수 학 강 의 록 제 32 권

# ON VERTEX OPERATOR ALGEBRAS

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## Lecture Notes

# On Vertex Operator Algebras

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# Introduction

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The phenomena of vertex operator algebras have existed both in physical field theory and in infinite-dimensional Lie theory for several decades. The concept of a vertex algebra was proposed by Borcherds [B], and the slightly revised concept "a vertex operator algebra" was formulated by Frenkel, Lepowsky and Meurman [FLM1-3]. Their motivations were to solve the problem of the moonshine representation of the Monster, the largest simple finite sporadic group. Vertex operator algebras are fundamental algebraic structures in conformal field theory, which plays important roles in statistical mechanics and string theory.

Algebraically, vertex operator algebras are natural one-variable analogues of commmutative associative algebras with an identity element. They are also natural generalizations of affine Lie algebras. In certain sense, Lie algebras and commutative algebras are reconciled in vertex operator algebras. The aim of these lectures on vertex operator algebras is to introduce certain important techniques and examples in vertex operator algebras so that nonexperts may get into this field. In Chapter 1, we present a new definition of vertex operator algebras in terms of mathematically rigorous version of the "duality" in physics and prove its equivalence to the definition given by Frenkel, Lepwosky and Meurman. This equivalence could be used to simplify many existing proofs of the Jacobi identity for various vertex operator algebras. In Chapter 2, we give the construction of the vertex operator algebras and modules associated with affine Lie algebras in the spirit of our new definition. In Chapter 3, we present the construction of vertex operator algebras and modules from integral even lattices.

# Chapter 1

In this chapter, we shall present a new definition of a vertex operator algebra and prove its equivalence to the one given by Frenkel, Lepowsky and Meurman ([FLM3]).

## 1. Calculus of Formal Variables

One of the key techniques in infinite-dimensional algebras is using of generating functions, where the formal delta function plays fundamental roles. In these notes, all the variables are formal and commute with each other. All the vector spaces are assumed over $\mathbf{C}$, the field of complex numbers. We denote by $\mathbf{Z}$ the ring of integers and by $\mathbf{Q}$ the field of rational numbers.

Let $z_1$ and $z_2$ be two formal variables. We have the following convention of binomial expansions:

$$(z_1 - z_2)^n = \sum_{l=0}^{\infty} \binom{n}{l} (-1)^l \frac{z_2^l}{z_1^{-n+l}} \qquad \text{for } n \in \mathbf{Z}. \tag{1.1}$$

In other words, the notion "$(z_1 - z_2)^n$" is always interpreted as the above formal series. In particular,

$$(z_1 - z_2)^n = (-z_2 + z_1)^n \qquad \text{if and only if } n \geq 0. \tag{1.2}$$

It can be proved that

$$n(z_1 - z_2)^{n-1} = \partial_{z_1}(z_1 - z_2)^n = \sum_{l=0}^{\infty} (-1)^l \binom{n}{l} l \frac{z_2^{l-1}}{z_1^{l-n}} = -\partial_{z_2}(z_1 - z_2)^n. \tag{1.3}$$

We define the "formal delta function" by

$$\delta(z) = \sum_{l \in \mathbf{Z}} z^l. \tag{1.4}$$

In terms of the delta function, we have the following "expansion of zero:"

$$\frac{1}{z_1 - z_2} - \frac{1}{z_2 - z_1} = z_2^{-1} \delta\left(\frac{z_1}{z_2}\right). \tag{1.5}$$

Let $V$ be a vector space and let

$$V[[z^{-1}, z]] = \{\sum_{n \in \mathbf{Z}} v_n z^n \mid v_n \in V\}, \;\; V[z^{-1}, z]] = \{\sum_{n=n_0}^{\infty} v_n z^n \mid v_n \in V, n_0 \in \mathbf{Z}\} \tag{1.6}$$

and $V[z^{-1},z]$ be the set of formal Laurent series with coefficients in $V$, the set of formal Laurent series truncated below and the set of Laurent polynomials, respectively. The product $f(z)g(z)$ of a Laurent series $f(z) \in \mathbf{C}[[z^{-1},z]]$ and $g(z) \in V[[z^{-1},z]]$ can be defined as usual only if they are truncated from the same direction. The product of a Laurent polynomial in $\mathbf{C}[z^{-1},z]$ with any Laurent series makes sense. For instance, for $f(z_1,z_2) \in \mathbf{C}[z_1^{-1},z_2^{-1};z_1,z_2]$, we have:

$$\delta\left(\frac{z_1}{z_2}\right) f(z_1,z_2) = \delta\left(\frac{z_1}{z_2}\right) f(z_1,z_1) = \delta\left(\frac{z_1}{z_2}\right) f(z_2,z_2), \tag{1.7}$$

because of

$$\delta\left(\frac{z_1}{z_2}\right) z_1^n = \sum_{m\in\mathbf{Z}} z_1^{m+n}/z_2^m \overset{l=m+n}{=} \sum_{l\in\mathbf{Z}} z_1^l/z_2^{l-n} = \delta\left(\frac{z_1}{z_2}\right) z_2^n, \qquad \text{for } n \in \mathbf{Z}. \tag{1.8}$$

**Lemma 1.1.** *We have the following properties on the delta function:*

$$z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) = z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \quad \partial_{z_1}\left[z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\right] = -\partial_{z_2}\left[z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\right]. \tag{1.9}$$

*Proof.* Note that

$$z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) = \sum_{n\in\mathbf{Z}} z_1^n/z_2^{n+1} \overset{m=-n-1}{=} \sum_{m\in\mathbf{Z}} z_2^m/z_1^{m+1} = z_1^{-1}\delta\left(\frac{z_2}{z_1}\right), \tag{1.10}$$

$$\partial_{z_1}\left[z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\right] = \sum_{n\in\mathbf{Z}} n z_1^{n-1}/z_2^{n+1} \overset{m=n-1}{=} \sum_{n\in\mathbf{Z}}(m+1)z_1^m/z_2^{m+2} = -\partial_{z_2}\left[z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\right]. \quad \square \tag{1.11}$$

**Lemma 1.2** *For $f(z) \in V[[z^{-1},z]]$, the following analogous Taylor's expansion holds:*

$$e^{z_0\frac{d}{dz}}(f(z)) = f(z+z_0). \tag{1.12}$$

*Proof.* By the fact that

$$e^{z_0\frac{d}{dz}}(z^n) = \sum_{m=0}^{\infty} \frac{n(n-1)\cdots(n-m+1)}{m!} z_0^m z^{n-m} = (z+z_0)^n, \qquad n \in \mathbf{Z}. \quad \square \tag{1.13}$$

**Lemma 1.3.** *The following equation on the delta function holds:*

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right). \tag{1.14}$$

*Proof.* By (1.1) and (1.2) and Lemma 1.1,

$$\begin{aligned}
& z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) \\
= & \sum_{n\in\mathbf{Z}}[(z_1-z_2)^n - (-z_2+z_1)^n]z_0^{-n-1} \\
= & \sum_{m=0}^{\infty}[(z_1-z_2)^{-m-1} - (-z_2+z_1)^{-m-1}]z_0^m \\
= & \sum_{m=0}^{\infty}\frac{z_0^m}{m!}\partial_{z_2}^m[(z_1-z_2)^{-1} - (-z_2+z_1)^{-1}] \\
= & e^{z_0\partial_{z_2}}(z_2^{-1}\delta(z_1/z_2)) \\
= & e^{-z_0\partial_{z_1}}(z_2^{-1}\delta(z_1/z_2)) \\
= & z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right). \qquad \square
\end{aligned} \tag{1.15}$$

Note that we also have:

$$z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) = e^{-z_0\partial_{z_1}}[z_2^{-1}\delta(z_1/z_2)] = e^{z_0\partial_{z_2}}[z_1^{-1}\delta(z_2/z_1)] = z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) \tag{1.16}$$

by Lemma 1.1.

## 2. Definitions of a Vertex Operator Algebras

The following definition of a vertex operator algebra was given in [FLM3]. *A vertex operator algebra is a* $\mathbf{Z}$*-graded vector space* $V = \bigoplus_{n\in\mathbf{Z}} V^{(n)}$ *(graded by weights) equipped with a linear map* $Y(\cdot,z):\ V\to(\mathrm{End}\,V)[[z,z^{-1}]]$ *such that for any* $u,\ v\in V$,

$$u_n v = 0 \qquad \text{for } n \text{ sufficiently large,} \tag{1.17}$$

*where* $Y(u,z)=\sum_{n\in\mathbf{Z}} u_n z^{-n-1}$*; the Jacobi identity holds:*

$$\begin{aligned}
& z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) \\
= & z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2);
\end{aligned} \tag{1.18}$$

*there are two distinguished elements* $\mathbf{1},\ \omega\in V$ *satisfying*

$$Y(\mathbf{1},z)v=v,\quad Y(v,z)\mathbf{1}\in V[[z]] \text{ and } \lim_{z\to 0}Y(v,z)\mathbf{1}=v \ \ (\text{creation property}); \tag{1.19}$$

$$L(m)L(n)-L(n)L(m)=(m-n)L(m+n)+\frac{1}{12}(m^3-m)\delta_{m+n,0}(\mathrm{rank}\,V), \tag{1.20}$$

$$\frac{d}{dz}Y(v,z) = Y(L(-1)v,z) \tag{1.21}$$

*and*

$$L(0)w = lw = (\textit{weight}\, w)w \ \ \textit{for} \ \ l \in \mathbf{Z} \ \ \textit{and} \ \ w \in V^{(l)}, \tag{1.22}$$

*where* $Y(\omega, z) = \sum_{n\in\mathbf{Z}} L(n)z^{-n-2}$ *and* $\textit{rank}\, V \in \mathbf{C}$. This completes the definition. We may denote the vertex operator algebra as $(V, Y, \mathbf{1}, \omega)$. Here we have relaxed two less important conditions in FLM's original definition. Below, we shall give a new definition. It is our wish that the new definition would be easier to be understood by nonexperts.

Traditionally, an *algebra* $\mathcal{A}$ is a vector space with a bilinear map $m(\cdot,\cdot) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. For each $a \in \mathcal{A}$, we define the left multiplication operator $L_a \in \mathrm{End}\,\mathcal{A}$ by $L_a(b) = m(a,b)$ for $b \in \mathcal{A}$. Note that we have a linear map $L : \mathcal{A} \to \mathrm{End}\,\mathcal{A}$ defined by $a \mapsto L_a$ for $a \in \mathcal{A}$. Thus we can use such a map $L$ to define an algebra. An element $\mathbf{1} \in \mathcal{A}$ is called an *identity element* if

$$L_a(\mathbf{1}) = L_{\mathbf{1}}(a) = a \qquad \text{for any} \ \ a \in \mathcal{A}. \tag{1.23}$$

One can easily verify that if $\mathcal{A}$ has an identity element, then it is unique. An algebra $\mathcal{A}$ is called *associative* if

$$L_a L_b = L_{L_a(b)} \qquad \text{for any} \ \ a, b \in \mathcal{A}. \tag{1.24}$$

An algebra $\mathcal{A}$ is called *commutative* if

$$L_a(b) = L_b(a) \qquad \text{for any} \ \ a, b \in \mathcal{A}. \tag{1.25}$$

The simplest interesting algebra is a commutative associative algebra with an identity element $\mathbf{1}$. If an algebra $\mathcal{A}$ contains an identity element, then the commutativity (1.25) is equivalent to

$$L_a L_b = L_b L_a \qquad \text{for} \ \ a, b \in \mathcal{A}. \tag{1.26}$$

As the traditional algebraic theory becomes more and more mature, one naturally thinks about establishing a one-variable algebraic theory. Of course, the phenomena of a one-variable algebraic theory have existed in physical field theory and infinite-dimensional Lie algebras for a long time. A "quantum group" can be viewed as a one-variable algebra. But here we are tackling axiomatic approaches.

What are the simplest natural axioms for a one-variable algebra? The answers, of course, are "commutativity, associativity and identity element." Then the question is "in what senses?" For any two vector spaces $U$ and $W$, we denote by $\mathrm{LM}(U, W)$ the set of all linear maps from $U$ to $W$.

The following is our new definition of a vertex operator algebra.

*A vertex operator algebra is a family $(V, Y(\cdot, z), \mathbf{1}, \omega)$, where $V$ is a vector space, $\mathbf{1}$ and $\omega$ are two special elements of $V$, and $Y(\cdot, z): V \to LM(V, V[z^{-1}, z]])$ is a linear map satisfying the following condition: for $u, v \in V$,*

$$L(l)L(n) - L(n)L(l) = (l-n)L(l+n) + \frac{(l^3 - l)}{12}\delta_{l+n,0}(rank\, V), \quad rank\, V \in \mathbf{C}, \tag{1.27}$$

*where $L(z) = Y(\omega, z) = \sum_{n\in\mathbf{Z}} L(n)z^{-n-2}$,*

$$L(-1)Y(v,z) - Y(v,z)L(-1) = \frac{d}{dz}Y(v,z); \tag{1.28}$$

$$Y(\mathbf{1}, z)v = v, \quad Y(v,z)\mathbf{1} = e^{zL(-1)}v; \tag{1.29}$$

$$V = \bigoplus_{n\in\mathbf{Z}}\{w \in V \mid L(0)w_l - w_l L(0) = (n-l-1)w_l,\ Y(w,z) = \sum_{l\in\mathbf{Z}} w_l z^{-l-1}\}, \tag{1.30}$$

$$(z_1 - z_2)^m Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^m Y(v, z_2)Y(u, z_1) \tag{1.31}$$

*for some positive integer $m$.*

## 3. Equivalence of the Two Definitions

The major difference between FLM's definition and ours is their Jacobi identity (1.18) and our commutativity (1.31). In practical, it is much more difficult to prove (1.18) than to prove (1.31). In many existing proofs of the Jacobi identity, the eqivalence of these two definitions has been essentially implicitly repeated again and again. Our point of view is that the Jacobi identity is better serving as working identity rather than an axiom. The equivalence of these two definition shows a "reconciliation of Lie algebras and commutative algebras." Let us first prove that FLM's definition implies ours.

We introduce the operator of taking residue:

$$\mathrm{Res}_z = \text{taking the coefficient of } z^{-1} \text{ in an expression.} \tag{1.32}$$

For instance $\mathrm{Res}_{z_1}(z_1z_2^3 + z_1^{-1}z_2^7 - z_1^{-3}z^4) = z_2^7$.

For $u \in V$, by (1.18) and (1.21),

$$\begin{aligned}
&[L(-1), Y(u,z_2)] \\
&= L(-1)Y(u,z_2) - Y(u,z_2)L(-1) \\
&= \mathrm{Res}_{z_0,z_1} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(L(z_0)u, z_2) \\
&= \mathrm{Res}_{z_0,z_1} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y(L(z_0)u, z_2) \\
&= \mathrm{Res}_{z_0} Y(L(z_0)u, z_2) \\
&= Y(L(-1)u, z) \\
&= \frac{d}{dz_2} Y(u, z_2).
\end{aligned} \tag{1.35}$$

Hence our axiom (1.28) holds. Moreover, by (1.9), (1.16) and (1.18-19) ,

$$\begin{aligned}
&Y(u,z_0)\mathbf{1} \\
&= \lim_{z_2\to 0} Y(Y(u,z_0)\mathbf{1}, z_2)\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) Y(Y(u,z_0)\mathbf{1}, z_2)\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u,z_0)\mathbf{1}, z_2)\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} \left[z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1)Y(\mathbf{1},z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y(\mathbf{1},z_2)Y(u,z_1)\right]\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} \left[z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\right] Y(u,z_1)\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(u,z_1)\mathbf{1} \\
&= \lim_{z_2\to 0} \mathrm{Res}_{z_1} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) Y(u,z_2+z_0)\mathbf{1} \\
&= \lim_{z_2\to 0} Y(u,z_2+z_0)\mathbf{1} \\
&= \lim_{z_2\to 0} e^{z_0\frac{d}{dz_2}}(Y(u,z_2))\mathbf{1} \\
&= \lim_{z_2\to 0} e^{z_0 \mathrm{ad}_{L(-1)}}(Y(u,z_2))\mathbf{1} \\
&= \lim_{z_2\to 0} e^{z_0L(-1)}Y(u,z_2)e^{-z_0L(-1)}\mathbf{1} \\
&= \lim_{z_2\to 0} e^{z_0L(-1)}Y(u,z_2)\mathbf{1} \\
&= e^{z_0L(-1)}u,
\end{aligned} \tag{1.36}$$

where we have used the fact $L(-1)\mathbf{1} = 0$ obtained from the second expression in (1.19) because $L(z) = Y(\omega, z) = \sum_{n\in\mathbf{Z}} L(n)z^{-n-1}$. This proves the second equation in our axiom (1.29). Furthermore, by (1.16), (1.18) and (1.22),

$$\begin{aligned}
& [L(0), Y(u, z_2)] \\
= \;& L(0)Y(u, z_2) - Y(u, z_2)L(0) \\
= \;& \mathrm{Res}_{z_0,z_1} z_1 z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(L(z_0)u, z_2) \\
= \;& \mathrm{Res}_{z_0,z_1} z_1 z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y(L(z_0)u, z_2) \\
= \;& \mathrm{Res}_{z_0}(z_2 + z_0)Y(L(z_0)u, z_2) \\
= \;& z_2 Y(L(-1)u, z) + Y(L(0)u, z_2) \\
= \;& z_2 \frac{d}{dz_2} Y(u, z_2) + nY(u, z_2) \\
= \;& z_2 \sum_{l\in\mathbf{Z}}(-l-1)u_l z_1^{-l-2} + n\sum_{l\in\mathbf{Z}} u_l z_1^{-l-1} \\
= \;& \sum_{l\in\mathbf{Z}}(n-l-1)u_l z_1^{-l-1}
\end{aligned} \tag{1.37}$$

for $u \in V^{(n)}$. Thus our axiom (1.30) is satisfied.

For $u, v \in V$, we let $m$ be a positive integer such that

$$u_n(v) = 0 \quad \text{for} \quad n > m, \qquad\qquad Y(u, z) = \sum_{n\in\mathbf{Z}} u_n z^{n-1}. \tag{1.38}$$

By the Jacobi identity (1.18) and (1.3), (1.14), we have

$$\begin{aligned}
& [Y(u, z_1), Y(v, z_2)] \\
= \;& Y(u, z_1)Y(v, z_2) - Y(v, z_2)Y(u, z_1) \\
= \;& \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2) \\
= \;& \mathrm{Res}_{z_0}\left[z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2 - z_2}{-z_0}\right)\right] Y(Y(u, z_0)v, z_2) \\
= \;& \sum_{n\in Z} Y(u_n v, z_2)[(z_1 - z_2)^{-n-1} - (-z_2 + z_1)^{-n-1}] \\
= \;& \sum_{n=0}^{m} Y(u_n v, z_2)[(z_1 - z_2)^{-n-1} - (-z_2 + z_1)^{-n-1}].
\end{aligned} \tag{1.39}$$

Note that

$$(z_1 - z_2)^{m+1}[(z_1 - z_2)^{-n-1} - (-z_2 + z_1)^{-n-1}] = (z_1 - z_2)^{m-n} - (-z_2 + z_1)^{m-n} = 0 \tag{1.40}$$

for $n = 0, 1, ..., m$. Thus

$$(z_1 - z_2)^{m+1}[Y(u, z_1), Y(v, z_2)] = 0. \tag{1.41}$$

Thus our axiom (1.31) holds. Our remaining axioms are the same as in FLM's. Therefore we have proved that FLM's definition implies ours.

Next we shall show that our definition of a vertex operator algebra implies FLM's. We only need to prove the Jacobi identity and (1.21-22). Note that the second equation in (1.29) implies

$$u_{-1}\mathbf{1} = u, \;\; u_n\mathbf{1} = 0 \;\text{ for } u \in V,\; 0 \le n \in \mathbf{Z}, \qquad Y(u, z) = \sum_{l \in \mathbf{Z}} u_l z^{-l-1}. \tag{1.42}$$

Since

$$\sum_{n \in \mathbf{Z}} L(n) z^{-n-2} = L(z) = Y(\omega, z) = \sum_{n \in \mathbf{Z}} \omega_n z^{-n-1}, \tag{1.43}$$

we have $\omega_0 = L(-1)$ and $\omega_1 = L(0)$. Thus

$$L(-1)\mathbf{1} = 0 = L(0)\mathbf{1}. \tag{1.44}$$

Let

$$V^{(n)} = \{w \in V \mid [L(0), w_l] = (n - l - 1)w_l,\; Y(w, z) = \sum_{l \in \mathbf{Z}} w_l z^{-l-1}\}. \tag{1.45}$$

For $u \in V^{(n)}$, by (1.30), we have

$$L(0)u = L(0)u_{-1}\mathbf{1} = [L(0), u_{-1}]\mathbf{1} = (n - (-1) - 1)u_{-1}\mathbf{1} = nu. \tag{1.46}$$

Thus (1.22) holds.

For any $u \in V$, by (1.28) and (1.30), we have:

$$\begin{aligned} & \frac{d}{dz} Y(u, z)\mathbf{1} \\ = \;& [L(-1), Y(u, z)]\mathbf{1} \\ = \;& L(-1)Y(u, z)\mathbf{1} \\ = \;& L(-1)e^{zL(-1)}u \\ = \;& e^{zL(-1)}L(-1)u \\ = \;& Y(L(-1)u, z)\mathbf{1}, \end{aligned} \tag{1.48}$$

For $u, v \in V$, by (1.31), there exists a positive integer $m$ such that

$$(z_1 - z_2)^m [Y(u, z_1), Y(v, z_2)] = (z_1 - z_2)^m [Y(L(-1)u, z_1), Y(v, z_2)] = 0. \tag{1.49}$$

$$\begin{aligned}
& (z_1 - z_2)^{m+1} Y(L(-1)u, z_1) Y(v, z_2)\mathbf{1} \\
= \; & (z_1 - z_2)^{m+1} Y(v, z_2) Y(L(-1)u, z_1)\mathbf{1} \\
= \; & (z_1 - z_2)^{m+1} Y(v, z_2) \partial_{z_1} Y(u, z_1)\mathbf{1} \\
= \; & \partial_{z_1}[(z_1 - z_2)^{m+1} Y(v, z_2) Y(u, z_1)\mathbf{1}] - (m+1)(z_1 - z_2)^m Y(v, z_2) Y(u, z_1)\mathbf{1} \\
= \; & \partial_{z_1}[(z_1 - z_2)^{m+1} Y(u, z_1) Y(v, z_2)\mathbf{1}] - (m+1)(z_1 - z_2)^m Y(u, z_1) Y(v, z_2)\mathbf{1} \\
= \; & (z_1 - z_2)^{m+1} \left( \frac{d}{dz_1} Y(u, z_1) \right) Y(v, z_2)\mathbf{1},
\end{aligned} \tag{1.50}$$

$$\begin{aligned}
& (z_1 - z_2)^m e^{z_2 L(-1)} Y(u, z_1 - z_2) v \\
= \; & (z_1 - z_2)^m [e^{z_2 L(-1)} Y(u, z_1 - z_2) e^{-z_2 L(-1)}] e^{z_2 L(-1)} v \\
= \; & (z_1 - z_2)^m e^{z_2 \mathrm{ad}_{L(-1)}} [Y(u, z_1 - z_2)] e^{z_2 L(-1)} v \\
= \; & (z_1 - z_2)^m e^{z_2 \partial_{z_1}} [Y(u, z_1 - z_2)] e^{z_2 L(-1)} v \\
= \; & (z_1 - z_2)^m Y(u, z_1) Y(v, z_2)\mathbf{1} \\
= \; & (z_1 - z_2)^m Y(v, z_2) Y(u, z_1)\mathbf{1} \\
= \; & (z_1 - z_2)^m Y(v, z_2) e^{z_1 L(-1)} u \\
= \; & (z_1 - z_2)^m e^{z_1 L(-1)} [e^{-z_1 L(-1)} Y(v, z_2) e^{z_1 L(-1)}] u \\
= \; & (z_1 - z_2)^m e^{z_1 L(-1)} e^{-z_1 \mathrm{ad}_{L(-1)}} (Y(v, z_2)) u \\
= \; & (z_1 - z_2)^m e^{z_1 L(-1)} e^{-z_1 \partial_{z_2}} (Y(v, z_2)) u \\
= \; & (z_1 - z_2)^m e^{z_1 L(-1)} Y(v, z_2 - z_1) u \\
= \; & (z_1 - z_2)^m e^{z_1 L(-1)} Y(v, -z_1 + z_2) u.
\end{aligned} \tag{1.51}$$

Here we have used:

$$\mathrm{ad}_{L(-1)}(Y(w, z)) = [L(-1), Y(w, z)] = \frac{d}{dz} Y(w, z), \tag{1.52}$$

which is (1.28) in our definition of a vertex operator algebra. The last equation in (1.51) holds because the expression only involves positive powers of $z_2$ by the first expression, which implies that all the expressions cannot contain negative powers of $(z_2 - z_1)$.

Note (1.50) implies

$$\begin{aligned} Y(L(-1)u,z_1)e^{z_2L(-1)}v &= Y(L(-1)u,z_1)Y(v,z_2)\mathbf{1} \\ &= \left(\frac{d}{dz_1}Y(u,z_1)\right)Y(v,z_2)\mathbf{1} \\ &= \left(\frac{d}{dz_1}Y(u,z_1)\right)e^{z_2L(-1)}v, \end{aligned} \tag{1.53}$$

whose constant terms with respect to $z_2$ is:

$$Y(L(-1)u,z_1)v = \frac{d}{dz_1}Y(u,z_1)v. \tag{1.54}$$

Since $v$ is an arbitrary element, we have:

$$Y(L(-1)u,z_1) = \frac{d}{dz_1}Y(u,z_1), \tag{1.55}$$

that is, (1.21) holds. Moreover, (1.51) implies

$$e^{z_2L(-1)}Y(u,z_1-z_2)v = e^{z_1L(-1)}Y(v,-z_1+z_2)u, \tag{1.56}$$

which is

$$Y(u,z_1-z_2)v = e^{(z_1-z_2)L(-1)}Y(v,-z_1+z_2)u. \tag{1.57}$$

Denoting $z = z_1 - z_2$, we obtain the following "skew-symmetry" of vertex operators:

$$Y(u,z)v = e^{zL(-1)}Y(v,-z)u. \tag{1.58}$$

Next, we shall prove the "associativity" of vertex operator algebras from our definition. For $u, v, w \in V$, let $m$ and $n$ be positive integers such that

$$(z-z')^m[Y(u,z),Y(w,z')] = 0, \quad (z-z')^n[Y(v,z),Y(w,z')] = 0. \tag{1.59}$$

Note that

$$\begin{aligned} &(z_1+z_2-z_3)^m(z_2-z_3)^nY(u,z_1+z_2)Y(v,z_2)Y(w,z_3)\mathbf{1} \\ = \;&(z_1+z_2-z_3)^m(z_2-z_3)^nY(w,z_3)Y(u,z_1+z_2)Y(v,z_2)\mathbf{1} \\ = \;&(z_1+z_2-z_3)^m(z_2-z_3)^nY(w,z_3)Y(u,z_1+z_2)e^{z_2L(-1)}v \\ = \;&(z_1+z_2-z_3)^m(z_2-z_3)^ne^{z_2L(-1)}e^{-z_2\mathrm{ad}_{L(-1)}}(Y(w,z_3))e^{-z_2\mathrm{ad}_{L(-1)}}(Y(u,z_1+z_2))v \end{aligned}$$

$$
\begin{aligned}
&= (z_1+z_2-z_3)^m(z_2-z_3)^n e^{z_2L(-1)}e^{-z_2\partial_{z_3}}(Y(w,z_3))e^{-z_2\partial_{z_1}}(Y(u,z_1+z_2))v\\
&= (z_1+z_2-z_3)^m(z_2-z_3)^n e^{z_3L(-1)}e^{(z_2-z_3)L(-1)}Y(w,z_3-z_2)(Y(u,z_1)v)\\
&= (z_1+z_2-z_3)^m(z_2-z_3)^n e^{z_3L(-1)}e^{(z_2-z_3)L(-1)}Y(w,-z_2+z_3)(Y(u,z_1)v)\\
&\overset{(1.57)}{=} (z_1+z_2-z_3)^m(z_2-z_3)^n e^{z_3L(-1)}Y((Y(u,z_1)v),z_2-z_3)w\\
&= (z_1+z_2-z_3)^m(z_2-z_3)^n e^{-z_3\partial_{z_2}}e^{z_3L(-1)}Y((Y(u,z_1)v),z_2)w\\
&= e^{-z_3\partial_{z_2}}[e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n]]e^{z_3L(-1)}Y((Y(u,z_1)v),z_2)w\\
&= e^{z_3L(-1)}e^{-z_3\partial_{z_2}}[(z_1+z_2)^m z_2^n Y((Y(u,z_1)v),z_2)w], \qquad (1.60)
\end{aligned}
$$

where we have used the fact that only positive powers of $z_3$ are involved in the above expressions. Thus

$$
\begin{aligned}
&(z_1+z_2)^m z_2^n Y((Y(u,z_1)v),z_2)w\\
&= e^{z_3\partial_{z_2}}e^{-z_3L(-1)}[(z_1+z_2-z_3)^m(z_2-z_3)^n Y(u,z_1+z_2)Y(v,z_2)Y(w,z_3)\mathbf{1}]\\
&= e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n e^{-z_3L(-1)}Y(u,z_1+z_2)Y(v,z_2)e^{z_3L(-1)}w]\\
&= e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n e^{-z_3\mathrm{ad}_{L(-1)}}(Y(u,z_1+z_2))e^{-z_3\mathrm{ad}_{L(-1)}}(Y(v,z_2))w]\\
&= e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n e^{-z_3\partial_{z_2}}(Y(u,z_1+z_2))e^{-z_3\partial_{z_2}}(Y(v,z_2))w]\\
&= e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n e^{-z_3\partial_{z_2}}[Y(u,z_1+z_2)Y(v,z_2)]w]\\
&= e^{z_3\partial_{z_2}}[(z_1+z_2-z_3)^m(z_2-z_3)^n]Y(u,z_1+z_2)Y(v,z_2)w\\
&= (z_1+z_2)^m z_2^n Y(u,z_1+z_2)Y(v,z_2)w. \qquad (1.61)
\end{aligned}
$$

Therefore, we have the following associativity:

$$
(z_1+z_2)^m Y(Y(u,z_1)v,z_2)w = (z_1+z_2)^m Y(u,z_1+z_2)Y(v,z_2)w. \qquad (1.62)
$$

The above associativity and the commutativity (1.31) consist of the "duality" in physics.

Let $u\in V^{(p)}, v\in V^{(q)}, w\in V^{(s)}$. Write

$$
Y(u,z)=\sum_{n\in\mathbf{Z}}u(n)z^{-n-p},\quad Y(v,z)=\sum_{n\in\mathbf{Z}}v(n)z^{-n-q}. \qquad (1.63)
$$

According to (1.30) and (1.46),

$$
[L(0),u(n)]=-nu(n);\quad \text{Similarly,}\quad [L(0),v(n)]=-nv. \qquad (1.64)
$$

By our definition, there exists a positive integer $k$ such that

$$u(n)w = v(n)w = 0 \qquad \text{for } k < n \in \mathbf{Z}. \tag{1.65}$$

Moreover, by (1.31) and (1.62), there exist a positive integer $m$ such that

$$(z_1 - z_2)^m Y(u, z_1)Y(v, z_2)w = (z_1 - z_2)^m Y(v, z_2)Y(u, z_1)w, \tag{1.66}$$

$$(z_0 + z_2)^m Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^m Y(Y(u, z_0)v, z_2)w. \tag{1.63}$$

Let

$$(z_1 - z_2)^m Y(u, z_1)Y(v, z_2)w = \sum_{l\in\mathbf{Z}} (z_1 - z_2)^m \xi_l(z_1, z_2), \tag{1.67}$$

with

$$\xi_l \in V^{(l)}[[z_1^{-1}, z_2^{-1}, z_1, z_2]], \qquad l \in \mathbf{Z}. \tag{1.68}$$

Then (1.46) and (1.64) imply

$$\begin{aligned}
&\sum_{l\in\mathbf{Z}} l(z_1 - z_2)^m \xi_l(z_1, z_2)\\
= \;& L(0)\sum_{l\in\mathbf{Z}} (z_1 - z_2)^m \xi_l(z_1, z_2)\\
= \;& L(0)(z_1 - z_2)^m Y(u, z_1)Y(v, z_2)w\\
= \;& \sum_{n_1,n_2=-k}^{\infty} (z_1 - z_2)^m L(0)u(-n_1)v(-n_2)wz_1^{n_1-p}z_2^{n_2-q}\\
= \;& \sum_{n_1,n_2=-k}^{\infty} (z_1 - z_2)^m \{[L(0), u(-n_1)]v(-n_2)w + u(-n_1)[L(0), v(-n_2)]w\\
& +u(-n_1)v(-n_2)L(0)w\}z_1^{n_1-p}z_2^{n_2-q}\\
= \;& \sum_{n_1,n_2=-k}^{\infty} (z_1 - z_2)^m (n_1 + n_2 + s)u(-n_1)v(-n_2)wz_1^{n_1-p}z_2^{n_2-q}\\
= \;& \sum_{l=s-2k} l(z_1 - z_2)^m [\sum_{n_1=-k}^{l+k-s} u(-n_1)v(s - l + n_1)wz_1^{n_1-p}z_2^{l-s-n_1-q}].
\end{aligned} \tag{1.69}$$

Thus

$$\xi_l(z_1, z_2) = \sum_{n_1=-k}^{l+k-s} u(-n_1)v(s - l + n_1)wz_1^{n_1-p}z_2^{l-s-n_1-q} \qquad \in V^{(l)}[z_1^{-1}, z_2^{-1}, z_1, z_2] \tag{1.70}$$

is a Laurent polynomial. Since

$$z_0^m Y(u, z_0 + z_2)Y(v, z_2)w = \sum_{l\in\mathbf{Z}} z_0^m \xi_l(z_0 + z_2, z_2), \tag{1.71}$$

we have:

$$Y(u, z_0 + z_2)Y(v, z_2)w = \sum_{l \in \mathbf{Z}} \xi_l(z_0 + z_2, z_2). \tag{1.72}$$

The fact in (1.70) allows us to multiply $\sum_{l\in\mathbf{Z}}(z_1 - z_2)^m \xi_l(z_1, z_2)$ to the equation (1.14). Thus

$$\begin{aligned}
& z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)w \\
= \;& z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left(\frac{z_1 - z_2}{z_0}\right)^m Y(u, z_1)Y(v, z_2)w \\
& -z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\left(\frac{z_2 - z_1}{-z_0}\right)^m Y(v, z_2)Y(u, z_1)w \\
= \;& z_0^{-m}\left[z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\right](z_1 - z_2)^m Y(u, z_1)Y(v, z_2)w \\
= \;& z_0^{-m} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\sum_{l\in\mathbf{Z}}(z_1 - z_2)^m \xi_l(z_1, z_2) \\
= \;& z_0^{-m} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\sum_{l\in\mathbf{Z}}(z_1 - z_2)^m \xi_l(z_1, z_2) \\
= \;& z_0^{-m} z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\sum_{l\in\mathbf{Z}}(z_2 + z_0 - z_2)^m \xi_l(z_2 + z_0, z_2) \\
= \;& z_1^{-m-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)(z_0 + z_2)^m \sum_{l\in\mathbf{Z}} \xi_l(z_2 + z_0, z_2) \\
= \;& z_1^{-m-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)(z_0 + z_2)^m Y(u, z_0 + z_2)Y(u, z_2)w \\
= \;& z_1^{-m-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)(z_0 + z_2)^m Y(Y(u, z_0)v, z_2) \\
= \;& z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) Y(Y(u, z_0)v, z_2) \\
= \;& z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2),
\end{aligned} \tag{1.73}$$

where we have used the fact:

$$\delta(z) = \sum_{n\in\mathbf{Z}} z^n = \sum_{n\in\mathbf{Z}} z^{n+m} = z^m\delta(z). \tag{1.74}$$

Therefore we have proved that our definition implies FLM's

**Theorem 1.4**. *FLM's definition (1.17-22) of a vertex operator algebras is equivalent to our definition (1.27-31).*

**Remark 1.5**. In practical, it is much easier to prove our axioms than FLM's. In next two chapters, we shall give two family of examples to justify this statement.

# Chapter 2

In this chapter, we shall mainly talk about the vertex operator algebras associated with affine Lie algebras and their irreducible modules.

1. Definitions of Lie Algebras and Examples

A Lie algebra $\mathcal{G}$ is a vector space with an operation $[\cdot,\cdot]$ satisfying

$$[u,v] = -[v,u] \qquad \textit{(skew symmetry)}, \tag{2.1}$$

$$[[u,v],w] + [[v,w],u] = [[w,u],v] = 0 \qquad \textit{(Jacobi Identity)} \tag{2.2}$$

for $u, v, w \in \mathcal{G}$. For a given element $u$ of $\mathcal{G}$, we define $\mathrm{ad}_u \in (\mathrm{End}\,\mathcal{G})$ by

$$\mathrm{ad}_u(v) = [u,v] \qquad \text{for } v \in \mathcal{G}. \tag{2.3}$$

Then (2.1) and (2.3) become

$$\mathrm{ad}_u(v) = -\mathrm{ad}_v(u), \tag{2.4}$$

$$\mathrm{ad}_u\mathrm{ad}_v - \mathrm{ad}_v\mathrm{ad}_u = \mathrm{ad}_{\mathrm{ad}_u(v)} \tag{2.5}$$

for $u, v \in \mathcal{G}$. One can see that the Jacobi identity (1.18) for a vertex operator algebra is a generalization of (2.5) and the skew-symmetry (1.58) is a generalization of (2.4). From this point of view, a vertex operator algebra is a one-variable generalization of a Lie algebra.

**Example 2.1**. Let $\mathcal{A}$ be an associative algebra. Define a new operation $[\cdot,\cdot]$ over $\mathcal{A}$ by

$$[u,v] = uv - vu, \qquad \text{for } u, v \in \mathcal{A}. \tag{2.7}$$

Then $(\mathcal{A}, [\cdot,\cdot])$ forms a Lie algebra. Let $\sigma \in (\mathrm{End}\,\mathcal{A})$ be an involutive anti-isomorphism, that is,

$$\sigma^2 = \mathrm{Id}_{\mathcal{A}}, \qquad \sigma(uv) = \sigma(v)\sigma(u) \qquad \text{for } u, v \in \mathcal{A}. \tag{2.8}$$

Set

$$\mathcal{A}^\sigma = \{u \in \mathcal{A} \mid \sigma(u) = -u\}. \tag{2.9}$$

Then $(\mathcal{A}^{\sigma}, [\cdot,\cdot])$ forms a Lie subalgebra of $(\mathcal{A}, [\cdot,\cdot])$. For instance, let $\mathcal{A} = M_{n\times n}$ be the algebra of $n \times n$ matrices. Then $(M_{n\times n}, [\cdot,\cdot])$ is called a *general linear Lie algebra.* Recall that a *simple Lie algebra* $\mathcal{L}$ is a Lie algebra in which there exists no proper nonzero subspace $I$ such that $[\mathcal{L}, I] \subset I$. Set

$$sl_n = \{u \in M_{n\times n} \mid \operatorname{tr} u = 0\}, \tag{2.10}$$

where $\operatorname{tr} u$ is the sum of diagonal entries of the matrix $u$. Then $(sl_n, [\cdot,\cdot])$ is a simple Lie subalgebra of $(M_{n\times n}, [\cdot,\cdot])$. Furthermore, for certain involutive anti-isomorphisms $\sigma$, the Lie subalgebras $(M^{\sigma}_{n\times n}, [\cdot,\cdot])$ are simple.

**Example 2.2.** Let $(\mathcal{A}, \circ)$ be any algebra. A map $d \in (\operatorname{End} \mathcal{A})$ is called a *derivation* of $(\mathcal{A}, \circ)$ if

$$d(u \circ v) = d(u) \circ v + u \circ d(v) \qquad \text{for } u, v \in \mathcal{A}. \tag{2.11}$$

Set

$$\operatorname{Der} \mathcal{A} = \text{the set of of all derivations of } \mathcal{A}. \tag{2.12}$$

Then $(\operatorname{Der} \mathcal{A}, [\cdot,\cdot])$ is a Lie subalgebra of $(\operatorname{End} \mathcal{A}, [\cdot,\cdot])$. For instance, let $\mathcal{A} = \mathbf{C}[z^{-1}, z]$ be the algebra of Laurent polynomials in $z$. Then

$$\operatorname{Der} \mathcal{A} = \mathcal{W} = \{f\frac{d}{dz} \mid f \in \mathcal{A}\} \tag{2.13}$$

and the Lie operation becomes:

$$\left[f\frac{d}{dz}, g\frac{d}{dz}\right] = (fg' - f'g)\frac{d}{dz} \qquad \text{for } f, g \in \mathbf{A}. \tag{2.14}$$

Moreover, $\mathcal{W}$ is a simple algebra which is called the *two-sided Witt algebra of rank 1.* Set

$$\ell_n = -z^{n+1}\frac{d}{dz} \qquad \text{for } n \in \mathbf{Z}. \tag{2.16}$$

Then $\{\ell_n \mid n \in \mathbf{Z}\}$ is a basis of $\mathcal{W}$ with the formula:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \qquad \text{for } m, n \in \mathbf{Z}. \tag{2.17}$$

In fact the Virasoro algebra (determined by (1.20)) is the unique one-dimensional central extension of $\mathcal{W}$.

If the algebra $(\mathcal{A}, \circ)$ is the eight-dimensional Cayley algebra, then Der $\mathcal{A}$ is the simple Lie algebra typed by $G_2$ (cf. [S]). When the algebra $(\mathcal{A}, \circ)$ is the twenty-seven-dimensional exceptional Jordan algebra, Der $\mathcal{A}$ is the simple Lie algebra typed by $F_4$ (cf. [S]).

## 2. Finite-Dimensional Simple Lie Algebras

Finite-dimensional simple Lie algebras (over $\mathbf{C}$) and their irreducible modules were classified by Killing and Cartan.

Let $\mathcal{G}$ be a finite-dimensional Lie algebra. We define the *Killing form* $\kappa(\cdot,\cdot)$ over $\mathcal{G}$ by

$$\kappa(u,v) = \mathrm{tr}(\mathrm{ad}_u \mathrm{ad}_v) \qquad \text{for } u, v \in \mathcal{G}. \tag{2.18}$$

It can be proved that $\kappa(\cdot,\cdot)$ is nondegenerate and invariant, that is,

$$\kappa([u,v],w) = \kappa(u,[v,w]) \qquad \text{for } u,v,w \in \mathcal{G}. \tag{2.19}$$

Moreover, there exists an abelian subalgebra $H \subset \mathcal{G}$ ($[H,H] = \{0\}$) such that

$$\mathcal{G} = H \oplus \bigoplus_{0 \neq \alpha \in H^*} \mathcal{G}_\alpha, \qquad \text{where } \mathcal{G}_\alpha = \{u \in \mathcal{G} \mid [h,u] = \alpha(h)u \text{ for } h \in H\}. \tag{2.20}$$

The above decomposition is called the *Cartan decomposition*, and the subalgebra $H$ is called a *Cartan subalgebra*. The expressions (2.19-20) imply that $\kappa|_{H\times H}$ is nondegenerate. We can identify $H^*$ with $H$ as follows: $\alpha \in H^* \leftrightarrow t_\alpha \in H$ with

$$\alpha(h) = \kappa(t_\alpha, h) \qquad \text{for } h \in H. \tag{2.21}$$

Set

$$\Delta = \{0 \neq \alpha \in H \mid \mathcal{G}_\alpha \neq \{0\}\}. \tag{2.22}$$

Then $\Delta$ spans $H$ and dim $\mathcal{G}_\alpha = 1$ for $\alpha \in \Delta$. Moreover, if $\alpha \in \Delta$, the only multiple of $\alpha$ in $\Delta$ are $\pm\alpha$ and $\kappa(\alpha,\alpha) > 0$. Define

$$\langle \alpha, \beta \rangle = \frac{2\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)} \qquad \text{for } \alpha, \beta \in \Delta. \tag{2.23}$$

Then we have

$$\langle \alpha, \beta \rangle \in \mathbf{Z}, \quad \beta - \langle \alpha, \beta \rangle \alpha \in \Delta \qquad \text{for } \alpha, \beta \in \Delta. \tag{2.24}$$

The set $\Delta$ is called the *root system* of $\mathcal{G}$. It turns out that $\Delta$ has only nine different classes.

If dim $H = n$, then there exist $\alpha_1, ..., \alpha_n \in \Delta$ such that any root $\beta \in \delta$ can be uniquely written as $\beta = \sum_{j=1}^{n} m_j\alpha_j$ with all $0 \leq m_j \in \mathbf{Z}$ or all $0 \geq m_j \in \mathbf{Z}$. The root $\beta$ is called a *positive root* in the former case and a *negative root* in the later case. All $\alpha_j$ are called *simple roots.* There exits a unique positive root $\theta$ such that

$$\theta - \beta = \sum_{j=1}^{n} m_j\alpha_j, \quad 0 \leq m_j \in \mathbf{Z}, \qquad \text{for any } \beta \in \Delta. \tag{2.25}$$

The root $\theta$ is called the *highest root.* It can be proved that

$$\langle\alpha_i, \alpha_j\rangle \leq 0, \ \langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle \leq 3, \qquad i, j = 1, ..., n, \ i \neq j. \tag{2.26}$$

## 3. PBW Theorem and the Representations

Let $\mathcal{G}$ be a Lie algebra. Set

$$T(\mathcal{G}) = \bigoplus_{m=0}^{\infty} T^m(\mathcal{G}), \qquad T^0(\mathcal{G}) = \mathbf{C}, \ T^m(\mathcal{G}) = \mathcal{G} \otimes ... \otimes \mathcal{G} \ (m \text{ copies}). \tag{2.27}$$

We define an algebraic operation "$\cdot$" on $T(\mathcal{G})$ by

$$(u_1 \otimes ... \otimes u_m) \cdot (v_1 \otimes ... \otimes v_n) = u_1 \otimes ... \otimes u_m \otimes v_1 \otimes ... \otimes v_n \ \in T^{m+n}(\mathcal{G}). \tag{2.28}$$

Then $(T(\mathcal{G}), \cdot)$ becomes an associative algebra. Let $J$ be the two sided ideal generated by

$$\{u \otimes v - v \otimes u - [u, v] \mid u, v \in \mathcal{G}\}. \tag{2.29}$$

Denote the quotient algebra by

$$U(\mathcal{G}) = T(\mathcal{G})/J. \tag{2.30}$$

The algebra $U(\mathcal{G})$ has the following universal property: for an associative algebra $\mathcal{A}$ and a Lie algebra homomorphism $\tau: \mathcal{G} \to \mathcal{A}$ (with $[a, b] = ab - ba$), there exists a unique associative algebra homomorphism $\tilde{\tau}: U(\mathcal{G}) \to \mathcal{A}$ such that $\tilde{\tau}|_{\mathcal{G}} = \tau$. The algebra $U(\mathcal{G})$ is thus called the *universal enveloping algebra* of $\mathcal{G}$.

We denote the image of $u_1 \otimes ... \otimes u_m$ in $T(\mathcal{G})$ by $u_1 \cdots u_m$. Suppose that $\{\xi_j \mid j \in \mathcal{I}\}$ is a basis of $\mathcal{G}$ with $\mathcal{I}$, an ordered index set. Then we have:

**Theorem 2.1** (Poincare-Birkhoff-Witt; PBW Theorem). *The set*

$$\{\xi_{i_1}^{m_1}\xi_{i_2}^{m_2} \cdots \xi_{i_k}^{m_k} \mid i_1 < i_2 \cdots < i_k, \ i_j \in \mathcal{I}, \ 0 \leq k \in \mathbf{Z}\} \tag{2.31}$$

*is a basis of* $U(\mathcal{G})$.

In particular, if $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_1$, $\mathcal{G}_2$ are subalgebras, then

$$U(\mathcal{G}) = U(\mathcal{G}_1)U(\mathcal{G}_2). \tag{2.32}$$

A vector space $M$ is called a *module* of a Lie algebra $\mathcal{G}$ if there exists a linear map $\pi : \mathcal{G} \to \mathrm{End}\, M$ such that

$$\pi([u,v]) = \pi(u)\pi(v) - \pi(v)\pi(u) \qquad \text{for } u, v \in \mathcal{G}. \tag{2.33}$$

The map $\pi$ is called a *representation* of $\mathcal{G}$. The module $M$ (or representation $\pi$) is called *irreducible* if there exists no proper nonzero subspace $I$ such that $\pi(\mathcal{G})(I) \subset I$.

Next we assume that $\mathcal{G}$ is a finite-dimensional simple Lie algebra with $H$ as a Cartan subalgebra. Let $M$ be a $\mathcal{G}$-module with the representation $\pi$. For $\xi \in \mathcal{G}, v \in M$, we denote $\pi(\xi)(v)$ by $\xi(v)$ or $\xi v$ when the context is clear. An element $\lambda \in H$ is called a *weight* of $M$ if there exists a nonzero vector $v_\lambda$ (*weight vector*) in $M$ such that

$$hv_\lambda = \kappa(h,\lambda)v_\lambda \qquad \text{for } h \in H. \tag{2.34}$$

Moreover, $\lambda$ is call *dominated integral* if

$$0 \leq \langle \alpha_i, \lambda \rangle = \frac{2\kappa(\alpha_i,\lambda)}{\kappa(\alpha_i,\alpha_i)} \in \mathbf{Z} \qquad \text{for all the simple roots } \alpha_i. \tag{2.35}$$

Let $\Delta_+$ and $\Delta_-$ be the sets of positive and negative roots, respectively. Set

$$\mathcal{G}_\pm = \bigoplus_{\alpha \in \Delta\pm} \mathcal{G}_\alpha, \qquad B_+ = H + \mathcal{G}_+. \tag{2.35}$$

Then $B_+$ and $\mathcal{G}_\pm$ are subalgebras of $\mathcal{G}$. Moreover,

$$\mathcal{G} = \mathcal{G}_- \oplus B_+. \tag{2.36}$$

Thus

$$U(\mathcal{G}) = U(\mathcal{G}_-)U(B_+). \tag{2.37}$$

A weight vector $v_\lambda$ of a $\mathcal{G}$-module is called a *highest-weight vector* if

$$\xi(v_\lambda) = 0 \qquad \text{for } \xi \in \mathcal{G}_+. \tag{2.38}$$

In this case, $\lambda$ is called a *highest weight.* If $M$ is generated by a highest weight vector $v_\lambda$, then

$$M = U(\mathcal{G})v_\lambda = U(\mathcal{G}_-)v_\lambda \tag{2.39}$$

and we call $M$ a *highest-weight module* ($\pi$ a *highest-weight representation*). Note that the second equality in (2.39) implies that a highest-weight irreducible modules has a unique highest weight and a unique highest-weight vector up to a constant multiple.

**Theorem 2.2.** *Any finite-dimensional irreducible module is generated by a highest weight vector associated with a dominated integral weight. Conversely, for any $\lambda \in H$ satisfying (2.35), there exists a unique finite-dimensional irreducible module with $\lambda$ as a highest weight.*

We define an action of $\mathcal{G}$ on $\mathbf{C}$ by

$$\xi(\mu) = 0 \qquad \text{for } \xi \in \mathcal{G},\ \mu \in \mathbf{C}. \tag{2.40}$$

Then $\mathbf{C}$ is the irreducible module corresponding to the weight 0. We call this module the *trivial module.* Besides, $(\mathcal{G}, \mathrm{ad})$ forms an irreducible module whose highest weight is the highest root.

Suppose that $\{e^1, e^2, ..., e^k\}$ is an orthonormal basis of $\mathcal{G}$ with respect to $\kappa(\cdot,\cdot)$. Write

$$[e^i, e^j] = \sum_{l=1}^{k} \lambda_{i,j}^l e^l, \qquad i,j = 1,2,...,k. \tag{2.41}$$

Then

$$[e^i, e^j] = -[e^i, e^j], \qquad \kappa([e^i, e^j], e^l) = \kappa(e^i, [e^j, e^l]) \tag{2.42}$$

imply

$$\lambda_{i,j}^l = -\lambda_{j,i}^l, \qquad \lambda_{i,j}^l = \lambda_{j,l}^i, \qquad i,j,l = 1,2,..,k. \tag{2.43}$$

The element

$$\Omega_{\mathcal{G}} = \sum_{i=1}^{k} e^i e^i \qquad \in U(\mathcal{G}). \tag{2.44}$$

is called the *Casimier element.*

**Theorem 2.3.** *The element $\Omega_{\mathcal{G}}$ is in the center of $U(\mathcal{G})$, that is,*

$$\xi\Omega_{\mathcal{G}} = \Omega_{\mathcal{G}}\xi \qquad \text{for } \xi \in U(\mathcal{G}). \tag{2.45}$$

*Proof.* It is enough to prove (2.45) for $\xi = e^j$. Note that

$$\begin{aligned}
& [e^j, \Omega_{\mathcal{G}}] \\
= & \sum_{i=1}^{k}([e^j, e^i]e^i + e^i[e^j, e^i]) \\
= & \sum_{i,l=1}^{k} \lambda_{j,i}^{l}(e^l e^i + e^i e^l]) \\
= & \sum_{i,l=1}^{k}(\lambda_{j,i}^{l} + \lambda_{j,l}^{i})e^l e^i \\
= & \sum_{i,l=1}^{k}(\lambda_{j,i}^{l} + \lambda_{i,j}^{l})e^l e^i \\
= & \ 0. \qquad \square
\end{aligned} \tag{2.46}$$

By the universal property of $U(\mathcal{G})$, any representation of $\mathcal{G}$ naturally extend to a representation of $U(\mathcal{G})$. If $(M, \pi)$ is a finite-dimensional irreducible $\mathcal{G}$-module, then $\pi(\Omega_{\mathcal{G}})$ is a constant map by Shur's Lemma. We denote

$$\mathrm{ad}(\Omega_{\mathcal{G}}) = \sum_{i=1}^{k} \mathrm{ad}_{e^i}\mathrm{ad}_{e^i} = 2\rho \mathrm{Id}_{\mathcal{G}}, \quad \Omega_{\mathcal{G}}|_{M_\lambda} = \rho_\lambda \mathrm{Id}_{M_\lambda}, \tag{2.47}$$

where $\rho$ is called the dual Coxeter number of $\mathcal{G}$. For instance, $\rho = n$ when $\mathcal{G} = sl_n$. Note that $\rho_0 = 0$. The reader may refer [H] for the details of the statements we presented in the above two sections.

4. Affine Lie Algebras and Vertex Operators.

Again let $\mathcal{G}$ be a finite-dimensional simple Lie algebra and the other setting are the same as the above sections. We set

$$\hat{\mathcal{G}} = \mathcal{G} \otimes \mathbf{C}[t^{-1}, t] \oplus \mathbf{C}\varsigma, \tag{2.48}$$

where $t$ is an indeterminate and $\varsigma$ is a base element. We define an operation $[\cdot,\cdot]$ on $\hat{\mathcal{G}}$ by

$$[\varsigma, \hat{\mathcal{G}}] = [\hat{\mathcal{G}}, \varsigma] = \{0\}, \tag{2.49}$$

$$[\xi \otimes t^m, \eta \otimes t^n] = [\xi, \eta] \otimes t^{m+n} + m\delta_{m+n,0}\kappa(\xi, \eta)\varsigma \qquad \text{for } \xi, \eta \in \mathcal{G},\ m, n \in \mathbf{Z}. \tag{2.50}$$

It can be easily verified that $(\hat{\mathcal{G}}, [\cdot,\cdot])$ forms a Lie algebra, which is called the *affine (Kac-Moody) Lie algebra* associated with $\mathcal{G}$ (cf. [K]). The subspace $\mathbf{C}\varsigma$ is the center of this algebra. For convenience, we denote $\xi \otimes t^n$ by $\xi(n)$ for $\xi \in \mathcal{G}$, $n \in \mathbf{Z}$. Let

$$\hat{\mathcal{G}}_{\pm} = \sum_{0<n\in\mathbf{Z}} \mathcal{G}(n), \quad \hat{B}_{+} = \mathcal{G}(0) + \mathbf{C}\varsigma + \hat{\mathcal{G}}_{+}. \tag{2.51}$$

Then $\hat{\mathcal{G}}_{\pm}$ and $\hat{B}_{+}$ are Lie subalgebras of $\hat{\mathcal{G}}$ and

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}_{-} + \hat{B}_{+}. \tag{2.52}$$

Thus

$$U(\hat{\mathcal{G}}) = U(\hat{\mathcal{G}}_{-})U(\hat{B}_{+}). \tag{2.53}$$

Let $M_\lambda$ be the finite-dimensional irreducible module with the highest weight $\lambda$ and a highest-weight vector $v_\lambda$. For $\chi \in \mathbf{C}$, we define the action of $\hat{B}_{+}$ on $M_\lambda$ by

$$\xi(n)(u) = \delta_{n,0}\xi(u), \quad \varsigma(u) = \chi \qquad \text{for } \xi \in \mathcal{G},\ 0 \leq n \in \mathbf{Z}. \tag{2.54}$$

Easily see that $M_\lambda$ becomes a $\hat{B}_{+}$-module. Consider the induced $\hat{\mathcal{G}}$-module:

$$M_{\lambda,\chi} = U(\hat{\mathcal{G}}) \otimes_{U(\hat{B}_{+})} M_\lambda. \tag{2.55}$$

By (2.53),

$$M_{\lambda,\chi} = U(\hat{\mathcal{G}}_{-}) \otimes_{\mathbf{C}} M_\lambda. \tag{2.56}$$

The number $\chi$ is called the *level* of the module $M_{\lambda,\chi}$.

For $\xi \in \mathcal{G}$, we define $\xi(z) \in LM(M_{\lambda,\chi}, M_{\lambda,\chi}[z^{-1}, z]])$ by:

$$\xi(z) = \sum_{n\in\mathbf{Z}} \xi(n) z^{-n-1}. \tag{2.57}$$

Moreover, we let

$$\xi^{+} = \sum_{n=0}^{\infty} \xi(n) z^{-n-1}, \quad \xi^{-} = \sum_{n=1}^{\infty} \xi(-n) z^{n-1}. \tag{2.58}$$

**Lemma 2.4.** *For $\xi, \eta \in \mathcal{G}$,*

$$[\xi(z_1), \eta^{+}(z_2)] = \frac{[\xi,\eta](z_1)}{z_2 - z_1} - \frac{\kappa(\xi,\eta)\chi}{(z_2 - z_1)^2}, \tag{2.59}$$

$$[\xi(z_1), \eta^{-}(z_2)] = \frac{[\xi,\eta](z_1)}{z_1 - z_2} + \frac{\kappa(\xi,\eta)\chi}{(z_1 - z_2)^2}. \tag{2.60}$$

*Moreover,*

$$\begin{aligned}[\xi(z_1),\eta(z_2)] &= z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)[\xi,\eta](z_1)+z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)\kappa(\xi,\eta)\chi \\ &= z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)[\xi,\eta](z_2)-z_2^{-1}\partial_{z_1}\delta\left(\frac{z_1}{z_2}\right)\kappa(\xi,\eta)\chi. \end{aligned} \tag{2.61}$$

*Proof.* Note that

$$\begin{aligned} &[\xi(z_1),\eta^+(z_2)] \\ =\ & \sum_{m\in\mathbf{Z}}\sum_{n=0}^{\infty}[\xi,\eta](m+n)z_1^{-m-1}z_2^{-n-1}+\sum_{n=0}^{\infty}-n\kappa(\xi,\eta)\chi z_1^{n-1}z_2^{-n-1} \\ =\ & \sum_{l\in\mathbf{Z}}\sum_{n=0}^{\infty}[\xi,\eta](l)z_1^{n-l-1}z_2^{-n-1}-\partial_{z_1}\sum_{n=0}^{\infty}\kappa(\xi,\eta)\chi z_1^{n-1}z_2^{-n-1} \\ =\ & (\sum_{l\in\mathbf{Z}}[\xi,\eta](l)z_1^{-l-1})(\sum_{n=0}^{\infty}z_1^{n}z_2^{-n-1})-\partial_{z_1}\frac{\kappa(\xi,\eta)\chi}{z_2-z_1} \\ =\ & \frac{[\xi,\eta](z_1)}{z_2-z_1}-\frac{\kappa(\xi,\eta)\chi}{(z_2-z_1)^2}, \end{aligned} \tag{2.62}$$

$$\begin{aligned} &[\xi(z_1),\eta^-(z_2)] \\ =\ & \sum_{m\in\mathbf{Z}}\sum_{n=1}^{\infty}[\xi,\eta](m-n)z_1^{-m-1}z_2^{n-1}+\sum_{n=0}^{\infty}n\kappa(\xi,\eta)\chi z_1^{-n-1}z_2^{n-1} \\ =\ & \sum_{l\in\mathbf{Z}}\sum_{n=1}^{\infty}[\xi,\eta](l)z_1^{-n-l-1}z_2^{n-1}+\partial_{z_2}\sum_{n=0}^{\infty}\kappa(\xi,\eta)\chi z_1^{-n-1}z_2^{n} \\ =\ & (\sum_{l\in\mathbf{Z}}[\xi,\eta](l)z_1^{-l-1})(\sum_{n=1}^{\infty}z_1^{-n}z_2^{n-1})+\partial_{z_2}\frac{\kappa(\xi,\eta)\chi}{z_1-z_2} \\ =\ & \frac{[\xi,\eta](z_1)}{z_1-z_2}+\frac{\kappa(\xi,\eta)\chi}{(z_1-z_2)^2}. \end{aligned} \tag{2.63}$$

Moreover,

$$\begin{aligned} &[\xi(z_1),\eta(z_2)] \\ =\ & [\xi(z_1),\eta^+(z_2)]+[\xi(z_1),\eta^-(z_2)] \\ =\ & \frac{[\xi,\eta](z_1)}{z_2-z_1}-\frac{\kappa(\xi,\eta)\chi}{(z_2-z_1)^2}+\frac{[\xi,\eta](z_1)}{z_1-z_2}+\frac{\kappa(\xi,\eta)\chi}{(z_1-z_2)^2} \\ =\ & \left(\frac{1}{z_2-z_1}+\frac{1}{z_1-z_2}\right)[\xi,\eta](z_1)+\left(\frac{1}{(z_1-z_2)^2}-\frac{1}{(z_2-z_1)^2}\right)\kappa(\xi,\eta)\chi \\ =\ & z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)[\xi,\eta](z_1)+z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)\kappa(\xi,\eta)\chi \\ =\ & z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)[\xi,\eta](z_1)-\partial_{z_1}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\kappa(\xi,\eta)\chi \\ =\ & z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)[\xi,\eta](z_2)-z_2^{-1}\partial_{z_1}\delta\left(\frac{z_1}{z_2}\right)\kappa(\xi,\eta)\chi \end{aligned} \tag{2.64}$$

by (1.5) and (1.7). □

Note that (2.64) implies

$$(z_1 - z_2)^2[\xi(z_1), \eta(z_2)] = 0. \tag{2.65}$$

Let $\{e^1, e^2, ..., e^k\}$ be an orthonormal basis of $\mathcal{G}$ with respect to the killing form $\kappa(\cdot,\cdot)$. Define operator

$$\mathcal{L}(z) = \sum_{i=1}^{k}[e^{i-}(z)e^i(z) + e^i(z)e^{i+}(z)] = \sum_{n\in\mathbf{Z}} \mathcal{L}(n)z^{-n-2} \tag{2.66}$$

on $M_{\lambda,\chi}$. The operator $\mathcal{L}(z)$ is called the *Segal-Sugawara operator.*

**Lemma 2.5.** *We have*

$$[\mathcal{L}(n), \xi(m)] = -2m(\rho + \chi)\xi(m+n) \qquad \textit{for } \xi \in \mathcal{G},\ m, n \in \mathbf{Z}. \tag{2.67}$$

*Proof.* For $j = 1, .., k$, by (2.41) and (2.43),

$$\begin{aligned}
&\sum_{i,l=1}^{k} \lambda_{j,i}^{l}(e^{i-}(z_1)e^{l+}(z_1) + e^{l-}(z_1)e^{i+}(z_1)) \\
= &\sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{i-}(z_1)e^{l+}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{l-}(z_1)e^{i+}(z_1) \\
= &\sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{i-}(z_1)e^{l\pm}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,l}^{i}e^{i-}(z_1)e^{l+}(z_1) \\
= &\sum_{i,l=1}^{k} (\lambda_{j,i}^{l} + \lambda_{j,l}^{i})e^{i-}(z_1)e^{l+}(z_1) \\
= &\sum_{i,l=1}^{k} (\lambda_{j,i}^{l} + \lambda_{i,j}^{l})e^{i-}(z_1)e^{l+}(z_1) \\
= &\ 0
\end{aligned} \tag{2.68}$$

$$\begin{aligned}
&2\sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{l+}(z_1)e^{i+}(z_1) \\
= &\sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{l+}(z_1)e^{i+}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{l+}(z_1)e^{i+}(z_1) \\
= &\sum_{i,l=1}^{k} \lambda_{j,i}^{l}e^{l+}(z_1)e^{i+}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,l}^{i}e^{i+}(z_1)e^{l+}(z_1) \\
= &\sum_{i,l=1}^{k} \lambda_{j,l}^{i}[e^{i+}(z_1), e^{l+}(z_1)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,l=1}^{k} \lambda_{j,l}^{i} \sum_{m,n=0}^{\infty} [e^i, e^l](m+n)] z_1^{-m-n-2} \\
&= \sum_{i,l=1}^{k} \lambda_{j,l}^{i} \sum_{p=0}^{\infty} (p+1)[e^i, e^l](p) z_1^{-p-2} \\
&= -\sum_{i,l=1}^{k} \lambda_{j,l}^{i} \frac{d}{dz_1} \sum_{p=0}^{\infty} [e^i, e^l](p) z_1^{-p-1} \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \frac{d}{dz_1} [e^i, e^l]^{+}(z_1), \qquad (2.69)
\end{aligned}
$$

$$
\begin{aligned}
& 2\sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{i-}(z_1) e^{l-}(z_1) \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{i-}(z_1) e^{l-}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{i-}(z_1) e^{l-}(z_1) \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{i-}(z_1) e^{l-}(z_1) + \sum_{i,l=1}^{k} \lambda_{j,l}^{i} e^{l-}(z_1) e^{i-}(z_1) \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} [e^{i-}(z_1), e^{l-}(z_1)] \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \sum_{m,n=1}^{\infty} [e^i, e^l](-m-n)] z_1^{m+n-2} \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \sum_{p=2}^{\infty} (p-1)[e^i, e^l](-p)] z_1^{p-2} \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \frac{d}{dz_1} \sum_{p=1}^{\infty} [e^i, e^l](-p)] z_1^{p-1} \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \frac{d}{dz_1} [e^i, e^l]^{-}(z_1). \qquad (2.70)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& 2\sum_{i,l=1}^{k} \lambda_{j,i}^{l} [e^{i-}(z_1) e^{l}(z_1) + e^{l}(z_1) e_i^{+}(z_1)] \\
&= 2\sum_{i,l=1}^{k} \lambda_{j,i}^{l} (e^{i-}(z_1) e^{l+}(z_1) + e^{l-}(z_1) e^{i+}(z_1)) \\
&\quad +2\sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{i-}(z_1) e^{l-}(z_1) + 2\sum_{i,l=1}^{k} \lambda_{j,i}^{l} e^{l+}(z_1) e^{i+}(z_1) \\
&= \sum_{i,l=1}^{k} \lambda_{j,i}^{l} \frac{d}{dz_1} [e^i, e^l](z_1) \\
&= \sum_{i=1}^{k} \frac{d}{dz_1} [e^i, [e^j, e^i]](z_1)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{d}{dz_1}[\sum_{i=1}^{k}\mathrm{ad}_{e^i}\mathrm{ad}_{e^i}(e^j)](z_1)\\
&= -2\rho\frac{d}{dz_1}e^j(z_1). \qquad (2.71)
\end{aligned}
$$

Moreover, by Lemma 2.4,

$$
\begin{aligned}
&\lambda^l_{j,i}\left[\frac{e^l(z_1)e^i(z_2)}{z_1-z_2}+\frac{e^i(z_2)e^l(z_1)}{z_2-z_1}\right]\\
&= \lambda^l_{j,i}\left[\frac{e^l(z_1)e^{i-}(z_2)}{z_1-z_2}+\frac{e^l(z_1)e^{i+}(z_2)}{z_1-z_2}+\frac{e^{i+}(z_2)e^l(z_1)}{z_2-z_1}+\frac{e^{i-}(z_2)e^l(z_1)}{z_2-z_1}\right]\\
&= \lambda^l_{j,i}[\frac{[e^l(z_1),e^{i-}(z_2)]}{z_1-z_2}+\frac{e^{i-}(z_2)e^l(z_1)}{z_1-z_2}+\frac{e^l(z_1)e^{i+}(z_2)}{z_1-z_2}\\
&\quad+\frac{[e^{i+}(z_2),e^l(z_1)]}{z_2-z_1}+\frac{e^l(z_1)e^{i+}(z_2)}{z_2-z_1}+\frac{e^{i-}(z_2)e^l(z_1)}{z_2-z_1}]\\
&= \lambda^l_{j,i}\left[\frac{[e^l,e^i](z_1)}{(z_1-z_2)^2}+\left[\frac{1}{z_1-z_2}+\frac{1}{z_2-z_1}\right][e^{i-}(z_2)e^l(z_1)+e^l(z_1)e^{i+}(z_2)]-\frac{[e^l,e^i](z_1)}{(z_2-z_1)^2}\right]\\
&= \lambda^l_{j,i}\left[z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)[e^l,e^i](z_1)+z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)[e^{i-}(z_2)e^l(z_1)+e^l(z_1)e^{i+}(z_2)]\right]. \qquad (2.72)
\end{aligned}
$$

Therefore, by Lemma 2.4,

$$
\begin{aligned}
&[e^j(z_1),\mathcal{L}(z_2)]\\
&= \sum_{i=1}^{k}[[e^j(z_1),e^{i-}(z_2)]e^i(z)+e_i^-(z_2)[e^j(z_1),e^i(z_2)]+[e^j(z_1),e^i(z_2)]e^{i+}(z_2)+e^i(z_2)[e^j(z_1),e^{i+}(z_2)]]\\
&= \sum_{i,l=1}^{k}\left[\frac{\lambda^l_{j,i}}{z_1-z_2}e^l(z_1)e^i(z_2)+z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\lambda^l_{j,i}[e^{i-}(z_2)e^l(z_1)+e^l(z_1)e^{i+}(z_2)]+\frac{\lambda^l_{j,i}}{z_2-z_1}e^i(z_2)e^l(z_1)\right]\\
&\quad+\frac{\chi e^j(z_2)}{(z_1-z_2)^2}e^j(z_2)+z_1^{-1}\partial_{z_2}\left[\delta\left(\frac{z_2}{z_1}\right)\right]\chi[e^{j-}(z_2)+e^{j+}(z_2)]-\frac{\chi e^j(z_2)}{(z_2-z_1)^2}e^j(z_2)\\
&= \sum_{i,l=1}^{k}z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)\lambda^l_{j,i}[e^l,e^i](z_1)+2\sum_{i,l=1}^{k}z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\lambda^l_{j,i}[e^{i-}(z_2)e^l(z_1)+e^l(z_1)e^{i+}(z_2)]\\
&\quad+2z_1^{-1}\partial_{z_2}\left[\delta\left(\frac{z_2}{z_1}\right)\right]\chi e^j(z_2)\\
&= z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)\sum_{i=1}^{k}[[e^j,e^i],e^i](z_1)+z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\sum_{i,l=1}^{k}2\lambda^l_{j,i}[e^{i-}(z_1)e^l(z_1)+e^l(z_1)e^{i+}(z_1)]\\
&\quad+2z_1^{-1}\partial_{z_2}\left[\delta\left(\frac{z_2}{z_1}\right)\chi e^j(z_2)\right]-2z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\partial_{z_2}\chi e^j(z_2)\\
&= 2\rho z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)e^j(z_1)-z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)2\rho\frac{d}{dz_1}e^j(z_1)\\
&\quad+2z_1^{-1}\partial_{z_2}\left[\delta\left(\frac{z_2}{z_1}\right)\chi e^j(z_1)\right]-2z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\frac{d}{dz_1}\chi e^j(z_1)\\
&= 2(\rho+\chi)\left[z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right)e^j(z_1)-z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\frac{d}{dz_1}e^j(z_1)\right]. \qquad (2.73)
\end{aligned}
$$

Finally

$$
\begin{aligned}
& [\mathcal{L}(n), e^j(m)] \\
= \ & -[e^j(m), \mathcal{L}(n)] \\
= \ & -\mathrm{Res}_{z_1,z_2} z_1^m z_2^{n+1} [e^j(z_1), \mathcal{L}(z_2)] \\
= \ & -2(\rho+\chi)\mathrm{Res}_{z_1,z_2} z_1^m z_2^{n+1} \left[ z_1^{-1}\partial_{z_2}\delta\left(\frac{z_2}{z_1}\right) e^j(z_1) - z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\frac{d}{dz_1}e^j(z_1)\right] \\
= \ & -2(\rho+\chi)\mathrm{Res}_{z_1}[(-n-1)z_1^{m+n}e^j(z_1) - z_1^{m+n+1}\frac{d}{dz_1}e^j(z_1)] \\
= \ & -2(\rho+\chi)[(-n-1)e^j(m+n) - (-m-n-1)e^j(m+n)] \\
= \ & -2(\rho+\chi)me^j(m+n).
\end{aligned}
\tag{2.74}
$$

Thus (2.67) holds because $\{e^j \mid j = 1, ..., k\}$ is a basis of $\mathcal{G}$. □

In the rest of this chapter, we always assume that

$$
\rho + \chi \neq 0. \tag{2.75}
$$

Moreover, we let:

$$
L(z) = \frac{\mathcal{L}(z)}{2(\rho+\chi)}(z) = \sum_{l\in\mathbf{Z}} L(n) z^{-n-1}. \tag{2.76}
$$

By the above Lemma,

$$
[L(m), \xi(n)] = -n\xi(m+n) \qquad \text{for } \xi \in \mathcal{G},\ m, n \in \mathbf{Z}. \tag{2.77}
$$

In particular, we have:

$$
[L(-1), \xi(z)] = \frac{d}{dz}\xi(z), \quad [L(0), \xi(n)] = -n\xi(n), \qquad \text{for } \xi \in \mathcal{G},\ n \in \mathbf{Z}. \tag{2.78}
$$

Note that $M_0 = \mathbf{C}v_0$ is one-dimensional. We denote $1 \otimes v_0 \in M_{0,\chi}$ by $\mathbf{1}$. Then

$$
M_{0,\chi} = U(\hat{\mathcal{G}}_-)\mathbf{1}. \tag{2.79}
$$

Since

$$
L(0) = \frac{1}{2(\rho+\chi)} \sum_{i=1}^{k} (2 \sum_{0<m\in\mathbf{Z}} e^i(-m)e^i(m) + e^i(0)e^i(0)), \tag{2.80}
$$

$$
L(-1) = \frac{1}{(\rho+\chi)} \sum_{i=1}^{k}\sum_{m=0}^{\infty} e^i(-m-1)e^i(m), \tag{2.81}
$$

we have:

$$L(-1)\mathbf{1} = L(0)\mathbf{1} = 0, \quad L(0)v_\lambda = \frac{1}{2(\rho+\chi)}\Omega_{\mathcal{G}} v_\lambda = \frac{\rho_\lambda}{2(\rho+\chi)} v_\lambda \tag{2.82}$$

(cf. (2.47)). Note that (2.78) and (2.82) imply

$$M_{\lambda,\chi} = \bigoplus_{m=0}^{\infty} M_{\lambda,\chi}^{(m+\rho_\lambda/2(\rho+\chi))}, \qquad M_{\lambda,\chi}^{(\mu)} = \{v \in M_{\lambda,\chi} \mid L(0)v = \mu v\}. \tag{2.83}$$

Set

$$M_{0,\chi}^0 = \mathbf{C1}, \ \ M_{0,\chi}^m = M_{0,\chi}^{m-1} + \mathrm{span}\{\xi_1(-n_1)\cdots\xi_m(-n_m) \mid \xi_j \in \mathcal{G}, \ 0 < n_j \in \mathbf{Z}\}, \tag{2.84}$$

for $0 < m \in \mathbf{Z}$. Note we have

$$M_{0,\chi} = \bigcup_{m=0}^{\infty} M_{0,\chi}^m. \tag{2.85}$$

We define a linear map $Y(\cdot, z) : M_{0,\chi} \to LP(M_\lambda, M_\lambda[z^{-1}, z]]$ by induction on $M_{0,\chi}^m$. When $m = 0$, define

$$Y(\mathbf{1}, z) = \mathrm{Id}_{M_{\lambda,\chi}}. \tag{2.86}$$

Suppose that we have defined $Y(v, z)$ for $v \in M_{0,\chi}^p$. Note that any element of $M_{0,\chi}^{p+1} \setminus M_{0,\chi}^p$ is a linear combination the elements of the form $\xi(-n)v$ with $\xi \in \mathcal{G}, \ v \in M_{0,\chi}^p$ and $0 < n \in \mathbf{Z}$. Thus we only need to define:

$$Y(\xi(-n)v, z) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}\xi^-(z)}{dz^{n-1}} Y(v,z) + Y(v,z)\frac{d^{n-1}\xi^+(z)}{dz^{n-1}}\right]. \tag{2.87}$$

Hence we have defined $Y(u, z)$ for $u \in M_{0,\chi}^{p+1}$. Note here $\lambda$ is arbitrary and $Y(\cdot, z)$ can be distinguished by specifying the space it acts.

**Lemma 2.6.** *The map $Y(\cdot, z)$ is well-defined.*

**Proof.** First note that $Y(u, z)$ is well defined for any $u \in M_{0,\chi}^1$ by our definition and

$$Y(\xi(-n)\mathbf{1}, z) = \frac{1}{(n-1)!}\frac{d^{n-1}\xi(z)}{dz^{n-1}} \qquad \text{for } \ \xi \in \mathcal{G}, \ 0 < n \in \mathbf{Z}. \tag{2.88}$$

In particular,

$$Y(\xi(-1)\mathbf{1}, z) = \xi(z). \tag{2.89}$$

For convenience, we identify $\xi(-n)\mathbf{1}$ with $\xi(-n)$ and $\xi(-1)\mathbf{1}$ with $\xi$ when the context is clear.

*Claim 1. For any $u \in M^1_{0,X}$,*

$$Y(u,z)\mathbf{1} = e^{zL(-1)}u. \tag{2.90}$$

Note that (2.78) implies

$$\xi^-(z) = e^{z\mathrm{ad}_{L(-1)}}(\xi(-1)) = e^{zL(-1)}\xi(-1)e^{-zL(-1)} \qquad \text{for } \xi \in \mathcal{G}. \tag{2.91}$$

Hence (2.90) holds for $u = \xi(-1)\mathbf{1} = \xi$ by (2.82). Note that any element of $\hat{\mathcal{G}}_-\mathbf{1}$ is a linear combination of the coefficients of $z$ in $\{\xi^-(z)\mathbf{1} \mid \xi \in \mathcal{G}\}$. Moreover, (2.88) implies

$$Y(\xi^-(x)\mathbf{1},z) = \sum_{n=0}^{\infty} x^n Y(\xi(-n-1)\mathbf{1},z) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\frac{d}{dz}\right)^n \xi(z) = \xi(z+x) \tag{2.92}$$

for $\xi \in \mathcal{G}$. Thus

$$Y(\xi^-(x)\mathbf{1},z)\mathbf{1} = \xi(z+x)\mathbf{1} = e^{(z+x)L(-1)}\xi = e^{zL(-1)}(e^{zL(-1)}\xi) = e^{zL(-1)}\xi^-(x)\mathbf{1}. \tag{2.93}$$

Thus (2.90) holds for $u \in \hat{\mathcal{G}}_-\mathbf{1}$. When $u = \mathbf{1}$, (2.90) trivially holds. We have proved (2.90).

For any $u \in M^1_{0,X}$, we denote

$$Y(u,z) = \sum_{n\in\mathbf{Z}} u_n z^{-n-1}. \tag{2.94}$$

*Claim 2. For any $u \in M^1_{0,X}$,*

$$(u_{-m}\mathbf{1})_{-1} = u_{-m}, \qquad \textit{for } 0 \le m \in \mathbf{Z}. \tag{2.95}$$

It is enough to prove it for $u = \xi(-n-1)$ for $\xi \in \mathcal{G}$, $0 \le n \in \mathbf{Z}$. For $n = 0$, it directly follows from (2.88). For $\xi \in \hat{\mathcal{G}}_-$,

$$\xi(-n-1)_{-m} = \binom{m+n-1}{n}\xi(-m-n) = \binom{m+n-1}{n}(\xi(-m-n)\mathbf{1})_{-1} = (\xi(-n-1)_{-m}\mathbf{1})_{-1}. \tag{2.96}$$

For any $w \in M_{0,X}$, we set

$$w^-(z) = \sum_{m=1}^{\infty} w_{-m} z^{m-1}, \quad w^+(z) = \sum_{m=0}^{\infty} w_m z^{-m-1}. \tag{2.97}$$

By a simple calculation, we have:

$$\mathrm{Res}_{z_1}(z_1-z)^{-1}Y(w,z_1) = w^-(z), \quad \mathrm{Res}_{z_1}(z-z_1)^{-1}Y(w,z_1) = w^+(z). \tag{2.98}$$

*Claim 3.* *For any* $u, v \in M^1_{0,X}$,

$$[Y(u,z_1), v^+(z_2)] = -Y(v^+(z_2 - z_1)u, z_1), \quad [Y(u,z_1), v^-(z_2)] = Y(v^+(-z_1+z_2)u, z_1). \tag{2.99}$$

It is clear when $u = \mathbf{1}$ or $v = \mathbf{1}$ since $Y(\mathbf{1}, z) = \mathrm{Id}_{M_{\lambda,X}}$. When $u, v \in \hat{\mathcal{G}}_-$, (2.99) is equivalent to (2.59) and (2.60). Since for $\xi \in \mathcal{G}$, $0 \leq n \in \mathbf{Z}$,

$$\begin{aligned} Y(L(-1)\xi(-n-1), z) &= (n+1)Y(\xi(-n-2), z) \\ &= (n+1)\frac{1}{(n+1)!}\frac{d^{n+1}\xi(z)}{dz^{n+1}} \\ &= \frac{d}{dz}Y(\xi(-n-1), z) \end{aligned} \tag{2.100}$$

by (2.88), we have

$$Y(L(-1)u, z) = \frac{d}{dz}Y(u,z) \qquad \text{for } u \in M^1_{0,X}. \tag{2.101}$$

Moreover, by (2.78),

$$[L(-1), Y(\xi^-(x)\mathbf{1}, z)] = [L(-1), \xi(z+x)] = \frac{d}{dz}\xi(z+x) = \frac{d}{dz}Y(\xi^-(x)\mathbf{1}, z) \tag{2.102}$$

for $\xi \in \mathcal{G}$. Hence we have:

$$[L(-1), Y(u,z)] = \frac{d}{dz}Y(u,z) \qquad \text{for } u \in M^1_{0,X}. \tag{2.103}$$

Therefore,

$$Y(u, z+x) = e^{x(d/dz)}Y(u,z) = Y(e^{xL(-1)}u, z) \qquad \text{for } u \in M^1_{0,X}. \tag{2.104}$$

Note that by (2.88),

$$\xi(-n-1)^{\pm}(z) = \frac{1}{n!}\frac{d^n\xi^{\pm}(z)}{dz^n} \qquad \text{for } \xi \in \mathcal{G}. \tag{2.105}$$

Now we want to prove (2.99) for $u = \xi^-(x)\mathbf{1}$, $v = \eta^-(y)\mathbf{1}$ for $\xi, \eta \in \mathcal{G}$. By (2.59-60) and (2.104),

$$\begin{aligned} &[Y(u,z_1), v^+(z_2)] \\ &= [\xi(z_1+x), \eta^+(z_2+y)] \end{aligned}$$

$$
\begin{aligned}
&= -Y(\eta^+(z_2+y-z_1-x)\xi, z_1+x) \\
&= -e^{z_1\partial_{z_1}}Y(\eta^+(z_2+y-z_1-x)\xi, z_1) \\
&= -Y(e^{xL(-1)}\eta^+(z_2+y-z_1-x)\xi, z_1) \\
&= -Y([e^{xL(-1)}\eta^+(z_2+y-z_1-x)e^{-xL(-1)}]e^{-xL(-1)}\xi, z_1) \\
&= -Y(e^{x\mathrm{ad}_{L(-1)}}[\eta^+(z_2+y-z_1-x)]\xi^-(x)\mathbf{1}, z_1) \\
&= -Y(e^{x\partial_{z_2}}[\eta^+(z_2+y-z_1-x)]\xi^-(x)\mathbf{1}, z_1) \\
&= -Y(\eta^+(z_2-z_1+y)\xi^-(x)\mathbf{1}, z_1) \\
&= -Y([\eta^-(y)\mathbf{1}]^+(z_2-z_1)\xi^-(x)\mathbf{1}, z_1) \\
&= -Y(v^+(z_2-z_1)u, z_1). \qquad (2.106)
\end{aligned}
$$

Thus the first equation in (2.99) holds. We can similarly prove the second.

*Claim 4. For $u, v \in M^1_{0,X}$, we have:*

$$
[u^-(z), v^-(z)] = ([u_{-1}, v_{-1}]\mathbf{1})^-(z), \quad [u^+(z), v^+(z)] = -([u_{-1}, v_{-1}]\mathbf{1})^+(z). \qquad (2.107)
$$

Note by the second equation in (2.99),

$$
[u_{-1}, v_{-1}]\mathbf{1} = \mathrm{Res}_{z_1,z_2} z_1^{-1} z_2^{-1} Y(v^+(-z_1+z_2)u, z_1)\mathbf{1} = \mathrm{Res}_{z_1} z_1^{-1} Y(v^+(-z_1)u, z_1)\mathbf{1} \quad (2.108)
$$

for $u, v \in M^1_{0,X}$.

$$
\begin{aligned}
Y([u_{-1}, v_{-1}]\mathbf{1}, z) &= \mathrm{Res}_{z_1} z_1^{-1} Y(v^+(-z_1)u, z+z_1) \\
&= \mathrm{Res}_{z_1} z_1^{-1} e^{z_1\partial_z} Y(v^+(-z_1)u, z) \\
&= \sum_{m=0}^{\infty} \frac{(-\partial_z)^{m+1}}{(m+1)!} Y(v_m u, z). \qquad (2.109)
\end{aligned}
$$

By (2.97),

$$
[[u_{-1}, v_{-1}]\mathbf{1}]^{\pm}(z) = \sum_{m=0}^{\infty} \frac{(-\partial_z)^{m+1}}{(m+1)!} (v_m u)^{\pm}(z). \qquad (2.110)
$$

On the other hand, by (2.98)

$$
\begin{aligned}
&[u^-(z), v^-(z)] \\
&= \mathrm{Res}_{z_1} (z_1 - z)^{-1} [Y(u, z_1), v^-(z)]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Res}_{z_1}(z_1-z)^{-1}Y(v^+(-z_1+z)u,z_1)\\
&= \mathrm{Res}_{z_1}\sum_{m=0}^{\infty}(-1)^{m+1}(z_1-z)^{-m-2}Y(v_m u,z_1)\\
&= \mathrm{Res}_{z_1}\sum_{m=0}^{\infty}(-1)^{m+1}\frac{\partial_z^{m+1}}{(m+1)!}(z_1-z)^{-1}Y(v_m u,z_1)\\
&= \sum_{m=0}^{\infty}\frac{(-\partial_z)^{m+1}}{(m+1)!}(v_m u)^-(z)\\
&= [[u_{-1},v_{-1}]\mathbf{1}]^-(z),
\end{aligned}
\tag{2.111}
$$

$$
\begin{aligned}
&[u^+(z),v^+(z)]\\
&= \mathrm{Res}_{z_1}(z-z_1)^{-1}[Y(u,z_1),v^+(z)]\\
&= -\mathrm{Res}_{z_1}(z-z_1)^{-1}Y(v^+(z-z_1)u,z_1)\\
&= -\mathrm{Res}_{z_1}\sum_{m=0}^{\infty}(z-z_1)^{-m-2}Y(v_m u,z_1)\\
&= -\mathrm{Res}_{z_1}\sum_{m=0}^{\infty}(-1)^{m+1}\frac{\partial_z^{m+1}}{(m+1)!}(z-z_1)^{-1}Y(v_m u,z_1)\\
&= \sum_{m=0}^{\infty}-\frac{(-\partial_z)^{m+1}}{(m+1)!}(v_m u)^+(z)\\
&= -[[u_{-1},v_{-1}]\mathbf{1}]^+(z).
\end{aligned}
\tag{2.112}
$$

Finally, let us prove the well-definedness. Since $\xi(-n) = (\xi(-n)\mathbf{1})_{-1}$, any element of $M_{0,X}$ can be written as a linear comination of the elements of the form $u^1_{-1}u^2_{-1}\cdots u^m_{-1}\mathbf{1}$ with $u^j \in M^1_{0,X}$. Hence (2.87) is equivalent to:

$$
Y(u_{-1}v,z) = u^-(z)Y(v,z) + Y(v,z)u^+(z) \qquad \text{for } u \in \hat{\mathcal{G}}_-,\ v \in M^p_{0,X}. \tag{2.113}
$$

Let

$$
\mathcal{I} = \{(i,-n) \mid i = 1,\dots,k,\ 0 < n \in \mathbf{Z}\} \tag{2.114}
$$

and define

$$
(i_1,-n_1) \le (i_2,-n_2) \text{ if } i_1 \le i_2 \text{ or } n_1 \lneq n_2. \tag{2.115}
$$

Denote

$$
e^{(i,-n)} = e^i(-n). \tag{2.116}
$$

Thus $\{e^j \mid j \in \mathcal{I}\}$ is a basis of $\hat{\mathcal{G}}_-$. Suppose that $Y(v,z)$ is well defined for any $v \in M_{0,\chi}^{m-1}$. Since (2.113) is linear in $u$, we only need to prove that if

$$\sum_{j\in\mathcal{I}} e^j_{-1} v^j = 0, \qquad v_j \in M_{0,\chi}^{m-1}, \tag{2.117}$$

then

$$\sum_{i=1}^{k}\sum_{n_i=1}^{q_i}[(e^{j-}(z)Y(v_j,z) + Y(v_j,z)e^{j+}(z)] = 0. \tag{2.118}$$

In order to prove this, let us introduce a new notion:

$$\mathcal{J} = \{\mathbf{J} = \{j_1, j_2, ..., j_m\} \mid j_l \in \mathcal{I}\}. \tag{2.119}$$

By the PBW Theorem, (2.117) can be written as:

$$\sum_{\mathbf{J}\in\mathcal{J}}\sum_{i=1}^{m} \mu_{\mathbf{J},i} e^{j_i}_{-1} e^{j_1}_{-1} \cdots e^{j_{i-1}}_{-1} e^{j_{i+1}}_{-1} \cdots e^{j_m}_{-1}\mathbf{1} + \sum_{j\in\mathcal{I}} e^j_{-1} u^j = 0 \tag{2.120}$$

with

$$u^j \in M_{0,\chi}^{m-2},\ \mu_{\mathbf{J},i} \in \mathbf{C},\ \sum_{i=1}^{m}\mu_{\mathbf{J},i} = 0. \tag{2.121}$$

Let

$$w = -\sum_{j\in\mathcal{I}} e^j_{-1} u^j. \tag{2.122}$$

By our assumption:

$$Y(w,z) = -\sum_{j\in\mathcal{I}}[e^{j-}(z)Y(u^j,z) + Y(u^j,z)e^{j+}(z)]. \tag{2.123}$$

Set

$$\phi_i = e^{j_2}_{-1} \cdots e^{j_{i-1}}_{-1} e^{j_{i+1}}_{-1} \cdots e^{j_m}_{-1}\mathbf{1}, \qquad i = 2, 3, ..., m. \tag{2.124}$$

On the other hand,

$$\begin{aligned} w &= \sum_{\mathbf{J}\in\mathcal{J}}\sum_{i=1}^{m} \mu_{\mathbf{J},i} e^{j_i}_{-1} e^{j_1}_{-1} \cdots e^{j_{i-1}}_{-1} e^{j_{i+1}}_{-1} \cdots e^{j_m}_{-1}\mathbf{1} \\ &= \sum_{\mathbf{J}\in\mathcal{J}}[\sum_{i=2}^{m} \mu_{\mathbf{J},i}[e^{j_i}_{-1}, e^{j_1}_{-1}]\phi_i + e^{j_1}_{-1}[(\mu_{\mathbf{J},1} + \mu_{\mathbf{J},2})e^{j_2}_{-1}\phi_2 + \sum_{i=3}^{m}\mu_{\mathbf{J},i} e^{j_i}_{-1}\phi_i]. \end{aligned} \tag{2.125}$$

Moreover,

$$w^1 = (\mu_{\mathbf{J},1} + \mu_{\mathbf{J},2})e^{j_2}_{-1}\phi_2 + \sum_{i=3}^{m}\mu_{\mathbf{J},i} e^{j_i}_{-1}\phi_i \ \in\ M_{0,\chi}^{m-2}, \tag{2.126}$$

because of $\sum_{i=1}^{k} \mu_{\mathbf{J},i} = 0$. By our inductive assumption:

$$\begin{aligned}
& Y(e^{j_1}_{-1}w^1, z) \\
= \; & e^{j_1-}(z)Y(w^1,z) + Y(w^1,z)e^{j_1+}(z) \\
= \; & (\mu_{\mathbf{J},1} + \mu_{\mathbf{J},2})[e^{j_1-}(z)e^{j_2-}(z)Y(\phi_2,z) + e^{j_1-}(z)Y(\phi_2,z)(z)e^{j_2+}(z) \\
& + e^{j_2-}(z)Y(\phi_2,z)(z)e^{j_1+}(z) + Y(\phi_2,z)(z)e^{j_2+}(z)e^{j_1+}(z)] + \sum_{i=3}^{m} \mu_{\mathbf{J},i}[e^{j_1-}(z)e^{j_i-}(z)Y(\phi_i,z) \\
& + e^{j_1-}(z)Y(\phi_i,z)(z)e^{j_i+}(z) + e^{j_i-}(z)Y(\phi_i,z)(z)e^{j_1+}(z) + Y(\phi_i,z)(z)e^{j_i+}(z)e^{j_1+}(z)] \\
= \; & \mu_{\mathbf{J},1}[e^{j_1-}(z)e^{j_2-}(z)Y(\phi_2,z) + e^{j_1-}(z)Y(\phi_2,z)(z)e^{j_2+}(z) + e^{j_2-}(z)Y(\phi_2,z)(z)e^{j_1+}(z) \\
& + Y(\phi_2,z)(z)e^{j_2+}(z)e^{j_1+}(z)] + \sum_{i=2}^{m} \mu_{\mathbf{J},i}[e^{j_i-}(z)e^{j_1-}(z)Y(\phi_i,z) + e^{j_1-}(z)Y(\phi_i,z)(z)e^{j_i+}(z) \\
& + e^{j_i-}(z)Y(\phi_i,z)(z)e^{j_1+}(z) + Y(\phi_i,z)(z)e^{j_1+}(z)e^{j_i+}(z)] \\
& + \sum_{i=2}^{m} \mu_{\mathbf{J},i}[[e^{j_1-},(z)e^{j_i-}(z)]Y(\phi_i,z) + (\phi_i,z)(z)[e^{j_i+}(z), e^{j_1+}(z)]]. \qquad (2.127)
\end{aligned}$$

Thus by (2.95) and (2.107),

$$\begin{aligned}
& Y(w,z) \\
= \; & \sum_{\mathbf{J}\in\mathcal{J}} [\sum_{i=2}^{m} \mu_{\mathbf{J},i}[([e^{j_i}_{-1}, e^{j_1}_{-1}]\mathbf{1})^-(z)Y(\phi_i,z) + Y(\phi_i,z)([e^{j_i}_{-1}, e^{j_1}_{-1}]\mathbf{1})^+(z)] + \sum_{\mathbf{J}\in\mathcal{J}} Y(e^{j_1}_{-1}w^1,z)] \\
= \; & \sum_{\mathbf{J}\in\mathcal{J}} \{\mu_{\mathbf{J},1}[e^{j_1-}(z)e^{j_2-}(z)Y(\phi_2,z) + e^{j_1-}Y(\phi_2,z)(z)e^{j_2+}(z) + e^{j_2-}Y(\phi_2,z)(z)e^{j_1+}(z) \\
& + Y(\phi_2,z)(z)e^{j_2+}(z)e^{j_1+}(z)] + \sum_{i=2}^{m} \mu_{\mathbf{J},i}[e^{j_i-}(z)e^{j_1-}(z)Y(\phi_i,z) \\
& + e^{j_i-}(z)Y(\phi_i,z)(z)e^{j_1+}(z) + Y(\phi_i,z)(z)e^{j_1+}(z)e^{j_i+}(z)]\} \\
= \; & \sum_{\mathbf{J}\in\mathcal{J}} \{\mu_{\mathbf{J},1}[e^{j_1-}(z)Y(e^{j_2}_{-1}\phi_2,z) + Y(e^{j_2}_{-1}\phi_2,z)(z)e^{j_1+}(z)] \\
& + \sum_{i=2}^{m} \mu_{\mathbf{J},i}[e^{j_i-}(z)Y(e^{j_1}_{-1}\phi_i,z) + Y(e^{j_1}_{-1}\phi_i,z)(z)e^{j_i+}(z)]\}, \qquad (2.128)
\end{aligned}$$

which is equivalent to (2.118) by (2.120). Thus we have proved that $Y(|_{M^m_{0,X}}, z)$ is well defined. By induction, we have proved our lemma. □

**Lemma 2.7**. *For* $u \in M_{0,X}$,

$$[L(-1), Y(u,z)] = \frac{d}{dz}Y(u,z), \;\; [L(0), Y(u,z)] = z\frac{d}{dz}Y(u,z) + Y(L(0)u,z). \qquad (2.129)$$

*Proof.* Note that (2.124) holds for $u \in M^1_{0,X}$ by (2.77-78), (2.82) and (2.103). Assume it holds for $u \in M^{m-1}_{0,X}$. Let $u = v_{-1}w$ with $v \in (\hat{\mathcal{G}}_- \bigcap M^{(p)}_{0,X})$ and $w \in (M^{m-1}_{0,X} \bigcap M^{(q)}_{0,X})$. It can be proved that $u \in M_{0,\chi}^{(p+q)}$.

$$\begin{aligned}
&[L(-1), Y(u,z)] \\
= \;& [L(-1), v^-(z)Y(w,z) + Y(w,z)v^+(z)] \\
= \;& [L(-1), v^-(z)]Y(w,z) + v^-(z)[L(-1), Y(w,z)] \\
& + [L(-1), Y(w,z)]v^+(z) + Y(w,z)[L(-1), v^+(z)] \\
= \;& \frac{dv^-(z)}{dz}Y(w,z) + v^-(z)\frac{d}{dz}Y(w,z) + \left(\frac{d}{dz}Y(w,z)\right)v^+(z) + Y(w,z)\frac{dv^+(z)}{dz} \\
= \;& \frac{d}{dz}[v^-(z)Y(w,z) + Y(w,z)v^+(z)] \\
= \;& \frac{d}{dz}Y(u,z). \qquad (2.130)
\end{aligned}$$

$$\begin{aligned}
&[L(0), Y(u,z)] \\
= \;& [L(0), v^-(z)Y(w,z) + Y(w,z)v^+(z)] \\
= \;& [L(0), v^-(z)]Y(w,z) + v^-(z)[L(0), Y(w,z)] \\
& + [L(0), Y(w,z)]v^+(z) + Y(w,z)[L(0), v^+(z)] \\
= \;& z\frac{dv^-(z)}{dz}Y(w,z) + v^-(z)z\frac{d}{dz}Y(w,z) + \left(z\frac{d}{dz}Y(w,z)\right)v^+(z) + Y(w,z)z\frac{dv^+(z)}{dz} \\
& (p+q)[v^-(z)Y(w,z) + Y(w,z)v^+(z)] \\
= \;& z\frac{d}{dz}Y(u,z) + Y(L(0)u,z). \quad \square \qquad (2.131)
\end{aligned}$$

**Lemma 2.8.** *For $u, v \in M_{0,\chi}$ and $w \in M_{\lambda,\chi}$, there exist positive integers $m, n$ such that*

$$(z_1 - z_2)^m [Y(u,z_1), Y(v,z_2)] = 0, \qquad (2.132)$$

$$(z_0 + z_2)^n Y(u, z_0 + z_2)Y(v,z_2)w = (z_0 + z_2)^n Y(Y(u,z_0)v, z_2)w. \qquad (2.133)$$

*Proof.* Note that for any $w^1 \in M_{0,\chi}^{(i)}$ and $w^2 \in M_{\lambda,\chi}^{(j+\rho_\lambda/2(\rho+\chi))}$, by (2.129),

$$w_l^1 w^2 = 0 \qquad \text{for } m(w^1, w^2) = i + j \leq l \in \mathbf{Z}. \qquad (2.134)$$

We may assume that $u, v, w$ are homogeneous and $u \in M_{0,\chi}^p$ and $v \in M_{0,\chi}^q$.

*Step 1.* $p = 1$.

If $u \in \mathbf{C1}$, the lemma trivially holds because of $Y(\mathbf{1}, z) = \mathrm{Id}_{M_{\lambda,\chi}}$. Hence we may assume $u \in \hat{\mathcal{G}}_{-}\mathbf{1}$. Suppose $q = 0$. Then we may assume $v = \mathbf{1}$. Note that (2.132) trivially holds. Moreover,

$$\begin{aligned}
& (z_0+z_2)^{m(u,w)}Y(u,z_0+z_2)Y(\mathbf{1},z_2)w \\
= \;& (z_0+z_2)^{m(u,w)}Y(u,z_0+z_2)w \\
= \;& \sum_{l=1-m(u,w)}^{\infty} u_{-l}w(z_0+z_2)^{m(u,w)+l-1} \\
= \;& \sum_{l=1-m(u,w)}^{\infty} u_{-l}w(z_2+z_0)^{m(u,w)+l-1} \\
= \;& (z_2+z_0)^{m(u,w)}Y(u,z_2+z_0)w \\
= \;& (z_2+z_0)^{m(u,w)}Y(u^{-}(z_0)\mathbf{1},z_2)w \\
= \;& (z_2+z_0)^{m(u,w)}Y(Y(u,z_0)\mathbf{1},z_2)w. \qquad (2.135)
\end{aligned}$$

Suppose that the Lemma holds for $q < j$ and $j \geq 1$. Then we may assume $v = v^1_{-1}v^2$ with $v^1 \in \hat{\mathcal{G}}_{-}$ and $v^2 \in M^{j-1}_{0,\chi}$. By assumption, there exists a positive integer $m > m(v^1, u)$ such that

$$(z_1-z_2)^m[Y(u,z_1),Y(v^2,z_2)] = (z_1-z_2)^m[Y(v_l u,z_1),Y(v^2,z_2)] = 0, \qquad (2.136)$$

for $0 \leq l < m(v^1, u)$, where $v_l u \in \hat{\mathcal{G}}_{-}\mathbf{1}$. Then by (2.99), we have:

$$\begin{aligned}
& (z_1-z_2)^m[Y(u,z_1),Y(v,z_2)] \\
= \;& (z_1-z_2)^m[Y(u,z_1),v^{1-}(z_2)Y(v^2,z_2)+Y(v^2,z_2)v^{1+}(z_2)] \\
= \;& (z_1-z_2)^m[Y(u,z_1),v^{1-}(z_2)]Y(v^2,z_2)+(z_1-z_2)^m v^{1-}(z_2)[Y(u,z_1),Y(v^2,z_2)] \\
& +(z_1-z_2)^m[Y(u,z_1),Y(v^2,z_2)]v^{1+}(z_2)+(z_1-z_2)^m Y(v^2,z_2)[Y(u,z_1),v^{1+}(z_2)] \\
= \;& (z_1-z_2)^m Y(v^{1+}(-z_1+z_2)u,z_1)Y(v^2,z_2)-(z_1-z_2)^m Y(v^2,z_2)Y(v^{1+}(z_2-z_1)u,z_1) \\
= \;& (z_1-z_2)^m[Y(v^{1+}(-z_1+z_2)u,z_1)-Y(v^{1+}(z_2-z_1)u,z_1)]Y(v^2,z_2) \\
= \;& 0. \qquad (2.137)
\end{aligned}$$

Thus (2.132) holds. Furthermore, there exists a positive integer $n$ such that

$$(z_0+z_2)^n Y(u,z_0+z_2)Y(v^2,z_2) = (z_0+z_2)^n Y(Y(u,z_0)v^2,z_2), \qquad (2.138)$$

$$(z_0+z_2)^n Y(v_l^1 u, z_0+z_2)Y(v^2,z_2) = (z_0+z_2)^n Y(Y(v_l^1 u, z_0)v^2, z_2) \tag{2.139}$$

for $0 \leq l < m(v^1, u)$. Again by (2.99),

$$[Y(u,z_0), v_{-1}^1] = \mathrm{Res}_{z_2} z_2^{-1} Y(v^+(-z_0+z_2)u, z_0) = Y(v^+(-z_0)u, z_0), \tag{2.140}$$

$$\begin{aligned}
& (z_0+z_2)^n Y(Y(u,z_0)v,z_2) \\
= \ & (z_0+z_2)^n Y(Y(u,z_0)v_{-1}^1 v^2, z_2) \\
= \ & (z_0+z_2)^n [Y([Y(u,z_0), v_{-1}^1]v^2, z_2) + Y(v_{-1}^1 Y(u,z_0)v^2, z_2)] \\
= \ & (z_0+z_2)^n [Y(Y(v^{1+}(-z_0)u, z_0)v^2, z_2) + v^{1-}(z_2)Y(Y(u,z_0)v^2,z_2) + Y(Y(u,z_0)v^2,z_2)v^{1+}(z_2)] \\
= \ & (z_0+z_2)^n [Y(v^{1+}(-z_0)u, z_0+z_2)Y(v^2,z_2) + v^{1-}(z_2)Y(u,z_0+z_2)Y(v^2,z_2) \\
& +Y(u,z_0+z_2)Y(v^2,z_2)v^{1+}(z_2)] \\
= \ & (z_0+z_2)^n [[Y(u,z_0+z_2), v^{1-}(z_2)]Y(v^2,z_2) + v^{1-}(z_2)Y(u,z_0+z_2)Y(v^2,z_2) \\
& +Y(u,z_0+z_2)Y(v^2,z_2)v^{1+}(z_2)] \\
= \ & (z_0+z_2)^n [Y(u,z_0+z_2)v^{1-}(z_2)Y(v^2,z_2) + Y(u,z_0+z_2)Y(v^2,z_2)v^{1+}(z_2)] \\
= \ & (z_0+z_2)^n Y(u,z_0+z_2)Y(v,z_2).
\end{aligned} \tag{2.141}$$

Thus (2.133) holds.

*Step 2.* $p > 1$

We prove by induction on $p$. Suppose that the lemma holds for $p < i$. When $p = i$, we may assume $u = u_{-1}^1 u^2$ with $u^1 \in \hat{\mathcal{G}}_{-}\mathbf{1}$ and $u^2 \in M_{0,X}^{i-1}$. By the proof of the Jacobi identity in the last chapter, the Jacobi identity holds for $u^1$ and $v$. Thus

$$\begin{aligned}
& [Y(u^1,z_1), Y(v,z_2)] \\
= \ & \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(u^1,z_0)v,z_2) \\
= \ & \mathrm{Res}_{z_0}\left(\frac{1}{z_1-z_0-z_2} + \frac{1}{z_2-z_1+z_0}\right) Y(Y(u^1,z_0)v,z_2) \\
= \ & \mathrm{Res}_{z_0}\left(\frac{1}{z_1-z_2-z_0} - \frac{1}{-z_2+z_1-z_0}\right) Y(Y(u^1,z_0)v,z_2) \\
= \ & Y(u^{1+}(z_1-z_2)v,z_2) - Y(u^{1+}(-z_2+z_1)v,z_2).
\end{aligned} \tag{2.142}$$

Thus we have the following generalization of (2.99):

$$[u^{1+}(z_1), Y(v, z_2)] = Y(u^{1+}(z_1 - z_2)v, z_2), \;\; [u^{1-}(z_1), Y(v, z_2)] = -Y(u^{1+}(-z_2 + z_1)v, z_2). \tag{2.143}$$

By assumption, there exists a positive integer $m > m(u^1, v)$ such that

$$(z_1 - z_2)^m [Y(u^2, z_1), Y(v, z_2)] = (z_1 - z_2)^m [Y(u^2, z_1), Y(u^1_l v, z_2)] = 0 \tag{2.144}$$

for $0 \leq l < m(u^1, v)$. Now we have:

$$\begin{aligned}
&(z_1 - z_2)^m [Y(u, z_1), Y(v, z_2)] \\
= \;& (z_1 - z_2)^m [u^{1-}(z_1) Y(u^2, z_1) + Y(u^2, z_1) u^{1+}(z_1), Y(v, z_2)] \\
= \;& (z_1 - z_2)^m [u^{1-}(z_1), Y(v, z_2)] Y(u^2, z_1) + (z_1 - z_2)^m u^{1-}(z_1) [Y(u^2, z_1), Y(v, z_2)] \\
& + (z_1 - z_2)^m [Y(u^2, z_1), Y(v, z_2)] u^{1+}(z_1) + (z_1 - z_2)^m Y(u^2, z_1) [u^{1+}(z_1), Y(v, z_2)] \\
= \;& -(z_1 - z_2)^m Y(u^{1+}(-z_2 + z_1)v, z_2) Y(u^2, z_1) + (z_1 - z_2)^m Y(u^2, z_1) Y(u^{1+}(z_1 - z_2)v, z_2) \\
= \;& (z_1 - z_2)^m Y(u^2, z_1) [Y(u^{1+}(z_1 - z_2)v, z_2) - Y(u^{1+}(-z_2 + z_1)v, z_2)] \\
= \;& 0.
\end{aligned} \tag{2.145}$$

This proves (2.132).

By assumption again, there exists a positive integer $n$ such that

$$(z_0 + z_2)^n Y(u^2, z_0 + z_2) Y(v, z_2) = (z_0 + z_2)^n Y(Y(u^2, z_0)v, z_2), \tag{2.146}$$

$$(z_0 + z_2)^n Y(u^2, z_0 + z_2) Y(u^1_l v, z_2) = (z_0 + z_2)^n Y(Y(u^2, z_0) u^1_l v, z_2), \tag{2.147}$$

for $0 \leq l < m(u^1, v)$. Note that by (2.95) and (2.143),

$$\begin{aligned}
&(z_0 + z_2)^n Y(Y(u, z_0)v, z_2) w \\
= \;& (z_0 + z_2)^n [Y(u^{1-}(z_0) Y(u^2, z_0)v, z_2) w + Y(Y(u^2, z_0) u^{1+}(z_0)v, z_2) w] \\
= \;& (z_0 + z_2)^n [Y((u^{1-}(z_0)\mathbf{1})_{-1} Y(u^2, z_0)v, z_2) w + Y(u^2, z_0 + z_2) Y(u^{1+}(z_0)v, z_2) w] \\
= \;& (z_0 + z_2)^n [u^{1-}(z_2 + z_0) Y(Y(u^2, z_0)v, z_2) w + Y(Y(u^2, z_0)v, z_2) u^{1+}(z_2 + z_0) w] \\
& + (z_0 + z_2)^n Y(u^2, z_0 + z_2) [u^{1+}(z_0 + z_2), Y(v, z_2)] w \\
= \;& (z_0 + z_2)^n [u^{1-}(z_2 + z_0) Y(u^2, z_0 + z_2) Y(v, z_2) w + Y(u^2, z_0 + z_2) Y(v, z_2) u^{1+}(z_2 + z_0) w]
\end{aligned}$$

$$+(z_0+z_2)^n Y(u^2,z_0+z_2)[u^{1+}(z_0+z_2)Y(v,z_2)-Y(v,z_2)u^{1+}(z_0+z_2)]w$$
$$= (z_0+z_2)^n[u^{1-}(z_2+z_0)Y(u^2,z_0+z_2)Y(v,z_2)+Y(u^2,z_0+z_2)u^{1+}(z_0+z_2)Y(v,z_2)]w$$
$$= (z_0+z_2)^n Y(u,z_0+z_2)Y(v,z_2)w. \tag{2.148}$$

Thus (2.133) holds. □

Let

$$\omega = \frac{1}{2(\rho+\chi)}\sum_{i=1}^{k} e^i_{-1}e^i_{-1}\mathbf{1}. \tag{2.149}$$

**Theorem 2.8.** *The family* $(M_{0,\chi}, Y(\cdot,z), \mathbf{1}, \omega)$ *forms a vertex operator algebra.*

*Proof.* Note that by (2.76) and (2.87),

$$Y(\omega,z) = L(z). \tag{2.150}$$

Moreover, (2.82), (2.86), (2.129) and Lemma 2.8 imply that we only need to (1.27) and the second equation in (1.29). Easily verify

$$L(0)\omega = 2\omega,\quad L(1)\omega = 0,\quad L(2)\omega = \frac{k\chi}{2(\rho+\chi)}\mathbf{1}. \tag{2.151}$$

Since (2.82), (2.129) and Lemma 2.8 imply the Jacobi identity (1.18), we have:

$$[L(z_1), L(z_2)]$$
$$= \mathrm{Res}_{z_0} z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(\omega,z_0)\omega, z_2)$$
$$= z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) Y(L(-1)\omega,z_2) - 2z_2^{-1}\partial_{z_1}\delta\left(\frac{z_1}{z_2}\right) Y(\omega,z_2) - z_2^{-1}\partial_{z_1}^3\delta\left(\frac{z_1}{z_2}\right)\frac{k\chi}{2(\rho+\chi)}Y(\mathbf{1},z_2)$$
$$= z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\frac{dL(z_2)}{dz_2} - 2z_2^{-1}\partial_{z_1}\delta\left(\frac{z_1}{z_2}\right) L(z_2) - z_2^{-1}\partial_{z_1}^3\delta\left(\frac{z_1}{z_2}\right)\frac{k\chi}{2(\rho+\chi)}, \tag{2.152}$$

whose component form is (1.27) with the rank $= k\chi/2(\rho+\chi)$. Moreover, the first equation in (2.129) imply

$$u_{-n-1} = \frac{\mathrm{ad}^n_{L(-1)}}{n!}(u_{-1}),\qquad Y(u,z) = \sum_{l\in\mathbb{Z}} u_l z^{-l-1}\qquad \text{for } u\in M_{0,\chi}. \tag{2.153}$$

Hence

$$u^-(z) = \sum_{n=0}^{\infty} u_{-n-1}z^n = e^{z\mathrm{ad}_{L(-1)}}(u_{-1}) = e^{zL(-1)}u_{-1}e^{-zL(-1)}\qquad \text{for } u\in M_{0,\chi}. \tag{2.154}$$

Now for any $v = v^1_{-1}v^2_{-1}\cdots v^j_{-1}\mathbf{1}$ with $v^i \in \hat{\mathcal{G}}_{-}\mathbf{1}$, we have:

$$\begin{aligned}
& Y(v,z)\mathbf{1} \\
= \;& v^{1-}(z)v^{2-}(z)\cdots v^{j-}(z)\mathbf{1} \\
= \;& [e^{zL(-1)}v^1_{-1}e^{-zL(-1)}][e^{zL(-1)}v^2_{-1}e^{-zL(-1)}]\cdots[e^{zL(-1)}v^j_{-1}e^{-zL(-1)}]\mathbf{1} \\
= \;& e^{zL(-1)}v^1_{-1}v^2_{-1}\cdots v^j_{-1}\mathbf{1} \\
= \;& e^{zL(-1)}v.
\end{aligned} \tag{2.155}$$

Since any element of $M_{0,\chi}$ is a linear combination of the elements like the above $v$, (2.155) implies the second equation in (1.29). □

An *ideal* of a vertex operator algebra $(V, Y(\cdot,z), \mathbf{1}, \omega)$ is a subspace of $V$ such that

$$Y(u,z)I \subset I[z^{-1};z]] \qquad \text{for any } u \in V. \tag{2.156}$$

An algebra $(V, Y(\cdot,z), \mathbf{1}, \omega)$ is called *simple* if the only ideals of $V$ are $V$ and $\{0\}$. Since $L(0)I \subset I$, an ideal $I$ is always a graded subspace. Moreover, by the skew symmetry (1.58),

$$Y(v,z)V \subset I \qquad \text{for any } v \in I. \tag{2.157}$$

Thus $M_{0,\chi}$ has a unique maximal proper ideal $J_{0,\chi}$ because it is a highest-weight $\hat{\mathcal{G}}$-module. Furthermore, (2.157) implies that we have the induced vertex operator map $Y(\cdot,z)$ (we still use the same notation which can always be distinguished by specifying the space it acts) on

$$V_\chi = M_{0,\chi}/J_{0,\chi}. \tag{2.158}$$

Note $J_{0,\chi}\bigcap(\mathbf{C}\omega + \mathbf{C}\mathbf{1}) = \{0\}$ by the last equation in (2.151). Thus we can identify the elements of $\mathbf{C}\omega + \mathbf{C}\mathbf{1}$ with their image in $V_\chi$. Easily see:

**Theorem 2.10.** *The family $(V_\chi, Y(\cdot,z), \mathbf{1}, \omega)$ forms a simple vertex operator algebra.*

A *module* of a vertex operator algebra $(V, Y(\cdot,z), \mathbf{1}, \omega)$ is a graded space $W = \bigoplus_{l\in\mathbf{C}} W^{(l)}$ with a linear map $Y^W(\cdot,z)\, V \to LM(W, W[z^{-1};z]])$ such that (2.132-133) hold for $u, v \in V$, $w \in W$ and

$$Y^W(\mathbf{1},z) = \mathrm{Id}_W, \quad L(0)u_n w = (p+q-n-1)u^W_n w, \quad Y^W(u,z) = \sum_{l\in\mathbf{Z}} u^W_l z^{-l-1}, \tag{2.159}$$

when $u \in V^{(p)}$ and $w \in W^{(q)}$. A *submodule* $W_1$ of $W$ is a subspace such that

$$Y^W(u,z)W_1 \subset W_1[z^{-1},z]] \qquad \text{for any } u \in V. \tag{2.160}$$

A module $W$ is called *irreducible* of the only submodules of $W$ are $W$ and $\{0\}$. By (2.83), (2.129) and Lemma 2.8,

**Theorem 2.11.** *The family $(M_{\lambda,\chi}, Y(\cdot,z))$ is a module of the vertex operator algebra $(M_{0,\chi}, Y(\cdot,z), \mathbf{1}, \omega)$.*

Since any submodule of $M_{\lambda,\chi}$ is a graded subspace by (2.159) and $M_{\lambda,\chi}$ is a highest-weight $\hat{\mathcal{G}}$-module, there exists a unique maximal proper submodule $J_{\lambda,\chi}$. We denote the induced map of $Y(\cdot,z)$ on

$$V_{\lambda,\chi} = M_{\lambda,\chi}/J_{\lambda,\chi} \tag{2.161}$$

by $Y^\lambda(\cdot,z)$. Then $(V_{\lambda,\chi}, Y^\lambda(\cdot,z))$ is an irreducible module of the vertex operator algebra $(M_{0,\chi}, Y(\cdot,z), \mathbf{1}, \omega)$. We remark that in general, we do not have:

$$Y(J_{0,\chi}, z)M_{\lambda,\chi} \subset J_{\lambda,\chi}[z^{-1}; z]]. \tag{2.162}$$

**Theorem 2.12.** *When $\chi$ is a positive integer, (2.162) holds if and only if $\langle\theta,\lambda\rangle \le \chi$, where $\theta$ is the highest root of $\mathcal{G}$. Therefore, $(V_{\lambda,\chi}, Y^\lambda(\cdot,z))$ is an irreducible module of the simple vertex operator algebra $(V_\chi, Y(\cdot,z), \mathbf{1}, \omega)$ when $\chi$ is a positive integer and $\langle\theta,\lambda\rangle \le \chi$.*

*Proof.* Suppose that $\chi$ is a positive integer. Let $e_\theta \in \mathcal{G}_\theta$ be a root vector (cf. (2.20)). Let

$$w^0 = (e_\theta(-1))^{\chi+1}\mathbf{1}, \quad W_\theta = \text{span}\{\xi(0)w^0 \mid \xi \in \mathcal{G}\}. \tag{2.163}$$

Then $W_\theta$ forms a finite-dimensional irreducible module of $\mathcal{G}$ with the action:

$$\xi(v) = \xi(0)(v) \qquad \text{for } \xi \in \mathcal{G},\ v \in W_\theta. \tag{2.164}$$

Moreover, the highest weight of $W_\theta$ is $(\chi+1)\theta$ and $w^0$ is a corresponding highest-weight vector.

Note that by our definition (2.87) (or (2.113)) of vertex operators,

$$\text{Res}_z z^{-1} Y(w^0, z)v_\lambda = (e_\theta(-1))^{\chi+1} v_\lambda. \tag{2.165}$$

Moreover by [K],

$$J_{0,\chi} = U(\hat{\mathcal{G}}_-)W_\theta. \tag{2.166}$$

Thus if (2.162) holds, we have $(e_\theta(-1))^{\chi+1}v_\lambda \in J_{\lambda,\chi}$, which implies $\langle\theta,\lambda\rangle \leq \chi$ by [H].

Next we assume that $\langle\theta,\lambda\rangle \leq \chi$. This implies that

$$\lambda - (\chi+1)\theta \text{ is not a weight of } M_\lambda \tag{2.167}$$

(cf. [H]). By (2.166),

$$\xi(n)v = 0 \qquad \text{for } \xi \in \mathcal{G},\ v \in W_\theta,\ 0 < n \in \mathbf{Z}. \tag{2.168}$$

Then by the Jacobi identity,

$$[\xi(z_1), Y(v,z_0)] = z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)Y(\xi(0)v,z) \qquad \text{for } \xi \in \mathcal{G},\ v \in W_\theta, \tag{2.168}$$

which implies

$$[\xi(m), v(n)] = u(m+n), \qquad u = \xi(0)v, \tag{2.169}$$

where $Y(w,z) = \sum_{l\in\mathbf{Z}} w(l)z^{-l-\chi-1}$ for $w \in W_\theta$. Since $L(0)w = (\chi+1)w$ by (2.78) and (2.82), we have $[L(0), w(l)] = lw(l)$ for $w \in W_\theta,\ l \in \mathbf{Z}$. Thus we have:

$$w(n)v = 0 \qquad \text{for } w \in W_\theta,\ v \in M_\lambda,\ 0 < n \in \mathbf{Z}. \tag{2.170}$$

We want to prove (2.170) for $n = 0$. By (2.169), the subspace of

$$M = \text{span}\{w(0)v \mid w \in W_\theta,\ v \in M_\lambda\} \tag{2.171}$$

is a $\mathcal{G}$-submodule of $M_\lambda$. Suppose that (2.170) does not hold for $n = 0$, then $M = M_\lambda$ because $M_\lambda$ is an irreducible $\mathcal{G}$-module. In particular, $v_\lambda \in M$. Since $v_\lambda$ is a highest-weight vector, we can prove by induction on the weights of $W_\theta$ that

$$v_\lambda = \sum_{1=1}^{j} w^i(0)v^i, \quad w^i(0)v^i \neq 0, \qquad \text{where } w^i,\ v^i \text{ are weight vectors} \tag{2.172}$$

with $\text{weight}(w^l) < \text{weight}(w^1) = (\chi+1)\theta,\ 1 < l$.

This implies that $\lambda - (\chi+1)\theta$ is a weight of $M_\lambda$. A contradiction.

Set

$$M' = \mathrm{span}\{w(-n)v \mid v \in M_\lambda, w \in W_\theta, 0 < n \in \mathbf{Z}\} \ \subset \bigoplus_{l=1}^{\infty} M_{\lambda,\chi}^{(l+\rho_\lambda/2(\rho+\chi))}. \tag{2.173}$$

Moreover, (2.169) shows that

$$\hat{B}_+(M') \subset M'. \tag{2.174}$$

Thus

$$U(\hat{\mathcal{G}}_-)M' = U(\hat{\mathcal{G}})M' \tag{2.175}$$

is a proper graded $\hat{\mathcal{G}}$-submodule of $M_{\lambda,\chi}$, which is equivalent to that it is a proper $M_{0,\lambda}$-submodule. Thus

$$U(\hat{\mathcal{G}}_-)M' \subset J_{\lambda,\chi}. \tag{2.176}$$

Hence we have proved that

$$Y(w,z)M_\lambda \subset J_{\lambda,\chi}[z^{-1};z]] \qquad \text{for any } \ w \in W_\theta. \tag{2.177}$$

Set

$$J = \{w \in J_{0,\chi} \mid Y(w,z)M_\lambda \subset J_{\lambda,\chi}[z^{-1};z]]\}. \tag{2.178}$$

Then $W_\theta \subset J$. Moreover, by the associativity (2.133), for $w \in J$, $u \in M_{0,\chi}$ and $v \in M_\lambda$, there exits a positive integer $m$ such that

$$(z_0+z_2)^m Y(Y(u,z_0)w,z_2)v = (z_0+z_2)^m Y(u,z_0+z_2)Y(w,z_2)v \ \in J_{\lambda,\chi}[z_0^{-1},z_2^{-1};z_0,z_2]]. \tag{2.179}$$

This implies

$$Y(u,z_0)w \in J[z_0^{-1};z_0]], \tag{2.180}$$

which implies that $J$ is a $M_{0,\chi}$-submodule. By (2.166), we have $J = J_{0,\lambda}$. Next Set

$$M_1 = \{v \in M_{\lambda,\chi} \mid Y(w,z)v \in J_{\lambda,\chi}[z^{-1};z]], \text{ for } w \in J_{0,\chi}\}. \tag{2.181}$$

By the associativity (2.133), we can similarly prove $M_1 = M_{\lambda,\chi}$. This completes the proof of Theorem 2.12. □

**Remark 2.12.** (a) The modules $\{V_{\lambda,\chi} \mid \langle\theta,\lambda\rangle \leq \chi\}$ are the all the irreducible modules of $V_\chi$ in terms of FLM's definitions of a module of a vertex operator algebra (cf. [FLM3] or [FHl]).

(b) Frenkel and Zhu [FZ] proved Theorem 2.10 and Theorem 2.12 by Zhu's associative algebra $A(V)$ (cf. [Z]).

(c) Dong and Lepowsky proved Theorem 2.12 by the tensors of level-one $\hat{\mathcal{G}}$-modules.

# Chapter 3

In this chapter, we shall present the construction of vertex operator algebras and modules associated with integral lattices. Let us first recall some basic definitions.

A *(rational) lattice* $L$ is a free abelian group of finite rank with a $\mathbf{Q}$-valued symmetric $\mathbf{Z}$-bilinear form $\langle\cdot,\cdot\rangle$. The lattice $L$ is called *integral* (respectively, *even*) if $\langle L, L\rangle \subset \mathbf{Z}$ (respectively, $\langle\alpha,\alpha\rangle \in 2\mathbf{Z}$ for all $\alpha \in L$). Moreover, if $0 < \langle\alpha,\alpha\rangle$ for all $0 \neq \alpha \in L$, then we say that $L$ is *positive definite.*

Let $L$ be an integral even lattice with the symmetric nondegenerate $\mathbf{Z}$-bilinear form $\langle\cdot,\cdot\rangle$. Set

$$H = \mathbf{C} \otimes_{\mathbf{Z}} L \tag{3.1}$$

and extend $\langle\cdot,\cdot\rangle$ to $H$ canonically. Let

$$\mathcal{L} = \{h \in H \mid \langle h,\alpha\rangle \in \mathbf{Z}, \text{ for } \alpha \in L\}. \tag{3.2}$$

Then $\mathcal{L}$ is a rational lattice and $L$ is a sublattice of $\mathcal{L}$ under the identification $\alpha \leftrightarrow 1 \otimes \alpha$. When $L$ is positive definite, $\mathcal{L}/L$ is a finite group. In particular, when $L$ is a root lattice ($\mathbf{Z}$-span of the root system) of a simple Lie algebra of type A, D and E, $\mathcal{L}$ is the corresponding weight lattice. Let $\{\alpha_1, \alpha_2, ..., \alpha_q\}$ be a basis of $L$. We define a $\mathbf{Z}$-bilinear map $F(\cdot,\cdot)$ : $L \times L \to \{1, -1\}$ by:

$$F(\alpha_i,\alpha_j) = \begin{cases} (-1)^{\langle\alpha_i,\alpha_j\rangle} & \text{if } i < j \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

Then we have:

$$F(\alpha,\beta)F(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle} \qquad \text{for } \alpha,\ \beta \in L. \tag{3.4}$$

Choose representatives $\{\lambda_j \mid j \in \mathcal{I}\}$ from the cosets in $\mathcal{L}$, that is,

$$\mathcal{L} = \bigcup_{j\in\mathcal{I}} \lambda_j + L \qquad \text{is a disjoint union,} \tag{3.5}$$

such that the representative of $L$ is 0. We extend $F(\cdot,\cdot)$ to $L \times \mathcal{L}$ by

$$F(\alpha,\beta+\lambda_j) = F(\alpha,\beta) \qquad \text{for } j \in \mathcal{I},\ \alpha,\beta \in L. \tag{3.6}$$

Viewing $H$ as an abelian Lie algebra, we associate with it an affine Lie algebra:

$$\hat{H} = H \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \tag{3.7}$$

with the Lie operation $[\cdot, \cdot]$:

$$[\hat{H}, c] = 0, \qquad [\alpha(l), \beta(j)] = \langle \alpha, \beta \rangle l \delta_{l+j,0} c, \tag{3.8}$$

where

$$\alpha(l) = \alpha \otimes t^l \qquad \text{for } \alpha,\ \beta \in H;\ \ l,\ j \in \mathbf{Z}. \tag{3.9}$$

We set

$$\hat{H}_{\pm} = \text{span}\,\{\alpha(l) \mid \alpha \in H,\ 0 < \pm l \in \mathbf{Z}\}. \tag{3.10}$$

Define $\mathbf{C}\{\mathcal{L}\}$ to be the $\mathbf{C}$-vector space with a basis

$$\{\iota(\gamma) \mid \gamma \in \mathcal{L}\}, \tag{3.11}$$

and for each $\alpha \in L$, we define $\hat{\alpha} \in \text{End}\,\mathbf{C}\{\mathcal{L}\}$ by

$$\hat{\alpha}[\iota(\gamma)] = F(\alpha, \gamma)\iota(\alpha + \gamma) \qquad \text{for } \gamma \in \mathcal{L}. \tag{3.12}$$

We denote $\hat{L} = \{\hat{\alpha} \mid \alpha \in L\}$. Set

$$\hat{B} = \hat{H}_{+} + H(0) + \mathbf{C}c. \tag{3.13}$$

Make $\mathbf{C}$ a (one-dimensional) $\hat{B}$-module as follows:

$$\hat{H}_{+}(1) = 0,\ \ H(0)(1) = 0,\ \ c(1) = 1. \tag{3.14}$$

Form the following induced $\hat{H}$-module:

$$S(\hat{H}_{-}) = U(\hat{H}) \otimes_{U(\hat{B})} \mathbf{C}. \tag{3.15}$$

Here $U(\cdot)$ denotes the universal enveloping algebra, $S(\cdot)$ denotes the symmetric algebra, and we have used the PBW theorem to make the identification. Let

$$V_{\mathcal{L}} = S(\hat{H}_{-}) \otimes_{\mathbf{C}} \mathbf{C}\{\mathcal{L}\}. \tag{3.16}$$

We let $\hat{H}_- + \hat{H}_+ + \mathbf{C}c$ act on the first factor of $V_{\mathcal{L}}$ and let $\hat{L}$ act on the second. We identify $H$ with $H(0)$ and denote $\eta \otimes \iota(\gamma)$ by $\eta\iota(\gamma)$ for $\eta \in S(\hat{H}_-)$, $\gamma \in \mathcal{L}$. For $h \in H$, $\gamma \in \mathcal{L}$, $\eta \in S(\hat{H}_-)$, we define

$$h(\eta\iota(\gamma)) = \langle h, \gamma\rangle \eta\iota(\gamma), \quad z^h(\eta\iota(\gamma)) = z^{\langle h,\gamma\rangle}\eta\iota(\gamma), \tag{3.17}$$

where $z$ is a formal variable. We define the action of $\hat{H}$ linearly on $V_{\mathcal{L}}$. Then $V_{\mathcal{L}}$ is an $\hat{H}$-module.

For any $h \in H$, we define

$$h^+(z) = \sum_{l=0}^{\infty} h(l)z^{-l-1}, \quad h^>(z) = \sum_{l=1}^{\infty} h(l)z^{-l-1}, \quad h^-(z) = \sum_{l=1}^{\infty} h(-l)z^{l-1}, \tag{3.18}$$

$$h(z) = h^+(z) + h^-(z). \tag{3.19}$$

Note that for $\xi, \eta \in H$,

$$[\xi(0), \eta(z)] = 0, \quad [\xi^{\pm}(z_1), \eta^{\pm}(z_2)] = 0, \tag{3.20}$$

$$[\xi^+(z_1), \eta^-(z)] = [\xi^>(z_1), \eta^-(z_2)] = \frac{\langle \xi, \eta\rangle}{(z_1 - z_2)^2} \tag{3.21}$$

by (3.8). In particular,

$$[\xi(z_1), \eta(z_2)] = \frac{\langle \xi, \eta\rangle}{(z_1 - z_2)^2} - \frac{\langle \xi, \eta\rangle}{(z_2 - z_1)^2} = -\langle \xi, \eta\rangle z_1^{-1}\delta\partial_{z_1}\left(\frac{z_1}{z_2}\right). \tag{3.22}$$

For any $M \subset \mathcal{L}$, we define

$$\mathbf{C}\{M\} = \text{span}\,\{\iota(\alpha) \mid \alpha \in M\}, \quad V_M = S(\hat{H}_-) \otimes \mathbf{C}\{M\}. \tag{3.23}$$

For $v = \xi^1(-n_1)\cdots\xi^k(-n_k)\iota(\alpha) \in V_L$ with $n_j > 0$, we shall define the operator $Y(v, z)$ on $V_{\mathcal{L}}$ by induction. We define:

$$Y(\iota(\alpha), z) = e^{\int \alpha^-(z)dz} e^{\int \alpha^>(z)dz} \hat{\alpha} z^{\alpha}, \quad \text{where} \int z^l dz = \frac{z^{l+1}}{l+1}, \quad l \neq -1. \tag{3.24}$$

Suppose we have defined $Y(\xi^2(-n_2)\cdots\xi^k(-n_k)\iota(\alpha), z)$. Then we define:

$$\begin{aligned} Y(v, z) &= \frac{d^{n_1-1}\xi^{1-}(z)/dz^{n_1-1}}{(n_1-1)!} Y(\xi^2(-n_2)\cdots\xi^k(-n_k)\iota(\alpha), z) \\ &\quad + Y(\xi^2(-n_2)\cdots\xi^k(-n_k)\iota(\alpha), z)\frac{d^{n_1-1}\xi^{1+}(z)/dz^{n_1-1}}{(n_1-1)!}. \end{aligned} \tag{3.25}$$

This definition is independent of the order of $\xi^1(-n_1)\cdots\xi^k(-n_k)$ because of (3.20), that is, $Y(v,z)$ is well defined. We extend $Y(\cdot,z)$ linearly on $V_L$.

For $u = \xi^1(-n_1)\cdots\xi^k(-n_k)\iota(\gamma) \in V_{\mathcal{L}}$, we define the weight

$$\mathrm{wt}\, u = \sum_{j=1}^{k} n_j + \frac{\langle\gamma,\gamma\rangle}{2}. \tag{3.26}$$

Then we have

$$V_{\mathcal{L}} = \bigoplus_{l\in\mathbf{Q}} \mathcal{L}^{(l)}, \quad \mathcal{L}^{(l)} = \{u \in V_{\mathcal{L}} \mid \mathrm{wt}\, u = l\}. \tag{3.27}$$

We let

$$\mathbf{1} = \iota(0). \tag{3.28}$$

Suppose that $\{h^j \mid j = 1, ..., q\}$ is an orthonormal basis of $H$. Let

$$\omega = \frac{1}{2}\sum_{j=1}^{q}(h^j(-1))^2. \tag{3.29}$$

**Theorem 3.1.** *The family $(V_L, Y(|_{V_L}, z), \mathbf{1}, \omega)$ defined in the above forms a vertex operator algebra and each $(V_{\lambda_j+L}, Y(|_{V_{\lambda_j+L}}, z))$ is an irreducible module of $(V_L, Y(|_{V_L}, z), \mathbf{1}, \omega)$.*

*Proof.* We identify $\hat{H}_-$ with $\hat{H}_-\mathbf{1}$ if we can distinguish them clearly from the context. For $u \in V_L$, we let

$$Y(u,z) = \sum_{l\in\mathbf{Z}} u_l z^{-l-1}, \quad u^+(z) = \sum_{n=0}^{\infty} u_n z^{-n-1}, \quad u^-(z) = \sum_{n=1}^{\infty} u_{-n} z^{n-1}. \tag{3.30}$$

By our definition (3.24-25),

$$Y(h(-n-1),z) = \frac{d^n h(z)/dz^n}{n!}, \quad h(-n-1)^{\pm}(z) = \frac{d^n h^{\pm}(z)/dz^n}{n!}, \tag{3.31}$$

for $h \in H,\ 0 \le n \in \mathbf{Z}$. Hence

$$h(-n-1)_{-m} = \binom{m+n-1}{n} h(-m-n), \quad h(-n-1)_i = 0, \quad h(-n-1)_{l+n} = \binom{-l-1}{n} h(l), \tag{3.32}$$

for $0 \le i < n,\ m, l \in \mathbf{Z},\ 0 < m, 0 \le l$. Moreover,

$$L(-1) = \omega_0 = \sum_{i=1}^{q}\sum_{m=0}^{\infty} h^i(-m-1)h^i(m), \tag{3.33}$$

$$L(0) = \omega_1 = \sum_{i=1}^{q}\sum_{m=1}^{\infty} [h^i(-m)h^i(m) + (h^i(0))^2/2]. \tag{3.34}$$

Hence

$$L(0)\iota(\gamma) = \frac{\langle\gamma,\gamma\rangle}{2}\iota(\gamma), \quad L(-1)\iota(\gamma) = \sum_{i=1}^{q}\langle\gamma, h^i\rangle h^i(-1)\iota(\gamma) = \gamma(-1)\iota(\gamma) \tag{3.35}$$

for $\gamma \in \mathcal{L}$,

$$[L(-1), Y(u,z)] = \frac{d}{dz}Y(u,z), \quad [L(0), h(m)] = -mh(m) \tag{3.36}$$

for $u \in \hat{H}_-$, $h \in H$, $m \in \mathbf{Z}$,

$$[L(-1), z^\alpha] = [L(0), z^\alpha] = 0, \quad [L(-1), \hat{\alpha}] = \alpha(-1)\hat{\alpha}, \quad [L(0), \hat{\alpha}] = (\alpha(0) + \langle\alpha,\alpha\rangle/2)\hat{\alpha}, \tag{3.37}$$

for $\alpha \in L$. Furthermore, for $\alpha \in L$,

$$[L(-1), \int \alpha^-(z)dz] = \sum_{m=1}^{\infty}[L(-1), \alpha(-m)]z^m/m = \alpha^-(z) - \alpha(-1), \tag{3.38}$$

$$[L(-1), \int \alpha^>(z)dz] = \sum_{m=1}^{\infty} -[L(-1), \alpha(m)]z^{-m}/m = \alpha^+(z), \tag{3.39}$$

$$[L(0), \int \alpha^-(z)dz] = z\alpha^-(z), \quad [L(0), \int \alpha^>(z)dz] = z\alpha^>(z) = Y(\alpha(-1)\iota(\alpha), z) = Y(L(-1)\iota(\alpha), z). \tag{3.40}$$

By (3.36-39), it is easy to verify that

$$[L(-1), Y(\iota(\alpha), z)] = \frac{d}{dz}Y(\iota(\alpha), z), \tag{3.40}$$

$$[L(0), Y(\iota(\alpha), z)] = z\frac{d}{dz}Y(\iota(\alpha), z) + Y(L(0)\iota(\alpha), z), \qquad \alpha \in L. \tag{3.41}$$

Note that by (3.33),

$$u = \xi^1(-n_1)\cdots\xi^j(-n_j)\iota(\alpha) = \xi^1(-n_1)_{-1}\cdots\xi^j(-n_j)_{-1}\iota(\alpha) \tag{3.42}$$

for $\xi^i \in H$, $\alpha \in L$, $0 < n_i \in \mathbf{Z}$. Moreover, by (3.32), (3.35), (3.40) and the same arguments as in the proof of Lemma 2.7, we can prove (2.129) for the above $u$ by induction on $j$. Thus (2.129) holds for any $u \in V_L$ (which is a linear combination of the elements like (3.39)). We can prove (2.155) for $v = u$ in (3.39) as (2.153-155).

By (3.34-35),

$$L(0)w = lw \qquad \text{for } w \in V_{\mathcal{L}}^{(l)}. \tag{3.43}$$

Hence for any $u \in V_L^{(j_1)}$ and $w \in V_{\mathcal{L}}^{(j_2)}$ (cf. (3.27-28)),

$$u_n w = 0 \qquad \text{for } m(u,w) = j_1 + j_2 \le n \in \mathbf{Z}, \tag{3.44}$$

because (2.129) holds for any $u \in V_L$. Let $\xi, \eta \in H$ and let $w \in V_{\mathcal{L}}$ be homogeneous. By (3.22) and (3.31),

$$\begin{aligned} &[Y(\xi^-(x), z_1), Y(\eta^-(y), z_2)] \\ = \ & [\xi(z_1 + x), \eta(z_2 + y)] \\ = \ & \frac{\langle \xi, \eta \rangle}{(z_1 + x - z_2 - y)^2} - \frac{\langle \xi, \eta \rangle}{(z_2 + y - z_1 - x)^2}. \end{aligned} \tag{3.45}$$

Hence

$$(z_1 + x - z_2 - y)^2 [Y(\xi^-(x), z_1), Y(\eta^-(y), z_2)] = 0. \tag{3.46}$$

Moreover,

$$\begin{aligned} & (z_0 + z_2 + x)^{m(\xi,w)} Y(Y(\xi^-(x), z_0)\eta^-(y), z_2)w \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} Y(\xi(z_0 + x)\eta^-(y), z_2)w \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} [Y(\xi^-(z_0 + x)\eta^-(y), z_2) + \frac{\langle \xi, \eta \rangle}{(z_0 + x - y)^2} Y(\mathbf{1}, z)]w \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} [\xi^-(z_2 + z_0 + x) Y(\eta^-(y), z_2) \\ & + Y(\eta^-(y), z_2)\xi^+(z_2 + z_0 + x) + \frac{\langle \xi, \eta \rangle}{(z_0 + x - y)^2} Y(\mathbf{1}, z)]w \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} [\xi^-(z_2 + z_0 + x) Y(\eta^-(y), z_2)w \\ & + \eta(z_2 + y)\xi^+(z_2 + z_0 + x)w + \frac{\langle \xi, \eta \rangle}{(z_0 + x - y)^2} w] \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} [\xi^-(z_0 + z_2 + x) Y(\eta^-(y), z_2)w \\ & + \eta(z_2 + y)\xi^+(z_0 + z_2 + x)w + \frac{\langle \xi, \eta \rangle}{(z_0 + x - y)^2} w] \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} [\xi(z_2 + z_0 + x) Y(\eta^-(y), z_2)w \\ & + [\eta(z_2 + y), \xi^+(z_0 + z_2 + x)]w + \frac{\langle \xi, \eta \rangle}{(z_0 + x - y)^2} w] \\ = \ & (z_0 + z_2 + x)^{m(\xi,w)} Y(\xi^-(x), z_0 + z_2) Y(\eta^-(y), z_2)w. \end{aligned} \tag{3.47}$$

Note that (3.46-47) imply that the duality (2.132-133) holds for $u, v \in \hat{H}_-$ and $w \in V_{\mathcal{L}}$.

For $h \in H, \alpha \in L$, by (3.8), (3.12) and (3.21),

$$[h(z), \hat{\alpha}] = \langle h, \alpha \rangle \hat{\alpha} z^{-1}, \ \ [h(z_1), z_2^{\alpha}] = 0, \tag{3.48}$$

$$[h(z_1), \int \alpha^-(z_2)dz_2] = \frac{\langle h, \alpha\rangle}{z_1 - z_2} - \langle h, \alpha\rangle z_1^{-1}, \ \ [h(z_1), \int \alpha^>(z_2)dz_2] = \frac{\langle h, \alpha\rangle}{z_2 - z_1}. \tag{3.49}$$

Hence,

$$[h(z_1), Y(\iota(\alpha), z_2)] = \left(\frac{\langle h, \alpha\rangle}{z_1 - z_2} + \frac{\langle h, \alpha\rangle}{z_2 - z_1}\right) Y(\iota(\alpha), z_2). \tag{3.50}$$

This implies:

$$(z_1 + x - z_2)[Y(h^-(x)\mathbf{1}, z_1), Y(\iota(\alpha), z_2)] = (z_1 + x - z_2)[h(z_1 + x), Y(\iota(\alpha), z_2)] = 0. \tag{3.51}$$

Thus the commutativity (3.132) holds for $u \in \hat{H}_-$, $v = \iota(\alpha)$ with $\alpha \in L$. Furthermore, for $h \in H, \alpha \in L$ and $w \in V_{\mathcal{L}}$,

$$\begin{aligned}
& (z_0 + x + z_2)^{m(h,w)} Y(Y(h^-(x)\mathbf{1}, z_0)\iota(\alpha), z_2)w \\
= \ & (z_0 + x + z_2)^{m(h,w)} Y(h(z_0 + x)\iota(\alpha), z_2)w \\
= \ & (z_0 + x + z_2)^{m(h,w)}[Y(h^-(z_0 + x)\iota(\alpha), z_2) + \langle h, \alpha\rangle(z_0 + x)^{-1}Y(\iota(\alpha), z_2)]w \\
= \ & (z_0 + x + z_2)^{m(h,w)}[h^-(z_2 + z_0 + x)Y(\iota(\alpha), z_2)w \\
& +Y(\iota(\alpha), z_2)h^+(z_2 + z_0 + x)w + \langle h, \alpha\rangle(z_0 + x)^{-1}Y(\iota(\alpha), z_2)w] \\
= \ & (z_0 + x + z_2)^{m(h,w)}[h^-(z_0 + z_2 + x)Y(\iota(\alpha), z_2)w \\
& +Y(\iota(\alpha), z_2)h^+(z_0 + z_2 + x)w + \langle h, \alpha\rangle(z_0 + x)^{-1}Y(\iota(\alpha), z_2)w] \\
= \ & (z_0 + x + z_2)^{m(h,w)}[h(z_0 + z_2 + x)Y(\iota(\alpha), z_2)w \\
& +[Y(\iota(\alpha), z_2), h^+(z_0 + z_2 + x)]w + \langle h, \alpha\rangle(z_0 + x)^{-1}Y(\iota(\alpha), z_2)w] \\
= \ & (z_0 + x + z_2)^{m(h,w)} Y(h^-(x)\mathbf{1}, z_0 + z_2)Y(\iota(\alpha), z_2)w,
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
& (z_0 + z_2)^{m(\iota(\alpha),w)} Y(Y(\iota(\alpha), z_0)h^-(x)\mathbf{1}, z_2)w \\
= \ & (z_0 + z_2)^{m(\iota(\alpha),w)}[Y(h^-(x)Y(\iota(\alpha), z_0)\mathbf{1}, z_2) + Y([Y(\iota(\alpha), z_0), h^-(x)]\mathbf{1}, z_2)]w \\
= \ & (z_0 + z_2)^{m(\iota(\alpha),w)}[h^-(z_2 + x)Y(Y(\iota(\alpha), z_0)\mathbf{1}, z_2) \\
& +Y(Y(\iota(\alpha), z_0)\mathbf{1}, z_2)h^+(z_2 + x) + \frac{\langle h, \alpha\rangle}{-z_0 + x}Y(Y(\iota(\alpha), z_0)\mathbf{1}, z_2)]w \\
= \ & (z_0 + z_2)^{m(\iota(\alpha),w)}[h^-(z_2 + x)Y(e^{z_0L(-1)}\iota(\alpha), z_2) \\
& +Y(e^{z_0L(-1)}\iota(\alpha), z_2)h^+(z_2 + x) + \frac{\langle h, \alpha\rangle}{-z_0 + x}Y(e^{z_0L(-1)}\iota(\alpha), z_2)]w \\
= \ & (z_0 + z_2)^{m(\iota(\alpha),w)}[h^-(z_2 + x)Y(\iota(\alpha), z_2 + z_0)w
\end{aligned}$$

$$
\begin{aligned}
&+Y(\iota(\alpha), z_2+z_0)h^+(z_2+x)w + \frac{\langle h,\alpha\rangle}{-z_0+x}Y(\iota(\alpha), z_2+z_0)w] \\
= \ & (z_0+z_2)^{m(\iota(\alpha),w)}[h^-(z_2+x)Y(\iota(\alpha), z_0+z_2)w \\
&+Y(\iota(\alpha), z_0+z_2)h^+(z_2+x)w + \frac{\langle h,\alpha\rangle}{-z_0+x}Y(\iota(\alpha), z_0+z_2)w] \\
= \ & (z_0+z_2)^{m(\iota(\alpha),w)}[[h^-(z_2+x), Y(\iota(\alpha), z_0+z_2)]w \\
&+Y(\iota(\alpha), z_0+z_2)h(z_2+x)w + \frac{\langle h,\alpha\rangle}{-z_0+x}Y(\iota(\alpha), z_0+z_2)w] \\
= \ & (z_0+z_2)^{m(\iota(\alpha),w)}Y(\iota(\alpha), z_0+z_2)h(z_2+x)w \\
= \ & (z_0+z_2)^{m(\iota(\alpha),w)}Y(\iota(\alpha), z_0+z_2)Y(h^-(x)\mathbf{1}, z_2)w. \qquad (3.53)
\end{aligned}
$$

Thus the associativity (2.133) holds for $w \in V_{\mathcal{L}}$ and $u \in \hat{H}_-, v = \iota(\alpha)$ or $v \in \hat{H}_-, u = \iota(\alpha)$.

Let $\alpha, \beta \in L$ and let $w = \xi^1(-n_1)\cdots\xi^j(-n_j)\iota(\gamma) \in V_{\mathcal{L}}$. Note that

$$
\begin{aligned}
&[\int \alpha^>(z_1)dz_1, \int \beta^-(z_2)dz_2] \\
= \ & \sum_{m=1}^{\infty}[\alpha(m), \beta(-m)]\frac{z_1^{-m}}{-m}\frac{z_2^m}{m} \\
= \ & \sum_{m=1}^{\infty} -\frac{\langle\alpha,\beta\rangle}{m}\left(\frac{z_2}{z_1}\right)^m \\
= \ & \langle\alpha,\beta\rangle \ln\left(1-\frac{z_2}{z_1}\right) \\
= \ & \ln\left(1-\frac{z_2}{z_1}\right)^{\langle\alpha,\beta\rangle}. \qquad (3.54)
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
&e^{\int \alpha^>(z_1)dz_1}e^{\int \beta^-(z_2)dz_2}e^{-\int \alpha^>(z_1)dz_1} \\
= \ & e^{\mathrm{ad}\int \alpha^>(z_1)dz_1}(e^{\int \beta^-(z_2)dz_2}) \\
= \ & e^{\ln\left(1-\frac{z_2}{z_1}\right)^{\langle\alpha,\beta\rangle}}e^{\int \beta^-(z_2)dz_2} \\
= \ & \left(1-\frac{z_2}{z_1}\right)^{\langle\alpha,\beta\rangle}e^{\int \beta^-(z_2)dz_2}. \qquad (3.55)
\end{aligned}
$$

Moreover,

$$
z^\alpha\hat{\beta} = z^{\langle\alpha,\beta\rangle}\hat{\beta}z^\alpha. \qquad (3.56)
$$

Now we have:

$$
Y(\iota(\alpha), z_1)Y(\iota(\beta), z_2)
$$

$$
\begin{aligned}
&= e^{\int \alpha^-(z_1)dz_1} e^{\int \alpha^>(z_1)dz_1} \hat{\alpha} z_1^\alpha e^{\int \beta^-(z_2)dz_2} e^{\int \beta^>(z_2)dz_2} \hat{\beta} z_2^\beta \\
&= e^{\int \alpha^-(z_1)dz_1} \left(e^{\int \alpha^>(z_1)dz_1} e^{\int \beta^-(z_2)dz_2} e^{-\int \alpha^>(z_1)dz_1}\right) e^{\int \alpha^>(z_1)dz_1 + \int \beta^>(z_2)dz_2} \hat{\alpha} z_1^\alpha \hat{\beta} z_2^\beta \\
&= \left(1 - \frac{z_2}{z_1}\right)^{\langle \alpha, \beta\rangle} z_1^{\langle \alpha,\beta\rangle} e^{\int \alpha^-(z_1)dz_1 + \int \beta^-(z_2)dz_2} e^{\int \alpha^>(z_1)dz_1 + \int \beta^>(z_2)dz_2} \hat{\alpha}\hat{\beta} z_1^\alpha z_2^\beta \\
&= F(\alpha,\beta)(z_1 - z_2)^{\langle \alpha,\beta\rangle} e^{\int \alpha^-(z_1)dz_1 + \int \beta^-(z_2)dz_2} e^{\int \alpha^>(z_1)dz_1 + \int \beta^>(z_2)dz_2} \widehat{(\alpha+\beta)} z_1^\alpha z_2^\beta. \qquad (3.57)
\end{aligned}
$$

For convenience, we let

$$
G(z_1, z_2) = e^{\int \alpha^-(z_1)dz_1 + \int \beta^-(z_2)dz_2} e^{\int \alpha^>(z_1)dz_1 + \int \beta^>(z_2)dz_2} \widehat{(\alpha+\beta)} z_1^\alpha z_2^\beta \qquad (3.58)
$$

and $m(\alpha,\beta) = |\langle \alpha,\beta\rangle| + 1$.Then

$$
\begin{aligned}
& (z_1 - z_2)^{m(\alpha,\beta)} [Y(\iota(\alpha), z_1), Y(\iota(\beta), z_2)] \\
&= (z_1 - z_2)^{m(\alpha,\beta)} [F(\alpha,\beta)(z_1 - z_2)^{\langle \alpha,\beta\rangle} - F(\beta,\alpha)(z_2 - z_1)^{\langle \alpha,\beta\rangle}] G(z_1, z_2) \\
&= (z_1 - z_2)^{m(\alpha,\beta)} F(\alpha,\beta) [(z_1 - z_2)^{\langle \alpha,\beta\rangle} - F(\alpha,\beta)^{-1} F(\beta,\alpha)(z_2 - z_1)^{\langle \alpha,\beta\rangle}] G(z_1, z_2) \\
&= (z_1 - z_2)^{m(\alpha,\beta)} F(\alpha,\beta) [(z_1 - z_2)^{\langle \alpha,\beta\rangle} - (-1)^{\langle \alpha,\beta\rangle}(z_2 - z_1)^{\langle \alpha,\beta\rangle}] G(z_1, z_2) \\
&= F(\alpha,\beta)[(z_1 - z_2)^{m(\alpha,\beta) + \langle \alpha,\beta\rangle} - (-z_2 + z_1)^{\langle \alpha,\beta\rangle + m(\alpha,\beta)}] G(z_1, z_2) \\
&= 0. \qquad (3.59)
\end{aligned}
$$

Therefore, the commutativity (2.132) holds for $u = \iota(\alpha), v = \iota(\beta)$.

Next we note

$$
\int (z_2 + z_0)^{-1} dz_0 = \sum_{l=0}^{\infty} z_2^{-1}(-1)^l \left(\frac{z_0}{z_2}\right)^l = \sum_{l=0}^{\infty} -\frac{1}{l+1}\left(-\frac{z_0}{z_2}\right)^{l+1} = \ln\left(1 + \frac{z_0}{z_2}\right). \qquad (3.60)
$$

For $j \neq -1$,

$$
\begin{aligned}
& \int (z_2 + z_0)^j dz_0 + \int z_2^j dz_2 \\
&= \sum_{l=0}^{\infty} \int \binom{j}{l} z_2^j \left(\frac{z_0}{z_2}\right)^l dz_0 + \frac{z_2^{j+1}}{j+1} \\
&= \sum_{l=0}^{\infty} \binom{j}{l} \frac{z_2^{j+1}}{j+1} \left(\frac{z_0}{z_2}\right)^{l+1} + \frac{z_2^{j+1}}{j+1} \\
&= \frac{(z_2 + z_0)^{j+1}}{j+1} \\
&= \int (z_2 + z_0)^j d(z_2 + z_0). \qquad (3.61)
\end{aligned}
$$

Let

$$m(\alpha, w) = \sum_{i=1}^{j} n_j + |\langle \alpha, \gamma \rangle| + 1. \tag{3.62}$$

We have

$$\begin{aligned}
& (z_0+z_2)^{m(\alpha,w)} Y(Y(\iota(\alpha), z_0)\iota(\beta), z_2)w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} Y(e^{\int \alpha^-(z_0)dz_0}\iota(\alpha+\beta), z_2)w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} e^{\int \alpha^-(z_2+z_0)dz_0} Y(\iota(\alpha+\beta), z_2) e^{\int \alpha^+(z_2+z_0)dz_0} w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} e^{\int \alpha^-(z_2+z_0)dz_0 + \int (\alpha+\beta)^-(z_2)dz_2} e^{\int(\alpha+\beta)^>(z_2)dz_2} \\
& (\alpha+\beta)\hat{}\, z_2^{\alpha+\beta} e^{\int \alpha^>(z_2+z_0)dz_0 + \int \alpha(0)(z_2+z_0)^{-1}dz_0} w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} e^{(\int \alpha^-(z_2+z_0)dz_0 + \int \alpha^-(z_2)dz_2) + \int \beta^-(z_2)dz_2} \\
& e^{\int \beta^>(z_2)dz_2 + (\int \alpha^>(z_2)dz_2 + \int \alpha^>(z_2+z_0)dz_0)} (\alpha+\beta)\hat{}\, z_2^{\alpha+\beta} e^{\alpha \ln\left(1+\frac{z_0}{z_2}\right)} w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} e^{\int \alpha^-(z_2+z_0)d(z_2+z_0) + \int \beta^-(z_2)dz_2} \\
& e^{\int \beta^>(z_2)dz_2 + \int \alpha^>(z_2+z_0)^> d(z_2+z_0)} (\alpha+\beta)\hat{}\, z_2^{\beta}(z_2+z_0)^{\alpha} w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} e^{\int \alpha^-(z_0+z_2)d(z_0+z_2) + \int \beta^-(z_2)dz_2} \\
& e^{\int \beta^>(z_2)dz_2 + \int \alpha^>(z_0+z_2)^> d(z_0+z_2)} (\alpha+\beta)\hat{}\, z_2^{\beta}(z_0+z_2)^{\alpha} w \\
= \ & F(\alpha,\beta)(z_0+z_2)^{m(\alpha,w)} z_0^{\langle\alpha,\beta\rangle} G(z_0+z_2, z_2)w \\
= \ & (z_0+z_2)^{m(\alpha,w)} Y(\iota(\alpha), z_0+z_2) Y(\iota(\beta), z_2),
\end{aligned} \tag{3.63}$$

that is, the associativity (2.133) holds for $u = \iota(\alpha)$, $v = \iota(\beta)$. We can prove (2.132-133) for $u = \xi^1(-n_1)\cdots\xi^i(-n_i)\iota(\alpha)$ and $u = \eta^1(-m_1)\cdots\eta^j(-m_j)\iota(\beta)$ by the same arguments as in the proof of Lemma 2.8. Thus (2.132-133) hold. □

**Remark 3.2.** (1) When $L$ is a root lattice of types A, D and E, $V_L$ is the simple vertex operator algebra associated with the corresponding simple Lie algebra with level $\chi = 1$. That is, $V_L \cong V_1$ as vertex operator algebras.

(2) We have not used the "normal ordering" (cf. [FLM 3]) to define vertex operators.

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