lambda,\epsilon}$ and the coefficient a_ϵ and the definition of Prop.4.2.

(Sketch of the variational direct method)

Fix any $\eta > 0$, $\xi \in (0,\delta')$ and consider the minimizing problem of $\mathcal{H}_{\Omega,a_\epsilon}$ in $E(\xi,\eta,\epsilon)$. Let $\{\Phi_m\}_{m=1}^\infty \subset E(\xi,\eta,\epsilon)$ be a minimizing sequence. That is

$$\lim_{m\to\infty} \mathcal{H}_{\Omega,a_\epsilon}(\Phi_m) = \inf_{E(\xi,\eta,\epsilon)} \mathcal{H}_{\Omega,a_\epsilon}.$$

We can assume without loss of generality that $\mathcal{H}_{\Omega,a_\epsilon}(\Phi_m) \leqq \mathcal{H}_{\Omega,a_\epsilon}(\widetilde{\Phi}_{\lambda,\epsilon})$ for $m \geqq 1$. Let $\theta_m \in [0,2\pi)$ such that

$$\|\Phi_m - e^{i\theta_m}\widetilde{\Phi}_{\lambda,\epsilon}\|_{L^2(D(\rho))} = \inf_{0\leqq\theta<2\pi} \|\Phi_m - e^{i\theta}\widetilde{\Phi}_{\lambda,\epsilon}\|_{L^2(D(\rho))}$$

and $\Phi'(x) = e^{-i\theta_m}\Phi_m$. Then $(\Phi'_m - \widetilde{\Phi}_{\lambda,\epsilon})_{|D(\rho)} \in N(\Phi_\lambda)$ naturally holds for each m.

Note also $\mathcal{H}_{\Omega,a_\epsilon}(\Phi_m) = \mathcal{H}_{\Omega,a_\epsilon}(\Phi'_m)$. From Lemma 4.3, we have

$$\|\Phi'_m - \widetilde{\Phi}_{\lambda,\epsilon}\|^2_{L^2(D(\rho))} \leqq c'\epsilon(\delta/2)^{-1}.$$

Therefore we take $\epsilon_0 = \xi^2\delta/(4c')$, then

$$\sup_{m\geqq1} \|\Phi'_m - \widetilde{\Phi}_{\lambda,\epsilon}\|_{L^2(D(\rho))} < \xi \quad \text{for} \quad 0 < \epsilon < \epsilon_0. \tag{4.6}$$

This implies that the minimizing sequence $\{\Phi'_m\}_{m=1}^\infty$ can not approach the boundary. By the standard technique of the elliptic variational problem in the above situation, we can find a global minimizer $\widehat{\Phi}_\epsilon$ of that functional as an interior element of $E(\xi,\eta,\epsilon)$. We can also prove that

$$\lim_{\epsilon\to 0}\|\widehat{\Phi}_\epsilon - \widetilde{\Phi}_{\lambda,\epsilon}\|_{L^2(D(\rho))} = 0. \tag{4.7}$$

By applying the elliptic estimate, $\{\widehat{\Phi}_\epsilon\}_{\epsilon:\text{small}}$ is relatively compact in $C^2(\overline{\Omega}\setminus \cup_{k=1}^N B(p_k;3\rho/4))$ and so the above convergence in (4,7) is improved to classical one. That is

$$\widehat{\Phi}_\epsilon \longrightarrow \Phi_\lambda \quad \text{in} \quad C^1(\overline{D(\rho)}) \quad \text{as} \quad \epsilon\to 0.$$

Therefore (i) $\widehat{\Phi}_\epsilon \neq 0$ in $\overline{D(\rho)}$ and (ii) $\widehat{\Phi}_\epsilon$ is homotopic to Φ_λ as a map from $\overline{D(\rho)}$ into $\mathbb{C}\setminus\{0\}$.

Considering a map $\widehat{\Phi}_\epsilon$ from $B(p_k;\rho)$ into $\mathbb{C}$ with above properties (i)-(ii) with the property (**) concerning γ , we conclude that $\widehat{\Phi}_\epsilon$ must have at least one zero in $B(p_k;\rho)$ for each k by the aid of the degree argument of continuous maps. We conclude

$$Z[\widehat{\Phi}_\epsilon]\cap B(p_k;\rho)\neq\emptyset.$$

To get a solution which has exactly one point in each $B(p_k;\rho)$, we have to set up, in the first step, the following situation. By taking ρ_0 small and then taking $\lambda>0$ large,

$$\frac{d}{ds}\arg\Phi_\lambda(p_k+\rho(\cos s,\sin s)) \geqq 1/2 \qquad s\in[0,2\pi) \tag{4.9}$$

and $\Phi_\lambda(\partial B(p_k;\rho))$ encloses a star-shaped region with respect to the origin (actually it is almost the unit disk). From this condition and the smooth convergence $\widehat{\Phi}_\epsilon \to \Phi_\lambda$ in $\overline{D(\rho)}$ in (4.8), $\widehat{\Phi}_\epsilon$ satisfies the same condition as in (i),(ii) and (4.9). For each k, consider the variational problem

$$\mathcal{F}_k(\phi) = \inf_{\phi\in E_k}\int_{B(p_k;\rho)}\left\{\frac{a_\epsilon}{2}|\nabla\phi|^2+\frac{\lambda}{4}(1-|\phi|^2)^2\right\}dx \tag{4.10}$$

where

$$E_k=\{\phi\in H^1(B(p_k;\rho;\mathbb{C}))\mid \phi(x)=\widehat{\Phi}_\epsilon(x),\ x\in\partial B(p_k;\rho)\}$$

The important point is to note that $\widehat{\Phi}_\epsilon$ (restricted to $B(p_k;\rho)$) is a global minimizer of the above variational problem. Thus we are in a position to apply the Proposition 5.1, which is a characterizing theorem for a single vortex of a global minimizer of GL type functional for a first kind boundary condition of winding number 1. We can clain that $\widehat{\Phi}_\epsilon$ has exactly one zero in $B(p_k;\rho)$.

§5. Baumann-Carlson-Phillips's result (modified version)

Let $G \subset \mathbb{R}^2$ be a contractible bounded domain with a smooth boundary and $g \in C^1(\partial G; \mathbb{C})$ satisfy the following condition:

(A) $g : \partial G \ni s \longmapsto \mathbb{C} \setminus \{0\}$ is widing number 1 and its image encloses a star-shaped domain with respect to the origin and $(d/ds)\arg\phi(s) > 0$ for $s \in \partial G$, where s is the canonical parameter which is anti-clockwise on ∂G.

Let us consider the following functional

$$\mathcal{J}_G(\Phi) = \int_G \left(\frac{a(x)}{2}|\nabla\Phi|^2 + \frac{b(x)}{4}K(|\Phi|^2) \right) dx \tag{5.1}$$

for Φ satisfying $\Phi_{|\partial G} = g$, where $a = a(x) > 0$, $b = b(x)$ are real valued Hölder continuous function in $\overline{G}$ and $K = K(r)$ is a real valued C^1 function in $\mathbb{R}$.

Proposition 5.1. Assume (A). Let Φ be a global minimizer to the above variational problem (5.1). Then Φ has exactly one zero in G.

Remark. The case $a = b \equiv 1$ is celebrated result due to Baumann, Carlson, Phillips [1]. The proof is quite similar as that in [1]. We need this modified version for the proof of the main result.

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ON THE INVERSE CONDUCTIVITY PROBLEM WITH FINITE MEASUREMENTS

HYEONBAE KANG AND JIN KEUN SEO

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ with a connected C^2 boundary. Let D be a subdomain of Ω. Assume D and Ω are both conductor of elasticity with conductivity coefficients 1 and $k(0 < k \neq 1)$, respectively. We are interested in determining the size and location of unknown object D from the relationship between a given flux g to the boundary of the body Ω and measurement of the voltage u on a portion of the boundary $\partial\Omega$. The voltage u satisfies the Neumann Problem

$$P[D,g]: \begin{cases} L_D u := \operatorname{div}((1+(k-1)\chi_D)\nabla u) = 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega, \quad \displaystyle\int_{\partial\Omega} u = 0, \end{cases}$$

where $0 < k \neq 1$ and χ_D is indicator function of D and ν is outerward unit normal vector. Let us define the Neumann-to-Dirichlet map $\Lambda_D : L^2(\partial\Omega) \to H^1(\partial\Omega)$ by

$$\Lambda_D(g) := u|_{\partial\Omega}, \quad g \in L_0^2(\Omega) := \{\psi \in L^2(\partial\Omega) : \int_{\partial\Omega} \psi = 0\}$$

where u is the solution of the Neumann problem $P[D,g]$. Clearly, the size and location of the object D influence on the relationship between the Neumann data g and the corresponding Dirichlet data $f = u|_\Omega$. Conversely, the pair (g, f) has some information on D. The inverse problem is to determine D from one (or two, three,...) pair (g, f). In this note, we will discuss about the following two important questions:

[Uniqueness] Does $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ on a portion of $\partial\Omega$ imply $D_1 = D_2$?

[Stability] If $\|\Lambda_{D_1}(g) = \Lambda_{D_2}(g)\|_{L^2(\partial\Omega)}$ is small, is $|D_1 \setminus D_2| + |D_2 \setminus D_1|$ small ?

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Here, we assume $\bar{D}_j \subset \Omega$ and $|E|$ is the Lebesgue measure of the set E.

However, in one dimension, $\Lambda_{D_1}(g) = \Lambda_{D_2}(g)$ for non-zero g if and only if $|D_1| = |D_2|$. Indeed, this uniqueness and stability do not hold in one dimension. But we have some uniqueness and stability result within a restricted class when $n \geq 2$. We will first explain the uniqueness result

Uniqueness Result

The uniqueness question with $m-$ measurements can be stated as follows:

Suppose that D_1 and D_2 are subdomains in Ω. Can we choose appropriate functions $f_1, f_2, .., f_m$ so that if $u_j^i, i = 1,2, j = 1, ..., m$, is the solution to $P[D_i, f_j]$, $u_j^1 = u_j^2, j = 1, ..., m$, on $\partial\Omega$ imply $D_1 = D_2$?

Throughtout this section, we assume that u_j are the solution of $P[D_j, g]$ $(j = 1, 2)$ and g is not identically zero. Let $\mu = k - 1$. By setting

$$u^e = u|_{\Omega\setminus\overline{D}} \text{ and } u^i = u|_D,$$

the equation $\text{div}((1 + \mu\chi_D)\nabla u) = 0$ may be written as

$$\Delta u^e = 0 \text{ in } \Omega \setminus \overline{D},$$

$$\Delta u^i = 0 \text{ in } D,$$

$$u^e = u^i \text{ on } \partial D$$

$$\frac{\partial u^e}{\partial\nu} = k\frac{\partial u^i}{\partial\nu} \text{ on } \partial D.$$

From integrating by part, we obtain the following useful identity

$$\int_\Omega (1 + \mu\chi_{D_1})|\nabla(u_1 - u_2)|^2 dx + \mu \int_{D_2\setminus D_1} |\nabla u_2|^2 dx$$
$$= \int_{\partial\Omega} (\Lambda_{D_1}(g) - \Lambda_{D_2}(g))g d\sigma + \mu \int_{D_1\setminus D_2} |\nabla u_2|^2 dx.$$

So, if $D_1 \subset D_2$, $\mu > 0$ and $\Lambda_{D_1}(g) = \Lambda_{D_1}(g)$, then it must be

$$\nabla u_2 = 0 \text{ in } D_2 \setminus D_1.$$

If $D_2 \setminus \bar{D}_1$ contains a non-empty openset, it follows from unique continuation theorem that u_2 is constnat in Ω. But this is not possible from the condition that g is not identically zero.) So, one can arrive the following:

Theorem. *If $D_1 \subset D_2$ and $\Lambda_{D_1}(g) = \Lambda_{D_1}(g)$, then $D_1 = D_2$.*

It is easy to see that if $u_1 = u_2 = f$ on $\partial\Omega$, then $D_1 \cap D_2$ is not empty set. Indeed, it follows from the harmonic continuation that

$$u_1 = u_2 \text{ in } \Omega \setminus \overline{D_1 \cup D_2}.$$

By the maximum principle, we have

$$\max_{\Omega} |u_1 - u_2| \leq \max_{\partial(D_1 \cap D_2)} |u_1 - u_2|$$

Hence, if eithor $D_1 = D_2$ or $D_1 \cap D_2 = \emptyset$, $u_1 = u_2$ in Ω. It follows from unique continuation that $D_1 \cap D_2 = \emptyset$ implies that g is identically zero. So, if g is not identically zero, it must be that $D_1 \cap D_2$ is non-empty.

A lot of research has been devoted on the uniqueness question within restricted classes of domains. Friedman and Isakov [FI] proved that if D_1 and D_2 are assumed to be convex polyhedrons so that diam$(D_i) <$ dist $(D_i, \partial\Omega), i = 1, 2$ and if, for nonzero $g \in L^2(\partial\Omega)$, the solution u_1 of $P[D_1, g]$ and the solution u_2 of $P[D_2, g]$ satisfy $u_1 = u_2$ on $\partial\Omega$, then $D_1 = D_2$. Barcelo, Fabes, and Seo [BFS] are able to remove the above distance restriction with an appropriately chosen Neumann data g. We will explain this uniqueness results for the class of polygon when $n = 2$.

Theorem. *Let D_1, D_2 be two polygons compactly contained in Ω. Let g be a nonzero piecewise continuous function on $\partial\Omega$ so that $\{x \in \partial\Omega : g(x) > 0\}$ is connected. Suppose $u_i, i = 1, 2$ are the solutions to the Neumann problem $N(D_i, g)$. If $u_1 = u_2$ on $\partial\Omega$, then*

$$\textit{convex hull } D_1 = \textit{ convex hull } D_2.$$

Seo [S] remove the convexity restriction on D_i when $n = 2$ with cost of two measurements.

Theorem. *Let D_1, D_2 be two polygons compactly contained in Ω. Let g_1, g_2 be two nonvanishing piecewise continuous functions on $\partial\Omega$ with average zero such that* *for each real* α, *the set* $\{z \in \partial\Omega : \psi_1(z) - \alpha\psi_2(z) \geq 0\}$ *is connected and* g_1 *is not identical to* αg_2. *Suppose* $u_i^j, i, j = 1, 2$ *are the solutions to the Neumann problem* $P(D_i, g_j)$. *If* $u_1^j = u_2^j (j = 1, 2)$ *on* $\partial\Omega$, *then*

$$D_1 = D_2.$$

Recently, we are able to answer the uniqueness questions within the class of disks or balls. (See [KS2] for balls.) Previously, Friedman and Isakov proved the uniqueness of the disk with one measurement when Ω is assumed to be the half space [FI]. Isakov and Powell [IP] extend this result to the union of disjoint disks contained in the half space under a certain condition. Kang and Seo [KS1] remove the condition that Ω is the half space, and prove that any disk contained in a Lipschitz domain Ω in $\mathbb{R}^2$ can be uniquely determined with one measurement. In the paper [KS1], we introduce a useful represention formula which states as follows:

The solution u to the Neumann problem $N[D, g]$ can be decomposed into the harmonic part and the refraction part, namely,

$$u = H + S_D h_D$$

where H is a harmonic function in Ω, S_D is the single layer potential on D, and h_D is uniquely and explicitly determined by the domain D and the harmonic part H. Moreover, H can be computed explicitly from the boundary measurement $(g, \Lambda_D(g))$ and H is decomposed by two different singular integarls. This representation seems to somehow inherit geometic properties of ∂D and can be applied to the inverse conductivity problem to find D. Moreover, this formula is so concrete that a numerical implementation is possible (see [KSS2]). We apply it to prove the uniqueness of the disk and the ball. The uniqueness of ball in three dimensional space has been open for a while. (see [I]) Precise statement of the result is as follows;

Theorem. *Let* Ω *be a bounded simply connected Lipschitz domain in* $\mathbb{R}^n (n = 2, 3, ..)$. *Let* $g \in L_0^2(\partial\Omega)$ *(not identically zero). If* $\Lambda_{B_1}(g) = \Lambda_{B_2}(g)$, *then* $B_1 = B_2$. *Here,* B_1 *and* B_2 *are disks or balls compactly contained in* Ω.

To prove this Theorem in $\mathbb{R}^3$, we investigate the smallest closed convex set E_j where the harmonic function $\mathcal{S}_{D_j}\phi_j|_{\mathbb{R}^3\setminus D_j}$ extends $\mathbb{R}^3 \setminus E_j$. Using special properties of the single layer potential on balls, we are able to derive $E_1 = E_2$ and $B_1 = B_2$.

Recently, Fabes, Kang and Seo deals with the uniqueness and stability within the class of small perturbation of disks. To explain this result, let us fix the notion of $\epsilon-$perturbations of disks. For a C^2 domain D being an $\epsilon-$perturbation of a disk means that there exists a disk $B \subset \Omega_0$ of the radius larger than a fixed number, say d_0, such that

$$\partial D : P + \epsilon\omega_\epsilon(P)\nu(P), \quad P \in \partial B$$

where $\|\omega_\epsilon\|_{C^1(\partial B)} \leq 1$ and $\nu(P)$ is the outward unit normal to ∂B at P. Let $\mathcal{C}[\epsilon]$ be the class of all $\epsilon-$perturbations of disks contained in the region $\Omega_0 := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta_0\}$ where δ_0 is a fixed positive number.

Theorem. *Let $g \in L^2_0(\partial\Omega)$ and $D_0 \in \mathcal{C}[\epsilon]$, and let $\Lambda_{D_0}(g) = f$. If $D \in \mathcal{C}[\epsilon]$ and $\Lambda_D(g) = \Lambda_{D_0}(g)$, we show that*

$$|D_0 \Delta D| \leq C\epsilon$$

for some constant depending on (f, g) not on D or ϵ.

Of course this is not a uniqueness result. What it says instead is that if the boundary measuements are the same then two domains must be very close. For this reason we call this result an approximate identification. It seems that this approximate identification of a domain is quite meaningful in a practical sense.

There are some local uniqueness results(see [BFI], [P], [AIP]) which we will not explain here.

Stability Result

In the paper [BFI], Bellout, Friedman, and Isakov obtained a local stability result and local uniqueness results when $n = 2$ under the assumption that two objects are sufficiently close.

Theorem [BFI]. *Let $n = 2$. Let ∂D_0 be piecewise analytic boundary given by $x = f(s)$ and let*

$$\partial D_h : x = f(s) + hd_h(s)\nu(s) \quad \text{a.e.}$$

where $|d_h| + |\nabla d_h| \le C$ and $d_h \to 0$ as $h \to 0$. Assume that the set $\{d_h \neq 0\}$ consist of a finite number of components. Suppose that the Neumann data $g \in C^2(\partial\Omega)$ have a unique local maximum and unique local minimum on $\partial\Omega$. Then

$$dist(D_h, D_0) \le C \int_{\partial\Omega} |\Lambda_{D_h}(g) - \Lambda_D(g)|$$

where C may depend on the family $\{d_h\}$.

Kang, Seo, and Sheen [KSS1] obtain first global stability estmate (log type) when D_j is assumed to be disk. Fabes, Kang, and Seo[FKS] obtain hölder type gloal stability estimate for the class of small perturbation of disks under a certain condition on g. The result for disks states as follows.

Theorem. *Let g be a Neumann data with the the following condition (N).*

(N1) *There exists a positive number M such that $|g'(P)| > M$ if $|g(P)| < M$, $P \in \partial\Omega$. (Here, g' means the tangential derivative on $\partial\Omega$.)*

(N2) *$\{P \in \partial\Omega : g(P) \ge 0\}$ and $\{P \in \partial\Omega : g(P) \le 0\}$ are nonempty connected subsets of $\partial\Omega$.*

There exist $\alpha > 0$ and C depending only on δ_0 and d_0 such that

$$|D_1 \Delta D_2| \le C \|\Lambda_{D_1}(g) - \Lambda_{D_2}(g)\|^{\alpha}_{L^\infty(\partial\Omega)} \tag{3.1}$$

for every disks $D_1, D_2 \in \mathcal{C}[0]$.

The condition (N) guarantee the existence of the lower bound of $|\nabla u|$ which depends only on M when $D \in \mathcal{C}[\epsilon]$ and u is the weak solution to $P[D, g]$. (See [KSS1].)

To extend this global stability result for diskes to small perturbation of diskes, we need uniform stability of the solutions to $P[D, g]$ under $C^{1,\alpha}$ perturbation of the domain D. If $h > 0$ and D_h be a C^2 subdomain of Ω such that the $C^{1,\alpha}$ distance between D and D_h are less than h. Let u and u_h be weak solutions to the Neumann Problems $P[D, g]$ and $P[D_h, g]$, respectively. We prove that

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch.$$

This type of stability question of the solutions to the diffraction equation was considered by Bellout, Friedman, and Isakov in relation to the stability question of the inverse conductivity problem [BF, BFI]. They proved the L^p-stability when $p < 4$.

The proof of the diect stabilty estimate is based on our earlier result on the representation of solutions to $P[D, g]$. In [KS] authors proved that the solution of $P[D, g]$ can be represented as sum of a function harmonic in Ω and a single layer potential of a certain function on ∂D. Then, standard Schauder estimates and precise local estimates of integral operators arising from the layer potentials and their derivatives lead us to the estimate.

Using this uniform stability estimate, we obtain the global uniqueness and stability for the class $\mathcal{C}[\epsilon]$. To be precise, we prove that, for a Neumann data g satisfying the conditon (N),

$$|D_1 \Delta D_2| \leq C \left(\|\Lambda_{D_1}(g) - \Lambda_{D_2}(g)\|^{\alpha}_{L^\infty(\partial\Omega)} + \epsilon \right)$$

for every $D_1, D_2 \in \mathcal{C}[\epsilon]$ and for some constants C and $\alpha < 1$ independent of ϵ.

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Hyeonbae Kang, Department of Mathematics, Korea University, Seoul 136-701, Korea
E-mail address: kang@semi.korea.ac.kr

Jin Keun Seo, Department of Mathematics, Yonsei University, Seoul 120-749, Korea
E-mail address: seoj@bubble.yonsei.ac.kr

A comparison result for the state constraint problem of differential games

Shigeaki Koike

Department of Mathematics
Saitama University
255 Shimo-Okubo Urawa
Saitama 338 Japan

e-mail: skoike@rimath.saitama-u.ac.jp

1 Introduction

In this abstract, we consider the following first-order PDE of max-min type under a boundary condition which will be derived from state constraint requirement:

$$u(x,y) + H(x,y,D_x u(x,y), D_y u(x,y)) = 0 \text{ in } \Omega \times \Omega, \tag{1.1}$$

where the (upper) Hamiltonian is

$$H(x,y,p,q) \equiv \max_{a\in A} \min_{b\in B} \{-\langle f(x,a),p\rangle - \langle g(y,b),q\rangle - h(x,y,a,b)\}$$

for $(x,y,p,q) \in \overline{\Omega} \times \overline{\Omega} \times \mathbf{R}^n \times \mathbf{R}^n$. Here A and B are compact sets in $\mathbf{R}^m$, for some integer m, Ω is an open bounded set in $\mathbf{R}^n$, $f : \overline{\Omega} \times A \to \mathbf{R}^n$ and $g : \overline{\Omega} \times B \to \mathbf{R}^n$ are given functions, and $u : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ is unknown. Also, D_x and D_y denote the partial derivatives with respect to $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$, respectively.

It is well-known that this kind of min-max PDE arises when we discuss differential games. More precisely, we can expect that the unique solution of (1.1) under certain boundary condition must be the "value function" of the corresponding differential game.

A particular interest for game theorists is the case when $h \equiv 1$, which is called "pursuit evasion game". Via the viscosity solution theory, Aliziary de Roquefort first studied a simple case of the pursuit evasion game under state constraint requirement. Recently, following the setting in [8], Bardi, Koike and Soravia in [3] discuss the uniqueness and representation formula of solutions and the convergence of numerical schemes for more general pursuit evasion games.

Our aim here is to give a key idea of the uniqueness result in [3] and to show that it works even for the above slightly more general PDEs than those in [3].

2 Value function

In this section, we recall the value function for (1.1) with state constraint requirement.

First, we give our regularity assumption on given functions:

$$(A0) \qquad \begin{cases} f \in C(\overline{\Omega} \times A; \mathbf{R}^n), \ \sup_{a \in A} \|f(\cdot, a)\|_{W^{1,\infty}(\Omega;\mathbf{R}^n)} < \infty, \\ g \in C(\overline{\Omega} \times B; \mathbf{R}^n), \ \sup_{b \in B} \|g(\cdot, b)\|_{W^{1,\infty}(\Omega;\mathbf{R}^n)} < \infty, \\ \text{and } h \in C(\overline{\Omega} \times \overline{\Omega} \times A \times B; \mathbf{R}). \end{cases}$$

Let us recall the state equation corresponding to (1.1): Given measurable functions $\alpha \in \mathcal{A} \equiv \{\alpha : [0,\infty) \to A \text{ measurable}\}$ and $\beta \in \mathcal{B} \equiv \{\beta : [0,\infty) \to B \text{ measurable}\}$, we denote by $(X(\cdot; x, \alpha), Y(\cdot; y, \beta))$ the unique solution of

$$\begin{cases} X'(t) = f(X(t), \alpha(t)) & \text{for } t > 0, \\ Y'(t) = g(Y(t), \beta(t)) & \text{for } t > 0, \\ (X(0), Y(0)) = (x, y). \end{cases} \tag{2.1}$$

We define the subsets of $\mathcal{A}$ and $\mathcal{B}$, which involve the state constraint requirement: For $(x, y) \in \overline{\Omega} \times \overline{\Omega}$,

$$\mathcal{A}(x) = \{\alpha \in \mathcal{A} \mid X(t; x, \alpha) \in \overline{\Omega} \text{ for all } t \geq 0\},$$

and

$$\mathcal{B}(y) = \{\beta \in \mathcal{B} \mid Y(t; y, \beta) \in \overline{\Omega} \text{ for all } t \geq 0\}.$$

We will suppose certain hypotheses which imply that $\mathcal{A}(x) \neq \emptyset$ for $x \in \overline{\Omega}$, and $\mathcal{B}(y) \neq \emptyset$ for $y \in \overline{\Omega}$.

Next, we introduce the set of strategies:

$$\Gamma(x,y) \equiv \left\{ \gamma : \mathcal{A}(x) \to \mathcal{B}(y) \;\middle|\; \begin{array}{c} \text{If } \alpha = \hat{\alpha} \text{ a.e. in } (0,t) \text{ for } t > 0, \alpha, \hat{\alpha} \\ \in \mathcal{A}(x), \text{ then } \gamma[\alpha] = \gamma[\hat{\alpha}] \text{ a.e. in } (0,t). \end{array} \right\}$$

Using these notations, we define the value function: $V : \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ by

$$V(x,y) = \sup_{\gamma \in \Gamma(x,y)} \inf_{\alpha \in \mathcal{A}(x)} \int_0^\infty e^{-t} h(X(t;x,\alpha), Y(t;y,\gamma[\alpha]), \alpha(t), \gamma[\alpha](t))dt.$$

At least formally, we can verify that V satisfies (2.1) in $\Omega \times \Omega$. See [5] and [4] for instance.

Following [7], we shall adapt the notion of $A(x)$ and $B(y)$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$:

$$A(x) = \left\{ a \in A \;\middle|\; \begin{array}{c} \text{There is } r > 0 \text{ such that } X(t;x',a) \in \overline{\Omega} \\ \text{for } t \in [0,r], x' \in B(x,r) \cap \overline{\Omega} \end{array} \right\},$$

and

$$B(y) = \left\{ b \in B \;\middle|\; \begin{array}{c} \text{There is } r > 0 \text{ such that } Y(t;y',b) \in \overline{\Omega} \\ \text{for } t \in [0,r], y' \in B(y,r) \cap \overline{\Omega} \end{array} \right\}.$$

We note that $A(x) = A$ and $B(y) = B$ provided $x \in \Omega$ and $y \in \Omega$, respectively.

In what follows, we suppose the hypothesis:

(A1) $\qquad A(x) \neq \emptyset$ for $x \in \partial\Omega$, and $B(y) \neq \emptyset$ for $y \in \partial\Omega$.

We shall consider the PDE in $\overline{\Omega} \times \overline{\Omega}$ for the boundary value problem of (1.1):

$$u(x,y) + \mathcal{H}(x,y,D_x u(x,y), D_y u(x,y)) = 0 \text{ on } \overline{\Omega} \times \overline{\Omega}, \tag{2.2}$$

where

$$\mathcal{H}(x,y,p,q) \equiv \max_{a \in A(x)} \min_{b \in B(y)} \{-\langle f(x,a),p\rangle - \langle g(y,b),q\rangle - h(x,y,a,b)\}$$

for $(x,y,p,q) \in \overline{\Omega} \times \overline{\Omega} \times \mathbf{R}^n \times \mathbf{R}^n$.

Definition. A function u in $\overline{\Omega}\times\overline{\Omega}$ is called a viscosity subsolution (resp., supersolution) of (2.2) if, for any $\phi \in C^1(\overline{\Omega}\times\overline{\Omega})$, $(x_0, y_0) \in \arg\max(u^* - \phi)(x,y)$ (resp., $(x_0,y_0) \in \arg\min(u_* - \phi)(x,y)$) yields

$$u^*(x_0,y_0) + \mathcal{H}_*(x_0, y_0, D_x\phi(x_0,y_0), D_y\phi(x_0,y_0)) \le 0$$

$$(\text{resp., } u_*(x_0,y_0) + \mathcal{H}^*(x_0, y_0, D_x\phi(x_0,y_0), D_y\phi(x_0,y_0)) \ge 0)\,.$$

A function u in $\overline{\Omega}\times\overline{\Omega}$ is called a viscosity solution of (2.2) if it is a viscosity sub- and supersolution of (2.2).

Remark. Because of the lower semicontinuity of the mappings $x \in \overline{\Omega} \to A(x) \subset 2^A$ and $y \in \overline{\Omega} \to B(y) \subset 2^B$, it is easy to observe that the following properties hold: For $(p,q) \in \mathbf{R}^n \times \mathbf{R}^n$,

$$\mathcal{H}_*(x,y,p,q) = \sup_{a\in A(x)} \min_{b\in B}\{-\langle f(x,a),p\rangle - \langle g(y,b),q\rangle - h(x,y,a,b)\} \text{ on } \partial\Omega\times\overline{\Omega},$$

and

$$\mathcal{H}^*(x,y,p,q) = \max_{a\in A} \inf_{b\in B(y)}\{-\langle f(x,a),p\rangle - \langle g(y,b),q\rangle - h(x,y,a,b)\} \text{ on } \overline{\Omega}\times\partial\Omega.$$

For the other cases, $\mathcal{H}_*$ and $\mathcal{H}^*$ are equal to H.

The sub- and superscripts $*$ in the above denote the lower and upper semicontinuous envelopes, respectively. See [2] for the standard notations.

We shall omit the terminology "viscosity" since we only treat viscosity solutions.

3 Comparison Principle

In order to show that "subsolutions $\le$ supersolutions in $\overline{\Omega}\times\overline{\Omega}$", we will adapt two different methods; the one is Soner's technique in [9] which is available only when the sub- or supersoultion has some continuity condition. The other was originally developed by Dupuis-Ishii for oblique boundary value problems. We will barrow a "test" function from [8] for this technique.

We refer to [6] and references therein for the latter technique. We also refer to [7] for a similar one to the state constraint problem.

We will combine these ideas in the argument below.

However, to our knowledge, it seems hard to obtain the comparison principle for (2.2) directly without assuming any continuity for sub- or supersolutions.

On the other hand, the formal value function may be expected to be the unique solution under some hypotheses and also it may be possible to show that it is a continuous solution of (2.2).

Therefore, we will suppose that the value function is a continuous solutions of (2.2) even for the comparison principle. In [3], we give some sufficient condition which yields that the value function is a continuous solution of (2.2).

We remark here that, although we suppose the existence of continuous value function, our comparison result holds among possibly discontinuous solutions. Therefore, it is impossible to have a discontinuous solution under our hypotheses since our comparison principle implies the continuity of solutions.

Following [7], we shall suppose the following:

$$(A2) \qquad \begin{cases} \text{For any } x \in \partial\Omega, \text{ there are } r, \theta \in (0,1), \text{ and } \xi \in \mathrm{co}f(x, A(x)) \\ \text{such that } x' + B(t\xi, t\theta) \subset \Omega \text{ for } t \in (0,r) \text{ and } x' \in B(x,r). \end{cases}$$

and

$$(A2') \qquad \begin{cases} \text{For any } y \in \partial\Omega, \text{ there are } r, \theta \in (0,1), \text{ and } \eta \in \mathrm{co}g(y, B(y)) \\ \text{such that } y' + B(t\eta, t\theta) \subset \Omega \text{ for } t \in (0,r) \text{ and } y' \in B(y,r). \end{cases}$$

Our comparison result is as follows:

Theorem (Comparison principle) Assume that $(A0)$, $(A1)$, $(A2)$ and $(A2')$ hold. Let u and v be an upper semicontinuous subsolution and a lower semicontinuous supersolution of (2.2), respectively. Assume that the value function V is a continuous solution of (2.2).

Then, $u \le v$ in $\overline{\Omega} \times \overline{\Omega}$. More precisely, $u \le V \le v$ in $\overline{\Omega} \times \overline{\Omega}$.

Before going to the proof, we prepare to a result from [8]:

Lemma (Lemma 6.1 in [8]) Assume that $(A0)$, $(A1)$ and $(A2)$ hold. Fix $x_0 \in \partial\Omega$, and $\xi \in \mathbf{R}^n$ and $\theta \in (0,1)$ in $(A2)$. There exist constants

$c_0,\ C_0,\ C_1,\ r > 0$ and $\varphi \in C^1(\mathbf{R}^n)$ such that

$$\begin{cases} (1) & c_0|z|^2 \le \varphi(z) \le C_0|z|^2 \text{ for } z \in \mathbf{R}^n, \\ (2) & |D\varphi(z)| \le C_1|z| \text{ for } z \in \mathbf{R}^n, \\ (3) & \langle \xi', D\varphi(z)\rangle \le 0 \text{ if } z \in \bigcup_{t>0} B(t\xi, \theta t) \text{ and } \xi' \in B(\xi, r). \end{cases}$$

Sketch of Proof of Theorem: We shall only show $u \le V$ in $\overline{\Omega} \times \overline{\Omega}$ since the other assertion can be done similarly.

As in the standard proof of comparison principle for viscosity solutions, we shall suppose

$$\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \{u(x,y) - V(x,y)\} \equiv \Theta > 0.$$

Then, we will get a contradiction.

Let $(x_0, y_0) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of $u - V$ over $\overline{\Omega} \times \overline{\Omega}$. We may suppose that it is unique in view of the standard perturbation technique.

We shall only give a brief proof of the hardest case;

$$(x_0, y_0) \in \partial\Omega \times \partial\Omega.$$

Following the standard argument, we consider the function Φ on $\overline{\Omega} \times \overline{\Omega} \times \overline{\Omega} \times \overline{\Omega} \to \mathbf{R}$ by

$$\Phi_\epsilon(x, y, x', y') = u(x,y) - V(x', y') - \phi_\epsilon(x, y, x', y')$$

for a C^1 function ϕ_ϵ, which will be chosen later. We will derive a contradiction at the maximum point $(x_\epsilon, y_\epsilon, x'_\epsilon, y'_\epsilon)$ of Φ_ϵ over $\overline{\Omega} \times \overline{\Omega} \times \overline{\Omega} \times \overline{\Omega}$.

In our argument below, we will avoide the cases when $x_\epsilon \in \partial\Omega$ and $y'_\epsilon \in \partial\Omega$ for small $\epsilon > 0$. For this purpose, our "test" function here is chosen by

$$\phi_\epsilon(x, y, x', y') = \frac{\varphi(x - x')}{\epsilon^4} - \frac{\langle \xi, x - x'\rangle}{\epsilon} + \left|\frac{y - y'}{\epsilon} + \eta\right|^2,$$

where ξ and η come from the assumptions $(A2)$ and $(A2')$ for $x = x_0$ and $y = y_0$, respectively, and φ is the function in Lemma associated with $x_0 \in \partial\Omega$ and ξ.

Let $(x_\epsilon, y_\epsilon, x'_\epsilon, y'_\epsilon)$ be the maximum point of Φ_ϵ with the above ϕ_ϵ. We shall omit the subscript ϵ in what follows.

The standard argument with the continuity of V (with respect to y variable in this case) gives the properties:

$$\begin{cases} (i) & \lim_{\epsilon\to 0}(x,y,x',y') = (x_0,y_0,x_0,y_0), \\ (ii) & \lim_{\epsilon\to 0} \frac{|x-x'|^2}{\epsilon^4} = 0, \\ (iii) & \lim_{\epsilon\to 0} \left|\frac{y-y'}{\epsilon} + \eta\right| = 0, \\ (iv) & \lim_{\epsilon\to 0} u(x,y) = u(x_0,y_0). \end{cases} \tag{3.1}$$

Recalling $(A2')$ with (iii) of (3.1), we can check that $y' \in \Omega$ for small $\epsilon > 0$.

Now, we shall show that $x \in \Omega$ for small $\epsilon > 0$. In fact, otherwise, the definition of u yields that, for any $a \in A(x_0)$,

$$C \geq \min_{b\in B}\left\{-\left\langle f(x,a), \frac{D\varphi(x-x')}{\epsilon^4} - \frac{\xi}{\epsilon}\right\rangle - \left\langle g(y,b), \frac{2(y-y'+\epsilon\eta)}{\epsilon^2}\right\rangle\right\}.$$

Notice that C includes the "h" term.

Hence, taking the same convex combination as for $\xi \in \mathrm{co}f(x_0, A(x_0))$, we find $\xi' \in B(\xi, r)$ (with $r > 0$ in Lemma) such that

$$C\left(1 + \frac{o(1)}{\epsilon}\right) \geq -\left\langle \xi', \frac{D\varphi(x-x')}{\epsilon^4} - \frac{\xi}{\epsilon}\right\rangle \geq \frac{\nu\langle\xi', \xi\rangle}{\epsilon}$$

for some $\nu > 0$. Here, we have used (iii) of (3.1) and (3) of Lemma.

Thus, for small $\epsilon > 0$, we get a contradiction.

Therefore, the definitions of sub- and supersolutions, we have

$$\begin{aligned} u(x,y) - V(x',y') \leq\ & H(x',y',D_x\phi(x,y,x',y'),D_y\phi(x,y,x',y')) \\ & -H(x,y,D_x\phi(x,y,x',y'),D_y\phi(x,y,x',y')). \end{aligned}$$

It is sufficient to check that the right hand side of the above goes to 0 as $\epsilon \to 0$. This assertion can be checked by the standard calculation with (3.1). We leave it to the reader. (See [3].) QED

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THE RIEMANN PROBLEM FOR A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE BY THE VISCOSITY APPROACH

Choon-Ho Lee

Abstract. We prove the existence of solutions of the Riemann problem for a system of conservation laws of mixed type using the viscosity method which was developed by Dafermos and characterized the solutions of that problem.

0. Introduction

In this paper we have a concern with the initial value problem

$$U_t + F(U)_x = 0, \tag{0.1}$$

$$U(x,0) = \begin{cases} U_- & x \le 0 \\ U_+ & x > 0, \end{cases} \tag{0.2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and U_-, U_+ are assigned vectors in $\mathbb{R}^n$, which is called the Riemann problem. For $n = 1$ equation (0.1) is a generalized form known as Burger's equation

$$u_t - uu_x = \mu u_{xx} \tag{0.3}$$

with the viscosity term. Cole[1] and Hopf[5] found a transform

$$u = -2\mu \frac{v_x}{v}$$

which converts (0.3) into a linear heat equation

$$v_t = \mu v_{xx}. \tag{0.4}$$

They investigate some properties of solutions of (0.1) using the explicit solution of (0.4) as $\mu \to 0$. This idea was developed by Kalashnikov[7] and Tupciev[11][12]. They used to the viscosity term ϵu_{xx} in the general form

$$u_t + f(u)_x = 0, \tag{0.5}$$

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and proved the existence of solution of (0.5) by vanishing the viscosity term. This method is available in case that f is not genuinely nonlinear in the sense of Lax[8].

In this paper we study the existence and properties of solutions of the Riemann Problem for a 2×2 system of conservation laws of the mixed type

$$\begin{aligned} u_t - f(v)_x &= 0, \\ v_t - g(u)_x &= 0 \end{aligned} \tag{0.6}$$

with the initial data

$$(u,v)(x,0) = \begin{cases} (u_+, v_+) & x > 0, \\ (u_-, v_-) & x < 0. \end{cases} \tag{0.7}$$

Here we assume
(I) $f \in C^2(\mathbb{R})$ is a strictly increasing convex function.
(II) $g \in C^2(\mathbb{R})$ and there exist α, β, η with $\alpha < \eta < \beta$ such that

$$\begin{aligned} &g'(u) \geq 0 \text{ if } u \notin (\alpha, \beta) \text{ and } g'(u) < 0 \text{ for } u \in (\alpha, \beta), \\ &g''(u) < 0 \text{ if } u < \eta \text{ and } g''(u) > 0 \text{ if } u > \eta. \end{aligned}$$

(III) $g(u) \to \pm\infty$ as $u \to \pm\infty$.

If $f(v) = v$, then the typical model of equation (0.6) is the one-dimensional isothermal motion of a compressible elastic fluid or solid in the Lagrangian coordinates. In this case the existence of solutions to the Riemann problem (0.6), (0.7) has been studied by Dafermos [2], Dafermos and DiPerna [3], Fan [4], James [6], Slemrod [10]. Our approach was based on a vanishing "viscosity" term pursued by Kalashnikov [7], Tupchiev [11] [12]. Their idea is to replace (0.6) with the system

$$\begin{aligned} u_t - f(v)_x &= \epsilon t u_{xx}, \\ v_t - g(u)_x &= \epsilon t v_{xx} \end{aligned} \tag{0.8}$$

for $x \in \mathbb{R}$, $t > 0$ and construct solutions as the limit of the solutions of (0.8), (0.7) as $\epsilon \to 0+$. Since the system is invariant under the transformation $(x,t) \to (ax, at)$, where $a > 0$, (0.8) and (0.7) admit solutions of the form $(u_\epsilon(\xi), v_\epsilon(\xi))$, where $\xi = \frac{x}{t}$. A simple computation shows that $u = u_\epsilon(\xi), v = v_\epsilon(\xi)$ are solutions of (0.8) and (0.7) if they satisfy

$$\begin{aligned} -\xi u' - f(v)' &= \epsilon u'', \\ -\xi v' - g(u)' &= \epsilon v'' \end{aligned} \tag{0.9}$$

with the boundary conditions

$$(u,v)(\pm\infty) = (u_\pm, v_\pm) \tag{0.10}$$

where $' = \frac{d}{d\xi}$ and $'' = \frac{d^2}{d\xi^2}$. We shall call the boundary value problem (0.9) and (0.10) the problem (P_ϵ). Similarly the initial value problem (0.6) and (0.7) are called the Riemann problem (P).

This paper consists of three sections. In Section 1 we establishes that if the data are in different phases there is a solution of P_ϵ which exhibits one change of phase. In order to prove the results, we use the arguments of Dafermos [1] and Slemrod [6]. In Section 2 we discuss the existence of solution to the Riemann problem to give conditions on which solutions of P_ϵ possess limits. In order to prove this we use the uniform bounded variation of solutions of the problem (P_ϵ) and the Helley's theorem -the convergence on the bounded variation functions. In Section 3 we study the properties of solution of the Riemann problem (P).

Throughout this paper we always assume Assumptions (I) and (II) unless other mentions it.

1. Existence of solutions of Problem (P_ϵ)

In this section we will prove the existence of solutions of Problem (P_ϵ). In order to prove this we first consider the following equation

$$\begin{aligned} \epsilon u'' &= -\xi u' - \mu f(v)', \\ \epsilon v'' &= -\xi v' - \mu g(u)', \\ (u,v)(\pm L) &= (u_\pm, v_\pm), \end{aligned} \tag{1.1}$$

where $L > 1$, and $0 \le \mu \le 1$. Using the Leray-Schauder fixed point theorem, we can prove

Lemma 1.1. *Let $u_- < \alpha$, $u_+ > \beta$. If there exists a constant M such that every possible solution of (1.1) with $u'(\xi) > 0$ when $\alpha \le u(\xi) \le \beta$ satisfies the a priori estimate*

$$\sup_{|\xi|\le L} (|u(\xi)| + |v(\xi)|) \le M, \tag{1.2}$$

then problem (P_ϵ) has a solution with $u'(\xi) > 0$ if $\alpha \le u(\xi) \le \beta$.

In order to obtain the *a priori* estimate we use the behavior of solutions of (1.1) which is modified by Lee [9]. The following lemma is originated by Dafermos [1] and Slemrod [10].

Lemma 1.2. *Let $(u(\xi), v(\xi))$ be a solution of (1.1) on $[-L, L]$, $\mu > 0$ with $u'(\xi) >$ 0. Then we have:*

(1) *On any subinterval (l_1, l_2) for which $g'(u(\xi)) > 0$ one of the following holds:*

(i) $u(\xi)$ and $v(\xi)$ are constant on (l_1, l_2).

(ii) $v(\xi)$ is a strictly increasing(or decreasing) function with no critical points in (l_1, l_2); $u(\xi)$ has, at most, one critical point in (l_1, l_2) that necessarily must be a maximum(or minimum).

(iii) $u(\xi)$ is a strictly increasing (or decreasing) function with no critical point in (l_1, l_2); $v(\xi)$ has, at most, one critical point in (l_1, l_2) that necessarily must be a maximum(or minimum).

(2) *On any subinterval* (l_1, l_2) *for which* $g'(u(\xi)) < 0$ *the graph of* $v = v(u)$ *is convex(or concave) at points where* $u'(\xi) > 0$*(or* $u'(\xi) < 0$*).*

(3) *If* $\alpha \le u(\xi) \le \beta$*, then* u *and* v *can have no local maxima or minima at* ξ *for which* $u(\xi) = \alpha$ *or* $u(\xi) = \beta$*.*

If the initial data lies in the different phases, the properties of solutions of (1.1) is as follow.

Lemma 1.3. *Assume that* $u_- < \alpha$*,* $u_+ > \beta$ *and let* $u(\xi)$*,* $v(\xi)$ *be a solution of (1.1) with* $\mu > 0$ *for which* $u'(\xi) > 0$ *when* $\alpha \le u(\xi) \le \beta$*. Then one of the followings holds:*

(1) *No extremal points:* $u(\xi)$*,* $v(\xi)$ *have no local maxima or minima on* $[-L, L]$*. They are non-constant and monotone,* u *being monotone increasing.*

(2) *One extremal point: (a)* $u(\xi)$ *has a minimum at some* ξ_-*,* $u(\xi_-) < u_-$*;* $v(\xi)$ *is decreasing on* $[-L, L]$*. (b)* $u(\xi)$ *has a maximum at some* ξ_+*,* $u(\xi_+) > u_+$*;* $v(\xi)$ *is decreasing on* $[-L, L]$*. (c)* $v(\xi)$ *has a maximum at some* η_- *(or* η_+*);* $u(\eta_-) < \alpha$ *(or* $u(\eta_+) > \beta$*) and* $u(\xi)$ *is increasing on* $[-L, L]$*. (d)* $v(\xi)$ *has a minimum at some* η*;* $\alpha < u(\eta) < \beta$ *and* $u(\xi)$ *is increasing on* $[-L, L]$*.*

(3) *Two extremal points: (a)* $v(\xi)$ *has a local maximum at* η_- *(or* η_+*) and a local minimum at* η*,* $u(\xi)$ *is increasing on* $[-L, L]$ *and* $u_- < u(\eta_-) < \alpha$ *(or* $u_+ > u(\eta_+) > \beta$*),* $\alpha < u(\eta) < \beta$*. (b)* $u(\xi)$ *has a minimum at* ξ_-*,* $u(\xi_-) < u_-$*;* $v(\xi)$ *has a local minimum at* η*,* $\eta > \xi_-$*,* $\alpha < u(\eta) < \beta$*. (c)* $u(\xi)$ *has a maximum at* ξ_+*,* $u(\xi_+) > u_+$*;* $v(\xi)$ *has a local minimum at* η*,* $\eta < \xi_+$*,* $\alpha < u(\eta) < \beta$*.*

(4) *Three extremal points: (a)* $v(\xi)$ *has local maxima at* η_-*,* η_+ *and a local minimum at* η*,* $\eta_- < \eta < \eta_+$*;* $u(\xi)$ *is increasing with* $u_- < u(\eta_-) < \alpha$*,* $\alpha < u(\eta) < \beta$*,* $\beta < u(\eta_+) < u_+$*. (b)* $u(\xi)$ *has a minimum at* ξ_-*,* $u(\xi_-) < u_-$ *and maximum at* ξ_+*,* $u(\xi_+) > u_+$ *and* $v(\xi)$ *has a local minimum at* η*,* $\xi_- < \eta < \xi_+$*,* $\alpha < u(\eta) < \beta$*. (c)* $u(\xi)$ *has a minimum at* ξ_-*,* $u(\xi_-) < u_-$*,* $v(\xi)$ *has a local minimum at* η*,* $\alpha < u(\eta) < \beta$ *and a local maximum at* η_+*,* $\eta < u(\eta_+) < u_+$*,* $\xi_- < \eta < \eta_+$*. (d)* $u(\xi)$ *has a maximum at* ξ_+*,* $u(\xi_+) > u_+$*,* $v(\xi)$ *has a local maximum at* η_-*,* $u_- < u(\eta_-) < \alpha$*, and a local minimum at* η*,* $\alpha < u(\eta) < \beta$*.*

Combining Lemma 1.2 and 1.3, we have the *a priori* estimate of solutions of (1.1)

Lemma 1.4. *Assume* $v_+ < v_-$ *and* $u_-, u_+ < \alpha$*(or* $v_- < v_+$ *and* $u_-, u_+ > \beta$*). Then there is a constant* M *such that every possible solution of (1.1),* $0 \le \mu \le 1$*, satisfies the a priori estimate (1.2), where* M *depends at most on* $u_-, u_+, v_-, v_+, \epsilon, f, g$ *and is independent of* μ *and* L*.*

Lemma 1.1 and 1.4 imply

Theorem 1.5. *If* $u_- < \alpha$*,* $u_+ > \beta$ *(or* $u_- > \alpha$*,* $u_+ < \beta$*), there are solutions* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *of* (P_ϵ) *satisfying* $u'(\xi) > 0$ *when* $\alpha \le u(\xi) \le \beta$*.*

2. Uniform boundedness of solutions $\{(u_\epsilon(\xi), v_\epsilon(\xi))\}$

In this section we state that the solutions $\{(u_\epsilon(\xi), v_\epsilon(\xi))\}$ of (1.1) is uniformly boundedness.

Lemma 2.1. *Under the assumption I and II, the set $\{(u_\epsilon(\xi), v_\epsilon(\xi))\}$ is uniformly bounded independent of ϵ.*

Using Lemma 2.1 and Helly's theorem, we can prove the existence of weak solution of (1.1)

Theorem 2.2. *Under the assumption I and II, there exists a weak solution of (0.6) and (0.7).*

3. Properties of Solutions

¿From Section 2 we have known that the set $\{(u_\epsilon(\xi), v_\epsilon(\xi)\}$ is uniformly bounded in ϵ. We will consider $u_\epsilon(\xi)(v_\epsilon(\xi)$, respectively) as a multivalued function of $v_\epsilon(u_\epsilon$, respectively). For convenience, we parameterized the curve $v = V(u)$ by $(U(s), V(s))$ where s is the length of the arc of $v = V(u)$ joining (u_-, v_-) and the point $(U(s), V(s))$. Since the curve $v = V(u)$ does not intersect itself, the parameterization is bijective. In this kind of parameterization, s increases when ξ increases. We call the curve $(U(s), V(s))$ the base curve of the solution $(u(\xi), v(\xi))$.

Now we study the discontinuities of $(u(\xi), v(\xi))$. Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. Denote C_{ξ_0} by the portion of the base curve in the (u, v)-plane that connects points $(u(\xi_0-), v(\xi_0-))$ and $(u(\xi_0+), v(\xi_0+))$. We fix $(\bar{u}, \bar{v}) \in C_{\xi_0}$. For n large, we define $\xi_{\epsilon_n}(u; \bar{u}, \bar{v})$ to be the branch of the inverse function of $u = u_{\epsilon_n}(\xi)$ for which

$$v_{\epsilon_n}(\xi_{\epsilon_n}(\bar{u}; \bar{u}, \bar{v})) \to \bar{v} \tag{3.1}$$

as $n \to \infty$. For n large, we define ξ_{ϵ_n}, $\hat{u}_{\epsilon_n}$, $\hat{v}_{\epsilon_n}$ by the relations

$$\xi_{\epsilon_n} = \xi_{\epsilon_n}(\bar{u}) + \epsilon\zeta, \tag{3.2}$$

$$\hat{v}_{\epsilon_n}(\zeta) = v_{\epsilon_n}(\xi_n), \tag{3.3}$$

$$\hat{u}_{\epsilon_n}(\zeta) = u_{\epsilon_n}(\xi_n), \tag{3.4}$$

Lemma 3.1. *Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. Then for any $(\bar{u}, \bar{v}) \in C_{\xi_0}$ it follows that*
(a) if $U(s)$ is increasing (decreasing) at $(\bar{u}, \bar{v})$, then

$$-\xi_0(\bar{u} - u(\xi_0-)) - (f(\bar{v}) - f(v(\xi_0-))) \geq 0 \quad (\leq 0).$$

(b) if $V(s)$ is increasing (decreasing) at $(\bar{u}, \bar{v})$, then

$$-\xi_0(\bar{v} - v(\xi_0-)) - (g(\bar{u}) - g(u(\xi_0-))) \geq 0 \quad (\leq 0).$$

Moreover we can change all ξ_0- to ξ_0+.

Next theorem show that the solution $u(\xi)$ of (0.6) must lie in the α-phase $(-\infty, \alpha]$ or β-phase $[\beta, \infty)$.

Theorem 3.2. *$u(\xi)$ takes no value in (α, β) and may take at most one of α and β as a value.*

Since the base curve $(U(s), V(s))$ is oriented in the direction in which s increase, we can talk about the right and left sides of $(U(s_0), V(s_0))$ for the portions of the curve with $s < s_0$ and $s > s_0$ respectively. We define

(3.5.a)
$$S(U(s), V(s)) = \begin{cases} 1 & \text{if both } U(s) \text{ and } V(s) \text{ are strictly increasing or strictly decreasing at } s, \\ -1 & \text{if both } U(s) \text{ and } -V(s) \text{ are strictly increasing or strictly decreasing at } s, \\ 0 & \text{otherwise} \end{cases}$$

(3.5.b)
$$\begin{aligned} S(U(s_0), V(s_0); +) &= \lim_{s \to s_0+} S(U(s), V(s)) \\ S(U(s_0), V(s_0); -) &= \lim_{s \to s_0-} S(U(s), V(s)) \end{aligned}$$

If $U(s)$ or $V(s)$ attains a local extremum at $s = s_\alpha (or s = s_\beta)$ in the region $u < \alpha$ (or $u > \beta$), we set
$$(u_\alpha, v_\alpha) = (U(s_\alpha), V(s_\alpha))$$
$$(\text{ or } (u_\beta, v_\beta) = (U(s_\beta), V(s_\beta)).)$$

(u_1, v_1) is called *a constant state* of $(u(\xi), v(\xi))$ if $(u(\xi), v(\xi))$ is constant in some interval of $\mathbb{R}$.

Lemma 3.3. *The solution $(u(\xi), v(\xi))$ has no constant state other than (u_-, v_-), (u_+, v_+) and possibly (u_α, v_α) and (u_β, v_β).*

Let ξ_0 be a point of discontinuity of solution $(u(\xi), v(\xi))$. Then

(3.6)
$$\begin{aligned} S(u(\xi_0-), v(\xi_0-); +)\sqrt{-f'(v(\xi_0-))g(u(\xi_0-))} &\geq \xi_0 \\ &\geq S(u(\xi_0+), v(\xi_0+); -)\sqrt{-f'(v(\xi_0+))g(u(\xi_0+))}. \end{aligned}$$

If $u(\xi)$ or $v(\xi)$ is strictly monotone from the left at $\xi_0 \in \mathbb{R}$, then

(3.7a)
$$\xi_0 = S(u(\xi_0-), v(\xi_0-); -)\sqrt{-f'(v(\xi_0-))g(u(\xi_0-))}.$$

If $u(\xi)$ or $v(\xi)$ is strictly monotone from the right at $\xi_0 \in \mathbb{R}$, then

(3.7b)
$$\xi_0 = S(u(\xi_0+), v(\xi_0+); +)\sqrt{-f'(v(\xi_0+))g(u(\xi_0+))}.$$

Combining these facts and Lemma 3.3, we have

Theorem 3.4. *Let ξ_0 be a point of discontinuity of $(u(\xi), v(\xi))$. If $(u(\xi_0-), v(\xi_0-))$ (or $(u(\xi_0+), v(\xi_0+))$) is differ from (u_-, v_-), (u_+, v_+), (u_α, v_α), (u_β, v_β), then ξ_0 is a contact discontinuity from the left (or right).*

Corollary 3.5. *(a) At least one of* $(u(0-), v(0-))$ *and* $(u(0+), v(0+))$ *is a constant state of* $(u(\xi), v(\xi))$. *Furthermore,* $\xi = 0$ *is either a point of continuity of* $(u(\xi), v(\xi))$ *or the phase boundary (at which the shock jumps from one phase to another).*
(b) Besides the constant states and the phase boundary, $(u(\xi), v(\xi))$ *consists of shocks and simple waves of the first kind for* $\xi < 0$ *and of the second kind for* $\xi > 0$.

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EXISTENCE OF BOUNDARY OPTIMAL CONTROL FOR THE BOUSSINESQ EQUATIONS

HYUNG-CHUN LEE

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ABSTRACT. This paper deals with Neumann boundary optimal control problems associated with the Boussinesq equations (including the solid media). These problems are first put into an appropriate mathematical formulation. Then the existence of optimal solutions is proved.

1. INTRODUCTION

We study boundary optimal control problem for an steady natural convection fluid. The control is heat flux on the portion of the flow boundary.

We consider the nondimensional Boussinesq equations (including the solid media) as follows:

$$-Pr\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + Pr\ Ra\,\frac{\mathbf{g}}{|\mathbf{g}|}\,T + \mathbf{f} \quad \text{in } \Omega_f, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_f, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega_f, \tag{1.3}$$

$$\mathbf{u} \equiv \mathbf{0} \quad \text{in } \Omega - \Omega_f = \Omega_s, \tag{1.4}$$

$$-\nabla\cdot(\kappa\nabla T) + (\mathbf{u}\cdot\nabla)T = Q \quad \text{in } \Omega, \tag{1.5}$$

$$T = T_b \quad \text{on } \Gamma_D, \tag{1.6}$$

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and

$$\frac{\partial T}{\partial n} = g \quad \text{on } \Gamma_C, \tag{1.7}$$

where $\Gamma_D = \partial\Omega \setminus \Gamma_C$ and $\mathbf{u}$ denotes the velocity field, p the pressure field, T the temperature, and control g. The domains Ω_f and Ω_s are disjoint polyhedral domains in R^2, $\Omega =$ interior $(\bar{\Omega}_f \cup \bar{\Omega}_s)$. $\mathbf{g}$ is the gravitational force vector, κ is the thermal conductivity coefficients function such that $\kappa \equiv \kappa_f$ in Ω_f and $\kappa \equiv \kappa_s$ in Ω_s. $\mathbf{n}$ denotes the outward unit normal to Ω and Pr and Ra denote the Prandtl and Rayleigh numbers, respectively. The data functions $\mathbf{f}$, Q, T_b are assume to be known. Note that as a results of our assumptions about the flow, the mechanical equations (1.1)-(1.4) are fully coupled with the thermal equations (1.5)-(1.7).

We now define the optimal control problem. Seek $(\mathbf{u}, T, p, g) \in \mathbf{H}_0^1(\Omega_f) \times H^1(\Omega) \times L_0^2(\Omega_f) \times L^2(\Gamma_C)$ such that the cost functional

$$\mathcal{J}(\mathbf{u}, T, p, g) = \frac{1}{2\delta}\int_{\Omega_s} |T - T_d|^2 \, d\Omega + \frac{\gamma}{2}\int_{\Gamma_C} |g|^2 \, d\Gamma \tag{1.8}$$

is minimized subject to (1.1)-(1.7) where T_d is some desired temperature distribution and Ω_s is a portion of Ω. The nonnegative parameters δ and γ can be used to change the relative importance of the two terms appearing in the defintion of $\mathcal{J}$ as well as to act a penalty parameter. Incidently, the appearance of the control g in the cost functional is necessary because we are not imposing an a priori limits on the size of this control.

We close this section by introducing some of notation used in subsequenct sections. We introduce some function spaces and their norms, along with some related notation used in subsequent sections (for details see [1]). Throughout, C will denote a positive constant whose meaning and value changes with context. We define the Sobolev space $H^m(\Omega)$ for nonnegative integer m by

$$H^m(\Omega) := \{T \in L^2(\Omega) | D^\alpha T \in L^2(\Omega) \text{ for } 0 \leq |\alpha| \leq m\} \tag{1.9}$$

where $D^\alpha T$ is the weak (or distributional) partial derivative, α is a muti-index, $|\alpha| = \sum_i \alpha_i$. Clearly, $H^0(\Omega) = L^2(\Omega)$. The norm associated with $H^m(\Omega)$ that we use is $||\cdot||_m$, given by

$$||T||_m = \left\{ \sum_{0\le|\alpha|\le m} ||D^\alpha T||_0^2 \right\}^{\frac{1}{2}}. \tag{1.10}$$

One particular subspace

$$H_D^1(\Omega) = \left\{ S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_C \right\}. \tag{1.11}$$

The usual inner product associated with $H^m(\Omega)$ will be denoted by $(\cdot,\cdot)_m$.

For vector valued functions, we define the Sobolev space $\mathbf{H}^m(\Omega)$ by

$$\mathbf{H}^m(\Omega) := \{\mathbf{u} \mid u_i \in H^m(\Omega), i = 1, 2\}, \tag{1.12}$$

where $\mathbf{u} = \{u_1, u_2\}$, and its associated norm $||\cdot||$ is given by

$$||\mathbf{u}||_m = \left\{ \sum_{i=1}^{2} ||u_i||_m^2 \right\}^{\frac{1}{2}}. \tag{1.13}$$

We also define a subspace of $L^2(\Omega)$

$$L_0^2(\Omega) := \{q \in L^2(\Omega) | \int_\Omega q d\Omega = 0\}. \tag{1.14}$$

In all subspaces, we use norms induced by the original spaces. We also make use of the well-known space $\mathbf{L}^4(\Omega)$ equipped with the norm $||\cdot||_{\mathbf{L}^4(\Omega)}$.

We denote

$$H_C^1(\Omega) = \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_C\}, \tag{1.15}$$

$$H_D^1(\Omega) = \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_D\}, \tag{1.16}$$

We also define the solenoidal spaces

$$V := \{\mathbf{u} \in \mathbf{H}_0^1(\Omega_f) | \nabla \cdot \mathbf{u} = 0\}.$$

If Ω is bounded and has a Lipschitz continuous boundary (these are kinds of domains under consideration here), Sobolev's emmbedding theorem yields that $H^1(\Omega)$

$\hookrightarrow\!\!\hookrightarrow L^4(\Omega)$, where $\hookrightarrow\!\!\hookrightarrow$ denotes compact embedding, *i.e.* a constant C exists such that

$$\|u\|_{L^4(\Omega)} \leq C\|u\|_1. \tag{1.17}$$

Obviously a similar result holds for the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^4(\Omega)$.

We introduce the following bilinear and trilinear forms, for $\mathbf{u}, \mathbf{v}$ and $\mathbf{w} \in \mathbf{H}^1(\Omega_f)$, $T, S, R \in H^1(\Omega)$,

$$a(\mathbf{u},\mathbf{v}) = \int_{\Omega_f} \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega_f), \tag{1.18}$$

$$b(\mathbf{v},q) = -\int_{\Omega_f} q\nabla\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_f),\ \forall q \in L^2(\Omega), \tag{1.19}$$

$$c(\mathbf{u},\mathbf{w},\mathbf{v}) = \int_{\Omega_f} (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega_f), \tag{1.20}$$

$$A(T,S) = \int_{\Omega} \kappa\nabla T\cdot\nabla S\, d\Omega \quad \forall T,S \in H^1(\Omega), \tag{1.21}$$

$$C(\mathbf{u},T,S) = \int_{\Omega_f} (\mathbf{u}\cdot\nabla)TS\, d\Omega \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_f),\ \forall T,S \in H^1(\Omega), \tag{1.22}$$

$$d(T,\mathbf{v}) = \int_{\Omega_f} T\mathbf{e}_2\cdot\mathbf{v}\, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_f),\ \forall T \in H^1(\Omega), \tag{1.23}$$

where $\mathbf{e}_2 = \mathbf{g}/|\mathbf{g}|$.

2. The weak formulation of the Boussinesq equations

The weak form of the constraint equations (1.1)-(1.6) is then given as follows: seek $\mathbf{u} \in \mathbf{H}_0^1(\Omega_f)$, $p \in L_0^2(\Omega_f)$ and $T \in H^1(\Omega)$ such that

$$Pr\ a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = Pr\ Ra\ d(T,\mathbf{v}) + <\mathbf{f},\mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega_f), \tag{2.1}$$

$$b(\mathbf{u},q) = 0 \quad \forall q \in L_0^2(\Omega_f), \tag{2.2}$$

$$A(T,S) + C(\mathbf{u},T,S) = <Q,S> + \kappa_f(g,S)_{\Gamma_D} \quad \forall S \in H_C^1(\Omega), \tag{2.3}$$

and

$$T = T_b \quad \text{on } \Gamma_D, \tag{2.4}$$

Lemma 2.1. *There are constants $C_{1,2,3,4}$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega_f)$ and for all $T, S \in H^1(\Omega)$:*

$$|A(T,S)| \leq \max(\kappa_f, \kappa_s)|T|_{1,\Omega}|S|_{1,\Omega} \quad \forall T, S \in H^1(\Omega), \tag{2.5}$$

$$A(T,T) \geq \min(\kappa_f, \kappa_s)|T|_{1,\Omega}^2 \quad \forall T, S \in H^1(\Omega), \tag{2.6}$$

$$|C(\mathbf{u}, T, S)| \leq C_1||\mathbf{u}||_{L^4(\Omega_f)}|T|_{1,\Omega}||S||_{L^4(\Omega)} \quad \forall T, S \in H^1(\Omega) and \forall \mathbf{u} \in \mathbf{V}, \tag{2.7}$$

$$C(\mathbf{u}, T, T) = 0 \quad if \quad \mathbf{u} \in \mathbf{V}, \tag{2.8}$$

$$|a(\mathbf{u}, \mathbf{v})| \leq C||\mathbf{u}||_1||\mathbf{v}||_1, \qquad a(\mathbf{u}, \mathbf{u}) \geq C||\mathbf{u}||_1^2, \tag{2.9}$$

$$|c(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq C_2||\mathbf{u}||_1||\mathbf{v}||_1|\mathbf{w}|_1, \tag{2.10}$$

$$|c(\mathbf{u}, \mathbf{v}, \mathbf{v})| = 0, \quad if \, \mathbf{u} \in \mathbf{V}, \tag{2.11}$$

and

$$|d(\mathbf{u}, T)| \leq C_3||T||_{0,2}||\mathbf{v}||_{0,2} \leq C_4|T|_1|\mathbf{u}|_1 \tag{2.12}$$

where $||\cdot||_{0,2}$ and $||\cdot||_{0,4}$ denote the $L^2(\Omega)$ and $L^4(\Omega)$ norm respectively.

We now show that solution always exists for data $Q \in H^{-1}(\Omega)$ and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_f)$. Further, that solution is unique for either small data or an equivalent restriction on the Rayleigh and Prandtl number. The existence and uniqueness of the solution for the homogeneous case can be found in many papers (*e.g.* [3]). We extend their work with suitable modifications.

Lemma 2.2 (Leray-Schauder). *Let V be a Banach space, and let $G : [0,1] \times V \to V$ be a continuous, compact map, such that $G(0, v) = v_0$ is independent of $v \in V$.*

Suppose there exists $M < \infty$ such that, for all $(\sigma, x) \in [0,1] \times V$,

$$G(\sigma, x) = x \implies ||x|| < M.$$

Then the map $G_1 : V \to V$ given by $G_1(v) = G(1, v)$ has a fixed point.

Proposition 2.3. *For every $g \in L^2(\Gamma_C)$, $Q \in H^{-1}(\Omega)$ and $\mathbf{f} \in \mathbf{H}^{-1}(\Omega_f)$ (2.1)-(2.4) has a solution $(\mathbf{u}, T, p) \in \mathbf{H}_0^1(\Omega_f) \times H^1(\Omega) \times L_0^2(\Omega_f)$ satisfying the estimates*

$$||\mathbf{u}||_1 + ||T||_1 \le C(||\mathbf{f}||_{-1} + ||Q||_{-1} + ||g||_{1/2,\Gamma_C}) \tag{2.13}$$

and

$$||p||_0 \le C(||\mathbf{f}||_{-1} + ||Q||_{-1} + ||g||_{1/2,\Gamma_C} + ||\mathbf{u}||_1) \tag{2.14}$$

Proof. Given $T_b \in H^{1/2}(\Gamma_D)$, by the virtue of the trace theorem, we may choose a T_0 in $H^1(\Omega)$, satisfy $T_0 = T_b$ on Γ_D and $||T_0||_1 \le C\ ||T_b||_{1/2,\Gamma_D}$. By setting $\hat{T} = T - T_0$, we may see that seeking a $(\mathbf{u}, T, p) \in \mathbf{H}_0^1(\Omega_f) \times H^1(\Omega) \times L_0^2(\Omega_f)$ satisfying (2.1)-(2.4) is equivalent to seeking a $(\mathbf{u}, \hat{T}, p) \in \mathbf{H}_0^1(\Omega_f) \times H_D^1(\Omega) \times L_0^2(\Omega_f)$ satisfying

$$Pr\ a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = Pr\ Ra\ d(\hat{T}, \mathbf{v}) + < \mathbf{f}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega_f), \tag{2.15}$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega_f). \tag{2.16}$$

$$A(\hat{T}, S) + C(\mathbf{u}, \hat{T}, S) = < Q, S > + \kappa_f(g, S)_{\Gamma_C} \quad \forall S \in H_C^1(\Omega), \tag{2.17}$$

Thus, without loss of generality, we can consider only for the case of homogeneous Dirichlet boundary condtion.

For $\mathbf{u} \in \mathbf{V}$, $A(\cdot,\cdot) + C(\mathbf{u},\cdot,\cdot)$ is a continuous, elliptic, bilnear form on $H_D^1(\Omega) \times H_D^1(\Omega)$ by (2.5)-(2.8) of lemma 2.1. Thus, for given $g \in L^2(\Omega)$ and $Q \in H^{-1}(\Omega)$, by the Lax-Milgram lemma there is a unique solution $T \in H_D^1(\Omega)$ satisfying (2.3) and the estimate

$$||T||_1 \le C(||g||_{0,\Gamma_C} + ||Q||_{-1}) \tag{2.18}$$

Thus, we may define a mapping $F : V \to H^1_D(\Omega)$ by $F(\mathbf{u}) = T$. In fact, $|F(\mathbf{u})|_1 \leq \min\{\kappa_f, \kappa_s\}||Q||_{-1}$. The theorem will be proved if one can show that there is at least on $\mathbf{u} \in \mathbf{V}$ such that

$$Pr\ a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) = Pr\ Ra\ d(F(\mathbf{u}),\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.19}$$

From inequalities (2.9) it follows that $a(\cdot,\cdot)$ is a continuous elliptic bilinear form on $\mathbf{V} \times \mathbf{V}$ and $|-c(\mathbf{u},\mathbf{u},\mathbf{v}) + d(F(\mathbf{u}),\mathbf{v})| \leq (C_2||\mathbf{u}||_1^2 + Pr\ Ra\ C_4||F(\mathbf{u})||_1)\ ||\mathbf{v}||_1$, for all $\mathbf{v} \in \mathbf{V}$ follows from (2.10)-(2.12). Thus we may define a mapping $G : \mathbf{V} \to \mathbf{V}$ by

$$Pr\ a(\mathbf{u},\mathbf{v}) = -c(\mathbf{u},\mathbf{u},\mathbf{v}) + Pr\ Ra\ d(F(\mathbf{u}),\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.20}$$

Clearly $\mathbf{u}$ is a solution of (2.19) if it is a solution of

$$G(\mathbf{u}) = \mathbf{u}. \tag{2.21}$$

Now, we may apply the Leray-Schauder Principle to prove the existence of the solution to (2.21). First we verify the complete continuity of G. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}$. Set $\mathbf{w} = G(\mathbf{u}_2) - G(\mathbf{u}_1)$.

Substracting the equations obtained from (2.20) by substituting $\mathbf{u}_2$ and $\mathbf{u}_1$ for $\mathbf{u}$ and $\mathbf{w}$ for $\mathbf{v}$, we get

$$Pr\ a(\mathbf{w},\mathbf{w}) = -c(\mathbf{u}_2 - \mathbf{u}_1;\mathbf{u}_2,\mathbf{w}) - c(\mathbf{u}_1;\mathbf{u}_2 - \mathbf{u}_1,\mathbf{w}) + Pr\ Ra\ d(F(\mathbf{u}_2) - F(\mathbf{u}_1),\mathbf{w}). \tag{2.22}$$

Now, we estimte $|F(\mathbf{u}_2) - F(\mathbf{u}_1)|_1$. Substitute $\mathbf{u}_2$ and $\mathbf{u}_1$ in (2.3) and subtract to get

$$\begin{aligned} &A(F(\mathbf{u}_2) - F(\mathbf{u}_1), S) \\ &\quad = -C(\mathbf{u}_2 - \mathbf{u}_1; F(\mathbf{u}_2), S) - C(\mathbf{u}_1; F(\mathbf{u}_2) - F(\mathbf{u}_1), S) \quad \forall S \in H^1_D(\Omega). \end{aligned} \tag{2.23}$$

Substituting $F(\mathbf{u}_2) - F(\mathbf{u}_1)$ for S and using (2.6)-(2.8)

$$|F(\mathbf{u}_2) - F(\mathbf{u}_1)|_1 \leq C||Q||_{-1}\ ||\mathbf{u}_2 - \mathbf{u}_1||_{0,4}. \tag{2.24}$$

Thus,

$$|\mathbf{w}|_1 \leq Pr^{-1}[||\mathbf{u}_2||_{0,4} + ||\mathbf{u}_1||_{0,4} + Pr\ Ra\ C||Q||_{-1}]\ ||\mathbf{u}_2 - \mathbf{u}_1||_{0,4} \tag{2.25}$$

Since $\mathbf{H}_0^1(\Omega_f)$ is compactly imbedded in $\mathbf{L}^4(\Omega_f)$ and hence so is $\mathbf{V}$. It follows that G is absolutly continuous.

Now, we define $G(\sigma, \mathbf{v}) = \sigma G(\mathbf{v})$ for all $(\sigma, \mathbf{v}) \in [0,1] \times V$. Clearly, $G(0, \mathbf{v}) = \mathbf{0}$ is independent of $\mathbf{v}$.

Suppose $\sigma \in (0, 1]$ and $\mathbf{v} \in \mathbf{V}$ satisfies $\sigma G(\mathbf{v}) = \mathbf{v}$. Then

$$\sigma^{-1}\ Pr\ a(\mathbf{v}, \mathbf{v}) = -c(\mathbf{v}; \mathbf{v}, \mathbf{v}) + Pr\ Ra\ d(F(\mathbf{v}), \mathbf{v}). \tag{2.26}$$

From the above fact, we have

$$|\mathbf{v}|_1 \leq \sigma\ Ra\ C_4 |F(\mathbf{v})|_1 \leq C||f||_{-1} \tag{2.27}$$

which complete the proof. □

3. The optimization problem and the existence of an optimal solution

We state the optimal control problem. We look for a $(\mathbf{u}, T, p, g) \in \mathbf{H}_0^1(\Omega_f) \times H^1(\Omega) \times L_0^2(\Omega_f) \times L^2(\Gamma_C)$ such that the cost functional

$$\mathcal{J}(\mathbf{u}, T, p, g) = \frac{1}{2\delta} \int_{\Omega_s} |T - T_d|^2\, d\Omega + \frac{\gamma}{2} \int_{\Gamma_C} |g|^2\, d\Gamma \tag{3.1}$$

subject to the constraints

$$Pr\ a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = Pr\ Ra\ d(T, \mathbf{v}) + \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega_f), \tag{3.2}$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega_f), \tag{3.3}$$

$$A(T, S) + C(\mathbf{u}, T, S) = \langle Q, S \rangle - \kappa_f (g, S)_{\Gamma_C} \quad \forall S \in H_D^1(\Omega), \tag{3.4}$$

and

$$T = T_b \quad \text{on } \Gamma_D. \tag{3.5}$$

The *admissibility set* $\mathcal{U}_{ad}$ is defined by

$$\mathcal{U}_{ad} = \{(\mathbf{v}, S, q, z) \in \mathbf{H}^1_0(\Omega_f) \times H^1(\Omega) \times L^2_0(\Omega_f) \times L^2(\Gamma_C) : \\ (3.2) - (3.5) \text{ are satisfied}\}. \tag{3.6}$$

Then, the constraint minimization probelm (3.1)-(3.3) can be stated as follows:

find $(\mathbf{u}, T, p, g) \in \mathcal{U}_{ad}$ such that

$$\mathcal{U}(\mathbf{u}, T, p, g) \leq \mathcal{U}(\mathbf{v}, S, q, z), \quad \forall (\mathbf{v}, S, q, z) \in \mathcal{U}_{ad}$$

satisfying

$$||\mathbf{u} - \mathbf{v}||_1 + ||T - S||_1 + ||p - q||_0 + ||g - z||_{0,\Gamma_C} \leq \epsilon \tag{3.7}$$

We now show the existence of optimal solution. The existence of an optimal solution can be proved based on the *a priori* estimates (2.13) and (2.14) and standard techniques.

Theorem 3.1. *There is an optimal solution* $(\mathbf{u}, T, p, g) \in \mathcal{U}_{ad}$ *to the problem (3.7).*

Proof. $\mathcal{U}_{ad}$ is apparently nonempty because of lemma 2.2. Thus we may choose a minimizing sequence $\{\mathbf{u}^{(n)}, T^{(n)}, p^{(n)}, g^{(n)}\}$ in $\mathcal{U}_{ad}$ such that

$$\lim_{n\to\infty} \mathcal{J}(\mathbf{u}^{(n)}, T^{(n)}, p^{(n)}, g^{(n)}) = \inf_{(\mathbf{v},S,q,z)\in\mathcal{U}_{ad}} \mathcal{J}(\mathbf{v}, S, q, z). \tag{3.8}$$

By the definition of $\mathcal{U}_{ad}$

$$Pr\ a(\mathbf{u}^{(n)}, \mathbf{v}) + c(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}) + b(\mathbf{v}, p^{(n)}) \\ = Pr\ Ra\ d(T^{(n)}, \mathbf{v}) + < \mathbf{f}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega_f), \tag{3.9}$$

$$b(\mathbf{u}^{(n)}, q) = 0 \quad \forall q \in L^2_0(\Omega_f). \tag{3.10}$$

$$A(T^{(n)}, S) + C(\mathbf{u}^{(n)}, T^{(n)}, S) = < Q, S > + \kappa_f(g^{(n)}, S) \quad \forall S \in H^1_0(\Omega), \tag{3.11}$$

and

$$T = T_b \quad \text{on } \Gamma_D. \tag{3.12}$$

From (3.1), we easily see that $\{||g^{(n)}||_{0,\Gamma_C}\}$ is uniformly bounded. Also, by (2.13) and (2.14) we have that $\{||\mathbf{u}^{(n)}||_1\}$, $\{||T^{(n)}||_1\}$ and $\{||p^{(n)}||_0\}$ are uniformly bounded. We may then extract subsequences such that

$$\begin{gathered} g^{(n)} \rightharpoonup g \text{ in } L^2(\Gamma_C), \\ \mathbf{u}^{(n)} \rightharpoonup \mathbf{u} \text{ in } \mathbf{H}_0^1(\Omega_f) \quad \text{and} \quad \nabla\mathbf{u}^{(n)} \rightharpoonup \nabla\mathbf{u} \text{ in } \mathbf{L}^2(\Omega_f), \\ T^{(n)} \rightharpoonup T \text{ in } H^1(\Omega) \quad \text{and} \quad \nabla T^{(n)} \rightharpoonup \nabla T \text{ in } \mathbf{L}^2(\Omega), \\ p^{(n)} \rightharpoonup p \text{ in } L^2(\Omega_f), \\ \mathbf{u}^{(n)} \to \mathbf{u} \text{ in } \mathbf{L}^4(\Omega_f), \end{gathered}$$

for some $(\mathbf{u}, T, p, g) \in \mathbf{H}_0^1(\Omega_f) \times H^1(\Omega) \times L_0^2(\Omega_f) \times L^2(\Gamma_C)$. The last convergence result above follows from the compact embedding $\mathbf{H}^1(\Omega_f) \hookrightarrow\hookrightarrow \mathbf{L}^4(\Omega_f)$. We may pass to the limit in (3.9)-(3) to determine that $(\mathbf{u}, T, p, g)$ satisfies (3.2)-(3.5). Indeed, the only troublesome term when one passes to the limit is the nonlinearity $c(\cdot,\cdot,\cdot)$. However, note that

$$c(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}) = \int_\Gamma (\mathbf{u}^{(n)} \cdot \mathbf{n})\mathbf{u}^{(n)} \cdot \mathbf{v} \, d\Gamma - \int_\Omega \mathbf{u}^{(n)} \cdot \nabla\mathbf{v} \cdot \mathbf{u}^{(n)} \, d\Omega \quad \forall C^\infty(\bar{\Omega}_f).$$

Then, since $\mathbf{u}^{(n)} \to \hat{\mathbf{u}}$ in $\mathbf{L}^2(\Omega_f)$ and $\mathbf{u}^{(n)}|_\Gamma \to \hat{\mathbf{u}}|_\Gamma$ in $\mathbf{L}^2(\Gamma)$, we have that

$$\begin{aligned} \lim_{k\to\infty} c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) &= \int_\Gamma (\hat{\mathbf{u}} \cdot \mathbf{n})\hat{\mathbf{u}} \cdot \mathbf{v} \, d\Gamma - \int_\Omega \hat{\mathbf{u}} \cdot \nabla\mathbf{v} \cdot \hat{\mathbf{u}} \, d\Omega \\ &= c(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in C^\infty(\bar{\Omega}_f). \end{aligned}$$

Then, since $C^\infty(\bar{\Omega}_f)$ is dense in $\mathbf{H}^1(\Omega)$, we also have that

$$\lim_{k\to\infty} c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = c(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_f).$$

Thus we have shown that $(\mathbf{u}, T, p, g)$ indeed satisfies (3.2)-(3.5) so that $(\mathbf{u}, T, p, g) \in \mathcal{U}_{ad}$.

Finally, it is easy to see that $\mathcal{J}(\cdot,\cdot,\cdot,\cdot)$ is weakly lower semicontinuous so that

$$\mathcal{J}(\mathbf{u}, T, p, g) = \inf_{(\mathbf{v},S,q,z)\in\mathcal{U}_{ad}} \mathcal{J}(\mathbf{v}, S, q, z). \tag{3.13}$$

Thus an optimal solution belonging to $\mathcal{U}_{ad}$ exists. □

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DEPARTMENT OF MATHEMATICS, AJOU UNIVERSITY SUWON, KOREA
E-mail address: hclee@madang.ajou.ac.kr

EXISTENCE AND MULTIPLICITY RESULTS OF POSITIVE RADIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS IN AN EXTERIOR DOMAIN

YONG-HOON LEE

ABSTRACT. We prove various aspects about the existence of positive radial solutions for semilinear elliptic problems in an exterior domain using the method of upper and lower solutions and several fixed point index arguments.

1. INTRODUCTION

In this paper, we consider the nonexistence, existence, uniqueness or multiplicity of positive radial solutions for semilinear elliptic problems of the form

$$\Delta u + \mu g(|x|)f(u(x)) = 0, \text{ in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{1.2}$$

$$u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{1.3}$$

where $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\},\ r_o > 0,\ n \geq 3$ and μ is a positive real parameter.

We introduce some terminology to facilitate the statement of propositions. We say that given problem (P) holds *Prop A for solutions* if there exists $\mu_f > 0$ such that (P) has at least two solutions, at least one solution or none according to $0 < \mu < \mu_f$, $\mu = \mu_f$ or $\mu > \mu_f$, (P) holds *Prop B for solutions* if there exists $\mu_f > 0$ such that (P) has at least one solution or none according to $0 < \mu < \mu_f$ or $\mu > \mu_f$, and we say that (P) holds *Prop C for solutions* if (P) has a solution for all $\mu > 0$.

Now let us give some conditions on g and f for precise description.

(H_1) $g : [r_o, \infty) \to (0, \infty)$ is continuous and $\int_{r_o}^{\infty} rg(r)dr < \infty$.

(H_2) $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, $f_0 \equiv \lim_{u \to 0^+} \frac{f(u)}{u} = 0$ and $f_\infty \equiv \lim_{u \to \infty} \frac{f(u)}{u} = \infty$.

(H_3) $f : [0, \infty) \to (0, \infty)$ is continuous, nondecreasing and $f_\infty = \infty$.

For Ω an annulus, problem (1.1),(1.2) has been studied by Bandle, Coffman and Marcus [1], Garazia [4], Lin [10], Santanilla [14], Nagasaki and Suzuki [11] and Pacard [13].

Among them, Lin [10] considered the problem when $g \equiv 1$ and proved that the problem holds Prop A for positive radial solutions if $f > 0$ on $[0, \infty)$ and $f_\infty = \infty$, and the problem holds Prop C for positive radial solutions if f satisfies (H_2).

For Ω an exterior domain, works related to problem (1.1), (1.2) include Noussair and Swanson [12], Bandle and Marcus [2], Santanilla [15] and Ha and Lee [6].

In particular, when g satisfies (H_1) and $f(u) = e^u$, Ha and Lee [6] proved that problem (1.1)~(1.3) holds Prop A for positive radial solutions.

In the present work, we mainly present similar results as Lin [10] that under assumptions (H_1) and (H_2), problem (1.1)~(1.3) holds Prop A for positive radial solutions and under assumptions (H_1) and (H_3), problem (1.1)~(1.3) holds Prop C for positive radial solutions.

Since we are looking for radial solutions, we may reduce problem (1.1)~(1.3), via suitable transformations to the following problems of ordinary differential equations

$$u''(t) + \mu q(t)f(u(t)) = 0, \quad 0 < t < 1 \tag{1.4}$$

$$u(0) = 0 = u(1), \tag{1.5}$$

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where q is singular at 1.

Problem (1.4),(1.5) itself also has some references, for example, Choi [3] proved that the problem holds Prop B for positive solutions when $f(u) = e^u$, and q is singular at 0 with certain growth restriction. Wong [16] generalized Choi's result that the problem holds Prop B of positive solutions if f satisfies (H_2'), together with a condition like (b_2) in Section 4 and q is singular at 0 with somewhat stronger growth restriction than Choi's. Ha and Lee [6] also generalized Choi's result that the problem holds Prop A for positive solutions if $f(u) = e^u$, q is singular at 0 with $\int_0^1 sq(s)ds < \infty$. Zhang [17] proved that the problem holds Prop C for positive solutions if $f(u) = u^p$, $0 < p < 1$ and q is singular at 0 and 1 with $\int_0^1 s(1-s)q(s)ds < \infty$.

As we indicated in the introductory comment for problem (1.1)~(1.3), we shall prove that problem (1.4), (1.5) holds Prop A (Prop C) for positive solutions if f satisfies (H_2) $\big((H_3)\big)$ and q is singular at 0 and/or 1 with a suitable integrability condition. These extend the results of Choi, Wong and Ha and Lee, and provide results we expect for problem (1.1)~(1.3). Our techniques of proofs for problem (1.4),(1.5) mainly use the method of upper and lower solutions and several fixed point index theories.

2. PRELIMINARIES

Let us consider

$$\Delta u + \mu g(|x|)f(u(x)) = 0, \quad \text{in } |x| > r_o \tag{1.1}$$

$$u = 0, \quad \text{if } |x| = r_o, \tag{1.2}$$

$$u \to 0, \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $r_o > 0$ and $n \geq 3$.

Since we are concerned with radial solutions, applying series of transformations $r = |x|$, $s = r^{2-n}$ and $t = (r_o^{2-n} - s)/r_o^{2-n}$, we can rewrite problem (1.1)~(1.3) as

$$\begin{aligned} &z''(t) + \mu q(t)f(z(t)) = 0, \\ &z(0) = 0 = z(1), \end{aligned}$$

where $q(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} g\big(r_o(1-t)^{\frac{-1}{n-2}}\big)$. Thus by (H_1), $q : [0,1) \to (0,\infty)$ is continuous and singular at 1 satisfying $\int_0^1 (1-s)q(s)ds < \infty$, and problem (1.1)~(1.3) with condition (H_1) are reduced to problem (1.4),(1.5) with q and conditions on q described above.

We now present a theorem on upper and lower solutions for the singular problems we are dealing with. Consider the problem

$$\begin{aligned} &u''(t) + F(t, u(t)) = 0, \\ &u(0) = a, \ u(1) = b, \end{aligned} \tag{2.1}$$

where $F : D \to \mathbf{R}$ is a continuous function and $D \subset (0,1) \times \mathbf{R}$. A solution $u(\cdot)$ of (2.1) means a function $u \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ such that $(t, u(t)) \in D$ for all $t \in (0,1)$ and $u''(t) + F(t,u(t)) = 0$ for all $t \in (0,1)$ with $u(0) = a$ and $u(1) = b$.

Definition 2.1. $\alpha \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ is called a *lower solution* of (2.1) if $(t,\alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{aligned} &\alpha''(t) + F(t,\alpha(t)) \geq 0, \ t \in (0,1) \\ &\alpha(0) \leq a, \ \alpha(1) \leq b. \end{aligned}$$

We also define an upper solution $\beta \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ if β satisfies the reverse of the above inequalities.

If α and $\beta \in C([0,1],\mathbf{R})$ are such that $\alpha(t) \leq \beta(t)$, for all $t \in [0,1]$, we define the set $D_\alpha^\beta = \{(t,x) \in (0,1) \times \mathbf{R} : \alpha(t) \leq x \leq \beta(t)\}$. The following is a fundamental theorem of the method of upper and lower solutions for problem (2.1) due to Habets and Zanolin [7].

Proposition 2.1. *Let α and β be a lower and an upper solution for (2.1) such that*
(a_1) $\alpha(t) \le \beta(t)$ *for all* $t \in [0,1]$
(a_2) $D_\alpha^\beta \subset D$.
Assume also that there is a function $h \in C((0,1),(0,\infty))$ *such that*
(a_3) $|F(t,x)| \le h(t)$, *for all* $(t,x) \in D_\alpha^\beta$ *and*
(a_4) $\int_0^1 s(1-s)h(s)ds < \infty$.
Then (2.1) has at least one solution u such that

$$\alpha(t) \le u(t) \le \beta(t), \quad \text{for all } t \in [0,1].$$

Remark. It is easy to see that if we assume, instead of (a_4), the condition $\int_0^1 sh(s)ds < \infty$, then the solution u we find belonds to $C^1((0,1])$. Similarly, if $\int_0^1 (1-s)h(s)ds < \infty$, then $u \in C^1([0,1))$.

We also state some properties for fixed point index which are well known and crucial in our arguments, see Guo and Lakshmikantham [5] for proof and further discussion of the fixed point index.

Proposition 2.2. *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets in E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \ u \in K \cap \partial\Omega_1 \ \text{ and } \ \|Tu\| \ge \|u\|, \ u \in K \cap \partial\Omega_2.$$

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Proposition 2.3. *Let E be a Banach space, K a cone in E and Ω bounded open in E. Let $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Tx \ne \lambda x$, for all $x \in K \cap \partial\Omega$ and all $\lambda \ge 1$. Then*

$$i(T, K \cap \Omega, K) = 1.$$

Proposition 2.4. *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is a compact map such that $Tx \ne x$ for $x \in \partial K_r$. If $\|x\| \le \|Tx\|$, for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

3. EXISTENCE AND UNIQUENESS

We expect Prop C for positive solutions, if f satisfies (H_2). Therefore, it is enough to consider the following problem

$$u''(t) + q(t)f(u(t)) = 0, \quad 0 < t < 1 \tag{3.1}$$

$$u(0) = 0 = u(1). \tag{1.5}$$

For more general approach, we assume that q is singular at 0 and 1. Our main existence result is

Theorem 3.1. *Assume (H_2) and*
(H) $q \in C((0,1),(0,\infty))$ *satisfies* $\int_0^1 s(1-s)q(s)ds < \infty$.
Then (3.1), (1.5) has at least one positive solution.

Proof. First, it is well known that the problem (3.1), (1.5) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)q(s)g(u(s))ds,$$

where $G(t,s)$ is the Green's function corresponding to the linear homogeneous problem explicitly written by

$$G(t,s) = \begin{cases} s(1-t) & \text{for } 0 \le s \le t \\ t(1-s) & \text{for } t \le s \le 1. \end{cases}$$

Thus (3.1), (1.5) is equivalent to the fixed point equation

$$u = Tu$$

in $E = C([0,1])$, where $T : E \to E$ is given by

$$Tu(t) = \int_0^1 G(t,s)q(s)g(u(s))ds.$$

By the condition (H), T is completely continuous on the cone of nonnegative functions in E. We define a cone K in E by

$$K = \{u \in E | u(t) \geq 0,\ t \in [0,1],\ \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{4}\|u\|_\infty\}.$$

Then it is not hard to check $T(K) \subset K$.

Second, by (H), we may choose $\eta > 0$ so that $\eta \int_0^1 s(1-s)q(s)ds \leq 1$. Since $f_0 = 0$, there exists $R_1 > 0$ such that $g(u) \leq \eta u$, for $0 < u \leq R_1$. Let $\Omega_1 = \{u \in E : \|u\|_\infty < R_1\}$, then Ω_1 is bounded open in E and $0 \in \Omega_1$. Moreover, let $u \in K \cap \partial\Omega_1$, then $u \in K$, $\|u\|_\infty = R_1$, and thus

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)q(s)f(u(s))ds \\
&\leq \int_0^1 s(1-s)q(s)f(u(s))ds \\
&\leq \eta \int_0^1 s(1-s)q(s)u(s)ds \\
&\leq \eta \int_0^1 s(1-s)q(s)\|u\|_\infty ds \leq \|u\|_\infty.
\end{aligned}$$

Therefore

$$\|Tu\|_\infty \leq \|u\|_\infty, \quad \text{for all } u \in K \cap \partial\Omega_1.$$

Third, choose $\mu > 0$ such that $\frac{\mu}{4}\int_{1/4}^{3/4} G(\frac{1}{2}, s)ds > 1$. Since $f_\infty = \infty$, there exists $R > 0$ such that $q_o f(u) \geq \mu u$, for all $u \geq R$, where $q_o = \min_{t \in [1/4,3/4]} q(t)$. Let $R_2 = \max\{2R_1, 4R\}$ and $\Omega_2 = \{u \in E : \|u\|_\infty < R_2\}$, then Ω_2 is bounded open in E and $\bar{\Omega}_1 \subset \Omega_2$. We show $\|Tu\|_\infty \geq \|u\|_\infty$, for all $u \in K \cap \Omega_2$, so let $u \in K$ and $\|u\|_\infty = R_2$, then $\min_{t \in [1/4,3/4]} u(t) \geq 1/4\|u\|_\infty \geq R$. Thus $q(t)f(u(t)) \geq \mu u(t)$, for all $t \in [1/4, 3/4]$ and

$$\begin{aligned}
Tu(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2}, s)q(s)f(u(s))ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)q(s)f(u(s))ds \\
&\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)u(s)ds \\
&\geq \frac{\mu}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2}, s)\|u\|_\infty ds \\
&> \|u\|_\infty.
\end{aligned}$$

Therefore $\|Tu\|_\infty \geq \|u\|_\infty$, for $u \in K \cap \partial\Omega_2$, and by Proposition 2.2, T has a fixed point u in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $R_1 \leq \|u\|_\infty \leq R_2$. Futhermore, since $G(t,s)q(s) > 0$ for all $s \in (0,1)$, it follows that $u > 0$ on $(0,1)$, and this completes the proof.

We have a similar result as Theorem 3.1 when q is singular at 0. The proof generally follows that of Theorem 3.1, and it is enough to check the second part in the proof of Theorem 3.1.

Theorem 3.2. ([8]) *Assume* (H_2) *in Theorem 3.1 and*
(H') $q \in C((0,1],(0,\infty))$ *satisfies* $\int_0^1 sq(s)ds < \infty$.
Then $(3.1),(1.5)$ *has at least one positive solution.*

Since the problem having the singularity at 0 and the problem having the singularity at 1 are equivalently transformed, we get the following corollary;

Corollary 3.1. *Assume* (H_2) *in Theorem 3.1 and*
(H'') $q \in C([0,1),(0,\infty))$ *satisfies* $\int_0^1 (1-s)q(s)ds < \infty$.
Then $(3.1),(1.5)$ *has at least one positive solution.*

We now state the uniqueness of positive solution for problem $(3.1),(1.5)$. Let u be a positive solution of $(3.1),(1.5)$ and let $L_u = \max_{t\in[0,1]} g(u(t))$. Then $q(t)g(u(t)) \leq L_n q(t)$ and $\int_0^1 |u''(t)|dt \leq L_n \int_0^1 q(t)dt < \infty$, provided by $q \in L^1[0,1]$. Thus both $u'(0^+)$ and $u'(1^-)$ exist and consequently, all positive solutions of $(3.1),(1.5)$ are of $C^1[0,1]\cap C^2(0,1)$. Based on this fact, we obtain the existence of a unique positive solution for $(3.1),(1.5)$ as follows. One may refer to Lee [8] for the proof.

Theorem 3.3. ([8]) *Assume* (H_2) *and also assume*
(H''') $q \in C((0,1),(0,\infty))$ *satisfies* $\int_0^1 q(s)ds < \infty$.
(H_4) *f is increasing and* $\frac{f(u)}{u}$ *is strictly monotone.*
Then $(3.1),(1.5)$ *has a unique positive solution.*

Let $p > 1$ and let $f(u) = u^p$. Then f obviously satisfies (H_2) in Theorem 3.1 and (H_4) in Theorem 3.3. Thus we obtain the existence and uniqueness results of positive solutions for the following Emden-Fowler problem

$$u''(t) + q(t)u(t)^p = 0, \quad 0 < t < 1 \tag{3.2}$$
$$u(0) = 0 = u(1). \tag{1.5}$$

Corollary 3.2. *Let* $p > 1$ *and assume* (H) *in Theorem 3.1.*
Then $(3.2),(1.5)$ *has at least one positive solution.*
Moreover, assume (H''') *in Theorem 3.2.*
Then $(3.2),(1.5)$ *has a unique positive solution.*

Let us consider the semilinear elliptic problems of the form;

$$\Delta u + |x|^{-\lambda} g(|x|) f(u(x)) = 0, \text{ in } \Omega, \tag{3.3}$$
$$u = 0, \qquad \text{if } |x| = r_o, \tag{1.2}$$
$$u \to 0 \qquad \text{as } |x| \to \infty, \tag{1.3}$$

where $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\}$ and $n \geq 3$. For any real number λ satisfying $\lambda < 2(n-1)$, we prove the existence of positive radial solutions for $(3.3) \sim (1.3)$ if f satisfies (H_2) and g satisfiesthe following condition;
(H_1') $g \in C([r_o,\infty),(0,\infty))$ satisfies $\int_{r_o}^\infty x^{1-\lambda} g(x)dx < \infty$.
By the transformations in Section 2, we can rewrite $(3.3) \sim (1.3)$ as (1.4),(1.5) with

$$q(t) = \frac{r_o^{2-\lambda}}{(n-2)^2}(1-t)^{\frac{-2(n-1)+\lambda}{n-2}} f(r_o(1-t)^{\frac{-1}{n-2}}).$$

Thus by (H_1'), $q : [0,1) \to (0,\infty)$ is continuous and singular at 1 satisfying $\int_0^1 (1-s)q(s)ds < \infty$. Therefore by Corollary 3.1, we obtain an existence result for problem $(3.3) \sim (1.3)$ as follows.

Corollary 3.3. *Let* $\lambda < 2(n-1)$ *and assume* (H_1') *and* (H_2).
Then $(3.3) \sim (1.3)$ *has at least one positive radial solution for all* $0 < r_o < \infty$.

It is easy to check that if $\int_{r_o}^\infty x^{n-1-\lambda} f(x)dx < \infty$, then $\int_0^1 q(s)ds < \infty$. Thus we obtain a uniqueness result for the problem too.

Corollary 3.4. *Let $\lambda < 2(n-1)$, and assume (H_2) and (H_4). Moreover, assume*
(H_1'') $g \in C([r_o, \infty), (0, \infty))$ *satisfies* $\int_{r_o}^{\infty} x^{n-1-\lambda} g(x)dx < \infty$.
Then $(3.3) \sim (1.3)$ *has a unique positive radial solution for all* $0 < r_o < \infty$.

Example 3.1. If $f(u) = u^p$, $p > 1$ or $f(u) = e^u - u$, then $(3.3) \sim (1.3)$ has at least one positive radial solution or a unique positive radial solution provided $\int_{r_o}^{\infty} x^{1-\lambda} g(x)dx < \infty$ or $\int_{r_o}^{\infty} x^{n-1-\lambda} g(x)dx < \infty$, respectively.

4. NONEXISTENCE, EXISTENCE AND MULTIPLICITY

Let us consider the problem

$$(1.4) \qquad u''(t) + \mu q(t) f(u(t)) = 0, \quad 0 < t < 1$$
$$(4.1_\lambda) \qquad u(0) = \lambda = u(1),$$

where $\lambda \geq 0$. We first state an existence theorem for the above problem.

Theorem 4.1. *Assume* (H) *and also assume*
(b_1) $f : [0, \infty) \to (0, \infty)$ *is a continuous.*
(b_2) $I_0 \equiv \sup_{c \in (0,\infty)} \int_0^c \frac{du}{\sqrt{F(c) - F(u)}} < \infty$, *where* $F(u) = \int_0^u f(s)ds$.
Then for each $\lambda \geq 0$, *there exists a positive real number* μ_λ *such that* $(1.4), (4.1_\lambda)$ *holds Prop B for positive solutions.*

We need a lemma to prove the theorem. Let us consider the problem with k a positive real parameter.

$$(4.2) \qquad u''(t) + k f(u(t)) = 0, \quad 0 < t < 1$$
$$(4.3) \qquad u(0) = a \geq 0, \ u(1) = b \geq 0.$$

Lemma 4.1. *If* f *satisfies* (b_1) *and* (b_2), *then* $(4.2), (4.3)$ *does not have a positive solution for all* $k \geq 2{I_0}^2 + 1$.

Proof. Let $k \geq 2I_0^2 + 1$ and suppose that $(4.2), (4.3)$ has a positive solution $u(t)$, and let $u_o \equiv u(t_o) = \max_{t \in [0,1]} u(t)$. Then we obtain

$$\sqrt{2k} = \int_a^{u_o} \frac{du}{\sqrt{F(u_o) - F(u)}} + \int_b^{u_o} \frac{du}{\sqrt{F(u_o) - F(u)}}$$
$$\leq 2 \int_0^{u_o} \frac{du}{\sqrt{F(u_o) - F(u)}}$$

Thus

$$(4.4) \qquad k \leq 2\Big(\int_0^{u_o} \frac{du}{\sqrt{F(u_o) - F(u)}}\Big)^2 \leq 2I_0^2.$$

This contradicts to $k \geq 2I_0^2 + 1$. Similarly, we can get contradictions when u attains its maximum at $t = 0$ or $t = 1$ and the proof is done.

Proof of Theorem 4.1.. First, we show that for each $\lambda \geq 0$, $(1.4), (4.1_\lambda)$ has a positive solution for some μ using the method of upper and lower solutions. $\beta(t) = \lambda + \int_0^1 G(t,s) q(s) ds$, the solution of

$$u''(t) + q(t) = 0, \quad 0 < t < 1$$
$$u(0) = \lambda = u(1)$$

satisfies

$$\beta''(t) + \mu q(t) f(\beta(t)) = q(t)(\mu f(\beta(t)) - 1) \leq 0,$$

for $\mu \leq \frac{1}{M_\beta}$, where $M_\beta = \max_{t \in [0,1]} f(\beta(t))$, and $G(t,s)$ the Green function given in Section 3. This shows that $\beta(t)$ is an upper solution of $(1.4), (4.1_\lambda)$ for $\mu \leq \frac{1}{M_\beta}$. On the other hand, $\alpha \equiv \lambda$ is obviously a lower solution of (1.4), (4.1_λ) and $\alpha(t) \le \beta(t)$ for all $t \in [0,1]$. Thus by Proposition 2.1, the problem has a solution for $0 < \mu \le \frac{1}{M_\beta}$. Let $\mu_1 > 0$ be fixed and assume that (1.4), (4.1_λ) has a positive solution u_1 for $\mu = \mu_1$, then for all $\mu \in (0, \mu_1)$, the problem also has a positive solution, since u_1 and λ are upper and lower solutions respectively. For fixed λ, let $\mu_\lambda = \sup\{\mu > 0 : (1.4), (4.1_\lambda)$ has a positive solution for $\mu\}$, then $\mu_\lambda \ge \frac{1}{M_\beta}$. We show that $\mu_\lambda < \infty$. Suppose $\mu_\lambda = \infty$, for some $\lambda \ge 0$, then we may choose a sequence of parameters (μ_n) with $\mu_n \to \infty$ such that the problem has a positive solution u_n for each μ_n. Consider the following equation on the interval $[\frac{1}{4}, \frac{3}{4}]$,

$$y''(t) + \mu_n q_o f(y(t)) = 0, \quad t \in [\frac{1}{4}, \frac{3}{4}] \tag{4.5}$$

$$y(\frac{1}{4}) = u_n(\frac{1}{4}),\ y(\frac{3}{4}) = u_n(\frac{3}{4}), \tag{4.6}$$

where $q_o = \min_{1/4 \le t \le 3/4} q(t)$. Then u_n and α, the straight line connecting $(\frac{1}{4}, u_n(\frac{1}{4}))$ and $(\frac{3}{4}, u_n(\frac{3}{4}))$ are upper and lower solutions of (4.5), (4.6) respectively. Thus for each n, (4.5), (4.6) has a positive solution. Since $\lim_{n\to\infty} \mu_n = \infty$, the above conclusion contradicts to Lemma 4.1 and the proof is done.

Under the assumptions of Theorem 4.1, we know that (1.4), (1.5) has at least one solution for $\mu \in (0, \mu_0)$. The existence of the second solution for the same value of μ will be proved, under additional conditions on f, by using fixed point index arguments. The first step in this direction is to prove *a priori* boundedness of possible positive solutions for (1.4), (1.5). Define $I(c)$ as

$$I(c) = \int_0^c \frac{du}{\sqrt{F(c) - F(u)}}, \quad \text{where } F(u) = \int_0^u f(s)ds.$$

One may refer to Lee [9] that condition (H_3) implies that $I(0) = 0$, $I_0 < \infty$, $I(c) \to 0$ as $c \to \infty$, and that there is no point $c > 0$ such that $I(c) = 0$. Therefore,

$$\frac{1}{I(c)} < \infty \text{ for } c > 0 \quad \text{and} \quad \frac{1}{I(c)} \to \infty \text{ as } c \to \infty \text{ or } c \to 0. \tag{4.7}$$

Lemma 4.2. *Assume (H_3). Let $k > 0$ be fixed and assume that* (4.2), (4.3) *has a positive solution for k, then there exists $M(k) > 0$ such that for all $k^* \ge k$ and for all possible positive solutions u of* (4.2), (4.3) *for k^*, one has*

$$\|u\|_\infty < M(k).$$

Proof. Let k be fixed and assume that (4.2), (4.3) has a positive solution. Let $k^* \ge k$ and u be a positive solution of the problem for k^*. Then by (4.4),

$$\frac{1}{I(u_o)} \le \sqrt{\frac{2}{k}},$$

where $u_o = \max_{t\in[0,1]} u(t)$. The above inequality and (4.7) imply that u_o is bounded above and its upper bound depends of k, but not of k^*. The proof is complete.

Lemma 4.3. *Assume (H) and (H_3). Let $\mu > 0$ be fixed and assume that* (1.4), (1.5) *has a positive solution, then there exists $M_\mu > 0$ such that for all $\mu^* \ge \mu$ and for all possible positive solutions u of* (1.4), (1.5) *for μ^*, one has*

$$\|u\|_\infty < M_\mu.$$

Proof. Let μ be given and assume that (1.4), (1.5) has a positive solution for μ. Suppose that the conclusion is not true, then we may choose a sequence (μ_n), not necessarily distinct, and positive solution u_n of (1.4),(1.5) for each μ_n such that $\mu_n \ge \mu$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Consider problem (4.5), (4.6) again. By similar arguments in the proof of Theorem 4.1, for each μ_n, (4.5), (4.6) has a positive solution y_n. Since $\lim_{n\to\infty} \|u_n\|_\infty = \infty$, $\lim_{n\to n} \|y_n\|_\infty = \infty$ and this contradicts to Lemma 4.2.

We now state the main theorem.

Proposition 4.1. ([6]) *Let $f : [0,\infty) \to (0,\infty)$ be continuous and nondecreasing. Let μ and μ_o, $0 < \mu < \mu_o$, and $M > 0$ be given. Then there exist $\bar{\mu} \in (\mu, \mu_o)$ and $\lambda_o \in (0,1)$ such that*

$$\mu f(u+\lambda) < \bar{\mu} f(u),$$

for all $u \in [0, M]$ and all $\lambda \in (0, \lambda_o)$.

The proof is simply done by uniform continuity of f on $[0, M+1]$.

Lemma 4.4. *Assume (H) and (H_3). Let $\mu \in (0, \mu_0)$ be given, where μ_0 is in Theorem 4.1. Then there exists $\lambda > 0$ such that $(1.4), (4.1_\lambda)$ has a positive solution for given μ and λ.*

Proof. Let $\mu \in (0, \mu_0)$ be given, and let u be a positive solution of $(1.4), (1.5)$ known to exist by Theorem 4.1. For μ, μ_0 and M_μ given in Lemma 4.3, we may choose $\bar{\mu} \in (\mu, \mu_0)$ and $\lambda > 0$, by Proposition 4.1, such that

$$\mu f(u+\lambda) < \bar{\mu} f(u),$$

for all $u \in [0, M_\mu]$. We know by Theorem 4.1 that $(1_{\bar{\mu}})$ has a positive solution $\bar{u}$ which, without loss of generality, satisfies $u(t) \leq \bar{u}(t)$ for all $t \in [0,1]$. It also satisfies by Lemma 4.3 that $0 \leq \bar{u}(t) < M_\mu$ on $[0,1]$. Let $u_\lambda(t) = \bar{u}(t) + \lambda$, then u is a lower solution of $(1.4), (4.1_\lambda)$ for μ and λ. On the other hand,

$$\begin{aligned} u_\lambda''(t) + \mu q(t) f(u_\lambda(t)) &= \bar{u}''(t) + \mu q(t) f(\bar{u}(t) + \lambda) \\ &= q(t)\big[\mu f(\bar{u}(t)+\lambda) - \bar{\mu} f(\bar{u}(t))\big] \leq 0 \end{aligned}$$

and $u_\lambda(0) = \lambda = u_\lambda(1)$. Thus u_λ is an upper solution of $(1.4), (4.1_\lambda)$ and obviously $u(t) < u_\lambda(t)$ on $[0,1]$. Therefore $(1.4), (4.1_\lambda)$ has a solution between u, and u_λ and the proof is complete.

We can set up an operator equation for fixed point arguments by the same way as in Section 3 so that problem (1.4),(1.5) is equivalent to the fixed point equation $u = Tu$ in $E = C([0,1])$, where $T : E \to E$ is given by $Tu(t) = \mu \int_0^1 G(t,s) q(s) f(u(s)) ds$. Then T is completely continuous on the cone of nonnegative functions in E and defining again a cone K in E by $K = \{u \in E | u(t) \geq 0,\ t \in [0,1]\ \min_{t \in [\frac14, \frac34]} u(t) \geq \frac14 \|u\|_\infty\}$. We get $T(K) \subset K$.

Proof of Theorem 4.2. Let $\mu \in (0, \mu_0)$ be given, then by Lemma 4.4, there exists $\lambda > 0$ such that $(1.4), (4.1_\lambda)$ has a positive solution u_λ for given μ and λ. Let $\Omega = \{u \in X : -M_\mu < u(t) < u_\lambda(t),\ t \in [0,1]\}$, where M_μ is given in Lemma 4.3, then Ω is bounded open in E, $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ is condensing, since it is completely continuous. Let $u \in K \cap \partial\Omega$, then there exists $t_o \in [0,1]$ such that $u(t_o) = u_\lambda(t_o)$ and

$$\begin{aligned} 0 \leq Tu(t_o) &= \int_0^1 \mu G(t_o, s) q(s) f(u(s)) ds \\ &< \lambda + \int_0^1 \mu G(t_o, s) q(s) f(u_\lambda(s)) ds \\ &= u_\lambda(t_o) \leq \nu u_\lambda(t_o) = \nu u(t_o), \end{aligned}$$

for all $\nu \geq 1$. Thus $Tu \neq \nu u$, for all $u \in K \cap \partial\Omega$ and all $\nu \geq 1$ and by Proposition 2.3,

$$i(T, K \cap \Omega, K) = 1.$$

Next, let us choose $M > 0$ such that

$$\frac{M}{4} \int_{\frac14}^{\frac34} \mu G(\frac12, s) ds > 1,$$

and let $q_o = \min_{t \in [\frac14, \frac34]} q(t)$. Since $f_\infty = \infty$, we may choose $R_1 > 0$ such that $q_o f(u) \geq Mu$, for all $u \geq R_1$. Let $R = \max\{M_\mu, 4R_1, \|u_\lambda\|_\infty\}$, then by Lemma 4.3, $Tu \neq u$ for $u \in \partial K_R$. Furthermore, if $u \in \partial K_R$, then

$$\min_{t \in [\frac14, \frac34]} u(t) \geq \frac14 \|u\|_\infty \geq R_1.$$

Thus $q(t)f(u(t)) \geq q_o f(u(t)) \geq Mu(t)$, for all $t \in [\frac{1}{4}, \frac{3}{4}]$ and

$$\begin{aligned}
Tu(\frac{1}{2}) &= \int_0^1 \mu G(\frac{1}{2}, s)q(s)f(u(s))ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s)q(s)f(u(s))ds \\
&\geq M \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s)u(s)ds \\
&\geq \frac{M}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s)\|u\|_\infty ds \\
&> \|u\|_\infty.
\end{aligned}$$

Therefore $\|Tu\|_\infty \geq \|u\|_\infty$ and by Proposition 2.4,

$$i(T, K_R, K) = 0.$$

Consequently by the additivity of the fixed point index,

$$0 = i(T, K_R, K) = i(T, K \cap \Omega, K) + i(T, K_R \setminus \overline{K \cap \Omega}, K).$$

Since $i(T, K \cap \Omega, K) = 1$, $i(T, K_R \setminus \overline{K \cap \Omega}, K) = -1$ and thus, T has a fixed point on $K \cap \Omega$ and another on $K_R \setminus \overline{K \cap \Omega}$.

Finally, to show the existence of a solution at $\mu = \mu_0$, choose an increasing sequence (μ_n) such that $\mu_n \to \mu_0$ and each (1.4), (1.5) has a positive solution for $\mu = \mu_n$. Let u_n be a solution of (1.4), (1.5) for μ_n, then by Lemma 4.3 and Arzela-Ascoli Theorem, (u_n) has a subsequence converging to $u \in C[0,1]$. Writing (1) in integrating form and applying Lebesgue Convergence Theorem, we can easily show that u is a solution of (1.4), (1.5) for μ_0 and the proof is complete.

Example 4.1. Let us consider the problem

$$\begin{aligned}
&u''(t) + \mu q(t)(u(t)^p + \epsilon) = 0, \quad 0 < t < 1 \\
&u(0) = 0 = u(1),
\end{aligned}$$

where $p > 1$ and $\epsilon > 0$. Let $f(u) = u^p + \epsilon$, then f satisfies (H_3), thus by Theorem 4.2, there exists $\mu_0 > 0$ such that problem has at least two positive solution for $0 < \mu < \mu_0$, at least one positive solution for $\mu = \mu_0$, and no solution for $\mu > \mu_0$, if q satisfies (H).

If q is singular at $t = 0$ $(t = 1)$, then Theorem 4.2 is valid modifying (H_1) suitably. We state the facts as the following corollary without proof.

Corollary 4.1. *Assume (H_3) and if q is singular at 0 and satisfies*

(H') $\quad q : (0, 1] \to (0, \infty)$ *is continuous and* $\int_0^1 sq(s)ds < \infty$.

Then problem (1.4),(1.5) holds Prop A for positive solutions.

Similarly, assume (H_3) and if q is singular at 1 and satisfies

(H'') $\quad q : [0, 1) \to (0, \infty)$ *is continuous and* $\int_0^1 (1-s)q(s)ds < \infty$.

Then problem (1.4),(1.5) holds Prop A for positive solution.

We conclude this section describing one of the aims of this paper for problem (1.1)~(1.3). Consider

(1.1) $\quad \Delta u + \mu g(|x|)f(u(x)) = 0, \quad$ in $|x| > r_o$

(1.2) $\quad u = 0, \quad$ if $|x| = r_o$,

(1.3) $\quad u \to 0, \quad$ as $|x| \to \infty$,

where $r_o > 0$ and $n \geq 3$.

We know that problem (1.1)~(1.3) can be reduced to problem (1.4),(1.5) and condition (H_1) corresponds to condition (H'') via the trasformations we used. Therefore by Corollary 4.1, we obtain the following corollary.

Corollary 4.2. *Assume (H) and (H_3), then problem (1.1)~(1.3) holds Prop A for positive radial solutions.*

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DEPARTMENT OF MATHEMATICS PUSAN NATIONAL UNIVERSITY PUSAN 609-735, KOREA

E-MAIL: YHLEE@ HYOWON.PUSAN.AC.KR

On regularity for heat flows for p-harmonic maps

Masashi Misawa

Department of Computer Science and Information Mathematics,
Faculty of Electro-Communications,
parindent0pt The University of Electro-Communications

We establish a regularity for solutions to the evolution problem associated to p-harmonoc maps, provided the target manifold has nonpositive sectional curvature.

0 Introduction.

Let M, N be compact, smooth orientable Riemannian manifolds of dimension m, l with metrics g, h, respectively and suppose that $\partial M,\ \partial N = \emptyset$. Since N is compact, we assume that N is isometrically embedded into a Euclidean space R^n for some n. For a C^1-map $u : M \to N \subset R^n$, we considera the p-energy functional given by

$$I(u) = \int_M \frac{1}{p} |Du|^p dM, \tag{0.1}$$

where $p \geq 2$ and, in local coordinates on M, with $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, $|g| = \det(g_{\alpha\beta})$ and $D_\alpha = \partial / \partial x^\alpha \quad (\alpha = 1, \cdots, m)$,

$$dM = \sqrt{|g|} dx, \quad |Du|^2 = \sum_{\alpha,\beta=1}^{m} \sum_{i=1}^{n} g^{\alpha\beta} D_\alpha u^i D_\beta u^i.$$

The Euler-Lagrange equation of a variational functional I is

$$-\triangle_p u + A_p(u)(Du, Du) = 0, \tag{0.2}$$

where $\triangle_p$ denotes the differential operator on M

$$\triangle_p u = \frac{1}{\sqrt{|g|}} D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} |Du|^{p-2} D_\beta u \right)$$

and, with the second fundamental form $A(u)(Du, Du)$ of N in R^n at u,

$$A_p(u)(Du, Du) = |Du|^{p-2} g^{\alpha\beta} A(u)(D_\alpha u, D_\beta u).$$

Here and in what follows, the summation notation over repeated indices is adopted.

For $q > 1$, we now define a set of Sobolev mappings between M and N, denoted by $W^{1,q}\ (M,\ N)$, as a space of maps $u : M \to R^n$ belonging to usual Sobolev space $W^{1,q}(M, R^n)$ such that $u(x) \in N$ for almost everywhere $x \in M$. To look for maps belonging to $W^{1,p}(M, N)$ satisfying (0.2) in the distribution sence, we are concerned with heat flows

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$u(t) \in W^{1,p}(M,N)$, $0 \le t < \infty$, for the p-energy (0.1) with a given map $u_0 \in W^{1,p}(M,N)$ where the heat flows are prescribed by Cauchy problem for a system of second order nonlinear partial differential equations of parabolic type

$$\partial_t u - \triangle_p u + A_p(u)(Du,Du) = 0 \quad \text{in} \quad (0,\infty)\times M, \tag{0.3}$$

$$u(0,x) = u_0(x), \quad x \in M. \tag{0.4}$$

The partial regularity of minimizing p-harmonic maps has been widely discussed (see [12,18] for $p = 2$, [11, 14] for $p > 1$). We also recall that the partial regularity of p-harmonic maps of C^1−class are investigated in [17] for $p = 2$ and [10, 13] for $p > 2$. On the other hand, Struwe[19], Chen and Struwe[3] have proved the global exstence and partial regularity for a weak solution to the evolution problem for harmonic maps(p = 2).

To study the partial regularity of weak solutions, one needs to establish so-called the monotonicity formula for solutions. The monotonicity formula has been obtained for stationary p-harmonic maps (refer to [17, 10]) and C^1−weak solutions to heat flows for harmonic maps (refer to [19,3]). However it remains open problem whether the mnotonicity type inequality holds for C^1−weak solutions to heat flows for p-harmonic maps($p > 1$) or not, so that it has been difficult to investigate the partial regularity of weak solutons to heat flows for p-harmonic maps. Here we note that the global existence of a weak solution to the heat flow for p-harmonic maps($p \ge 2$) has recently shown in the case that the target manifold is a sphere, $N = S^{n-1}$[1] and that the $C^{1,\mu}$−regularity of solutions of degenerate parabolic systems with only principal terms was accomplished in [6,7,8,9] ([20] for corresponding elliptic systems). In this paper we show the compacteness for weak solutions to (0.3) of C^1−class with the same initial value, provided the sectional curvature of the target manifold is nonpositive.

For simplicity, we assume that the domain manifold is an Euclidean space, $M = R^m$ and that the metric is flat, $(g^{\alpha\beta})$ = the identity matrix.

We are now interested in weak solutions of (0.3) and (0.4): $u \in L^\infty((0,+\infty); W^{1,p}(R^m, N))$ satisfying, for almost all t_1, t_2, $0 \le t_1, t_2 < +\infty$, and all $\varphi \in L^p_{\text{loc}}((0,+\infty); W^{1,p}(R^m, R^n)) \cap L^\infty_{\text{loc}}((0,+\infty)\times R^m, R^n)$, $\partial_t\varphi \in L^2_{\text{loc}}((0,+\infty)\times R^m, R^n)$ and the support of which is compactly contained in $[0,+\infty)\times R^m$

$$\int_{\{t\}\times R^m} u\cdot\varphi\Big|_{t=t_1}^{t=t_2} dx + \int_{(t_1,t_2)\times R^m} \{-u\cdot\partial_t\varphi + |Du|^{p-2}Du\cdot D\varphi + \varphi\cdot A_p(u)(Du,Du)\}dz = 0 \tag{0.5}$$

and satisfying the initial condition

$$|u(t) - u_0|_{W^{1,p}(R^m)} \to 0, \quad t \to 0. \tag{0.6}$$

To state our results, we need some preminalies: Let us introduce the parabolic metric

$$\delta(z_1,z_2) = \max\{|x_1 - x_2|,\ |t_1 - t_2|^{1/2}\}, \quad z_i = (t_i, x_i),\ i = 1,2 \tag{0.7}$$

and denote by $\mathrm{dist}_\delta(z, A)$ a distance between a point z with respect to δ respectively. Here we recall some function spaces: For a positive number T and a open set $\Omega \subset R^m$, Hölder space $C^{0,\mu}((0,T) \times \Omega, R^n)$, denoted the space of Hölder continuous functions in $(0,T) \times \Omega$ (with respect to the metric δ) with an exponent μ, the usual Lebegue space $L^q(\Omega) = L^q(\Omega, R^n)$, $q \geq 1$, and Sobolev spaces with $q \geq 1 : W^{1,q}(\Omega) = W^{1,q}(\Omega, R^n)$.

Then our main theorem is the following:

Theorem *Suppose the sectional curvature is nonpositive and* $u_0 \in W^{1,p}(R^m, N)$. *Let* $\{u_k\}$ *be a sequence of weak solutions* $u_k \in C^0((0,+\infty); C^1_{\mathrm{loc}}(R^m, R^n))$ *to* (0.3) *and* (0.4) *satisfying* $Du_k \in L^{2(p-1)}_{\mathrm{loc}}((0,+\infty); L^{2(p-1)}(R^m, R^{mn}))$. *Then there exist a subsequence* $\{u_k\}$ *and a map* $u : [0,+\infty) \times R^m \to R^n$ *such that*

$$\sup_{0 \leq t < +\infty} I(u(t)) \leq I(u_0), \quad \partial_t u \in L^2((0,+\infty) \times R^m, R^n), \tag{0.8}$$

and

$$Du_k \to Du \quad \text{weak-star in } L^\infty([0,+\infty); L^p(R^m, R^{mn})), \tag{0.9}$$

$$\partial_t u_k \to \partial_t u \quad \text{weakly in } L^2((0,+\infty) \times R^m, R^n), \tag{0.10}$$

$$\begin{aligned} u_k \to u \quad &\text{weakly in } W^{1,p}_{\mathrm{loc}}((0,+\infty) \times R^m, R^n) \\ &\text{and strongly in } C^0((0,+\infty); C^1_{\mathrm{loc}}(R^m, R^n)). \end{aligned} \tag{0.11}$$

Moreover u *is a weak solution to* (0.3), (0.4) *and there exists a positive number* $\alpha, 0 < \alpha < 1$ *such that* u, Du *belong to* $C^{0,\alpha}_{\mathrm{loc}}((0,+\infty) \times R^m)$ *and it holds that*

$$\partial_t u - \triangle_p u + A_p(u)(Du, Du) = 0 \quad \text{almost everywhere in } (0,+\infty) \times R^m. \tag{0.12}$$

Remark. We use the assumption $\{Du_k\} \subset L^{2(p-1)}_{\mathrm{loc}}((0,+\infty); L^{2(p-1)}(R^m, R^{mn}))$ in deriving the energy inequality (Lemma 2.2). The assumption is satisfied by weak solutions in the case $p = 2$.

Remark. We are able to make simple modification of the arguments for the proof of Theorem to obtain the same assertion as in Theorem in the case where the domain is a compact, smooth orientable Riemannian manifold(refer to [16]).

Some standard notations: For $z_0 = (t_0, x_0) \in (0,+\infty) \times M$ and $r, \tau > 0$

$$B_r(x_0) = \{x \in R^n : |x - x_0| < r\}, Q_{r,\tau}(z_0) = (t_0 - \tau,\ t_0) \times B_r(x_0).$$

and $Q_r(z_0) = Q_{r,r^2}(z_0)$. The center points x_0, z_0 are omitted when no confusion may arise.

1 Preliminary.

We now state some algebraic inequality, for convenience.

Lemma 1.1 *There exists a positive constant γ depending only on p such that, for all vectors $P, Q \in R^{mn}$ with $V(s) = Q + s(P-Q)$ for any s, $0 \le s \le 1$,*

$$(P-Q)\cdot(|P|^{p-2}P - |Q|^{p-2}Q) \ge |P-Q|^2 \int_0^1 |V(s)|^{p-2}ds, \tag{1.1}$$

$$||P|^{p-2}P - |Q|^{p-2}Q| \le \gamma |P-Q| \int_0^1 |V(s)|^{p-2}ds, \tag{1.2}$$

$$||P|^{p-1} - |Q|^{p-1}| \le \gamma |P-Q| \int_0^1 |V(s)|^{p-2}ds. \tag{1.3}$$

Proof. By a usual calculation, we have

$$\begin{aligned} &|P|^{p-2}P - |Q|^{p-2}Q \\ &= \int_0^1 \{|V(s)|^{p-2}(P-Q) + (p-2)|V(s)|^{p-4}V(s)\cdot(P-Q)V(s)\}ds, \end{aligned} \tag{1.4}$$

so that

$$\begin{aligned} &(P-Q)\cdot(|P|^{p-2}P - |Q|^{p-2}Q) \\ &= \int_0^1 \{|V(s))|^{p-2}|P-Q|^2 + (p-2)|V(s))|^{p-4}(V(s)\cdot(P-Q))^2\}ds. \end{aligned} \tag{1.5}$$

Noting that the second term in the integrated function in (1.5) is nonnegative, we have (1.1).

Applying Schwarz inequality for (1.4), we immediately obtain (1.2) with $\gamma = p-1$.

Since we may calculate

$$\begin{aligned} &|P|^{p-1} - |Q|^{p-1} \\ &= \int_0^1 \left\{ |V(s)|^{p-2}(P-Q)\cdot\frac{V(s)}{|V(s)|} + (p-2)|V(s)|^{p-3}(P-Q)\cdot V(s) \right\} ds, \end{aligned} \tag{1.6}$$

we have (1.3) by Schwarz inequality.

2 Energy estimates and Bochner formula.

Let $u_0 \in W^{1,p}(R^m, N)$. In this section we give a-priori estimates valid for weak solutions $u \in C^0((0,+\infty); C^1_{\rm loc}(R^m, R^n))$ of (0.3) and (0.4) such that $Du \in L^{2(p-1)}_{\rm loc}((0,+\infty); L^{2(p-1)}(R^m, R^{mn}))$.

First of all we have the following estimate (refer to [4,7,10,15]).

Lemma 2.1 *A function $|Du|^{p/2-1}Du$ has weak derivatives which lie in $L^2_{\rm loc}((0,+\infty)\times R^m)$ and there exists a positive constant γ depending only on m, p and N such that, for all $Q_{2r} = Q_{2r}(t_0, x_0) \subset (0,+\infty) \times R^m$,*

$$\begin{aligned} &\sup_{t_0-r^2\le t\le t_0} \int_{B_r\times\{t\}} |Du|^2 dx + \int_{Q_r} |Du|^{p-2}|D^2u|^2 dz \\ &\qquad \le \gamma r^{-2}\left(\int_{Q_{2r}} |Du|^2 dz + (1+|Du|^2_{L^\infty(Q_{2r})})\int_{Q_{2r}} |Du|^p dz\right). \end{aligned} \tag{2.1}$$

Proof. We now put, for a positive number h, $\Delta_{h,i}u(t,x) = (u(t,x+he^i) - u(t,x))/h$ and $\overline{\Delta}_{h,i}u(t,x) = (u(t,x) - u(t,x-he^i))/h$ as the difference quotients in the i−th direction $(i = 1,\cdots,m)$ and, as Steklov averagings on the t−variable, $d_hu(t,x) = \int_t^{t+h} u(s,x)ds/h$ and $\bar{d}_hu(t,x) = \int_{t-h}^t u(s,x)ds/h$. Let $\eta \in C^1_0(B_{3r/2})$ be $\eta = 1$ in B_r, $0 \le \eta \le 1$ and $|D\eta| \le 4/r$, and $\sigma \in C^1_0((t_0-(2r)^2, t_0])$ be $\sigma = 1$ in $(t_0 - r^2, t_0)$ and $|\partial_t\sigma| \le 4/r^2$. We put h, l as $0 < h < \varepsilon$, $0 < l < \mathrm{dist}\{\mathrm{supp}\ \eta, \partial B_r\}$, where ε is sufficiently small. Also let a cutoff function $\sigma^\tau_\varepsilon(t) \in C^1_0((t_0-(2r)^2, t_0-\varepsilon))$ be $\sigma^\tau_\varepsilon(t) = \sigma$ in $(t_0-(2r)^2, t_0-\tau-\varepsilon)$ with a sufficiently small $\tau > 0$. Taking $\bar{d}_h(\overline{\Delta}_{l,i}\ (\sigma^\tau_\varepsilon\eta^2 d_h(\Delta_{l,i}u)))$ as a test function in (0.3), we have, by a change of variables,

$$\begin{aligned} 0 &= \int_{Q_{2r}} \{\partial_t(d_h(\Delta_{l,i}u))\cdot d_h(\Delta_{l,i}u)\sigma^\tau_\varepsilon\eta^2 + d_h(\Delta_{l,i}(|Du|^{p-2}Du))\cdot D(\sigma^\tau_\varepsilon\eta^2 d_h(\Delta_{l,i}u)) \\ &\quad + \sigma^\tau_\varepsilon\eta^2 d_h(\Delta_{l,i}u)\cdot d_h(\Delta_{l,i}(|Du|^{p-2}A(u)(Du,Du)))\}dz \\ &= -\frac{1}{2}\int_{B_{2r}\times(t_0-(2r)^2,t_0-\tau-\varepsilon)} \eta^2|d_h(\Delta_{l,i}u)|^2\partial_t\sigma^\tau_\varepsilon dz \\ &\quad - \frac{1}{2}\int_{(t_0-\tau-\varepsilon,t_0-\varepsilon)}\int_{B_{2r}\times\{t\}} \eta^2|d_h(\Delta_{l,i}u)|^2\partial_t\sigma^\tau_\varepsilon dxdt \\ &\quad + \int_{Q_{2r}} \sigma^\tau_\varepsilon\eta^2 d_h(\Delta_{l,i}(|Du|^{p-2}Du))\cdot D(d_h(\Delta_{l,i}u))dz \\ &\quad - 2\int_{Q_{2r}} \sigma^\tau_\varepsilon\eta|D\eta||d_h(\Delta_{l,i}(|Du|^{p-2}Du))||d_h(\Delta_{l,i}u)|dz \\ &\quad - \left|\int_{Q_{2r}} \sigma^\tau_\varepsilon\eta^2 d_h(\Delta_{l,i}u)\cdot d_h(\Delta_{l,i}(|Du|^{p-2}A(u)(Du,Du)))dz\right|. \end{aligned} \tag{2.2}$$

Let τ, ε tend to zero so that we have

$$
\begin{aligned}
0 \geq & \frac{1}{2} \int_{B_{2r}\times\{t=t_0\}} \eta^2 |d_h(\Delta_{l,i}u)|^2 dx \\
& + \int_{Q_{2r}} \sigma\eta^2 d_h(\Delta_{l,i}(|Du|^{p-2}Du)) \cdot D(d_h(\Delta_{l,i}u)) dz \\
& - \int_{Q_{2r}} \eta^2 |\partial_t\sigma| |d_h(\Delta_{l,i}u)|^2 dz - 2\int_{Q_{2r}} \sigma\eta |D\eta| |d_h(\Delta_{l,i}(|Du|^{p-2}Du))| |d_h(\Delta_{l,i}u)| dz \\
& - \left| \int_{Q_{2r}} \sigma\eta^2 d_h(\Delta_{l,i}u) \cdot d_h(\Delta_{l,i}(|Du|^{p-2}A(u)(Du,Du))) dz \right|.
\end{aligned} \tag{2.3}
$$

Since $u \in C^0((0,+\infty); C^1_{\mathrm{loc}}(R^m))$, $\Delta_{l,i}u \in C^0((0,+\infty); C^1_{\mathrm{loc}}(R^m))$. Thus we find, for any sufficiently small $l > 0$, as $h \downarrow 0$,

$$
\begin{aligned}
& d_h(\Delta_{l,i}u) \to \Delta_{l,i}u \quad \text{in } C^0((0,+\infty); C^0_{\mathrm{loc}}(R^m)), \\
& d_h(\Delta_{l,i}(Du)) \to \Delta_{l,i}(D\dot{u}) \quad \text{in } C^0((0,+\infty); C^0_{\mathrm{loc}}(R^m)).
\end{aligned} \tag{2.4}
$$

Adopting (2.4) in (2.3), we obtain from letting h tend to zero

$$
\begin{aligned}
& \frac{1}{2} \int_{\{t=t_0\}\times B_{2r}} \eta^2 |(\Delta_{l,i}u)|^2 dx + \int_{Q_{2r}} \sigma\eta^2 \Delta_{l,i}(Du) \cdot \Delta_{l,i}(|Du|^{p-2}Du) dz \\
& \leq \int_{Q_{2r}} \eta^2 |\partial_t\sigma| |\Delta_{l,i}u|^2 dz + \int_{Q_{2r}} \sigma\eta |D\eta| |\Delta_{l,i}u| |\Delta_{l,i}(|Du|^{p-2}Du)| dz \\
& \quad + \left| \int_{Q_{2r}} \sigma\eta^2 \Delta_{l,i}u \cdot \Delta_{l,i}(|Du|^{p-2}A(u)(Du,Du)) dz \right|.
\end{aligned} \tag{2.5}
$$

To evaluate the each term in (2.5), we set the abbreviation: $u_\lambda = u + \lambda l \Delta_{l,i}u$ with $\lambda, l > 0$. By (1.1), (1.2) and Young's inequality, we have, for the second terms in the both side,

$$
\begin{aligned}
& \int_{Q_{2r}} \sigma\eta^2 \Delta_{l,i}(Du) \cdot \Delta_{l,i}(|Du|^{p-2}Du) dz \\
& \geq \int_{Q_{2r}} \sigma\eta^2 |\Delta_{l,i}(Du)|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda dz,
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
& \int_{Q_{2r}} \sigma\eta |D\eta| |\Delta_{l,i}u| |\Delta_{l,i}(|Du|^{p-2}Du)| dz \\
& \leq \frac{1}{4} \int_{Q_{2r}} \sigma\eta^2 |\Delta_{l,i}(Du)|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda dz \\
& \quad + \int_{Q_{2r}} \sigma |D\eta|^2 |\Delta_{l,i}u|^2 \int_0^1 |Du_\lambda|^{p-2} d\lambda dz.
\end{aligned} \tag{2.7}
$$

The last term is estimated similarily as in [10, Page 389-390], so that we have, with a positive constant γ depending on N,

$$\begin{aligned}
&\left|\Delta_{l,i}u \cdot \Delta_{l,i}(|Du|^{p-2}A(u)(Du,Du))\right| \\
&\le \gamma|\Delta_{l.i}u||\Delta_{l.i}(Du)|\int_0^1 |Du_\lambda|^{p-1}d\lambda + \gamma|\Delta_{l.i}u|^2\int_0^1 |Du_\lambda|^p d\lambda \\
&\le \frac{1}{4}|\Delta_{l.i}(Du)|^2\int_0^1 |Du_\lambda|^{p-2}d\lambda + \gamma|\Delta_{l.i}u|^2\int_0^1 |Du_\lambda|^p d\lambda.
\end{aligned} \tag{2.8}$$

By substitution of (2.6), (2.7) and (2.8) into (2.5), we derive from routine estimate

$$\begin{aligned}
&\int_{\{t=t_0\}\times B_{2r}} \sigma\eta^2|\Delta_{l,i}u|^2dx \\
&\quad + \int_{Q_{2r}} \sigma\eta^2|\Delta_{l,i}(Du)|^2\int_0^1 |Du_\lambda|^{p-2}d\lambda dz \\
&\le \gamma r^{-2}\int_{Q_{2r}} \eta^2|\Delta_{l,i}u|^2dz + \gamma r^{-2}\int_{(t_0-(2r)^2,t_0)\times B_{3r/2}} |\Delta_{l,i}u|^2\int_0^1 |Du_\lambda|^{p-2}d\lambda dz \\
&\quad + \gamma\int_{Q_{2r}} \sigma\eta^2|\Delta_{l,i}u|^2\int_0^1 |Du_\lambda|^p d\lambda dz.
\end{aligned} \tag{2.9}$$

As a result, we arrive at the desired estimate (2.1).

The following estimate is fundamental(refer [1,3,19]).

Lemma 2.2 (Energy inequality) *It holds*

$$\sup_{0\le t<+\infty} I(u(t)) + \int_0^T\int_{R^m} |\partial_t u|^2dxdt \le I(u_0). \tag{2.10}$$

Proof. First of all we observe that

$$\partial_t u \in L^2_{\mathrm{loc}}((0,+\infty)\times R^m). \tag{2.11}$$

For this purpose take a bounded domain $B \subset R^m$, a positive number T and sufficiently small positive numbers ε, h. We put a usual cutoff function $\eta \in C_0^\infty(B)$ and continuous piecewise linear functions $\sigma = \sigma_\delta$ for sufficiently small $\delta > 0$ such that $\sigma_\delta = 0$ on $(-\infty, \delta+\varepsilon)\cup(T-h-\delta,+\infty)$, $\sigma_\delta = 1$ on $(2\delta+\varepsilon, T-h-2\delta)$. Also put $\Delta_{h,t}u(s,x) = (u(s+h,x)-u(s,x))/h$.

Take a function $\varphi \in L^2((0,+\infty); W_0^{1,2}(R^m))$ satisfying $\varphi = 0$ in $R\setminus[0,+\infty)$. By substitution of a valid test function $\bar{d}_h\varphi$ into (0.5) with $t_1 = 0, t_2 = T+h$, we immediately have

$$\int_{(0,+\infty)\times R^m} \{\partial_t(d_h u)\cdot\varphi + d_h(|Du|^{p-2}Du)\cdot D\varphi + \varphi\cdot d_h(|Du|^{p-2}A(u)(Du,Du))\}dz = 0,$$

which implies that $d_h(|Du|^{p-2}Du)$ is weak differentiable in $(0,+\infty)\times R^m$ and that

$$\begin{aligned}\operatorname{div}\{d_h(|Du|^{p-2}Du)\} = \partial_t(d_h u) + d_h(|Du|^{p-2}A(u)(Du,Du))\\ \text{almost everywhere in } (0,+\infty)\times R^m.\end{aligned} \tag{2.12}$$

Multiplying (2.12) by $\sigma_\delta\eta^2\Delta_{h,t}u$ and integrating it in $(0,T)\times B$, we have

$$\begin{aligned}\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz &= \int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2\Delta_{h,t}u\cdot\operatorname{div}(d_h(|Du|^{p-2}Du))dz\\ &\quad-\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2\Delta_{h,t}u\cdot d_h(|Du|^{p-2}A(u)(Du,Du))dz.\end{aligned} \tag{2.13}$$

The first term of the right hand is estimated from above by

$$\begin{aligned}&\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u||d_h(\operatorname{div}(|Du|^{p-2}Du))|\\ &=(p-1)\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u||d_h(|Du|^{p-2}|D^2u|)|dz\\ &\le\frac14\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz+\gamma\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2 d_h((|Du|^{p-2}|D^2u|)^2)dz\\ &\le\frac14\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz+\gamma\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|Du|^{2(p-2)}|D^2u|^2dz.\end{aligned} \tag{2.14}$$

Similarily, the second term of the right hand side is bouded from above by

$$\begin{aligned}&\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u||d_h(|Du|^{p-2}A(u)(Du,Du))|dz\\ &\le\frac14\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz+\gamma\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2 d_h(|Du|^{p-2}A(u)(Du,Du))^2dz\\ &\le\frac14\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz+\gamma(N)\int_{(\varepsilon,+\infty)\times B}\sigma_\delta\eta^2|Du|^{2p}dz.\end{aligned} \tag{2.15}$$

By combination of (2.14) and (2.15) with (2.13), we have

$$\begin{aligned}\int_{(\varepsilon,T-h)\times B}\sigma_\delta\eta^2|\Delta_{h,t}u|^2dz &\le\gamma\int_{(\varepsilon,T-h)\times B}\sigma_\delta\eta^2|Du|^{2(p-2)}|D^2u|^2dz\\ &\quad+\int_{(\varepsilon,T-h)\times B}\sigma_\delta\eta^2|Du|^{2p}dz.\end{aligned}$$

Now we note that $Du\in L^\infty((0,T)\times B)$ and that $|Du|^{p/2-1}Du\in L^2((0,T)\times B)$ by Lemma 2.1, sn that, letting δ tend to zero, we obtain (2.11). Since $\partial_t u\in L^2_{\text{loc}}((0,+\infty)\times R^m)$ and $Du\in C^0_{\text{loc}}((0,+\infty)\times R^m)$, we find that

$$\operatorname{div}(|Du|^{p-2}Du)\in L^2_{\text{loc}}((0,+\infty)\times R^m)$$

ans that

$$\partial_t u - \operatorname{div}(|Du|^{p-2}Du) + |Du|^{p-2}A(u)(Du, Du) = 0 \quad \text{almost everywhere in } (0,+\infty)\times R^m. \tag{2.16}$$

Next, for $v \in C^\infty((t_1,t_2)\times B_r)$, $0 < t_1 < t_2 \le T$, and $\eta \in C_0^\infty(B_{2r})$ satisfying $\eta = 1$ in B_r and $|D\eta| \le 2/r$, we readily verify, by integration by parts,

$$\begin{aligned}
&\int_{(t_1,t_2)\times B_{2r}} \operatorname{div}(|Dv|^{p-2}Dv)\cdot \partial_t v\eta^2 dz = -\int_{(t_1,t_2)\times B_{2r}} (|Dv|^{p-2}Dv)\cdot D(\eta^2\partial_t v)dz \\
&= -\int_{(t_1,t_2)\times B_{2r}} \frac{1}{p}\partial_t |Dv|^p\eta^2 dz - 2\int_{(t_1,t_2)\times B_{2r}} |Dv|^{p-2}Dv\cdot(\partial_t v D\eta)\eta dz \\
&= -2\int_{(t_1,t_2)\times B_{2r}} |Dv|^{p-2}Dv\cdot(\partial_t v D\eta)\eta dz \\
&- \quad \frac{1}{p}\int_{\{t=t_2\}\times B_r} |Dv|^p\eta^2 dx + \frac{1}{p}\int_{\{t=t_1\}\times B_r} |Dv|^p\eta^2 dx.
\end{aligned} \tag{2.17}$$

Now, by Lemma 2.1 and the assumption $Du \in C^0_{\text{loc}}((0,+\infty)\times R^m)$, we observe, for all $a \in R$,

$$|Du|^a \in L^2_{\text{loc}}((0,+\infty); W^{1,2}_{\text{loc}}(R^m \cap \{|Du|>0\})) \cap L^\infty_{\text{loc}}((0,+\infty)\times R^m \cap \{|Du|>0\}). \tag{2.18}$$

In view of $Du = |Du|^{(2-p)/2}|Du|^{(p-2)/2}Du$, (2.18) immediately implies that

$$D^2 u \in L^2_{\text{loc}}((0,+\infty)\times R^m \cap \{|Du|>0\}).$$

To prove our lemma, we approximate u by a sequence of smooth maps $\{u_k\} \subset C^\infty_{\text{loc}}((0,+\infty)\times R^m)$ (refer to [10, pp 391-392]) such that Du_k converges to Du locally uniformly on $(0,+\infty)\times R^m$ and $\partial_t u_k$ converge to $\partial_t u$ in $L^2_{\text{loc}}((0,+\infty)\times R^m)$ and that, for arbitrary $a \in R$, as $k \to +\infty$,

$$\begin{aligned}
&\chi_{|Du_k|>0} \to \chi_{|Du|>0} \quad \text{in } L^1_{\text{loc}}((0,+\infty)\times R^m\ ), \\
&\operatorname{div}(|Du_k|^a Du_k) \to \operatorname{div}(|Du|^a Du) \quad \text{in } L^2_{\text{loc}}((0,+\infty)\times R^m \cap \{|Du|>0\}), \\
&D(|Du_k|^a) \to D(|Du|^a) \quad \text{in } L^2_{\text{loc}}((0,+\infty)\times R^m \cap \{|Du|>0\}\ ).
\end{aligned} \tag{2.19}$$

Since $D_\beta(|Du_k|^{p-2}D_\alpha u_k) = 0$ a.e. on $\{|Du_k| = 0\}$ for $\alpha, \beta = 1,\cdots,m$, the domains of integration in (2.17) with $v = u_k$ are restricted to the intersection of the domains with $\{|Du_k|>0\}$. Integrating (2.16) multiplied by $\eta^2\partial_t u$ in $(t_1,t_2)\times B_{2r}$ and applying (2.17) with $v = u_k$ and (2.19) to the resulting inequality, we have, for all $B_{2r} \subset R^m$ and t_1, t_2, $0 < t_1, t_2 \le T$,

$$\begin{aligned}
0 =&2\int_{(t_1,t_2)\times B_{2r}} |Du|^{p-2}Du\cdot(\partial_t u D\eta)\eta dz \\
&+ \frac{1}{p}\int_{\{t=t_2\}\times B_{2r}} |Du|^p dx - \frac{1}{p}\int_{\{t=t_1\}\times B_{2r}} |Du|^p dx + \int_{(t_1,t_2)\times B_{2r}} |\partial_t u|^2\eta^2 dz,
\end{aligned}$$

where we used the fact that $\partial_t u$ is orthogonal to $A(u)(Du, Du)$ in R^n. The validity is verified as follows: Note that there exists a tubular neighborhood $N_\varepsilon \subset R^n$ of N such that each point $y \in N_\varepsilon$ has a unique nearest point $\pi(y) \in N$ and that a map $\pi : N_\varepsilon \to N$ is C^1. Since $u(z) \in N$ for almost everywhere $z \in (0, +\infty) \times R^m$, $\pi(u) = u$ almost everywhere on $(0, +\infty) \times R^m$. Thus, by the chain rule for weak differentiation, we have

$$\partial_t u = \partial_t \pi(u) = (d\pi)_u(\partial_t u) \in T_u N \quad \text{almost everywhere in } (0, +\infty) \times R^m. \tag{2.20}$$

By Young's inequality, we have

$$\begin{aligned} &\frac{1}{p}\int_{\{t=t_2\}\times B_{2r}} |Du|^p dx + \frac{1}{2}\int_{(t_1,t_2)\times B_{2r}} \int_{B_{2r}} |\partial_t u|^2 \eta^2 dz \\ &\leq \frac{1}{p}\int_{\{t=t_1\}\times B_{2r}} |Du|^p dx + \gamma r^{-2} \int_{(t_1,t_2)\times B_{2r}} |Du|^{2(p-1)} dz. \end{aligned} \tag{2.21}$$

Noting the assumption $|Du|^{2(p-1)} \in L^1_{\text{loc}}((0, +\infty), L^1(R^m))$, by letting r tend to $+\infty$ in (2.21), we have the desired estimate.

Lemma 2.3(Bochner type estimate) *It holds, for any $\varphi \in L^2_{\text{loc}}((0, +\infty); W_0^{1,2}(R^m)) \cap W^{1,2}_{\text{loc}}((0, +\infty); L^2_{\text{loc}}(R^m))$ with $\varphi \geq 0$ in $(0, T) \times R^m$ and all t_1, t_2, $0 < t_1, t_2 < +\infty$,*

$$\begin{aligned} &\int_{\{t\}\times R^m} |Du|^2 \varphi dx \Big|_{t=t_1}^{t=t_2} - \int_{(t_1,t_2)\times R^m} |Du|^2 \partial_t \varphi dz \\ &+ \int_{(t_1,t_2)\times R^m} |Du|^{p-2}\left(\delta^{\alpha\beta} + (p-2)\frac{D_\alpha u \cdot D_\beta u}{|Du|^2}\right) D_\beta |Du|^2 D_\alpha \varphi dz \\ &+ 2\int_{(t_1,t_2)\times R^m} |Du|^{p-2}|D^2 u|^2 \varphi dz + \frac{p-2}{2}\int_{(t_1,t_2)\times R^m} |Du|^{p-4} |D|Du|^2|^2 \varphi dz \\ &= -\int_{(t_1,t_2)\times R^m} \varphi |Du|^{p-2} D_\beta(A(u)(Du, Du)) \cdot D_\beta u dz. \end{aligned} \tag{2.22}$$

Proof. Let $\varphi \in C^0((0, +\infty); C^1_0(R^m)) \cap C^1((0, +\infty); C^0(R^m))$ be a function with $\varphi \geq 0$ and take an interval $(t_1, t_2) \subset (0, +\infty)$ and a sequence $\{u_k\} \subset C^\infty_{\text{loc}}((0, +\infty) \times R^m)$ satisfying (2.19). By integration by parts and simple calculations (also see [10, pp 391-392]), we see that each u_k satisfies

$$\begin{aligned} &\int_{(t_1,t_2)\times R^m} \partial_t u_k \cdot \operatorname{div}(\varphi D u_k) dz \\ &= \int_{(t_1,t_2)\times R^m} \frac{1}{2}|Du_k|^2 \partial_t \varphi dz \\ &\quad - \int_{\{t=t_2\}\times R^m} \frac{1}{2}|Du_k|^2 \varphi dx + \int_{\{t=t_1\}\times R^m} \frac{1}{2}|Du_k|^2 \varphi dx, \end{aligned} \tag{2.23}$$

$$\int_{(t_1,t_2)\times R^m} \mathrm{div}(f'(|Du_k|^2)Du_k)\cdot \mathrm{div}(\varphi Du_k)dz$$
$$= \int_{(t_1,t_2)\times R^m} \frac{1}{2}(\delta^{\alpha\beta}|Du_k|^{p-2} + (p-2)|Du_k|^{p-4}D_\alpha u_k\cdot D_\beta u_k)D_\beta|Du_k|^2 D_\alpha\varphi dz$$
$$+ \int_{(t_1,t_2)\times R^m} \varphi|Du_k|^{p-2}|D^2u_k|^2dz + \frac{p-2}{4}\int_{(t_1,t_2)\times R^m} \varphi|Du_k|^{p-4}|D|Du_k|^2|^2dz. \tag{2.24}$$

Noting the convergnce (2.19) of u_k to u, we are able to let k tend to infinity in (2.23) and (2.24) to have (2.23) and (2.24) replaced u_k by u. Multiplying (2.16) by $\mathrm{div}(\varphi Du)$ and integrating the resulting equality, we have

$$0 = -\int_{(t_1,t_2)\times R^m} \mathrm{div}(\varphi Du)\cdot(\partial_t u - \mathrm{div}(f'(|Du|^2)Du) + |Du|^{p-2}A(u)(Du,Du))dz. \tag{2.25}$$

A substitution of (2.23) and (2.24) with replacing u_k by u into (2.25) gives

$$-\int_{(t_1,t_2)\times R^m} \frac{1}{2}|Du|^2\partial_t\varphi dz$$
$$+ \int_{\{t=t_2\}\times R^m} \frac{1}{2}|Du|^2\varphi dx - \int_{\{t=t_1\}\times R^m} \frac{1}{2}|Du|^2\varphi dx$$
$$+ \int_{(t_1,t_2)\times R^m} \frac{1}{2}(\delta^{\alpha\beta}|Du|^{p-2} + (p-2)|Du|^{p-4}D_\alpha u\cdot D_\beta u)D_\beta|Du|^2D_\alpha\varphi dz \tag{2.26}$$
$$+ \int_{(t_1,t_2)\times R^m} \varphi|Du|^{p-2}|D^2u|^2dz + \frac{p-2}{4}\int_{(t_1,t_2)\times R^m} \varphi|Du|^{p-4}|D|Du|^2|^2dz$$
$$= -\int_{(t_1,t_2)\times R^m} \varphi D_\beta(|Du|^{p-2}A(u)(Du,Du))\cdot D_\beta u dz.$$

Noting that $D_\beta u$ ($\beta = 1,\cdots,m$) is orthogonal to $A(u)(Du,Du)$ in R^n (see the proof of (2.20)), we observe

$$D_\beta(|Du|^{p-2}A(u)(Du,Du))\cdot D_\beta u = |Du|^{p-2}D_\beta(A(u)(Du,Du))\cdot D_\beta u. \tag{2.27}$$

Combining (2.26) with (2.27), we have (2.22).

3 Proof of Theorem.

Now we give the proof of our Theorem. We make assumption that the sectional curvature of the target manifold is nonpositive, which has not needed until now. Firstly we derive Harnack type estimate by the technique of DeGiorgi (refer to [6,7,8]).

Let $\{u_k\} \subset C^0((0,+\infty); C^1_{\rm loc}(R^m, R^n))$ be a sequence of weak solutions to (0.3) and (0.4) satisfying $\{Du_k\} \subset L^{2(p-1)}_{\rm loc}((0,+\infty); L^{2(p-1)}(R^m, R^{mn}))$.

Lemma 3.1 *Then there exists a positive constant γ depending only on N, p and m such that, for each $u = u_k$ and any $Q_r \subset (0,+\infty) \times R^m$,*

$$\sup_{Q_{r/2}} |Du|^2 \le \max\left\{\gamma 2^{m+1}\left(\frac{1}{|Q_r|}\int_{Q_r} |Du|^p dz\right), 2^{2/(p-2)}\right\}. \tag{3.1}$$

Proof of Lemma 3.1. We know that, for any smooth vectorfield ν satisfying $\nu(v) \in (T_v N)^\perp$ for any $v \in N$,

$$\sum_{i,j=1}^n \frac{d\nu^j}{dv^i}(v) V^i W^j = -\nu(v) \cdot A(v)(V,W) \quad \text{for any } V, W \in T_v N. \tag{3.2}$$

Noting that $D_\alpha u \in T_u N$ and $A(u)(D_\alpha u, D_\beta u) \in (T_u N)^\perp$ $(\alpha, \beta = 1, \cdots, m)$, we have, for $\alpha, \beta, \gamma, \bar\gamma = 1, \cdots, m$,

$$\begin{aligned} D_\alpha(A(u)(D_\gamma u, D_{\bar\gamma} u)) \cdot D_\beta u &= D_\alpha u \cdot \frac{d}{du}(A(u)(D_\gamma u, D_{\bar\gamma} u)) \cdot D_\beta u \\ &= -A(u)(D_\gamma u, D_{\bar\gamma} u) \cdot A(u)(D_\alpha u, D_\beta u). \end{aligned} \tag{3.3}$$

Since the sectional curvature of the target manifold is nonpositive, we obtain, from (3.3),

$$|Du|^{p-2} \sum_{\alpha,\beta=1}^m D_\beta(A(u)(D_\alpha u, D_\alpha u)) \cdot D_\beta u \ge 0, \tag{3.4}$$

so that, from (2.52), it follows that, for any $\varphi \in L^2_{\rm loc}((0,\infty); W^{1,2}_0(R^m)) \cap W^{1,2}_{\rm loc}((0,+\infty); L^2_{\rm loc}(R^m))$ with $\varphi \ge 0$ in $(0,+\infty) \times R^m$ and all t_1, t_2, $0 < t_1, t_2 < +\infty$,

$$\begin{aligned} &\int_{\{t\}\times R^m} |Du|^2 \varphi dx \Big|_{t=t_1}^{t=t_2} - \int_{(t_1,t_2)\times R^m} |Du|^2 \partial_t \varphi dz \\ &+ \int_{(t_1,t_2)\times R^m} |Du|^{p-2}\left(\delta^{\alpha\beta} + (p-2)\frac{D_\alpha u \cdot D_\beta u}{|Du|^2}\right) D_\beta |Du|^2 D_\alpha \varphi dz \\ &+ 2\int_{(t_1,t_2)\times R^m} |Du|^{p-2}|D^2 u|^2 \varphi dz + \frac{p-2}{2}\int_{(t_1,t_2)\times R^m} |Du|^{p-4} |D|Du|^2|^2 \varphi dz \le 0. \end{aligned} \tag{3.5}$$

Thus we are able to proceed with our estimates similarly as in [6, pp 234-235, pp 238-240, Theorem 5.1] to have (3.1).

Proof of Theorem. The validity of (0.8), (0.9) and (0.10) immediately follows from our energy inequality (2.10). The first assertion in (0.11) is obtained from Lemma 2.2 and Sobolev imbedding theorem..

We now consider the validity of the latter statement in Theorem.

Applying our energy inequality (2.10) for (3.1) in Lemma 3.1, we have that, for any region $Q \subset (0, +\infty) \times R^m$, with a positive constant γ depending only on $N, p, m, |Q|$ and $I(u_0)$,

$$|Du_k| \leq \gamma \text{ in } Q. \tag{3.6}$$

Thus we observe from (3.6) that,with a positive constant $\bar{\gamma}$ depending only on $N, p, m, |Q|$ and $I(u_0)$,

$$|A_p(u_k)(Du_k, Du_k)| \leq \gamma(N)|Du_k|^p \leq \bar{\gamma} \quad \text{in } Q, \tag{3.7}$$

which implies that the nonlinear term $A_p(u_k)(Du_k, Du_k)$ in the equation (0.3) is uniformly bounded in Q with respect to k. By (2.16) with $u = u_k$ and (3.7), we argue similarily as in the proof of Theorem 1.1 in [6, pp 245-256, pp 275-291] (see also [2,7,8,9]) to observe that each Du_k is locally Hölder continuous in Q, independently of k. On the other hand, similarily as the proof of Theorem 1 in [2] (also see the proof of Theorem 1 in [5]), by (2.16) with $u = u_k$ and (3.7), we also find that each u_k is locally Hölder continuous in Q, independently of k. Thus we see by Ascoli-Arzela theorem that there exists a subsequence $\{u_k\}$ such that, as $k \to +\infty$, u_k and Du_k converge uniformly on Q to u and Du respectively and that u, Du are locally Hölder continuous in Q. As a rsesult we obtain the second assertion in (0.11). Passing to the limit $k \to +\infty$ in (0.5) for $\{u_k\}$ with φ the support of which is contained in Q, we find that u satisfies (0.5) in Q. Since u, Du are locally Hölder continuous in Q, we are able to argue similarily as in the proof of Lemma 2.1 and 2.2 to have (0.12).

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Notes on Decay Properties of Nonstationary Navier-Stokes Flows in $\mathbb{R}^n$

Tetsuro MIYAKAWA

Graduate School of Mathematics
Kyushu University 36
Hakozaki, Fukuoka 812, Japan
E-mail : miyakawa@math.kyushu-u.ac.jp

1. Introduction and main result

We consider the incompressible Navier-Stokes system of equations in $\mathbb{R}^n$, $n \geq 2$:

$$\text{(NS)} \qquad \begin{aligned} &\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \Delta\boldsymbol{u} - \nabla p \quad (x\in\mathbb{R}^n,\ t>0) \\ &\nabla\cdot\boldsymbol{u} = 0 \qquad\qquad\qquad\quad\ (x\in\mathbb{R}^n,\ t\geq 0) \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{a}, \qquad \lim_{|x|\to\infty}\boldsymbol{u} = 0 \end{aligned}$$

for unknown velocity $\boldsymbol{u} = (u^1, \cdots, u^n)$, unknown (scalar) pressure p, and a given initial velocity $\boldsymbol{a} \in \boldsymbol{L}^2$ such that $\nabla\cdot\boldsymbol{a} = 0$. Here and in what follows,

$$\nabla = (\partial_1, \cdots, \partial_n), \qquad \partial_j = \partial/\partial x^j \quad (j = 1, \cdots, n), \qquad \partial_t = \partial/\partial t,$$

$$\Delta\boldsymbol{u} = \sum_{j=1}^n \partial_j^2\boldsymbol{u}, \qquad \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \sum_{j=1}^n u^j\partial_j\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = \sum_{j=1}^n \partial_j u^j.$$

Introducing the heat semigroup

$$(e^{-tA}\boldsymbol{a})(x) = \int E_t(x-y)\boldsymbol{a}(y)dy, \qquad E_t(x) = (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right),$$

and the bounded projector $P : \boldsymbol{L}^r \to \boldsymbol{L}^r_\sigma$ $(1 < r < \infty)$ onto the subspace $\boldsymbol{L}^r_\sigma$ of solenoidal vector fields, one can transform problem (NS) into the integral equation :

$$\text{(IE)} \qquad \boldsymbol{u}(t) = e^{-tA}\boldsymbol{a} - \int_0^t e^{-(t-\tau)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\tau)d\tau.$$

The standard results as given in [2, 3, 4, 6, 8, 9, 10, 11, 18] together ensure the existence of a weak solution $\boldsymbol{u}$ with the following decay properties:

(*a*). If $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma$ and $\|e^{-tA}\boldsymbol{a}\|_2 \le C(1+t)^{-\alpha}$ for some $\alpha > 0$, then

$$\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\beta} \qquad \text{with} \quad \beta = \min(\alpha, (n+2)/4).$$

(*b*). If $1 < r \le n/(n-1)$ and $\boldsymbol{a} \in \boldsymbol{L}^r_\sigma \cap \boldsymbol{L}^2_\sigma$, then

$$\|\boldsymbol{u}(t)\|_q \le C(1+t)^{-(n/r-n/q)/2} \qquad \text{for all} \quad r \le q \le 2.$$

(*c*). If $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$ and $\|e^{-tA}\boldsymbol{a}\|_1 \le C(1+t)^{-\gamma}$ for some $\gamma > 0$, then

$$\|\boldsymbol{u}(t)\|_1 \le C(1+t)^{-\delta} \qquad \text{with} \quad \delta = \min(1/2, \gamma).$$

The results above suggest that if $\boldsymbol{a}$ and $\boldsymbol{u}$ are smooth, then

(*d*). $|\boldsymbol{a}| \sim |x|^{-\alpha}$ for some $n/2 \le \alpha < n$ would imply $|\boldsymbol{u}| \sim |x|^{-\beta}t^{-\gamma}$ with $\beta + 2\gamma = \alpha$.

(*e*). Even if $\alpha \ge n$ in the above, we would have $|\boldsymbol{u}| \sim |x|^{-\beta}t^{-\gamma}$ with $\beta+2\gamma = \min(\alpha, n+1)$.

Takahashi [14] has recently proved (*d*) and (*e*) for smooth and bounded weak solutions, which are known to exist globally in time for a specific class of smooth and small initial data. However, the smoothness and boundedness of general weak solutions remain open when $n \ge 3$.

In this paper we shall verify the above properties in terms of some norms which can be regarded as substitutes for the $\boldsymbol{L}^p$-(quasi-)norms, $0 < p \le 1$. To state the results, we recall that the Hardy space $\boldsymbol{H}^p = \boldsymbol{H}^p(\mathbb{R}^n)$, $0 < p \le 1$, of vector-valued distributions is defined to be the set of all n-tuples of tempered distributions $\boldsymbol{f} = (f^1, \cdots, f^n)$ such that

$$\sup_{t>0} |\varphi_t * \boldsymbol{f}| \in L^p(\mathbb{R}^n), \quad \text{with (quasi-)norm} \quad \|\boldsymbol{f}\|_{H^p} = \left\| \sup_{t>0} |\varphi_t * \boldsymbol{f}| \right\|_p,$$

where φ_t is the standard Friedrichs mollifier and $\|\cdot\|_p$ is the L^p-(quasi-)norm. It is well known ([13, 16, 17]) that $\boldsymbol{H}^1$ is a Banach space with norm $\|\cdot\|_{H^1}$ and, when $0 < p < 1$, $\boldsymbol{H}^p$ is a complete metric space with metric $d(\boldsymbol{f}, \boldsymbol{g}) = \|\boldsymbol{f} - \boldsymbol{g}\|^p_{H^p}$. Furthermore, we know ([13]) that

$$(\boldsymbol{H}^1)^* = \text{BMO}, \quad (\text{VMO})^* = \boldsymbol{H}^1, \quad \text{and} \quad (\boldsymbol{H}^p)^* = \mathfrak{C}^\alpha \quad \text{with} \quad \alpha = n(1/p - 1),$$

where BMO is the space of functions of bounded mean oscillation, VMO the closure of $\boldsymbol{C}^\infty_c(\mathbb{R}^n)$ in BMO, and $\mathfrak{C}^\alpha$ is the homogeneous Hölder-Zygmund space of order α ([13, 16, 17]). Using these duality relations, we introduce the norms

$$\text{(1.1)} \qquad \|\boldsymbol{f}\|^*_p = \sup_{g \ne 0} \left| \int \boldsymbol{f} \cdot \boldsymbol{g}\, dx \right| / [\boldsymbol{g}]_\alpha \qquad (\alpha = n(1/p - 1), \ \ 0 < p < 1),$$

where $[\cdot]_\alpha$ is the seminorm of the space $\mathfrak{C}^\alpha$ and the supremum is taken over all $\boldsymbol{g}$ in the closure $\mathfrak{C}^\alpha_0$ of $C^\infty_c(\mathbb{R}^n)$ in $\mathfrak{C}^\alpha$. We then introduce the Banach spaces

$$\begin{aligned} \boldsymbol{X}^p &= (\mathfrak{C}^\alpha_0)^* \quad (0 < \alpha = n(1/p-1) < 1) \quad \text{with norm } \|\cdot\|_{X^p} = \|\cdot\|^*_p. \\ \boldsymbol{X}^1 &= \boldsymbol{H}^1 \quad \text{with norm } \|\cdot\|_{X^1} = \|\cdot\|_{H^1}. \end{aligned}$$

We denote by $\mathfrak{C}^\alpha_\sigma$ and $\boldsymbol{X}^p_\sigma$ the closure of the space $\boldsymbol{C}^\infty_{0,\sigma}$ of compactly supported smooth solenoidal vector fields in $\mathfrak{C}^\alpha_0$ and $\boldsymbol{X}^p$, respectively. As will be shown in Section 2, one can show the Helmoltz decomposition

$$\mathfrak{C}^\alpha_0 = \mathfrak{C}^\alpha_\sigma \oplus \mathfrak{C}^\alpha_\pi, \qquad \boldsymbol{X}^p = \boldsymbol{X}^p_\sigma \oplus \boldsymbol{X}^p_\pi,$$

where the subscript π means the spaces consisting of functions of the form ∇q for some scalar distribution q. Furthermore, note that

$$\boldsymbol{H}^p \subset \boldsymbol{X}^p \quad \text{with continuous injection.}$$

Employing the norms $\|\cdot\|_{X^p}$, we can now state our result in the following way.

Theorem. *Given an $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^1$, there exists a weak solution $\boldsymbol{u}$ of* (NS) *such that:*

(i) *Let $n/(n+1) < p < r \le 1$. If $\boldsymbol{a} \in \boldsymbol{X}^p_\sigma$, then $\boldsymbol{u}(t) \in \boldsymbol{X}^p_\sigma \cap \boldsymbol{X}^r_\sigma$ for all $t \ge 0$, and*

$$\|\boldsymbol{u}(t)\|_{X^r} = O(t^{-(n/p-n/r)/2}) \qquad \text{as } t \to \infty, \tag{1.2}$$

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_{X^p} = 0. \tag{1.3}$$

(ii) *Let $0 < p \le n/(n+1) < r \le 1$. Even if $\boldsymbol{a} \in \boldsymbol{H}^p$, we can only show that*

$$\boldsymbol{u}(t) \in \boldsymbol{X}^r \quad \text{for all } t \ge 0, \tag{1.4}$$

and

$$\|\boldsymbol{u}(t)\|_{X^r} = O(t^{-(n+1-n/r)/2}) \qquad \text{as } t \to \infty. \tag{1.5}$$

When $p = r = n/(n+1)$, assertion (1.5) has to be slightly modified by introducing a new function space via the real interpolation method. The precise statement on this case will be given in Section 2 (see (2.16) and (2.17)). The result above covers those given in [8, 9].

It is possible to understand that our Theorem above reflects the (possible) asymptotic behavior of $\boldsymbol{u}$ as stated in (d) and (e). To see this, observe that the norm $\|\cdot\|_{X^p}$ can be regarded as a substitute for the usual $\boldsymbol{L}^p$-(quasi-) norm. Indeed, if we write

$$f_\lambda(x) = f(x/\lambda)$$

for an arbitrary $\lambda > 0$, then we easily see that

$$\|f_\lambda\|_{X^p} = \lambda^{n/p}\|f\|_{X^p} \qquad \text{and} \qquad \|f_\lambda\|_p = \lambda^{n/p}\|f\|_p.$$

Thus, if we formally replace $\|\cdot\|_{X^p}$ by $\|\cdot\|_p$ and if we apply the formal correspondence

$$\|f\|_p < +\infty \quad \Longleftrightarrow \quad |f(x)| \sim |x|^{-n/p},$$

then it is easy to deduce properties (d) and (e) from the statements of our Theorem.

Schonbek and Schonbek [12] discuss the large time behavior of the moments

$$\int |x|^m |\boldsymbol{u}(x,t)|^2 dx \qquad (0 \le m \le n)$$

of smooth solutions $\boldsymbol{u}$ of (NS) corresponding to a specific class of initial data $\boldsymbol{a}$. Writing $\|f\|_{2,m} = (\int |x|^m |f(x)|^2 dx)^{1/2}$, we easily see that

$$\|f_\lambda\|_{2,m} = \lambda^{(m+n)/2}\|f\|_{2,m} \qquad (\lambda > 0),$$

and so the norm $\|\cdot\|_{2,m}$ has the same scaling property as $\|\cdot\|_{2n/(m+n)}$. The assumption $0 \le m \le n$ implies $1 \le 2n/(m+n) \le 2$. Using this cprrespondence, one can exactly state the decay result of [12] in terms of the results (a)–(c) described above. Our Theorem further suggests that the result of [12] would be true even for $n \le m < n+2$, although this case is not treated in [12].

Chen and Miyakawa [4] deal with the Cauchy problem for equations of motion of an incompressible rotating fluid; and deduce the decay results of $\boldsymbol{L}^2$ and $\boldsymbol{L}^1$-norms of exactly the same form as obtained in [8, 18] for the Navier-Stokes system. As will be shown in the next section, the proof of our Theorem is based only on the decay rates of $\boldsymbol{L}^2$ and $\boldsymbol{L}^1$-norms, so the results of this paper hold also for weak solutions of the equations treated in [4].

In Section 3 we consider the same problem as in Theorem for a perturbation problem of the stationary Navier-Stokes flows $\boldsymbol{w}$ in $\mathbb{R}^n$ satisfying

$$|\boldsymbol{w}| \le C/(1+|x|), \qquad |\nabla \boldsymbol{w}| \le C/(1+|x|)^2.$$

Invoking the properties of the semigroup which solves the corresponding linearized problem, we show that the same result as in the above Theorem holds provided that $n/(n+1) < p < 1$. Our result in this case will be stated in Theorem 3.4.

2. Proof of Theorem

Recall that (see [7, 15]) every weak solution $\boldsymbol{u}$ of (NS) satisfies

$$
\begin{aligned}
\langle \boldsymbol{u}(t), \boldsymbol{\varphi} \rangle &= \langle e^{-tA}\boldsymbol{a}, \boldsymbol{\varphi} \rangle - \int_0^t \langle \boldsymbol{u} \cdot \nabla \boldsymbol{u}, e^{-(t-s)A} P\boldsymbol{\varphi} \rangle ds \\
&= \langle e^{-tA}\boldsymbol{a}, \boldsymbol{\varphi} \rangle - \int_0^{t/2} \langle \boldsymbol{u} \cdot \nabla \boldsymbol{u}, e^{-(t-s)A} P\boldsymbol{\varphi} \rangle ds \\
&\quad - \int_{t/2}^t \langle \boldsymbol{u} \cdot \nabla \boldsymbol{u}, e^{-(t-s)A} P\boldsymbol{\varphi} \rangle ds
\end{aligned} \tag{2.1}
$$

for all $\boldsymbol{\varphi} \in \boldsymbol{C}_c^\infty(\mathbb{R}^n)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing. Here,

$$P\boldsymbol{\varphi} = (I + R \otimes R) \cdot \boldsymbol{\varphi}$$

is the bounded projector onto the subspace of solenoidal vector fields and $R = (R_1, \cdots, R_n)$ are the Riesz transforms ([13]), which are written via the Fourier transform as

$$\widehat{R_j f}(\xi) \equiv \int e^{-ix\cdot\xi}(R_j f)(x)dx = \frac{i\xi_j}{|\xi|}\hat{f}(\xi), \qquad (i = \sqrt{-1},\ \ j = 1, \cdots, n).$$

Recall also that (see [8]) the operators P and e^{-tA} are bounded on both of VMO and $\mathfrak{C}_0^\alpha$, and we have the estimates

$$\|e^{-tA}\boldsymbol{a}\|_{H^q} \le Ct^{-(n/p-n/q)/2}\|\boldsymbol{a}\|_{H^p} \quad (0 < p \le q), \qquad [e^{-tA}\boldsymbol{a}]_{\mathrm{VMO}} \le C[\boldsymbol{a}]_{\mathrm{VMO}}, \tag{2.2}$$

$$[e^{-tA}\boldsymbol{a}]_\beta \le Ct^{-(\beta-\alpha)/2}[\boldsymbol{a}]_\alpha, \qquad [e^{-tA}\boldsymbol{a}]_\alpha \le Ct^{-\alpha/2}[\boldsymbol{a}]_{\mathrm{VMO}} \qquad (0 < \alpha \le \beta). \tag{2.3}$$

(i) If $\boldsymbol{a} \in \boldsymbol{L}_\sigma^2 \cap \boldsymbol{H}_\sigma^p$ for some p with $n/(n+1) < p < 1$, then (2.2) and the boundedness of the semigroup $\{e^{-tA}\}_{t\ge 0}$ in $\boldsymbol{L}_\sigma^2$ together imply that

$$\|e^{-tA}\boldsymbol{a}\|_2 \le C(1+t)^{-n/4-n(1/p-1)/2}.$$

Thus, the results in [6, 10, 11, 18] ensure the existence of a weak solution $\boldsymbol{u}$ such that

$$\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4-n(1/p-1)/2}. \tag{2.4}$$

A result of [8] then shows that the solution $\boldsymbol{u}$ satisfies

$$\|\boldsymbol{u}(t)\|_1 \le C\|\boldsymbol{u}(t)\|_{H^1} \le C(1+t)^{-n(1/p-1)/2}. \tag{2.5}$$

Here we recall the *Helmholtz decomposition* of the spaces $\boldsymbol{X}^p$, $\mathfrak{C}_0^\alpha$ and VMO:

$$\boldsymbol{X}^p = \boldsymbol{X}_\sigma^p \oplus \boldsymbol{X}_\pi^p, \qquad \mathfrak{C}_0^\alpha = \mathfrak{C}_\sigma^\alpha \oplus \mathfrak{C}_\pi^\alpha, \qquad \mathrm{VMO} = \mathrm{VMO}_\sigma \oplus \mathrm{VMO}_\pi$$

where σ means solenoidality and π means the subspace of vector fields which are the gradients of scalar functions. All these decompositions are immediately deduced from the fact that the operator P is bounded on $\boldsymbol{X}^p$, $\mathfrak{C}^\alpha$ and VMO, respectively. Moreover, we know (see [8]) that the space $\boldsymbol{C}_{0,\sigma}^\infty$ of smooth solenoidal vector fields with compact support in $\mathbb{R}^n$ is dense in VMO$_\sigma$. The same is true of the spaces $\boldsymbol{X}^p$ and $\mathfrak{C}_0^\alpha$, as shown in the following

Lemma 2.1. (i) *Let $0 < \alpha < 1$. Then the space $\boldsymbol{C}_{0,\sigma}^\infty$ is dense in $\mathfrak{C}_\sigma^\alpha$.*

(ii) *Let $n/(n+1) < p < 1$. Then $\boldsymbol{C}_{0,\sigma}^\infty$ is dense in $\boldsymbol{X}_\sigma^p$.*

Proof. (i) First we note (see [16]) that $(\mathfrak{C}_0^\alpha)^*$ equals the homogeneous Besov space $\dot{\boldsymbol{B}}_{1,1}^{-\alpha}$. Suppose $\boldsymbol{f} \in (\mathfrak{C}_0^\alpha)^* = \dot{\boldsymbol{B}}_{1,1}^{-\alpha}$ annihilates $\boldsymbol{C}_{0,\sigma}^\infty$. By a theorem of De Rham, $\boldsymbol{f} = \nabla q$ for some scalar distribution q, and by the definition of $\dot{\boldsymbol{B}}_{1,1}^{-\alpha}$ (see [16]), we may assume that $q \in \dot{\boldsymbol{B}}_{1,1}^{1-\alpha}$. Since A is an isomorphism between $\dot{\boldsymbol{B}}_{1,1}^{\beta+2}$ and $\dot{\boldsymbol{B}}_{1,1}^{\beta}$ for all $\beta \in \mathbb{R}$, we have

$$\begin{aligned}\partial_j q &= AA^{-1}\partial_j q = -\partial_k A^{-1}\partial_j \partial_k q \\ &= -R_j R_k \partial_k q = [(I-P)\nabla q]_j\end{aligned}$$

for $j = 1, \cdots, n$. So we conclude that $\boldsymbol{f} = \nabla q \in \boldsymbol{X}_\pi^p$ with $\alpha = n(1/p - 1)$. Since $\boldsymbol{X}_\pi^p$ is defined to be the annihilator of $\mathfrak{C}_\sigma^\alpha$, it follows that $\boldsymbol{f} = 0$ on $\mathfrak{C}_\sigma^\alpha$. The result now follows from the Hahn–Banach theorem.

(ii) This time, we have $(\boldsymbol{X}^p)^* = (\dot{\boldsymbol{B}}_{1,1}^{-\alpha})^* = \dot{\boldsymbol{B}}_{\infty,\infty}^{\alpha} = \mathfrak{C}^\alpha$, with $\alpha = n(1/p-1)$, and so $\boldsymbol{f} = \nabla q \in \mathfrak{C}^\alpha$. Hence we may assume $q \in \mathfrak{C}^{\alpha+1}$. Since A is an isomorphism between $\mathfrak{C}^{\beta+2}$ and $\mathfrak{C}^\beta$ for all $\beta \in \mathbb{R}$, the same calculation as in the proof of (i) shows that $\boldsymbol{f}$ is in $N(P)$, where P is regarded as a bounded projector on $\mathfrak{C}^\alpha$. Hence, $\boldsymbol{f} = 0$ on $\boldsymbol{X}_\sigma^p$ and this proves the result.

Suppose now that $\boldsymbol{a} \in \boldsymbol{L}_\sigma^2 \cap \boldsymbol{H}_\sigma^p$ for some $n/(n+1) < p < 1$ and write

$$1/p = 1/2 + 1/q, \qquad 1 < q < \infty.$$

A result of [5] then implies $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in \boldsymbol{H}^p$ if $\boldsymbol{u} \in \boldsymbol{L}_\sigma^q$, with estimate

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{u}\|_{H^p} \le C\|\boldsymbol{u}\|_q \|\nabla \boldsymbol{u}\|_2. \tag{2.6}$$

From (2.1), (2.2) and (2.6) it follows that, with $p < r < 1$ and $\beta = n(1/r - 1)$,

$$\begin{aligned}|\langle \boldsymbol{u}(t), \boldsymbol{\varphi}\rangle| &\le C\left((1+t)^{-(n/p-n/r)/2} + \int_0^{t/2} (t-s)^{-(n/p-n/r)/2}\|\boldsymbol{u}\|_q \|\nabla \boldsymbol{u}\|_2 ds\right) \cdot [\boldsymbol{\varphi}]_\beta \\ &\quad + C\int_{t/2}^t \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H^r} ds \cdot [\boldsymbol{\varphi}]_\beta.\end{aligned}$$

Since

$$\|\boldsymbol{u}(s)\|_q \le C(1+s)^{-(n/p-n/q)/2} = C(1+t)^{-n/4},$$

we see by (2.6) that, with $1/r' = 1/r - 1/2$,

$$\begin{aligned}
\|\boldsymbol{u}(t)\|_{X^r} &\le C\left((1+t)^{-(n/p-n/r)/2} + t^{-(n/p-n/r)/2}\int_0^{t/2} \|\boldsymbol{u}\|_q \|\nabla \boldsymbol{u}\|_2 ds\right) \\
&\quad + C\int_{t/2}^t \|\boldsymbol{u}\|_{r'}\|\nabla \boldsymbol{u}\|_2 ds \\
&\le C\left((1+t)^{-(n/p-n/r)/2} + t^{-(n/p-n/r)/2}\int_0^{t/2} (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds\right) \\
&\quad + C\int_{t/2}^t \|\boldsymbol{u}\|_1^\theta \|\boldsymbol{u}\|_2^{1-\theta}\|\nabla \boldsymbol{u}\|_2 ds,
\end{aligned}$$

where $\theta = 2(1/r - 1)$. Direct calculation using (2.4) and (2.5) gives

$$\begin{aligned}
\|\boldsymbol{u}(s)\|_1^\theta \|\boldsymbol{u}(s)\|_2^{1-\theta} &\le C(1+s)^{-\theta n(1/p-1)/2}(1+s)^{-(1-\theta)(n/4+n(1/p-1)/2} \\
&= C(1+s)^{-(n/p-n/r)/2-n/4}.
\end{aligned}$$

We thus obtain

$$\|\boldsymbol{u}(t)\|_{X^r} \le C\left((1+t)^{-(n/p-n/r)/2} + t^{-(n/p-n/r)/2}\int_0^t (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds\right). \tag{2.7}$$

Suppose first that $n \ge 3$. Since $n/2 > 1$, we have

$$\int_0^t (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds \le \left(\int_0^\infty (1+s)^{-n/2} ds\right)^{1/2}\left(\int_0^\infty \|\nabla \boldsymbol{u}\|_2^2 ds\right)^{1/2} \le C\|\boldsymbol{a}\|_2.$$

Here we have used that $\int_0^\infty \|\nabla \boldsymbol{u}\|_2^2 ds \le \frac{1}{2}\|\boldsymbol{a}\|_2^2$, which follows from the energy inequality

$$\|\boldsymbol{u}(t)\|_2^2 + 2\int_0^t \|\nabla \boldsymbol{u}\|_2^2 ds \le \|\boldsymbol{a}\|_2^2 \qquad \text{for all} \quad t \ge 0.$$

Hence, we obtain (1.2) from (2.7) if $n \ge 3$.

When $n = 2$, we have

$$\begin{aligned}
\int_0^t (1+s)^{-1/2}\|\nabla \boldsymbol{u}\|_2 ds &\le \int_0^\infty (1+s)^{-1/2}\|\nabla \boldsymbol{u}\|_2 ds \\
&= \left(\int_0^1 + \int_1^\infty\right)(1+s)^{-1/2}\|\nabla \boldsymbol{u}\|_2 ds
\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty (1+s)^{-1/2}\|\nabla \boldsymbol{u}\|_2 ds &= \sum_{\ell=0}^\infty \int_{2^\ell}^{2^{\ell+1}} (1+s)^{-1/2}\|\nabla \boldsymbol{u}\|_2 ds \\ &\le \sum_{\ell=0}^\infty \left(\int_{2^\ell}^{2^{\ell+1}} (1+s)^{-1} ds\right)^{1/2} \left(\int_{2^\ell}^{2^{\ell+1}} \|\nabla \boldsymbol{u}\|_2^2 ds\right)^{1/2} \\ &\le C\sum_{\ell=0}^\infty \|\boldsymbol{u}(2^\ell)\|_2 \le C\sum_{\ell=0}^\infty (1+2^\ell)^{1/2-1/p} < +\infty.\end{aligned}$$

Here we have used (2.4) and the fact that

$$\int_s^t \|\nabla \boldsymbol{u}\|_2^2 d\tau \le \frac{1}{2}\|\boldsymbol{u}(s)\|_2^2 \qquad (0 \le s \le t),$$

which follows from the energy equality, valid for $n = 2$ (see [7, 15]):

$$\|\boldsymbol{u}(t)\|_2^2 + 2\int_s^t \|\nabla \boldsymbol{u}\|_2^2 d\tau = \|\boldsymbol{u}(s)\|_2^2 \qquad (0 \le s \le t).$$

This, together with (2.7), implies (1.2) for $n = 2$.

To prove (1.3), we note that (2.1) implies

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{X^p} &\le C\left(\|e^{-tA}\boldsymbol{a}\|_{H^p} + \int_0^t \|e^{-(t-s)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{H^p} ds\right) \\ &\le C\left(\|e^{-tA}\boldsymbol{a}\|_{H^p} + \int_0^M \|e^{-(t-s)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{H^p} ds\right) \\ &\quad + C\int_M^t \|\boldsymbol{u}\|_r \|\nabla \boldsymbol{u}\|_2 ds,\end{aligned}$$

with $0 < M < t$ and $1/r = 1/p - 1/2$. Inserting $\|\boldsymbol{u}(s)\|_r \le C(1+s)^{-n/4}$ gives

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{X^p} &\le C\left(\|e^{-tA}\boldsymbol{a}\|_{H^p} + \int_0^M \|e^{-(t-s)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{H^p} ds\right) \\ &\quad + C\int_M^t (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds.\end{aligned}$$

Here we take an arbitrary $\varepsilon > 0$ and then choose $M > 0$ so that

$$C\int_M^\infty (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds < \varepsilon,$$

which is possible since

$$\int_0^\infty (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds < +\infty \tag{2.8}$$

as we have shown in the proof of (1.2). On the other hand, the functions $\varphi \in \mathscr{S}$ such that $\int x^\alpha \varphi dx = 0$ for every multi-index α are dense in the spaces $\boldsymbol{H}^p$ $(0 < p \le 1)$ (see [13]). This, together with (2.2) and the uniform boundedness of the operators $e^{-tA} : \boldsymbol{H}^p \to \boldsymbol{H}^p$, implies that

$$\lim_{t\to\infty} \|e^{-tA}\boldsymbol{a}\|_{H^p} = 0 \qquad \text{for all } \ 0 < p \le 1.$$

We thus conclude that

$$\lim_{t\to\infty} \|e^{-(t-s)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{H^p} = 0$$

provided $P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \in \boldsymbol{H}^p$ for some $n/(n+1) < p \le 1$. Furthermore, we have

$$\|e^{-(t-s)A}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{H^p} \le C\|\boldsymbol{u}\|_r\|\nabla\boldsymbol{u}\|_2 \le C(1+s)^{-n/4}\|\nabla\boldsymbol{u}\|_2$$

with $C > 0$ independent of s and the right-hand side is integrable with respect to s over the interval $[0, M]$. We can thus apply the dominated convergence theorem to get

$$\limsup_{t\to\infty} \|\boldsymbol{u}(t)\|_{X^p} \le C\int_M^\infty (1+s)^{-n/4}\|\nabla\boldsymbol{u}\|_2 ds < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves (1.3).

(ii) To treat the case $0 < p \le n/(n+1)$, we need the weak Hardy space $\boldsymbol{H}^p_w$. To this end, recall that a function f on $\mathbb{R}^n$ is in the weak L^p space, denoted $L^p_w(\mathbb{R}^n)$ $(0 < p < \infty)$, if and only if

$$\|f\|_{p,w} \equiv \sup_{t>0} t|\{x : |f(x)| > t\}|^{1/p} < +\infty.$$

A (vector-valued) tempered distribution $\boldsymbol{f}$ is in the weak Hardy space $\boldsymbol{H}^p_w = \boldsymbol{H}^p_w(\mathbb{R}^n)$ if

$$\|\boldsymbol{f}\|_{H^p_w} \equiv \left\|\sup_{t>0}|\varphi_t * \boldsymbol{f}|\right\|_{p,w} < +\infty,$$

where φ_t denotes the Friedrichs mollifier. Obviously, $\boldsymbol{H}^p \subset \boldsymbol{H}^p_w$ with continuous injection; and, as is well known (see [13]), we have the following characterization:

$$\boldsymbol{H}^p_w = (\boldsymbol{H}^{p_0}, \boldsymbol{H}^{p_1})_{\theta,\infty}, \qquad (p_0 \ne p_1, \quad 0 < \theta < 1, \quad 1/p = (1-\theta)/p_0 + \theta/p_1),$$

where the right-hand side is the real interpolation space ([1]). Thus, if $0 < p_0 < p < p_1 < 1$ and if we write $\alpha_0 = n(1/p_0 - 1)$, $\alpha_1 = n(1/p_1 - 1)$, $1/p = (1-\theta)/p_0 + \theta/p_1$, then the space

$$\boldsymbol{Y} = (\mathfrak{C}^{\alpha_0}_0, \mathfrak{C}^{\alpha_1}_0)_{\theta,1} \tag{2.9}$$

is defined independent of the choice of p_0 and p_1, and we see that

$$\boldsymbol{H}^p_w \subset \boldsymbol{X}^{n/(n+1)} \equiv \boldsymbol{Y}^* \qquad \text{with continuous injection.} \tag{2.10}$$

Applying the real interpolation theory ([1]) gives

$$\begin{aligned} &\|e^{-tA}\boldsymbol{a}\|_{H^p_w} \le C\|\boldsymbol{a}\|_{H^p_w}, \qquad \|e^{-tA}\boldsymbol{a}\|_{H^q} \le Ct^{-(n/p-n/q)/2}\|\boldsymbol{a}\|_{H^p_w} \quad (0 < p < q \le 1), \\ &\|e^{-tA}\boldsymbol{a}\|_{X^q} \le Ct^{-(n/p-n/q)/2}\|\boldsymbol{a}\|_{X^p} \quad (n/(n+1) < p < q \le 1) \end{aligned} \tag{2.11}$$

and, with $\gamma = n+1-n/q \geq 0$,

$$\|e^{-tA}\boldsymbol{a}\|_{X^q} \leq Ct^{-\gamma/2}\|\boldsymbol{a}\|_{X^{n/(n+1)}} \leq Ct^{-\gamma/2}\|\boldsymbol{a}\|_{H_w^{n/(n+1)}} \qquad (n/(n+1) \leq q). \tag{2.12}$$

Now let $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^p_\sigma$ for some p with $0 < p \leq n/(n+1)$. As shown in [8, 18], we have

$$\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-n/4-1/2}, \qquad \|\boldsymbol{u}(t)\|_1 \leq C(1+t)^{-1/2}. \tag{2.13}$$

On the other hand, it is proved in [5, 8] that if

$$1 + 1/n = 1/2 + 1/q, \quad 1 \leq q < \infty, \quad n \geq 2,$$

and if $\boldsymbol{u} \in \boldsymbol{L}^q_\sigma$, then $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \in \boldsymbol{H}^{n/(n+1)}_w$ and we have the estimate

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H_w^{n/(n+1)}} \leq C\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{u}\|_2. \tag{2.14}$$

It should be noticed here that (see [8, 9])

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} \in \dot{\boldsymbol{H}}^{n/(n+1)} \qquad \text{implies} \qquad \boldsymbol{u} \equiv 0.$$

Thus the result (2.14) is optimal.

Now fix $n/(n+1) < r < 1$. Invoking (2.9)–(2.14), we estimate (2.1) as follows:

$$\begin{aligned}|\langle \boldsymbol{u}(t), \boldsymbol{\varphi}\rangle| \;&\leq\; C\left((1+t)^{-(n/p-n/r)/2} + \int_0^{t/2}(t-s)^{-n(1+1/n-1/r)/2}\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{u}\|_2 ds\right)\cdot[\boldsymbol{\varphi}]_\beta \\ &\quad + C\int_{t/2}^t \|\boldsymbol{u}\|_{r'}\|\nabla\boldsymbol{u}\|_2 ds\cdot[\boldsymbol{\varphi}]_\beta\end{aligned}$$

where $1/r' = 1/r - 1/2$ and $\beta = n(1/r-1)$. But,

$$\|\boldsymbol{u}(t)\|_q \leq C(1+t)^{-n(1+1/n-1/q)/2} = C(1+t)^{-n/4},$$

so we get

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{X^r} \;&\leq\; C\left((1+t)^{-(n/p-n/r)/2} + t^{-n(1+1/n-1/r)/2}\int_0^{t/2}(1+s)^{-n/4}\|\nabla\boldsymbol{u}\|_2 ds\right) \\ &\quad + C\int_{t/2}^t \|\boldsymbol{u}\|_{r'}\|\nabla\boldsymbol{u}\|_2 ds.\end{aligned}$$

Since $1 < 1/r < 1 + 1/n$, it follows that $1/2 < 1/r' < 1/2 + 1/n \leq 1$. Thus, writing $\theta = 3 - 2/r$, we obtain

$$\begin{aligned}\|\boldsymbol{u}(s)\|_{r'} \;&\leq\; \|\boldsymbol{u}(s)\|_1^{1-\theta}\|\boldsymbol{u}(s)\|_2^\theta \leq C(1+s)^{-(1-\theta)/2}(1+s)^{-\theta(n/4+1/2)} \\ &=\; C(1+s)^{-1/2-n\theta/4} \leq C(1+s)^{-n(1+1/n-1/r)/2-n/4}.\end{aligned}$$

Therefore,

$$\|\boldsymbol{u}(t)\|_{X^r} \le C\left((1+t)^{-(n/p-n/r)/2} + t^{-n(1+1/n-1/r)/2}\int_0^t (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds\right). \tag{2.15}$$

Assertion (1.5) now follows from (2.8) and (2.15). This proves (ii).

Similarly, one can show the following, which extends (1.5) to the case $r = p = n/(n+1)$.

Corollary. *If* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma \cap \boldsymbol{X}^{n/(n+1)}$, *then*

$$\boldsymbol{u}(t) \in \boldsymbol{X}^{n/(n+1)} \quad \textit{for all} \quad t \ge 0, \tag{2.16}$$

and

$$\|\boldsymbol{u}(t)\|_{X^{n/(n+1)}} \le C \quad \textit{for all} \quad t \ge 0, \tag{2.17}$$

with $C > 0$ *independent of* t.

The space $\boldsymbol{X}^{n/(n+1)}$ is employed as a substitute for $\boldsymbol{H}^{n/(n+1)}$ or $\boldsymbol{L}^{n/(n+1)}$. We know nothing about the existence of a weak solution $\boldsymbol{u}$ which *never* decays in $\boldsymbol{X}^{n/(n+1)}$ as $t \to \infty$. The assumption on the initial data $\boldsymbol{a}$ holds, *e.g.*, if $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^{n/(n+1)}_w$. Indeed, in this case the real interpolation theory for Hardy spaces as given in [16, 17] implies $\boldsymbol{a} \in \boldsymbol{H}^1 \subset \boldsymbol{L}^1$.

Proof of Corollary. It suffices to prove (2.17). We take

$$0 < \alpha_0 < 1 < \alpha_1 \quad \text{and} \quad 0 < \theta < 1 \quad \text{with} \quad (1-\theta)\alpha_0 + \theta\alpha_1 = 1$$

so that

$$\boldsymbol{Y} = (\mathfrak{C}^{\alpha_0}_0, \mathfrak{C}^{\alpha_1}_0)_{\theta,1}.$$

Interpolating between the operators $e^{-tA} : \mathrm{VMO} \to \mathfrak{C}^{\alpha_k}_0$, $k = 0, 1$, we see that

$$\|e^{-tA}\boldsymbol{a}\|_Y \le Ct^{-1/2}[\boldsymbol{a}]_{\mathrm{VMO}}$$

and therefore

$$\|e^{-tA}\boldsymbol{a}\|_1 \le C\|e^{-tA}\boldsymbol{a}\|_{H^1} \le Ct^{-1/2}\|\boldsymbol{a}\|_{X^{n/(n+1)}}, \qquad \|e^{-tA}\boldsymbol{a}\|_2 \le Ct^{-n/4-1/2}\|\boldsymbol{a}\|_{X^{n/(n+1)}}.$$

Since $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$ by assumption, it follows that

$$\|e^{-tA}\boldsymbol{a}\|_1 \le C(1+t)^{-1/2}, \qquad \|e^{-tA}\boldsymbol{a}\|_2 \le C(1+t)^{-n/4-1/2}.$$

Hence, by [8, 18] there exists a weak solution $\boldsymbol{u}$ of (NS) such that

$$\|\boldsymbol{u}(t)\|_1 \le C(1+t)^{-1/2}, \qquad \|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4-1/2}. \tag{2.18}$$

We can now prove (2.17). This time, we apply (2.9), (2.11) and (2.14) to (2.1), to get

$$|\langle \boldsymbol{u}(t), \boldsymbol{\varphi}\rangle| \le \Big(\|e^{-tA}\boldsymbol{a}\|_{X^{n/(n+1)}} + C\int_0^t \|\boldsymbol{u}\|_q \|\nabla \boldsymbol{u}\|_2 ds\Big)\cdot \|\boldsymbol{\varphi}\|_Y$$

with $1/q = 1/2 + 1/n$. Since (2.18) implies $\|\boldsymbol{u}(s)\|_q \le C(1+s)^{-n/4}$, we obtain

$$\|\boldsymbol{u}(t)\|_{X^{n/(n+1)}} \le \|e^{-tA}\boldsymbol{a}\|_{X^{n/(n+1)}} + C\int_0^t (1+s)^{-n/4}\|\nabla \boldsymbol{u}\|_2 ds.$$

Since $\|e^{-tA}\boldsymbol{a}\|_{X^{n/(n+1)}} \le C\|\boldsymbol{a}\|_{X^{n/(n+1)}}$, we get (2.17) from (2.8). This proves the Corollary.

3. Decay of perturbations of stationary flows in $\mathbb{R}^n$, $n \ge 3$

In this section we assume $n \ge 3$ and study the decay properties of solutions $\boldsymbol{u}$ of

$$\partial_t \boldsymbol{u} + (A+B)\boldsymbol{u} + P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = 0 \quad (t>0), \qquad \boldsymbol{u}(0) = \boldsymbol{a}, \tag{3.1}$$

where

$$A\boldsymbol{u} = -\Delta \boldsymbol{u}, \qquad B\boldsymbol{u} = P(\boldsymbol{w}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{w}),$$

and $\boldsymbol{w}$ is a smooth solenoidal vector field on $\mathbb{R}^n$ satisfying

$$|\boldsymbol{w}| \le C/(1+|x|), \qquad |\nabla \boldsymbol{w}| \le C/(1+|x|)^2. \tag{3.2}$$

As is shown in [8, 9], there exist stationary flows $\boldsymbol{w}$ in $\mathbb{R}^n$ with decay property (3.2); and in this case problem (3.1) is the perturbation problem for stationary flows as treated in [3, 9].

In terms of the semigroup $\{e^{-tL}\}_{t\ge 0}$, with $L = A+B$, we can write (3.1) in the form

$$\boldsymbol{u}(t) = e^{-tL}\boldsymbol{a} - \int_0^t e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\tau)d\tau \tag{3.3}$$

or, with $0 \le s \le t$,

$$\langle \boldsymbol{u}(t), \boldsymbol{\varphi}\rangle = \langle \boldsymbol{u}(s), e^{-(t-s)L^*}\boldsymbol{\varphi}\rangle - \int_s^t \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}\boldsymbol{\varphi}\rangle d\tau \qquad (\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}). \tag{3.4}$$

To deduce our desired result (Theorem 3.4 below), we need first to discuss properties of the semigroup $\{e^{-tL^*}\}_{t\ge 0}$ in the homogeneous Hölder-Zygmund spaces $\mathfrak{C}^\alpha$, $0<\alpha<1$. To this end, we establish the following lemma. In what follows we write

$$\|\boldsymbol{w}\| = \sup |x|\cdot|\boldsymbol{w}(x)| \qquad \text{and} \qquad \|\nabla\boldsymbol{w}\| = \sup |x|^2|\nabla\boldsymbol{w}(x)|,$$

and $[\,\cdot\,]_\alpha$ denotes the norm of $\mathfrak{C}^\alpha_0$ if $\alpha > 0$. Furthermore, we understand that $[\,\cdot\,]_0 = [\,\cdot\,]_{\mathrm{VMO}}$.

Lemma 3.1. (i) *Let $n/(n+1) < p \le q \le 1$. For $0 < \omega < \pi/2$, there exists a constant $\eta = \eta_{p,\omega} > 0$ such that if*

$$\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\| \le \eta,$$

then the estimate

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_{H^q} \le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-(n/p-n/q)/2} \tag{3.5}$$

holds for $\lambda \in \mathbb{C}\setminus\{0\}$ *satisfying* $|\arg\lambda| \le \pi-\omega$.

(ii) *Let* $0<\alpha\le\beta<1$. *For* $0<\omega<\pi/2$, *there is a constant* $\eta'=\eta'_{\alpha,\omega}>0$ *such that if*

$$\|\boldsymbol{w}\|+\|\nabla\boldsymbol{w}\|\le\eta,$$

then the estimate

$$[(\lambda+L^*)^{-1}\boldsymbol{a}]_\beta \le C[\boldsymbol{a}]_\alpha/|\lambda|^{1-(\beta-\alpha)/2} \tag{3.6}$$

holds for $\lambda \in \mathbb{C}\setminus\{0\}$ *satisfying* $|\arg\lambda| \le \pi-\omega$.

(iii) *Let* $n/(n+1)<p\le q\le 1$. *For* $0<\omega<\pi/2$ *there is a constant* $\mu>0$ *such that if*

$$\|\boldsymbol{w}\|+\|\nabla\boldsymbol{w}\|\le\mu,$$

then we have

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_{X^q} \le C\|\boldsymbol{u}\|_{X^p}/|\lambda|^{1-(n/p-n/q)/2} \tag{3.7}$$

for $\lambda \in \mathbb{C}\setminus\{0\}$ *satisfying* $|\arg\lambda| \le \pi-\omega$.

Proof. Assertion (iii) is immediately obtained via duality from (ii), and assertion (ii) follows from (i) via duality. So we need only prove (i). Since $(n/p-n/q)/2<1$, the Hardy–Littlewood–Sobolev inequality for fractional integrals in Hardy spaces ([13]) gives

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_{H^q} \le C\|A^\gamma(\lambda+L)^{-1}\boldsymbol{u}\|_{H^p} \qquad (\gamma=(n/p-n/q)/2).$$

To estimate the right-hand side we invoke the (formal) Neumann series expansion

$$A^\gamma(\lambda+L)^{-1}\boldsymbol{u} = A^\gamma(\lambda+A)^{-1}\sum_{k=0}^{\infty}[-B(\lambda+A)^{-1}]^k\boldsymbol{u}.$$

Applying the Mikhlin multiplier theorem gives

$$\|A^\gamma(\lambda+L)^{-1}\boldsymbol{u}\|_{H^p}^p \le C^p|\lambda|^{p(\gamma-1)}\left\|\sum_{k=0}^{\infty}[-B(\lambda+A)^{-1}]^k\boldsymbol{u}\right\|_{H^p}^p.$$

Since the quantity $\|\cdot\|_{H^p}^p$ satisfies the triangle inequality, and since (see [9])

$$\|B(\lambda+A)^{-1}\boldsymbol{u}\|_{H^p}^p \le C^p(\|\boldsymbol{w}\|^p+\|\nabla\boldsymbol{w}\|^p)\|\boldsymbol{u}\|_{H^p}^p,$$

we see that if $\|\boldsymbol{w}\|+\|\nabla\boldsymbol{w}\|$ is small enough, then

$$\left\|\sum_{k=0}^{\infty}[-B(\lambda+A)^{-1}]^k\boldsymbol{u}\right\|_{H^p}^p \le \sum_{k=0}^{\infty}[C^p(\|\boldsymbol{w}\|^p+\|\nabla\boldsymbol{w}\|^p)]^k\|\boldsymbol{u}\|_{H^p}^p \le C\|\boldsymbol{u}\|_{H^p}^p.$$

This shows (i).

Now, $\mathfrak{C}_0^\alpha$ and VMO are Banach spaces, so we can invoke the representation

$$e^{-tL^*} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + L^*)^{-1} d\lambda$$

with an appropriate choice of Γ, to deduce part (i) of the following

Corollary 3.2. (i) *We have the estimates*

$$[e^{-tL^*}\boldsymbol{a}]_\beta \le Ct^{-(\beta-\alpha)/2}[\boldsymbol{a}]_\alpha \qquad \text{for } 0 \le \alpha \le \beta < 1. \tag{3.8}$$

Here we understand that $[\,\cdot\,]_0 = [\,\cdot\,]_{\mathrm{VMO}}$.

(ii) *Let* $n/(n+1) < p \le 1$. *If* $\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|$ *is small depending on* p, *we have*

$$\lim_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_{X^p} = 0 \qquad \text{for all } \boldsymbol{a} \in \boldsymbol{X}_\sigma^p. \tag{3.9}$$

Proof. We need only show (ii). Note that the semigroup $\{e^{-tL}\}_{t\ge 0}$ is defined on $\boldsymbol{X}^p$ as the dual semigroup of $\{e^{-tL^*}\}_{t\ge 0}$. Thus, there holds the estimate

$$\|Le^{-tL}\| \le Ct^{-1}. \tag{3.10}$$

In view of (3.10), it suffices to show that

$$\overline{R(L)} = \boldsymbol{X}^p \tag{3.11}$$

in order to get (3.9). Indeed, if $\boldsymbol{a} \in R(L)$ with $\boldsymbol{a} = L\boldsymbol{b}$, then (3.10) yields

$$\|e^{-tL}\boldsymbol{a}\|_{X^p} = \|Le^{-tL}\boldsymbol{b}\|_{X^p} \le Ct^{-1}\|\boldsymbol{b}\|_{X^p} \to 0 \qquad \text{as } t \to \infty. \tag{3.12}$$

Now, if (3.11) holds, then for each $\boldsymbol{a} \in \boldsymbol{X}^p$ and each $\varepsilon > 0$, there is a $\boldsymbol{b} \in R(L)$ such that $\|\boldsymbol{a} - \boldsymbol{b}\|_{X^p} < \varepsilon$. From (3.12) and

$$\begin{aligned}\|e^{-tL}\boldsymbol{a}\|_{X^p} &\le \|e^{-tL}(\boldsymbol{a}-\boldsymbol{b})\|_{X^p} + \|e^{-tL}\boldsymbol{b}\|_{X^p} \\ &\le C\|\boldsymbol{a}-\boldsymbol{b}\|_{X^p} + \|e^{-tL}\boldsymbol{b}\|_{X^p} \le C\varepsilon + \|e^{-tL}\boldsymbol{b}\|_{X^p},\end{aligned}$$

it follows that

$$\limsup_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_{X^p} \le C\varepsilon + \lim_{t\to\infty} \|e^{-tL}\boldsymbol{b}\|_{X^p} = C\varepsilon,$$

which shows the desired result (3.9).

It thus remains to prove (3.11). To do so we prepare

Lemma 3.3. *We have the estimate*

$$\|B\boldsymbol{u}\|_{X^p} \le C(\|\boldsymbol{w}\|^p + \|\nabla \boldsymbol{w}\|^p)^{1/p}\|A\boldsymbol{u}\|_{X^p}. \tag{3.13}$$

Admitting Lemma 3.3 for a moment, we shall complete the prof of (3.11). Let

$$\widehat{\boldsymbol{D}} = \text{the completion of } D(A) \text{ in the norm } \|\boldsymbol{u}\|_{A,p} = \|A\boldsymbol{u}\|_{X^p}$$

and consider in the space $\widehat{\boldsymbol{D}}$ the equation

$$L\boldsymbol{u} = \boldsymbol{f} \tag{3.14}$$

for any given $\boldsymbol{f} \in \boldsymbol{X}^p$. Equation (3.14) is rewritten as

$$\boldsymbol{u} = A^{-1}(\boldsymbol{f} - B\boldsymbol{u}) \equiv \Phi\boldsymbol{u}$$

and by Lemma 3.3 the affine map Φ satisfies

$$\begin{aligned} \|\Phi\boldsymbol{u}\|_{A,p} &\le \|\boldsymbol{f}\|_{X^p} + C(\|\boldsymbol{w}\|^p + \|\nabla\boldsymbol{w}\|^p)^{1/p}\|\boldsymbol{u}\|_{A,p} \\ \|\Phi\boldsymbol{u} - \Phi\boldsymbol{v}\|_{A,p} &\le C(\|\boldsymbol{w}\|^p + \|\nabla\boldsymbol{w}\|^p)^{1/p}\|\boldsymbol{u} - \boldsymbol{v}\|_{A,p}. \end{aligned}$$

Thus, by the contraction mapping principle, equation (3.14) has a unique solution $\boldsymbol{u} \in \widehat{\boldsymbol{D}}$ for any given $\boldsymbol{f} \in \boldsymbol{X}^p$ provided that $\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\|$ is small enough.

To complete the proof of (3.11), we have to show that A is injective on $\boldsymbol{X}^p$. But, this is obvious since $\boldsymbol{X}^p = \dot{\boldsymbol{B}}_{1,1}^{-\alpha}$ with $\alpha = n(1/p-1)$ and, for all $s \in \mathbb{R}$, the operator A is isomorphic from $\dot{\boldsymbol{B}}_{1,1}^{s}$ onto $\dot{\boldsymbol{B}}_{1,1}^{s-2}$.

Proof of Lemma 3.3. We already know that (see [9])

$$\|BA^{-1}\boldsymbol{v}\|_{H^p} \le C(\|\boldsymbol{w}\|^p + \|\nabla\boldsymbol{w}\|^p)^{1/p}\|\boldsymbol{v}\|_{H^p}.$$

Applying duality argument two times yields (3.13).

Using the above results, we can prove the following result in the same way as in Section 2.

Theorem 3.4. *Let $n/(n+1) < p < 1$. There exists a number $\mu > 0$ depending only on n and p such that if $\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\| \le \mu$, then for each $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^p$, problem (3.1) has a weak solution $\boldsymbol{u}$ satisfying*

$$\|\boldsymbol{u}(t)\|_{X^r} = O(t^{-(n/p-n/r)/2}) \qquad \text{as } t \to \infty \tag{3.15}$$

for all r such that $p < r \le 1$, and

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_{X^p} = 0. \tag{3.16}$$

The case $p = n/(n+1)$ remains an open problem. The proof of Theorem 3.4 is the same as in the previous section, if we use the estimates

$$(3.17) \qquad \|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4-\beta/2}, \qquad \|\boldsymbol{u}(t)\|_{H^1} \le C(1+t)^{-\beta/2} \qquad (\beta = n(1/p-1)),$$

which were deduced in [9]. The details are omitted here.

Remark. In (3.17), it is impossible to replace the $\boldsymbol{H}^1$-norm $\|\cdot\|_{H^1}$ by the $\boldsymbol{L}^1$-norm $\|\cdot\|_1$. If one wants to use $\|\cdot\|_1$ instead of $\|\cdot\|_{H^1}$, then, as we have shown in [9], one must replace assumption (3.2) on $\boldsymbol{w}$ by

$$(3.2') \qquad \boldsymbol{w} \in \boldsymbol{L}^{(n,1)}(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}^{(n/2,1)}(\mathbb{R}^n),$$

where $\boldsymbol{L}^{(p,q)}$ stands for the Lorentz space ([1, 13, 16, 17]) with norm $\|\cdot\|_{(p,q)}$. In this case, the statement of Theorem 3.4 holds with $\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|$ replaced by $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)}$. The details are given in [9].

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STABILITY AND BIFURCATIONS OF TRAVELLING WAVES OF REACTION DIFFUSION EQUATIONS

SHUNSAKU NII

ABSTRACT.
Stability problem of travelling waves of bistable type is treated. Especially, travelling waves which emerge from heteroclinic loop via heteroclinic bifurcation is focused.

1. TRAVELLING WAVES AND STABILITY

Consider a one-dimensional reaction diffusion system:

$$u_t = Bu_{xx} + F(u), \tag{1.1}$$

where $t \geq 0$, $x \in \mathbb{R}$, $u(x,t) \in \mathbb{R}^n$, B is an $n \times n$ positive diagonal matrix and $F\colon \mathbb{R}^n \to \mathbb{R}^n$.

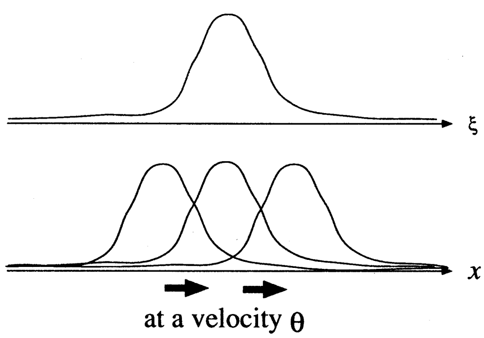
Set $\xi = x - \theta t$, then on the moving frame (ξ, t) this equation is expressed as follows.

$$u_t = Bu_{\xi\xi} + \theta u_\xi + F(u). \tag{1.2}$$

A steady state solution of the equation (1.2); *i.e.* a solution $u(\xi)$ of the equation

$$Bu_{\xi\xi} + \theta u_\xi + F(u) = 0, \tag{1.3}$$

corresponds to a solution $u(x,t) = u(x - \theta t)$ of (1.1) which translates at a constant velocity θ preserving its profile. This kind of solutions are called travelling waves.



In this paper we restrict our attention to travelling waves of bistable type, which means $u(\xi)$ satisfies the boundary condition

$$\lim_{\xi \to \pm\infty} u(\xi) = u_\pm, \tag{1.4}$$

Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Urawa, 338, Japan

where we assume that $u \equiv u_\pm$ are stable steady state constant solutions of (1.1). In such a case, $(u(\xi), u'(\xi))$ becomes a heteroclinic solution connecting $(u_-, 0)$ and $(u_+, 0)$ of the following first order system.

$$\begin{cases} u' = v \\ v' = -B^{-1}F(u) - \theta B^{-1}v. \end{cases} \qquad (\,' = \tfrac{d}{d\xi}) \tag{1.5}$$

The existence problem of such travelling waves or, equivalently, the existence problem of heteroclinic (homoclinic) orbits of (1.5), already commands a large body of literature, and it is also one of the main sources of motivation for the development of bifurcation theory for homoclinic or heteroclinic orbits of vector fields.

In this paper, we investigate stability of travelling waves. Formally, we define stability of a wave in the following way:

Definition 1.1. *A travelling wave $u_0(\xi)$ is said to be asymptotically stable relative to* (1.2) *if there exists a neighborhood N of u_0 in $BU(\mathbb{R}, \mathbb{R}^n)$ such that each solution $u(\xi, t)$ of* (1.2) *that starts in N at $t = 0$ satisfies*

$$\|u(\xi, t) - u_0(\xi + k)\|_\infty \to 0 \quad (t \to +\infty) \tag{1.6}$$

for some $k \in \mathbb{R}$ depending on $u(\xi, t)$, where $BU(\mathbb{R}, \mathbb{R}^n) = \{v : \mathbb{R} \to \mathbb{R}^n | bounded\ uniformly\ continuous\}$.

Remark 1.1.
If $u_0(\xi)$ is a travelling wave, then so is $u_0(\xi + k)$.

Thanks to the well-known fact of infinite-dimensional dynamical system below, linearised eigenvalue problem suffices to prove stability of a given wave.

Fact ([1],[4] and [9]).
$u_0(\xi)$ is asymptotically stable if the spectrum $\sigma(L)$ of the linearisation L of (1.2) *at $u_0(\xi)$ (i.e. $LP := BP_{\xi\xi} + \theta P_\xi + DF(u_0(\xi))\,P$) satisfies the following:*

(1) *there exists $\beta < 0$ such that $\sigma(L) \setminus \{0\} \subset \{\lambda | Re\lambda < \beta\}$;*
(2) 0 *is a simple eigenvalue* (0 *is an eigenvalue corresponding to translation*).

Remark 1.2.
If $u_\pm$ are stable for (1.1) *then there exists a simple closed curve K and a constant $\beta < 0$ such that*

$$\sigma(L) \cap \{\lambda | Re\lambda > \beta\} \subset K^\circ, \tag{1.7}$$

where K° is the interior enclosed by K. Moreover $\sigma(L) \cap K^\circ$ consists of isolated eigenvalues with finite multiplicity.

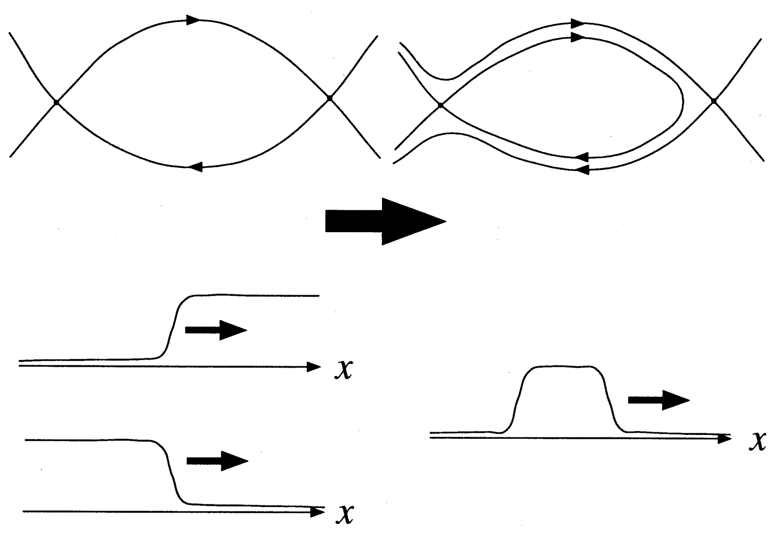
The equation (1.5) may undergo various bifurcations. Especially, if a heteroclinic orbit or a homoclinic orbit emerges through the bifurcation, a travelling wave corresponding to it appear. In this situation, it is natural to expect that there should be some relation between the structure of the bifurcation and stability of the wave which is produced by the bifurcation.

Yanagida and Maginu's early work [18] treated stability of double pulse solutions corresponding to double homoclinic orbits generated from a Shil'nikov type homoclinic orbit. In that paper, crossing direction of stable and unstable manifold of the equilibrium when the wave speed θ varies played and essential role. This result was, recently, generalized by Alexander and Jones [2] [3].

The topic of this paper is stability of tarvelling waves corresponding to homoclinic or heteroclinic orbits bifurcating from what is called a heteroclinic loop.

2. Stability of pulses

This section is devoted to stability of travelling waves which correspond to homoclinic orbits bifurcating from a heteroclinic loop.



First, we summarise the frame work. Consider a system of ordinary differential equations on $\mathbb{R}^{2n}$ depending on a parameter $\mu \in \mathbb{R}^k$ $(k \geq 2)$:

$$\dot{x} = f(x) + g(x;\mu) \qquad (\dot{} = \tfrac{d}{dt}), \tag{2.1}$$

where f and g are smooth and $g(x;0)=0$. Assume (2.1) has three equilibria $\mathbf{O}_1$, $\mathbf{O}_2$ and $\mathbf{O}_3$ and the eigenvalues

$$-\eta^i_{n-1}(\mu),\dots,-\eta^i_1(\mu),-\rho^i(\mu),\nu^i(\mu),\kappa^i_1(\mu),\dots,\kappa^i_{n-1}(\mu)$$

of linearisations of F at each equilibrium satisfy

$$-Re\eta^i_{n-1}(0)\le\dots\le-Re\eta^i_1(0)<-\rho^i(0)<0<\nu^i(0)<Re\kappa^i_1(0)\le\dots\le Re\kappa^i_{n-1}(0).$$

Also assume that for $\mu=0$ the system

$$\dot{x}=f(x) \tag{2.2}$$

has heteroclinic orbits Γ_i of (2.2) from $\mathbf{O}_i$ to $\mathbf{O}_{i+1}$ $(i=1,2)$ simultaneously.

In what follows, we consider bifurcations of these heteroclinic orbits under the following non-degeneracy conditions.

(1) For each i, the heteroclinic orbit $\Gamma_i=\{h_i(t)\}$ is tangent to the eigenspace associated to the eigenvalue $\nu^i(0)$ of lenearisation of f at $\mathbf{O}_i$ as $t\to-\infty$ and the eigenspace to $-\rho^{i+1}(0)$ of $\mathbf{O}_{i+1}$ as $t\to+\infty$.

(2) For $\mu=0$, the unstable manifold $\mathfrak{W}^u(\mathbf{O}_i)$ and the stable manifold $\mathfrak{W}^s(\mathbf{O}_{i+1})$ $(i=1,2)$ has one-dimensional intersection *i.e.* for all $p\in\Gamma_i$

$$\dim\left(T_p\mathfrak{W}^u(\mathbf{O}_i)\cap T_p\mathfrak{W}^s(\mathbf{O}_{i+1})\right)=1.$$

(3) For $\mu\doteq0$, $\mathfrak{W}^u(\mathbf{O}_i)$ is transverse to the $(n+1)$-dimensional ν-stable manifold $\mathfrak{W}^{\nu,s}(\mathbf{O}_{i+1})$ $(i=1,2)$ which is invariant and is tangent to the sum of the eigenspaces corresponding to $\nu^{i+1}(0)$, $-\rho^{i+1}(0)$ and $-\eta^{i+1}_j$ $(1\le j\le n-1)$. Also $\mathfrak{W}^s(\mathbf{O}_{i+1})$ is transverse to the $(n+1)$-dimensional $(-\rho)$-unstable manifold corresponding to the eigenvalues $-\rho^i(0)$, $\nu^i(0)$ and $\kappa^i_k(0)$ $(1\le k\le n-1)$.

(4) For a non-trivial bounded solution $\hat{q}^i(t)$ $(i=1,2)$ of the linear system of ordinary differential equations

$$\dot{\hat{z}}=-{}^tDf(h_i(t))\hat{z}\qquad(i=1,2), \tag{2.3}$$

the vectors given by the integrals

$$\int_{-\infty}^{+\infty}\hat{q}^i(s)\cdot\frac{\partial}{\partial\mu}g(h_i(s);0)ds \tag{2.4}$$

are linearly independent, and hence non-zero.

Remark 2.1.
The bounded solution $\hat{q}^i(t)$ is unique up to multiplication by constants.

Under these conditions, the following holds.

Proposition 2.1 (Kokubu [11]).
If the conditions (1) − (4) *are satisfied, then there exist two hypersurfaces* M_i $(i = 1, 2)$ *of codimension* 1 *in a sufficiently small neighborhood of* $\mu = 0$ *in* $\mathbb{R}^k$, *so that each of* M_i *consists of parameter values* μ *for which the system has a heteroclinic orbit* Γ_i. *Moreover* M_1 *and* M_2 *intersect transversely at* $\mu = 0$.

The analysis of the existence of a heteroclinic orbit from $\mathbf{O}_1$ to $\mathbf{O}_3$ is divided into two cases.

(i) $\nu^2(0) \neq \rho^2(0)$ (the case of non-critical eigenvalues.)
(ii) $\nu^2(0) = \rho^2(0)$ (the case of critical eigenvalues.)

Proposition 2.2 (Kokubu [11]).
For the case of non-critical eigenvalues, there exists a hypersurface M *of codimension* 1 *with the boundary*

$$\partial M = M_1 \cap M_2$$

in a sufficiently small neighborhood of $\mu = 0$ *in* $\mathbb{R}^k$, *so that* M *consists of parameter values* μ *for which the system has a heteroclinic orbit* $\Gamma = \{h(t)\}$ *from* O_1 *to* O_3. *Moreover,*

(a) *if* $\nu^2(0) < \rho^2(0)$, *then* M *is tangent to* M_2 *at* $\mu = 0$;
(b) *if* $\nu^2(0) > \rho^2(0)$, *then* M *is tangent to* M_1 *at* $\mu = 0$.

For the case of critical eigenvalues, we impose a further condition:

(5) The set $\{\mu | \nu^2(\mu) = \rho^2(\mu)\}$ forms a surface Π in the parameter space $\mathbb{R}^k$ and is transverse to both of M_1 and M_2 at $\mu = 0$.

Proposition 2.3 (Kokubu [11]). *For the case of critical eigenvalues, there exists a hypersurface* M *of codimension* 1 *with the boundary*

$$\partial M = M_1 \cap M_2$$

in a sufficiently small neighborhood of $\mu = 0$ *in* $\mathbb{R}^k$, *so that* M *consists of parameter values* μ *for which the system has a heteroclinic orbit* $\Gamma = \{h(t)\}$ *from* O_1 *to* O_3. *Moreover* M *is tangent to neither* M_1 *nor* M_2 *at* $\mu = 0$ *in* Π.

For some reaction diffusion equations, this kind of bifurcations take place. In the sequel, let us assume that the system (1.5) undergoes the bifurcation as in Propositions 2.2 and 2.3 with $\mathbf{O}_1 = \mathbf{O}_3$. In such a case, the two heteroclinic orbits Γ_1 and Γ_2 form a loop called a heteroclinic loop and the orbit $\Gamma = \{h(\xi) | \xi \in \mathbb{R}\}$ bifurcating from the loop is a homoclinic orbit.

The structure of heteroclinc loop is classified according to the following two types of twisting.

Let $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ (resp. $\left(\hat{V}_{s,1}^0, \cdots, \hat{V}_{s,1}^{n-1}\right)$) be a basis of the unstable (resp. stable) eigenspace of $\mathbf{O}_1$ satisfying

$$\hat{V}_{u,1}^0 = \lim_{\xi \to -\infty} \frac{h_1(\xi)}{|h_1(\xi)|} \quad , \quad \hat{V}_{s,1}^0 = \lim_{\xi \to +\infty} \frac{h_2(\xi)}{|h_2(\xi)|}$$

and

$$\det\left(\hat{V}_{u,1}^0 \cdots \hat{V}_{u,1}^{n-1} \; \hat{V}_{s,1}^0 \cdots \hat{V}_{s,1}^{n-1}\right) > 0.$$

Then, $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ determines the orientation of the local unstable manifold of $\mathbf{O}_1$ and it propagates to the orientation of the global unstable manifold $\mathfrak{W}^u(\mathbf{O}_1)$ of $\mathbf{O}_1$. Let $\left(\hat{V}_{u,2}^0, \cdots, \hat{V}_{u,2}^{n-1}\right)$ be a basis of the unstable eigenspace of $\mathbf{O}_2$ with

$$\hat{V}_{u,2}^0 = \lim_{\xi \to -\infty} \frac{h_2(\xi)}{|h_2(\xi)|} \quad ,$$

and let

$$\hat{V}_{s,2}^0 = \lim_{\xi \to +\infty} \frac{h_1(\xi)}{|h_1(\xi)|} \quad .$$

From the assumption (2) for the heteroclinic orbit Γ_1, the tangent space $T_{h_1(\xi)}\mathfrak{W}^u(\mathbf{O}_1)$ is tangent to the space spanned by $\hat{V}_{s,2}^0, \hat{V}_{u,2}^1, \cdots, \hat{V}_{u,2}^{n-1}$ in the limit of $\xi \to +\infty$. We determine the orientation of $\left(\hat{V}_{u,2}^1, \cdots, \hat{V}_{u,2}^{n-1}\right)$ so that the orientation of $\left(\hat{V}_{s,2}^0, \hat{V}_{u,2}^1, \cdots, \hat{V}_{u,2}^{n-1}\right)$ is compatible with that of $\mathfrak{W}^u(\mathbf{O}_1)$, and then the orientation of $\mathfrak{W}^u(\mathbf{O}_2)$ is naturally determined by that of $\left(\hat{V}_{u,2}^0, \hat{V}_{u,2}^1, \cdots, \hat{V}_{u,2}^{n-1}\right)$. Similarly the orientation of $\mathfrak{W}^u(\mathbf{O}_2)$ again determines the orientation of the unstable eigenspace of $\mathbf{O}_1$, but this orientation is not necessarily compatible with that of $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ which we defined at the beginning.

Definition 2.1. *The heteroclinic loop consisting of Γ_1 and Γ_2 is said to be non-twisted with respect to the strong unstable direction if the above orientation is compatible with that of $\left(\hat{V}_{u,1}^0, \cdots, \hat{V}_{u,1}^{n-1}\right)$ defined at the beginning, otherwise it is said to be twisted with respect to the strong unstable direction.*

Twisting with respect to the strong stable direction is similarly defined.

We have another definition of twisting, which is directly related to the structure of bifurcation. (See for example B.Deng [5] and references in it.)

Let $q^2(t)$ be a solution of the variational equation

$$\dot{z} = Df(h_2(t))z \tag{2.5}$$

along $h_2(t)$, for which the next limits exist and are non-zero, and $q^2(t)$ points to $\mathbf{O}_1$ along the heteroclinic orbit Γ_1 in the limit of $t \to -\infty$:

$$\lim_{t\to-\infty} |q^2(t)|e^{\rho^2(0)t}, \quad \lim_{t\to+\infty} |q^2(t)|e^{-\nu^1(0)t}. \tag{2.6}$$

Here, we may assume

$$\lim_{\xi\to-\infty} q^2(\xi)e^{\rho^2\xi} = -\hat{V}^0_{s,2}$$

and

$$\lim_{\xi\to+\infty} q^2(\xi)e^{-\nu^1\xi} = c\hat{V}^0_{u,1}$$

for some non-zero c.

Definition 2.2. *The heteroclinic orbit Γ_2 is non-twisted if c is positive, and twisted if c is negative.*

With these two kinds of twisting, we define the sign σ of $\Gamma = \{h(\xi)\}$ as follows.

Definition 2.3. $\sigma = +1$ *if either of the following holds.*

(1) *The heteroclinic loop is non-twisted with respect to the strong unstable direction and Γ_2 is non-twisted.*

(2) *The heteroclinic loop is twisted with respect to the strong unstable direction and Γ_2 is twisted.*

Otherwise $\sigma = -1$.

Remark 2.2.
If either of the following is satisfied, then $\sigma = +1$, otherwise $\sigma = -1$.

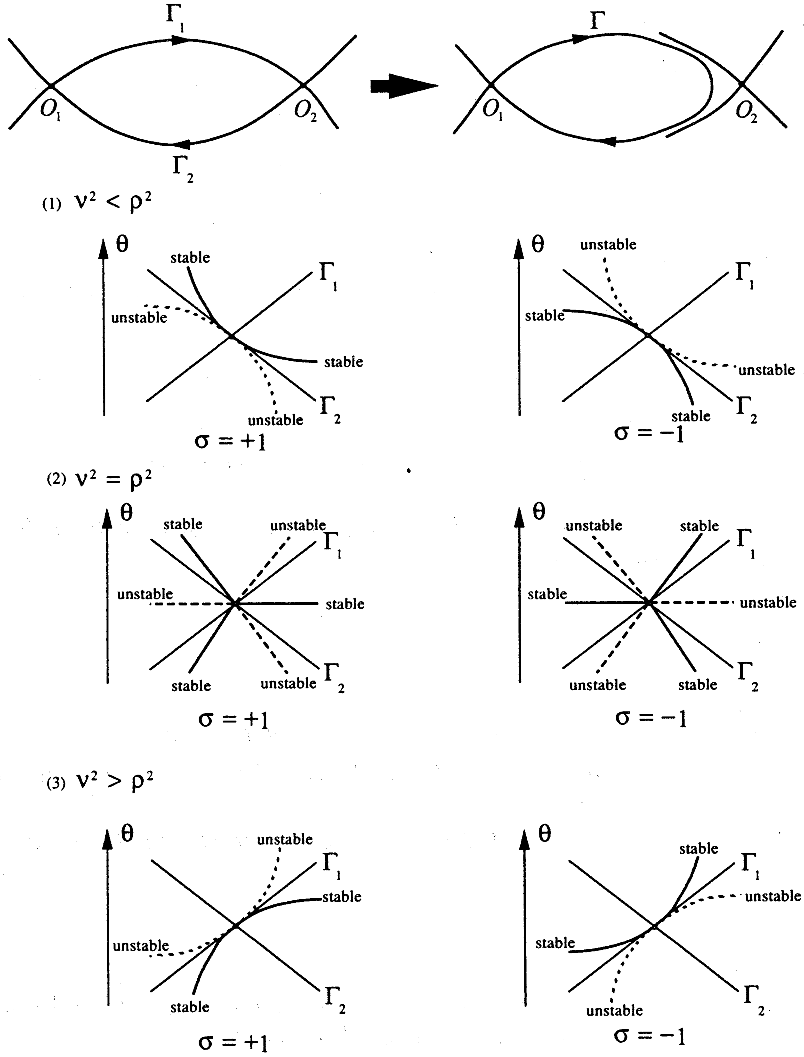
(1) *The heteroclinic loop is non-twisted with respect to the strong stable direction and Γ_1 is non-twisted.*

(2) *The heteroclinic loop is twisted with respect to the strong stable direction and Γ_1 is twisted.*

Theorem 2.1 (Nii[12]).
If one of the travelling waves which correspond to heteroclinic orbits Γ_1 and Γ_2 is unstable, then the wave corresponding to Γ is also unstable.

If both of them are stable, then provided that σ is given, then the stability of the travelling wave corresponding to the homoclinic orbit is determined by the bifurcation diagram which includes speed of travelling wave as a bifurcation parameter. i.e. Stability of the wave is determined as in the figure below.

Here, each line labeled Γ_i' expresses the bifurcation curve on which the heteroclinic orbit Γ_i persists. If the homoclinic orbit Γ bifurcates along one of the thick curves labeled 'Stable', then the wave corresponding to Γ is stable, whereas if the homoclinic *orbit* Γ *bifurcates along one of the broken curves labeled 'Unstable', then the wave corresponding to* Γ *is unstable.*



3. Stability of multiple fronts

The bifurcation near heteroclinic loop can be a little more complicated. For instance, when the heteroclinic orbits Γ_1 and Γ_2 are both twisted, then countably many heteroclinic orbits also bifurcate from the heteroclinic loop. In fact, this is exactly the case for FitzHugh-Nagumo equations.

The following system is called FitzHugh-Nagumo equations.

$$\begin{cases} u_t = u_{xx} + f(u) - w \\ w_t = \varepsilon(u - \gamma w), \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}$ $t \geq 0$ and $u(x,t), w(x,t) \in \mathbb{R}$, and $1 \gg \varepsilon > 0, \gamma > 0$ are parameters. In what follows, the non-linear term $f(u)$ is assumed to be a smooth cubic-like function of u satisfying the conditions:

(1) $f(0) = f(a) = f(1) = 0$, for some constant a with $0 < a < 1$.
(2) $f'(0) < 0$ and $f'(1) < 0$.
(3) $f(u) > 0$ if $u \in (-\infty, 0) \cup (a, 1)$ and $f(u) < 0$ if $u \in (0, a) \cup (1, +\infty)$.
(4) $\int_0^1 f(u)du > 0$.

In this paper we shall restrict our attention to large $\gamma > 0$ so that the system (3.1) has three spatially homogeneous stationary solutions $(u, w) \equiv (u_1, w_1) := (0, 0)$, $(u_\dagger, w_\dagger)$ and (u_2, w_2). Here u_* and w_* $(* = 1, 2$ or $\dagger)$ are constants which satisfy

$$\begin{cases} f(u_*) - w_* = 0 \\ u_* - \gamma w_* = 0, \end{cases} \quad i = 1, 2 \text{ or } \dagger$$

$$0 = u_1 < u_\dagger < u_2 < 1.$$

Again, let $\xi = x - \theta t$ be a moving frame for some constant c, then in (ξ, t) coordinate, (3.1) is expressed as

$$\begin{cases} u_t = u_{\xi\xi} + \theta u_\xi + f(u) - w \\ w_t = \theta w_\xi + \varepsilon(u - \gamma w). \end{cases} \tag{3.2}$$

The equation of travelling wave is

$$\begin{cases} u_{\xi\xi} + \theta u_\xi + f(u) - w = 0 \\ \theta w_\xi + \varepsilon(u - \gamma w) = 0 \end{cases} \tag{3.3}$$

or in form of first order equations,

$$\begin{cases} u' = v \\ v' = -\theta v - f(u) + w \qquad ({}' = \frac{d}{d\xi}) \\ w' = -\frac{\varepsilon}{\theta}(u - \gamma w). \end{cases} \tag{3.4}$$

Notice that $a_1 := (u_1, 0, w_1) = (0, 0, 0)$ and $a_2 := (u_2, 0, w_2)$ are equilibria of (3.4).

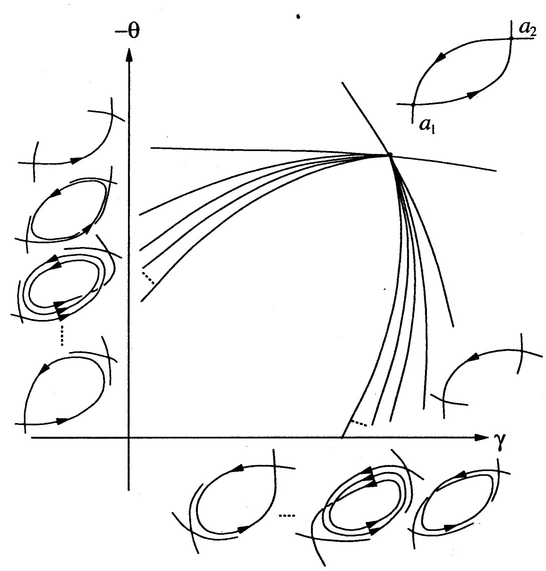
It is well known that (3.4) has a heteroclinic solution $z_1(\xi)$ from a_1 to a_2 ($z_2(\xi)$ from a_2 to a_1) for certain parameter values. This solution corresponds to a travelling wave of (3.1) which satisfies

$$\lim_{\xi\to-\infty}(u(\xi),w(\xi))=(u_1,w_1)\lim_{\xi\to+\infty}(u(\xi),w(\xi))=(u_2,w_2)$$

$$\left(\lim_{\xi\to-\infty}(u(\xi),w(\xi))=(u_2,w_2)\lim_{\xi\to+\infty}(u(\xi),w(\xi))=(u_1,w_1)\text{ respectively}\right).$$

This wave is called travelling front, or simple front (travelling back or simple back respectively).

Deng [6] proved that for certain parameter value $\mu_0=(\gamma(\varepsilon),\theta(\varepsilon),\varepsilon)$, the system (3.4) has heteroclinic solutions z_1 and z_2 simultaneously forming what is called a heteroclinic loop Furthermore, there is a sequence of N-heteroclinic solutions $\{z_{(N),1}(\xi)\}_{N=1}^{\infty}$ from a_1 to a_2 ($\{z_{(N),2}(\xi)\}_{N=1}^{\infty}$ from a_2 to a_1) which correspond to travelling waves called N-fronts (N-backs respectively) bifurcating from the heteroclinic loop, together with homoclinic solutions to a_1 and a_2 which correspond to travelling pulses.



Concerning the stability of these waves, the following is proven.

Theorem (Nii [13] [14]).
Assume that the system (3.4) *is linear in some small neighborhoods of equilibria* a_i $(i=1,2)$, *then the N-front (N-back) bifurcating from the heteroclinic loop at* $\mu=\mu_0=(\gamma_0(\varepsilon),c_0(\varepsilon),\varepsilon)$ *is stable for* $\mu\approx\mu_0$.

Remark 3.1.
B.Sandstede [15] *proved same result. The proof was based on what is called Lin's method.*

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WELL-POSEDNESS OF DISPERSIVE SYSTEM ON WATER WAVES INTERACTION

TAKAYOSHI OGAWA
GRADUATE SCHOOL OF POLYMATHEMATICS
NAGOYA UNIVERSITY
NAGOYA 464-01 JAPAN

1. Interaction Phenomena

This note is concerned with the time local well-posedness of the system of dispersive equations which describes the interaction phenomena appearing in the waver wave theory. Let $\phi(t,x,y)$ be the fluid velocity potential and $\eta(t,x)$ be the surface displacement. Then the motion of the fluid surface is described by the following system of equations.

$$\begin{aligned}
&\Delta\phi = 0, \quad x \in R, \quad -h < y < \eta(x), \\
&\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = 0, \quad y = \eta(x), \\
&\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \eta - \frac{C}{(1+(\eta_x)^2)^{3/2}}\frac{\partial^2\eta}{\partial^2 x} = 0,
\end{aligned}$$

where $y = \eta(x)$ and $t \geq 0$.

In order to extract interaction phenomena, so called multiple-scaling expansion is employed: Introducing scaled variables (t_k, x_k, y_k), where $t_k = \varepsilon^k t$, $x_k = \varepsilon^k x$ $(k = 1, 2, \cdots)$, unknown functions are expanded in the following forms:

$$\begin{aligned}
\phi(t,x,y) &= \sum_n \varepsilon^{\mu(n)}\phi_n(t_1, t_2, .., x_1, x_2, ..) \\
\eta(t,x) &= \sum_n \varepsilon^{\mu(n)}\eta_n(t_1, t_2, .., x_1, x_2, ..)
\end{aligned}$$

We then suppose that the first approximation of the surface displacement is given by

$$\eta_1(t_1,x_1) = S(t_1,x_1)e^{i(kx-\omega t)} + \bar{S}(t_1,x_1)e^{-i(kx-\omega t)} + L(t_1,x_1)$$

where the first two term describes highly oscillating wave and $L(t,x)$ denotes the slowly drifting long wave. Comparing to the same order terms of $\varepsilon^{\mu(n)}$ we obtain an interaction equation (Kaupman [22], Grimshaw [19]):

$$\begin{cases} i\partial_t S + \partial_x^2 S = \alpha L S, \\ \partial_t L + \partial_x L = \beta\partial_x(|S|^2), \end{cases}$$

(special case is the modified Zakharov system (Yajima-Oikawa [43]).

Slightly general form of the equation is obtained by Djordjevic-Redekopp [15] and Benney [7] [8]:

$$\begin{cases} i(\partial_t S + c_g \partial_x S) + \partial_x^2 S = \alpha L S + \gamma |S|^2 S, \\ \partial_t L + c_l \partial_x L = \beta \partial_x (|S|^2), \end{cases}$$

c_g, c_l, α, β, γ are real constants. When the long wave L is governed by a dispersive equation, two kinds of models are suggested: Coupled Schrödinger-KdV equation (Kawahara-Sugimoto-Kakutani [24]):

$$\begin{cases} i(\partial_t S + c_g \partial_x S) + \partial_x^2 S = \alpha L S, \\ \partial_t L + c_l \partial_x L + \partial_x^3 L + \partial_x L^2 = \beta \partial_x (|S|^2), \end{cases}$$

and an interaction on surface between two phase flow (Funakoshi-Oikawa [16]):

$$\begin{cases} i\partial_t S + \partial_x^2 S = \alpha L S, \\ \partial_t L + \nu D_x \partial_x L = \beta \partial_x (|S|^2), \end{cases}$$

where $\nu > 0$ and $D_x = H\partial_x$ with $Hu = \mathcal{F}^{-1}((-i)sgn(\xi)\hat{u})$ being the Hilbert transform.

Common structures among those equations are summarized as follows:

- Short wave envelope "S" is governed by the Schrödinger type equation
- Long wave "L" is subject to the dispersive or wave equation with the drift effect.
- Common coupling nonlinearities.

We note that the Benney's equation is solvable by "inverse scattering method" (Yajima-Oikawa [43], Ma [32])

2. Well-posedness

On a view of the theory of evolution equation, it is desirable to show the well-posedness of those systems. By using Galilei-Gauge transform

$$\begin{cases} u(t,x) = \sqrt{|\alpha\beta|} e^{it(c_g - c_l)/2 - i(c_g - c_l)x^2/4} S(t, x + c_l t), \\ v(t,x) = \beta L(t, x + c_l t), \end{cases}$$

Tsutsumi-Hatano [39] simplified the second system and consider the Cauchy problem in the following form:

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma |u|^2 u, \qquad t, x \in R, \\ \partial_t v = \partial_x (|u|^2), \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x). \end{cases}$$

They showed the time local well-posedness Benney's' equation in the Sobolev space $H^{m+1/2} \times H^m$ where $m = 0, 1, 2, \cdots$. Analogous observation was done for the coupled Schrödinger-KdV equation (M.Tsutsumi [37]):

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in R, \\ \partial_t v + \partial_x^3 v + \partial_x v^2 = \partial_x(|u|^2) \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases}$$

Here we precisely define the meaning of "well-posedness" in the Sobolev space in H^s. For $s \geq 0$ we let

$$H^s(R) = \{u \in L^2(R); \langle \xi \rangle^s \hat{u}(\xi) \in L^2\},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.
To consider the weaker solution, we solve the corresponding integral equation to the Cauchy problem of interaction system. For example to Benney's equation,

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')\Big\{u(t')v(t') + \gamma|u(s)|^2 u(t')\Big\}dt', \tag{2.1}$$

$$v(t) = v_0 + \int_0^t \beta\partial_x|u(t')|dt', \tag{2.2}$$

where $U(t) = e^{it\partial_x^2}$ is the free Schrödinger evolution group.

Definition. (Well posedness in H^s) The equation (B) is (time locally) well-posed in H^s if for any $(u_0, v_0) \in H^s \times H^{s-1/2}$, there exists a time interval $T = T(u_0, v_0)$ and unique pair of solutions (u, v) of the integral equations (2.1)-(2.2) such that

- $(u, v) \in C([0,T]; H^s) \cap X \times C([0,T]; H^{s-1/2}) \cap Y$, where X and Y are properly chosen subspaces in $C([0,T]; H^s)$ and $C([0,T]; H^{s-1/2})$ respectively.
- (u, v) is unique in the above space,
- (u, v) is continuously depending on (u_0, v_0).

Under this framework, Tsutsumi-Hatano established the local $H^{m+1/2}$ well-posedness in [?] and [39]. In fact by observing three conservation laws:

$$\begin{array}{ll} \circ & \|u(t)\|_2 = \|u_0\|_2, \\ \circ & P(u(t), v(t)) = P(u_0, v_0)(\text{momentum}), \\ \circ & E(u(t), v(t)) = E(u_0, v_0)(\text{energy}), \end{array}$$

where

$$\begin{array}{l} P(u,v) = \|v(t)\|_2^2 + 2Im\int_R u(t)\partial_x \bar{u}(t)dx, \\ E(u,v) = \|\partial_x u\|_2^2 + \frac{\gamma}{2}\|u\|_4^4 + \int_R v|u|^2 dx, \end{array}$$

they also proved the time global well-posedness to the Benney system for $H^{3/2}$ initial data. Since the largest space where Tsutsumi-Hatano obtained the well-posedness was $H^{1/2} \times L^2$, the following natural question arose: Can we obtain the well-posedness in larger space than $H^{1/2}$?

Concerning the Cauchy problem of the nonlinear Schrödinger equation with a single power nonlinearity:

$$\begin{cases} i\partial_t u + \partial_x^2 u = \gamma|u|^{p-1}u, & t, x \in R, \\ u(x,0) = u_0(x), \end{cases}$$

the well-posedness in the Sobolev spaces was obtained one by one in the last two decades. For example,

H^2 solvability:	Baillon-Cazenave-Figueria [1]
H^1 solvability:	Ginibre-Velo [18]
L^2 well-posedness:	Y.Tsutsumi [40]
H^m well-posedness ($m \in N$):	Kato [23]
H^s well-posedness:	Cazenave-Weisslar [13].

We should also note that for the smooth polynomial nonlinearity up to third order, it is shown by Kenig-Ponce-Vega [29] that the well-posedness in the negative order Sobolev space.

The crucial point of the solvability is how to choose the proper subspace X and Y in the definition. This is strongly related to the explicit form of the nonlinearity and to find a suitable estimate for the linear Schrödinger evolution operator $U(t) = e^{it\partial_x^2}$ is essential if the initial data is in a weaker class. Initially the L^p-L^q type estimate and the Strichartz type estimate were utilized for the proof of well-posedness:

- (1) L^p-$L^{p'}$ estimate.

$$\|U(t)u_0\|_p \le Ct^{-n/2(1-2/p)}\|u_0\|_{p'}.$$

- (2) Strichartz estimate (Ginibre-Velo [18], Yajima[42], Cazenave-Weissler [13]).

$$\|U(t)u_0\|_{L^r(R;L^p)} \le C\|u_0\|_2.$$
$$\|U(\cdot) * f\|_{L^r(R;L^p)} \le C\|f\|_{L^{r'}(R;L^{p'})}$$

Later the local smoothing property was found and used for the well-posedness:

- (3) Local smoothing effect (Sjölin [36], Vega [41], Constantin-Saut [14]).

$$\|D^{1/2}U(t)u_0\|_{L^2(-T,T;L^2)} \le C\|u_0\|_2.$$

- (4) Inhomogeneous Kato's smoothing effect (Kenig-Ponce-Vega [25]

$$\|DU(t) * f\|_{L^\infty_x(R;L^2_T)} \le C\|f\|_{L^1_x(R;L^2_T)}.$$

The well-posedness in $H^{1/2}$ by Tsutsumi-Hatano used the local smoothing properties (3) and (4) as well as the time derivative version of them: For simplicity, we consider the simplified equation:

$$u(t) = U(t)u_0 - i\int_0^t U(t-\tau)u(\tau)v(\tau)d\tau,$$

$$v(t) = v_0 + \int_0^t \partial_x(|u(\tau)|^2)d\tau$$

Suppose that $\partial_x u$ be a point wise function (for example, in $L^\infty(R; L^2(0,T))$). The smoothing effect of $U(t)$ gives $D_x^{1/2}$ gain and for the inhomogeneous term, $\int_0^t U(t-t')F(t')dt'$ gives a full derivative ∂_x. Therefore,$u_0 \in H^{1/2}$ is a sufficient (almost necessary) condition to obtain the contraction mapping associated with the integral equation . If we assume $u_0 \in H^s$ $(s < 1/2)$, we have a regularity gap (so called derivative loss) and can not treat the nonlinear term with a full derivative $\partial_x u$.

3. Commutator Method

One possible method to avoid the above mentioned difficulty is to employ the commutator argument. If we pass the half derivative $D_x^{1/2}$ from

$$\int_0^t \partial_x |u(\tau)|^2 d\tau$$

into u then the half derivative $D_x^{1/2}$ can be absorbed by the smoothing properties of $U(t)$ in the inhomogeneous part. Define $\widetilde{D}_x^s = HD_x^s$ with $(0 \le s < 1)$, we arrange the nonlinearity (here we omit $\gamma|u|^2u$):

$$\begin{aligned}\int_0^t U(t-t')\Big\{ v_0 u(t') + N_{cm}(u(t')) + (\widetilde{D}_x^{1/2} u) \int_0^{t'} D_x^{1/2}(|u(\tau)|^2) d\tau \\ - \widetilde{D}_x^{1/2}\Big(u \int_0^{t'} D_x^{1/2}(|u(\tau)|^2) d\tau\Big)\Big\} dt',\end{aligned}$$

where

$$\begin{aligned}N_{cm}(u) = u \int_0^t \partial_x(|u(\tau)|^2) d\tau - (\widetilde{D}_x^{1/2} u) \int_0^t D_x^{1/2}(|u(\tau)|^2) d\tau\Big). \\ + \widetilde{D}_x^{1/2}\Big(u \int_0^t D_x^{1/2}(|u(\tau)|^2) d\tau\Big).\end{aligned}$$

Note that the second and the commutator term $N_{cm}(u)$ includes up to the half derivative. This enable us to treat the equation under the weaker initial data. According to the commutation estimates in Kenig-Ponce-Vega [25], it is possible to establish the semi well-posedness of the Benney's equation up to $u_0 \in H^s$ where $0 < s < 1/2$ (Bekiranov-Ogawa-Ponce [2]). However, the extremal case, $s = 0$ with the full well-posedness for the full system (B) was left for the proof.

4. Main Results

To cover all non-negative exponent $s \ge 0$ and to treat the full system we introduce the Fourier restriction norm used by Bourgain [9],[10],[11] as auxiliary function spaces X and Y.

$$\begin{aligned}\|u\|_{X_b^s} &\equiv \int_{R^2} \langle \tau + \xi^2 \rangle^{2b} \langle \xi \rangle^{2s} |\hat{u}(\tau,\xi)|^2 d\xi d\tau, \\ \|v\|_{Y_b^s} &\equiv \int_{R^2} \langle \tau + c\xi \rangle^{2b} \langle \xi \rangle^{2s} |\hat{v}(\tau,\xi)|^2 d\xi d\tau.\end{aligned}$$

For the Benney's equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in R, \\ \partial_t v + c\partial_x v = \partial_x(|u|^2), \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \qquad (B)$$

we have:

Theorem 1 [4]. Let $s \geq 0$ and $1/2 < b < 3/4$. For $(u_0, v_0) \in H^s \times H^{s-1/2}$ Benney's equation (B) is locally well-posed i.e., $^\exists T = T(u_0, v_0) > 0$ $^\exists (u, v) \in C([0,T); H^s) \times C([0,T); H^{s-1/2})$: unique solution of (B) with

$$u \in X_b^s \quad v \in Y_b^{s-1/2}$$

and the map from (u_0, v_0) to (u, v) is the Lipschitz continuous from $H^s \times H^{s-1/2}$ to $C([0,T); H^s) \times C([0,T); H^{s-1/2})$.

For the two phase flow equations

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in R, \\ \partial_t v + \nu\partial_x D_x v = \partial_x(|u|^2), \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \qquad (FO)$$

we use the following space instead of $Y_b^{s-1/2}$.

$$\|v\|_{Z_b^s} \equiv \int_{R^2} \langle \tau + \nu\xi|\xi| \rangle^{2b} \langle \xi \rangle^{2s} |\hat{v}(\tau, \xi)|^2 d\xi d\tau.$$

Theorem 2 [4]. For $s \geq 0$, $1/2 < b < 3/4$ and $|\nu| < 1$, (FO) is locally well-posed with $(u_0, v_0) \in H^s \times H^{s-1/2}$, i.e., $^\exists T = T(u_0, v_0) > 0$ $^\exists (u, v) \in C([0,T); H^s) \times C([0,T); H^{s-1/2})$: unique solution of (FO) and

$$u \in X_b^s \quad v \in Z_b^{s-1/2}.$$

∘ Remark 1:

Structure of the nonlinear terms requires the regularity difference $\frac{1}{2}$. Let (u, v) be a solution pair of the Benney's equation. Then setting

$$u_\lambda(x,t) = \lambda^{3/2} u(\lambda x, \lambda^2 t),$$
$$v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^2 t),$$

(ignoring $\gamma|u|^2 u$). (u_λ, v_λ) solves Benney's equation with initial data

$$u_{\lambda 0} = \lambda^{3/2} u_0(\lambda x),$$
$$v_{\lambda 0} = \lambda^2 v_0(\lambda x).$$

u_λ and $s-1/2$ to v_λ

$$\|D_x^s u_\lambda\|_2^2 = \lambda^{2+2s}\|D_x^s u\|_2^2$$
$$\|D_x^{s-1/2} v_\lambda\|_2^2 = \lambda^{2+2s}\|D_x^{s-1/2} v\|_2^2$$

The difference of order 1/2 is required to keep them equivalent under the scaling.

o Remark 2:

A similar result for Schrödinger-KdV system is possible (Bekiranov-Ogawa-Ponce [3]):

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu + \gamma|u|^2 u, & t, x \in R, \\ \partial_t v + \partial_x^3 v + \partial_x v^2 = \partial_x(|u|^2), & \\ u(x,0) = u_0(x), & \\ v(x,0) = v_0(x), & \end{cases} \qquad (SK)$$

i.e.,$H^s \times H^{s-1/2}$ local well-posedness ($s \geq 0$). This improves the previous result by M.Tsutsumi [37].

o Remark 3:

An analogous result for the Zakharov system (c.f. [45]):

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in R, \\ \partial_t^2 v - \partial_x^2 v = \partial_x^2(|u|^2), & \\ u(x,0) = u_0(x), & \\ v(x,0) = v_0(x), \partial_t v(x,0) = v_1(x) & \end{cases} \qquad (Z)$$

is considered by Bourgain [12] and Ginibre-Tsutsumi-Velo [17].

o Remark 4:

For (FO), if $|\nu| = 1$, our method does not work well. A sort of cancellation prevents to establish the crucial estimate. We are expecting that if $s > 0$ then a similar result hold for (FO) with $\nu = \pm 1$.

o Remark 5:

In view of Theorem 1 and 2, whether the second equation (long wave) is dispersive type or wave equation, it does not concern on the well-posedness result. In the other words, any smoothing effect in the second equation does not give any effect to obtain up to $L^2 \times H^{-1/2}$ solutions.

o Remark 6:

We are expecting so far that the case $s = 0$ is optimal. For example we can show that the crucial estimate in our result does not hold for $s < 0$.

Proposition 3. If $s < 0$ then there is a counter example of the following estimate:

$$\|uv\|_{X^s_{b-1}} \leq C\|u\|_{X^s_b}\|v\|_{Z_b^{s-1/2}}.$$

One application of the well-posedness for the equations is a limiting problem in the system (FO).

$$\begin{cases} i\partial_t u + \partial_x^2 u = vu, & t, x \in R, \\ \partial_t v + \nu\partial_x D_x v = \partial_x(|u|^2), \end{cases} \quad (FO)$$

Passing the parameter $\nu \to 0$, the solution strongly converges to the solution without the dispersive term $\partial_x D_x v$. Namely:

Theorem 4 [4] As $\nu \to 0$ in (FO) the $L^2 \times H^{-1/2}$ solution converges to the solution of Benney's equation with $c = 0$. i.e., Let (u_ν, v_ν) be $L^2 \times H^{-1/2}$ solution of (FO) and (u, v) be of (B) with $c = 0$ with the same initial data. Then

$$\|u_\nu - u\|_{C(0,T;L^2)} \to 0,$$
$$\|v_\nu - v\|_{C(0,T;H^{-1/2})} \to 0$$

as $\nu \to 0$.

The system (FO) describes the model under the deep water flow, and (B) with $c = 0$ is for the shallow flow, Theorem 4 states that the solution in the deep flow equation (FO), is approximately getting close to the shallow setting solution as parameter $\nu \to 0$.

In the regular case, the similar result is relatively easy to show. However our Theorem 4 proves that the system is stable even in the weaker space $L^2 \times H^{-1/2}$. This stability stems from the smoothing properties not only by free Schrödinger evolution operator but the nonlinear coupling term in the second equation. In fact, the nonlinear coupling $\partial_x|u|^2$ has a better smoothing property itself than other term. One can observe that if we multiply the solution u with the complex conjugate of u, a sort of cancellation happens and the singularity is disappear. Therefore to obtain the weaker solution for (FO) or (SK), we do not need the dispersive properties for the second equation but only need the Schrödinger part and this special structure of nonlinearity.

5. Basic Strategy

We basically follows the idea introduced by Bourgain [9], [10]. Consider the linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = F, & t, x \in R, \\ u(0) = u_0, \end{cases} \quad (LS)$$

The corresponding integral equation:

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')F(t')dt'. \tag{5.1}$$

Introduce a cut off function, $\psi_\delta = \delta\psi(t/\delta)$, where

$$\psi(t) = \begin{cases} 1: & \text{if } |t| < 1 \\ 0: & \text{if } |t| > 2. \end{cases}$$

By the space-time Fourier transform, the integral equation (5.1) is expressed as the follows:

$$\psi_\delta(t)u(t) = c\int_{R^2} e^{it\tau+ix\xi}\delta\hat{\psi}(\delta(\tau+\xi^2))\widehat{u_0}d\tau d\xi$$
$$-\psi_\delta\int_{R^2} e^{it\tau+ix\xi}\Big\{\frac{1-e^{it(\tau+\xi^2)}}{i(\tau+\xi^2)}\hat{\hat{F}}\Big\}d\tau d\xi$$

Roughly speaking, the second term in RHS $\simeq \langle\tau+\xi^2\rangle^{-1}$ and this yields a smoothing effect except the point $\tau=-\xi^2$. Along with the characteristic $\tau=-\xi^2$, there is no smoothing effect, however, the characteristics from nonlinear terms fill this gap. To see this, let F be replaced by uv for the short wave equation. We consider the first approximation:

$$u \longrightarrow \frac{1}{\langle\tau+\xi^2\rangle^b}\times\text{smooth cut off}$$
$$v \longrightarrow \frac{\langle\xi\rangle^{1/2}}{\langle\tau+c\xi\rangle^b}\times\text{smooth cut off}$$

and also take into account of the smoothing effect from linear inhomogeneous term:

$$\langle\tau+\xi^2\rangle^{-1}$$

the nonlinear term is approximately given by the convolution of the coupling nonlinear term:

$$\langle\tau+\xi^2\rangle^{-1} * (\langle\xi\rangle^{1/2}\langle\tau+c\xi\rangle^{-1}).$$

Therefore we shall investigate

$$\psi_\delta\int_0^t U(t-t')u(t')v(t')dt' \longrightarrow \frac{1}{\langle\tau+\xi^2\rangle}\Big(\frac{1}{\langle\tau+\xi^2\rangle^b} * \frac{\langle\xi\rangle^{1/2}}{\langle\tau+c\xi\rangle^b}\Big)$$

and

$$\psi_\delta\int_0^t W(t-t')\partial_x|u(t')|^2dt' \longrightarrow \frac{i\xi}{\langle\tau+c\xi\rangle}\Big(\frac{1}{\langle\tau+\xi^2\rangle^b} * \frac{1}{\langle\tau-\xi^2\rangle^b}\Big)$$

respectively.

Utilizing the norms

$$\|\langle\tau+\xi^2\rangle^b\hat{\hat{f}}\|_{L^2_\tau(L^2_\xi)} \equiv \|f\|_{X^0_b}$$
$$\|\langle\tau+c\xi\rangle^b\langle\xi\rangle^{-1/2}\hat{\hat{g}}\|_{L^2_\tau(L^2_\xi)} \equiv \|g\|_{Y_b^{-1/2}}$$

we apply the Banach fix point theorem into a map defined by integral equations:

$$\Psi(u,v) = \psi_\delta U(t)u_0 - i\psi_\delta\int_0^t U(t-t')u(t')v(t')dt'$$
$$\Xi(u,v) = \psi_\delta W(t)v_0 - i\psi_\delta\int_0^t W(t-t')\partial_x|u(t')|^2dt'$$

where $W(t)=e^{-ct\partial_x}$.

Choose δ small and consider $t\in[0,\delta)$, the map $(u,v)\to(\Psi,\Xi)$ is shown to be contraction on the spaces and conclude the existence and well-posedness results.

6. Linear and Nonlinear Estimates

More specifically, we states some lemma which leads to the our conclusions. Recall that $\psi(t)$ be a cut off defined in the previous section. and $\psi_\delta = \delta\psi(t/\delta)$ for some $\delta > 0$..

Lemma 5. ([Bourgain], [Kenig-Ponce-Vega]) Let $b \in (1/2, 1)$, $s \in R$, $\delta \in (0,1)$ and $U(t) = e^{it\partial_x^2}$ be the Schrödinger evolution group.

$$\|\psi_\delta U(t)u_0\|_{X_b^s} \le C\delta^{(1-2b)/2}\|u_0\|_{H^s},$$
$$\|\psi_\delta U(\cdot) * F\|_{X_b^s} \le C(1+\delta^{(1-2b)/2})\|F\|_{X_{b-1}^s},$$

where $U(\cdot) * F(t) = \int_0^t U(t-t')F(t')dt'$.

Analogous estimates for $W(t) = e^{-ct\partial_x}$ and $V(t) = e^{-\nu t\partial_x D_x}$ also hold for the same exponents.

Lemma 6. ([Kenig-Ponce-Vega]) Let $b \in (1/2, 1)$, $s \in R$ and $\delta \in (0,1)$.

$$\|\psi_\delta F\|_{X_b^s} \le C\delta^{(1-2b)/2}\|F\|_{X_b^s}$$

if $0 < a < b < 1/2$

$$\|\psi_\delta F\|_{X_{-b}^s} \le C\delta^{(b-a)/4(1-a)}\|F\|_{X_{-a}^s}$$

We give three nonlinear estimate in the norm X_b^s and Y_b^s which is the crucial to obtain our results.

Lemma 7. For $s \ge 0$, $a \le 0$ $b \in (1/2, 1)$

$$(1) \qquad \||u|^2u\|_{X_b^s} \le C\|u\|_{X_b^s}^3,$$
$$(2) \qquad \|\partial_x|u|^2\|_{Y_a^{s-1/2}} \le C\|u\|_{X_b^s}^2.$$
$$(3) \qquad \|uv\|_{X_a^s} \le C\|u\|_{X_b^s}\|v\|_{Y_b^s},$$

Lemma 7 (1) follows from a simple application of the Strichartz estimate:

$$\|U(t)u_0\|_{L^6(R;L^6)} \le C\|u_0\|_2.$$

Lemma 7 (2) is an Immediate corollary of the following lemma.

Lemma 8. For $s \ge 0$, $a \le 0$, $b \in (1/2, 1)$,

$$\|\partial_x|u|^2\|_{L_t^2(H_x^{s-1/2})} \le C\|u\|_{X_b^s}^2.$$

Note that Lemma 8 shows the nonlinear term of the 2nd equation, $\partial_x|u|^2$ has slightly better property. Related estimate is known as a special quadratic form estimate in the context of nonlinear wave equations. (Klainerman-Machedon [30]).

Since they are all bilinear estimate, it is easy to see:

Lemma 9.

$$\||u|^2u - |u'|^2u'\|_{X_b^s} \le C(\|u\|_{X_b^s} + \|u'\|_{X_b^s})^2\|u-u'\|_{X_b^s},$$
$$\|\partial_x(|u|^2 - |u'|^2)\|_{Y_a^{s-1/2}} \le C(\|u\|_{X_b^s} + \|u\|_{X_b^s})\|u-u'\|_{X_b^s},$$
$$\|uv - u'v'\|_{X_a^s} \le C\|u-u'\|_{X_b^s}\|v\|_{Y_b^s} + \|u'\|_{X_b^s}\|v-v'\|_{Y_b^s}.$$

A similar way, estimates for equation (FO) also hold.

Lemma 10. Analogous estimates as in Lemma 7 & 9 hold for $v \in Z_b^s$ where C depends on $(1-|\nu|)^{-1}$.

Proposition 11. If $\nu = \pm 1$ and $s < 0$, there is a counter example of the above estimates corresponding Lemma 7-(3) for $v \in Z_b^s$.

Gathering Lemma 5-7, if $t < \delta$, the maps

$$\Psi(u,v) = \psi_1 U(t)u_0 - i\psi_1 \int_0^t U(t-t')\psi_\delta uv dt'$$

$$\Xi(u,v) = \psi_1 W(t)v_0 - i\psi_1 \int_0^t W(t-t')\psi_\delta \partial_x |u|^2 dt'$$

satisfies the following estimates:

$$\begin{aligned} \|\Psi(u,v)\|_{X_b^0} &\le C\|u_0\|_{L^2} + C\|\psi_\delta uv\|_{X_{b-1}^0} \\ &\le C\|u_0\|_{L^2} + C\delta^\varepsilon \|u\|_{X_b^0}\|v\|_{Y_b^{-1/2}} \\ \|\Xi(u,v)\|_{Y_b^{-1/2}} &\le C\|v_0\|_{H^{-1/2}} + C\|\psi_\delta \partial_x |u|^2\|_{Y_{b-1}^{-1/2}} \\ &\le C\|v_0\|_{H^{-1/2}} + C\delta^\varepsilon \|u\|_{X_b^0}^2. \end{aligned}$$

Hence the map $(u,v) \to (\Psi(u,v), \Xi(u,v))$ is bounded if

$$\begin{aligned} \|u\|_{X_b^0} &\le 2C\|u_0\|_2 \equiv M \\ \|v\|_{Y_b^{-1/2}} &\le 2C\|v_0\|_{H^{-1/2}} \equiv N \end{aligned}$$

and δ is small. Similarly we have

$$\|\Psi(u,v) - \Psi(u',v')\|_{X_b^0} + \|\Xi(u,v) - \Xi(u',v')\|_{Y_b^{-1/2}}$$
$$\le \frac{1}{2}(\|u-u'\|_{X_b^0} + \|v-v'\|_{Y_b^{-1/2}}).$$

This shows that Φ and Ξ give us a contraction mapping and the well-posedness will be shown.

7. Out-line of Proof of Lemma 8

Finally we briefly sketch the proof of Lemma 8. For simplicity, we show the case $s = 0$. Let

$$\begin{aligned} f(\tau,\xi) &= \langle \tau + \xi^2 \rangle^b \hat{u}, \\ f^*(\tau,\xi) &= \langle \tau - \xi^2 \rangle^b \hat{\bar{u}}, \\ g(\tau,\xi) &= \langle \tau + c\xi \rangle^b \langle \xi \rangle^{-1/2} \hat{v}. \end{aligned}$$

Recalling a convolution estimate:

$$\int \frac{d\sigma}{\langle \sigma - a \rangle^p \langle \sigma - b \rangle^q} \le \frac{C}{\langle a - b \rangle^r}$$

for $r = \min(p,q)$ with $p + q > 1 + r$, Note

$$\begin{aligned}(\sigma+\eta^2)-(\sigma-\tau+(\xi-\eta)^2) &= \tau-\xi^2+2\eta\xi \\ &\simeq 2\eta\xi.\end{aligned}$$

$$\|\partial_x|u|^2\|_{L_t^2(H_x^{-1/2})} = \|i\xi\langle\xi\rangle^{-1/2}\widehat{|u|^2}\|_{L_\tau^2(L_\xi^2)}$$

$$= \|i\xi\langle\xi\rangle^{-1/2}\Big(\frac{f}{\langle\tau+\xi^2\rangle^b} * \frac{f^*}{\langle\tau-\xi^2\rangle^b}\Big)\|_{L_\tau^2(L_\xi^2)}$$

$$\le \Big\||\xi|^{1/2}\Big(\int\frac{d\sigma d\eta}{\langle\sigma+\eta^2\rangle^{2b}\langle\tau-\sigma-(\xi-\eta)^2\rangle^{2b}}\Big)^{1/2} \times (|f|^2 * |f^*|^2)^{1/2}\Big\|_{L_\tau^2(L_\xi^2)}$$

$$\le \||f|^2 * |f^*|^2\|_{L_\tau^1(L_\xi^1)} \times$$

$$\||\xi|^{1/2}\Big(\int\frac{d\sigma d\eta}{\langle\sigma+\eta^2\rangle^{2b}\langle\tau-\sigma-(\xi-\eta)^2\rangle^{2b}}\Big)^{1/2}\|_{L_\tau^\infty L_\xi^\infty}$$

$$\le \|f\|_{L_\tau^2(L_\xi^2)}\|f^*\|_{L_\tau^2(L_\xi^2)}$$

$$\times \||\xi|^{1/2}\Big(\int\frac{d\eta}{\langle\tau-\xi^2+2\eta\xi\rangle^{2b}}\Big)^{1/2}\|_{L_\tau^\infty L_\xi^\infty}$$

$$\le C\|f\|^2_{L_\tau^2(L_\xi^2)} = C\|u\|^2_{X_b^s}.$$

For the proof of Lemma 7 (3), see Bekiranov-Ogawa-Ponce [4].

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SPATIAL CRITICAL POINTS NOT MOVING ALONG THE HEAT FLOW AND A BALANCE LAW

SHIGERU SAKAGUCHI

ABSTRACT. We consider solutions of the heat equation, in domains in $\mathbb{R}^N$, and their spatial critical points. In particular, we show that a solution u has a spatial critical point not moving along the heat flow if and only if u satisfies some balance law. Furthermore, in the case of Dirichlet, Neumann, and Robin homogeneous initial-boundary value problems on bounded domains, we prove that if the origin is a spatial critical point never moving for sufficiently many compactly supported initial data satisfying the balance law with respect to the origin, then the domain must be a ball centered at the origin.

§1. Introduction.

This note is a summary of my recent work with Magnanini [MS]. We consider spatial critical points of solutions of the heat equation. Among spatial critical points, *hot spots* have been studied by Chavel and Karp. At each time, a hot spot is a point where the solution attains its spatial maximum. In [CK] they considered the Cauchy problem for the heat equation in Riemannian manifolds and studied the location and the limit of the hot spots of the nonnegative solution as time goes to infinity. The problem for the initial-boundary value problems on unbounded domains in $\mathbb{R}^N$ is considered in [JS].

There is a conjecture by Klamkin [Kl 1] concerned with the location of the hot spots of solutions of the initial-boundary value problems on bounded convex domains in $\mathbb{R}^N$. The conjecture, modified by Kawohl, is that if the hot spot does not move in time for positive constant initial data under the homogeneous Dirichlet boundary condition, then the convex domain must have some sort of symmetry. Partial answers to this conjecture have been given by Gulliver and Willms, and Kawohl (see [GW, Ka]).

Motivated by Chavel, Karp, and Klamkin, we consider the following problem:

Determine when and how the spatial critical point does not move.

To understand this problem more precisely, let us consider the unique solution $u(x,t)$ of the one-dimensional Cauchy problem for the heat equation:

$$\partial_t u = \partial_x^2 u \quad \text{in} \quad \mathbb{R} \times (0, \infty) \quad \text{and} \quad u(x,0) = \varphi(x) \quad \text{in} \quad \mathbb{R}, \tag{1.1}$$

where φ is a nonzero bounded function. Then the set of spatial critical points $C(t)$ of u is defined by

$$C(t) = \{ \ x \in \mathbb{R} \ ; \ \partial_x u(x,t) = 0 \ \}. \tag{1.2}$$

Let us suppose that there exist a point $x_0 \in \mathbb{R}$ and a nonempty time interval (t_1, t_2) such that $x_0 \in \cap_{t \in (t_1,t_2)} C(t)$. Namely, x_0 is a spatial critical point not moving in (t_1, t_2). Then, by analyticity of u we get

$$\partial_x u(x_0, t) = 0 \quad \text{for any} \quad t \in (0, \infty). \tag{1.3}$$

Define the function $v(x,t)$ by

$$v(x,t) = \begin{cases} u(x,t) & \text{if} \ \ x < x_0 \\ u(2x_0 - x, t) & \text{if} \ \ x \geqq x_0 \end{cases} \tag{1.4}$$

Then v is a solution of the heat equation on $\mathbb{R} \times (0, \infty)$, and v satisfies the following:

$$v(x,t) = v(2x_0 - x, t) \quad \text{for any} \quad (x,t) \in \mathbb{R} \times (0, \infty), \tag{1.5}$$

$$v(x,t) = u(x,t) \qquad \text{for any} \quad (x,t) \in (-\infty, x_0) \times (0, \infty). \tag{1.6}$$

By using the spatial analyticity of u and v we get

$$u \equiv v \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

Namely,

$$u(x,t) = u(2x_0 - x, t) \quad \text{for any} \quad (x,t) \in \mathbb{R} \times (0, \infty). \tag{1.7}$$

This implies that $\varphi(x) = \varphi(2x_0 - x)$ for almost every $x \in \mathbb{R}$. Consequently, in the one-dimensional Cauchy problem the existence of a spatial critical point not moving in some time interval implies the symmetry of the initial data. Conversely, the symmetry of the bounded initial data implies the existence of a spatial critical point not moving along the heat flow, since this point of symmetry is exactly the spatial critical point not moving in the whole time interval. The same argument as above works in the one-dimensional initial-boundary value problems on bounded intervals. For instance, in the case of the homogeneous Dirichlet initial-boundary value problem with nonnegative initial data φ , the existence of a spatial critical point not moving in some time interval is equivalent to the kind of symmetry described above for φ.

In the sequel, we will address this problem in higher spatial dimension. In this case, it is easy to show that the existence of a not moving spatial critical point does not imply any symmetry of the initial data. For example, let λ_1 and $\psi_1(x)$ be respectively the first eigenvalue and a first eigenfunction of $-\Delta$ under the homogeneous Dirichlet boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$. Then $u(x,t) = e^{-\lambda_1 t}\psi_1(x)$ is a solution of the homogeneous Dirichlet initial-boundary value problem with initial data $u(x,0) = \psi_1(x)$. Here, any critical point of ψ_1 is a not moving spatial critical point of u. However, if Ω is not symmetric, then ψ_1 is not symmetric.

Our first result is proved easily by using the explicit representation of the solution, but it suggests a general principle. For brevity, let us take the origin as a not moving spatial critical point of the solution.

Theorem 1. *Let u be the unique solution of the Cauchy problem for the heat equation*

$$\partial_t u = \Delta u \quad in \quad \mathbb{R}^N \times (0,\infty), \quad and \quad u(x,0) = \varphi(x) \quad in \quad \mathbb{R}^N,$$

where $N \geqq 1$ and φ is a bounded function in $\mathbb{R}^N$. Then the following three conditions are equivalent:

(i) $\nabla u(0,t) = 0$ *for any* $t \in (0,\infty)$,

(ii) *There exists a nonempty interval* $(t_1,t_2) \subset (0,\infty)$ *such that* $\nabla u(0,t) = 0$ *for any* $t \in (t_1,t_2)$,

(iii) $\displaystyle\int_{S^{N-1}} \omega\varphi(r\omega)\, d\omega = 0$ *for almost every* $r \geqq 0$.

Here, $\omega = (\omega_1,...,\omega_N)$ is a vector in the standard (N-1)-dimensional unit sphere S^{N-1} in $\mathbb{R}^N$, $d\omega$ is the volume element of S^{N-1}, and ∇ denotes the spatial gradient.

Condition (iii) in Theorem 1 can be regarded as a *balance law*. Precisely this condition means that for almost every $r \geqq 0$ the first moments of the function $\varphi(r\bullet)$ on S^{N-1} with respect to the origin of $\mathbb{R}^N$ are zero. In particular, if the spatial dimension $N = 1$, this balance law implies symmetry. We also want to emphasize that this balance law is a general principle, in the sense that, as is proved in the following theorem, on any domain in $\mathbb{R}^N$ the balance law for a solution of the heat equation is a necessary and sufficient condition for the existence of a spatial critical point not moving along the heat flow.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^N$ containing the origin 0, and let (a, b) be a nonempty interval. Suppose that $u = u(x, t)$ satisfies*

$$\partial_t u = \Delta u \quad in \quad \Omega \times (a, b).$$

Then, $\nabla u(0, t) = 0$ for any $t \in (a, b)$ if and only if

$$\int_{S^{N-1}} \omega u(r\omega, t)\, d\omega = 0 \quad for\ any \quad (r, t) \in [0, d_*) \times (a, b),$$

where $d_* = \text{dist}\,(0, \partial\Omega)$.

Now, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and let $0 \in \Omega$. Consider the following initial-boundary value problem:

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in} \quad \Omega, \\ (1 - \alpha)\dfrac{\partial u}{\partial \nu} + \alpha u = g & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases} \tag{1.8}$$

Here, φ is a bounded function on Ω, ν denotes the exterior normal unit vector to $\partial\Omega$, α is a constant with $0 \leqq \alpha \leqq 1$, and $g = g(x, t)$ is a given continuous function on $\partial\Omega \times [0, \infty)$. When $0 < \alpha < 1$, problem (1.8) is known as Robin's problem, and is reduced to Dirichlet and Neumann problems when $\alpha = 1$ and $\alpha = 0$, respectively.

When Ω is a ball centered at the origin, the balance law for initial and boundary data is also a sufficient condition to ensure that the origin is a spatial critical point not moving along the heat flow.

Theorem 3. *Let Ω be a ball in $\mathbb{R}^N$ centered at the origin with radius $R > 0$. Suppose that for almost every $r \in [0, R)$ and for any $t \in [0, \infty)$*

$$\int_{S^{N-1}} \omega \varphi(r\omega)\, d\omega = \int_{S^{N-1}} \omega g(R\omega, t)\, d\omega = 0.$$

Let u be the solution of (1.8). Then u satisfies

$$\nabla u(0, t) = 0 \quad for\ any \quad t \in (0, \infty).$$

This result suggests the following question:

If the origin is always a spatial critical point not moving for sufficiently many compactly supported initial data satisfying the balance law, must the domain Ω be a ball centered at the origin?

Our last result answers this question.

Theorem 4. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$, and let $B_\delta(0)$ be a ball centered at the origin with radius $\delta > 0$ such that $B_\delta(0) \subset \Omega$. Consider the initial-boundary value problem (1.8) with $g \equiv 0$. Assume that for any C^∞ initial data φ with support in $B_\delta(0)$ and satisfying $\int_{S^{N-1}} \omega\varphi(r\omega)\, d\omega = 0$ for any $r \in [0, \delta)$ the corresponding solution u is such that*

$$\nabla u(0,t) = 0 \quad \textit{for any} \quad t \in (0, \infty).$$

Then $\Omega = B_R(0)$ for some $R > 0$.

It is worth to mention that as in Theorem 1 the conclusion of Theorem 4 still holds if we assume that $\nabla u(0,t) = 0$ only on some time interval, since problem (1.8) with $g \equiv 0$ is solved by an eigenfunction expansion (see [I, Theorem 15.3, pp. 121–122] for example) and in particular $\nabla u(0,t)$ is a real analytic function of t on $(0, \infty)$.

For proofs of the theorems we refer to [MS].

§2. Remarks.

Let us give several remarks concerning the theorems.

1. Let us consider the Cauchy problem for the heat equation

$$\partial_t u = \Delta u \text{ in } \mathbb{R}^N \times (0, \infty), \text{ and } u(x,0) = \varphi(x) \text{ in } \mathbb{R}^N.$$

Suppose that the initial data φ is a bounded function having compact support. Let $z \in \mathbb{R}^N$ satisfy

$$\int_{S^{N-1}} \omega\varphi(z + r\omega)\, d\omega = 0 \text{ for almost every } r \geqq 0. \tag{2.1}$$

Namely, suppose that φ satisfies the balance law with respect to the point z. By multiplying (2.1) by r^N and integrating the resulting equation with respect to r from 0 to ∞, we get

$$\int_{\mathbb{R}^N} y\varphi(z + y)\, dy = 0.$$

Since φ has compact support, we obtain

$$z \int_{\mathbb{R}^N} \varphi(x)\, dx = \int_{\mathbb{R}^N} x\varphi(x)\, dx. \tag{2.2}$$

Therefore, when $\int_{\mathbb{R}^N} \varphi(x)\, dx \neq 0$, by Theorem 1 at most one point z determined by (2.2) can be a spatial critical point not moving along the heat flow, and when $\int_{\mathbb{R}^N} \varphi(x)\, dx = 0$ and $\int_{\mathbb{R}^N} x\varphi(x)\, dx \neq 0$, by Theorem 1 there is no spatial critical point not moving along the heat flow, namely, *every spatial critical point of the solution must move.* For example, when φ is nonzero and nonnegative, then the point z determined by (2.2) is called the Euclidean center of mass of φ, say m_φ. In this case, with the help of the explicit representation of the solution, it is known that the set of spatial critical points of the solution is contained in the closed convex hull of the support of φ for any $t > 0$, and it consists of one point, the hot spot, after a finite time, and further it tends to m_φ as $t \to \infty$ (see [CK, Theorem 1, p. 274] and [JS, Introduction, pp. 810–811]). Consequently, if φ satisfies the balance law with respect to m_φ, then m_φ is only a spatial critical point not moving along the heat flow. On the other hand, if φ does not satisfy the balance law with respect to m_φ, then every spatial critical point of the solution must move along the heat flow.

Similarly, let us consider the initial-boundary value problem for the heat equation in the half space $\mathbb{R}^N_+ = \{ x = (x', x_N) \in \mathbb{R}^N ; x' \in \mathbb{R}^{N-1}$ and $x_N > 0 \}$ for nonzero nonnegative initial data having compact support under the homogeneous Dirichlet boundary condition. This problem is equivalent to the Cauchy problem for the initial data φ having compact support with $\varphi(x', -x_N) = -\varphi(x', x_N) \leqq 0$ $(x_N \geqq 0)$. In this case, the similar fact concerning the spatial critical points of the solution is known in [JS, Theorem 1, p.812]. Of course, for any such initial data φ we have $\int_{\mathbb{R}^N} \varphi(x)\, dx = 0$ and $\int_{\mathbb{R}^N} x\varphi(x)\, dx \neq 0$, therefore, by (2.2) and Theorem 1, in this initial-boundary value problem every spatial critical point of the solution must move along the heat flow.

2. Consider the function $v \in C^2(\Omega)$ satisfying

$$-\Delta v = \lambda v \quad \text{in} \quad \Omega$$

for some constant $\lambda \in \mathbb{R}$ and for some domain Ω in $\mathbb{R}^N$. For example, let v be any eigenfunction of $-\Delta$. Then, by Theorem 2 any critical point z of v satisfies the balance law

$$\int_{S^{N-1}} \omega v(z + r\omega)\, d\omega = 0 \quad \text{for any } r \in [0, d_*),$$

where $d_* = \text{dist}\,(z, \partial\Omega)$. Indeed, if we put $u(x,t) = e^{-\lambda t} v(z + x)$, then u satisfies the heat equation in $\tilde{\Omega} \times (0, \infty)$, where $\tilde{\Omega} = \{ x \in \mathbb{R}^N ; x = y - z$, and $y \in \Omega \}$. Therefore Theorem 2 is applied to u.

3. Theorems 3 and 4 have their elliptic counterparts respectively. Precisely, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and let $0 \in \Omega$. Consider the problem:

$$\begin{cases} -\Delta u = \varphi \quad \text{in} \quad \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial \nu} + \alpha u = g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

Here φ is a bounded function on Ω, ν denotes the exterior normal unit vector to $\partial\Omega$, α is a constant with $0 \leqq \alpha \leqq 1$, and g is a given continuous function on $\partial\Omega$.

When $\alpha = 0$, in the case of the Neumann boundary condition, we assume that

$$\int_{\Omega} \varphi(x)\,dx + \int_{\partial\Omega} g(x)\,d\sigma = 0, \tag{2.4}$$

where $d\sigma$ denotes the surface measure of $\partial\Omega$.

The elliptic counterpart of Theorem 3 is

Theorem 5. *Let $\Omega = B_R(0)$ for some $R > 0$. Suppose that*

$$\int_{S^{N-1}} \omega\varphi(r\omega)\,d\omega = \int_{S^{N-1}} \omega g(R\omega)\,d\omega = 0 \quad \text{for almost every} \quad r \in [0, R).$$

Let u be a weak solution of (2.3). Suppose that u belongs to $C^1(\Omega) \cap C^0(\overline{\Omega})$ when $\alpha = 1$, and that it belongs to $C^1(\overline{\Omega})$ when $0 \leqq \alpha < 1$. Then u satisfies

$$\nabla u(0) = 0.$$

The following is the elliptic counterpart of Theorem 4 proved along the same line as in the proof of Theorem 4.

Theorem 6. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$, and let $B_\delta(0)$ be a ball centered at the origin with radius $\delta > 0$ such that $B_\delta(0) \subset \Omega$. Consider the boundary value problem (2.3) with $g \equiv 0$. Assume that for any C^∞ function φ with support in $B_\delta(0)$ and satisfying $\int_{S^{N-1}} \omega\varphi(r\omega)\,d\omega = 0$ for any $r \in [0, \delta)$ (and further (2.4) with $g \equiv 0$ when $\alpha = 0$) the corresponding solution u is such that $\nabla u(0) = 0$. Then $\Omega = B_R(0)$ for some $R > 0$.*

4. Let us state two theorems for centrosymmetry which correspond to Theorem 3 and Theorem 4. The centrosymmetry has been mentioned in [Kl 1].

Theorem 7. *Suppose that Ω is centrosymmetric with respect to the origin (that is, if $x \in \Omega$, then $-x \in \Omega$), and that $\varphi(x) = \varphi(-x)$ for any $x \in \Omega$, and $g(x,t) = g(-x,t)$ for any $(x,t) \in \partial\Omega \times [0,\infty)$. Let u be the solution of (1.8). Then $u(x,t) = u(-x,t)$ for any $(x,t) \in \Omega \times (0,\infty)$. In particular $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$.*

Proof. Let $v(x,t) = u(-x,t)$. Then v is also a solution of (1.8). Uniqueness of the solution implies that $v \equiv u$, which proves this theorem. □

Theorem 8. *Let Ω be a bounded C^2 domain in $\mathbb{R}^N$, and let $B_\delta(0)$ be a ball centered at the origin with radius $\delta > 0$ such that $B_\delta(0) \subset \Omega$. Consider the initial-boundary value problem (1.8) with $g \equiv 0$. Suppose that $\alpha = 1$ or $\alpha = 0$ (that is, we only consider the Dirichlet or Neumann problem). Assume that for any C^∞ initial data φ with support in $B_\delta(0)$ and satisfying $\varphi(x) = \varphi(-x)$ for any $x \in B_\delta(0)$ the corresponding solution u is such that*

$$\nabla u(0,t) = 0 \quad \text{for any} \quad t \in (0,\infty).$$

Then Ω is centrosymmetric with respect to the origin.

Of course these theorems have their elliptic counterparts respectively, as in the previous remark 3.

5. Instead of spatial critical points let us consider spatial zero points. Namely, instead of $\nabla u(\bullet, t) = 0$ consider $u(\bullet, t) = 0$ for each time t. Then, along the similar arguments we can get all theorems by replacing the balance law $\int_{S^{N-1}} \omega u(r\omega, t)\, d\omega = 0$ or $\int_{S^{N-1}} \omega\varphi(r\omega)\, d\omega = 0$ by $\int_{S^{N-1}} u(r\omega, t)\, d\omega = 0$ or $\int_{S^{N-1}} \varphi(r\omega)\, d\omega = 0$. Especially, the corresponding proof of the symmetry results are much easier.

6. There are related symmetry results due to Alessandrini [A 1,2] which proved another older conjecture of Klamkin [Kl 2]. We quote a theorem of [A 2] (see [A 2, Theorem 1.3, p. 254]).

Theorem (Alessandrini). *Let Ω be a bounded domain in $\mathbb{R}^N$ and let all of its boundary points be regular with respect to the Laplacian. Let $\varphi \in L^2(\Omega)$ with $\varphi \not\equiv 0$ and let $u = u(x,t)$ be the weak solution of*

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{2.5}$$

If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\bullet, t)$ is constant on every level surface $\{ x \in \Omega \,;\, u(x,\tau) = \text{const.} \}$ of $u(\bullet,\tau)$ in Ω, then one of the following two cases occurs.

(A) *φ is an eigenfunction of $-\Delta$ under the homogeneous Dirichlet boundary condition.*

(B) *Ω is a ball.*

Moreover, if case (B) occurs, then $u(\bullet, t)$ is radially symmetric for every $t \geqq 0$ and u never vanishes in $\Omega \times [\tau, \infty)$.

Roughly speaking, this theorem shows that if all the level surfaces are invariant with respect to the time variable t under the homogeneous Dirichlet boundary condition, then the initial data is an eigenfunction or the domain is a ball.

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Department of Mathematical Sciences, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama-shi, Ehime 790-77 Japan.

Curved moving planes and applications

Takashi Suzuki*

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1 Introduction

In 1979, Gidas, Ni, and Nirenberg [3] applied the method of moving planes of Alexandroff [1] and established some results concerning qualitative features of solutions for the semilinear elliptic boundary value problem

$$-\Delta u = f(u), \quad u > 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

Here, $\Omega \subset \mathcal{R}^n$ denotes a bounded domain and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ the usual Laplacian for $x = (x_1, x_2, \cdots, x_n) \in \Omega$. The following theorem is a typical example.

Theorem 1 *Let $\Omega = \{x \in \mathcal{R}^n \mid |x| < 1\}$ be the unit ball and $f \in C^1$. Then, any solution $u \in C^2\left(\overline{\Omega}\right)$ of (1) is radially symmetric: $u = u\left(|x|\right)$, and satisfies*

$$u_r < 0 \qquad on \quad 0 < r = |x| \leq 1. \tag{2}$$

Let us recall the proof. First, a family of moving hyper-planes $\{T_\lambda\}$ are prepared. Then, the reflection $S_\lambda : x \mapsto x^\lambda$ with respect to T_λ induces the transformation $u^\lambda(x) = u(x^\lambda)$, which satisfies the same equation. The comparison between u and u^λ, based on the maximum principle and the continuation argument with respect to λ give the above conclusion.

The argument still works if $\{T_\lambda\}$ is replaced by a family of concentric spheres and S_λ by the Kelvin transformation. This method, called the moving sphere method, was adopted by several authors recently ([4], [7], [6]). We refer to the following theorem of [10] as a typical example.

*Deparatment of Mathematics, Osaka University, Machikaneyamacho 1-1, Toyonakashi, 560, Japan

Theorem 2 *Let $\Omega = \{x \in \mathcal{R}^n \mid a < |x| < 1\}$ be an annulus with $0 < a < 1$ and $n > 2$, and consider the problem (1) with $f(u) = u^{\frac{n+2}{n-2}}$. Then, any solution $u \in C^2(\overline{\Omega})$ satisfies*

$$u(x) = \left(\frac{|x|}{\sqrt{a}}\right)^{2-n} u\left(a\frac{x}{|x|^2}\right) \qquad (x \in \Omega)$$

and

$$\left(|x|^{\frac{n-2}{2}}u\right)_r < 0 \qquad \left(\sqrt{a} < r = |x| < 1\right). \tag{3}$$

On the other hand, the authors obtained the following theorem ([9], [8]).

Theorem 3 *Let $\Omega = \{x \in \mathcal{R}^n \mid |x| < 1\}$ be the unit ball and and $f(r,s)$ be continuous on $[0,1] \times [0,\infty)$. Suppose that it is C^1 in s and also the property*

$$r \in (0,1) \mapsto (1-r^2)^{\frac{n+2}{2}} f\left(r, (1-r^2)^{-\frac{n-2}{2}} s\right)$$
$$: \textit{decreasing for each } \; s > 0. \tag{4}$$

Then, any solution $u \in C^2(B) \cap C(\overline{B})$ of

$$-\Delta u = f(|x|, u), \quad u > 0 \quad \textit{in} \quad B \qquad \textit{and} \qquad u = 0 \quad \textit{on} \quad \partial B \tag{5}$$

is radially symmetric and satisfies

$$\left((1-r^2)^{\frac{n-2}{2}}u\right)_r < 0 \qquad \textit{for} \quad 0 < r = |x| < 1.$$

This theorem is applicable to $f(|x|, u) = K(|x|)\, u^\sigma$ for instance, if $1 < \sigma < \frac{n+2}{n-2}$ and $\left(1-r^2\right)^{(n+2-\sigma(n-2))/2} K(r)$ is decreasing in $r \in (0,1)$. In connection with this, it should be noted that the method [3] still works for (5), provided that $f(r,s)$ is nonincreasing in s. This proves that any classical solution of (5) is radially symmetric and satisfies (2) if (4) is replcaed by

$$r \in (0,1) \mapsto f(r,s) \qquad : \text{nonincreasing for each } \; s > 0. \tag{6}$$

In the case of $n = 2$, the assumption (4) implies $f(r,s) \geq 0$. Restricted to the positive nonlinearity, conversely, the property of (4) is weaker than that of (6). Actually, the condition (12) is motivated by [11] concerning two-dimensional domains.

Details are not desribed here, but taking the infinitesimal generator of the Möbius transformations implies a partial result of theorem 3. Proceeding to the general case, we provide $\Omega = \{|x| < 1\}$ with the Poincaré metric $ds^2 = \left(1 - |x|^2\right)^2 dx^2$ because the standard hyper-planes cannot be applied for the situation (4). We take a family of geodesic hyperplanes to introduce the transformation of reflection. This transformation is isometric with respect to that metric, under which the corresponding Laplace-Beltrami operator Δ_g is invariant. Writing the equation in terms of Δ_g, we have seen that the argument of the continuation used for the proof of Theorem 1 works and Theorem 3 follows.

The purpose of the present paper is to show that the moving sphere method is interpreted as a variant of the above argument of ours. Namely, we can show that the Kelvin transformation is regarded as an isometry with respect to a certain metric, under which the corresponding Laplace-Beltrami operator is invariant. Consequently, a general form of Theorem 2 can be proven by this observation.

2 Kelvin Transform and Coulomb Metric

The well-known property of the Kelvin transformation $y = x/|x|^2$ is expressed as

$$\Delta_y U = |x|^{n+2}\Delta_x u \qquad \text{for} \quad U(y) = |x|^{n-2}u(x). \tag{7}$$

This section is devoted to an underlying geometrical structure, which leads to a generalized form of this transformation. The key observation is the following.

Lemma 4 *The transformation $x \mapsto y = x/|x|^2$ is an isometry with respect to the metric $ds^2 = dx^2/|x|^2$.*

In fact, the Laplace-Beltrami operator Δ_g on this Riemannian space $(\mathcal{R}^n, ds^2)$ is given as

$$\Delta_g v = |x|^2\Delta v - (n-2)x\cdot\nabla v. \tag{8}$$

See [5]. Therefore, writing $u(x) = |x|^{-\frac{n-2}{2}}v(x)$ we obtain

$$\begin{aligned} |x|^{\frac{n+2}{2}}\Delta_x u &= |x|^2\Delta v - (n-2)x\cdot\nabla v - \frac{(n-2)^2}{4}v \\ &= \Delta_g v - \frac{(n-2)^2}{4}v. \end{aligned} \tag{9}$$

From Lemma 4, this operator Δ_g is invariant under the transformation $x \mapsto y$ so that

$$\Delta_g v - \frac{(n-2)^2}{4}v = \Delta_g V - \frac{(n-2)^2}{4}V$$

for $V(y) = v(x)$. Then, similarly to the first equality of (9) this quantity is equal to $|y|^{\frac{n+2}{2}}\Delta_y U$ for $V(y) = |y|^{\frac{n-2}{2}}U(y)$. This leads to (7). We call $ds^2 = dx^2/|x|^2$ the Coulomb metric.

Lemma 4 is proven in the following way. First, the metric is radially symmetric about the origin. Therefore, the geodisc segment ℓ connecting x and

$y = x/|x|^2$ lies in the half line L starting from the origin, containing x and y. That is, $\ell \subset L$.

On the other hand ℓ is divided into two parts of the same length at z, the crossing point of ℓ and $S^{n-1} \equiv \{|x| = 1\}$. This is a consequence of

$$\int_{|x|}^{1} \frac{ds}{s} = \int_{1}^{|y|} \frac{ds}{s} = -\log|x|.$$

Crucial is that the unit sphere $T = S^{n-1}$ is a geodesic hyper-plane, that is, any geodesic curve connecting any two points on T lies in T. This implies that L is perpendicular to T and the transformation $x \mapsto y$ is a reflection with respect to T. Therefore, it is an isometry and the proof is completed.

The key fact mentioned above is a consequence of the following proposition.

Proposition 5 *The sphere* $T_\lambda = \{|x| = \lambda\}$ *is a geodisc hyper-plane with respect to the metric* $ds^2 = \rho(|x|)\, dx^2$ *if and only if*

$$(\log \rho)'(\lambda) = -2/\lambda. \tag{10}$$

As is described above, if the condition (10) is satisfied, then the transformation

$$y = \sigma(|x|)\, x \qquad \text{with} \qquad \int_{t}^{\lambda} \rho(s)^{1/2} ds = \int_{\lambda}^{\sigma(t)} \rho(s)^{1/2} ds \tag{11}$$

is an isometry. In particular, the associated Laplace-Beltrami operator is invariant under this transformation.

We note that the case $\rho(r) = 1/r^2$ admits the relation (10) for any $\lambda > 0$. This induces a family of isomeric transformations defined by (11), namely

$$x \mapsto \lambda^2 x/|x|^2.$$

Then, the corresponding Laplace-Beltrami operator Δ_g descibed as (8) is invariant under this transformation, and the argument of moving planes works.

To conclude the section, we give the proof of Proposition 5 .

First we note that the geodesic curve $x = x(t)$ is a solution of

$$\ddot{x} + \frac{1}{2}(\log \rho)'(|x|)\, |x|^{-1} \left\{ \left(\frac{d}{dt}|x|^2 \right) \dot{x} - |\dot{x}|^2\, x \right\} = 0. \tag{12}$$

See [5]. If $x(t) \in T_\lambda$ is a geodesic, we have

$$|x|^2 = \lambda^2, \qquad x \cdot \dot{x} = 0, \qquad \text{and} \qquad \ddot{x} \cdot x + |\dot{x}|^2 = 0.$$

Therefore, operating $x\cdot$ to (12), we have (10) by $|\dot{x}| \neq 0$.

Conversely, let (10) hold. Given $a, b \in T_\lambda$, we can take a geodesic curve $x(t) \in T_\lambda$ satisfying (12). To see this, let a', b' be the crossing points with S^{n-1} and the half lines connecting a, b with the origin, respectively. In use of the rotation of the axes, we can suppose that a', b' are located on the circle $S^1 \subset S^{n-1}$ indicated by $x_1 = x_2 = \cdots = x_{n-2} = 0$ and $|x| = 1$. Then, those two points are connected by an orbit expressed as

$$\omega(t) = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cos t \\ \sin t \end{pmatrix}.$$

We have $|\omega| = |\dot{\omega}| = 1$ and $\ddot{\omega} + \omega = 0$. Therefore, the orbit $x(t) = \lambda\omega(t) \in T_\lambda$ connects a and b, satisfying

$$|x| = |\dot{x}| = \lambda \qquad \text{and} \qquad \ddot{x} + x = 0.$$

This means (12) under the assumptions of (10), and $x(t)$ is a geodesic curve. □

3 A symmetry and mononicity theorem

The rest of the present paper is devoted to the following equation on the annulus $A = \{x \in \mathcal{R}^n \mid a < |x| < 1\}$.

$$-\Delta u = f\left(x, |x|^{\frac{n-2}{2}} u, (x \cdot \nabla)\left(|x|^{\frac{n-2}{2}} u\right)\right), \quad u > 0 \quad \text{in } A \tag{13}$$

Here, $f = f(x, s, q)$ is a continuous function on $\overline{A} \times [0, \infty) \times [0, \infty)$, C^1 in s and q, and even with respect to q:

$$f(x, s, -q) = f(x, s, q) \qquad (x \in A, s \geq 0, q \in \mathcal{R}).$$

We suppose the properties

$$r \in (a, \sqrt{a}) \mapsto f(r\omega, s, q) \qquad : \text{non-decreasing} \tag{14}$$

and

$$r \in (\sqrt{a}, 1) \mapsto f(r\omega, s, q) \qquad : \text{non-increasing} \tag{15}$$

for each $\omega \in S^{n-1}$, $s \geq 0$, and $q \in \mathcal{R}$. Then, we can prove the following theorems. If they are applied for

$$f(x, s, q) = |x|^{-\frac{n+2}{2}} s^{\frac{n+2}{n-2}},$$

then Theorem 2 follows.

Theorem 6 *Suppose, furthermore, that*

$$r^{\frac{n+2}{2}} f(r\omega, s, q) \le r_*^{\frac{n+2}{2}} f(r_*\omega, s, q) \qquad in \quad r \in (1, \sqrt{a}), \tag{16}$$

where $\omega \in \mathcal{S}^{n-1}$, $q \in \mathcal{R}$, *and* $r_* = a/r$. *Then, any* $u \in C^2(A) \cap C(\overline{A})$ *satisfying (13) and* $u = 0$ *on* $|x| = 1$ *has the properties*

$$u(x) \le \left(\frac{|x|}{\sqrt{a}}\right)^{2-n} u\left(a\frac{x}{|x|^2}\right) \tag{17}$$

on $\sqrt{a} < |x| < 1$ *and (3).*

Theorem 7 *Similarly, let*

$$r^{\frac{n+2}{2}} f(r\omega, s, q) \ge r_*^{\frac{n+2}{2}} f(r_*\omega, s, q) \qquad in \quad r \in (\sqrt{a}, 1)$$

for $a < r < \sqrt{a}$, $\omega \in \mathcal{S}^n$, $s \ge 0$, *and* $q \in \mathcal{R}$. *Then, any* $u \in C^2(A) \cap C(\overline{A})$ *satisfying (13) and* $u = 0$ *on* $|x| = a$ *has the properties (17) on* $a < |x| < \sqrt{a}$ *and*

$$\left(|x|^{\frac{n-2}{2}} u\right)_r > 0 \qquad \left(a < |x| < \sqrt{a}\right).$$

We shall only show Theorem 6. Given $\lambda \in (\sqrt{a}, 1)$, we set

$$T_\lambda = \{|x| = \lambda\} \qquad \text{and} \qquad \Sigma_\lambda = \{\lambda < |x| < 1\}.$$

Writing $x^\lambda = \lambda^2 \cdot x/|x|^2$, we have $|x| > |x^\lambda|$ for $x \in \Sigma_\lambda$.

The function $v(x) = |x|^{\frac{n-2}{2}} u(x)$ solves

$$|x|^2 \Delta v - (n-2) x \cdot \nabla v - \frac{(n-2)^2}{4} v + |x|^{\frac{n+2}{2}} f(x, v, (x \cdot \nabla) v) = 0$$

on A. In use of the Coulomb metric $ds^2 = dx^2/|x|^2$, this is written as

$$\Delta_g v - \frac{(n-2)^2}{4} v + |x|^{\frac{n+2}{2}} f(x, v, (x \cdot \nabla) v) = 0 \qquad \text{in} \quad A.$$

Let $v^\lambda(x) = v(x^\lambda)$. From the description given above, we have $\Delta_g v^\lambda = \Delta_g v$. On the othe hand, $x \cdot \nabla = r\partial_r$ for $r = |x|$ and hence $y \cdot \nabla_y v^\lambda = -x \cdot \nabla_x v$. Therefore, the property $f(x, s, -q) = f(x, s, q)$ implies the relation

$$\Delta_g v^\lambda - \frac{(n-2)^2}{4} v^\lambda + \left|x^\lambda\right|^{\frac{n+2}{2}} f\left(x^\lambda, v^\lambda, (x \cdot \nabla) v^\lambda\right) = 0 \qquad \text{in} \quad A.$$

We show that the assumption on f guarantees

$$\left|x^\lambda\right|^{\frac{n+2}{2}} f(x^\lambda, s, q) \ge |x|^{\frac{n+2}{2}} f(x, s, q) \tag{18}$$

for $\sqrt{a} < \lambda < |x| < 1$, $s \geq 0$, and $q \geq 0$. This follows from

$$r_1^{\frac{n+2}{2}} f(r_1\omega, s, q) \geq r^{\frac{n+2}{2}} f(r\omega, s, q) \tag{19}$$

for $\omega \in S^{n-1}$, $\sqrt{a} < r < 1$, $r_* = a/r < r_1 < r$.

In fact, if $r_1 \geq \sqrt{a}$, then (19) is a consequence of (15). Otherwise, $r_* < r_1 < \sqrt{a}$ so that

$$\begin{aligned} r_1^{\frac{n+2}{2}} f(r_1\omega, s, q) &\geq r_*^{\frac{n+2}{2}} f(r_*\omega, s, q) \\ &\geq r^{\frac{n+2}{2}} f(r\omega, s, q) \end{aligned}$$

by (14) and (16). The inequality (18) has been proven.

Therefore, the function $w_\lambda = v^\lambda - v$ satisfies

$$\begin{aligned} &\Delta_g w - \frac{(n-2)^2}{2} w_\lambda \\ &+|x|^{\frac{n+2}{2}} \left(f\left(x, v^\lambda\right), (x \cdot \nabla) v^\lambda \right) - f\left(x, v, (x \cdot \nabla) v\right)\right) \leq 0 \end{aligned}$$

on $\Sigma_\lambda = \{\lambda < |x| < 1\}$. Writing

$$b_\lambda(x) = \int_0^1 f_s\left(x, tv^\lambda(x) + (1-t)v(x), tx \cdot \nabla v^\lambda(x) + (1-t)x \cdot \nabla v(x)\right) dt,$$

$$d_\lambda(x) = \int_0^1 f_q\left(x, tv^\lambda(x) + (1-t)v(x), tx \cdot \nabla v^\lambda(x) + (1-t)x \cdot \nabla v(x)\right) dt,$$

and $\beta_\lambda(x) = d_\lambda(x)x$, we obtain

$$\Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda + |x|^{\frac{n+2}{2}} \left(c_\lambda(x) + \beta_\lambda(x) \cdot \nabla\right) w_\lambda \leq 0 \qquad \text{on} \quad \Sigma_\lambda.$$

We apply the identity (9) for $z_\lambda(x) = |x|^{-\frac{n-2}{2}} w_\lambda(x)$. Then we obtain the following lemma.

Lemma 8 *Under the assumptions of Theorem 6, each $\lambda \in (\sqrt{a}, 1)$ admits the inequality*

$$\Delta z_\lambda + b_\lambda(x) z_\lambda + \beta_\lambda(x) \cdot \nabla z_\lambda \leq 0 \qquad on \quad \Sigma_\lambda, \tag{20}$$

where $z_\lambda(x) = |x|^{-\frac{n-2}{2}} \left(v^\lambda - v\right)$.

4 Proof of Theorem

Once Lemma 8 is proven, Theorem 6 follows from the standard argument ([2]). We shall sketch the proof for completeness.

Putting

$$\Lambda \equiv \{\lambda \in (a,1) \mid z_\lambda > 0 \ \text{ in } \ \Sigma_\lambda\},$$

we see that the desired consequence follows from $\Lambda = (\sqrt{a}, 1)$. Let $\lambda \in \Lambda$. We have $z_\lambda = 0$ on T_λ, and $z_\lambda > 0$ in Σ_λ. Therefore, Hopf's boundary lemma can be applied by (20) so that

$$\frac{\partial z_\lambda}{\partial \nu} < 0 \qquad \text{on} \quad T_\lambda, \tag{21}$$

where ν denotes the outer unit normal vector on T_λ from Σ_λ.

The coefficents $b_\lambda(x)$ and $\beta_\lambda(x)$ are uniformely bounded. For r_0 close to 1, the maximum principle holds for the equation (20) on any subdomain of $A \setminus \overline{B}_{r_0}$ and for any λ. Here and henceforth, $B_{r_0} = \{x \in \mathcal{R}^n \mid |x| < r_0\}$. This implies $[r_0, 1) \subset \Lambda$.

We note the following lemma.

Lemma 9 *If $\lambda \notin \Lambda$, there exists some $x_0 \in \Sigma_\lambda \cap \overline{B}_{r_0}$ such that $z_\lambda(x_0) \le 0$.*

Proof: As we have proven, $\lambda < r_0$ and hence $\Sigma_\lambda \cap \overline{B}_{r_0} \neq \emptyset$. Suppose

$$z_\lambda(x) > 0 \qquad \text{on} \quad \Sigma_\lambda \cap \overline{B_{r_0}}.$$

Then we get

$$\Delta z_\lambda + b_\lambda(x) z_\lambda + \beta_\lambda(x) \cdot \nabla z_\lambda \le 0 \qquad \text{in} \quad \Sigma_\lambda \setminus \overline{B}_{r_0},$$

and

$$z_\lambda \ge 0 \qquad \text{on} \quad \partial\left(\Sigma_\lambda \setminus \overline{B}_{r_0}\right).$$

Now the maximum principle guarantees $z_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B}_{r_0}$.

However, we have $z_\lambda > 0$ in Σ_{r_0} and hence $z_\lambda > 0$ in Σ_λ. This means $\lambda \in \Lambda$, a contradiction. □

We show that Λ is left-open. Let $\lambda_0 \in \Lambda$. If not, there exists a sequence $\{\lambda_n\}$ satisfying

$$\lambda_0 - \frac{1}{n} < \lambda_n \le \lambda_0 \qquad \text{and} \qquad \lambda_n \notin \Lambda.$$

Therefore, Lemma 9 guarantees the existence of $x_n \in \Sigma_{\lambda_n} \cap \overline{B}_{r_0}$ satisfying

$$z_{\lambda_n}(x_n) \le 0.$$

We have $z_{\lambda_n} = 0$ on T_{λ_n}. We have a point y_n satisfying

$$\frac{\partial z_{\lambda_n}}{\partial r}(y_n) \le 0 \tag{22}$$

on the segment connecting x_n and $\lambda_n \frac{x_n}{|x_n|}$.

Taking a subsequence if necessary, we may suppose the existence of some $x_0 \in \overline{\Sigma}_{r_0} \cap \overline{B}_{r_0}$ satisfying $x_n \to x_0$. This implies $z_{\lambda_0}(x_0) \leq 0$.

Here, $\lambda_0 \in \Lambda$ and hence $x_0 \in T_{\lambda_0}$. In particular, $y_n \to x_0$ and $\frac{\partial z_{\lambda_0}}{\partial r}(x_0) \leq 0$ follows from (22). However, this is equivalent to

$$\frac{\partial z_{\lambda_0}}{\partial \nu}(x_0) \geq 0,$$

which contradicts to (21) valid for $\lambda = \lambda_0 \in \Lambda$.

The final stage is to show that Λ is left-closed. In fact, let $\{\lambda_n\} \subset \Lambda$ be a sequence satisfying $\lambda_n \downarrow \lambda_1 > \sqrt{a}$. Then, we have

$$\Delta z_{\lambda_1} + (b_{\lambda_1}(x) + \beta_{\lambda_1}(x)\nabla)\, z_\lambda \leq 0,$$

$$z_{\lambda_1} \geq 0, \qquad \text{and} \qquad z_{\lambda_1} \not\equiv 0 \qquad \text{in } \Sigma_{\lambda_1}.$$

Actually, the last relation follows from $z_{\lambda_1} > 0$ on $|x| = 1$. Therefore, the maximum principle implies $z_{\lambda_1} > 0$ in Σ_{λ_1}, or equivalently, $\lambda_1 \in \Lambda$.

In this way, $\Lambda = (\sqrt{a}, 1)$ and the proof has been completed. □

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Spiky patterns and their stability in a reaction-diffusion system

Izumi Takagi

Mathematical Institute, Tôhoku University
Aoba, Aoba-ku, Sendai 980-77, Japan

1. Introduction

We consider the stability of stationary solutions to the following reaction-diffusion system due to A. Gierer and H. Meinhardt ([GM]):

$$(1.1)\qquad \begin{cases} A_t = \epsilon^2 A_{xx} - A + \dfrac{A^p}{H^q} + \sigma_0, \\ \tau H_t = D H_{xx} - H + \dfrac{A^r}{H^s} \end{cases}$$

for $x \in (0,1)$ and $t > 0$, under homogeneous Neumann boundary conditions

$$(1.2)\qquad A_x = H_x = 0 \qquad \text{at } \ x = 0, 1.$$

Here, $A = A(x,t) > 0$ and $H = H(x,t) > 0$ represent the respective concentrations of biochemicals called an *activator* and an *inhibitor*; ϵ, τ and D are positive constants, σ_0 is a nonnegative constant, and the exponents (p,q,r,s) are assumed to satisfy

$$(1.3)\qquad 0 < \frac{p-1}{q} < \frac{r}{s+1} \qquad (p > 1,\ q > 0,\ r > 0,\ s \geq 0).$$

This reaction-diffusion system was proposed to model pattern formation in developmental biology. It is interpreted that changes in cells or tissues begin at the place where the activator concentration is high.

When $\sigma_0 = 0$, the system without diffusion

$$(1.4)\qquad \begin{cases} \dfrac{dA}{dt} = -A + \dfrac{A^p}{H^q}, \\ \tau \dfrac{dH}{dt} = -H + \dfrac{A^r}{H^s} \end{cases}$$

has a unique steady-state $(A,H) = (1,1)$, which is stable if $0 < \tau < \dfrac{s+1}{p-1}$ and is unstable if $\tau > \dfrac{s+1}{p-1}$. Due to the boundary conditions, this steady-state is also a stationary solution to (1.1) & (1.2). It is known that even if the constant steady-state $(A, H) = (1, 1)$ is stable as a stationary solution to (1.4), it becomes unstable as a solution to (1.1) & (1.2) if the ratio ϵ^2/D is sufficiently small (Turing's diffusion-driven instability). Numerical simulations suggest that in such situations there is a stationary solution which exhibits a spiky pattern.

As a first step in studying (1.1) & (1.2), we consider the limit $D \to \infty$. Heuristically, as $D \to \infty$, $H_{xx}(x,t) \to 0$, and hence $H(x,t) \to \xi(t)$ because of (1.2). The equation for ξ is obtained by integrating the second equation in (1.1) over $0 < x < 1$ and using (1.2). This results in the following so-called *shadow system*:

$$(1.5) \qquad \begin{cases} A_t = \epsilon^2 A_{xx} - A + \dfrac{A^p}{H^q} + \sigma_0 & \text{for } x \in (0,1), \\ \tau \dfrac{d\xi}{dt} = -\xi + \dfrac{1}{\xi^s} \displaystyle\int_0^1 A^r \, dx, & \\ A_x = 0 \quad \text{at } x = 0,\ 1. & \end{cases}$$

Concerning stationary solutions to (1.5), we have the following

Proposition 1.1. *There is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then (1.5) has a solution $(A(x;\epsilon), \xi(\epsilon))$ satisfying*

i) $A_x(x;\epsilon) < 0$ on $(0,1)$,

ii) $A(x;\epsilon) \to \sigma_0$ as $\epsilon \downarrow 0$ locally uniformly in the interval $(0,1]$;

iii) $A(0;\epsilon) \to +\infty$ as $\epsilon \downarrow 0$;

iv) $\xi(\epsilon) \to +\infty$ as $\epsilon \downarrow 0$.

A stationary solution $(A(x), \xi)$ to (1.5) is said to be of *mode k* if $A'(x)$ has exactly $k-1$ zeros in the interval $(0,1)$.

Proposition 1.2. *Let $A(x;\epsilon)$ be as in Proposition 1.1 and put*

$$\tilde{A}(x;\epsilon) := \begin{cases} A(2m - x;\epsilon) & \text{if } 2m-1 < x < 2m \ ; \\ A(x - 2m;\epsilon) & \text{if } 2m < x < 2m+1 \ , \end{cases}$$

for $m = 1, 2, 3, \ldots$. For each natural number k, set

$$A_k^+(x;\epsilon) := \tilde{A}(kx+1; k\epsilon), \qquad A_k^-(x;\epsilon) := \tilde{A}(kx; k\epsilon), \qquad \text{and} \qquad \xi_k(\epsilon) := \xi(k\epsilon)$$

for $x \in [0,1]$ and for $\epsilon \in (0, \epsilon_0/k)$. Then both of

$$(A_k^+(x;\epsilon), \xi_k(\epsilon)) \quad \text{and} \quad (A_k^-(x;\epsilon), \xi_k(\epsilon))$$

are stationary solutions to (1.5) of mode k.

For the proof of Proposition 1.1 and Proposition 1.2, see [T].

To state our results on the stability of these stationary solutions to the shadow system, we introduce the following two quantities:

$$\alpha := \frac{qr}{p-1} - (s+1) \tag{1.6}$$

$$\beta := \frac{qr}{p-1}\left(\frac{1}{p-1} - \frac{1}{2r}\right). \tag{1.7}$$

Note that $\alpha > 0$ because of (1.3), while $\beta < 0$ if $r < \frac{1}{2}(p-1)$, $\beta = 0$ for $r = \frac{1}{2}(p-1)$, and $\beta > 0$ for $r > \frac{1}{2}(p-1)$. The following four theorems are obtained in a joint work with Wei-Ming Ni and Eiji Yanagida [NTY]. The first three of them are concerned with stationary solutions of mode one, i.e., solutions with only one peak at the boundary:

Theorem 1.3. (Instability) *If $\tau > \beta$, then the stationary solution $(A(\cdot;\epsilon), \xi(\epsilon))$ is unstable, provided that $\alpha > 0$ and $\epsilon > 0$ are sufficiently small.*

Theorem 1.4. (Stability, I) *If $1 < p < 5$, r is sufficiently close to 2 (so that, in particular, $\beta > 0$), and $0 < \tau + \max\{C_0\alpha, C_1\sqrt{\alpha}\} < \beta$, then $(A(\cdot;\epsilon), \xi(\epsilon))$ is stable, as long as ϵ is sufficiently small. Here, C_0 and C_1 are positive constants depending only on (p, q, r).*

Theorem 1.5. (Stability, II) *If $1 < p < 5$, r is sufficiently close to $p+1$, and $\tau_* < \tau < \beta - C_1\sqrt{\alpha}$, then $(A(\cdot;\epsilon), \xi(\epsilon))$ is stable whenever α and ϵ are sufficiently small. Here, τ_* and C_1 are positive constants depending only on (p, q, r).*

For the shadow system, stationary solutions with interior peaks are turned out to be unstable:

Theorem 1.6. *If $k \geq 2$, then both of the stationary solutions of mode k, $(A_k^{\pm}(\cdot;\epsilon), \xi_k(\epsilon))$, are unstable for ϵ sufficiently small.*

2. Characteristic Equation

To make the arguments transparent we restrict ourselves to considering the case $\sigma_0 = 0$. We point out, however, that the case $\sigma_0 > 0$ is treated as a perturbation of the case $\sigma_0 = 0$ and no essential difficulty comes in. As is well-known, if the eigenvalue problem

$$\begin{cases} \epsilon^2 a'' - a + \dfrac{pA(x;\epsilon)^{p-1}}{\xi(\epsilon)^q} a - \dfrac{qA(x;\epsilon)^p}{\xi(\epsilon)^{q+1}} h = \lambda a, \quad \text{in } (0,1) \\ -h + r\displaystyle\int_0^1 \dfrac{A(x;\epsilon)^{r-1}}{\xi(\epsilon)^s} a\,dx - s\int_0^1 \dfrac{A(x,\epsilon)^r}{\xi(\epsilon)^{s+1}}\,dx h = \tau\lambda h, \\ a'(0) = a'(1) = 0. \end{cases}$$

has no eigenvalue in the left half plane $\operatorname{Re}\lambda \geq 0$, then the stationary solution $(A(x;\epsilon), \xi(\epsilon))$ is stable, whereas if there is an eigenvalue with positive real part, then it is unstable. To study this eigenvalue problem it is convenient to introduce the scaling

$$A(x;\epsilon) = \xi(\epsilon)^{q/(p-1)} u_\epsilon(x), \quad a(x) = \xi(\epsilon)^{q/(p-1)}\phi(x), \quad h = \xi(\epsilon)\eta.$$

Then by a straightforward computation we see that the eigenvalue problem above is equivalent to the following:

$$\epsilon^2\phi'' - \phi + pu_\epsilon^{p-1} - qu_\epsilon^p\eta = \lambda\phi \qquad \text{in } (0,1), \tag{2.1}$$

$$-(1+s)\eta + \frac{r\int_0^1 u_\epsilon^{r-1}\phi\,dx}{\int_0^1 u_\epsilon^r\,dx} = \lambda\tau\eta, \tag{2.2}$$

$$\phi'(0) = \phi'(1) = 0. \tag{2.3}$$

In what follows, $\mathcal{L}_\epsilon$ denotes the linearized operator defined by the right hand sides of (2.1)–(2.2) subject to boundary conditions (2.3).

Lemma 2.1. *Under homogeneous Neumann boundary conditions, the spectrum of the linearized operator*

$$L_\epsilon = \epsilon^2\frac{d^2}{dx^2} - 1 + pu_\epsilon$$

consists of the eigenvalues $\{l_{j,\epsilon}\}_{j=0}^\infty$ satisfying

$$l_{0,\epsilon} \geq \delta_0 > 0 > -\delta_0 \geq l_{1,\epsilon} > l_{2,\epsilon} > \cdots > l_{j,\epsilon} > l_{j+1,\epsilon} > \cdots \downarrow -\infty,$$

provided that ϵ is sufficiently small. Here, δ_0 is a positive constant independent of ϵ. If $\varphi_{j,\epsilon}$ is a normalized eigenfunction belonging to $l_{j,\epsilon}$, then $\{\varphi_{j,\epsilon}\}_{j=0}^\infty$ forms a complete orthonormal system of $L^2(0,1)$. Moreover, $\varphi_{j,\epsilon}(x)$ has exactly $j-1$ zeros in the closed interval $[0,1]$.

By making use of this lemma and the observation that $Lu_\epsilon = (p-1)u_\epsilon^p$, we obtain

Proposition 2.2. *(i) $l_{0,\epsilon}$ is not an eigenvalue of $\mathcal{L}_\epsilon$.*

(ii) $\lambda \in \mathbb{C} \setminus \{l_{j,\epsilon}\}$ is an eigenvalue of $\mathcal{L}_\epsilon$ if and only if it satisfies the characteristic equation

$$\chi(\lambda,\epsilon) := \alpha + \lambda\left(\frac{qr}{p-1}\frac{\int_0^1 u_\epsilon^{r-1}(L_\epsilon-\lambda)^{-1}u_\epsilon\,dx}{\int_0^1 u_\epsilon^r\,dx} - \tau\right) = 0. \tag{2.4}$$

We observe that $w_\epsilon(z) := u_\epsilon(\epsilon z)$ satisfies

$$\begin{cases} w_\epsilon'' - w_\epsilon + w_\epsilon^p = 0 & \text{in } (0,1/\epsilon), \\ w_\epsilon'(0) = w_\epsilon'(1/\epsilon) = 0 \end{cases}$$

and as $\epsilon \to 0$, it converges to w, which is the unique positive solution to

$$\begin{cases} w'' - w + w^p = 0 & \text{in } (0,+\infty), \\ w'(0) = 0, \quad \lim_{z\to\infty} w(z) = 0. \end{cases}$$

Under the boundary conditions $\phi'(0) = 0$, $\lim_{z\to\infty}\phi(z) = 0$, the linearized operator

$$L := \frac{d^2}{dz^2} - 1 + pw(z)^{p-1}$$

has a finite number of eigenvalues $l_0 > 0 > l_1 > \cdots > l_n > -1$, which are simple. By the eigenfunction expansion and by a uniform decay estimate of the function $(L_\epsilon-\lambda)^{-1}u_\epsilon$, we can prove

Lemma 2.3. *(i) The characteristic function $\chi(\lambda,\epsilon)$ is meromorphic in $\lambda \in \mathbb{C}\setminus\{l_{j,\epsilon}\}_{j=0}^\infty$ and admits the following expansion:*

$$\chi(\lambda,\epsilon) = \alpha + \lambda\left(\sum_{j=0}^\infty \frac{c_{j,\epsilon}}{l_{j,\epsilon}-\lambda} - \tau\right),$$

where

$$c_{j,\epsilon} = \frac{qr}{(p-1)\int_0^1 u_\epsilon^r\,dx}\int_0^1 u_\epsilon^{r-1}\varphi_{j,\epsilon}\,dx \int_0^1 u_\epsilon\varphi_{j,\epsilon}\,dx.$$

(ii) As $\epsilon \to 0$,

$$\chi(\lambda,\epsilon) \to \chi_0(\lambda) := \lambda\left(\frac{qr\int_0^\infty w^{r-1}(L-\lambda)^{-1}w\,dz}{(p-1)\int_0^\infty w^r\,dz} - \tau\right) + \alpha$$

locally uniformly in $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\sqrt{\lambda+1} > \max\{1-r,0\}\} \setminus \{\lambda = l_j\}_{j=0}^n$.

Finally, we make a crucial observation on the sign of $c_{j,\epsilon}$ and compute the exact value of $\chi_0'(0)$:

Lemma 2.4.

(i) $c_{0,\epsilon} > 0$.

(ii) If $r = 2$, *then* $c_{j,\epsilon} \geq 0$ *for all* $j = 0, 1, 2, \ldots$.

(iii) If $r = p + 1$, *then* $c_{j,\epsilon} \leq 0$ *for all* $j = 1, 2, 3, \ldots$.

(iv) $\chi_0'(0) = \beta - \tau$.

3. Proof of Theorems

Again we consider only the case $\sigma_0 = 0$. By Lemma 2.3 and Lemma 2.4, we see that, in the small neighborhood of $\lambda = 0$, the characteristic equation can be approximated by

$$\chi_0(\lambda) = a(\lambda)\lambda^k + (\beta - \tau)\lambda + \alpha = 0$$

in which $a(0) \neq 0$ and $k \geq 2$. By elementary reasoning we get the following

Lemma 3.1.

(i) If α *is sufficiently small and* $\beta < \tau$, *then* $\chi_0(\lambda) = 0$ *has at least two positive roots in the interval* $(0, l_1)$.

(ii) If $a(0) > 0$, $k = 2$, $\beta - \tau > 0$, *then there is a* $\delta_1 > 0$ *such that, for* $\alpha > 0$ *satisfying* $3\alpha/(\beta - \tau) < \delta_1$ *and* $a(0)\alpha < (\beta - \tau)^2$, *the equation* $\chi_0(\lambda) = 0$ *has two negative roots in the small neighborhood of* $\lambda = 0$.

(iii) In addition to the assumptions of (ii), suppose that $c_{1,\epsilon} < 0$. *Then there is a* $\delta_1 > 0$ *such that if* $\beta - \tau < 4a(0)\delta_1$ *and* $8a(0)\alpha < (\beta - \tau)^2$, *then* $\chi(\lambda, \epsilon) = 0$ *has at least three negative roots in the interval* $(l_{1,\epsilon}, 0)$, *provided that* ϵ *is sufficiently small.*

Clearly Theorem 1.3 is an immediate consequence of Lemma 3.2 (i). By making use of Lemma 2.4 (ii) and (iii), it is not difficult to see that (i) if $r = 2$, then we have necessarily $k = 2$ and $a(0) > 0$ and that (ii) if $r = p + 1$ and $1 < p < 5$, then we have $k = 2$ and $a(0) > 0$. Therefore, Lemma 3.2 (ii) implies that if $r = 2$, then the characteristic equation $\chi(\lambda, \epsilon) = 0$ has at least two negative roots in the interval $(l_{j,\epsilon}, 0)$, provided that α and ϵ are sufficiently small. To finish the proof of Theorem 1.4, we need to show that $\chi(\lambda, \epsilon) = 0$ has no root in the half plane $\mathrm{Re}\,\lambda \geq 0$. By the Rouche theorem there is a natural number m such that $\chi(\lambda, \epsilon)$ and the rational function

$$\chi_m(\lambda, \epsilon) := \alpha + \lambda \left(\sum_{j=0}^{m} \frac{c_{j,\epsilon}}{l_{j,\epsilon} - \lambda} - \tau \right)$$

has the same number of zeros in the half plane $\operatorname{Re}\lambda > l_{1,\epsilon}$. Since we may assume that $c_{j,\epsilon} > 0$ for all $j = 0, 1, 2, \ldots$, it turns out that $\chi_m(\lambda, \epsilon)$ has at least m zeros in the interval $(-\infty, l_{1,\epsilon})$. This implies that $\chi_m(\lambda, \epsilon)$ can have at most two zeros in the half plane $\operatorname{Re}\lambda > l_{1,\epsilon}$ because $\chi_m(\lambda, \epsilon) = 0$ has exactly $m + 2$ roots in $\mathbb{C}$. Therefore, $\chi(\lambda, \epsilon) = 0$ cannot have any root in the half plane $\operatorname{Re}\lambda \geq 0$. Once the assertion of the theorem is proved for $r = 2$, we have by continuity that it holds true also for r sufficiently close to 2. The proof of Theorem 1.5 is carried out along the same line as in that of Theorem 1.4.

To prove Theorem 1.6, we put

$$L^{\pm}_{k,\epsilon} := \epsilon^2 \frac{d^2}{dx^2} - 1 + p u^{\pm}_{k,\epsilon}(x)^{p-1}$$

in which we define $u^{\pm}_{k,\epsilon}(x) = \xi_k(\epsilon)^{-q/(p-1)} A^{\pm}_k(x; \epsilon)$. Then we have the following

Lemma 3.2. *If ϵ is sufficiently small, then the operator $L^{\pm}_{k,\epsilon}$ under homogeneous Neumann boundary conditions has exactly k positive eigenvalues $l^k_{0,\epsilon} > l^k_{1,\epsilon} > \cdots > l^k_{k-1,\epsilon} > 0$.*

We need to distinguish between the case where k is odd and the case where k is even. When $k = 2m$ is even, we see that the eigenfunction $\varphi^k_{m,\epsilon}(x)$ belonging to $l^k_{m,\epsilon}$ has a symmetry such that

$$\int_0^1 u^{\pm}_{k,\epsilon}(x)^{r-1} \varphi^k_{m,\epsilon}(x)\, dx = 0.$$

Hence, we see that $\lambda = l^k_{m,\epsilon} > 0$ is an eigenvalue of the linearized operator $\mathcal{L}^{\pm}_{k,\epsilon}$ and $(\phi, \eta) = (\varphi^k_{m,\epsilon}, 0)$ is an eigenvector (see (2.1)–(2.3)). This shows the instability of $(A^{\pm}_k(\cdot; \epsilon), \xi_k(\epsilon))$ in the case where k is even. When $k = 2m - 1$ is odd, we see that the second eigenfunction $\varphi^k_{1,\epsilon}(x)$ vanishes at a point close to $x = (k-2)/k$ and it is approximated in each subinterval $[(i-1)/k, i/k]$, $(i = 1, \ldots, k)$, by the scaled first eigenfunction $\varphi_{0,\epsilon}$ of L_ϵ. From this we see that, in the expansion of the characteristic function $\chi_k(\lambda, \epsilon)$ corresponding to $(A^{\pm}_k(\cdot; \epsilon), \xi_k(\epsilon))$, the coefficient of the second term is positive: $c_{1,\epsilon} > 0$. Therefore,

$$\lim_{\lambda \uparrow l^k_{0,\epsilon}} \chi_k(\lambda, \epsilon) = +\infty,$$

$$\lim_{\lambda \downarrow l^k_{1,\epsilon}} \chi_k(\lambda, \epsilon) = -\infty,$$

whence follows that $\chi_k(\lambda, \epsilon) = 0$ has a positive root in the interval $(l^k_{1,\epsilon}, l^k_{0,\epsilon})$. Thus, $(A^{\pm}_k(\cdot; \epsilon), \xi_k(\epsilon))$ is unstable. This completes the proof of Theorem 1.6.

4. References

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L^p-continuity of wave operators for Schrödinger operators and its applications

KENJI YAJIMA

Department of Mathematical Sciences, University of Tokyo,
Komaba, Meguroku, Tokyo 153, Japan

Let $H = -\triangle + V$ be the Schrödinger operator on $L^2(\mathbf{R}^m)$. There are many cases that we need estimate the norm of functions $f(H)$ of the operator H between suitable function spaces, e.g. between L^p spaces. For example, the solution $u(t)$ of the time dependent Schrödinger equation

$$i\partial u/\partial t = (-\triangle + V)u, \qquad u(0) = \phi$$

is given by $u(t) = e^{-itH}\phi$ via the exponential function e^{-itH} and the L^p–L^q estimates, hence the Strichartz type estimates of the solutions follow from the norm estimates of the corresponding operator e^{-itH} from $L^p(\mathbf{R}^m)$ to $L^q(\mathbf{R}^m)$. Likewise, the solution of the wave equations with potentials

$$\partial^2 u/\partial t^2 - \triangle u + V(x)u = 0, \quad u(0) = \phi, \quad u_t(0) = \psi$$

may be written in the form $u(t) = (\cos t\sqrt{H})\phi + (\sin t\sqrt{H}/\sqrt{H})\psi$ and L^p–L^q estimates of the solutions reduce to those of the operators $\cos t\sqrt{H}$ and $\sin t\sqrt{H}/\sqrt{H}$.

The purpose of my talk is to present a method to obtain L^p–L^q estimates of functions $f(H)$, more precisely a method that reduces the estimates $f(H)$ to those of the corresponding function $f(H_0)$ of the free Schrödinger operator $H_0 = -\triangle$. The latter operator may be expressed explicitly by using the Fourier transform:

$$f(H_0)u(x) = \int e^{ix\xi} f(\xi^2)\hat{u}(\xi)d\xi$$

and may be estimated by using various means. Here and hereafter the Fourier transform $\hat{u}(\xi)$ is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{m/2}} \int e^{-ix\xi} u(x)dx.$$

The method is based on the scattering theory and we briefly review the latter theory. We assume that the potential V is short range, viz., for some $\epsilon > 0$, the multiplication operator $\langle x\rangle^{1+\epsilon}V$ is compact from $H^2(\mathbf{R}^m)$ to $L^2(\mathbf{R}^m)$. Then, the following limits exist (cf. [1], [12]):

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u\,, \quad u \in L^2(\mathbf{R}^m), \tag{1}$$

and the operators $W_\pm$ are called wave operators for the pair (H, H_0). The wave operators are isometries on $L^2(\mathbf{R}^m)$ as the strong limits of unitary operators. A deeper result also in [1] and [12] is the completeness of the wave operators, that is, the image of $W_\pm$ is equal to the continuous spectral subspace $L^2_c(H)$ for H. Actually the singular continuous spectrum of H is absent and the image of $W_\pm$ is equal to the absolutely continuous spectral subspace $L^2_{ac}(H)$ for H. Write $P_c(H)$ for the orthogonal projection onto $L^2_c(H)$. The completeness result implies that the limits

$$Z_\pm u = \lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_c(H)u, \quad u \in L^2(\mathbf{R}^m)$$

also exist and $Z_\pm = W_\pm^*$, where $*$ means the adjoint operator.

The important property of the wave operators which especially concerns us here is the intertwining property: $W_\pm$ intertwine the continuous part of H and H_0:

$$Hu = W_\pm H_0 W_\pm^* u, \qquad u \in L^2_c(H).$$

It follows then that, for any Borel functions f,

$$f(H)P_c(H) = W_\pm f(H_0)W_\pm^*, \quad f(H_0) = W_\pm^* f(H)P_c(H)W_\pm \tag{2}$$

on $L^2(R^m)$. The relations (2) immediately imply that, once we know that $W_\pm$ are bounded from a suitable function space to another, the estimate of the operator norm of $f(H)P_c(H)$ between such spaces may be derived from the corresponding estimate of $f(H_0)$. For example, if we know that $W_\pm$ is bounded in the Sobolev spaces $W^{k,p}$, then $f(H)P_c(H)$ and $f(H_0)$ have equivalent norms between these spaces:

$$C^{-1}\|f(H_0)\|_{B(W^{k,p},W^{k',q})} \le \|f(H)P_c(H)\|_{B(W^{k,p},W^{k',q})} \le C\|f(H_0)\|_{B(W^{k,p},W^{k',q})}. \tag{3}$$

Note that the constant C in (3) is independent of the functions f.

In this paper, we show, under suitable conditions on $V(x)$, that $W_\pm$ are bounded in $L^p(R^m)$ for any $1 \le p \le \infty$, and give its applications to the L^p-L^q estimates on the solutions of Schrödinger equations, wave and Klein-Gordon equations with potentials, and to the "Fourier multiplier theorems" for the generalized eigenfunction expansions associated with the Schrödinger operator H.

We assume that the spatial dimension $m \ge 3$ and $V(x)$ satisfies the following assumption, where $\mathcal{F}$ is the Fourier transform, $\langle x\rangle = (1+|x|^2)^{1/2}$, $\ell = [(m-1)/2]$ is the smallest integrer greater than $(m-3)/2$ if $m \ge 4$ and $\ell = 0$ if $m = 3$, and $m_* = (m-1)/(m-2)$. For multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, $D^\alpha = D_1^{\alpha_1}\cdots D_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Assumption The potential $V(x)$ is real valued and for any $|\alpha| \le \ell$, $\langle x\rangle^{2\sigma}D^\alpha V \in L^2(\mathbf{R}^m)$ for some $\sigma > 1/m_*$. Moreover V satisfies <u>one</u> of the following conditions:

1. $\|\langle x\rangle^{2\sigma}\langle D\rangle^{\ell}V\|_{L^2}$ is sufficiently small.

2. For some $p_0 > m/2$ (or $p_0 = 2$ if $m = 3$) and $\delta > 3m/2 + 1$, there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbf{R}^m}\left(\int_{|x-y|\le 1}|D^\alpha V(y)|^{p_0}dy\right)^{1/p_0} \le C\langle x\rangle^{\delta}, \qquad |\alpha|\le \ell.$$

The assumption implies $\langle x\rangle^{\sigma}V \in L^{q_0}(\mathbf{R}^m)$ for some $q_0 > m/2$ (or $q_0 = 2$ if $m = 3$) and V is short-range in the sense as mentioned above. It follows that H and H_0 with the domain $W^{2,2}(R^m)$ are selfadjoint in $L^2(R^m)$, the wave operators (1) exist and are complete. The wave operators are a fortiori bounded in $L^2(R^m)$.

Theorem 1 *Let V satisfy Assumption and let zero be neither eigenvalue nor resonance of H. Then, for any $1 \le p \le \infty$, $W_\pm$ and $Z_\pm$ originally defined on $L^2 \cap L^p$ can be extended to bounded operators in L^p and*

$$\|W_\pm f\|_{L^p} \le C_p\|f\|_{L^p}, \qquad \|Z_\pm f\|_{L^p} \le C_p\|f\|_{L^p}, \qquad f \in L^2\cap L^p. \tag{4}$$

Remark 1 *Zero is said to be resonance of H if there exists a solution u of $-\Delta u(x) + V(x)u(x) = 0$ such that $\langle x\rangle^{-\gamma}u(x) \in L^2(R^m)$ for any $\gamma > 1/2$ but not for $\gamma = 0$. Under the Assumption, it is well known ([7], [15]) that zero can never be a resonance of H if $m \ge 5$; and that zero is neither eigenvalue nor resonance of H if $\|\langle x\rangle^{2\sigma}\langle D\rangle^{\ell}V\|_{L^2}$ is sufficiently small.*

Remark 2 *If zero is resonance of H* Theorem 1 *never holds. If zero is eigenvalue of H, then* Theorem 1 *does not hold in general. This can be seen by comparing the results of Jensen-Kato([7]) or Murata ([15]) with* Theorem 3 *below.*

We list some immediate consequences of Theorem 1. For Banach spaces X and Y we write $B(X,Y)$ for the space of bounded operators from X to Y, $B(X) = B(X,X)$.

Theorem 2 *Let the assumption of* Theorem 1 *be satisfied, and $1 \le p, q \le \infty$. Then, for any Borel functions f on $\mathbf{R}^1$,*

$$C^{-1}\|f(H_0)\|_{B(L^p,L^q)} \le \|f(H)P_c(H)\|_{B(L^p,L^q)} \le C\|f(H_0)\|_{B(L^p,L^q)}, \tag{5}$$

where the constant C does not depend on f.

Remark 3 *The intermediate results that also follow from* (2) *and* Theorem 1*:*

$$\|f(H)P_c(H)u\|_{L^p} \le C\|f(H_0)W_\pm^* u\|_{L^p}, \qquad \|f(H_0)u\|_{L^p} \le C\|f(H)P_c(H)W_\pm u\|_{L^p} \tag{6}$$

are also of use, where the constant C is independent of Borel f or u.

We should mention here the boundedness of the wave operators $W_\pm$ between weighted L^2 spaces has been shown by Isozakai [6] when $W_\pm$ are restricted to the part of strictly positive energy. We also mention the works of Melin [14] and Jensen-Nakamura [8]. The wave operators are in fact not the only operators which satisfy the interwining property (2) and the L^p continuity. Indeed, Melin [14] has constructed a family of such operators $A_\theta, \theta \in S^{m-1}$ when m *is odd and* V *is smooth and small.* Thus, his A_θ may as well be used to obtain the estimates (5) for such case. It is not clear to us, however, whether his results immediately lead to the boundedness of $W_\pm$. Jensen-Nakamura [8] studies the mapping property of $f(H)$ between Besov spaces when f is smooth and vanishes at infinity.

Combining the well known Kato's L^p–L^q estimate [10] for e^{-itH_0} with Theorem 2 we obtain the following estimate for the propagator e^{-itH} of the time dependent Schrödinger equation. This is a generalization of Journe-Soffer-Sogge [9].

Theorem 3 *Let the assumption of* Theorem 1 *be satisfied. Then, for any $2 \le p \le \infty$ and $1/p + 1/q = 1$, there exists a constant C_p such that for all $t \ne 0$*

$$\|e^{-itH}P_c(H)f\|_{L^p} \le C_p|t|^{m(1/p-1/2)}\|f\|_{L^q}, \qquad f \in L^2L^q. \tag{7}$$

As mentioned above. Theorem 2 can be applied to the wave and Klein-Gordon equations with potentials

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + V(x)u + \mu^2 u = 0, \quad u(0,x) = \phi(x), \;\; u_t(0,x) = \psi(x). \tag{8}$$

Combining Theorem 2 with the well-known L^p–L^q estimates for the free wave and Klein-Gordon equations (cf. Strichartz [20], Pecher [16]), we obtain the following result. The statement for the wave equation is a generalization of Beals and Strauss [2].

Theorem 4 *Let the assumption of* Theorem 1 *be satisfied. Then, for any $2 \le p \le 2(m+1)/(m-1)$ and $1/p+1/q = 1$, there exists a constant $C > 0$ such that for any ϕ, $\psi \in L_c^2(H) \cap L^q$ with $\sqrt{H+\mu^2}\phi \in L^q$, the solution $u(t,x)$ of (8) satisfies*

$$\|u(t,\cdot)\|_{L^p} \le C|t|^{1+m(1/p-1/q)}\left(\|\sqrt{H+\mu^2}\phi\|_{L^q} + \|\psi\|_{L^q}\right), \qquad |t| \ge 1. \tag{9}$$

It is well known that estimates (7) or (9) lead to various space-time integrability properties of the propagators of corresponding equations which are important in non-linear analysis. We omit here, however, the detailed discussion into such direction and content ourselves by showing an inequality of Strichartz type [21] as a prototype (cf. Ginibre-Velo [5], Yajima [25], Pecher [17] and Brenner [3]).

Theorem 5 *Let the assumption of* Theorem 1 *be satisfied and let* ϕ, $\psi \in L^2_c(H)$ *be such that* $(H+\mu^2)^{1/4}\phi \in L^2(R^m)$ *and* $(H+\mu^2)^{-1/4}\psi \in L^2(R^m)$. *Then the solution* u *of (8) belongs to* $L^p(R^{m+1}_{t,x})$ *with* $p = 2(m+1)/(m-1)$ *and*

$$\|u\|_{L^p(R^{m+1}_{t,x})} \le C(\|(H+\mu^2)^{1/4}\phi\|_{L^2} + \|(H+\mu^2)^{-1/4}\psi\|_{L^2}). \tag{10}$$

Proof Write $B_0 = \sqrt{H_0+\mu^2}$ and $B = \sqrt{H+\mu^2}$. By Strichartz's inequality (cf. [21], Corollary 2), we have

$$\|\{\cos(tB_0)/B_0^{1/2}\}\phi\|_{L^p(R^{m+1})} + \|\{\sin(tB_0)/B_0^{1/2}\}\phi\|_{L^p(R^{m+1})} \le C\|\phi\|_{L^2(R^m)}.$$

Applying (6) to $\{\cos(tB)/B^{1/2}\}P_c(H)$ at every fixed t, we have with

$$\begin{aligned}\|\{\cos(tB)/B^{1/2}\}P_c(H)\phi\|^p_{L^p(R^{m+1})} &= \int_{-\infty}^{\infty} \|\{\cos(tB)/B^{1/2}\}P_c(H)\phi\|^p_{L^p(R^m_x)}dt \\ \le C_p^p \int_{-\infty}^{\infty} \|\{\cos(tB_0)/B_0^{1/2}\}W_\pm^*\phi\|^p_{L^p(R^m_x)}dt &\le C_p^p C^p\|\phi\|^p_{L^2}.\end{aligned}$$

This implies $\|\cos(tB)P_c(H)\phi\|_{L^p(R^{m+1})} \le C\|B^{1/2}\phi\|_{L^2}$. The norm $\|\sin(tB)P_c(H)\psi\|_{L^p(R^{m+1})}$ may be estimated in a similar fashion and we obtain (10). ∎

We can make Theorem 2 more precise as follows. When H admits the generalized eigenfunction expansions, the result may be called the 'Fourier multiplier theorem' for the expansion formula. For $j = 1,\dots,m$, define $D_j^\pm = W_\pm D_j W_\pm^*$, where $D_j = -i\partial/\partial x_j$. We call $D^\pm = (D_1^\pm,\dots,D_m^\pm)$ the asymptotic momentum operators. $D_j^\pm$ are commuting selfadjoint operators in $L^2(R^m)$ and, for any Borel function f on R^m, $f(D^\pm)$ can be defined by functional calculus. We have $f(D^\pm)P_c(H) = W_\pm f(D)W_\pm^*$ and the application of Theorem Theorem 1 yields the following

Theorem 6 *Let the assumption of* Theorem 1 *be satisfied and* $1 \le p, q \le \infty$. *Then, there exists a constant* C *independent of* u *and Borel functions* f *such that*

$$\|f(D^\pm)P_c(H)u\|_{L^p} \le C\|f(D)W_\pm^*u\|_{L^p}, \quad \|f(D)u\|_{L^p} \le C\|f(D^\pm)P_c(H)W_\pm u\|_{L^p}, \tag{11}$$

$$C^{-1}\|f(D)\|_{B(L^p,L^q)} \le \|f(D^\pm)P_c(H)\|_{B(L^p,L^q)} \le C\|f(D)\|_{B(L^p,L^q)}. \tag{12}$$

Remark 4 *When* $f(\xi)$ *is a function of* $|\xi|^2$, $f(\xi) = \tilde{f}(|\xi|^2)$, *we have* $f(D) = \tilde{f}(H_0)$ *and* $f(D^{\pm})P_c(H) = \tilde{f}(H)P_c(H)$. *Hence Theorem* Theorem 2 *follows from* Theorem 6.

We relate Theorem 6 with the Fourier multiplier theorem for the generalized Fourier transform associated with H. For simplicity, we assume (2) of Assumption and that zero is neither resonance nor eigenvalue of H. We write $\phi_0(x,\xi) = e^{ix\cdot\xi}$ and $\hat{u}(\xi) = \mathcal{F}u(\xi)$. M_γ is the multiplication operator with $\langle x\rangle^{-\gamma}$ and $R(z) = (H-z)^{-1}$, $R_0(z) = (H_0 - z)^{-1}$ are resolvents.

Kato and Kuroda [11] have shown the followings: For $\gamma > 1$, $B(L^2)$-valued function $M_\gamma R(z) M_\gamma$ of $z \in \mathbf{C}^1 \setminus [0,\infty)$ has continuous boundary values $M_\gamma R(\lambda \pm i0)M_\gamma$ on $[0,\infty)$; and the functions defined by $\phi_\pm(\cdot\,,\xi) = (1 - R(\xi^2 \pm i0)V)\phi_0(\cdot\,,\xi)$ are outgoing (incoming for $-$ sign) generalized eigenfunctions of H in the sense that they are solutions of $(-\Delta + V(x))\phi_\pm(x,\xi) = |\xi|^2\phi_\pm(x,\xi)$ satisfying the outgoing (incoming) radiation condition:

$$\phi_\pm(x,\xi) = \phi_0(x,\xi) + \frac{e^{\pm i|x||\xi|}}{|x|^{(m-1)/2}}\left(f(\hat{x},\xi) + O(|x|^{-1})\right)$$

as $|x| \to \infty$ with fixed $\hat{x} = x/|x|$. Define the generalized Fourier transform by

$$\mathcal{F}_{\pm,H}f(\xi) = (2\pi)^{-m/2}\int_{R^m} \overline{\phi_\pm(x,\xi)}f(x)dx. \tag{13}$$

Then, $\mathcal{F}_{\pm,H}$ are unitary from $L^2_c(H)$ onto $L^2(\mathbf{R}^m)$ and vanish on the point spectral subspace $L^2_p(H)$ for H; $\mathcal{F}_{\pm,H}$ diagonalizes H_c, viz., $\mathcal{F}_{\pm,H}H_c\mathcal{F}^*_{\pm,H}g(\xi) = |\xi|^2 g(\xi)$; moreover $W_\pm$ can be expressed in terms of $\mathcal{F}_{\pm,H}$:

$$W_\pm f(x) = \mathcal{F}^*_{\pm,H}\mathcal{F}f(x) = (2\pi)^{-m/2}\int_{R^m}\phi_\pm(x,\xi)\hat{f}(\xi)d\xi. \tag{14}$$

Note that the unitarity of $\mathcal{F}_{\pm,H}$ implies the generalized eigenfunction expansions:

$$f(x) = (2\pi)^{-m/2}\int_{R^m}\phi_\pm(x,\xi)\mathcal{F}_{\pm,H}f(\xi)d\xi, \qquad f \in L^2_c(H). \tag{15}$$

For a function f write M_f for the multiplication operator with $f(\xi)$ (this is a little abuse of notation but should not cause any confusion). In virtue of (14), we have

$$f(D^\pm)P_c(H) = W_\pm f(D)W_\pm^* = \mathcal{F}^*_{\pm,H}\mathcal{F}f(D)\mathcal{F}^*\mathcal{F}_{\pm,H} = \mathcal{F}^*_{\pm,H}M_f\mathcal{F}_{\pm,H},$$

that is, $f(D^\pm)$ is nothing but the Fourier multiplier M_f for the generalized Fourier transform $\mathcal{F}_{\pm,H}$. Thus Theorem 6 immediately implies the following

Theorem 7 *Let* V *satisfy the condition (2) of* Assumption *and let zero be neither eigenvalue nor resonance of* H. *Let* $1 \le p, q \le \infty$. *Then there exists a constant* $C > 0$ *independent of Borel functions* f *on* R^m *such that*

$$C^{-1}\|f(D)\|_{B(L^p,L^q)} \le \|\mathcal{F}^*_{\pm,H}M_f\mathcal{F}_{\pm,H}\|_{B(L^p,L^q)} \le C\|f(D)\|_{B(L^p,L^q)}. \tag{16}$$

Remark 5 *The argument above shows that estimate (16) remains valid if either one of* $\mathcal{F}^*_{\pm,H}$ *or* $\mathcal{F}_{\pm,H}$ *may be replaced by corresponding operator for another Schrödinger operator* $H' = H_0 + V'$ *satisfying the condition of the Theorem 7.*

Theorem 7 and the well know (ordinary) Fourier multiplier theorem(cf. Taylor [22]) yield the following L^p boundedness of the multiplier in the generalized Fourier transform.

Corollary 8 *Let* V *satisfy the condition (2) of* Assumption *and let zero be neither eigenvalue nor resonance of* H. *Suppose that* $P(\xi)$ *satisfies*

$$\sup_{R>0} R^{-m} \int_{R<|\xi|<2R} ||\xi|^{|\alpha|}\partial_\xi^\alpha P(\xi)|^2 d\xi < \infty, \quad |\alpha| \le [m/2]+1, \tag{17}$$

where $[m/2]$ *is the greatest integer* $\le m/2$. *Then* $\mathcal{F}^*_{\pm,H} M_P \mathcal{F}_{\pm,H} \in B(L^p)$ *for any* $1 < p < \infty$. *In particular, if* $P(\xi) = f(\xi^2)$ *satisfies (17),* $f(H)P_c(H) \in B(L^p)$.

Remark 6 *The same remark as in* Remark 5 *applies and Corollary 8 remains valid if one of* $\mathcal{F}^*_{\pm,H}$ *or* $\mathcal{F}_{\pm,H}$ *may be repalved by the corresponding operator for another Schrödinger operator* $H' = H_0 + V'$.

We outline the proof of Theorem 1 for W_+. The proof for W_- is similar. $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ are the resolvents of H_0 and H, respectively and $R^\pm(\lambda) = R(\lambda \pm i0)$, $R_0^\pm(\lambda) = R_0(\lambda \pm i0)$ are their boundary values on the upper and lower banks of $\mathbf{C} \setminus [0, \infty)$.

We start with the Duhamel formula

$$e^{-itH_0} = e^{-itH} + i\int_0^t e^{-i(t-s)H} V e^{-isH_0} ds.$$

Multiplying both sides with e^{itH} and taking the limit $t \to \infty$ yield

$$W_+ u = u + i\int_0^\infty e^{isH} V e^{-isH_0} u ds. \tag{18}$$

Rewriting the last integral by using the Plancherel formula produces the well known stationary representation formula of the wave operator ([12], [18]):

$$W_+ u = u - \frac{1}{2\pi i}\int_0^\infty R^-(\lambda) V \{R_0^+(\lambda) - R_0^-(\lambda)\} u d\lambda. \tag{19}$$

Expanding $R^-(\lambda) = \sum_{n=0}^\infty (-1)^n R_0^-(\lambda)(V R_0^-(\lambda))^n$ in (19) yields the formal expansion

$$W_+ u = u + \sum_{n=1}^\infty (-1)^n W_n u, \quad W_n u = \frac{1}{2\pi i}\int_{-\infty}^\infty (R_0^-(\lambda) V)^n R_0^+(\lambda) u d\lambda, \tag{20}$$

which of course is a disguised form of the iteration of the Duhamel identity in (18):

$$W_+ u = u + i\int_0^\infty e^{itH_0} V e^{-itH_0} u dt + i^2 \int_{0\le t_1 \le t_2 \le \infty} e^{it_1 H_0} V e^{-i(t_1 - t_2)H_0} V e^{-it_2 H_0} u dt_1 t_2 + \cdots.$$

Let $X = (X_1, \dots, X_m)$ be the vector of multiplication operators by the variables x_j. If we write

$$V(X) = \int e^{iX\xi}\hat{V}(\xi)d\xi,$$

and use the identity $e^{itH_0}e^{iX\xi}e^{-itH_0}u(x) = e^{iX\xi}e^{it(2D\xi+\xi^2)}u(x) = e^{ix\xi+it\xi^2}u(x+2t\xi)$,

$$W_1u(x) = \frac{i}{2}\int_{[0,\infty)\times\mathbf{R}^m} e^{ix\xi}e^{it\xi^2/2}\hat{V}(\xi)u(x+t\xi)dtd\xi. \tag{21}$$

Introducing the polar coordinates $\xi = r\omega, 0 < r < \infty, \omega \in \Sigma$, Σ being the unit sphere, and changing the variable $(t, r, \omega) \to ((t - 2x\omega)/r, r, \omega)$, we arrive at the formula

$$W_1u(x) = \int_{[2x\omega,\infty)\times\Sigma} \widehat{K}_V(t,\omega)u(t\omega + x_\omega)dtd\omega \tag{22}$$

where $x_\omega = x - 2(x\omega)\omega$ is the reflection of x along the ω-axis and

$$\widehat{K}_V(t,\omega) = \frac{i}{2}\int_0^\infty \hat{V}(r\omega)r^{m-2}e^{itr/2}dr.$$

It follows by Minkowski inequality and the fact that $x \to x_\omega$ is measure preserving that

$$\|W_1u\|_{L^p(\mathbf{R}^m)} \le \|\widehat{K}_V\|_{L^1(\mathbf{R}\times\Sigma)}\|u\|_{L^p(\mathbf{R}^m)}. \tag{23}$$

By Hölder's inequality, we have for any $\sigma > 1/m_*$,

$$\|\widehat{K}_V\|_{L^1(\mathbf{R}\times\Sigma)} \le \int_\Sigma \left(\int_\mathbf{R} \langle t\rangle^\sigma |\widehat{K}_V(t,\omega)|^{m-1}dt\right)^{1/(m-1)} d\omega$$

By using Hausdorff-Young's inequality in the first step, Hölder's inequality in the second, and again Hölder's inequality and Plancherel's formula in the last, we estimate

$$\int_\Sigma \left(\int_\mathbf{R} |\widehat{K}_V(t,\omega)|^{m-1}dt\right)^{1/(m-1)} d\omega$$
$$\le C_m \int_\Sigma \left(\int_0^\infty |\hat{V}(r\omega)|^{m_*}r^{m-1}dr\right)^{1/m_*} d\omega \le C_m\|\hat{V}\|_{L^{m_*}} \le C_m\|\langle D\rangle^\rho V\|_{L^2}$$

for $\rho > (m-3)/2$ (or $\rho = 0$ if $m = 3$). Likewise,

$$\int_\Sigma \left(\int_\mathbf{R} |t\widehat{K}_V(t,\omega)|^{m-1}dt\right)^{1/(m-1)} d\omega \le C_m\|\langle D\rangle^\rho(\langle x\rangle V)\|_{L^2}$$

It follows by interpolation that

$$\|\widehat{K}_V\|_{L^1(\mathbf{R}\times\Sigma)} \le C\|\langle x\rangle^\sigma\langle D\rangle^\rho V\|_{L^2(\mathbf{R}^m)}\|u\|_{L^p(\mathbf{R}^m)}. \tag{24}$$

(Actually Plancherel's formula with respect to the radial variables implies

$$\int_\Sigma\int_0^\infty |\widehat{K}_V(t,\omega)|^2dtd\omega \le \pi\int_\Sigma\int_0^\infty |\hat{V}(r\omega)r^{m-2}|^2drd\omega = \int |\xi|^{m-3}|\hat{V}(\xi)|^2d\xi$$

and similar estimate for the t-derivative of $\widehat{K}_V(t,\omega)$. Hence Hölder's inequality and the interpolation inequality imply the following estimate which is slightly better than (24) as far as the regularity is concerned:

$$\|\widehat{K}_V\|_{L^1([0,\infty)\times\Sigma)} \le C\|\langle x\rangle^\sigma V\|_{H^{(m-3)/2}(\mathbf{R}^m)}, \quad \sigma > 1/2$$

Nonethless, the method used to obtain (24) produces better estmates for W_n when $n \ge 2$ and this is why we employ the former method in what follows.)

Repeating the computation which lead to the indenty (21), we write $W_n f(x)$ in the following form:

$$W_n f(x) = \int_{[0,\infty)^n\times\mathbf{R}^{nm}} e^{ix\xi_1} e^{i\sum_{j=1}^n t_j\xi_j^2/2} \prod_{j=1}^n \hat{V}(\xi_{j-1}-\xi_j) f(x+\sum_{j=1}^n t_j\xi_j) dT d\Xi \tag{25}$$

where $dT = dt_1\dots dt_n, d\Xi = d\xi_1\dots d\xi_n$. Then, manipulating in a way similar to that is used for deriving (22), we obtain,

$$W_n f(x) = \int_{[0,\infty)^{n-1}\times I\times\Sigma^n} \hat{K}_n(T,\Omega) f(x_{\omega_1}+\rho) dT d\Omega, \tag{26}$$

where $d\Omega = d\omega_1\dots d\omega_n$, $I = (2x\omega_1,\infty)$, is the range of the integration by the variable t_n and $\rho = t_1\omega_1+\cdots+t_n\omega_n$. Here $K_n(\xi_1,\dots,\xi_n) = (i/2)^n\prod_{j=1}^n \hat{V}(\xi_{j-1}-\xi_j)$, $k_0 = 0$, and

$$\hat{K}_n(T,\Omega) = \int_{[0,\infty)^n} e^{i\sum_{j=1}^n t_j s_j/2}(s_1\cdots s_n)^{m-2} K(s_1\omega_1,\cdots,s_n\omega_n) ds_1\cdots ds_n.$$

Since $x \to x_{\omega_1}+\rho$ is an isometry, it follows by Minkowski's inequality that

$$\|W_n f\|_{L^p} \le 2\|\hat{K}_n\|_{L^1([0,\infty)^n\times\Sigma^n)}\|f\|_{L^p}, \qquad 1\le p\le\infty. \tag{27}$$

On the other hand, the estimate similar to that is used to derive (24) implies

$$\|\hat{K}_n\|_{L^1([0,\infty)^n, L^1(\Sigma^n))} \le (C_m\|\langle D\rangle^\rho(\langle x\rangle^{2\sigma}V)\|_{L^2})^n, \tag{28}$$

and by combining (27) with (28), we obtain

$$\|W_n\|_{B(L^p)} \le (C_m\|\langle D\rangle^\rho(\langle x\rangle^{2\sigma}V)\|_{L^2})^n. \tag{29}$$

Thus, the series in (20) converges in the operator norm of $B(L^p)$ and W_+ is bounded in $L^p(R^m)$, if $\|\langle D\rangle^\rho(\langle x\rangle^{2\sigma}V)\|_{L^2} < C_m^{-1}$. This proves Theorem 1 when $\|\langle D\rangle^\rho(\langle x\rangle^{2\sigma}V)\|_{L^2}$ is small.

When $\|\langle D\rangle^\rho(\langle x\rangle^{2\sigma}V)\|_{L^2}$ is not small, (20) no longer converges in norm and the argument above breaks down. Using the identity $R^-(\lambda) = R_0^-(\lambda) - R_0^-(\lambda)VR^-(\lambda)$, we write $W_+u = u - W_1u + \widetilde{W}_2u$, where W_1 is as above and

$$\widetilde{W}_2 u = \frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)VR^-(\lambda)V\{R_0^+(\lambda)-R_0^-(\lambda)\}u d\lambda. \tag{30}$$

We wish to show that $\widetilde{W}_2$ is bounded in $L^p(\mathbf{R}^m)$ by proving that the integral kernel $\widetilde{W}_2(x,y)$ satisfies the well known criterion for the L^p boundedness:

$$\max\left\{\sup_{x\in\mathbf{R}^m}\int_{\mathbf{R}^m}|\widetilde{W}_2(x,y)|dy,\quad \sup_{y\in\mathbf{R}^m}\int_{\mathbf{R}^m}|\widetilde{W}_2(x,y)|dx\right\}<\infty. \tag{31}$$

The integral kernel $\widetilde{W}_2(x,y)$ is given by

$$\widetilde{W}_2(x,y)=\frac{1}{2\pi i}\int_0^\infty \langle R^-(k^2)V(G_{+,y,k}-G_{-,y,k}),VG_{+,x,k}\rangle dk^2, \tag{32}$$

where $G_{\pm,y,k}(x)=G_\pm(x-y,k)$ are that of $R_0^\pm(k^2)$ or the incoming-outgoing fundamental solutions of $-\triangle-k^2$, which satisfy $G_\pm(x,k)\sim Ce^{\pm ik|x|}|x|^{-(m-1)/2}k^{(m-3)/2}$ as $|x|\to\infty$, and where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the coupling between suitable function spaces. Thus crude estimations would only yield

$$|\text{the integrand of (32)}|\le Ck^{m-3}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2} \tag{33}$$

and we are evidently faced with the following two difficulties:
(1) **High energy difficulty**: The integral (32) does not converge absolutely at $k=\infty$;
(2) **Low energy difficulty**: Even if we restrict the integral (32) to a compact interval via a smooth cut off function of k, (33) produces only $|\widetilde{W}_2(x,y)|\le C\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2}$ which is far from (31). For improving the decay property of $\widetilde{W}_2(x,y)$ by exploiting the oscillation property of $G_\pm(x,k)$, we apply the integration by parts with respect to k. However, the singularity at $k=0$ of $G_\pm(x,k)$ prevents us from doing this as many times as necessary.

For separating the two difficulties, we decompose $\widetilde{W}_2$ into the low and the high energy parts and treat them separately. We decompose by using the cut off functions $\phi_1\in C_0^\infty(R^1)$ and $\phi_2\in C^\infty(R^1)$ such that $\phi_1(\lambda)^2+\phi_2(\lambda)^2=1$, and $\phi_1(\lambda)=1$ for $|\lambda|\le 1$ and $\phi_1(\lambda)=0$ for $|\lambda|\ge 2$. Note that $W_\pm=\sum_{j=1}^2\phi_j(H)W_\pm\phi_j(H_0)$ thanks to the intertwining property of $W_\pm$. We show that $W_{2,low}\equiv\phi_1(H)\widetilde{W}_2\phi_1(H_0)$ and $W_{2,high}\equiv\phi_2(H)\widetilde{W}_2\phi_2(H_0)$ are both bounded in L^p. Of course it is well known that $\phi_j(H_0)$ is bounded in L^p. It can be shown that the integral kernel of $\phi_1(H)$ is bounded by $C_N(1+|x-y|)^{-N}$, hence $\phi_j(H_0)$ is also L^p bounded. (Incidentaly the statement on the integral kernel of $\phi_1(H)$ was a conjecture of Simon [19].)

For evading the singular behavior of the resolvent kernel at zero energy, we split $R^-(\lambda)=R^-(0)+\tilde{R}^-(\lambda)$ and single out the contribution of $R^-(0)$. We decompose $W_{2,low}u=W_{2,low}^{(1)}u+W_{2,low}^{(2)}u$ accordingly. In virtue of the orthogonality of Hardy functions in the upper and the lower half planes, we have

$$W_{2,low}^{(1)}u=\phi_1(H)\left\{\frac{1}{2\pi i}\int_{-\infty}^\infty R_0^-(\lambda)VR^-(0)VR_0^+(\lambda)d\lambda\right\}\phi_1(H_0)u; \tag{34}$$

and using the identity $(R_0^+(\lambda) - R_0^-(\lambda))\phi_1(H_0) = (R_0^+(\lambda) - R_0^-(\lambda))\phi_1(\lambda)$, we write

$$W_{2,low}^{(2)} u = \frac{1}{2\pi i} \int_0^\infty \phi_1(H) R_0^-(\lambda) V \tilde{R}^-(\lambda) V (R_0^+(\lambda) - R_0^-(\lambda)) \tilde{\phi}_1(\lambda) \phi_1(H_0) u d\lambda, \tag{35}$$

where $\tilde{\phi}_1 \in C_0^\infty(\mathbf{R})$ is such that $\tilde{\phi}_1(\lambda)\phi_1(\lambda) = \phi_1(\lambda)$. What is important here is to observe that, if we write $R^-(0)$ as an integral operator of the form

$$R(0)f(x) = \int K(x, x-y) f(x-y) dy$$

and if we set $M_y(x) = V(x)K(x, x-y)V(x-y)$, $W_{2,low}^{(1)}$ can be expressed as a superposition

$$W_{2,low}^{(1)} u = -\int_{R^m} \phi_1(H) W_1(M_y) \phi_1(H_0) u_y dy, \qquad u_y(x) = u(x-y), \tag{36}$$

where $W_1(M_y)$ is defined by (22) with M_y in place of V. One can show under the condition of the Theorem that $M_y(x)$ satisfies

$$\int_{\mathbf{R}^m} \|\langle x \rangle^\sigma M_y\|_{H^\rho(\mathbf{R}^m)} dy < \infty \tag{37}$$

for some $\sigma > 1/m_*$ and, applying (24) to (36), we see that $W_{2,low}^{(1)}$ is bounded in L^p.

We set $G_{\pm,x,k}(y) = e^{\pm ik|x|}\tilde{G}_{\pm,x,k}(y)$ to make the oscillation property of $G_{+,x,k}(y)$ explicit and write the integral kernel of $W_{2,low}^{(2)}$ in the form $W_{2,low}^{(2)}(x,y) = W_{2,low}^{(2),+}(x,y) - W_{2,low}^{(2),-}(x,y)$:

$$W_{2,low}^{(2),\pm}(x,y) = \frac{1}{2\pi i} \int_0^\infty e^{-ik(|x| \mp |y|)} \langle \tilde{R}^-(k^2) V \tilde{G}_{\pm,y,k}, V \tilde{G}_{+,x,k} \rangle \tilde{\phi}_1(k^2) dk^2, \tag{38}$$

where we ignored the harmless factors $\phi_1(H_0)$ and $\phi_1(H)$ (recall (32)). We apply the integration by parts with respect to k, choosing ℓ suitably:

$$\begin{aligned} W_{2,low}^{(2),\pm}(x,y) &= \frac{1}{2\pi i} \int_0^\infty \frac{D_k^\ell e^{-ik(|x| \mp |y|)}}{(|y| \mp |x|)^\ell} \langle \tilde{R}^-(k^2) V \tilde{G}_{\pm,y,k}, V \tilde{G}_{+,x,k} \rangle \tilde{\phi}_1(k^2) dk^2 \\ &= \frac{1}{\pi i} \int_0^\infty \frac{e^{-ik(|x| \mp |y|)}}{(|x| \mp |y|)^\ell} D_k^\ell \{ k \langle \tilde{R}^-(k^2) V \tilde{G}_{\pm,y,k}, V \tilde{G}_{+,x,k} \rangle \tilde{\phi}_1(k^2) \} dk, \end{aligned} \tag{39}$$

and gain the addition decay factor $(|x| \mp |y|)^{-\ell}$. Here the boundary terms do not appear for $\ell \leq [(m+3)/2]$ and the integral converges absolutely because $\tilde{R}^-(k^2)$ vanishes at $k = 0$. In this way it is possible to show that

$$|W_{2,low}^{(2),\pm}(x,y)| \leq C(1 + ||x| \mp |y||)^{-(m+2)/2} \langle x \rangle^{-(m-1)/2} \langle y \rangle^{-(m-1)/2} \tag{40}$$

and $W_{2,low}^{(2)}(x,y)$ indeed satisfies the criterion (31). Though the splitting of $R^-(\lambda)$ as above is unnecessary when m is odd because of the simpler structure of $G_\pm(x,k)$, it makes the proof of the theorem simpler even in that case.

To show that $W_{2,high} = \phi_2(H)\widetilde{W}_2\phi_2(H_0)$ is also bounded in L^p, we need to overcome the high energy difficulty mentioned above. We expand $R^-(k^2)$ in (30) as

$$R^-(k^2) = \sum_{n=0}^{2N-1}(-1)^n R_0^-(k^2)(VR_0^-(k^2))^n + (R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N \tag{41}$$

and decompose $\widetilde{W}_2$ into $2N+1$ summands accordingly: $\widetilde{W}_2 = \sum_{n=2}^{2N+2}(-1)^n W^{(n)}$. As we have proven $W^{(2)} = W_2, \dots, W^{(2N+1)} = W_{2N+1}$ are all bounded in L^p. On the hand, by using the well know results on the mapping property of the resolvent, we see by letting N sufficiently large, $F_N(k^2) = (R^-(k^2)V)^N R^-(k^2)(VR_0^-(k^2))^N$ can be made to decay as rapidly as desired as $k \to \infty$ as an operator valued function between suitable function spaces and the integrals

$$W_{high}^{(2N+2),\pm}(x,y) = \frac{1}{2\pi i}\int_0^\infty e^{-ik(|x|\pm|y|)}\langle F_N(k^2)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k}\rangle\tilde{\phi}_2(k^2)dk^2 \tag{42}$$

converge absolutely, where $\tilde{\phi}_2 \in C^\infty(\mathbf{R})$ is such that $\tilde{\phi}_2(\lambda) = 0$ near $\lambda = 0$ and $\tilde{\phi}_2(\lambda)\phi_2(\lambda) = \phi_2(\lambda)$. Set $W_{high}^{(2N+2)}(x,y) = W_{high}^{(2N+2),+}(x,y) - W_{high}^{(2N+2),-}(x,y)$ and write $W_{high}^{(2N+2)}$ for the integral operator with this kernel. Then $\phi_2(H)W^{(2N+2)}\phi_2(H_0) = \phi_2(H)W_{high}^{(2N+2)}\phi_2(H_0)$ and virtually the same technique as was used for proving (40) for $W_{2,low}^{(2),\pm}(x,y)$ yields

$$|W_{high}^{(2N+2),\pm}(x,y)| \le C(1+||x| \mp |y||)^{-(m+2)/2}\langle x\rangle^{-(m-1)/2}\langle y\rangle^{-(m-1)/2},$$

and $W_{high}^{(2N+2)}(x,y)$ satisfies the criterion (31).

We refer the readers to our papers [25], [26] and [27] for the details and possible extensions.

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Critical Exponents in Semilinear Diffusion Equations

Eiji Yanagida
Graduate School of Mathematical Sciences,
University of Tokyo, Meguro-ku, Tokyo 153, Japan
e-mail : yanagida@ms.u-tokyo.ac.jp

This article is based on a joint work with N. Mizoguchi [5, 6, 8].

We are concerned with the Cauchy problem with source

$$(1) \qquad \begin{cases} u_t = u_{xx} + |u|^{p-1}u & \text{in } \mathbf{R} \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}, \end{cases}$$

and the problem with absorption

$$(2) \qquad \begin{cases} u_t = u_{xx} - |u|^{p-1}u & \text{in } \mathbf{R} \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}, \end{cases}$$

where $p > 1$. Since the pioneering work of Fujita [2], many results have been obtained concerning the critical exponent for the behavior of *positive* solutions. However there is no result concerning the critical exponent for solutions with sign changes. See, e.g., the survey paper of Levine [3].

Our aim is to determine critical exponents when the number of sign changes is prescribed for initial data. To describe our result precisely, we introduce the following definitions. For a function u on $\mathbf{R}$ with $u \not\equiv 0$, define the number of sign changes $z(u)$ by the supremum of j such that the inequalities

$$u(x_i) \cdot u(x_{i+1}) < 0, \quad i = 1, 2, \cdots, j,$$

hold for some $-\infty < x_1 < x_2 < \cdots < x_{j+1} < +\infty$. We denote by Σ_k the set of initial data with $z(u_0) = k$ for which (1) has a time-local classical solution. Finally, put

$$p_k = 1 + \frac{2}{k+1}, \quad k = 0, 1, 2, \cdots.$$

First we give our result for (1) concerning the blowup and global existence of solutions.

Theorem 1

(a) *If $1 < p \leq p_k$, then any nontrivial solution of (1) with $u_0 \in \Sigma_k$ blows up in finite time.*

(b) *If $p > p_k$, then there exists a nontrivial global solution of (1) with $u_0 \in \Sigma_k$.*

Next we consider (2). It is easy to show that any bounded solution of (2) satisfies

$$|u(x,t)| \leq Ct^{-1/(p-1)}, \quad t > 0,$$

for some constant $C > 0$. We say that a solution u decays fast $t \to \infty$ if

$$\lim_{t \to \infty} t^{1/(p-1)} u(x,t) = 0 \quad \text{uniformly in } \mathbf{R}.$$

Otherwise the solution is said to decay slowly.

Our second result is stated as follows.

Theorem 2

(a) *If $1 < p < p_k$, then any nontrivial solution of (2) with $u_0 \in \Sigma_k$ decays slowly as $t \to \infty$.*

(b) *If $p \geq p_k$, then there exists a nontrivial solution of (2) with $u_0 \in \Sigma_k$ which decays fast as $t \to \infty$.*

Our strategy to show the above theorems is as follows. We introduce the similarity variables

$$v(y,s) = (t+1)^{1/(p-1)} u(x,t)$$

with $x = (t+1)^{1/2}y$ and $t = e^s - 1$. Then (1) and (2) are written as

$$(3) \qquad \begin{cases} v_s = v_{yy} + \dfrac{y}{2}v_y + \dfrac{1}{p-1}v \pm |v|^{p-1}v & \text{in } \mathbf{R} \times (0,\infty), \\ v(y,0) = u_0(y) & \text{in } \mathbf{R}. \end{cases}$$

Set

$$\rho(y) = \exp(y^2/4),$$

and let $H^1_\rho(\mathbf{R})$ be the Sobolev space with the weight $\rho(y)$ defined by

$$H^1_\rho(\mathbf{R}) = \{\, v \mid \int_{\mathbf{R}} (v^2 + v_y^2)\rho dy < \infty \}.$$

Let L be a linear operator defined by

$$L\varphi = \varphi_{yy} + \frac{y}{2}\varphi_y + \frac{1}{p-1}\varphi,$$

and consider the eigenvalue problem

$$L\varphi = \lambda\varphi \quad \text{in } H^1_\rho.$$

We denote by λ_j and φ_j the jth eigenvalue and its associated eigenfunction, respectively. Then we have

$$\lambda_j = \frac{1}{p-1} - \frac{j+1}{2}$$

and

$$\varphi_j(y) = \frac{d^j}{dy^j}\exp(-y^2/4)$$

for $j = 0, 1, 2, \cdots$. Thus the exponent p_k is related with λ_k as

$$\begin{cases} \lambda_k > 0 & \text{if } 1 < p < p_k, \\ \lambda_k = 0 & \text{if } p = p_k, \\ \lambda_k < 0 & \text{if } p > p_k. \end{cases}$$

Using the above properties of L and applying the theory of infinite dimensional dynamical systems to (3), we can prove Theorems 1 and 2 when initial data u_0 are restricted in H^1_ρ. Using this result, we construct certain special solutions and compare them with other solutions. Then we make use of the nonincrease of intersection numbers [1, 4] to prove Theorems 1 and 2 for general initial data. See [5, 6, 8] for details.

Finally, we remark that our method is applicable to problems on the half line $(0, \infty)$ with the Dirichlet or Neumann boundary condition at $x = 0$. We also refer to [7] for some related results on a bounded interval.

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