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SOME TOPICS ON CARNOT-CARATHEODORY METRICS

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Some topics on Carnot-Caratheodory Metrics

-Contact structures-

Dedicated to the memory of my wife, Keiko

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PREFACE

Let M be a smooth manifold and let H be a subbundle of the tangent bundle TM (sometimes H is called a polarization). Vector fields in H are said to be H-horizontal or simply horizontal. A piecewise smooth curve in M is said tobe (H-) horizontal with respect to H if the tangent vectors to this curve are H-horizontal. The metric defined in terms of the horizontal curves in M is called Carnot-Caratheodory metric (C-C metric,or sub-Riemannian metric etc).

When H is bracket-generating, we may sometimes define a C-C metric. For examples, a contact structure satisfies that $TM = H + [H, H]$ where $[H, H] \simeq \mathbb{R}$, and has a sub-Riemannian metric on M.

In $1960'$ s S.Sasaki defined an (almost) contact metric structure and many geometers have studied this topics. The contact structure in this note is different from his one. And N.Tanaka (1975) studied the harmonic theory of CR-structures, which is similar to M.Rumin's one.

1996, I talked this topics in Santiago Compostrela and had a chance to write it in Lecture Note Series in Seoul National University. Recently, some papers and books about this topics are published and so I try to rewrite the new version with respect to view of

I.Kupka, Géométrie sous-riemannienne, Séminarire Bourbaki, 48 ème année, 1995-96, no.817,351-380

This note is dedicated to the memory of my wife, Keiko, who died suddenly on October 24,1997.

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Contents

§1.1 Carnot-Caratheodory metrics (Sub-Riemannian structures).

1.1 Definition. *A sub-Riemannian structure (or abbreviated to SR-structure) on a manifold M is a pair (E, g) where E is a distribution with the constant rank and g is a Riemannian metric on E.*

Remark 1.1. *Given (E, g), we can define a metric g^* on T^*M, i.e., a C^∞-function $g^* : T^*M \longrightarrow \mathbb{R}$ whose restriction to each fiber T^*_mM is a positive semi-definite form : for $z \in T^*_mM$,*

$$g^*(z) = \max\{\frac{< v, z >^2}{g(v)} \mid v \in E[m] \setminus \{0\}\},$$

where $< -, - >$ is the canonical pairing $TM \times_M T^*M \longrightarrow \mathbb{R}$. If we give g^*, we also determine E and g, i.e., E is the annihilator of the kernel of g^* and, for $v \in E[m]$,

$$g(v) = \min\{\lambda \in \mathbb{R} \mid < v, z >^2 \leq \lambda g^*(z) \; \forall \; z \in T^*_mM\}.$$

Remark 1.2. *By Remark 1.1, we can generalize the notion of SR-structure : We choose $g^* : T^*M \longrightarrow \mathbb{R}$, which is a positive semi-definite quadratic form for each fiber of T^*M. We suppose the following condition ; if E is the sheaf generated by vector fields V such that, for all m in the domain of V, all $z \in T^*_mM$ with $g^*(z) = 0$, $< V[m], z >= 0$, $\Gamma(E)$ satisfies the rank condition.*

g^* defines a structure (E, g) on M. E is a subset of TM, but, in general, not a subbundle as followings ; we set

$$E := \bigcup_{m \in M} E[m],$$

$$E[m] \subset T_mM, \; E[m] := \{v \mid < v, z >= 0 \; for \; z \in T^*_mM : \; g^*(z) = 0\}$$

and the function $g : E \longrightarrow \mathbb{R}$ by setting

$$g(v) := \min\{\lambda \in \mathbb{R} \mid < v, z >^2 \leq \lambda g^*(z) \; \forall \; z \in T^*_mM\}$$

if $v \in E[m]$.

1.2 Horizontal curves.

A horizontal curve is an absolutely continuous curve $\phi : I \longrightarrow M$, I interval, such that, for almost all $t \in I$, the tangent vector $\frac{d\phi}{dt}(t)$ belongs to $E[\phi(t)]$.

1.3 Length of a horizontal curve.

For a horizontal curve $\phi : I \longrightarrow M$, we define its length $L(\phi)$ by

$$L(\phi) := \int_I \sqrt{g\left(\frac{d\phi}{dt}(t)\right)} dt \leq \pm\infty$$

If I is compact, $L(\phi) < \infty$.

1.4 Energy of a horizontal curve.

We define the energy $E(\phi)$ of a horizontal curve $\phi : I \longrightarrow M$ by means of

$$E(\phi) := \frac{1}{2}\int_I g(\frac{d\phi}{dt}(t))dt \leq \pm\infty$$

Example 1.1.

Let $M := \mathbb{R}^2$ be the Euclidean plane, and let E be the set of piecewise linear curves where each segment is either vertical or horizontal. Then the corresponding distance between the points $v_1 := (x_1, y_1)$ and $v_2 := (x_2, y_2)$ is equal to $|x_1 - x_2| + |y_1 - y_2|$ where we use Euclidean metric of $\mathbb{R}^2$ for g.

Let M be the Euclidean 3-space $\mathbb{R}^3$, and let E be the standard contact subbundle which is the kernel of the (contact) 1-form $\eta := dz + xdy$ in $\mathbb{R}^3$. This means that the tangent space $E_{v_0} \subset T_{v_0}\mathbb{R}^3 = \mathbb{R}^3$ is given at each $v_0 := (x_0, y_0, y_0) \in \mathbb{R}^3$ by $z + x_0 y = 0$.

• E is generated by the following two independent vector fields $\partial_1 := \partial/\partial x$ and $\partial_2 := \partial/\partial y - x\partial/\partial z$. These field do not commute.

In fact,

$$\begin{aligned}[\partial_1, \partial_2] &= [\partial/\partial x, \partial/\partial y - x\partial/\partial z] \\ &= [\partial/\partial x, -x\partial/\partial z] \\ &= -\partial/\partial z\end{aligned}$$

and so these fields ∂_1, ∂_2 and $[\partial_1, \partial_2]$ span the tangent bundle $T\mathbb{R}^3$ at each point $v \in \mathbb{R}^3$.

§1.2 Bracket generating, Hörmander theorem.

Let E be a smooth distribution and let $U \subset M$ be an open set. Let $\Gamma(E, U)$ be the set of all smooth sections of E on U. For each positive integer k, we set $\Gamma_k(E, U)$ the set of all vector fields X on U such that X is a linear combination with smooth coefficients of iterated brackets of degree $\geq k$ of members of $\Gamma(E, U)$, and $\Gamma_k(E) := \bigcup_U \Gamma_k(E, U)$. For $p \in M$, we set $\Gamma_k(p) := \{X(p) \mid X \in \Gamma_k(E)\}$. And we set

$$E_\infty(p) := \bigcup_{k=1}^{\infty} E_k(p).$$

1.5 Definition. *A smooth distribution E on a manifold M is bracket generating (or sometimes Hörmander condition, nonholonomic) if $E_\infty = T_pM$ for all $p \in M$. If k is a positive integer, a smooth distribution E such that $E_k = T_pM$ for all $p \in M$ is called k-generating.*

Let $P : \Gamma(M)(:=$ smooth functions on $M) \longrightarrow \Gamma(M)$ be a second order partial differential operator (abbreviated by 'PDE ') with real smooth coefficients. We assume that there is a subbundle E of TM such that P is of form

$$P = \sum_{j=1}^{N} X_j^2 + \text{lower order term}$$

on any sufficient small open set $U \subset M$, where $X_1, \cdots, X_N$ span E on U. Then we have

1.6 Theorem (Hörmander). *If E satisfies Hörmander's condition, then P is hypoelliptic, i.e., any distributional solution u of the equation $Pu = f$ is also smooth when f is smooth.*

For a proof, see [Hö.1.2]. Hörmander has shown the hypoellipticity for more general PDEs. (See, [O]). We assume that E satisfies Hörmander's condition. The leading term of P defines a sub-Riemannian metric g as follows ;

$$g(x) := \sum_{j=1}^{N} X_j(x) \otimes X_j(x).$$

This is an analogue of the relation between a Riemannian metric and the leading term of its Laplacian.

1.7 Definition. *A smooth distribution E on M is said to be strongly bracket-generating (abbreviated to ABG) i.e., $k = 2$ at a point $p \in M$, if $E(p) + [X, E](p) = T_pM$ for every $X \in \Gamma(E)$ such that $X(p)$ is defined and is not equal to* 0.

• The case where $E \neq TM$ but E is SBG, is the contact case. Let $E \subset TM$ be a smooth bundle (poralization) spanned by smooth vector fields $X_1, X_2, \cdots, X_m$. Every E of rank $E = n_1$ can be spanned by $m \leq n_1 + n$ fields ($n :=$ dim M) and locally we need only $m = n_1$ fields. In fact, we work locally, and so $m = n_1$ sufficies. We denote successive commutators of our vector fields by X_i for suitable indices $i > m$ and for each X_i the number deg X_i which is the degree of the corresponding commutator.
• deg $X_i = 1 \Longleftrightarrow i \leq m$,
• deg $X_i = 2 \Longleftrightarrow X_i = [X_\mu, X_\nu]$, $1 \leq \mu, \nu \leq m$. In the followings, we assume that E is strongly backet-generating. Let N be the annihilator of E in T^*M. Then N is a smooth subbundle of rank $k(:= m - q)$ in T^*M.

1.8 Lemma. *Every Riemannian metric on E gives rise to a unique Riemannian metric on N.*

Proof. Let $p \in M$ with $0 \neq \omega_p \in T_pN$. Let ω be a local section of N through ω_p. If X, Y are local sections of E near p, then we have

$$d\omega(X, Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\} = -\frac{1}{2}\omega([X, Y])$$

in particular, $d\omega(X_p, Y_p)$ only depends on ω_p, not on the choices of the extension ω. Since the commutators of sections of E span TM, the restriction of $d\omega$ to E_p does not vanish. This means that there is a natural injective bundle map j of N into the exterior product $E^* \wedge E^*$. Since a Riemannian metric on E induces a Riemannian metric on $E^* \wedge E^*$, this metric can be pulled back via j to a metric on N. □

For $\omega \in \Gamma(N)$, we define a map $j\omega : E \longrightarrow T^*M$ by

$$(j\omega(X))(Y) := d\omega(X, Y)$$

Since E is strongly bracket-generating, $j\omega$ is an injective bundle map and $j\omega(E)$ is complementary to N.

1.9 Lemma. *Let $\omega^1, \cdots, \omega^k$ be a local orthonormal basis with respect to the metric of Lemma 1.8. Then, for every $i \in \{1, \cdots, k\}$, the $(m-1)$-dimensional subspace of T_p^*M, which is spaned by $j\omega^i(E_p) \cup \{\omega_p^j \mid i \neq j\}$, only depends on $\omega_p^1, \cdots, \omega_p^k$ and is transverse to $\{\lambda\omega_p^i \mid \lambda \in \mathbb{R}\}$.*

Proof. It is evident that $A_p^i = \ll (j\omega^i(E_p) \cup \{\omega_p^j \mid i \neq j\} \gg$ is transverse to ω_p^i. To show that A_p^i only depends on $\omega_p^1, \cdots, \omega_p^k$, let $\overline{\omega}_p^1, \cdots, \overline{\omega}_p^k$ be another local orthonormal basis of N near p with $\overline{\omega}_p^i = \omega_p^i$. Then there is a smooth function $[g_{ij}]$ of a neighbouhood of p in M into $SO(k)$ such that $[g_{ij}](p) = \text{Id}$ and $\overline{\omega}^i = \sum_j g_{ij}\omega^j$. We have

$$[dg_{ij}]_p = 0, \quad j = 1, \cdots, n$$

Let $X^1, \cdots, X^q$ be a local orthonormal basis of E near p and let $\sigma^j = j\omega^i(X^j)$. Then

$$\begin{aligned} dg_{ij} &= \sum \alpha a_\alpha^{ij}\sigma^\alpha + \sum_\beta b_\beta^{ij}\omega^\beta \quad \text{smooth functions } a_\alpha^{ij},\ b_\beta^{ij} \\ d\overline{\omega}^i &= \sum_j dg_{ij} \wedge \omega^j + \sum_j g_{ij} d\omega^j \\ &= \sum_{\sigma,j} a\alpha^{ij}\sigma^\alpha \wedge \omega^j + \sum_{\beta,j} b_\beta^{ij}\omega^\beta \wedge \omega^j + \sum g_{ij} d\omega^j \end{aligned}$$

Since $a\alpha^{jj} = 0$ for $j = 1, \cdots, n$, this implies that

$$(j\overline{\omega}^i)(X) = (j\omega^i)(X) + \sum_{i\neq j}\sum_\alpha a_\alpha^{ij}(X)\omega^j \text{ for every } X \in E_p,$$

i.e., $(j\overline{\omega}^i)(X) - (j\omega^i)(X) \in \ll \omega^j \mid i \neq j \gg$ as claimed □

1.10 Corollary. *If E is strongly bracket generationg, then every Riemannian metric $< -, - >_E$ on E can intrinsically be extended to a Riemannian metric on M.*

Proof. Let $p \in M$. Lemma 1.9 implies that, for every $i \in \{1, \cdots, k\}$, the choice of an orthonormal basis $\omega_p^1, \cdots, \omega_p^k$ on N_p determines an $(m-1)$-dimensional subspace of T_p^*M, which annihilates a 1-dimensional subspace A^i of T_pM transverse to the kernel of ω_p^i. Let $Z^i \in A^i$ be such that $\omega_p^i(Z_i) = 1$.

The vectors $Z^1, \cdots, Z^k$ span a k-dimensional subspace of T_pM which is complementary to E_p. Thus we can define an extension $g(\omega_p^1, \cdots, \omega_p^k)$ of $< -, - >_{E,p}$ by choosing the vectors $Z^1, \cdots, Z^k$ orthonormal and perpendicular to E_p.

Now the space of orthonormal basis of N_p can be identified with $O(k)$. Let μ be the normalized Haar measure on $O(k)$ (which satisfies $\mu(O(k)) = 1$), and define

$$< X, Y >_p := \int_{O(k)} g(\xi)(X, Y)d\mu(\xi) \quad X, Y \in T_pM.$$

Then $< -, - >_p$ is a scalar product on T_pM extending the product on E_p and moreover is a scalar intrinsically by $(E, < -, - >_E)$. □

The Riemannian metric $< -, - >$ on M defined in 1.10 will be called the canonical extension of $< -, - >_E$.

1.11 Lemma.

$$\theta \frac{D}{dt}\lambda(X)'(t)|_{t=0} = \widetilde{P}ad^*(\widetilde{P}\theta X, (1-\tilde{P})\theta X) \\ - (1-\widetilde{P})ad^*(\widetilde{P}\theta X, \widetilde{P}\theta X).$$

Proof. Let $\phi = \lambda(X)$. Then $\theta\phi'(0) = \widetilde{P}\theta X$, $\frac{d}{dt}\theta\phi'(t)|_{t=0} = \widetilde{P}ad^*(\widetilde{P}\theta X, \theta X)$, and, by definition, $\theta\frac{D}{dt}\phi'(t) = \frac{d}{dt}\theta\phi'(t) - ad^*(\theta\phi'(t), \theta\phi'(t))$. Together with this yields the claim. □

1.12 Corollary. *If E is strongly bracket-generating, then every geodesic (with respect to d_E) is uniquely determined by its tangent and its covariant derivative at p. In particular, $\exp_p$ (with respect to d_E) is intrinsically defined.*

Proof. For $Y \in E_p$, the map $\alpha_Y : E_p^{\perp} \longrightarrow E_p$, $Z \longmapsto \widetilde{P}a_p(\theta Y, \theta Z)$ is independent of the choices of the local trivialization of TM near p, and its inverse. The claim follows from Lemma 1.11 and the fact that every Riemannian metric on a strongly generated distribution can intrinsically be extended to a Riemannian metric on M. □

1.13 Chow's connectivity theorem. *Let (E, g) be a SR-structure. Every two points in M can be joined by a horizontal curve in M.*

Remark 1.3. *This theorem is independent of the choices of SR-metrics, and also it holds for a generalized SR-structure in the sense of Remark 1.2. In this case, a horizontal curve means an absolutely continuous map $\phi : I \longrightarrow M$ such that, for $\forall\, t \in I$,*

$$\frac{d\phi}{dt}(t) \in E[\phi(t)].$$

1.14 Connectivity Theorem for the contact polarization E. *Every two points in $\mathbb{R}^3$ can be joined by a smooth E-horizontal curve.*

Proof. Take a curve $\underline{c} := (x(t), y(t))$, $t \in [0,1]$, in the (x,y)-plane joining two given points (x_1, y_1) and (x_2, y_2) and the formal area "bounded" by $\underline{c}$ defined by the integral

$$\int_{\underline{c}} x dy = \int_0^1 x(t)y'(t)dt$$

is equal to a given number a. (We easily find such $\underline{c}$, i.e., among curves of constant curvature). Then we take the horizontal lift of $\underline{c} := (x(t), y(t))$ to $\mathbb{R}^3$ by letting $z(t) = z_1 - \int_0^t x(t)y'(t)dt$ for a given value z_1 of z. Then the lifted curve $c := (x(t), y(t), z(t))$ is indeed horizontal as $dz(t) = z'(t)dt = -x(t)y'(t)dt = -x(t)dy(t)$ and it joints the given points (x_1, y_1, z_1) and $(x_2, y_2, z_2 := z_1 + a)$. □

Example 1.2.

In $\mathbb{R}^2$, we consider the system of $T\mathbb{R}^2$ generated by

$$X := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Y := \begin{bmatrix} 0 \\ f(x) \end{bmatrix},$$

where f is a C^∞-function defined by

$$f \begin{cases} = 0, & \text{negative } x \\ > 0, & \text{positive } x \end{cases}$$

Chow ′ theorem does not hold but any two points are ″accessible ″.

This result appears in the 1909-papers by Caratheodory on formalism of the classical thermodynamics where horizontal curves roughly correspond to adiabatic processes.

The metric structure on the SR-structures. For a SR-structure (M, E, g), we can define a distance function d_E by means of Chow ′ theorem. For $(x, y) \in M \times M$,

$$d_E(x, y) := \min\{L(\phi) \mid \phi : [a, b] \longrightarrow M \text{ horizontal } \phi(a) = x,\ \phi(b) = y\}.$$

This number is well defined by Chow ′s theorem. It is clear that d_E satisfies the axiomes of distance. We can also show that $d_E(x, y) > 0$ if $x \neq y$ as followings ; every point $m \in M$ has a relativly compact open neighbourhood V and a Riemannian metric G on V, such that $d_E(x, y) \geq d_G(x, y)$ for $\forall\ (x, y) \in V \times V$, where d_G is a metric on g. We obtain the following fundamental theorem ;

1.15 Fundamental theorem. *A SR-structure on a manifold M defines a metric on M which induces the original topology of M. In particular, M is paracompact.*

Remark 1.4. *i) Theorem 1.15 also holds for generalized SR-structures in the sense of Remark 1.2.*

ii) In $M := \mathbb{R}^n$, *if* g^* *is bounded, for all* $x, y \in M$,

$$d_E(x, y) = \sup\{|\phi(x) - \phi(y)| \mid \phi \in C^\infty(M),\ g^*(d\phi) \leq 1,\ \phi \textit{with compact support}\}.$$

Although the metric d_E may be considered to be the same as Riemannian metric, but it is very different each other. For example, in the Riemannian case, there is an open neighbourhood of the diagonal Δ_M of $M \times M$ so that the distance function is of class C^∞, but this phenomenon does not always occur in the sub-Riemannian case.

Since M is paracompact, M has a Riemannian metric. The distance functions defined by Riemannian metrics are equivalent on all compact sets of M, but it does not hold in a sub-Riemannian case.

Contact C-C metric on (M, E)**.** The E-horizontal curves c in $\mathbb{R}^3$ are the lifts of curves in the (x, y)-plane, such that the z-coordinate of c is equal to the formal area of the (x, y)-projection $\underline{c}$ of c. If two points v_1 and v_2 in $\mathbb{R}^3$ lie on the same vertical line (or z-line), i.e., have equal (x, y)-coordinates then the (x, y)-projections $\underline{c}$ of curves c joining these points are closed in the (x, y)-plane and so the (formal) area of these curves $\underline{c}$ is bounded by

$$\text{area } \underline{c} \leq \text{const}(\text{length } \underline{c})^2 \leq \text{const}(\text{length } c)^2$$

where const $= (4\pi)^{-1}$.

It follows that the C-C distance between v_1 and v_2 is bounded from below by the Euclidean distance as followings ;

$$(*) \qquad d_E \geq \text{const}^{-1/2}(d_{\text{can}})^{1/2}$$

since the Euclidean distance d_{can} between our points is equal to $z_1 - z_2 =$ area $\underline{c}$. We also have the upper bound

$$d_E \leq \alpha \text{const}^{-1/2} d_{\text{can}}^{1/2}$$

where α is a certain positive function depending on the Euclidean norms of the points v_1 and v_2. (We may take $\alpha := (1 + ||v_1|| + ||v_2||)^2)$.

In fact, We can join v_1 and v_2 by a curve c in $\mathbb{R}^3$ which projects to a curve $\underline{c}$ in $\mathbb{R}^2$ such that

$$d_{\text{can}} = \text{area } \underline{c} = (4\pi)^{-1}(\text{length } \underline{c})^2$$

while

$$d_E \leq \text{length } c \leq \alpha \text{length } \underline{c}.$$

• The C-C metric is locally equivalent to $\sqrt{\text{Euclidean metric}}$ on every vertical line in $\mathbb{R}^3$.

• The C-C metric is locally equivalent to the Euclidean metric on every horizontal curve.

Heisenberg group in view of the contact example.

Let G be the 3-dimensional Heisenberg group which can be defined as the only simply-connected nilpotent non-abelian Lie group and let $\mathfrak{g}$ be its Lie algebra which admits a basis $x,\ y, z$ with

$$[x, z] = [y, z] = 0,\ [x, y] = z.$$

We introduce a polarization $E \subset TG$ by taking the left translations of the (x, y)-plane

$$E_0 \subset \mathfrak{g} = T_{id}G.$$

• There is a diffeomorphism between G and $\mathbb{R}^3$ sending this E to the standard contact subbundle in $\mathbb{R}^3$.

Next, we take a left invariant metric g on G and let d_E be the C-C metric defined with E and g.

• d_E is a metric.

In fact, there are two methods to prove this statement.

(1) We look at the homeomorphism (projection)

$$G \longrightarrow \mathbb{R}^2 = G/\text{center},$$

where the center of G is equal to the one parameter subgroup obtained by the exponentiation of the (central) line spanned by z in $\mathfrak{g}$. The (contact) geometry of this projection of $G = \mathbb{R}^3$ to $\mathbb{R}^2$ is identified to the (x, y)-projection of Example 1.1 and the proof of the connectivity theorem 1.14 applies.

(2) We give the Lie group theoretic proof of connectivity. We consider the one parameter groups G_x and G_y of right translations of G corresponding to x and y in $\mathfrak{g}$. The orbits of these subgroups are obviously tangent to E. On the other hand, G_x and G_y, as subgroups in G, generate G since the derived Lie algebra generated x and y is equal to $\mathfrak{g}$. It follows that every two points in G can be joined by a piecewise smooth curve whose every segment is a piece of the orbit of G_x or of G_y. $\square$

Connectivity theorem for general Lie groups. Let G be any connected Lie group and let E_0 be a linear subspace in $\mathfrak{g}$. This E_0 define a left invariant polarization $E \subset TG$ and so we define the C-C metric dist_E on G. This is a metric if and only if the derived Lie algebra of E_0 is equal to $\mathfrak{g}$. Furthermore, if any Riemannian metric used in the definition of dist_E were left invariant, then dist_E is also left invariant on G.

§**1.3 Chow connectivity theorem.** Let $X_1, \cdots, X_m$ be smooth vector fields on a connected manifold M such that the derived Lie algebra of these vector fields span each tangent space T_vM $(v \in M)$. Then every two points in M can be joined by a piecewise smooth curve in M where each piece is a segment of an integral curve of one of the fields X_i.

• (Lie group) Let $\mathfrak{L}$ be the Lie algebra generated by the fields $X_i(i = 1, \cdots, m)$ and let $G \subset$ Diff M be the subgroup of diffeomorphisms generated by the one-parameter subgroups corresponding to $X_i(i = 1, \cdots, m)$. The theorem claims that G is transitive on M provided $\mathfrak{L}$ spans TM. This is immediate if $\mathfrak{L}$ is of finite dimensional, for $\mathfrak{L}$ can be identified with $\mathfrak{g}$ (which makes G finite dimensional as well), and the condition "$\mathfrak{L}$ spans TM" amounts to surjectivity of differential of the orbit map $G \longrightarrow M \ : g \longmapsto g(v_0)$, $(v_0 \in M)$. In fact, this argument applies to the infinite dimensional case as well.

• (polarization) If the dimension of the span $E_v \subset T_vM$ of the fields X_i at v is independent of v, the span E of these fields is a subbundle in TM, i.e., a polarization of M in our sense where the orbits of X_i are horizontal. Thus Chow theorem implies the connectivity property for E-horizontal curves.

• (polarization defined by 1-forms) We can define a polarization $E \subset TM$ as the zeros of a system of 1-forms on M. This suggests a dual approach to the connectivity property of E which does not directly use orbits of vector fields tangent to E but it appeals to leaves of 1-dimensional foliations obtained by intersecting E with submanifolds $W \subset M$ with codim $W = \text{rank } E - 1$.

Let E be a contact subbundle on a 3-dimensional manifold M and $W_0 \subset M$ is a curve transverse to E, we take 2-dimensional cylinders $W_\epsilon \subset M$, $\epsilon > 0$ around W_0 and $E \cap TW_\epsilon$ give us (spiral) curves in W_ϵ tangent to E which closely follow W_0 for small ϵ.

Proof of Chow connectivity theorem. We are given vector fields $X_1, \cdots, X_m$ on a connected manifold M which derived Lie algebra is equal to TM, and we want to join a pair of points in M by a piecewise smooth curve where each piece is a segment of an integral curve of some field X_i, $(i = 1, \cdots, m)$. Since we works locally, we may assume that the fields X_i generate a one-parameter groups of diffeomorphisms of M and we must join given points by piecewise orbit curves. In other words, we must prove that the group G of diffeomorphisms of M generated by those subgroups is transitive on M.

1.16 Trivial Lemma. *If G contains one-parameter subgroups, say, $Y_1(t), \cdots, Y_p(t)$ where the corresponding vector fields $Y_1, Y_2, \cdots, Y_p$ span TM (without taking commutators), then G is transitive on M.*

Proof. This follows from the implicit function theorem. Namely, for each $v \in M$ we consider the composed action map $E_v : \mathbb{R}^p \longrightarrow M$ defined by

$$(t_1, \cdots, t_p) \longmapsto Y_1(t) \circ Y_2(t) \circ \cdots, \circ Y_p(t_p)(v).$$

The differential of E_v at the origin $0 \in \mathbb{R}^p$ sends $\mathbb{R}^p$ onto the span of the fields Y_i in T_vM and so is surjective in our case. Thus the orbit $G(v)$ is open in M for each $v \in M$ by the implicit function theorem and $G(v) = M$ since M is connected. □

Let $X(t)$ be a one-parameter group on M and let Y be a vector field. Let us look at the transport $X_*(t)Y$ of Y by $X(t)$. We note that for small $t \longrightarrow 0$,

$$X_*(t)Y = Y + t[X, Y] + o(t),$$

and conclude that, since the commutators of X_i span TM, there are vector fields Y_j, $(j = 1, \cdots, p \geq m)$ on M which span TM and such that

(i) $Y_i = X_i$ for $i = 1, \cdots, m$,

(ii) each field Y_j for $j > m$ is equal to $(X_i(t_j))_* Y_{j'}$, i.e., the transport of some $Y_{j'}$, $j' < j$, by the flow $X_i(t)$ at $t = t_j$ (where i also depends on j).

Finally, we note that the one-parameter group $Y_j(t)$ are contained in G since the transport of a field Y by $X(t)$ corresponds to the conjugation of the one parameter group $Y(\tau)$,

$$Y(\tau) \longmapsto X(t)Y(\tau)X^{-1}(t)$$

for $Y \longmapsto X_*(t)Y$ and the proof follows by the Trivial Lemma. □

Improving Chow theorem into the smooth case.

In the original Chow theorem curves joining given points must necessary consists of pieces of orbits of different fields and so they cannot be made smooth. But if we have a smooth polarization H, where the commutators of H-horizontal fields span TM, we may slightly improve the results as followings ;

1.17 Chow theorem in the smooth case. *Every two points in M can be joined by a smooth H-horizontal curve in M, i.e., by a smooth immersion $f : [0,1] \longrightarrow M$ with $f(0) := v_0$ and $f(1) := v$ and $f'(t) \in H$ for $t \in [0,1]$.*

Proof. Either of our two proofs of the Chow theorem provides a smooth family Φ of piecewise smooth curves issuing from v_0, say $\phi(t) \in \Phi, t \in [0,1]$, such that the map $\Phi \longrightarrow V$ defined by $\phi \longmapsto \phi(1)$ contains a given point v in its image, i.e., $\phi(1) = v$ for a certain ϕ. The curve ϕ consists of segments of orbit of certain H-horizontal vector fields $Y_1, \cdots, Y_k$ and when these fields are fixed, then ϕ is uniquely determined by the lengths of the segments. In fact, these lengths serve as coordinates in Φ and so the curves in Φ close to ϕ are obtained by slightly varying these lengths, called $\ell_i := \ell_i(\phi)$, $i = 1, \cdots, k$. Next, let us smoothly interpolate between Y_i and Y_{i+1} for all $i = 1, \cdots, k$. Namely, we introduce a smooth family of fields $Y_t = Y_t(\ell_1, \cdots, \ell_k)$, $k \in [0, L_k]$ for $L_k := \sum_{i=1}^k \ell_i$, such that

(i) $Y_t = Y_{i+1}$ for $t \in [L_i + \epsilon, L_{i+1} + \epsilon]$, for $L_i := \ell_1 + \cdots + \ell_i$ and small $\epsilon > 0$,

(ii) $||Y_t|| \leq$ constant for some constant ≥ 0 independent of ϵ.

Now we define $f(t)$ as the integral curve of the field Y_t issuing from v_0, i.e., $f(0) = v_0$ and $f'(t) = Y_t$ at $v = f(t)$, and observe that $f \longmapsto \phi$ for $\epsilon \longrightarrow 0$. It easily follows that the map $f \longmapsto f(1)$ contains v in its image for a sufficiently small ϵ. □

A new proof of the Chow theorem and the Hölder bound on the C-C metric. It is well known the following ; approximate expression for the commutator of one-parameter groups $X_1(t)$ and $X_2(t)$ in Diff(M) in terms of the one-parameter group corresponding to the Lie brackets of the fields X_1 and X_2,

$$(*) \qquad [X_1(t), X_2(t)]^0 := X_1(t) \circ X_2(t) \circ X_1^{-1}(t) \circ X_2^{-1}(t) = [X_1, X_2](t^2) + o(t^2)$$

where the additive notation refers to some Euclidean structure in a relevant neighbourhood and so we should note that $X_i^{-1}(t) = X_i(-t)$, $(i = 1, 2)$.

Proof of $(*)$. The following elementary formulas hold ;

$$(\tau Y)(t) = Y(t\tau). \tag{1.1}$$

$$(X + \tau Y)(t) = X(t) \circ Y(\tau t) + o(t\tau) = Y(\tau t) \circ X(t) + o(t\tau) \text{ for } t, \tau \to 0. \tag{1.2}$$

$$X_1(t)X_2(\tau)X_1^{-1}(t) = (X_2 + \tau[X_1, X_2])t + o(t\tau) \tag{1.3}$$

Note that (1.1) is obvious, (1.2) follows from the Taylor expansion for the composition $X(t) \circ Y(\tau t)$, and (1.3) is implied by (1.1), (1.2) and the definition of Lie bracket.

Now we have $(*)$ in the form

$$X_1(t) \circ X_2(t) \circ X_1^{-1}(t) = [X_1, X_2](t^2) \circ X_2(t) + o(t^2)$$

by applying first (1.3) and then (1.2) and (1.1) to the left hand side.

Next we observe that $(*)$ by induction implies

$$[X_1(t), [X_2(t), X_3(t)]^0]^0 = [X_1, [X_2, X_3]](t^3) + o(t^3)$$

$$\cdots\cdots$$

$$[X_1(t), [\cdots, X_d(t)]^0 \cdots]^0 = [X_1, [\cdots, X_d], \cdots](t^d) + o(t^d).$$

□

1.18 Proposition. *For arbitrary degree,*

$$\text{C-C metric} \lesssim (\text{Riemann. dist})^{1/d}$$

Proof. We consider the simplest case where dim $M = 3$ and TM is generated by X_1, X_2 and $Y = [X_1, X_2]$. We denote by $Y^0(t)$ the one-parameter family (not a subgroup) of diffeomorphisms defined by

$$Y^0(t) := \begin{cases} [X_1(|t|^{1/2}), X_2(|t|^{1/2})]^0 & \text{for } t \geq 0 \\ [X_2(|t|^{1/2}), X_1(|t|^{1/2})]^0 & \text{for } t \leq 0 \end{cases}$$

and we observe that the composed map

$$F^0 : (t_1, t_2, t_3) \longmapsto X_1(t) \circ X_2(t) \circ Y^0(t)(v)$$

sends the box $B(\epsilon) \subset \mathbb{R}^3$ defined by $|t_1| \leq \epsilon$, $|t_2| \leq \epsilon$, $|t_3| \leq \epsilon^2$ into the ϵ'-C-C ball in M around v for $\epsilon' \approx \epsilon$ (in fact, for $\epsilon' \leq 10\epsilon$.) It remains to show that the image of this box contains a Riemannian δ-ball around v for $\delta \gtrsim \epsilon^2$. We compare F^0 with the composed map

$$F : (t_1, t_2, t_3) \longmapsto X_1(t) \circ X_2(t) \circ Y(t)(v),$$

for which the image of the ϵ-box is δ-large by the implicit function theorem. In fact, the F-image of the ϵ^2 cube defined by $|t_1| \leq \epsilon^2$, $i = 1, 2, 3$, contains the required δ-ball. Then we observe with $(*)$ that $F^0 = F + o(\epsilon^2)$ in the ϵ^2-cube. It follows by elementary topology that the F^0-image of the ϵ^2-cube is essentially as large as the F-image. □

§1.4 The tangent cone of the C-C metric spaces.

Let G be the Heisenberg group, i.e., the simply connected 3 dimensional nilpotent Lie group (diffeomorphic to $\mathbb{R}^3$). And let X and Y generate the Lie algebra $\mathfrak{g}$ so that $X, Y, Z := [X, Y]$ is the basis of $\mathfrak{g}$. There is a family of automorphisms $\{\phi_t\}$ of $\mathfrak{g}$, whose representation with respect to the basis X, Y, Z is

$$\begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{bmatrix}.$$

Take the left-invariant Riemannian metric g on G for which X, Y, Z are orthonormal. On $\mathfrak{g}$, this metric is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The metric $(1/t^2)g$ is isometric to $(1/t^2)\phi_t^*(g)$ (ϕ_t provides the isometry), which is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^2 \end{bmatrix}.$$

Then, as $t \longrightarrow 0$, the lengths of vectors transverse to the distribution spanned by X and Y (considered as left-invariant vector fields on G) become infinite, while the lengths of horizontal vectors remains unchanged. In the limit, only horizontal curves have finite lengths, and the sequence of metric spaces $(G, g/t^2)$ converges to the metric (G, d_c).

1.19 Theorem(Pansu[P]). *If G is a nilpotent Lie group with left-invariant Riemannian metric g, then*

$$\lim_{t \to +\infty} (G, g/t) = (\overline{G}, d_c),$$

where $\overline{G}$ is a nilpotent Lie group and d_c is a C-C metric on $\overline{G}$. If G is graded, then $\overline{G} = G$; otherwise $\overline{G}$ is the graded nilpotent Lie group associated to G.

Here, the limit is means the Hausdorff limit of a sequence of metric spaces ;

1.20 Definition. *The Hausdorff distance $H_X(A, B)$ between two compact subsets A, B of a metric space X is defined by*

$$\inf\{\epsilon \mid B \subset N_\epsilon(A), A \subset N_\epsilon(B)\},$$

where N_ϵ is the ϵ-neighbourhood.

The Hausdorff distance $H(A, B)$ between two "abstract" compact metric spaces A, B is equal to

$$\inf_X H_X(A,\ B)$$

where the infimum is taken over all isometric imbeddings of the pair A, B into all possible metric spaces X. Note that such metric spaces exist ; e.g., $X := A \times B$.

A sequence $\{A_i\}$ of compact metric spaces is said to converge in the sense of Hausdorff to a metric space B if $\lim_{i \to \infty} H(A_i, B) = 0$. The more practical definition is in the following ;

1.21 Theorem(M.Gromov[Gr1]). *A sequence $\{A_i\}$ of compact metric spaces converges to B if and only if there is a sequence of positive real numbers $\epsilon_i \longrightarrow 0$ such that, for each i, there is an ϵ_i-dense net $\Gamma_i \subset A_i$ and an ϵ_i-dense net $\Gamma_i' \subset B$ which is ϵ_i-quasi-isometric to Γ_i.*

An ϵ-dense net in a space A means a set of points with the property that each point of A is within distance ϵ of some point of the set. An ϵ-quasi-isometry between two metric spaces is a mapping which preserves distances up to a factor $1 + \epsilon$.

If the spaces A_i are not compact, convergence means that for each $R > 0$, the balls of radius R about fixed base points in A_i converge to the ball of radius R about a fixed point in B.

1.22 Definition(M.Gromov[Gr1]). *The sequence $\{A_i\}$ is uniformly compact if*

(i) the diameters, diam(A_i) are uniformly bounded,

(ii) For any $\epsilon > 0$, the minimum number of ϵ-balls needed to cover A_i is bounded (uniformly in i).

"Uniformly compact" is the necessary and sufficient condition for the existence of a convergent subsequence of a sequence of compact metric spaces.

1.23 Definition. *The tangent cone of a metric space (M, d) at a point $x \in M$ is $T_x M := \lim_{\lambda \to \infty}(M, \lambda \cdot d)$ if the limit exists. x is chosen as the base point for all the metric spaces $(M, \lambda \cdot d)$.*

In the case of Heisenberg group, we see, in the canonical coordinate,

$$d_c((0,0,0),(0,0,z)) = \sqrt{z}.$$

Thus d_c is, in general, not smooth.

1.24 Lemma(Metivier[Me]). *Let Ω be a neighbourhood of $p \in M$. Suppose that $v_i := \dim(E_i(x))$ is constant for each i ($x \in \Omega$) and that $\dim(E_r(x)) := n = \dim\ M$ for some r. (Assume r is minimal). Then for any $x_0 \in \Omega$, there exist neighbourhoods $\Omega_1 \subset \Omega_0 \subset \Omega$ of x_0, a neighbourhood U_0 of the origin 0 in $\mathbb{R}^n$, and a C^∞ map $\theta : \overline{\Omega}_1 \times \Omega_0 \longrightarrow \mathbb{R}^n$ such that ;*

(i) For each $x \in \overline{\Omega}_1$ the map $\theta_x : y \longmapsto \theta(x, y)$ is a C^∞ diffeomorphism from Ω_0 to $\theta_x(\Omega_0) = U_0$, and $\theta_x(x) = 0$,

(ii) For each $x \in \overline{\Omega}_1$, the vector fields $X_{i,x} := \theta x, *X_i$, $i = 1, \cdots, k$ are of degree ≤ 1 ar 0,

(iii) If $\widehat{X}_{i,x}$ denotes the homogeneous part of degree one of $X_{i,x}$, then the vector fields $\widehat{X}_{i,x}$ generate a nilpotent Lie algebra of dimension n. Furthermore, let $\widehat{E}_i(\xi) := (\xi, \widehat{X}_{1,x}, \cdots, \widehat{X}_{k,x})$. Then $\dim \widehat{E}_i(\xi) = v_i$ for all $\xi \in \mathbb{R}^n$, $i = 1, \cdots, r$.

(iv) The vector fields $\widehat{X}_{i,x}$ and $X_{i,x}$ depend smoothly on $x \in \Omega_1$.

Now, we define a one-parameter group of dilations of M (locally). Let X_I be the m-fold commutator $[X_{i_1}, \cdots, [X_{i_{m-1}}, X_{i_m}], \cdots]$ for a multi-index $I := (i_1, \cdots, i_m)$. We may choose from among the X_I's a subset $\{Y_j\}$, $j = 1, \cdots, n$, of vector fields such that $\{Y_i\}_{i \in I}$ is a basis of T_xM for all $x \in \Omega$. Thus, any point $x \in \Omega$ (or in a smaller neighbourhood, again denoted by Ω) may be uniquely written in the form

$$x = \exp\left(\sum_{i=1}^{n} a_i Y_i\right)$$

for some real numbers a_i. The a_i are the normal coordinates of x. We define the dilation γ_r in terms of normal coordinates as followings

$$(\gamma_r x)_i := r^{[i]} a_i,$$

where $[i] = k$ if $\dim(E_{k-1}) < i \leq \dim(E_k)$. The $\widehat{X}_{i,x}$ are homogeneous with respect to γ_r.

We may choose, for each k, $1 \leq k \leq r$, a subset $\{\widehat{X}_{jk,x}\}, j = 1, 2, \cdots$ of the commutators of the $\widehat{X}_{i,x}$'s which yields a basis for $E_k(x)/E_{k-1}(x)$. A vector field Y on $\mathbb{R}^n$ may be written

$$Y = \sum_{j,k} a_{jk} \widehat{X}_{jk,x}, \quad a_{jk} \in C^\infty(M).$$

If we expand the a_{jk}'s in their Taylor series about 0 in normal coordinates, Y will be exhibited as a formal sum of homogeneous differential operators. Y is of degree $\leq \lambda$ if each term in this formal sum is homogeneous of degree $\leq \lambda$.

Let D_r be the distribution spanned by $\{\gamma_{r*}(X_i)\}$ and d_r be the associated C-C metric. And let D_∞ be the distribution spanned by $\{\widehat{X}_i\}$ and let d_∞ be its associated metric. Let $B_r(k)$ and $S_r(k)$ be the ball and sphere of radius k in the metric d_r, $1 \leq r \leq \infty$.

1.25 Lemma. *d_r converges, in the sense of Hausdorff, to d_∞ as $r \longrightarrow \infty$.*

The quasi-isometric distance (X, Y) between two metric spaces (X, d_X) and (Y, d_Y) is defined as the logarithm of the infimum of the metric dialation of all homeomorphisms $f : X \longrightarrow Y$. If X and Y are not homeomorphic, then $(X, Y) := \infty$.

1.26 Lemma. *The quasi-isometric distance between (M, rd_1) and (M, d_r) tends to zero as $r \longrightarrow \infty$.*

1.27 Lemma. *There is a function $F(\rho) > 0$ defined for $\rho > 0$ such that $F(\rho) \longrightarrow 0$ as $\rho \longrightarrow 0$ and $d_1(p, q) < \rho$ implies $d_r(p, q) < F(\rho)$ for all $r \geq R$ and for any p, q in any compact ball B. This R may depend on ρ but not on p and q.*

Proof. We recall the main idea in the proof of Chow theorem. We choose a linearly independent set from among the X_I's which spans T_xM. Denote the multi-index subscripts appearing in this set by $I_1, I_2, \cdots, I_n$. To each multi-index I we associate a flow ϕ_I on M as followings ; If $I := i$, set $\phi_I(t) := \exp(tX_i)(x)$, and $I := (i, J)$, set $\phi_I(t) := \phi_J(-\sqrt{t}) \circ \phi_i(-\sqrt{t}) \circ \phi_J(\sqrt{t}) \circ \phi_i(\sqrt{t})$ (Here, (i, J) denotes the multi-index obtained by appending an i to the beginning of the multi-index J). Now, we define a map $\phi : \mathbb{R}^n \longrightarrow M$ by

$$\phi(t_1, t_2, \cdots, t_n) := \phi_{I_n}(t_n) \circ \phi_{I_{n-1}}(t_{n-1}) \circ \cdots \circ \phi_{I_1}(t_1).$$

Note that $\phi(0) = x$. It is easy to check that ϕ is a C^1-map and that $\phi_*(\partial/\partial t_j)|_{t=0} = X_{I_j}$ for $j = 1, \cdots, n$. The inverse function theorem implies that ϕ is a C^1 diffeomorphism near the origin. Moreover, by the construction of ϕ, $\phi(t)$ is the endpoint of a horizontal curve, and so any point near $x \in M$ may be reached by a horizontal curve.

If we apply this construction to a local basis of vector fields for D_∞, we see that some Riemannian ball $B_x(\epsilon)$ about $x \in M$ is contained in the image under ϕ of some ball $B(\delta)$ in $\mathbb{R}^n$. Now it is clear that we may choose a local orthonormal basis $\{X_i^r\}$ for D_r which depends continuously on r. Then we may construct a map $\phi^r : \mathbb{R}^n \longrightarrow M$ associated to each basis $\{X_i^r\}$, and it is clear that $\phi^r|_B$ depends continuously on the vector fields used to define it, and so $\phi^r|_B$ depends continuously on r. Thus, for large r, $\phi^r(B)$ contains $B(\epsilon/2, x)$, e.g., $\rho := \epsilon/2$ and $F(\rho) := \delta$ we see that

$$d(q, x) < \rho \Longrightarrow d_r(q, x) < F(\rho)$$

for large r. Clearly, we may take $\delta \longrightarrow 0$ as $\epsilon \longrightarrow 0$ and the estimate is obviously uniform on compact sets in M, and so Lemma 1.27 is proved. □

Proof of Lemma 1.25. Lemma 1.27 implies that y' will be close to y with respect to the metric d_r for large r. We associate to any piecewise smooth curve c_1 joining x to y which is tangent a.e. to D_{r_1} a curve c_2 of the same length which joins x to a point y' and which is tangent a.e. to D_{r_2}. If r_1 and r_2 are large, y' will be close to y. The procedure is as followings ;

The curve c_1 satisfies

$$\dot{c}_1(t) = \sum_{i=1}^{n} a_i(t) X_i^{r_1}(c_1(t)), \quad c_1(0) = x,\ c_1(T) = y$$

for a.e. t, $0 \le t \le T$. The curve c_2 satisfies

$$\dot{c}_2(t) = \sum_{i=1}^{n} a_i(t) X_i^{r_2}(c_2(t)), \quad c_2(0) = x,\ c_1(T) = y$$

for a.e. t, $0 \le t \le T$. Since we may assume that $\{X_i^r\}$ is an orthonormal set for all r, we have $||\dot{c_1}(t)|| = ||\dot{c_2}(t)||$ and so length(c_1) = length(c_2). An elementary estimate based on the Gronwall lemma shows that y' is Riemannian close to y if r_1 and r_2 are sufficiently large. Thus there is a d_{r_2}-short path from y to y', and so,

$$d_{r_2}(x, y) \le d_{r_1}(x, y) + \epsilon(R) \quad \text{for } r_1, r_2 \ge R,$$

where $\epsilon(R) \longrightarrow 0$ as $R \longrightarrow \infty$. Similarly, we see that

$$d_{r_1}(x, y) \le d_{r_2}(x, y) + \epsilon(R).$$

The estimates are uniform for all $x, y \in B$ if B is compact, so $H((B, d_{r_1}), (B, d_{r_2})) \longrightarrow 0$ as r_1, and $r_2 \longrightarrow \infty$. In particular, letting $r_1 := \infty$ we have

$$\lim_{r \to \infty} H((B, d_r), (B, d_\infty)) = 0.$$

This completes the proof of Lemma 1.25. □

Proof of Lemma 1.26. We may identify a neighbourhood in M with a neighbourhood of $0 \in \mathbb{R}^n$ via θ. Let $B_1(1)$ be the C-C ball centered at 0.

First, we show that up to bounded distortion, γ_r, applied to curves or vectors in $\gamma_{1/r}(B_1(1))$ which are tangent to D, multiplies length by r. Let $x_0 \in S_1(1)$. To estimate the C-C distance of $\gamma_{1/r}(x_0)$ from 0, we need to estimate the actions of γ_r on vectors in D whose base points lie in $\gamma_{1/r}(B(1))$. Let $y \in B(1)$ and let $V \in D(\gamma_{1/r}(y))$. Then

$$V = \sum_i v_i \widehat{X}_{i,x}|_{\gamma_{1/r}(y)} + \sum_i v_i R_i|_{\gamma_{1/r}(y)}, \quad v_i \in \mathbb{R}$$

where $R_i := X_{i,x} - \widehat{X}_{i,x}$ is a vector field of degree ≤ 0. Thus

$$\gamma_{r*}(v) = r \sum_i v_i \widehat{X}_{i,x} + \sum_i v_i \gamma_{r*} R_i(\gamma_{1/r}(y))$$

where $\gamma_{r*}(\widehat{X}_{i,x}) = r \cdot \widehat{X}_{i,x}$. Now the definition of local degree implies that if R_i has degree ≤ 0, then the length of $\gamma_{r*}(R_i(\gamma_{1/r}(y)))$ remains bounded as $r \longrightarrow \infty$.

(In deed, The homogeneous terms in the formal expansion of R_i as a sum of homogeneous operators (with respect to γ_r) look like $a_{jk,\ell}\widehat{X}_{jk,x}$ if a_{jk} has the formal expansion $a_{jk} = \sum_{i=0}^{\infty} a_{jk,\ell}$, where $a_{jk,\ell}$ is a function homogeneous of degree ℓ. Since

$$a_{jk,\ell}(\gamma_{1/r}(y)) = r^{-1} a_{jk,\ell}(y) \text{ and } \gamma_{r*}(\widehat{X}_{jk,x}(\gamma_{1/r}(y))) = r^k \widehat{X}_{jk,x'},$$

we have

$$\gamma_{r*}(a_{jk,\ell}\widehat{X}_{jk,x}(\gamma_{1/r}(y))) = r^{k-1} a_{jk,\ell}\widehat{X}_{jk}(y).$$

"R_i is of local degree ≤ 0" means $k - 1 \leq 0$, and so such a term remains bounded as $r \longrightarrow \infty$. This implies the result.)

Since $R_i(0) = 0$, we have $||R_i(\gamma_{1/r}(y)|| \longrightarrow 0$ as $r \longrightarrow \infty$, where $|| - ||$ denotes the Riemannian length. Therefore we have

$$\frac{1}{r}\frac{||\gamma_{r*}(V)||}{||V||} = \frac{1}{r}\frac{||r \sum_i v_i \widehat{X}_{i,x}|_y + \sum_i v_i \gamma_{r*} R_i(\gamma_{1/r}(y)))||}{|| \sum_i v_i \widehat{X}_{i,x}|_{\gamma_{1/r}(y)} + \sum_i v_i R_i(\gamma_{1/r}(y))||} \longrightarrow 1$$

as $r \longrightarrow \infty$, and so this expression is bounded above and below by $1/c$ an c respectively for some $c > 1$, for all sufficiently large r.

Second, from this estimate on vectors we have the estimate on curves. If $p : [0, 1] \longrightarrow \mathbb{R}^n$ is a curve joining 0 to $\gamma_{1/r}(x_0)$ which is tangent to the distribution D a.e. and which lies in $\gamma_{1/r}(B(1))$, then $\gamma_r(p)$ is a curve joining 0 to x_0. Therefore its length is bounded below by a positive constant, and with the above inequality on vectors, we see that

$$(*) \qquad \text{const.} \leq \text{length}(\gamma_r(p)) \leq \text{length}(p).$$

Lemma 1.25 implies that $B_\infty(k) \subset B_r(k+\delta)$ for all large r and some δ. Also, it is clear that $B_1(1) \subset B_\infty(p)$ for some k, and so $B(1) \subset B_r(k+\delta)$ for all large r. This shows that we may choose a piecewise-smooth curve $\overline{p}$ tangent to D_r and joining 0 to x_0, of length $\leq K+\delta =$ constant. Then $\overline{p} = \gamma_{1/r}(p)$ is tangent to D, joining 0 to $\gamma_{1/r}(x_0)$ and satisfies

$$(**) \qquad \text{length}(p) \leq \frac{\text{const.}}{r} \text{ for some constant}$$

Therefore we have

$$\lim_{r\to\infty} \frac{\text{length}(\gamma_r(p))}{r\text{length}(p)} = 1,$$

which proves Lemma 1.26. □

Remark 1.5. *(*) and (**) imply the following inequality ; for some $c > 1$ and all large r,*

$$B_1(1/cr) \subset \gamma_{1/r}(B_1(1)) \subset B_1(c/r).$$

1.28 Lemma. *If X and Y are two metric spaces with finite diameters, then*

$$\frac{H(X,Y)}{diam(X) + diam(Y)} \leq (X,Y).$$

Proof. Let (X,d_1) and (Y,d_2) be two metric spaces with finite diameters. If $(X,Y) < \infty$, then there is a homeomorphism $f : X \longrightarrow Y$ whose distortion is arbitrarily close to $e^{(X,Y)}$. By identifying Y and X by f, we have a single X with two metrics d_1 and d_2. We may imbed each of these metric spaces isometrically into a third metric space ; namely, let $C^0(X)$ be continuous functions on X with metric induced by the sup norm. A point $x \in X$ is sent to the point $F_i(x) = d_i(x,-) \in C^0(X)$, $i = 1,2$. For any x_1, $x_2 \in X$,

$$\left|\log\left(\frac{d_1(x_1,x_2)}{d_2(x_1,x_2)}\right)\right| \leq (X,Y)$$

and

$$\max\{d_1(x_1,x_2), d_2(x_1,x_2)\} \leq \text{diam}(X) + \text{diam}(Y)$$

It follows that

$$|d_1(x_1,x_2) - d_2(x_1,x_2)| \leq (1 - e^{-(X,Y)})(\text{diam}(X) + \text{diam}(Y)).$$

Thus we have

$$H(X,Y) \leq (\text{diam}(X) + \text{diam}(Y))(X,Y).$$

□

A distribution D is said to be generic if, for each i, $\dim E_i(x)$ is independent of the point $x \in M$, i.e., D is a subbundle of TM.

1.29 Theorem(J.Mitchell[M]). *For a generic distribution D on M, the tangent cone of (M, d_c) on M is isometric to (G, d_c), where G is a nilpotent Lie group with left-invariant C-C metric.*

§1.5 Isoperimetric inequality and Sobolev's inequalty.

1.30 Boxed Theorem. *The C-C balls $B_m(\rho)$ in M around $m \in M$ are uniformly equivalent to the exponential images of the boxes. This means that there are strictly positive continuous functions $C := C(m)$ and $\rho = \rho_0(m)$, such that*

$$\exp_m Box(C^{-1}\rho) \subset B_m(\rho) \subset \exp_m Box(C\rho)$$

for all $m \in M$ and $\rho \leq \rho_0(m)$.

This theorem immediately implies the universal bound on the Riemannian volume of concentric C-C balls in a compact manifold M ;

1.31 Corollary. *Fix a volume element on M. For all compact K of M, there is a constant C such that, for all $m \in K$ and all positive real number ρ with $B(m, 2\rho) \subset K$,*

$$vol\, B(m, 2\rho) \leq C\ vol\, B(m, \rho).$$

We can easily estimate the Hausdorff dimensions by this Corollary. We replace C-C-balls by boxes. For example, Let $m \in M$ be a regular point. We can choose a field of adapted framing $(e_1, e_2, \cdots, e_d)$ $d := \dim\ M$ on a neightbourhood V of m. We have the following equality by means of boxes associated to a field $(e_1, \cdots, e_d)$;

$$\dim_{\mathrm{Hau}} M = h(m),$$

where we define an invariant $h(m) := \sum_{k \geq 1} k \dim\{\mathcal{E}^k[m]/\mathcal{E}^{k-1}[m]\}$ $(\mathcal{E}^0[m] := 0)$. Here and hereafter, we set $\mathcal{E} := \Gamma(E) :=$ the sections of E, and $\mathcal{E}^k := [\mathcal{E}, \mathcal{E}^{k-1}]$ $(k \geq 2)$, where the stalk $[\mathcal{E}, \mathcal{E}^{k-1}](m)$ at $m \in M$ is defined by

$$[\mathcal{E}, \mathcal{E}^{k-1}](m) := \{[V, W] \mid V \in \mathcal{E},\ W \in \mathcal{E}^{k-1}\}.$$

If the SR-structure on M is regular, M is of same dimension at all points of M, and so $\dim_{\mathrm{Hau}}$ is of constant value of $h(m)$. In general, for a structure which is neccesarily regular, we have

$$\dim_{\mathrm{Hau}} M \leq \sup\{h(m) \mid m \in M\}$$

And, for a compact subset $K \subset M$ with topological dimension $\geq \dim\ M - 1$, Gromov showed that,if $(\mathcal{D}, g)$ is regular,

$$\dim_{\mathrm{Hau}} K \geq \dim_{\mathrm{Hau}} M - 1.$$

In fact, we choose an open set U of M which is projected onto a manifold M' of dimension $\dim\ M - 1$ by π such that $\pi(K \cap U) = U'$ and the fibers of π are horizontal curves. We use boxes on a field of framings whose vectors are tangent to fibers of π. Then we proved the claim. □

In particular, we have

1.32. *If C-C-structure is contact, and K is a compact subset of M of topological dimension at least equal to $\frac{\dim M+1}{2}$, then*

$$\dim_{\mathrm{Hau}} K \geq \frac{\dim\ M+3}{2}$$

1.33 Isoperimetric Inequality. *Let M be a connected compact manifold with a C-C-structure. Suppose that (E,g) is regular. For a compact domain whose boundary is a hypersurface H and with $mes_{N-1}H \geq \frac{1}{2} mes_N M$,*

$$mes_{N-1}H \geq C(mes_N D)^{\frac{N-1}{N}}.$$

where, mes_{N-1} denotes the Hausdorff measure of dimension $N-1$, and C is a constant independent of D.

M.Gromov([Gr1]) showed this inequality by using Corollary 1.31 together with the following Lemma of Vitali type. Let mes $:=$ mes$_N$.

1.34 Vitali$'$ covering lemma. *For each $\lambda > 1$ there are positive numbers $\mu > 0$ and $\delta > 0$, such that for every measurable subset $D \subset M$ of measure μ there are balls $B_i := B(m_i, R_i)$ $i = 1, \cdots, v$ around some points $m_i \in M$ satisfying the following properties.*
(1) B_i are mutually disjoint ; moreover, the concentric balls $B_i' := B(m_i, \lambda R_i)$ are also mutually disjoint,
(2) B_i contain at least δ-part of the total measure of D, i.e.,

$$\sum_{i=1}^{m} mes(B_i \cap D) \geq \delta mes\, D,$$

(3) The intersection $B_i \cap D$ is δ-substantial in each ball,i.e.,

$$mes(B_i \cap D) \geq \delta mes(B_i)$$

(4) The intersection of D with the (larger) λR_i-balls B_i' are somewhat smaller than B_i, i.e.,

$$mes(D \cap B_i') \leq \frac{1}{2} mes(B_i) \quad i = 1, \cdots, v.$$

(5)

$$mes(D \cap B_i) \leq R_i \lambda mes_{N-1}(H \cap B_i') \quad i = 1, \cdots, v.$$

Proof. Consider concentric balls $B(m, R_j)$ for $m \in D$ of radii $R_j := 2^{-j} R_0$ for $R_0 :=$ diam M and $j = 1, 2, \cdots$ If m is a density point of D, mes$(D \cap B(m, R_j)) \geq \frac{1}{2}mes(B(m, R_j))$ for large j. If $\delta > 0$ is small and $\mu < \delta$mes$(B(m_1, R_1))$, there is first j, i.e., j_0, such that mes$(B(m, R_{j_0}) \cap D) \geq \delta$mes$(B(m, R_{j_0}))$. Furthermore, by making μ and δ smaller, we arrive at the situation where $\lambda R_{j_0} < R_1$ and the intersection of D with $B(m, \lambda R_{j_0})$ is somewhat smaller than B_j in the sense of (4). Thus for each density point $m \in D$ we constructed a ball $B(m, R := R_m)$ satisfying the above (3) and (4) and now we select the required B_i among them. We start with the ball $B_1 := B(m_1, R_{m_1})$ for the point m_1 where the function R_m assumes its maximum on D (note that R_m takes finitely many values). Then we take the point $m_2 \in D$ outside $B(m_1, 2\lambda R_{m_1})$ where again R_m is maximal on $D \setminus B(m_1, 2\lambda R_{m_1})$. Clearly the ball $\lambda B_2 := B(m_2, \lambda R_{m_2})$ does not intersect $\lambda B_1 := B(m_1, \lambda R_{m_1})$. Then we take the maximal ball outside $2\lambda B_1 \cup 2\lambda B_2$ for B_3 and so on. The resulting balls B_i satisfying (1), (3) and (4). Furthermore, the concentric balls $2\lambda B_i$ cover D, (it is trivial) and so by the doubling property these B_i contain definite part of D, i.e., satisfy (2) with some $\delta' > 0$ which may be somewhat smaller than the one used above.

To conclude the proof we must ensure that all (or almost all) points $m \in D$ are density points with respect to the C-C distance. In fact, D is open or at least contains an open dense set of full measure. □

Remark. *The above proof does not need equiregularity of M if " mes " is understood in the Riemannian (Lebesgue) sense.*

1.35 Isoperimetric inequality in compact domains $M_0 \subset M$. *There is constants $\mu > 0$ and $C \geq 0$ (depending on M_0) such that every domain D inside M_0 with mes $D \leq \mu$ bounded by a closed hypersurface H satisfies*

$$(++) \qquad mes(D) \leq C(mes_{N-1}H)^{\frac{N}{N-1}}$$

where $N := \dim_{\mathrm{Hau}} M$.

Proof. Take B_i from Lemma 1.34 with a sufficiently large λ and apply (5) to each intersection $D \cap B_i$. Then we note that each (small) ball $B := B(R)$ has $\mathrm{mes}(B) \gtrsim \mathbb{R}^N$ and since the measure of $D \cap B$ is compatible with $\mathrm{mes}(B)$ (5) gives us the bounded measure $\mathrm{mes}_{N-1}(H \cap B') \gtrsim \mathbb{R}^{N-1}$, which implies that

$$R\mathrm{mes}_{N-1}(H \cap B') \lesssim (\mathrm{mes}_{N-1}(H \cap B'))^{\frac{N}{N-1}}.$$

Thus we have ((5) of R), for all i,

$$\mathrm{mes}(D \cap B_i) \leq C_{M_0}\mathrm{mes}_{N-1}(H \cap \lambda B_i).$$

Since the balls B_i exhaust (an essential part of) D and the balls λB_i do not intersect owing to (1), the claim holds. □

Remarks.

(1) *If M is a compact manifold, we must restrict D in size. For example, if M is a closed connected manifold, then $(++)$ also holds true with $C := C(M)$ for all $D \subset M$ with $mes(D) \leq \frac{1}{2}mes(M)$.*

(2) *The inequality $(++)$ and its proof can be transplanted to the asymptotic frame work, that is, for a smooth automorphism $A : TM \longrightarrow TM$, we set $g_t := A^*g$. In this case, it can be used for evaluating the Sobolev constant and the first eigenvalue of (M, g_t) for $t \longrightarrow +\infty$.*

The isoperimetric inequalities give one Sobolev inequalities. Let $f : M \longrightarrow \mathbb{R}$ be a C^1-function (which is supposed for simplicity). Let $d_E f : D \longrightarrow \mathbb{R}$ be the restriction of the differential $df : TM \longrightarrow \mathbb{R}$ to E. For all $m \in M$, $d_E f(m)$ belongs to $\mathcal{E}^*[m]$, dual to $\mathcal{E}[m]$, which is isomorphic to $T^*_m M/\mathcal{E}^0[m]$, where $\mathcal{E}^0[m]$ is the annihilator of $\mathcal{E}[m]$. We can measure $d_E f$ by means of the cometric g^*. Let $< d_E f >_{g^*}$ be the norm of $d_E f$ with respect to g^*.

For a C^1-map $M \longrightarrow V$ between Riemannian manifolds, $< d_E f(m) >$ is set as the norm of the map $df(m) : \mathcal{E}[m] \longrightarrow T_{f(m)}V$ with respect to the metrics $\mathcal{E}[m]$ and $T_m Y$.

The isoperimetric inequality yields the following Sobolev inequality for the L_p-norm, $p = \frac{N}{N-1}$ of a function f on M in terms of the L_1-norm of $d_E f$;

$$(*) \qquad \int_M |f(m)|^{N/N-1} \mathrm{dvol} \leq \mathrm{const} \left(\int_M < d_E f(m) >_{g^*} \mathrm{dvol} \right)^{N/N-1}$$

This implies the bound

$$(*\mathrm{q}) \qquad \|f\|_{L_p} \leq \mathrm{const}_q < d_E f >_{L_q}$$

for all q in the interval $1 \leq q < N$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$ as follows. Apply (*) to $|f|^a$ for $a = \frac{p(N-1)}{N}$, we have

$$\|f\|^a_{L_p} \lesssim \| |f|^{a-1} < d_E f >_{L_1} \|$$

where $\lesssim$ means that $'' \leq \mathrm{const}''_q$. Then we use the Hölder inequality

$$\| |f|^{a-1} < d_E f > \|_{L_1} \lesssim \| |f|^{a-1} \|_{L_b} < d_E f >_{L_q}$$

for $b = (1 - \frac{1}{q})^{-1}$, and note that

$$\| |f|^{a-1} \|_{L_b} = \|f\|^c_{L_p}$$

for $c := pb^{-1}$ Thus $\|f\|^{a-c}_{L_p} \lesssim < d_E f >_{L_q}$, which yields $(*q)$ since $a - c = 1$ with our choice of a, b and c.

The inequality (*) for $q > 1$ can be also derived from the following estimates for convolution integrals. Let M be a nilpotent group and $K(v)$ be a function (convolution kernel), such that

$$|K(m)| \leq (\mathrm{dist}(0, m))^{-(N-1)}, \quad m \in M$$

Then

$$(**\mathrm{q}) \qquad \|K * f\|_{L_p} \leq \mathrm{const}_q \|f\|_{L_q}$$

for all q in the interval $1 < q < N$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$.

This is classical for $M := \mathbb{R}^N$.

Finally, we recall the Green form $\omega_0(m)$. Let M be a nilpotent group with a self-homotopy $A : M \longrightarrow M$. A closed $(n-1)$-form ω_0 on $M \setminus \{0\}$, 0 stands for the identity element, is said to be a Green form if it is

(1) E-horoizontal,
(2) A-invariant,
(3) closed and non-exact,
(4) $\|\omega_0\|_m \leq \mathrm{const}\ \mathrm{dist}^{-(N-1)}(0, m), \quad m \in M$

Note that the non-exactness of ω_0 makes

$$\int_S \|\omega_0\|_S ds \geq \left| \int_H \omega_0 \right| = c_0 \neq 0$$

for a fixed c_0 and all smooth closed hypersurfaces H around the origin, where ds refers to the $(N-1)$-dimensional Hausdorff measure on H.

Example. *If $M := \mathbb{R}^n$, such an ω_0 may be obtained as the radial pull-back of the volume form on $S^{n-1} \subset \mathbb{R}^n$.*

We consider smooth maps $p : M \longrightarrow B^{n-1}$ with E-horizontal fibers. Note that such a p pulls back (necessarily closed)$(n-1)$-forms on B^{n-1} to closed $(n-1)$-forms ω on M which vanish on E, i.e., vanish on every (local) hyperplane field containing E, and such forms are called E-horizontal.

1.36 Linear Lemma. *If the subbundle $E \subset TM$ Lie generates TM, then every $(n-1)$-dimensional de Rham cohomology class in M can be represented by a closed horizontal $(n-1)$-forms ω.*

First proof. Passing to the double cover of M if necessary, we assume that M is oriented and so the cohomology $H^{n-1}(M;\mathbb{R})$ is dual to $H_1(M;\mathbb{R})$. Then every integral class in $H_1(M;\mathbb{R})$ can be realized by a closed horizontal curve c which gives one a closed $(n-1)$-current, called c^*, representing the class $[c]^* \in H^{n-1}(M;\mathbb{R})$ where the latter $*$ denotes the Poincaré duality. Now, in order to pass from currents to forms one needs some smoothing or diffusion of currents preserving E-horizontality. This is easy if M admits a transitive action of a connected group G preserving E as one can diffuse the current c^* by taking $\int_G c^* d\mu$, where $d\mu$ is a smooth measure with a compact support on G (localized near id $\in G$). For example, this diffusion is available if our polarization E is a constant structure. In the general case, the diffusion is achieved with a smooth family of horizontal curves, say $c_b \subset M$, $b \in B$, such that the corresponding map $S' \times B \longrightarrow M$ (for c_b parametrized by the circle S') is a submersion. The existence of such family is proven in the same manner as of an individual c. $\square$

Second proof. We assume that M is oriented and take a non-vanishing oriented volume form Ω on M. Then the interior product with Ω establishes an isomorphism between vector fields X and exterior $(n-1)$-forms, i.e.,

$$X \longleftrightarrow X \cdot \Omega,$$

and similarly bivectors correspond to $(n-2)$-forms

$$X \wedge Y \longleftrightarrow (X \wedge Y) \cdot \Omega$$

Closed $(n-1)$-forms correspond to divergence free vector fields, where the divergence δX of X is the function defined by the equality

$$L_X \Omega = (\delta X)\Omega$$

where L_X denotes the Lie derivative. We recall the formula

$$d((X \wedge Y) \cdot \Omega) = [X,Y] \cdot \Omega + Y \cdot L_X \Omega - X \cdot L_Y \Omega$$

which implies that the field

$$[X,Y]+\delta(X)Y-\delta(Y)X$$

has zero divergence and, moreover, corresponds to an exact $(n-1)$-form. Then, for all functions a, the field

$$a[X,Y]+(Xa+a\delta(X))Y-\delta(aY)X$$

corresponds to an exact $(n-1)$-form, or in other word, $a[X,Y]$ is equal to $(Xa+a\delta(X))Y-\delta(aY)X$ modulo(the fields corresponding to) exact forms. (The latter expression is antisymmetric in X and Y since $Xa+a\delta(X)=\delta(aX)$).

We note that $(n-1)$-forms vanishing on $E\subset TM$ corresponds to vector fields sitting on E. Thus, to show the Lemma 1.36, we must find a divergence free E-horizontal field in a given cohomology class. We pick up some fields $X_1,\cdots,X_s$ spanning E (here s may be greater than rank E) add to these X_i their successive commutators, say, $X_j,\ j:=s+1,\cdots,r$, which span TM and note that every cohomology class in $H^{n-1}(M;\mathbb{R})$ can be represented by a divergence free field of the form $\sum_{i=1}^r a_iX_i$. But the above formulas allows us replace every (commutator) term in this sum with $i>s$ by (cohomologically) equivalent lower terms and thus we have a desired divergence free representative of the form $\sum_{i=1}^s a_i'X_i$. □

1.37 Lemma. *Every M admits a Green form.*

Proof. Divide $M\setminus\{0\}$ by the (infinite cyclic) group $\{A'\}$ generated by A, take some E-horizontal closed non-exact $(n-1)$-form $\overline{\omega}$ on the quotient space $(M\setminus\{0\})/\{A'\}$ (which exists according to Linear Lemma) and pull $\overline{\omega}$ back to M for the quotient map

$$M\longrightarrow\{0\}\longrightarrow M\setminus\{0\}/\{A'\}$$

□

Let $X(m)$ be the divergence free vector field associated to the Green form $\omega_0(m)$. Then we have

$$\|X(m)\|\leq \text{const}(\text{dist}(0,m))^{-(N-1)}.$$

We note that every function f on M decaying at ∞ can be reconstructed from d_Ef by convolution with $X(m)$, as $f(0)=\int_M df(X(m))dvol$, and so

$$(**q)\Longrightarrow(*q)\quad\text{for } q>1.$$

Remark 1.6. *It is easily proved from an estimate of vol($B_1(\epsilon)$) (here vol. means Riemannian volume) that*

$$C^{-1}\epsilon^Q\leq vol(B_1(\epsilon))\leq C\epsilon^Q$$

for some $C>1$ and Q is the Hausdorff dimension of the metric space (M,d_c), i.e.,

$$Q=\sum_i i(\dim(E_i)-\dim(E_{i-1})).$$

Remark 1.7. *The Hausdorff Q-dimensional measure μ^Q is commensurate with Lebesgue measure (on $B_1(1)$) ;*

$$\left(\frac{V_Q}{C\cdot 2^Q}\right)\mu\leq\mu^Q\leq(C\cdot V_Q)\mu,$$

where μ is the Lebesgue measure and V_Q is the volume of unit ball in $\mathbb{R}^Q$.

§2.1 Minimizing curves and Geodesics.

2.1 Definition. *Let (M, E, g) be a sub-Riemannian manifold. A minimizing curve is a horizontal curve $\phi : [a, b] \longrightarrow M$ such that $L(\phi) \leq L(\psi)$ for all horizontal curves $\psi : [c, d] \longrightarrow M$ with $\psi(c) = \phi(a)$, $\psi(d) = \phi(b)$.*

If ϕ is minimizing, then the restrictions of ϕ to all closed intervals are the same.

In the Riemannian case, every minimizing curve is the projection of an extremal. These curves can be defined by two methods - one is dependent on the square of Lagrangian, and other is dependent on Hamiltonian. We shall use Hamiltonian formalism. To use the Lagrangian formalism, it is neccesary to introduce the multplifiers of Lagrangian, which is not easy to define in the sub-Riemannian case. If $H := \frac{1}{2}g^*$, g^* is the cometric of g, then extremals are the trajectories of Hamiltonian field $\vec{H}$ of H (for the canonical symplectic structure on T^*M).

For a SR-structure (E, g), the situations are very different from the Riemannian case. We have two cases of minimizing curves in terms of theory of variations. One generalizes the Riemannian cases ; they are projections of trajectories of hamilton field $\vec{H}$ of $H = \frac{1}{2}g^*$ on which H does not vanish, where g^* is the cometric of g. Other is founded by Montgomery ([Mo]).

Let E^0 be the annihilator of E which is a vector subbundle of T^*M. Generally, the restriction ω_{E^0} of the canonical symplectic form ω of T^*M to E^0 (more preicisely, to $TE^* \times_{E^0} TE^0$) is not symplectic.

2.2 Definition. *A characteristic curve of E^0 is an absolute continuous curve $\phi : I \longrightarrow E^0$, I is an interval not reducing to one point, such that, for almost all $t \in I$, $\frac{d\phi}{dt}(t)$ belongs to the kernel of $\omega_{E^0}|_{\phi(t)}$.*

This class of minimizing curves consists of horizontal curves which are projections of characteristic curves included in $E^0 \setminus O_M$. Note that the intersection of these classes of minimizing curves is not empty ; A minimizing currve may have two liftings into T^*M, one is a trajectory of $\vec{H}$, other is a characteristic curve of E^0.

Example 2.1.

We consider canonical coordinates (x, y, z) on $M := \mathbb{R}^3$. We set

$$E := \ker(dz - \frac{y^2}{2}dx)$$
$$g := dx^2 + dy^2|_E,$$
$$\phi : \mathbb{R} \longrightarrow M,\ \phi(t) := (t, 0, 0)$$

If $p, q, r : T^*M \longrightarrow \mathbb{R}$ are dual coordinates of dx, dy, dz, a trajectory of $\vec{H}$, contained in $TM \setminus O_M$, which is a lifting of ϕ is $\Phi(t) : x = t, y = z = 0, p = 1, q = r = 0$ (it is not a lifting trajectory of $\vec{H}$). A charactersistic curve in $E^0 \setminus O_M$, a lifting of ϕ, is $\Psi(t) : x = t, y = z = 0, p = q = 0, r = 1$.

Contact structures.

If SR-structure is a contact structure, E^0 is a real line bundle and $E^0 \setminus O_M$ is a symplectic submanifold of T^*M. The we have exceptional geodesics.

In the followings, an ordinary geodesic, or simply, a geodesic is the projection of a trajectory of $\overrightarrow{H}$ contained in $TM \setminus O_M$ and an exceptional geodesic is a horizontal curve which is the projection of a characteristic of E^0 contained in $E^0 \setminus O_M$.

The notion of characteristic curves is not so geometric..

We work locally, and so let M be an open set in $\mathbb{R}^d$ and let E be the kernel of a vectorial form $\omega : TM \longrightarrow \mathbb{R}^c$. We extend the SR-metric $g : E \longrightarrow \mathbb{R}$ to a Riemannian metric on M such that ω is the isomorphism on orthogonal vectors to E. Let $H^1(M; a, b)$, $a, b \in M$ be the space of curves $\phi : [0,1] \longrightarrow M$ with finite energy and $\phi(0) := a$ and $\phi(1) := b$, and let $\mathrm{Hor}(E, a, b)$ be the subset of horizontal curves in $H^1(M, a, b)$;

$$\mathrm{Hor}(E, a, b) := \widetilde{\omega}^{-1}(0),$$

where $\widetilde{\omega}^{-1}(\phi) = \omega(\frac{d\phi}{dt}).H^1(M, a, b)$ and $L^2([0,1]; \mathbb{R}^c)$ are manifolds modelled as Hilbert spaces and $\widetilde{\omega}$ is a C^∞-map. Then exceptional geodesics are singularities of map $\widetilde{\omega}$ on the kernel(manifold) of $\widetilde{\omega} = 0$.

§2.2 Ordinary geodesics.

A fundamental difference between (ordinary) geodesics in the sub-Riemannian case and in the Riemannian case is as followings ; in the Riemannian case, a geodesic is not determined by its initial vector but is determined by its initial covector. This is the initial point of a trajectory of $\overrightarrow{H}$ lifting the geodesic. More precisely, if z is its initial covector, then the initial tangent vector is determined by the condition ;

$$< w, v >_g = < w, z > \quad \forall\, w \in E[\pi_{T^*M}(z)].$$

In the Riemannian case, this condition determines z if v is known, and the correspondence $v \longmapsto z$ is nothing but the Legendre transformation associated to the metric g. In the sub-Riemannian case, this claim does not hold and the parameters of z which is not determined by v are differential invariants of order ≥ 2 of the geodesic. We can interpret this fact by means of curves. We can state the minimality of small arcs of ordinary geodesics as in Riemannian case.

2.3 Proposition. *Let $\phi : [a, b] \longrightarrow M$ be an ordinary geodesic of the SR-structure (E, g) parametrized by arc-length. For all $t \in [a, b]$, there is an $\epsilon > 0$ whose restriction to $[a, b] \cap [t - \epsilon, t + \epsilon]$ is minimizing.*

Exponential maps.

For all $m \in M$, we can define an exponential map $\exp_m : O \longrightarrow M$ such that $O \cap \{H = 0\} = 0_m$ (zeros of $T_m M$), where O is an open set of M. Note that the set $\{H = 0\}$ in $T^*_m M$ is nothing but the annihilator $E^0[m]$ of $E[m]$. For $z \in O,\ z \neq 0_m$, we have

$$\exp_m(z) = \pi_{T^*M_m}(\phi(\sqrt{2H(z)}, \overline{z})$$

if $\phi(\sqrt{2H(z)},\overline{z})$ is defined, where $t \longmapsto \phi(t,\overline{z})$ is the trajectory of $\overrightarrow{H}$ such that $\dfrac{d\phi}{dt}(0,\overline{z}) = \overline{z} = \dfrac{z}{\sqrt{2H(z)}}$ and $\exp_m(0_m) = m$.

This map does not have good properties in the Riemannian case ;

• The image of $\exp_m$ is not contained in O, and, in particular, $\exp_m$ is not a local diffeomorphism at $0_m \subset T^*_m M$.
• If $z \in \{H = \frac{1}{2}\} \cap T^*_m M$, a conjugate point $pc(z)$ of z is defined to be the point $\exp_m(t_c z)$, if it exists, such that $\exp_m$ is a local diffeomorphism at tz for all t; $0 < t < t_c$ and is not so at $t = t_c$. Then there is a compact set $K \subset T^*_m M$ such that, for $z \notin K \cap \{H = \frac{1}{2}\}$, pc$(z)$ is defined and tends m when z tends to ∞ in $\{H = \frac{1}{2}\} \cap T^*_m M$.

§2.3 Exceptional geodesics.

A horizontal characteristic is defined to be a characteristic which is projected onto a horizontal curve. We distinguished these horizontal characteristics according to the rank of the distribution $\mathcal{D}$. In the case where the rank is of odd,for a generic distribution $\mathcal{D}$, there is an open set O_0 in $\mathcal{D}^0$ whose complementary os of codimension 1 in $\mathcal{D}^0$ and there is a line field δ on O tangent to $\mathcal{D}^0$, such that all horizontal characteristics contained in O are integral curves of δ. In the case whrere the rank is of even, there is generically a saturated set Σ of $\mathcal{D}^0$, of codimension 1 in $\mathcal{D}^0$, and there is a line field δ tangent to Σ on the union of saturated sets of codimension 1 of Σ ; Every horizontal characteristcs are contained in Σ and if it is contained in O, it is an integral curve of δ. In the followings, we say the horizontal characteristics contained in O to be generic, and call their projections generic exceptional geodesics. " exceptional" geodesics are related to their rigidity.

2.4 Definition. *A horizontal C^1-curve $\phi : [a,b] \longrightarrow M$ is rigid if there is a neighbourhood U of ϕ in the space $C^1([a,b];M)$ with C^1-topology such that, if ψ is a horizontal C^1-curve in U with $\psi(a) = \phi(a)$, $\psi(b) = \phi(b)$, then there is a C^1-diffeomorphism $f : [a,b] \longrightarrow [a,b]$ such that $\psi = \phi \circ f$.*

Minimality and rigidity of small arcs of exceptional geodesics for the distribution of rank 2..

2.5 Proposition. *Suppose that the distribution on M is of rank 2. Let $\phi : [a,b] \longrightarrow M$ be a generic exceptional geodesic parametrized by the arc-length. There is an $\epsilon > 0$ whose restriction to the subinterval of $[a,b]$ with length $\leq \epsilon$ is minimal and rigid.*

For proof,see [L-S] appendix.

§2.4 Critical points of the energy integral.

For a fixed point $p \in M$, let $\mathcal{C}_p$ and $\mathcal{C}_p^E$ be the set of smooth curves on M and the set of horizontal smooth curves on M respectively. $\mathcal{C}_p^E$ is a submanifold of $\mathcal{C}_p$. Let $e_1, \cdots, e_n$ be an orthonormal framing on a some neighborhood U of p such that $e_1 \cdots e_k$ is the basis of E. And let $\omega^1, \cdots, \omega^n$ be the dual framing to $e_1, \cdots, e_n$. Then $\omega := (\omega^1, \cdots, \omega^n)$ is a $\mathbb{R}^n$-valued 1-form. The tangent space $T_c\mathcal{C}_p$ at c is the set of vector fields along c with $p = c(0)$. For $X \in T_c\mathcal{C}_p^E$, there is a horizontal variation curve $\alpha : (-\epsilon, \epsilon) \times [0,1] \to M$. If we set

$$D_c X := \frac{d}{ds}\omega(\frac{\partial \alpha}{\partial t})|_{s=0} = \frac{d}{dt}\omega(x) - 2d\omega(\dot{c}, X) \tag{2.1}$$

(This definition is independent of the choices of α), then that X is an element of $T_\gamma C_p^E$ is equivalent to that D_cX is a curve on $\mathbb{R}^k \subset \mathbb{R}^n$. On the other hand, for the endpoint map $\pi : C_p^E \to M : c \longmapsto c(1)$, its differential $\pi_*(c) : T_cC_p^E \longrightarrow T_{c(1)}M : X \longmapsto X(1)$ is not necessarily surjective. In fact, if c is one point, then image of π_* is equal to the fiber E_p of E. Thus the set $C_{pq}^E := \pi^{-1}(q)$ of all horizontal curves joining p and q does not have the structure of manifolds. Now, we define the tangent space $T_cC_{pq}^E$ to C_{pq}^E at c by the set of all variation vector fields $X := \frac{\partial \alpha}{\partial s}|_{s=0}$ corresponding to the Dirichlet variation problem $\alpha : (-\epsilon, \epsilon) \times [0,1] \longrightarrow M$.

- This set is contained in $\ker \pi_*(c)$.

2.6 Lemma([Ki]). *If c is regular (i.e. not abnormal), then*

$$T_cC_{pq}^E = \ker \pi_*(c).$$

In the following, we assume that horizontal curves are regular. We consider the Dirichlet variation problem $\alpha : (-\epsilon, \epsilon) \times [0,1] \longrightarrow M$ *of a horizontal curve c. Let X be the corresponding variation vector field. Then, the energy integral* $E(c) := \frac{1}{2}\int_0^1 < \omega(\dot{c}), \omega(\dot{c}) > dt$ *of c can be written as*

$$\frac{1}{2}\frac{d}{ds}E(c_s)|_{s=0} = \int_0^1 < D_cX, \omega(\dot{c}) > dt,$$

where $c_s(t) := \alpha(s,t)$.

We take an inner product g_c *on* T_cC_p:

$$g_c(X,Y) := \int_0^1 < D_cX, D_cY > dt.$$

Let J_c^E *be the orthogonal complement to the subspace* $\ker \pi_*(c) = T_cC_{pq}^E$ *in* $T_cC_p^E$ *with respect to the inner product* g_c.

- *That c is a geodesic is equivalent to that*

$$D_c^{-1}\omega(\dot{c}) \in J_c^E.$$

If C_{pq} is the submanifold of C_p (consisting of curves not necessarily horizontal joining p and q), then the tangent space T_cC_{pq} consists of vector fields along c with value 0 at endpoints. Let J_c be the orthogonal complement to T_cC_{pq} in T_cC_p. If we define a $\mathfrak{gl}(m;\mathbb{R})$-valued 1-form a by

$$a(u)\omega(v) := -2d\omega(u,v),$$

then, we have, for $X \in T_cC_{pq}$

$$\begin{aligned} g_c(X,Y) &= \int_0^1 < \frac{d}{dt}\omega(X) + a(\dot{c})\omega(X), D_cY > dt \\ &= -\int_0^1 < \omega(X), \frac{d}{dt}D_cY - a^*(\dot{c})D_cY > dt, \end{aligned}$$

where $a^*(\dot{c})$ denotes the transpose of $a(\dot{c})$. Then we have

$$J_c = \{Y \in T_cC_p | \frac{d}{dt} D_cY - a^*(\dot{c})D_cY = 0\} \tag{2.2}$$

On the other hand, if we set the orthogonal projection

$$P := D_c^{-1} \circ \widetilde{P} \circ D_c : T_cC_p \to T_cC_p^E$$

(where $\widetilde{P}$ is the orthogonal projection $\mathbb{R}^n \longrightarrow \mathbb{R}^k$), then we have

$$PJ_c \subset J_c^E.$$

2.7 Lemma. *If c is regular, then*

$$J_c^E = PJ_c.$$

Namely, for $\forall X \in J_c^E$, there is a map $\varphi : [0,1] \longrightarrow \mathbb{R}^n$, such that

$$\begin{cases} D_cX = P\varphi \\ \varphi' - a^*(\dot{c})\varphi = 0 \end{cases}$$

2.8 Theorem(Hamenstädt)([H]). *If c is a regular geodesic, there is a map $\varphi : [0,1] \longrightarrow \mathbb{R}^n$ such that*

$$\begin{aligned} \omega(c) &= P\varphi \\ \varphi' - a^*(\dot{c})\varphi &= 0. \end{aligned} \tag{2.3}$$

For a variation curve $\alpha : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times [0,1] \longrightarrow M$ of a geodesic c, if we set $X := \partial\alpha/\partial s|_{(s,u)=(0,0)}$ and $Y := \partial\alpha/\partial u|_{(s,u)=(0,0)}$, then

$$\begin{aligned} &\frac{\partial^2}{\partial s \partial u} E(c_{s,u})|_{(s,u)=(0,0)} \\ &= \int_0^1 \{< D_cX, D_cY - a^*(Y)\varphi > - < \theta(X), 2A^*(\dot{c}, Y)\varphi >\} dt, \end{aligned} \tag{2.4}$$

where $c_{s,t}(t) := \alpha(s,u,t)$ and A^* is the $\mathfrak{gl}(m;\mathbb{R})$-valued 2-form defined by

$$A^*(u,v) := da^*(u,v) - \frac{1}{2}[a^*(u), a^*(v)].$$

We define the index form $I_c(X,Y)$ as regarding the right hand side of (2.4) as a symmetric bilinear form on on $T_cC_p^H$. Now, we assume that $Y \in T_cC_p^H$ satisfies

$$I_c(X,Y) = 0 \quad Y \in T_cC_{pq}^H.$$

If we take $\psi_0 : [0,1] \longrightarrow \mathbb{R}^m$ so that

$$\begin{aligned} \psi_0' - a^*(\dot{c})\psi_0 &= -2A^*(\dot{c}, Y)\varphi \\ \psi_0(0) &= 0, \end{aligned}$$

then

$$\begin{aligned} I_c(X,Y) &= \int_0^1 < D_cX, D_cY - a^*(Y)\varphi - \psi_0 > dt \\ &= \int_0^1 < D_cX, D_cY - P(a^*(Y)\varphi + \psi_0) > dt \\ &= g_c(X, D_c^{-1}(D_cY - P(a^*(Y)\varphi + \psi_0))). \end{aligned}$$

Thus we have $D_c^{-1}(D_cY - P(a^*(Y)\varphi + \psi_0)) \in J_c^E$. Therefore, there is $\psi_1 : [0,1] \longrightarrow \mathbb{R}^m$ such that

$$\begin{aligned} D_cY - P(a^*(Y)\varphi + \psi_0) &= P\psi_1 \\ \psi_1' - a^*(\dot{c})\psi_1 &= 0 \end{aligned}$$

Summing up, we have

2.9 Definition([Ki]). *A vector field Y is a Jacobi field along c if there is a $\psi : [0,1] \longrightarrow \mathbb{R}^m$ $(\psi = \psi_0 + \psi_1)$ such that*

$$(2.5) \qquad \begin{aligned} D_c Y &= P(a^*(Y)\varphi + \psi) \\ \psi' - a^*(\dot{c})\psi &= 2A^*(\dot{c}, Y)\varphi. \end{aligned}$$

2.10 Proposition([Ki]). *If (c_s, φ_s) is the solution to (2.3) with the initial condition $c_s(0) = p$, $\varphi_s = u + sv$ $(u, v \in \mathbb{R}^m,\ |s| < \epsilon)$, then its variation vector field $Y := \partial c_s/\partial s|_{s=0}$ is a Jacobi field satisfying $Y(0) = 0, \psi(0) = v$.*

If Y is identically zero, (2.5) implies that $P\psi$ is identically zero and satisfies that $\psi' - a^*(\dot{c})\psi = 0$. Since c is normal, ψ is identically zero. Therefore, a Jacobi field Y is uniquely determined with initial condition $(Y(0), \psi(0))$.

Appendix. The first conjugate locus of Heisenberg group ([N-S], [Pa]).

Let H be the Heisenberg group with Lie algebra $\mathfrak{h}$,i.e.,

$$H := \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

We fix a left invariant metric on H, and choose an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{h}$ as followings ;

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = 0$$

It follows from Cambel-Hausdorff$'$ formula that for $g, h \in H$

$$\begin{aligned} x_1(gh) &= x_1(g) + x_1(h) \\ x_2(gh) &= x_2(g) + x_2(h) \\ x_3(gh) &= x_3(g) + x_3(h) + \frac{1}{2}\left(x_1(g)x_2(h) - x_1(h)x_2(g)\right) \end{aligned}$$

Since X_1, X_2, X_3 are left invariant,

$$(2.6) \qquad \begin{aligned} X_1 &= \frac{\partial}{\partial x_1} + \frac{1}{2}x_2\frac{\partial}{\partial x_3} \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3} \\ X_1 &= \frac{\partial}{\partial x_3} \end{aligned}$$

Let $\sigma(t)$ be a geodesic through the identity $e \in H$, i.e., $\sigma(t) := \sum_{i=1}^{3} a_i(t)X_i(\sigma(t))$, $a_i(t)$ are smooth functions. The fundamental fact on Lie groups implies that

$$(2.7) \qquad \begin{aligned} \frac{da_1}{dt} &= -a_3a_2 \\ \frac{da_2}{dt} &= a_3a_2 \\ \frac{da_1}{dt} &= 0 \end{aligned}$$

Since a_3 is constant and

$$\begin{bmatrix} \frac{da_1}{dt} \\ \frac{da_1}{dt} \end{bmatrix} = a_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

(2.8)
$$\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} = \begin{bmatrix} \cos a_3 t & -\sin a_3 t \\ \sin a_3 t & \cos a_3 t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

is the solution of the ordinary differential equation (2.7) with the initial condition $a_1, a_2 \in \mathbb{R}$. We set $\sigma(t) := (x_1(t), x_2(t), x_3(t))$. Since $\dot\sigma(t) = \sum_i a_i(t) X_i(\sigma(t))$, $x_i(t)$ $(1 \le i \le 3)$ are a solution of ordinary differential equations

$$\begin{cases} \frac{dx_1}{dt} & = a_1(t) \\ \frac{dx_2}{dt} & = a_2(t) \\ \frac{dx_3}{dt} & = a_3(t) - \frac{1}{2}\left(a_1(t)x_2 - a_2(t)x_1\right) \end{cases}$$

by (2.6). It is evident to see that if $a_3 \neq 0$,

$$\begin{aligned} x_1(t) &= \frac{a_1}{a_3}\sin a_3 t + \frac{a_2}{a_3}(\cos a_3 t - 1) \\ x_2(t) &= \frac{a_2}{a_3}\sin a_3 t + \frac{a_1}{a_3}(\cos a_3 t - 1) \\ x_3 &= a_3(t) - \frac{1}{2}\left(a_1(t)x_2 - a_2 x_1\right) \end{aligned}$$

is the solution of (2.9) with the initial condition $\frac{dx_i}{dt}(0) = a_i, x_i(0) = 0$ $(1 \le i \le 3)$, and if $a_3 = 0$,

$$\begin{cases} x_1 & = a_1 t \\ x_2 & = a_2 t\ . \\ x_3 & = 0 \end{cases}$$

Now, we consider Jacobi fields J along a geodesic $\sigma(t)$ such that $J(0) = 0$ and $\nabla_{\dot\sigma} J)(0) = w$. Let T_eH be the tangent space of H at the identity $e \in H$ and $\mathrm{Exp}_e : T_eH \longrightarrow H$ the exponential map. We consider the variation field of the one-parameter family of geodesics $\mathrm{Exp}_e(t(\dot\sigma(0) + sw))$. By (2.10),for a geodesic $\sigma(t)$ with $\dot\sigma(0) = (a_1, a_2, a_3)$, $a_3 \neq 0$, a basis of Jacobi fields along the geodesic $\sigma(t)$ is given by

$$\begin{aligned} J_1 &= t\dot\sigma(t) = \left(t\frac{dx_1}{dt}, t\frac{dx_2}{dt}, t\frac{dx_3}{dt}\right) \\ J_2 &= \left(\frac{1}{a_3}\sin a_3 t, -\frac{1}{a_3}(\cos a_3 t - 1), \frac{a_1}{a_3}t - \frac{a_1}{a_3}\sin a_3 t\right) \\ J_3 &= \left(\frac{1}{a_3}(\cos a_3 t - 1), \frac{1}{a_3}\sin a_3 t, \frac{a_2}{a_3}t - \frac{a_2}{a_3^2}\sin a_3 t\right) \end{aligned}$$

By (2.10),(2.11), for a geodesic $\sigma(t)$ with $\dot\sigma(0) = (a_1, a_2, 0)$, a basis of Jacobi fields along the geodesic $\sigma(t)$ is given by

$$\begin{cases} J_1 & = (a_1 t, a_2, 0) \\ J_2 & = (-a_2 t, a_1 t, 0) \\ J_3 & = (0, 0, t) \end{cases}$$

From (2.13) we see that there are no conjugate point along a geodesic $\sigma(t) := (a_1 t, a_2 t, 0)$. Now we compute the first conjugate point along a geodesic $\sigma(t)$ with $\dot\sigma(0) = (a_1, a_2, a_3)$, $a_3 \neq 0$, where $a_1^2 + a_2^2 + a_3^2 = 1$.

2.11 Lemma. *We define a function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by*

$$f(\theta) := \begin{vmatrix} \sin\theta & \cos\theta - 1 & a_1\cos\theta - a_2\sin\theta \\ 1-\cos\theta & \sin\theta & a_1\sin\theta + a_2\cos\theta \\ a_1(\theta - \sin\theta) & a_2(\theta-\sin\theta) & \frac{1+a_3^2}{2} - \frac{1-a_3^2}{2}\cos\theta \end{vmatrix}$$

Then we have

$$f(\theta) = \{2(1-\cos\theta) - (1-a_3^2)\theta\sin\theta\}$$
$$f(\theta) = f(-\theta)$$
$$f(\theta) > 0 \ (0 < |\theta| < 2a)$$

Proof. Our first claims a straightforward computation. Note that

$$f'(\theta) = (1+a_3^2)\sin\theta - (1-a_3^2)\theta\cos\theta$$
$$f''(\theta) = a_3^2\cos\theta + (1-a_3^2)\theta\sin\theta$$

For $0 < \theta < \frac{\pi}{2}$, $f''(\theta) > 0$ and so $f'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ since $f'(0) = 0$. Oviously $f'(\theta) > 0$ for $\frac{\pi}{2} < \theta < \pi$ and so $f(\theta) > 0$ for $< \theta < \pi$. It is evident that $f(\theta) > 0$ for $\pi \leq \theta < 2\pi$. □

From (2.12) and Lemma 2.7,setting $\theta := a_3 t$, we see that the first conjugate point along a geodesic $\sigma(t)$ with $\dot\sigma(t) = (a_1, a_2, a_3)$, $a_3 \neq 0$ is given by

$$\sigma(\frac{2\pi}{|a_3|}) = (0, 0, \frac{\pi(1+a_3^2)}{|a_3^2|}).$$

2.12 Theorem. *For a left invariant metric on the Heisenberg group H, the first conjugate locus of the identity element of H is contained in the center of H and given by*

$$\left\{(0,0,\pm\frac{\pi(1+a_3^2)}{a_3^2}) \mid 0 < a_3^2 \leq 1\right\} = \{(0,0,\pm s\pi) \mid s \geq 2\}.$$

Remark. *Let $\sigma(t)$ be a geodesic and $q := \sigma(t)$ a conjugate point to $\sigma(0)$. The dimension of Jacobi fields J such that $J(0) = J(t_0) = 0$ is called the order of the conjugate point q of the geodesic $\sigma(t)$. The order of the first conjugate point of a geodesic $\sigma(t)$ in H with $\sigma(0) := (a_1, a_2, a_3)$ $a_3 \neq 0$ is given as followings ;*

$$\begin{cases} 1 & \textit{if}\, a_3 \neq 1 \\ 2 & \textit{if}\, a_3 = 1 \end{cases}$$

2.13 Theorem. *Let H be the Heisenberg group with a left invariant metric. For each $g \in H$ the cut locus of g coincides with the first conjugate locus of g. Moreover the cut locus of the identity element of H is contained in the center of H.*

Proof. Since H acts as an isometry group via left translations, it is sufficient to see our claim at the identity element $e \subset H$. Let $C(t)$ be a geodesic through e which is contained in the center of H and let x be the first conjugate point to e along $C(t)$. Then the point x is also a cut point of $C(t)$ with respect to $e \in H$. Now,we consider a geodesic $\sigma(t)$ with $\dot{\sigma}(0) = (a_1, a_2, a_3)$ through e which is not contained in the center of H and let x be the first conjugate point to e along $\sigma(t)$. We claim that $\sigma(t)$ realizes the distance between e and x. It follows from Theorem 2.8 that the first conjugate point x is contained in the center of H. Set $x := (0, 0, s\pi)$. We may assume that $1 > a_3 > 0$. Then $s > 2$. By (2.10), we have

$$a_3 t_0 = 2\pi m \quad (m \in \mathbb{Z}), \ s\pi = a_3 t_0 + \frac{1}{2a_3}(a_1^2 + a_2^2)t_0 = \frac{1}{2a_3}(1 + a_3^2)t_0$$

Since x is the first conjugate point to e along $\sigma(t)$, the length of $\sigma(t)$ from e to x is given by $t_0 = \sqrt{s-1}2\pi$. Since $\sqrt{s-1}2\pi < s\pi$, the length $\sigma(t)$ from e to x is less than that of the geodesic $C(t) := (0, 0, t)$ $(0 \leq t \leq s\pi)$ contained in the center of H. Let $\tau(t)$ be a minimal geodesic from e to x. Then x is also a conjugate point to e along $\tau(t)$ by (2.10) and Lemma 2.7. Since $\tau(t)$ is minimal, x is also the first conjugate point to e along $\tau(t)$. In this case we have $t_2 = \sqrt{s-1}2\pi$ where $x := \tau(t_2)$ by the same computation as above and so $\sigma(t)$ and $\tau(t)$ have the same length. This implies that if x is the first conjugate point to e along $\sigma(t)$ then $\sigma(t)$ realizes the distance $d(e, x)$ and so the point x is also a cut point to e. □

§2.5 Hopf-Rinow's theorem.

M is said to be complete if it is complete as a metric space.

2.14 Theorem(Hopf-Rinow). *(i) If M is complete, then every geodesic can be extended indefinitely, and any two points can be joined by a geodesic,*

(ii) Assume that E is SBG. If there is a point x_0 such that every geodesic from $_0$ can be indefinitely extended, then M is complete.(Here, M is assumed to be connected).

Proof. (i) Let $c(t)$ be a geodesic on the interval $0 \leq t < T$, parametrized by arc-length. Then $\mathrm{d}(c(t_1, c(t_2)) \leq |t_1 - t_2|$, and so the completeness implies that there is a point x_0 such that $\lim_{t\to T} c(t) = x_0$. By extending c to $[0, T]$ we have a continuous map, and so its image is compact in M. By the method of the analytic continuation, there are curves of length minimizing joining two points and they are geodesics.

(ii) By the similar way in Riemannian geometry it is proved that every point of M can be joined to x_0 by a length minimizing geodesic.

Given a Cauchy sequence $\{x_j\}$, we consider a sequence $\{c_j\}$ of length minimizing geodesics parametrized by arc length joining x_0 to x_j. By Arzela-Ascoli theorem argument, it is proved that,by passing to a subsequence if necessary, there is a uniform limit $c(t) := \lim_{j\to\infty} c_j(t)$ on a small interval $0 \leq t \leq \epsilon$. Now, $c(t)$ is a geodesic and so we set $c(t) =: \exp_{x_0}(tu)$ for some unit cotangent vector u. similarly $c_j(t) =: \exp_{x_0}(tu_j)$ for some unit cotangent vector u_j.

We must show that u is the limit of $\{u_j\}$. It is not clear because the unit sphere in the cotangent space is not compact. The SBG implies that there is $t_0 > 0$ small enough such that $\exp_{x_0}$ is a local diffeomorphism in a neighbourhood of $t_0 u$. Then by using the uniqueness of the length minimizing geodesics $c_j(t)$, $0 \leq t \leq t_0$ we have $t_0 u$ as the limit of $t_0 u_j$. Finally, the continuity of $\exp_{x_0}$ implies that $\lim_{j\to\infty} \exp_{x_0}(t_j u_j) = \exp_{x_0}(Tu)$ if $t_j \longrightarrow T$, and choosing $t_j := \mathrm{d}(x_0, x_j)$ we have $\lim_{j\to\infty} x_j = \exp_{x_0}(Tu)$, proving completeness. □

2.15 Corollary. *Assume that E is SBG, and let $T > 0$ be such that the closed ball of radius T about the point x_0 is complete. Then the subset of the unit sphere* *in the cotangent space of all u such that the geodesic* $\exp_{x_0}(tu)$ *on* $[0, T]$ *is length minimizer,is compact.*

Since the topology of M is locally Euclidean, the completeness of M is equivalent to the compactness of all closed balls.

2.16 Theorem. *The completeness of* M *is equivalent to the existence of a sequence of functions* $\phi_j : M \longrightarrow \mathbb{R}$ *satisfying*
(i) ϕ_j *has compact support,*
(ii) $\lim_{j\to\infty} \phi_j(x) = 1$ *pointwise for each* $x \in M$,
(iii) $|\phi_j(x) - \phi_j(y)| \leq \epsilon_j d(x, y)$ *for all* $x, y \in M$ *for a sequence* $\epsilon_j \longrightarrow 0$.

Proof. Assume that M is complete. Take $\phi_j(x) = h_j(\mathrm{d}(x_0, x))$, where h_j is the real function taking value one on $[0, j]$ and zero on $[2j, \infty]$ and linear in between. Then (i) follows from completeness, (ii) is clear, and (iii) follows from

$$|\phi_j(x) - \phi_j(y)| \leq j^{-1}|\mathrm{d}(x_0, x) - \mathrm{d}(x_0, y)| \leq j^{-1}\mathrm{d}(x, y).$$

Conversely, suppose that such functions exist, and let $\{x_k\}$ be a Cauchy sequence. Choose j large enough such that $\phi_j(x_1)$ is close to one, say $\phi_j(x_1) \geq 1/2$, and so that $\epsilon_j \mathrm{d}(x_1, x_k) \leq 1/4$ for all k (since $\{x_k\}$ is Cauchy, $\mathrm{d}(x_1, x_k)$ is bounded). Then by (iii) we have $\phi_j(x_k) \geq 1/4$ for all k, and so the sequence $\{x_k\}$ lies in the compact support of ϕ_j. Then the compactness shows that $\{x_j\}$ has a limit. □

2.17 Theorem. *Let* M *be a sub-Riemannian manifold. If there is a Riemannian contraction of the metric with respect to which* M *is complete, then* M *is complete in the given sub-Riemannian metric.*

Proof. Let $\{x_j\}$ be a Cauchy sequence with respect to d. Then it is a Cauchy sequence with respect to d_R (since $\mathrm{d}_R \leq d$) and so there is $x \in M$ such that $x_j \longrightarrow x$ in the d_R metric. But topologically the two metrics are equivalent, and so $x_j \longrightarrow x$ in the d metric. □

Example 2.2. *Let* $M := \mathbb{R}^n$ *and* $|g^{jk}(x)| \leq c(1 + |x|)$ *for all* j, k, *and all* x. *Then it contracts to a Riemannian metric satisfying the same estimate, and the Riemannian metric is complete. Thus* M *is complete.*

§3.1 Rumin Complexes.

Let M be a smooth manifold of dimension $2n+1$ with a contact structure, i.e., a hyperplane bundle $E \subset TM$ such that there is a 1−form η on M such that ker $\eta = E$, and $d\eta|_E$ is non-degenerate. Let $\Lambda^* M$ be the graded algebra of differential forms on M. Let I^* be the ideal generated by η, i.e.,

$$I^* := \{\eta \wedge \beta + d\eta \wedge \gamma \mid \beta, \gamma \in \Lambda^* M\}.$$

Let J^* be the annihilator of I^*, i.e.,

$$J^* := \{\alpha \in \Lambda^* M \mid \eta \wedge \alpha = d\eta \wedge \alpha = 0\}.$$

• I^* and J^* are independent of choices of the contact form η, and stable by the exterior differentiation d.

We consider the complexes induced from the de Rham complex on $\Lambda^* M/I^*$ and J^*. Let d_E be the differential operator induced by d on E. Since $d\eta|_E$ is a symplectic form, the following lemma is useful;

Lefschetz lemma. *The operator* $L : \Lambda^k E \longrightarrow \Lambda^{k+2} E,\ \alpha \longmapsto d\eta \wedge \alpha$ *is injective for* $k \le n-1$, *and surjective for* $k \ge n-1$, *where* $2n :=$ *rank* E.

Then we have

$$\Lambda^k M/I^k = \{0\} \quad \text{for } k \ge n+1$$
$$J^k = \{0\} \quad \text{for } k \le n.$$

Thus we have the Rumin complex ;

3.1 Theorem (Rumin,[R1,R2]). *There is a second order differential linear operator* $D : \Lambda^n M/I^n \longrightarrow J^{n+1}$ *such that*

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{d_E} \Lambda^1 M/I^1 \xrightarrow{d_E} \cdots \xrightarrow{d_E} \Lambda^n M/I^n$$

$$\xrightarrow{D} J^{n+1} \xrightarrow{d_E} J^{n+2} \xrightarrow{d_E} \cdots \xrightarrow{d_E} J^{2n+1} \longrightarrow 0$$

is a resolution of $\mathbb{R}$. *The cohomology of this complex is equal to the cohomology of the de Rham complex of* M.

In order to construct D, it is sufficient to define its lifting $\tilde{D} : \Lambda^n M/\{\eta \wedge \alpha \mid \alpha \in \Lambda^{n-1} M\} \simeq \Lambda^n E \longrightarrow J^{n+1}$ as followings ;

3.2 Lemma. *Let* $\alpha \in \Lambda^n E$. *There is a unique lifting* $\tilde{\alpha}$ *of* α *in* $\Lambda^n M$ *such that* $d\tilde{\alpha} \in J^{n+1}$. *Then we set*

$$\tilde{D}\alpha = d\tilde{\alpha}.$$

Proof. If $\overline{\alpha}$ is any lifting of α, there is $\beta \in \Lambda^{n-1} E$ such that $\tilde{\alpha} = \overline{\alpha} + \eta \wedge \beta$ satisfies $\eta \wedge d\tilde{\alpha} = 0$, i.e.,$(d\overline{\alpha} + d\eta \wedge \beta)|_E = 0$. The operator L is an isomorphism in deg. $n-1$ and so $\tilde{\alpha}$ is uniquely determined. It is obvious that $d\eta \wedge d\tilde{\alpha} = d(\eta \wedge d\tilde{\alpha}) = 0$. Thus we have $d\tilde{\alpha} \in J^{n+1}$. □

Therefore, taking the quotient, we have $D : \Lambda^n M/I^n \simeq \Lambda^n E/\{d\eta \wedge \beta \mid \beta \in \Lambda^{n-2}M\} \to J^{n+1}$. In deed, let $\overline{d\eta \wedge \beta}$ be the projection of $d\eta \wedge \beta$ in $\Lambda^n E$. We have

$$d(d\eta \wedge \beta - \eta \wedge d\beta) = 0 \in J^{n+1}.$$

Then $d\eta \wedge \beta - \eta \wedge d\beta$ is the requested lifting of $\overline{d\eta \wedge \beta}$ in $\Lambda^n M$ and so

$$\tilde{D}(\overline{d\eta \wedge \beta}) = d(d\eta \wedge \beta - \eta \wedge d\beta) = 0.$$

Thus we have D by taking the quotient of $\tilde{D}$.

D is a second order differential operator with respect to the derivations along E and 1st order one with respect to liftings. The d_E is of order 1 with respect to derivations in the horizontal directions. Moreover, by construction, D and d_E are independent of the contact structure E and also the choices of contact forms η.

Now, we show the (local) exactness of Rumin complex.

(i) Let $\alpha \in \Lambda^k M/I^k$ such that $d_E\alpha = 0$, i.e., there is a lifting $\tilde{\alpha}$ of α in $\Lambda^k M$ such that $d\tilde{\alpha} = \eta \wedge \beta + d\eta \wedge \gamma \in I^{k+1}$. Then we have

$$\begin{aligned} d(\tilde{\alpha} - \eta \wedge \gamma) &= \eta \wedge \beta + d\eta \wedge \gamma - d\eta \wedge \gamma + \eta \wedge d\gamma \\ &= \eta \wedge (\beta + d\gamma) \end{aligned}$$

where $\tilde{\alpha} - \eta \wedge \gamma$ is another lifting of α. Thus it is reduced to $d\tilde{\alpha} = \eta \wedge \beta$. The differentiation of the above implies that

$$0 = d\eta \wedge \beta - \eta \wedge d\beta.$$

Thus, restricting on E, we have

$$d\eta \wedge \beta|_E = L(\beta|_E) = 0.$$

By the injectivity of L for $k \leq n-1$, we have $\beta|_E = 0$. Then we can write β as

$$\beta = \eta \wedge \delta$$

and so $d\tilde{\alpha} = \eta \wedge \beta = 0$. Therefore there is locally $\mu \in \Lambda^{k-1}M$ such that $\tilde{\alpha} = d\mu$. Projecting μ into $\overline{\mu} \in \Lambda^{k-1}M/I^{k-1}$, we have $\alpha = d_E\overline{\mu}$. Therefore the local exactness of deg. $k \leq n-1$ holds.

(ii) Let $\alpha \in J^k$ satisfy $d_E\alpha (= d\alpha) = 0$. Then there is locally $\beta \in \Lambda^{k-1}M$ such that $\alpha = d\beta$. By the surjectivity of L in deg.$k-3 \geq n-1$, i.e., deg $k \geq n+2$, we can write β as

$$\beta = \eta \wedge \gamma + d\eta \wedge \mu,$$

Thus we have

$$\beta - d(\eta \wedge \mu) = \eta \wedge \gamma + d\eta \wedge \mu - d\eta \wedge \mu + \eta \wedge d\mu.$$

Then

$$\beta' := \eta \wedge (\gamma + d\mu)$$

is a desired one. Indeed, it is immediate that $d\beta' = d\beta = \alpha$ and $\eta \wedge \beta' = 0$. Since $\alpha \in J^k$, we have

$$\begin{aligned} d\eta \wedge \beta' &= d(\eta \wedge \beta') + \eta \wedge d\beta' \\ &= d \cdot 0 + \eta \wedge \alpha = 0, \end{aligned}$$

so that $\beta' \in J^{k-1}$. Therefore the local exactness in deg. $k \geq n+2$ holds.

(iii) The case in deg. n. Let $\alpha \in \Lambda^n M/I^n$ satisfy $D\alpha = 0$. Then, by definition, there is a lifting $\tilde{\alpha}$ of α such that $d\tilde{\alpha} = 0$. Therefore there is $\beta \in \Lambda^{n-1}M$ such that $\tilde{\alpha} = d\beta$, and so, by projection, $\alpha = d_E\overline{\beta}$.

(iv) The case in deg.$n+1$. Let $\alpha \in J^{n+1}$ satisfy $d_E\alpha (= d\alpha) = 0$. Therefore there is locally $\beta \in \Lambda^n M$ such that $\alpha = d\beta \in J^{n+1}$ and so, by projection, $\alpha = D(\overline{\beta})$. □

§3.2. The hypoellipticity of the contact complex, Hodge Theory.

For a contact form η, there is a unique vector field T transverse to E satisfying the equation $\eta(T) = 1$ and $\mathcal{L}_T\eta = 0$. T is called the Reeb field associated to the contact form η.

We choose a Riemannian metric g on M such that T is normal to E and $|T|_g = 1$. To get a metric associated to the symplectic form $d\eta$, we give a complex structure J on E ; J is an endomorphism of E satisfying

$$\begin{aligned} J^2 &= -Id, \\ d\eta(X, JY) &= -d\eta(JX, Y), \quad \forall X, \forall Y \in E, \\ d\eta(X, JX) &> 0 \quad \forall X \in E \setminus \{0\} \end{aligned}$$

• Such a J can be chosen globally on M if E is orientable.

Proof. (Weinstein [W]) Let g be any Riemannian metric on M. Restricting on E, we can write $d\eta$ as

$$d\eta|_E = g|_E(A-, -),$$

where A is antisymmetric. We decompose A as

$$A = S \cdot J,$$

where S is positive definite symmetric and J is orthogonal, i.e., J is a complex structure. □

We define a metric g_E on E by

$$g_E := d\eta|_E(-, J-).$$

Then we have an adapted metric defined on TM by

$$g_\eta = \eta \otimes \eta + d\eta(-, J-),$$

here, we extend J on TM by $JT := T$. We can canonically extend this metric to $\Lambda^* M$.

3.3 Remark.

In 1960's a (almost) contact metric structure on a $(2n+1)$-dimensional manifold M has defined by S.Sasaki ([S]); there are a Riemannian metric g and a tensor field J such that

$$\begin{aligned} g(T, X) &= \eta(X) \\ 2g(X, JY) &= d\eta(X, Y) \quad \forall X, \forall Y \in TM \\ J^2 X &= -X + \eta(X)T \end{aligned}$$

Our contact structure is different from his one in the sense that we use a Riemannian metric on M auxiliary. Moreover, we can consider his one as a 1-dimensional Riemannian foliation, in fact, S.Yorozu ([Y]) has studied the "basic" cohomology from the point of view. Unfortunately, we can not catch it in detail. see Theorem 4.4.

We can also identify the quotient spaces $\Lambda^k M / I^k$ $(k \leq n)$ with $J^k := \{\alpha \in \Lambda^k M \mid \iota_T(\alpha) = 0, \Lambda\alpha = 0\}$ where Λ is the adjoint operator of $L := d\eta \wedge -$.

By means of the above metric g_η, we can define the $*$-operator by ;

$$\alpha \wedge *\beta := (\alpha, \beta)\eta \wedge (d\eta)^n, \quad \forall\alpha, \forall\beta \in \Lambda^* M.$$

• J^k and J^{2n+1-k} are mutually dual by $*$-operator.

Proof. Let $\alpha \in J^{2n+1-k}$, i.e., $\eta \wedge \alpha = 0$ and $d\eta \wedge \alpha = 0$. Then we have

$$(\eta \wedge \beta, *\alpha)\mathrm{dvol} = \eta \wedge \beta \wedge *(*\alpha), \quad \forall\beta \in \Lambda^{k-1} M.$$

Since $*^2\alpha = \alpha$, we have $(\eta \wedge \beta, *\alpha) = 0$. By the same way,

$$(d\eta \wedge \gamma, *\alpha)\mathrm{dvol} = d\eta \wedge \gamma \wedge *(*\alpha) = 0 \quad \forall\gamma \in \Lambda^{k-2} M.$$

Thus $*\alpha$ is orthogonal to $I^k = (J^k)^\perp$. □

We define the formal adjoints of the operators d_E and D by

$$\delta_E := (-1)^k * d_E * \quad \text{on } J^k \ (k \neq n+1)$$
$$D^* := (-1)^{n+1} * D * \quad \text{on } J^{n+1}.$$

That is, for $\alpha \in J^k$ and $\beta \in J^{k+1}$,

$$(d_E\alpha, \beta) = (\alpha, \delta_E\beta) \text{ and } (D\alpha, \beta) = (\alpha, D^*\beta) \quad \text{when } k = n.$$

Hereafter, we shall use the same notation for the local inner product and the global inner product.

Proof. For $k \neq n$,

$$(d_E\alpha, \beta) - (\alpha, \delta_E\beta) = \int_M d_E\alpha \wedge *\beta + (-1)^k \alpha \wedge d_E * \beta.$$

It is sufficient to prove that

$$d(\alpha \wedge *\beta) = d_E\alpha \wedge (*\beta) + (-1)^k \alpha \wedge d_E(*\beta).$$

Now, we have, by definition, that $d_E = d$ in degree $k > n$ and $\mathrm{im}(d_E - d) \subset I^*$ in degree $k < n$. Moreover, forms in I^* are annihilated by exterior products of forms in J^*. Thus the requested formulas hold for $k \neq n$.

In the case that $k = n$, it follows from the definition of D that there are $\overline{\alpha} := \alpha + \eta \wedge \mu$ and $\overline{*\beta} := *\beta + \eta \wedge \nu$ such that $D\alpha = d\overline{\alpha}$ and $D(*\beta) = d(\overline{*\beta})$. Then we have

$$\begin{aligned}(D\alpha, \beta) - (\alpha, D^*\beta) &= \int_M d\overline{\alpha} \wedge (*\beta) + (-1)^n \alpha \wedge d(\overline{*\beta}) \\ &= \int_M d\overline{\alpha} \wedge \overline{*\beta} + (-1)^n \overline{\alpha} \wedge d(\overline{*\beta}) - D\alpha \wedge \eta \wedge \nu \\ &\quad - (-1)^n \eta \wedge \mu \wedge D(*\beta) \\ &= \int_M d(\overline{\alpha} \wedge \overline{*\beta}) = 0\end{aligned}$$

since, by definition of D, $D\alpha \wedge \eta = D(*\beta) \wedge \eta = 0$. □

• Rumin complex is not elliptic.

In fact, on the Heisenberg group H^3, the Laplacian is written as

$$\Delta_E f := (d_E\delta_E + \delta_E d_E)f = -(X^2 + Y^2)f, \ f \in C^\infty(H^3).$$

3.4 Definition (maximal hypoellipticity). *Let P be an operator on functions or forms on M of degree k with respect to the contact field E. P is maximal hypoelliptic if there are local estimates*

$$(*) \qquad ||f||_{2,k,E} \leq K \cdot (||Pf||_2 + ||f||_2)$$

where K is a constant independent of f with compact support in a neighbourhood U, $||f||_2$ is the L_2- norm of f and

$$||f||_{2,k,E} = \sum_{\ell < k} ||X_{i_1} \cdot X_{i_2} \cdots X_{i_l} \cdot f||_2,$$

here $2n$ vector fields $\{X_1, X_2, \cdots, X_{2n}\}$ are chosen so that they generate E on U.

• The estimates (*) mean that P controls the maximum of derivatives.

3.5 Theorem. *Let M be a contact manifold with an associated metric. The following Laplacians are maximal hypoelliptic ;*

(i) $\Delta_E := (n-k)d_E\delta_E + (n-k+1)\delta_E d_E$ *on* J^k $(0 \leq k \leq n-1)$.

(ii) $\Delta_E := (d_E\delta_E)^2 + D^*D$ *on* J^n,

(iii) $\Delta_E := D^*D + (\delta_E d_E)^2$ *on* J^{n+1},

(iv) $\Delta_E := (n-k+1)d_E\delta_E + (n-k)\delta_E d_E$ *on* J^k $(n+2 \leq k \leq 2n+1)$.

3.6 Corollary. *The weak solutions of $\Delta_E\alpha = \beta$, $\beta \in C^\infty(M)$ are of class C^∞. The Laplacian Δ_E induces an orthogonal decomposition of C^∞-forms with compact support ;*

$$J^k = \ker\, \Delta_E \oplus im\, \Delta_E,$$

where ker Δ_E *is of finite dimensional.*

There is a unique coclosed representative in each class of d_E-cohomology with compact support. In particular, if M is compact its cohomology can be represented by Δ_E-harmonic forms.

We define d_H on $\Lambda^*E := \{\alpha \in \Lambda^*M \mid \iota_T\alpha = 0\}$ by

$$d_H := \Pi \circ d$$

where Π is the orthogonal projection onto Λ^*E.

• d_H is a 1st order derivative with respect to E.

• Moreover, we have

$$(3.1) \qquad d_H^2 = -L\mathcal{L}_T = -\mathcal{L}_T L,$$

where $\mathcal{L}_T$ is the Lie derivative with respect to T.

Proof. Let $\alpha \in \Lambda^*E$. Then we have

$$d_H\alpha = d\alpha - \eta \wedge \iota_T d\alpha = d\alpha - \eta \wedge \mathcal{L}_T\alpha.$$

On the other hand,

$$dd_H\alpha = -d\eta \wedge \mathcal{L}_T\alpha + \eta \wedge d\mathcal{L}_T\alpha,$$

and

$$\eta \wedge d\mathcal{L}_T\alpha \in \ker \Pi = \eta \wedge \Lambda^* M, \; d\eta \wedge \mathcal{L}_T\alpha \in \mathrm{im}\, \Pi.$$

In fact,

$$\iota_T(d\eta \wedge \mathcal{L}_T\alpha) = \iota_T d\eta \wedge \mathcal{L}_T\alpha + d\eta \wedge \iota_T\mathcal{L}_T\alpha = 0 + d\eta \wedge \mathcal{L}_T\iota_T\alpha = 0.$$

By projection, we have thus $d_H^2\alpha = -d\eta \wedge \mathcal{L}_T\alpha = -L\mathcal{L}_T\alpha = -\mathcal{L}_T L\alpha$. □

The complex structure J induces a decomposition of forms on E with respect to their bidegrees ;

$$\Lambda^k E \otimes \mathbb{C} = \sum_{p+q=k} \Lambda^{p,q} E.$$

We denote the projection of $\Lambda^* E \otimes \mathbb{C}$ onto $\Lambda^{p,q}E$ by $\Pi^{p,q}$. For an operator P on $\Lambda^* E \otimes \mathbb{C}$, we set

$$P^{k,\ell} := \sum_{p,q} \Pi^{p+k,q+\ell} \circ P \circ \Pi^{p,q}$$

for $k, \ell \in \mathbb{Z}$.

Then we have

$$d_H = d_H^{1,0} + d_H^{0,1} + T(J), \tag{3.2}$$

where $T(J)$ is a tensor and equal to zero if the complex structure is integrable, i.e.,

$$[E^{1,0}, E^{1,0}] \subset E^{1,0}.$$

Proof. For $\alpha \in \Lambda^k E$, and $X_0, X_1, \cdots, X_k \in E$, we have

$$\begin{aligned} &d_H\alpha(X_0, X_1, \cdots, X_k) \\ &= \sum_{0 \le i \le k} (-1)^i X_i \cdot (\alpha(X_0, \cdots, \widehat{X_i}, \cdots X_k)) \\ &+ \sum_{0 \le i < j < \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k) \end{aligned} \tag{3.3}$$

It follows that

$$d_H \Lambda^{p,q} E \subset \Lambda^{p+1,q}E + \Lambda^{p,q+1}E + \Lambda^{p+2,q-1}E + \Lambda^{p-1,q+2}E,$$

i.e., $d_H = d_H^{1,0} + d_H^{0,1} + d_H^{2,-1} + d_H^{-1,2}$. $d_H^{2,-1}$ and $d_H^{-1,2}$ depend only on the second part of (3.3), i.e., algebraic. □

Let P be a linear differential operator on $\Lambda^* E$. We define the degree of P by the maximal degree of horizontal derivatives of components of forms α in a local expression of $P\alpha$. For example, ∂_T is of second order defined by

$$\partial_T f := T \cdot f, \; f \in C^\infty(M).$$

In fact, $Tf = (Y_1X_1 - X_1Y_1)f + Xf$ where we choose $X_1 \in E$ of norm 1, $Y_1 = JX_1 \in E$ and $X = T - [Y_1, X_1] \in E$.

In the following, let $o(k)$ be the linear differential operator on Λ^*E if deg.$< k$. We set ;

$$\partial_H := d_H^{1,0} \quad \overline{\partial_H} := d_H^{0,1}.$$

Then (3.2) can be written of form ;

$$d_H = \partial_H + \overline{\partial_H} + o(1). \tag{3.2'}$$

By the same way, we have

$$\mathcal{L}_T = (\mathcal{L}_T)^{0,0} + o(1) \tag{3.4}$$

where $o(1)$ is a tensor and equal to zero if the Reeb flow preserves the complex structure, i.e., $\mathcal{L}_T J = 0$.

The equations (3.2'), (3.1) and (3.4) imply that

$$\partial_H^2 = o(2), \overline{\partial_H}^2 = o(2) \tag{3.5}$$

$$\partial_H \overline{\partial_H} + \overline{\partial_H} \partial_H = -L\mathcal{L}_T + o(1), \tag{3.6}$$

where o are zero if the complex structure is integrable and invariant by T. In this case, $\overline{\partial_H}$ defines a complex (see, N.Tanaka[T]).

Now, we define the adjoint of the operator d_H. $\Lambda^k E$ can be identified with $\{\alpha \in \Lambda^{2n+1-k}M \mid \eta \wedge \alpha = 0\} \simeq \eta \wedge \Lambda^{2n-k}E$ by the $*$-operator. Then we define $*_E : \Lambda^* E \longrightarrow \Lambda^{2n-*}E$ defined by

$$\alpha \wedge *_E \beta := (\alpha, \beta)(d\eta)^n.$$

We have

$$*_E \beta = *(\eta \wedge \beta) \text{ for } \beta \in \Lambda^k E \ (k \geq n),$$

and so

$$* = (-1)^k \eta \wedge *_E \text{ on } \Lambda^k E \ (k \leq n).$$

Thus the adjoint of d_H can be written as

$$\delta_H = - *_E d_H *_E . \tag{3.7}$$

We set

$$d_H^J := J^{-1} d_H J,$$

and so its adjoint is

$$\delta_H^J = J^{-1} \delta_H J.$$

We have the following relations on $\Lambda^* E$;

$$[\Lambda, d_H] = -\delta_H^J + o(1), \quad [\Lambda, d_H^J] = \delta_H + o(1). \tag{3.8}$$

Proof. Choose a local orthonormal framing $\{X_1, JX_1, \cdots, X_n, JX_n\}$ on E. They give local expressions of the above operators on $\Lambda^* E$ which are similar in Kählerian geometry. The only different terms of d_H are algebraic functions of the bracket of X_i, JX_i. By the same way (3.7) makes us calculate the commutator $[\Lambda, d_H]$. □

We rewrite N.Tanaka's formulas of $\Delta_H := d_H \delta_H + \delta_H d_H$, $\Delta_{\partial_H} := \partial_H \partial_H^* + \partial_H^* \partial_H$ and $\Delta_{\overline{\partial}_H} := \overline{\partial}_H \overline{\partial}_H^* + \overline{\partial}_H^* \overline{\partial}_H$.

3.7 Proposition. *(i)* $\Delta_{\partial_H} - \Delta_{\overline{\partial_H}} = \sqrt{-1}(k-n)\mathcal{L}_T + o(2)$ *on* $\Lambda^k E$,

(ii) $\Delta_H = \Delta_H^{0,0} + o(2)$, *i.e.,* Δ_H *preserves the bidegrees up to* o.

Proof. (i) (3.8) implies that

$$\begin{aligned}\Delta_{\partial_H} &= \partial_H^*\partial_H + \partial_H\partial_H^* \\ &= (\sqrt{-1}[\Lambda, \overline{\partial_H}])\partial_H + \partial_H(\sqrt{-1}[\Lambda, \overline{\partial_H}]) + o(2) \\ &= \sqrt{-1}\Lambda\overline{\partial_H}\partial_H - \sqrt{-1}\overline{\partial_H}\Lambda\partial_H + \sqrt{-1}\partial_H\Lambda\overline{\partial_H} - \sqrt{-1}\partial_H\overline{\partial_H}\Lambda + o(2).\end{aligned}$$

By the same way, we also have

$$\Delta_{\overline{\partial_H}} = -\sqrt{-1}\Lambda\partial_H\overline{\partial_H} + \sqrt{-1}\partial_H\Lambda\overline{\partial_H} - \sqrt{-1}\overline{\partial_H}\Lambda\partial_H + \sqrt{-1}\overline{\partial_H}\partial_H\Lambda + o(2).$$

It follows from (3.6) that

$$\begin{aligned}\Delta_{\partial_H} - \Delta_{\overline{\partial_H}} &= \sqrt{-1}\Lambda(\overline{\partial_H}\partial_H + \partial_H\overline{\partial_H}) - \sqrt{-1}(\partial_H\overline{\partial_H} + \overline{\partial_H}\partial_H)\Lambda + o(2) \\ &= \sqrt{-1}\Lambda(-\mathcal{L}_T L) + \sqrt{-1}\mathcal{L}_T L\Lambda + o(2) \\ &= -\sqrt{-1}[\Lambda, L]\mathcal{L}_T + o(2) \\ &= \sqrt{-1}(k-n)\mathcal{L}_T + o(2).\end{aligned}$$

(ii) Since $d_H = \partial_H + \overline{\partial_H} + o(1)$, we have

$$\Delta_H = \Delta_H^{0,0} + \Delta_H^{-1,1} + \Delta_H^{1,-1} + o(2)$$

where

$$\begin{aligned}\Delta_H^{1,-1} &= \overline{\partial_H^*}\partial_H + \partial_H\overline{\partial_H^*} + o(2) \\ &= (-\sqrt{-1}[\Lambda, \partial_H])\partial_H + \partial_H(-\sqrt{-1}[\Lambda, \partial_H]) + o(2) \\ &= -\sqrt{-1}\Lambda\partial_H^2 + \sqrt{-1}\partial_H\Lambda\partial_H - \sqrt{-1}\partial_H\Lambda\partial_H + \sqrt{-1}\partial_H^2\Lambda + o(2) \\ &= o(2),\end{aligned}$$

because of (3.5). It is similar for $\Delta_H^{-1,1}$. □

• Δ_H is an almost "scalar," i.e., modulo o on each $\Lambda^{p,q}E$.

In fact, let $\{X_k, Y_k := JX_k\}$ be an orthonormal framing on a neighbourhood of a point. We set

$$Z_k := (X_k - \sqrt{-1}Y_k)/\sqrt{2} \in E^{1,0}$$
$$Z_{\overline{k}} := (X_k + \sqrt{-1}Y_k)/\sqrt{2} \in E^{0,1}$$

and θ^k and $\theta^{\overline{k}}$ are their dual framing.

For $\alpha := \sum_{I,J} \alpha_{I,J}\theta^I \wedge \theta^{\overline{J}} \in \Lambda^* E \otimes \mathbb{C}$, we have

$$\iota_k\alpha = \alpha(Z_k, -),\ e_k\alpha = \theta^k \wedge \alpha,\ \partial_k\alpha = \sum_{I,J}(Z_k\alpha_{I,J})\theta^I \wedge \theta^{\overline{J}}.$$

By the same way, we set the conjugate operators by $e_{\overline{k}}$, $\iota_{\overline{k}}$, and $\partial_{\overline{k}}$. Then we have the following relations ;

$$\partial_{\overline{k}}\partial_k - \partial_k\partial_{\overline{k}} \simeq \sqrt{-1}\partial_T,\ L = \sqrt{-1}\sum_{k=1}^{n} e_k e_{\overline{k}},\ \Lambda = \sqrt{-1}\sum_{k=1}^{n} \iota_k \iota_{\overline{k}}$$

$$\partial_H \simeq \sum_{k=1}^{n} \partial_k e_k\ (= \sum_{k=1}^{n} e_k \partial_k),\ \partial_H^* \simeq -\sum_{k=1}^{n} \partial_{\overline{k}}\iota_k\ (= -\sum_{k=1}^{n} \iota_k \partial_{\overline{k}}),\ \mathcal{L}_T \simeq \partial_T$$

here, $A \simeq B$ means that A and B are of the same order p, i.e., $A = B + o(p)$.

3.8 Proposition. *On $\Lambda^{p,q}E$,*

$$\begin{aligned}\Delta_{\partial_H} &\simeq -\left(1 - \frac{p}{n}\right)\sum \partial_{\overline{k}}\partial_k - \frac{p}{n}\sum \partial_k\partial_{\overline{k}} \\ &\simeq -\frac{1}{2}\sum(X_k^2 + Y_k^2) + \sqrt{-1}\left(p - \frac{n}{2}\right)\partial_T\end{aligned}$$

and

$$\begin{aligned}\Delta_H &\simeq -\left(1 + \frac{p-q}{n}\right)\sum \partial_k\partial_{\overline{k}} - \left(1 - \frac{p-q}{n}\right)\sum \partial_{\overline{k}}\partial_k \\ &\simeq -\sum \frac{1}{2}(X_k^2 + Y_k^2) + \sqrt{-1}(p-q)\partial_T.\end{aligned}$$

Proof.

$$\begin{aligned}\Delta_{\partial_H} &= \partial_H^*\partial_H + \partial_H\partial_H^* \\ &\simeq -(\sum_k \partial_{\overline{k}}\iota_k)(\sum_\ell e_\ell\partial_\ell) - (\sum_\ell e_\ell\partial_\ell)(\sum_k \partial_{\overline{k}}\iota_k) \\ &\simeq -\sum_{k\neq\ell} \partial_{\overline{k}}\partial_\ell(\iota_k e_\ell + e_\ell\iota_k) - \sum \partial_{\overline{k}}\partial_k\iota_k e_k - \sum \partial_k\partial_{\overline{k}} e_k\iota_k.\end{aligned}$$

Using the identity

$$e_k\iota_\ell + \iota_\ell e_k = \delta_{\ell k},$$

we have

$$\begin{aligned}\Delta_{\partial_H} &\simeq -\sum \partial_{\overline{k}}\partial_k + \sum(\partial_{\overline{k}}\partial_k - \partial_k\partial_{\overline{k}})e_k\iota_k \\ &\simeq -\sum \partial_{\overline{k}}\partial_k + \sqrt{-1}\partial_T(\sum e_k\iota_k).\end{aligned}$$

Now, the identity

$$e_k\iota_k\theta^I \wedge \theta^{\overline{J}} = \begin{cases} \theta^I \wedge \theta^{\overline{J}} & \text{if } k \in I \\ 0 & \text{otherwise}\end{cases}$$

holds. Then

$$\sum e_k\iota_k = p \cdot Id \text{ on } \Lambda^{p,q}E,$$

and so

$$\Delta_{\partial_H} \simeq -\sum \partial_{\overline{k}}\partial_k + \sqrt{-1}p\partial_T.$$

Thus we have the requested formula for Δ_{∂_H} using the relations

$$\sqrt{-1}\partial_T = n^{-1}\sum \partial_{\overline{k}}\partial_k - \partial_k\partial_{\overline{k}}$$

and

$$\sum \partial_{\overline{k}}\partial_k + \partial_k\partial_{\overline{k}} = \sum (X_k^2 + Y_k^2).$$

Finally it suffice to note that from Proposition 3.7

$$\Delta_H \simeq \Delta_{\partial_H} + \Delta_{\overline{\partial_H}}.$$

□

The Laplacian $-\sum(X_k^2+Y_k^2)+\sqrt{-1}\lambda T$ acting on functions is maximal hypoelliptic for $|\lambda| \neq n, n+2, n+4, \cdots$. In particular, we have

3.9 Corollary. *Δ_H is maximal hypoelliptic on $\Lambda^{p,q}E$ for p and $q < n$.*

We will discuss the regularity of Δ_E after we prove the identities between Δ_E and Δ_H. First, we rewrite these operators by means of d_E, δ_E and D.

3.10 Proposition.

(3.9) $\quad$ *(i)$d_E \simeq d_H + (n-k+1)^{-1}L\delta_H^J$ and $\delta_E = \delta_H$ on J^k $(0 \leq k \leq n-1)$,*

(3.10) $\quad$ *(ii)$\iota_T D \simeq \mathcal{L}_T - d_H\delta_H^J$ on J^n*

Proof. (i) Let $\alpha \in J^k$ $(0 \leq k \leq n-1)$. $d_E\alpha$ is, by definition, the orthogonal projection of $d_H\alpha$ onto ker Λ. Lefschetz decomposition theorem implies

$$d_E\alpha = \sum_{s\geq 0} a_s L^s\Lambda^s d_H\alpha \; (a_s; \text{the universal constants}).$$

We show that $\Lambda^s d_H\alpha = o(1)\alpha$ $(s \geq 2)$. In deed, (3.8) and $\alpha \in J^k \subset \ker\Lambda$ imply

$$\begin{aligned}\Lambda d_H\alpha &= d_H\Lambda\alpha - \delta_H^J\alpha + o(1)\alpha \\ &= -\delta_H^J\alpha + o(1)\alpha.\end{aligned}$$

Hence, $[\Lambda, \delta_H^J] = 0$ implies that $\Lambda^2 d_H\alpha = o(1)\alpha$. In order to compute a_1, note that $\Lambda d_E\alpha = 0 = \Lambda(d_H + a_1 L\Lambda d_H)\alpha + o(1)\alpha$ and $[\Lambda, L] = (n-k+1)\mathrm{Id}$ in deg.$k-1$, which imply

$$a_1 = -(n-k+1)^{-1}.$$

The relation (3.9) can be proved by using (3.8). Finally, using $[\delta_H, \Lambda] = 0$, we have

$$\delta_E = \delta_H \text{ on } J^k.$$

(ii) Let $\alpha \in J^n$. Then there is uniquely β such that

$$D\alpha = d(\alpha + \eta \wedge \beta).$$

β is the solution of $L\beta = -d_H\alpha$, i.e., $\beta = -\Lambda d_H\alpha$ holds. In fact, since $\alpha \in \ker\Lambda = \ker L$ in deg.n,

$$Ld_H\alpha = d_H L\alpha = 0,$$

and so

$$L\Lambda d_H\alpha = \Lambda L d_H\alpha + d_H\alpha.$$

Thus, using (3.8), we have

$$\iota_T D\alpha = \mathcal{L}_T\alpha + d_H\Lambda d_H\alpha.$$

□

3.11 Proposition. *Δ_E preserves " almost" bidegrees, i.e., $\Delta_E \simeq \Delta_E^{0,0}$.*

Proof. (i) In the case that $k \le n-1$. (3.8) and the above relations imply that

$$d_E\delta_E \simeq d_H\delta_H + (n-k+2)^{-1}L\delta_H^J\delta_H$$

and

$$\begin{aligned}\delta_E d_E &\simeq \delta_H d_H + (n-k+1)^{-1}\delta_H L\delta_H^J \\ &\simeq \delta_H d_H + (n-k+1)^{-1}(L\delta_H\delta_H^J - d_H^J\delta_H^J) \\ &\simeq \delta_H d_H + (n-k+1)^{-1}(L\delta_H\delta_H^J + d_H^J\Lambda d_H) \\ &\simeq \delta_H d_H + (n-k+1)^{-1}(L\delta_H\delta_H^J + \Lambda d_H^J d_H - \delta_H d_H)\end{aligned}$$

Thus we have

$$\begin{aligned}\Delta_E &= (n-k)d_E\delta_E + (n-k+1)\delta_E d_E \\ &\simeq (n-k)\Delta_H + \frac{n-k}{n-k+2}L\delta_H^J\delta_H + L\delta_H\delta_H^J + \Lambda d_H^J d_H\end{aligned}$$

On the other hand, we proved that $\Delta_H \simeq \Delta_H^{0,0}$. By Proposition 3.7 and (3.5) we see that

$$(*)\qquad \begin{aligned}d_H^J d_H &\simeq \sqrt{-1}(\overline{\partial_H} - \partial_H)(\partial_H + \overline{\partial_H}) \\ &\simeq \sqrt{-1}(\overline{\partial_H}\partial_H - \partial_H\overline{\partial_H} \simeq -d_H d_H^J,\end{aligned}$$

which means that $d_H^J d_H$ and $d_H d_H^J$ are of same type (1,1) up to o. Thus

$$\Delta_E \simeq \Delta_E^{0,0}.$$

(ii) In the case that $k = n$. We rewrite operators d_E, δ_E and $\iota_T D$ by means of $\partial_H, \partial_H^*, \overline{\partial_H}$ and $\overline{\partial_H^*}$. These formulas need to prove the hypoellipticity of Δ_E in deg.n. (3.8) and (3.9) imply that

$$(3.11)\qquad \begin{aligned}d_E\delta_E &\simeq d_H\delta_H + \frac{1}{2}L\delta_H^J\delta_H \simeq d_H\delta_H + \frac{1}{2}(-d_H + \delta_H^J L)\delta_H \\ &\simeq \frac{1}{2}d_H\delta_H + \frac{1}{2}\delta_H^J(\delta_H L + d_H J) \quad (\text{since } L|_{J^n} = 0 = \Lambda|_{J^n}) \\ &\simeq \frac{1}{2}(d_H\delta_H + \delta_H^J d_H^J).\end{aligned}$$

On the other hand, we have $D^*D = D^*(\eta \wedge \iota_T D) = (\iota_T D)^*(\iota_T D)$ by (3.10).
Recall that

$$\begin{aligned}&d_H \simeq \partial_H + \overline{\partial_H},\ \delta_H \simeq \partial_H^* + \overline{\partial_H^*} \\ &d_H^J = J^{-1}d_H J \simeq \sqrt{-1}(\overline{\partial_H} - \partial_H),\ \delta_H^J = J^{-1}\delta_H J \simeq \sqrt{-1}(\partial_H^* - \overline{\partial_H^*}).\end{aligned}$$

These relations yield

$$(3.12)\qquad \begin{aligned}(d_E\delta_E)^{1,-1} &\simeq \partial_H\overline{\partial_H} \\ (d_E\delta_E)^{-1,1} &\simeq \overline{\partial_H}\partial_H^* \\ (d_E\delta_E)^{0,0} &\simeq \frac{1}{2}\Delta_H \\ (\iota_T D)^{1,-1} &\simeq \sqrt{-1}\partial_H\overline{\partial_H^*} \\ (\iota_T D)^{-1,1} &\simeq -\sqrt{-1}\overline{\partial_H}\partial_H^* \\ (\iota_T D)^{0,0} &\simeq \mathcal{L}_T + \sqrt{-1}(\overline{\partial_H}\overline{\partial_H^*} - \partial_H\partial_H^*),\end{aligned}$$

here, we use the fact that $\overline{\partial_H}\partial_H^* \simeq -\partial_H^*\overline{\partial_H}$ and $\overline{\partial_H^*}\partial_H \simeq -\partial_H\overline{\partial_H^*}$ when we develop that $\Delta_H \simeq (\Delta_H)^{0,0}$. Summing up, we have by (3.5)

$$\begin{aligned}\Delta_E^{2,-2} &\simeq (d_E\delta_E)^{1,-1}(d_E\delta_E)^{1,-1} + ((\iota_T D)^*)^{1,-1}(\iota_T D)^{1,-1} \\ &\simeq -\overline{\partial_H^*}\partial_H\partial_H\overline{\partial_H^*} + (-\sqrt{-1}\overline{\partial_H^*}\partial_H)(\sqrt{-1}\partial_H\overline{\partial_H^*}) \\ &= o(4).\end{aligned}$$

By the same way, we see that

$$\begin{aligned}\Delta_E^{1,-1} &\simeq (d_E\delta_E)^{1,-1}(d_E\delta_E)^{0,0} + (d_E\delta_E)^{0,0}(d_E\delta_E)^{1,-1} \\ &+ ((\iota_T D)^*)^{1,-1}(\iota_T D)^{0,0} + ((\iota_T D)^*)^{0,0}(\iota_T D)^{1,-1} \\ &\simeq \partial_H\overline{\partial_H^*}\frac{1}{2}\Delta_H + \frac{1}{2}\Delta_H\partial_H\overline{\partial_H^*} \\ &+ \sqrt{-1}\partial_H\overline{\partial_H^*}(\mathcal{L}_T + \sqrt{-1}(\overline{\partial_H\partial_H^*} - \partial_H\partial_H^*)) \\ &+ (-\mathcal{L}_T - \sqrt{-1}(\overline{\partial_H\partial_H^*} - \partial_H\partial_H^*))(\sqrt{-1}\partial_H\overline{\partial_H^*}) \\ &\simeq \frac{1}{2}(\partial_H\overline{\partial_H^*}\Delta_H + \Delta_H\partial_H\overline{\partial_H^*}) \\ &- \partial_H\overline{\partial_H^*\partial_H\partial_H^*} + \partial_H\overline{\partial_H^*}\partial_H\partial_H^* + \overline{\partial_H\partial_H^*}\partial_H\overline{\partial_H^*} - \partial_H\partial_H^*\partial_H\overline{\partial_H^*}.\end{aligned}$$

Note that

$$\partial_H\overline{\partial_H^*}\partial_H\partial_H^* \simeq -\overline{\partial_H^*}\partial_H\partial_H\partial_H^* = o(4)$$

and

$$\overline{\partial_H\partial_H^*}\partial_H\overline{\partial_H^*} \simeq -\overline{\partial_H}\partial_H\overline{\partial_H^*\partial_H^*} = o(4).$$

It follows that

$$\begin{aligned}-\partial_H\overline{\partial_H^*\partial_H\partial_H^*} &\simeq -\partial_H\overline{\partial_H^*}(\overline{\partial_H\partial_H^*} + \overline{\partial_H^*\partial_H}) \\ &\simeq -\partial_H\overline{\partial_H^*}\Delta_{\overline{\partial_H}} \simeq -\frac{1}{2}\partial_H\overline{\partial_H^*}\Delta_H, \text{ (cf Prop. 3.7)}\end{aligned}$$

and, similarly

$$\begin{aligned}-\partial_H\partial_H^*\partial_H\overline{\partial_H^*} &\simeq -(\partial_H\partial_H^* + \partial_H^*\partial_H)\partial_H\partial_H^* \\ &\simeq -\Delta_{\partial_H}\partial_H\overline{\partial_H^*} \simeq -\frac{1}{2}\Delta_H\partial_H\overline{\partial_H^*}.\end{aligned}$$

Thus we have

$$\Delta_H^{1,-1} = o(4).$$

By the conjugation,

$$\Delta^{-2,2} = \Delta_H^{-1,1} = o(4).$$

Finally, we have

$$\Delta_H \simeq \Delta_H^{0,0}.$$

(iii) In the case of deg. $\geq n+1$, it suffices to note that

$$\Delta_E * = *\Delta_E,$$

here, $* : \Lambda^{p,q}E \cap \ker\ L = J^{p,q} \longrightarrow \eta \wedge \Lambda^{n-q,n-p}M \cap \ker\ L =: J^{n-q,n-p}$. □

3.12 Definition. *Let A and B be linear differential operators (LDO) defined on a vector bundle over a contact manifold. We define $A \gtrsim B$ if there exists an LDO P such that $A \simeq B + P^*P$.*

The criterion of hypoellipticity by Helffer-Nourrigat implies that

3.13 Proposition. *If $A \gtrsim B \gtrsim 0$, then the maximal hypoellipticity of B gives one of A.*

For $f \in C^\infty(M)$ and $\alpha := \sum \alpha_{I,J}\theta^I \wedge \theta^{\overline{J}} \in \Lambda^* E$, we set

$$\Delta_K f := -\sum_{i=1}^{n}(X_i^2 + Y_i^2)f.$$

(called Kohn Laplacian) and

$$\Delta_K \alpha := \sum_{I,J} \Delta_K(\alpha_{I,J})\theta^I \wedge \theta^{\overline{J}}.$$

• (By the criterion of Helffer-Nourrigat) Δ_K is maximal hypoelliptic and

$$n\Delta_H \gtrsim \Delta_K \quad \text{on } \Lambda^{p,q}E \ (p, q < n). \tag{3.13}$$

Proof. It is seen from Proposition 3.8 that

$$\Delta_H \simeq -\left(1 + \frac{p-q}{n}\right)\sum \partial_k \partial_{\overline{k}} - \left(1 - \frac{p-q}{n}\right)\sum \partial_{\overline{k}}\partial_k.$$

and

$$\sum(\partial_{\overline{k}}\partial_k + \partial_k\partial_{\overline{k}}) = \sum(X_k^2 + Y_k^2).$$

Then we have

$$n\Delta_H \simeq \Delta_K - (n-1+p-q)\sum \partial_k\partial_{\overline{k}} - (n-1+q-p)\sum \partial_{\overline{k}}\partial_k,$$

i.e.,

$$n\Delta_H \simeq \Delta_K + P^*P,$$

where

$$P : \Lambda^{p,q}E \longrightarrow \Lambda^{1,0}E \otimes \Lambda^{p,q}E + \Lambda^{0,1}E \otimes \Lambda^{p,q}E, \alpha \longmapsto (\lambda_{1,0}\partial\alpha, \lambda_{0,1}\overline{\partial}\alpha),$$

and $\lambda_{1,0}^2 = n-1+q-p$, $\lambda_{0,1}^2 = n-1+p-q$. □

3.14 Proposition. *(i) $(n-k+2)\Delta_E \gtrsim (n-k)(n-k+1)\Delta_H$ on J^k $(k \leq n-1)$,*
(ii) $4\Delta_E \gtrsim \Delta_H^2$ on $J^{p,q}$ $(p+q=n,\ p,q<n)$,
(iii) $\Delta_E \simeq (\Delta_K + \sqrt{-1}(n+1)\partial_T)^2$ on $J^{n,0}$ (resp. $(\Delta_K - \sqrt{-1}(n+1)\partial_T)^2$ on $J^{0,n}$).
(iv) The above formulas hold by replacing k by $2n+1-k$.

In bidegree $\neq (n,0)$ and $(0,n)$, the maximal hypoellipticity of Δ_H implies one of Δ_E. In this case, $P_{n+1} := \Delta_K + \sqrt{-1}(n+1)\partial_T$ and its adjoint P_{-n-1} are also hypoelliptic although they are not positive (Folland [F]).

In order to use the criterion of Helffer-Nourrigat, note that

$$\partial_{\overline{k}} P_{n+1} \simeq P_{n-1}\partial_{\overline{k}} \ (1 \le k \le n)$$
$$\partial_{\overline{k}} P_\lambda \simeq P_{\lambda-2}\partial_{\overline{k}} \ \text{(in general)}.$$

If $\Pi(P_{n+1})f = 0$, then $\Pi(P_{n-1})\Pi(\partial_{\overline{k}})f = 0$. Since $P_{n-1} \gtrsim \frac{\Delta_K}{n}$, P_{n-1} is hypoelliptic. Thus we have

$$\Pi(\partial_{\overline{k}})f = 0.$$

It follows that

$$P_{n+1} \simeq \frac{1}{n}\sum \partial_{\overline{k}}\partial_k - \left(2 + \frac{1}{n}\right)\sum \partial_k\partial_{\overline{k}}, \ (\text{by } -\partial_k \simeq \partial_{\overline{k}}^*).$$

Hence

$$\Pi(\partial_k)f = 0.$$

Finally, $\Pi(X)f = 0$, $\forall X \in E$, and thus $f = 0$.
The proof of Proposition (3.14).
(i) We define Δ'_E on J^k by

$$\Delta'_E := (n-k+2)d_E\delta_E + (n-k+1)\delta_E d_E.$$

Then

$$(\Delta'_E)^{0,0} \simeq (n-k+1)\Delta_H.$$

In fact, Proposition 3.9 and (3.8) yield

$$\begin{aligned}
\Delta'_E &\simeq (n-k+2)(d_H\delta_H + (n-k+2)^{-1}L\delta_H^J\delta_H) \\
&\quad + (n-k+1)(\delta_H d_H + (n-k+1)^{-1}\delta_H L\delta_H^J) \\
&\simeq (n-k+1)(d_H\delta_H + \delta_H d_H) + d_H\delta_H + \delta_H L\delta_H^J + L\delta_H^J\delta_H \\
&\simeq (n-k+1)\Delta_H + d_H\delta_H - d_H\delta_H^J + L(\delta_H^J\delta_H + \delta_H\delta_H^J).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(d_H\delta_H - d_H^J\delta_H^J)^{0,0} &\simeq ((\partial_H + \overline{\partial_H})(\partial_H^* + \overline{\partial_H^*}) - \sqrt{-1}(\overline{\partial_H} - \partial_H)\sqrt{-1}(\partial_H^* - \overline{\partial_H^*}))^{0,0} \\
&\simeq \partial_H\partial_H^* + \overline{\partial_H\partial_H^*} - \partial_H\partial_H^* - \overline{\partial_H\partial_H^*} = o(2)
\end{aligned}$$

and

$$(\delta_H^J\delta_H + \delta_H\delta_H^J)^{-1,-1} \simeq \sqrt{-1}(\partial_H^*\overline{\partial_H^*} - \overline{\partial_H^*}\partial_H^* + \overline{\partial_H^*}\partial_H^* - \partial_H^*\overline{\partial_H^*}) = o(2).$$

Finally, for $0 \le k \le n-1$

$$\begin{aligned}
(n-k+2)\Delta_E &\simeq (n-k+2)\Delta_E^{0,0} \ (\text{cf. Prop. 3.10}) \\
&\simeq (n-k+2)\left((n-k)d_E\delta_E + (n-k+1)\delta_E d_E\right)^{0,0} \\
&\simeq (n-k)(\Delta'_E)^{0,0} + 2(n-k+1)(\delta_E d_E)^{0,0} \\
&\simeq (n-k+1)\left[(n-k)\Delta_H + 2((d_E^{1,0})^*(d_E^{1,0}) + (d_E^{0,1})^*(d_E^{0,1}))\right],
\end{aligned}$$

i.e.,

$$(n-k+2)\Delta_E \gtrsim (n-k)(n-k+1)\Delta_H.$$

(ii) In deg.n, we have

$$\Delta_E \simeq \Delta_E^{0,0} \simeq \left((d_E\delta_E)^2 + (\iota_T D)^*(\iota_T D)\right)^{0,0}.$$

Developing the above, we see that

$$\begin{aligned}\Delta_E \simeq (d_E\delta_E)^{0,0}(d_E\delta_E)^{0,0} &+ \sum_{k=-1,1} ((d_E\delta_E)^{k,-k})^*(d_E\delta_E)^{k,-k} \\ &+ \sum_{k=-1,0,1} ((\iota_T D)^{k,-k})^*(\iota_T D)^{k,-k}\end{aligned}$$

here, by (3.12), we have

$$2(d_E\delta_E)^{0,0} \simeq \Delta_H,$$

so that

$$4\Delta_E \gtrsim \Delta_H^2.$$

(iii) In bidegree $(n,0)$, we write all terms by developing of Δ_E. First, we have

$$\Delta_H \simeq \Delta_{\partial_H} + \Delta_{\overline{\partial_H}} \simeq 2\Delta_{\partial_H} \text{ in deg. } n \text{ cf. Prop. 3.7).}$$

Since $\partial_H = 0$ in bidegree $(n,0)$ $(\Lambda^{n+1,0}E = 0)$, we may set

$$\Delta_H \simeq 2\partial_H\partial_H^*.$$

On the other hand, (3.12) gives rise to

$$\begin{aligned}((\iota_T D)^{1,-1})^*(\iota_T D)^{1,-1} &\simeq (-\sqrt{-1}\cdot\overline{\partial_H}\partial_H^*)(\sqrt{-1}\partial_H\overline{\partial_H^*}) \\ &= o(4)\ (\overline{\partial_H^*} = 0 \text{ in bideg. } (n,0)).\end{aligned}$$

By the same way, we have

$$\begin{aligned}((d_E\delta_E)^{1,-1})^*(d_E\delta_E)^{1,-1} &\simeq (\overline{\partial}_H\partial_H^*)(\partial_H\overline{\partial_H^*}) \\ &= o(4)\end{aligned}$$

and

$$\begin{aligned}((\iota_T D)^{-1,1})^*(\iota_T D)^{-1,1} &\simeq ((d_E\delta_E)^{-1,1})^*(d_E\delta_E)^{-1,1} \\ &\simeq (\partial_H\overline{\partial_H^*})(\overline{\partial_H}\partial_H^*) \\ &\simeq \partial_H(\overline{\partial_H^*}\overline{\partial_H} + \overline{\partial_H}\overline{\partial_H^*})\partial_H^*\ (\overline{\partial_H^*} = 0 \text{ on } \Lambda^{n-1,0}M). \\ &\simeq \partial_H\Delta_{\overline{\partial_H}}\partial_H^*.\end{aligned}$$

Now, by Proposition 3.7,

$$\Delta_{\overline{\partial_H}} \simeq \Delta_{\partial_H} + \sqrt{-1}\mathcal{L}_T \quad \text{on } (n-1)-\text{forms}.$$

Hence

$$(*)\qquad \begin{aligned}((\iota_T D)^{-1,1})^*(\iota_T D)^{-1,1} &\simeq \partial_H(\Delta_{\partial_H} + \sqrt{-1}\mathcal{L}_T)\partial_H^* \\ &\simeq \sqrt{-1}\mathcal{L}_T\partial_H\partial_H^* + \Delta_{\partial_H}\partial_H\partial_H^* \\ &\simeq \sqrt{-1}\mathcal{L}_T\partial_H\partial_H^* + (\partial_H\partial_H^*)^2\end{aligned}$$

And, we also have

$$(\iota_T D)^{0,0} \simeq \mathcal{L}_T + \sqrt{-1}(\overline{\partial_H\partial_H^*} - \partial_H\partial_H^*) \simeq \mathcal{L}_T - \sqrt{-1}\partial_H\partial_H^*.$$

Thus

$$\begin{aligned}\Delta_E &\simeq (\partial_H\partial_H^*)^2 + (-\mathcal{L}_T + \sqrt{-1}\partial_H\partial_H^*)(\mathcal{L}_T - \sqrt{-1}\partial_H\partial_H^*) \\ &\quad + 2(\sqrt{-1}\mathcal{L}_T\partial_H\partial_H^* + (\partial_H\partial_H^*))^2 \\ &\simeq 4(\partial_H\partial_H^*)^2 - \mathcal{L}_T^2 + 4\sqrt{-1}\mathcal{L}_T\partial_H\partial_H^* \\ &\simeq (2\partial_H\partial_H^* + \sqrt{-1}\mathcal{L}_T)^2,\end{aligned}$$

i.e.,

$$\Delta_E \simeq (\Delta_H + \sqrt{-1}\mathcal{L}_T)^2,$$

where $\Delta_H \simeq \Delta_K + \sqrt{-1}n\mathcal{L}_T$ on bideg.$(n,0)$.

(iv) The identities in deg.$\geq n+1$ can be shown by the $*$-operator. □

We can have simple expressions in deg. $(n,0)$ and $(0,n)$. For it, we consider two operators;

$$P^{\pm} := d_E\delta_E \pm (\sqrt{-1})^{n+1} * D.$$

• $P^{\pm}$ are self-adjoint ; $\Delta_E = (P^{\pm})^2 = (P^-)^2$. (Note that $D^2 = (-1)^n * D*$ and $*^2 = 1$.)

• P^+ and P^- do not preserve bidegrees but preserve the pairs $J^{p,q} \oplus J^{p+1,q-1}$. More precisely, we have

3.15 Proposition. *P^+ (resp.P^-) preserves the spaces*

$$J^{\lambda+2k,n-\lambda-2k} \oplus J^{\lambda+2k+1,n-\lambda-2k-1}$$

up to o, where $k \in \mathbb{Z}$, $\lambda = (n+1)(n+2)/2 + 1$ (resp.$(n+1)(n+2)/2$).

Proof. Note that $*D = *(\eta \wedge \iota_T D) = *_E(\iota_T D)$. Since ker L = ker Λ in deg.n, we have $J^n = \iota_T J^{n+1}$. And $\iota_T D$ has values in $J^n = \Lambda^n E \cap \ker \Lambda$. On such forms, $*_E$ can be written by

$$(-1)^{n(n+1)/2}\, C\alpha := \sum(\sqrt{-1})^{p-q}\alpha^{p,q}.$$

(3.12) implies that on $J^{p,q}$

$$\begin{aligned}(P^+)^{1,-1} &\simeq (1 + (-1)^{(n+1)(n+2)/2+q+1})\partial_H\overline{\partial_H^*}, \\ (P^-)^{-1,1} &\simeq (1 + (-1)^{(n+1)(n+2)/2+q})\overline{\partial_H}\partial_H^*\end{aligned}$$

and the similar formulas for P^{-1} also hold by replacing q by $q+1$. Thus we have proved. □

Since $J^{0,n+1} = J^{n+1,0} = 0$, Proposition (3.15) means that $J^{n,0}$ and $J^{0,n}$ are preserved by P^+ and P^- up to o. Thus Δ_E can be written as the square of scalar operators.

§3.3 The spectral sequence of the Rumin Complex ([J],[JKa]).

Let M be a smooth manifold of dimension $2n-1$. Let Q be a subbundle of TM and $Q^{\perp}$ be a subbundle of $T^{*}M$ orthogonal to Q.

First, we assume that Q is of codimension 1. Following Rumin, define the ideals I and J of the graded differential algebra $\Omega := \Lambda M$ as follows ; I is the ideal generated by $\Gamma(Q^{\perp})$ and $d\Gamma(Q^{\perp})$, and J is the annihilator of $\Gamma(Q^{\perp})$ and $d\Gamma(Q^{\perp})$. Then we have $J \subset I$.

3.16 Lemma. *The injection $J \longrightarrow I$ induces an isomorphism in cohomology ;*

$$H^{k}(J) \simeq H^{k}(I).$$

Proof. It is sufficient to show that I/J is acyclic. Let τ be a local base of the bundle $Q^{\perp}$. If $\alpha := \tau \wedge \beta + d\tau \wedge \gamma$ is an element of I, we set $s(\alpha) := \tau \wedge \gamma$. s is well-defined as a map of I/J onto itself, and satisfies $s(d\alpha) + ds(\alpha) = \alpha$ for $\alpha \in I/J$. □

3.17 Corollary. *We define the boundary operator $D : H^{k}(\Omega/I) \longrightarrow H^{k+1}(J)$ by $D[\alpha] := [d\tilde{\alpha}]$ where $\alpha \in \Omega$ satisfies that $d\alpha \in I$, and $\tilde{\alpha} \in \Omega$ satisfies that $\alpha - \tilde{\alpha} \in I$ and $d\tilde{\alpha} \in J$. Then we have the exact sequence ;*

$$\cdots \xrightarrow{D} H^{k}(J) \to H^{k}(\Omega) \to H^{k}(\Omega/I) \xrightarrow{D} H^{k+1}(J) \to \cdots$$

When (M,Q) is a contact manifold, the operator D is non-zero only in degree n and defines an operator (written by the same letter) $D : \Omega^{m-1}/I^{m-1} \longrightarrow J^{m}$. This is just the operator appeared in the Rumin complex which is given in Theorem 3.1.

Now, we assume that the subbundle Q of TM is of codimension q (≥ 1). For all ℓ $(0 \leq \ell \leq q)$, we define the ideals I_{ℓ} and J_{ℓ} of Ω as followings ; I_{ℓ} is the ideal generated by $\Gamma(\wedge^{\ell+1}Q^{\perp})$ and $d(\Gamma(\wedge^{\ell+1}Q^{\perp}))$, and J_{ℓ} is the annihilator of $\Gamma(\wedge^{q-\ell+1}Q^{\perp})$ and $d(\Gamma(\wedge^{q-\ell+1}Q^{\perp}))$. Note that $I_{q} = 0$ and $J_{0} = \Omega$. The ideals I_{ℓ} and J_{ℓ} are stable under d, and graded : we have $I_{\ell} = \oplus I_{\ell}^{k}$ where $I_{\ell}^{k} := I_{\ell} \cap \Omega^{k}$ and $J_{\ell} := \oplus J_{\ell}^{k}$ where $J_{\ell}^{k} = J_{\ell} \cap \Omega^{k}$. We also have that $J_{\ell+1} \subset I_{\ell} \subset J_{\ell}$.

3.18 Lemma. *The injection of $J_{\ell+1}$ into I_{ℓ} induces an isomorphism in cohomology ; $H^{k}(J_{\ell+1}) \simeq H^{k}(I_{\ell})$.*

With the filtration $\Omega \supset I_{0} \supset \cdots \supset I_{q-1} \supset 0$, we will associate a spectral sequence which converges to the cohomology of M. By means of the isomorphisms in Lemma 3.18, we have the spectral sequence associated with (M,Q) by Forman ; for $r \geq 1$,

$$\begin{aligned} E_{r}^{\ell} &= \operatorname{im}(H(J_{\ell}/I_{\ell} \longrightarrow H(J_{\ell-r+1}/I_{\ell})) \\ &= (I_{\ell} + J_{\ell} \cap d^{-1}(I_{\ell+r-1}))/(I_{\ell} + J_{\ell} \cap d(J_{\ell-r+1})) \end{aligned}$$

and have boundary operators corresponding to $d_{r}^{\ell} : E_{r}^{\ell} \longrightarrow E_{r}^{\ell+r}$.

Note that $E_{1}^{\ell} = H(C_{\ell}, d_{\ell})$ where we set

$$C_{\ell} := J_{\ell}/I_{\ell} \quad C_{\ell}^{k} := J_{\ell}^{k}/I_{\ell}^{k}$$

and $d_{\ell} : C_{\ell}^{k} \longrightarrow C_{\ell}^{k+1}$ is induced by d. Then d_{1}^{ℓ} are Rumin operators which generalize $D_{\ell} : H^{k}(C_{\ell}, d_{\ell}) \longrightarrow H^{k+1}(C_{\ell+1}, d_{\ell+1})$.

§3.4 L^2-cohomology of complex hyperbolic space.

Let Δ be the Laplace-Beltrami operator on the complex hyperbolic space B^{2n} of complex dimension n and $\mathcal{H}^k$ the Hilbert space of L^2-forms of degree k which are harmonic, i.e., in the kernel of Δ. We have (cf.Gromov.1.2B and 1.4A)

3.19 Theorem. *For any $\alpha \in \Lambda^k(B^{2n})$, $k \neq n$, we have*

$$<\Delta\alpha, \alpha> \geq c_k < \alpha, \alpha> \qquad c_k > 0.$$

In particular, $\mathcal{H}^k = 0$ for $k \neq n$. Moreover, for $k = n$, 0 is an isolated point in the spectrum of Δ.

Let L be the operator of the exterior multiplication by the Kähler form, and L^* its adjoint. Recall that a form is primitive if it is in the kernel of L^*.

3.20 Corollary. *L^2-harmonic forms are primitive.*

Proof. Since L^* is bounded and commutes with Δ, it maps L^2-harmonic k-forms to L^2-harmonic $(k-2)$-forms and so we have the claim since $\mathcal{H}^k = 0$ for $k \neq n$. □

3.21 Theorem (cf. Corollary 3.6). *We have the Hodge decomposition ;*

$$L^2\Lambda^k = \mathcal{H}^k \oplus d(L^2\Lambda^{k-1}) \oplus d^*(L^2\Lambda^{k+1})$$

where $d(L^2\Lambda^{k-1})$ and $d^*(L^2\Lambda^{k+1})$ are closed. The orthogonal projection onto $\mathcal{H}^k$ induces an isomorphism of the (unreduced) L^2-cohomology

$$(\ker d \cap L^2\Lambda^k)/d(L^2\Lambda^{k-1})$$

onto $\mathcal{H}^k$. In particular, the (unreduced) L^2-cohomology vanishes in all degrees except n.

3.22 Proposition (cf. Gromov[Gr2], Lemma 1.1 A). *Let α and β be L^2-forms of degrees r and s respectively such that $r + s = 2n - 1$. Assume that $d\alpha$ is also in L^2 and β is closed. Then $\int d\alpha \wedge \beta = 0$.*

The space $\mathcal{H}^n$ decomposes into the sum of $\mathcal{H}^{p,q}$ of L^2-harmonic forms of bidegree (p, q) for $p + q = n$.

3.23 Lemma. *Let ϕ be a smooth $(n-1)$-form on the sphere S^{2n-1} such that $\tau \wedge d\phi = 0$. Then there is a smooth $(n-1)$-form ψ on $\overline{B^{2n}}$ such that*

(1) $\psi|_{S^{2n-1}} = \phi$,
(2) *$d\psi$ is L^2 with respect to the Bergman metric on B^{2n}.*

Moreover, if $d\psi = 0$, then we may choose ψ such that $d\psi$ has compact support in B^{2n}.

Proof. Consider the projection $\pi : z \longmapsto \frac{z}{\|z\|}$ from $B^{2n} \setminus \{0\}$ to S^{2n-1}. Let ψ be any smooth form on B^{2n} which coincides with $\pi^*\phi$ outside a neighbourhood U of 0. It extends to a smooth form on $\overline{B^{2n}}$ whose restriction to S^{2n-1} is ϕ. (Note that if $d\phi = 0$ then $d\psi$ has support in U). Since $\tau \wedge d\phi = 0$, we have outside U

$$\pi^*\tau \wedge d\psi = 0, \text{i.e., } (\tau - \overline{\tau}) \wedge d\psi = 0,$$

from which we get

$$\tau \wedge \overline{\tau} \wedge d\psi = 0$$

and also (recall that $\tau + \overline{\tau} = d(\|z\|^2)$)

$$d\tau \wedge d\psi = 0.$$

Then, by the formula for the Kähler form, the n-form $d\psi$ on B^{2n} satisfies the conditions $Ld\psi = 0$ and $L^*d\psi = 0$ outside U. (Note that on forms of degree n, we have $LL^* = L^*L$, so that the kernels of L and L^* cincide).

3.24 Proposition. *Let ω be a primitive (p,q)-form, $p+q=n$, on a Kähler manifold of complex dimension n. Then,*

$$*\omega = (-1)^{\frac{n(n+1)}{2}} i^{p-q}\omega.$$

We apply this Proposition to $\omega := d\psi$ to have

$$\int_{B^{2n}} \omega \wedge *\omega < \infty.$$

Thus, we can prove Lemma 3.23. □

3.25 Lemma. *Let ω be a smooth, closed n-form on $\overline{B^{2n}}$ such that $\omega|_{S^{2n-1}}$ is 0. Then, $\omega = d\sigma$ where σ is L^2 for the Bergman metric on B^{2n}.*

3.26 Proposition. *Let σ be a p-form on B^{2n}. Then*

$$\|\sigma\|^2 \leq (1-r^2)^p \|\sigma\|^2_{\text{eucl}}$$

where $\|\sigma\|$ (resp. $\|\sigma\|_{\text{eucl}}$) is the norm of σ with respect to the Bergman (resp. the Euclidean) metric and $r := \|z\|$.

Proof of Proposition 3.26. It follows immediately from the formula of the metric that if $\sigma = \alpha + \beta \wedge \tau + \gamma \wedge \overline{\tau} + \delta \wedge \tau \wedge \overline{\tau}$ with $\alpha, \beta, \gamma, \delta$ orthogonal to τ and to $\overline{\tau}$, then

$$\begin{aligned}\|\sigma\|^2 = (1-r^2)^p \|\alpha\|^2_{\text{eucl}} &+ (1-r^2)^{p+1} \left(\|\beta \wedge \tau\|^2_{\text{eucl}} + \|\gamma \wedge \overline{\tau}\|^2_{\text{eucl}}\right) \\ &+ (1-r^2)^{p+2} \|\delta \wedge \tau \wedge \overline{\tau}\|^2_{\text{eucl}}.\end{aligned}$$

□

Proof of Lemma 3.25. We construct a smooth $(n-1)$-form σ on $\overline{B^{2n}}$ such that

(1) $d\sigma = \omega$ outside a neighbourhood of the origin,
(2) $i_{\frac{\partial}{\partial r}}\sigma = 0$,
(3) $\sigma|_{S^{2n-1}} = 0$.

Such a σ is obtained as followings ; write

$$\omega = \omega_0(r) + dr \wedge \omega_1(r)$$

where $\omega_0(r)$ and $\omega_1(r)$ are forms on S^{2n-1} depending on $r := \|z\|$ as a parameter. Then the conditions on ω become $d\omega_0(r) = 0, \omega_0'(r) = d\omega_1(r)$ and $\omega_0(1) = 0$. Let $\sigma := \sigma_0(r)$ where $\sigma_0'(r) = \omega_1(r)$ and $\sigma_0(1) = 0$. Then σ satisfies (1),(2) and (3).

Note that the above conditions (2) and (3) imply that $\|\sigma\|_{\text{eucl}}$ vanishes on S^{2n-1}. Specializing Proposition 3.26 to the case $p = n-1$ and noting the relation between volume forms ;

$$d\text{vol} = (1-r^2)^{-(n+1)}d\text{vol}_{\text{eucl}},$$

and so we have

$$\int_{B^{2n}} \|\sigma\|^2 d\text{vol} = \int_{B^{2n}} (1-r^2)^{-(n+1)}\|\sigma\|^2_{\text{eucl}} d\text{vol}_{\text{eucl}}.$$

The latter integral is finite because σ is smooth on $\overline{B^{2n}}$ with all the components of σ vanishing on S^{2n-1} so that $\|\sigma\|^2_{\text{eucl}} \leq \text{const.}(1-r^2)^2$. Thus ω is the sum of $d\sigma$, with σ in L^2, and of a closed n-form with compact support which,since $H^n_c(B^{2n}) = 0$, may be written $d\mu$, where μ is compactly supported. This proves that ω is in $d(L^2)$. □

3.27 Theorem. *Let α be a smooth $(n-1)$-form on S^{2n-1} and let ϕ be the unique $(n-1)$-form congruent to α modulo τ such that $\tau \wedge d\phi = 0$. Then let ψ be any $(n-1)$-form on $\overline{B^{2n}}$ with restriction ϕ to S^{2n-1}, satisfying $d\psi \in L^2$. Such a ψ exists, and the class of $d\psi$ in the degree n L^2-cohomology space* $(\ker_{L^2} d)/d(L^2)$ *is well defined and uniquely determined by α. We set it $S\alpha$. If $d\phi = 0$ (and, in particular, if $\alpha \in I^{n-1}$), we have $S\alpha = 0$.*

Proof. Lemma 3.2 ensures the existence of ϕ and Lemma 3.23 the existence of ψ. It is evident that $d\psi \in \ker_{L^2} d$. The choice of ψ in Lemma 3.23 is not unique. However, if ψ_1 is another choice, then the difference $\psi_1 - \psi$ is an $(n-1)$-form vanishing when restricted to the sphere. By Lemma 3.25, $d(\psi_1 - \psi) = d\sigma$ with σ in L^2 so that $d\psi_1 - d\psi \in d(L^2)$. We conclude that $d\psi$ is uniquely determined modulo $d(L^2)$. If $\alpha \in I^{n-1}$ then, by Lemma 3.2, $d\psi = 0$. In this case, by Lemma 3.23, $d\psi$ may be chosen with compact support. But, since $H^n_{\mathbb{C}}(B^{2n}) = 0$, we have $d\psi = d\chi$ where χ has compact support, and so in L^2. Then we have $S\alpha = 0$. □

3.28 Proposition. *Let α_1 and α_2 be elements of Λ^{n-1}/I^{n-1}. Then we have*

$$\int_{B^{2n}} S\alpha_1 \wedge S\alpha_2 = \int_{S^{2n-1}} \alpha_1 \wedge D\alpha_2$$

Proof. In view of Proposition 3.22, the left hand side is well defined. Now we have

$$\int_{B^{2n}} S\alpha_1 \wedge S\alpha_2 = \int_{B^{2n}} d\psi_1 \wedge d\psi_2 = \int_{B^{2n}} d(\psi_1 \wedge d\psi_2) = \int_{S^{2n-1}} \alpha_1 \wedge D\alpha_2.$$

□

3.29 Corollary. *The kernel of S coincides with the kernel of D.*

Proof. By Proposition 3.28, if $S\alpha = 0$, then for any β in Λ^{n-1}/I^{n-1}, we have $\int \beta \wedge D\alpha = 0$, and so $D\alpha = 0$. □

Let P be the orthogonal projection onto the space $\mathcal{H}^n$ of L^2-harmonic n-forms and $P_{p,q}$ the orthogonal projection onto the space of L^2-hramonic (p,q)-forms. It follows from Theorem 3.21 that P identifies the L^2-cohomology with $\mathcal{H}^n$. We set $S\alpha := Pd\psi$ and $S_{p,q}\alpha := P_{p,q}d\psi$ with ψ chosen as in Theorem 3.27, which are called Szegö map to the space of harmonic n-forms $\mathcal{H}^n$ (resp. to the space of harmonic (p,q)-forms $\mathcal{H}^{p,q}$).

Chapter 4. Vanishing Theorems on a Contact Manifold

§4.1 A connection adapted to contact complexes.

Let $g_\eta := \eta\otimes\eta + d\eta(-, J-)$ be the adapted metric on a contact manifold. Then adapted connection ∇ must satisfy that $\nabla g_\eta = 0$ and $\nabla\eta\otimes\eta = 0$ ($\Longleftrightarrow \nabla\eta = 0$).

The Levi-Civita connection does not have this property (see. Remark 3.3). In fact, $\nabla\eta = 0$ implies that $\nabla_X Y \in E\,\forall X, \forall Y \in E$. The torsion Tor. of ∇ does not vanish ; on $E\times E$,

$$\eta(\mathrm{Tor}(X,Y)) = \eta(\nabla_X Y - \nabla_Y X - [X,Y]) = d\eta(X,Y).$$

Moreover, we have

$$(*)\qquad \begin{aligned} 2(\nabla_X\eta)(Y) &= d\eta(X,Y) - \eta(\mathrm{Tor}(X,Y)) - (\mathrm{Tor}(T,X),Y) \\ &\quad - (\mathrm{Tor}(T,Y),X) - (J(\mathcal{L}_T J)(Y),X) \end{aligned}$$

where $(\mathcal{L}_T J)(X) = [T, JX] - J[T,X]$ is the Lie derivative of the complex structure with respect to the Reeb field T. Since

$$\mathcal{L}_T g_\eta = d\eta(-, (\mathcal{L}_T J)-) = -g_\eta(-, J(\mathcal{L}_T J)-),$$

$\mathcal{L}_T J$ measures the deformation of the metric by the flow of T.

On the other hand, $\forall X, \forall Y$,

$$(J(\mathcal{L}_T J)(X),Y) = (J(\mathcal{L}_T J)(Y),X).$$

Thus, decomposing (*) into symmetric and antisymmetric parts with respect to X and Y, $\nabla\eta = 0$ if and only if

$$\eta(\mathrm{Tor}(X,Y)) = d\eta(X,Y)$$

and $X \longmapsto \mathrm{Tor}(T,X) + \frac{1}{2}J(\mathcal{L}_T J)(X)$ is antisymmetric for g_η.

In the following, we choose the connection which satisfies

$$\mathrm{Tor}(X,Y) = d\eta(X,Y)T \text{ for } X, Y \in E$$

and

$$\mathrm{Tor}(T,X) = -\frac{1}{2}J(\mathcal{L}_T J)(X).$$

- This connection preserves the complex structure J, i.e., $\nabla J = 0$ if J is integrable. In general, $\nabla_T J = 0$ holds.

- (Webster[We]) The unique metric connection preserving η and J with the torsion $\mathrm{Tor}(T,X)$ satisfies $\mathrm{Tor}(T,JX) = -J\mathrm{Tor}(T,X)$, i.e.,$\mathrm{Tor}(T,E^{0,1}) \subset E^{1,0}$.

See,[FGR] for the holonomy groups of the above connection, and [FGR],FG] for sub-Riemannian symmetric spaces.

In the following, we suppose that a complex structure J is integrable. Such (η, J, g) on a contact manifold is called a pseudohermitian structure.

§4.2. Vanishing theorems on a pseudohermitian structures.

Since J is integrable, we have

$$d_H = \partial_H + \overline{\partial_H}.$$

And for $\alpha \in \Lambda^k E \otimes \mathbb{C}$,

$$(\mathcal{L}_T\alpha)(X_1, \cdots, X_k) = -\sum_{i=1}^{k} \alpha(X_1, \cdots, [T, X_i], \cdots, X_k) + \mathcal{L}_T(\alpha(X_1, \cdots, X_k)),$$

then we have

$$\mathcal{L}_T^{1,-1}\alpha = \sum \alpha(\cdots, \Pi^{0,1}\mathrm{Tor}(T, -), \cdots) \tag{4.1}$$

and

$$\mathcal{L}_T^{-1,1}\alpha = \sum \alpha(\cdots, \Pi^{1,0}\mathrm{Tor}(T, -), \cdots)$$

where $\Pi^{1,0}$ (resp.$\Pi^{0,1}$) is the projection of $E \otimes \mathbb{C}$ onto $E^{1,0}$ (resp. $E^{0,1}$).

Developing $d_H^2 = -L\mathcal{L}_T$, we have

$$\partial_H^2 = -L\mathcal{L}_T^{1,-1} \text{ and } \overline{\partial_H}^2 = -L\mathcal{L}_T^{-1,1}. \tag{4.2}$$

4.1 Lemma.

$$[\Lambda, d_H] = -\delta_H^J, \ \textit{and}\ [\Lambda, d_H^J] = \delta_H. \tag{4.3}$$

Proof. Let $\{X_1, Y_1 := JX_1, \cdots, X_n, Y_n := JX_n\}$ be an orthonormal framing at $x \in M$ whose brackets at x are colinear of T. In fact, we take an orthonormal base $\{X_1, Y_1, \cdots, X_n, Y_n\}$ at x and translate them parallel along geodesics at x. Such parallel vector fields are orthonormal and satisfies $Y_i = JX_i$ since the pseudohermitian connection is metrical and preserves η and J. On the other hand, at x, $\nabla_X Y = 0$ for frame fields. Then we have $0 = \nabla_X Y - \nabla_Y X = [X, Y] + \mathrm{Tor}(X, Y)$, and so $[X, Y] = -d\eta(X, Y)T$. Such a framing is said to be normal in the following. □

We have easily from the proof of Proposition 3.7,

$$\Delta_H - \Delta_H^{0,0} = \sqrt{-1}(\partial_H^2 - \overline{\partial_H}^2)\Lambda - \sqrt{-1}\Lambda(\partial_H^2 - \overline{\partial_H}^2). \tag{4.4}$$

In fact, we consider $(\Delta_H\alpha, \alpha)$ for $\alpha \in J^k = \Lambda^k E \cap \ker\ \Lambda$. Then, using (4.2) and $[\Lambda, L] = n - k$, we have

$$((\Delta_H - \Delta_H^{0,0})\alpha, \alpha) = (n-k)(R_1\alpha, \alpha), \tag{4.5}$$

here, we set $R_1\alpha := \sqrt{-1}(\mathcal{L}_T^{1,-1} - \mathcal{L}_T^{-1,1})\alpha$ for $\alpha \in J^k$. By (4.1), we can also have the following formulas ;

$$\begin{aligned} R_1\alpha(X_1, \cdots, X_k) &= -\sum_{i=1}^{k} \alpha(X_1, \cdots, J\mathrm{Tor}(T, X_i), \cdots, X_k) \\ &= -\frac{1}{2}\sum_{i=1}^{k} \alpha(X_1, \cdots, (\mathcal{L}_T J)(X_i), \cdots, X_k) \end{aligned} \tag{4.6}$$

Moreover, we have

- $R_1 J\alpha = -JR_1\alpha$,
- $(R_1(J\alpha), J\alpha) = -(R_1\alpha, \alpha)$.

Now, we express $\Delta_E - \Delta_E^{0,0}$. It follows from the proofs of Propositions 3.10 and 3.11,

$$\Delta_E = (n-k)\Delta_H + \frac{n-k}{n-k+2} L\delta_H^J \delta_H + L\delta_H \delta_H^J + \Lambda d_H^J d_H. \tag{4.7}$$

Since we consider only the case that $((\Delta_E - \Delta_E^{0,0})\alpha, \alpha)$ for $\alpha \in J^k = \Lambda^k E \cap \ker\, \Lambda$, it is sufficient to write $d_H^J d_H - (d_H^J d_H)^{1,1} = -\sqrt{-1}(\partial_H^2 - \overline{\partial_H}^2)$. By the above calculation, we have

$$((\Delta_E - \Delta_E^{0,0})\alpha, \alpha) = (n-k)(n-k+1)(R_1\alpha, \alpha). \tag{4.8}$$

Then Proposition 3.14(i) and (4.3) imply that

$$\left(\frac{n-k+2}{n-k+1}\right) \Delta_E^{0,0} = (n-k)\Delta_H^{0,0} + 2(d_E^{1,0})^* d_E^{1,0} + 2(d_E^{0,1})^* d_E^{0,1}. \tag{4.9}$$

Thus we write $\Delta_H^{0,0}$ by means of the pseudohermitian connection,

4.2 Lemma (Weitzenböck-Tanaka Formula for $\Lambda^{p,q}E$).

$$\Delta_H^{0,0} = \left(1 + \frac{p-q}{n}\right) (\nabla^{0,1})^* \nabla^{0,1} + \left(1 - \frac{p-q}{n}\right) (\nabla^{1,0})^* \nabla^{1,0} + R_2, \tag{4.10}$$

where $\nabla^{1,0} := \nabla_{\Pi^{1,0}}$ and $\nabla^{0,1} := \nabla_{\Pi^{0,1}}$, and R_2 is algebraic which is a "trace" of the curvature of the pseudohermitian connection.

Proof. We recall the calculation of Proposition 3.8. Choose a local framing $\{X_k, Y_k := JX_k\}$, $k = 1, \cdots, n$ normal at $x \in M$. Then we set ;

$$Z_k := \frac{1}{\sqrt{2}}(X_k - \sqrt{-1}Y_k) \in E^{1,0}, \quad Z_{\overline{k}} := \frac{1}{\sqrt{2}}(X_k + \sqrt{-1}Y_k) \in E^{0,1}$$

and θ^k, $\theta^{\overline{k}}$ its dual. For $\alpha := \sum_{I,J} \alpha_{I,J} \theta^I \wedge \theta^{\overline{J}}$,

$$\nabla_{Z_i} \alpha = \sum (Z_k \cdot \alpha_{I,J}) \theta^I \wedge \theta^{\overline{J}},$$

since $\nabla_{Z_i} \theta^j = 0$ at x.

On the other hand, the Lie brackets at x of any vectors of the framing are colinear to T. Cartan$'$ formula (3.3) implies that

$$\partial_H \alpha = \sum (Z_k \cdot \alpha_{I,J}) \theta^k \wedge \theta^I \wedge \theta^{\overline{J}} = \sum e_k \nabla_k \alpha.$$

This expression means that $\partial_H\alpha$ is tensorial in Z_k, and so well-defined in a neighbourhood of x, i.e.,

$$\partial_H = \sum_{k=1}^{n} e_k \nabla_k.$$

And then we have $\partial_H^* = -\sum \nabla_k^* \iota_k$. It is evident that $(\nabla_k)^* = -\nabla_{\overline{k}}$ at x. Thus we have in a neighbourhood of x,

$$\partial_H^* = -\sum \iota_k \nabla_{\overline{k}}.$$

Finally, we have $[Z_{\overline{k}}, Z_\ell] = \sqrt{-1}\delta_{k\ell}T$, and so

$$\nabla_{\overline{k}}\nabla_\ell - \nabla_\ell\nabla_{\overline{k}} - \delta_{k\ell}\nabla_{\sqrt{-1}T} = R(Z_{\overline{k}}, Z_\ell). \tag{4.11}$$

We shall write $\Delta_H^{0,0}$ on $\Lambda^{p,q}E$. First, since $d_H = \partial_H + \overline{\partial_H}$, we have $\Delta_H^{0,0} = \Delta_{\partial_H} + \Delta_{\overline{\partial_H}}$. Proposition 3.8 implies that

$$\begin{aligned} \Delta_{\partial_H} &= \partial_H\partial_H^* + \partial_H^*\partial_H \\ &= (\sum e_k\nabla_k)(\sum -\iota_\ell\nabla_{\overline{\ell}}) - (\sum \iota_\ell\nabla_{\overline{\ell}})(e_k\nabla_k) \\ &= -\sum e_k\iota_\ell\nabla_k\nabla_{\overline{\ell}} - \sum e_k\iota_\ell\nabla_{\overline{\ell}}\nabla_k \text{ at } x. \end{aligned}$$

By using the relation $\iota_\ell e_k + e_k\iota_\ell = \delta_{k\ell}$, we have

$$\begin{aligned} \Delta_{\partial_H} &= \sum e_k\iota_\ell(\nabla_{\overline{\ell}}\nabla_k - \nabla_k\nabla_{\overline{\ell}}) - \sum \nabla_{\overline{k}}\nabla_k \\ &= \sum e_k\iota_\ell(\delta_{k\ell}\nabla_{\sqrt{-1}T} + R(\overline{\ell}, k)) - \sum \nabla_{\overline{k}}\nabla_k \\ &= (\sum e_k\iota_k)\nabla_{\sqrt{-1}T} + \sum e_k \wedge \iota_\ell R(\overline{\ell}, k) - \sum \nabla_{\overline{k}}\nabla_k. \end{aligned}$$

Set

$$\Delta_{\partial_H} := p\nabla_{\sqrt{-1}T} + (\nabla^{1,0})^*\nabla^{1,0} + \sum e_k\iota_\ell R(\overline{\ell}, k) \text{ (tensorial).}$$

Similarly we have

$$\Delta_{\overline{\partial_H}} := -q\nabla_{\sqrt{-1}T} + (\nabla^{0,1})^*\nabla^{0,1} + \sum e_{\overline{k}}\iota_{\overline{\ell}} R(\ell, \overline{k}).$$

On the other hand, (4.9) implies that

$$n\nabla_{\sqrt{-1}T} = \sum \nabla_{\overline{k}}\nabla_k - \nabla_k\nabla_{\overline{k}} - R(\overline{k}, k) \text{ at } x$$

Set

$$n\nabla_{\sqrt{-1}T} := -(\nabla^{1,0})^*\nabla^{1,0} + (\nabla^{0,1})^*\nabla^{0,1} - \sum R(\overline{k}, k). \tag{4.12}$$

Therefore, we have the requested formula (4.10) with

$$R_2 := \sum_{1\le k,\ell\le n} (e_k\iota_\ell R(\overline{\ell}, k) + e_{\overline{k}}\iota_{\overline{\ell}} R(\ell, \overline{k})) + \frac{p-q}{n} \sum_{1\le k\le n} R(\overline{k}, k) \tag{*}$$

for all orthonormal framings $\{Z_k\}$. $\square$

4.3 Remark.

By (4.10), R_2 is self-adjoint. Let $R^{1,0}$ and $R^{0,1}$ be two components of R_2 in $E \otimes \mathbb{C}$;

$$R^{1,0}V = \sum_{k=1}^{n} R(\overline{k}, k)\Pi^{1,0}V \text{ and } R^{0,1}V = \sum_{k=1}^{n} R(\overline{k}, k)\Pi^{0,1}V.$$

We extend it to $\alpha \in \Lambda^k M \otimes \mathbb{C}$;

$$(R^{1,0}\alpha)(V_1, \cdots, V_k) := -\sum_{i=1}^{k} \alpha(V_1, \cdots, R^{1,0}V_i, \cdots, V_k),$$

where $V_1, \cdots, V_k \in E$. We extend $R^{0,1}$ by the same way. It is easily proved that

$$R(\overline{\ell}, \ell)k = R(\overline{\ell}, k)\ell$$
$$\iota_\ell R(\overline{\ell}, k) = R(\overline{\ell}, k)\iota_\ell - \iota_{R(\overline{\ell},\ell)k}$$
$$-\sum_{k,\ell} e_k \iota_{R(\overline{\ell},\ell)k} = R^{1,0}$$

We put these relations into (*) and so we have

$$R_2 = \sum_{k,\ell}(e_k R(\overline{\ell}, k)\iota_\ell + e_{\overline{k}} R(\ell, \overline{k})\iota_{\overline{\ell}}) + \left(1 + \frac{q-p}{n}\right) R^{1,0} - \left(1 - \frac{q-p}{n}\right) R^{0,1}..$$

□

4.4 Theorem (Vanishing theorem of $H^k(M, \mathbb{R})$, $k \neq n, n+1$).

$$H^k(M, \mathbb{R}) = 0 \ (k < n)$$

if, for $\forall \alpha \in J^k \setminus \{0\}$, $((R_2 + (n-k+2)R_1)\alpha, \alpha) > 0$.

OPEN PROBLEM:. *In the case* $k = n$, *we have no informations.*

It is sufficient to use the Weizenböck formulas (4.8), (4.9) and (4.10). In fact, $R_2 J\alpha = JR_2\alpha$ and $R_1 J\alpha = -JR_1\alpha$. The positivity condition can be written as

$$(R_2\alpha, \alpha) > (n-k+2)|(R_1\alpha, \alpha)| \ \forall \alpha \in J^k \setminus \{0\},$$

that is, R_2 controls R_1.

We write down in the case of degree 1. First, by (4.6), we have $R_1 = -(1/2)\mathcal{L}_T J$. R_1 measures the deformation of the complex structure and the metric by the Reeb flow. $R_1 = 0$ if and only if this flow is Riemannian. For R_2, (*) is reduced the simple form ; $\forall \ell, k \in [1, n]$, $R(\overline{\ell}, k)\iota_k \alpha = 0 = R(\ell, \overline{k})\iota_{\overline{\ell}}\alpha$. Then we have, on $J^1 \simeq \Lambda^1 E$,

$$R_2\alpha = -\left(1 - \frac{1}{n}\right)\left(\sqrt{-1}\sum_{\ell} R(\overline{\ell}, \ell)\right) J\alpha = \left(1 - \frac{1}{n}\right)\sum_{\ell=1}^{n} R(Y_\ell, X_\ell)J\alpha.$$

R_2 can be viewed as a "Ricci curvature" with respect to a contact structure.

From $R(\bar{\ell}, \ell)k = R(\bar{\ell}, k)\ell$ it follows that

$$\begin{aligned}\frac{2n}{n-1}(R_2\alpha, \alpha) &= \sum_{i=1}^{n} K(\alpha^\sharp, X_i) + K(\alpha^\sharp, Y_i)\\ &\quad + K(J(\alpha^\sharp), X_i) + K(J(\alpha^\sharp), Y_i)\\ &= \mathrm{Ric}_E(\alpha^\sharp, \alpha^\sharp) + \mathrm{Ric}_E(J\alpha^\sharp, J\alpha^\sharp),\end{aligned}$$

where $K(U, V) := (R(V, U)U, V)$ is the sectional curvature of the plane $\{U, N\}$.

§4.3 Transversally Kählerian structures.

In the case that $R_1 = 0$, the complex structure, the Riemannian metric and also the pseudohermitian connection and its curvature are invariant by the Reeb flow. It is natural to assume that the orbit space $N := M/(\mathrm{T})$ of Reeb flows has the structure of manifold, i.e., all flows are closed and have the same period. Then we identify M with the total space of the S^1-bundle over N. We denote the canonical projection $M \longrightarrow N$ by π. Then we have

4.5 Proposition. *For $k \le n$,*

$$\begin{aligned}\{\alpha \in J^k M \mid \Delta_E \alpha = 0\} &= \{\alpha \in \Lambda^k M \mid \Delta\alpha = 0\}\\ &= \pi^*\{\alpha \in \Lambda^k N \mid \Delta\alpha = \Lambda\alpha = 0\}.\end{aligned}$$

Proof. We denote the above three spaces by $\mathcal{H}_E, \mathcal{H}_{\mathrm{dR}}$ and $\pi^*\mathcal{H}_N$ respectively. It is sufficient to prove that $\mathcal{H}_E \subset \mathcal{H}_{\mathrm{dR}}$ when $k < n$. The equalities follows from that $\dim \mathcal{H}_E = \dim \mathcal{H}_{\mathrm{dR}} = \dim H^k(M, \mathbb{R})$. It follows from (4.1), (4.2), (4.4) and (4.7) that $\Delta_H = \Delta_H^{0,0}$ and $\Delta_E = \Delta_E^{0,0}$ if $\mathcal{L}_T J = 0 = \mathrm{Tor}(T, -)$. By (4.9) we have that $\Delta_E \alpha = 0$ implies $\Delta_H \alpha = 0$. In fact, since Δ_H preserves the bidegree, each component $\alpha^{p,q} \in J^{p,q}$ of α is harmonic, i.e., $d_H \alpha^{p,q} = 0 = \delta_H \alpha^{p,q}$. Taking the decomposition of d_H and δ_H, we have

$$\partial_H \alpha = \overline{\partial_H}\alpha = \partial_H^* \alpha = \overline{\partial_H}^* \alpha = 0,$$

i.e.,

$$\Delta_{\partial_H}\alpha = 0 = \Delta_{\overline{\partial_H}}\alpha.$$

Then we have

$$\Delta_{\partial_H} - \Delta_{\overline{\partial_H}} = \sqrt{-1}(k - n)\mathcal{L}_T,$$

which implies $\mathcal{L}_T \alpha = 0$. We write d and δ by means of d_H and δ_H

$$d_H = \mathrm{proj}_{\ker \iota_T} d = d - \eta \wedge \iota_T d = d - \eta \wedge \mathcal{L}_T \tag{4.13}$$

$$\delta_H = \mathrm{proj}_{\ker \iota_T} \delta = \delta - \eta \wedge \iota_T \delta = \delta - \eta \wedge \Lambda \tag{4.14}$$

since $\iota_T \delta + \delta \iota_T = \Lambda$ is the adjoint relation of $d(\eta \wedge -) + \eta \wedge d = L$. We have $\delta\alpha = d\alpha = 0$.

We must prove that harmonic forms on M are in J^k in degree $k < n$, i.e., $\iota_T \alpha = \Lambda\alpha = 0$. This means that they are invariant by the flow of T. Thus they are pullbacks of π of primitive forms on N.

We shall prove the case of degree n, and prove in deg. $k < n$ by the same way. Let $\alpha \in \mathcal{H}_E$.

- Harmonic forms are invariant by the Riemannian flow.

Then we show that $\mathcal{L}_T\alpha = 0$.

In fact, the flow ϕ_t^T generated by T is an isometry, i.e., $(\phi_t^T)^* = (\phi_t^T)^{-1} = \phi_t^{-T}$. Then we have, by definition, that $\mathcal{L}_T^* = -\mathcal{L}_T$. Thus $L\mathcal{L}_T = \mathcal{L}_T L$ implies that $\Lambda\mathcal{L}_T = \mathcal{L}_T\Lambda$. Moreover, $\mathcal{L}_T\iota_T = \iota_T\mathcal{L}_T = \iota_T d\iota_T$, and so $\mathcal{L}_T$ preserves the spaces J^k of the contact complex. On the other hand, $\mathcal{L}_T\alpha$ is a Δ_E-harmonic. (3.10) implies that

$$\mathcal{L}_T\alpha = \iota_T D\alpha + d_H\delta_H^J\alpha = d_E\delta_E^J\alpha.$$

$\mathcal{L}_T\alpha$ is the harmonic representative of the zero class of d_E-cohomology, and so $\mathcal{L}_T\alpha = 0$.

The identities in Proposition 3.11 and 3.13 must be equal since they depend on the elementary formulas $\partial_H^2 \simeq \overline{\partial_H}^2 = o(2)$ and $[d_H, \Lambda] \simeq -\delta_H^J$ which are exact in this cases. In particular, in J^n,

$$\begin{aligned}\Delta_E &= \Delta_E^{0,0}\\ &= \frac{\Delta_H^2}{4} + \sum_{-1\le k\le 1} ((d_E\delta_E)^{k,-k})^*(d_E\delta_E)^{k,-k} + ((\iota_T D)^{k,-k})^*(\iota_T D)^{k,-k}.\end{aligned}$$

Thus we have that $\Delta_H\alpha = 0$, which implies, by (4,13) and (4.14),that

$$\begin{aligned}d\alpha &= d_H\alpha + \eta\wedge\mathcal{L}_T\alpha = 0,\\ \delta\alpha &= \delta_H\alpha + \eta\wedge\Lambda\alpha = 0.\end{aligned}$$

That is, $\alpha \in \mathcal{H}_{\mathrm{dR}}$. □

- $H^1(M, \mathbb{R}) = H^1(N, \mathbb{R})$.

Then we have $H^1(M, \mathbb{R}) = 0$ if the Ricci curvature of N is positive. In fact, this condition can also be obtained by using the pseudohermitian connection ∇^H and the contact complex of M.

In fact we can show that ∇^H is the pull-back of the connection ∇^N of the Levi-Civita connection of N, i.e., $\nabla_X^H Y \in E$, $\forall X, Y$ and $\pi\nabla_X^H Y = \nabla_{\pi X}^N \pi Y$. This means that ∇^H can be characterized as followings ; it preserves $g|_E$, η and torsion $\mathrm{Tor}(T, X) = \mathcal{L}_T JX = 0$. Thus we have $\pi R^H(X, Y)Z = R^N(\pi X, \pi Y)\pi Z$. Therefore the positivity of the Ricci curvature of ∇^N is equivalent to the positivity of R_2.

Appendix ; Geodesics on a contact manifold.

It is natural to study geodesics in a contact manifold M from points of Carnot-Caratheodory metrics. We use the pseudohermitian connection on M. For two points , $x_2 \in M$, let $\gamma_u(t)$, $t, u \in [0,1]$ be a family of curves with $x_1 := \gamma_u(0)$ and $x_2 := \gamma_u(1)$. Let $V := \partial\gamma/\partial u$ and $X := \partial\gamma/\partial t$. We consider the energy functional

$$E(\gamma) := \int_0^1 ||\dot\gamma(t)||^2 dt$$

and the length

$$\ell(\gamma) := \int_0^1 ||\dot\gamma(t)|| dt.$$

The minimum of E is equal to one of ℓ up to parametrizations.

$$\begin{aligned}
\frac{\partial}{\partial u}E(\gamma_u)|_{u=0} &= V\int_0^1 (X,X)dt \\
&= 2\int_0^1 (\nabla_V X, X)dt \\
&= 2\int_0^1 (\nabla_X V + (\mathrm{Tor}(V,X),X))dt \\
&= 2\int_0^1 X(V,X) - (V,\nabla_X X) + (\mathrm{Tor}(V,X),X)dt \\
&= 2\int_0^1 -(V,\nabla_X X) + (\mathrm{Tor}(V,X),X)dt,
\end{aligned}$$

where

$$\begin{aligned}
(\mathrm{Tor}(V,X),X) &= \mathrm{Tor}(\eta(V))T + V_E, X) \\
&= \eta(V)\mathrm{Tor}(T,X) + d\eta(V_E,X)T \\
&= \eta(V)(\mathrm{Tor}(T,X),T).
\end{aligned}$$

- $E(\gamma)$ is a minimum in parametrizations of γ if and only if

$$(\nabla_X X, X) = 0 = X||\dot\gamma(t)||^2.$$

In the following, we assume that this condition is satisfied. We decompose $\nabla_X X$ as

$$\nabla_X X = \alpha(t)JX + Y,$$

where Y is orthogonal to X and JX. Then

$$\frac{\partial}{\partial u}E(\gamma_u)|_{u=0} = 2\int_0^1 -\alpha(t)(V,JX) - (V,Y) + \eta(V)(\mathrm{Tor}(T,X),X)dt.$$

Recall that

$$\begin{aligned}
d\eta(V,X) &= V\eta(X) - X\eta(V) - \eta([V,X]) \\
&= X\eta(V) = -(V,JX).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\partial}{\partial u}E(\gamma_u)|_{u=0} &= 2\int_0^1 -\alpha(t)(V,JX) - (V,Y) + \eta(V)(\mathrm{Tor}(T,X),X)dt \\
&= 2\int_0^1 \eta(V)(X\alpha(t) + (\mathrm{Tor}(T,X),X)) - (V,Y)dt.
\end{aligned}$$

Conversely, if γ is not constant, all vector fields V along γ with value 0 at endpoints satisfying

$$X\eta(V) = (V, JX),$$

can be integrated in the family of curves with endpoints. Therefore $E(\gamma)$ is stationary if and only if

$$Y = 0, \text{ and } X\alpha(t) + (\mathrm{Tor}(T, X), X) = 0.$$

Summing up, we have

4.A.1 Proposition (Equations of C-C geodesics). *γ is a C-C geodesic if and only if $\nabla_X X$, $X := \dot\gamma(t)$ is colinear to JX, i.e.,*

$$\nabla_X X = -\alpha JX,$$

where

$$X\alpha(t) = -(\mathit{Tor}(T, X), X) = \frac{1}{2}(J(\mathcal{L}_T J)X, X).$$

4.A.2 Example.

Let H^3 be the Heisenberg group of dimension 3. We give H^3 the pseudohermitian structure invariant by translations. The projection $\pi : H^3 \longrightarrow H^3/ < T >= \mathbb{R}^2$ is a Riemannian submersion. Indeed, η is projected to the area form $A := (1/2)(xdy - ydx)$. A simple closed curve in $\mathbb{R}^2$ can be lifted to a simple closed "Legendre" curve if and only if its area is zero. And Legendre curves joining x_1 and x_2 in H^3 are liftings of curves πx_1 and πx_2 with fixed area. Thus C-C geodesics are minimal for fixed area, i.e.,arcs of circles. Their tangent vectors $X := \dot\gamma(t)/||\dot\gamma(t)||$ satisfy, by definition, $\nabla_X X = kJX$ where k is the curvature of a directed circle and ∇ is the Levi-Civita connection of $\mathbb{R}^2$. Since the pseudohermitian connection is projected onto ∇ and the complex structure is invariant by T, we see, by equations of geodesics, that

$$\nabla_X X = kJX,$$

where

$$Xk = -(\mathrm{Tor}(T, X), X) = 0.$$

□

- A pseudohermitian structure is complete if the C-C distance is complete.

In this case, it is equivalent to

- The pseudohermitian connection is geodesically complete.

CHAPTER 5. GEOMETRY OF PFAFFIAN SYSTEMS

§5.1 Horizontal cohomology.

Let M be a connected compact manifold, and let H be a smooth subbundle of TM. Let $H_1 := H + [H, H]$ be the subbundle of TM consisting of vector field X of local form ;

$$X := Y_0 + [Y_1, Y_2], \ Y_0, \ Y_1, \ Y_2 \in H.$$

Then there is an anti-symmetric bilinear map $\mu(-,-) : H \times H \longrightarrow H_1/H$ defined by

$$\mu(X, Y) := [X, Y] \bmod (H), \tag{5.1}$$

• (5.1) is well defined.

• If M is the total space of a principal fiber bundle and H comes from a connection, then μ is just the curvature of the connection.

Suppose that the vector bundle H_1/H is of rank k_1, then $\mu(-,-)_x$ is $\mathbb{R}^{k_1}$-valued 2-form on H_x, thus it determines k_1 elements of $\Lambda^2 H_x$, which are denoted by $\theta^1, \cdots, \theta^{k_1}$. Thus we can write $\mu_x := (\theta^1, \cdots, \theta^{k_1})$. Let $I_x(\theta^1, \cdots, \theta^{k_1})$ or simply I_x be the exterior algebraic ideal in ΛH_x generated by $\theta^1, \cdots, \theta^{k_1}$.

If H has non-degeneracy $r > 0$, i.e., there is the biggest number $r > 0$ such that, for $(r-1)$-forms $a_1, \cdots, a_{k_1}$ on H_x,

$$a_1 \wedge \theta^1 + \cdots + a_{k_1} \wedge \theta^{k_1} \neq 0$$

unless $a_1 = \cdots = a_{k_1} = 0$, then H is two-step generating, i.e., $H_1 := H + [H, H] = TM$. A contact stracture H is a spacial case of 2-step generating one.

5.1 Definition. *H is strongly bracket generating if for, $\forall X \in H_x$, $X \neq 0$, the induced map $H_x \longrightarrow TM_x/H_x; Y \longmapsto \mu(X, Y)$ is submersion.*

• (Weinstein) Let M be the total space of a principal fiber bundle and let H come from a connection. Then H is strongly bracket generating if and only if M is flat.

5.2 Lemma. *If H is strongly bracket generating and (M, H) is not a 3-dimensional contact manifold, then H is two-step generating.*

Proof. Assume otherwise, i.e., there are 1-forms $a_1, \cdots, a_{k_1}$, which are not all zero, such that

$$a_1 \wedge \theta^1 + \cdots a_{k_1} \wedge \theta^{k_1} = 0. \tag{5.2}$$

Without loss of generality we assume that $a_1, \cdots, a_k$ are linearly independent at $x \in M$. Choose a coordinate system $\{x_i\}$ such that $a_1 = dx_1, \cdots, a_{k_1} = dx_{k_1}$ at x. Write

$$\theta^i := \sum_{\ell,k} \theta^i_{\ell,k} dx_\ell \wedge dx_k = 0,$$

then from (5.2) at x we have

$$\sum_{\ell \geq k_1+1, k \geq k_1+1} \theta^i_{\ell,k} dx_\ell \wedge dx_k = 0,$$

which contradicts to the strongly bracket generating of H. $\square$

Now, we shall define partial connections which is a generalization of Levi-Civita connection to sub-Riemannian metrics.

5.3 Definition. *Assume that there is a subbundle K in TM complementary to H, i.e., $TM = H \otimes K$ and $\pi : TM \longrightarrow H$ the projection. A bilinear map*

$$(X, Y) \in H_x \times C^\infty(H) \longmapsto D_X Y \in H_x$$

depending smoothly on x, is a partial connection if

(1) $$D_X(fY) = < df, X > Y + f D_X Y, \ X, Y \in C^\infty(H), \ f \in C^\infty M$$

where $< -, - >$ is the pairing between T^*M and TM.

(2) $$D_X Y - D_Y X = \pi[X, Y], \ X, Y \in C^\infty(M)$$

(3) $$X(Y, Z) = (D_X Y, Z) + (Y, D_X Z)$$

• Suppose that M is the total space of a fiber bundle $W \longrightarrow M \longrightarrow B$ over a Riemannian manifold and H comes from a connection on the fiber bundle, then horizontally lifting the Levi-Civita connection on B to H, we have a partial connection.

• For given H, K and $(-, -)$ on H, there is a unique partial connection.

An orthonormal framing $\{e_i\}$ for H is normal at a given point $x_0 \in M$ if $D_{e_j} e_i(x_0) = 0$.

• Such a normal framing always exists, and so $\pi[e_i, e_j](x_0) = 0$.

The (partial) curvature of the partial connection is a trilinear map

$$R : C^\infty(H) \times C^\infty(H) \times C^\infty(H) \longrightarrow C^\infty(H)$$

defined by

$$R(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{\pi[X,Y]} Z.$$

• $R(X, Y)Z$ is not a tensor in the " usual " sense.

5.4 Lemma. *Let X, Y and Z be smooth horizontal vector fields on M and f a smooth function. Then we have*

$$R(fX, Y)Z = fR(X, Y)Z, \quad R(X, Y)fZ = (\mu(X, Y)f)Z + fR(X, .Y)Z.$$

Remark. In general there is no partial connection and volume form canonically associated with the sub-Riemannian metric. However, if H is a contact structure, then there is a natural volume form dvol and a complementary bundle K to H defined as follows ; let η be the 1-form such that $\eta = 0$ defines H and

$$(X,Y) := d\eta(X, JY), \; X, Y \in H \tag{5.3}$$

where J is an endomorphism of H such that det $J = 1$.
- Such a 1-form η uniquely exists.
- We set

$$K := \{X \mid d\eta(X, -) = 0\} \tag{5.4}$$

and dvol $:= \eta \wedge (d\eta)^n$. The induced partial connection D is called the canonical partial connection of the sub-Riemannian metric.

Let $\Lambda^* M$ be the sheaf of smooth differential forms on M, and $\Lambda_N M$ be the subsheaf consisting of ω such that if H is locally defined by k 1-forms $\omega_1 = \omega_2 = \cdots = \omega_k = 0$, then

$$\omega = \sum (f_i \wedge \omega_i + g_i \wedge d\omega_i),$$

where f_i and g_i are smooth differential forms.

There is a natural filtration $\Lambda^*_N M$ and $d(\Lambda^k_N M) \subset \Lambda^{k+1}_N M$. $\Lambda^*_N M$ is a algebraic and differential ideal of $\Lambda^* M$. Let $d^k_N : \Lambda^k_N M \to \Lambda^{k+1}_N M$ be the restriction of d. Let $\Lambda^*_H M$ be the quotient sheaf $\Lambda^* M / \Lambda^*_N M$, which has a natural filtration $\Lambda_H M := \oplus \Lambda^k_H M$ and a natural operator

$$d_H := d^k_H : \Lambda^k_H \longrightarrow \Lambda^{k+1}_H M$$

defined as the followings; if $p_H : \Lambda^* M \longrightarrow \Lambda^*_H M$ is the projection,

$$d_H p_H(\omega) := p_H(d\omega).$$

In the following, we assume the technical condition ; $\Lambda^k_H M$ satisfies the condition (L) if $\omega \in \Lambda^k_H M$ satisfies $\omega(x) = 0$ for every $\in M$ then $\omega = 0$ (as a cross-section of $\Lambda^k M$).

5.5 Lemma. *Suppose that H satisfies the following condition ; there are 1-forms $\omega_1, \cdots, \omega_k$ such that H is defined by $\omega_1 = \cdots = \omega_k = 0$ locally, and $d\omega_{k_1+1}, \cdots, d\omega_k$ can be uniquely written as*

$$d\omega_{k_1+i} = \sum_{j=1}^{k} f_i^j \wedge \omega_j + \sum_{j=1}^{k_1} g_i^j \wedge \omega_j, \quad i = 1, \cdots, k - k_1,$$

where f_i^j, g_i^j are smooth forms, then $\Lambda^2_H M$ satisfies condition (L).

5.6 Corollary. *If H is two-step generating, then $\Lambda^2_H M$ satisfies condition (L).*

We shall determine the stalk of $\Lambda^k_H M$ over $x \in M \Lambda^k_H T_x M$ explicity. Clearly, if $k = 1$, $\Lambda^1_H T_x M = H_x$. However, for $k \geq 2$, $\Lambda^k_H T_x M$ is not freely generated by H_x.

5.7 Lemma. *Suppose that H_1/H is of rank k_1, $\mu_x := (\theta^1, \cdots, \theta^{k_1})$. Then the stalk of $\Lambda^2_H M$ over $x \in M$ is*

$$\Lambda^2_H T_x M = \Lambda^2(H_x)/span(\theta^1, \cdots, \theta^{k_1}).$$

Proof. Choose a subbundle V_1 in TM which is complementary to H. Suppose that H_x is spanned by $e_1, \cdots, e_m$, V_1 spanned by $b_1, \cdots, b_k$, and

$$[e_i, e_j] := \sum c^\ell_{ij}(x) b_\ell(x) \mod (e_1, \cdots, e_m),\ c^\ell_{ij} = -c^\ell_{ji}.$$

Then we can choose a local coordinate neighbourhood $(x_1, \cdots, x_m, y_1, \cdots, y_k)$ such that H is determined by $\omega_1 = \cdots = \omega_k = 0$, where

$$\omega_\ell := \begin{cases} dy_\ell - \sum c^\ell_{ij} x_i dx_j + O(y^2 + x^2) & \ell = 1, \cdots, k_1 \\ O(x^2 + y^2), & \ell = k_1 + 1, \cdots, k \end{cases}$$

Here $O(x^2 + y^2)$ denotes 1-form $\sum f_i dx_i + \sum g_j dy_j$ where $f_i := O(x^2 + y^2)$ and $g_j := O(x^2 + y^2)$. And so

$$d\omega_\ell := \begin{cases} -\sum c^\ell_{ij} dx_i \wedge dx_j + O(|y| + |x|) & \ell = 1, \cdots, k_1 \\ O(|x| + |y|), & \ell = k_1 + 1, \cdots, k \end{cases}$$

Thus lemma is easily followed. □

The above result can be easily generalized to $k > 2$,

5.8 Lemma. *The stalk of $\Lambda_H M$ over $x \in M$ is*

$$\Lambda_H T_x M = \Lambda H_x / I_x(\theta^1, \cdots, \theta^{k_1}.$$

• This corresponds to J^*.

According to Ginzburg and Rumin, we consider the exact sequence

$$0 \longrightarrow H^1_N(M) \longrightarrow H^1(M) \longrightarrow H^1_H(M)$$
$$\longrightarrow H^2_N(M) \longrightarrow H^2(M) \longrightarrow H^2_H(M) \longrightarrow \cdots,$$

where

$$H^k_N(M) := \ker\ d^k_N / \mathrm{im}\ d^{k-1}_N$$
$$H^k_H(M) := \ker\ d^k_H / \mathrm{im}\ d^{k-1}_H$$

• (Rumin[R1.2]) Let (M, H) be a contact manifold of dimension $2n+1$. Then we have $H^k_H M \simeq H^k(M)$ for $k = 1, \cdots, n-1$.

We shall generalize the above result to a two-step generating subbundle H.

5.9 Lemma. *If every $x \in M$ admits a neighbourhood U such that $H^k_N(U) = 0$ for $k = 0, 1, \cdots, r+1 < n$, then $H^k(M)$ is isomorphic to $H^k_H(M)$, for $k = 1, \cdots, r$.*

Proof. We have the commutative exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda(U \cup V) & \longrightarrow & \Lambda(U) \oplus \Lambda(V) & \longrightarrow & \Lambda(U \cap V) & \longrightarrow & 0 \\ & & \downarrow p_H & & \downarrow p_H & & \downarrow p_H & & \\ 0 & \longrightarrow & \Lambda_H(U \cup V) & \longrightarrow & \Lambda_H(U) \oplus \Lambda_H(V) & \longrightarrow & \Lambda_H(U \cap V) & \longrightarrow & 0 \end{array}$$

so

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(U \cup V) & \longrightarrow & H^1(U) \oplus H^1(V) & \longrightarrow & H^1(U \cap V) & \longrightarrow & \cdots \\ & & \downarrow p_H & & \downarrow p_H & & \downarrow p_H & & \\ 0 & \longrightarrow & H^1_H(U \cup V) & \longrightarrow & H^1_H(U) \oplus H^1_H(V) & \longrightarrow & H^1_H(U \cap V) & \longrightarrow & \cdots \end{array}$$

and by a standard argument we can prove the lemma. □

5.10 Lemma. *If at every $x \in M$, H_x has non-degeneracy r, then $H^1_N(M) = \cdots = H^r_N(M) = 0$.*

Proof. Fix some point in M. Then there is a coordinate system (x_i, y_j) and k 1-forms $\omega_1, \cdots, \omega_k$ such that H is defined by $\omega_1 = \cdots = \omega_k = 0$, where

$$\omega_j = dy_j - \sum c^j_{i\ell} x_i dx_\ell + O(|x|^2 + |y|^2),\ j = 1, \cdots, k$$

and

$$d\omega_j = \theta^j + O(|x| + |y|),\ j = 1, \cdots, k.$$

Now let α_s be a closed s-form ($s \le r$) of the form $\sum f_i \wedge \omega_i + \sum g_i \wedge d\omega_i$. Then

$$d\alpha_r = \sum df_i \wedge \omega_i + \sum((-1)^{s-1} f_i + dg_i) \wedge d\omega_i,$$

thus by assumption we have $(-1)^{s-1} f_i + dg_i \equiv 0 \bmod << \omega >>$, where $<< \omega >>$ is the algebraic ideal generated by $\omega_1, \cdots, \omega_k$. It need only to prove that for an s-form $\alpha := \sum_{i_1 < \cdots < i_k} f_J \wedge \omega_{i_1} \wedge \cdots \omega_{i_k}$, $d\alpha = 0$ if and only if $\alpha = 0$. Here, f_J is an $(s-k)$-form $f_J := \sum h_J dx_{j_1} \wedge \cdots \wedge dx_{j_{s-k}}$.

$$\begin{aligned} d\alpha = &\sum df_J \wedge \omega^J + \sum (-1)^{s-i} f_J \wedge d\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \\ &+ \cdots + \sum (-1)^{s-u-1} f_J \wedge \omega_{i_1} \wedge \cdots \wedge d\omega_{i_u} \wedge \cdots \wedge \omega_{i_{s-k}}, \end{aligned}$$

which implies that $\sum_{j \ge k} f_{(1,2,\cdots,k-1,j)} \wedge d\omega_j = 0$. Again by assumption that $r \ge s$, we have $f_{(1,2,\cdots,k-1,j)} = 0$. By the similar way, $f_J = 0$ for any J. □

5.11 Corollary. *Under the same condition as in Lemma 5.10, $H^i_H(M) \equiv H^i(M)$, for $i = 1 \cdots, r-1$.*

Let N be a (smooth) manifold. A map $f : N \longrightarrow M$ is said to be horizontal if the pull back f^*H of H by f is zero.

Let $I^q := [0,1] \times \cdots \times [0,1]$ ($q-$times). Let $C_q(M)$ be the free abelian group generated by q-singular cubes $f : I^q \longrightarrow M$, and $C_{q,H}(M)$ be the subgroup generated by horizontal ones, and we set

$$C(M) := \oplus C_q(M), \quad C_H(M) := \oplus C_{q,H}CM).$$

We define the k-th horizontal singular homology group by

$$H_{k,H}(M) := \frac{\ker \delta^k}{\mathrm{im}\delta^{k-1}},$$

where δ is the restriction of the boundary operators to $C_H(M)$. There is a well defined pairing between $H_H(M)$ and $H_{q,H}(M)$. Suppose that f represents a k-th horizontal singular homology and ω does a k-th horizontal cohomology, then we define

$$< [f], [\omega] > := \int_f \omega. \tag{5.5}$$

5.12 Lemma. *The pairing (5.5) is well defined.*

Proof. Let ω' (resp. f') represent the same element as ω (resp. f). Then there is a horizontal k such that $f' = f + \delta k$. Without loss of generality we assume that H is defined by k 1-forms $e_1 = \cdots = e_k = 0$ within the image of f, f' and k. Then we have $\omega' = \omega + \sum h_i \wedge e_i + g_i \wedge de_i$, and

$$\int_{f'} \omega' = \int_{f'} (\omega' - \omega) + \int_{f'} \omega = \int_{f'} (\omega' - \omega) + \int_k d\omega + \int_f \omega \tag{5.6}$$

Now the first term above is

$$\int_{f'} h_i \wedge e_i + g_i \wedge de_i = \int_{f'} g_i \wedge de_i = (-1)^{\deg(g_i)} \int_{f'} dg_i \wedge e_i = 0.$$

For the second term above, note that $d\omega = \sum h'_i \wedge e_i + g'_i \wedge de_i$, and so

$$\int_k d\omega = \int_k g'_i \wedge de_i = \int_k dg'_i \wedge e_i - (\int_{f'} - \int_f) g'_i \wedge e_i = 0.$$

Therefore, we have

$$\int_f \omega = \int_{f'} \omega'.$$

□

• (Thom) If (M, E) is a contact manifold of dimension $2n+1$, $H_{k,H}(M) \simeq H_k(M)$, $k = 1, \cdots, n-1$, and the pairing (5.5) is nondegenerate modulo torsion elements.

§5.2 Characteristic classes of horizontal connections.

Let V be a vector bundle over M, and $H^* \subset T^*M$ the subbundle dual to H.

5.13 Definition. *A horizontal connection is a linear smooth map*

$$D : C^\infty(V) \longrightarrow C^\infty(H^* \otimes V)$$

which satisfies

$$D(fs) = d_E f \otimes s + f Ds, \quad f \in C^\infty(M),\ s \in C^\infty(H).$$

Example 1. Let $TM := E \oplus K$ be a splitting where K is a vector bundle over M, and let $\pi_K : TM \longrightarrow K$ the projection onto K. We define the horizontal connection

$$Ds := \sum_i \pi_K[s, e_i] \otimes e^i, \quad s \in C^\infty(K),$$

where e_i is a local framing for K.

Example 2. Let M be the total space of a fiber bundle $W \longrightarrow M \longrightarrow B$ and let H come from a connection, and let D_B be the Levi-Civita connection on B, and $\overline{D}$ be the horizontal lift of D_B. Then we define

$$Ds := \sum (\overline{D}_{e_i} s) \otimes e^i, \quad s \in C^\infty(H)$$

where e_i is an orthonormal framing for H.

• D is a horizontal connection.

Example 3. Let $D_X a$, $X \in H$, $a \in C^\infty(H)$ be a partial connection for the sub-Riemannian metric.

• The partial connection is a horizontal connection.

If an orthogonal framing e_i spans H, we define a horizontal connection $D : C^\infty(H) \longrightarrow C^\infty(H^*) \otimes C^\infty(H)$ by

$$Ds := \sum e^i \otimes D_{e_i}(s) \tag{5.7}$$

where e^i are the dual framing of e_i.

• (5.7) is well defined.

Set $s := \sum f_i s_i$, then we have

$$Ds = \sum \omega_{ij} \otimes s_j, \tag{5.8}$$

and $\omega_{ij} \in \Lambda^1_H(M)$. The connection 1-form relative to the local framing s_i is the matrix valued horizontal 1-form $\omega = [\omega_{ij}]$.

We extend D to be a derivation map

$$C^\infty(\Lambda^p_E(M) \otimes V) \longrightarrow C^\infty(\Lambda^{p+1}_E(M) \otimes V)$$

by

$$D(\theta_p \otimes s) := d_E\theta_p \otimes s + (-1)^p \theta_p \wedge Ds. \tag{5.9}$$

Then we have

$$\begin{aligned} D^2 fs) &= D(d_E f \otimes s + fDs) \\ &= d_E^2 f \otimes s - d_E f \wedge Ds + d_E f \wedge Ds + fD^2 s \\ &= fD^2 s \end{aligned}$$

Set $D^2(s)(x_0) := \Omega(x_0)s(x_0)$. Ω is called the curvature for the horizontal connection D. In tems of a local framing s_i,

$$\Omega = d_E\omega - \omega \wedge \omega. \tag{5.10}$$

• Ω is well defined, i.e., $\Omega' = h\Omega h^{-1}$, $h \in GL(\mathbb{C}^k)$. We say $P : \mathrm{End}(\mathbb{C}^k) \longrightarrow \mathbb{C}$ is an invariant polynomial map, if $P(hAh^{-1}) = P(A)$ for any $h \in GL(\mathbb{C}^k)$. Set $P(D) := P(\Omega)$.

5.14 Theorem. *Let P be an invariant polynomial map.*

(a) $d_E P(D) = 0$,

(b) Given two connections D_0 and D_1, we have a differential form $TP(D_0, D_1)$ so that

$$P(D_1) - P(D_0) = d_E\{TP(D_1, D_0)\}.$$

The proof is similar to the usual Chern-Weil theory.

Let V is a real vector bundle with a fiberwise metric $< -, - >_V$. A horizontal connection D is said to be sub-Riemannian if

$$d < s_1, s_2 >_V = < Ds_1, s_2 >_V + < s_1, Ds_2 >_V, \quad s_1, s_2 \in C^\infty(V).$$

If D is a sub-Riemannian connection, we define the total horizontal Pontragin class as

$$p(D) := \det\left(I + \frac{1}{2\pi}\Omega\right) = p_1(D) + p_2(D) + \cdots$$

where $p_k(D)$ is the $4k$-form, called the $4k$-th horizontal Pontragin class. Moreover, if the vector bundle V has even rank $2r$, then we can define

$$e(D) := \frac{(-1)^r}{2^q \pi^r r!} \sum \epsilon_{i_1 \cdots i_{2r}} \sum \theta_{i_1 i_2} \wedge \cdots \wedge \theta_{i_{2r-1} i_{2r}}.$$

Similarly, we can define the secondary invariants. Namely we have

5.15 Theorem. *Let P be an invariant polynomial map. Let D_τ be a family of horizontal connections with curvatures $\Omega(\tau)$, which satisfy*

$$P_H(P(\Omega(\tau), \cdots, \Omega(\tau)) = 0,$$
$$P_H\left(P(\frac{\partial D_\tau}{\partial \tau}, \Omega(\tau), \cdots, \Omega(\tau))\right) = 0$$

Then the horizontal cohomology class $TP(D_\tau, D_0) \in H_H(M)$ is independent of τ.

In the following, let D be the partial connection associated with a splitting $TM = H \oplus K$, and we assume the technical condition (L)

5.16 Theorem. *Suppose that $\Lambda_E^2 M$ satisfies the condition (L). Then the curvature of the horizontal connection (7.7) can be written in terms of the partial curvature as followings ;*

(5.11)
$$\Omega s = \sum_{i,j} P_H(e^i \wedge e^j \otimes R(e_i, e_j)s)$$

Moreover, if p_k and P_k are the k-th Pontragian class of $H \longrightarrow M$ and k-th Pontragin polynomial respectively, then

$$P_k = P_H(p_k).$$

Proof. By the technical condition (L), we only need to prove (5.11) at a point x_0. Note that the right hand side of (7.11) is defined independent of a local framing e_i. And so we prove (5.11) for a local framing e_i normal at x_0. Now,

$$\Omega s(x_0) = \sum P_H(d_H e^i \otimes D_{e_i} s)(x_0) + \sum_{i<j} e^i \wedge e^j \otimes R(e_i, e_j)s(x_0).$$

We need to prove $de^i(e_j, e_k)(x_0) = 0$. In fact,

$$de^i(e_j, e_k) = \frac{1}{2}\left(e_j(e^i(e_k)) - e_k(e^i(e_j)) - e^i([e_j, e_k])\right)(x_0) = 0,$$

Thus we have $(d_H e^i \otimes D_{e_i} s)(x_0) = 0$. $\square$

Remark. Let I_x be generated by $\theta^1, \cdots, \theta^k$ which are orthonormal with respect to the inner product on $\Lambda^2 H$,

$$\theta^r = \sum_{i,j} \theta^r_{ij} e^i \wedge e^j,$$

where e_i is an orthonormal framing for H, then (7.11) can be written as

$$\Omega = \sum_{i,j} \left(R(e_i, e_j) - \sum_{\ell,k,r} R(e_\ell, e_k)\theta^r_{\ell k}\theta^r_{ij} \right) \otimes e^i \wedge e^j.$$

Then we have

$$R(e_i, e_j) - \sum_r \sum_{\ell,k} \theta^r_{\ell k}\theta^r_{ij} R(e_\ell, e_k) \tag{5.12}$$

is a tensor.

In deed, we need only to prove that in view of Lemma 7.4

$$\mu(e_i, e_j) - \sum_r \sum_{\ell,k} \theta^r_{\ell k}\theta^r_{ij} \mu(e_{ell}, e_k) = 0. \tag{5.13}$$

If H is given by 1-forms $\omega_1 = \cdots = \omega_k = 0$, where

$$d\omega_i \equiv \theta^i \bmod(e^j)$$

then $[e_i, e_j] = 2\sum_r \theta^r_{ij} n_r \bmod (e_r)$, where n_r is the dual vector field to ω_r. Thus we have

$$\mu(e_i, e_j) = 2\sum_r \theta^r_{ij} n_r.$$

There

$$\begin{aligned}\mu(e_i, e_j) - \sum_r \sum_{\ell,k} \theta^r_{\ell k}\theta^r_{ij}\mu(e_\ell, e_k) &= 2\sum \theta^r_{ij} n_r - 2\sum_r \sum_{\ell,k} \sum_t \theta^r_{\ell k}\theta^r_{ij}\theta^t_{\ell k} n_t \\ &= 2\sum \theta^r_{ij} n_r - 2\sum_r \sum_{\ell,k} \sum_t \delta_{rt}\theta^r_{ij} n_t = 0.\end{aligned}$$

We can write the horizontal Pontragin classes in terms of the second jets of the sub-Riemannian metric, moreover, if H is contact, the construction is canonical and the lower horizontal Potragin classes are in fact the Pontragin classes of H.

We define a tri-linear map $T : H \otimes H \otimes H \longrightarrow H$ by

$$T(x, y, z) := R(x, y)z - \sum \frac{1}{4}(\theta^r, \overline{x} \wedge \overline{y})(\theta^r, e^i \wedge e^j)R(e_i, e_j)z$$

Here $\overline{x}$ denotes the dual to $x \in H$ in H^*. T is a well defined tensor. Indeed, note that $\theta^r_{ij} = (\theta^r, e^i \wedge e^j)/2$, expand $x = (x, e_1)e_1 + \cdots + (x, e_m)e_m$ and similarly we expand y.

§5.3. The Hodge Theory of $H^1(M)$ for a Paffian system.

Let ∇ be the Levi-Civita connection of (M, g). The relation between the Levi-Civita connection of g and the partial connection of the sub-Riemannian metric is

$$D_X Y = \pi \nabla_X Y, \; X \in H, \; Y \in C^\infty(H), \tag{5.14}$$

where $\pi : TM \longrightarrow H$ is the projection.

For two horizontal forms ω_1, ω, of the same degree their inner product

$$(\omega_1, \omega_2)_0 := \int_M (\omega_1, \omega_2) \mathrm{dvol}$$

where $(-,-)$ is the inner product induced on ΛH_x. We define the dual δ_H of d_H with respect to $(-,-)$ and define the Laplacian Δ_H by

$$\Delta_H := d_H \delta_H + \delta_H d_H.$$

For $\omega \in \Lambda_H M$, its weighted Sobolev norm is denoted by

$$||\omega||_1^2 := (\omega, \omega)_1 = \int_M \sum_i (D_{e_i}\omega, D_{e_i}\omega) \mathrm{dvol},$$

where $\{e_i\}$ is an orthonormal framing on H. In the following, we assume that I_x is generated by $\theta^1, \cdots, \theta^k$ which are orthonormal with respect to the induced inner product on $\Lambda^2 H$.

5.17 Lemma. *If $\{e_i\}$ is an orthonormal framing, $\{y_1, \cdots, y_k\}$ is an orthonormal framing for $K := H^\perp$, then if ω is a horizontal 1-form or 2-form,*

$$\begin{aligned} d_H\omega &= \sum_i e^i \wedge D_{e_i}\omega - \sum \left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega \right) \theta^r, \\ \delta_H \omega &= -\sum_i \iota_{e_i} D_{e_i}\omega - D^0\omega \end{aligned} \tag{5.16}$$

where D^0 is the 0-th order operator

$$D^0 := \sum_j p_H(\iota_{y_j} \nabla_{y,j}). \tag{5.17}$$

Remark. *D^0 is only depend on dvol, g and K. In particular, if H is contact, then D^0 is canonically defined tensor, thus it is another invariant of the sub-Riemannian metric.*

Proof. Let $p_1 : \Lambda M \longrightarrow \Lambda H$ and $p_2 : \Lambda H \longrightarrow \Lambda_H M$ be the orthogonal projections respectively. Then $p_H := p_2 \circ p_1$ and define $\overline{d} := p_1 d$. Then, by (5.14), we can rewrite $\overline{d}$ as

$$\overline{d} := \sum_i e_i \wedge D_{e_i}. \tag{5.18}$$

Thus, for horizontal 1-forms or 2-forms,

$$d_H\omega = p_2\bar{d}\omega = \bar{d}\omega - \sum_r (\theta^r, \bar{d}\omega)\theta^r.$$

And so (5.15) is proved. Next we compute δ_H. Let δ be the adjoint of d with respect to g,

$$\begin{aligned}\delta_H\omega &= p_1\delta\omega \\ &= p_1(\sum \iota_{e_i}\nabla_{e_i}\omega + \iota_{y_i}\nabla_{y_i}\omega \\ &= \sum \iota e_i D_{e_i}\omega + p_1\iota_{(y_i}\nabla_{y_i}\omega).\end{aligned}$$

□

5.18 Lemma. *If, for any $x,\ y \in C^\infty(H^\perp)$, $\nabla_x y \in C^\infty(H^\perp)$, then $D^0 = 0$.*

Remark. *If $H^\perp$ is an integrable subbundle (e.g., H is contact), then $D^0 = 0$ if every leaf of $H^\perp$ is totally geodesic with respect to g.*

5.19 Lemma. *For a horizontal 1-form ω,*

$$\begin{aligned}-\Delta^1_H\omega = &\sum D_{e_i}D_{e_i}\omega - D_{D_{e_i}e_i}\omega + \sum_{i,j} e^i \wedge \iota_{e_j}R(D_{e_i}, D_{e_j})\omega \\ &+ D^0\left(\sum_i e^i \wedge D_{e_i}\omega\right) + \sum e^i \wedge D_{e_i}D_0\omega \\ &- \sum_{r,j} e_j\left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\right)\iota_{e_j}\theta^r - \sum_r\left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\right)\iota_{e_j}D_{e_j}\theta^r\end{aligned}$$

Proof. Choose an orthonormal framing $\{e_i\}$, normal at $x_0 \in M$. Using (5.18), we have

$$\Delta^1_H = (\bar{d}\delta + \delta\bar{d})\omega - \delta(\sum(\omega, \theta^r)\theta^r). \tag{5.19}$$

□

5.20 Corollary. *If M is the total space of a fiber bundle $W \longrightarrow M \longrightarrow B$ over a Riemannian manifold with totally geodesic fibers and the sub-Riemannian metric is the horizontal lifting of the Riemannian metric on B, then*

(5.20)

$$\begin{aligned}-\Delta^1_H\omega = &\sum D_{e_i}D_{e_i}\omega - D_{D_{e_i}e_i}\omega + \sum_{i,j} e^i \wedge \iota_{e_j}R(D_{e_i}, D_{e_j})\omega \\ &- \sum_{i,j} e_j\left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\right)\iota_{e_j}\theta^r - \sum_r\left(\theta^r, \sum_i e^i \wedge D_{e_i}\omega\right)\iota_{e_j}D_{e_j}\theta^r\end{aligned}$$

where D is the horizontal lift of the Levi-Civita connection on B.

We suppose that I_x is generated by $\theta^1, \cdots, \theta^k$ such that

$$\theta^r := \sum_{i,j}\theta^r_{ij}e^i \wedge e^j,\ \theta^r_{ij} = -\theta^r_{ji}.$$

Without loss of generality, we assume that they are orthonormal,

$$\sum_{i,j} \theta^r_{ij}\theta^s_{ij} = \delta_{rs}.$$

We define the technical quantities ;

$$\text{(5.21)} \qquad \lambda_1(x) := \max\left|\sum_r \frac{2\sum_{i,j,s,t}\theta^r_{ij}\theta^r_{st}(u_{si},u_{tj}) - \sum_{i,j,s,t}\theta^r_{ij}\theta^r_{st}(u_{st},u_{ij})}{\sum_{s,i}|u_{si}|^2}\right|$$

$$\text{(5.22)} \qquad \lambda_2(x) := \max\left|\sum_{ru} \frac{\sum_{i,j,\ell,s,k,t}\theta^r_{ij}\theta^r_{\ell k}\theta^u_{st}\theta^u_{i\ell}(u_{sj},u_{tk})}{\sum_{s,i}|u_{si}|^2}\right|$$

5.21 Lemma. *$\lambda_1(x)$, $\lambda_2(x)$ are depend only on I_x (and independent of the choices of $\theta^1,\cdots,\theta^r,e_1,\cdots,e_n$).*

By a simple calculation we have the proof.

5.22 Theorem. *If, at every point $x \in M$, $1-\lambda_1(x)-2\lambda_2(x) > 0$, then Δ^1_H is hypoelliptic.*

Proof. By definition, it is sufficient to prove that there is a positive $\delta_0 > 0$ such that

$$\text{(5.22)} \qquad (\Delta^1_H\omega,\omega)_0 \geq \delta_0(\omega,\omega)_1 - N(\omega,\omega)_0.$$

Now,

$$\begin{aligned}(\Delta^1_H\omega,\omega) &= (d_H\omega,d_H\omega)_0 + (\delta_H\omega,\delta_H\omega)_0 \\ &= (\bar{d}\omega,\bar{d}\omega)_0 - \sum(\bar{d}\omega,\theta^r)^2_0 + (\delta_H\omega,\delta_H\omega)_0 \\ &= ((\bar{d}\delta_H+\delta\bar{d})\omega,\omega)_0 - \sum_r(\bar{d}\omega,\theta^r)^2_0.\end{aligned}$$

Modulo a 0-th operators, $\delta_H := \sum_i \iota_{e_i}D_{e_i}$, thus modulo 1st order operators, we have

$$\bar{d}\delta+\delta\bar{d} = \sum D_{e_i}D_{e_i} + \sum_{i,j} e^i\wedge\iota_{e_i}R(D_{e_i},D_{e_i}).$$

Let $\omega := \sum_i u_ie^i$. Let O_1 denote a sum of terms of the form $(D_{e_i}u_j,u_k)_0$, which is bounded (for any positive $\epsilon > 0$) by

$$|O_1(\omega)|_0 \leq \epsilon||\omega||^2_1 + N_\epsilon||\omega||^2_0.$$

Now

$$(\theta^r,\bar{d}\omega)^2 = \left(\sum_{i,j}\theta^r_{ij}D_{e_i}u_j\right)^2 + O_1.$$

Thus

$$\text{(5.24)} \qquad \begin{aligned}(\Delta_H\omega,\omega)_0 &= \sum_{i,j}(D_{e_i}u_i,D_{e_i}u_i) + \sum_{i,j}(R(D_{e_i},D_{e_j})u_i,u_j)_0 \\ &\quad - \sum_r\left(\sum_{i,j}\theta^r_{ij}D_{e_i}u_j\right)^2_0 + O_1\end{aligned}$$

By integration by parts the second term above is

$$\sum_{i,j}\left(R(D_{e_i},D_{e_j})u_i,u_j\right) = \sum_{i,j}\sum_{\ell,k}(\theta^r_{\ell k}\theta^r_{ij}R(D_{e_i},D_{e_i})u_i,u_j)_0 + O_1 \tag{5.25}$$
$$= 2\sum \theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_i, D_{e_k}u_j)_0 + O_1$$

Here we used the fact modulo 0-th order operators,

$$R(D_{e_i},D_{e_j}) = \sum_r\sum_{\ell,k}\theta^r_{\ell k}\theta^r_{ij}R(D_{e_l},D_{e_k}) \tag{5.26}$$

Using integration by parts repeatedly, the third term in (5.24) is

$$\sum_r\left(\sum_{i,j}\theta^r_{ij}D_{e_i}u_j\right)^2_0$$
$$= \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_j, D_{e_\ell}u_k)_0$$
$$= \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_j, D_{e_\ell}u_k)_0 - \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(R(D_{e_i},D_{e_\ell})u_j,u_k)_0 + O_1$$
$$= \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_j, D_{e_\ell}u_k)_0 - \sum_{ij\ell kru}\theta^u_{i\ell}\theta^u_{st}\theta^r_{\ell k}\theta^r_{ij}(R(D_{e_s},D_{e_t})u_j,u_k)_0 + O_1 \tag{5.27}$$
$$= \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_j, D_{e_\ell}u_k)_0 - \sum_{ij\ell kru}\theta^u_{i\ell}\theta^u_{st}\theta^r_{\ell k}\theta^r_{ij}(R(D_{e_s},D_{e_t})u_j,u_k)_0 + O_1$$
$$= \sum_{ij\ell kr}\theta^r_{\ell k}\theta^r_{ij}(D_{e_i}u_j, D_{e_\ell}u_k)_0 + \sum_{ij\ell kru}\theta^u_{i\ell}\theta^u_{st}\theta^r_{\ell k}\theta^r_{ij}(D_{e_t}u_j, D_{e_s})u_k)_0 + O_1$$

Here we have used (5.26) again. Inserting (5.25) and (5.27) into (5.24), we have

$$(\Delta_H\omega)_0 \geq \sum_{ij}(D_{e_i}u_i, D_{e_j}u_j)_0 - 2\sum\theta^r_{\ell k}\theta^r_{ij}(D_{e_\ell}u_i, D_{e_k}u_j)_0$$
$$+ \sum_{ij\ell kr}\theta^r_{ij}\theta^r_{\ell k}(D_{e_\ell}u_j, D_{e_i}u_k)_0$$
$$- \sum_{ij\ell kru}\theta^r_{ij}\theta^r_{\ell k}\theta^u_{i\ell}\theta^u_{st}(D_{e_s}u_j, D_{e_t}u_k)_0$$
$$+ \sum_{ij\ell kru}\theta^r_{ij}\theta^r_{\ell k}\theta^u_{i\ell}\theta^u_{st}(D_{e_t}u_j, D_{e_s}u_k)_0 + O_1$$
$$\geq (1-\lambda_1-2\lambda_2)\sum_{ij}(D_{e_i}u_j, D_{e_i}u_j)_0 + O_1$$

Therefore we proved (5.23). □

5.23 Corollary. *If H has non-degeneracy ¿ 0, and $1-\lambda_1(x)-\lambda_2(x)>0$, then*

$$H^1(M)=\{\omega\in\Lambda^1_H M \mid d_H\omega=\delta_H\omega=0\}.$$

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