수 학 강 의 록 제 37 권

HOMOTOPY K3 SURFACES AND GLUING RESULTS IN SEIBERG-WITTEN THEORY

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HOMOTOPY K3 SURFACES AND GLUING RESULTS IN SEIBERG-WITTEN THEORY: THREE LECTURES FOR THE GARC

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Forward

In October 1994, Seiberg and Witten introduced a new set of equations into gauge theory. These equations greatly simplify and clarify the technical details of gauge theory. In these lectures we will reprove several results which were first proved using Donaldson theory with the newer Seiberg-Witten theory. In particular, we will show that there are topological 4-manifolds which admit infinitely many differential structures. In addition we will show that there are smooth 4-manifolds which may not be built with complex pieces. With these results as motivation, we will proceed to take a closer look at the mathematics behind gluing formula. These notes will not, however, include an introduction to the Seiberg-Witten equations. For an introduction to the Seiberg-Witten equations, we recommend our earlier notes [A1], the book by Morgan [M], and the forthcoming book by Salamon [S].

Before the advent of gauge theory, the theory of 4-manifolds was largely dominated by the study of specific examples. Complex surfaces form a very rich collection of 4-manifolds. Complex surfaces may be grouped into rational surfaces, elliptic surfaces, and surfaces of general type. Elliptic surfaces have structure which simplifies their study and are a fascinating collection of 4-manifolds.

In the first lecture we will discuss the topology of elliptic surfaces, beginning with the K3 surface. We will also discuss two methods for constructing new 4-manifolds from old: the log-transform and rational blow downs. Using these two methods we will construct all of the elliptic surfaces with the homotopy type of K3, and an additional collection of manifolds that were studied by Gompf and Mrowka.

With the advent of Donaldson theory, it became possible to prove that there are 4-manifolds which are homeomorphic but not diffeomorphic. New examples of this phenomenon were found as people became more proficient at computing Donaldson invariants. For a while it was conjectured that every 4-manifold could be expressed as a connected sum, $X_1\#X_2\#\dots\#X_n$, where each of the X_i are complex or anti-complex or S^4. Gompf and Mrowka constructed counter-examples to this conjecture and proved that their examples were indeed counter-examples.

In the second lecture we will compute the Seiberg-Witten invariants of the elliptic surfaces homotopy equivalent to K3 and of the Gompf-Mrowka examples. This will verify that there are infinitely many distinct differential structures on the K3 surface. It will also show that not every 4-manifold is built out of complex parts.

There are many approaches to computing the Seiberg-Witten invariants. We will use the rational blow-down approach of Fintushel and Stern. This approach has a very topological feel. It is an example of a neck-stretching result or pinching technique. The first neck-stretching result was Donaldson's vanishing theorem. Donaldson also proved that the simplest invariant was unchanged under the connected sum with a negative definite 4-manifold, for example, with $\overline{\mathbb{C}P^2}$. A connected sum with $\overline{\mathbb{C}P^2}$ is called a blow-up. It became important to understand the effect of a blow-up on the Donaldson invariants. Many people worked on this problem: Donaldson, Taubes, Morgan, Bryan, Leness, Ozsvath, Fintushel, and Stern. Kronheimer and Mrowka developed a blow-up formula for the Seiberg-Witten invariants in their paper [KM]. The second lecture ends with an outline of a proof of the rational blow-down formula.

The third lecture begins by discussing a method to organize gluing results, called Floer theory, as developed by Austin and Braam. The lecture proceeds to discuss this theory for the Seiberg-Witten invariants. In the Seiberg-Witten case, this produces a homology theory which is a topological invariant for 3-manifolds when $b_1 \geq 2$. We conclude by working out the Seiberg-Witten Floer theory for manifolds of positive scaler curvature and for Euclidean manifolds.

I would like to thank Suhyoung Choi, Hyuk Kim, Seoul National University and the GARC for the invitation to gives these Lectures and for their hospitality. I would also like to thank Rob Kirby for several blackboard discussions, and Ron Fintushel for sharing his joint work with Ron Stern on rational blow-downs.

Lecture 1
Handlebody Decomposition

In this section, we will describe the K3 surface both algebraically and topologically. We will also use a cut-and-paste technique to construct an infinite collection of 4-manifolds with the homotopy type of the K3 surface. By M.Freedman's work, all of these manifolds are homeomorphic. We will see that these manifolds are not all diffeomorphic, and that some of them do not admit complex structures, even though they are irreducible.

The K3 surface is an algebraic variety, so an algebraic description will consist of the solution sets of polynomial equations glued together by algebraic relations. The topological description will be a handle body decomposition.

The K3 surface is an example of an elliptic surface. An elliptic surface or elliptic fibration is a smooth algebraic variety together with a map onto $\mathbb{C}P^1$ so that the fibers are elliptic curves. Even though the total space is nonsingular, the fibers might have singularities. Some people allow the base to be any nonsingular algebraic curve and allow rational double points in the total space. The smallest elliptic surface is the half Kummer surface, denoted $E(1)$.

We will begin by describing this surface in detail, because it may be used as a building block to build all other elliptic fibrations. In equations it is given by:

$$\begin{aligned} E(1) = \{([x:y:t],z) \in \mathbb{C}P^2 \times \mathbb{C} \mid x^2t + y^3 + (z^5-1)t^3 = 0\} \\ \cup \ \{([u:v:s],w) \in \mathbb{C}P^2 \times \mathbb{C} \mid u^2s + v^3 + (w-w^6)s^3 = 0\} \end{aligned}$$

related by $zw = 1$, $uz^3 = x$, $vz^2 = y$, $s = t$ with projection map $\pi : E(1) \to \mathbb{C}P^1$; $([x:y:t],z) \mapsto [z:1]$ and $([u:v:s],w) \mapsto [1:w]$.

The implicit function theorem may be used to show that $E(1)$ is a complex manifold with complex dimension two. In particular, $E(1)$ is a smooth 4-manifold. Most points in $\mathbb{C}P^1$ are regular values of π. In fact, let $[p:q]$ be any point except $[1:w]$ when $w = w^6$. Thus we may write $[p:q] = [z:1]$, and points in $\pi^{-1}([z:1])$ are given by

$$x^2 + y^3 + (z^5 - 1) = 0 \qquad \text{if } t \neq 0 \,,$$

or $[1:0:0]$. The map π is regular at $[z:1]$ if $d\pi$ is surjective, that is, if we can solve for either dx or dy in the linearization:

$$5z^4dz + 3y^2dy + 2xdx = 0 \,.$$

Clearly this may be done unless $x = 0$ and $y = 0$, which implies that $z^5 = 1$, but we have already excluded these points. We will analyze the singular points later in this section. If

$[z:1]$ is a regular value, there is a small disk about $[z:1]$, so that $\pi^{-1}(D_\epsilon) = \pi^{-1}([z:1]) \times D_\epsilon$. The map,

$$p : \pi^{-1}([z:1]) = \{[x:y:t] \mid x^2t + y^3 + (z^5 - 1)t^3 = 0\} \to \mathbb{C}P^1; [x:y:t] \mapsto [x:t]$$

is clearly a 3-fold branched covering with branch points $[1:0]$, $[\pm(1-z^5)^{\frac{1}{2}} : 1]$. Figure 1 is a picture of this cover.

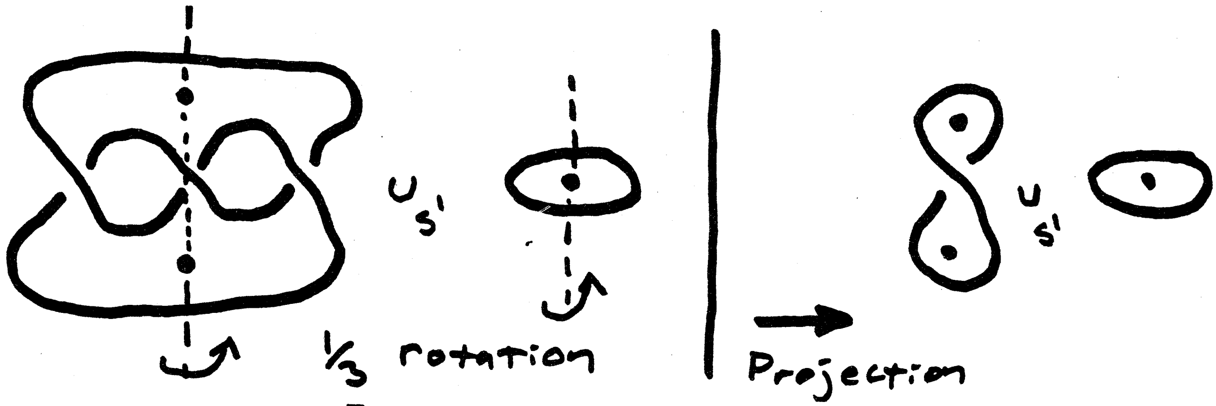

FIGURE 1: 3-FOLD BRANCHED COVER

By inspection, the surface in this branched cover is orientable and has Euler characteristic zero. It is, therefore, a torus. Thus $\pi^{-1}(D_\epsilon) \cong T^2 \times D^2$.

We will now digress to give a handle decomposition of $T^2 \times D^2$ and a large subset of $E(1)$. When W_1 is an n-dimensional manifold with boundary, adding a k-handle to W_1 constructs a new manifold by $W_2 = W_1 \cup_{S^{k-1} \times D^{n-k}} D^k \times D^{n-k}$. Here $S^{k-1} \times D^{n-k} \hookrightarrow (\partial D^k) \times D^{n-k} \hookrightarrow \partial(D^k \times D^{n-k})$, and we just pick any embedding of $S^{k-1} \times D^{n-k}$ into ∂W_1. The set $D^k \times \{0\} \subseteq D^k \times D^{n-k}$ is called the core of the handle. If we add handles to the empty set and end up with a manifold diffeomorphic to X^n, we will have a handle decomposition of X^n. Figure 2 is a handle decomposition of T^2.

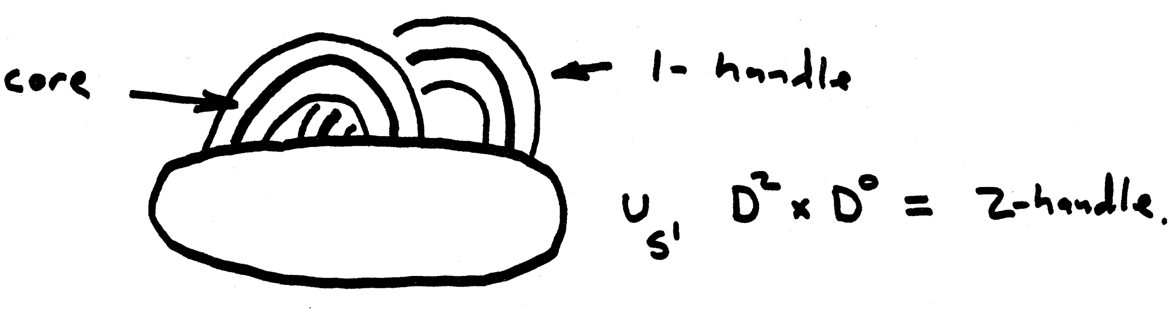

FIGURE 2: T^2

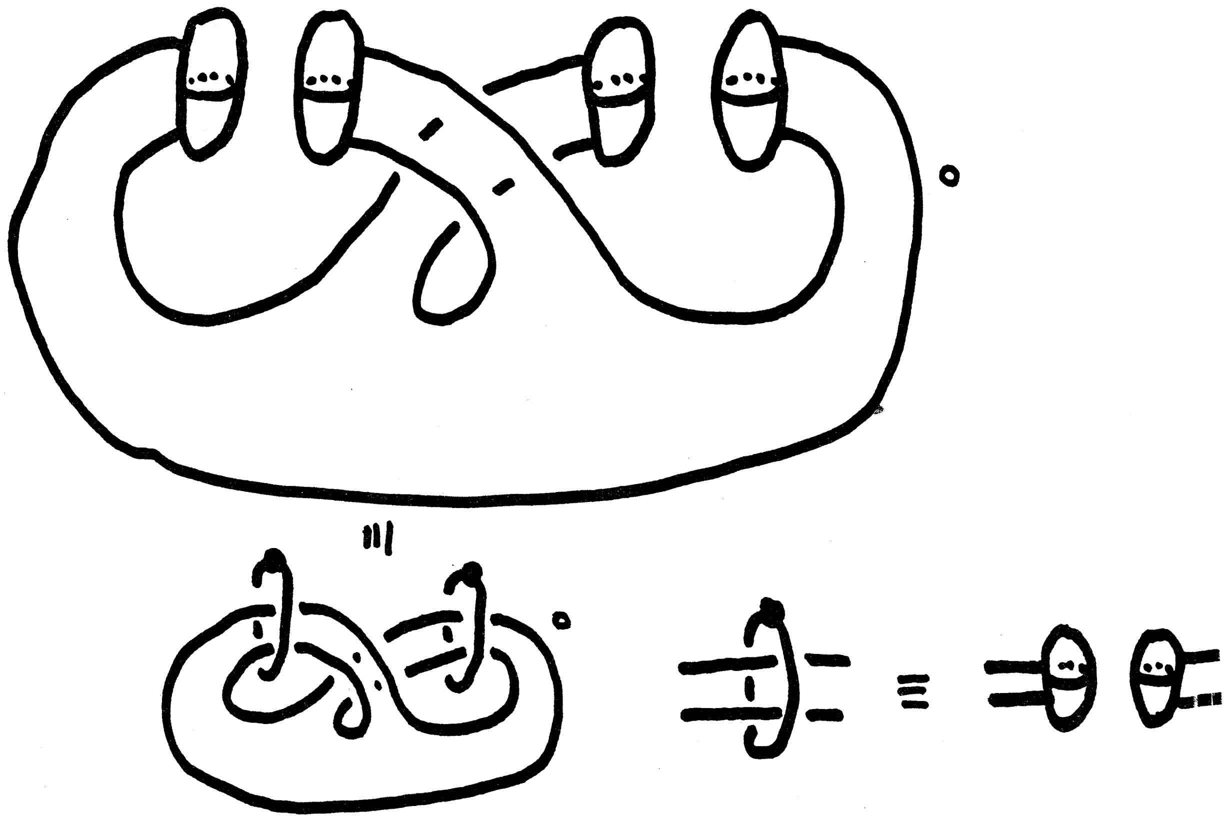

FIGURE 3: $T^2 \times D^2$

In order to get a handle decomposition of $T^2 \times D^2$; we just multiply the above handle decomposition by D^2. This is drawn in figure 3. The boundary of a 4-dimensional 0-handle is S^3 which we visualize as $\mathbb{R}^3 \cup \{\infty\}$. All of the other handles may be glued onto the $\mathbb{R}^3$. A 1-handle is a $D^1 \times D^3$ attached along an $S^0 \times D^3$ which is drawn as a set of balls in figure 3. It is a useful notation to replace a pair of balls by an unknotted circle with a dot on it. A 2-handle is a $D^2 \times D^2$ attached along a $S^1 \times D^2$. To visualize the attaching map we draw $S^1 \times \{0\}$. The curve $S^1 \times \{1\}$ is parallel to $S^1 \times \{0\}$, but it might link it some number of times. This linking number is called the framing and it is usually written next to the $S^1 \times \{0\}$ [A1],[HKK],[R].

There is a large open set inside $E(1)$ which is of fundamental importance in the theory of 4-manifolds. We will now describe this set. The set,

$$V = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 - 1 = 0\}$$

is clearly an open dense subset of $E(1)$. Let

$$V' = \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0\} .$$

When $|x| \gg 1$, $|y| \gg 1$ and $|z| \gg 1$, the 1 in the definition of V may be disregarded. That is, V and V' are diffeomorphic near ∞. This is just an application of the implicit function theorem.

Now let $\Sigma(2,3,5) = V' \cap S_1(0)$ and note that the map $h : \mathbb{R} \times \Sigma(2,3,5) \to V' - \{0\}$; $h(t,x,y,z) = (t^{15}x, t^{10}y, t^6 z)$ is a diffeomorphism. This implies that

$$V = E_8 \cup_{\Sigma(2,3,5)} [1,\infty) \times \Sigma(2,3,5) .$$

The manifold E_8 is a compact manifold with boundary. It is called the E_8-plumbing or E_8 manifold. There is an obvious 3-fold branched covering map

$$p_3 : V \to \mathbb{C}^2;\; p_3(x,y,z) = (x,z) .$$

The branch set is $S = \{(x,z) \in \mathbb{C}^2 \mid x^2 + z^5 - 1 = 0\}$. The 5-fold branched cover $S \to \mathbb{C}$; $(x,z) \mapsto x$, shows that S is a once punctured surface of genus 2. The branch set intersects a large 3-sphere $\{(x,z) \mid (|x| = 10 \text{ and } |z| \le 10) \text{ or } (|x| \le 10 \text{ and } |z| = 10)\}$ only inside the solid torus $\{(x,z) \mid |x| = 10 \text{ and } |z| \le 10\}$ in the shape of a right-hand $5,2$ torus knot, Figure 4.

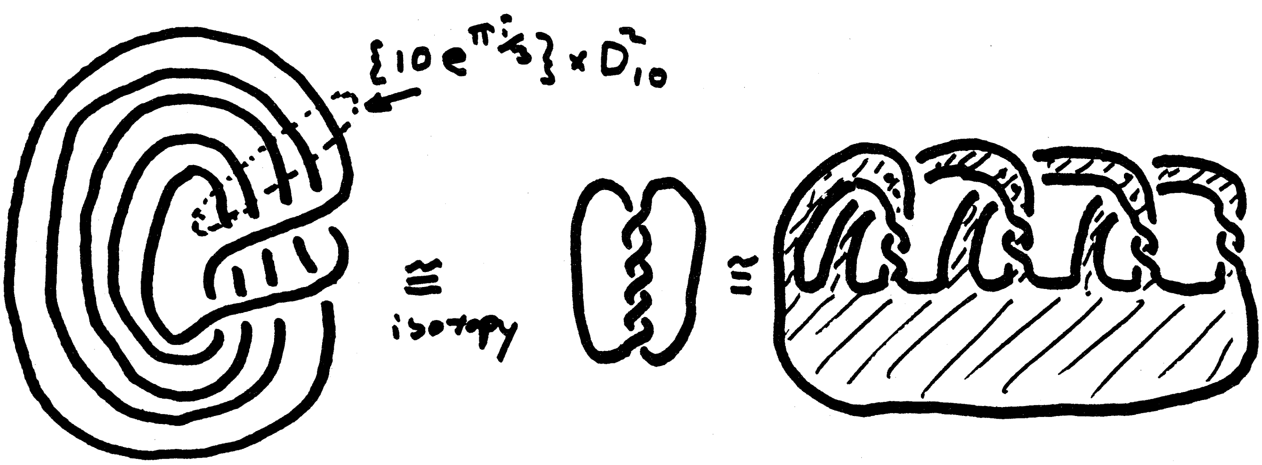

FIGURE 4: THE BRANCH SET

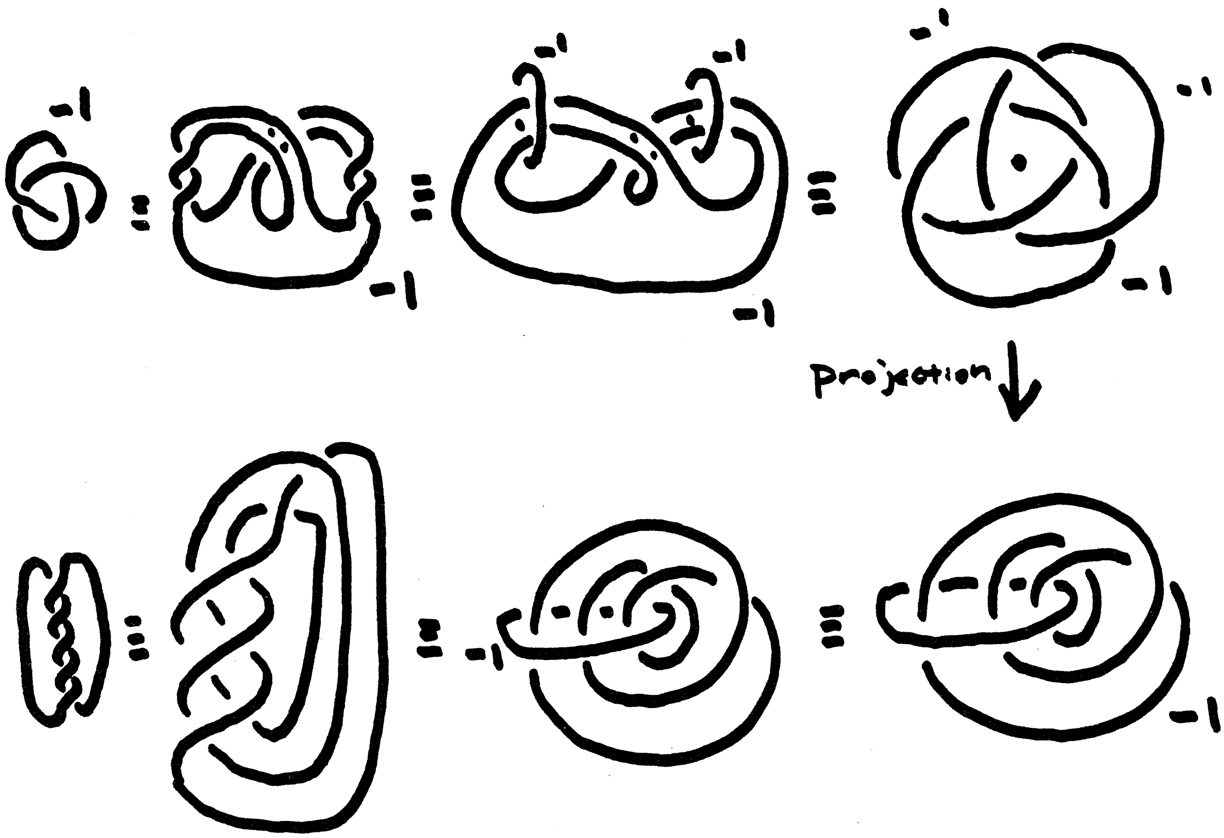

FIGURE 5: $\Sigma(2,3,5)$

This shows that E_8 is the 3-fold branched cover D^4 branched along the Seifert surface in figure 4 pushed into the interior of D^4. In particular, $\Sigma(2,3,5)$ is the 3-fold branched cover of S^3 with branch set $T(5,2)$. A Rolfson twist will untie the branch set [R]. After the branch set is untied, it is easy to construct the branched cover as drawn in figure 5. In the total space of the branched cover, the branch set is the axis of symmetry together with the point at infinity. The Boromean rings may be isotoped until they are reminiscent of our picture of $T^2 \times D^2$. This is not an accident and we will discuss it further after describing E_8. Blowing down the two components that correspond to 1-handles in $T^2 \times D^2$ shows that $\Sigma(2,3,5)$ is -1 surgery on a left trefoil.

To construct E_8, we will use the technique of Akbulut and Kirby for constructing handle decompositions of branched covers [AK]. Begin with an example in one lower dimension. To describe the 3-fold branched cover of $D^2 \times I$ branched along $\{0\} \times I$, first note that the branch set may be isotoped into the boundary relative to the endpoints. Cutting out the image of the isotopy will produce a fundamental domain for the cover. The cover may be built by gluing three of these fundamental domains together. Finally, the cover may be decomposed into a neighborhood of the branch set; neighborhoods of the remaining parts of the copies of the image of the isotopy, and three copies of $D^2 \times I$ minus the isotopy. See figure 6.

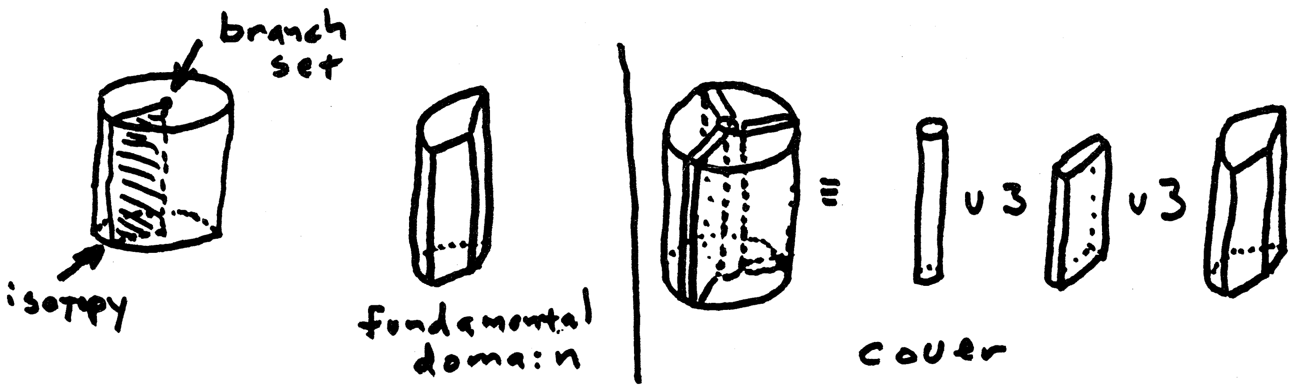

FIGURE 6: BUILDING A BRANCH COVER

The same decomposition works for any branch cover when the branch set may be isotoped into the boundary. In the case of E_8, D^4 minus the isotopy is diffeomorphic to D^4. There is a $I \times$ 0-handle in the neighborhood of the isotopy minus the core, glued along $S^0 \times D^3$. The first two of these glue together with the three D^4's to make one 0-handle. The final $I \times$ 0-handle becomes a 1-handle in E_8. A 1-handle in the branch set becomes a 1-handle in the part of the isotopy outside of a neighborhood of the branch set. This in turn becomes a $I \times D^1 \times D^2$ attached along $S^0 \times D^1 \times D^2 \cup I \times S^0 \times D^2$. In other words, there is a 2-handle in E_8 for each 1-handle in the branch set. Figure 7 is a handle decomposition of E_8. The handles in E_8 coming from the three copies of the isotopy are black. The neighborhood of the branch set is just D^2 times the branch set. The 0-handle therefore becomes a $D^2 \times D^2$ attached along $S^1 \times D^2$ drawn in green. The 1-handle becomes a $D^2 \times D^1 \times D^1$ attached along $S^1 \times D^1 \times D^1 \cup D^2 \times S^0 \times D^1 = S^2 \times D^1$, i.e. a 3-handle. The $S^1 \times D^1 \times \{0\}$ part of the 3-handles consist of the cores of the black 2-handles together with the twisted strips.

The 3-handles pass geometrically once over some 2-handles, thus we may cancel 2-handle/3-handle pairs. The green 2-handle cancels the 1-handle, leaving the simple description of E_8 in figure 8.

The second picture in figure 8 is the description that arises from the obvious 2-fold cover. The second picture is the usual picture of E_8.

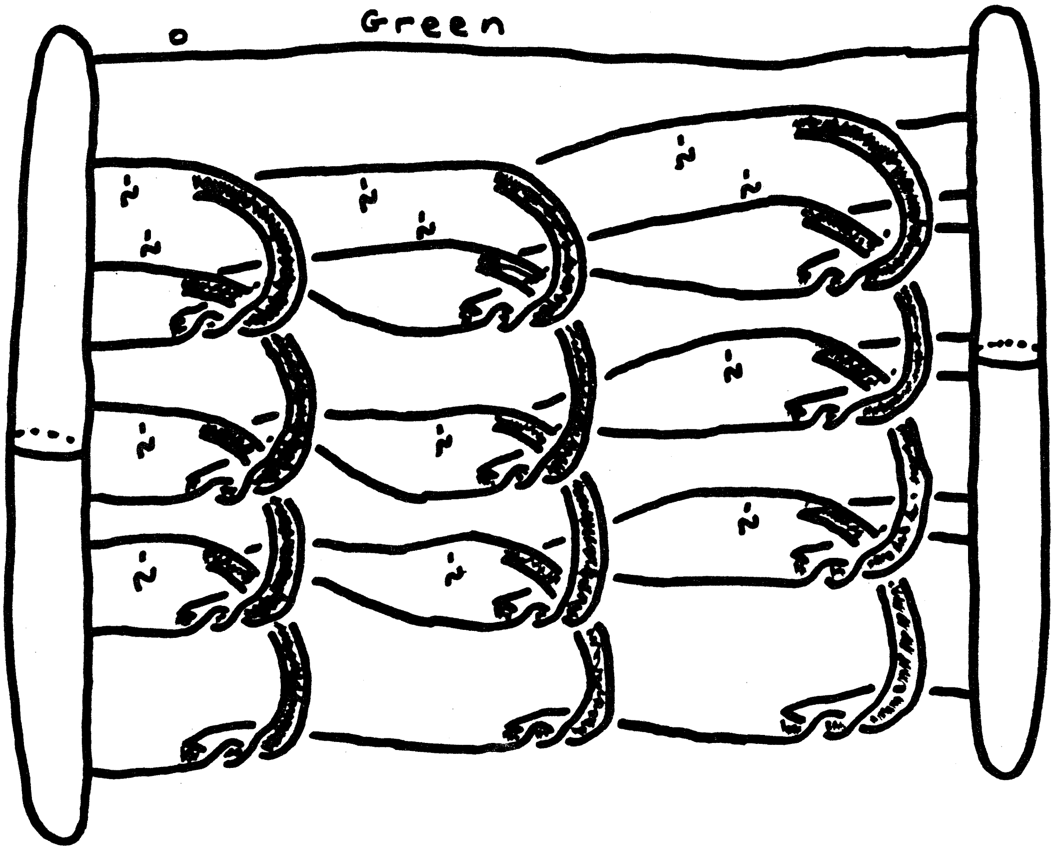

FIGURE 7: $\mathbb{Z}_3$ ACTION ON E_8

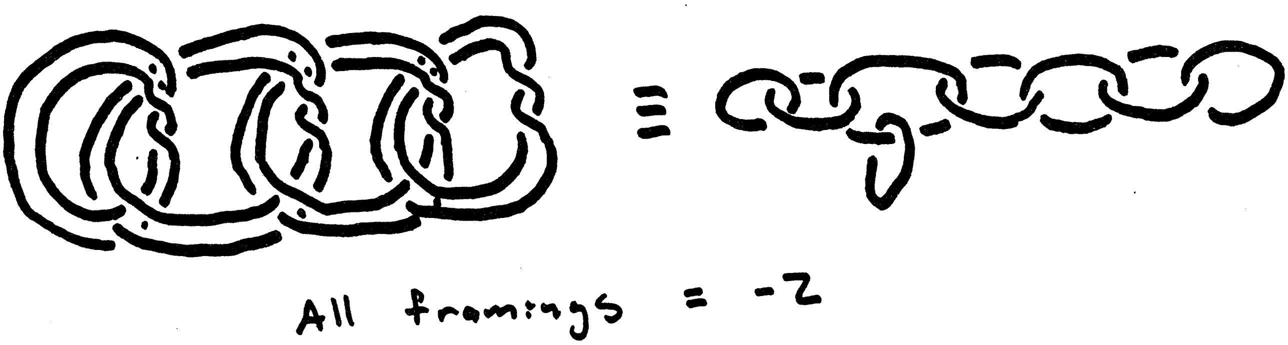

FIGURE 8: E_8

Now that we have a good picture of V, we will describe the remainder of $E(1)$. A section of $E(1)$ may be defined by

$$\sigma : \mathbb{C}P^1 \to E(1); \quad [z:1] \mapsto ([1:0:0], z)$$

in the first chart and $[1,0] \mapsto ([1:0:0],0)$ in the second chart. The remainder of $E(1)$ consists of the section and the fiber over [1:0]. A neighborhood of the union of a section and this fiber is called a nucleus. The fiber over [1:0] is given by

$$\{[u:v:s] \in \mathbb{C}P^2 \mid u^2 s + v^3 = 0\} \; .$$

The real part of this fiber in the $v-u$ plane is drawn in figure 9.

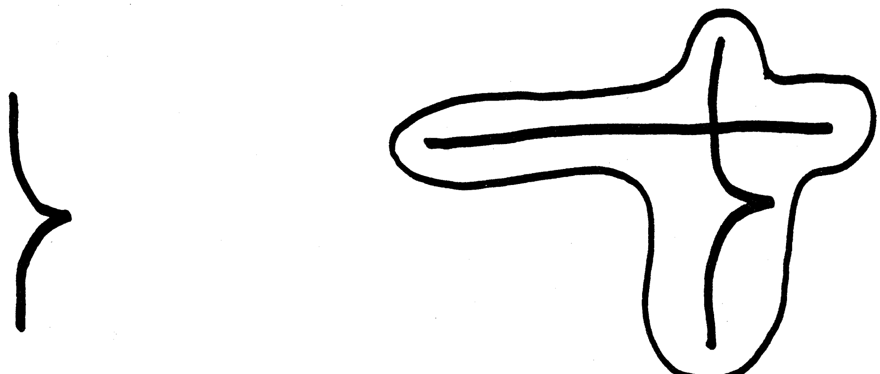

FIGURE 9: A CUSP FIBER AND NUCLEUS

There are cusp fibers over the points $[\lambda:1]$ where $\lambda^5 = 1$ and [1:0]. Algebraic geometers usually draw $E(1)$ as a section together with six cusp fibers (figure 10).

FIGURE 10: ALGEBRAIC GEOMETRY PICTURE OF $E(1)$

The same argument that showed that V' is an open cone on $\Sigma(2,3,5)$ proves that the cusp fiber minus the point at infinity is an open cone on the right-hand trefoil. Thus, the cusp fiber is geometrically a 2-sphere with one singular point. A neighborhood of the cusp fiber may be cut into a 4-disk around the singular point plus the remaining part of the neighborhood. In other words, the cusp neighborhood is constructed by attaching a 2-handle along the right trefoil. One other way to describe the cusp neighborhood is $\pi^{-1}(D^2)$ where D^2 is a small disk around [1:0]. The inverse image of $[1:\frac{1}{10}]$ describes the framing of the 2-handle. The answer is that the handle is zero framed. Adding two

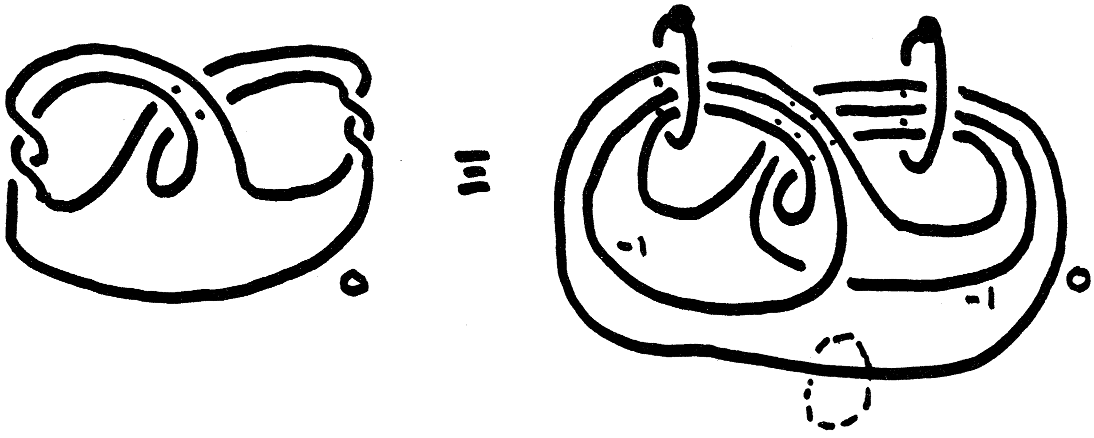

FIGURE 11: THE CUSP NEIGHBORHOOD

1-handle/ 2-handle pairs demonstrates that the cusp neighborhood may be obtained by adding two 2-handles to $T^2 \times D^2$. See figure 11.
Looking at figures 2 and 3 shows that the section intersects the cusp neighborhood in the dotted circle. The nucleus is the union of the cusp neighborhood and a 2-handle which is the part of the section outside of the cusp neighborhood. To compute the framing, notice that the image of the section is given by $S = \{([1:0:0], z)\} \cup \{([1:0:0], w)\}$. Linearizing the definition equations of $E(1)$ gives:

$$dt + 3y^2dy + 3t^2(z^5 - 1)dt + t^3 5z^4 dz = 0 \ , \quad \text{etc.}$$

On S this becomes $dt = 0$ and $ds = 0$. It follows that the tubular neighborhood of S is given by $N(S) = \{([1:dy:0], z)\} \cup \{([1:dv:0], w)\}$ where

$$\begin{aligned}([1:dv:0], z) &= ([z^3 : z^2 dv : 0], \frac{1}{z}) \\ &= ([1 : \frac{1}{z}\ dv : 0], \frac{1}{z}) \ .\end{aligned}$$

Since the transition function is $\frac{1}{z}$ the framing is $e(N(S))[S] = -1$.

Putting the preceding discussion together gives the picture of the nucleus in figure 12. A 1-handle glued to the 4 disk creates an $S^1 \times D^3$ which has boundary $S^1 \times S^2$. This is exactly the same as the boundary of a $D^2 \times S^2$ which is a 2-handle attached to D^4 in the trivial way. This means that if we replace every 1-handle with a zero-framed 2-handle, the boundary will not change. Replacing the 1-handles and blowing down the -1 framed components in the nucleus generates the picture of its boundary in figure 12. Notice that this is the same as the boundary of E_8, but with the opposite orientation. Thus $E(1)$ is just the union of the nucleus and E_8.

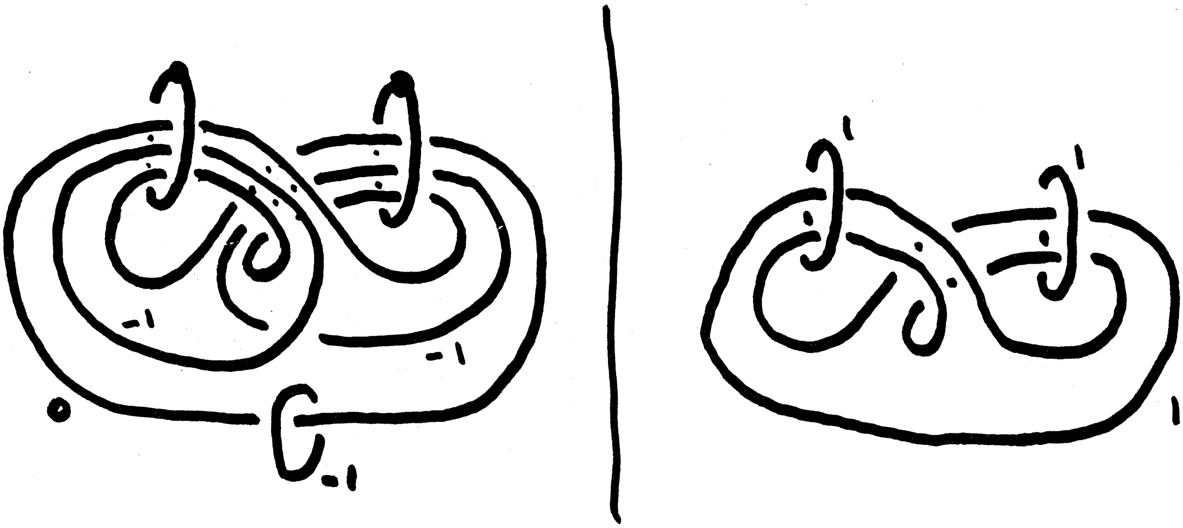

FIGURE 12: THE NUCLEUS AND ITS BOUNDARY

Handle decompositions of 4-manifolds are often very complicated. For example, since $H_2(E(1)) = \mathbb{Z}^{10}$, the smallest handle decomposition of $E(1)$ would have ten 2-handles, requiring a picture of a ten component link. It is, therefore, a good idea to write a 4-manifold as a union of several 4-manifolds with boundary, and describe these pieces with handle decompositions. The pieces may then be moved around. The nucleus is the union of a $T^2 \times D^2$ and a "buffer zone" created by adding three 2-handles to $[0,1] \times T^3$. We will draw a picture of $[0,1] \times M^3$, where M is a 3-manifold, by drawing a Dehn surgery description of M and putting an I on each component. With this convention, we draw a picture of the buffer zone, B, in figure 13.

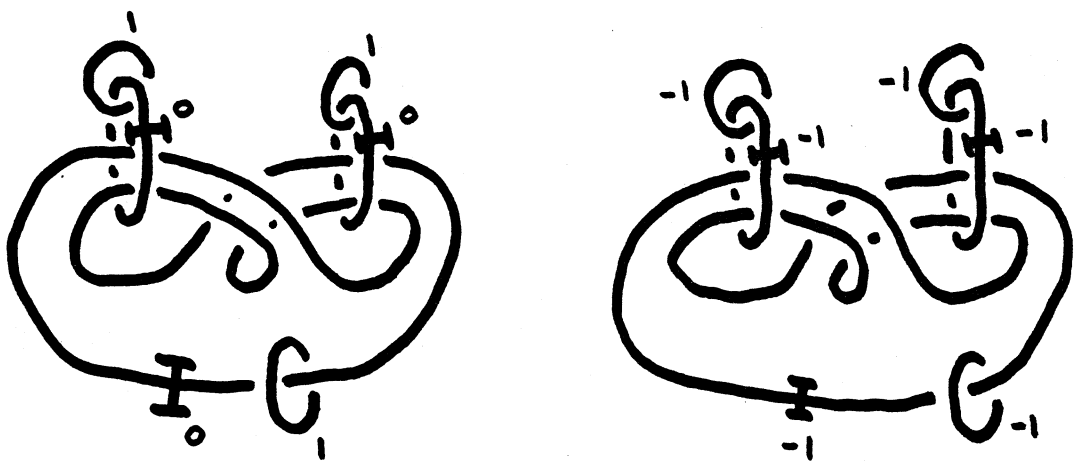

FIGURE 13: THE BUFFER ZONE, B

Interchanging the order of components changes a k-handle $D^k \times D^{n-k}$ into an $n-k$-handle, $D^{n-k} \times D^k$. Performing this on an entire handle decomposition is called turning the manifold upside down. Given a manifold W with boundary $\partial W = M \amalg -N$, we can turn W upside down by doubling along M, $D(W) = W \cup_M -W$, and converting W into a Dehn surgery description of M. If W only has 0-handles, 1-handles and 2-handles, then we just have to figure out how to add new 2-handles, because there is no real choice for adding 3-handles and 4-handles. As figure 14 demonstrates, the new 2-handles are just zero framed meridians of the old 2-handles.

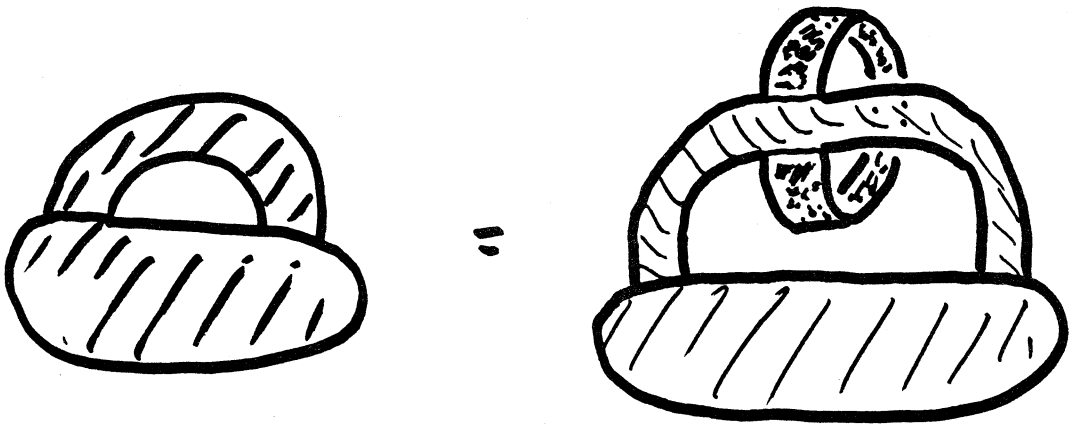

FIGURE 14: DOUBLING

Applying this technique to the buffer zone means adding zero framed meridians to all of the +1-framed 2-handles, then blowing down the +1-framed 2-handles.

In the decomposition, $E(1) = E_8 \cup_{\Sigma(2,3,5)} B \cup_{T^3} T^2 \times D^2$, $B \cup_{T^3} T^2 \times D^2$ is the nucleus. In order to see $E_8 \cup_{\Sigma(2,3,5)} B$, note that B is the total space of a 3-fold branched cover. The manifold, $E_8 \cup B$ is obtained by adding the three 2-handles to E_8. We can construct a handle decomposition of $E_8 \cup B$ by first constructing a handle decomposition of the quotient, then passing to the cover. Figure 15 is a more symmetrical picture of B.

By passing to the quotient of the obvious $\mathbb{Z}_3$ action, we get the first picture in figure 16. This is exactly the same as figure 5 except the quotient has one extra 2-handle which misses the boundary. Repeating the construction described around figures 6 and 7 produces the handle decomposition of $E_8 \cup B$ in figure 17.

We are now ready to construct a K3 surface. A K3 surface is a compact, complex analytic, simply connected surface, X, with $c_1(X) = 0$. Not all K3 surfaces are the same

FIGURE 15: *B* REARRANGED

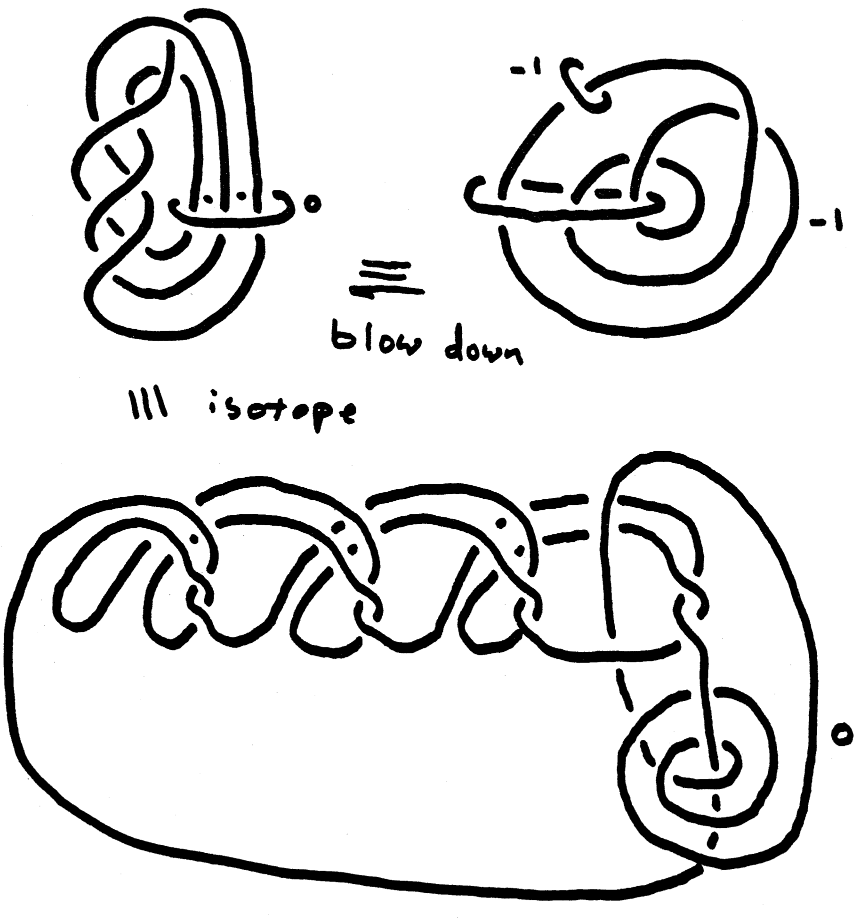

FIGURE 16: *B* AS A 3-FOLD BRANCHED COVER

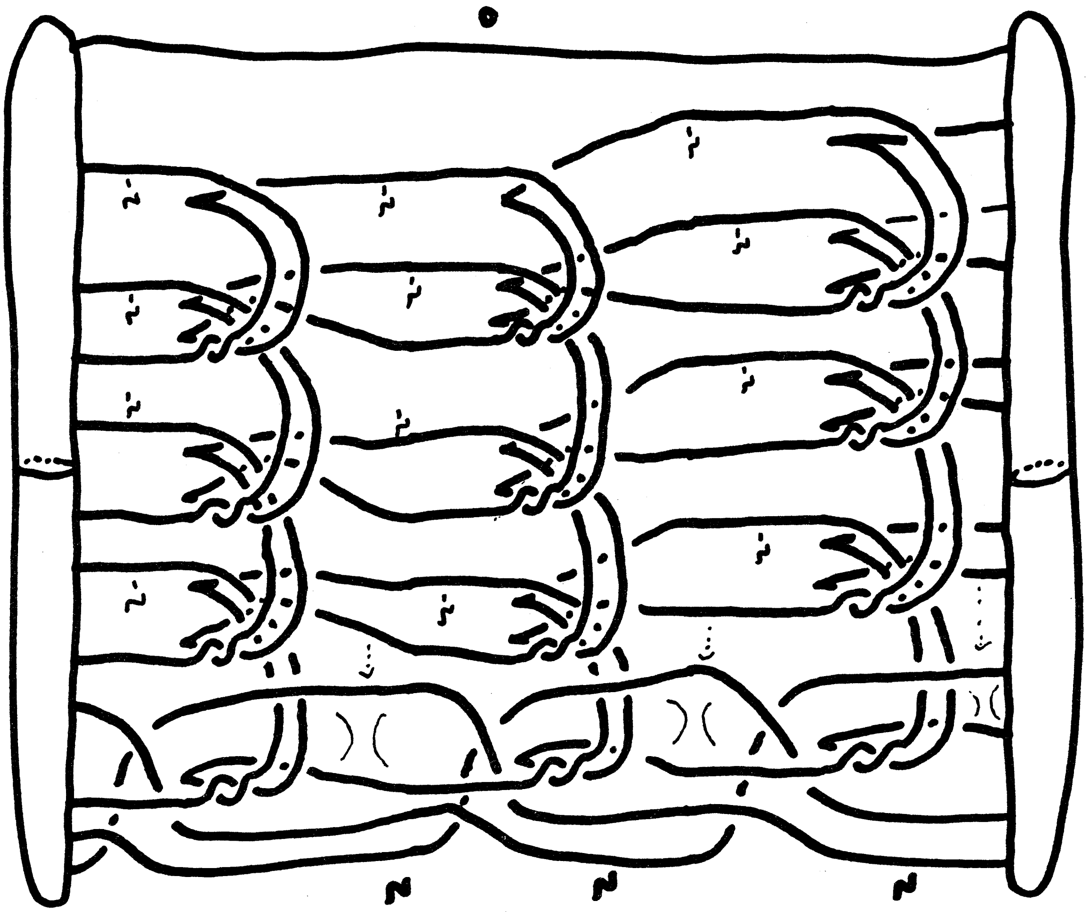

FIGURE 17: $E_8 \cup B$

as complex surfaces, but they are all the same as smooth 4-manifolds. Let

$$\begin{aligned}E(2) = \{&(([x:y:t],[p:q] \in \mathbb{C}P^2 \times (\mathbb{C}P^1 - \{[\pm 1 : 1]\}) \mid \\ &(p^2-q^2)^5x^2t + (p^2-q^2)y^3 + ((11p^2-10q^2)^5 - (p^2-q^2)^5)t^3 = 0\} \\ &\cup \{(([u:v:s],[p:q]) \in \mathbb{C}P^2 \times (\mathbb{C}P^1 - \{[\pm\sqrt{110} : 11]\})) \mid \\ &(11p^2-10q^2)^6u^2s + (11p^2-10q^2)^6v^3 + ((11p^2-10q^2)^5(p^2-q^2) + (p^2-q^2)^6)s^3 = 0\}\end{aligned}$$

glued together in an appropriate way. Projection onto the $\mathbb{C}P^1$ factor shows that $E(2)$ is an elliptic fibration. The map

$$r : E(2) \to E(1); ([x:y:t];[p:q]) \mapsto ([x:y:t], \frac{11p^2-10q^2}{p^2-q^2}) ,$$

on the first chart and

$$([u:v:s],[p:q]) \mapsto ([u:v:s], \frac{p^2-q^2}{11p^2-10q^2}) ,$$

on the second chart is a two fold branched cover. The easy way to see this is to get rid of the homogeneous coordinates and solve for p^2 as $p^2 = \frac{z-10}{z-11}$. The branch set is therefore the pair of regular fibers over $z = 10$ and $z = 11$ in $E(1)$ or [0:1] and [1:0] in $E(2)$. We could also construct the n-fold branched cover of $E(1)$, branched over a pair of regular fibers. The n-fold cover is the n^{th} elliptic surface, $E(n)$.

We can now easily give a topological description of $E(2)$ and all of the $E(n)$. Divide $E(1)$ into two parts, before taking the 2-fold branched cover. All of the fibers with $|z-10| \leq 2$ are regular so they combine into a $T^2 \times D^2$. The remainder of $E(1)$ is $E_8 \cup B$ which is disjoint from the branch set. Since E_8 may be constructed with no 2-handles (figure 8), $\pi_1(E_8) = 1$. VanKampen's theorem and figure 13 show that $\pi_1(B) = 1$, thus $\pi_1(E_8 \cup B) = 1$. It follows that $E(2)$ contains two disjoint copies of $E_8 \cup B$. The branched cover of $T^2 \times D^2$ is just the product of T^2 and the n-fold branched cover of D^2 branched at two points. Because the n^{th} fold branched cover of D^2 is $S^2 - nD^2$, $E(n) = T^2 \times (S^2 - nD^2) \cup n(E_8 \cup B)$. See figure 18.

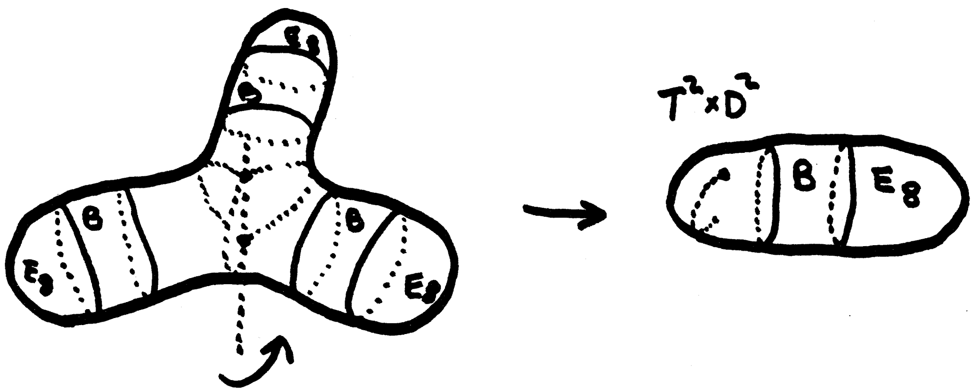

FIGURE 18: $E(3)$

Each of the $E(n)$ have a nucleus which is the union of a section and a cusp fiber. Call the nucleus of $E(n)$, N_n. In particular,

$$\begin{aligned} E(2) &= E_8 \cup B \cup (T^2 \times (S^2 - D^2)) \cup B \cup E_8 \\ &= E_8 \cup B \cup (T^3 \times I) \cup B \cup E_8 \\ &= E_8 \cup_\Sigma B \cup_{T^3} B \cup_\Sigma E_8 , \end{aligned}$$

and there is an N_2 embedded in $B \cup_{T^3} B$.

Since B is obtained by adding three 2-handles to $T^3 \times I$ (figure 13), $B \cup_{T^3} B$ is obtained from $T^3 \times I$ by attaching six 2-handles to $T^3 \times I$, three to a side. We can construct a handle decomposition of $T^3 \times I$ from a handle decomposition of T^3. Think of T^3 as $[-1,1]^3/\sim$. A 0-handle is $[-\frac{1}{2},\frac{1}{2}]^3$. Tubular neighborhoods of the x, y and z axes in T^3 minus the 0-handle are three 1-handles. Tubular neighborhoods of the xy, yz and zx planes in T^3 minus the 0-handle and 1-handles form three 2-handles. The leftover part is a 3-handle. Any 1-handle is equivalent to a pair of 1-handles with a 2-handle passing geometrically one time over each (figure 19).

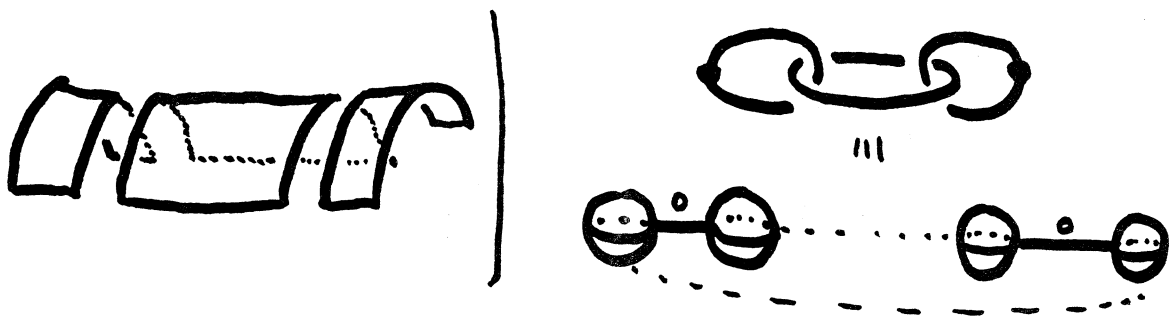

FIGURE 19: A COMPLICATED 1-HANDLE

The 2-handles which are added to $T^3 \times I$ to produce $B \cup_{T^3} B$ are attached along $\{((x,0,.1),\epsilon)\}$, $\{((.1,y,0),\epsilon)\}$ and $\{((0,.1,z),\epsilon)\}$ for $\epsilon = 0$ or $\epsilon = 1$. They are all -1 framed, when the framing is compared with the T^2 sections of $T^3 \times I$. The 3-handle in T^3 is attached mostly to the boundary of $[-\frac{1}{2},\frac{1}{2}]^3$, so the 3-handle in $T^3 \times I$ is attached mostly along $(\partial[-\frac{1}{2},\frac{1}{2}]^3) \times \{0\}$. This separates the boundary of the 0-handle into two parts, the inside of this cube and the outside of of this cube. The inside corresponds to the left boundary of $T^3 \times I$ and the outside corresponds to the right boundary. In our picture of a handle decomposition of $B \cup_{T^3} B$, we draw all of the attaching maps in ∂D^4 which is the union of a large 3-disk and a small 3-disk. See figure 20.

In figure 20 there are two 1-handles labeled with r, and four 2-handles labeled with r. These should be colored red. The one 1-handle and four 2-handles labeled b should be left alone and the remaining handles should be colored green. With this convention the red handles together with the z-axis are drawn in figure 21. Sliding the vertical red handle over the other two vertical handles produces a handle decomposition of an N_2 embedded in $B \cup_{T^3} B$ (figure 21).

By sliding on the y-axis, we see that the green handles may form a disjoint nucleus. In fact, by adding a cusp fiber from the right side and a cusp fiber from the left side, we could extend the black handles into a third disjoint nucleus.

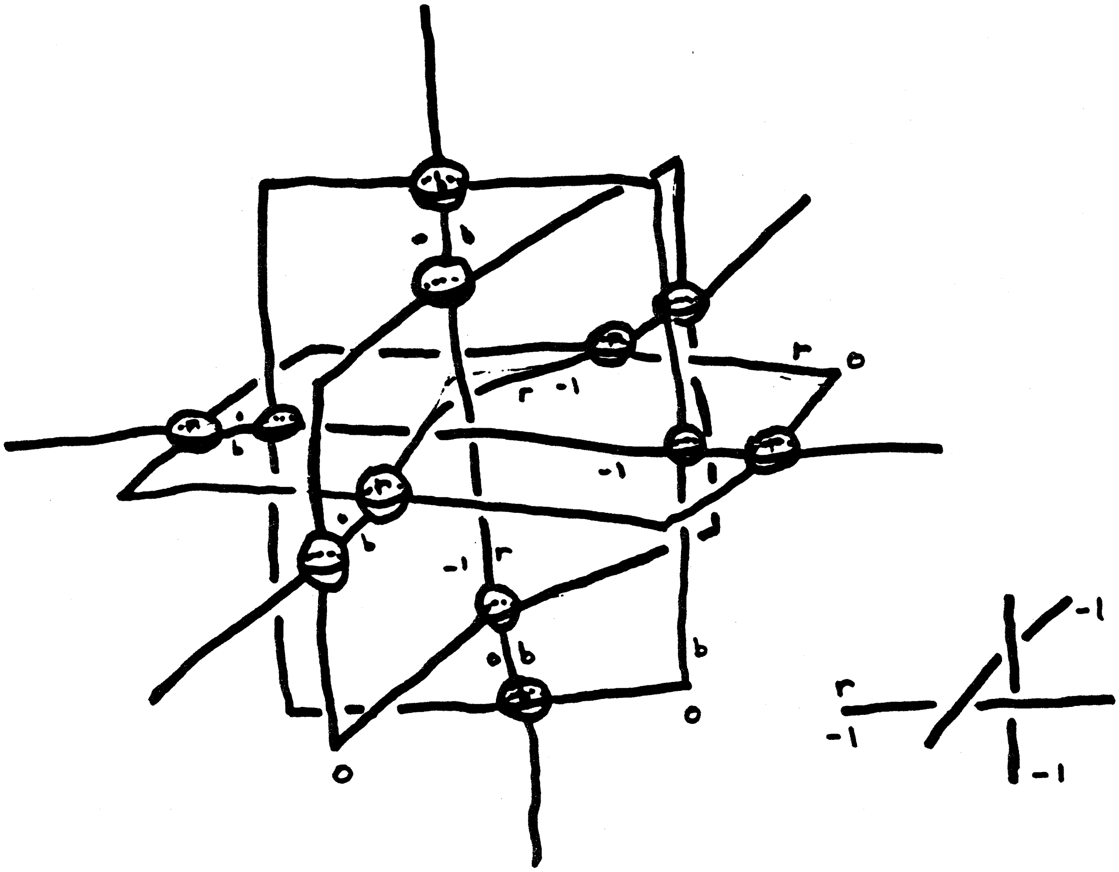

FIGURE 20: $B \cup_{Y^3} B$

We will now discuss several cut-and-paste techniques which may be used to construct new 4-manifolds. The first method is called blowing up a point. Let D_R be the disk of radius R in $\mathbb{C}^2$ and define

$$\hat{D}_R = \{[a:b:c] \in \mathbb{C}P^2 \mid |c|^2 \leq R^2 |ab|^2 (|a|^2 + |b|^2)^{-1}\} \, .$$

Let $S = \{[x:y:0] \in \mathbb{C}P^2\}$. It is clear that $h : D_R - \{0\} \to \hat{D}_R - S$; $(x,y) \mapsto [y:x:xy]$ is a biholomorphic map. By definition, when p is a point in a complex surface, X, there is an R and a biholomorphic map, $\phi : D_R \to N(p) \subseteq X$. Note that $X \cong (X - \{p\}) \cup_{D_R - \{0\}} -D_R$ as complex manifolds. The orientation on D_R is important in this construction. The blow-up of X is $\hat{X} \equiv (X - \{p\}) \cup_{D_R - \{0\}} -\hat{D}_R$. This construction is complex, so $\hat{X}$ has a natural complex structure. Topologically, we may think of $-D_R$ as a 0-handle in X, so to build $\hat{X}$ we remove $-D_R$ and add $-\hat{D}_R$. By letting $R \to 0$ it is evident that $\hat{D}_R$ is a tubular neighborhood of S in $\mathbb{C}P^2$. This is just the Euler class one disk bundle over S^2. Thus $\hat{D}_R$ is just a 0-handle with a 1-framed 2-handle attached and $-\hat{D}_R$ is a 0-handle with

FIGURE 21: THE RED NUCLEUS

a -1-framed 2-handle attached. Thus $\hat{X}$ is just X with an additional trivial -1-framed 2-handle.

The next technique is called a log transform. Every regular elliptic curve is equivalent to $\mathbb{C}/\mathbb{Z}[1,\omega]$. This implies that given any regular elliptic fibration over a disk, $\pi_1 : Y_1 \to D$ (think $Y_1 = \{([x:y:t],z) \mid |z| \le \frac{1}{2},\ x^2t + y^3 + (z^5-1)t^3 = 0\}$) there is a holomorphic map $\omega : D \to \mathbb{C}$ and a fiber preserving biholomorphic map $Y_1 \to Y_2$ where

$$Y_2 = \{(\lambda, z) \in \mathbb{C}^2 \mid |z| \le \frac{1}{2}\}/\sim \qquad (\lambda, z) \sim (\lambda + 1, z) \sim (\lambda + \omega(z), z)\ .$$

The periods of the Weierstrass $\wp$-function are non-elementary holomorphic functions of the coefficients of the associated cubic. Our function $\omega(z)$ is just $\frac{\omega_2(z)}{\omega_1(z)}$ where ω_1 and ω_2 are the periods of the $\wp$-function. Let

$$Y_3 = \{(\lambda, z) \in \mathbb{C}^2 \mid |z| \le \frac{1}{2}^{1/p}\}/\sim$$

$$(\lambda, z) \sim (\lambda+1, z) \sim (\lambda+\omega(z^p), z) \sim (\lambda + \frac{k}{p}, \exp(2\pi i \frac{k}{p})z)\ .$$

The map $f : Y_3 - \pi_3^{-1}(0) \to Y_2 - \pi_2^{-1}(0)$; $f(\lambda, z) = \left(\lambda - \frac{1}{2\pi i}\ln z, z^p\right)$ is fiber preserving, biholomorphic with inverse

$$f^{-1}(\lambda, z) = \left(\lambda + \frac{1}{2\pi i p}\ln z, \exp(1/p \ln z)\right)\ .$$

This implies that we may replace Y_2 with Y_3 inside any complex surface. This new surface is called the p-log transform of the old surface. A p-log transform may still be performed when the projection maps are smooth but not holomorphic. In this case there is no reason that the resulting manifold should be complex.

In order to get a topological description of a p-log transform, look at the case $\omega(z) \equiv i$. In this case $Y_2 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times D$, $(\lambda, z) \mapsto (\text{Im } \lambda, \text{Re } \lambda, z)$ and $Y_3 = \tilde{Y}_3/\mathbb{Z}_p$ where $\tilde{Y}_3 \equiv \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times D$, $(\lambda, z) \mapsto (\text{Im } \lambda, \text{Re } \lambda, z)$ is just Y_3 without the third equivalence $((\lambda, z) \sim (\lambda + \frac{k}{p}, \exp(2\pi i \frac{k}{p}) \cdot z))$. The group $\mathbb{Z}_p$ acts by this third equivalence which is clearly trivial on the first factor of $\mathbb{R}/\mathbb{Z}$. This means that a p-log transform is just $\mathbb{R}/\mathbb{Z}$ cross the change in the last two coordinates. Forgetting the first factor of $\mathbb{R}/\mathbb{Z}$, we see that $\tilde{Y}_3$ is just a solid torus ($S^1 \times D^2$) with $\mathbb{Z}_p$ acting by 1/p rotation in both factors (figure 22).

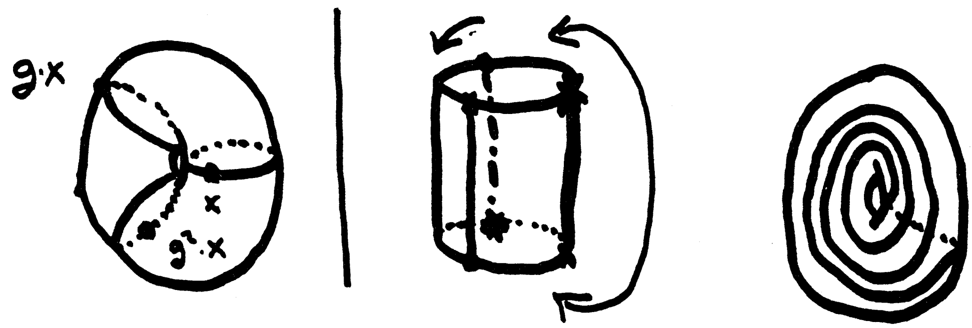

FIGURE 22: $\tilde{Y}_3$ AND Y_3 FOR $p = 3$

It is evident that Y_3 and Y_2 are both diffeomorphic to $T^2 \times D^2$, so the only difference between X and the p-log transform is the gluing map of this $T^2 \times D^2$. A p-log transform is just S^1 cross Dehn surgery. As an example consider the 3-log transform of N_2, which we will call $N_2(3)$. The nucleus, N_2, is obtained by adding three 2-handles to $T^2 \times D^2$ (figure 21), so $N_2(3)$ will be obtained by adding three 2-handles to $T^2 \times D^2$. In N_2, one 2-handle is attached along the fiber of $pt \times S^1 \times D^2$ with -1 framing relative to the natural framing coming from $pt \times \partial(S^1 \times D^2)$. Thus $N_2(3)$ has a 2-handle attached along the fiber in the Y_3 fibration of $pt \times S^1 \times D^2$ with framing -1 relative to $pt \times \partial(S^1 \times D^2)$. One 2-handle in N_2 is attached along $* \times \{1\} \times \partial D^2$ with framing -2. The image of this in Y_3 is $* \times S^1 \times \{1\}$. The third 2-handle in N_2 is attached along the first S^1 factor with -1 framing. This is unchanged in $N_2(3)$. Putting this together gives the picture of $N_2(3)$ in figure 23.

The vertical 1-handle is the first S^1 factor and $pt \times \partial(S^1 \times D^2)$ is the horizontal plane together with part of the outside of the horizontal 1-handle. This picture may be simplified and generalized as in figure 24. Cancel the vertical -1-framed 2-handle and 1-handle to get

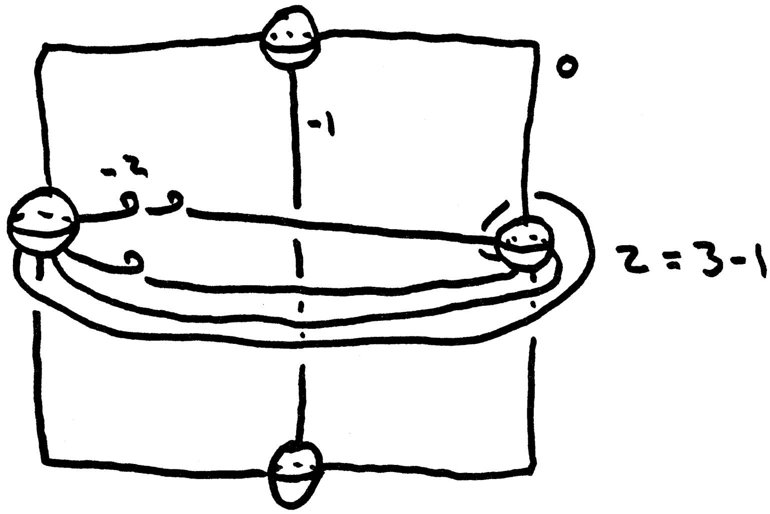

FIGURE 23: $N_2(3)$

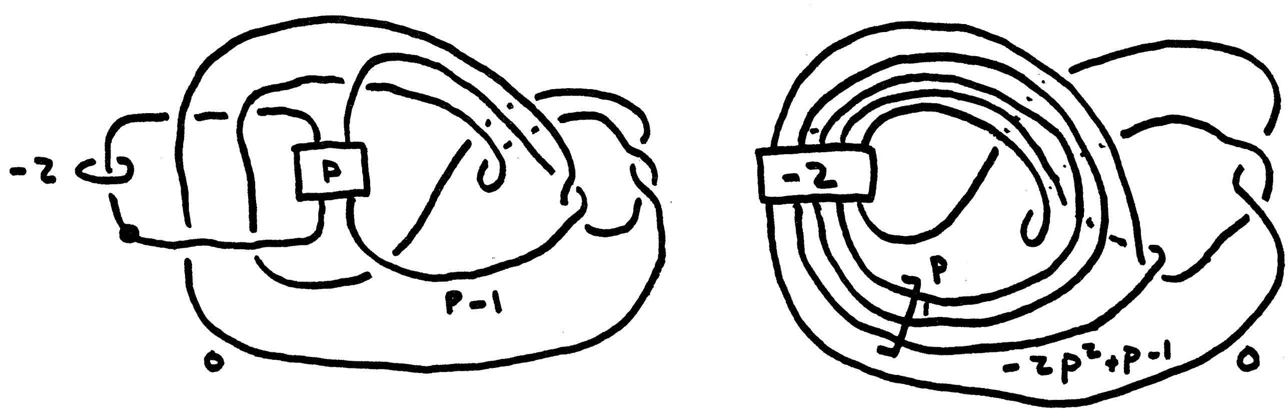

FIGURE 24: $N_2(p)$

the general p-case in the first part of figure 24. Cancel the horizontal −2-framed 2-handle/ 1-handle pair to get the second part. By construction, $\partial N_2(p) = \partial N_2$ since we are only changing the interior of N_2 with a p-log transform. The second picture in figure 24 shows that $\pi_1(N_2(p)) = 1$ since there are no 1-handles, and

$$Q_{N_2(p)} = \begin{pmatrix} 0 & 1 \\ 1 & -2p^2 + p - 1 \end{pmatrix} .$$

The bracket with a p-label is used to indicate that there is a cable with p strands.

The final cut-and-paste technique is a generalization of the blow up called a rational blow up. A blow up replaces a disk D^4 with the neighborhood of a sphere. A rational blow up replaces a rational homology disk with the neighborhood of a configuration of spheres. Let $R(p)$ be the 4-manifold in figure 25.

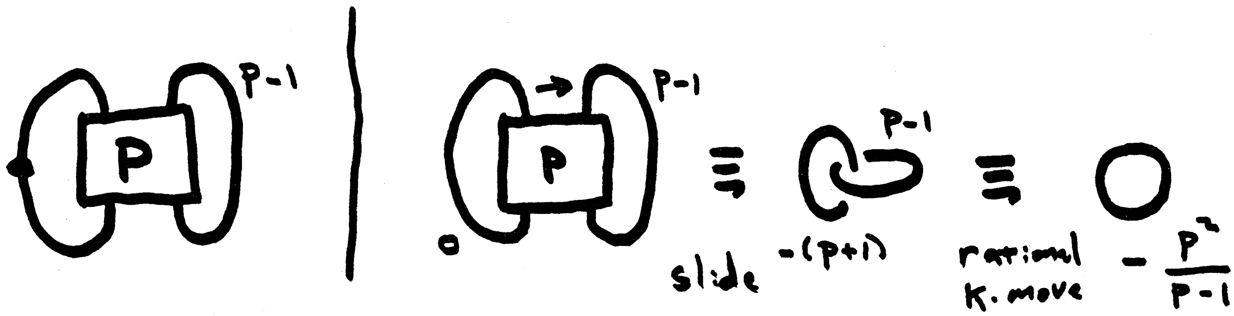

FIGURE 25: $R(p)$ AND $\partial R(p) = L(p^2, p-1)$

It is evident that $\pi_1(R(p)) = \mathbb{Z}_p$ and $H_k(R(p)) = 0$ for $k \geq 2$, so that $H_*(R(p);\mathbb{Q}) = H_*(D^4;\mathbb{Q})$.

Let $C(p)$ be the manifold in figure 26. A sequence of rational K-moves shows that $\partial C(p) = L(p^2, p-1)$. The cores of the 2-handles in $C(p)$ coned off to different cone points in the 0-handle is a configuration of spheres which intersect one another. The manifold $C(p)$ is just a neighborhood of those spheres.

FIGURE 26: $C(p)$

A p-rational blow up of a manifold just replaces a copy of $R(p)$ in the manifold with a copy of $C(p)$. A p-rational blow down replaces $C(p)$ with $R(p)$. A p-log transform is a combination of $p-1$ blow ups followed by a rational blow down. Figures 27 and 28 demonstrate this.

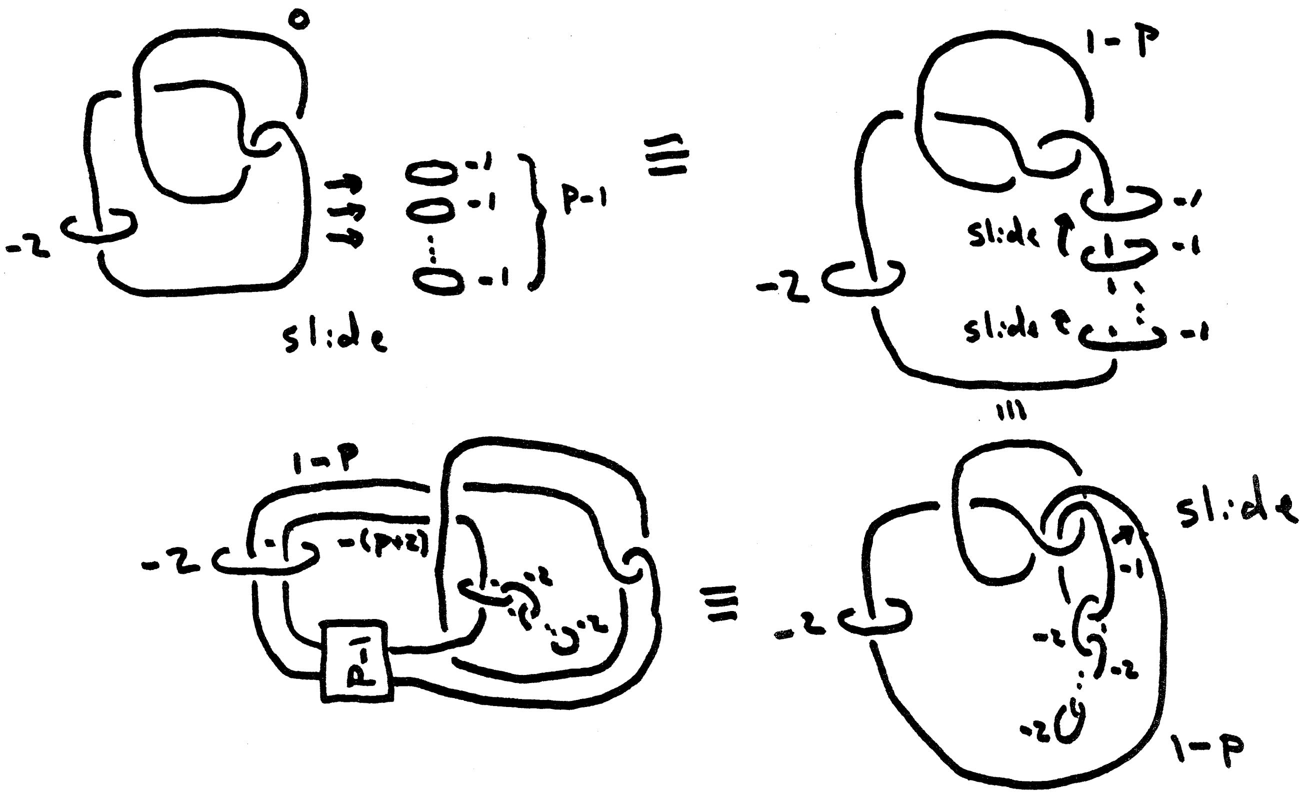

FIGURE 27: $N_2\#(p-1)\overline{\mathbb{C}P^2}$

Figure 27 shows N_2 blown up $p-1$ times. The 0-framed 2-handle is slid over all of the −1-framed handles, then the lowest −1-framed is slid over the next −1-framed handle going up, and so on. This ends by sliding the top −1-framed 2-handle over the $(1-p)$-framed 2-handle. In this last picture there is an obvious copy of $C(p)$. We need to cut out this $C(p)$ and glue in a copy of $R(p)$. Cutting out $C(p)$ amounts to putting an I-lable on each of the components in $C(p)$. In figure 28, we draw $(N_2\#(p-1)\overline{\mathbb{C}P^2}) - C(p)$ and rearrange the new boundary until it looks like the boundary of $R(p)$.
From the last picture in figure 28 we see that $((N_2\#(p-1)\overline{\mathbb{C}P^2}) - C(p)) \cup R(p) = N_2(p)$ as drawn in figure 24.

We will now use these cut-and-paste techniques to construct many smooth manifolds with the homotopy type of $K3$. Let $E(2;p)$ be the result of a p-log transform applied to a regular fiber of $E(2)$. This amounts to removing the red N_2 from figure 20 and gluing in $N_2(p)$, i.e.

$$E(2;p) = (E(2) - N_2) \cup_{\partial N_2} N_2(p) \ .$$

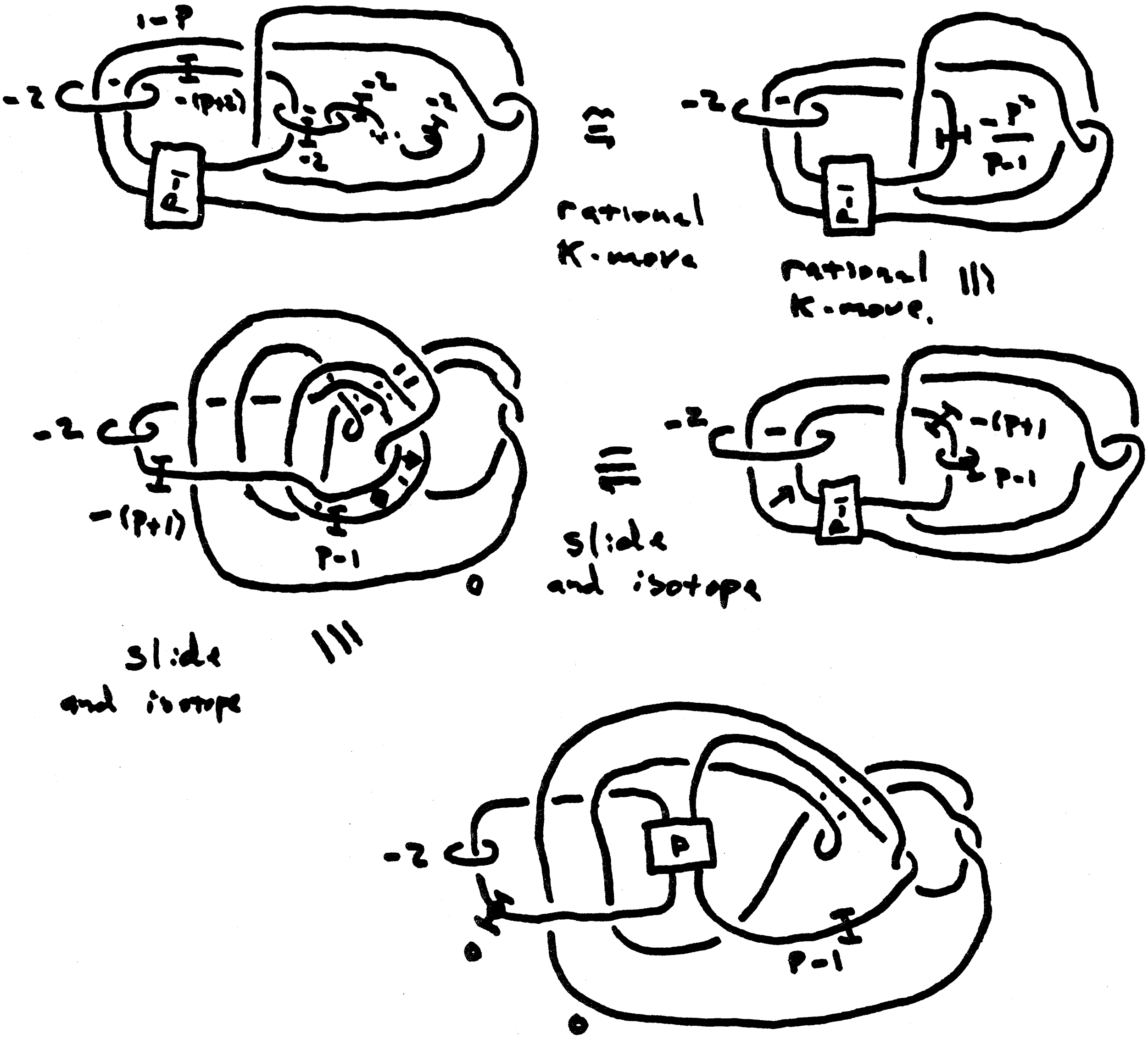

FIGURE 28: $(N_2\#(p-1)\overline{\mathbb{C}P^2}) - C(p)$

By looking at the intersection form of $N_2(p)$, we see that $E(2;p) \simeq E(2)$ if and only if p is odd. Recall that the two quadratic forms

$$\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$$

are equivalent over the integers if and only if $n \equiv m \pmod 2$.

We may apply a p-log transform to one fiber and a q-log transform to a different fiber to create a new manifold $E(2;p,q)$. In order to draw a picture of $E(2;p,q)$, we will do both log transforms inside the nucleus, N_2. To see the result of a q-log transform applied to $N_2(p)$, first redraw $N_2(p)$ so that a regular fiber is easily visible. Figure 29 shows a new picture of N_2 in which two regular fibers are visible. To construct figure 29, we double the 1-handles with the method from figure 20. Also double the 0-framed 2-handle. In other words, we replace the 2-handle with two copies of itself and a 3-handle glued to a sphere constructed from one copy of the core of each of the doubled 2-handles.

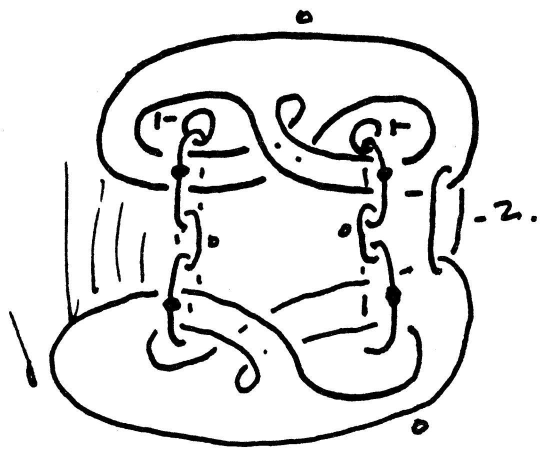

FIGURE 29: N_2 WITH AN EXTRA FIBER

Now we repeat the description of a p-log transform described around figures 22, 23 and 24 to get the new picture of $N_2(p)$ with an obvious regular fiber in figure 30.
After sliding the -1-framed 2-handle over the 0-framed 2-handle in the doubled 1-handle, we may apply a q-log transform to the regular fiber. This creates the manifold $N_2(p,q)$ in figure 31.
Define

$$E(2;p,q) = (E(2) - N_2) \cup_{\partial N_2} N_2(p,q) \ .$$

The fundamental group of $N_2(p,q)$ may be computed from figure 31 with VanKampen's theorem. Label the 1-handles from left to right as a_1, a_2, a_3, a_4. The fundamental group of $N_2(p,q)$ is generated by the a_k with relations coming from the 2-handles. As an example, the p-framed 2-handle contributes the relation $a_2 a_1^{-p}$. The final answer is $\pi_1(N_2(p,q)) = \mathbb{Z}_{gcd(p,q)}$. In the same way we can compute the intersection pairing of $N_2(p,q)$. This form is even if and only if both p and q are odd. We conclude that $E(2;p,q)$ has the homotopy type of $K3$ if and only if $p \equiv q \equiv 1 \pmod 2$ and $gcd(p,q) = 1$.

Kodaira proved that every complex surface with the homotopy type of a $K3$ surface is deformation equivalent to one of $E(2)$, $E(2;p)$, or $E(2;p,q)$ [K],[BPV]. In particular,

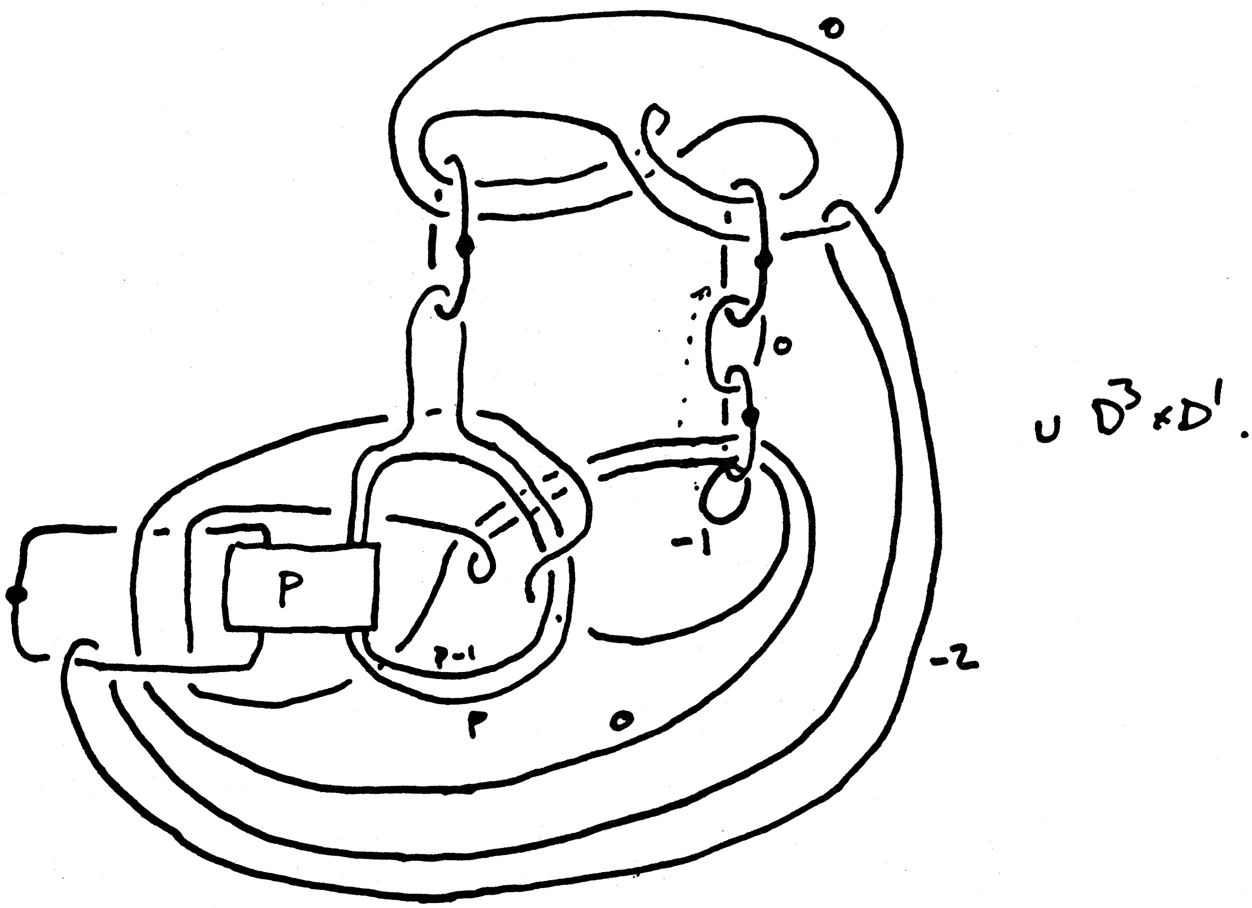

FIGURE 30: A REGULAR FIBER IN $N_2(p)$

any complex surface with this homotopy type is diffeomorphic to one of $E(2)$, $E(2;p)$, or $E(2;p,q)$. Freedman proved that all of these manifolds are homeomorphic. In the next section we will prove that they are not all diffeomorphic.

There is one more interesting construction. Perform a p-log transform in the red nucleus and a q-log transform in the green nucleus in figure 20 to create the Gompf-Mrowka manifold,

$$GM(p,q) = (E(2) - N_2^{\text{red}} - N_2^{\text{green}}) \cup_{2\partial N_2} (N_2(p) \cup N_2(q)) \ .$$

This situation is similar to Dehn surgery in a Seifert fiber space. Dehn surgery along fibers will construct new Seifert fiber spaces, but Dehn surgery at random will not produce a

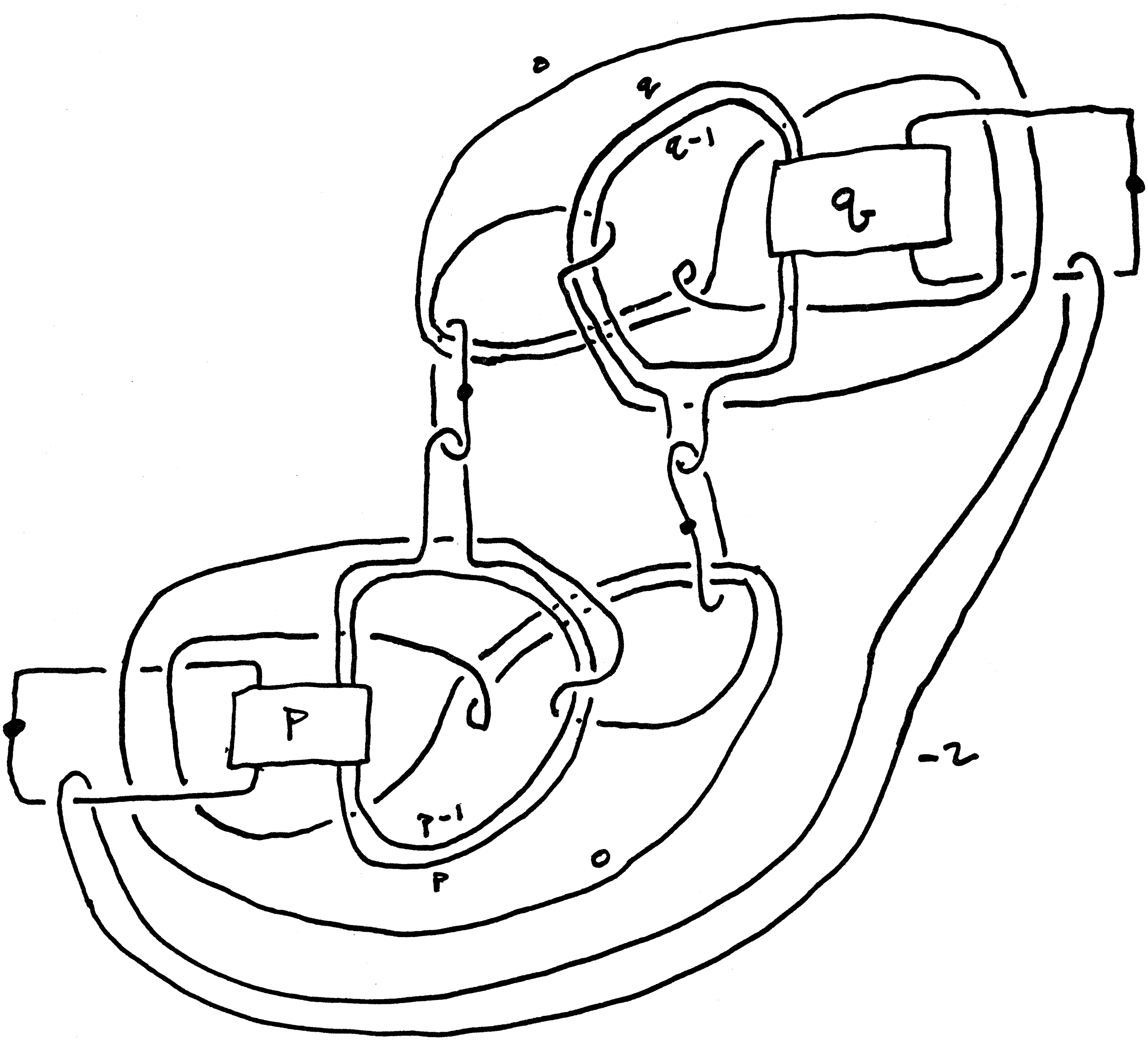

FIGURE FIGURE 31: $N_2(p, q)$

Seifert fiber space. Log transforms along fibers in an elliptic surface will produce complex manifolds but log transforms applied at random will not produce complex manifolds. The manifolds $E(2; p, q)$ is constructed by a pair of log transforms on fibers, but $GM(p, q)$ is constructed by a p-log transform on a fiber and a q-log transform on the image of a fiber in the natural $\mathbb{Z}_3$ action on $E_8 \cup B$. See figure 32.

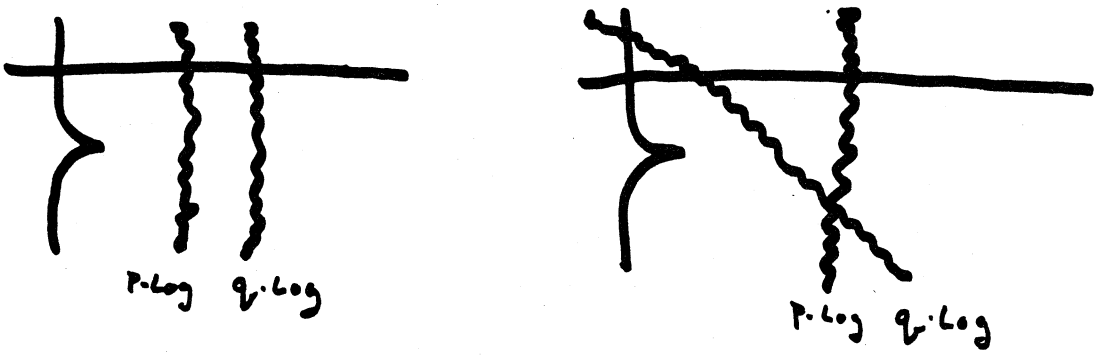

FIGURE 32: $E(2; p, q)$ vs $GM(p, q)$

In the next section we will prove that $GM(3,5)$ is not diffeomorphic to any connected sum of the form $X_1 \# X_2 \# \ldots X_n$ where either X_k or $\bar{X}_k$ is complex for each k and no X_k is diffeomorphic to S^4.

Lecture 2

In the previous lecture we constructed a large collection of interesting 4-manifolds. In this lecture we will use some of the main theorems from gauge theory to study these manifolds. Any gauge theory starts with a usually nonlinear system of partial differential equations which depend on the geometry of some underlying manifold. The space of solutions to these differential equations mod out by the relevant symmetry group is called a moduli space. The moduli space encodes some subtle information about the underlying manifold. We will first list some of the results about manifolds which have been proved with moduli spaces. After some sample applications of these theorems, we will consider some of the technical details involved in the proofs.

The first type of topological result proved via moduli spaces, was a non-existence theorem. A typical argument would be to assume that some smooth 4-manifold has a given intersection form and prove that the moduli space would have to be an impossible object. For example, any smooth 4-manifold with intersection form $E_8 \oplus E_8$ would have a compact 1-dimensional moduli space with one boundary point. Since there is no compact 1-manifold with boundary the point, there can be no smooth 4-manifold with intersection form $E_8 \oplus E_8$. The first theorem along these lines was Rochlin's theorem.

Theorem. *If X is a smooth spin 4-manifold, then* $16 \,|\text{Sign } X$.

The most surprising theorem along these lines is Donaldson's Theorem A.

Theorem A. *If X is a smooth, simply connected 4-manifold with definite intersection form Q_X, then Q_X is diagonalizable over $\mathbb{Z}$.*

The most recent theorem in this direction is Furuta's 10/8 ths theorem [F].

Theorem. *If X is a smooth spin 4-manifold with* $\text{Sign}\, X \neq 0$,

$$rk\ H_2(x) \geq \frac{10}{8}|\,\text{Sign}\ X| + 2\ .$$

The second application of gauge theory was to define invariants of smooth 4-manifolds. For example, the Seiberg-Witten moduli space associated to a Spin_c structure is a finite collection of signed points. When $b_2^+(X) \geq 2$, the number of these points counted with sign depends only on the first Chern class of the Spin_c structure. This number defines a function

$$n : H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

called the Seiberg-Witten invariant. This function is non-zero only on a finite subset of $H^2(X;\mathbb{Z})$, called the set of basic classes of X.

For an invariant to be useful, there must be some method to compute it. It is not hard to compute the Seiberg-Witten invariant for Kähler manifolds. Recall some definitions. An almost complex structure is a map $J : TX \to TX$ so that $J^2 = -I$. When X is a complex manifold, multiplication by i is an almost complex structure. This example is called a complex structure. A metric, g, is compatible with a complex or almost complex structure if $g(Jx, Jy) = g(x, y)$. If h is any metric, $g(x, y) = \frac{1}{2}(h(x, y) + h(Jx, Jy))$ will be a compatible metric. Given an almost complex structure and a compatible metric, we may define a non-degenerate 2-form by $\omega(x, y) = g(x, Jy) = g(Jx, J^2y) = -g(y, Jx) = -\omega(y, x)$. A complex manifold is called Kähler if this form is closed, $d\omega = 0$.

Using an almost complex structure, the complexified tangent bundle, $TX \otimes_{\mathbb{R}} \mathbb{C}$, may be split into the i eigenspace of J, $T^{(1,0)}X$, and the $-i$ eigenspace of J, $T^{(0,1)}X$. Let $\Lambda^{p,q}X = \Lambda^p(T^{(1,0)}X)^* \otimes \Lambda^q(T^{(0,1)}X)^*$. Any almost complex manifold has a natural Spin$_c$-structure with positive spinners $\Lambda^{0,0}X \oplus \Lambda^{0,2}X$, negative spinners, $\Lambda^{0,1}X$, and determinant line bundle $\Lambda^{0,2}X$. The first Chern class of the bundle $(\Lambda^{0,2}X)^*$ is called the canonical class, K.

With the above notations we may state the following theorem of Witten [W].

Theorem. *If X is a Kähler manifold with $b^2_+ > 1$, then*

$$n(K) = 1 \ , \qquad n(-K) = (-1)^{(\mathrm{Sign}(X) + \chi(X))/4} \ .$$

Furthermore, $n(L) \neq 0$ implies that $\frac{1}{2}(L - K) \in H^2(X; \mathbb{Z})$,

$$0 \leq \frac{1}{2}(L - K) \cdot [\omega] \leq -K \cdot [\omega]$$

with equality only if $L = K$ or $L = -K$.

Corollary. *The only basic class of $E(2)$ is 0 and $n(0) = 1$.*

PROOF. It is well known that $E(2)$ is Kähler. In fact, using the techniques from Lecture 1 it is possible to prove that $E(2)$ is diffeomorphic to $\{[\omega : x : y : z] \in \mathbb{C}P^3 \mid \omega^4 + x^4 + y^4 + z^4 = 1\}$. Now,

$$\begin{aligned} c_1(K_{E(2)}) &= -c_1(\Lambda^{0,2}E(2)) \\ &= -c_1(T^{(0,1)}E(2)) \\ &= +c_1(TE(2)) \\ &= +c_1(TP^3 \mid_{E(2)}) - c_1(NE(2)) \\ &= 4h - 4h = 0 \ . \end{aligned}$$

Here h is the generator of $H^2(\mathbb{C}P^3;\mathbb{Z})$, and $c_1(NE(2)) = 4h$ because $E(2)$ has degree 4. $\square$

It is almost possible to think about the Seiberg-Witten invariant axiomatically. The invariant associates to any smooth 4-manifold a function

$$n : H^2(X;\mathbb{Z}) \to \mathbb{Z} .$$

For $E(2)$ this function is 1 at 0 and 0 everywhere else. There are a few formula describing how this invariant changes under cut-and-paste operations. These formula may be used to compute the Seiberg-Witten invariants for many 4-manifolds and conceivably all 4-manifolds.

All of these formula are proved with neck-stretching arguments. We will first state the formula, then apply the formula to the examples from Lecture 1, then discuss the proofs of these formula.

Theorem (Vanishing). *If $\pi_1(Z) = 1$,*

$$Z = X\#Y \ , \quad \textit{and} \quad b^2_+(X) \geq b_2^+(Y) \geq 1 \ ,$$

then the Seiberg-Witten invariant,

$$n_Z : H^2(Z;\mathbb{Z}) \to \mathbb{Z} \quad \textit{is zero.}$$

Theorem (Blow-up formula). *If every basic class of X satisfies $K^2[X] = 2\chi(X) + 3\mathrm{Sign}(X)$, then the only basic classes of $X\#\overline{\mathbb{C}P^2}$ are of the form $K \pm E$, and $n_{X\#\overline{\mathbb{C}P^2}}(K \pm E) = n_x(K)$ where E is the exceptional divisor (generator of $H^2(\overline{\mathbb{C}P^2})$).*

We should remark that the condition $K^2[X] = 2\chi(X) + 3\mathrm{Sign}(X)$ is equivalent to the Seiberg-Witten moduli space having dimension zero. A manifold is said to have simple type if all basic classes satisfy $K^2[X] = 2\chi(X) + 3\mathrm{Sign}(X)$. It is conjectured that every manifold with $b^2_+ > 1$ has simple type. We will see that all of the examples in Lecture 1 have simple type. It is true that all basic classes are characteristic. A class, c, is called characteristic if $Q(c,x) \equiv Q(x,x) \pmod 2$. The final gluing formula is Fintushel and Stern's rational blow-down formula [FS].

Theorem (Rational blow-down). *Let $Y = X \cup C(p)$ and $Z = X \cup R(p)$. If $K_Y \in H^2(Y;\mathbb{Z})$ and $K_Z \in H^2(Z;\mathbb{Z})$ are characteristic elements so that $K_Y^2[Y] \geq 2\chi(Y) + 3\mathrm{Sign}(Y)$ and $i_Y^* K_Y = i_Z^* K_Z$ where $i_Y : X \to Y$ and $i_Z : X \to Z$, then*

$$n_Y(K_Y) = n_Z(K_Z) \ .$$

It is probably necessary to include a formula for the Seiberg-Witten invariant of a surface sum in order to have a complete set of axioms for the Seiberg-Witten invariant. If X is a possibly disconnected four manifold with two disjointly embedded surfaces of genus g and opposite self intersection numbers, then we may define the surface sum of X to be $X\#_{F_1,F_2} = (X - N(F_1 \cup F_2)/\partial N(F_1) = \partial N(F_2)$. One indication that a surface sum formula would be important is its importance in symplectic topology. Cliff Taubes has shown that the canonical class of a symplectic manifold is a basic class with Seiberg-Witten invariant ± 1. If F_1 and F_2 are symplectly embedded surfaces in a symplectic manifold, then $X\#_{F_1,F_2}$ is a symplectic manifold. Each of the previous Seiberg-Witten glueing formulae has a symplectic counterpart. There is no symplectic connected sum; there is a symplectic blow-up. Recently, J. Etnyre proved that a rational blow down may be performed symplectically, provided that the spheres in the configuration are symplectically embedded [E].

As a first application of these theorems, we will compute the Seiberg-Witten invariants of $E(2;5)$. Four applications of the blow-up formula show that $\pm e_1 \pm e_2 \pm e_3 \pm e_4$ are the only basic classes of $E(2)\#4\overline{\mathbb{CP}^2}$. Write the intersection form of $E(2)\#4\overline{\mathbb{CP}^2}$ as $2E_8 \oplus 3H \oplus 4\langle -1\rangle$ with basis vectors $x_1, \ldots, x_{20}$ for the first $2E_8 \oplus 2H$, f and s for the third H and $e_1, \ldots, e_4$ for $4\langle -1\rangle$. The third H is represented by the red N_2 in figures 20 and 21. In particular, $f^2 = 0$, $s^2 = -2$, and $f \cdot s = 1$. The configuration $C(5)$ embeds into $N_2\#4\langle -1\rangle$ representing the elements $u_0 = f + 2e_1 + e_2 + e_3 + e_4$, $u_1 = e_2 - e_1$, $u_2 = e_3 - e_2$ and $u_3 = e_4 - e_3$. See Figure 27. Over the rationals, the cohomology splits as

$$H^2(E(2)\#4\overline{\mathbb{CP}^2}) = H^2(X) \oplus H^2(C(5)) \ .$$

It follows that $i_Y^* K$ is just the projection of K into $H^2(X)$. In other words,

$$i_Y^* K = K + a_0 u_0 + \cdots + a_3 u_3$$

where the a_n are the unique rational numbers so that $i_Y^* K \cdot u_n = 0$. In our case, we get

$$\begin{aligned} i_Y^*(e_1 + e_2 + e_3 + e_4) &= \tfrac{4}{5}\, f \ , \\ i_Y^*(e_1 + e_2 + e_3 - e_4) &= i_Y^*(e_1 + e_2 - e_3 + e_4) \\ &= i_Y^*(e_1 - e_2 + e_3 + e_4) = i_Y^*(-e_1 + e_2 + e_3 + e_4) = \tfrac{2}{5}\, f \ , \end{aligned}$$

and

$$i_Y^*(e_1 + e_2 - e_3 - e_4) = i_Y^*(e_1 - e_2 + e_3 - e_4) = i_Y^*(-e_1 + e_2 + e_3 - e_4) = 0 \ .$$

With rational coefficients,

$$H^2(E(2;5)) = H^2(X) \oplus H^2(R(5)) = H^2(X) .$$

so it appears that $0, \pm\frac{2}{5}f$, and $\pm\frac{4}{5}f$ are the candidates for basic classes on $E(2;5)$. Even though $\frac{1}{5}f$ looks like a rational class, it is really an integral class in $H^2(E(2;5))$. In Figure 24, $\frac{1}{5}f$ is the 0-framed 2-handle. It is a multiple fiber. Nearby regular fibers wrap around $\frac{1}{5}f$ five times as in Figure 22. After realizing that $\frac{1}{5}f$ is an integral class, it is easy to check that $e_1+e_2+e_3+e_4$ and $\frac{4}{5}f$ satisfy the conditions of the rational blow-down formula. In fact the previous discussion generalizes to give:

Fact. *If $p = 2k+1$, then $\pm\frac{2n}{p}\ f$ are basic classes for $E(2;p)$ when $0 \le n \le k$.*

Since $E(2;p)$ has p basic classes, and $E(2;q)$ has q basic classes, we see that $E(2;p)$ is diffeomorphic to $E(2;q)$ if and only if $p = q$. All of the $E(2;p)$ are homeomorphic, so $E(2;p)$ is an infinite family of different differential structures on the topological K3 manifold.

The 0-framed 2-handles in Figure 31 represent $\frac{1}{p}f$ and $\frac{1}{q}f$. Repeating the preceding argument of blowing up then rationally blowing down to the regular fiber in Figure 30 proves:

Fact. *The basic classes for $E(2;2n+1,2m+1)$ are $\left(\pm\frac{2k}{2n+1} \pm \frac{2\ell}{2m+1}\right) f$ for $0 \le k \le n$ and $0 \le \ell \le m$.*

Notice that the dimension of the space spanned by the basic classes is zero or one in all of these examples. This is in fact true for any minimal complex surface.

As a final example, we will compute the Seiberg-Witten invariaints of $GM(3,5)$. Pick a basis for $H^2(E(2)) = 2E_8 \oplus 3H$ with f_r and s_r representing the classes in the red nucleus and f_g and s_g representing the classes in the green nucleus. The blow-up rational blow-down procedure implies:

Fact. *The basic classes of $GM(3,5)$ are 0, $\pm\frac{2}{3}f_r$, $\pm\frac{2}{5}f_g$, $\pm\frac{2}{3}f_r \pm \frac{2}{5}f_g$, $\pm\frac{4}{5}f_g$, and $\pm\frac{2}{3}f_r \pm \frac{4}{5}f_g$.*

Notice that the basic classes of $GM(3,5)$ span a two-dimensional space. It follows that $GM(3,5)$ is not diffeomorphic to any of the manifolds $E(2)$, $E(2;p)$, $E(2;p,q)$. Work of Kodaira shows that any complex manifold homotopy equivalent to $\pm E(2)$ is diffeomorphic to one of $E(2)$, $E(2;p)$ or $E(2;p,q)$ [K],[BPV]. It follows that $GM(3,5)$ is not complex with either orientation.

More than this is true. It was conjectured that every smooth 4-manifold is diffeomorphic to a connected sum, $X_1\#X_2\ldots\#X_n$ so that either X_k is complex, or $\bar{X}_k$ is complex, or X_k is S^4. The manifold $GM(3,5)$ is a counter-example to this conjecture. Since $X\#S^4 \cong X$, we may assume that the connected sum decomposition has no S^4 factors, unless the manifold is S^4. As we have seen, neither $GM(3,5)$ nor $\overline{GM(3,5)}$ is complex, so for $GM(3,5)$ to satisfy the conjecture, it would have to be a nontrivial connected sum $GM(3,5) = Y_1\#\ldots\#Y_n$. By Van Kampen's theorem each of the Y_k are simply connected. If one of the Y_k had an odd intersection form (i.e. $x \in H^2(Y_k)$, $Q_{Y_k}(x,x) \equiv 1 \pmod 2$) then $GM(3,5)$ would have an odd intersection form. Since $GM(3,5)$ has an even intersection form, we must conclude that all of the Y_k are even and simply connected, therefore spin. By Donaldson's Theorem A there is no smooth 4-manifold with a non-trivial even definite intersection form. It follows from the classification of even unimodular indefinite forms that $Q_{Y_k} = n_k E_8 \oplus m_k H$. Rochlin's theorem implies that each of the n_k is even, say $n_k = 2k$. The Meyer-Vietoris sequence shows that

$$Q_{GM(3,5)} = \bigoplus_k 2p_k E_8 \oplus m_k H \ .$$

This gives

$$22 = \text{rk } Q_{GM(3,5)} = 2 \cdot 8 \sum_k |p_k| + 2 \sum_k |m_k|$$

and

$$-16 = \text{Sign } Q_{GM(3,5)} = -16 \sum_k p_k \ .$$

The first equation implies that at most one of the p_k may be non-zero, say $p_1 \neq 0$. The second equation implies that $p_1 = 1$. Furuta's 10/8'ths theorem and the first equation imply that $Q_{Y_1} = 2E_8 \oplus 3H$ and $H^2(Y_k) = 0$ for $k > 1$. If Y_2 had a complex structure, there would be a $c_1 \in H^2(Y_2)$ so that $c_1^2[Y_2] = 2\chi(Y_2) + 3\ \text{Sign}(Y_2)$, but this is impossible because $\pi_1(Y_2) = 1$ and $H^2(Y_2) = 0$ imply that $c_1^2[Y_2] = 0$, $\chi(Y_2) = 2$ and Sign $Y_2 = 0$.

With these applications as motivation, we will now look at some of the technical details involved in the proofs of the gluing theorems. All of the gluing theorems are proved with a neck-stretching or neck-pinching technique. In order to study the moduli space on $Z = X \cup ([0,T] \times M) \cup Y$, look at the case when T is very large. Decay estimates, compactness results and the implicit function theorem are used to described the moduli space of Z in terms of the moduli spaces of X and Y. After the moduli space of Z is understood, a gluing theorem for the invariants may be proved.

Leon Simon proved a general theorem about the decay rate of solutions to evolution equations [Si]. We will sketch a proof of a decay estimate for the Seiberg-Witten equations on a cylinder $\mathbb{R} \times M$. Without presenting all of the details, remember the general structure of the Seiberg-Witten equations on a cylinder. Start with a Spin$_c$ structure on M. This amounts to a complex line bundle, L, a complex plane bundle, W, some multiplication maps:

$$c : T^*M \otimes W \to W$$
$$[\,\cdot\,,\,\cdot\,] : W \otimes \bar{W} \to T^*M$$

and a method for constructing a differential operator $\partial_A : \Gamma(W) \to \Gamma(W)$ from a connection A on L [A1],[A2],[S]. A 1-parameter family of connections on L may be thought of as a connection on the pull-back of L to $\mathbb{R} \times M$. A 1-parameter family of spinners on M is a spinner on $\mathbb{R} \times M$. With this notation the Seiberg-Witten equations on $\mathbb{R} \times M$ are:

$$\frac{\partial A}{\partial t} = 2i[\psi, \bar{\psi}] - *F_A \ ,$$
$$\frac{\partial \psi}{\partial t} = -\partial_A \psi .$$

Under mild assumptions about a solution to these equations, we will prove that the solution decays exponentially as $t \to \infty$. The basic idea is to define an energy functional and show that it satisfies a differential inequality.

Let

$$E(T) = \int_T^\infty \int_M [|\nabla_{\underline{A}}\psi|^2 + \frac{1}{8}|\psi|^4 + \frac{s}{4}|\psi|^2 \frac{1}{2}|F_{\underline{A}}|^2]\, dvol_M\, dt \ .$$

The underline is used to indicate that we are thinking of the object as an object on $\mathbb{R} \times M$. Now compute

$$\frac{dE}{dT} = -\int_M [|\nabla_{\underline{A}}\psi|^2 + \frac{1}{8}|\psi|^4 + \frac{s}{4}|\psi|^2 \frac{1}{2}|F_{\underline{A}}|^2]\, dvol_M\, dt \ .$$

and

$$|\nabla_{\underline{A}}\psi|^2 = \left| \frac{\partial \psi}{\partial t} \otimes dt + \nabla_A \psi \right|^2$$
$$= |\partial_A \psi|^2 + |\nabla_A \psi|^2$$

and

$$\frac{1}{2}|F_{\underline{A}}|^2 = \frac{1}{2}|F_A + dt \wedge \frac{\partial A}{\partial t}|^2$$
$$= \frac{1}{2}|F_A|^2 + \frac{1}{2}| * F_A - 2i[\psi, \bar{\psi}]|^2$$
$$= \langle *F_A, 2i[\psi, \bar{\psi}]\rangle - \frac{1}{8}|\psi|^4 + | * F_A - 2i[\psi, \bar{\psi}]|^2 \ .$$

Plugging into the expression for $\frac{dE}{dT}$ gives

$$\begin{aligned}\frac{dE}{dT} &= -\int_M |\partial_A\psi|^2 + |\nabla_A\psi|^2 + \frac{s}{4}|\psi|^2 + \langle *F_A, 2i[\psi,\bar\psi]\rangle \\ &\quad + |*F_A - 2i[\psi,\bar\psi]|^2 \, dvol_M\, dt \\ &= -\int_M |\partial_A\psi|^2 + \langle \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi - \frac{1}{2} * F_A\cdot\psi, \psi\rangle \\ &\quad + |*F_A - 2i[\psi,\bar\psi]|^2 \, dvol_M\, dt \\ &= -\int_M 2|\partial_A\psi|^2 + |*F_A - 2i[\psi,\bar\psi]|^2 \, dvol_M\, dt \ \equiv -J.\end{aligned}$$

In the above computation, we used the Bochner-Weitzenboch formula:

$$\partial_A^*\partial_A\psi = \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi - \frac{1}{2} * F_A\cdot\psi \ .$$

We will now bound everything by J in order to derive a differential inequality. Assume that A,ψ converges to a nondegenerate limit in the sense that

$$\left\| \begin{bmatrix} A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\|_{L^\infty} \longrightarrow 0 \quad \text{as} \quad T\to\infty$$

with $J(A_\infty,\psi_\infty) = 0$, and there is a constant $\delta > 0$ so that $\delta\|x\| \le \|Lx\|$, where

$$L\begin{bmatrix} u \\ a \\ \phi \end{bmatrix} \equiv \begin{bmatrix} d * a \\ *da + idu - 4i[\phi,\psi_\infty] \\ \partial_{A_\infty}\phi - \frac{1}{2}c(a,\psi_\infty) \end{bmatrix} \ .$$

The function L is just the linearization of the right-hand side of the Seiberg-Witten equations about the point A_∞,ψ_∞. The Seiberg-Witten equations have a large group of symmetry, so we are adding an extra gauge fixing condition:

$$d^*(A - A_\infty) = 0 \ .$$

This is not the most geometrically natural gauge fixing condition. We will discuss gauge fixing further in lecture 3. Compute

$$\begin{aligned}\left\| \begin{bmatrix} A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\|_{L^2}^2 &\le \delta^{-2} \left\| \begin{bmatrix} d^*(A - A_\infty) \\ *d(A - A_\infty) - 4i[\psi - \psi_\infty, \bar\psi_\infty] \\ \partial_{A_\infty}(\psi - \psi_\infty) - \frac{1}{2}c(A - A_\infty, \psi_\infty) \end{bmatrix} \right\|_{L^2}^2 \\ &\le 2\delta^{-2} \left\| \begin{bmatrix} 0 \\ *F_A - 2i[\psi,\bar\psi] \\ \partial_A\psi \end{bmatrix} \right\|_{L^2}^2 \\ &\quad + K\|\psi - \psi_\infty\|_{L^\infty}^2 \left\| \begin{bmatrix} 0 \\ A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\|_{L^2}^2\end{aligned}$$

for some large constant K independent of A and ψ. Pick T_0 so that $t > T_0$ implies $K\|\psi - \psi_\infty\|^2_{L^\infty} \le \frac{2}{3}$ and rearrange the above inequality to get:

$$\left\| \begin{bmatrix} 0 \\ A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\|^2_{L^2} \le 3\delta^{-2} \left\| \begin{bmatrix} 0 \\ *F_A - 2i[\psi, \bar{\psi}] \\ \partial_A \psi \end{bmatrix} \right\|^2_{L^2} \le 3\delta^{-2} J \; .$$

We may now bound E by a constant times J,

$$\begin{aligned} E(T) &= \int_T^\infty \int_M |\nabla_{\underline{A}}\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{8}|\psi|^4 + \frac{1}{2}|F_A|^2 \, dvol_M \, dt \\ &= \int_T^\infty \int_M d * \langle \nabla_{\underline{A}}\psi, \psi \rangle - \frac{1}{2} F_A \wedge F_A \\ &\quad + [\langle \nabla_{\underline{A}}\psi, \psi \rangle + \frac{s}{4}|\psi|^2 - \frac{1}{2}\langle F^+_{\underline{A}}\psi, \psi \rangle \\ &\quad + |F^+_{\underline{A}}|^2 - 2\langle F^+_{\underline{A}}, i[\psi, \psi] \rangle + \frac{1}{8}|\psi|^4] \, dvol_M \, dt \\ &= \int_T^\infty \int_M d[\langle \nabla_{\underline{A}}\psi, \psi \rangle - \frac{1}{2}(A - A_\infty) \wedge F_A] \\ &+ [| \not{D}_{\underline{A}}\psi|^2 + |F^+_{\underline{A}} - i[\psi, \bar{\psi}]|^2] \, dvol_M \, dt \\ &= \int_M \frac{1}{2}(A - A_\infty) \wedge F_A - *(\partial_A \psi, \psi) \\ &\le \frac{1}{2}\delta^{-1} \left| \left\langle \begin{bmatrix} 0 \\ A - A_\infty \\ \psi - \psi_\infty \end{bmatrix}, L \begin{bmatrix} 0 \\ A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\rangle \right| + \text{const} \left\| \begin{bmatrix} 0 \\ A - A_\infty \\ \psi - \psi_\infty \end{bmatrix} \right\|^2_{L^2} \\ &\le C \, J \end{aligned}$$

In the above computation, we used the 4-dimensional Bochner-Weitzenboch formula:

$$\not{D}^*_{\underline{A}} \not{D}_{\underline{A}}\psi = \nabla^*_{\underline{A}}\nabla_{\underline{A}}\psi + \frac{s}{4}\psi - \frac{1}{2}F^+_{\underline{A}} \cdot \psi$$

and the fact that the 4-dimensional Seiberg-Witten equations may be written as

$$\begin{aligned} F^+_{\underline{A}} &= i[\psi, \bar{\psi}] \\ \not{D}_{\underline{A}}\psi &= 0 \; . \end{aligned}$$

Putting together the inequalities for E and $\frac{dE}{dT}$ gives

$$\frac{dE}{dT} \le -\frac{1}{C} E \; .$$

A bootstrapping argument starting with this differential inequality will prove that $A - A_\infty$ and $\psi - \psi_\infty$ decay exponentially.

We will now look at the other components of a gluing argument. The implicit function theorem states that under suitable hypothesis, for every point x_0, so that $f(x_0)$ is small, there is a nearby point x_1 so that $f(x_1) = 0$. The compactness results show that the hypothesis holds for the Seiberg-Witten equations. One version of the implicit function theorem is [AMR]:

Theorem. *If $f \in C^2(E, F)$ and $Df|_{x_0}$ is an isomorphism so that*

$$\|D^2 f|_x\| \leq K \quad for \quad \|x - x_0\| \leq R \, ,$$

let

$$R_1 = \min\{\frac{1}{2}K^{-1}\|Df|_{x_0}^{-1}\|^{-1}, R\}$$
$$R_2 = \min\{R_1^{-1}, \frac{1}{2}\|Df|_{x_0}^{-1}\|^{-1}(\|Df|_{x_0}\| + KR_1)^{-1})^{-1}\} \quad and$$
$$R_3 = \frac{1}{2}R_2\|Df|_{x_0}\|$$

then f maps $D_{R_2}(x_0)$ diffeomorphically to a subset of F containing $D_{R_3}(f(x_0))$.

If A_X, ψ_X is an irreducible, nondegenerate solution to the Seiberg-Witten equations on $X \cup [0, \infty) \times M$, approaching A_∞, ψ_∞, and A_Y, ψ_Y is a good solution on $Y \cup (-\infty, 0] \times M$ approaching A_∞, ψ_∞. Then the decay estimate will prove that there is an approximate solution on $X \cup [0, T] \times M \cup Y$ given by A_X, ψ_X on $X \cup [0, \frac{1}{2}T] \times M$ and A_Y, ψ_Y on $[\frac{1}{2}T, T] \times M \cup Y$. The implicit function theorem will then show that there is a nearby solution on $X \cup [0, T] \times M \cup Y \equiv Z$.

For many applications the case when A_X, ψ_X is irreducible and nondegenerate, and A_Y, ψ_Y is reducible and degenerate is more useful. In particular, assume that there is only one solution to the Seiberg-Witten equations on $\mathbb{R} \times M$ and assume that this solution is reducible, say $(A_M, 0)$. This happens when M has a metric with positive scalar curvature. Also assume that $Y \cup ((-\infty, 0] \times M)$ has exactly one Seiberg-Witten solution with boundary value $(A_M, 0)$ and assume that this solution is reducible, say $(A_Y, 0)$. If $b_+^2(Y) \geq 1$ this will not happen for generic metrics. The most interesting case therefore is when $b_+^2(Y) = 0$. Let the moduli space of Seiberg-Witten solutions on $X \cup ([0, \infty) \times M)$ with boundary value $(A_M, 0)$ be $\mathcal{M}_X(A_M, 0)$. We will assume that $\mathcal{M}_X(A_M, 0)$ consists entirely of nondegenerate irreducible solutions.

With the above assumptions one would generically find that $\dim \mathcal{M}_X(A_M, 0) > \dim \mathcal{M}_Z$, so that gluing a solution from X to the solution on Y will not always produce a solution on Z. Looking at this situation in more detail, the Seiberg-Witten equations together with a global gauge-fixing condition define a map:

$$F : \Gamma(\Lambda^1 Z \oplus W_+) \to \Gamma(\Lambda^0 Z \oplus \Lambda^2_+ Z \oplus W_-)$$

and S^1 acts on $\Gamma(\Lambda^1 Z \oplus W_+)$ as constant gauge transformations. The Seiberg-Witten moduli space over Z is $\mathcal{M}_Z = F^{-1}(0)/S^1$. Pick T large and $\varepsilon > 0$ small once and for all and define the space of approximate solutions to be $F^{-1}(D_\varepsilon(0))/S^1$. There is a map

$$\varphi : \mathcal{M}_X(A_M, 0) \to F^{-1}(D_\varepsilon(0))/S^1$$

sending A, ψ to an element which is A, ψ when restricted to $X \cup [0, T-1] \times M$ and $(A_Y, 0)$ on Y. This map is well defined because of the decay estimates. If CoKer $T_{(A,\psi)}F = 0$ for all (A, ψ) in $F^{-1}(D_\varepsilon(0))/S^1$ then the implicit function theorem would imply that there is a map from $F^{-1}(D_\varepsilon(0))/S^1$ to $\mathcal{M}_Z$. Unfortunately, CoKer$T_{(A,\psi)}F \neq 0$. This allows us to define an S^1-equivariant vector bundle over $F^{-1}(D_\varepsilon(0))$ with fiber CoKer$T_{(A,\psi)}F$. We will denote this vector bundle by CoKer TF and the induced bundle over $F^{-1}(D_\varepsilon(0))/S^1$ by CoKer TF/S^1. The pull-back construction produces a bundle φ^* CoKer TF/S^1 over $\mathcal{M}_X(A_M, 0)$.

We may fix the problem with the implicit function theorem for a price. Pick a map, σ : CoKer $TF \to \Gamma(\Lambda^0 \oplus \Lambda^2_+ Z \oplus W_-)$ so that $TF \oplus \sigma$ is a surjection. This will allow us to study $(F \oplus \sigma)^{-1}(0)$. There is a natural inclusion, $F^{-1}(D_\varepsilon(0))/S^1 \to (F \oplus \sigma)^{-1}(D_\varepsilon(0))/S^1$. By the implicit function theorem, there is a map taking almost $F \oplus \sigma$ solutions to nearby $F \oplus \sigma$ solutions, say

$$\pi : (F \oplus \sigma)^{-1}(D_\varepsilon(0))/S^1 \to (F \oplus \sigma)^{-1}(0)/S^1 \ .$$

To see what this says about the original problem, let $p_1 : (F \oplus \sigma)^{-1}(0) \to \Gamma(\Lambda^1 Z \oplus W_+)$ and $p_2 : (F \oplus \sigma)^{-1}(0) \to$ CoKer TF be projections. In this case, $p_2 \circ \pi \circ i$ is a section of CoKer TF/S^1 which induces a section $\psi : \mathcal{M}_X(A_M, 0) \to \varphi^*$ CoKer TF/S^1. A moment's thought shows that $p_1 \circ \pi \circ i \circ \phi$ takes $\psi^{-1}(0)$ into $\mathcal{M}_Z$. In fact this map shows that $\psi^{-1}(0)$ is diffeomorphic to $\mathcal{M}_Z$.

The zeros of a section of a vector bundle are Poincaré dual to the Euler class of the bundle. There is a natural principal S^1 bundle over $\mathcal{M}_X(A_M, 0)$ given by framed solutions, $\hat{\mathcal{M}}_X(A_M, 0) = F_X^{-1}(0) \to \mathcal{M}_X(A_M, 0)$. The bundle φ^* CoKer TF/S^1 is associated to the bundle of framed solutions. We may easily compute the fiber dimension of

ϕ^* CoKer TF/S^1. Namely,

$$\begin{aligned} d_z &\equiv \dim \mathcal{M}_Z \\ &= \text{Index } TF_Z - 1 \\ &= \text{Index } TF_X + \text{Index } TF_Y - 1 \\ &= \dim \mathcal{M}_X(A_M, 0) + \dim \text{Ker } TF_Y - \dim \text{CoKer } TF_Y \ . \end{aligned}$$

So $\dim$ CoKer $TF_Y = d_x - d_z$.

The base point fibration is defined to be $\beta = \hat{\mathcal{M}}_X(A_M, 0) \times_{S^1} \mathbb{R}^2$. A bit of work will show that

$$\varphi^* \text{ CoKer } TF/S^1 \cong \bigoplus_1^{\frac{1}{2}(d_x - d_z)} \beta \ .$$

We are now in a position to give Fintushel and Stern's proof of the rational blow-down formula. Recall that a mu class is defined by $\mu = e(\beta)$ and that the Seiberg-Witten invariant is given by $n_Z = \mu^{d_Z/2} \cap [\mathcal{M}_Z]$. The first step is to prove that with the bundles specified as in the theorem there is only one solution on $C(p)$ and only one solution on $R(p)$ and the solutions are reducible. Then

$$\begin{aligned} n_{X \cup C(p)} &= \mu^{d_{X \cup C}/2} \cap [\mathcal{M}_{X \cup C}] \\ &= \mu^{d_{X \cup C}/2} \cap \mu^{\frac{1}{2}(d_X - d_{X \cup C})} \cap [\mathcal{M}_X(A, 0)] \\ &= \mu^{d_X/2} \cap [\mathcal{M}_X(A, 0)] \\ &= \mu^{d_{X \cup R}/2} \cap [\mathcal{M}_{X \cup R}] = n_{X \cup R(p)} \ . \end{aligned}$$

Lecture 3

In this lecture we will look at part of the gluing process in detail. There is a standard method for organizing the details of a general gluing theorem. This method is known as Floer theory. In the proof of the rational blow-down formula in the last lecture it was important to notice that there was only one solution to the Seiberg-Witten equations on $\mathbb{R} \times L(p^n, p-1)$. For a general 3-manifold, M, the Seiberg-Witten equations will have many solutions over $\mathbb{R} \times M$. The Floer homology is a natural way to organize all of the solutions over $\mathbb{R} \times M$. One other important ingredient in the proof of the rational blow-down formula was the action of the constant gauge transformations. We will include this group action into our discussion by studying equivariant Floer homology as pioneered by Austin and Braam. Even though we will review the definition of equivariant Floer homology, we recommend Austin and Braam's original article [AB]. M. Marcolli and B. Wang have worked out the equivariant version of Seiberg-Witten-Floer theory [MW].

Equivariant Floer theory starts with some standard pieces of data. Let G be a compact Lie group, which acts on a manifold X. Let $f : X \to S^1$ be G equivariant with a set of critical points R. Given a continuous indexing function, $\mu : \mathbb{R} \to \mathbb{Z}$, define $R_k = \mu^{-1}(k)$. The space of flows is defined to be:

$$\bar{\mathcal{M}}(R_n, R_m) \equiv \{\phi : \mathbb{R} \to X \mid \dot{\phi} = -\text{grad } f\,|_\phi, \phi(-\infty) \in R_n, \phi(\infty) \in R_m\}$$

The indexing function should be compatible with the space of flows in the sense that $\dim \bar{\mathcal{M}}(R_n, R_m) = n - m + \dim R_n$. The real numbers act on $\bar{\mathcal{M}}$ by $(t \cdot \phi)(s) = \phi(s+t)$. Denote the quotient by $\mathcal{M}(R_n, R_m) = \bar{\mathcal{M}}(R_n, R_m)/\mathbb{R}$. We call the maps

$$u^{n-m} : \mathcal{M}(R_n, R_m) \to R_n;\ \phi \mapsto \phi(-\infty)$$
$$\ell_{n-m} : \mathcal{M}(R_n, R_m) \to R_m;\ \phi \mapsto \phi(\infty)\ .$$

The upper and lower end point maps respectively. Under reasonable assumptions, the maps u and ℓ are fibrations. Denote the Lie algebra of G by $\}$ and its dual by $\}^*$. We will also need orientations on the critical submanifolds, and the flow spaces.

With the above notation, let

$$\Omega^n(R_k)^G \equiv \bigoplus_{p+2q=n} \Gamma(\Lambda^p R_k \otimes \text{Sym}^q \}^*)^G$$

be the complex of G-invariant forms where

$$(\alpha \cdot g)(X_1, \ldots, X_p, A_1, \ldots, A_q) = \alpha(g_* X_1, \ldots g_* X_p, g A_1 g^{-1}, \ldots, g A_q g^{-1})\ .$$

Let $d_G : \Omega^*(R_k)^G \overset{\text{deg } 1}{\longrightarrow} \Omega^*(R_k)^G$;

$$\begin{aligned} &d_G\alpha(X_1,\ldots,X_p,A_1,\ldots,A_q) \\ &\quad \equiv \sum_k (-1)^{k+1} X_k(\alpha(X_1,\ldots,\hat{X}_k,\ldots,X_p,A_1,\ldots,A_q)) \\ &\quad + \sum_{n<m} (-1)^{n+m} \alpha([X_n,X_m],X_1,\ldots,\hat{X}_n,\ldots,\hat{X}_m,\ldots,X_p,A_1,\ldots,A_q) \\ &\quad - \sum_k \alpha(A_k^\dagger, X_1,\ldots,X_p,A_1,\ldots,\hat{A}_k,\ldots,A_q) \; . \end{aligned}$$

Here $A_k^\dagger$ is the vector field $\frac{d}{dt} x \cdot \exp(tA_k)|_{t=0}$. Finally, define $C^k(X,f) = \oplus_{n+m=k} \Omega^n(R_m)^G$ and $\delta : C^*(X,f) \overset{\text{deg } 1}{\longrightarrow} C^*(X,f)$;

$$\delta\alpha = d_G\alpha + \sum_{n>0} (-1)^{\text{form deg } \alpha} u_*^n \ell_n^* \alpha \; .$$

The homology of the complex $(C^*(X,f),\delta)$ is the equivariant Floer homology of X, f.

In order to get a feel for what the equivariant Floer homology measures, we will look at several special cases. When $G = 1$, f is a constant map, and the indexing function is zero, $R_0 = X$ and the equivariant Floer complex reduces to the ordinary DeRham complex. When $G = 1$ and f is a Morse function, the critical points are isolated. Let H_p be the Hessian of the critical point p and define the index of p to be $\mu(p) = \frac{1}{2}(\text{rank } H_p - \text{Sign } H_p)$. In this case, the equivariant Floer complex reduces to the usual Morse complex. If $G = 1$ and f is non-constant and not Morse, then we are in the so-called Bott-Morse case. The equivariant Floer homology still computes the homology of X from the homology of the critical points of f. A good specific example is $X = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$, $f : X \to S^1$; $f(x,y,z) = \exp(2\pi i z^2)$. At the other extreme, when X is a point and G is arbitrary, the equivariant Floer homology of (X, f) is just the ring of G-invariant polynomials. If H is a closed subgroup of G and $X = G/H$, then the equivariant Floer homology of X is just the ring of H-invariant polynomials. Floer's important insight was that this homology is well defined when X is infinite dimensional, even if the Hessian has infinitely many negative modes. The key is to define an index or even a relative index. Cohen, Jones and Segal are working on a homotopy theoretic interpretation of this semi-infinite homology [CJS].

Returning to the Seiberg-Witten equations, let $\xi : P \times_{\text{Spin}_c(3)} sp_1 \to T^*M$ be a Spin$_c$-structure on a 3-manifold. Call the bundle of spinnors W, the associated line bundle L, and the space of connections on L, $\mathcal{A}_L$ [A2]. Pick a base point $x_0 \in M$ and a pair $(A_L, \psi_L) \in \mathcal{A}_L \times \Gamma(W)$. We can now describe the relevant configuration space:

$$X = S^1 \times \mathcal{A}_L \times \Gamma(W)/\sim$$

where $(\lambda, A, \psi) \sim (\lambda g(x_0), A - 2g^{-1}dg, \psi g^{-1})$ for any $g : M \to S^1$. The Lie group S^1 acts on X by multiplication in the first factor. Define a function $F : X \to S^1$ by

$$F_\eta(\lambda, A, \psi) = \exp(\frac{i}{\pi} \int_M \tfrac{1}{2}(A - A_L) \wedge (F_A - 2i\eta) - *(\partial_A \psi, \psi)) .$$

Here η is any divergence-free 1-form on M. In [A1] we showed that F is well defined and computed,

$$\text{grad } F|_{\lambda,A,\psi} = \begin{bmatrix} 0 \\ *F_A - 2i[\psi, \bar{\psi}] - i\eta \\ \partial_A \psi \end{bmatrix} .$$

Following the definition of equivariant Floer homology, we will first look at the space of critical points of F. This space, $R = \{(\lambda, A, \psi) \mid \text{grad } F|_{\lambda,A,\psi} = 0\}$ is sometimes called the framed Seiberg-Witten character variety. We studied this space in detail in [A2]. Here we will use a different linear model of this space which is more natural. The results in [A2] easily transfer to this slightly different set-up.

The first step is to study the linearization of R. Let $\mathcal{G} = \text{Maps}(M, S^1)$. Since X is a quotient space, the tangent space to X at (λ, A, ψ) will be the quotient

$$\mathbb{R} \times \Lambda(\Gamma^1 M \oplus W)/(\text{Im}(dL_{(\lambda,A,\psi)} : T_e\mathcal{G} \to \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W))) .$$

This is just CoKer L_0, where $L_0 = dL_{(\lambda,A,\psi)}$. After identifying $T_e\mathcal{G}$ with $\Gamma(\Lambda^0 M)$, we see that

$$\begin{aligned} L_0(u) &= \frac{d}{dt}(\lambda e^{iu(x_0)t}, A - 2e^{-iut}de^{iut}, \psi e^{-iut})|_{t=0} \\ &= (u(x_0), -2du, -\psi iu) . \end{aligned}$$

The last step might look a bit strange because of the identification of $T_\lambda S^1 \times T_A \mathcal{A}_L$ with $\mathbb{R} \times \Lambda(\Gamma^1 M)$. It is useful to compute the formal adjoint of L_0 because CoKer L_0 = Ker L_0^*. A short computation shows that

$$\begin{aligned} &L_0^* : \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W) \to \Gamma(\Lambda^0 M) \\ &L_0^*(t, a, \phi) = t\delta_{x_0} - 2d^*a + 2\langle \psi i, \phi \rangle . \end{aligned}$$

We may proceed in the same way to compute the linearization of grad F. Let $L_1 : \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W) \to \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W)$,

$$\begin{aligned} L_1(s, a, \phi) &= \frac{d}{dt}(0, *F_A + i * dat - 2i[\psi + t\phi, \overline{\psi + t\phi}] - i\eta, \partial_{A+ita}(\psi + i\phi)) \\ &= (0, *da - 4[\phi, \bar{\psi}], \partial_A \phi - \tfrac{1}{2}c(a, \psi)) . \end{aligned}$$

Putting this all together, we may conclude that R is locally diffeomorphic to the first cohomology of the complex:

$$0 \to \Lambda(\Gamma^0 M) \xrightarrow{L_0} \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W) \xrightarrow{L_1} \mathbb{R} \times \Lambda(\Gamma^1 M \oplus W) \xrightarrow{L_2} \Lambda(\Gamma^0 M) \to 0$$

where $L_2 = L_0^*$. This conclusion follows from the implicit function theorem, provided that $H^2(L_*) = 0$.

Due to the singular nature of the operators, L_0 and L_2, it is easier to study the situation without the S^1 factor in X, and later modify the answer to include the S^1 factor using ad hoc arguments. In [A2], we showed that $H^0(L_*) \oplus H^2(L_*) = 0$ for generic choices of Spin$_c$-structure and perturbation η as long as $H^1(M) \neq 0$. This leaves the case when $H^1(M) = 0$. In this situation reducible solutions may not be avoided, so we have to analyze the linear model at a solution with $\psi \equiv 0$. Still disregarding the S^1 factor, we have

$$H^2(L^*) = \text{CoKer } \partial_A \oplus \frac{\text{Ker}(d^* : \Omega^1(M) \to \Omega^0(M))}{\text{Im}(*d : \Omega^1(M) \to \Omega^0(M))}\ .$$

The diagrams

$$\begin{array}{ccc} \Omega^2(M) & \xrightarrow{d} & \Omega^3(M) \\ \uparrow\wr & & \uparrow\wr \\ \Omega^1(M) & \xrightarrow[d^*]{} & \Omega^0(M) \end{array}\ , \qquad \begin{array}{ccc} \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) \\ \uparrow\wr & & \uparrow\wr \\ \Omega^1(M) & \xrightarrow[*d]{} & \Omega^1(M) \end{array}\ ,$$

show that the second factor is $H^2(M) = H^1(M) = 0$. On the trivial line bundle, any solution with $\psi \equiv 0$ would have $\mathcal{A}$ trivial so that $H^2(L_*) = \text{CoKer } \partial = 0$. For generic Spin$_c$-structures, CoKer $\partial = 0$. We will look at this in more detail when we study the relationship between the Spin$_c$-structure and the equivariant Floer homology.

The geometric meaning of the gauge fixing condition, $L_0^* = 0$, is a slice that is perpendicular to the action of the gauge group. The extra S^1 in the domain of F_η is included to make the gauge group act freely. One useful idea is to replace the gauge fixing condition, $L_0^* = 0$, with $d^*(A - A_0)$. This is the condition that we used in lecture 2. It is also used by Furuta in the proof of the 10/8 ths theorem. Even though this second gauge fixing condition does not have a natural geometric interpretation, it is often better. One advantage is that the operator is non-singular. The other advantage is that the second gauge fixing condition is invariant under constant gauge transformations, so the extra S^1 does not have to be put in by hand.

The upshot of the above discussion is that the space, R, is a finite disjoint union of circles with possibly one extra point. At least for generic perturbation and Spin$_c$-structure.

The space of flows from (λ_1, A_1, ψ_1) to (λ_2, A_2, ψ_2) is just the Seiberg-Witten moduli space with boundary values $(A_1, \psi_1), (A_2, \psi_2)$ provided that $\lambda_1 = \lambda_2$ or $\psi_1 \equiv 0$ or $\psi_2 \equiv 0$. In order to define an index, collapse the deformation complex of R to

$$D_{A,\psi} : \Lambda(\Gamma^0 M \oplus \Lambda^1 M \oplus W) \to \Lambda(\Gamma^0 M \oplus \Lambda^1 M \oplus W)$$

$$D_{A,\psi} = L_0^* \oplus L_1 \oplus L_0 = \begin{bmatrix} 0 & -2d^* & 2\langle \psi i, - \rangle \\ -2d & *d & -4[-, \bar{\psi}] \\ -\psi i & -\frac{1}{2}c(a, -) & \partial_A \end{bmatrix}$$

The operator $D_{A,\psi}$ has a discrete real spectrum. Let $\delta_{A,\psi} = \frac{1}{2} \inf\{|\lambda| \mid \lambda \in \text{Spec } D_{A,\psi} - \{0\}\}$. If A_t, ψ_t is a path, define the spectral flow, $SF(D_{A_0,\psi_0}, D_{A_1,\psi_1})$, to be the number of eigenvalues which change from negative to positive minus the number which change from positive to negative. In truth, the spectral flow counts the number of eigenvalues that cross a line from $-\delta_{A_0,\psi_0}$ to δ_{A_1,ψ_1}. This convention assures that the spectral flow is the dimension of the space of flows. We define the index to be

$$\mu(\lambda, A, \psi) = SF(D_{A_*,\psi_*}, D_{A,\psi}) \ .$$

The equivariant Floer homology defined with the above input is a topological invariant when $\dim H^1(M) > 1$. This is very similar to the four-dimensional situation. When $b_+^2 > 1$, there are diffeomorphism invariants, but when $b_+^2 = 1$ the invariants depend on a choice of chamber. Equivariant Floer theory as defined above is called Down theory. When the configuration space X is finite dimensional the homology of the complex defined with a function f is isomorphic to the cohomology of the complex defined with the function $-f$. The cohomology computed from the $-f$ complex is not the same when X is infinite dimensional, so call the $-f$ homology the up theory. A third Floer theory may be defined by cancelling the S^1 factor and disregarding the reducibles in the configuration space. This is called irreducible theory. It is possible to combine the three different theories into one theory which is a topological invariant independent of $H^1(M)$. This is current research of Froyshov, and Kronheimer-Mrowka. By including a spectral-flow counter-term, Y. Lim was able to define an invariant in the case when $H^1(M) = 0$ and show that his invariant is equivalent to Casson's invariant [L].

We will look at the down theory in greater detail. To prove that the equivariant Floer complex is a complex, and to prove that the homology is an invariant, it is necessary to have a gluing theorem. The basic gluing theorem is:

Theorem. *For T sufficiently large there is a proper embedding*

$$G : \mathcal{M}(R_a, R_b) \times_{R_b} \mathcal{M}(R_b, R_c) \times [T, \infty) \to \mathcal{M}(R_a, R_c)$$

onto an end of $\mathcal{M}(R_a, R_c)$*. Moreover, every end of* $\mathcal{M}(R_a, R_c)$ *is of this form.*

Since $\mathcal{M}(R_a, R_b) = \bar{\mathcal{M}}(R_a, R_b)/\mathbb{R}$, it will be helpful to have a preferred representative in each equivalence class. Recall that for $(\lambda, A, \psi) \in \mathcal{M}(R_a, R_b)$,

$$E(\lambda, A, \psi) = \int_{-\infty}^{\infty}\int_M |\nabla_{\underline{A}}\psi|^2 + \frac{s}{4}|\psi|^2 + \tfrac{1}{8}|\psi|^4 + \tfrac{1}{2}|F_A|^2 d\mathrm{vol}_M \; dt \; .$$

In the proof of the exponential decay we showed that

$$E(\lambda, A, \psi) = \int_M \tfrac{1}{2}(A_\infty - A_\infty) \wedge F_{A_{-\infty}} - *(\partial_{A_{-\infty}}\psi_{-\infty}, \psi_{-\infty}) \; .$$

In particular the energy only depends on the boundary values, allowing us to denote it by $E(a, b)$. Let $\lambda = \frac{1}{2}\min_{a\neq b} E(a, b)$, and define a slice, $s : \mathcal{M}(R_a, R_b) \to \bar{\mathcal{M}}(R_a, R_b); s(x) = (A, \psi, p)$ by the condition

$$\int_{-\infty}^{0}\int_M |\nabla_{\underline{A}}\psi|^2 + \frac{s}{4}|\psi|^2 + \tfrac{1}{8}|\psi|^4 + \tfrac{1}{2}|F_A|^2 \, d\mathrm{vol}_M \, dt = \lambda \; .$$

As long as R_b is irreducible, $\mathcal{M}(R_a, R_b) \times_{R_b} \mathcal{M}(R_b, R_c)$ consists of pairs (x_1, x_2) where $s(x_1) = (p_1, A_1, \psi_1)$, $s(x_2) = (p_2, A_2, \psi_2)$ and $p_1 = p_2$. Let ρ be a smooth cut-off function with

$$\rho(t) = \begin{cases} 1 & \text{if } t < -1 \\ 0 & \text{if } t > 1 \; . \end{cases}$$

Given s, define

$$A^s(t) = \rho(t)A_1(t+s) + (1-\rho(t))A_2(t-s) \; ,$$
$$\psi^s(t) = \rho(t)\psi_1(t+s) + (1-\rho(t))\psi_2(t-s) \; .$$

The pair (A^s, ψ^s) is an approximate solution to the Seiberg-Witten equations built with (A_1, ψ_1) on the left side and (A_2, ψ_2) on the right side separated by distance $2s$.

We will look for an element in $\mathcal{M}(R_a, R_c)$ of the form $(p, A, \psi) = (p, A^s + a, \psi^s + \phi)$. The flow equations,

$$\frac{\partial A}{\partial t} = 2i[\psi, \bar{\psi}] - *F_A \; , \qquad \frac{\partial \psi}{\partial t} = -p_A\psi$$

become

$$\mathcal{L}_{(A^s,\psi^s)}\begin{bmatrix} a \\ \phi \end{bmatrix} - \begin{bmatrix} 2i[\phi, \bar{\phi}] \\ \frac{1}{2}c(a, \phi) \end{bmatrix} = \begin{bmatrix} 2i[\psi^s, \bar{\psi}^s] - *F_{A^s} - \frac{\partial A^s}{\partial t} \\ -\partial_{A^s}\psi^s - \frac{\partial \psi^s}{\partial t} \end{bmatrix} \; ,$$

after substitution. Here,

$$\mathcal{L}_{(A^s,\psi^s)} = \begin{bmatrix} \frac{\partial}{\partial t} + *d & -4i[\;,\bar{\psi}^s] \\ -\frac{1}{2}c(\;,\psi^s) & \frac{\partial}{\partial t} + \partial_{A_s} \end{bmatrix} \; .$$

In order to solve the flow equations we will find a right inverse to the operator $\mathcal{L}_{(A^s,\psi^s)}$. As a nondegeneracy condition we will assume that there are bounded operators, P_1 and P_2, so that

$$\mathcal{L}_{(A_1,\psi_1)}P_1x = x \quad \text{and} \quad \mathcal{L}_{(A_2,\psi_2)}P_2x = x \ .$$

An approximate right inverse to $\mathcal{L}_{(A^s,\psi^s)}$ is given by

$$(Qx)(t) = \rho(Kt-2)P_1(\rho(t)x(t+s)) + \rho(-Kt-2)P_2((1-\rho(t))x(t-s)) \ .$$

Now define

$$P = \sum_{n=0}^{\infty} Q(I - \mathcal{L}_{(A^s,\psi^s)}Q)^n \ .$$

For K large depending only on the Spin$_c$-structure, $\|I - \mathcal{L}_{(A^s,\psi^s)}Q\|$ will be small so that P is well defined. Finally, let $Px = \begin{bmatrix} R_1x \\ R_2x \end{bmatrix}$ and

$$f(x) \equiv x - \begin{bmatrix} 2i[R_2x, \overline{R_2x}] \\ \frac{1}{2}c(R_1x, R_2x) \end{bmatrix} .$$

The map $Df|_0 = I$ is clearly an isomorphism. The a priori bounds on solutions to the Seiberg Witten equations show that $D^2f|_x$ is bounded in a large ball. The term

$$\begin{bmatrix} 2i[\psi^s, \bar{\psi}^s] - *F_{A^s} - \frac{\partial A^s}{\partial t} \\ -\partial_{A^s}\psi^s - \frac{\partial \psi^s}{\partial t} \end{bmatrix}$$

is close to zero when s is large because (A_1, ψ_1) and (A_2, ψ_2) converge exponentially to a common boundary value. The implicit function theorem implies that there is a unique small x_s so that

$$f(x_s) = \begin{bmatrix} 2i[\psi^s, \bar{\psi}^s] - *F_{A^s} - \frac{\partial A^s}{\partial t} \\ -\partial_{A^s}\psi^s - \frac{\partial \psi^s}{\partial t} \end{bmatrix} .$$

The gluing map is defined by

$$G((p, A_1, \psi_1), (p, A_2, \psi_2), s) = (p, A^s + R_1x_s, \psi^s + R_2x_s) \ .$$

To prove that all ends are accounted for, look at a Cauchy sequence in $\mathcal{M}(R_a, R_c)$, say x_k. Let $s(x_k) = (A_k, \psi_k, p_k)$. By the compactness theorem, (A_k, ψ_k) converges to a Seiberg-Witten solution, (A_-, ψ_-, p_-). The solution (A_-, ϕ_-, p_-) is not necessarily in $\mathcal{M}(R_a, R_c)$, but it will be in $\mathcal{M}(R_a, R_x)$ for some x. To find the other half, rescale the slice by requiring the energy up to 0 to be $E(a, x) + \lambda$. This will produce a solution (A_+, ψ_+, p_+) in $\mathcal{M}(R_x, R_y)$. In the best case, $y = c$, and $\|x_k - G((A_-, \psi_-, p_-), (A_+, \psi_+, p_+), k)\| \to 0$

as $k \to \infty$. The implicit function theorem will show that the x_k are in the image of the gluing map when k is large. If we are not in the best case, we change the energy at $t = 0$ and repeat the argument to get a new segment. This process will stop because the moduli space has finite energy.

We are now ready to work out a few examples. As a first example, assume that the metric induced from the Spin$_c$ structure has strictly positive scalar curvature. The a priori bound implies that $\psi \equiv 0$ for small perturbations.

Lemma (A priori bound). *If $(A, \psi, p) \in R)$ then* $|\psi|^2 \leq \max\{0, 2|\eta| - s\}$.

In this case, the equations for the framed Seiberg-Witten character variety simplify to

$$*F_A = i\eta \ .$$

Chern-Weil theory implies that $c_1(L) = -\frac{1}{2\pi i} F_A = -\frac{1}{2\pi}[*\eta]$. This condition fails to hold for generic perturbations if $b_1(M) \neq 0$. It follows that $R = \emptyset$ for generic perturbations when $b_1 \neq 0$. This proves the following:

Theorem. *If the Spin$_c$ structure on M has positive scalar curvature and $b_1(M) \neq 0$, then*

$$SWF^*(M, L) = 0 \ .$$

Corollary. *If X^4 has a nontrivial Seiberg-Witten invariant, then no 3-manifold which admits a metric of positive scalar curvature with* $\mathrm{Im}(H_2(M) \to H_2^+(M)) \neq 0$ *embeds into X.*

We should make some remarks about this corollary at this point. The basic reason that this result is true is that any 3-manifold which embeds into a 4-manifold with a non-trivial Seiberg-Witten invariant must have some connection and spinnor that solve the three dimensional Seiberg-Witten equations. The previous theorem states that any 3-manifold with positive scaler curvature and $b_1 \geq 1$ does not admit any solutions to the *generically perturbed* Seiberg-Witten equations. This is why we must assume that the embedding is homologically non-trivial. In fact, $S^1 \times S^2$ embeds into any 4-manifold as the boundary of a regular neighborhood of a trivial S^2. With this in mind, we will make a short aside to review the neck-stretching argument while paying attention to the perturbation.

The neck-stretching argument in [KM] and [A1] is proved by studying the energy functional from lecture 2. With the perturbation this functional is:

$$E(A, \psi) = \int_x |\nabla_A \psi|^2 + \frac{s}{4}|\psi|^2 + 2|[\psi, \bar{\psi}] + \eta^+|^2 + \frac{1}{2}|F_A|^2 dvol.$$

As in lecture 2, this may be rewritten,

$$E(A,\psi) = \int_X |\not{\partial}_A\psi|^2 + |F_A^+ - i[\psi,\bar{\psi}] - i\eta^+|^2 dvol$$
$$+ 2\pi^2 c_1(L)\cap[X] - 4\pi\int_X (-\frac{1}{2\pi i}F_A\wedge\eta^+ .$$

By Hodge theory, Chern-Weil theory and the Bianchi identity, $-\frac{1}{2\pi i}F_A = c_1(L) + d\alpha$. Integration by parts will show that this last integral is a topological invariant, $-4\pi(c_1(L)\cup[\eta^+])\cap[X]$, provided, $d\eta^+ = 0$. Careful inspection of the transversality arguments shows that harmonic perturbations are sufficiently generic to force a regular moduli space when the metric is varied. Since the Seiberg-Witten invariant of X is non-trivial, there is a solution to the Seiberg-Witten equations with any metric. In particular, pick a metric so that $[0,T]\times M$ embeds with T large. Since there is a topological bound on the energy, and the energy in a segment $[a,b]\times M$ is $J_{\{b\}\times M} - J_{\{a\}\times M}$, where $J_{\{t\}\times M}$ is a monotone function, it follows that there is an interval, $[a,a+1]$, so that $J_{\{a+1\}\times M} - J_{\{a\}\times M}$ is small. This in turn implies that there is a solution to the three dimensional Seiberg-Witten equations on M:

$$*F_A = 2i[\psi,\bar{\psi}] + 2i\rho$$
$$\not{\partial}_A\psi = 0.$$

The perturbations are related by:

$$\eta^+|_{N(M)} = dt\wedge\rho + *\rho \quad \text{or} \quad \rho = i_{\frac{\partial}{\partial t}}\eta^+.$$

The condition, $d\eta^+ = 0$ insures that ρ is divergence free and independent of t. The only way to be sure that the solution on M is not reducible ($\psi\equiv 0$) is to assume that $c_1(L)\neq -\frac{1}{\pi}[*\rho]$. In other words, we assume that there is an $[F]\in H_2(M)$ so that,

$$c_1(L)\cap[F] \neq -\frac{1}{\pi}[*\rho]\cap[F] = -\frac{1}{\pi}[\eta^+]\cap[F].$$

On the other hand, if $b_1(M) = 0$ then the Chern-Weil condition will always hold. In fact we can see that the Seiberg-Witten character variety is exactly one point. Since $b_1(M) = 0$, we may write the connection A as ia. With this substitution, the Seiberg-Witten equations become $da = *\eta$. Recall that η is divergence free so that $d*\eta = 0$, i.e. $*\eta$ is closed. Every closed 2-form is exact because $H^2(M) = H^1(M) = 0$. This implies that there is an a that solves the equations. To see that this solution is unique, let a_1 and a_2 be two solutions, so that $d(a_1-a_2) = da_1 - da_2 = *\eta - *\eta = 0$, i.e., $a_1 - a_2$ is closed. The difference is therefore exact: $a_1 - a_2 = d\theta$, so $ia_2 = ia_1 - 2g^{-1}dg$ where $g = \exp(i\theta/2)$. Thus, $R = \{(\theta, 0, 1)\}$. Pick $\mu \equiv 0$. Then the equivariant Floer complex is:

$$\begin{aligned} C(X;F) &= \bigoplus_{n+m=*} \Omega^n(R_m)^{S_1} \\ &= \Omega(R_0)^{S_1} \\ &= \bigoplus_{p+2q=*} \Gamma(\Lambda^p R_0 \otimes \mathrm{Sym}^q \mathbb{R})^{S_1} \\ &= \begin{cases} \mathrm{Sym}^q \mathbb{R} & \text{if } 2q = * \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

This proves

Theorem. *If the Spin$_c$ structure on M has positive scalar curvature and $b_1(M) = 0$, then*

$$SWF^*(M, L) = \begin{cases} \mathbb{R} & \text{if } * = 2q \\ 0 & \text{otherwise} \end{cases} .$$

We can also compute the Seiberg-Witten Floer homology for Euclidean manifolds. As an example, let $\Delta'(2,3,6)$ be the group of isometries of $\mathbb{R}^2$ generated by reflections about the lines through the points (0,0), $(0, \frac{1}{2})$, $(\sqrt{3}/2, 0)$. Let $\Delta(2,3,6)$ be the orientation preserving subgroup, so that $\Delta(2,3,6)$ is generated by rotations with angles $\frac{2\pi}{2}$, $\frac{2\pi}{3}$, and $\frac{2\pi}{6}$ about the points indicated in Figure 33.

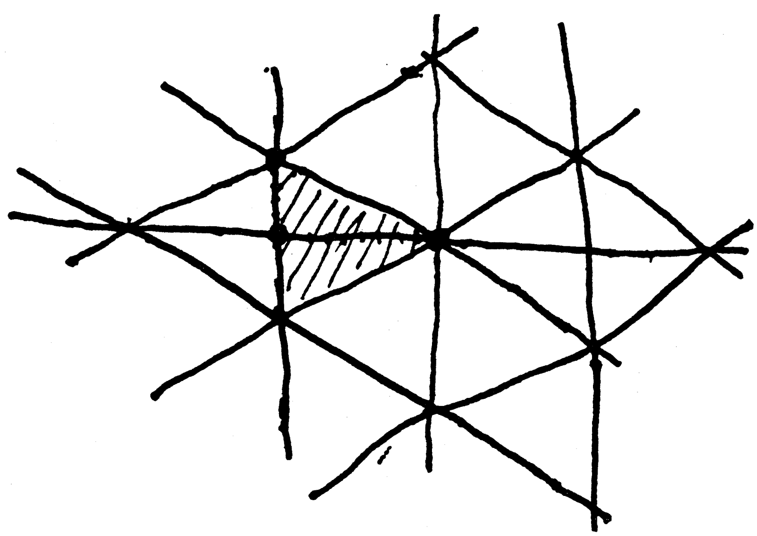

FIGURE 33: $\Delta(2,3,6)$

By working with Figure 33, it is possible to prove that $\Delta(2,3,6)$ has the presentation

$$\Delta(2,3,6) = \langle a,b,c \mid abc = a^2 = b^3 = c^6 = 1\rangle \ .$$

Identifying $\mathbb{R}^2$ with the affine subspace $\{(x,y,z) \in \mathbb{R}^3 \mid z = 1\}$, we may represent a, b and c as linear maps. The group $\Delta(2,3,6)$ also acts on $\mathbb{R}^1 \times S^1 \cong \{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \|u-v\| = 1\}$ in a natural way. The action on $\mathbb{R}^2 \times S^1$ is free, so the quotient is a smooth 3-manifold. The universal cover of this 3-manifold is the same as the universal cover of $\mathbb{R}^2 \times S^1$, namely, $\mathbb{R}^3$. The fundamental group of the 3-manifold is an extension, $\tilde{\Delta}(2,3,6)$, of $\Delta(2,3,6)$:

$$1 \to \mathbb{Z} \to \tilde{\Delta}(2,3,6) \to \Delta(2,3,6) \to 1 \ .$$

Denote the manifold $\mathbb{R}/\tilde{\Delta}(2,3,6)$ by $\Sigma(2,3,6)$. A bit of work will produce the presentation

$$\tilde{\Delta}(2,3,6) = \langle p,q,r \mid pqr = p^2 = q^3 = r^6\rangle \ .$$

Thinking of $\mathbb{R}^3$ as the affine subspace, $\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_4 = 1\}$, we may represent p, q and r by:

$$p = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$r = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

These matrices induce an action on $\mathbb{R}^2 \times S^1 = \mathbb{R}^3/\mathbb{Z} = \{(x_1,\ldots,x_4) \mid x_4 = 1\}/\sim$ where $(x_1,x_2,x_3,x_4) \sim (x_1,x_2,x_3+1,x_4)$. By considering the $\mathbb{R}^2$ factor we can recover the action of $\Delta(2,3,6)$ on $\mathbb{R}^2$.

The manifold $\Sigma(2,3,6)$ is a Seifert fiber space. The regular fiber represents the element pqr in $\tilde{\Delta}(2,3,6)$. The regular fiber may be seen as the set of points with a fixed, generic value of x_1 and x_2. There are three singular fibers corresponding to $(x_1,x_2) = (0,0)$, $(x_1,x_2) = (0,\frac{1}{2})$ and $(x_1,x_2) = (\frac{\sqrt{3}}{2},0)$.

We can define a one parameter family of Spin_c-structures on $\Sigma(2,3,6)$ by

$$\xi_a : \Sigma(2,3,6) \times \mathrm{Spin}_c(3)\tilde{\times} sp_1 \to T^*\Sigma(2,3,6) \ ;$$

$$\xi_a[(x_1,x_2,x_3,1,1), y_1 i + y_2 j + y_3 k] = y_1 dx_1 + y_2 dx_2 + a y_3 dx_3 \ .$$

The induced metric on $\Sigma(2,3,6)$ is: $g_a = dx_1^2 + dx_2^2 + a^2 dx_3^2$, which is always flat. The length of the regular fiber is a. These Spin$_c$-structures are flat, so we may not conclude that $\psi \equiv 0$ as in the positive scalar curvature case. However, the a priori bound will imply that ψ is small for small perturbations. This means that $c_1(L) = -\frac{1}{2\pi i}F_A = -\frac{1}{\pi} * [\psi, \bar{\psi}] - \frac{1}{2\pi} * \eta$ is small. Since $c_1(L)$ is an integral class, it must be zero. Thus there are no solutions on non-trivial bundles and

$$SWF^*(\Sigma(2,3,6), L) = 0 \quad \text{for } L \text{ non-trivial.}$$

On the trivial bundle there is a unique solution to the Seiberg-Witten equations.

Lemma. *There is a unique solution to the 3-dimensional Seiberg-Witten equations when the scalar curvature is zero and the perturbation is harmonic.*

PROOF. The pair (A, ψ) solves the 3D Seiberg-Witten equations if and only if $I_\eta(A,\psi) = 0$, where

$$I_\eta(A,\psi) = \int_M |*F_A - 2i[\psi,\bar{\psi}] - i\eta|^2 + 2|\partial_A \psi|^2 d\text{vol} .$$

We can rewrite $I_\eta(A,\psi)$ as:

$$\begin{aligned} I_\eta(A,\psi) = \int_M & |F_A|^2 + |2i[\psi,\bar{\psi}] - i\eta|^2 \\ & - 2\langle *F_A, 2i[\psi,\bar{\psi}]\rangle - 2\langle *F_A, i\eta\rangle \\ & + 2\langle \partial_A^* \partial_A \psi, \psi\rangle \, d\text{vol} . \end{aligned}$$

Using the Bochner-Weitzenboch formula,

$$\partial_A^* \partial_A \psi = \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi - \tfrac{1}{2} * F_A \cdot \psi$$

will simplify the integral. The $-\frac{1}{2} * F_A \cdot \psi$ will cancel the $-2\langle *F_A, 2i[\psi,\bar{\psi}]\rangle$ term and $s = 0$. Furthermore, we may decompose $F_A = (da + d^*\alpha + \omega)i$ where ω is harmonic. By the comments preceding this lemma, $\omega = 0$. By the Bianchi identity, $d^*\alpha = 0$. Thus,

$$\begin{aligned} \int_M \langle *F_A, i\eta\rangle \, d\text{vol} &= \int_M \langle *dai, i\eta\rangle \, d\text{vol} \\ &= \int_M \langle ai, d^* * i\eta\rangle \, d\text{vol} = 0 \end{aligned}$$

since η is harmonic. The final expression for $I_\eta(A,\psi)$ is

$$\begin{aligned} I_\eta(A,\psi) = \int_M & |F_A|^2 + |2i[\psi,\bar{\psi}] - i\eta|^2 \\ & + 2|\nabla_A \psi|^2 \, d\text{vol} . \end{aligned}$$

By inspection, $I_\eta(A,\psi) = I_{-\eta}(A,\psi j)$. It follows that (A,ψ) is a solution to the η-equations if and only if $(A,\psi j)$ is a solution to the $-\eta$ equations. This shows that $F_A \equiv 0$ for any solution.

On $\Sigma(2,3,6)$, for example, the only harmonic 1-forms are $cdx_3 = \eta$. It is an easy algebraic exercise to show that the only solutions to

$$2[\psi,\bar{\psi}] = -cdx_3$$

are $\psi = (c/a)^{\frac{1}{2}}(1+j)e^{i\theta}$, so up to gauge equivalence, we may take $\psi = (c/a)^{\frac{1}{2}}(1+j)$. The equation, $\partial_A\psi = 0$ now uniquely specifies A. □

This lemma proves that the framed Seiberg-Witten character variety is diffeomorphic to S^1 for the given Spin_c-structures. The equivariant Floer complex becomes

$$\begin{aligned} C^n &= \bigoplus_{p+2q+r=n} \Gamma(\Lambda^p R_r \otimes \mathrm{Sym}^q\mathbb{R})^{S^1} \\ &= \bigoplus_{p+2q=n} \Gamma(\Lambda^p S^1 \otimes \mathrm{Sym}^q\mathbb{R})^{S^1} . \end{aligned}$$

The only equivariant forms on S^1 are constants and constant multiples of $d\theta$. It follows that C^{2k+1} is spanned by $d\theta \otimes \left(\frac{\partial^*}{\partial\theta}\right)^k$ for $k \geq 0$ and C^{2k} is spanned by $1 \otimes \left(\frac{\partial^*}{\partial\theta}\right)^k$ for $k \geq 0$. In this basis, the complex becomes

$$0 \to \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{-1} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{-2} \mathbb{R} \xrightarrow{0} \dots .$$

Thus

$$SWF^*(\Sigma(2,3,6),\underline{\mathbb{C}}) = \begin{cases} \mathbb{R} & \text{if } * = 0 \\ 0 & \text{otherwise} . \end{cases}$$

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