수 학 강 의 록 제 35 권

# INTRODUCTION TO KNOT THEORY

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# INTRODUCTION TO KNOT THEORY

Youn W. Lee

Contents

## Preface

This note represents the lectures I gave in a course, "Introduction to knot theory," in the Fall semester of 1990 at the Seoul National University. The lectures were intended to give a graduate student a brief exposure to introductory knot theory.

The topics covered in the note are far from complete and biased by my taste. Reading the note requires a basic knowledge of algebraic topology and elementary techniques in manifold theory. Proofs are given in full detail, within the scope of the note, whenever they are attempted.

I would like to take this opportunity to thank Professors Hyuk Kim and Hyunkoo Lee for encouraging me to write the note, especially the latter, for reading the manuscript and making valuable suggestions. I am also indebted to the students who listened to the lectures and Ms. Chaeun Park who typed the note. Finally, I thank the Department of Mathematics, Seoul National University, for making the publication of the note possible.

## 1. Basic definitions and a theorem of Lickorish

We introduce some of the basic definitions and notation we use in this note. The study of knots and links plays an important role in the study of 3-manifold: this can be explained by a theorem by Lickorish which says that any orientable, closed, connected 3-manifold is obtained by doing a framed surgery along a link in $S^3$, the 3-dimensional sphere. We give a proof of this theorem later. We consider everything in smooth category. Note that there is no difference between the smooth and piecewise linear category in the dimensions we are interested in.

A *knot* is defined to be a (smoothly) embedded circle in $S^3$. A *link* is defined to be a finite collection of disjointly embedded circles in $S^3$. If each component of a link is oriented (by an arrow), the link is called an *oriented link.* Two links $L_1$ and $L_2$ are *equivalent* if there exists a diffeomorphism $f$ of $S^3$ such that $f(L_1) = L_2$. Another equivalence relation often considered in knot theory is that of isotopy: two links $L_1$ and $L_2$ are *isotopic* if there exists a diffeomorphism $f$ of $S^3$ isotopic to the identity map such that $f(L_1) = L_2$. Here we remark that every orientation preserving diffeomorphism of $S^3$ is isotopic to the identity map. The ultimate goal in the study of knots and links is to classify them under some equivalence relation: an approach to this goal is to discover as many invariants as possible by which we can distinguish distinct knots and links. The main purpose of this note is to introduce various known invariants to the reader.

It is convenient to consider a link as a subset of $\mathbf{R}^3$, the 3-dimensional Euclidean space, rather than a subset of $S^3$. Consider $S^3 \cong \mathbf{R}^3 \cup \{\infty\}$ as the one-point compactification of $\mathbf{R}^3$. Since every link can be isotoped away from $\infty$ (general position), we may assume that a link is a subset of $\mathbf{R}^3$. Furthermore, the new point of view does not affect the equivalence relations discussed above.

We state a theorem which makes it possible to draw a diagram of any link on a plane. A proof of the theorem is an application of the Thom transversality theorem. Assume that $\mathbf{R}^3$ has the usual $x$, $y$ and $z$ axes.

**Theorem 1.1.** *Let $L$ be a link in $\mathbf{R}^3$. Then $L$ is isotopic to a link $L'$ such that $L' \subset \{(x, y, z) : z > 0\}$ and the projection of $L'$ onto $x$-$y$ plane is an immersion with double points only.*

*We call $L'$ a link projection.*

We have some of the well known links in Figure 1.

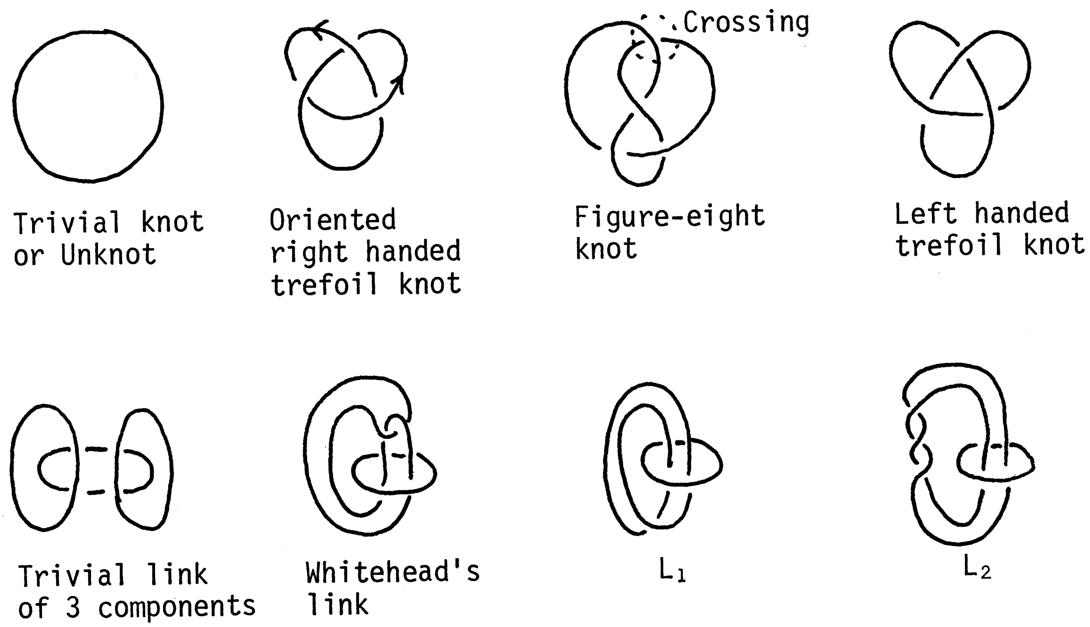



FIGURE 1.

A knot $K$ is called a *trivial knot* if it is isotopic to the knot in Figure 1 with no crossing.

**Remark.** A knot is trivial if and only if the knot bounds an embedded 2-dimensional disk in $S^3$.

Any embedded circle in an orientable 3-manifold has a compact tubular neighborhood diffeomorphic to $S^1 \times D^2$, where the circle corresponds to $S^1 \times \{0\}$. Similarly a link in an orientable 3-manifold has a disjoint union of copies of $S^1 \times D^2$

embedded in the 3-manifold as a tubular neighborhood. In general a tubular neighborhood of a submanifold is unique up to isotopy of the ambient manifold.

Suppose that $L$ is a link of $n$ components in an orientable 3-manifold $M$ and suppose that

$$f : \bigvee_{1\leq i\leq n}(S^1 \times D^2)_i \to M$$

is an embedding onto a tubular neighborhood of $L$ such that

$$L = f(\bigvee_{1\leq i\leq n}(S^1 \times \{0\})_i).$$

Let

$$M' = (M - f(\bigvee_{1\leq i\leq n} S^1 \times \mathring{D}^2)) \bigcup_{f|} (\bigvee_{1\leq i\leq n}(D^2 \times S^1)_i),$$

where the identification is made by $f \mid \bigvee_{1\leq i\leq n}(\partial D^2 \times S^1)_i$. We call $M'$ *the result of a surgery on $M$ with framing $f$.*

**Example 1.** Let $L$ be the trivial knot, and given an integer $n$ define

$$f_n : S^1 \times D^2 \to S^3$$

as in the figure 2.

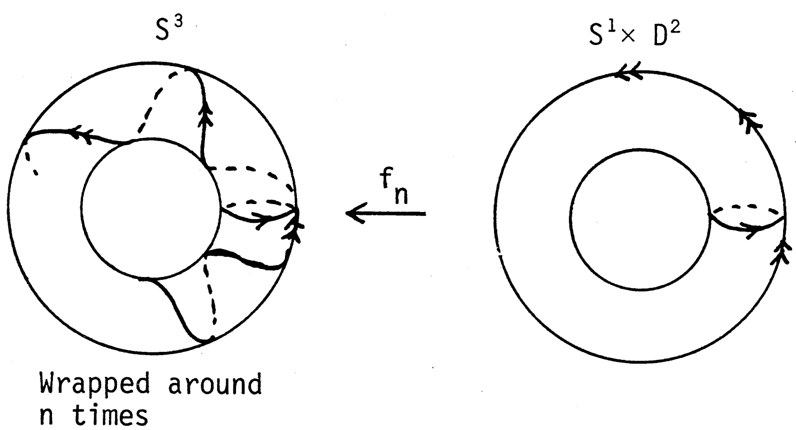



FIGURE 2.

Denote by $S^3(L; f_n)$ the result of the framed surgery on $S^3$ with framing $f_n$. It can be seen that $S^3(L; f_0)$ is diffeomorphic to $S^1 \times S^2$, $S^3(L; f_1)$ to $S^3$, $S^3(L; f_2)$ to $\mathbf{R}P^3$, and $\pi_1(S^3(L; f_n))$ is isomorphic to $\mathbf{Z}_n$.

**Theorem 1.2 (Lickorish).** *Let $M$ be an orientable, closed, connected 3-manifold. Then $M$ can be obtained from $S^3$ by a framed surgery along a link in $S^3$.*

*Proof.* Every orientable closed 3-manifold bounds an orientable compact 4-manifold [20]. Let $W$ be an orientable, connected, compact 4-manifold with $\partial W = M$. Let $D_0^4$ be an embedded 4-dimensional disk in the interior of $W$. By Morse function theory [19], we may assume that $W$ is obtained from $D_0^4$ by attaching handles of dimensions $0, 1, 2, 3$ and $4$. We may further assume that the handles are attached in the order of their dimension. Let $W_0$ be the union of $D_0^4$ and all the 0-handles in the above handle decomposition of $W$, and let $W_i$, $1 \leq i \leq 4$, be the union of $W_{i-1}$ and all the $i$-handles in $W$. Note that $W_0$ is a disjoint union of 4-disks, and $W_1$ and $W$ have the same number of components. Since $W$ is connected, so is $W_1$. This implies that all the 0-handles can be cancelled with the equal number of 1-handles. By inverting the handle decomposition of $W$ it can be shown that all the 4-handles can be cancelled with the equal number of 3-handles. Hence $W$ admits a new handle decomposition, with only one 0-handle $D_0^4$, 1-handles, 2-handles and 3-handles. Define $W_i$, $1 \leq i \leq 3$, as before.

We claim that $\partial W_1$ can be obtained from $\partial D_0^4 \cong S^3$ by a framed surgery on a link (trivial) on $S^3$. To see this let $V_0$ be a connected, orientable, 4-manifold with a connected boundary. Suppose that $V_1$ is an orientable 4-manifold obtained from $V_0$ by attaching one 1-handle. Since $\partial V_0$ is connected, we may assume that the attaching is done on an embedded 3-disk in $\partial V_0$. From the fact that $V_1$ is orientable, $V_1$ is diffeomorphic to $V_0 \# S^1 \times D^3$ (boundary connected sum). Hence $\partial V_1$ is diffeomorphic to $\partial V_0 \# S^1 \times S^2$. But in the above example $S^1 \times S^2$ is obtained from $S^3$ by a framed surgery along the trivial knot. This implies that $\partial V_0 \# S^1 \times S^2$ can be obtained from $\partial V_0$ by a framed surgery. Continuing with the proof of the claim, if $W_1$ is obtained from $D_0^4$ by attaching $k$ 1-handles, then the above discussion implies that

$$\partial W_1 \cong (\partial D_0^4) \# \bigvee_{1 \le i \le k} (S^1 \times S^2)_i,$$

where $\cong$ means "is diffeomorphic to." Therfore, $\partial W_1$ can be obtained from $S^3$ by a framed surgery along a link.

By applying the above argument to the inverted handle decomposition of $W$, we may assume that $\partial W_2$ is obtained from $\partial W_3$ by a framed surgery along a link in $\partial W_3$. It follows immediately from the definition of framed surgery that if a 3-manifold $N'$ is obtained from $N$ by a framed surgery, then $N$ is obtained from $N'$ by a framed surgery. Therefore, $\partial W_3$ is obtained from $\partial W_2$ by a framed surgery. Since $\partial W_2$ is obviously obtained from $\partial W_1$ by a framed surgery, we conclude that $\partial W = \partial W_3 = M$ is obtained from $S^3$ by a sequence of framed surgeries. But a standard argument using general position implies that the framed surgeries can be done at the same time. Thus $M$ is obtained from $S^3$ by a framed surgery along a link in $S^3$.

## 2. Knot complement and knot group

Given a knot $K$, let $V$ denote a tubular neighborhood of $K$. Then $S^3 - \mathring{V}$ is a compact 3-manifold that is homotopy equivalent to $S^3 - K$. We call either one of these two manifolds the *complement* of $K$. Clearly, equivalent knots have diffeomorphic complements, so any topological invariant of the complement of a knot is an invariant of the equivalence class containing the knot. We mention that if the complements of two knots are diffeomorphic, then the knots are equivalent. A proof of this assertion has been completed recently by Gordon and Luecke [6]. In applying the assertion, one faces the task of deciding whether or not two 3-manifolds are diffeomorphic, which is a difficult problem itself.

The next theorem characterizes the trivial knot in terms of the complement. We leave the proof as an excercise.

**Theorem 2.1.** *A knot $K$ is trivial if and only if its complement is diffeomorphic to $S^1 \times D^2$.*

**Remark.** The links $L_1$ and $L_2$ in Figure 1 have the diffeomorphic complements but they are not equivalent.

We now turn to algebraic topological invariants of knot complements. From the Mayer-Vietoris sequence associated to $S^3 = (S^3 - \mathring{V}) \cup V$, we obtain

$$H_0(S^3 - \mathring{V}; \mathbf{Z}) \cong H_1(S^3 - \mathring{V}; \mathbf{Z}) \cong \mathbf{Z}, \quad H_i(S^3 - \mathring{V}; \mathbf{Z}) \cong 0 \text{ if } i \neq 0 \text{ or } 1.$$

Therefore, the homology group of knot complements can not distinguish distinct knots.

Given a knot (or a link) $K$, define the *knot group* of $K$ to be

$$\pi_1(S^3 - \mathring{V}, x_0) \cong \pi_1(S^3 - K, x_0),$$

where $x_0$ is a base point. A knot group does not depend up to isomorphism on the choice of base point since the knot complement is pathwise connected. We study

Wirtinger presentations of knot groups. We first recall Van Kampen's theorem [17].

Suppose that the space $X$ is a union of two open subsets $X_1$ and $X_2$. Let $X_0 = X_1 \cap X_2$, and suppose that $X_i$ is non-empty and pathwise connected for $i = 0, 1, 2$. Suppose that the fundamental groups of these spaces with respect to a base point in $X_0$ have the following presentation.

$$\pi_1(X_1) = \langle x_1, \cdots ; r_1, \cdots \rangle$$
$$\pi_1(X_2) = \langle y_1, \cdots ; s_1, \cdots \rangle$$
$$\pi_1(X_0) = \langle z_1, \cdots ; t_1, \cdots \rangle$$

If $i_1 : X_0 \to X_1$ and $i_2 : X_0 \to X_2$ denote the inclusion maps, then

$$\pi_1(X) \cong \langle x_1, \cdots , y_1, \cdots ; r_1, \cdots , s_1, \cdots , i_{1*}(z_1) = i_{2*}(z_1), \cdots \rangle.$$

Let $K$ be a knot projection (the argument works for links as well). Consider $K$ as the union of two kind of arcs ; under arcs and over arcs. A connected arc $u_j$ in $K$ is called an *under arc* if

(i) $u_j$ passes under every crossing it encounters,

(ii) the endpoints of $u_j$ are not projected as double points onto the $x$-$y$ plane,

(iii) $u_j$ meets at least one crossing, and

(iv) $u_j$ is maximal with properties (i), (ii) and (iii) in the sense that there is no connected arc $u$ in $K$ containing $u_j$ such that $u$ satisfies (i), (ii) and (iii) and encounters more crossings than $u_j$ does.

We call each component of the closure of $K - \cup u_j$ an *over arc*. Note that under arcs do not cross under arcs (similarly for over arcs). A projection of the right handed trefoil knot is given in Figure 3, where under arcs, $\{u_j\}$, are given by broken curves and over arcs, $\{v_i\}$, are given by solid curves.

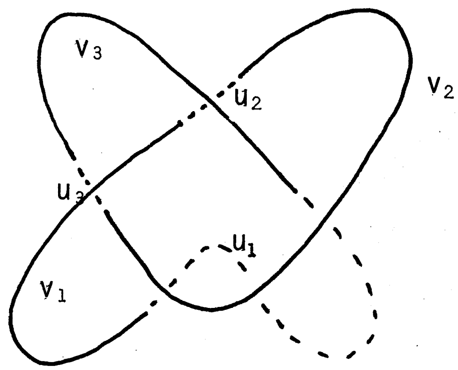



FIGURE 3.

Now let

$$\begin{aligned} D^3 &= \{(x,y,z) \in \mathbf{R}^3 : x^2+y^2+z^2 \le 1\}, \\ D^3_+ &= \{(x,y,z) \in D^3 : z \ge 0\}, \\ D^3_- &= \{(x,y,z) \in D^3 : z \le 0\}, \\ D^2 &= D^3_+ \cap D^3_-. \end{aligned}$$

Without loss of generality we may assume that $K$ is contained in the interior of $D^3$ such that each over arc in $K$ is contained in the interior of $D^3_+$ except for the two endpoints and each under arc is contained in $D^2$. Let $V$ be a thin tubular neighborhood of $K$ contained in the interior of $D^3$. Then

$$\pi_1(S^3 - \mathring{V}) \cong \pi_1(\mathbf{R}^3 - \mathring{V}) \cong \pi_1(D^3 - \mathring{V}).$$

We compute $\pi_1(D^3 - \mathring{V})$ by the Van Kampen's theorem. Now

$$D^3 - \mathring{V} = (D^3_+ - \mathring{V}) \bigcup (D^3_- - \mathring{V}) \text{ and } (D^3_+ - \mathring{V}) \bigcap (D^3_- - \mathring{V}) = D^2 - \mathring{V}.$$

Observe that $D^3_+ - \mathring{V}$ is diffeomorphic to a solid torus with holes, where each hole corresponds uniquely to an over arc, $D^3_- - \mathring{V}$ is diffeomorphic to the 3-disk, and $D^2 - \overset{\circ}{V}$ is diffeomorphic to a 2-disk with holes, where each hole corresponds uniquely to an under arc.

With respect to a base point $p$ in $D^2 - \overset{\circ}{V}$,

$$\pi_1(D^3_+ - \overset{\circ}{V}) = \langle x_1, x_2, \cdots \rangle, \text{ a free group on generators } \{x_i\},$$

where there is a one to one correspondence between $\{x_i\}$ and $\{v_i\}$,

$\pi_1(D^3_- - \overset{\circ}{V}) \cong \{1\}$, and $\pi_1(D^2 - \overset{\circ}{V}) = \langle r_1, r_2, \cdots \rangle$, a free group on generators $\{r_j\}$,

where there is a one to one correspondence between $\{r_j\}$ and $\{u_j\}$. Let

$$i_1 : D^2 - \overset{\circ}{V} \to D^3_+ - \overset{\circ}{V} \text{ and } i_2 : D^2 - \overset{\circ}{V} \to D^3_- - \overset{\circ}{V}$$

denote the natural inclusion maps, then by the Van Kampen's theorem

$$\pi_1(D^3 - \overset{\circ}{V}) \cong \langle x_1, x_2, \cdots ; i_{1*}(r_1) = 1,\ i_{2*}(r_2) = 1, \cdots \rangle.$$

This presentation is called a *Wirtinger presentation* of the knot group of $K$.

We now work out a Wirtinger presentation of the right handed trefoil knot using the projection in Figure 3. With the generators represented by the loops in Figure 4,

$$\pi_1(D^3_+ - \overset{\circ}{V}) = \langle x_1, x_2, x_3 \rangle, \quad \pi_1(D^2 - \overset{\circ}{V}) = \langle r_1, r_2, r_3 \rangle.$$

In Figure 4, assume that the page is a part of the $x$-$y$ plane and the positive direction of $z$-axis is toward us. A reader should understand that a thin tubular neighborhood of the knot is missing. A careful study of Figure 4 should show

$$i_{1*}(r_1) = x_2^{-1} x_3^{-1} x_2 x_1,$$
$$i_{1*}(r_2) = x_2^{-1}(x_2 x_3^{-1} x_1^{-1} x_3) x_2,$$
$$i_{1*}(r_3) = x_2^{-1} x_3^{-1}(x_1 x_3 x_1^{-1} x_2^{-1}) x_3 x_2.$$

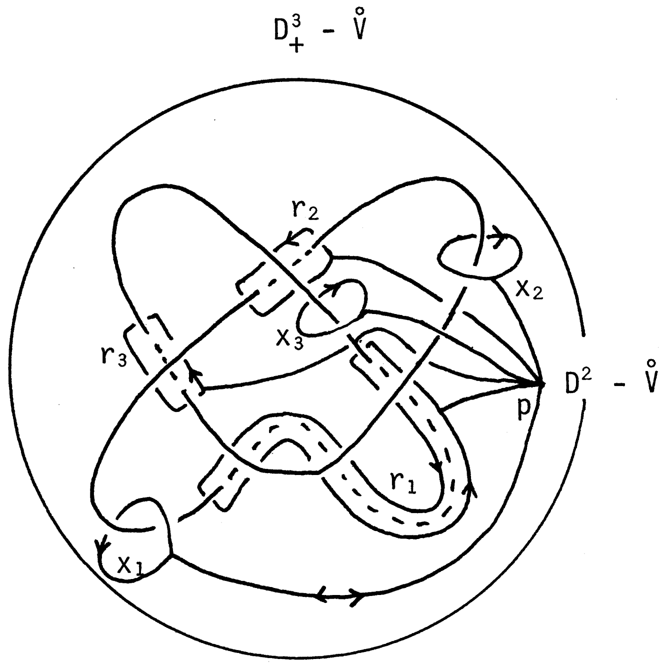



FIGURE 4.

From the fact that a relation in a presentation of a group can be replaced with another conjugate to it without affecting the group, and by eliminating redundant generators,

$$\begin{aligned}\pi_1(S^3 - K) &\cong \langle x_1, x_2, x_3; x_2^{-1}x_3^{-1}x_2x_1 = x_2x_3^{-1}x_1^{-1}x_3 = x_1x_3x_1^{-1}x_2^{-1} = 1\rangle \\ &\cong \langle x_1, x_2; x_2^{-1}x_1^{-1}x_2^{-1}x_1x_2x_1 = x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1 = 1\rangle \\ &\cong \langle x_1, x_2; x_1x_2x_1 = x_2x_1x_2\rangle.\end{aligned}$$

We use this computation to show that the right handed trefoil knot is not equivalent to the trivial knot. Add relation $x_1^2 = 1$ to get an epimorphism from $\langle x_1, x_2; x_1x_2x_1 = x_2x_1x_2\rangle$ to $\langle x_1, x_2; x_1^2 = x_2^2 = (x_1x_2)^3 = 1\rangle \cong S_3$, where $S_3$ is the group of permutations on 3 letters. The trivial knot group is isomorphic to $\mathbf{Z}$, and any homomorphic image of $\mathbf{Z}$ is cyclic. Since $S_3$ is not a cyclic group, it follows that the right handed trefoil knot is not equivalent to the trivial knot.

**Example 2.** (Figure-eight knot group)

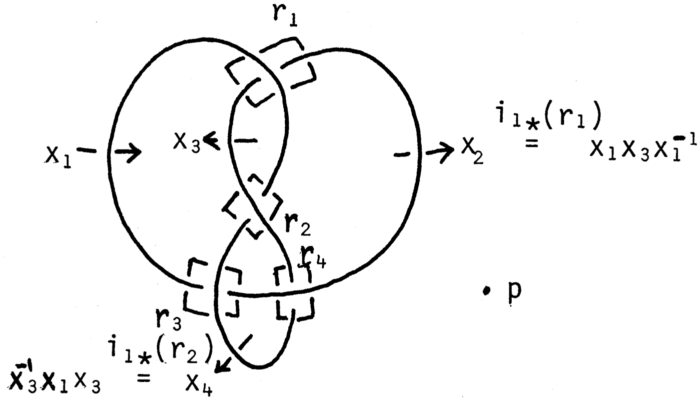



FIGURE 5.

In any Wirtinger presentation of a knot group, one relation is always redundant. From Figure 5, the figure eight knot group is isomorphic to

$$\langle x_1, x_3; x_1^{-1} \cdot x_3^{-1} x_1^{-1} x_3 \cdot x_1 x_3^{-1} x_1^{-1} \cdot x_3^{-1} x_1 x_3 \overset{i_{1*}(r_3)}{=} 1 \rangle.$$

By adding $x_1^2 = x_3^2 = 1$ (notice that $x_1^2 = 1$ implies $x_3^2 = 1$), we see that the group has $\langle x_1, x_3; x_1^2 = x_3^2 = (x_1 x_3)^5 = 1 \rangle$ as a homomorphic image. This group is an index 2 subgroup of the reflection group associated to the spherical triangle (2,2,5), in particular, the group is not abelian.

The knots, *Granny knot* and *square knot*, in Figure 6 have the isomorphic knot groups (excercise) but they are not equivalent, which we show later.

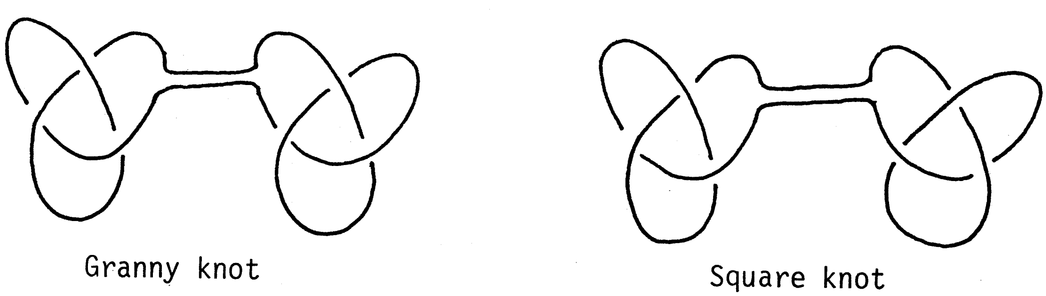



FIGURE 6.

## 3. Dehn's Lemma and framed surgery

We give three important theorems by Papakyriakopoulis in 3-manifold theory without proof. The reader may refer to [9] for proofs.

(**Dehn's Lemma**) Suppose that $J \subset \partial M$ is an embedded circle in the boundary of a 3-manifold $M$ and $J$ is homotopically trivial in $M$. Then $J$ is the boundary of a properly embedded 2-disk in $M$.

(**Loop theorem**) Suppose that $M$ is a 3-manifold and the homomorphism from $\pi_1(\partial M)$ to $\pi_1(M)$, induced by the inclusion map, has a non-trivial kernel. Then there exists an embedded circle in $\partial M$ representing a non-trivial element in the kernel such that the circle bounds a properly embedded 2-disk in $M$.

(**Sphere theorem**) Let $M$ be an orientable 3-manifold with $\pi_2(M) \not\cong \{0\}$. Then there exists an embedded 2-sphere $S$ in $M$ such that $S$ is not contractible in $M$.

Before applying the above theorems to our study, we introduce the notion of preferred meridian and longitude of a knot. Let $K$ be a knot and $V$ a tubular neighborhood of $K$. Then there exists an embedded circle $\mu$ in $\partial V \cong S^1 \times S^1$ unique up to isotopy such that $\mu$ bounds a properly embedded 2-disk in $V$. We call $\mu$ or an element of $H_1(S^1 \times S^1; \mathbf{Z})$ represented by $\mu$ a *preferred meridian* of $K$. Let

$$i_1 : \partial V \to V \quad \text{and} \quad i_2 : \partial V \to S^3 - \mathring{V}$$

be the inclusion maps. Then there exists a short exact sequence

$$0 \to H_1(\partial V; \mathbf{Z}) \xrightarrow{f=(i_{1*}, -i_{2*})} H_1(V; \mathbf{Z}) \oplus H_1(S^3 - \mathring{V}; \mathbf{Z}) \to 0.$$

Choose an element $x \in H_1(\partial V; \mathbf{Z})$ such that $\{\mu, x\}$ is a basis for $H_1(\partial V; \mathbf{Z})$. Assuming that $H_1(V; \mathbf{Z})$ and $H_1(S^3 - \mathring{V}; \mathbf{Z})$ have been identified with $\mathbf{Z}$, we have

$$f(\mu) = (0, m) \text{ and } f(x) = (p, q) \quad \text{for some integers } m,\ p \text{ and } q.$$

Since $f$ is an isomorphism, $m \cdot p = \pm 1$. Now $\text{Ker}(i_{2*}) \cong \mathbf{Z}$ is generated by $\pm(q\mu - mx)$. We define one of these two generators of $\text{Ker}(i_{2*})$ to be a *preferred longitude* $\lambda$ of $K$. So it is either $q\mu - mx$ or $-q\mu + mx$. Since $m = \pm 1$, $\lambda$ can be represented by an embedded circle in $\partial V$, and we may further assume that $\mu$ intersets $\lambda$ exactly at one point. The above discussion also shows that $\mu$ is a generator of $H_1(S^3 - \mathring{V}; \mathbf{Z}) \cong \mathbf{Z}$.

In Figure 7, preferred meridians and longitudes are drawn for the right handed trefoil knot and the figure eight knot:

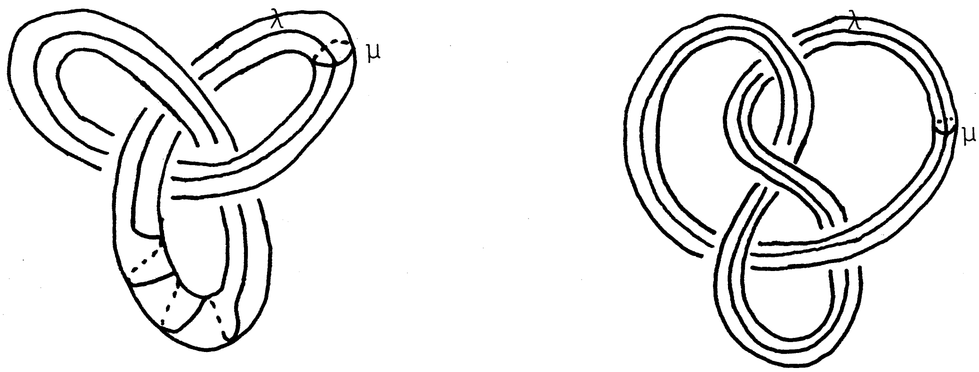



FIGURE 7.

**Theorem 3.1.** *A knot $K$ is trivial if and only if the knot group of $K$ is isomorphic to* $\mathbf{Z}$.

*Proof.* If $K$ is trivial, then the complement of $K$, $S^3 - \mathring{V}$ is diffeomorphic to $S^1 \times D^2$ by Theorem 2.1. Hence the knot group of $K$ is isomorphic to $\mathbf{Z}$.

Suppose that $\pi_1(S^3 - \mathring{V}) \cong \mathbf{Z}$. Then

$$H_1(S^3 - \mathring{V}; \mathbf{Z}) \cong \pi_1(S^3 - \mathring{V}) \cong \mathbf{Z}.$$

Let $\lambda$ be a preferred longitude for $K$. Since $\lambda$ is homologous to 0 in $S^3 - \mathring{V}$, $\lambda$ is homotopically trivial. By Dehn's lemma $\lambda$ bounds a properly embedded 2-disk $D$

in $S^3 - \overset{\circ}{V}$. It can be easily seen that $K$ and $\lambda$ bound a properly embedded annulus $A$ in $V$. Now $A \cup D$ is a smoothly embedded 2-disk with boundary $K$. Therefore, $K$ is trivial.

We leave the proof of next two theorems to the reader.

**Theorem 3.2.** *A knot $K$ is trivial if and only if the knot group of $K$ is abelian.*

**Theorem 3.3.** *A knot $K$ is not trivial if and only if $i_* : \pi_1(\partial V) \to \pi_1(S^3 - \overset{\circ}{V})$ is injective, where $i : \partial V \to S^3 - \overset{\circ}{V}$ is the inclusion map.*

**Theorem 3.4.** *For any knot $K$, $\pi_i(S^3 - \overset{\circ}{V}) \cong \{0\}$ for $i \geq 2$, and the knot group of $K$ is torsion-free.*

*Proof.* Suppose that $\pi_2(S^3 - \overset{\circ}{V}) \not\cong \{0\}$. Then by the Sphere theorem there exists an embedded 2-sphere $S$ in $S^3 - \overset{\circ}{V}$ such that $S$ is not contractible in $S^3 - \overset{\circ}{V}$. From the well known theorem that every embedded 2-sphere in $S^3$ bounds two 3-disks whose interiors do not intersect, we see that there exists an embedded 3-disk $D$ in $S^3 - \overset{\circ}{V}$ with $\partial D = S$. This is a contradiction. Hence $\pi_2(S^3 - \overset{\circ}{V}) \cong \{0\}$. To show that other homotopy groups are trivial, let $\tilde{X}$ be the universal covering space of $S^3 - \overset{\circ}{V}$. From the homotopy long exact sequence associated to the covering projection, we see that $\pi_2(\tilde{X}) \cong \{0\}$. On the other hand, $H_i(\tilde{X}; \mathbf{Z}) = \{0\}$, $i \geq 3$, since $\tilde{X}$ is a 3-manifold with non-empty boundary. By the Hurewicz isomorphism theorem, $\pi_i(\tilde{X}) \cong \{0\}$ for $i \geq 3$. It follows again from the homotopy long exact sequence that $\pi_i(S^3 - \overset{\circ}{V}) \cong \{0\}$ for $i \geq 3$.

To prove the second assertion, suppose that $\pi_1(S^3 - \overset{\circ}{V})$ contains a torsion element. Then there exists a prime number $m$, where the cyclic group of order $m$, $\mathbf{Z}_m$, is a subgroup of $\pi_1(S^3 - \overset{\circ}{V})$. Since $\pi_i(\tilde{X}) \cong 0$ for all $i$, $\tilde{X}/\mathbf{Z}_m$ is a $K(\mathbf{Z}_m, 1)$ space. Hence $H_i(\tilde{X}/\mathbf{Z}_m; \mathbf{Z}) \cong H_i(\mathbf{Z}_m; \mathbf{Z})$, where the group on the right hand side is the $i^{th}$ homology group of $\mathbf{Z}_m$ with coefficients $\mathbf{Z}$. It is well known [15] that

$H_i(\mathbf{Z}_m;\mathbf{Z})$ is not trivial for infinitely many $i$'s but $H_i(\tilde{X}/\mathbf{Z}_m;\mathbf{Z}) \cong \{0\}$ for $i \geq 3$. Therefore, a knot group is torsion-free.

We consider connected sums of oriented manifolds as an introduction to connected sums of knots.

**(Connected sum of oriented manifolds)**

Let $M$ and $N$ be connected, oriented, closed $n$-manifolds. Suppose that $D_1 \subset M$ and $D_2 \subset N$ are embedded $n$-disks. We give $\partial(M - \mathring{D}_1)$ and $\partial(N - \mathring{D}_2)$ the orientation compatible with that of $M - D_1$ and $N - D_2$, respectively. Define a *connected sum* of $M$ with $N$ by

$$M\#N = (M - \mathring{D}_1)\bigcup\nolimits_f(N - \mathring{D}_2),$$

where $f$ is an orientation reversing diffeomorphism from $\partial(M - \mathring{D}_1)$ to $\partial(N - \mathring{D}_2)$. The connected sum does not depend on the choice of $D_1$ and $D_2$ but depends on $f$. For example, when $M = N = S^7$, the connected sum produces 7-manifolds all homeomorphic to $S^7$ but distinct as smooth manifolds [18], [26].

The connected sum also depends on the orientation. Let $\mathbf{C}P^2$ be the complex projective 2-space with the natural orientation, and let $\overline{\mathbf{C}P^2}$ be the same manifold with the reversed orientation. Then the signature, $\sigma(\mathbf{C}P^2\#\overline{\mathbf{C}P^2})$, of $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ is equal to 0 whereas $\sigma(\mathbf{C}P^2\#\mathbf{C}P^2) \neq 0$. This implies that $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ is not even homotopy equivalent to $\mathbf{C}P^2\#\mathbf{C}P^2$.

Finally we remark that the connected sum operation is well defined for manifolds of dimension less than 5 by a theorem of Cerf [1].

**(Connected sum of oriented knots)**

Let $K$ be an oriented knot such that there exsits an embedded 2-sphere $S$ in $\mathbf{R}^3$ with $K$ transverse regular to $S$. Suppose that $K \cap S$ consists of two distinct points $p$ and $q$. Let $\alpha$ be an embedded arc on $S$ from $p$ to $q$, and let $D$ be the embedded 3-disk in $\mathbf{R}^3$ with $\partial D = S$ (see Figure 8.)

Define $K_1 = (K \cap D) \cup \alpha$ and $K_2 = (K - D) \cup \alpha$, and orient $K_1$ and $K_2$ compatibly with $K$. After smoothing $K_1$ and $K_2$ near $p$ and $q$, we regard $K_1$ and $K_2$ as oriented knots. We call $K$ a *connected sum*, $K_1 \# K_2$, of $K_1$ and $K_2$. One can show that the connected sum operation is well-defined for the isotopy (respecting orientations on knots) classes of oriented knots.

The next theorem says that we can not untangle knots by taking connected sums.

**Theorem 3.5.** *Let $K_1$ and $K_2$ be oriented knots. If $K_1 \# K_2$ is trivial, then both $K_1$ and $K_2$ are trivial.*

*Proof.* Put $K = K_1 \# K_2$. Let $S$ be an embedded 2-sphere in $\mathbf{R}^3$ and $D$ the embedded 3-disk in $\mathbf{R}^3$ such that $\partial D = S$ as in Figure 8. Let $C = \mathbf{R}^3 - \mathring{D}$.

Let $V$ be a tubular neighborhood of $K$. Then $\mathbf{R}^3 - V = (D - V) \cup (C - V)$, and $(D - V) \cap (C - V)$ is diffeomorphic to an annulus. Denote the annulus by $A$. Let

$$i_1 : A \to D - V \quad \text{and} \quad i_2 : A \to C - V$$

be the inclusion maps. From the Van Kampen's theorem it follows that $i_{1*}$ and $i_{2*}$ are monomorphisms from $\pi_1(A) \cong \mathbf{Z} =< x >$ to $\pi_1(D - V)$ and $\pi_1(C - V)$, respectively, where $x$ is represented by the loop in Figure 8. Furthermore,

$$\pi_1(D - V) \cong \pi_1(S^3 - K_1) \quad \text{and} \quad \pi_1(C - V) \cong \pi_1(S^3 - K_2).$$

Hence $\pi_1(\mathbf{R}^3 - V) \cong \pi_1(S^3 - K)$ is an amalgamated free product [16] of $\pi_1(D - V)$ and $\pi_1(C - V)$ by the Van Kampen's theorem since $i_{1*}$ and $i_{2*}$ are monomorphisms. Therefore, $\pi_1(S^3 - K_2)$ and $\pi_1(S^3 - K_2)$ can be considered as subgroups of $\pi_1(S^3 - K)$. This implies that

$$\pi_1(S^3 - K_1) \cong \pi_1(S^3 - K_2) \cong \mathbf{Z}.$$

By Theorem 3.1, $K_1$ and $K_2$ are trivial.

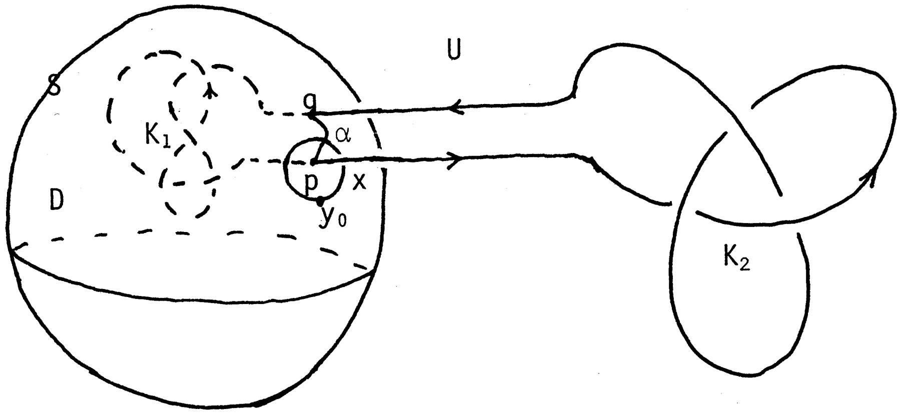



FIGURE 8.

A knot is defined to be a *prime knot* if it is not a connected sum of two non-trivial knots.

**Remark.** A list of prime knots with the number of crossings less than or equal to 10 is given in the knot table in [24].

We state two theorems without proof.

**Theorem (Waldhausen [30]).** *Let $K_1$ and $K_2$ be knots. Let $\mu_i$ and $\lambda_i$ be, respectively, a preferred meridian and longitude for $K_i$, $i = 1, 2$. If there exists an isomorphism $f : \pi_1(S^3 - \mathring{V}(K_1)) \to \pi_1(S^3 - \mathring{V}(K_2))$ such that $f(\mu_1) = \mu_2$ and $f(\lambda_1) = \lambda_2$, then $K_1$ is equivalent to $K_2$.*

**Theorem (Whitten [31]).** *Prime knots with the isomorphic knot groups have homeomorphic knot complements.*

This theorem, together with the theorem by Gordon and Luecke [6] mentioned in Chapter 2, implies that prime knots are completely determined by their knot groups.

**(Dehn surgery)**

Suppose that $K \subset S^3$ is a knot and $V$ a tubular neighborhood of $K$. Given a diffeomorphism $f : \partial D^2 \times S^1 \to \partial V$, define

$$S^3(K; f) = (S^3 - \overset{\circ}{V}) \bigcup_f D^2 \times S^1.$$

We call $S^3(K; f)$ the result of a *Dehn surgery* on $S^3$ using $f$.

Let $p \in S^1$ be a fixed point and let $\alpha = \partial D^2 \times \{p\}$. Let $A$ be a small, closed arc in $S^1$ containing $p$ in the interior. Let $D$ denote the closure of $(D^2 \times S^1 - D^2 \times A)$. Then $D$ is diffeomorphic to the 3-disk, and

$$S^3(K; f) = (S^3 - \overset{\circ}{V}) \bigcup_{f | \partial D^2 \times A} (D^2 \times A) \bigcup_{\partial D} D.$$

Observe that in the first union a 2-handle is attached and in the second a 3-handle is attached. Therefore, $S^3(K; f)$ is completely determined by $f \,|\, \partial D^2 \times A$, and thus by $f(\alpha)$.

To specify $f(\alpha)$ in $\partial V$, first orient $K$. Then, orient a preferred meridean $\mu$ and longitude $\lambda$ such that the linking number of $\mu$ with $K$ is equal to 1, and the orientation of $\lambda$ is in the direction of that of $K$. Now there exist integers $a$ and $b$, unique up to sign, such that $f_*(\alpha) = \pm(b\mu + a\lambda)$ in $H_1(\partial V; \mathbf{Z})$ with the $\pm$ signs depending on the orientation on $\alpha$. Note that $a$ and $b$ are relatively prime since $f$ is a diffeomorphism. We call $b/a$ the *surgery coefficient* of the Dehn surgery. If $a = 0$, we use $\infty$ for $b/a$. One should note that the surgery does not depend on the choice of orientations of $K$ and $\alpha$.

We can easily extend the Dehn surgery to a surgery along a link by specifying a surgery coefficient for each component of the link. We remark that if all the surgery coefficients are integers for a link, then the Dehn surgery on the link is just a framed surgery which we discussed in Chapter 1, and vice versa.

**Remark.** For any knot $K$ and integer $n$, $S^3(K; \frac{1}{n})$ is a homology 3-sphere, i.e.,

$$H_*(S^3(K; \frac{1}{n}); \mathbf{Z}) \cong H_*(S^3; \mathbf{Z}).$$

If $K$ is the trivial knot, then $S^3(K;\frac{1}{n}) \cong S^3$ for all $n$.

We say that a Dehn surgery on a knot is *trivial* if the surgery coefficient is equal to 1/0. A knot $K$ is defined to have propery $P$ if every non-trivial Dehn surgery on $K$ yields a non-simply-connected 3-manifold. Clearly, the trivial knot does not have the property $P$ by the above remark. It has been known that various knots, for example, connected sums of non-trivial knots, have property $P$. It is a long standing conjecture that every non-trivial knot has property $P$. If the conjecture were true, one could not construct a counter example to the 3-dimensional Poincaré conjecture by doing a Dehn surgery along a knot.

**Example 3.** Let $K$ be the right handed trefoil knot. We study $S^3(K;n/1)$ by computing $\pi_1(S^3(K;n/1))$. When $n=1$, $S^3(K;1/1)$ is known as the Poincaré homology 3-sphere.

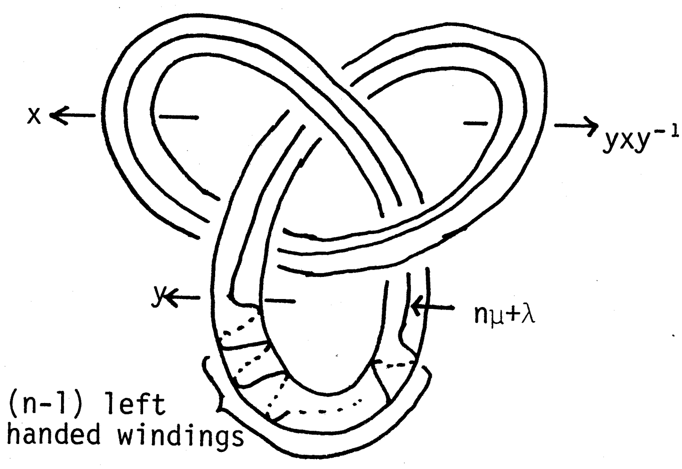



FIGURE 9.

With the generators given in Figure 9,

$$\pi_1(S^3 - \mathring{V}) \cong \langle x, y; yxy = xyx \rangle.$$

So

$$\pi_1(S^3(K;n/1)) \cong \langle x,y; yxy = xyx, y^{-2}x^{-1}y^{-n+4}x^{-1} = 1\rangle.$$

It follows easily

$$H_1(S^3(K;n/1);\mathbf{Z}) \cong \langle y; y^n = 1\rangle.$$

Hence $S^3(K;n/1)$ is a homology 3-sphere if and only if $n = \pm 1$.

Suppose that $n = 1$. Put $M = S^3(K;1)$. Then

$$\pi_1(M) \cong \langle x,y; yxy = xyx, y^{-2}x^{-1}y^3x^{-1} = 1\rangle.$$

Let $z = xy$ or $y = x^{-1}z$. Then

$$yxy = xyx \Rightarrow x^{-1}(xyxy) = (xy)x \Rightarrow x^{-1}z^2 = zx \Rightarrow z^3 = xzxz = (xz)^2.$$

Now $y = x^{-1}z = zxz^{-1}$. Hence

$$\begin{aligned} y^{-2}x^{-1}y^3x^{-1} &= zx^{-2}(z^{-1}x^{-1}z)x^3z^{-1}x^{-1} = z(x^{-2}xz^{-1})x^3z^{-1}x^{-1} \\ &= z^{-1}x^4(z^{-1}x^{-1}) = z^{-1}x^5z^{-2}. \end{aligned}$$

So

$$\pi_1(M) \cong \langle x,z; (zx)^2 = z^3 = x^5\rangle.$$

This group is known as the *binary icosahedral group* [2], and it has 120 elements. One can show that there is a 2 to 1 epimorphism from $\pi_1(M)$ to $A_5$, the alternating group on 5 letters.

Now we suppose $n = -1$, and let $N = S^3(K;-1)$. Then

$$\pi_1(N) \cong \langle x,y; yxy = xyx, y^{-2}x^{-1}y^5x^{-1} = 1\rangle.$$

Let $z = xy$. Then it follows that

$$\pi_1(N) \cong \langle x,z; (zx)^2 = z^3 = x^7\rangle.$$

This group, $\langle x,z; (zx)^2 = z^3 = x^7 = 1\rangle$, is the hyperbolic polyhedral group associated to the hyperbolic triangle $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$ [2], and it acts on the hyperbolic plane with a compact fundamental domain. Therefore, the group is infinite, and so is $\pi_1(N)$.

**(Kirby Calculus)**

Let $L = \{K_1(r_1), K_2(r_2), \ldots, K_n(r_n)\}$ be a framed link of $n$ components in $S^3$, where each surgery coefficient $r_i$ is an integer. Let $S^3(L; r_1, \ldots, r_n)$ be the result of framed surgery along $L$. Now construct $W$ by

$$W(L; r_1, \cdots, r_n) = D^4 \bigcup\nolimits_f \bigvee\nolimits_{1 \leq i \leq n} (D^2 \times D^2)_i,$$

where $W$ is a 4-manifold obtained from $D^4$ by attaching $n$ 2-handles using the attaching map

$$f : \bigvee\nolimits_{1 \leq i \leq n} (\partial D^2 \times D^2)_i \to \partial D^4 = S^3,$$

that is induced by the framing, $r_1, r_2, \cdots, r_n$. From the construction,

$$S^3(L; r_1, r_2, \cdots, r_n) \cong \partial W(L; r_1, r_2, \cdots, r_n).$$

We orient $W$ such that the orientation restricts to the standard orientation of $D^4$. We also orient each component of $L$ arbitrarily. For each $i$, let $CK_i$ be the cone of $K_i$ with respect to the center of $D^4$. Then $CK_i \cup_f (D^2 \times \{0\})_i$ is a topologically embedded 2-sphere in $W$. We orient this sphere by orienting $(D^2 \times \{0\})_i$ so that the orientation restricts to that of $K_i$. Let $a_i \in H_2(W; \mathbf{Z})$ be the unique element determined by this oriented sphere, $i = 1, 2, \cdots, n$. Then $\{a_1, a_2, \cdots, a_n\}$ is a basis for $H_2(W; \mathbf{Z})$ from the fact that $W$ is a homotopy type of bouquet of these $n$ 2-spheres. We further orient $(\{0\} \times D^2)_i$ for each $i$ such that the orientation induces the orientation of $W$ together with the orientation already defined on $(D^2 \times \{0\})_i$. This orientation and the inclusion homomorphism:

$$H_2((\{0\} \times D^2)_i, (\{0\} \times \partial D^2)_i; \mathbf{Z}) \to H_2(W, \partial W; \mathbf{Z})$$

determines a unique element of $H_2(W, \partial W; \mathbf{Z})$, $i = 1, 2, \cdots, n$. We denote this element by $a'_i$. Using an excision isomorphism, one can shows that $\{a'_1, a'_2, \cdots, a'_n\}$ is a free abelian basis for $H_2(W, \partial W; \mathbf{Z})$.

It is not hard to see that the intersection number

$$\langle a_i, a_j \rangle = \text{link}(K_i, K_j) \text{ if } i \neq j,$$
$$\langle a_i, a_i \rangle = r_i.$$

Let $A$ be the matrix representing the intersection pairing of $H_2(W; \mathbf{Z})$ with respect to the ordered basis $(a_1, a_2, \cdots, a_n)$. We have the following commutative diagram [27].

$$\begin{array}{ccccccccc} 0 & \to & H^1(\partial W;\mathbf{Z}) & \to & H^2(W,\partial W;\mathbf{Z}) & \to & H^2(W;\mathbf{Z}) & \to & H^2(\partial W;\mathbf{Z}) & \to & 0 \\ & & \downarrow \cap[\partial W] & & \downarrow \cap[W,\partial W] & & \downarrow \cap[W,\partial W] & & \downarrow \cap[\partial W] & & \\ 0 & \to & H_2(\partial W,\mathbf{Z}) & \to & H_2(W;\mathbf{Z}) & \xrightarrow{j_*} & H_2(W,\partial W;\mathbf{Z}) & \to & H_1(\partial W;\mathbf{Z}) & \to & 0 \end{array}$$

The vertical maps are isomorphisms. It follows from the diagram that the matrix $A$ is the representation of homomorphism $j_*$ with respect to the ordered bases $(a_1, a_2, \cdots, a_n)$ and $(a'_1, a'_2, \cdots, a'_n)$. Note that $H_1(\partial W; \mathbf{Z})$ is isomorphic to the cokernel of $j_*$. Therefore, $H_1(\partial W : \mathbb{Z}) \cong 0$ if and only if $A$ is unimodular.

We introduce two operations $\theta_1$ and $\theta_2$ on framed links.

Operation $\theta_1$ : Given a frame link $L$ we add to or subtract from $L$ an unknot with framing 1 or -1, which is separated from the other components of $L$ by an embedded 2-sphere in $S^3$. If we denote the new link by $L'$, then $\partial W(L) \cong \partial W(L')$ since $\partial W(L)\#S^3 \cong \partial W(L')$ or $\partial W(L) \cong \partial W(L')\#S^3$ dependending whether or not the unknotted circle is added or subtracted. But note that $W(L) \ncong W(L')$.

Let $K_1$ and $K_2$ be two knots in $S^3$. Let $f : I \times I \to S^3$ be an embedding such that $f(I \times I) \cap K_i = f(\{i\} \times I)$, $i = 0, 1$. Then define

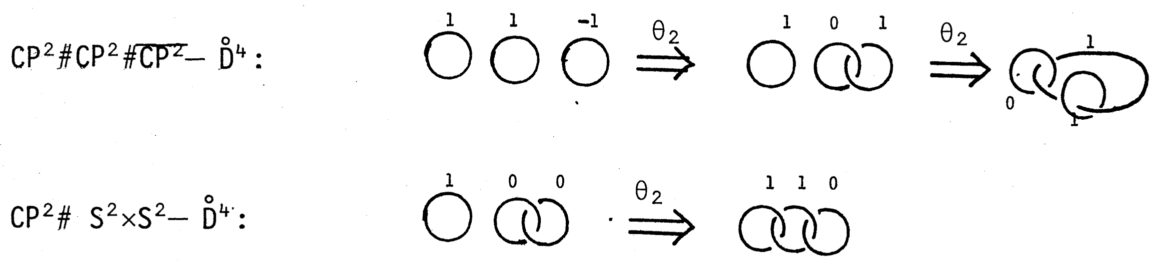
$$K_1 \#_f K_2 = (K_1 \bigcup K_2 - f(I \times I)) \bigcup f(I \times \{0, 1\}).$$

We call $f$ a *band.*

Operation $\theta_2$ : Let $K_i$ and $K_j$ be two components in $L$. Replace $K_j$ with $K'_j = \tilde{K}_i \#_f K_j$, where $\tilde{K}_i$ is obtained by pushing $K_i$ off itself by a small isotopy using the framing of $K_i$, and $f$ is any band missing the rest of $L$. Orient $K'_j$ compatibly with $K_j$. We also assign a new framing $r'_j = r_j + r_i \pm 2$ link $(K_i, K_j)$

to $K'_j$, where + is used when the orientation of $K'_j$ is in the same direction as that of $K_i$ and − is used otherwise. Let $L' = \{K_1(r_1), \cdots, K'_j(r'_j), \cdots, K_n(r_n)\}$. It can be seen that $W(L) \cong W(L')$ since $W(L')$ is obtained from $W(L)$ by sliding the $j^{th}$ 2-handle along the band $f$ over the $i^{th}$ 2-handle.

**Example 4.** The following diagram shows that $\mathbf{C}P^2 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ is diffeomorphic to $\mathbf{C}P^2 \# S^2 \times S^2$. Both manifolds are given in terms of 2-handle attaching on $D^4$.




Suppose that framed links $L$ and $L'$ are related as in Figure 10. One can show that by a sequence of operations $\theta_1$ and $\theta_2$, $L$ can be deformed to $L'$. Hence $\partial W(L) \cong \partial W(L')$.

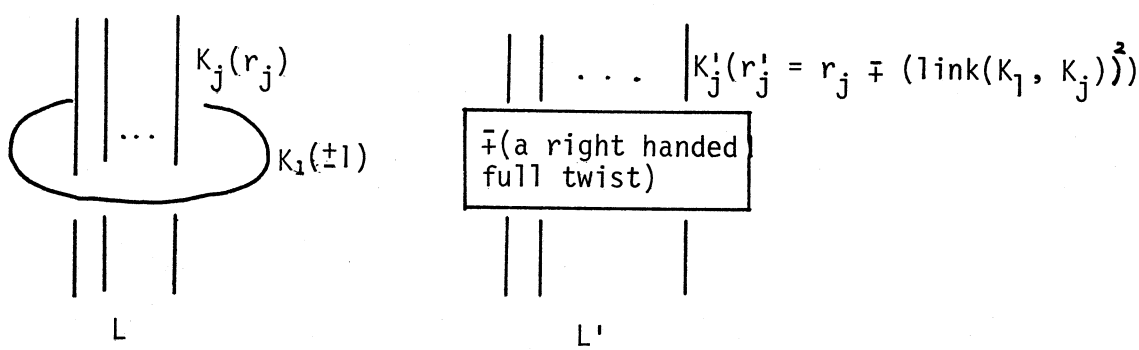



FIGURE 10.

**Example 5.** Using the move described in Figure 10 as many times as necessary, one can show that the 3-manifold, obtained by surgery on any one of the framed links in Figure 11, is diffeomorphic to Poincaré homology 3-sphere.

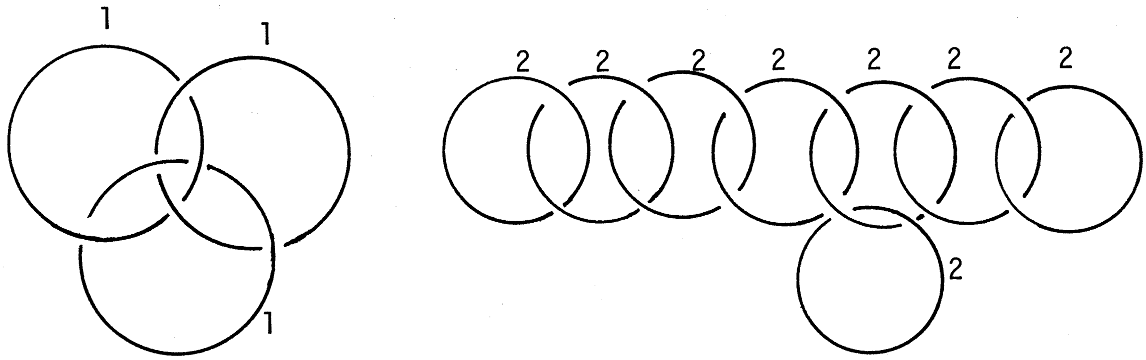



FIGURE 11.

We end this chapter by stating a deep theorem of Kirby [13].

**Theorem.** *Given two framed links $L$ and $L'$, $\partial W(L) \cong \partial W(L')$ (preserving the orientation) if and only if $L$ can be deformed to $L'$ by a sequence of operations $\theta_1$ and $\theta_2$.*

## 4. Seifert surface.

Let $K$ be a knot. Choose an isomorphism from $H_1(S^3 - \overset{\circ}{V}; \mathbf{Z}) \to \mathbf{Z}$. Since $S^1$ is a $K(\mathbf{Z}, 1)$ space, there exists a map $f : S^3 - \overset{\circ}{V} \to S^1$ realizing the isomorphism. Let $p \in S^1$ be a fixed point. Approximate $f$ by a map $g$ transverse regular to $p$. Then $g^{-1}(p)$ is a 2-dimensional orientable surface since its normal bundle in $S^3 - \overset{\circ}{V}$ is a pull-back of the trivial normal bundle of $\{p\}$ in $S^1$. The boundary of $g^{-1}(p)$ is a union of embedded circles in $\partial V$ and at least one component is essential (homotopically non-trivial) in $\partial V$; otherwise, $g$ sends a meridian to a contractible loop in $S^1$. Each component of $\partial g^{-1}(p)$ which is inessential in $\partial V$ bounds a disk in $\partial V$. By attaching a disk to $g^{-1}(p)$ along each inessential boundary component and by pushing the disk into $S^3 - \overset{\circ}{V}$ (the inner-most disk the farthest), we may assume that $g$ is transverse regular to $p$ and each boundary component of $g^{-1}(p)$ is essential in $\partial V$. This implies that all the components of $\partial g^{-1}(p)$ are homologous to each other in $\partial V$, i.e., they are parallel curves in $\partial V$. From the fact that $g_* : H_1(S^3 - \overset{\circ}{V}; \mathbf{Z}) \to \mathbf{Z}$ is an isomorphism, it follows that $\partial g^{-1}(p)$ has odd number of components. Notice that any two adjacent (in $\partial V$) components bound an annulus in $\partial V$. Attach the annulus to $g^{-1}(p)$ along any pair of adjacent components, and push it into $S^3 - \overset{\circ}{V}$ to get a new orientable surface with two less boundary components. By repeating the process, we obtain an orientable surface $F$ whose boundary is a circle, say $\lambda$, in $\partial V$. Since $\lambda$ bounds $F$, $\lambda$ is homologous to 0 in $H_1(S^3 - \overset{\circ}{V}; \mathbf{Z})$, thus $\lambda$ is a preferred longitude of $K$. Now there exists a properly embedded annulus $A$ in $V$ with $\partial A = K \cup \lambda$. Since $F$ is embedded in $S^3 - \overset{\circ}{V}$ properly, $A \cup F$ is an orientable surface in $S^3$ with boundary $K$.

Given a knot $K$, an orientable connected surface $F$ embedded in $S^3$ is called a *Seifert surface* of $K$ if $\partial F = K$.

The above argument shows that every knot has a Seifert surface. It can be modified to show that every link has a Seifert surface.

(**Canonical Seifert surface**)

Let $K$ be a projection of a knot (the construction works for links with some modification). Orient $K$ and change the diagram of $K$ at each crossing as follows.

The diagram becomes a union of disjoint circles in the $x$-$y$ plane (see Figure 12.)

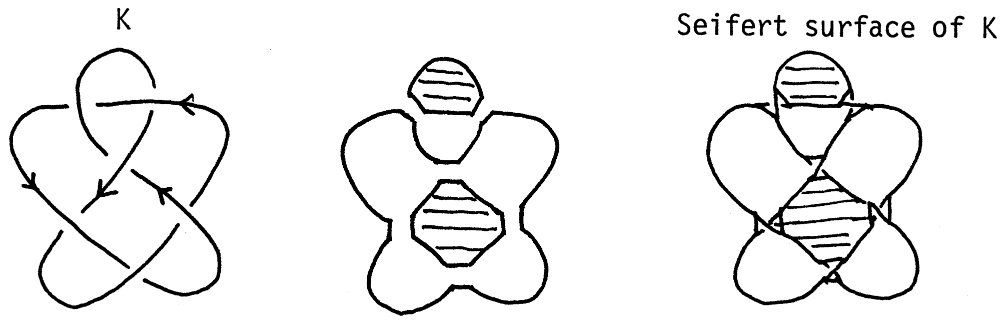



FIGURE 12.

We push the circles in the positive direction of $z$-axis, and put them at different levels such that if a circle is contained in the interior of another, then the inner circle is placed higher than the outer one. A circle which is not contained in the interior of another is placed on the $x$-$y$ plane. After this adjustment the circles bound disjointly embedded disks in $\mathbf{R}^3$, all parallel to the $x$-$y$ plane. Join the disks by bands of half-twist, where there is one band for each crossing and the direction of the twist of each band corresponds to that of the crossing. Let $F$ be the resulting surface. Clearly, $F$ is connected and $\partial F = K$. To see that $F$ is orientable, observe that each disk in the above decomposition of $F$ as disks and bands has a boundary that has an orientation induced from that of $K$. Now orient each disk compatibly with its boundary. Finally, observe that bands are attached to the disks such a way that the orientation on disks extends to an orientation of $F$. This orientation of $F$ is compatible with that of $K$.

The surface $F$ constructed above is called the *canonical Seifert surface* of the projection $K$. The surface does not depend on an orientation of $K$ but the canonical Seifert surface of a link projection depends on the orientation of the components.

The *genus* $g(K)$ of a knot $K$ is defined to be the minimum genus of all Seifert surfaces of $K$. Clearly, a knot $K$ is trivial if and only if $g(K) = 0$. It is not difficult to show that

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

**Remark.** There is an algorithm of Haken [8] for deciding whether or not two knots are equivalent. It can be used to compute the genus of a knot but the algorithm is too complex to be practical.

**(Cyclic covers of knot complements)**

Let $K$ be a knot and $F$ a Seifert surface of $K$. Denote $S^3 - \mathring{V}(K)$ by $X$ and $F \cap X$ by $F$. We may assume that $F$ is properly embedded in $X$. Now $F$ has a thin tubular neighborhood in $X$ diffeomorphic to $F \times [-1, 1]$. Identify the neighborhood with $F \times [-1, 1]$ such that $F \times \{0\}$ is identified with $F$. Let $Y$ be the space (manifold with corners) obtained by splitting $X$ along $F$. Let

$$N^+ = Y \bigcap F \times [0, 1] \quad \text{and} \quad N^- = Y \bigcap F \times [-1, 0] \quad \text{(see figure 13).}$$

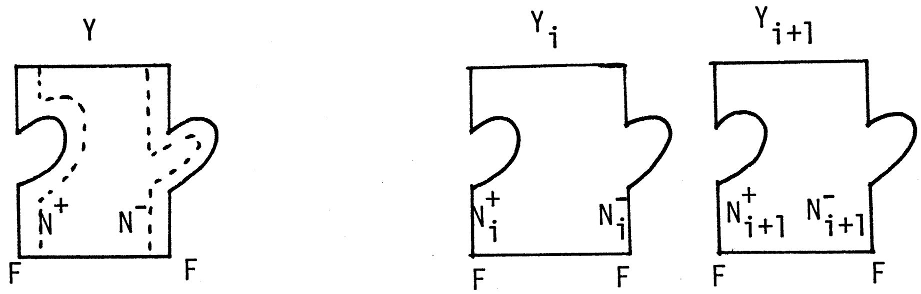



FIGURE 13.

For each integer $i$ let $Y_i$ be a copy of $Y$. For any positive integer $k \geq 2$ define $\tilde{X}_k = \cup_{0 \leq i \leq k-1} Y_i$, where $Y_i$ is joined to $Y_{i+1}$, $0 \leq i \leq k-2$, in such a way that $N_i^-$ is joined to $N_{i+1}^+$ along $F$ and $N_{k-1}^-$ is joined to $N_0^+$ along $F$. Then $\tilde{X}_k$ is a compact 3-manifold with a boundary diffeomorphic to $S^1 \times S^1$. We call $\tilde{X}_k$ the $k^{th}$ *cyclic cover* of $X$.

We also define $\tilde{X} = \cup_{i \in \mathbf{Z}} Y_i$, where $N_i^-$ is joined to $N_{i+1}^+$ along $F$ for all $i$. $\tilde{X}$ is called the *infinite cyclic cover* of $X$.

The next theorem shows that these covering spaces do not depend on the choice of Seifert surfaces. From the construction, there exists a natural $k$-fold covering projection $p : \tilde{X}_k \to X$, and an infinite covering projection $\tilde{p} : \tilde{X} \to X$. Let

$$i_* : \pi_1(X) \to H_1(X; \mathbf{Z}) \stackrel{\text{mod}\, k}{\longrightarrow} \mathbf{Z}_k \quad \text{and} \quad j_* : \pi_1(X) \to H_1(X; \mathbf{Z}) \cong \mathbf{Z}$$

be the natural homomorphisms.

**Theorem 4.1.** *Both covering projections $p$ and $\tilde{p}$ are regular projections, and*

$$p_*(\pi_1(\tilde{X}_k)) = Ker(i_*) \quad \text{and} \quad \tilde{p}_*(\pi_1(\tilde{X})) = Ker(j_*).$$

*Proof.* From the construction, the group of covering transformations of $p$ (or $\tilde{p}$) acts transitively on each fiber. Hence both projections are regular. Let $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}_k$ (or $\tilde{X}$) such that $p(\tilde{x}_0) = x_0$ (or $\tilde{p}(\tilde{x}_0) = x_0$). Then $p_*(\pi_1(\tilde{X}_k, \tilde{x}_0))$ and $\tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are normal subgroups of $\pi_1(X, x_0)$ [27].

Now

$$\pi_1(X, x_0)/\tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong \text{(The group of the covering transformations of } \tilde{p}) \cong \mathbf{Z}.$$

Hence the commutator subgroup of $\pi_1(X, x_0)$ is contained in $\tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Denote $\pi_1(X, x_0)$ by $G$ and the commutator subgroup by $G'$. Then we have

$$1 \to \tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0))/G' \to G/G' \to G/\tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0)) \to 1.$$

Since $G/G' \cong G/\tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong \mathbf{Z}$, $\quad \tilde{p}_*(\pi_1(\tilde{X}, \tilde{x}_0)) = G' = \mathrm{Ker}(j_*)$.

Similarly, it follows that $p_*(\pi_1(\tilde{X}_k)) = \mathrm{Ker}(i_*)$.

The above theorem implies that $\tilde{X}_k$ and $\tilde{X}$ are invariants of equivalence classes of knots, since the covering spaces correspond to specific normal subgroups of $\pi_1(X)$.

**Example 6.** Let $K$ be the figure eight knot and $X = S^3 - \overset{\circ}{V}(K)$. Let $F$ be the canonical Seifert surface of $K$ drawn in Figure 14, and choose generators $a$ and $b$ of $H_1(F; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$ as in the diagram. Note that $F$ deformation retracts to the bouquet of circles, $a \cup b$. Hence we may choose $\{\alpha, \beta\}$ as a basis for $H_1(Y; \mathbf{Z})$ as in the figure, where $Y = S^3 - F$. Then $\alpha$ and $\beta$ have the property that link $(a, \alpha) = \mathrm{link}(b, \beta) = 1$. We call $(\alpha, \beta)$ the *dual basis* of $(a, b)$.

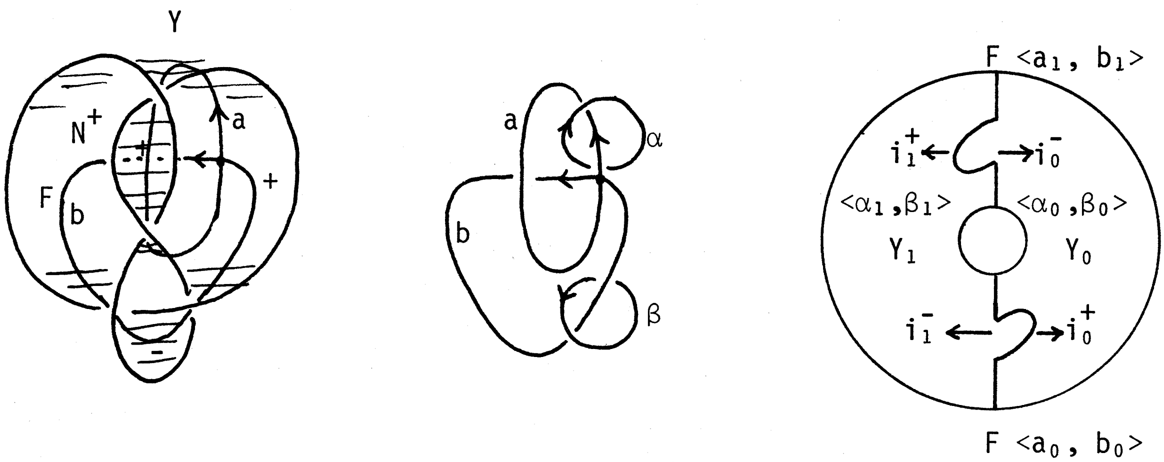



FIGURE 14.

We remark that if $F$ is a canonical Seifert surface of a knot, then $S^3 - F$ has the homotopy type of a bouquet of circles.

Let $i^+ : F \subset N^+ \subset Y$ and $i^- : F \subset N^- \subset Y$ be the natural inclusions. From the diagram, we have

$$i_*^+(a) = -\alpha, \quad i_*^+(b) = -\alpha + \beta, \quad i_*^-(a) = -\alpha - \beta, \quad i_*^-(b) = \beta.$$

If we let $M = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$, then

$$(i_*^+(a)\ i_*^+(b)) = (\alpha\ \beta)M \quad \text{and} \quad (i_*^-(a)\ i_*^-(b)) = (\alpha\ \beta)M^T,$$

where $M^T$ is the transpose of $M$.

**Remark.** Suppose that $x$ and $y$ are basis elements of $H_1(F; \mathbf{Z})$ for a Seifert surface $F$ of a knot $K$, and $x^*$ is the dual element of $x$ in $H_1(Y; \mathbb{Z})$. Then the coefficient of $x^*$ of $i_*^+(y)$ is equal to $\text{link}(x, y^+)$, where $y^+$ is obtained by pushing $y$ off $F$ in the positive direction of the tubular neighborhood.

We are now ready to compute the homology groups of $\tilde{X}_2 = Y_0 \cup Y_1$. Let

$$i_j^+ : F \subset N_j^+ \subset Y_j \quad \text{and} \quad i_j^- : F \subset N_j^- \subset Y_j$$

be the natural inclusions for $j = 0, 1$. With coefficients $\mathbf{Z}$, we have an exact sequence,

$$\begin{aligned} 0 \to H_2(\tilde{X}) \to H_1(F_0 \vee F_1) &\xrightarrow{i_*} H_1(Y_0) \oplus H_1(Y_1) \to H_1(\tilde{X}_2) \\ &\xrightarrow{\partial} H_0(F_0 \vee F_1) \to H_0(Y_0) \oplus H_0(Y_1) \to H_0(\tilde{X}_2) \to 0, \end{aligned}$$

where $F_j$ denotes the copy of $F$ in $N_j^+$, $j = 0, 1$. Observe that $\text{Im}(\partial) \cong \mathbf{Z}$. Hence $H_1(\tilde{X}_2) \cong \text{Coker}(i_*) \oplus \mathbf{Z}$. If we choose $\{a_j, b_j\}$ and $\{\alpha_j, \beta_j\}$ as bases for $H_1(F_j)$ and $H_1(Y_j)$, respectively, for $j = 0, 1$, then $\{a_0, b_0, a_1, b_1\}$ and $\{\alpha_0, \beta_0, \alpha_1, \beta_1\}$ are bases for $H_1(F_0 \vee F_1)$ and $H_1(Y_0) \oplus H_1(Y_1)$, respectively. Then

$$\begin{aligned} i_*(a_0) &= i_{0*}^+(a_0) - i_{1*}^-(a_0) = -\alpha_0 - (-\alpha_1 - \beta_1) \\ i_*(b_0) &= i_{0*}^+(b_0) - i_{1*}^-(b_0) = -\alpha_0 + \beta_0 - \beta_1 \\ i_*(a_1) &= i_{0*}^-(a_1) - i_{1*}^+(a_1) = -\alpha_0 - \beta_0 - (-\alpha_1) \\ i_*(b_1) &= i_{0*}^-(b_1) - i_{1*}^+(b_1) = \beta_0 - (-\alpha_1 + \beta_1) \end{aligned}$$

In matrix notation,

$$i_*(a_0\ b_0\ a_1\ b_1) = (\alpha_0\ \beta_0\ \alpha_1\ \beta_1)\begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$
$$= (\alpha_0\ \beta_0\ \alpha_1\ \beta_1)\begin{pmatrix} M & M^T \\ -M^T & -M \end{pmatrix}$$

The cokernel of $i_*$ is isomorphic to quotient group of $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ modulo the subgroup generated by the column vectors of the above matrix. By a sequence of column operations we obtain

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -2 & -3 & -5 \end{pmatrix}$$

It follows that the cokernel of $i_*$ is isomorphic to $\mathbf{Z}_5$, and $i_*$ is injective. So we conclude

$$H_1(\tilde{X}_2;\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}_5 \quad \text{and} \quad H_2(\tilde{X}_2;\mathbf{Z}) = \{0\}.$$

If the matrix $M$ is found for a knot $K$ as in the above example using a Seifert surface $F$ of $K$ ($F$ does not need to be a canonical Seifert surface), then $M$ is called a *Seifert matrix* of $K$. The matrix clearly depends on $F$, on the choice of positive direction of the normal bundle of $F$, and on the choice of basis of $H_1(F;\mathbf{Z})$. Then the matrix $\begin{pmatrix} M & M^T \\ -M^T & -M \end{pmatrix}$ becomes a presentation matrix of $H_1(\tilde{X}_2;\mathbf{Z})$, where $\tilde{X}_2$ is the cyclic double cover of $S^3 - K$.

Continuing the example, we compute $H_*(\tilde{X}_3;\mathbf{Z})$. A presentation matrix for $H_1(\tilde{X}_3;\mathbf{Z})$ is given by

$$\begin{pmatrix} M & O & -M^T \\ -M^T & M & O \\ O & -M^T & M \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

The matrix can be reduced to the following matrix by column operations.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 & 4 \end{pmatrix}$$

Therefore,

$$H_1(\tilde{X}_3; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_4 \quad \text{and} \quad H_2(\tilde{X}_3; \mathbf{Z}) \cong 0.$$

**Remark.** There are knots with $H_2(\tilde{X}_k; \mathbf{Z}) \ncong \{0\}$. For example, if $\tilde{X}_6$ is the 6-fold cyclic cover of the complement of the right handed trefoil knot, then $H_2(\tilde{X}_6; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$.

**(Computation of $H_*(\tilde{X}; \mathbf{Z})$).**

Using the notation already introduced,

$$\tilde{X} = \bigcup_{j \in \mathbf{Z}} Y_j \quad \text{and} \quad F_j \subset N_j^+ \subset Y_j,$$

and we have the natural inclusions $i_j^+$ and $i_j^-$ of $F$ into $Y_j$.

Let $\Lambda$ be the ring of Laurent polynomials,

$$\Lambda = \mathbf{Z}[t, t^{-1}] = \{a_n t^{-n} + a_{n-1} t^{-(n-1)} + \cdots + a_1 t^{-1} + a_0 + b_1 t + \cdots + b_m t^m\}.$$

Let $\tau$ be the covering transformations of $\tilde{p} : \tilde{X} \to X$ moving $Y_0$ to $Y_1$. We regard $H_*(\tilde{X}; \mathbf{Z})$ as a $\Lambda$-module by defining for any $x \in H_*(\tilde{X}; \mathbf{Z})$,

$$\begin{aligned} &(a_n t^{-n} + \cdots + a_0 + b_1 t + \cdots + b_m t^m) \cdot x \\ &= a_n \tau_*^{-n}(x) + \cdots + a_1 \tau_*^{-1}(x) + a_0 x + b_1 \tau_*(x) + \cdots + b_m \tau_*^m(x). \end{aligned}$$

We have now Mayer-Vietoris exact sequence of $\Lambda$-modules and $\Lambda$-homomorphisms:

$$\begin{aligned} 0 \to H_2(\tilde{X}) &\xrightarrow{\partial_2} H_1(\textstyle\bigvee_{j \in \mathbf{Z}} F_j) \xrightarrow{i_*} H_1(\textstyle\bigvee_{j \in \mathbf{Z}} Y_j) \to H_1(\tilde{X}) \\ &\xrightarrow{\partial_1} H_0(\textstyle\bigvee_{j \in \mathbf{Z}} F_j) \xrightarrow{\iota_*} H_0(\textstyle\bigvee_{j \in \mathbf{Z}} Y_j) \to H_0(\tilde{X}) \to 0 \end{aligned}$$

Here the homology groups are taken with coefficients $\mathbf{Z}$. If we let 1 be a generator of $H_0(F_1)$, then $l_*(1) = -1+t$. Hence $l_*$ is injective and $H_0(\tilde{X}) \cong \Lambda/(-1+t) \cong \mathbf{Z}$, where $(-1+t)$ is the ideal in $\Lambda$ generated by $-1+t$. Since $\tilde{X}$ is connected, this is what we should expect. Finally, $H_1(\tilde{X}) \cong \mathrm{Coker}(i_*)$.

**Example 7.** We again take the figure eight knot. Under the notation of the previous example, we have

$$\begin{aligned} i_*(a_1) &= i_{1*}^+(a_1) - i_{0*}^-(a_1) = -t\alpha_0 - (-\alpha_0 - \beta_0), \\ i_*(b_1) &= i_{1*}(b_1) - i_{0*}^-(b_1) = -t\alpha_0 + t\beta_0 - \beta_0. \end{aligned}$$

Note that $H_1(\bigvee_{j\in\mathbf{Z}} F_j)$ is a free $\Lambda$-module on generators $(a_1, b_1)$, and $H_1(\bigvee_{j\in\mathbf{Z}} Y_j)$ is also a free $\Lambda$-module on generators $(\alpha_0, \beta_0)$. Therefore, $H_1(\tilde{X})$ is isomorphic to the quotient module of $\Lambda \oplus \Lambda$ modulo the $\Lambda$-submodule generated by the column vectors of the matrix

$$\begin{pmatrix} -t+1 & -t \\ 1 & t-1 \end{pmatrix} = tM - M^T.$$

In other words, the matrix is a presentation matrix of $H_1(\tilde{X})$. By column operations, the matrix reduces to $\begin{pmatrix} -t+1 & t^2-3t+1 \\ 1 & 0 \end{pmatrix}$. Hence

$$H_1(\tilde{X}; \mathbf{Z}) \cong \Lambda/(-t^2+3t-1), \quad H_2(\tilde{X}; \mathbf{Z}) = \{0\}.$$

Note that as an abelian group $H_1(\tilde{X}; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$.

In general if $M$ is a Seifert matrix of a knot $K$, then $tM - M^T$ is a presentation matrix of $H_1(\tilde{X}; \mathbf{Z})$, where $\tilde{X}$ is the infinite cyclic cover of the complement of $K$.

**(Alexander polynomial)**

Let $K$ be a knot and $F$ a Seifert surface of $K$ of genus $g$. Let $B = (a_1, b_1, a_2, b_2, \cdots, a_g, b_g)$ be an ordered basis for $H_1(F; \mathbf{Z})$ which is isomorphic to the direct sum of $2g$ copies of $\mathbf{Z}$. With the standard orientation on $S^3$, the Alexander duality [27] gives a canonical isomorphism,

$$\Phi : H_1(S^3 - F; \mathbf{Z}) \to H^2(S^3, F; \mathbf{Z}) \to H^1(F; \mathbf{Z}) \to H_1(F; \mathbf{Z}).$$

Let $\alpha_i = \Phi^{-1}(a_i)$ and $\beta_i = \Phi^{-1}(b_i)$, $1 \le i \le g$. Then $\alpha_i$ and $\beta_i$ are represented by embedded circles in $S^3 - F$ such that $\text{link}(a_i, \alpha_i) = \text{link}(b_i, \beta_i) = 1$.

Choose a positive direction in a tubular neighborhood of $F$ in $S^3$, i.e., choose a trivialization of a tubular neighborhood of $F$ in $S^3$. Let

$$i^+ : F \subset N^+ \subset S^3 - F = Y$$

be the natural inclusion map defined in the previous section. Let $M_B$ be the matrix representing $i_*^+ : H_1(F; \mathbf{Z}) \to H_1(Y; \mathbf{Z})$ with respect to the basis $B = (a_1, b_1, \cdots, a_g, b_g)$ and its dual basis $B^* = (\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g)$. So

$$(i_*^+(a_1), i_*^+(b_1), \cdots, i_*^+(a_g), i_*^+(b_g)) = (\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g) M_B.$$

We call $M_B$ a *Seifert matrix* of $K$. We may also call $M_B^T$ a *Seifert matrix* of $K$ since $i_*^-$ would be represented by $M_B^T$.

Suppose that $B' = (a'_1, b'_1, \cdots, a'_g, b'_g)$ is another ordered basis for $H_1(F; \mathbf{Z})$. Then there exists a $2g \times 2g$ unimodular matrix $A(|A| = \pm 1)$ such that

$$\begin{pmatrix} a'_1 \\ b'_1 \\ \vdots \\ a'_g \\ b'_g \end{pmatrix} = A \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_g \\ b_g \end{pmatrix}.$$

We have

$$\text{link}\left( \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_g \\ b_g \end{pmatrix}, (\alpha_1 \beta_1 \cdots \alpha_g \beta_g) \right) = I.$$

By multiplying the above identity by $A$ from the left and by $A^{-1}$ from the right, we get

$$\operatorname{link}\left(A\begin{pmatrix}a_1\\b_1\\\vdots\\a_g\\b_g\end{pmatrix}, (\alpha_1\beta_1\cdots\alpha_g\beta_g)A^{-1}\right) = \operatorname{link}\left(\begin{pmatrix}a'_1\\b'_1\\\vdots\\a'_g\\b'_g\end{pmatrix} (\alpha_1\beta_1\cdots\alpha_g\beta_g)A^{-1}\right) = I.$$

Hence the dual basis $B'^*$ of $B'$ is equal to $(\alpha_1\beta_1\cdots\alpha_g\beta_g)A^{-1}$. This implies

$$\begin{aligned} i_*^+(a'_1b'_1\cdots a'_gb'_g) &= i_*^+((a_1b_1\cdots a_gb_g)A^T) = i_*^+(a_1b_1\cdots a_gb_g)A^T \\ &= (\alpha_1\beta_1\cdots\alpha_g\beta_g)M_BA^T = (\alpha'_1\beta'_1\cdots\alpha'_g\beta'_g)AM_BA^T. \end{aligned}$$

Thus, we have proved:

**Theorem 4.2.** *Under the above notation, if bases $B = (a_1b_1\cdots a_gb_g)$ and $B' = (a'_1b'_1\cdots a'_gb'_g)$ are related by $B'^T = AB^T$, then the corresponding Seifert matrices are related by $M_{B'} = AM_BA^T$.*

We define *Alexander polynomial* of a knot $K$ as the determinant of $tM - M^T$, where $M$ is a Seifert matrix of $K$. The Alexander polynomial for a link is defined the same way. But we require that the link is oriented and that $M$ is obtained from a Seifert surface that can be oriented consistently with the link. For example, if $F$ is the shaded surface in the figure,

L =  and F = 

then $F$ can not be oriented consistently with $L$. Note that every canonical Seifert surface of an oriented link can be oriented conistently with the orientation of the link.

**Theorem 4.3.** *The Alexander polynomial is well-defined for equivalence classes of knots up to multiplication by $t^n$, where $n$ is a positive integer.*

*Proof.* Let $K$ be a knot. We first assume that $K$ is given by a fixed embedded circle in the equivalence class of $K$. We divide the argument into several parts.

(1) Choice of a positive direction of the tubular neighborhood of a Seifert surface $F$, when $F$ and a basis for $H_1(F;\mathbf{Z})$ are given:

Let $B = (a_1, b_1, \ldots, a_g, b_g)$ be a basis for $H_1(F;\mathbf{Z})$, and let $M_B$ the Seifert matrix representing $i_*^+ : H_1(F;\mathbf{Z}) \to H_1(Y;\mathbf{Z})$. Suppose that $M_B = (m_{ij})$. Then $m_{ij} = \text{link}(c, d^+)$, where $c$ and $d$ are $i^{th}$ and $j^{th}$ vectors in $(a_1, b_1, \cdots, a_g, b_g)$, respectively, and $d^+$ is obtained by pushing $d$ off $F$ in the positive direction of the tubular neighborhood of $F$. Let $\overline{M}_B = (\overline{m}_{ij})$ be the Seifert matrix obtained by reversing the positive direction. Then

$$\overline{m}_{ij} = \text{link}(c, d^-) = \text{link}(d^-, c) = \text{link}(d, c^+) = m_{ji}.$$

Hence $\overline{M}_B = M_B^T$. So

$$|t\overline{M} - \overline{M}^T| = |tM^T - M| = |(tM^T - M)^T| = |tM - M^T|.$$

Hence if the positive direction is reversed, the polynomial remains unchanged.

(2) Choice of a basis for $H_1(F;\mathbf{Z})$, when $F$ and a positive direction of a tubular neighborhood of $F$ are given:

Suppose that $B$ and $B'$ are bases for $H_1(F;\mathbf{Z})$. By theorem 4.2, $M_{B'} = AM_BA^T$ for some unimodular matrix $A$. Then

$$|tM_{B'} - M_{B'}^T| = |tAM_BA^T - (AM_BA^T)^T| = |A|\,|tM_B - M_B^T|\,|A^T| = |tM_B - M_B^T|.$$

There is no change in the Alexander polynomial under different choices of bases for $H_1(F;\mathbf{Z})$.

(3) Choice of a Seifert surface:

Let $F$ and $F'$ be Seifert surfaces of $K$. By an isotopy make $F$ transverse regular to $F'$ such that $F \cap F'$ is a union of $K$ and circles contained in the interior of both

$F$ and $F'$. This is possible since the preferred longitude is unique up to an isotopy. Note that an isotopy of a Seifert surface does not change the Alexander polynomial. Now $S^3 - F \cup F'$ is an open 3-manifold with more than one component. Let $C$ be the closure of a component. Let $F_0 = C \cap F$ and $F'_0 = C \cap F'$. Then we may regard $C$ as a cobordism between $F_0$ and $F'_0$ (the cobordism has corners along $F_0 \cap F'_0$.) By Morse function theory, $F'_0$ is obtained from $F_0$ by attaching handles of dimensions 0,1,2 and 3. Since $C$ is connected we can eliminate 0-handles with equal numbers of 1-handles, and similarly eliminate all 3-handles. So we may assume that $F'_0$ is obtained from $F_0$ by attaching 1-and 2-handles. Since a 2-handle is a 1-handle when viewed from $F'_0$, it suffices to show

$$\begin{aligned}&(\text{Alexander polynomial associated to } F')\\ &= t(\text{Alexander polynomial associated to } F)\end{aligned}$$

when $F'$ is obtained from $F$ by attaching a single 1-handle. Then it follows for any two Seifert surfaces $F$ and $F'$,

$$\begin{aligned}&(\text{Alexander polynomial associated to } F')\\ &= t^n(\text{Alexander polynomial associated to } F)\end{aligned}$$

for some integer $n$.

Assume that $F'$ is obtained from $F$ by attaching a single 1-handle. Let $B = (a_2, b_2, \cdots, a_g, b_g)$ be a basis for $H_1(F; \mathbf{Z})$. Choose $a_1$ and $b_1$ in $H_1(F'; \mathbf{Z})$ as in Figure 15. Then $B' = (a_1, b_1, a_2, b_2, \cdots, a_g, b_g)$ is a basis for $H_1(F'; \mathbf{Z})$ by regarding the elements $a_2, b_2, \cdots, a_g$ and $b_g$ as elements of $H_1(F'; \mathbf{Z})$.

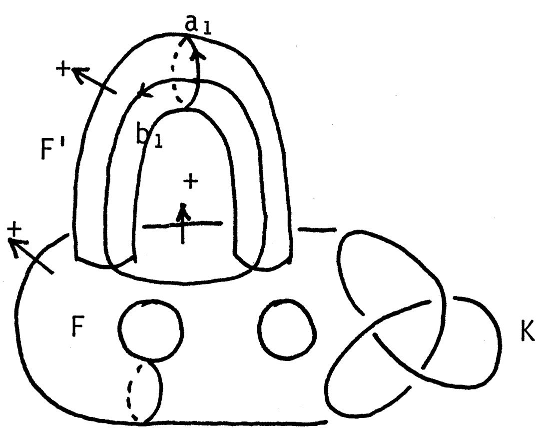



FIGURE 15.

Choose positive directions of the tubular neighborhoods of $F$ and $F'$ so that they agree over $F \cap F'$ and the positive direction along $F$ points into the 1-handle over the two disks where the handle is attached. Let $M_B$ and $M_{B'}$ be the Seifert matrices associated to $F$ and $F'$, respectively. Then we have

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_B & \\ 0 & \times & & & \end{pmatrix},$$

where an entry denoted by $\times$ is unspecified.

So $|tM_{B'} - M_{B'}^T| = t|tM_B - M_B^T|$.

We finally show that two embedded circles $K$ and $K'$ in the equivalence class of a knot give the identical Alexander polynomial if Seifert surfaces are chosen properly. Since $K$ is equivalent to $K'$, there exists a diffeomorphism $f$ of $S^3$ onto itself with $f(K) = K'$. Let $F$ be a Seifert surface of $K$. Then $f(F)$ is a Seifert surface of $K'$. Choose positive directions for the tubular of neighborhoods of $F$ and $f(F)$ so that they are preserved under $f$. Let $B$ be a basis for $H_1(F;\mathbf{Z})$. Then $B' = f_*(B)$ is a basis for $H_1(F';\mathbf{Z})$.

(4) If $f$ is orientation preserving, then the Seifert matrix associated to $F$ with respect to $B$ is equal to the Seifert matrix associated to $F'$ with respect to $B'$. Hence we have the identical Alexander polynomial.

(5) Suppose that $f$ is orientation reversing. If $x$ and $y$ are two basis elements in $B$, then

$$\text{link}(x^+, y) = -\text{link}(f_*(x)^+, f_*(y))$$

as Figure 16 shows. Hence $M_B = -M_{B'}$. So

$$|tM_{B'} - M_{B'}^T| = |-tM_B + M_B^T| = (-1)^{2g}|tM_B - M_B^T| = |tM_B - M_B^T|.$$

Therefore, the Alexander polynomial does not change.

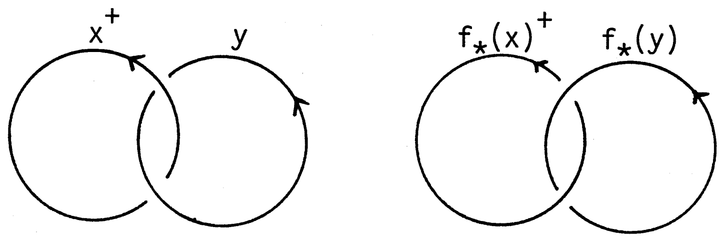



FIGURE 16.

**Remark.** Theorem 4.3 is true for oriented links if we restrict the equivalence class to the isotopy (preserving orientations on links) classes of oriented links. The restriction on the Seifert surfaces ensure that $F \cap F'$ is a union circles in the interior of both $F$ and $F'$ except for the link itself in part (3) of the proof of the theorem : if we let $V$ denote a thin tubular neighborhood of a component of link $L$, then by homological considerations $V \cap F$ is unique up to isotopy of $\partial V$ for any oriented Seifert surface $F$.

**(Signature of knots)**

Let $K$ be a knot and $M$ be a Seifert matrix of $K$. We define the *signature* $\sigma(K)$ of $K$ as the signature of the symmetric matrix $M + M^T$.

Recall that every symmetric matrix over $\mathbf{R}$ is congruent to a diagonal matrix. The *signature* of the symmetric matrix is defined to be the number of positive entries minus the number of negative entries of the diagonalized matrix.

**Theorem 4.4.** *The signature of knots is well-defined on the isotopy classes of knots. If $rK$ denotes the mirror image of a knot $K$, then $\sigma(rK) = -\sigma(K)$.*

*Proof.* We prove the theorem by ckecking the steps of the proof of theorem 4.3.

(1) If positive direction of a tubular neighborhood of a Seifert matrix is reversed, then the Seifert matrix is changed from $M$ to $M^T$. Clearly,

$$\sigma(M + M^T) = \sigma(M^T + (M^T)^T).$$

Hence there is no change in the signature.

(2) Suppose that $B$ and $B'$ are two ordered bases for $H_1(F;\mathbf{Z})$, where $F$ is a Seifert surface of $K$. Then $M_{B'} = AM_BA^T$ for some unimodular matrix $A$. So

$$M_{B'} + M_{B'}^T = AM_BA^T + AM_B^TA^T = A(M_B + M_B^T)A^T.$$

The signature does not change since congruent symmetric matrices have the identical signature.

(3) Suppose that

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_B & \\ 0 & \times & & & \end{pmatrix}.$$

We need to show that

$$\sigma(M_{B'} + M_{B'}^T) = \sigma(M_B + M_B^T).$$

Now

$$M_{B'} + M_{B'}^T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_B + M_B^T & \\ 0 & \times & & & \end{pmatrix}$$

This matrix is congruent to

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \times & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & M_B + M_B^T & \\ 0 & 0 & & & \end{pmatrix}.$$

To see this, add a proper multiple of the first column of $M_{B'} + M_{B'}^T$ to each column (with column number $\geq 3$) of the matrix, and do the corresponding row operations. So

$$\sigma(M_{B'} + M_{B'}^T) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & \times \end{pmatrix}\right) + \sigma(M_B + M_B^T) = \sigma(M_B + M_B^T)$$

since $\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & \times \end{pmatrix}\right) = 0$.

(4) Suppose that $K$ and $K'$ are isotopic knots. Then there exists an orientation preserving diffeomorphism $f$ of $S^3$ onto itself with $f(K) = K'$. Hence $K$ and $K'$ have the identical Seifert matrices, so $\sigma(K) = \sigma(K')$.

The argument so far proves that signature is well defined for the isotopy classes of knots.

(5) To show the second assertion of the theorem, observe that there exists an orientation reversing diffeomorphism $f$ of $S^3$ such that $f(K) = rK$. So if $M$ is a Seifert matrix of $K$, then $-M$ is a Seifert matrix of $rK$. Therefore,

$$\sigma(rK) = \sigma((-M) + (-M)^T) = -\sigma(M + M^T) = -\sigma(K).$$

We denote the Alexander polynomial of a knot $K$ by $\Delta_K(t)$. For the trivial knot $K$ we define $\Delta_K(t) = 1$. One should note that this is consistent with the definition of Seifert matrix, i.e., if one computes $\Delta_K(t)$ for the trivial knot $K$ from a Seifert surface of $K$ with genus greater than 0, then $\Delta_K(t) = t^n$ for some positive integer $n$.

**Example 8.** (1) Let $K$ be the left handed trefoil knot. A Seifert matrix of $K$ is equal to $M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ from Figure 17.

$$M + M^T = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \overset{\text{congruence}}{\sim} \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

$$|tM - M^T| = \begin{vmatrix} t-1 & -t \\ 1 & t-1 \end{vmatrix} = t^2 - t + 1.$$

Hence

$$\sigma(K) = 2 \text{ and } \Delta_K(t) = t^2 - t + 1.$$

The signature of the right handed trefoil knot (the mirror image of the left handed trefoil knot) is equal to $-2$ by Theorem 4.4.

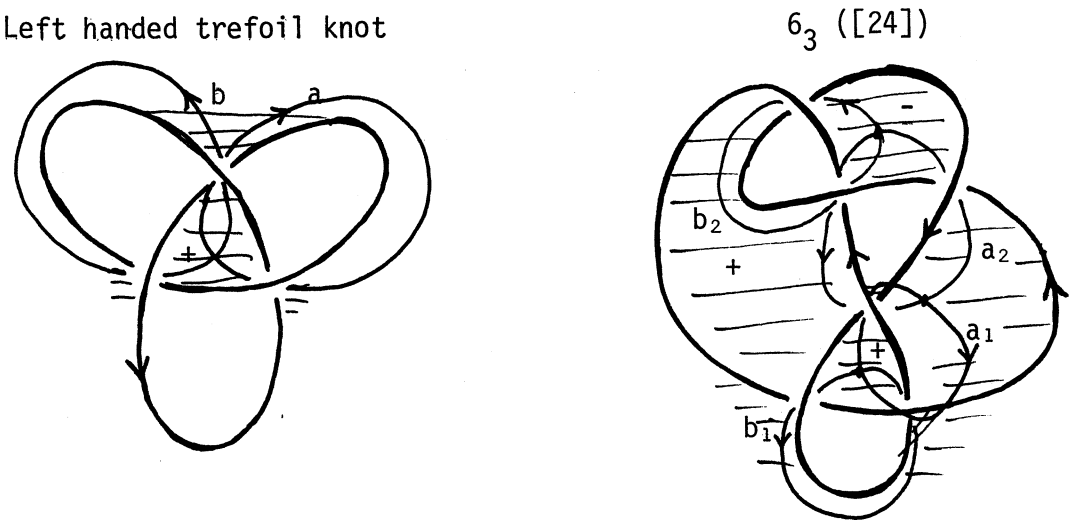



FIGURE 17.

(2) Let $K$ be $6_3$ knot in Figure 17. Using the basis elements given in the diagram, a Seifert matrix of $K$ is equal to

$$M = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One way to check that $(a_1, b_1, a_2, b_2)$ is indeed a basis for the first homology group of the Seifert surface is to compute $\det(M - M^T)$. If the determinant is equal to $\pm 1$, then $(a_1, b_1, a_2, b_2)$ is a basis. The statement holds in general and the converse is also true. This will be explained in the subsequent section. In our case, $\det(M - M^T) = 1$.

Now

$$|tM - M^T| = t^4 - 3t^3 + 5t^2 - 3t + 1,$$

$$\begin{pmatrix} -2 & -1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{5}{2} & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{13}{6} & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hence

$$\Delta_K(t) = t^4 - 3t^3 + 5t^2 - 3t + 1 \quad \text{and} \quad \sigma(K) = 0.$$

(3) Now we consider the oriented link $L$ given in Figure 18. Using the basis $(a, b, c)$ in the figure, we find that a Seifert matrix of $L$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence we have

$$\Delta_L(t) = -t^3 + 2t^2 - 2t + 1 \quad \text{and} \quad \sigma(L) = 1.$$

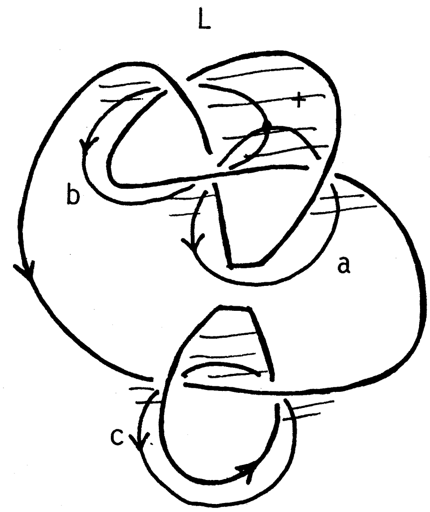



FIGURE 18.

## 5. Properties of Alexander polynomial and signature

Let $L$ be a link and $F$ a Seifert surface of $L$. Suppose that a positive direction for a tubular neighborhood of $F$ is chosen. Then orient $F$ such that the orientation of $F$ followed by the positive direction of the tubular neighborhood of $F$ gives the standard orientation of $\mathbf{R}^3$. Let $B$ be a basis for $H_1(F;\mathbf{Z})$, and let $x$ and $y$ be two basis elements in $B$. Figure 19 shows that

$$\text{link}(x, y^+) - \text{link}(x^+, y) = -\langle x, y\rangle,$$

the intersection number of $x$ with $y$. Therefore, if we let $M_B$ be the Seifert matrix, and $N_B$ the matrix representing the intersection pairing for $H_1(F;\mathbf{Z})$ with respect to $B$, then $M_B - M_B^T = -N_B$.


FIGURE 19.

**Theorem 5.1.** *If $K$ is a knot, then $\Delta_K(1) = 1$, and if $L$ is a link of more than one component, then $\Delta_L(1) = 0$.*

*Proof.* Let $M_B$ be a Seifert matrix of $K$. Then from the above discussion

$$\Delta_K(1) = |M_B - M_B^T| = |-N_B| = (-1)^{2g}|N_B| = |N_B|,$$

where $g$ is the genus of the Seifert surface from which $M_B$ is computed. We note that $N_B$ is congruent to the $2g \times 2g$ matrix

$$J = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & O & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & O & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Hence $|N_B| = 1$ and so $\Delta_K(1) = 1$.

Suppose that $F$ is a Seifert surface of a link $L$ with $n$ ($\geq 2$) components, and that the genus of $F$ is equal to $g$. Then it can be shown that $N_B$ is congruent to $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Hence $|N_B| = 0$, and $\Delta_L(1) = 0$.

**Corollary 5.2.** *Let $M$ be a Seifert matrix of a knot $K$. Then $|M + M^T|$ is an odd integer, in particular, $M + M^T$ is non-singular.*

*Proof.* Let $\Delta_K(t)$ be the Alexander polynomial obtained from $M$. Then

$$\Delta_K(1) \equiv \Delta_K(-1) \pmod 2.$$

So

$$|M + M^T| \equiv 1 \pmod 2.$$

From the definition, it is clear that the signature of a non-singular even dimensional symmetric matrix is even. Hence it follows from the above corollary that the signature of a knot is always an even integer.

**Theorem 5.3.** *Let*

$$\Delta_K(t) = a_0 + a_1 t + \cdots + a_n t^n \quad (a_0 \neq 0,\ a_n \neq 0)$$

*be an Alexander polynomial of a knot $K$. Then*

$$a_0 + a_1 t + \cdots + a_n t^n = a_n + a_{n-1} t + \cdots + a_0 t^n$$

*as polynomials in $t$, i.e., the coefficients of $\Delta_K(t)$ are symmetric.*

*Proof.* Suppose that $M$ is a $2g \times 2g$ Seifert matrix of $K$. By theorem 4.3,

$$|tM - M^T| = t^l(a_0 + a_1 t + \cdots + a_n t^n)$$

for some non-negative integer $l$. Then

$$|t^{-1}M - M^T| = t^{-l}(a_0 + a_1 t^{-1} + \cdots + a_n t^{-n}) = t^{-l-n}(a_0 t^n + a_1 t^{n-1} + \cdots + a_n).$$

On the other hand,

$$\begin{aligned} |t^{-1}M - M^T| &= t^{-2g}|M - tM^T| = t^{-2g}|M^T - tM| = t^{-2g}|tM - M^T| \\ &= t^{-2g} \cdot t^l(a_0 + a_1 t + \cdots + a_n t^n). \end{aligned}$$

Hence

$$-l - n = -2g + l \text{ and } a_0 + a_1 t + \cdots + a_n t^n = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$$

as desired.

In the above theorem, if $K$ is an oriented link with $c$ components, then the proof the theorem shows that

$$a_0 + a_1 t + \cdots + a_n t^n = (-1)^{c-1}(a_n + a_{n-1} t + \cdots + a_0 t^n).$$

In the notation of theorem 5.3, the integer $n$ is called the degree, $d(K)$, of $K$. Clearly, $d(K) \leq 2 \cdot g(K)$. If $K$ is the $6_3$ knot, then $d(K) = 4$. On the other hand, the knot has a canonical Seifert surface of genus 2 (see Figure 17). Hence $g(K) = 2$.

Let $K_1$ and $K_2$ be two knots. Then $K_1 \# K_2$ has a Seifert surface $F \cong F_1 \# F_2$ (where $\#$ is a boundary connected sum), where $F_i$ is a Seifert surface of $K_i$, $i = 1, 2$. We may further assume that there exists an embedded 2-sphere $S$ in $\mathbf{R}^3$ such that $F \cap S$ is the arc at which the connected sum is made. Hence there exists Seifert matrices $M$, $M_1$ and $M_2$ of $K_1 \# K_2$, $K_1$, and $K_2$, respectively, such that $M = M_1 \oplus M_2$ (direct sum of two matrices). Hence we have the following theorem.

**Theorem 5.4.** *For any knots $K_1$ and $K_2$,*

$$\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t) \quad \text{and} \quad \sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2).$$

**Example 9.** We can now show that Granny knot is not equivalent to the square knot. Observe that if $K$ denotes the right handed trefoil knot, then the Granny knot can be considered to be $K \# K$ and square knot to be $K\#(rK)$. So by the above theorem and a previous computation,

$$\sigma \text{ (Granny knot) } = -2 - 2 = -4, \quad \sigma \text{ (square knot) } = -2 + (2) = 0.$$

Hence these two knots are not equivalent.

**Example 10.** Let $K$ be the knot in the solid torus $V$ in Figure 20. Choose an oriented, preferred meridian $\mu$ and longitude $\lambda$ in $\partial V$ as shown in the diagram. Let $K_1$ be an arbitrary knot, and let $\mu'$ and $\lambda'$ (respectively) be the oriented, preferred meridian and longitude of $K_1$.

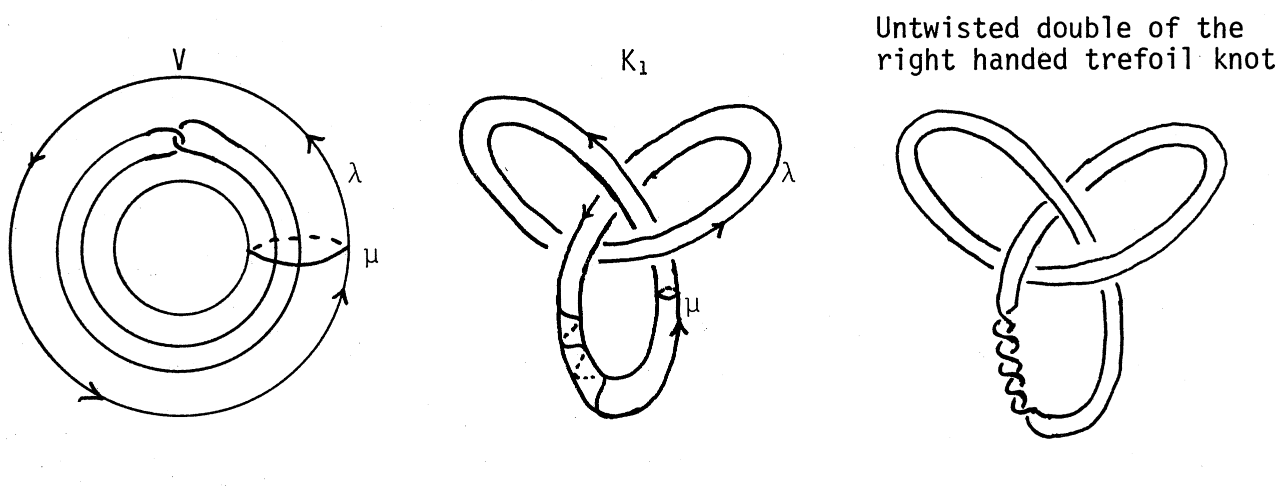



FIGURE 20.

Given an integer $m$, let $h_m : V \to V(K_1)$ be a diffeomorphism with

$$h_m(\mu) = \mu', \quad h_m(\lambda) = m\mu' + \lambda'.$$

Let $K_2 = h_m(K)$. We call $K_2$ an *m-twisted double* of $K_1$. A computation of the knot group of $K_2$ shows that $K_2$ is not trivial for all $m$ if $K_1$ is not.

We compute the Alexander polynomial of $K_2$. Choose a Seifert surface $F$ of $K_2$ as in Figure 21, and choose a basis, $(a, b)$ of $H_1(F; \mathbf{Z})$ as in the diagram. Then a Seifert matrix of $K_2$ is $\begin{pmatrix} -1 & -1 \\ 0 & m \end{pmatrix}$. Therefore,

$$\Delta_{K_2}(t) = -mt^2 + (1 + 2m)t - m.$$

Hence if $m = 0$ ($K_2$ is called an *untwisted double* in this case), then $\Delta_{K_2}(t) = t$.

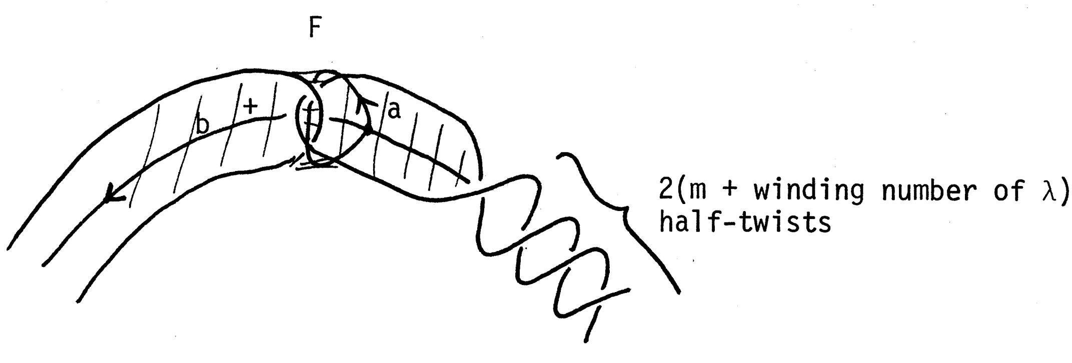



FIGURE 21.

## (Unknotting number)

For a knot $K$, define the unknotting number $u(K)$ of $K$ to be the minimum number of crossing changes that turn $K$ into the trivial knot. Note that every projection of a knot can be turned into a projection of the trivial knot by a finite number of crossing changes : orient the projection and travel along the projection in the direction of the orientation from a base point. As we go, change each crossing so that the crossing is passed under when it is encountered the first time. The result is a projection of the trivial knot. This shows that unknotting number is well-defined.

There is no known algorithm for the computation of the unknotting numbers. It is not even known whether or not the unknotting number is additive under the connected sum of knots except when both knots have unknotting number 1 [25]. We can compute the unknotting numbers in some cases by relating them to other computable knot invariants. We investigate one such case.

Let $K_+$, $K_-$ and $K_0$ denote the three oriented projections as indicated in Figure 22 when they are identical except in a neighborhood of a crossing where they differ as shown.

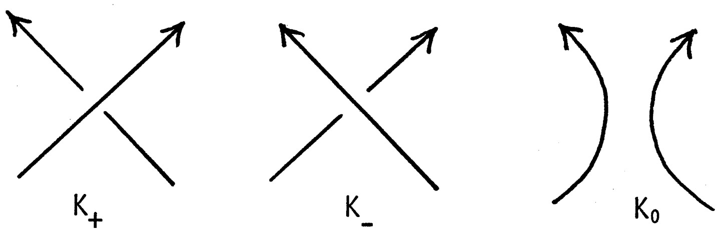



FIGURE 22.

Note that if $K_+$ is a knot, then $K_-$ is again a knot and $K_0$ is a 2-component link.

**Theorem 5.5.** *For any two knots* $K_+$ *and* $K_-$,

$$\sigma(K_-) = \sigma(K_+) + 2 \quad \text{or} \quad 0.$$

*Proof.* Let $F$ and $F'$ be the canonical Seifert surfaces of $K_+$ and $K_-$, respectively, where $K_-$ is given as in Figure 23.

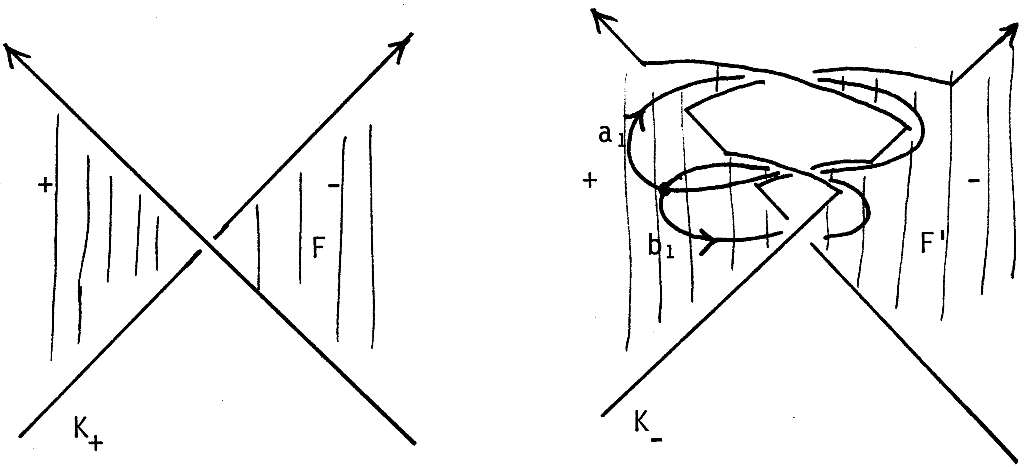



FIGURE 23.

Let $(a_2, b_2, \cdots, a_g, b_g)$ be a basis for $H_1(F; \mathbf{Z})$. Choose $(a_1, b_1, a_2, b_2, \cdots, a_g, b_g)$ as a basis for $H_1(F'; \mathbf{Z})$, where $a_1$ and $b_1$ are given in Figure 23. Let $M$ be the Seifert matrix of $K_+$ associated to the above basis, and the positive direction of the normal bundle of $F$ as indicated in the figure. Then a Seifert matrix $M'$ of $K_-$ has the following expression.

$$M' = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \times & \cdots & \times \\ 0 & 0 & & & \\ \vdots & \vdots & & M & \\ 0 & 0 & & & \end{pmatrix}.$$

Then

$$M' + M'^T = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M + M^T & \\ 0 & \times & & & \end{pmatrix}.$$

Since $M + M^T$ is non-singular (Corollary 5.2), a linear combination of row vectors of $M + M^T$ equals the second row of $M' + M'^T$ with entries whose column number is greater than 2. This implies that $M' + M'^T$ is congruent to

$$\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & \times & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & M + M^T & \\ 0 & 0 & & & \end{pmatrix}.$$

Hence

$$\sigma(M' + M'^T) = \sigma\begin{pmatrix} 2 & 1 \\ 1 & \times \end{pmatrix} + \sigma(M + M^T).$$

Now $\sigma\begin{pmatrix} 2 & 1 \\ 1 & \times \end{pmatrix} = 2$ or $0$, so we obtain the conclusion of the theorem.

**Corollary 5.6.** *For any knot $K$, $|\sigma(K)| \leq 2u(K)$.*

**Example 11.** Let $K$ be $7_3$ knot (Figure 24). Then $\sigma(K) = -4$. We also see that two crossing changes turn $7_3$ into the trivial knot. So by the corollary $u(K) = 2$.

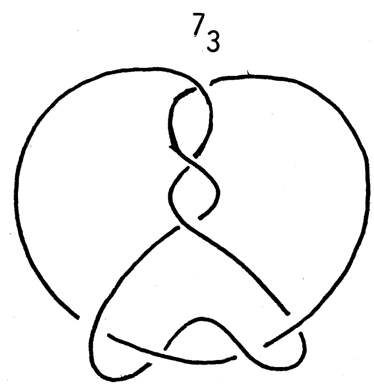



FIGURE 24.

### (Normalized Alexander polynomials of $K_+$, $K_-$ and $K_0$)

Let $L$ be an oriented link with $c$ components, and let

$$\Delta_L(t) = a_0 + a_1 t + \cdots + a_n t^n \quad (a_0 \neq 0,\ a_n \neq 0)$$

be an Alexander polynomial of $L$. By Theorem 5.3 and the remark following the theorem, if we define $\tilde{\Delta}_L(t)$ by $\Delta_L(t) = t^{\frac{n}{2}}\,\tilde{\Delta}_L(t)$, then

$$\tilde{\Delta}_L(t^{-1}) = (-1)^{c-1}\,\tilde{\Delta}_L(t).$$

We call $\tilde{\Delta}_L(t)$ the *normalized Alexander polynomial* of $L$.

Recall Example 8(2), (3). We may consider $6_3$ knot as $K_+$ and the link $L$ as $K_0$. Notice that $K_-$ is the trivial knot. Now

$$\begin{aligned}
\tilde{\Delta}_{K_+}(t) &= t^2 - 3t + 5 - 3t^{-1} + t^{-2},\\
\tilde{\Delta}_{K_-}(t) &= 1,\\
\tilde{\Delta}_{K_0}(t) &= -t^{\frac{3}{2}} + 2t^{\frac{1}{2}} - 2t^{-\frac{1}{2}} + t^{-\frac{3}{2}}.
\end{aligned}$$

We have

$$\tilde{\Delta}_{K_+}(t) - \tilde{\Delta}_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K_0}(t) = 0.$$

The next theorem shows that this identity holds not by an accident.

**Theorem 5.6.** *For any $K_+$, $K_-$ and $K_0$, we have*

$$\tilde{\Delta}_{K_+}(t) - \tilde{\Delta}_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K_0}(t) = 0.$$

*Proof.* Let $F_+$, $F_-$ and $F_0$ be the canonical Seifert surfaces of $K_+$, $K_-$ and $K_0$, respectively, as in Figure 25. Choose positive directions of the normal bundles of the surfaces as in the figure.

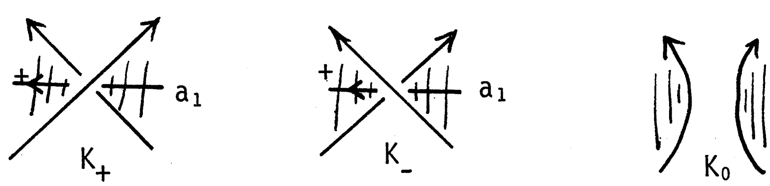



Figure 25.

Let $(b_1, b_2, \ldots, b_n)$ be a basis for $H_1(F_0; \mathbf{Z})$. By choosing an embedded circle $a_1$ passing through the band exactly once we may assume that $(a_1, b_1, b_2, \ldots, b_n)$ is a basis for $H_1(F_+; \mathbf{Z})$ and $H_1(F_-; \mathbf{Z})$. We let $M_+$, $M_-$ and $M_0$ be the Seifert matrices of $K_+$, $K_-$ and $K_0$, respectively, associated to these bases and the given positive directions of the normal bundles of the surfaces. Then there exist integers $r, x, y, \cdots, z, x', y', \cdots, z'$ such that

$$M_+ = \begin{pmatrix} r & x & y & \cdots & z \\ x' & & & & \\ y' & & & & \\ \vdots & & M_0 & & \\ z' & & & & \end{pmatrix}, \quad M_- = \begin{pmatrix} r+1 & x & y & \cdots & z \\ x' & & & & \\ y' & & & & \\ \vdots & & M_0 & & \\ z' & & & & \end{pmatrix}.$$

Therefore,

$$|tM_- - M_-^T| = \begin{vmatrix} rt - r & xt - x' & \cdots & zt - z' \\ x't - x & & & \\ \vdots & & tM_0 - M_0^T & \\ z't - z & & & \end{vmatrix} + \begin{vmatrix} t-1 & xt - x' & \cdots & zt - z' \\ 0 & & & \\ \vdots & & tM_0 - M_0^T & \\ 0 & & & \end{vmatrix}.$$

So there exist $l, m, n \in \mathbf{Q}$ such that

$$t^l \tilde{\Delta}_{K_-}(t) = t^m \tilde{\Delta}_{K_+}(t) + (t-1) \cdot t^n \cdot \tilde{\Delta}_{K_0}(t).$$

Differentiate the identity in $t$, and replace $t$ with 1. Using the fact that $\tilde{\Delta}'_K(1) = 0$ for any knot $K$ and $\tilde{\Delta}_L(1) = 0$ for any link $L$ with more than one component, we obtain $l = m$. So we have

$$(A) \qquad \tilde{\Delta}_{K_-}(t) = \tilde{\Delta}_{K_+}(t) + (t-1)t^{n-m}\,\tilde{\Delta}_{K_0}(t)$$

If $\tilde{\Delta}_{K_0}(t) = 0$, then $\tilde{\Delta}_{K_-}(t) = \tilde{\Delta}_{K_+}(t)$, and we have

$$\tilde{\Delta}_{K_+}(t) - \tilde{\Delta}_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K_0}(t) = 0.$$

So suppose that $\tilde{\Delta}_{K_0}(t) \neq 0$. By replacing $t$ with $t^{-1}$ in (A),

$$\tilde{\Delta}_{K_-}(t^{-1}) = \tilde{\Delta}_{K_+}(t^{-1}) + (t^{-1} - 1)t^{-(n-m)}\,\tilde{\Delta}_{K_0}(t^{-1})$$

This implies

$$(B) \qquad \tilde{\Delta}_{K_-}(t) = \tilde{\Delta}_{K_+}(t) - (t^{-1} - 1)t^{-(n-m)}\,\tilde{\Delta}_{K_0}(t)$$

From (A) and (B),

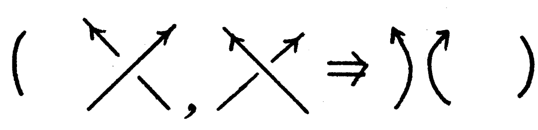
$$(t-1)t^{n-m} = -(t^{-1} - 1)t^{-(n-m)}.$$

This forces $n - m = -\frac{1}{2}$. Now (A) is equivalent to

$$\tilde{\Delta}_{K_+}(t) - \tilde{\Delta}_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K_0}(t) = 0.$$

**Remark.** We can compute inductively the normalized Alexander polynomial of a link using the identity in the above theorem.

Suppose that a crossing in a link $K$ is labelled $i$. Define $\varepsilon_i = 1$ if the crossing is a positive crossing and $\varepsilon_i = -1$ if the crossing is a negative crossing . Let $\sigma_i K$ be the link obtained from $K$ by a crossing change at the crossing. Define $\eta_i K$ to be the link obtained from $K$ by splitting $K$ at the crossing: .



Under this notation, the identity in Theorem 5.6 is equivalent to

$$(*) \qquad \tilde{\Delta}_{\sigma_i K}(t) = \tilde{\Delta}_K(t) + \varepsilon_i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{\eta_i K}(t).$$

**Theorem 5.7.** *If $K$ is an oriented link of two components, $C_1$ and $C_2$, then $\tilde{\Delta}'_K(1) = -l(K)$, where $l(K)$ denotes link$(C_1, C_2)$.*

*Proof.* By changing some of the crossings between $C_1$ and $C_2$, unlink $C_1$ from $C_2$. Let $L$ be the new link. From $(*)$, it follows $\tilde{\Delta}_L(t) = 0$. So $\tilde{\Delta}'_L(1) = 0$. Label the crossings we have changed 1 through $k$. Then $K = \sigma_k \sigma_{k-1} \cdots \sigma_1 L$. Suppose that the theorem holds for $\sigma_i \cdots \sigma_1 L$. Then

$$\tilde{\Delta}_{\sigma_{i+1}\sigma_i \cdots \sigma_1 L}(t) = \tilde{\Delta}_{\sigma_i \cdots \sigma_1 L}(t) + \varepsilon_{i+1}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{\eta_{i+1}\sigma_i \cdots \sigma_1 L}(t).$$

By differentiating in $t$,

$$\begin{aligned}\tilde{\Delta}'_{\sigma_{i+1} \cdots \sigma_1 L}(t) = \tilde{\Delta}'_{\sigma_i \cdots \sigma_1 L}(t) &+ \varepsilon_{i+1}(\frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{3}{2}})\tilde{\Delta}_{\eta_{i+1}\sigma_i \cdots \sigma_1 L}(t) \\ &+ \varepsilon_{i+1}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}'_{\eta_{i+1}\sigma_i \cdots \sigma_1 L}(t).\end{aligned}$$

Notice that $\tilde{\Delta}'_{\eta_{i+1}\sigma_i \cdots \sigma_1 L}(1) = 0$, for $\eta_{i+1}\sigma_i \cdots \sigma_1 L$ is a knot.

Hence

$$\tilde{\Delta}'_{\sigma_{i+1} \cdots \sigma_1 L}(1) = -l(\sigma_i \cdots \sigma_1 L) + \varepsilon_{i+1} = -l(\sigma_{i+1} \cdots \sigma_1 L).$$

Use an induction on $i$ to complete the proof.

**Corollary.** *For any $K_+$, $K_-$ and $K_0$ ($K$ is an oriented knot),*

$$\tilde{\Delta}''_{K_+}(1) - \tilde{\Delta}''_{K_-}(1) = 2l(K_0).$$

*Proof.* Differentiate the identity in Theorem 5.6 twice in $t$, and apply Theorem 5.7.

## 6. $S$-reduced Seifert matrices

Most of the knot invariants we have studied come from the Seifert matrices of the knots. But a Seifert matrix itself is not an invariant of a knot, unless we modify the matrix under an equivalence relation.

Let $K$ be a knot and $F$ a Seifert surface of $K$. Suppose that a positive direction of the normal bundle of $F$ is chosen, and $F$ is oriented consistently with the positive direction. Let $B = (a_1, b_1, \cdots, a_g, b_g)$ be an ordered basis for $H_1(F;\mathbf{Z})$, and let $M_B$ be the Seifert matrix obtained from these data.

Suppose that $\det(M_B) = 0$.

Then $\mathrm{Ker}(i_*^+)$ is a non-trivial subgroup of $H_1(F;\mathbf{Z})$, where $i_*^+ : H_1(F;\mathbf{Z}) \to H_1(S^3 - F;\mathbf{Z})$ is the homomorphism which $M_B$ represents (refer to Chapter 4). Choose a primitive element $a_1'$ in $\mathrm{Ker}(i_*^+)$. Then there exists new basis $B' = (a_1', b_1', \cdots, a_g', b_g')$ for $H_1(F;\mathbf{Z})$ such that $\langle a_1', b_1' \rangle = 1$, and for any $u \in \{a_1', b_1'\}$ and $v \in \{a_2', b_2', \cdots, a_g', b_g'\}$, $\langle u, v \rangle = 0$. Now $M_{B'}$ is congruent to $M_B$, and there exists a $(2g-2) \times (2g-2)$ matrix $M_1$ such that

$$M_{B'} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_1 & \\ 0 & \times & & & \end{pmatrix}.$$

If $\det(M_1) = 0$, then we apply the same process to $M_1$ to reduce it further. Hence we can show inductively that a Seifert matrix $M$ is congruent to a matrix of the form

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_r & \\ 0 & \times & & & \end{pmatrix},$$

where $M_r$ is a non-singular matrix.

We call $M_r$ a *$S$-reduction* of $M$ [21] [29]. Note that an $S$-reduction of a Seifert matrix could be empty.

**Remark.** It can be easily seen that for a knot $K$, $\Delta_K(t) = 1$ if and only if an $S$-reduction of any Seifert matrix of $K$ is empty. Furthermore, for each knot $K$, $d(K)$ (degree of Alexander polynomial of $K$) is always even.

Let $M$ and $M_1$ be even dimensional square matrices over an integral domain $R$ with 1. Suppose that $M$ and $M_1$ are related by

$$M = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & & & \\ \vdots & \vdots & & M_1 & \\ 0 & \times & & & \end{pmatrix}.$$

Then we call $M_1$ a *column reduction* of $M$, $M_1 = C_r M$, and $M$ a column enlargement of $M_1$, $M = C_e M_1$. If $M_1$ is obtained from $M$ by a sequence of column reductions, we still call $M_1$ a column reduction of $M$. Define a *row reduction*, $M_1 = R_r M$, and an enlargement, $M = R_e M_1$, similarly, where each entry of the first row and column of $M$ is 0 except that the 2nd row of the first column is 1.

### (Symplectic bases and symplectic matrices)

Let $F$ be an oriented Seifert surface of a knot. A basis $B = (a_1, b_1, \cdots, a_g, b_g)$ of $H_1(F; \mathbf{Z})$ is called a *symplectic basis* if

$$\left\langle \begin{pmatrix} a_1 \\ b_1 \\ \vdots \\ a_g \\ b_g \end{pmatrix}, (a_1, b_1, \cdots, a_g, b_g) \right\rangle = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 0 & & \\ & & \ddots & & \\ 0 & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Note that such a basis always exists. Denote the matrix in the right hand side of the above identity by $J$. Suppose that $B'$ is also a symplectic basis for $H_1(F; \mathbf{Z})$. Then these exists an unimodular matrix $A$ such that $B' = BA$. It follows that $A^T J A = J$. We call any unimodular matrix $A$ over an integral domain satisfying this identity a *symplectic matrix.* Two matrices $M$ and $M'$ are called *symplectically congruent* if $M' = A^T M A$ for some symplectic matrix $A$.

**Lemma 6.1.** *Suppose that non-singular matrices, $M_r$ and $M'_r$, are column reductions of matrices, $M$ and $M'$, respectively, over an integral domain. If $M$ is simplectically congruent to $M'$, then $M_r$ is simplectically congruent to $M'_r$. In particular,* $\dim(M_r) = \dim(M'_r)$ *and* $\det(M_r) = \det(M'_r)$.

*Proof.* Let $A$ be a $2n \times 2n$ symplectic matrix such that

$$A^T M A = M' \text{ or } A^T M = M' A^{-1} \quad \cdots (*).$$

From $A^T J A = J$, $A^{-1} = J^T A^T J$. This implies that $A^{-1}$ is obtained from $A^T$ (or from $A$) by simply rearranging the entries with $\pm$, singns added: for a $2 \times 2$ matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $D(U) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. For any even dimensional matrix $V$, we define $D(V)$ to be the matrix obtained from $V$ be applying the operation $D$ to each $2 \times 2$ submatrix when $V$ is partitioned into disjoint $2 \times 2$ submatrices. Then $A^{-1} = D(A^T)$.

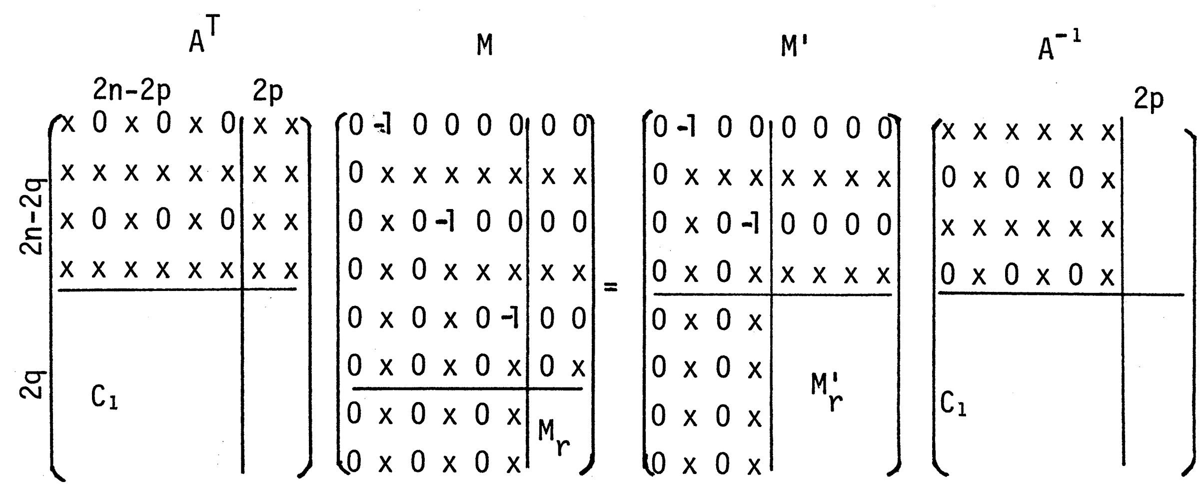
Suppose that dimensions of $M_r$ and $M'_r$ are $2p$ and $2q$, respectively. Without loss of generality we assume that $p \le q$. Let $A = (a_{ij})$.

First, compare $(1,1)$ entries of both sides of $(*)$. This gives $a_{21} = 0$. Next, compare $(1,3)$ entries to get $a_{41} = 0$. By repeating the steps, obtain

$$a_{2i,(2j-1)} = 0 \quad \text{for } 1 \le i \le n-p, \quad 1 \le j \le n-q.$$

To illustrate the information we have obtained on the entries of $A$, we look at an example, with $n = 4$, $p = 1$ and $q = 2$.

(*)




Let $C_1$ be the 1st column of $A^{-1}$ consisting of the last $2q$ entries. The identity $(*)$ implies that $M'_r \cdot C_1 = 0$. Since $M'_r$ is non-singular $C_1 = 0$. Equivalently, the last $2q$ entries of the 2nd column of $A^T$ are equal to 0. Similarly, it follows that the last $2q$ entries of any $(2i)^{th}$ column of $A^T$, $1 \leq i \leq n-p$, are 0.

Now any $(2i)^{th}$ column vector of $A^T$, $1 \leq i \leq n-p$, can have non-zero components only at $(n-q)$ rows common to all $i$. If $n-q < n-p$, then these column vectors are not linearly independent, thus $A^T$ can not be non-singular. So $n-q \geq n-p$, which implies that $p = q$.

Let $B$ be the $2p \times 2p$ submatrix of $A$ consisted of the last $2p$ columns and rows. The identity $(*)$ and the information we have on the entries of $A$ imply that $B^T M_r B = M_r'$ and $B^T J B = J$. Therefore, $M_r$ is symplectically congruent to $M_r'$.

**Lemma 6.2.** *Let $M = R_e C_e M_1$ be a Seifert matrix (over* $\mathbf{Z}$*). Then $M$ is congruent to $C_e R_e M_1$.*

*Proof.* Let $B = (a_1, b_1, a_2, b_2, \cdots, a_g, b_g)$ be the basis associated to $M$. Let $m_{23}$ be the $(2,3)$ entry of $M$. Let

$$B' = (a_2 - m_{23}a_1, b_2, a_1, b_1 + m_{23}b_2, a_3, b_3, \cdots, a_g, b_g).$$

It is easy to check that $B'$ is a basis, and with respect to $B'$, we obtain a new Seifert matrix:

$$M' = \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \times & \times & \cdots & \cdots & \cdots & \times \\ 0 & \times & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 1 & \times & \times & \cdots & \times \\ \vdots & \times & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & & M_1 & \\ 0 & \times & 0 & \times & & & \end{pmatrix}$$

Now $M'$ is congruent to $M$, and this completes the proof.

**Lemma 6.3.** *Suppose that $M = R_e M_1$ is a Seifert matrix associated to a symplectic basis, where $M_1$ is non-singular. Then $M_1$ is a column reduction of a matrix which is symplectically congruent to $M$ over an integral domain containing $det(M_1)$ as a unit.*

*Proof.* Suppose that $M$ represents $i_*^+ : H_1(F;\mathbf{Z}) \to H_1(S^3 - F;\mathbf{Z})$ with respect to a symplectic basis $B = (a_1, b_1, \cdots, a_g, b_g)$. Let $R$ be an integral domain in which $\det(M_1)$ is a unit. Then there exist $u_1$, $u_i$, and $v_i$, $2 \leq i \leq g$, in $R$ such that if we let

$$b_1' = b_1 + u_1 a_1 + \sum_{2 \leq i \leq g} (u_i a_i + v_i b_i),$$

then $i_*^+(b_1') = 0$. Let

$$B' = (b_1', -a_1, a_2 - v_2 a_1, b_2 + u_2 a_1, \cdots, a_g - v_g a_1, b_g + u_g a_1).$$

Check that $B'$ is a symplectic basis for $H_1(F;\mathbf{Z})$. With respect to $B'$, $i_*^+$ is represented by $C_e M_1$. This matrix is symplectically congruent to $M$ over $R$, thus the proof is completed.

**(Seifert matrices of oriented knots)**

Let $K$ be an oriented knot and $F$ a Seifert surface of $K$. Orient $F$ such that it induces the orientation of $K$. We choose a positive direction of the normal bundle of $F$ such that the orientation on $F$ followed by the positive direction of the normal bundle agrees with the standard orientation of $\mathbf{R}^3$. Choose a basis $B$ for $H_1(F;\mathbf{Z})$. A Seifert matrix with respect to $B$ is called a Seifert matrix of the oriented knot $K$. An $S$-reduction of a Seifert matrix of an oriented knot is called a *non-singular Seifert matrix* of the knot.

We now give a proof of the following theorem of Trotter [28] [29].

**Theorem 6.4.** *If $M_r$ and $M_r'$ are non-singular Seifert matrices of an oriented knot $K$, then $M_r$ is congruent to $M_r'$ over an integral domain in which* $\det(M_r)$ *is a unit.*

*Proof.* By the proof of theorem 4.3, it suffices to prove the theorem when $M_r$ and $M_r'$ are $S$-reductions of Seifert matrices $M$ and $M'$ of $K$ corresponding to Seifert surfaces $F$ and $F'$, respectively, where $F'$ is obtained from $F$ by a single 1-handle attaching. There are two cases to consider as in Figure 26 depending on whether or not the handle is attached on the positive side of $F$.

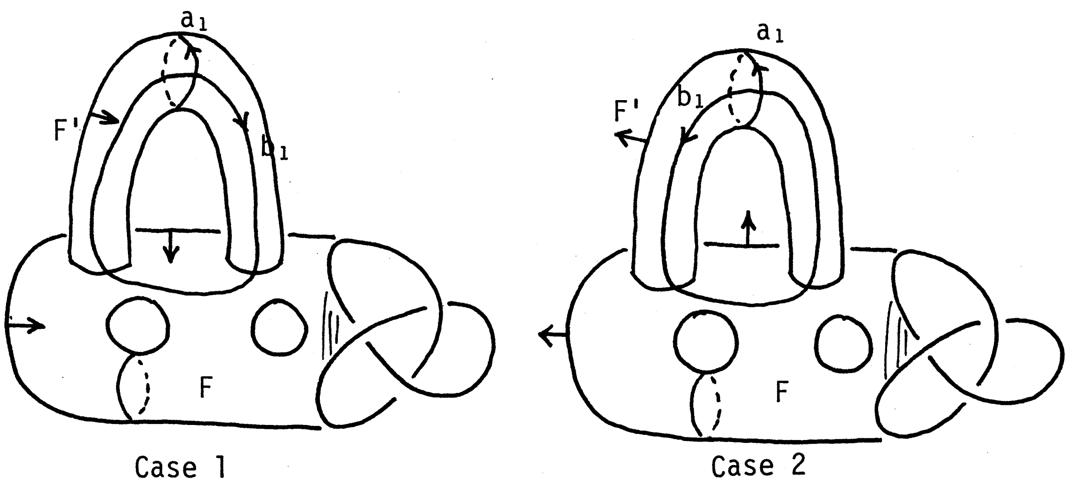



FIGURE 26.

Case 1. By choosing a basis for $H_1(F';\mathbb{Z})$ properly, we see that

$$M' \underset{\mathbf{Z}}{\sim} C_e M \text{ (congruence over } \mathbf{Z}).$$

Hence $C_e^p M_r' \underset{\mathbf{Z}}{\sim} C_e^q M_r$ for some integers $p$ and $q$. By choosing symplectic bases, we see that there exist non-singular matrices $\overline{M}_r'$ and $\overline{M}_r$ such that $M_r' \sim \overline{M}_r'$,

$M_r \sim \overline{M}_r$ and $C_e^p\overline{M}'_r$ is symplectically congruent to $C_e^q\overline{M}_r$ over $\mathbf{Z}$. By lemma 6.1, $\overline{M}'_r$ is symplectically congruent to $\overline{M}_r$. Therefore, $M'_r \underset{\mathbf{Z}}{\sim} M_r$.

Case 2. In this case, we have $M' \underset{\mathbf{Z}}{\sim} R_e M$. Then there exist integers $p$ and $q$ such that

$$C_e^p M'_r \underset{\mathbf{Z}}{\sim} R_e C_e^q M_r \underset{\mathbf{Z}}{\sim} C_e^q R_e M_r \quad \text{by Lemma 6.2.}$$

Let $R$ be an integral domain in which $\det(M_r)$ is a unit. The last matrix in the above equation is congruent to $C_e^{q+1}M_r$ over $R$ by Lemma 6.3. By choosing proper symplectic bases, it follows that there exist non-singular matrices $\overline{M}'_r$ and $\overline{M}_r$ such that $\overline{M}'_r \sim M'_r$, $\overline{M}_r \sim M_r$ and $C_e^p\overline{M}'$ is symplectically congruent to $C_e^{q+1}\overline{M}_r$ over $R$. Again apply Lemma 6.1 to conclude that $\overline{M}'_r$ is symplectically congruent to $\overline{M}_r$ over $R$, thus $M'_r$ is congruent to $M_r$ over $R$.

(**Oriented cobordism**)

Let $K$ be an oriented knot, and let $F$ and $F'$ be oriented (consistently with $K$) Seifert surfaces of $K$. We say that $F'$ is positively cobordant to $F$ if $F'$ can be obtained from $F$ up to isotopy of $\mathbf{R}^3$ by adding ambient 1-handles (more precisely, by doing ambient 0-surgeries) to the negative side of $F$, and by adding 2-handles to the positive side of $F$.

The next theorem follows from the proof of Theorem 6.4.

**Theorem 6.5.** *Let $F$ and $F'$ be oriented Seifert surfaces of an oriented knot $K$. Suppose that $M_r$ and $M'_r$ are non-singular Seifert matrices induced from $F$ and $F'$, respectively. If $F'$ is positively cobordant to $F$, then $M_r$ is congruent to $M'_r$ over $\mathbf{Z}$.*

**Example 12.** Let $K$ be the $10_3$ knot oriented as in Figure 27, and let $F$ and $F'$ be the oriented Seifert surfaces. Both surfaces consist of a disk and two twisted bands; horizontal and vertical ones. In $F$, the disk is behind the vertical band, and in $F'$, the disk is in front of the vertical band.

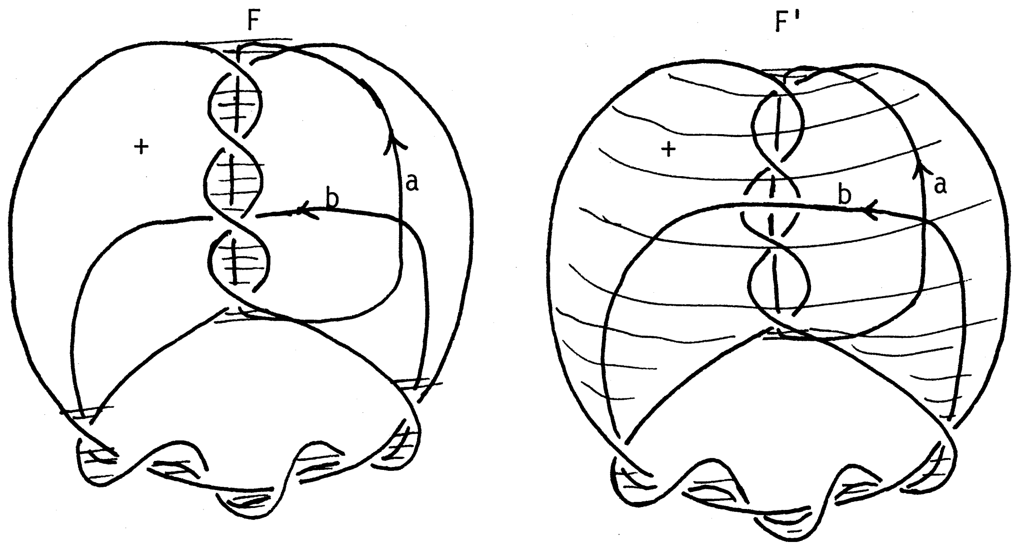



FIGURE 27.

Choosing $(a, b)$ as a basis for the 1st homology group of both surfaces as in the figure, we obtain two non-singular Seifert matrices

$$M = \begin{pmatrix} -2 & -1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix}$$

corresponding to $F$ and $F'$, respectively. We prove that $F$ is not positively cobordant to $F'$ by showing that $M$ is not congruent to $M'$ over $\mathbf{Z}$. If $M$ is congruent to $M'$, then there exists a unimodular matrix $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ such that $A^T M A = M'$. This implies that $-2p^2 - rp + 3r^2 + 2 = 0$. So $-(4p + r)^2 + 25r^2 + 4^2 = 0$. Hence $25r^2 + 4^2$ must be the square of an integer but this is only possible by elementary number theory when $r = 0$. This forces $p = s = \pm 1$ and $q = \mp\frac{1}{2}$, thus showing that $M$ is not congruent to $M'$ over $\mathbb{Z}$.

In theorem 6.4, if $\det(M_r)$ is a prime number, then $M_r$ is congruent to $M'_r$ over $\mathbf{Z}$ [29].

Suppose that $K$ is an oriented knot. Let $\overline{K}$ be the same knot with the reversed orientation. We say that $K$ is invertible if there exists an orientation preserving diffeomorphism $f$ of $\mathbf{R}^3$ such that $f(K) = \overline{K}$, respecting the orientation. In general, if $M$ is a Seifert matrix of $K$ then $M^T$ is a Seifert matrix of $\overline{K}$. Therefore, if $M$ is a non-singular Seifert matrix of an oriented knot $K$, $\det(M)$ is prime, and $M$ is not congruent to $M^T$ over $\mathbf{Z}$, then $K$ is not invertible. It is shown [29] that if $M = \begin{bmatrix} 5 & 1 \\ 2 & 11 \end{bmatrix}$, then $M$ is not congruent to $M^T$ over $\mathbf{Z}$. Therefore, any oriented knot with a Seifert matrix equal to $M$ is not invertible. In Figure 28, one such knot is given.

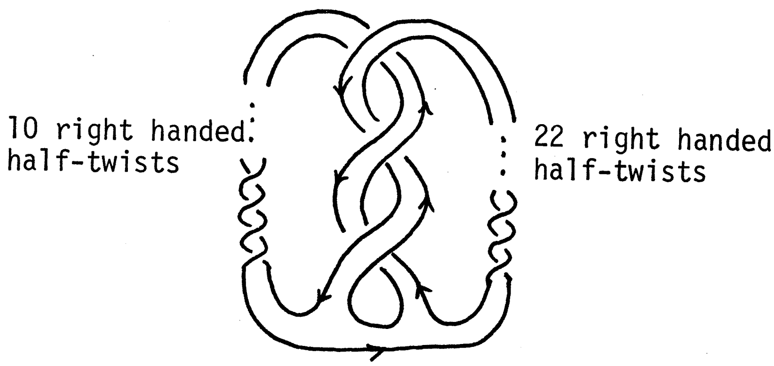



FIGURE 28.

## 7. Concordance, Signature and Arf invariant

In this chapter we show that the concordance classes of oriented knots form an abelian group under the connected sum operation, and prove that signature and Arf invariants give homomorphisms from this group to $\mathbf{Z}$ and $\mathbf{Z}_2$, respectively.

(**Concordance**)

Two oriented knots $K_0$ and $K_1$ are *concordant* if there exists an embedding $h : S^1 \times I \to S^3 \times I$ such that

$$h(S^1 \times \{i\}) = K_i \times \{i\} \subset S^3 \times I,\ i = 0, 1,\ \text{respecting the orientation,}$$

where $S^1$ is given an orientation. Here $h$ is not required to preserve the $I$-level.

A knot is called a ***slice knot*** if it is concordant to the trivial knot. It is clear that a knot $K$ is a slice knot if and only if $K \subset S^3 = \partial D^4$ bounds a properly, smoothly, embedded 2-disk $D$ in $D^4$.

**Remark.** Note that every knot bounds topologically embedded 2-disk in $D^4$.

**Example 13.** We show the $6_1$ knot $K$ (Figure 29) is a slice knot. The knot bounds a singular disk $f : D^2 \to S^3$ with double point set, $\alpha \cup \beta$ as in the figure.

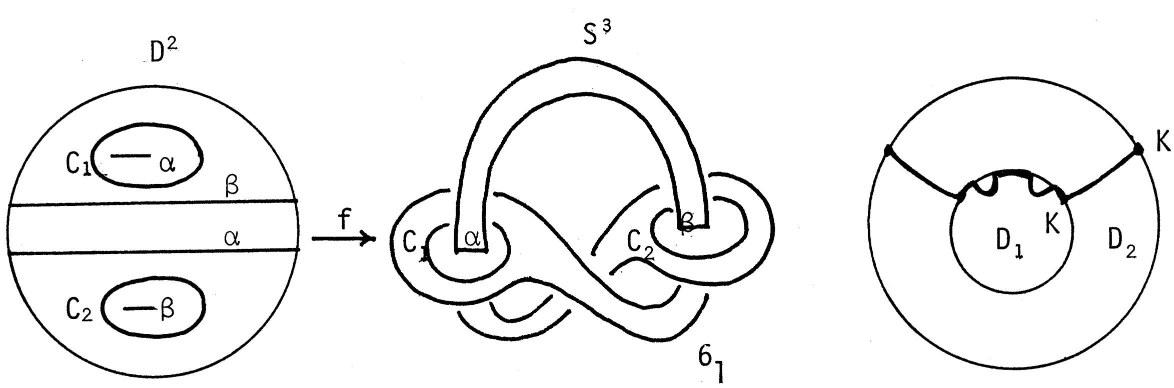



FIGURE 29.

Choose two small embedded disks $C_1$ and $C_2$ in the interior of $D^2$ as in the figure. Let $D_2$ be a 4-disk of radius 2, and $D_1$ the concentric disk of radius 1 in $D_2$.

Identify $S^3$ with $\partial D_1$. By pushing $f(\mathring{C}_1)$ and $f(\mathring{C}_2)$ into $D_1$ while leaving $f(\partial C_1)$ and $f(\partial C_2)$ on $\partial D_1$, we find an embedding (not proper) $g : D^2 \to D_1$ such that $g(\partial D^2) = K$. Now there exists an embedding

$$h : S^1 \times I \to D_2 - \mathring{D}_1 \cong S^3 \times I$$

such that $h$ respects $I$-level, with

$$h(S^1 \times \{0\}) = K \subset \partial D_1 \quad \text{and} \quad h(S^1 \times \{1\}) \subset \partial D_2,$$

and $g(D) \cup h(S^1 \times I)$ is a proper embedding of $D^2$ into $D_2$. Since $h(S^1 \times \{1\})$ is equivalent to $K$, $K$ bounds a properly embedded 2-disk in $D_2$, thus $K$ is a slice knot.

A knot $K$ is called a *ribbon knot* if there exists a singular embedding

$$f : D^2 \to S^3, \quad f(\partial D^2) = K,$$

with double points only such that for each component $A$ of the double point set, $f^{-1}(A)$ is a union of two arcs; one is in the interior of $D^2$ and the other intersects $\partial D^2$ at the two endpoints.

Example 13 shows that every ribbon knot is a slice knot. It is not known whether or not the converse is true.

**Theorem 7.1.** *If knots $K_0$ and $K_1$ are concordant, then $K_0 \# r\overline{K}_1$ ($rK_1$ is the mirror image of $K_1$ and $\overline{K}_1$ is $K_1$ with the reversed orientation) is a slice knot. In particular, for every knot $K$, $K \# r\overline{K}$ is a slice knot.*

*Proof.* Let $h : S^1 \times I \to S^3 \times I$ be a concordance from $K_0$ to $K_1$. Let $x \in S^1$ be a fixed point. By an isotopy of $S^3 \times \{1\}$ onto itself we can assume that $\pi h(x \times \{1\}) =$

$\pi h(x \times \{0\})$, where $\pi\colon S^3 \times I \to S^3$ is the projection. Define $f : \{x\} \times I \to S^3 \times I$ by $f(x,t) = (\pi h(x,0), t)$. Since $f$ and $h \,|\, \{x\} \times I$ are homotopic embeddings of an 1-manifold into a 4-manifold, they are isotopic [7]. Hence after composing with a diffeomorphism of $S^3 \times I$ onto itself, we may assume that $h(x,t) = (\pi h(x,0), t)$ for $t \in I$.

Let $D^1$ be a closed tubular neighborhood of $x$ in $S^1$. Again, after an isotopy, we may assume that $h \,|\, D^1 \times I$ is a product embedding. Therefore, there exists an embedding

$$F : (D^3 \times I, D^1 \times I) \to (S^3 \times I, h(S^1 \times I))$$

of pairs such that $F(D^1 \times \{0\})$ and $F(D^1 \times \{0\})$ are arcs in $K_0 \times \{0\}$ and $K_1 \times \{1\}$, respectively, where we consider $(D^3, D^1)$ as the standard disk pair.

Let

$$D^4 = S^3 \times I - F(\mathring{D}^3 \times I), \quad D = h(S^1 \times I) - F(\mathring{D}^1 \times I).$$

We may consider

$$\partial D^4 = S^3 \# \overline{S}^3 \ (\overline{S}^3 \text{ is } S^3 \text{ with the orientation reversed}), \ \partial D = K_0 \# r\overline{K}_1,$$

$D$ is a properly embedded 2-disk in $D^4$, and $\partial D \subset \partial D^4$. Therefore, $K_0 \# r\overline{K}_1$ is a slice knot.

Since every knot is concordant to itself the second statement follows immediately.

**Remark.** By slightly modifying the proof of Theorem 7.1, it can be shown that connected sum respects concordance relation. Hence the concordance classes of oriented knots form an abelian group, with the trivial knot as the identity and the mirror image of a knot with the reversed orientation as its inverse. The next theorem shows that the signature of knots induces a homomorphism from this group to $\mathbf{Z}$.

**Theorem 7.2.** *If $K$ is a slice knot, then $\sigma(K) = 0$.*

*Proof.* Let $D \subset D^4$ be a properly embedded 2-disk with $\partial D = K$. Since $D$ is contractible, $D$ has a tubular neighborhood diffeomorphic to $D \times D^2$ (see Figure 30). Let $X = D^4 - D \times \overset{\circ}{D}{}^2$. Identify $\partial X \cap D \times D^2$ with $D \times S^1$, and denote by $f : D \times S^1 \to S^1$ the projection to the second factor. There exists an extension $g$ of $f$ over $X$ since the obstruction to extending lies in a trivial group,

$$H^{i+1}(X, \partial X \bigcap D \times D^2; \pi_i(S^1)) \cong H^{i+1}(D^4, D \times D^2; \pi_i(S^1)) = \{0\},$$

where the isomorphism is an excision isomorphism. We may further assume that $g$ is a product near $D \times S^1$. Let $p$ be a point in $S^1$. Clearly, $g$ is transverse regular to $p$ in a collar neighborhood of $D \times S^1$ in $X$. Approximate $g$ by $h$ transverse regular to $p$ so that $g = h$ near $D \times S^1$.

Now $h^{-1}(p)$ is an orientable 3-manifold in $X$ such that

$$h^{-1}(p) \bigcap D \times S^1 = D \times \{p\}.$$

Let $W_0$ be the component of $h^{-1}(p)$ containing $D \times \{p\}$. Let

$$W_1 = \{(x, \theta p) \in D \times D^2 : 0 \leq \theta \leq 1\}.$$

Let $W = W_0 \cup W_1$ and $F = W \cap \partial D^4$. Then $F$ is an orientable surface with $\partial F = K$ and $\partial W = F \cup D$ identified along $K$. At this moment $F$ may have several components.

Let $F_0$ be the component of $F$ with $\partial F_0 = K$. Each component (close 2-surface) in $F - F_0$ bounds a unique 3-dimensional submanifold in $S^3$ which does not contain $K$. Choose a minimal component of $F - F_0$ in the sense that the 3-manifold it bounds does not contain any other component of $F$. Let $Y$ be the 3-manifold which the minimal component bounds. Attach $Y$ to $W$ to close up the boundary component, and push $Y$ into $D^4$ by a small distance to get a new 3-manifold (we

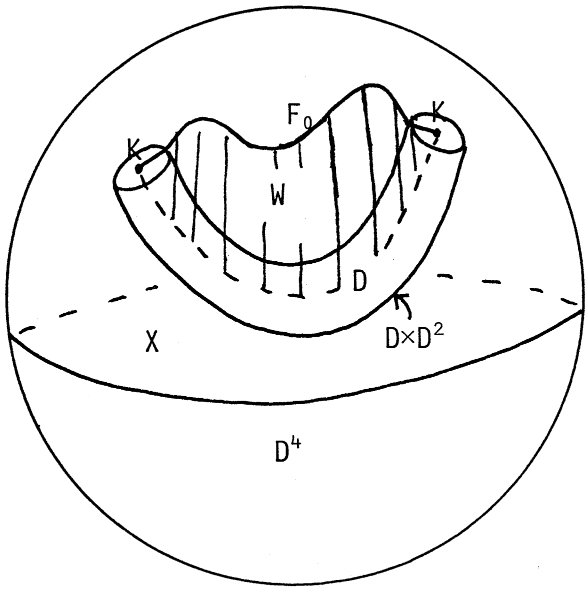



FIGURE 30.

call it $W$ again) which has one less boundary components. Repeat the process to modify $W$ so that $W \cap \partial D^4 = F$ is connected. Then $F$ is a Seifert surface of $K$ and $\partial W = F \cup D$.

Let $i : \partial W \to W$ be the inclusion map. Then we have a commulative diagram of long exact sequences,

$$\begin{array}{ccccccccc}
\longrightarrow & H^1(W) & \longrightarrow & H^1(\partial W) & \xrightarrow{\delta} & H^2(W,\partial W) & \longrightarrow & H^2(W) \\
 & \big\downarrow \cap[W,\partial W] & & \big\downarrow \cap[\partial W] & & \big\downarrow \cap[W,\partial W] & & \big\downarrow \\
\longrightarrow & H_2(W,\partial W) & \xrightarrow{\partial} & H_1(\partial W) & \xrightarrow{i_*} & H_1(W) & \longrightarrow & H_1(W,\partial W)
\end{array}$$

where the homology and cohomology groups are taken over $\mathbf{R}$ and the vertical maps are duality isomomorphisms. From the exactness of the bottom sequence,

$$H_1(\partial W) \cong \mathrm{Ker}(i_*) \oplus \mathrm{Im}(i_*).$$

Furthermore, $H^1(W) = \mathrm{Hom}(H_1(W), \mathbb{R})$,

$$H^1(\partial W) = \mathrm{Hom}(H_1(\partial W), \mathbf{R}), \quad H^2(W, \partial W) = \mathrm{Hom}(H_2(W, \partial W), \mathbf{R}),$$

and, $\delta = \mathrm{Hom}(\partial)$. Therefore, $\dim(H_1(\partial W)) - \dim(\mathrm{Im}(\partial)) = \dim(\mathrm{Ker}(\delta))$. From the commutativity of the diagram, $\dim(\mathrm{Ker}(\delta)) = \dim(\mathrm{Ker}(i_*))$. This implies that

$$\dim(\mathrm{Ker}(i_*)) = (1/2)\dim(H_1(\partial W)).$$

Since $H_1(F) \cong H_1(\partial W)$, there exists a basis $B = (a_1, a_2, \cdots, a_g, b_1, \cdots, b_g)$ of $H_1(F)$ such that $h_*(a_i) = 0$, $1 \le i \le g$, where $h : F \to W$ is the inclusion map.

Choose a positive direction of the normal bundle of $W$ in $D^4$ and restrict it to a positive direction of the normal bundle of $F$ in $S^3$. With respect to this positive direction, we claim that link $(a_i, a_j^+) = 0$ for $1 \le i,\ j \le g$, which we show later. So a Seifert matrix of $K$ has the form, $M = \begin{pmatrix} O & D \\ C & E \end{pmatrix}$.

So

$$M + M^T = \begin{pmatrix} O & D + C^T \\ C + D^T & E + E^T \end{pmatrix}.$$

Let $U = D + C^T$ and $V = E + E^T$. Since $M + M^T$ is non-singular, $U$ is non-singular. We have

$$\begin{pmatrix} O & U \\ U^T & V \end{pmatrix} = \begin{pmatrix} O & U \\ U^T & \frac{1}{2}V + \frac{1}{2}V \end{pmatrix} \overset{\text{congruence}}{\sim} \begin{pmatrix} O & U \\ U^T & O \end{pmatrix}$$

$$\sim \begin{pmatrix} U^{-1} & O \\ O & -I \end{pmatrix} \begin{pmatrix} O & U \\ U^T & O \end{pmatrix} \begin{pmatrix} (U^T)^{-1} & O \\ O & -I \end{pmatrix}$$

$$= \begin{pmatrix} O & -I \\ -I & O \end{pmatrix} \sim \begin{pmatrix} -2I & -I \\ -I & O \end{pmatrix} \sim \begin{pmatrix} -2I & O \\ O & \frac{1}{2}I \end{pmatrix}$$

Therefore,

$$\sigma(K) = \sigma(M + M^T) = 0.$$

To show the above claim, let $G_i$ and $G_j$ be properly embedded singular surfaces in $W$ such that $\partial G_i = a_i$ and $\partial G_j = a_j$. Such surfaces exist since $a_i \in \mathrm{Ker}(h_*)$. We let $G_j^+$ be the surface obtained from $G_j$ by pushing it off $W$ in the positive normal direction of $W$. It is well known that $\mathrm{link}(a_i, a_j^+) = \langle G_i, G_j^+ \rangle$. But $\langle G_i, G_j^+ \rangle = 0$ since $G_i \cap G_j^+ = \phi$. This finishes the proof of the claim.

**Remark.** The proof of the above theorem shows that if $K$ is a slice knot, then there exists a polynomial $p(t) \in \mathbf{Z}[t]$ such that $\Delta_K(t) = p(t) \cdot p(t^{-1})$, in particular, $\Delta_K(-1) = \pm$ (square of an integer). For example,

$$\Delta_{6_1}(t) = -2 + 5t - 2t^2 = t(t-2)(\frac{1}{t} - 2).$$

If $K$ is the figure eight knot, then

$$\Delta_K(t) = -1 + 3t - t^2, \ \Delta_K(-1) = -5.$$

Therefore, the figure eight knot is not a slice knot (Note that $\sigma(K) = 0$.)

(**Arf invariant**) [22]

Let $f : S^2 \to M^4$ be a piecewise linear embedding into a closed, oriented, simply connected, smooth 4-manifold. Suppose that $f$ is differentiable except at one point $x \in S^2$. Let $D^4$ be a smoothly embedded 4-disk in $M$ centered at $f(x)$ such that $f(S^2) \cap \partial D^4$ is a knot $K$ in $\partial D^4$. Let $\xi = [f] \in H_2(M; \mathbf{Z})$. Suppose that the mod 2 reduction of the Poincare dual of $\xi$ is the 2nd Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbf{Z}_2)$ of $M$. Under these assumptions $f : S^2 \to M$ is called *admissible* for the knot $K$. Define the *Arf invariant*

$$\varphi(K) = \frac{\xi \cdot \xi - \sigma(M^4)}{8} \pmod 2,$$

where $\xi \cdot \xi$ denotes the self intersection number of $\xi$ and $\sigma(M)$ denotes the signature (index) of $M$. We first see why $\frac{\xi \cdot \xi - \sigma(M)}{8}$ is an integer.

(**Unimodular symmetric bilinear form**)[10]

Let $V$ be a finitely generated free abelian group, and let $h : V \otimes V \to \mathbf{Z}$ a unimodular symmetric bilinear form (the determinant of the matrix representing $h$ is equal to $\pm 1$). Then there exists an element $\xi \in V$ (not unique) such that $h(x, x) \equiv h(x, \xi) \pmod 2$ for all $x \in V$. Such an element $\xi$ is called a *characteristic* *element*. For example if $V = \langle a \rangle \oplus \langle b \rangle$ and $h$ is represented by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to the basis $(a, b)$, then $\xi = a$ is a characteristic element. Given a symmetric bilinear form $(h, V)$, let $\sigma(h)$ denote the signature of $h$.

**Theorem.** *Under the above notation*

$$h(\xi, \xi) - \sigma(f) \equiv 0 \quad (mod\ 8).$$

A symmetric bilinear form is defined to be even if $h(x, x) \equiv 0 \pmod 2$ for all $x \in V$. A symmetric bilinear form is even if and only if all the diagonal entries of a matrix representing $h$ are even. For an even form $h$, $0 \in V$ is a characteristic element. Hence we have

**Corollary.** *If $(h, V)$ is an even form, then $\sigma(f) \equiv 0$ (mod 8), in particular,* $\dim(V) \geq 8$.

Note that there exists an even form represented by the $8 \times 8$ matrix $E_8$.

For a closed, oriented, 4-manifold $M$ ($M$ only needs to be a topological manifold), let $(h', H^2(M; \mathbf{Z}_2))$ be the pairing induced by the cup product. There exists a unique element $w_2 \in H^2(M; \mathbf{Z}_2)$ such that $h'(x, x) = h'(w_2, x)$ for all $x \in H^2(M; \mathbf{Z}_2)$. By the Wu formula, $w_2$ is equal to the 2nd Stiefel-Whitney class $w_2(M)$. Suppose that there exists $\xi \in H_2(M; \mathbf{Z})$ such that the mod 2 reduction of the Poincaré dual of $\xi$ is equal to $w_2(M)$ (such $\xi$ may not exist), then $\xi$ is a characteristic element for the unimodular symmetric bilinear form

$$h : H_2(M; \mathbf{Z}) \otimes H_2(M; \mathbf{Z}) \to \mathbf{Z}$$

induced by Poincaré duality. We denote $x \cdot y$ for $h(x, y)$. Then by the above theorem $\xi \cdot \xi - \sigma(M) \equiv 0 \pmod 8$. This shows that in the definition of $\varphi(K)$, $(\xi \cdot \xi - \sigma(M))/8$ is an integer. Before we prove that $\varphi$ is well defined we state some other important theorems in 4-dimensional manifold theory.

**Theorem (Rohlin).** *If $M$ is an oriented, closed, smooth 4-manifold with $w_2(M) = 0$, then $\sigma(M) \equiv 0$ (mod 16).*

This theorem is a corollary of the next theorem which was proved later.

**Theorem (Kervaire, Milnor).** *Let $M$ be a closed, oriented, smooth 4-manifold such that the pairing $(h, H_2(M; \mathbf{Z}))$ has a characteristic element $\xi$. If $\xi$ can be represented by a smoothly embedded 2-sphere in $M$, then*

$$\xi \cdot \xi - \sigma(M) \equiv 0 \quad (\textit{mod 16}).$$

**Theorem (Freedman).** *There exists an oriented, closed, topological 4-manifold $M$ with $w_2(M) = 0$ and $\sigma(M) = 8$. Such a manifold $M$ does not admit a smooth (or $PL$) structure.*

The next theorem shows that Arf invariant is well defined up to concordance.

**Theorem 7.3.** *If knots $K_0$ and $K_1$ are concordant, and if the Arf invariant is defined for them, then $\varphi(K_0) = \varphi(K_1)$.*

*Proof.* Let $f_i : S^2 \to M_i$ be an admissible map for $K_i$, $i = 0, 1$. Let $D_i^4$ be an embedded 4-disk in $M_i$ centered at the singular value $f_i(x)$ with $\partial D_i^4 \cap f_i(S^2) = K_i$. Suppose that $h : S^1 \times I \to S^3 \times I$ is a concordance from $K_0$ to $K_1$. Let $N = h(S^1 \times I)$. We consider $N$ as a properly embedded submanifold of $D_0^4 - \frac{1}{2}\mathring{D}_0^4$ by identifying $D_0^4 - \frac{1}{2}\mathring{D}_0^4$ with $S^3 \times I$ such that $S^3 \times \{0\} = \partial D_0^4$. Let

$$X_0 = M_0 - \frac{1}{2}\mathring{D}_0^4 \quad \text{and} \quad X_1 = M_1 - \mathring{D}_1^4.$$

Both manifolds have the natural orientation induced from $M_0$ and $M_1$.

Let $M = X_0 \cup_g X_1$, where $g$ is an orientation preserving diffeomorphism from $\partial X_0$ to $\partial X_1$ such that $g(K_1 \subset \partial \frac{1}{2} D_0^4) = K_1 \subset \partial D_1^4$ (See Figure 31)

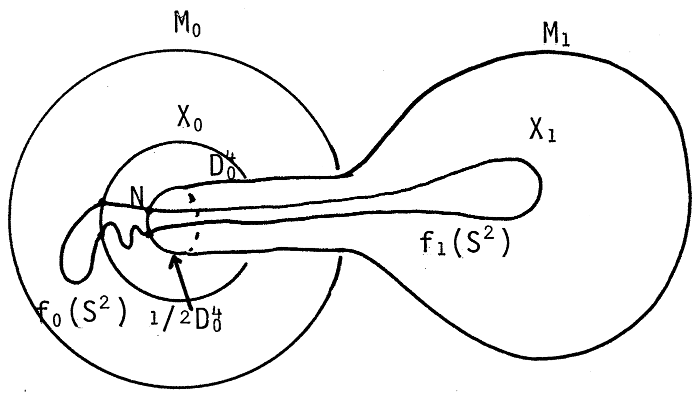



FIGURE 31.

Orient $M$ so that it is compatible with $M_0$. Then $M \cong M_0 \# \overline{M}_1$, where $\overline{M}_1$ is $M_1$ with the orientation reversed. Define a smooth embedding $f : S^2 \to M$ by

$$f(S^2) \bigcap X_0 = (f_0(S^2) - \overset{\circ}{D}{}_0^4) \bigcup N \quad \text{and} \quad f(S^2) \bigcap X_1 = f_1(S^2) - \overset{\circ}{D}_1.$$

Furthermore, define $f$ such that $f^{-1} f_0$ preserves the orientation, and $f^{-1} f_1$ reverses the orientation.

Let $\xi_i = [f_i] \in H_2(M_i; \mathbf{Z})$, $i = 0, 1$. Regard $\xi_i$ as an element of $H_2(X_i; \mathbf{Z})$ since the homomorphism: $H_2(X_i; \mathbf{Z}) \to H_2(M_i; \mathbf{Z})$, induced by the inclusion map, is an isomorphism. Identify $H_2(M; \mathbf{Z})$ with $H_2(X_0; \mathbf{Z}) \oplus H_2(X_1; \mathbf{Z})$. Then $\xi = [f] = \xi_0 - \xi_1$. Let $w_i = w_2(M_i)$. We regard $w_i$ as elements of $H^2(X_i; \mathbf{Z}_2)$. Then $w_2(M) = w_0 + w_1$. From naturality, the mod 2 reduction of the Poincaré dual of $\xi$ is $w_2(M)$. Therefore, $\xi$ is a characteristic element of $(h, H_2(M; \mathbf{Z}))$.

By the above mentioned theorem of Kervaire and Milnor,

$$\xi \cdot \xi - \sigma(M) \equiv 0 \pmod{16}.$$

So

$$(\xi_0 - \xi_1) \cdot (\xi_0 - \xi_1) - (\sigma(M_0) - \sigma(M_1)) \equiv \xi_0 \cdot \xi_0 + \xi_1 \cdot \xi_1 - \sigma(M_0) + \sigma(M_1) \equiv 0 \pmod{16}.$$

Then

$$(\xi_0 \cdot \xi_0)_{M_0} - \sigma(M_0) \equiv (\xi_1 \cdot \xi_1)_{M_1} - \sigma(M_1) \pmod{16}.$$

Hence

$$\frac{(\xi_0 \cdot \xi_0)_{M_0} - \sigma(M_0)}{8} \equiv \frac{(\xi_1 \cdot \xi_1)_{M_1} - \sigma(M_1)}{8} \pmod{2}$$

as desired.

**Remark.** Theorem 7.3 implies that the Arf invariant does not depend on the choice of admissible map.

**Theorem 7.4.** *For any knot $K$, $\varphi(K)$ is defined.*

*Proof.* Let $W = W(K;1)$ be the result of a handle attaching on $D^4$ along $K$ using the framing 1 (See Kirby Calculus of Chapter 3). Then

$$H_0(W;\mathbf{Z}) \cong H_2(W;\mathbf{Z}) \cong \mathbf{Z}, \text{ and } H_i(W;\mathbf{Z}) \cong \{0\} \text{ if } i \neq 0 \text{ or } 2.$$

Furthermore, $\partial W$ is a homology 3-sphere. Every homology 3-sphere bounds a parallelizable, simply connected 4-manifold [12]. Let $X$ be such a manifold which $\partial W$ bounds. Let $M = W \cup X$. $M$ is an orientable, closed 4-manifold. Orient $M$ consistently with the natural orientation on $W$. Regard $S^2 = D^2_+ \cup D^2_-$, and let $x \in D^2_+$ be the north pole of $S^2$. Let $CK$ be the cone of $K$ in $D^4$ with respect to the center of $D^4$. Define a piecewise linear embedding

$$f : S^2 \to M = D^4 \bigcup D^2 \times D^2 \bigcup X$$

by $f(D^2_+) = CK$, $f(x) =$ center of $D^4$, $f(\partial D^2_+) = K$ and $f(D^2_-) = D^2 \times \{0\}$.

We claim that $f$ is an admissible map for the knot $K$. From the construction, $f(S^2) \cap \partial D^4 = K$. Let $\xi = [f(S^2)] \in H_2(M;\mathbf{Z})$. Then $\xi$ is a generator of $H_2(M;\mathbf{Z}) \cong \mathbf{Z}$ and $\xi \cdot \xi = 1$. Let $w$ be the mod 2 reduction of the Poincaré dual of $\xi$. We need to show that $w = w_2(M)$.

Let $u \in H^2(M;\mathbf{Z})$. Then there exists $\eta \in H_2(M;\mathbf{Z})$ such that the mod 2 reduction of the Poincaré dual of $\eta$ is equal to $u$ since $H^1(M;\mathbf{Z}) = \{0\}$. Express $\eta = k\xi + \overline{\eta}$, where $k \in \mathbf{Z}$ and $\overline{\eta} \in H_2(X;\mathbf{Z})$. Now $\overline{\eta} \cdot \overline{\eta} = 0$ since $X$ is parallelizable, and $\overline{\eta} \cdot \xi = 0$. In modulo 2,

$$(u \cup u)([M]) \equiv \eta \cdot \eta \equiv (k\xi + \overline{\eta}) \cdot (k\xi + \overline{\eta}) \equiv k + \overline{\eta} \cdot \overline{\eta} \equiv k \equiv (k\xi + \overline{\eta}) \cdot \xi = (u \cup w)([M]).$$

By the Wu formula [27] $w = w_2(M)$.

**Theorem 7.5.** *For any two knots $K_1$ and $K_2$,*

$$\varphi(K_1\#K_2) = \varphi(K_1) + \varphi(K_2).$$

*Proof.* Let $f_i : S^2 \to M_i$ be an admissible map for $K_i$, for $i = 1, 2$. Let $D_i^4$ be embedded 4-disks containing the singular point of $f_i$ such that $K_i = f_i(S^2) \cap \partial D_i^4$. Form the connected sum $M = M_1^4 \# M_2^4$ in such a way that $D = (D_1^4 \cup D_2^4) \cap M$ becomes an embedded 4-disk in $M$, $f_1(S^2)\#f_2(S^2)$ is embedded (piecewise linearly) in $M$, and $f_1(S^2)\#f_2(S^2)$ intersects $\partial D^4$ in $K_1\#K_2$. Define an embedding $f : S^2 \to M$ such that $f(D_+^2) = C(K_1\#K_2)$ (see the proof of theorem 7.4 for the notation) with

$$f(x) = \text{ center of } D^4, \quad f(D_-^2) = [f_1(S^2)\#f_2(S^2)] - \mathring{D}^4.$$

It follows that $f$ is admissible for $K_1\#K_2$ and $\xi = [f] \in H_2(M; \mathbf{Z})$ is equal to

$$\xi_1 + \xi_2 \in H_2(M; \mathbf{Z}) \cong H_2(M_1; \mathbf{Z}) \oplus H_2(M_2; \mathbf{Z}),$$

where $\xi_i = [f_i] \in H_2(M_i; \mathbf{Z})$. Since

$$\sigma(M_1\#M_2) = \sigma(M_1) + \sigma(M_2),$$

the theorem follows.

We have now shown that the Arf invariant gives a homomorphism from the concordance classes of knots to $\mathbf{Z}_2$, in particular, the Arf invariant of any slice knot is 0. The next example shows that the Arf invariant of the right handed trefoil knot is 1.

**Example 14.** Let $M$ be the projective complex 2-space. Then

$$M = (D_+^2 \times D^2 \bigcup\nolimits_\alpha D_-^2 \times D^2) \bigcup\nolimits_{S^3} D^4,$$

where $\alpha : \partial D^2_- \times D^2 \to \partial D^2_+ \times D^2$ is the diffeomorphism given by

$$\alpha(e^{i\theta_1}, re^{i\theta_2}) \doteq (e^{i\theta_1}, re^{i(\theta_1+\theta_2)}).$$

Orient $M$ so that it restricts to the standard orientation of $D^2_+ \times D^2$.

Let

$$\gamma = [D^2_+ \times \{0\} \bigcup D^2_- \times \{0\}] \in H_2(M; \mathbf{Z}),$$

where $\gamma$ is oriented consistently with $D^2_+ \times \{0\}$. Then $\gamma$ is a generator of $H_2(M; \mathbf{Z}) \cong \mathbf{Z}$. To find $\gamma \cdot \gamma$, let

$$\gamma' = C(\alpha(\partial D^2_- \times \{p\})) \bigcup_\alpha D^2_- \times \{p\},$$

where $p$ is a point in $D^2$ with $|p| = \frac{1}{2}$, and the cone is taken with respect to the center of $D^2_+ \times D^2_-$. Clearly, $[\gamma] = [\gamma']$ in $H_2(M; \mathbf{Z})$, and $\gamma$ intersects $\gamma'$ transversally at one point, $(0,0) \in D^2_+ \times D^2_-$. Hence

$$\gamma \cdot \gamma = \gamma \cdot \gamma' = \langle D^2_+ \times \{0\}, C\alpha(\partial D^2_- \times \{p\}) \rangle = \text{link}(\partial D^2_+ \times \{0\}, \alpha(\partial D^2_- \times \{p\})) = 1$$

(see Figure 32.) This implies that $\sigma(M) = 1$.

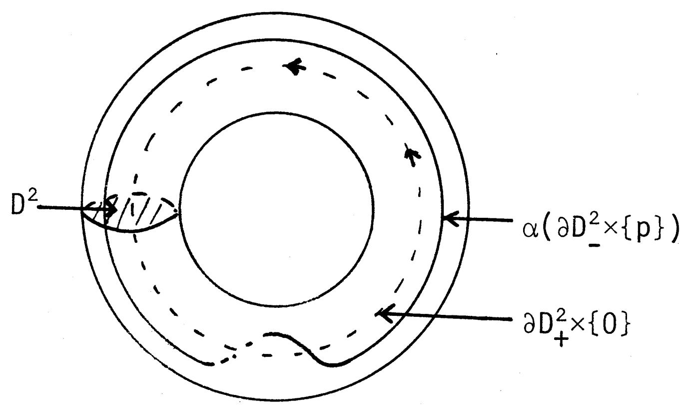



FIGURE 32.

Let $L$ be the knot in $\partial D^2_- \times D^2$ as in Figure 33, and let $K = \alpha(L) \subset \partial D^2_+ \times D^2$. Clearly, $L$ is the trivial knot in $\partial(D^2_- \times D^2) \cong S^3$, and $K$ the right handed trefoil knot in $\partial(D^2_+ \times D^2)$. Let $D$ be a properly embedded 2-disk in $D^2_- \times D^2$ such that $\partial D = L$, and let $CK$ be the cone of $K$ in $D^2_+ \times D^2$ with respect to the center $(0,0)$ of $D^2_+ \times D^2$. Define $f : S^2 \to M$ by

$$f(D^2_+) = CK, \quad f(D^2_-) = D, \quad f(\partial D^2_+) = K \quad \text{and} \quad f(x) = (0,0),$$

where $S^2 = D^2_+ \cup D^2_-$ and $x \in D^2_+$ is the north pole of $S^2$.

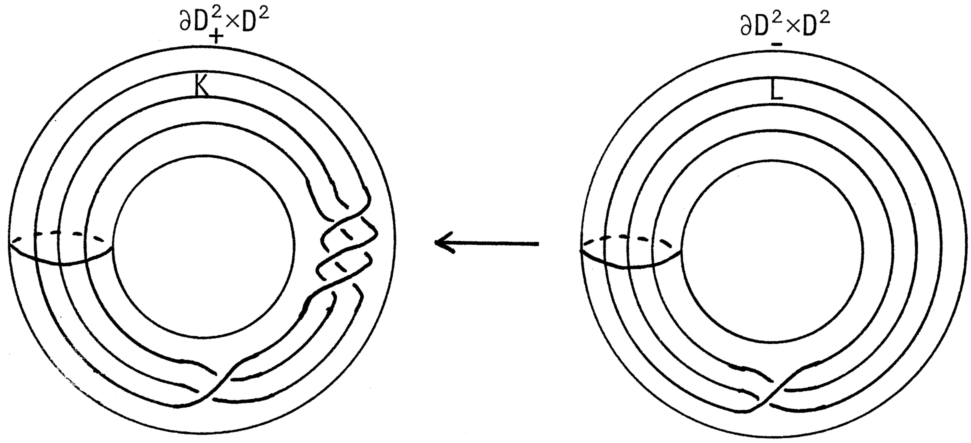



FIGURE 33.

Let $\xi = [f] \in H_2(M; \mathbf{Z})$. From the construction, $\xi = 3\gamma$ with a proper orientation on $f$. For any integer $k$,

$$k\gamma \cdot k\gamma \equiv k^2 \equiv k \equiv k\gamma \cdot 3\gamma \pmod 2.$$

Hence the mod 2 reduction of the Poincaré dual of $3\gamma$ is $w_2(M)$. Therefore, $f$ is admissible for $K$ (right handed trefoil knot). So

$$\varphi(K) = \frac{3\gamma \cdot 3\gamma - \sigma(M)}{8} = \frac{9-1}{8} = 1 \pmod 2.$$

We finish this chapter after stating two theorems which relate the Arf invariant to other invariants.

**Theorem (Robertello).** *For any knot $K$,*

$$\varphi(K) \equiv \frac{1}{2}\tilde{\Delta}''_K(1) \quad (mod\ 2).$$

For any oriented homology 3-sphere $M$, let $\mu(M)$ be the $\mu$-invariant. By definition,

$$\mu(M) \equiv \frac{\sigma(W)}{8} \pmod 2,$$

where $W$ is a paralellizable 4-manifold [12], $\partial W = M$. By Rohlin's theorem $\mu(M)$ is well-defined.

**Theorem (Gordon).** *For any knot $K$,*

$$\varphi(K) = \mu(S^3(K;1)),$$

*where $S^3(K;1)$ is the result of frame 1 surgery along $K$.*

## 8. Generalized polynomial

A new polynomial invariant for oriented links was discovered by Jones in 1985. Jones polynomial $V_K(t)$ for an oriented link $K$ satisfies the skein relation,

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{K_0}(t) = 0.$$

This identity is similar to the one which the normalized Alexander polynomial satisfies (Theorem 5.6).

It turned out that both Jones and Alexander polynomials are contained in a polynomial with two variables, the "Generalized polynomial". The generalized polynomial was discovered in 1985 by several people; Freyd, Yetter, Hoste, Lickorish, Millett and Ocneanu [4]. In this note we follow the treatment of the polynomial by Lickorish and Millett in [14].

The next theorem characterizes completely the generalized polynomial.

**Theorem 8.1.** *To each oriented link $K$ a unique element $K(l,m) \in \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$ can be associated so that $K(l,m)$ depends only on the isotopy (preserving orientations on links) class of $K$; if $\mathcal{U}$ is the trivial knot, then $\mathcal{U}(l,m) = 1$; and*

$$(*) \qquad lK_+(l,m) + l^{-1}K_-(l,m) + mK_0(l,m) = 0$$

We first study some properties of the generalized polynomial before we turn to the proof of the theorem.

### (Properties of generalized polynomial)

(1) For each link $K$, define

$$\overline{V}_K(t) = K(it^{-1}, i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})).$$

Put $l = it^{-1}$ and $m = i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$ in $(*)$:

$$it^{-1}K_+(it^{-1}, i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})) + i^{-1}tK_-(it^{-1}, i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})) + i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})K_0(it^{-1}, i(t^{\frac{1}{2}} - t^{\frac{1}{2}})) = 0.$$

This implies

$$t^{-1}\overline{V}_{K_+}(t) - t\overline{V}_{K_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\overline{V}_{K_0}(t) = 0.$$

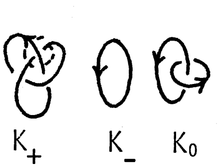
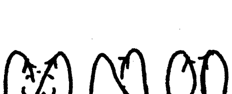
Since $\overline{V}_{\mathcal{U}}(t) = 1$, $\overline{V}_K(t) = V_K(t)$.

Similarly,

$$\tilde{\Delta}_K(t) = K(i, i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})) \text{ for any link } K.$$

(2) $K(l, m)$ can be computed for any oriented link $K$ using $(*)$ and $\mathcal{U}(l, m) = 1$.

**Example 15.**

| | $\tilde{\Delta}_K(t)$ | $V_K(t)$ | $K(\ell, m)$ |
|---|---|---|---|
|  $K_+$ $K_-$ $K_0$ | $\tilde{\Delta}_{K_+}(t)=1-(t^{1/2}-t^{-1/2})\cdot(-1)\cdot(t^{1/2}-t^{-1/2})= t^{-1}-1+t.$ | $V_{K_+}(t)= -t^4+t^3+t$ | $K_+(\ell, m)=-2\ell^{-2}-\ell^{-4}+\ell^{-2}m^2$ |
|  $K_+$ $K_-$ $K_0$ | $\tilde{\Delta}_{K_+}(t)-0+(t^{1/2}-t^{-1/2})=0.$ $\tilde{\Delta}_{K_+}(t)=-(t^{1/2}-t^{-1/2})$ | $V_{K_+}(t)=t^{5/2}+t^{1/2}$ | $\ell K_+(\ell, m)-\ell^{-1}(\ell+\ell^{-1}\cdot m^{-1}+m=0.$ $K_+(\ell, m)=\ell^{-2}(\ell+\ell^{-1}\cdot m^{-1}-\ell^{-1}m$ |
|  $K_+$ $K_-$ $K_0$ | $1-1+(t^{1/2}-t^{-1/2})\tilde{\Delta}_{K_0}(t)=0.$ $\tilde{\Delta}_{K_0}(t)=0$ | $t^{-1}-t+(t^{1/2}-t^{-1/2})\cdot V_{K_0}(t)=0.$ $V_{K_0}(t)=t^{1/2}+t^{-1/2}$ | $\ell+\ell^{-1}+mK_0(\ell, m)=0$ $K_0(\ell, m)=-(\ell+\ell^{-1})m^{-1}$ |

(3) Under the notation introduced in the paragraph preceding Theorem 5.7, the identity $(*)$ is equivalent to

$$(*) \qquad (\sigma_i K)(l,m) = -l^{2\varepsilon_i} K(l,m) - l^{\varepsilon_i} m(\eta_i K)(l,m)$$

(4) If $K$ is the trivial link of $c$ components, then

$$K(l,m) = \mu^{c-1}, \quad \text{where } \mu = -(l+l^{-1})m^{-1}.$$

We prove the statement by inducting on the number of components. If $c=1$, then $K(l,m) = 1 = \mu^0$. So the assertion holds. Suppose that the assertion holds for the trivial link of $(c-1)$ components. With $K_+$, $K_-$ and $K_0 = K$ as in Figure 34 we have by $(*)$:

$$l\mu^{c-2} + l^{-1}\mu^{c-2} + mK(l,m) = 0.$$

So

$$K(l,m) = -(l+l^{-1})m^{-1} \cdot \mu^{c-2} = \mu^{c-1},$$

thus proving the assertion.

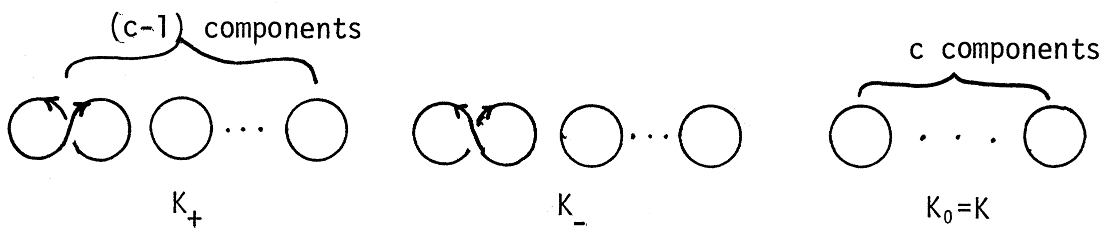
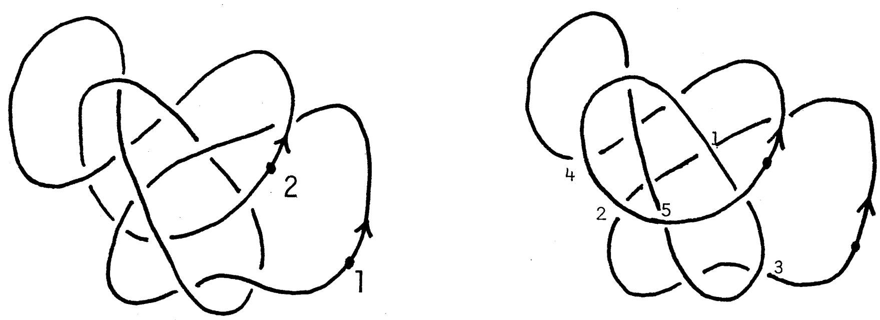



FIGURE 34.

We assume from now on that for each link, an ordering is given to the components of the link and a base point is specified for each component. Given such a link $K$, beginning at the base point of the first component of $K$ and proceeding in the direction of the orientation of $K$, change those crossings necessary so that each crossing is first encountered as an under-crossing. Continue the procedure with the rest of the components in the prescribed order. This results in a trivial link (see Figure 35.). We denote this link by $\alpha(K)$, and call the link the *standard ascending projection* associated to $K$. Label those crossings that have been changed 1 through $k$ so that $K = \sigma_k\sigma_{k-1}\cdots\sigma_1\alpha(K)$.




K $\qquad$ $\alpha(K)$

Figure 35.

(5) For any link $K$, the lowest power of $m$ in $K(l,m)$ is equal to $1-c$, where $c$ is the number of components of $K$, and the powers of $l$ and $m$ are either all even or odd depending upon whether $c$ is odd or even, respectively.

We prove the assertion by an induction on the number of crossings in $K$. If there are no crossings in $K$, then

$$K(l,m) = [-(l+l^{-1})m^{-1}]^{c-1}.$$

Hence the lowest power of $m$ in $K(l,m)$ is equal to $1-c$, and also the second statement holds. Assume that the assertion holds for projections with the number of crossings less than $n$. Suppose that $K$ has $n$ crossings and $K = \sigma_k\sigma_{k-1}\cdots\sigma_1\alpha(K)$. The assertion holds for $\alpha(K)$. Suppose that the assertion holds for $\sigma_{i-1}\cdots\sigma_1\alpha(K) = L$. Then

$$(\sigma_i L)(l,m) = -l^{2\varepsilon_i}L(l,m) - l^{\varepsilon_i}m(\eta_i L)(l,m).$$

By assumption, the lowest power of $m$ in $L(l,m)$ is equal to $1-c$. On the other hand, since $\eta_i L$ has less than $n$ crossings, by the induction hypothesis, the lowest power of $m$ in $(\eta_i L)(l,m)$ is equal to $2-c$ if the $i^{th}$ crossing is between two distinct components of $L$, or equal to $-c$ if the crossing is in a component of $L$. Hence the lowest power of $m$ in $l^{\varepsilon_i} m(\eta_i L)(l,m)$ is equal to $3-c$ or $1-c$. So the lowest power of $m$ in $(\sigma_i L)(l,m)$ is equal to $1-c$.

Finally, observe that the parity of powers of $l$ and $m$ in $-l^{2\varepsilon_i} L(l,m)$ is the same as that of $l$ and $m$ in $L(l,m)$, and from the induction hypothesis, the parity of powers of $l$ and $m$ in $l^{\varepsilon_i} m(\eta_i L)(l,m)$ is the same as that of $l$ and $m$ in $L(l,m)$, not depending upon whether $\eta_i L$ has $c-1$ components or $c+1$ components. Hence the parity of powers of $l$ and $m$ in $(\sigma_i L)(l,m)$ is the same as that of powers of $l$ and $m$ in $L(l,m)$, thus proving the induction hypothesis for link projections with $n$ crossings

(6) Reversing the orientation of every component leaves the polynomial unchanged.

This property follows immediately from the observation that the sign of each crossing does not change if the orientation of every component is reversed.

(7) For any link $K$,

$$(rK)(l,m) = K(l^{-1},m).$$

Hence if $K$ is amphicherial ($rK$ is isotopic to $K$), then $K(l,m)$ is symmetric in $l$ and $l^{-1}$.

To prove the assertion, we again induct on the number of crossings in $K$. If $K$ has no crossings, then the assertion holds clearly. Assume that the assertion holds for links with less than $n$ crossings. Suppose that $K$ has $n$ crossings.

Let $K=\sigma_k \cdots \sigma_1 \alpha(K)$. Suppose that the assertion holds for $L=\sigma_{i-1} \cdots \sigma_1 \alpha(K)$. We label the crossings in $r\alpha(K)$, corresponding to the crossings labelled $1, 2, \cdots, k$ in $\alpha(K)$ by $1', 2', \cdots, k'$, respectively. Then $\varepsilon_{i'} = -\varepsilon_i$ and $rK = \sigma_{k'} \cdots \sigma_{1'} r\alpha(K)$. Now

$$\begin{aligned}
(r\sigma_i L)(l,m) &= (\sigma_{i'} rL)(l,m) \\
&= -l^{2\varepsilon_{i'}} rL(l,m) - l^{\varepsilon_{i'}} m(\eta_{i'} rL)(l,m) \\
&= -(l^{-1})^{2\varepsilon_i} L(l^{-1},m) - (l^{-1})^{\varepsilon_i} (\eta_i L)(l^{-1},m) \\
&= \sigma_i L(l^{-1},m).
\end{aligned}$$

Therefore, the induction hypothesis holds for links with $n$ crossings.

(8) If $K_1$ and $K_2$ are links separated by an embedded 2-sphere, then

$$(K_1 \vee K_2)(l,m) = \mu \cdot K_1(l,m) \cdot K_2(l,m).$$

Furthermore,

$$(K_1 \# K_2)(l,m) = K_1(l,m) \cdot K_2(l,m).$$

Note that connected sums of links are not well-defined. So the second identity gives many non-isotopic links with the identical generalized polynomial.

To prove the first assertion, let $L = K_1 \vee K_2$ and let $c_i$ be the number of components in $K_i$, $i = 1,2$. If $L$ has no crossing, then

$$L(l,m) = \mu^{c_1+c_2-1} = \mu\mu^{c_1-1}\,\mu^{c_2-1} = \mu K_1(l,m)K_2(l,m).$$

So the assertion holds. Suppose that $L = \sigma_k \cdots \sigma_1 \alpha(L)$. To finish the proof, induct on the number of crossing changes as before.

The second assertion follows immediately from the first. We may assume that $(K_1\#K_2)_+$ and $(K_1\#K_2)_-$ are as given in Figure 36.

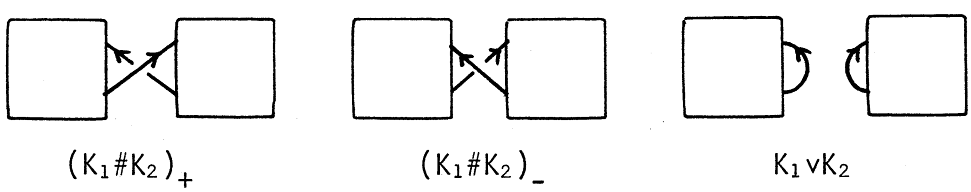



FIGURE 36.

Then

$$l(K_1\#K_2)(l,m) + l^{-1}(K_1\#K_2)(l,m) + m(K_1 \vee K_2)(l,m) = 0$$

$$((K_1\#K_2)(l,m))(l + l^{-1}) + m \cdot (-\frac{l+l'}{m}) \cdot K_1(l,m) \cdot K_2(l,m) = 0.$$

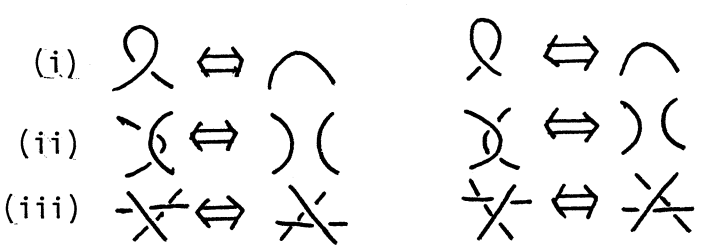
Hence

$$(K_1 \# K_2)(l,m) = K_1(l,m)K_2(l,m).$$

(9) It is shown in [14] that there exist pairs of knots with the same generalized polynomial but different genus, different unknotting numbers, and different signatures. There also exists a knot (with 11 crossings) whose mirror image can not be distinguished by the Jones polynomial but is distinguished by the generalized polynomial.

**(Existence and uniqueness of the generalized polynomial)**

Here we recall the following well-known fact: if $K$ and $K'$ are isotopic projections of a link, then $K$ can be deformed to $K'$ by a sequence of three Reidemeister moves:




*Proof of Theorem 8.1.* [14]

We use induction on the number of crossings in the link projections to prove the existence and uniqueness of the generalized polynormial. Let $\mathcal{L}_n$ be the set of all oriented link projections with the number of crossings less than or equal to $n$. We assume that each element of $\mathcal{L}_n$ is ordered and based, thus two projections, with different ordering or basepoints of the same projection, are regarded as distinct elements of $\mathcal{L}_n$.

Induction hypothesis $(n-1)$: (1) Assume that to each $K \in \mathcal{L}_{n-1}$, there is an associated element $\mathcal{P}(K) \in \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$ which is independent of a choice of base points and ordering of components, is independent under Reidemeister moves in

$\mathcal{L}_{n-1}$, and satisfies the identity $(*)$. (2) If $\mathcal{U}^c$ is an ascending projection of $c$ components, then $\mathcal{P}(\mathcal{U}^c) = \mu^{c-1}$, where $\mu = -(l + l^{-1})m^{-1}$.

The induction hypothesis (0) is automatically satisfied if we define $\mathcal{P}(K) = \mu^{c-1}$, where $K \in \mathcal{L}_0$ and $K$ has $c$ components. Note that every element of $\mathcal{L}_0$ is an ascending projection.

Assume induction hypothesis $(n-1)$. Suppose that $K \in \mathcal{L}_n$. If $K$ is an ascending projection, then define $\mathcal{P}(K) = \mu^{c-1}$, where $c$ is the number of components in $K$. If $K$ is not an ascending projection (with prescribed base points and an ordering of components), then $K = \sigma_k \sigma_{k-1} \cdots \sigma_1 \alpha(K)$, where $\alpha(K)$ is the standard ascending projection associated to $K$. Here we allow that a crossing is changed more than once. Supposing that $\mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K))$ is already defined, let

$$\mathcal{P}(\sigma_i \cdots \sigma_1 \alpha(K)) = -l^{2\varepsilon_i} \mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K)) - l^{\varepsilon_i} m \mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K)).$$

Notice that $\mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K))$ has been uniquely defined since

$$\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K) \in \mathcal{L}_{n-1}.$$

So the above equation defines $\mathcal{P}(K)$ recursively for each $K \in \mathcal{L}_n$.

Assuming that $\mathcal{P}(K)$ satisfies the induction hypothesis $(n)$, we finish the proof of the theorem. Let $K$ and $K'$ be isotopic projections. Then there exists a large integer $n_0$ such that $K, K' \in \mathcal{L}_{n_0}$, and Reidemeister moves deform $K$ to $K'$ in $\mathcal{L}_{n_0}$. By the induction hypothesis $(n_0)$, $\mathcal{P}(K) = \mathcal{P}(K')$. Furthermore, $\mathcal{P}(K)$ satisfies $(*)$, and $\mathcal{P}(\mathcal{U}) = 1$.

We prove the induction hypothesis $(n)$ by a sequence of five lemmas.

**Lemma 8.2.** *Let $K \in \mathcal{L}_n$ and $K = \sigma_k \cdots \sigma_1 \alpha(K)$. Then $\mathcal{P}(K)$ does not depend on a choice of crossings and the order of operations $\sigma_i$.*

*Proof.* We first show that the interchange of two adjacent operations, $\sigma_{i+1}$ and $\sigma_i$, in $\sigma_k \cdots \sigma_1 \alpha(K)$ does not change $\mathcal{P}(K)$. Let $L = \sigma_{i-1} \cdots \sigma_1 \alpha(K)$.

(1) First assume that the crossings labelled $i$ and $i+1$ are distinct. Then

$$\begin{aligned}\mathcal{P}(\sigma_{i+1}\sigma_i L) &= -l^{2\varepsilon_{i+1}}\mathcal{P}(\sigma_i L) - l^{\varepsilon_{i+1}} m\mathcal{P}(\eta_{i+1}\sigma_i L)\\ &= -l^{2\varepsilon_{i+1}}[-l^{2\varepsilon_i}\mathcal{P}(L) - l^{\varepsilon_i} m\mathcal{P}(\eta_i L)]\\ &\quad - l^{\varepsilon_{i+1}} m[-l^{2\varepsilon_i}\mathcal{P}(\eta_{i+1}L) - l^{\varepsilon_i} m\mathcal{P}(\eta_i\eta_{i+1}L)]\\ &= l^{2(\varepsilon_i+\varepsilon_{i+1})}\mathcal{P}(L) + l^{2\varepsilon_{i+1}+\varepsilon_i} m\mathcal{P}(\eta_i L) + l^{\varepsilon_{i+1}+2\varepsilon_i} m\mathcal{P}(\eta_{i+1}L)\\ &\quad + l^{\varepsilon_{i+1}+\varepsilon_i} m^2\mathcal{P}(\eta_i\eta_{i+1}L).\end{aligned}$$

The expression is symmetric in $i$ and $i+1$. Hence

$$\mathcal{P}(\sigma_{i+1}\sigma_i L) = \mathcal{P}(\sigma_i\sigma_{i+1}L)$$

by the induction hypothesis $(n-1)$. Then, again by the induction hypothesis,

$$\mathcal{P}(\sigma_k\cdots\sigma_{i+1}\sigma_i\cdots\sigma_1\alpha(K)) = \mathcal{P}(\sigma_k\cdots\sigma_i\sigma_{i+1}\cdots\sigma_1\alpha(K)).$$

(2) Assume that the $i^{th}$ crossing is the same as the $(i+1)st$ crossing. Then

$$\mathcal{P}(\sigma_{i+1}\sigma_i L) = l^{2(\varepsilon_{i+1}+\varepsilon_i)}\mathcal{P}(L) + l^{2\varepsilon_{i+1}+\varepsilon_i} m\mathcal{P}(\eta_i L) - l^{\varepsilon_{i+1}} m\mathcal{P}(\eta_i L).$$

Since

$$\varepsilon_{i+1} = -\varepsilon_i, \quad \mathcal{P}(\sigma_{i+1}\sigma_i L) = \mathcal{P}(L).$$

Hence

$$\begin{aligned}\mathcal{P}(\sigma_k\cdots\sigma_{i+1}\sigma_i\cdots\sigma_1\alpha(K)) &= \mathcal{P}(\sigma_k\cdots\sigma_i\sigma_{i+1}\cdots\sigma_1\alpha(K))\\ &= \mathcal{P}(\sigma_k\cdots\sigma_{i+2}\sigma_{i-1}\cdots\sigma_1\alpha(K)).\end{aligned}$$

Now suppose that $\{\overline{1},\overline{2},\cdots,\overline{j}\}$ is the minimal set of crossings necessary to be changed once to obtain $K$ from $\alpha(K)$. Then for any set of crossings $1,2,\cdots,k$ with $K=\sigma_k\cdots\sigma_1\alpha(K)$, (1) and (2) imply

$$\mathcal{P}(\sigma_{\overline{j}}\cdots\sigma_{\overline{1}}\alpha(K)) = \mathcal{P}(\sigma_k\cdots\sigma_1\alpha(K)),$$

thus showing the independence of $\mathcal{P}(K)$ on a choice of crossings.

**Lemma 8.3.** *Let $K \in \mathcal{L}_n$. Then $\mathcal{P}(K)$ does not depend on the choice of base points.*

*Proof.* It suffices to show that the base point of a component can be moved accross a crossing leaving the polynomial unchanged. Let $b_1$ and $b_2$ be base points in component $c_i$ accross a crossing, where the crossing is between components $c_i$ and $c_j$. Let $K_1$ and $K_2$ be the projections corresponding to the base points $b_1$ and $b_2$, respectively. If $i \neq j$, then $\alpha(K_1) = \alpha(K_2)$. Hence the polynomial is unchanged. Suppose that $i = j$. By interchanging the role of $b_1$ and $b_2$, we may assume that the direction from $b_1$ to $b_2$ agrees with the orientation of $c_i$. As shown in Figure 37, $\alpha(K_1)$ differ from $\alpha(K_2)$ only at one crossing, say crossing $k$.

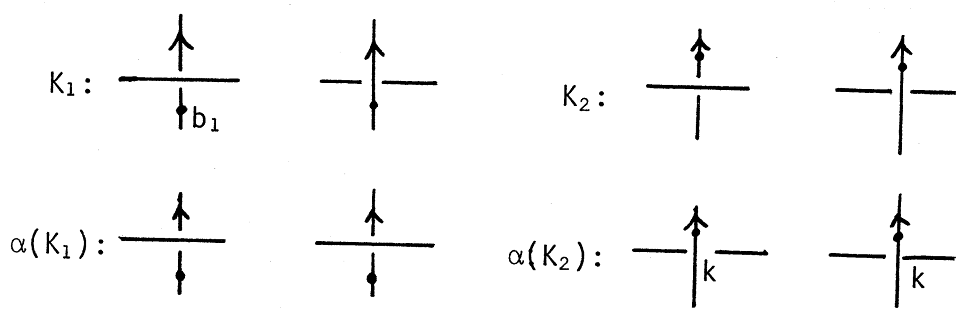



FIGURE 37.

Therefore, if $K_1 = \sigma_j \cdots \sigma_1 \alpha(K_1)$, then $K_2 = \sigma_j \cdots \sigma_1 \sigma_k \alpha(K_2)$. To prove $\mathcal{P}(K_1) = \mathcal{P}(K_2)$, it suffices to show that

$$\mathcal{P}(\sigma_k \alpha(K_2)) = \mathcal{P}(\alpha(K_1)) = \mu^{c-1},$$

where $c$ is the number of components in $K_1$. Using the observation that $\eta_k \alpha(K_2)$ is the trivial link of $c+1$ components with less than $n$ crossings, we have

$$\begin{aligned}\mathcal{P}(\sigma_k \alpha(K_2)) &= -l^{2\varepsilon_k} \mu^{c-1} - l^{\varepsilon_k} m \mu^c \\ &= \mu^{c-1}(-l^{2\varepsilon_k} - l^{\varepsilon_k}(-l - l^{-1})) \\ &= \mu^{c-1}(-l^{2\varepsilon_k} + l^{\varepsilon_k + 1} + l^{\varepsilon_k - 1}) \\ &= \mu^{c-1}.\end{aligned}$$

**Lemma 8.4.** *The identity* $(*)$ *holds for links in* $\mathcal{L}_n$.

*Proof.* First note that $K_+$ and $K_-$ have the same standard ascending projection, say $L$. We label the crossing in $L$ at which $K_+$ differ from $K_-$ by 1. There are two cases to consider:

(1) $K_+$ agrees with $\alpha(K_+)$ at the 1st crossing.

(2) $K_+$ does not agree with $\alpha(K_+)$ at the 1st crossing.

We only give an argument for the 1st Case since the 2nd Case is about the same. Now there exist crossings labelled $2, \cdots, k$ in $\alpha(K_+)$ such that $K_+ = \sigma_k \cdots \sigma_2 \alpha(K)$. Then $K_- = \sigma_1 \sigma_k \cdots \sigma_2 \alpha(K)$. By definition,

$$\mathcal{P}(K_-) = -l^2 \mathcal{P}(K_+) - lm\mathcal{P}(\eta_1 K_+).$$

Since $\eta_1 K_+ = K_0$,

$$l\mathcal{P}(K_+) + l^{-1}\mathcal{P}(K_-) + m\mathcal{P}(K_0) = 0.$$

**Lemma 8.5.** *For links in* $\mathcal{L}_n$, $\mathcal{P}(K)$ *is invariant under Reidemeister moves which do not increase the number of crossings beyond* $n$.

*Proof.*

(i) First Reidemeister move: Put a base point as in Figure 38.

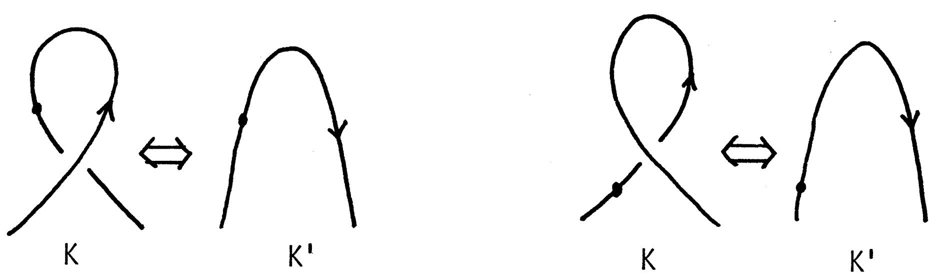



FIGURE 38.

Then $\alpha(K)$ is the same as $K$ at the crossing in the figure. There exist crossings $1, 2, \cdots, k$ different from this crossing in $\alpha(K)$ such that $\sigma_k \cdots \sigma_1 \alpha(K) = K$. Furthermore, $\sigma_k \cdots \sigma_1 \alpha(K') = K'$. By induction hypothesis

$$\mathcal{P}(\alpha(K)) = \mathcal{P}(\alpha(K')) = \mu^{c-1},$$

where $c$ is the number of components in $K$. Suppose that

$$\mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K)) = \mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K')).$$

By the induction hypothesis,

$$\mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K)) = \mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K')).$$

So

$$\begin{aligned} P(\sigma_i \cdots \sigma_1 \alpha(K)) &= -l^{2\varepsilon_i} \mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K)) - l^{\varepsilon_i} m \mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K)) \\ &= -l^{2\varepsilon_i} \mathcal{P}(\sigma_{i-1} \cdots \sigma_1 \alpha(K')) - l^{\varepsilon_i} m \mathcal{P}(\eta_i \sigma_{i-1} \cdots \sigma_1 \alpha(K')) \\ &= \mathcal{P}(\sigma_i \cdots \sigma_1 \alpha(K')). \end{aligned}$$

Hence by inducting on $i$, we obtain $\mathcal{P}(K) = \mathcal{P}(K')$.

(ii) Second Reidemeister move: We first introduce notation. Let $a_1$ and $a_2$ be disjoint arcs in a link $K$. Then we say that $a_1 < a_2$ if we encounter $a_1$ before $a_2$ when we travel $K$ according to the orientation, ordering and the choice of base points. In Figure 39, 2nd Reidemeister moves are described between two arcs $a_1$ and $a_2$. If $a_1 < a_2$, then there exist crossings $1, 2, \cdots, k$ in $\alpha(K)$ distinct from the crossings $p$ and $q$ such that $K = \sigma_k \cdots \sigma_1 \alpha(K)$ and $K' = \sigma_k \cdots \sigma_1 \alpha(K')$. Using the induction hypothesis, it follows that $\mathcal{P}(K) = \mathcal{P}(K')$ as in (i).

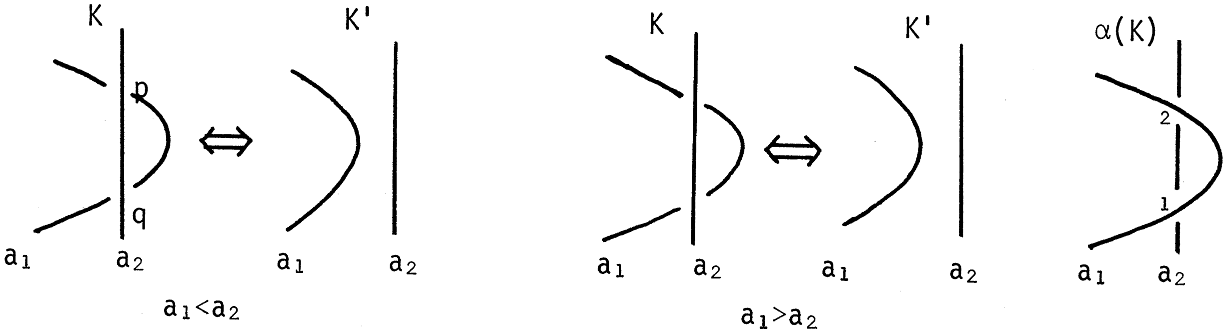



FIGURE 39.

Suppose that $a_1 > a_2$. Then there exist crossings labelled $3, 4, \cdots k$ in $\alpha(K)$ such that $K = \sigma_k \cdots \sigma_3\sigma_2\sigma_1\alpha(K)$ and $K' = \sigma_k \cdots \sigma_3\alpha(K')$. Hence it suffices to show that $\mathcal{P}(\sigma_2\sigma_1\alpha(K)) = \mu^{c-1}$, where $c$ is the number of components in $K$.

$$\begin{aligned}\mathcal{P}(\sigma_2\sigma_1\alpha(K)) &= l^{2(\varepsilon_2+\varepsilon_1)}\mu^{c-1} + l^{2\varepsilon_2+\varepsilon_1}m\mathcal{P}(\eta_1\alpha(K)) - l^{\varepsilon_2}m\mathcal{P}(\eta_2\sigma_1\alpha(K)) \\ &= \mu^{c-1} + l^{\varepsilon_2}m(\mathcal{P}(\eta_1\alpha(K)) - \mathcal{P}(\eta_2\sigma_1\alpha(K)))\end{aligned}$$

since $\varepsilon_1 \cdot \varepsilon_2 = -1$.

By the induction hypothesis, it follows that $\mathcal{P}(\eta_1\alpha(K)) = \mathcal{P}(\eta_2\sigma_1\alpha(K))$ as shown in Figure 40 and thus $\mathcal{P}(\sigma_2\sigma_1\alpha(K)) = \mu^{c-1}$.

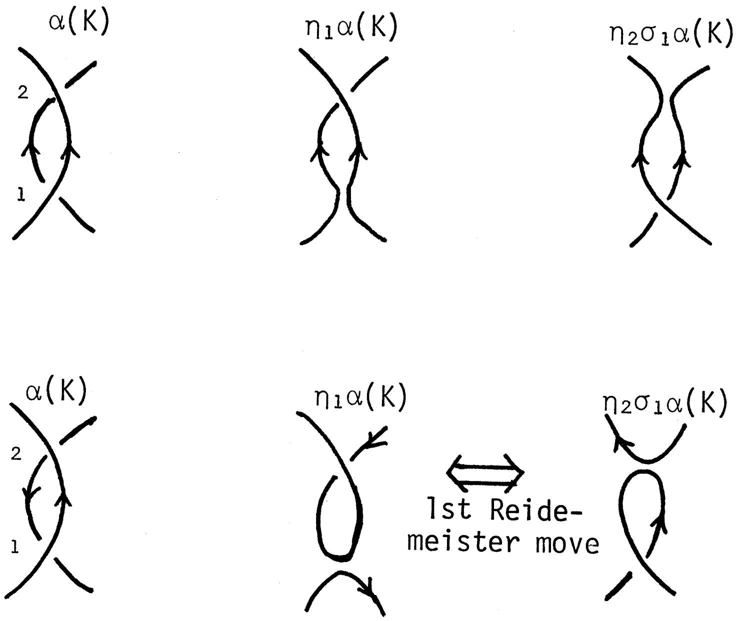



Figure 40.

(iii) Third Reidemeister move: Let $K \in \mathcal{L}_n$, and $a_1, a_2$ and $a_3$ be the three arcs involved in the Reidemeister move. Let $\tau K$ be the result of the move as in Figure 41. We first prove that it suffices to show that $\mathcal{P}(K) = \mathcal{P}(\tau K)$ assuming $a_1 < a_2 < a_3$.

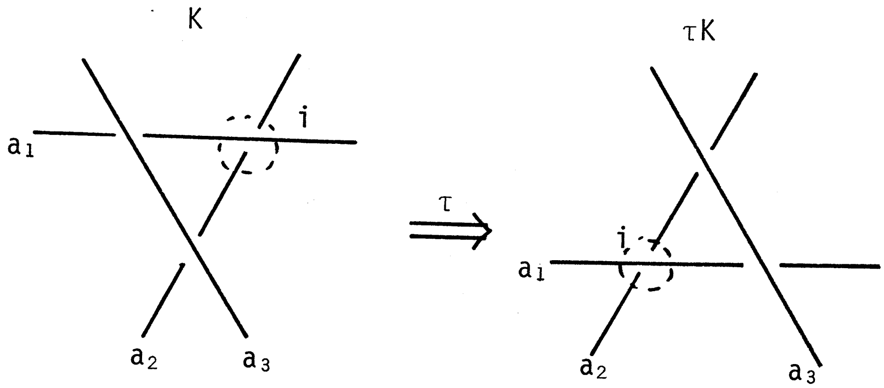



FIGURE 41.

Let $i$ be the circled crossing in Figure 41. Then

$$\mathcal{P}(\sigma_i K) = -l^{2\varepsilon_i}\mathcal{P}(K) - l^{\varepsilon_i} m \mathcal{P}(\eta_i K),$$
$$\mathcal{P}(\sigma_i \tau K) = -l^{2\varepsilon_i}\mathcal{P}(\tau K) - l^{\varepsilon_i} m \mathcal{P}(\eta_i \tau K),$$
$$\mathcal{P}(\sigma_i \tau K) = \mathcal{P}(\tau \sigma_i K).$$

And by Figure 42,

$$\mathcal{P}(\eta_i K) = \mathcal{P}(\eta_i \tau K).$$

Therefore, $\mathcal{P}(K) = \mathcal{P}(\tau K)$ if and only if $\mathcal{P}(\sigma_i K) = \mathcal{P}(\tau \sigma_i K)$. This observation implies that it suffices to show $\mathcal{P}(K) = \mathcal{P}(\tau K)$ assuming $a_1 < a_2 < a_3$.

Suppose $a_1 < a_2 < a_3$. Then $K$, $\tau K$, $\alpha(K)$ and $\alpha(\tau K)$ are identical in the support of $\tau$. Hence there exist a set of crossings $1, 2, \cdots, k$ outside of the support of $\tau$ such that $K = \sigma_k \cdots \sigma_1 \alpha(K)$ and $\tau K = \sigma_k \cdots \sigma_1 \alpha(\tau K)$. Now induct on the number of crossing changes using the induction hypothesis as in (i) to finish the proof.

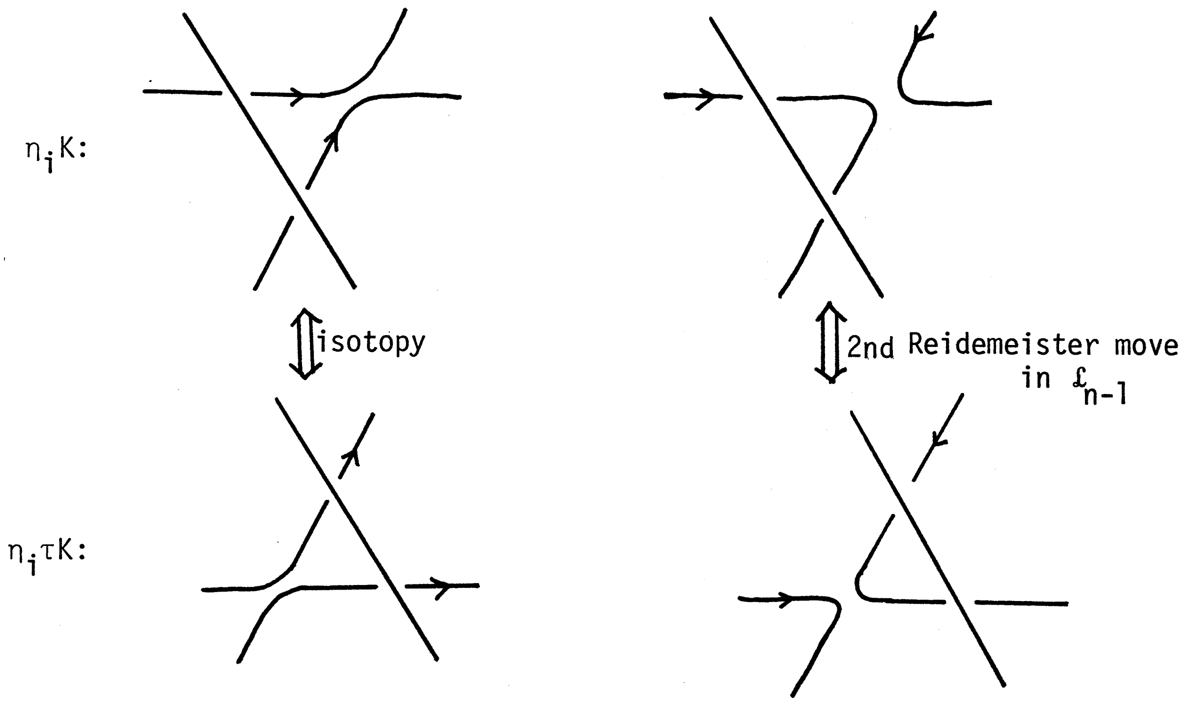



Figure 42.

We define a loop in a link projection as a simple closed curve which is a projection of a sub-arc of the link (Figure 43); a loop could be the projection of a whole component of the link with no self crossing, or it has a double point as endpoints; but a loop may have many crossings.

**Lemma 8.6.** *Let $K \in \mathcal{L}_n$. Then $\mathcal{P}(K)$ is independent of the ordering of components of $K$.*

*Proof.* Let $K \in \mathcal{L}_n$ and $K'$ be the link $K$ with a different ordering of components of $K$. We need to show $\mathcal{P}(K) = \mathcal{P}(K')$.

Suppose that $K = \bigvee_{1\le l\le m} K_l$, a disjoint union of connected diagrams, where each $K_l$ is connected when projected onto the plane and it does not have any common crossings with $K_{l'}$ if $l \neq l'$. Then from the definition, it follows that

$$\mathcal{P}(K) = \mu^{m-1}\Pi_{1\le l\le m}\mathcal{P}(K_l).$$

Furthermore,

$$K' = \vee_{1\le l\le m}K'_l \quad \text{and} \quad \mathcal{P}(K') = \mu^{m-1}\Pi_{1\le i\le m}\mathcal{P}(K'_l).$$

Hence it suffices to prove $\mathcal{P}(K_l) = \mathcal{P}(K'_l)$ for each $l$.

Assume that $K$ is a connected diagram. Suppose that

$$K = \sigma_k\cdots\sigma_1\alpha(K), \quad \sigma(K') = \sigma_{\bar{j}}\cdots\sigma_{\bar{1}}\alpha(K) \quad \text{and} \quad K' = \sigma_{i'}\cdots\sigma_{1'}\alpha(K').$$

Then we have

$$\begin{array}{ccc} \alpha(K) & \xrightarrow{\sigma_k\cdots\sigma_1} & K \\ {\scriptstyle \sigma_{\bar{j}}\cdots\sigma_{\bar{1}}}\big\downarrow & & \\ \alpha(K') & \xrightarrow{\sigma_{i'}\cdots\sigma_{1'}} & K' \end{array}$$

So

$$\mathcal{P}(K) = \mathcal{P}(\sigma_k\cdots\sigma_1\alpha(K)), \quad \mathcal{P}(K') = \mathcal{P}(\sigma_{i'}\cdots\sigma_{1'}\alpha(K')).$$

Since $\sigma_{i'}\cdots\sigma_{1'}\sigma_{\bar{j}}\cdots\sigma_{\bar{1}}\alpha(K) = K$ with the ordering ignored, by Lemma 8.2,

$$\mathcal{P}(K) = \mathcal{P}(\sigma_{i'}\cdots\sigma_{1'}\sigma_{\bar{j}}\cdots\sigma_{\bar{1}}\alpha(K)).$$

Therefore, to show $\mathcal{P}(K) = \mathcal{P}(K')$, it suffices to prove $\mathcal{P}(\sigma_{\bar{j}}\cdots\sigma_{\bar{1}}\alpha(K)) = \mu^{c-1}$, where $c$ is the number of components in $K$.

In $\alpha(K')$, let $L$ be a minimal loop in the sense that it does not contain any other loops in its interior. If $L$ has a double point ($L$ is not a projection of a component.), we denote the point by $p$. If $L$ has no crossing other than $p$, then $L$ is a loop with a single crossing $p$ since the diagram of $K$ is connected. In this case, eliminate the loop by the 1st Reidemeister move to get a projection $K''$ with less than $n$ crossings. By Lemma 8.5,

$$\mathcal{P}(\alpha(K')) = \mathcal{P}(\sigma_{\bar{j}} \cdots \sigma_{\bar{1}}\alpha(K)) = \mathcal{P}(\alpha(K'')) = \mu^{c-1}$$

by the induction hypothesis.

If $L$ has no double point, let $p$ be an arbitrary point on $L$ away from the crossings on $L$. We assume now $L$ has crossings with other arcs of $\alpha(K')$. Let $D$ be the disk which $L$ bounds. Near $D$, $\alpha(K')$ is a union of $L$ and short arcs. If no pair of these short arcs intersect more than once in $D$, then there exists a short arc $t$ whose crossing points with $L$ are the farthest from $p$, i.e., any other short arc has at least one crossing point with $L$ closer to $p$ than one of the crossing points of $t$ with $L$. Since $\alpha(K')$ is an ascending projection, we can push $t$ off $L$ away from $p$, not increasing the number of crossings, by the 2nd and 3rd Reidemeister moves. By the induction hypothesis and Lemma 8.5, $\mathcal{P}(\sigma_{\bar{j}} \cdots \sigma_{\bar{1}}\alpha(K)) = \mu^{c-1}$. So finally, we assume that there exists a pair of short arcs intersecting more than once in $D$. Such pairs bound disks in $D$. Choose a minimal such disk, say $D'$, i.e., $D'$ does not contain any other disk of this type (see Figure 43.)

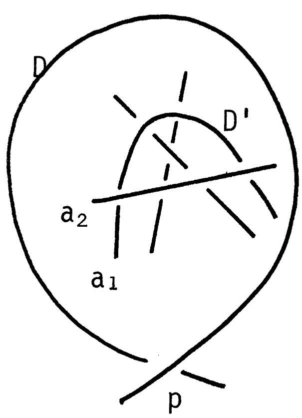



FIGURE 43.

Let $a_1$ and $a_2$ be the arcs bounding $D'$. By 2nd and 3rd Reidemeister moves, we can eliminate the crossings between $a_1$ and $a_2$ to reduce the number of crossings from $\alpha(K')$. It follows that

$$\mathcal{P}(\sigma_{\bar{j}} \cdots \sigma_{\bar{1}} \alpha(K)) = \mu^{c-1}.$$

This finishes the proof of the lemma.

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