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ON VOLUMES OF SOME HYPERBOLIC 3 - MANIFOLDS

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On volumes of some hyperbolic 3-manifolds

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PREFACE

These notes are some fragments of a series of lectures and seminars talks given at Seoul National University between September 1995 and March 1996 under the auspices of the Global Analysis Research Center. The talks were devoted to some problems in the theory of hyperbolic 3-manifolds and 3-orbifolds.

Inspired by W. Thurston lecture notes "The Geometry and Topology of 3-manifolds" this theory was widely developed in last twenty years. For basic definitions and main results of the theory we refer to beautiful textbooks appeared in last years: R. Benedetti, C. Petronnio "Lectures on Hyperbolic Geometry" and J. Ratcliffe "Foundations of Hyperbolic Manifolds".

The present notes devoted to application of the theory to some interesting examples. More exactly, we discuss volumes and other properties of some series of hyperbolic 3-manifolds.

Firstly in chapter 1 we shortly recall famous results of E. Andreev, E. Vinberg, C. Hodgson and I. Rivin on the existence of polyhedra in the Lobachevsky space.

In chapter 2, following to J. Milnor, E. Vinberg and R. Kellerhals, we find volumes of some families of hyperbolic polyhedra such like tetrahedra, pyramids, prisms and antiprisms, in terms of the Lobachevsky function.

In chapter 3 we recall a remarkable Thurston–Jørgensen theorem on a structure of the set of volumes of hyperbolic 3-manifolds.

We recall, that the first example of a closed orientable hyperbolic 3-manifold was constructed by F. Löbell in 1931. In chapter 4 we construct a series of manifolds which generalize Löbell's example and discuss their isometries and volumes.

The chapter 5 devoted to Fibonacci manifolds. these manifolds were introduced by H. Helling, A. Kim and J. Mennicke and have many interesting properties. We consider these manifolds from different point of view. In particular, we give description of Fibonacci manifolds as branched coverings of the 3-sphere and by Dehn surgery. We discuss their Heegaard genus and equivariant Heegaard genus. Moreover we show that their volumes correspond to limit ordinals in Thurston–Jørgensen theorem.

In chapter 6 we discuss the ten smallest known hyperbolic manifolds $\mathcal{M}_1$, ..., $\mathcal{M}_{10}$, which were founded by C. Hodgson and J. Weeks using famous computer program "SnapPea". It is interesting that all of them can be obtained by Dehn surgeries on the Whitehead link. We also discuss some properties of the smallest known Weeks–Matveev–Fomenko manifold $\mathcal{M}_1$ and of the Meyerhoff–Neumann manifold $\mathcal{M}_3$.

I wish to thank all those who made the lectures and seminars at GARC possible. It is an even greater pleasure to thank Proffessor Hyunkoo Lee, Professor Hyuk Kim, Professor Suhyoung Choi and all partipitiences of the seminars for their hospitality that make my visit to Seoul enjoyable and remarkable.

Contents

Chapter 1

Polyhedra in the Lobachevsky space

In this chapter we recall some criteria of existence of polyhedra in the Lobachevsky space $\mathbb{H}^3$ due to E. Andreev [5], E. Vinberg [84], and C. Hodgson and I. Rivin [34].

1.1 Acute-angled polyhedra

Let $\mathbb{H}^2$ denotes the Lobachevsky plane and $\mathbb{H}^n$, $n \geq 3$, denotes the n-dimensional Lobachevsky space.

We recall ([87, p.60]), that a convex k-gon with angles $\theta_1, \ldots, \theta_k$ exists on the Lobachevsky plane if and only if

$$\theta_1 + \ldots + \theta_k < (k - 2)\pi, \tag{1.1}$$

and depends (up to a motion) on $(k-3)$ parameters. The situation is completely different in spaces of larger dimension.

Let us consider a *convex polyhedron* P in $\mathbb{H}^n$, that is an intersection of finitely many half-spaces:

$$P = \bigcap_{i=1}^{k} \alpha_i^-, \tag{1.2}$$

where α_i^- is a half-space bounded by the hyperplane α_i. It may be assumed always, that non of the half-spaces α_i^- contains the intersection of all others. We will be interested in polyhedra of finite volume. A family of half-spaces $\{\alpha_1^-, \ldots, \alpha_k^-\}$ is said to be *acute-angled* if for any distinct indices i, j either the hyperplanes α_i and α_j intersect and the dihedral angle $\alpha_i^- \cap \alpha_j^-$ does not exceed $\pi/2$, or $\alpha_i^+ \cap \alpha_j^+ = \emptyset$. A convex polyhedron P is said to be *acute-angled* if the set of half-spaces $\{\alpha_1^-, \dots, \alpha_k^-\}$ from (1.2) is an acute-angled family of half-spaces.

A *combinatorial type* of convex polyhedra of finite volume in the space $\mathbb{H}^n$ is the set of all polyhedra whose closures in $\mathbb{H}^n$ are combinatorially isomorphic to a given bounded convex polyhedra $\mathcal{P}$ in the Euclidean n-dimensional space.

Theorem 1.1 ([5]) *A bounded acute-angled polyhedron in the space $\mathbb{H}^n$, $n \geq 3$, is determined by its combinatorial type and its dihedral angles uniquely up to a motion.*

As we remarked above, the problem of the existence of a polygon in $\mathbb{H}^2$ with given angles has a satisfactory solution if the inequality (1.1) takes place. In the 3-dimensional case existence conditions for an acute-angled polyhedron are given by Andreev's theorem.

Theorem 1.2 ([5]) *Let P be a compact convex polyhedron in $\mathbb{H}^3$ with faces F_i and dihedral angles $\alpha_{ij} \leq \pi/2$ between faces F_i and F_j. Call a circular sequence of k edge-adjacent faces of P such that no three of these faces have a common point, a k-prismatic element. Then P has trivalent vertices, and dihedral angles α_{ij} satisfy the following system of inequalities depending only of the combinatorial type:*

(1) $0 < \alpha_{ij} \leq \pi/2$;

(2) if $F_i \cap F_j \cap F_k$ is a vertex, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} > \pi$;

(3) if F_i, F_j, F_k form a 3-prismatic element, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$;

(4) if F_i, F_j, F_k, F_l form a 4-prismatic element, then $\alpha_{ij}+\alpha_{jk}+\alpha_{kl}+\alpha_{li} < 2\pi$;

(5) if F_s is a quadrilateral with the sides in cyclic order α_{is}, α_{js}, α_{ks}, α_{ls}, then

$$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$$

and

$$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi.$$

Moreover, these conditions are sufficient for an abstract polyhedron P with trivalent vertices, but not a simplex, to be realizable as a compact convex polyhedron in $\mathbb{H}^3$ with dihedral angles α_{ij}.

Analogous theorem take place in the case of a finite-volume acute-angled polyhedron in $\mathbb{H}^3$.

Theorem 1.3 ([6]) *Let P be an abstract three-dimensional polyhedron not a simplex such that at every vertex three or four faces meet. The following conditions are necessary and sufficient for the existence in $\mathbb{H}^3$ of a convex polyhedron of finite volume of the combinatorial type P with the dihedral angles $\alpha_{ij} \leq \pi/2$:*

(1) $0 < \alpha_{ij} \leq \pi/2$;

(2) if $F_i \cap F_j \cap F_k$ is a vertex, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \geq \pi$ and if $F_i \cap F_j \cap F_k \cap F_l$ is a vertex, then $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} = 2\pi$;

(3) if F_i, F_j, F_k form a 3-prismatic element, then $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$;

(4) if F_i, F_j, F_k, F_l form a 4-prismatic element, then $\alpha_{ij}+\alpha_{jk}+\alpha_{kl}+\alpha_{li} < 2\pi$;

(5) if P is a triangular prism with bases F_1 and F_2, then

$$\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{23} + \alpha_{24} + \alpha_{25} < 3\pi;$$

(6) if among the faces F_i, F_j, F_k we have F_i and F_j, F_i and F_k adjacent, but F_i and F_k not adjacent, but concurrent in one vertex (we may say: they touch each other) and all three do not meet in one vertex, then $\alpha_{ij} + \alpha_{jk} < \pi$.

1.2 The Gram matrix

A characterization of a convex acute-angled polyhedron in $\mathbb{H}^n$ in terms of the Gram matrix was given by E. Vinberg [84]. Let us consider the pseudo-Euclidean space $\mathbb{R}^{n,1}$ of the vector model of the Lobachevsky space $\mathbb{H}^n$. Let us assume, that the polyhedron $P \subset \mathbb{H}^n$ is represented in the form (1.2), and consider for each $i = 1, \ldots, k$ the unit vector e_i of $\mathbb{R}^{n,1}$ orthogonal to the hyperplane α_i and direct away from P. It means, that P is the intersection in $\mathbb{R}^{n,1}$ of $\mathbb{H}^n$ with the convex polyhedral cone

$$K(P) = \{x \in \mathbb{R}^{n,1} \mid (x, e_i) \leq 0, \qquad i = 1, \ldots, k\} \tag{1.3}$$

The Gram matrix of the system of vectors $\{e_1, \ldots, e_k\}$ is said to be the *Gram matrix of the polyhedron P*.

We recall, that a square matrix A is said to be *decomposable* if by some permutation of the rows and the same permutation of the columns it can be brought to the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B and C are square matrices.

Theorem 1.4 ([84]) *Any indecomposable symmetric matrix of signature* $(n, 1)$ *with* 1*'s along the main diagonal and non-positive entries off it is the Gram matrix for some convex polyhedron in the space* $\mathbb{H}^n$. *This polyhedron is defined uniquely up to a motion.*

In particular, Theorem 1.4 admits to consider a case of a tetrahedron, which is not covered by Theorem 1.2. According to Theorem 1.4, the statement analogous to Theorem 1.2 will be true in the case of a tetrahedron, if we add to conditions (1)–(5) an extra condition that the determinant of the Gram matrix must be negative.

1.3 The Gauss map

To explain the approach of C. Hodgson and I. Rivin [34], [35], we need the following definition.

Let P be a compact convex polyhedron in Euclidean space E^3. The *Gauss Map* G from P to the unit sphere S^2 is a set-valued function which assign to each point $p \in P$ the set of outward unit normals to support planes to P at p. Thus, the whole of a *face* f of P is mapped under G to a single point which is outward unit normal to f. An *edge* e of P is mapped to a geodesic segment $G(e)$ on S^2, whose length is easily seen to be the exterior dihedral angle at the edge e. A *vertex* v of P is mapped by G to a spherical polygon $G(v)$, whose sides are the images under G of edges incident to v and whose angles are easily seen to be the angles supplementary to the planar angles of the faces incident to v; that is, $G(e_1)$ and $G(e_2)$ meet at angle $\pi - \alpha$ whenever e_1 and e_2 meet at angle α. In other words, $G(v)$ is exactly the "spherical polar" of the link of v in P. (The *link* of a vertex v is the intersection of an infinitesimal sphere centred at v with P.)

We remark, that in a graph-theoretical sense the image $G(P)$ under the Gauss Map of P is dual to P, while metrically, it is the unit sphere S^2.

Example 1.1. If P is a cube in E^3, then $G(P)$ is an octahedron on S^2 with all edges of the length $\frac{\pi}{2}$ which faces are right-angled triangles. In particular, the sum of angles of $G(P)$ around any vertex $G(f)$ of $G(P)$ is equal to 2π.

Let us apply a similar consideration to a convex polyhedron P in $\mathbb{H}^3$. Associate to each vertex v of P a spherical polygon $G(v)$ spherically polar to the link of the vertex v in P. Glue the resulting polygons together into a closed surface, using the rule that faces $G(v_1)$ and $G(v_2)$ are gluing isometrically whenever vertices v_1 and v_2 share an edge.

The resulting metric space $G(P)$ is topologically the sphere S^2 and the complex is still combinatorially dual to P. But metrically it is no longer the sphere.

Example 1.2. Let P be a such called "Lambert cube" $L(n,n,n)$, $n \geq 3$, in $\mathbb{H}^3$, i.e. a combinatorial cube with three dihedral angles π/n, corresponding to mutually orthogonal non-intersecting edges, and the angle $\pi/2$ for all other edges. The Gaussian image $G(P)$ is an octahedron with three edges of the length π/n and all other of the length $\pi/2$. The sum of angles of $G(P)$ around any vertex $G(f)$ of $G(P)$ is equal to $3(\pi - \frac{\pi}{2}) + (\pi - \frac{\pi}{n}) > 2\pi$.

In general case, the sum of angles of a bounded hyperbolic n-gon f is less then $2\pi(n-2)$. Hence the sum of angles around a vertex $G(f)$ is greater then 2π. Thus vertices $G(f)$ of $G(P)$ are a *cone-like singularities* (or a *cone-points*) with *cone angle* greater then 2π. In other words, an image $G(P)$ is a *cone-fold* with the underlying space S^2.

The following theorem gives a precise characterization of those cone-folds that can arise as an image $G(P)$ for a compact convex polyhedron P in $\mathbb{H}^3$.

Theorem 1.5 ([34]) *A metric space (M, g) homeomorphic to S^2 can arise as the Gaussian image $G(P)$ of a compact convex polyhedron P in $\mathbb{H}^3$ if and only if the following conditions hold:*

- *(a) The metric g has constant curvature 1 away from a finite collection of cone points c_i.*
- *(b) The cone angles at c_i are greater than 2π.*
- *(c) The lengths of closed geodesics of (M, g) are all strictly greater than 2π.*

Furthermore the metric of $G(P)$ determines the hyperbolic polyhedron P uniquely (up to a motion).

We remark, that this characterization is a generalization of Andreev's theorem for compact acute-angled polyhedron. It is shown in [32] how Andreev's theorem follows from above theorem.

Example 1.3. Let P be a regular icosahedron with dihedral angles $2\pi/3$. Then Gaussian image $G(P)$ is a cone-fold with underlying space S^2 and with singular cone-points as vertices of the regular dodecahedron. Using formulae of spherical trigonometry, we see that cone angles are equal to $3\arccos(\sqrt{5}/3)$.

Second result from [34] is a generalization of the Andreev's theorem for acute-angled polyhedron of finite volume in the case of ideal polyhedron (with all vertices on the sphere an infinity).

We recall, that the following well-known characterization of convex polyhedra was proved by Steinitz: *a graph is the one-skeleton of a convex polyhedron in* $\mathbb{E}^3$ *if and only if it is a 3-connected planar graph.* We will call graphs satisfying the criteria of Steinitz's theorem *polyhedral graphs.*

Theorem 1.6 ([34]) *Let P be a polyhedral graph with weights $w(e)$ assigned to the edges. Let P^* be the planar dual of P, where the edge e^* dual to e is assigned to dual weight $w^*(e^*) = \pi - w(e)$. Then P can be realized as a convex polyhedron in $\mathbb{H}^3$ with all vertices on the sphere at infinity and with dihedral angle $w(e)$ at every edge e if and only if the following conditions hold:*

(a) $0 < w^*(e^*) < \pi$ *for all edges e^* of P^*.*

(b) If the edges $e_1^*, \ldots, e_k^*$ form the boundary of a face of P^*, then

$$w^*(e_1^*) + \cdots + w^*(e_k^*) = 2\pi.$$

(c) If the edges $e_1^*, \ldots, e_k^*$ form a simple circuit which does not bound a face of P^*, then

$$w^*(e_1^*) + \cdots + w^*(e_k^*) > 2\pi.$$

It is interesting to remark, that above theorem is closely connected with well-known question of Jacob Steiner. In 1832 he asked the following question [34] : *In which cases does a convex polyhedron have a (combinatorial) equivalent which is inscribed in, or circumscribed about, a sphere?*

Let us consider the Klein model of $\mathbb{H}^3$ in the unit ball B^3. In this model hyperbolic lines and planes are represented by Euclidean lines and planes respectively. Convexity is also preserved. Thus hyperbolic convex polyhedra with all vertices on the sphere at infinity correspond precisely to convex Euclidean polyhedra inscribed in the sphere $S^2 = \partial B^3$. Therefore a polyhedron is of inscritible type exactly when it admits an edge-weighting that satisfies the condition in above theorem.

Furthermore [34], as a polyhedron is inscrutable if and only if its planar dual is circumscribable, the above theorem admits to get answer to the second part of Steiner's question too.

Chapter 2

Volumes of polyhedra

In this chapter we discuss volumes of polyhedra in the Lobachevsky space $\mathbb{H}^3$. We will start from ideal tetrahedra. Their volumes will be obtained in terms of the Lobachevsky function $\Lambda(x)$. Next we will describe volumes of some other polyhedra in $\mathbb{H}^3$. Most of results of this chapter are well-known and well-discussed in literature. And we will be use approaches on J. Milnor [57], E. Vinberg [3] and R. Kellerhals [38].

2.1 An ideal tetrahedron

2.1.1 The volume of an infinite cone

For our aims it will be convenient to use the upper half space model for the Lobachevsky space. We recall, that the *upper half model* for $\mathbb{H}^{n+1}$ consists of all $(n+1)$-tuples $(x_1, \ldots, x_n, y)$ with $y > 0$, provided with the Riemannian metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2 + dy^2}{y^2}.$$

We recall that, in general case, if

$$ds^2 = \sum_{i,j=1}^{m} g_{ij}\, dx_i\, dx_j$$

is a Riemannian metric, then the n-dimensional volume element is

$$dV = \sqrt{g}\, dx_1 \cdots dx_m,$$

where $g = \det(g_{ij})$.

Therefore in our case, the $(n+1)$-dimensional volume element in $\mathbb{H}^{n+1}$ is

$$dV_{n+1} = \frac{dx_1 \cdots dx_n dy}{y^{n+1}}$$

As a first example of a volume computation using the upper half space model, consider a region defined as a cone over the graph of a function. Let $f(x_1, \ldots, x_n)$ be a function $f : \mathbb{R}^n \to \mathbb{R}$, defined in some bounded region $\mathcal{D}$ of $\mathbb{R}^n$. The graph of f is the set

$$\mathcal{G} = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in \mathcal{D}\}.$$

Let us consider a region $\mathcal{C}$ of $\mathbb{R}^{n+1}$ defined as the *infinite cone* obtained by joining points of $\mathcal{G}$ with the infnity point ∞ , i.e. $\mathcal{C}$ is defined by an inequality of the form:

$$\mathcal{C} = \{(x_1, \ldots, x_n, y) : y \geq f(x_1, \ldots, x_n)\},$$

where $(x_1, \ldots, x_n) \in \mathcal{D}$.

Then for its volume we obtain:

$$V_{n+1}(\mathcal{C}) = \int_{\mathcal{C}} dV_{n+1} = \frac{1}{n} \int_{\mathcal{D}} \frac{dx_1 \cdots dx_n}{(f(x_1, \ldots, x_n))^n}.$$

In particular suppose that our graph $y = f(x_1, \ldots x_n)$ is defined by the equation $y = \sqrt{1 - x_1^2 - \cdots - x_n^2}$ and is the unit hemisphere in $\mathbb{R}^{n+1}$, that is the hyperplane in the half space model for $\mathbb{H}^{n+1}$. Then we obtain the following

Lemma 2.1 ([57]) *The volume of the infinite cone $\mathcal{C}$ in $\mathbb{H}^{n+1}$ obtained by joining a compact region $\mathcal{G}$ in the hyperplane $y = \sqrt{1 - x_1^2 - \ldots - x_n^2} > 0$ to the infinity point ∞ is given by the formula*

$$V_{n+1}(\mathcal{C}) = \frac{1}{n} \int_{\mathcal{D}} \frac{dx_1 \cdots dx_n}{(1 - x_1^2 - \ldots - x_n^2)^{n/2}}. \tag{2.1}$$

where $\mathcal{D}$ is the projection of $\mathcal{G}$ on $\mathbb{R}^n$.

As an example of using Lemma 2.1, we will obtain the well-known formula for the area of a triangle. Let us consider a triangle $T(0, \alpha, \beta)$ in the Lobachevsky plane $\mathbb{H}^2$ with one vertex at infinity (and the angle 0 in this vertex) and other vertices on the unit semicircle in $\mathbb{R}^2$ with angles α, β and first coordinates x' and x'', respectively. Then the triangle $T(0, \alpha, \beta)$ can be regarded as an infinite cone over a compact region on the hyperplane $y = \sqrt{1 - x^2}$ and a compact region $\mathcal{D}$ is the segment $[x', x'']$.

Thus according to (2.1) we have

$$V_2(T(0,\alpha,\beta)) = \int_{x'}^{x''} \frac{dx}{\sqrt{1-x^2}} = \arccos(x') - \arccos(x'') = \pi - \alpha - \beta. \quad (2.2)$$

If $T = T(\alpha,\beta,\gamma)$ is a bounded triangle in $\mathbb{H}^3$ with angles $\alpha,\beta,\gamma > 0$, $\alpha+\beta+\gamma < \pi$, then we can continue the edge between angles α and γ from the vertex corresponding to α to the infinity and consider triangles $T' = T(0,\delta,\pi-\gamma)$ and $T'' = T \cup T' = T(0,\alpha,\beta+\delta)$ for some δ. Then according to (2.2) we get

$$V_2(T) = V_2(T') - V_2(T'') = \pi - \alpha - \beta - \gamma.$$

Therefore we have the following well-known result.

Proposition 2.1 *If $T \subset \mathbb{H}^2$ is a triangle with angles α, β, γ, then the area of T is given by the formula:*

$$area(T) = V_2(T) = \pi - \alpha - \beta - \gamma.$$

We remark that the area $V_2(T)$ is maximal if $\alpha = \beta = \gamma = 0$ and T is the regular ideal triangle with all vertices at infinity. So $V_2^{max} = \pi$.

2.1.2 The volume of an infinite cone over a right-angled triangle

Now let us consider a three-dimensional case and use more traditional notations (x,y,z) instead (x_1,x_2,y) for coordinates in $\mathbb{R}^3$. In this notations $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 \, : \, z > 0\}$, and

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let a region $\mathcal{D}$ be the right-angled triangle in the unit 2-disk in $\mathbb{R}^2$ with one angle equals α and the length of the adjacent leg equals b, $0 < b < 1$, such that

$$\mathcal{D} = \{(x,y) \, : \, 0 \le x \le b, \, 0 \le y \le \tan\alpha\}.$$

Then

$$\mathcal{G} = \{(x,y,z) \, : \, (x,y) \in \mathcal{D}, \, z = \sqrt{1-x^2-y^2}\}$$

is the compact region on the unit hemisphere in $\mathbb{R}^3$, and let $\mathcal{C}$ be the infinite cone over $\mathcal{G}$ with the apex at infinity:

$$\mathcal{C} = \{(x,y,z) \, : \, 0 \le x \le b, \, 0 \le y \le \tan\alpha, \, z > \sqrt{1-x^2-y^2}\}.$$

According to Lemma 2.1 we have

$$V_3(\mathcal{C}) = \frac{1}{2}\int_{\mathcal{D}} \frac{dx dy}{1-x^2-y^2} = \frac{1}{2}\int_0^b dx \int_0^{x\tan\alpha} \frac{dy}{1-x^2-y^2}.$$

Using the identity

$$\int_0^c \frac{dy}{a^2-y^2} = \frac{1}{2a}\ln\left(\frac{a+c}{a-c}\right),$$

we get

$$\int_0^{x\tan\alpha} \frac{dy}{1-x^2-y^2} = \frac{1}{2\sqrt{1-x^2}}\ln\left(\frac{\sqrt{1-x^2}+x\tan\alpha}{\sqrt{1-x^2}-x\tan\alpha}\right),$$

and hence

$$V_3(\mathcal{C}) = \frac{1}{4}\int_0^b \frac{1}{2\sqrt{1-x^2}}\ln\left(\frac{\sqrt{1-x^2}+x\tan\alpha}{\sqrt{1-x^2}-x\tan\alpha}\right) dx. \tag{2.3}$$

Using the substitution $x = \cos\theta$ for $\beta \leq \theta \leq \pi/2$, where $\cos\beta = b$, we get $\sin\theta = \sqrt{1-x^2}$, and $d\theta = -dx/\sqrt{1-x^2}$. So from (2.3) we obtain

$$V_3(\mathcal{C}) = \frac{1}{4}\int_\beta^{\pi/2} \ln\left(\frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}\right) d\theta.$$

For our next aims it will be more convenient to express this result in terms of the following version of the ln sin integral. Let us consider a function

$$\Lambda(x) = -\int_0^x \ln|\,2\sin\zeta\,|\,d\zeta. \tag{2.4}$$

Then for indefinite integral we have

$$\int \ln\left(\frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}\right) d\theta = \Lambda(\theta-\alpha) - \Lambda(\theta+\alpha),$$

and we get

$$V_3(\mathcal{C}) = \frac{1}{4}\left(\Lambda\left(\frac{\pi}{2}-\alpha\right) - \Lambda\left(\frac{\pi}{2}+\alpha\right) - \Lambda(\beta-\alpha) + \Lambda(\beta+\alpha)\right). \tag{2.5}$$

We remark. that $\mathcal{C}$ is the tetrahedron in $\mathbb{H}^3$ founded by the triangle $\mathcal{G}$ on the semi-sphere and by three edges from vertices of the triangle to infinity. In addition, dihedral angles, corresponding to the bottom of the tetrahedra $\mathcal{C}$ are $\pi/2$, $\pi/2$ and β, where $\cos\beta = b$, and dihedral angles corresponding to the infinity vertex are α, $\pi/2-\alpha$ and $\pi/2$ (see Figure 2.1).

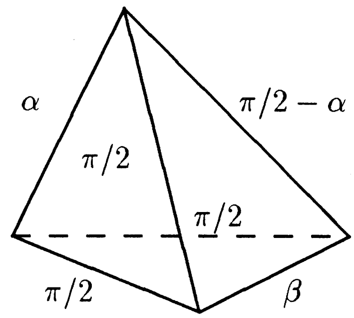

Figure 2.1. The tetrahedron $\mathcal{C}$.

As a tetrahedron in $\mathbb{H}^3$ is uniquely, up to isometry, determined by its dihedral angles, we will be use notation $T(\alpha,\beta,\gamma;\alpha',\beta',\gamma')$ for tetrahedron whose dihedral angles corresponding to some vertex are α,β,γ and angles at opposite edges are α',β',γ'. Thus in this notation $\mathcal{C} = T(\alpha,\frac{\pi}{2}-\alpha,\frac{\pi}{2};\beta,\frac{\pi}{2},\frac{\pi}{2})$ and we can rewrite formula (2.5) in the form:

$$V_3\left(T\left(\alpha,\frac{\pi}{2}-\alpha,\frac{\pi}{2};\beta,\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{4}\left(\Lambda\left(\frac{\pi}{2}-\alpha\right) - \Lambda\left(\frac{\pi}{2}+\alpha\right) - \Lambda(\beta-\alpha) + \Lambda(\beta+\alpha)\right). \tag{2.6}$$

In particular if $\beta=\alpha$, then the tetrahedron $T(\alpha,\frac{\pi}{2}-\alpha,\frac{\pi}{2};\alpha,\frac{\pi}{2},\frac{\pi}{2})$ has two vertices at infinity. For its volume from (2.6) we have:

$$V_3\left(T\left(\alpha,\frac{\pi}{2}-\alpha,\frac{\pi}{2};\alpha,\frac{\pi}{2},\frac{\pi}{2}\right)\right) = \frac{1}{2}\Lambda(\alpha). \tag{2.7}$$

2.1.3 The Lobachevsky function

As we see, formula (2.7) gives a simple geometric sense of the value of the function $\Lambda(\alpha)$ as the volume of the corresponding tetrahedron.

This function $\Lambda(x)$, defined by (2.4) , was introduced by J.Milnor [57] and called *the Lobachevsky function.* This function is related to the function

$$L(x) = -\int_0^x \ln\cos\zeta\, d\zeta,$$

which is traditionally called the Lobachevsky function, by the equation

$$L(x) = \Lambda\left(x-\frac{\pi}{2}\right) - x\ln 2.$$

We will eliminate some properties of $\Lambda(x)$ in the following

Proposition 2.2 *The Lobachevsky function $\Lambda(x)$ has the following properties:*

(1) *$\Lambda(x)$ is a continuous function.*

(2) *$\Lambda(x)$ is an odd function:*

$$\Lambda(-x) = -\Lambda(x). \tag{2.8}$$

(3) *The derivation $\Lambda'(x)$ exists for all $x \neq k\pi$, $k \in \mathbb{Z}$, and*

$$\Lambda'(x) = -\ln|\,2\sin x\,| = -\ln\left(\cos\left(x + \frac{\pi}{2}\right)\right) - \ln 2. \tag{2.9}$$

(4) $\Lambda''(x) = -\cot x$.

(5) *$\Lambda(x)$ is a periodic function with period π.*

(6) $\Lambda(\pi/2) = 0$.

(7) $\Lambda(2x) = 2\Lambda(x) + 2\Lambda(x + \pi/2)$.

(8) *In $[0, \pi/2]$ the function $\Lambda(x)$ has a maximum in the unique point $x = \pi/6$ and*

$$\Lambda^{max} = \Lambda(\pi/6) = 0.507\ldots.$$

(9) *For any $m \in \mathbb{Z}$*

$$\Lambda(mx) = m\sum_{k=0}^{m-1} \Lambda\left(x + \frac{k\pi}{m}\right). \tag{2.10}$$

Proof. Properties (1)-(4), (8) are obvious; for proving (5) see [10]; (6) follows from (2) and (5); (7) follows from (5) and (7); for proving (9) see [57], [10, p.99] or [65, p.466]. □

2.1.4 The volume of an ideal tetrahedron

Now we will find the volume of a tetrahedron in $\mathbb{H}^3$ with all vertices at infinity, or in another words, with all ideal vertices. We will call such tetrahedron to be an *ideal* tetrahedron.

Lemma 2.2 *Let $T(\alpha,\beta,\gamma;\alpha',\beta',\gamma')$ be an ideal tetrahedron in $\mathbb{H}^3$. Then $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$ and $\alpha+\beta+\gamma=\pi$.*

Proof. Because all vertices are ideal, the sums of dihedral angles at triples of edges which are incident to a vertex, are equal π. And we get four equalities from which the statement follows. □

Therefore an ideal tetrahedron is determined uniquely by three dihedral angles α, β, γ such that $\alpha+\beta+\gamma=\pi$ and we will be use notation

$$T(\alpha,\beta,\gamma) = T(\alpha,\beta,\gamma;\alpha,\beta,\gamma).$$

Theorem 2.1 ([57]) *If $T(\alpha,\beta,\gamma)$ is an ideal tetrahedron in $\mathbb{H}^3$, then its volume is given by the formula*

$$vol(T(\alpha,\beta,\gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Proof. Let us assume firstly, that all angles α, β and γ are acute. As we consider the upper half space model for $\mathbb{H}^3$, we can suppose that three ideal vertices of the tetrahedron $T = T(\alpha,\beta,\gamma)$ lie on the Euclidean plane $\mathbb{E}^2 = \{(x,y,z) : z=0\}$ and because $\alpha+\beta+\gamma=\pi$, they form an Euclidean triangle. Let us consider in the triangle perpendiculars to edges which pass through their midpoints. The common point for perpendiculars is the center of the circle in which our triangle in inscribed. Therefore central angles are twice of angles of the triangle. Let us connect the center with vertices of the triangle. Then we will get six triangles which are three pairs of equals Euclidean triangles with angles $(\alpha,\frac{\pi}{2},\frac{\pi}{2}-\alpha)$, $(\beta,\frac{\pi}{2},\frac{\pi}{2}-\beta)$ and $(\gamma,\frac{\pi}{2},\frac{\pi}{2}-\gamma)$. If we consider each of these Euclidean triangles as the base $\mathcal{D}$ for the infinite cone as above, then we will get six tetrahedra with two ideal vertices whose volumes can be calculated according to formula (2.7). Therefore, we have:

$$vol(T(\alpha,\beta,\gamma)) = 2\,vol\left(T\left(\alpha,\frac{\pi}{2},\frac{\pi}{2}-\alpha,\alpha,\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

$$+2\,vol\left(T\left(\beta,\frac{\pi}{2},\frac{\pi}{2}-\beta,\beta,\frac{\pi}{2},\frac{\pi}{2}\right)\right) + 2\,vol\left(T\left(\gamma,\frac{\pi}{2},\frac{\pi}{2}-\gamma,\gamma,\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$
$$= \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

If one of angles α, β , γ is not acute, we can suppose that it is true for γ. Then repeating above considerations, we will get, that the volume of the tetrahedron $T(\alpha,\beta,\gamma)$ is the sum of volumes of two pairs of tetrahedra which correspond to angles α and β, minus volumes of two tetrahedra corresponding to the angle $\pi - \gamma$. Then also

$$vol(T(\alpha,\beta,\gamma)) = \Lambda(\alpha) + \Lambda(\beta) - \Lambda(\pi-\gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where we used property (2.8). □

As it was shown in [77, p.4.4] (see also [65, p.477]), ideal tetrahedra in $\mathbb{H}^3$ can be parameterized by complex numbers z, $\mathrm{Im}z > 0$. More exactly, if we denote

$$z' = \frac{z-1}{z}, \quad z'' = \frac{1}{1-z},$$

then there is an ideal tetrahedron T in $\mathbb{H}^3$, unique up to a motion, with dihedral angles

$$\arg z, \quad \arg z', \quad \arg z''.$$

We denote such tetrahedra by T_z. As a consequence of Theorem 2.1 we get

$$vol(T_z) = \Lambda(\arg z) + \Lambda(\arg z') + \Lambda(\arg z'').$$

We remark, that $vol(T(\alpha,\beta,\gamma))$ is function on two variables, because $\alpha+\beta+\gamma = \pi$. So we can consider a function

$$f(\alpha,\beta) = vol(T(\alpha,\beta,\pi-\alpha-\beta)) = \Lambda(\alpha) + \Lambda(\beta) - \Lambda(\alpha+\beta).$$

Using (2.9) we get that this function has the extremal value for $\alpha = \beta = \pi/3$.

Corollary 2.1 *An ideal tetrahedron in $\mathbb{H}^3$ is of maximal volume if and only if it is regular tetrahedron $T(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$. In this case*

$$V_2^{max} = vol\left(T\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)\right) = 3\Lambda\left(\frac{\pi}{3}\right) = 2\Lambda\left(\frac{\pi}{6}\right) = 2\Lambda^{max} = 1.0149426\ldots.$$

We recall that in two-dimensional case the regular ideal triangle with angles $(0,0,0)$ also was of the maximal area. In general case the following theorem is true.

Theorem 2.2 ([25]) *In hyperbolic n-dimensional space, for $n \geq 2$, a simplex is of maximal volume if and only if it is ideal and regular.*

We recall [25], that if we denote the maximal volume of simplex in $\mathbb{H}^n$ by V_n^{max}, then

$$V_2^{max} = \pi = 3.1415926\ldots,$$

$$V_3^{max} = 2\Lambda^{max} = 1.0149416\ldots,$$

$$V_4^{max} = \frac{\pi}{3}\left(4\pi - 10\arccos\frac{1}{3}\right) = 0.2688956\ldots,$$

and the following asymptotic take place:

$$V_n^{max} \sim e\frac{\sqrt{n}}{n!}, \qquad \text{for}\, n \to \infty.$$

2.2 An ideal pyramid

Let $P(\alpha_1, \ldots, \alpha_n)$, $n \geq 3$, be an ideal (with all vertices at infinity) pyramid in $\mathbb{H}^3$ with an n-gon as the bottom and with dihedral angles $\alpha_1, \ldots, \alpha_n$ between the bottom and lateral faces (see Figure 2.2).

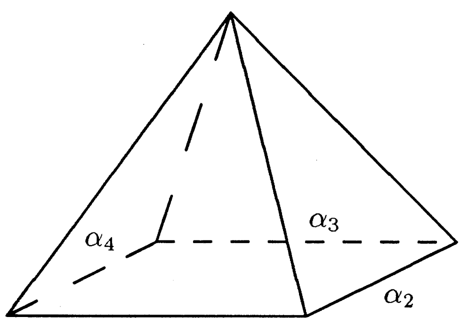

Figure 2.2. A prism $P(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Theorem 2.3 ([77, p.7.12]) *Let $P(\alpha_1, \ldots, \alpha_n)$, $n \geq 3$, be an ideal pyramid in $\mathbb{H}^3$. Then*

(i) $\alpha_1 + \ldots + \alpha_n = \pi$,

(ii) $vol(P(\alpha_1, \ldots, \alpha_n)) = \Lambda(\alpha_1) + \ldots + \Lambda(\alpha_n)$.

Proof. Let us use the induction by n. If $n = 3$, then $P(\alpha_1, \alpha_2, \alpha_3)$ is an ideal tetrahedron and this case was considered in Theorem 2.1. Let us suppose, that the statement is true for $n = k - 1$. If $P = P(\alpha_1, \ldots \alpha_k)$ is the pyramid whose bottom is a k-gon, then we can consider a segment with divides the k-gon in two parts: $(k-1)$-gon and a triangle. Let us denote by P' the ideal pyramid with the $(k-1)$-gonal bottom and by T the tetrahedron with the triangle as the bottom, such that $P = P' \cup T$. Let us denote by β the dihedral angle between the bottom of P' and the lateral face which is common with the tetrahedron T. Therefore, according to above notations, we have $P' = P(\alpha_1, \ldots, \alpha_{k-2}, \beta)$ and $T = T(\pi - \beta, \alpha_{k-1}, \alpha_k)$. Because by inductive hypothesis

$$\alpha_1 + \ldots + \alpha_{k-2} + \beta = \pi,$$

and

$$\pi - \beta + \alpha_{k-1} + \alpha_k = \pi,$$

we get

$$\alpha_1 + \ldots + \alpha_k = \pi.$$

For the volume $vol(P)$ of P according to the inductive hypothesis and by Theorem 2.1 we have

$$vol(P(\alpha_1, \ldots, \alpha_k)) = vol(P(\alpha_1, \ldots, \alpha_{k-2}, \beta)) + vol(T(\pi - \beta, \alpha_{k-1}, \alpha_k))$$

$$= \Lambda(\alpha_1) + \ldots + \Lambda(\alpha_{k-2}) + \Lambda(\beta) + \Lambda(\pi - \beta) + \Lambda(\alpha_{k-1}) + \Lambda(\alpha_k)$$

$$= \Lambda(\alpha_1) + \ldots + \Lambda(\alpha_k),$$

where we used that the Lobachevsky function is π-periodic and odd. □

Analogously to Corollary 2.1, for ideal pyramid we have:

Corollary 2.2 *An ideal pyramid $P(\alpha_1, \ldots, \alpha_n)$ is of maximal volume if and only if it is regular: $\alpha_1 = \ldots = \alpha_n = \pi/n$. In this case $V^{max} = n\,\Lambda(\pi/n)$.*

2.3 An ideal regular prism

Let us consider an ideal (with all vertices at infinity) prism in $\mathbb{H}^3$ with two equal regular n-gons as the top and the bottom. Suppose that the prism is *regular* in the sense that it is invariant under the dihedral group $\mathbb{D}_n$ of symmetries whose axe passes through centers of the top and the bottom. Let all dihedral angles at edges of the top and the bottom be equal to α and dihedral angles between lateral edges are equal to β (see Figure 2.3). Because all vertices of the prism are at infinity, we have $2\alpha + \beta = \pi$. Let us denote such ideal regular n-prism with n-gonal top and bottom and dihedral angle α by P_n^α.

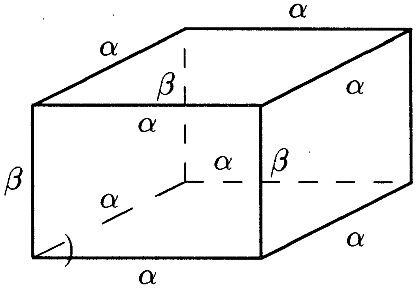

Figure 2.3. A prism P_4^α.

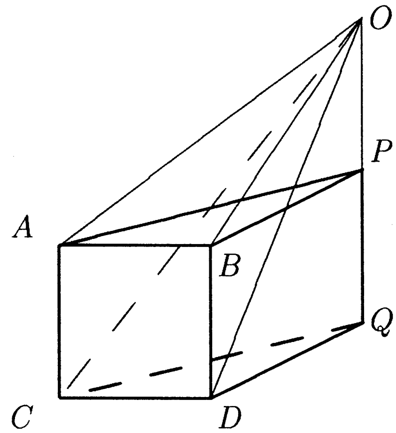

Figure 2.4. The piece Σ_n^α.

Theorem 2.4 ([77, p.7.16]) *If P_n^α is an ideal regular n-prism in $\mathbb{H}^3$ with dihedral angle α, then*

$$vol(P_n^\alpha) = n\left(\Lambda\left(\alpha+\frac{\pi}{n}\right) + \Lambda\left(\alpha-\frac{\pi}{n}\right) - 2\Lambda\left(\alpha-\frac{\pi}{2}\right)\right). \quad (2.11)$$

Proof. Let $\Sigma_n^\alpha = ABCDPQ$ be the $\frac{1}{n}$-piece of P_n^α (see Figure 2.4). By the construction, vertices P and Q are finite and vertices A, C, C, D are at infinity. We remark, that if we consider the upper half space model for $\mathbb{H}^3$, then vertices A, B, C and D lie on the Euclidean plane $\mathbb{E}^2$ and PQ is orthogonal to $\mathbb{E}^2$ Let us consider the infinite cone $\mathcal{C} = OQABCD$ over the pentagon $QABCD$ with the apex O at infinity which lies on the line PQ. If we denote by T' the tetrahedron $OPAB$, then $\mathcal{C} = \Sigma_n^\alpha \cup T'$. Let T be the tetrahedron $OQCD$. We remark, that T and T' are isometric because the prism P_n^α is regular. And we have the decomposition $\mathcal{C} = T \cup \mathcal{C}_1$, where $\mathcal{C}_1 = OABCD$ is the infinite cone over the quadrilateral $ABCD$. Comparing two decompositions $\mathcal{C} = \Sigma_n^\alpha \cup T' = T \cup \mathcal{C}_1$, and using that $vol(T) = vol(T')$, we get that $vol(\Sigma_n^\alpha) = vol(\mathcal{C}_1)$.

The infinite cone $\mathcal{C}_1$ is an ideal pyramid with the bottom $ABCD$ and its volume can be computed according to Theorem 2.3. Dihedral angles of $\mathcal{C}_1$ at edges AC and BD are equal to $\beta/2$. The dihedral angle at the edge AB is founded by the dihedral angle from Σ_n^α and by the dihedral angle from the ideal regular pyramid whose bottom is the regular n-gonal top of P_n^α. Thus the dihedral angle at AB is equal to $\alpha + \pi/n$. Using the item (i) of Theorem 2.3 we will get that the dihedral angle at the edge CD is equal to $\alpha - \pi/n$.

Therefore by Theorem 2.3 we get

$$vol(\Sigma_n^\alpha) = vol(\mathcal{C}_1) = \Lambda\left(\alpha+\frac{\pi}{n}\right) + \Lambda\left(\alpha-\frac{\pi}{n}\right) + 2\Lambda\left(\frac{\beta}{2}\right).$$

We recall that $2\alpha + \beta = \pi$, so $\beta/2 = \pi/2 - \alpha$. Using that the Lobachevsky function is odd, we can rewrite

$$vol(\Sigma_n^\alpha) = \Lambda\left(\alpha + \frac{\pi}{n}\right) + \Lambda\left(\alpha - \frac{\pi}{n}\right) - 2\Lambda\left(\alpha - \frac{\pi}{2}\right),$$

hence

$$vol(P_n^\alpha) = n\left(\Lambda\left(\alpha + \frac{\pi}{n}\right) + \Lambda\left(\alpha - \frac{\pi}{n}\right) - 2\Lambda\left(\alpha - \frac{\pi}{2}\right)\right)$$

and theorem is proved. □

Corollary 2.3 *An ideal regular n-prism P_n^α with dihedral angle α is of maximal volume if and only if*

$$\alpha = \arccos\left(\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{n}\right)\right).$$

Proof. By direct calculations using (2.11) and (2.9). □

Corollary 2.4 *An ideal regular 4-prism of maximal volume is an ideal regular cube with dihedral angles $\pi/3$ and its volume is equal*

$$V^{max}(P_4) = 10\,\Lambda\left(\frac{\pi}{6}\right) = 10\,\Lambda^{max} = 5.07\ldots.$$

Proof. According to Corollary 2.3 for the ideal regular 4-prism of maximal volume we get $\alpha = \pi/3$. In virtue of Theorem 2.4 we get

$$\begin{aligned} V^{max}(P_4) &= 4\left(\Lambda\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + \Lambda\left(\frac{\pi}{3} - \frac{\pi}{4}\right) - 2\Lambda\left(\frac{\pi}{3} - \frac{\pi}{2}\right)\right) \\ &= 4\left(\Lambda\left(\frac{7\pi}{12}\right) + \Lambda\left(\frac{\pi}{12}\right) + 2\Lambda\left(\frac{\pi}{6}\right)\right) \\ &= 2\Lambda\left(\frac{\pi}{6}\right) + 8\Lambda\left(\frac{\pi}{6}\right) = 10\Lambda\left(\frac{\pi}{6}\right), \end{aligned}$$

where we used the item (7) of Proposition 2.2 for the case $x = \pi/12$. □

We remark, that the formula from Corollary 2.4 can be obtained geometrically, if we divide the ideal $\frac{\pi}{3}$-cube $ABCDA'B'C'D'$ in five tetrahedra $AA'B'D'$, $CC'A'D'$, $ABCB'$, $ADCD'$ and $ACD'B'$ each of which is ideal regular and according to Corollary 2.2 has volume $2\Lambda(\pi/6)$.

2.4 An ideal regular antiprism

Let us consider an ideal (with all vertices at infinity) regular (with dihedral symmetry) antiprism in $\mathbb{H}^3$ with two equal regular n-gons as the top and the bottom with dihedral angles α and β and with equal triangles as the lateral faces. The antiprism can be regarded as a drum with triangular sides (see Figure 2.5 where for $n = 3$ the lateral boundary is shown).

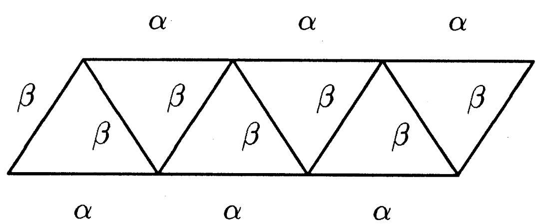

Figure 2.5.

Because the polyhedron is ideal, we have $2\alpha + 2\beta = 2\pi$. So we denote such ideal regular n-antiprism with n-gonal top and bottom and with dihedral angle α by $\mathcal{A}_n(\alpha)$. The following formula is essentially due to [77, p.6.43] were the case $\alpha = \pi/2$ was considered.

Theorem 2.5 *If $\mathcal{A}_n(\alpha)$ is an ideal regular n-antiprism in $\mathbb{H}^3$ with dihedral angle α, then*

$$vol(\mathcal{A}_n(\alpha)) = 2n\left(\Lambda\left(\frac{\alpha}{2}+\frac{\pi}{2n}\right) + \Lambda\left(\frac{\alpha}{2}-\frac{\pi}{2n}\right)\right) \qquad (2.12)$$

Proof. Let us consider the hexahedron $\Pi_n^\alpha = ABCDPQ$ which is the $\frac{1}{n}$-piece of $\mathcal{A}_n(\alpha)$ (see Figure 2.6). We recall that vertices P and Q are finite and all others vertices are at infinity.

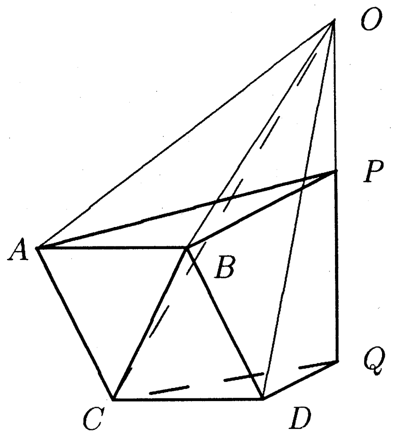

Figure 2.6.

Let O be an infinite point on the line PQ and let us consider the polyhedron $OABCDPQ$. There are two possible decompositions of this polyhedron: $OABCDPQ = T \cup \mathcal{C} = T' \cup \Pi_n^\alpha$, where $T = OCDQ$ and $T' = OABP$ are tetrahedra with one finite vertex and all other vertices at infinity, and $\mathcal{C} = OABCD$ is an ideal polyhedron which consists of two ideal tetrahedra $OABC$ and $OBCD$. Because the antiprism $\mathcal{A}_n(\alpha)$ is regular, tetrahedra T and T' are isometric, so $vol(T) = vol(T')$ and

$$vol(\Pi_n^\alpha) = vol(\mathcal{C}) = vol(OABC) + vol(OBCD).$$

We can calculate volumes of ideal tetrahedra $OABC$ and $OBCD$ according to Theorem 2.1. For this we need to know their dihedral angles. Let us consider the tetrahedron $OABC$. The dihedral angle at the edge AB is founded by the dihedral angle from the antiprism $\mathcal{A}_n(\alpha)$ (which is equal to α) and by the dihedral angle from the ideal regular pyramid with the apex O whose base is the top of the antiprism. This angle is equal to π/n by Theorem 2.3, item (i). Therefore $\angle AB = \alpha + \pi/n$. Because $\mathcal{A}_n(\alpha)$ is regular, we have $\angle AC = \angle BC = \frac{1}{2}(\pi - \alpha - \pi/n)$. Hence

$$vol(OABC) = \Lambda\left(\alpha + \frac{\pi}{n}\right) + 2\Lambda\left(\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2n}\right) = 2\Lambda\left(\frac{\alpha}{2} + \frac{\pi}{2n}\right),$$

where for the last step we used the item (7) of Proposition 2.2.

Analogously, for the tetrahedron $OBCD$ we have $\angle CD = \alpha - \pi/n$ and $\angle BC = \angle BD = \frac{1}{2}(\pi - \alpha + \pi/n)$. Whence

$$vol(OBCD) = \Lambda\left(\alpha - \frac{\pi}{n}\right) + 2\Lambda\left(\frac{\pi}{2} - \frac{\alpha}{2} + \frac{\pi}{2n}\right) = 2\Lambda\left(\frac{\alpha}{2} - \frac{\pi}{2n}\right).$$

Therefore

$$vol(\mathcal{A}_n(\alpha)) = n\,vol(\Pi_n^\alpha) = 2n\left(\Lambda\left(\frac{\alpha}{2}+\frac{\pi}{2n}\right) + \Lambda\left(\frac{\alpha}{2}-\frac{\pi}{2n}\right)\right)$$

and theorem is proved. □

Corollary 2.5 *An ideal regular n-antiprism $\mathcal{A}_n(\alpha)$ with the dihedral angle α is of maximal volume if and only if*

$$\alpha = \arccos\left(\cos\left(\frac{\pi}{n}\right) - \frac{1}{2}\right).$$

Proof. By direct calculations using (2.12) and (2.9). □

We remark, that a 2-antiprism is a tetrahedron and in this case we have Corollary 2.1.

Corollary 2.6 *The maximal volume ideal regular 3-antiprism is the regular octahedron with dihedral angles $\pi/2$ and its volume is equal to*

$$V^{max}(\mathcal{A}_3) = 8\,\Lambda\left(\frac{\pi}{4}\right).$$

Proof. According to Corollary 2.5 for the ideal regular 3-antiprism of maximal volume we have $\alpha = \pi/2$. In virtue of Theorem 2.5 we get

$$V^{max}(\mathcal{A}_3) = 6\left(\Lambda\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + \Lambda\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right)$$
$$= 6\left(\Lambda\left(\frac{5\pi}{12}\right) + \Lambda\left(\frac{\pi}{12}\right)\right) = 8\Lambda\left(\frac{\pi}{4}\right)$$

where we used the formula (2.10) for the case $m = 3$, $x = \pi/12$. □

Corollary 2.7 *The maximal volume of an ideal regular 5-antiprism is equal to*

$$V^{max}(\mathcal{A}_5) = 10\left(\Lambda\left(\frac{3\pi}{10}\right) + \Lambda\left(\frac{\pi}{10}\right)\right).$$

Proof. According to Corollary 2.5 for the ideal regular 5-antiprism of maximal volume we have

$$\alpha = \arccos\left(\cos\frac{\pi}{5} - \frac{1}{2}\right) = \arccos\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right) = \arccos\left(\frac{\sqrt{5}-1}{4}\right) = \frac{2\pi}{5}$$

and the formula hold directly by Theorem 2.5. □

2.5 A pyramid with the apex at infinity

In this section, following E.B.Vinberg [3] we will proof the formula for the volume of a pyramid in $\mathbb{H}^3$ with the apex at infinity. We start from the particular case of polyhedron of such type which is the a tetrahedron with three vertices at infinity and will use it for getting the formula for the general case.

2.5.1 A tetrahedron with three vertices at infinity

Let us consider a tetrahedron $T = OABC$ with vertices A, B, C at infinity with dihedral angles $\angle OA = \alpha$, $\angle OB = \beta$, $\angle OC = \gamma$, $\angle BC = a$, $\angle AC = b$, $\angle AB = c$ (see Figure 2.7).

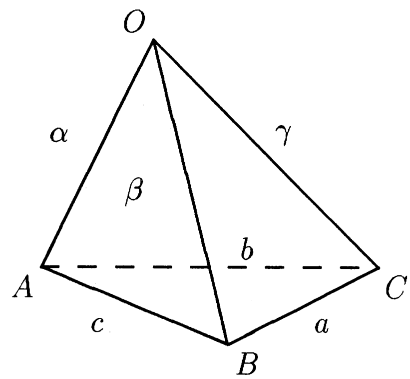

Figure 2.7.

Because vertices A, B, C are at infinity, the sum of the dihedral angles in these vertices is equal π. So we get

Lemma 2.3 *For a tetrahedron $T = OABC$ with infinite vertices A, B, C and with the dihedral angles as above, we have*

$$a = \frac{\pi + \alpha - \beta - \gamma}{2}, \quad b = \frac{\pi + \beta - \alpha - \gamma}{2}, \quad c = \frac{\pi + \gamma - \alpha - \beta}{2}.$$

Proof. Directly from equalities for the sum of dihedral angles coming at infinite points of T. □

As we see, the tetrahedron T is determined by the triple of angles (α, β, γ) or by the triple of angles (a, b, c).

Theorem 2.6 ([3, p.127]) *The volume of a tetrahedron T with three infinite vertices and dihedral angles α, β, γ, a, b and c as above, is given by the following formulae:*

$$vol(T) = vol(T(\alpha, \beta, \gamma, a, b, c))$$

$$= \frac{1}{2}\left(\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) + \Lambda(a) + \Lambda(b) + \Lambda(c) - \Lambda\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right)\right), \tag{2.13}$$

and

$$vol(T) = vol(T(\alpha,\beta,\gamma)) = \frac{1}{2}\left(\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) + \Lambda\left(\frac{\pi+\alpha-\beta-\gamma}{2}\right)\right.$$

$$\left. + \Lambda\left(\frac{\pi+\beta-\alpha-\gamma}{2}\right) + \Lambda\left(\frac{\pi+\gamma-\alpha-\beta}{2}\right) - \Lambda\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right)\right),$$

and

$$vol(T) = vol(T(a,b,c))$$

$$= \frac{1}{2}\left(\Lambda(a) + \Lambda(b) + \Lambda(c) - \Lambda(a+b) - \Lambda(a+c) - \Lambda(b+c) + \Lambda(a+b+c)\right).$$

Proof. Let us continue the edges of T to infinity, We will get three new infinite points A', B', C' which are opposite to infinite vertices A, B, C, respectively (see Figure 2.8).

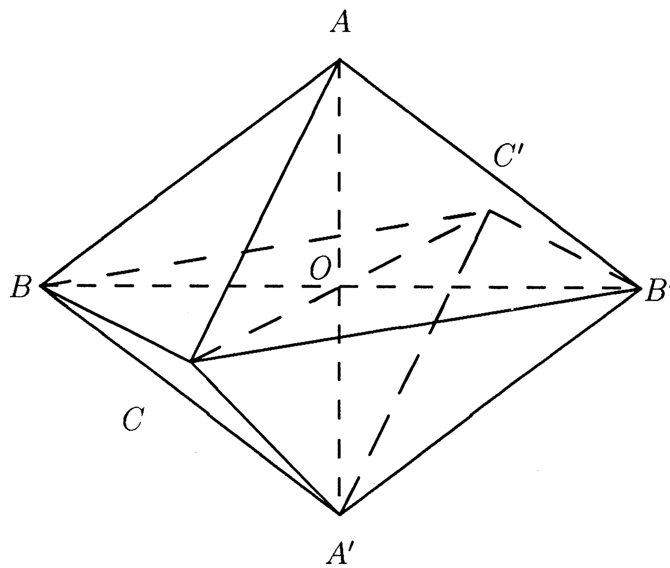

Figure 2.8.

The polyhedron $ABCA'B'C'$ is an octahedron with all vertices at infinity. Continuing faces of T, we will get a decomposition of the octahedron into eight tetrahedra with common finite vertex O and all other vertices at infinity.

By construction, these eight tetrahedra are such that there are four pairs of them which are symmetric in respect to the point O. Let us consider an ideal tetrahedron $T_1 = ABCB'$. There are three dihedral angles of T_1 which coincide with dihedral angles of $T : \angle BB' = \beta$, $\angle AB = c$, $\angle BC = a$, hence for opposite angles we have $\angle AC = \beta$, $\angle B'C = c$, $\angle AB' = a$. We remark that

$$T_1 = OABC \cup OAB'C = T \cup OAB'C.$$

Let us consider an ideal tetrahedron $T_2 = ABCA'$. Three of its dihedral angles coincide with dihedral angles of $T : \angle AA' = \alpha$, $\angle AB = c$, $\angle AC = b$, hence for opposite angles we have : $\angle BC = \alpha$, $\angle A'C = c$, $\angle A'B = b$. We remark, that

$$T_2 = OABC \cup OA'BC = T \cup OA'BC.$$

Let us consider a third ideal tetrahedron

$$T_3 = ACB'C' = OACB' \cup OAB'C'.$$

We remark, that tetrahedra $OAB'C'$ and $OA'BC$ are symmetric in respect to the point O. Therefore for volumes of above ideal tetrahedra we have:

$$vol(T_1) + vol(T_2) - vol(T_3) = vol(T) + vol(OAB'C)$$
$$+ vol(T) + vol(OA'BC) - vol(OACB') = 2\,vol(T). \qquad (2.14)$$

Because volumes of ideal tetrahedra can be calculated by Theorem 2.1, we get

$$vol(T_1) = \Lambda(\beta) + \Lambda(a) + \Lambda(c) \qquad (2.15)$$

and

$$vol(T_2) = \Lambda(\alpha) + \Lambda(b) + \Lambda(c). \qquad (2.16)$$

For the tetrahedron T_3 we remark, that the dihedral angle of T_3 at the edge CC' is equal to the compation angle at CC' in T, hence $\angle CC' = \pi - \gamma$, and opposite $\angle AB' = \pi - \gamma$. An angle at AC' in T_3 is equal to the angle at $A'C$ in the symmetric tetrahedron $OA'BC$. Hence $\angle AC' = c$ and opposite $\angle B'C = c$. Because the tetrahedron T_3 is ideal, we get

$$\angle B'C' = \pi - \angle AC' - \angle CC' = \pi - (\pi - \gamma) - c = \gamma - c,$$

and opposite $\angle AC = \gamma - c$. Therefore, using that the Lobachevsky function $\Lambda(x)$ is odd and π-periodic, by Lemma 2.3 we get:

$$vol(T_3) = \Lambda(\pi - \gamma) + \Lambda(c) + \Lambda(\gamma - c)$$

$$= -\Lambda(\gamma) + \Lambda(c) + \Lambda\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right) \tag{2.17}$$

From (2.14)–(2.17) we will get directly (2.13). Other formulae from the statement of the theorem follows in virtue of Lemma 2.3. □

We remark that if all vertices of a tetrahedron are at infinity, formula (2.13) coincides with the formula for volume of an ideal tetrahedron from Theorem 2.1.

2.5.2 The volume of a quadrilateral pyramid

Let $\mathcal{P}$ be a quadrilateral pyramid $OCDPQ$ with the apex C at infinity, the edge CD orthogonal to the base and with right angles at vertices P and Q in the quadrilateral $OPQD$. Denote dihedral angles coming at the vertex O by α, β and γ (see Figure 2.9).

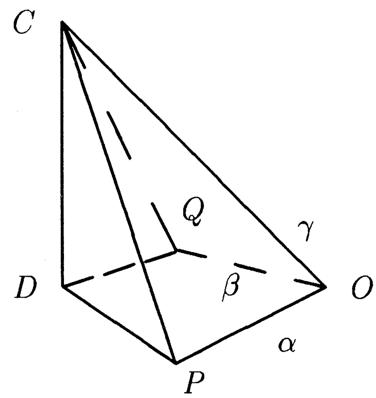

Figure 2.9.

Proposition 2.3 ([3, p.129]) *The volume of the quadrilateral pyramid $\mathcal{P}$ is given by the formula*

$$vol(\mathcal{P}) = \frac{1}{2}\left(\Lambda(\gamma) + \Lambda\left(\frac{\pi+\alpha-\beta-\gamma}{2}\right) + \Lambda\left(\frac{\pi+\beta-\alpha-\gamma}{2}\right)\right.$$

$$\left. + \Lambda\left(\frac{\pi+\alpha+\beta-\gamma}{2}\right) - \Lambda\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right)\right). \tag{2.18}$$

Proof. Continuing to infinity the two sides of the base of the pyramid $\mathcal{P}$ issuing from the vertex O and denoting new ideal vertices by A and B, we will get a tetrahedron $T = OABC$ which contains our pyramid $\mathcal{P}$ (see Figure 2.10).

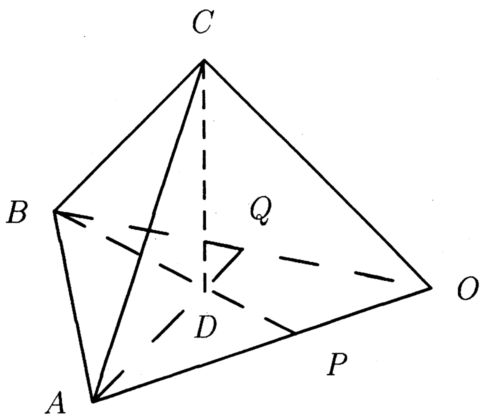

Figure 2.10.

Let us consider tetrahedra $T = OABC$, $T_1 = CDAP$, $T_2 = CDBQ$, $T_3 = ABCD$. Then

$$T = \mathcal{P} \cup T_1 \cup T_2 \cup T_3,$$

and

$$vol(\mathcal{P}) = vol(T) - vol(T_1) - vol(T_2) - vol(T_3). \tag{2.19}$$

Because the tetrahedron T is with three infinite vertices A, B, C and with dihedral angles α, β, γ incident to the vertex O, by Theorem 2.6 we get

$$vol(T) = \frac{1}{2}\left(\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma) + \Lambda\left(\frac{\pi + \alpha - \beta - \gamma}{2}\right)\right.$$

$$\left. + \Lambda\left(\frac{\pi + \beta - \alpha - \gamma}{2}\right) + \Lambda\left(\frac{\pi + \gamma - \alpha - \beta}{2}\right) - \Lambda\left(\frac{\alpha + \beta + \gamma - \pi}{2}\right)\right), \tag{2.20}$$

The tetrahedron T_1 has three right dihedral angles $\angle AD = \angle DP = \angle CP = \pi/2$ and vertices A and C at infinity, so $\angle AP = \angle CD = \alpha$, $\angle AC = \pi/2 - \alpha$, and according to formula (2.7) we get

$$vol(T_1) = \frac{1}{2}\Lambda(\alpha). \tag{2.21}$$

The tetrahedron T_2 has three right dihedral angles $\angle CQ = \angle DQ = \angle BD = \pi/2$. Because vertices B and C are at infinity, we get $\angle BQ = \angle CD = \beta$, $\angle BC = \pi/2 - \beta$, and according to (2.7), we have

$$vol(T_2) = \frac{1}{2}\Lambda(\beta). \tag{2.22}$$

Denote dihedral angle at the edge CD in the tetrahedron T_3 by ξ. Because the sum of all dihedral angles at the edge CD in T_1, T_2, $\mathcal{P}$ and T_3 is equal 2π, we get

$$\xi = 2\pi - \alpha - \beta - (\pi - \gamma) = \pi + \gamma - \alpha - \beta.$$

Using Theorem 2.6 for the tetrahedron T_3, which has vertices A, B, C at infinity, and by properties of the Lobachevsky function from Proposition 2.2, we have

$$vol(T_3) = \frac{1}{2}\left(\Lambda(\xi) - 2\Lambda\left(\frac{\xi}{2} + \frac{\pi}{2}\right)\right) = \Lambda\left(\frac{\xi}{2}\right) = \Lambda\left(\frac{\pi + \gamma - \alpha - \beta}{2}\right). \tag{2.23}$$

By substitution expressions from (2.20)–(2.23) in (2.19) and using that the Lobachevsky function is π-periodic and odd, we will get the formula (2.18). □

2.5.3 A pyramid with the apex at infinity

Because an arbitrary n-side pyramid $\mathcal{P}$ with the apex at infinity can be decomposed into n quadrilateral pyramids of the above type by dropping perpendiculars from its apex onto its base and onto the lines bounding the base, we will get the following theorem.

Theorem 2.7 ([3, p.130]) *Let $\mathcal{P}$ be n-side pyramid with the apex at infinity and with dihedral angles $\alpha_1, \ldots, \alpha_n$ corresponding to the base and with dihedral angles $\gamma_1, \ldots, \gamma_n$ at the side edges. Then*

$$vol(\mathcal{P}) = \frac{1}{2}\sum_{i=1}^{n}\left(\Lambda(\gamma_i) + \Lambda\left(\frac{\pi + \alpha_i - \alpha_{i+1} - \gamma_i}{2}\right) + \Lambda\left(\frac{\pi + \alpha_{i+1} - \alpha_i - \gamma_i}{2}\right)\right.$$

$$\left. + \Lambda\left(\frac{\pi + \alpha_i + \alpha_{i+1} - \gamma_i}{2}\right) - \Lambda\left(\frac{\alpha_i + \alpha_{i+1} + \gamma_i - \pi}{2}\right)\right). \qquad (2.24)$$

where $\alpha_{n+1} = \alpha_1$.

Proof. By Proposition 2.3. □

We remark, that all above proved formulae for volumes of tetrahedra and pyramids are particular cases of formula (2.24).

2.6 Orthoschemes

In this section we consider volumes of complete orthoschemes in the Lobachevsky space which are most basic objects in polyhedral geometry and are useful for decomposition of an arbitrary polyhedra. In general, an n-orthoscheme is a bounded n-simplex with vertices $P_0, \ldots, P_n$ such that

$$\text{span}\,(P_0, \ldots, P_i) \perp \text{span}\,(P_i, \ldots, P_n),$$

for $i = 1, \ldots, n-1$. In the 3-dimensional case we get a double-rectangular tetrahedron $P_0P_1P_2P_3$ such that the edge P_0P_1 is orthogonal to the plane $P_1P_2P_3$ and the edge P_2P_3 is orthogonal to the plane $P_0P_1P_2$ (see Figure 2.11).

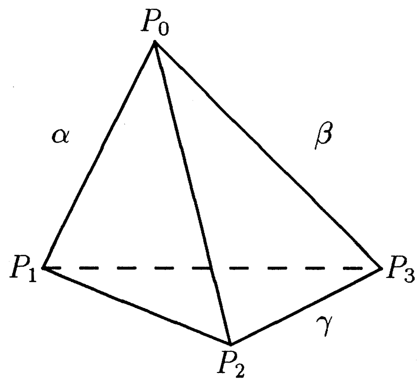

Figure 2.11.

By the definition, dihedral angles of the tetrahedron $P_0P_1P_2P_3$ at edges P_1P_2, P_1P_3 and P_0P_2 are right, and let us denote dihedral angles at edges P_0P_1, P_0P_3 and P_2P_3 by α, β and γ, respectively. The Gram matrix for the tetrahedron $P_0P_1P_2P_3$ is of the form

$$\begin{pmatrix} 1 & -\cos\alpha & 0 & 0 \\ -\cos\alpha & 1 & -\cos\beta & 0 \\ 0 & -\cos\beta & 1 & -\cos\gamma \\ 0 & 0 & -\cos\gamma & 1 \end{pmatrix}$$

with the determinant

$$\Delta = \sin^2\alpha\,\sin^2\gamma - \cos^2\beta,$$

that is negative by Theorem 1.4.

The volume of the double-rectangular tetrahedron can be calculated by the following formula which is essentially due to Lobachevsky.

Proposition 2.4 *Let $R = P_0P_1P_2P_3$ be a double-rectangular tetrahedron in $\mathbb{H}^3$ with acute angles α, β and γ. Then the volume $vol(R)$ of R is given by*

$$vol(R) = \frac{1}{4}\left(\Lambda(\alpha+\delta) - \Lambda(\alpha-\delta) + \Lambda\left(\frac{\pi}{2}+\beta-\delta\right) + \Lambda\left(\frac{\pi}{2}-\beta-\delta\right)\right.$$
$$\left. + \Lambda(\gamma+\delta) - \Lambda(\gamma-\delta) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\right), \tag{2.25}$$

where

$$0 \le \delta = \arctan\frac{\sqrt{\cos^2\beta - \sin^2\alpha\,\sin^2\gamma}}{\cos\alpha\,\cos\gamma} < \frac{\pi}{2}. \tag{2.26}$$

Proof. See [3, p.125], or [38], or [23, Ch.9]. □

As it was remarked in [38], we can generalize formulae (2.25) and (2.26) for more wide class of polyhedra. Let us consider the model of the Lobachevsky space $\mathbb{H}^3$ in the ball B^3. Let us assume, that one of vertices P_0 or P_3 (suppose for definitely that P_0) is an ideal point which lies outside B^3. We will be say that a polyhedron R is an *orthoscheme of degree 1*, if R can be obtained by cutting off the ideal vertex P_0 by the plane which is orthogonal to lines P_0P_1, P_0P_2 and P_0P_3. We will be say that R is an *orthoscheme of degree 2*, if R can be obtained by cutting two ideal vertices P_0 and P_3 of an ideal orthoscheme. We will be say that bounded polyhedron R is a *complete orthoscheme* if it is one of the following types: a double-rectangular tetrahedron (an orthoscheme of degree 0), an orthoscheme of degree 1, or an orthoscheme of degree 2.

Theorem 2.8 ([38]) *Let $R = R(\alpha, \beta, \gamma)$ be a complete orthoscheme with acute dihedral angles α, β, γ. Then the volume $vol(R)$ of R is given by*

$$vol(R) = \frac{1}{4}\Big(\Lambda(\alpha+\delta) - \Lambda(\alpha-\delta) + \Lambda\left(\frac{\pi}{2}+\beta-\delta\right) + \Lambda\left(\frac{\pi}{2}-\beta-\delta\right)$$

$$+\Lambda(\gamma+\delta) - \Lambda(\gamma-\delta) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\Big), \tag{2.27}$$

where

$$0 \le \delta = \arctan\frac{\sqrt{\cos^2\beta - \sin^2\alpha\,\sin^2\gamma}}{\cos\alpha\,\cos\gamma} < \frac{\pi}{2}. \tag{2.28}$$

Proof. See [38]. □.

2.7 Some computations

In above sections we described volumes of polyhedra in terms of the Lobachevsky function $\Lambda(x)$. For numeric computations it is very convenient to use the approximation of $\Lambda(x)$ which was given by D. Zagier [95]:

$$\Lambda(x) = x\,(9 - \log|\,2\sin x\,|) - \pi\sum_{n=1}^{4}\left[c_n\left(\frac{x}{\pi}\right)^{2n+1} + n\log\frac{n+\frac{x}{\pi}}{n-\frac{x}{\pi}}\right] + \epsilon, \tag{2.29}$$

with

$$c_1 = 0.14754863716,$$

$$c_2 = 0.00142852188,$$

$$c_3 = 0.00002919407,$$

$$c_4 = 0.00000076258,$$

and $|\, \epsilon \,| < 1.2 \cdot 10^{-11}$ for $|\, x \,| \leq \pi/2$.

As an example of calculations of volumes we give volumes of compact Coxeter tetrahedra and of bounded Coxeter tetrahedra in Figure 2.12 and Figure 2.13, respectively, where tetrahedra are given by their Coxeter schemes. These results we obtained in [55] by decomposition a Coxeter tetrahedra in double-rectangular tetrahedra and using Proposition 2.4.

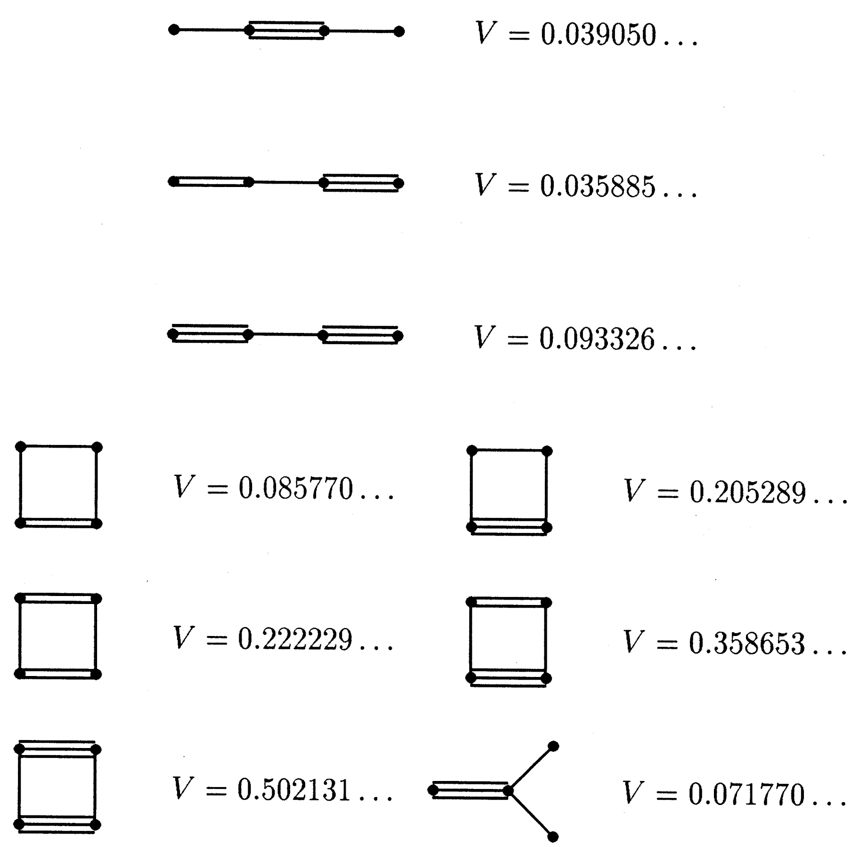

Figure 2.12.

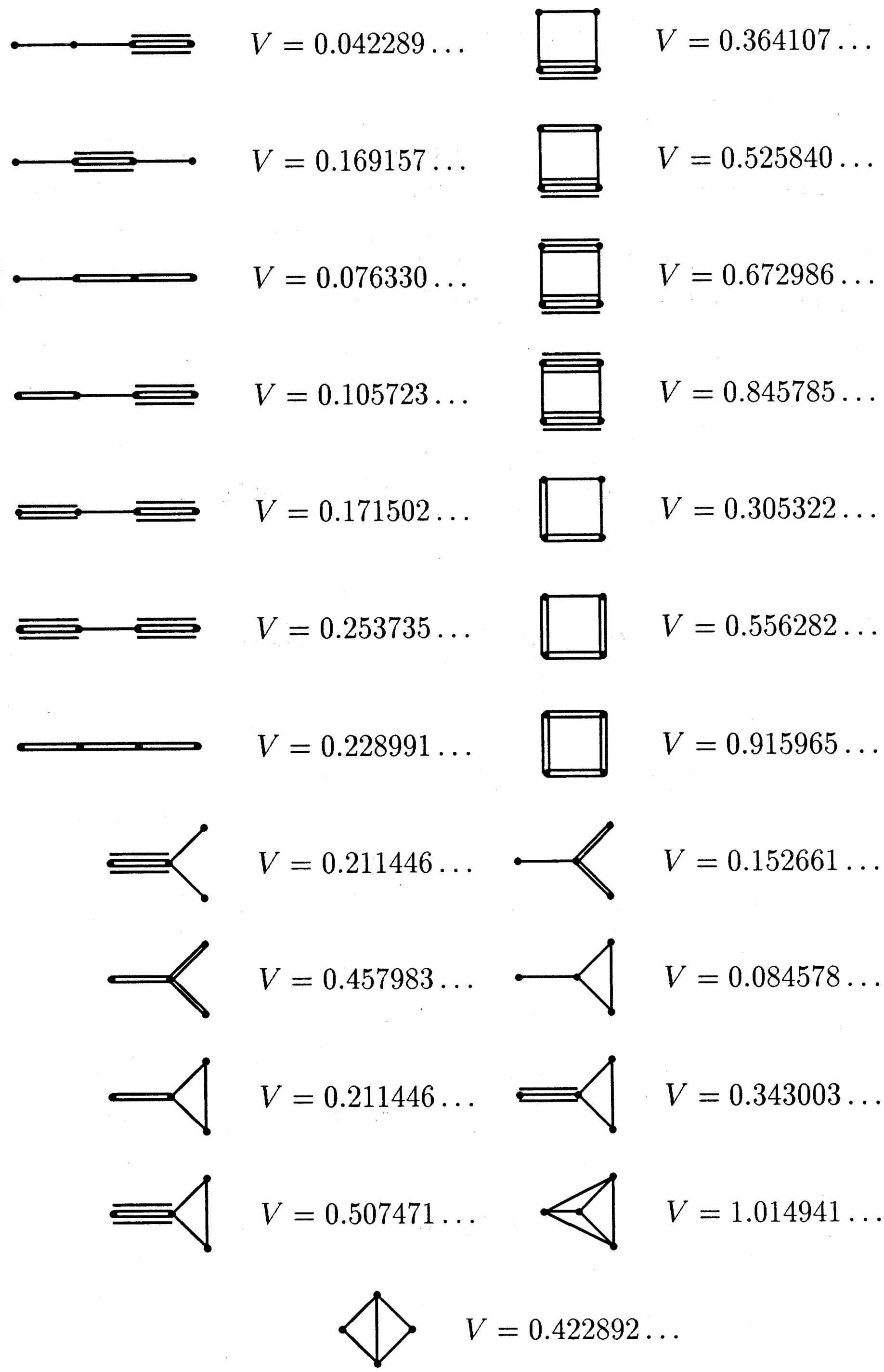
V = 0.042289...
V = 0.364107...
V = 0.169157...
V = 0.525840...
V = 0.076330...
V = 0.672986...
V = 0.105723...
V = 0.845785...
V = 0.171502...
V = 0.305322...
V = 0.253735...
V = 0.556282...
V = 0.228991...
V = 0.915965...
V = 0.211446...
V = 0.152661...
V = 0.457983...
V = 0.084578...
V = 0.211446...
V = 0.343003...
V = 0.507471...
V = 1.014941...
V = 0.422892...

Figure 2.13

In Table 2.1 we give volumes of Coxeter regular polyhedra in $\mathbb{H}^3$. These results can be obtained by a decomposition of a regular polyhedra in Coxeter tetrahedra (see also Corollaries 2.2, 2.4 and 2.6).

Table 2.1.

$\pi/3$–tetrahedron (ideal)	1.01494...	$= 2\Lambda(\pi/6)$
$2\pi/5$–cube	1.72248...	
$\pi/3$–cube (ideal)	5.07471...	$= 10\Lambda(\pi/6)$
$\pi/2$–octahedron (ideal)	3.66386...	$= 8\Lambda(\pi/4)$
$\pi/2$–dodecahedron	4.30621...	
$2\pi/5$–dodecahedron	11.19906...	
$\pi/3$–dodecahedron (ideal)	20.58020...	
$2\pi/3$–icosahedron	4.68603...	

Chapter 3

Volumes of hyperbolic manifolds

In this chapter we recall some properties of the volumes of hyperbolic manifolds. By a hyperbolic n-manifold M^n we mean an n-dimensional complete connected Riemannian manifold of constant negative curvature -1. According to Hopf-Kiling theorem [94, p.69], a hyperbolic n-manifold M^n can be obtained as a quotient space $M^n = \mathbb{H}^n/\Gamma$, where $\mathbb{H}^n$ is an n-dimensional Lobachevsky space and Γ is a discrete group of its isometries, acting without fixed points. The concept of the volume in $\mathbb{H}^n$ is carried over naturally to M^n. We will be consider the set $\mathcal{M}^n$ of n-dimensional orientable hyperbolic manifolds of finite volume.

Let us consider the volume function

$$v_n : \mathcal{M}^n \to \mathbb{R}_+$$

which makes correspondence between a manifold $M^n \in \mathcal{M}^n$ and its volume $vol(M^n)$. Denote

$$Vol_n = \{v_n(M^n) \,:\, M^n \in \mathcal{M}^n\},$$

that is a subset of $\mathbb{R}_+$.

In $n = 2$, then a manifold M^2 is a Riemannian surface and the structure of the set Vol_2 is described by the classical Gauss–Bonnet theorem. If M^2 is a hyperbolic surface of genus g with k punctures, then for its area we have:

$$vol(M^2) = 2\pi\,(2g - 2 + k).$$

Therefore Vol_2 is a discrete subset of $\mathbb{R}_+$ of the form $2\pi N$, where N is the set of integers, and Vol_2 can be pictured as in Figure 3.1.

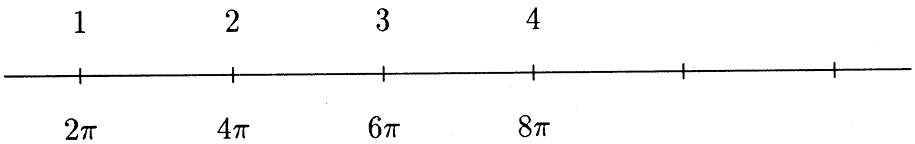

Figure 3.1.

The minimal area 2π is attained by two non-homeomorphic surfaces, a sphere with three punctures and a torus with one puncture. For given $v_0 = 2\pi n_0$, $n_0 \in N$, there is finitely many non-homeomorphic surfaces M^2 such that $vol(M^2) = v_0$. For each of them genus g and number of punctures k satisfy to the condition:

$$2g - 2 + k = n_0.$$

In particular, we see that for each even $n_0 \in N$, there are both compact and non-compact surfaces with the same area $v_0 = 2\pi n_0$.

Because for even dimensions $n = 2m \geq 2$, according to Gauss-Bonnet theorem,

$$v_{2m}(M^{2m}) = (-1)^m \frac{1}{2} v_{2m}(S^{2m}) \chi(M^{2m}),$$

we have that in this case the set Vol_{2m} is discrete also.

We recall that the first example of a compact hyperbolic 4-manifold was constructed by M. Davis [21] from a 4-dimensional polyhedron, so-called 120-cell. This polyhedron can be divided into 14400 congruent 4-simplices that are orthoschemes in terminology of section 2.6. Volumes of Coxeter 4-orthoschemes were calculated in [39]. Hence we get that the volume of Davis's 4-manifold is equal to $104\pi^2/3$ and according to Gauss–Bonnet theorem its Euler characteristic is equal to 26. More general, it was shown by J. Ratcliffe and S. Tschantz (see also [40]), that volumes of hyperbolic 4-manifolds belong to the set of the form $\frac{4}{3}\pi^2 N$.

The situation is similar for all $n \geq 4$. As it was shown by H. Wang [89], if $n \geq 4$, then for each real number x there are only finitely many (up to an isometry) n-dimensional hyperbolic manifolds with volume less or equal to x. Therefore, the set Vol_n is discrete for $n \geq 4$.

In the 3-dimensional case situation is completely different. In this case the following remarkable Thurston–Jøgensen theorem take place (see [77, Ch.6], [10, Ch.E], [24]).

Theorem 3.1 (Thurston–Jørgensen) *The volumes of 3-dimensional hyperbolic manifolds form a closed non-discrete set on the real line. This set is well* *ordered and its ordinal type is* ω^ω. *There are only finitely many manifolds with a given volume.*

The schematic picture of the set Vol_3 is shown in Figure 3.2.

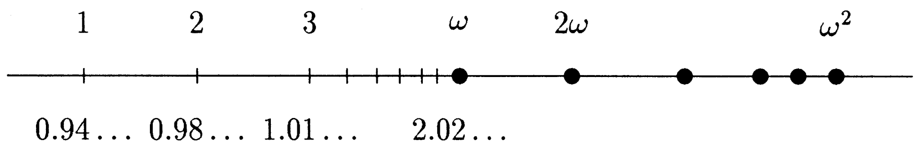

Figure 3.2

In particular, it follows from Thurston–Jørgensen theorem that there exists a smallest volume v_1, next smallest volume v_2, and so forth. This sequence $v_1 < v_2 < \ldots < v_k < \ldots$ has a limit point v_ω, which is a smallest volume of a complete hyperbolic manifold with one cusp. The next smallest manifold with one cusp has volume $v_{2\omega}$, and so forth. The first volume of a manifold with two cusps has volume v_{ω^2}, and so forth.

The smallest known volume manifold, whose volume is equal to $0.94\ldots$, was constructed by J. Weeks [90] and by S. V. Matveev and A. T. Fomenko [47]. Second smallest known value is $0.98\ldots$, and corresponding manifold was constructed by W. Thurston [77]. The third smallest value is $1.01\ldots$, that is the volume of Meyerhoff–Neumann manifold investigated in [56].

In [47] S. V. Matveev and A. T. Fomenko firstly conjectured the structure of the initial part of the set of volumes. In [36] C. Hodgson and J. Weeks refined the ten smallest known manifolds and their volumes, using famous computer program SnapPea [91]. These ten manifolds will be discussed in chapter 6. We will obtain them by Dehn surgeries on the Whitehead link and as two-fold branched coverings of the 3-sphere S^3.

The smallest value $2.02\ldots$ of volume of a non-compact manifold (which correspond to the first limit ordinal) is volume of the figure-eight knot complement [1]. In [77, p.6.26] W. Thurston constructed an example of two non-compact manifolds with different numbers of cusps, but with the same volume. He asked are there exist closed manifolds corresponding to limit ordinals. In section 5.5 we will show that Fibonacci manifolds have this property. In particular. we will obtain a closed manifold with volume equals to the volume of the figure-eight knot complement.

According to Thurston-Jørgensen theorem, there exist only finitely many manifolds with a given volume. But these finite numbers are not bounded. We will discuss it in section 4.3, where we will show, that for any integer

N there exists a right-angled polyhedron which is fundamental for at least N pairwise non-homeomorphic closed manifolds. Moreover, we will calculate their volumes.

Chapter 4

Löbell manifolds

This chapter is devoted to the first example of a closed orientable hyperbolic 3-manifold constructed in 1931 by F. Löbell and its generalizations.

We recall [87, p. 190], that in 1890, inspired by a number of examples due to Clifford, Klein formulated the problem of describing all connected compact Riemannian manifolds of constant curvature. Then Killing showed that these manifolds are Riemannian manifolds of the form X/Γ, where X is one of the spaces of constant curvature and Γ is a co-compact discrete group of its isometries, acting without fixed points (see [94, p. 69]). He called them *Clifford-Klein forms*. In the case of the negative curvature examples of spatial Clifford-Klein forms were unknown a long time [42, p.269-270].

Answering in the affirmative the question on the existence of spatial Clifford-Klein forms on constant negative curvature, F. Löbell in 1931 [45] constructed the first example of a closed orientable three-dimensional hyperbolic manifold. This example was obtained by gluing of eight copies of the right-angled 14-hedron with hexagonal top and bottom and with twelve pentagons on the lateral surface similar to the dodecahedron.

In this chapter we will carry over Löbell's construction to algebraic language and will obtain an infinite series of closed three-dimensional manifolds, both orientable and non-orientable, that generalize Löbell's example. We will discuss their volumes and isometries.

4.1 Construction of manifolds

In this section we will consider a construction of hyperbolic 3-manifolds from bounded right-angled polyhedra according to [78].

Let R be a bounded right-angled polyhedron in the Lobachevsky space $\mathbb{H}^3$. We recall, that according to Theorem 1.2, each planar trivalent graph without triangles and quadrilaterals can be realized as the 1-skeleton of a bounded right-angled polyhedron in $\mathbb{H}^3$. Let G be the group generated by reflections in faces of R. We remark that the stabilizer in G of each vertex of R is isomorphic to the eight-element abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_2^3$. In order to show how to obtain the manifold from eight copies of R, we consider an epimorphism $\varphi : G \to \mathbb{Z}_2^3$. We note that the group $\mathbb{Z}_2^3$ can be regarded as a vector space over the field $GF(2)$. Arguments close to those used in [4] for the right-angled dodecahedron, enable us to establish the following assertion.

Lemma 4.1 ([78]) *Let G be the group generated by reflections in faces of a right-angled polyhedron in $\mathbb{H}^3$. The kernel $Ker\ \varphi$ of an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ does not contain elements of finite order if and only if the images of the reflections in each three faces of R that are incident to a common vertex are linearly independent in the group $\mathbb{Z}_2^3$ regarded as the vector space.*

Proof. Indeed, if $g \in G$ is a non-trivial element with non-empty fixed point set, then for some $h \in G$, $hgh^{-1} \in \text{Stab}_G(v)$ for some vertex $v \in R$. Since the image of G under φ is abelian, we have $\varphi(g) = \varphi(hgh^{-1}) \neq 1$; for if it were, there will be a dependence between $\varphi(g_i)$, $\varphi(g_j)$ and $\varphi(g_k)$, where g_i, g_j and g_k are generators of G which are reflections in faces of R incident to v. Then g does not belong to Ker φ. □

We remark, that if we regard $\mathbb{Z}_2^3$ as a vector space, there are four elements such that each three of them are linearly independent. Let us denote them $\alpha = (1,0,0)$, $\beta = (0,1,0)$, $\gamma = (0,0,1)$ and $\delta = \alpha + \beta + \gamma = (1,1,1)$.

Lemma 4.2 ([78]) *Let G be the group generated by reflections in faces of a right-angled polyhedron in $\mathbb{H}^3$. If an epimorphism $\varphi : G \to \mathbb{Z}_2^3$ takes the generators of G into four elements of $\mathbb{Z}_2^3$ each three of which are linearly independent in $\mathbb{Z}_2^3$ regarded as the vector space, then $Ker\ \varphi$ does not contain elements that change the orientation.*

Proof. Without loss of generality we can suppose that the four elements of $\mathbb{Z}_2^3$ are α, β, γ and δ as above. Since G is generated by reflections, orientation-reversing elements in it are words of odd length in the generators of G. Consequently, the image of such element under the epimorphism φ is a word of odd length in α, β, γ and δ, and using that $\delta = \alpha + \beta + \gamma$, we will get a word of odd length in α, β and γ. But α, β and γ are linearly independent. Thus the orientation-reversing elements are not in Ker φ. □

As we see, if an epimorphism φ satisfies to conditions of Lemma 4.1 and Lemma 4.2, then $M = \mathbb{H}^3/\mathrm{Ker}\ \varphi$ is a closed orientable hyperbolic 3-manifold. Let us correspond to a face of the polyhedron R the element of $\mathbb{Z}_2^3$ which is the image under φ of the reflection in the face. Then the "coloring" of faces determined by an epimorphism satisfying to Lemma 4.1 and Lemma 4.2, is a four-coloring in "colors" α, β, γ and δ, such that colors of adjacent faces are different. Thus it is a classical graph-theoretical four-coloring of a map. Otherwise, each four-coloring of faces of a bounded right-angled polyhedron R in $\mathbb{H}^3$ determines a closed orientable hyperbolic 3-manifold obtained from eight copies of R.

We remark that the Löbell manifold [45] can be obtained in this way, and we consider it as a manifold from the following family.

Let $ABCA'B'C'$ be a combinatorial triangle prism, and we will draw the edge DE with vertices D and E lying on edges BB' and CC', respectively (see Figure 4.1).

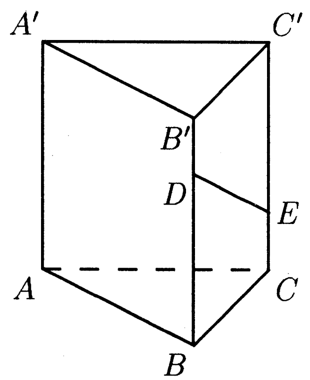

Figure 4.1. The piece $P(n)$.

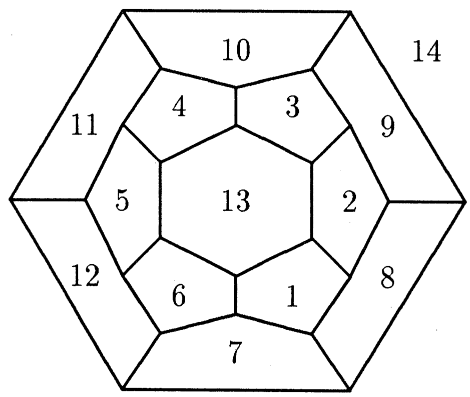

Figure 4.2. The polyhedron $R(6)$.

By Theorem 1.2, for any integer $n \geq 5$ there is a convex hexahedron $ABCA'B'C'DE$ in the Lobachevsky space $\mathbb{H}^3$, whose dihedral angles at edges AA', $B'D$ and EC are equal to π/n, $\pi/4$ and $\pi/4$, respectively, and all others angles are right. We denote this hexahedron by $P(n)$.

Let us consider the group $\Delta(n)$ generated by reflections in faces of $P(n)$. Elements of the group which leave the edge AA' fixed form the dihedral group $\mathbb{D}_n$ of order $2n$. Under the $\mathbb{D}_n$ action from $2n$ copies of $P(n)$ we will get a $(2n+2)$-hedron $R(n)$ with all angles right, whose top and bottom are regular n-gons and lateral surface consists of two cycles of n pentagons. In particular, $R(5)$ is the regular right-angled dodecahedron, and $R(6)$ is the right-angled 14-hedron, pictured in Figure 4.2, used by F. Löbell [45].

Let $G(n)$ be the group generated by reflections in faces of $R(n)$. Let us numerate faces of $R(n)$ according to Figure 4.2 such that the top is $(2n+1)$-face, the bottom is $(2n+2)$-face, pentagons of the first lateral level have numbers from 1 to n, and pentagons of the second lateral level have numbers from $(n+1)$ to $2n$. Then the group $G(n)$ has the following representation:

generators:

$$g_1, \ldots, g_{2n+2};$$

relations:

$$g_i^2 = 1, \qquad i = 1, \ldots, 2n+2;$$

$$g_{2n+1}g_i = g_i g_{2n+1}, \quad g_{2n+2}g_{n+i} = g_{n+i}g_{2n+2}, \quad i = 1, \ldots, n;$$

$$g_i g_{i+1} = g_{i+1} g_i, \qquad i = 1, \ldots, 2n-1;$$

$$g_i g_{n+i} = g_{n+i} g_i, \qquad i = 1, \ldots, n;$$

$$g_i g_{n+1+i} = g_{n+1+i} g_i, \qquad i = 1, \ldots, n-1;$$

$$g_1 g_n = g_n g_1, \quad g_{n+1} g_{2n} = g_{2n} g_{n+1}.$$

Definition 4.1 Let $\varphi_n : G(n) \to \mathbb{Z}_2^3$ be an epimorphism which kernel Ker φ_n is torsion-free. The hyperbolic 3-manifold $L(n) = \mathbb{H}^3/\text{Ker } \varphi_n$ is called a *manifold of Löbell type.* If, in addition, images of reflections in the top and the bottom coincides, then $L(n)$ is called a *standard manifold of Löbell type.*

We remark, that $L(n)$ depends of an epimorphism φ_n and not determined uniquely by n. In this terminology, the Löbell manifold constructed in [45] is a standard manifold for $n = 6$. The first example of a closed non-orientable hyperbolic 3-manifold constructed in 1980 by N. K. Al-Jubouri [4] is a manifold of Löbell type for $n = 5$.

As we remarked above, the existence of orientable manifolds of Löbell type is closely connected with the existence of corresponding four-colorings of faces of polyhedra.

Theorem 4.1 **([78])** *For any integer $n \geq 5$ there is an orientable manifold of Löbell type $L(n)$.*

Proof. For any $n \geq 5$ we specify an epimorphism $\varphi_n : G(n) \to \mathbb{Z}_2^3$ that we need in the definition of a manifold of Löbell type. Suppose that α, β, γ are linearly independent in $\mathbb{Z}_2^3$ and $\delta = \alpha + \beta + \gamma$.

If $n = 2k$, $k \geq 3$, is even, then for $i = 1, \ldots, k$ we define

$$\varphi_n(g_{2n+1}) = \varphi_n(g_{n+2i-1}) = \alpha, \quad \varphi_n(g_{2i-1}) = \beta,$$

$$\varphi_n(g_{2n+2}) = \varphi_n(g_{2i}) = \gamma, \quad \varphi_n(g_{n+2i}) = \delta.$$

The coloring of faces of $R(6)$ corresponding to φ_6 is shown in Figure 4.3.

If $n = 2k + 1$, $k \geq 2$, is odd, then for $i = 1, \ldots, k$ we define

$$\varphi_n(g_{2n+1}) = \varphi_n(g_{n+2i-1}) = \alpha, \quad \varphi_n(g_{2i-1}) = \varphi_n(g_{2n}) = \beta,$$

$$\varphi_n(g_{2n+2}) = \varphi_n(g_{2i}) = \gamma, \quad \varphi_n(g_{n+2i}) = \varphi_n(g_n) = \delta.$$

The coloring of faces of $R(5)$ corresponding to φ_5 is shown in Figure 4.4.

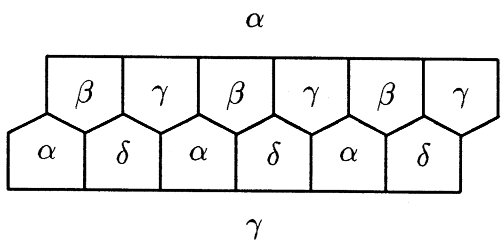

Figure 4.3.

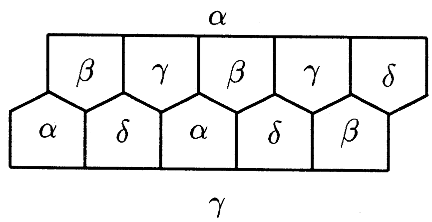

Figure 4.4.

It is obvious that conditions of Lemma 4.1 and Lemma 4.2 are satisfied. Therefore Ker φ_n does not contain elements of finite order and consists of orientation-preserving elements. Thus for $n \geq 5$ the manifold $\mathbb{H}^3/\text{Ker}\,\varphi_n$ is an orientable manifold of Löbell type. □

For obtain non-orientable manifolds we need "colorings" in more then four colors (which are also elements of $\mathbb{Z}_2^3$), satisfying to Lemma 4.1. Such colorings are possible also.

Theorem 4.2 **([78])** *For any integer $n \geq 5$ there is a non-orientable manifold of Löbell type $L(n)$.*

Proof. To obtain the necessary manifold we require that the kernel Ker ψ_n of the epimorphism $\psi_n : G(n) \to \mathbb{Z}_2^3$ should have no elements of finite order, but should contain orientation-reversing elements. Let us define:

$$\psi_n(g_{2n+1}) = \psi_n(g_{2n+2}) = \alpha,$$

$$\psi_n(g_j) = \varphi_n(g_j), \quad \psi_n(g_{n+j}) = \varphi_n(g_j) + \alpha, \quad j = 1, \ldots, n,$$

where φ_n is the epimorphism described in Theorem 4.1. From the explicit form of the epimorphism it is obvious that the condition of Lemma 4.1 is satisfied. Also, elements of the form $h_j = g_{2n+1}g_jg_{n+j}$, where $1 \leq j \leq n$, which are orientation-reversing as a product of the odd number of reflections, lie in the kernel Ker ψ_n. Thus for $n \geq 5$ the manifold $\mathbb{H}^3/\text{Ker } \psi_n$ is a non-orientable manifold of Löbell type. □

Remark. The theorem we have proved confirms the assertion of F. Löbell [45], that from eight copies of $R(6)$ one can obtain by a suitable gluing both an orientable and a non-orientable manifold.

The following figures present coloring corresponding to the Löbell manifold constructed in [45] (see Figure 4.5) and to the Al-Jubouri manifold constructed in [4] (see Figure 4.6).

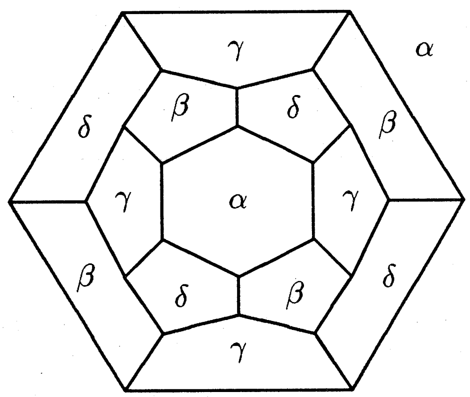

Figure 4.5.

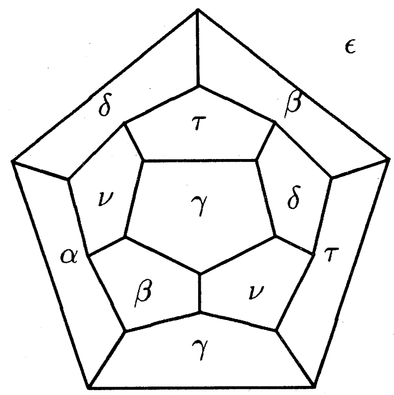

Figure 4.6.

where α, β, γ, δ are as above and $\epsilon = \alpha + \beta$, $\tau = \beta + \gamma$, $\nu = \alpha + \gamma$.

As we see, the Löbell manifold is a standard manifold in the sense of Definition 4.1. The following proposition devoted to the existence of standard manifolds of Löbell type.

Proposition 4.1 ([78]) *A standard orientable manifold of Löbell type* $L(n)$ *exists if and only if* $n = 3k$, $k \geq 2$. *It is unique for every* k.

Proof. To obtain the required manifold we need to impose on the epimorphism $\phi_n : G(n) \to \mathbb{Z}_2^3$ the following conditions : (1) $\phi_n(g_{2n+1}) = \phi_n(g_{2n+2})$; (2) the images of the reflections in any three faces of $R(n)$ that have a common vertex are linearly independent; (3) Ker ϕ_n does not contain orientation-reversing elements.

Let α, β and γ be linearly independent in $\mathbb{Z}_2^3$. Without loss of generality we may assume that

$$\phi_n(g_{2n+1}) = \phi(g_{2n+2}) = \alpha, \quad \phi_n(g_1) = \beta, \quad \phi_n(g_2) = \gamma.$$

Then from (2) and (3) it follows that $\phi_n(g_{n+2}) = \delta = \alpha + \beta + \gamma$. Similarly, if $\phi_n(g_2) = \gamma$, $\phi(g_{n+2}) = \delta$, then $\phi_n(g_{n+3}) = \beta$. By induction we obtain for $i = 1, \dots, n$:

$$\phi_n(g_i) = \phi_n(g_{n+i+2}) = \beta, \qquad i = 1 \pmod 3,$$

$$\phi_n(g_i) = \phi_n(g_{n+i-1}) = \gamma, \qquad i = 2 \pmod 3,$$

$$\phi_n(g_i) = \phi_n(g_{n+i-1}) = \delta, \qquad i = 0 \pmod 3.$$

Since to satisfy (2) reflections in adjacent faces must be mapped into different elements, n must be a multiple of 3. If $n = 3k$, $k \geq 2$, then the epimorphism

$$\phi_n(g_{2n+1}) = \phi_n(g_{2n+2}) = \alpha, \qquad \phi_n(g_{3i-2}) = \phi_n(g_{n+3i}) = \beta,$$

$$\phi_n(g_{3i-1}) = \phi_n(g_{n+3i-2}) = \gamma, \qquad \phi_n(g_{3i}) = \phi_n(g_{n+3i-1}) = \delta,$$

where $i = 1, \dots, k$, specifies a standard orientable manifold of Löbell type. The coloring of faces of $R(6)$ corresponding to ϕ_6 is shown in Figure 4.5. Since for each step of the construction of ϕ_n the image of the reflection in the next face is determined uniquely, for every $k \geq 2$ the standard orientable manifold of Löbell type $L(3k)$ is unique up to a change of basis in $\mathbb{Z}_2^3$ regarded as the vector space over $GF(2)$, that is up to an automorphism of the group. □

4.2 Naturally maximal groups

In this section we discuss naturally maximal groups in the sense of [49] and isometries of some manifolds of Löbell type.

We start from studying pairwise dispositions of fixed axes of elements of finite order in a discrete group.

Lemma 4.3 **([49])** *Let g and h be elliptic elements of orders n and m, $n \leq m$, respectively, which generate a discrete group. Suppose that the axes of the transformations g and h intersect in a point belonging to $\mathbb{H}^3$. Then one of the following cases takes place:*

1. $n = 2, m \geq 2$; **2.** $(n,m) = (3,3)$; **3.** $(n,m) = (3,4)$;
4. $(n,m) = (3,5)$; **5.** $(n,m) = (4,4)$; **6.** $(n,m) = (5,5)$.

Moreover, the following magnitude of the angle between the axes takes its values in the set $I(n,m)$, where

$$
\begin{array}{lrcl}
(i) & I(2,2) & = & \{\frac{k\pi}{l}, 1 \leq k < l,\ k \textit{ and } l \textit{ are coprime integer}\}, \\
 & I(2,3) & = & \{\gamma, \frac{\pi}{2}-\xi, \xi, \frac{\pi}{2}-\gamma, \frac{\pi}{2}, \frac{\pi}{2}+\gamma, \pi-\xi, \frac{\pi}{2}+\xi, \pi-\gamma\}, \\
 & I(2,4) & = & \{\frac{k\pi}{4}, k=1,2,3\}, \\
 & I(2,5) & = & \{\beta, \frac{\pi}{2}-\beta, \frac{\pi}{2}, \frac{\pi}{2}+\beta, \pi-\beta\} \\
 & I(2,m) & = & \{\frac{\pi}{2}\}, m \geq 6; \\
(ii) & I(3,3) & = & \{2\gamma, \pi-2\xi, 2\xi, \pi-2\gamma\}; \\
(iii) & I(3,4) & = & \{\xi, \pi-\xi\}; \\
(iv) & I(3,5) & = & \{\alpha, \alpha+2\gamma, \alpha+2\beta, \pi-\alpha\}; \\
(v) & I(4,4) & = & \{\pi/2\}; \\
(vi) & I(5,5) & = & \{2\beta, \pi-2\beta\}.
\end{array}
$$

Here elements of the set $I(n,m)$ are written down in increasing order, and the numbers α, β, γ and ξ are defined by conditions $\tan\alpha = 3-\sqrt{5}$, $\tan\beta = (\sqrt{5}-1)/2$, $\tan\gamma = (3-\sqrt{5})/2$, *and* $\tan\xi = \sqrt{2}$.

Proof. Let O be the point which is the intersection of the axes of the transformations g and h. Denote by P the plane which contains these axes, and let r be the reflection in the plane P. A transformation $w = gr$ is an orientation-reversing isometry which leaves the axe of g invariant. Hence w is a reflection and $w^2 = grgr = 1$, whence $r^{-1}gr = g^{-1}$. Analogously for the transformation h we have $r^{-1}hr = h^{-1}$. Then the group

$$G = \langle g,h,r\rangle = \langle g,h\rangle \leftthreetimes \langle r\rangle$$

is a semi-direct product of the group $\langle g,h\rangle$ which is discrete by the hypothesis and of the cyclic group of order two. Therefore G is also discrete. Moreover the group G has a fixed point O. According to the classification of finite subgroups of $SO(3)$ and its two-sheeted covering (see [94, p. 88]), the group G is a subgroup of one of the following group: cyclic, binary dihedral, binary tetrahedral, binary octahedral or binary icosahedral. A pairwise disposition of axes on reflection planes in binary polyhedral groups is well-known ([94, section 2.6]) and pictured in Figures 4.7–4.10.

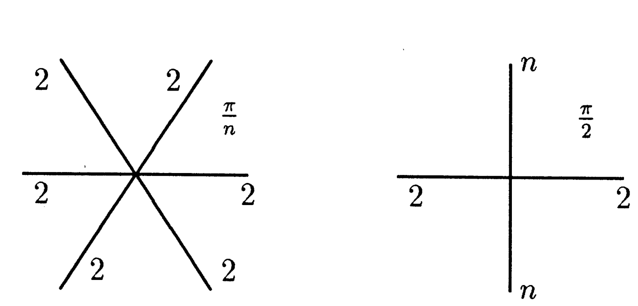

Figure 4.7. Dihedral case.

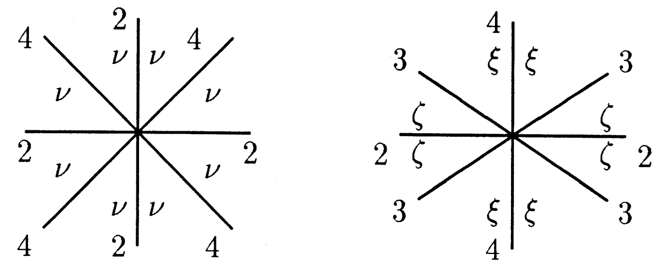

Figure 4.8. Octahedral case.

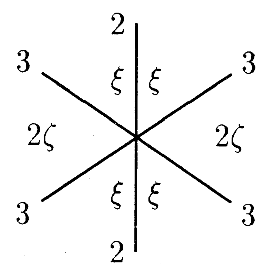

Figure 4.9.
Tetrahedral case.

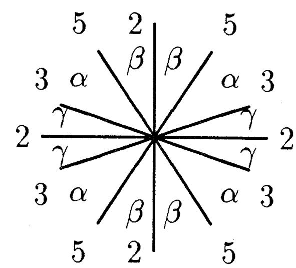

Figure 4.10.
Icosahedral case.

Angles between axes founded using formulas from spherical trigonometry (see, for example [3, p. 65]). Therefore in Figures 4.7–4.10 we have $\nu = \frac{\pi}{4}$; $\tan\xi = \sqrt{2}$, $\zeta + \xi = \frac{\pi}{2}$; $\tan\alpha = 3 - \sqrt{5}$, $\tan\beta = \frac{\sqrt{5}-1}{2}$, $\tan\gamma = \frac{3-\sqrt{5}}{2}$ and $\alpha + \beta + \gamma = \frac{\pi}{2}$. Looking over Figures 4.7–4.10, we will get cases (i) – (vi) of the statement of the lemma. □

Let us consider a compact convex Coxeter polyhedron P in the Lobachevsky space $\mathbb{H}^3$, i.e. its all dihedral angles are submultiplies of π. We denote by $\Delta = \Delta(P)$ the group generated by reflections in faces of P, and by $\Sigma = \Sigma(P)$ the symmetry group of the polyhedron P, consisting of all isometries of $\mathbb{H}^3$

which leave P invariant. It is well known [87, p. 199] that the group Δ is discrete and has P as its fundamental set in $\mathbb{H}^3$.

According to [49] we call the group $\Delta^* = \langle \Delta, \Sigma \rangle$, generated by Δ and Σ, the *natural extension* of the group Δ. The group Δ^* can be decomposed into the semidirect product $\Delta^* = \Delta \leftthreetimes \Sigma$ (see [87, p. 200]), where elements of Σ act on elements of Δ by conjugations. Therefore Δ^* is discrete, as a finite extension of the discrete group Δ.

Definition 4.2 We shall say that the group Δ is *naturally maximal* if $\Delta^* = \Delta \leftthreetimes \Sigma$ is a maximal discrete group, i.e. it is not contained as a proper subgroup in any discrete group of isometries of $\mathbb{H}^3$.

Let $\Delta(n)$ as above be the group generated by reflections in the faces of the hexahedron $P(n)$ (see Figure 4.1). It is obvious from Figure 4.1 that the symmetry group $\Sigma(n)$ of the hexahedron $P(n)$ is generated by the rotation τ of order two around the line which passes throught midpoints of edges AA' and DE. Then $\Delta^*(n) = \Delta(n) \leftthreetimes \langle \tau \rangle$.

Lemma 4.4 ([49]) *For $n \geq 6$ the group $\Delta(n)$ is naturally maximal.*

Proof. We will give a sketch of the proof of this statement which is essentially based on Lemma 4.3 and results of L. Greenberg [22] on the maximallity of triangle groups. Let g be an isometry of $\mathbb{H}^3$ such that a group $\langle \Delta^*(n), g \rangle$ is discrete, and denote by l the axe of order $n \geq 6$ which passes throught the edge AA' of $P(n)$ (see Figure 4.1). Without loss of generality we can assume that the intersection $d(l) \cap P(n)$ contains at least two points. Firstly we show that $g(l) = l$. It holds because according to the case $I(2, n)$ of Lemma 4.3, $g(l)$ is orthogonal to a pair of faces of $P(n)$. Therefore it is orthogonal to a triangle face, and we turn to an extension of the triangle group with the signature $(2, 4, n)$ studied in [22]. Secondly, by direct calculations of lengths and angles of $P(n)$, and using Lemma 4.3, we see that only case $g \in \Delta^*(n)$ can be realized by virtue of the assumption that the group $\langle \Delta^*(n), g \rangle$ is discrete. □

As well as above, we consider the $(2n+2)$-hedron $R(n)$ obtained by the action of the dihedral group $\mathbb{D}_n < \Delta(n)$ from $2n$ copies of $P(n)$. Let $G(n)$ be the group generated by reflections in faces of $R(n)$ and $S(n)$ be the group of symmetries of $R(n)$. Then

$$S(n) = \mathbb{D}_n \leftthreetimes \mathbb{Z}_2,$$

where the dihedral group $\mathbb{D}_n$ of order $2n$ is generated by reflections ρ and σ in faces $ABDB'A'$ and $ACEC'A'$ of $P(n)$, and $\mathbb{Z}_2 = \langle \tau \rangle$, where τ is the rotation of order two as above. Therefore

$$\Delta^*(n) = \langle \Delta(n), \tau \rangle = \langle G(n), \rho, \sigma, \tau \rangle = \langle G(n), S(n) \rangle = G(n) \leftthreetimes S(n)$$

is the semidirect product of the group generated by reflections and of the group of symmetries of its fundamental polyhedron. Thus we get

Corollary 4.1 *For $n \geq 6$ the group $G(n)$ is naturally maximal.*

The following statement gives the motivation why we consider a subclass of standard manifolds of Löbell type.

Lemma 4.5 *Let $L(n) = \mathbb{H}^3/\Gamma$, where $\Gamma = Ker\ \phi_n$, be a standard orientable manifold of Löbell type. Then Γ is a normal subgroup in $\langle G(n), S(n) \rangle$.*

Proof. Let ρ, σ and τ be generators of the group $S(n)$ as above. We recall, that according to Proposition 4.1, a standard orientable manifold $L(n)$ exist only for $n = 3k$, $k \geq 2$, and is determined by the epimorphism ϕ_n from the proof of Proposition 4.1. Let us firstly consider the case $n = 6$. The coloring of faces of $R(6)$ corresponding to the ϕ_6 is shown in Figure 4.11.

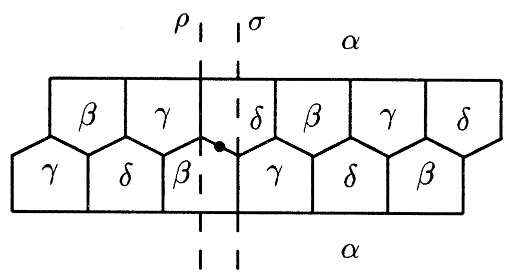

Figure 4.11.

Planes of reflections ρ and σ are correspond to dotted lines and the axe of the rotation τ corresponds to the dot. As we see, the group $S(6) = \langle \rho, \sigma, \tau \rangle$ can be regarded as a group of substitutions on the set of colors $\{\alpha, \beta, \gamma, \delta\}$ according to the following rules:

$$\begin{array}{llll} \rho: & \alpha \to \alpha & \sigma: \ \alpha \to \alpha & \tau: \ \alpha \to \alpha \\ & \beta \to \beta & \quad\ \ \beta \to \gamma & \quad\ \ \beta \to \delta \\ & \gamma \to \delta & \quad\ \ \gamma \to \beta & \quad\ \ \gamma \to \gamma \\ & \delta \to \gamma & \quad\ \ \delta \to \delta & \quad\ \ \delta \to \beta \end{array}$$

and this induces the action of $S(6)$ on

$$\mathbb{Z}_2^3 = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \alpha\gamma = \gamma\alpha,\ \beta\gamma = \gamma\beta \rangle$$

as a group of automorphisms and we consider a group $\Omega(6) = \langle \mathbb{Z}_2^3, S(6) \rangle$ which is the extension of the group by its automorphisms and can be decomposed in the semidirect product [8, p. 133] : $\Omega(6) = \mathbb{Z}_2^3 \leftthreetimes S(6)$.

Let us consider an epimorphism

$$\phi_6^* : \Delta^*(6) = G(6) \leftthreetimes S(6) \to \Omega(6) = \mathbb{Z}_2^3 \leftthreetimes S(6)$$

defined by the rule

$$\phi_6^*(x) = \phi_6(g) \cdot s,$$

where $x \in \Delta^*(6) = G(6) \leftthreetimes S(6)$ and so has a unique representation $x = g \cdot s$, where $g \in G(6)$ and $s \in S(6)$. One can check directly, that the epimorphism ϕ_6^* is defined correctly and that Ker $\phi_6^* =$ Ker ϕ_6. Hence, $\Gamma =$ Ker $\phi_6 =$ Ker ϕ_6^* is a normal subgroup of $G(6) \leftthreetimes S(6)$.

For arbitrary $n = 3k$, $k \geq 2$, using periodicity of the coloring corresponding to the epimorphism ϕ_n, the proof follows by the same arguments. □

Theorem 4.3 ([51]) *The isometry group of a standard orientable manifold $L(n)$ of Löbell type is isomorphic to*

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \leftthreetimes (\mathbb{D}_n \leftthreetimes \mathbb{Z}_2).$$

Proof. We recall [77], that the group of isometries of a hyperbolic 3-manifold is isomorphic to the group of outer automorphisms of its fundamental group. So, for $L(n) = \mathbb{H}^3/\Gamma_n$, where $\Gamma_n =$ Ker ϕ_n, we have:

$$Isom(L(n)) = N(\Gamma_n, Isom(\mathbb{H}^3)) \, / \, \Gamma_n,$$

where $N(\Gamma_n, Isom(\mathbb{H}^3))$ is the normalizer of Γ_n in the group of all isometries of the Lobachevsky space $\mathbb{H}^3$.

Because the manifold $L(n)$ is orientable standard, from Lemma 4.5 we have:

$$G(n) \leftthreetimes S(n) < N(\Gamma_n, Isom(\mathbb{H}^3)).$$

But according to Lemma 4.4, the group $G(n) \leftthreetimes S(n) = \Delta^*(n)$ is maximal discrete group. Therefore,

$$G(n) \leftthreetimes S(n) = N(\Gamma_n, Isom(\mathbb{H}^3)),$$

and

$$Isom(L(n)) = G(n) \leftthreetimes S(n) \, / \, \Gamma_n.$$

Similar to the proof of Lemma 4.5 we can consider the epimorphism

$$\phi_n^* : G(n) \leftthreetimes S(n) \to \mathbb{Z}_2^3 \leftthreetimes S(n)$$

defined by $\phi_n^*(x) = \phi_n(g) \cdot s$, for $x = g \cdot s$, $g \in G(n)$, $s \in S(n)$. By the same arguments we get $\Gamma = \mathrm{Ker}\ \phi_6 = \mathrm{Ker}\ \phi_6^*$. Therefore, using that $S(n) = \mathbb{D}_n \leftthreetimes \mathbb{Z}_2$, we have

$$Isom(L(n)) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \leftthreetimes (\mathbb{D}_n \leftthreetimes \mathbb{Z}_2)$$

and theorem is proved. □

Because the Löbell manifold constructed in [45] is standard orientable for $n = 6$, we have the following statement.

Corollary 4.2 **([49])** *The isometry group of the Löbell manifold constructed in [45] is isomorphic to*

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \leftthreetimes (\mathbb{D}_6 \leftthreetimes \mathbb{Z}_2).$$

The following statement is connected with arithmeticity of the group $G(n)$ (see [31], [61], [87] for nice discussions of arithmetic groups acting in the Lobachevsky space).

Lemma 4.6 *If $n \neq 5, 6, 7, 8, 10, 12, 18$, then the group $G(n)$ is non-arithmetic.*

Proof. Because groups $G(n)$ and $\Delta(n)$ are commensurable, they are arithmetic or non-arithmetic simultaneously. The Coxeter scheme of the group $\Delta(n)$ is shown in Figure 4.12.

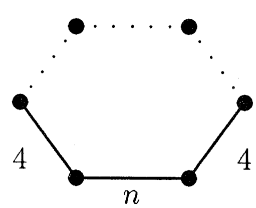

Figure 4.12.

According to [85], any Lanner subscheme of an arithmetic Coxeter scheme must be arithmetic too. The scheme of the group $\Delta(n)$ contains the scheme of a triangle group with signature $(2, 4, n)$. We recall, that all arithmetic triangle groups were listed in [74] and the group with signature $(2, 4, n)$ is arithmetic if and only if $n = 5, 6, 7, 8, 10, 12, 18$. Therefore, for other n the group $\Delta(n)$ and so the group $G(n)$ are non-arithmetic. □

We remark, that the Löbell manifold constructed in [45] is arithmetic. Indeed, its fundamental group is commensurable with the group $\Delta(6)$. Using the criterion of Vinberg for arithmeticity of groups generated by reflections [87, p.226], one can check by direct calculations that the group $\Delta(6)$ is arithmetic with the invariant trace field Q.

Theorem 4.4 *If $L(n) = \mathbb{H}^3/\Gamma$, $\Gamma = Ker\, \varphi$, is a non-arithmetic manifold of Löbell type, then its isometry group is isomorphic to*

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \leftthreetimes K,$$

where $K < S(n) = \mathbb{D}_n \leftthreetimes \mathbb{Z}_2$ consists of symmetries of the polyhedron $R(n)$ which act as substitutions on the coloring of faces of $R(n)$ corresponding to the epimorphism φ.

Proof. By Borel–Margulis theorem [14], all groups commensurable with a non-arithmetic group Γ are contained in a unique maximal one, which by Corollary 4.1 coincides with the group $G(n) \leftthreetimes S(n)$. Therefore

$$Isom(L(n)) = N(\Gamma, G(n) \leftthreetimes S(n)) \,/\, \Gamma.$$

If $s \in S(n)$ is such symmetry of $R(n)$ that does not act correctly as a substitution on the set of colors and hence does not extended up to an automorphism of the group $\mathbb{Z}_2^3$, then coset $G(n)/\Gamma$ are not invariant under the conjugation $s^{-1}G(n)s$ and $s^{-1}\Gamma s \neq \Gamma$. If K is the subgroup of all symmetries of $R(n)$ which act correctly on the coloring, then as well as above in the proof of Lemma 4.5, we can consider an epimorphism

$$\varphi^* : G(n) \leftthreetimes K \to \mathbb{Z}_2^3 \leftthreetimes K,$$

such that $\varphi^*(x) = \varphi(g) \cdot s$ for $x = g \cdot s$, $g \in G(n)$, $s \in K$. And by the same arguments we will get $\Gamma = \text{Ker}\, \varphi = \text{Ker}\, \varphi^*$. Therefore

$$Isom(L(n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \leftthreetimes K,$$

and theorem is proved. □

4.3 Volumes of Löbell manifolds

Lemma 4.7 (**[79]**) *Let G the non-arithmetic naturally maximal group generated by reflections in faces of a right-angled polyhedron R and S be its group of symmetries. Suppose Γ_1 and Γ_2 be the kernels of epimorphisms φ_1 and φ_2 satisfying the hypotheses of Lemma 4.1 and Lemma 4.2. Groups Γ_1 and Γ_2 are isomorphic if and only if there exists an element $s \in S$ such that $\Gamma_1 = s^{-1}\Gamma_2 s$.*

Proof. By virtue of Lemma 4.1 and Lemma 4.2, Γ_1 and Γ_2 are fundamental groups of closed orientable hyperbolic manifolds, so by the Mostow rigidity theorem they are isomorphic if and only if there exists an element $h \in Isom(\mathbb{H}^3)$ such that $\Gamma_1 = h^{-1}\Gamma_2 h$. We consider the commensurate of the group G,

$$\text{Comm } (G) = \{\gamma \in Isom(\mathbb{H}^3) \,:\, \gamma^{-1}G\gamma \text{ is commensurable with } G\}.$$

Since Γ_1 and Γ_2 are normal subgroups in G of index 8, and because $\Gamma_1 = h^{-1}\Gamma_2 h$ is a normal subgroup in $h^{-1}Gh$, then $h \in \text{Comm } (G)$. By hypothesis G is non-arithmetic, so by Borel–Margulis theorem [14] Comm (G) is discrete and all groups commensurable with G are contained in a unique maximal one. But $G^* = \langle G, S\rangle$ is a maximal discrete group by Corollary 4.1 and $G^* \subset \text{Comm } (G)$, so $G^* = \text{Comm } (G)$. Consequently $h = gs$, where $g \in G$, $s \in S$. Because Γ_1 and Γ_2 are normal subgroups of G, we get $\Gamma_1 = s^{-1}\Gamma_2 s$. □

Theorem 4.5 **([79])** *For any integer N there exist at least N pairwise non-homeomorphic manifolds of Löbell type with the same volume.*

Proof. We will show that for any $n = 6k$, $k \geq 4$, there exist at least $\frac{k+1}{2}$ pairwise non-homeomorphic orientable manifolds of Löbell type $L(n)$. This estimate is very quick, but enough for proving the theorem. Let us consider epimorphisms $f_0, f_1 : G(6) \to \mathbb{Z}_2^3$, defined according to colorings in Figures 4.13 and 4.14, respectively.

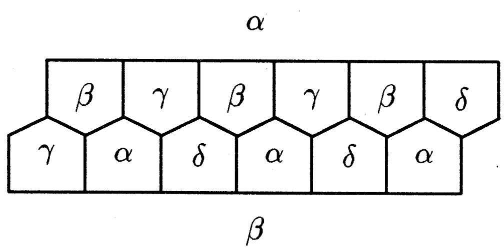

Figure 4.13. f_0-coloring.

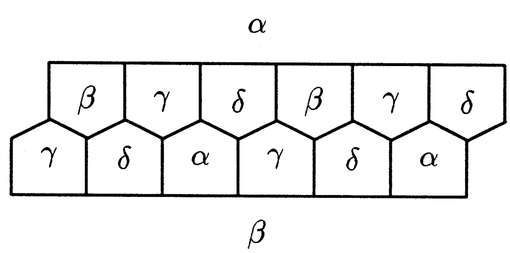

Figure 4.14. f_1-coloring.

It is obvious, that epimorphisms f_0 and f_1 satisfy conditions of Lemma 4.1 and Lemma 4.2. For any $k \geq 4$ we consider a set I_k that is a set of $(k+1)$ collections of k-tuples $(i_1, \ldots, i_k)$ of numbers 0 and 1 arranged in nondecreasing order and define the epimorphism

$$\varphi_{i_1,\ldots,i_k} : G(6k) \to \mathbb{Z}_2^3$$

by the following rules. If $j = 6(p-1) + q$, $1 \leq q \leq 6$, $1 \leq p \leq k$, then

$$\varphi_{i_1,\ldots,i_k}(g_j) \;=\; f_{i_p}(g_q), \quad \varphi_{i_1,\ldots,i_k}(g_{n+j}) \;=\; f_{i_p}(g_{6+q}),$$

and

$$\varphi_{i_1,\dots,i_k}(g_{2n+1}) = \alpha, \quad \varphi_{i_1,\dots,i_k}(g_{2n+2}) = \beta.$$

By other words, the epimorphism $\varphi_{i_1,\dots,i_k}$ is such that the coloring of lateral faces of $R(n)$ is obtained from k fragments pictured in Figures 4.13–4.14, which we take in correspondence with $i_p = 0$ or $i_p = 1$ for $p = 1, \dots, k$. It is obvious, that for any k-tuple $(i_1, \dots, i_k)$ the epimorphism $\varphi_{i_1,\dots,i_k}$ satisfies to conditions of Lemma 4.1 and Lemma 4.2, so $L_{i_1,\dots,i_k} = \mathbb{H}^3/\varphi_{i_1,\dots,i_k}$ is an orientable manifold of Löbell type which can be obtained from eight copies of the polyhedron $R(n)$. Therefore all manifolds $L_{i_1,\dots,i_k}$ have the same volume.

As we consider $n = 6k$, $k \geq 4$, according to Corollary 4.1, the group $G(n)$ is naturally maximal and by Lemma 4.6 $G(n)$ is non-arithmetic. By virtue of Lemma 4.7 two manifolds $L_{i_1,\dots,i_k}$ and $L_{j_1,\dots,j_k}$ are homeomorphic if and only if there exists a symmetry $s \in S(n)$ of the polyhedron $R(n)$ such that $s^{-1}\text{Ker}\ \varphi_{i_1,\dots,i_k} s = \text{Ker}\ \varphi_{j_1,\dots,j_k}$. We note that if the kernels Ker $\varphi_{i_1,\dots,i_k}$ and Ker $\varphi_{j_1,\dots,j_k}$ are conjugated by a symmetry s, then elements of the group $G(n)$ lying in one coset with respect to Ker $\varphi_{i_1,\dots,i_k}$, lie in one coset with respect to Ker $\varphi_{j_1,\dots,j_k}$ also. So faces of $R(n)$ colored in the same color under the epimorphism $\varphi_{i_1,\dots,i_k}$ are also colored in the same color under the epimorphism $\varphi_{j_1,\dots,j_k}$. Therefore, if two colorings of $R(n)$ are not equivalent by the group $S(n)$ (and by renaming of colors), then two corresponding manifolds of Löbell type are non-homeomorphic. We recall, that the group $S(n)$ can be decomposed in the semi-direct product $S(n) = \mathbb{D}_n \leftthreetimes \mathbb{Z}_2$, where elements of $\mathbb{D}_n$ leaves top and bottom of $R(n)$ invariant, and the involution τ change top and bottom. Let $(i_1, \dots, i_k)$ be an element of the set I_k and consider the upper level of the lateral faces. The color δ appeared $k + i_1 + \dots + i_k$ times and the color γ appeared $2k$ times. Therefore any two colorings corresponding to different elements of I_k cannot be equivalent under the dihedral group $\mathbb{D}_n$ action. Hence there exist at least $\frac{k+1}{2}$ pairwise non-homeomorphic manifolds of Löbell type with the same volume. □

Corollary 4.3 ([79]) *For any integer N there exists a right-angled polyhedron in $\mathbb{H}^3$ which is fundamental for at least N non-homeomorphic closed orientable manifolds.*

Proof. Let $\Gamma_1, \dots, \Gamma_{\frac{k+1}{2}}$ be subgroups of $G(n)$, $n = 6k$, of index 8, which are fundamental groups of different manifolds of Löbell type constructed in the theorem. It is clear from the construction (see Figure 4.13–4.14), that for any Γ_i, $i = 1, \dots, \frac{k+1}{2}$, elements of the set

$$\Sigma = \{1, g_{2n+1}, g_1, g_2, g_{2n+1}g_1, g_{2n+1}g_2, g_1g_2, g_{2n+1}g_1g_2\},$$

where g_{2n+1}, g_1, g_2 are generators of $G(n)$ as well as in Section 4.1, belong to different cosets $G(n)/\Gamma_i$. Therefore for any Γ_i we can take the polyhedron

$$Q(n) = \bigcup_{h \in \Sigma} h(R(n))$$

as its fundamental polyhedron, and $Q(n)$ is right-angled by construction. □

We recall, that according to the Thurston–Jørgensen theorem on volumes of hyperbolic manifolds, the number of 3-dimensional hyperbolic manifolds with the same volume is finite. As we can see from Theorem 4.5, these finite numbers are not bounded. Other series of examples which demonstrate this property, based on different interesting considerations were given by B.Apanasov and I.Gutsul [7], B.Zimmermann [100] in the case of compact manifolds, and by N.Wielenberg [92] in the case of non-compact manifolds.

The following theorem gives the exact formula for the volume of manifolds of Löbell type.

Theorem 4.6 *Let $L(n)$, $n \geq 5$, be a manifold of Löbell type. Then*

$$vol(L(n)) = 4n\left(2\Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{n}\right) + \Lambda\left(\theta - \frac{\pi}{n}\right) - \Lambda\left(2\theta - \frac{\pi}{2}\right)\right), \quad (4.1)$$

where $\theta = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right)$.

Proof. Let $L(n)$, $n \geq 5$, be a manifold of Löbell type, then according to the construction (see Section 4.1),

$$vol(L(n)) = 8 \cdot vol(R(n)) = 8 \cdot 2n \cdot vol(P(n)), \quad (4.2)$$

where $P(n)$ is the hexahedron $ABCA'B'C'DE$ (see Figure 4.1). We remark, that $P(n)$ can be regarded as an orthoscheme of degree 2 in the terminology of Section 2.6. Indeed, if we consider the model of the Lobachevsky space $\mathbb{H}^3$ in the ball, then we can regard the triangle ABC as a result of cutting off an ideal vertex, which is the intersection of lines $A'A$, DB and EC and lies outside the ball. Analogously, the triangle $A'B'C'$ is the result of cutting off an ideal vertex, which is the intersection of lines AA', DB' and EC' outside the ball. Because for our "ideal" tetrahedron with two vertices outside and two finite vertices D and E only three dihedral angles at lines AA', BD and $C'E$ are non-right, it is an orthoscheme of degree 2 in the terminology of Section 2.6, and its volume can be calculated due to Theorem 2.8.

For the hexahedron $P(n)$ we have $\alpha = \gamma = \pi/4$ and $\beta = \pi/n$, whence according to formula (2.27) we have:

$$vol(P(n)) = \frac{1}{4}\left(2\Lambda\left(\frac{\pi}{4}+\delta\right) - 2\Lambda\left(\frac{\pi}{4}-\delta\right) + \Lambda\left(\frac{\pi}{2}+\frac{\pi}{n}-\delta\right)\right.$$
$$\left.+\Lambda\left(\frac{\pi}{2}-\frac{\pi}{n}-\delta\right) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\right)$$
$$= \frac{1}{4}\left(2\Lambda\left(\delta+\frac{\pi}{4}\right) + 2\Lambda\left(\delta-\frac{\pi}{4}\right) + \Lambda\left(\frac{\pi}{2}+\frac{\pi}{n}-\delta\right)\right.$$
$$\left.+\Lambda\left(\frac{\pi}{2}-\frac{\pi}{n}-\delta\right) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\right), \tag{4.3}$$

were we used that the function $\Lambda(x)$ is odd. We recall, that according to Proposition 2.2, item (7),

$$\Lambda(2x) = 2\Lambda(x) + 2\Lambda\left(x+\frac{\pi}{2}\right).$$

Applying this formula for $x = \delta - \frac{\pi}{4}$, from (4.3) we will get, that:

$$vol(P(n)) = \frac{1}{4}\left(2\Lambda\left(2\delta-\frac{\pi}{2}\right) + \Lambda\left(\frac{\pi}{2}+\frac{\pi}{n}-\delta\right)\right.$$
$$\left.+\Lambda\left(\frac{\pi}{2}-\frac{\pi}{n}-\delta\right) + 2\Lambda\left(\frac{\pi}{2}-\delta\right)\right). \tag{4.4}$$

Let us denote $\theta = \frac{\pi}{2} - \delta$, then from (4.4) we have:

$$vol(P(n)) = \frac{1}{4}\left(2\Lambda(\theta) + \Lambda\left(\theta+\frac{\pi}{n}\right) + \Lambda\left(\theta-\frac{\pi}{n}\right) - \Lambda\left(2\theta-\frac{\pi}{2}\right)\right). \tag{4.5}$$

Now we will find θ. Because $\alpha = \gamma = \pi/4$ and $\beta = \pi/n$, from (2.28) we have

$$\delta = \arctan\sqrt{4\cos^2\left(\frac{\pi}{n}\right) - 1},$$

and using trigonometry formulae, we get

$$\delta = \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

Hence,

$$\theta = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right). \tag{4.6}$$

Therefore, from (4.2), (4.5) and (4.6), we get (4.1). □

Corollary 4.4 *The hyperbolic volume of the Löbell manifold $\mathcal{L}$ from [45] is given by*

$$vol(\mathcal{L}) = 24\left(2\Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{6}\right) + \Lambda\left(\theta - \frac{\pi}{6}\right) - \Lambda\left(2\theta - \frac{\pi}{2}\right)\right),$$

where $\theta = \frac{\pi}{2} - \arccos\frac{1}{\sqrt{3}}$. *And we have* $vol(\mathcal{L}) = 48.184368\ldots$.

From Theorem 4.5 we can find the asymptotic for the volume function of manifolds of Löbell type $L(n)$ for $n \to \infty$. As we see from (4.6), if $n \to \infty$, then $\theta \to \pi/6$ and because $\Lambda(x)$ is a continuous function, we get

Corollary 4.5 *If $L(n)$ is a manifold of Löbell type, then we have the following asymptotic:*

$$vol(L(n)) \sim 10\, V_3^{max}\, n, \quad n \to \infty. \tag{4.7}$$

We recall that the value $V_3^{max} = 2\Lambda(\pi/6) = 1.014\ldots$ is the maximal volume of a hyperbolic 3-simplex, that is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.

In the context of discussion of orbifolds of small volume, it is interesting to consider a value

$$\frac{vol(M)}{\mid Isom(M)\mid},$$

where $vol(M)$ is the volume of a hyperbolic manifold M and $\mid Isom(M)\mid$ is order of its isometry group.

According to Theorem 4.3 and Theorem 4.5, we get

Corollary 4.6 *If $L(n)$ is a standard orientable manifold of Löbell type, then we have the following asymptotic:*

$$\frac{vol(L(n))}{\mid Isom(L(n))\mid} \sim \frac{5}{16}\, V_3^{max}, \quad n \to \infty, \tag{4.8}$$

where $V_3^{max} = 2\Lambda(\pi/6) = 1.014\ldots$ is the maximal volume of a hyperbolic 3-simplex.

Chapter 5

Fibonacci manifolds

This chapter is devoted to three-dimensional compact orientable hyperbolic manifolds which are correspond to the Fibonacci groups. The Fibonacci groups

$$F(2,m) = \langle x_1, x_2, \dots, x_m \mid x_i x_{i+1} = x_{i+2}, \qquad i \bmod m \rangle$$

were introduced by J. Conway [17]. The first natural question connected with these groups is the question whether they finite or not [17]. It is known from [18], [27], [28], [75], that the group $F(2,m)$ is finite if and only if $m = 1,2,3,4,5,7$. Some algebraic generalizations of the groups $F(2,m)$ were considered in [37], [46] and [76]. A new stage in the investigation of the Fibonacci groups began with [28], where it was shown that the group $F(2,2n)$, $n \geq 4$, is isomorphic to a discrete co-compact subgroup of the full group of orientation-preserving isometries of the Lobachevsky space $\mathbb{H}^3$.

5.1 Construction of manifolds

We recall, that one of regular bounded polyhedra in the Lobachevsky space $\mathbb{H}^3$ is the icosahedron with dihedral angles equal $2\pi/3$ (see Table 2.1). In [9] L. A. Best constructed 3 closed orientable hyperbolic 3-manifolds whose fundamental set is the $2\pi/3$ -icosahedron. In [68] J. Richardson and J. Rubinstein corrected the list of examples from [9] and proved that there are exactly six pairwise non-homeomorphic orientable 3-manifolds which can be obtained from the $2\pi/3$ -icosahedron. Some their properties were studied in [64]. One of these manifolds was generalized by H. Helling, A. Kim and J. Mennicke in [28], were they constructed an infinite series of closed orientable hyperbolic manifolds, called Fibonacci manifolds.

Let us consider a polyhedron Y_n, $n \geq 4$, which consists of a central band of n-antiprism with triangle faces bounded above and below by n-pyramids. In particular, Y_5 is an icosahedron. The polyhedron Y_n has $(2n+2)$ vertices, $6n$ edges and $4n$ triangle faces. Let us denote its vertices by Q, R, $P_1, \ldots, P_{2n}$ and its faces by $F_1, \ldots, F_n$ and $F_1^*, \ldots, F_n^*$ similar to Figure 5.1, were the polyhedron Y_4 is shown.

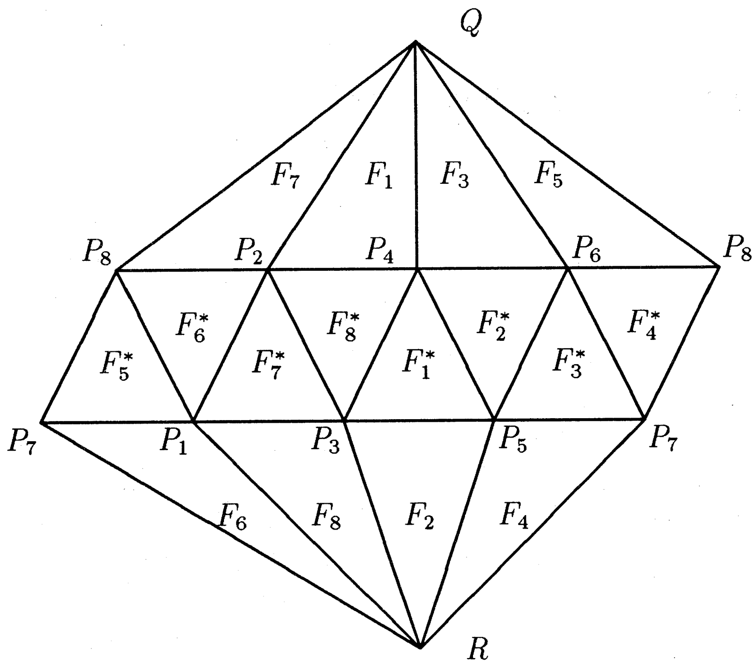

Figure 5.1.

Let us consider the following pairwise identifications of faces of Y_n:

$$s_i : F_i \longrightarrow F_i^*$$

defined for $i = 1, \ldots 2n$ by the following rules. If i is odd, then

$$s_i : QP_{i+1}P_{i+3} \longrightarrow P_{i+2}P_{i+3}P_{i+4}, \tag{5.1}$$

and if i is even, then

$$s_i : RP_{i+1}P_{i+3} \longrightarrow P_{i+2}P_{i+3}P_{i+4}. \tag{5.2}$$

Therefore we will get following cycles of equivalent edges. If i is odd, then

$$QP_{i+1} \xrightarrow{s_i} P_{i+2}P_{i+3} \xrightarrow{s_{i-1}^{-1}} P_iP_{i+2} \xrightarrow{s_{i-2}^{-1}} QP_{i+1}, \tag{5.3}$$

and if i is even, then

$$RP_{i+3} \xrightarrow{s_i} P_{i+2}P_{i+3} \xrightarrow{s_{i+1}} P_{i+4}P_{i+5} \xrightarrow{s_{i+2}^{-1}} RP_{i+2}. \tag{5.4}$$

From cycles (5.3) and (5.4) we have

$$s_i\, s_{i+1} = s_{i+2} \tag{5.5}$$

for any $i = 1, \ldots, 2n$, were all indices are by module $2n$.

Because we want to construct a hyperbolic manifold uniformized by the Fibonacci group with the polyhedron Y_n as its fundamental polyhedron, we will require that Y_n be a hyperbolic polyhedron whose boundary is $4n$ equal equilateral triangles and such that sums of dihedral angles corresponding to cycles of edges (5.3) and (5.4) are equal to 2π. Moreover we will require that the polyhedron Y_n has a cyclic symmetry of order n with the axe QR, similar to the icosahedron.

Let use the following notations for dihedral angles of Y_n:

$$\alpha = \angle QP_{2i} = \angle RP_{2i-1}, \qquad i = 1, \ldots, n,$$
$$\beta = \angle P_iP_{i+2}, \qquad i = 1, \ldots, 2n,$$
$$\gamma = \angle P_iP_{i+1}, \qquad i = 1, \ldots, 2n,$$

and denote by x the cosh of the length of an edge of Y_n.

Proposition 5.1 ([28]) *For $n \geq 4$ the polyhedron Y_n can be realized in the Lobachevsky space $\mathbb{H}^3$ with the following parameters:*

$$x = \frac{1}{4(1-\xi)} \left(4 + 2\xi - 4\xi^2 + (3 - 2\xi)\sqrt{2 + 2\xi}\right), \tag{5.6}$$

$$\cos\alpha = -\frac{2x\xi + 2\xi + 1}{2x + 1}, \tag{5.7}$$

$$\cos\beta = \frac{1}{\sqrt{2}(x-\xi)(2x+1)} \Big(\sqrt{2}\,x^2(1+\xi) - x(x+1)\sqrt{1+\xi}$$
$$-\sqrt{x^4(1-3\xi+2\xi^2) + x^3(1-8\xi+8\xi^2) + x^2(-1-6\xi+12\xi^2) + x(-1+8\xi^2)}$$
$$\overline{+\xi + 2\xi^2 + 2(x^2 - x^2\xi + x - 2x\xi - \xi)(x^2+x)\sqrt{2+2\xi}}\Big), \tag{5.8}$$

$$\cos\gamma = \frac{x - (x+1)\sqrt{2(1+\xi)}}{2x+1}, \tag{5.9}$$

where $\xi = \cos(2\pi/n)$, and in this case

$$\alpha + \beta + \gamma = 2\pi. \tag{5.10}$$

Proof. See [28], were formulae (5.6)–(5.9) obtained by a decomposition of the polyhedron Y_n in tetrahedra, and the formula (5.10) is proved. □

By Poincare theorem [87, p.164], using (5.3)–(5.5) and (5.10), we get that the polyhedron Y_n is fundamental for the group

$$F(2,2n) = \langle s_1, s_2, \dots, s_{2n} \mid s_i s_{i+1} = s_{i+2}, \qquad i \bmod 2n \rangle.$$

Therefore we get

Theorem 5.1 ([28]) *The Fibonacci group $F(2,2n)$, $n \geq 4$, is a discrete group of isometries of the Lobachevsky space $\mathbb{H}^3$ acting without fixed points, and the polyhedron Y_n with parameters given by (5.6)–(5.9) is its fundamental set.*

Definition 5.1 We will say the quotient space $M_n = HI^3/F(2,2n)$, $n \geq 4$, to be a hyperbolic *Fibonacci manifold.*

By the construction, the Fibonacci manifold M_n, $n \geq 4$, is a closed orientable hyperbolic 3-dimensional manifold. We remark, that the group $F(2,4)$ is isomorphic to $\mathbb{Z}_5$ and acts on the 3-sphere S^3 such that $M_2 = S^3/F(2,4)$ is a lens space $L(5,2)$. The group $F(2,6)$ is isomorphic to a 3-dimensional affine group and the manifold $M_3 = \mathbb{E}^3/F(2,6)$ is the Hantzche–Wendt manifold, studied in [98]. We will be call manifolds M_2 and M_3 Fibonacci manifolds too.

5.2 Fibonacci manifolds and the figure-eight knot

The closed connection between Fibonacci manifolds and the figure-eight knot was remarked by H. Hilden, M. Lozano and J. Montesinos [31]. By the other hand, the manifolds which can be obtained as coverings of the 3-sphere S^3 branched over the figure-eight knot were studied by J. Hempel [29].

By the construction, the polyhedron Y_n has the cyclic symmetry of order n with the axe QR. Let us denote this symmetry by ρ. Then for $i = 1, \dots, 2n$ we have

$$\rho : F_i \longrightarrow F_{i+2}, \qquad \rho : F_i^* \longrightarrow F_{i+2}^*,$$

where indices by module $2n$. The symmetry ρ induces an automorphism of the group $F(2,2n)$ such that

$$\rho : s_i \longrightarrow s_{i+2} = \rho^{-1} s_i \rho.$$

Let us consider the group $\Gamma_n = \langle F(2,2n), \rho \rangle$. The fundamental polyhedron for Γ_n is the $\frac{1}{n}$–piece of Y_n pictured in Figure 5.2.

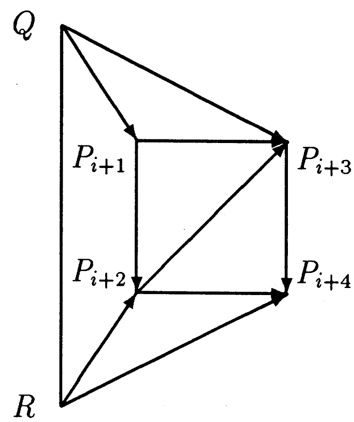

Figure 5.2.

Let us continue our considerations for fixed odd i. Edges of this piece are divided in cycles of equivalent under the group Γ_n action according to (5.1) and (5.2). The first cycle is

$$QP_{i+1} \xrightarrow{s_i} P_{i+2}P_{i+3} \xrightarrow{\rho s_{i+1}^{-1}} P_{i+2}P_{i+4} \xrightarrow{s_i^{-1}} QP_{i+3} \xrightarrow{\rho^{-1}} QP_{i+1}. \tag{5.11}$$

Hence

$$s_i \, \rho s_{i+1}^{-1} \, s_i^{-1} \, \rho^{-1} = 1,$$

and

$$\rho^{-1} \, s_i \, \rho = s_i \, s_{i+1} = s_{i+2}. \tag{5.12}$$

The second cycle is

$$RP_{i+4} \xrightarrow{s_{i+1}\rho^{-1}} P_{i+1}P_{i+3} \xrightarrow{\rho s_{i+2}\rho^{-2}} P_{i+1}P_{i+2} \xrightarrow{\rho} P_{i+3}P_{i+4} \xrightarrow{s_{i+1}^{-1}} RP_{i+2} \xrightarrow{\rho} RP_{i+4}. \tag{5.13}$$

Hence

$$s_{i+1} \, s_{i+2} \, \rho^{-1} \, s_{i+1}^{-1} \, \rho = 1,$$

and

$$\rho^{-1} \, s_{i+1} \, \rho = s_{i+1} \, s_i \, s_{i+1} = s_{i+1} \, s_{i+2} = s_{i+3}. \tag{5.14}$$

The third cycle is

$$QR \xrightarrow{\rho} QR, \tag{5.15}$$

whence

$$\rho^n = 1. \tag{5.16}$$

According to the Poincare theorem [87, p.164], the group Γ_n has the following representation:

$$\Gamma_n = \langle \rho, s_i, s_{i+1} \mid \rho^n = 1,\ \rho^{-1} s_i \rho = s_i s_{i+1},\ \rho^{-1} s_{i+1} \rho = s_{i+1} s_i s_{i+1} \rangle. \tag{5.17}$$

Let us express the generator s_i. From (5.17)

$$\rho^{-1} s_{i+1} \rho = s_{i+1} \rho^{-1} s_i \rho,$$

hence

$$s_i = \rho s_{i+1}^{-1} \rho^{-1} s_{i+1}.$$

Then we will get

$$s_{i+1}^{-1} \rho^{-1} s_{i+1} \rho = \rho s_{i+1}^{-1} \rho^{-1} s_{i+1} s_{i+1}.$$

Consider b such that $s_{i+1} = b\rho$, then

$$\rho^{-1} b^{-1} \rho^{-1} b \rho = b^{-1} \rho^{-1} b \rho b.$$

Therefore the group Γ_n has the following representation:

$$\Gamma_n = \langle \rho, b \mid \rho^n = b^n = 1,\ \rho^{-1} [b, \rho] = [b, \rho]\, b \rangle, \tag{5.18}$$

where $[b, \rho] = b^{-1}\rho^{-1}b\rho$ is the commutator of elements b and ρ. By the other hand, the group

$$\langle x, y \mid y^{-1} [x, y] = [x, y]\, x \rangle \tag{5.19}$$

is the group of the figure-eight knot, where generators x and y correspond to Figure 5.3.

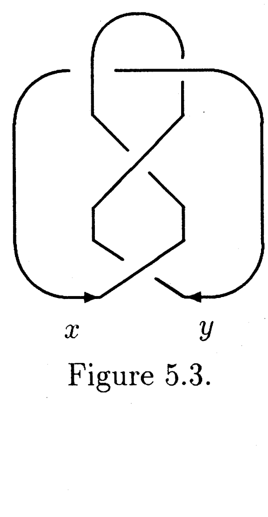

Figure 5.3.

Therefore the group Γ_n with the representation (5.18) is the group of the orbifold with the 3-sphere S^3 as underlying space and the figure-eight knot with branch index n as singular set. Let us denote this orbifold by $\mathcal{O}(n)$. So we get

Theorem 5.2 ([31]) *A Fibonacci manifold M_n, $n \geq 2$, is the n-fold regular covering of the orbifold $\mathcal{O}(n)$.*

We remark, that another proof of this statement based on the spine representation and surgery description of manifolds M_n was given by A. Cavicchioli and F. Spaggiari [16].

5.3 Volumes of Fibonacci manifolds

In this section we will calculate the volume of hyperbolic Fibonacci manifolds M_n, $n \geq 4$. By virtue of Theorem 5.2,

$$vol\,(M_n) \;=\; n\, vol\,(\mathcal{O}(n)), \tag{5.20}$$

where $\mathcal{O}(n)$ is the orbifold with the singular set the figure-eight knot.

Theorem 5.3 ([81]) *For $n \geq 4$ the hyperbolic volume of the orbifold $\mathcal{O}(n)$ is equal to*

$$vol\,(\mathcal{O}(n)) \;=\; 2\,(\Lambda(\beta+\delta)+\Lambda(\beta-\delta))\,, \tag{5.21}$$

where $\delta = \frac{\pi}{n}$, $\quad \beta = \frac{1}{2}\,\arccos\left(\cos(2\delta)-\frac{1}{2}\right)$.

Proof. . Consider the orbifold as the result of $(n,0)$-generalized surgery on the complement of the figure-eight knot according to Thurston's approach [77] (see also [62], [65]). We recall that the figure-eight knot complement can be obtained from two regular ideal tetrahedra. Let us consider a polyhedron $\mathcal{P}$ in $\mathbb{H}^3$ consisting of two ideal tetrahedra $T_z = ABCD$ and $T_w = ABCE$ in Figure 5.4.

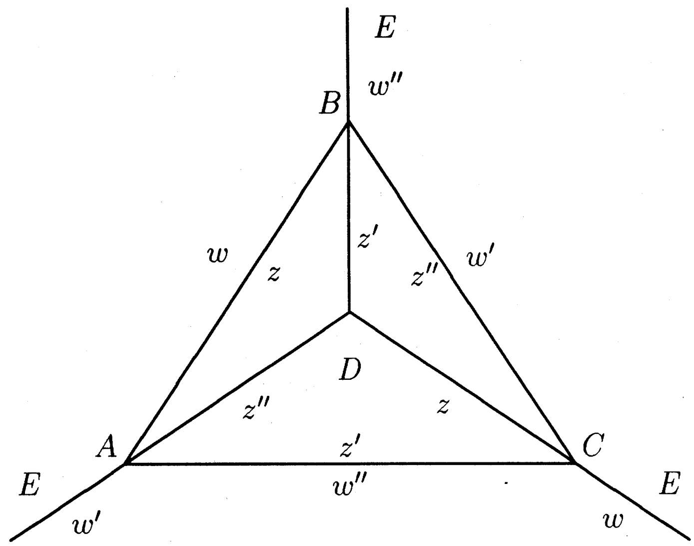

Figure 5.4.

Let us consider following transformations which identify pairs of faces of $\mathcal{P}$:

$$\begin{aligned} \alpha &: BCD \longrightarrow BEA, \\ \beta &: ADB \longrightarrow ACE, \\ \gamma &: CDA \longrightarrow ECB. \end{aligned}$$

According to [77] and [62], if the tetrahedra T_z and T_w is taken to be regular ideal tetrahedra in the Lobachevsky space, then one obtains the complete finite volume hyperbolic structure on the figure-eight-knot complement. By deforming the tetrahedra to differently shaped ideal hyperbolic tetrahedra, one obtains incomplete hyperbolic structures, whose metric completions are hyperbolic Dehn surgeries and generalized Dehn surgeries on the figure-eight knot (see also [65, ch.10]).

Without loss of generality we can assume that dihedral angles of the ideal tetrahedra are following (see Figure 5.4):

$$\angle AB = \angle CD = \arg z,\ \angle AC = \angle BD = \arg z',\ \angle BC = \angle AD = \arg z'' \quad (5.22)$$

for the tetrahedron T_z, and

$$\angle AB = \angle CE = \arg w,\ \angle AC = \angle BE = \arg w',\ \angle BC = \angle AE = \arg w'' \quad (5.23)$$

for the tetrahedron T_w, where we denote

$$\zeta' = 1 - \frac{1}{\zeta} \quad \text{and} \quad \zeta'' = \frac{1}{1-\zeta}. \quad (5.24)$$

By the action of transformations α, β, γ we will get two cycles of equivalent edges of $\mathcal{P}$:

$$BA \xrightarrow{\beta} EA \xrightarrow{\alpha^{-1}} CD \xrightarrow{\gamma} EC \xrightarrow{\beta^{-1}} BD \xrightarrow{\alpha} BA, \tag{5.25}$$

that implies

$$\beta\,\alpha^{-1}\,\gamma\,\beta^{-1}\,\alpha = 1; \tag{5.26}$$

and

$$AD \xrightarrow{\beta} AC \xrightarrow{\gamma} BE \xrightarrow{\alpha^{-1}} BC \xrightarrow{\gamma^{-1}} AD, \tag{5.27}$$

that implies

$$\beta\,\gamma\,\alpha^{-1}\,\gamma^{-1} = 1. \tag{5.28}$$

If both tetrahedra T_z and T_w are ideal regular, then all their dihedral angles are equal to $\pi/3$ and sums of dihedral angles corresponding to cycles (5.25) and (5.27) are equal 2π. By Poincare theorem [87, p.164] the group G of isometries of $\mathbb{H}^3$ generated by α, β and γ has the following representation:

$$G = \langle \alpha, \beta, \gamma \mid \beta\,\alpha^{-1}\,\gamma\,\beta^{-1}\,\alpha = 1,\ \beta\,\gamma\,\alpha^{-1}\,\gamma^{-1} = 1 \rangle. \tag{5.29}$$

Eliminating γ from the first relation in (5.28), we will get the representation

$$G = \langle \alpha, \beta \mid \beta \left[\alpha^{-1}, \beta\right] = \left[\alpha^{-1}, \beta\right] \alpha^{-1}, \rangle \tag{5.30}$$

that coincides with the representation (5.19) of the group of the figure-eight knot, and isometries α and β correspond to loops x^{-1} and y in Figure 5.3. In general case, from cycles (5.25) and (5.27) we get following equations for complex parameters of tetrahedra:

$$z\,w'\,z\,w\,z'\,w = 1,$$

that by (5.24) is equivalent to

$$z\,(z-1)\,w\,(w-1) = 1, \tag{5.31}$$

and analogously

$$z'\,z'\,w''\,w''\,z''\,w'\,z'' = 1,$$

that by (5.24) also is equivalent to (5.30).

Because all vertices A, B, C, D, E are equivalent in respect to the group G action, considering their links we will get the picture shown in Figure 5.5, where triangles and quadrilaterals A, B, C, D, E correspond to the same named vertices of the tetrahedra.

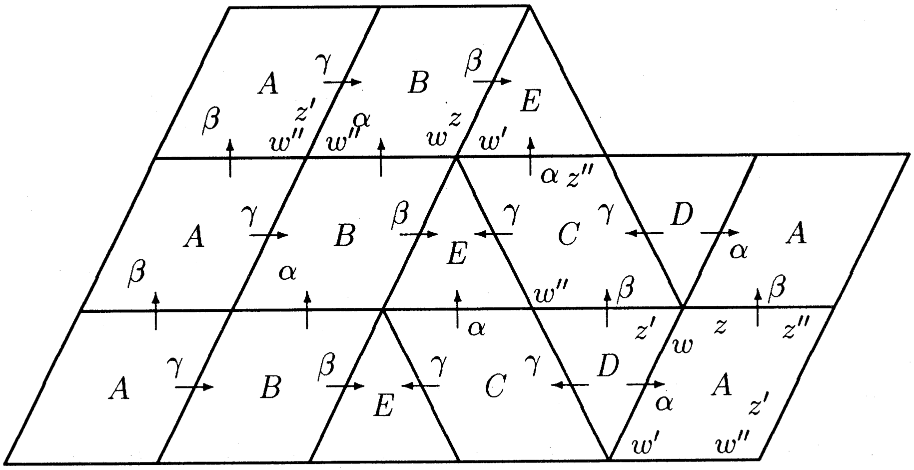

Figure 5.5.

Labels on arrows show the isometry which identifies edges of adjacent polygons. We recall that the figure-eight knot is pictured in Figure 5.3 and let us take the generator y as its meridian, and

$$l = x\,y\,x^{-1}\,y^{-1}\,y^{-1}\,x^{-1}\,y\,x$$

as its longitude (see [13, p.37] about their properties). We recall, that groups (5.19) and (5.29) are isomorphic for $x = \alpha^{-1}$, $y = \beta$. Therefore, the image of the longitude in the group G is equal to

$$l^* = \alpha^{-1}\,\beta\,\alpha\,\beta^{-1}\,\beta^{-1}\,\alpha\,\beta\,\alpha^{-1},$$

and if we recall, that from (5.28) $\gamma = \alpha\beta^{-1}\alpha^{-1}\beta$, we can rewrite

$$l^* = \alpha^{-1}\,\beta\,\alpha\,\beta^{-1}\,\gamma^{-1}. \tag{5.32}$$

The image of the meridian corresponds to the translation from the quadrilateral with the label A to the neighborhood quadrilateral with the label A according to the direction β. So we can express this translation in complex parameters:

$$\frac{1}{w''}\,z = z\,(1-w). \tag{5.33}$$

The image of the longitude corresponds to the translation from the quadrilateral A at the right lower corner to the quadrilateral A at the left upper corner according to arrows which give the expression for l^* in (5.31). So we can express this transformation in complex parameters:

$$w' \frac{1}{z'} w'' \frac{1}{z''} w' \, z \, w \, w'' \, z' \, w''$$

$$= z\,(z-1)\,\frac{1}{w\,(w-1)} = z^2\,(z-1)^2,$$

where we used (5.24) and (5.30).

Because we consider the $(n,0)$-generalized Dehn surgery, according to [77, p.4.18], [65, §10.5] from (5.30) and (5.32) we get that the orbifold $\mathcal{O}(n)$ can be obtained by completion of the incomplete hyperbolic structure on the union of two ideal tetrahedra T_z and T_w whose complex parameters z and w satisfy the system:

$$\begin{cases} z\,(z-1)\,w\,(w-1) = 1, \\ n\log(z\,(1-w)) = 2\pi i, \\ \operatorname{Im} z > 0, \\ \operatorname{Im} w > 0. \end{cases} \tag{5.34}$$

From this we obtain the following equation for w:

$$w^2 + \left(2i\sin\frac{2\pi}{n} - 1\right) w + e^{-\frac{2\pi i}{n}} = 0$$

which has the solutions

$$w = \frac{1}{2} - i\,\sin\left(\frac{2\pi}{n}\right) \pm i\,\sqrt{1 - \left(\cos\left(\frac{2\pi}{n}\right) - \frac{1}{2}\right)^2}.$$

If we denote $\varphi = \frac{2\pi}{n}$, $n \geq 4$, then we get

$$-1 < \cos\varphi - \frac{1}{2} < 1,$$

and we choose ψ, $0 < \psi < \pi$, such that $\cos\psi = \cos\varphi - \frac{1}{2}$. Hence

$$w = \frac{1}{2} + i\,(\pm\sin\psi - \sin\varphi),$$

and as we need $\operatorname{Im} z > 0$, we choose the solution with the sign " + " :

$$w = \frac{1}{2} + i\,(\sin\psi - \sin\varphi), \tag{5.35}$$

and hence

$$z = \frac{\cos\varphi + i\sin\varphi}{\frac{1}{2} - i(\sin\psi - \sin\varphi)}. \tag{5.36}$$

We remark that for $n \geq 5$ expressions (5.35) and (5.36) satisfy (5.34), but for $n = 4$ we get $\operatorname{Im} w < 0$ and the corresponding tetrahedron T_w is the "negative". It means that the volume of the orbifold is equal to the difference of volumes of T_z and T_w.

For finding the hyperbolic volume of the ideal tetrahedron T_w with complex parameter w we should know the values of the arguments of the following complex numbers:

$$\arg w, \qquad \arg\frac{w-1}{w}, \qquad \arg\frac{1}{1-w}.$$

Lemma 5.1 *With notation as above we have:*

$$\arg w = \arg\frac{1}{1-w} = \frac{\pi - \varphi - \psi}{2}, \qquad \arg\frac{w-1}{w} = \varphi + \psi.$$

Proof. From (5.35) by direct computations we have:

$$\tan(\arg w) = \frac{\sin\psi - \sin\varphi}{\frac{1}{2}} = \frac{\sin\psi - \sin\varphi}{\cos\psi - \cos\varphi}$$

$$= \cot\frac{\psi + \varphi}{2} = \tan\frac{\pi - \varphi - \psi}{2}.$$

Similarly, for the second complex parameter we see that

$$\frac{1}{1-w} = \frac{1}{\frac{1}{2} - i(\sin\psi - \sin\varphi)} = \frac{\frac{1}{2} + i(\sin\psi - \sin\varphi)}{\frac{1}{4} + (\sin\psi - \sin\varphi)^2},$$

and

$$\tan\left(\arg\frac{1}{1-w}\right) = \frac{\sin\psi - \sin\varphi}{\frac{1}{2}} = \tan\frac{\pi - \varphi - \psi}{2}.$$

Therefore

$$\arg w = \arg\frac{1}{1-w} = \frac{\pi - \varphi - \psi}{2}.$$

For the third complex number we remark, that

$$\arg w + \arg\frac{w-1}{w} + \arg\frac{1}{1-w} = \pi.$$

Hence

$$\arg\frac{w-1}{w} = \pi - (\pi - \varphi - \psi) = \varphi + \psi,$$

and this complete the proof of Lemma 5.1. □

From Lemma 5.1 and using the item (7) of the Proposition 2.2, we conclude that:

$$vol\,(T_w) = \Lambda(\varphi+\psi) + 2\Lambda\left(\frac{\pi-\varphi-\psi}{2}\right) = 2\Lambda\left(\frac{\varphi+\psi}{2}\right). \tag{5.37}$$

Now we consider the tetrahedron T_z with the complex parameter w.

Lemma 5.2 *With notation as above we have:*

$$\arg z = \arg\frac{1}{1-z} = \frac{\pi-\psi+\varphi}{2}, \qquad \arg\frac{z-1}{z} = \psi - \varphi.$$

Proof. Using the Lemma 5.1 from (5.36) we obtain:

$$\arg z = \arg\frac{e^{i\varphi}}{1-w} = \varphi + \frac{\pi-\varphi-\psi}{2} = \frac{\pi-\psi+\varphi}{2}.$$

Similarly,

$$\frac{z-1}{z} = 1 - \frac{1}{z} = 1 - \frac{\frac{1}{2} - i\,(\sin\psi - \sin\varphi)}{\cos\varphi + i\,\sin\varphi} = \frac{\cos\varphi - \frac{1}{2} + i\,\sin\psi}{\cos\varphi + i\,\sin\varphi}$$

$$= \frac{\cos\psi\, +\, i\,\sin\psi}{\cos\varphi\, +\, i\,\sin\varphi} = e^{i\,(\psi-\varphi)},$$

and

$$\arg\frac{z-1}{z} = \psi - \varphi.$$

Hence

$$\arg\frac{1}{1-z} = \pi - \arg z - \arg\frac{z-1}{z}$$

$$= \pi - \frac{\pi-\psi+\varphi}{2} - (\psi-\varphi) = \frac{\pi-\psi+\varphi}{2},$$

and the proof of the Lemma is complete.□

From Lemma 5.2 we obtain

$$vol\,(T_z) = \Lambda(\psi-\varphi) + 2\Lambda\left(\frac{\pi+\varphi-\psi}{2}\right) = 2\Lambda\left(\frac{\psi-\varphi}{2}\right) \tag{5.38}$$

Thus the volume of the orbifold $\mathcal{O}(n)$ is given by the formula:

$$vol\,(\mathcal{O}(n)) = vol\,(T_w) + vol\,(T_z) = 2\left(\Lambda\left(\frac{\psi+\varphi}{2}\right) + \Lambda\left(\frac{\psi-\varphi}{2}\right)\right).$$

In order to get the statement of Theorem 5.3, we denote $\delta = \frac{\varphi}{2}$ and $\beta = \frac{\psi}{2}$, then $\delta = \frac{\pi}{n}$ and $\beta = \frac{1}{2}\arccos\left(\cos(2\delta) - \frac{1}{2}\right)$. Therefore,

$$vol\,(\mathcal{O}(n)) = 2\,(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)),$$

and the proof of Theorem 5.3 is complete. □

According to (5.20), from Theorem 5.3 we obtain

Corollary 5.1 *For $n \geq 4$ the hyperbolic volume of the Fibonacci manifold M_n is equal to*

$$vol\,(M_n) = 2n\,(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)), \qquad (5.39)$$

where $\delta = \frac{\pi}{n}$, $\beta = \frac{1}{2}\arccos\left(\cos(2\delta) - \frac{1}{2}\right)$.

For some n the arguments of the Lobachevsky function from Theorem 5.3 have the simple expressions, and using properties of the Lobachevsky function given in Proposition 2.2 we get

Corollary 5.2 *For $n = 4$ we obtain*

$$vol\,(\mathcal{O}(4)) = \Lambda\left(\frac{\pi}{6}\right) = \Lambda^{max}.$$

Corollary 5.3 *For $n = 6$ we obtain*

$$vol\,(\mathcal{O}(6)) = \frac{8}{3}\,\Lambda\left(\frac{\pi}{4}\right).$$

Corollary 5.4 *For $n = 10$ we obtain*

$$vol\,(\mathcal{O}(10)) = 2\left(\Lambda\left(\frac{3\pi}{10}\right) + \Lambda\left(\frac{\pi}{10}\right)\right).$$

We remark the following interesting property of the figure-eight knot K. This is well-known that the volume of the hyperbolic manifold $S^3 \setminus K$ is equal to the doubled volume of the ideal regular tetrahedra. If we denote the orbifold $\mathcal{O}(n)$ by $K(n)$, then from Corollary 5.2 we get

$$4\,vol(K(4)) = vol(S^3 \setminus K).$$

Let in general case K be a hyperbolic knot and $K(n)$ be the hyperbolic orbifold with the 3-sphere as underlying space and the knot K with branch index n as singular set. Are there exist others such K and n that

$$n\, vol(K(n)) = vol(S^3 \setminus K) \ ?$$

If we redraw the figure-eight knot similar to Figure 5.6,

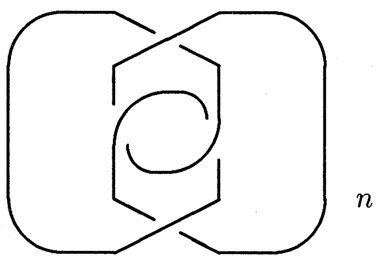

Figure 5.6.

then it is obvious that the orbifold $\mathcal{O}(n)$ has the rotation symmetry ρ of order two such that the axe of ρ and the singular set of $\mathcal{O}(n)$ are disjoint. The quotient space $\mathcal{O}(n)/\rho$ is the orbifold with the 3-sphere as underlying space and the 2-component link pictured in Figure 5.7 as singular set, whose components have branch indices 2 and n.

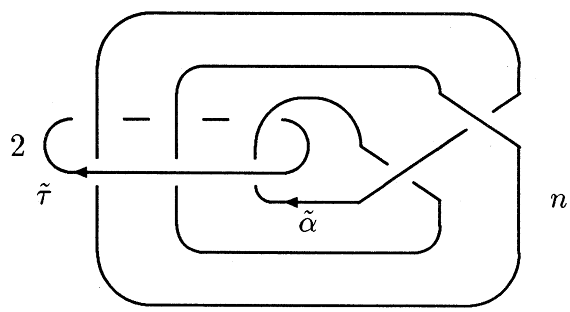

Figure 5.7.

The one component of this link can be regarded as the closure of the 3-string braid $\sigma_1\sigma_2^{-1}$. It was remarked in [77, p.6.48], that it is the 2-component link 6_2^2 in notations of [70], pictured in standard form in Figure 5.8.

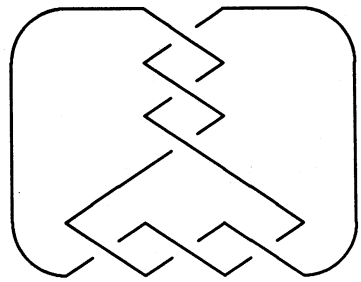

Figure 5.8.

Let us denote by $6_2^2(m,n)$, with $m,n \in N \cup \{\infty\}$, the orbifold with the 3-sphere as underlying space and the link 6_2^2 as singular set, whose components have branch indices m and n corresponding to $(m,0)$ and $(n,0)$-generalized Dehn surgeries. By the index ∞ we denote a removed component and in this case the orbifold is non-compact.

According to this notation, we have $6_2^2(2,n) = \mathcal{O}(n)/\rho$. So by Theorem 5.3 we get

Corollary 5.5 *For $n \geq 4$ the orbifold $6_2^2(2,n)$ is hyperbolic and*

$$vol\, 6_2^2(2,n) = \left(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)\right), \tag{5.40}$$

where $\delta = \frac{\pi}{n}$, $\beta = \frac{1}{2}\arccos\left(\cos(2\delta) - \frac{1}{2}\right)$.

5.4 Volumes of Turk's head links

In this section we consider the series of non-compact manifolds connected with the link 6_2^2. Denote by Th_n, $n \geq 2$, the closed 3-strings braid $\left(\sigma_1\sigma_2^{-1}\right)^n$. We note that members of the family Th_n are well-known. In particular, Th_2 is the figure-eight knot, Th_3 are the Borromean rings, Th_4 is the Turk's head knot 8_{18} and Th_5 is the knot 10_{123} according to the notation of [70]. It was shown in [77, p.6.48] that the manifold $S^3 \setminus Th_n$, $n \geq 2$ is hyperbolic and has a representation as an n–fold cyclic covering over the orbifold $6_2^2(n,\infty)$. In particular, for hyperbolic volumes we have:

$$vol\left(S^3 \setminus Th_n\right) = n\, vol\left(6_2^2(n,\infty)\right) \tag{5.41}$$

The computation of the volume of a manifold $S^3 \setminus Th_n$ will be based on the following theorem.

Theorem 5.4 *For $n \geq 2$ the orbifold $6_2^2(n,\infty)$ is hyperbolic and*

$$vol\,(6_2^2(n,\infty)) \;=\; 4\;(\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma))\,, \tag{5.42}$$

where $\gamma = \frac{\pi}{2n}$, $\alpha = \frac{1}{2}\,\arccos\left(\cos(2\gamma) - \frac{1}{2}\right)$.

Proof. Let us consider the link 6_2^2 pictured in Figure 5.7, where $\tilde{\alpha}$ and $\tilde{\tau}$ denote corresponding elements of the fundamental group $\pi_1(S^3 \setminus 6_2^2)$. By the Wirtinger algorithm [20] we obtain the following representation for $\pi_1(S^3 \setminus 6_2^2)$:

$$\langle \tilde{\alpha},\tilde{\tau} \mid \left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)\left(\tilde{\tau}^{2}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-2}\right)\left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)^{-1} = \tilde{\alpha}\,\rangle. \tag{5.43}$$

Choosing new generators u and r such that $\tilde{\alpha} = u^{-1}r^{-1}$ and $\tilde{\tau} = r^{-1}$ from (5.43) we will get

$$\pi_1(S^3 \setminus 6_2^2) = \langle u, r \mid \left(u\,r^{-1}\,u^{-2}\right)\left(r^{-1}\,u\,r^{-1}\,u^{-2}\,r\right)\left(u\,r^{-1}\,u^{-2}\right)^{-1} = u^{-1}\,r^{-1}\rangle,$$

and hence

$$\pi_1(S^3 \setminus 6_2^2) = \langle\, u, r \mid r^{-1}\left(u^2\,r\,u^{-1}\,r\,u^2\right)^{-1} r\left(u^2\,r\,u^{-1}\,r\,u^2\right) = 1\,\rangle. \tag{5.44}$$

Let us construct the fundamental domain for this group. Consider an ideal polyhedron $\mathcal{P} = ABCDEF\infty$ pictured in Figure 5.9

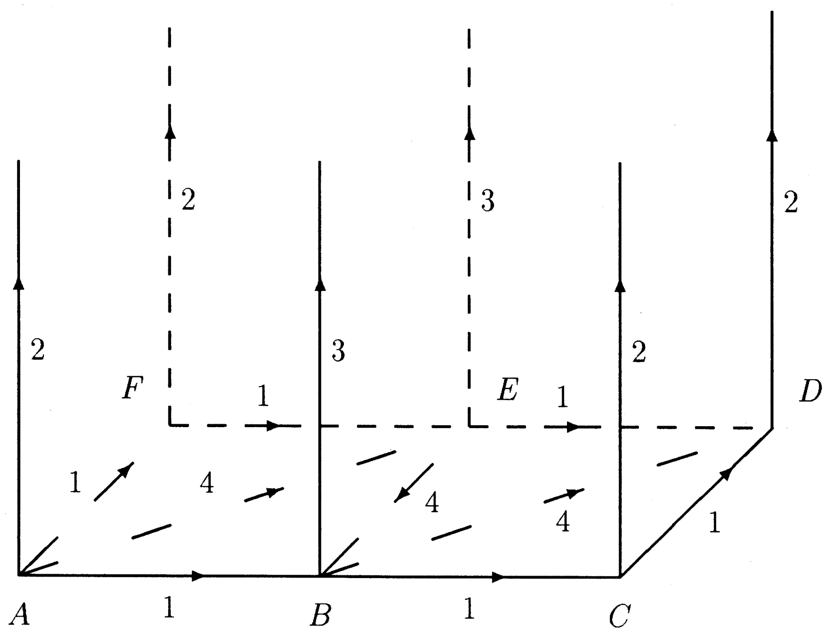

Figure 5.9.

which consists of four ideal tetrahedra

$$T_1 = AFE\infty, \quad T_2 = AEB\infty, \quad T_3 = DBE\infty, \quad T_4 = DCB\infty.$$

We recall that an ideal tetrahedron in $\mathbb{H}^3$ is uniquely (up to transformations (5.24)) determined by a complex parameter, and denote complex parameters corresponding to T_1, T_2, T_3 and T_4 by z_1, z_2, z_3 and z_4, respectively, and will be write T_{z_1}, T_{z_2}, T_{z_3} and T_{z_4}. So we can suppose that for the tetrahedron T_{z_1}:

$$\angle A\infty = \angle FE = \arg z_1, \ \angle F\infty = \angle AE = \arg z_1', \ \angle E\infty = \angle AF = \arg z_1''; \tag{5.45}$$

for the tetrahedron T_{z_2}:

$$\angle A\infty = \angle BE = \arg z_2, \ \angle E\infty = \angle AB = \arg z_2', \ \angle B\infty = \angle AE = \arg z_2''; \tag{5.46}$$

for the tetrahedron T_{z_3}:

$$\angle D\infty = \angle BE = \arg z_3, \ \angle B\infty = \angle DE = \arg z_3', \ \angle E\infty = \angle BD = \arg z_3''; \tag{5.47}$$

and for the tetrahedron T_{z_4}:

$$\angle D\infty = \angle BC = \arg z_4, \ \angle C\infty = \angle BD = \arg z_4', \ \angle B\infty = \angle CD = \arg z_4'', \tag{5.48}$$

where as well as above we use notations (5.24).

Let us consider isometries u, v, t, r of the Lobachevsky space $\mathbb{H}^3$ which identify the following pairs of faces of $\mathcal{P}$:

$$\begin{array}{llll} u: & \mathrm{A\,B\,E} & \longrightarrow & \mathrm{E\,D\,B}, \\ v: & \mathrm{A\,E\,F} & \longrightarrow & \mathrm{B\,D\,C}, \\ t: & \mathrm{A\,F}\,\infty & \longrightarrow & \mathrm{C\,D}\,\infty, \\ r: & \mathrm{A\,B\,C}\,\infty & \longrightarrow & \mathrm{F\,E\,D}\,\infty. \end{array}$$

Then we will get following cycles of equivalent edges, which are denoted by 1, 2, 3 and 4 in Figure 5.9.

For the cycle "2" we have:

$$A\infty \xrightarrow{t} C\infty \xrightarrow{r} D\infty \xrightarrow{t^{-1}} F\infty \xrightarrow{r^{-1}} A\infty, \tag{5.49}$$

whence

$$t\,r\,t^{-1}\,r^{-1} = 1, \tag{5.50}$$

and

$$z_1\,z_2\,z_4'\,z_4\,z_3\,z_1' = 1,$$

that by (5.24) is equivalent to

$$(z_1 - 1)\, z_2\, z_3\, (z_4 - 1) \;=\; 1. \tag{5.51}$$

For the cycle "3" we have:

$$B\infty \xrightarrow{r} E\infty \xrightarrow{r^{-1}} B\infty, \tag{5.52}$$

whence

$$r\, r^{-1} \;=\; 1, \tag{5.53}$$

and

$$z_2''\, z_3'\, z_4''\, z_1''\, z_2'\, z_3'' \;=\; 1,$$

that by (5.24) also is equivalent to (5.51).

For the cycle "1" we have:

$$AB \xrightarrow{u} ED \xrightarrow{r^{-1}} BC \xrightarrow{v^{-1}} AF \xrightarrow{t} CD \xrightarrow{v^{-1}} FE \xrightarrow{r^{-1}} AB, \tag{5.54}$$

whence

$$u\, r^{-1}\, v^{-1}\, t\, v^{-1}\, r^{-1} \;=\; 1, \tag{5.55}$$

and

$$z_2'\, z_3'\, z_4\, z_1''\, z_4''\, z_1 \;=\; 1,$$

that by (5.24) is equivalent to

$$\frac{z_1\, (z_2 - 1)\, (z_3 - 1)\, z_4}{(z_1 - 1)\, z_2\, z_3\, (z_4 - 1)} \;=\; 1,$$

and because the dominator is equal to 1 by (5.51), we get

$$z_1\, (z_2 - 1)\, (z_3 - 1)\, z_4 \;=\; 1. \tag{5.56}$$

For the cycle "4" we have:

$$AE \xrightarrow{u} EB \xrightarrow{u} BD \xrightarrow{v^{-1}} AE, \tag{5.57}$$

whence

$$u^2\, v^{-1} \;=\; 1, \tag{5.58}$$

and

$$z_1'\, z_2''\, z_2\, z_3\, z_3''\, z_4' \;=\; 1,$$

that by (5.24) and (5.51) is equivalent to (5.56).

Vertices of $\mathcal{P}$ found two cycles of equivalent: $\{\infty\}$ and $\{A, B, C, D, E, F\}$. For the first cusp we get the Figure 5.10,

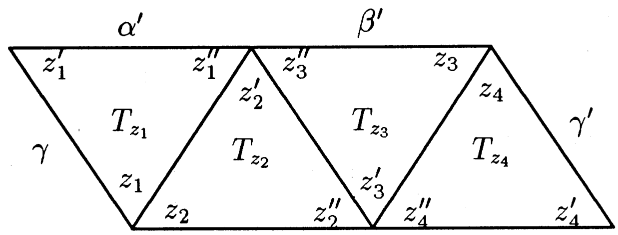

Figure 5.10.

where labels of triangles correspond to names of tetrahedra, and the isometry r identifies the side α with α', and the side β with β'; and the isometry t identifies the side γ with γ'. For the second cusp we have Figure 5.11,

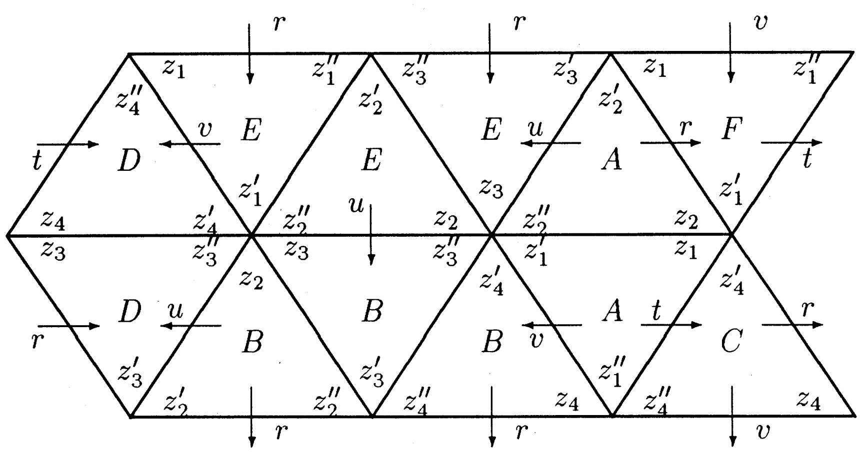

Figure 5.11.

where labels of triangles correspond to ideal vertices of the polyhedron $\mathcal{P}$.

Now we are interested in the volume of the orbifold $6_2^2(n,\infty)$ which can be obtained by $(n,0)$-generalized Dehn surgery on one of cusps of the hyperbolic manifold $S^3 \setminus 6_2^2$, and the second cusp is complete at the same time. Let us consider the cusp corresponding to the component of the links 6_2^2 with the meridian $\tilde{\tau}^{-1} = r$. Because the translation r takes the side α to the side α' and the side β to the side β' in Figure 5.10, we can express this translation in terms of complex parameters:

$$z_2 \frac{1}{z_1''} = z_2 (1 - z_1). \tag{5.59}$$

As we consider $(n,0)$-generalized Dehn surgery, we will get the structure of the cusp pictured in Figure 5.12,

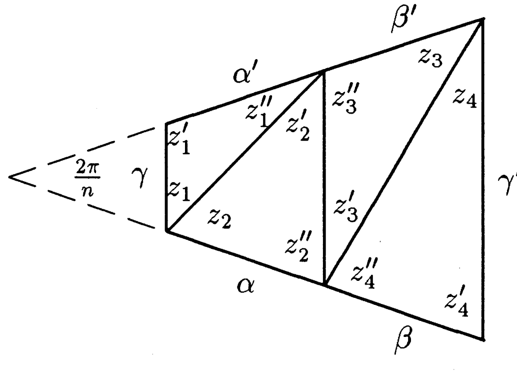

Figure 5.12.

and for complex parameters we will get

$$n \log(z_2(1 - z_1)) = 2\pi i. \tag{5.60}$$

For the second cusp we have meridian $\tilde{\alpha} = u^{-1}r^{-1}$, so according to Figure 5.11, we get for complex parameters:

$$z_2'' \frac{1}{z_1''} \frac{1}{z_2''} z_3 = z_3(1 - z_1),$$

and because this cusp remains complete, we will get the equation

$$\log(z_3(1 - z_1)) = 2\pi i. \tag{5.61}$$

Thus we get that the orbifold $6_2^2(n, \infty)$ can be obtained by completing of the incomplete hyperbolic structure on the union of the four tetrahedra whose complex parameters satisfy conditions (5.51), (5.56), (5.60) and (5.61), that is equivalent to the following system:

$$\begin{cases} (z_1 - 1)\, z_2\, z_3\, (z_4 - 1) = 1, \\ z_1\, (z_2 - 1)\, (z_3 - 1)\, z_4 = 1, \\ n \log(z_2\, (1 - z_1)) = 2\pi i, \\ \log(z_3\, (1 - z_1)) = 2\pi i. \end{cases} \tag{5.62}$$

If we denote $\zeta = \frac{1}{1 - z_1}$, then from (5.62) we get:

$$z_1 = \frac{\zeta - 1}{\zeta}, \quad z_2 = e^{\frac{2\pi i}{n}} \zeta, \quad z_3 = \zeta, \quad z_4 = 1 - \frac{1}{e^{\frac{2\pi i}{n}} \zeta}, \tag{5.63}$$

and the system (5.62) is reduced to the equation:

$$\left(e^{\frac{\pi i}{n}}\zeta + \frac{1}{e^{\frac{\pi i}{n}}\zeta} - \left(e^{\nu i} + e^{-\nu i}\right)\right)^2 = 1,$$

where $\nu = \frac{\pi}{n}$. Let we choose θ such that $e^{\frac{\pi i}{n}}\zeta = e^{\theta i}$. Then

$$(2\cos\theta - 2\cos\nu)^2 = 1,$$

hence

$$\cos\theta = \cos\nu \pm \frac{1}{2}.$$

Since $\cos\theta \leq 1$, we choose the sign "–":

$$\cos\theta = \cos\nu - \frac{1}{2}.$$

Substituting $\zeta = e^{i(\theta-\nu)}$ in (5.63) we obtain:

$$z_1 = 1 - \frac{1}{e^{i(\theta-\nu)}}, \quad z_2 = e^{i(\theta+\nu)}, \quad z_3 = e^{i(\theta-\nu)}, \quad z_4 = 1 - \frac{1}{e^{i(\theta+\nu)}}.$$

By direct computation we get the following result.

Lemma 5.3 *With notation as above we have*

$$(i) \qquad \arg z_1 = \arg\frac{z_1 - 1}{z_1} = \frac{\pi - \theta + \nu}{2}, \qquad \arg\frac{1}{1 - z_1} = \theta - \nu;$$

$$(ii) \qquad \arg z_2 = \theta + \nu, \qquad \arg\frac{z_2 - 1}{z_2} = \arg\frac{1}{1 - z_2} = \frac{\pi - \theta - \nu}{2};$$

$$(iii) \qquad \arg z_3 = \theta - \nu, \qquad \arg\frac{z_3 - 1}{z_3} = \arg\frac{1}{1 - z_3} = \frac{\pi - \theta + \nu}{2};$$

$$(iv) \qquad \arg z_4 = \arg\frac{z_4 - 1}{z_4} = \frac{\pi - \theta - \nu}{2}, \qquad \arg\frac{1}{1 - z_4} = \theta + \nu.$$

As the tetrahedron in $\mathbb{H}^3$ is uniquely defined by its dihedral angles, we see that $T_{z_1} = T_{z_3}$ and $T_{z_2} = T_{z_4}$. Therefore we conclude:

$$vol\left(6_2^2(n,\infty)\right) =$$

$$= 2\left(\Lambda(\theta+\nu) + \Lambda(\theta-\nu) + 2\Lambda\left(\frac{\pi-\theta-\nu}{2}\right) + 2\Lambda\left(\frac{\pi-\theta+\nu}{2}\right)\right) =$$

$$= 4 \left(\Lambda \left(\frac{\theta + \nu}{2} \right) + \Lambda \left(\frac{\theta - \nu}{2} \right) \right).$$

Remark that for $n = 2$ we again have the situation of "negative" tetrahedra. In this case the volume of the orbifold is equal to the difference of volumes of tetrahedra.

To finish the proof of the theorem, we denote $\gamma = \frac{\nu}{2}$, $\alpha = \frac{\theta}{2}$. Then $\gamma = \frac{\pi}{2n}$ and $\alpha = \frac{1}{2} \arccos \left(\cos(2\gamma) - \frac{1}{2} \right)$. Therefore

$$vol \left(6_2^2 \, (n, \infty) \right) = 4 \left(\Lambda \left(\alpha + \gamma \right) + \Lambda \left(\alpha - \gamma \right) \right),$$

that is (5.42) and theorem is proven.□

Corollary 5.6 *For $m \geq 2$ the hyperbolic volume of the non-compact manifold $S^3 \setminus Th_m$ is equal to*

$$vol \, (S^3 \setminus Th_m) = 4m \left(\Lambda(\alpha + \gamma) + \Lambda(\alpha - \gamma) \right), \tag{5.64}$$

where $\gamma = \frac{\pi}{2m}$, $\alpha = \frac{1}{2} \arccos (\, cos(2\gamma) - \frac{1}{2})$.

Comparing formula (5.64) with (2.12) and Corollary 2.5, one can see, that the volume of $S^3 \setminus Th_m$ is equal to the doubled value of ideal regular m-antiprism maximal volume.

5.5 Equality of volumes

Comparing formulae (5.39) and (5.64), we see that they are the same if $n = 2m$. Therefore, we get the following theorem.

Theorem 5.5 ([81]) *For $m \geq 2$ volumes of the compact Fibonacci manifold M_{2m} and the non-compact manifold the Turk's head link complement $S^3 \setminus Th_m$ are equal:*

$$vol \, (M_{2m}) = vol \left(S^3 \setminus Th_m \right). \tag{5.65}$$

Therefore volumes of the compact Fibonacci manifolds correspond to limit ordinals in the Thurston–Jørgensen theorem on volumes of hyperbolic 3-manifolds. In particular, the following assertions hold.

Corollary 5.7 *The volume of the manifold M_4 is equal to the volume of the complement of the figure-eight knot and corresponds to the first limit ordinal in Thurston–Jørgensen theorem.*

Corollary 5.8 *The volume of the manifold M_6 is equal to the volume of the complement of the Borromean rings.*

We recall, that many properties of hyperbolic manifolds are defined by arithmeticity or non-arithmeticity of their fundamental group (see [14], [61], [87]). As it was proved in [28] and [31], the manifold M_n is arithmetic for $n = 4, 5, 6, 8, 12$ and non-arithmetic otherwise. It is well-known [66] that Th_2, the figure-eight knot, is the only arithmetic knot. It is shown in [77] that Th_3, the Borromean rings, is also arithmetic. Using these results, from Theorem 5.5 for small values of m we will get:

Corollary 5.9 *There is the following table of arithmetic and non-arithmetic manifolds with equal volumes:*

m	M_{2m}	$S^3 \setminus Th_m$
2	*arithmetic*	*arithmetic*
3	*arithmetic*	*arithmetic*
4	*arithmetic*	*non-arithmetic*
5	*non-arithmetic*	*non-arithmetic*

Therefore there exist an arithmetic compact manifold M_8 and a non-arithmetic non-compact manifold $S^3 \setminus 8_{18}$ which have the same volume.

5.6 Fibonacci manifolds as two-fold coverings

The following theorem gives one more relation between the Fibonacci manifolds M_n and the Turk's head links Th_n.

Theorem 5.6 *For any $n \geq 2$ the Fibonacci manifold M_n is a two-fold covering of the three-dimensional sphere S^3 branched over the link Th_n.*

Proof. Let as well as above $\mathcal{O}(n)$ be an orbifold whose underlying space is the 3-sphere and singular set is the figure-eight knot with the branch index n (see Figure 5.6). As it was remarked above, the orbifold $\mathcal{O}(n)$ is the two-fold covering of the orbifold $6_2^2(2,n)$, whose singular set was pictured in Figure 5.7. Therefore, by Theorem 5.2 we get the following diagram of coverings:

$$M_n \xrightarrow{n} \mathcal{O}(n) \xrightarrow{2} 6_2^2(2,n). \tag{5.66}$$

We remark that the space X_n, where $X_2 = S^3$, $X_3 = \mathbb{E}^3$ and $X_n = \mathbb{H}^3$ for $n \geq 4$, is the universal covering for the Fibonacci manifold M_n, the orbifold $\mathcal{O}(n)$ and the orbifold $6_2^2(2,n)$. Let $F(2,2n)$, Γ_n and Ω_n be the fundamental groups of the manifold M_n, the orbifold $\mathcal{O}(n)$ and the orbifold $6_2^2(2,n)$ respectively. Therefore groups $F(2,2n)$, Γ_n, and Ω_n are discrete subgroups of the full isometry group of X_n. Moreover there are canonical isomorphisms $M_n = X_n \,/\, F(2,2n)$, $\mathcal{O}(n) = X_n \,/\, \Gamma_n$ and $6_2^2(2,n) = X_n \,/\, \Omega_n$. Hence the diagram (5.66) implies embeddings of subgroups

$$F(2,2n) \triangleleft \Gamma_n \triangleleft \Omega_n, \tag{5.67}$$

where $\mid \Omega_n : \Gamma_n \mid = 2$ and $\mid \Gamma_n : F(2,2n) \mid = n$.

For describing the group Ω_n we use the representation (5.44) for the fundamental group $\pi_1(S^3 \setminus 6_2^2)$:

$$\langle \tilde{\alpha}, \tilde{\tau} \mid \left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)\left(\tilde{\tau}^{2}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-2}\right)\left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)^{-1} = \tilde{\alpha} \rangle. \tag{5.68}$$

In this representation generators $\tilde{\alpha}$ and $\tilde{\tau}$ canonical correspond to arcs with the same labels on the link diagram of 6_2^2 in Figure 5.7.

According to [26] it follows from (5.68) that the group Ω_n of the orbifold $6_2^2(2,n)$ has the following representation:

$$\begin{aligned}\langle \alpha, \tau \mid \left(\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1}\right)\left(\tau^{2}\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-2}\right)\left(\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1}\right)^{-1} &= \alpha, \\ \alpha^n = \tau^2 &= 1\rangle,\end{aligned} \tag{5.69}$$

where generators α and τ of Ω_n correspond to generators $\tilde{\alpha}$ and $\tilde{\tau}$ of the group $\pi_1(S^3 \setminus 6_2^2)$.

Let us consider a group

$$\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1 \rangle \oplus \left\langle t \mid t^2 = 1 \right\rangle$$

and an epimorphism

$$\theta : \Omega_n \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2$$

defined by conditions

$$\theta(\alpha) = a, \quad \theta(\tau) = t. \tag{5.70}$$

By the construction of the two-fold cover $\mathcal{O}(n) \to 6_2^2(2,n)$ the loop τ from the group Ω_n lifts to a trivial loop in the group Γ_n. By the same reasons the loop α from the group Ω_n lifts to a loop, which generates a cyclic subgroup of the order n in the group Γ_n. Therefore,

$$\Gamma_n = \theta^{-1}(\mathbb{Z}_n) = \theta^{-1}\left(\langle a \mid a^n = 1\rangle\right). \tag{5.71}$$

For the $2n$-fold covering $M_n \to 6_2^2(2,n)$ both loops α and τ from the group Ω_n lift to trivial loops in the group $F(2,2n)$. Therefore

$$F(2,2n) \;=\; \theta^{-1}(1) \;=\; \operatorname{Ker}\theta. \tag{5.72}$$

Let T_n be a subgroup of Ω_n defined by the following condition:

$$\mathrm{T}_n \;=\; \theta^{-1}(\mathbb{Z}_2) \;=\; \theta^{-1}\left(\left\langle t \mid t^2 \;=\; 1\right\rangle\right). \tag{5.73}$$

Then we have a sequence of normal subgroups:

$$F(2,2n) \;\triangleleft\; \mathrm{T}_n \;\triangleleft\; \Omega_n, \tag{5.74}$$

where $\mid \Omega_n : \mathrm{T}_n \mid = n$ and $\mid \mathrm{T}_n : F(2,2n) \mid = 2$.

The group T_n is a subgroup of Ω_n. Hence it acts by isometries on the universal covering X_n and uniformize an orbifold $X_n \,/\, \mathrm{T}_n$. From (5.74) we get the following diagram of covers for orbifolds:

$$M_n = X_n \,/\, F(2,2n) \stackrel{2}{\longrightarrow} X_n \,/\, \mathrm{T}_n \stackrel{n}{\longrightarrow} 6_2^2(2,n) = X_n \,/\, \Omega_n. \tag{5.75}$$

Our next step is to describe the orbifold $X_n \,/\, \mathrm{T}_n$. First of all we will prove that the cover

$$p \;:\; X_n \,/\, \mathrm{T}_n \stackrel{n}{\longrightarrow} 6_2^2(2,n) = X_n \,/\, \Omega_n \tag{5.76}$$

is cyclic. We will use the following elementary lemma for it.

Lemma 5.4 *Let G, K, L be groups and $\theta : G \longrightarrow K \oplus L$ be an epimorphism. If $H = \theta^{-1}(K)$ then $H \triangleleft G$ and $G \,/\, H \cong L$.*

We will apply this lemma to the epimorphism $\theta : \Omega_n \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2$ which was defined by (5.70). Since $\mathrm{T}_n = \theta^{-1}(\mathbb{Z}_2)$, then $\mathrm{T}_n \triangleleft \Omega_n$ and $\Omega_n \,/\, \mathrm{T}_n \cong \mathbb{Z}_n$. It means that p is the regular n-fold cyclic cover. Moreover by (5.73) the cover p is branched over the component with the branch index n of the singular set of the orbifold $6_2^2(2,n)$.

It is known [71], that there is an involution in the symmetry group of 6_2^2 which changes two components (it is evident from Figure 5.8). Therefore the singular set of the orbifold $6_2^2(2,n)$ is equivalent to the link diagram in Figure 5.13.

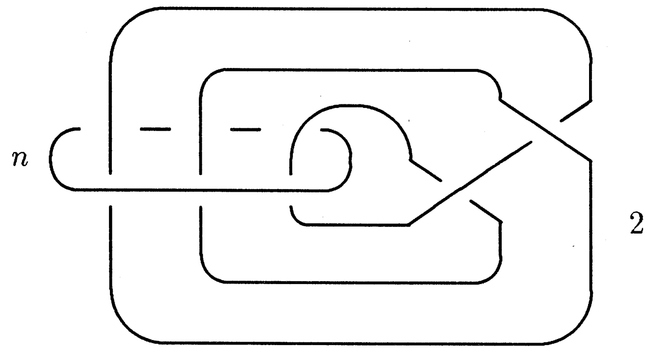

Figure 5.13.

In Figure 5.13 one can see that the component with the index n is unknotted. Therefore p is a standard cyclic cover of the 3-sphere which is the underlying space of the orbifold $6_2^2(2, n)$, branched over an unknotted circle. Hence the underlying space of the orbifold $X_n \,/\, \mathrm{T}_n$ is the 3-sphere too. Moreover the component with the index 2 of the singular set of $6_2^2(2, n)$ is the closed 3-string braid $\sigma_1\sigma_2^{-1}$. Therefore this component will lift to the closed 3-string braid $(\sigma_1\sigma_2^{-1})^n$ on the n-fold cyclic covering $X_n \,/\, \mathrm{T}_n$. It is the link Th_n according to our notation.

Summarizing we see that the orbifold $X_n \,/\, \mathrm{T}_n$ has the 3-sphere as its underlying space and the link Th_n with the branch index 2 as its singular set. For this orbifold we will be use notation $Th_n(2) = X_n \,/\, \mathrm{T}_n$.

Comparing diagrams (5.61) and (5.75) we conclude that the following diagram for covers holds (see Figure 5.14):

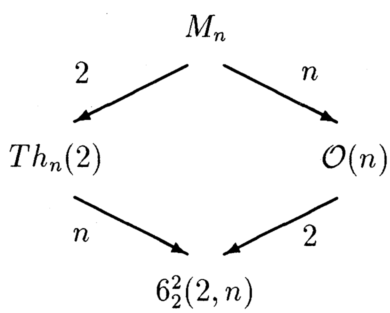

Figure 5.14.

From this diagram we see that the Fibonacci manifold M_n is the 2-fold covering of the orbifold $Th_n(2)$, and the theorem is proved. □

We recall that according to [13] a *π-orbifold* is an orbifold with the 3-sphere as its underlying space and a link with branch indices equal to 2 as its singular set.

Since for $n \geq 4$ the Fibonacci manifold M_n is hyperbolic, from Theorem 5.6 and Corollary 5.1 we get the following

Corollary 5.10 *For $n \geq 4$ the π-orbifold $Th_n(2)$ is hyperbolic and its volume is equal to*

$$vol\,(Th_n(2)) = n\,(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)),$$

where $\delta = \frac{\pi}{n}$, $\beta = \frac{1}{2}\arccos(\cos(2\delta) - \frac{1}{2})$.

5.7 Heegaard genus of Fibonacci manifolds

Following [96] we recall some well-known facts from the theory of 3-manifolds.

Let M^3 be a closed orientable 3-manifold. A pair $(H_g, H_g^{'})$ of handlebodies of genus g is called a *Heegaard splitting of genus g* of M^3 if $M^3 = H_g \cup H_g^{'}$ and $H_g \cap H_g^{'} = \partial H_g = \partial H_g^{'}$ is a closed orientable surface of genus g. The minimal genus among the genera of all Heegaard splittings of M^3 is called the *Heegaard genus* of M^3 and is denoted by $h(M^3)$. The three-dimensional sphere S^3 is the alone orientable manifold which Heegaard genus is equal to 0. The Heegaard genus is equal to 1 only for lens spaces and for manifold $S^2 \times S^1$. In particular, if a manifold M^3 admits Euclidean or hyperbolic structure, then $h(M^3) \geq 2$.

The minimal number of elements needed to generate the fundamental group $\pi_1(M^3)$ of a closed 3-manifold M^3 is called the *rank* of $\pi_1(M^3)$. For a 3-manifold M^3 we denote the rank of $\pi_1(M^3)$ by $r(M^3)$. The following inequality is valued : $r(M^3) \leq h(M^3)$ in obvious way [30]. In particular the Poincaré conjecture can be formulated in the following way : $r(M^3) = 0$ if and only if $h(M^3) = 0$. M. Boileau and H. Zieschang [12] have constructed a Seifert fiber space M^3 with the strictly inequality :

$$2 = r(M^3) < h(M^3) = 3.$$

It is obvious that the fundamental groups of a hyperbolic or Euclidean Fibonacci manifold is two-generated. We will show that for these manifolds Heegaard genus is equal two.

Proposition 5.2 ([83]) *For any $n \geq 3$ the Heegaard genus $h(M_n)$ of the Fibonacci manifold M_n is two.*

Proof. By Theorem 5.6 for any $n \geq 2$ the Fibonacci manifold M_n is the two-fold covering of S^3 branched over the Turk's head link Th_n. The link Th_n is a closed 3-string braid $(\sigma_1\sigma_2^{-1})^n$ and therefore has a 3-bridge presentation. Hence by Viro's theorem [88] (see also Section 6.3), $h(M_n) \leq 2$. But for $n \geq 4$ in virtue of Theorem 5.1 the manifold M_n is hyperbolic. The manifold M_3 coincides with the Hantzche–Wendt manifold [100] and admits an Euclidean structure. Therefore $h(M_3) = 2$ for $n \geq 3$. □

We remark that the same statement was proven in [16] starting from a spine representation of manifolds M_n.

Because the Fibonacci manifold M_2 is the two-fold covering of S^3 branched over the figure-eight knot and coincides with the lens space $L(5,2)$, we get $h(M_2) = 1$.

Let us make some remarks on above result.

First remark is connected with the classical $84(g-1)$–Hurwitz theorem. By this theorem the automorphism group of a compact Riemann surface of genus $g > 1$ is finite and bounded above by $84(g-1)$.

By Mostow rigidity theorem the isometry group of the compact hyperbolic 3-manifold is finite always. Moreover *a priori* it may be isomorphic to an arbitrary finite group [43].

In analogy to Hurwitz theorem for 2-dimensional case one can try to estimate the order of the isometry group of the hyperbolic 3-manifold in terms of Heegaard genus. But the following proposition show that it is impossible.

Proposition 5.3 *There are hyperbolic 3-manifolds of Heegaard genus 2 with an arbitrary large group of isometries.*

Proof. The Fibonacci hyperbolic manifold M_n, $n \geq 4$, has an orientation-preserving isometry of order n. This isometry is induced by the automorphism $s_i \to s_{i+2}$ with indices by mod $2n$, if we consider the standard representation for the fundamental group $\pi_1(M_n) = F(2,2n)$ (see section 5.1). As it was shown in Section 5.2, the quotient space of M_n by this isometry is the orbifold $\mathcal{O}(n)$ with the underlying space S^3 and the figure-eight knot as the singular set. Therefore by one hand the isometry group contains the cyclic group of the order n and so can be arbitrary large as $n \to \infty$. By the other hand, $h(M_n) = 2$ for $n \geq 4$. □

The second remark devoted to the relationship between Heegaard genus and volume of a hyperbolic 3-manifold.

Proposition 5.4 *There are hyperbolic 3-manifolds of Heegaard genus 2 with an arbitrary large volume.*

Proof. We recall, that according to Corollary 5.1, the volume of the hyperbolic Fibonacci manifold M_n is given by the formula

$$vol(M_n) = 2n\ (\Lambda(\beta + \delta) + \Lambda(\beta - \delta)),$$

where $\delta = \frac{\pi}{n}$ and $\beta = \frac{1}{2}\arccos(\cos(2\delta) - \frac{1}{2})$. Because the Lobachevsky function is continuous (see Proposition 2.2), we will get

$$\Lambda(\beta + \delta) + \Lambda(\beta - \delta) \longrightarrow 2\Lambda\left(\frac{\pi}{6}\right), \qquad n \to \infty.$$

We recall that $V_3^{max} = 2\Lambda(\frac{\pi}{6}) = 1.014\ldots$ is the volume of the ideal regular tetrahedron that is maximal volume of a simplex in $\mathbb{H}^3$. Therefore we have:

$$vol(M_n) \sim 2n\,V_3^{max}, \qquad n \to \infty.$$

Thus the volume $vol(M_n)$ of the hyperbolic Fibonacci manifold M_n is an arbitrary large as $n \to \infty$. By the other hand, in virtue of Proposition 5.1 the Heegaard genus $h(M_n)$ is equal two for $n \geq 4$. □

The third remark is connected with the genus of an invariant surface in the manifold M_n.

Proposition 5.5 *If $n > 10$ then each Heegaard surface of genus 2 in the Fibonacci manifold M_n is not invariant under the isometry group.*

Proof. Let (H_2, H_2') be a genus two Heegaard decomposition of a Fibonacci manifold M_n, $n > 10$. Denote by $S = H_2 \cap H_2' = \partial H_2 = \partial H_2'$ the corresponding Heegaard surface. Suppose that S is invariant under the isometry group of M_n. As in the proof of Proposition 5.2, we recall that for arbitrary n the manifold M_n has the orientation-preserving isometry $f : M_n \to M_n$ of order n, induced by the automorphism $s_i \to s_{i+2}$ of the fundamental group $F(2, 2n)$ of the manifold M_n. The restriction $f|_S : S \to S$ gives a topological automorphism of the surface S. From [41] there exists a suitable conformal structure on S which is invariant under $f|_S$. By Wiman theorem [93] the order of an automorphism of a Riemann surface S of genus g is bounded above by $4 \cdot g + 2$, that is equal ten in our case. Hence $n \leq 10$, that gives a contradiction. □

We recall, that in [99] B.Zimmermann defined an equivariant Heegaard genus for 3-manifolds. Let M be a closed orientable 3-manifold, and G be a finite group of its orientation-preserving homeomorphisms. As the *equivariant Heegaard genus* $g(M, G)$ of such G-action we define the minimal genus $g > 1$ of a Heegaard decomposition $M = H_g \cup H_g'$ of M invariant under G, i.e. G maps both handlebodies H_g and H_g' of the decomposition to itself.

Proposition 5.6 *There is a group G_n of order $4n$ consisting of orientation-preserving homeomorphisms of a Fibonacci manifold M_n, $n \geq 3$, such that the equivariant Heegaard genus of M_n in respect to G_n is $g(M_n, G_n) = n-1$.*

Proof. We recall, that the fundamental group of a Fibonacci manifold M_n is the Fibonacci group

$$F(2,2n) = \langle s_1, \ldots, s_{2n} \mid s_i \, s_{i+1} = s_{i+2}, \qquad i \bmod 2n \rangle . \tag{5.77}$$

Let as well as above, ρ be such symmetry of order n, that

$$\rho^{-1} \, s_i \, \rho = s_{i+2},$$

for $i = 1, \ldots, 2n$, where indices are by module $2n$. Hence ρ induces an automorphism of the group $F(2,2n)$ and we consider a group

$$\Gamma_n = \langle F(2,2n), \rho \rangle = F(2,2n) \leftthreetimes \langle \rho \rangle ,$$

which can be decomposed in a semi-direct product as an extension of a group by an automorphism. Here $\langle \rho \rangle$ is a cyclic group of order n generated by ρ. The presentation of the group Γ_n was given in (5.17) and (5.18) :

$$\Gamma_n = \Big\langle \rho, b \mid \rho^n = b^n = 1, \quad \rho^{-1} \, [b, \rho] = [b, \rho] \, b \Big\rangle , \tag{5.78}$$

where ρ and b are connected with generators $s_1, \ldots, s_{2n}$ from (5.77) by following equalities:

$$s_2 = b \, \rho, \tag{5.79}$$

$$s_1 = \rho \, s_2^{-1} \, \rho^{-1} s_2 = b^{-1} \, \rho^{-1} \, b \, \rho = [b, \rho] , \tag{5.80}$$

and

$$\rho^{-1} \, s_i \, \rho = s_{i+2} \tag{5.81}$$

for $i = 1, \ldots, 2n$. Hence

$$s_{2i+2} = \rho^{-i} \, (b \, \rho) \, \rho^i \tag{5.82}$$

and

$$s_{2i+1} = \rho^{-i} \, [b \, \rho] \, \rho^i \tag{5.83}$$

for $i = 0, \ldots, n-1$. In this case ρ and b correspond to loops in the group of the figure-eight knot pictured in Figure 5.15.

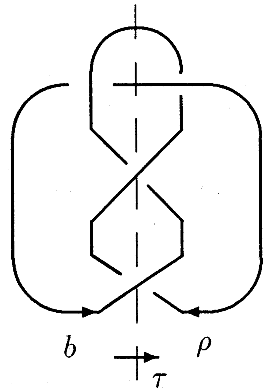

Figure 5.15.

So, the group Γ_n is the group of the orbifold $\mathcal{O}(n)$ whose underlying space is the 3-sphere S^3 and singular set is the figure-eight knot with branch index n. Let us consider an involution τ of S^3 whose axe is correspond to the dotted line in Figure 5.15 and intersects the figure-eight knot in two points. It is obvious, that τ is an involution of the orbifold $\mathcal{O}(n)$ and induces an automorphism of the group Γ by the equalities:

$$\tau^{-1}\,\rho\,\tau \;=\; b, \qquad \tau^{-1}\,b\,\tau \;=\; \rho. \tag{5.84}$$

Lemma 5.5 *The involution τ induces an automorphism of the group $F(2,2n)$.*

Proof. According to formulae (5.82) and (5.83), from (5.84) we get:

$$\tau^{-1}\,s_{2i+2}\,\tau \;=\; b^{-i}\,(\rho b)\,b^{i} \tag{5.85}$$

and

$$\tau^{-1}\,s_{2i+1}\,\tau \;=\; b^{-i}\,[\rho b]\,b^{i}. \tag{5.86}$$

We remark, that by (5.79) and (5.80):

$$\rho\,b \;=\; (b\rho)\,\rho^{-1}\,b^{-1}\,\rho\,b \;=\; s_2\,s_1^{-1}$$

and

$$[\rho, b] \;=\; [b,\rho]^{-1} \;=\; s_1^{-1}.$$

Because the group $F(2,2n)$ is a normal subgroup in the group Γ_n, the right sides of (5.85) and (5.86) also are elements from $F(2,2n)$. Moreover

$$\left(\tau^{-1}\,s_i\,\tau\right)\,\left(\tau^{-1}\,s_{i+1}\,\tau\right) \;=\; \tau^{-1}\,s_{i+2}\,\tau$$

for all $i = 1, \ldots, 2n$. Whence τ induces an automorphism of the group $F(2,2n)$ defined by (5.85) and (5.86). □

Therefore a group $\langle \rho, \tau \rangle$ generated by the symmetry ρ of order n and by the involution τ is a group of automorphisms of the group $F(2,2n)$. Let us consider a group

$$\Delta_n = \langle F(2,2n), \rho, \tau \rangle = F(2,2n) \leftthreetimes \langle \rho, \tau \rangle ,$$

which can be decomposed on the semi-direct product. In this case

$$\Delta_n = \Big\langle \rho, b, \tau \mid \rho^n = b^n = \tau^2 = 1, \quad \rho^{-1}[b,\rho] = [b,\rho]\, b, \quad \tau^{-1}\rho\tau = b \Big\rangle . \tag{5.87}$$

The quotient space $\mathcal{O}(n)/\tau$ of the orbifold $\mathcal{O}(n)$ by the involution τ is an orbifold whose underlying space is the 3-sphere S^3 and whose singular set is a spatial graph, pictured in Figure 5.16, which can be described as the torus knot 5_1 with a bridge AB. Points A and B are images of the intersection points of the singular set of $\mathcal{O}(n)$ with the axe of the involution τ. Two arcs of this graph, which are images of the axe of τ, have branch index 2, and the third, which is the image of the singular set of $\mathcal{O}(n)$, has the branch index n.

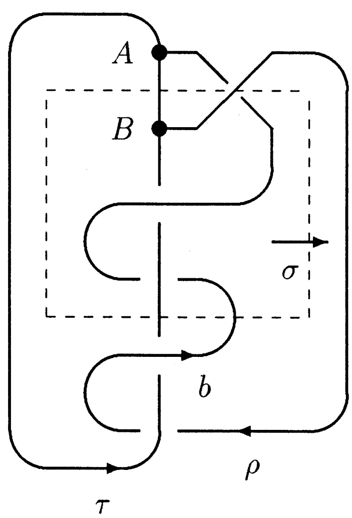

Figure 5.16.

The group Δ_n is the group of the orbifold $\mathcal{O}(n)/\tau$, whose generators ρ, b and τ are pictured in Figure 5.16. Indeed, the relation

$$\rho^{-1}[b,\rho] = [b,\rho]\, b$$

is a consequence of the fact that the loop around the bridge AB is the element of the order two in the group of the orbifold, and others relations from (5.87) hold by Wirtinger algorithm.

From Figure 5.16 we see, that the orbifold $\mathcal{O}(n)/\tau$ has an involution σ which acts on elements ρ and b by following rules:

$$\sigma^{-1}\, b\, \sigma \;=\; b^{-1} \tag{5.88}$$

and

$$\sigma^{-1}\, \rho\, \sigma \;=\; b\, \rho^{-1}\, b^{-1}. \tag{5.89}$$

Lemma 5.6 *The involution σ induces an automorphism of the group $F(2,2n)$.*

Proof. According to formulae (5.82) and (5.83), from (5.88) and (5.89) we get:

$$\begin{aligned}\sigma^{-1}\, s_{2i+2}\, \sigma \;&=\; \left(b\,\rho^{-1}\,b^{-1}\right)^{-i}\left(b^{-1}\,b\,\rho^{-1}\,b^{-1}\right)\left(b\,\rho^{-1}\,b^{-1}\right)^{i}\\ &=\; \left(b\,\rho^{-1}\,b^{-1}\right)^{-i}\, s_2^{-1}\,\left(b\,\rho^{-1}\,b^{-1}\right)^{i},\end{aligned} \tag{5.90}$$

and

$$\begin{aligned}\sigma^{-1}\, s_{2i+1}\, \sigma \;&=\; \left(b\,\rho^{-1}\,b^{-1}\right)^{-i}\left(b\,b\,\rho\,b^{-1}\,b^{-1}\,b\,\rho^{-1}\,b^{-1}\right)\left(b\,\rho^{-1}\,b^{-1}\right)^{i}\\ &=\; \left(b\,\rho^{-1}\,b^{-1}\right)^{-i}\, b\,(b\,\rho)\left(b^{-1}\,\rho^{-1}\,b\,\rho\right)(b\,\rho)^{-1}\,b^{-1}\left(b\,\rho^{-1}\,b^{-1}\right)^{i}\\ &=\; \left[b^{-1}\left(b\,\rho^{-1}\,b^{-1}\right)^{i}\right]^{-1} s_2\, s_1\, s_2^{-1}\left[b^{-1}\left(b\,\rho^{-1}\,b^{-1}\right)^{i}\right]\end{aligned} \tag{5.91}$$

Because the group $F(2,2n)$ is a normal subgroup in the group Γ_n, the right sides of (5.90) and (5.91) are elements of the group $F(2,2n)$. Moreover

$$\left(\sigma^{-1}\, s_i \sigma\right)\left(\sigma^{-1}\, s_{i+1}\sigma\right) \;=\; \sigma^{-1}\, s_{i+2}\sigma$$

for all $i = 1, \ldots, 2n$. Whence σ induces an automorphism of the group $F(2,2n)$ defined by (5.90) and (5.91). □

Therefore the group $\langle \rho, \tau, \sigma\rangle$ generated by the symmetry ρ of order n and by involutions τ and σ is a group of automorphisms of the group $F(2,2n)$. Let us consider a group

$$\Pi_n \;=\; \langle F(2,2n), \rho, \tau, \sigma\rangle \;=\; F(2,2n)\,\leftthreetimes\,\langle \rho, \tau\,\sigma\rangle \tag{5.92}$$

From (5.87), (5.88) and (5.89) we get the following representation of the group

$$\Pi_n = \Big\langle \rho, b, \tau, \sigma \mid \rho^n = b^n = \tau^2 = \sigma^2 = 1,\ \rho^{-1}\,[b,\rho] = [b,\rho]\,b,$$
$$\tau^{-1}\rho\tau = b,\ \sigma^{-1}b\sigma = b^{-1},\ \sigma^{-1}\rho\sigma = b\rho^{-1}b^{-1}\Big\rangle. \qquad (5.93)$$

The quotient space of the orbifold $\mathcal{O}(n)/\tau$ with the singular set in Figure 5.16 by the involution σ is the orbifold $(\mathcal{O}(n)/\tau)/\sigma$ whose underlying space is the 3-sphere and whose singular set is pictured in Figure 5.17.

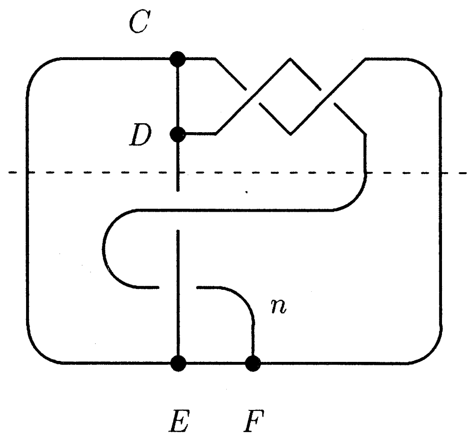

Figure 5.17.

Points C, E and F are images of the intersection of the singular set of the orbifold $\mathcal{O}(n)/\tau$ with the axe of involution σ, and the point D is the image of the point B from Figure 5.16. The singular set is a spatial graph with four vertices and six edges and combinatorial isomorphic to the 1-skeleton of a tetrahedron. So it is a *spatial tetrahedron.* By the other hand, if we delete edges CD and EF, we will get the figure-eight knot. So we can regard this singular set as a figure-eight knot with two bridges.

Next we will use results of D. McCullough, A. Miller and B. Zimmermann from [48], where the theory of group actions on handlebodies was developed.

Let us consider a decomposition of the orbifold $(\mathcal{O}(n)/\tau)\,\sigma$ according to the dotted line in Figure 5.17 :

$$(\mathcal{O}(n)/\tau)/\sigma = V_1 \cup V_2,$$

where orbifolds V_1 and V_2 have the 3-ball as underlying space and their singular sets contain points C, D and E, F, respectively. Let us consider preimages

$$\tilde{V_i} = G_n^{-1}(V_i),$$

where $G_n = \langle \rho, \tau, \sigma \rangle$, of the orbifolds V_i, $i = 1, 2$, in the Fibonacci manifold M_n. Therefore we get the decomposition $M_n = \tilde{V}_1 \cup \tilde{V}_2$, and G is a group of orientation-preserving homeomorphisms of $\tilde{V}_1$ and $\tilde{V}_2$.

By the construction, the stabilizers of points C and E are dihedral groups $\mathbb{D}_2$ of order 4, and the stabilizers of points D and F are dihedral groups $\mathbb{D}_n$ of order $2n$. According to [48], to each of orbifolds V_i, $i = 1, 2$, which is a handlebody orbifold, we correspond a "handlebody-orbifold" graph $\Gamma(G_{v_1}, n, G_{v_2})$ in Figure 5.18.

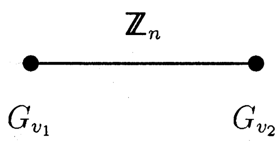

Figure 5.18.

Where G_{v_1} and G_{v_2} are stabilizers of vertices of the singular set of V_i and $\mathbb{Z}_n$ is the stabilizer of points of the common edge. In our case $n = 2$, $G_{v_1} = \mathbb{D}_2$ and $G_{v_2} = \mathbb{D}_n$. We recall [48], that if $\Gamma(G)$ is a graph corresponding to a finite group, its *Euler characteristic* is defined to be

$$\chi(\Gamma(G)) = \sum_{v \in V(\Gamma)} \frac{1}{|G_v|} - \sum_{e \in E(\Gamma)} \frac{1}{|G_e|}, \tag{5.94}$$

where G_v and G_e are groups corresponding to vertices and edgers of the graph $\Gamma(G)$. From (5.94) for the graph $\Gamma(\mathbb{D}_2, 2, \mathbb{D}_n)$ corresponding to the group of handlebody orbifold V_i, $i = 1, 2$, we get:

$$\chi(\Gamma(\mathbb{D}_2, 2, \mathbb{D}_n)) = \frac{1}{4} + \frac{1}{2n} - \frac{1}{2} = \frac{2 - n}{4n}. \tag{5.95}$$

According to the main theorem from [48], the Euler characteristic $\chi(\Gamma)$ of the graph corresponding to the handlebody orbifold $V_i = \tilde{V}_i / G_n$, the order of the group G_n and the genus $g > 1$ of the handlebody $\tilde{V}_i$ are satisfy to the equality

$$(1 - g) = |G_n| \chi(\Gamma). \tag{5.96}$$

Because $|G_n| = 4n$ and $\chi(\Gamma) = (2 - n)/4n$ from (5.95), we get that the genus of $\tilde{V}_i$ is equal $g = n - 1$. Therefore a Fibonacci manifold M_n, $n \geq 3$, has a Heegaard decomposition $M_n = \tilde{V}_1 \cup \tilde{V}_2$ of genus $(n - 1)$ such that both handlebodies $\tilde{V}_1$ and $\tilde{V}_2$ are invariant by the group $G_n = \langle \rho, \tau, \sigma \rangle$ action. Hence equivariant Heegaard genus $g(M_n, G_n) = n - 1$, and the proof of the Proposition is complete.□

5.8 Fibonacci manifolds obtained by Dehn surgery

In this section we will give also one description of Fibonacci manifolds M_n. We recall the following fundamental Lickorish's theorem.

Theorem 5.7 **([44])** *Every closed, orientable, connected 3-manifold may be obtained by surgery on a link in* S^3. *Moreover, one may always find such a surgery presentation in which the surgery coefficients are all* ± 1 *and the individual components of the link are unknotted.*

Proof. See [44] or [70, §9I]. □

For find such representation for Fibonacci manifolds, we use the approach of J.Montesinos [59]. We recall, that in virtue of Theorem 5.6, the Fibonacci manifold M_n, $n \geq 2$, can be obtained as the two-fold covering of the 3-sphere S^3 branched over the Turk's head link Th_n, that is the closure of the 3-strings braid $(\sigma_1 \sigma_2^{-1})^n$. Let us consider the projection of the Turk's head knot $Th_4 = 8_{18}$, pictured in Figure 5.19.

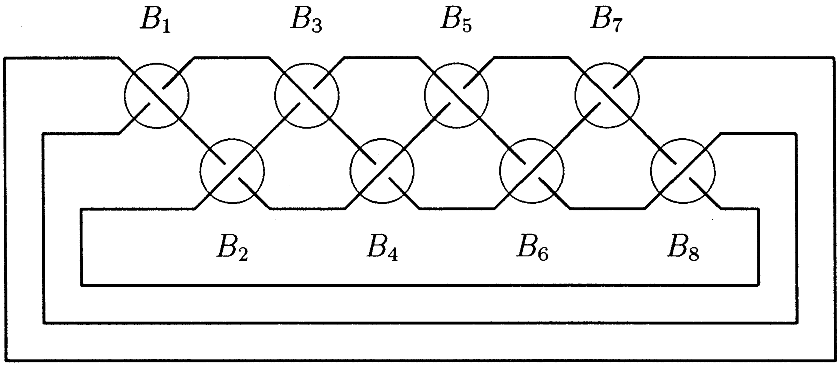

Figure 5.19.

Let us consider neighborhoods $B_1, \ldots, B_8$ of double-points of the diagram of Th_4, which are 3-balls with two arcs inside. As we see from Figure 5.19, each B_i is a $(+1)$-tangle if i is odd, and an (-1)-tangle if i is even.

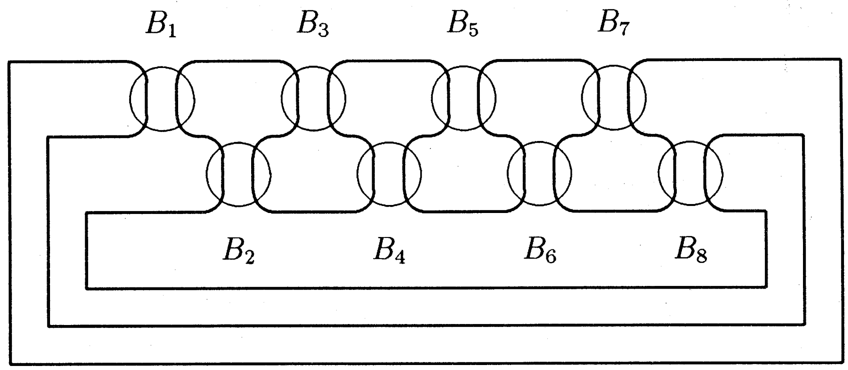

Figure 5.20.

If we replace each tangle B_i on a trivial tangle, we will get an unknotted closed curve $\mathcal{C}$ in S^3 (see Figure 5.20).

Using the approach from [59], we will redraw the curve $\mathcal{C}$ as a horizontal line. Because the two-fold coverings of B_i branched over $B_i \cap Th_4$ or $B_i \cap \mathcal{C}$ are solid tori, we see, similar to [59], that the two-fold covering of S^3 branched over Th_4 can be obtained by surgeries with parameters ± 1 on the links $L_1, \ldots, L_8$ in Figure 5.21.

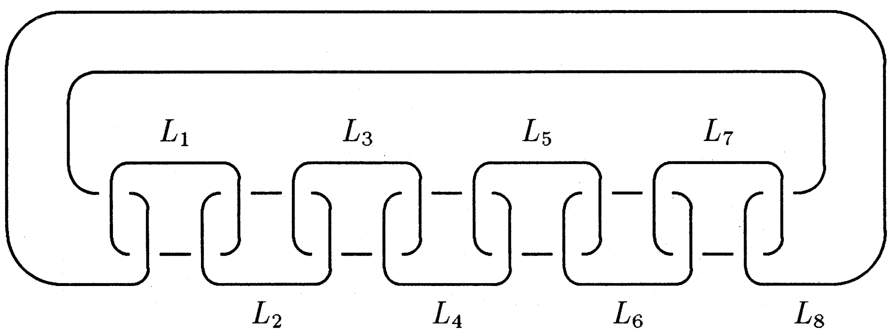

Figure 5.21.

Therefore the Fibonacci manifold M_4 can be obtained by Dehn surgery on the chain of circles $L_1 \cup \ldots \cup L_8$, by doing $(+1)$ surgery on circles with odd numbers, and by doing (-1) surgery on circles with even numbers. By the same arguments, for arbitrary n we will get

Proposition 5.7 *A Fibonacci manifold M_n, $n \geq 2$, can be obtained by Dehn surgery on the chain of circles $L_1 \cup \ldots \cup L_{2n}$, by doing $(+1)$ surgery on circles with odd numbers, and by doing (-1) surgery on circles with even numbers.*

By other considerations, this statement was proved by A. Cavicchioli and F. Spaggiari [16] using results of M. Takahashi [73]. Let us consider twists about unknotted components for circles with odd numbers. Then, as it was remarked in [16], according to the Kirby–Rolfsen calculus on framed links (see [70, Chapter 9]), we will get the alternating link which is a chain of four unknotted circles with all surgery coefficients equal to (-3) (see Figure 5.22).

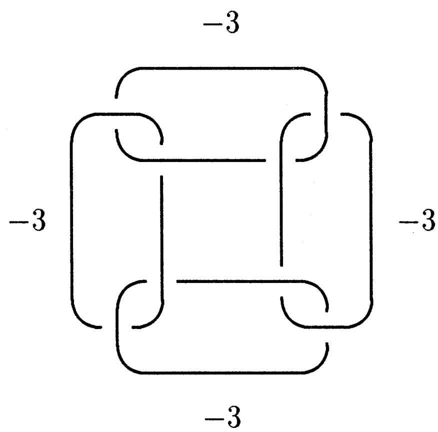

Figure 5.22.

By the same arguments one can get the same result for an arbitrary n. Let us denote the alternating link consisting of n linked unknotted circles similar to Figure 5.18 by $\mathcal{L}_n$. Then from above considerations we have

Proposition 5.8 *A Fibonacci manifold M_n, $n \geq 2$, can be obtained by (-3) surgeries on components of the link $\mathcal{L}_n$.*

We remark, that for $n = 2$ this property was discussed in [70, p.299]. æ

Chapter 6

Manifolds of small volume

We recall, that it follows from the Thurston–Jørgensen theorem that hyperbolic 3-manifolds can be ordered by their volumes, and there exists the smallest manifold. In [47] S. V. Matveev and A. T. Fomenko firstly conjectured the structure of the initial part of the set of volumes. The conjecture was based on numerous calculations of volumes, using computer programs. In [36] C. Hodgson and J. Weeks refined the ten smallest known manifolds and their volumes, using famous computer program SnapPea [91].

The smallest known manifold $\mathcal{M}_1$ in the list from [36], whose volume is equal to $0.94\ldots$, was constructed by J. Weeks [90] and by S. V. Matveev and A. T. Fomenko [47]. The manifold $\mathcal{M}_1$ can be described in the form $\mathcal{M}_1 = W(5,-2;5,-1)$, where $W(m,n;p,q)$ denotes the manifold obtained by (m,n) and (p,q) Dehn surgeries on components of the Whitehead link W. We remark, that the isometries of $\mathcal{M}_1$ were investigated by E. Molnar [58].

The second manifold $\mathcal{M}_2$, whose volume is equal to $0.98\ldots$, was constructed by W. Thurston [77] using $(5,-1)$ Dehn surgery on the figure-eight knot. We recall, that one can get the figure-eight knot by $(1,1)$ Dehn surgery on one component of W. So we can write $\mathcal{M}_2 = W(1,1;5,-1)$.

The third manifold $\mathcal{M}_3 = W(3,-2;6,-1)$ was described by R. Meyerhoff and W. Neumann [56]. It was proven in [80], that the volume of $\mathcal{M}_3$ is exactly equal to the volume of the regular ideal tetrahedron in the Lobachevsky space and it is equal to $1.01\ldots$.

The ten smallest manifolds $\mathcal{M}_1,\ldots,\mathcal{M}_{10}$ were obtained in [36] by using Dehn surgeries on different links in S^3 what was stipulared by the investigation of their length spectra. We remark, that all these manifolds can be obtained by surgeries on components of the Whitehead link and corresponding parameters are given in Table 6.1.

In this chapter we consider a family of compact 3-manifolds $W(m,n;p,q)$ which can be obtained by (m,n) and (p,q) Dehn surgeries on components of the Whitehead link W. According to the Montesinos theorem [59], any closed 3-manifold obtained by Dehn surgeries on a strongly invertible link can be presented as a 2-fold covering of the 3-sphere, branched over some link. In section 6.1 we apply the Montesinos algorithm for describing manifolds $W(m,n;p,q)$ as 2-fold branched coverings and for finding corresponding branch links (see Theorem 6.2). In section 6.2 we describe branch sets for the ten smallest hyperbolic 3-manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{10}$ (see Table 6.1). In section 6.3 we discuss the Heegaard genus of above manifolds. More exactly, using results of section 6.2, by the Viro theorem [88], we get that the Heegaard genus of above hyperbolic manifolds equals two (see Proposition 6.2). In section 6.4 we apply the criterion of B. Zimmermann [101] to the smallest known Weeks-Matveev-Fomenko manifold $\mathcal{M}_1$ to show that this manifold is maximally symmetric. In section 6.5 we will discuss the Meyerhoff–Neumann manifold whose volume corresponds to the third value.

6.1 Dehn surgeries on the Whitehead link

This section is devoted to describing of compact 3-manifolds $W(m,n;p,q)$ obtained by Dehn surgeries on the Whitehead link W (see Figure 6.1) as two-folds branched coverings of the 3-sphere. We remark, that non-compact 3-manifolds obtained by Dehn surgery on one component of W and their invariants were investigated by C. Hodgson, R. Meyerhoff and J. Weeks [33], the consistency relations and equations for surgery parameters were obtained by W. Nuemann and A. Reid [61].

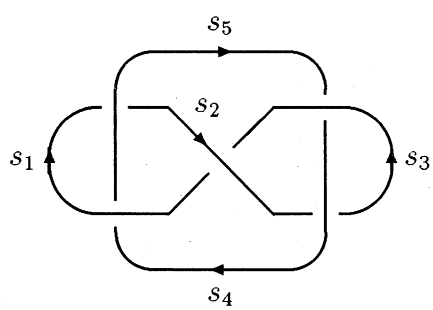

Figure 6.1. The Whitehead link.

We recall [60], that a link $\mathcal{L} \subset S^3$ is called *strongly invertible* if there is an orientation-preserving involution of S^3 which induces on each component of $\mathcal{L}$ an involution with two fixed points. Above mentioned involution is called a *strongly invertible involution* of the link. The following theorem of J. Montesinos [59] gives connection between two approaches for description of a manifold.

Theorem 6.1 ([59]) *Let M be a closed orientable 3-manifold that is obtained by doing surgeries on a strongly-invertible link L of n components. Then M is a 2-fold covering of S^3 branched over a link of at most $n+1$ components. Conversely, every 2-fold cyclic branched covering of S^3 can be obtained in this fashion.*

The proof of the theorem given in [59] is constructive and in particular it gives an algorithm for describing of the branch set of above 2-fold covering (see also examples of using this algorithm in [60]).

It is well-known that the Whitehead link W is strongly invertible, and a strongly invertible involution ρ of W is shown in Figure 6.2.

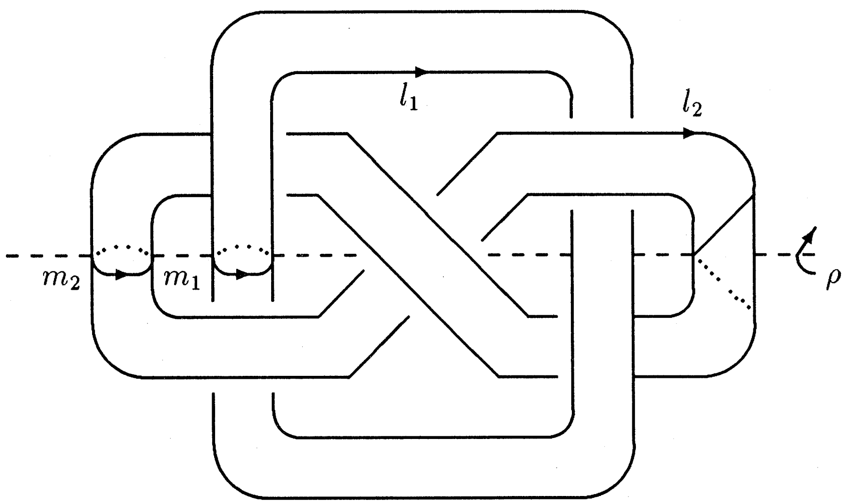

Figure 6.2.

Therefore we can apply the Montesinos algorithm to the Whitehead link W. Let us denote by $W(m,n;p,q)$ the compact 3-manifold which can be obtained by $\frac{m}{n}$ and $\frac{p}{q}$ Dehn surgeries on components of W. By Theorem 6.1 the manifold $W(m,n;p,q)$ is a 2-fold covering of S^3 branched over a link with at most three components. It follows from [59] that this covering is uniquely determined by the choose of a strongly invertible involution. Let us describe the branch set $\mathcal{L}(m,n;p,q)$ of the 2-fold covering $W(m,n;p,q)$ of S^3 corresponding to the involution ρ pictured in Figure 6.2.

Let $s_1, \ldots, s_5$ be Wirtinger generators of the fundamental group $\pi_1(S^3 \setminus W)$ as in Figure 6.1. We choose meridians m_1, m_2 and longitudes l_1, l_2 according to Figure 6.2 and such that longitudes represent elements of the second commutator group of $\pi_1(S^3 \setminus W)$ (see [15, p.37]). So we have :

$$\begin{aligned} m_1 &= s_5, & l_1 &= s_3 s_1^{-1}, \\ m_2 &= s_1, & l_2 &= s_5^{-1} s_4 s_2^{-1} s_1. \end{aligned}$$

Let V be a regular tubular neighborhood of the link W in S^3. Without loss of generality one can choose V, meridians m_1, m_2 and longitudes l_1, l_2 on the boundary of V to be invariant under the involution ρ. The quotient space of S^3 under ρ action is shown in Figure 6.3.

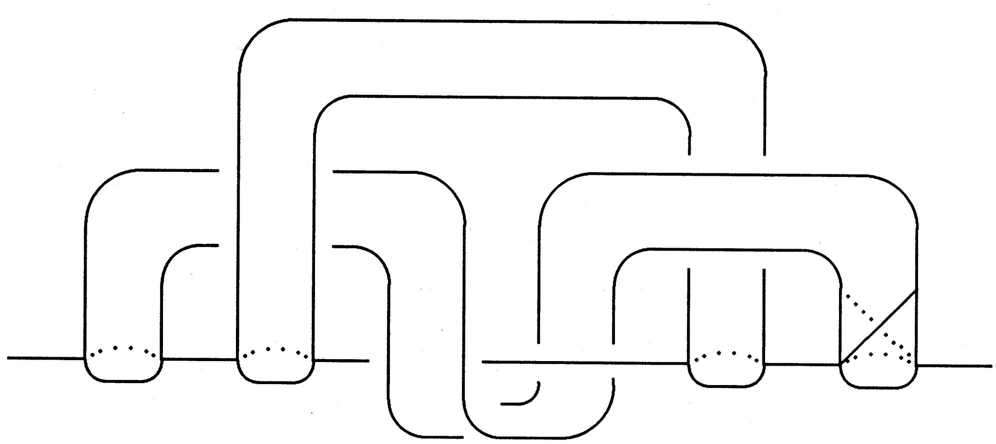

Figure 6.3. The quotient $(S^3 \setminus V)/\rho$.

The image of the tubular neighborhood V under the canonical projection $p : S^3 \to S^3/\rho$ consists of two 3-balls B_1 and B_2. Denote by $Fix(\rho)$ the axis of the involution ρ in S^3.. For each ball B_i the intersection $B_i \cap p(Fix(\rho))$ consists of two arcs. By the isotopy of B_i along the image $p(l_i)$ of the longitude l_i $(i = 1, 2)$ we get the following Figure 6.4:

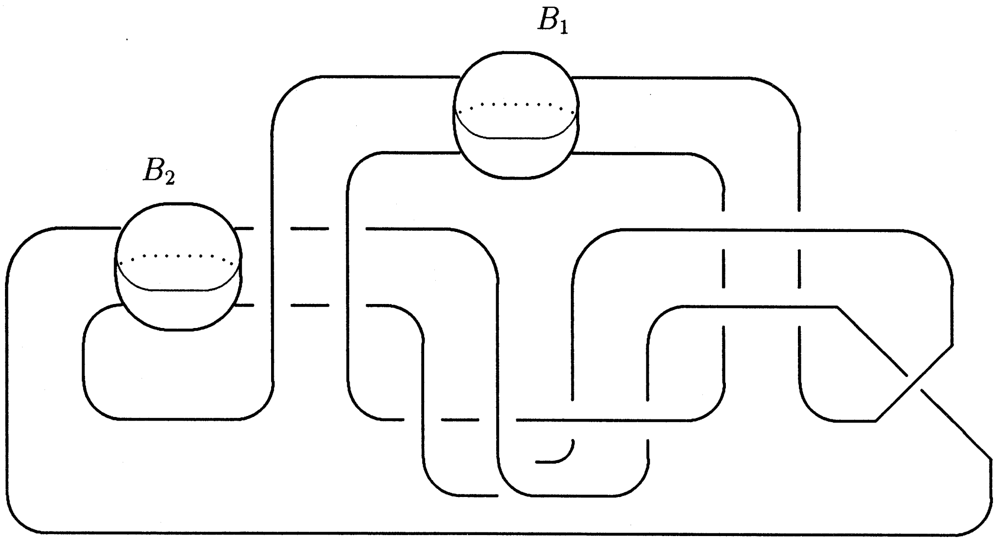

Figure 6.4.

The ball B_i with arcs $B_i \cap p(Fix(\rho))$ is a trivial tangle in terminology of [19] and [60]. By the Montesinos algorithm, for describing the link $\mathcal{L}(m,n;p,q)$ we need to replace these trivial tangles B_1 and B_2 by $\frac{m}{n}$ and $\frac{p}{q}$ rational tangles, respectively (see Figure 6.5).

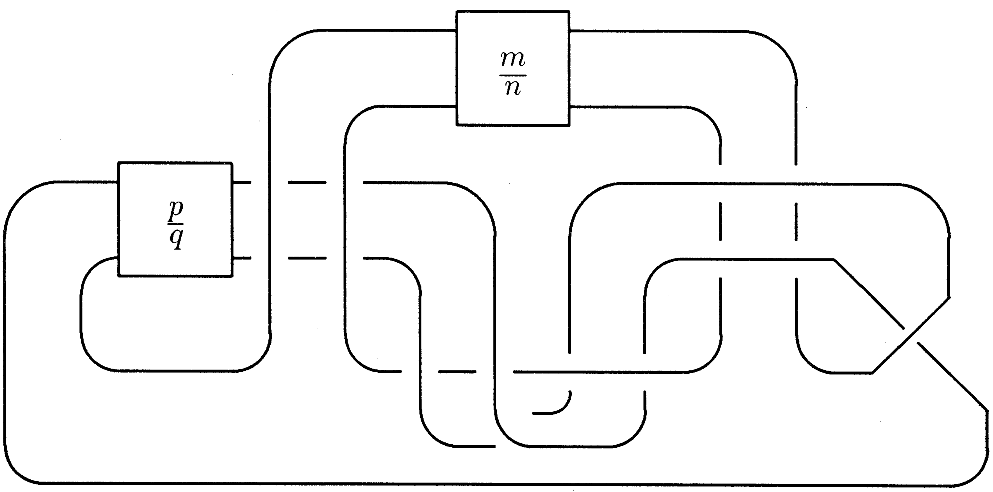

Figure 6.5. The link $\mathcal{L}(m,n;p,q)$.

In Figures 6.6–6.8, using Reidemeister's moves, we redraw the link $\mathcal{L}(m,n;p,q)$ in the more convenient form.

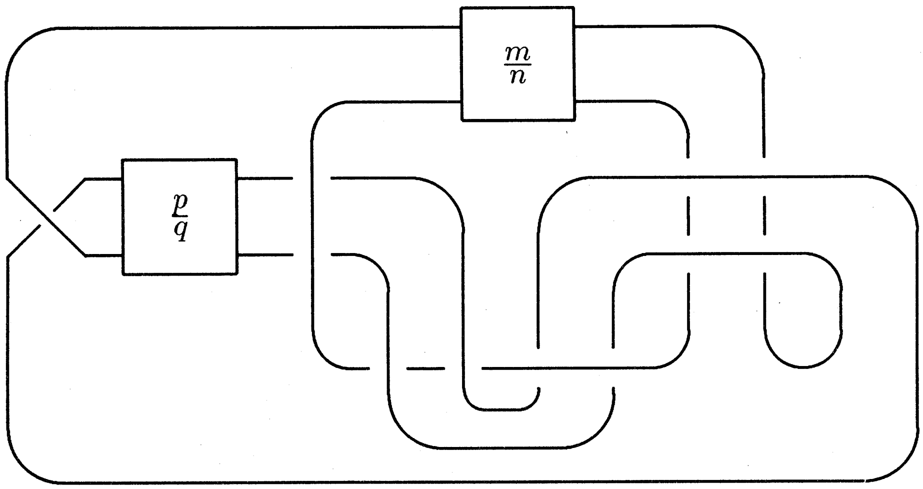

Figure 6.6. The link $\mathcal{L}(m, n; p, q)$.

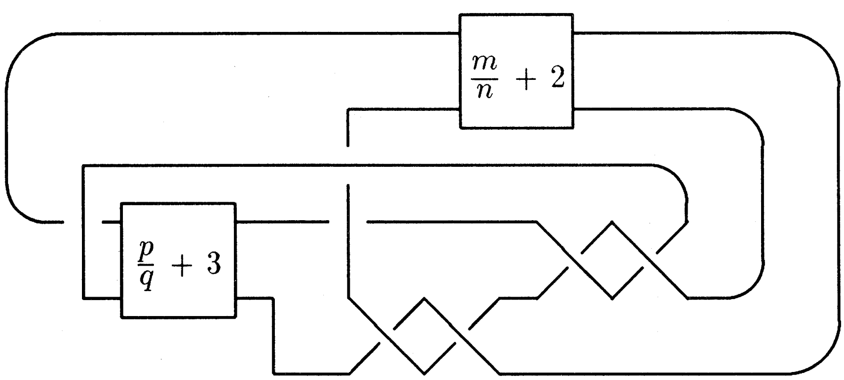

Figure 6.7. The link $\mathcal{L}(m, n; p, q)$.

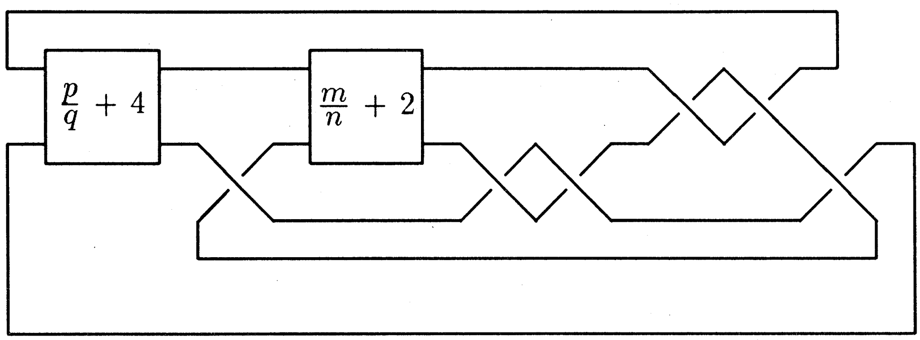

Figure 6.8. The link $\mathcal{L}(m,n;p,q)$.

As a consequence of above considerations we get the following theorem.

Theorem 6.2 ([52]) *Let $M = W(m,n;p,q)$ be a manifold obtained by (m,n) and (p,q) Dehn surgeries on the Whitehead link W. Then M is the two-fold covering of S^3 branched over the link $\mathcal{L}(m,n;p,q)$, pictured in Figure 6.8.*

We remark, that the basic polyhedron (in the sense of [19]) for the link $\mathcal{L}(m,n;p,q)$ is an octahedron. It means that the link $\mathcal{L}(m,n;p,q)$ can be obtained by replacing vertices of the octahedron by following rational tangles: $(p/q+4)$–tangle, $(m/n+2)$–tangle, 2–tangle, 1–tangle, 2–tangle and 1–tangle.

It is well-known, that components of the Whitehead link are symmetric. But unfortunately the presentation on the link $\mathcal{L}(m,n;p,q)$ in Figure 6.8 is not symmetric in respect to parameters of surgeries on components. So in Figure 6.9 we give a symmetric presentation of the link $\mathcal{L}(m,n;p,q)$, which is obtained from Figure 6.8 by Reidemeister's moves.

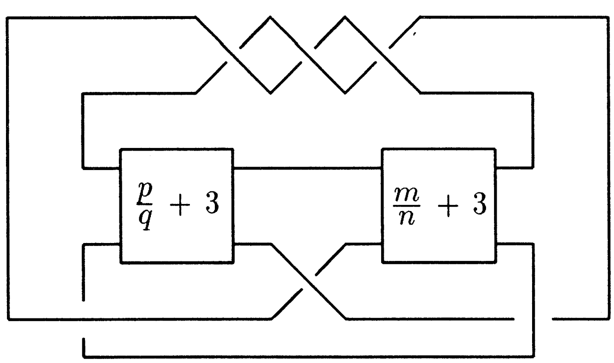

Figure 6.9. The link $\mathcal{L}(m,n;p,q)$.

6.2 The ten smallest manifolds $\mathcal{M}_1, \dots, \mathcal{M}_{10}$

In this section we will apply Theorem 6.2 to the ten smallest known compact orientable hyperbolic 3-manifolds $\mathcal{M}_1, \dots, \mathcal{M}_{10}$ from [36]. We remark, that all of them can be described in the form $W(m,n;p,q)$. The corresponding parameters of surgeries are given in the first column of Table 6.1. Therefore manifolds $\mathcal{M}_1, \dots, \mathcal{M}_{10}$ are 2-fold branched coverings of S^3. Let us denote corresponding branch sets by $\mathcal{L}_1, \dots, \mathcal{L}_{10}$.

For each $i = 1, \dots, 10$ we can consider an orbifold $\mathcal{L}_i(2)$ whose underlying space is S^3, the singular set is the link $\mathcal{L}_i$, and branch indices equal 2. The orbifold $\mathcal{L}_i(2)$ is a π-orbifold in the sense of [13]. Thus manifolds $\mathcal{M}_1, \dots, \mathcal{M}_{10}$ are two-fold coverings of π-orbifolds $\mathcal{L}_1(2), \dots, \mathcal{L}_{10}(2)$, whose volumes are given in the third column of Table 6.1.

Complements $S^3 \setminus \mathcal{L}_i$, $i = 1, \dots, 10$, are hyperbolic manifolds and their volumes can be found using the SnapPea program of J. Weeks [2], [90], [91] (see the fourth column in Table 6.1). In the last column there are given notations of links $\mathcal{L}_i$ according to tables from [15] and [70] which contain knots and links of small order. They are recognized using polynomial invariants of knots and Reidemeister's moves.

Table 6.1.

	$vol(\mathcal{M}_i)$	$vol(\mathcal{L}_i(2))$	$vol(S^3 \setminus \mathcal{L}_i)$	$\mathcal{L}_i$
$\mathcal{M}_1 = W(5,-2;5,-1)$	0.9427...	0.4713...	9.4270...	9_{49}
$\mathcal{M}_2 = W(1,1;5,-1)$	0.9813...	0.4906...	5.6387...	10_{161}
$\mathcal{M}_3 = W(3,-2;6,-1)$	1.0149...	0.5074...	8.1195...	10^2_{138}
$\mathcal{M}_4 = W(5,-1;5,-1)$	1.2637...	0.6318...	9.2505...	10_{155}
$\mathcal{M}_5 = W(1,1;6,-1)$	1.2844...	0.6422...	5.8430...	$11^2_?$
$\mathcal{M}_6 = W(1,1;1,-2)$	1.3985...	0.6992...	5.8296...	$14_?$
$\mathcal{M}_7 = W(1,-2;6,-1)$	1.4140...	0.7070...	5.9782...	$11^2_?$
$\mathcal{M}_8 = W(2,1;5,-1)$	1.4140...	0.7070...	7.7948...	$11^2_?$
$\mathcal{M}_9 = W(7,-3;5,-1)$	1.4236...	0.7118...	10.6933...	10_{162}
$\mathcal{M}_{10} = W(1,1;3,-2)$	1.4406...	0.7203...	7.1180...	$13_?$

In particular, as a consequence of Theorem 6.2 we get the following description of the ten smallest manifolds.

Corollary 6.1 *The ten smallest known closed orientable hyperbolic 3-manifolds $\mathcal{M}_1, \dots, \mathcal{M}_{10}$ are the two-fold coverings of S^3 branched over the links $\mathcal{L}_1, \dots, \mathcal{L}_{10}$, pictured in Figures 6.10–6.19.*

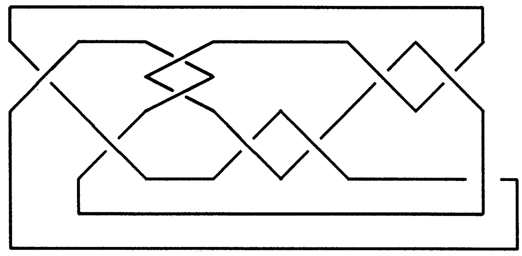

Figure 6.10. The knot $\mathcal{L}_1 = \mathcal{L}(5,-2;5,-1) = 9_{49}$.

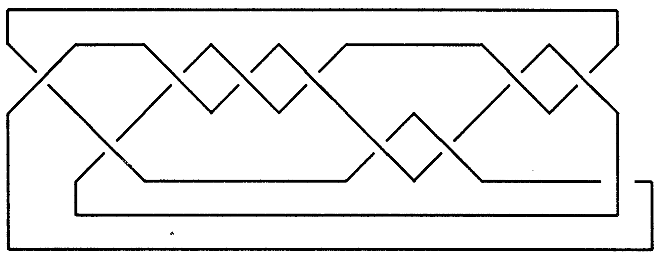

Figure 6.11. The knot $\mathcal{L}_2 = \mathcal{L}(1,1;5,-1) = 10_{161}$.

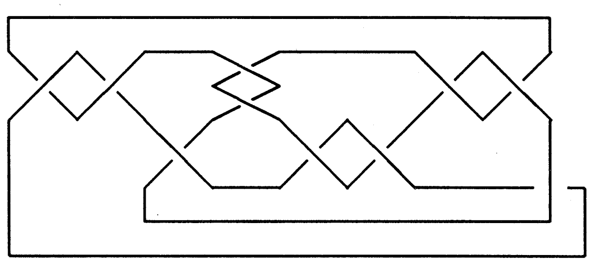

Figure 6.12. The link $\mathcal{L}_3 = \mathcal{L}(3,-2;6,-1) = 10^2_{138}$.

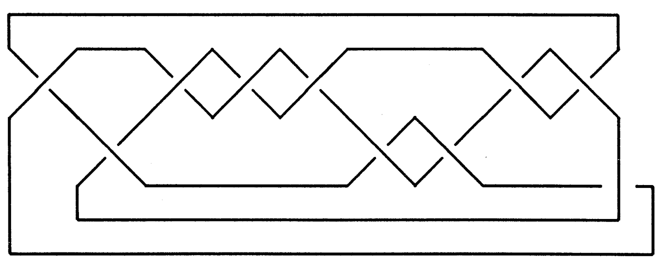

Figure 6.13. The knot $\mathcal{L}_4 = \mathcal{L}(5,-1;5,-1) = 10_{155}$.

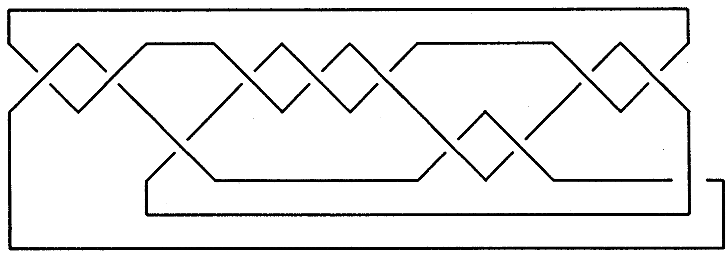

Figure 6.14. The link $\mathcal{L}_5 = \mathcal{L}(1,1;6,-1) = 11^2_?$.

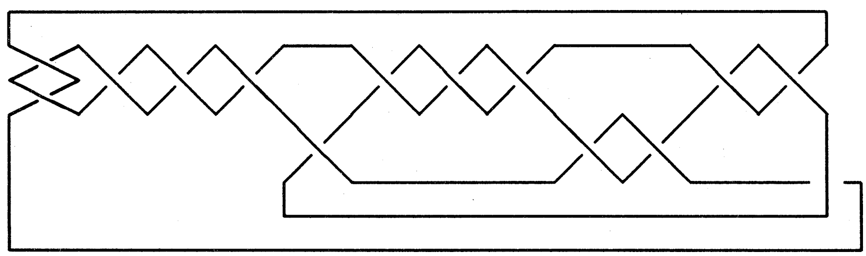

Figure 6.15. The knot $\mathcal{L}_6 = \mathcal{L}(1,1;1,-2) = 14_?$.

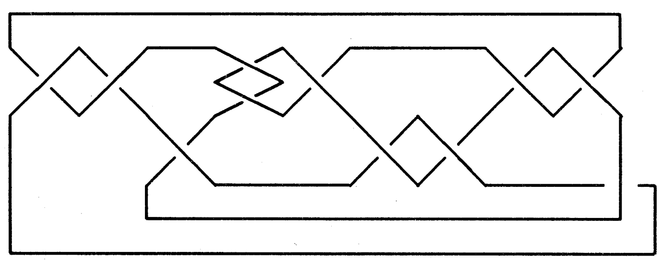

Figure 6.16. The link $\mathcal{L}_7 = \mathcal{L}(1,-2;6,-1) = 11^2_?$.

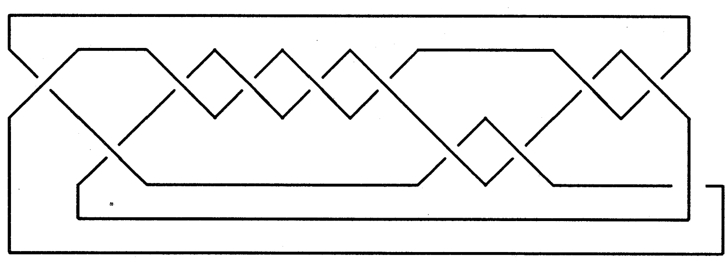

Figure 6.17. The link $\mathcal{L}_8 = \mathcal{L}(2,1;5,-1) = 11^2_?$.

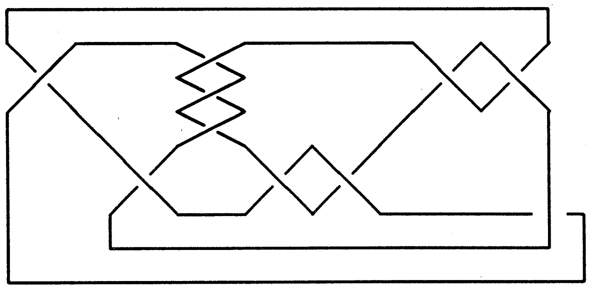

Figure 6.18. The knot $\mathcal{L}_9 = \mathcal{L}(7, -3; 5, -1) = 10_{162}$.

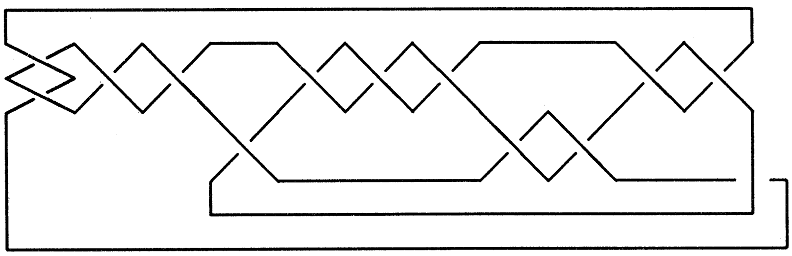

Figure 6.19. The knot $\mathcal{L}_{10} = \mathcal{L}(1, 1; 3, -2) = 13_?$.

6.3 The Heegaard genus of $W(m, n; p, q)$

Let $\mathcal{L}$ be a link in S^3. According to [96], a link $\mathcal{L}$ is said to have a *3-bridge presentation* if there is a genus 0 Heegaard splitting (B_1, B_2) of S^3 such that the link $\mathcal{L}$ intersects the 3-ball B_i $(i = 1, 2)$ in three unlinked arcs. That is there are three mutually disjoint discs in B_i each of which is bounded by one of the arcs considered and an arc on the boundary of B_i.

The following theorem of O. Viro [88] admits to estimate Heegaard genus $h(M)$ of a manifold M described as a two-fold covering of S^3.

Theorem 6.3 ([88]) *A closed orientable 3-manifold M^3 admits a Heegaard decomposition of genus 2 if and only if M^3 is a two-fold covering of S^3 branched over a link with a 3-bridge presentation.*

By Theorem 6.1 the manifold $M = W(m, n; p, q)$ is a two-fold covering of S^3 branched over the link $\mathcal{L}(m, n; p, q)$ (see Figure 6.20).

Let us consider the genus 0 Heegaard splitting (B_1, B_2) of S^3, where boundaries ∂B_1 and ∂B_2 correspond to the dotted line in Figure 6.20. Then for each

$i = 1, 2$ the intersection $\mathcal{L}(m, n; p, q) \cap B_i$ consists of three arcs, which can be isotopic to unlinked. Therefore $\mathcal{L}(m, n; p, q)$ admits a 3-bridge presentation. So as a consequence of Theorem 6.3 we have the following statement.

Proposition 6.1 ([83]) *Let $M = W(m, n; p, q)$ be a manifold obtained by (m, n) and (p, q) Dehn surgeries on the Whitehead link W. Then $h(M) \leq 2$.*

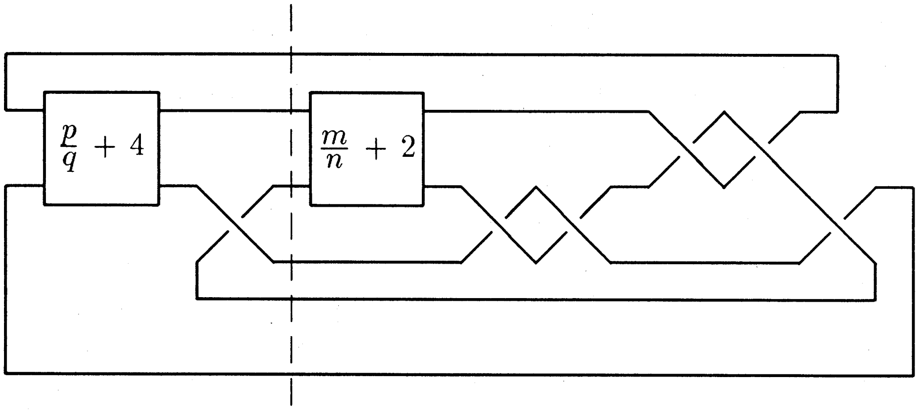

Figure 6.20. The link $\mathcal{L}(m, n; p, q)$.

We recall that $h(M) = 0$ or 1 if and only if M is the 3-sphere, the lens space or $S^2 \times S^1$ [96]. In each of these cases M does not admit a hyperbolic structure [72]. This gives the following refinement of above proposition.

Proposition 6.2 ([83]) *Let $M = W(m, n; p, q)$ be a hyperbolic manifold obtained by (m, n) and (p, q) Dehn surgeries on the Whitehead link W. Then $h(M) = 2$.*

We recall, that a manifold $W(1, 1; p, q)$ can be obtained by (p, q) Dehn surgery on the figure-eight knot. For this subset of manifolds the Heegaard genus was considered in [60] and [77].

In particular, from Proposition 6.2 we get the Heegaard genus of the ten smallest known hyperbolic 3-manifolds.

Corollary 6.2 *For manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{10}$ the Heegaard genus equals two.*

As the Heegaard genus of a manifold gives an estimate for the rank of its fundamental groups, we have

Corollary 6.3 *For manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_{10}$ the rank of $\pi_1(\mathcal{M}_i)$ equals two.*

6.4 $\mathcal{M}_1$ is a maximally symmetric manifold

We recall, that the maximal possible order of a finite group G of orientation-preserving homeomorphisms of the orientable 3-dimensional handlebody V_g of genus $g > 1$ is $12(g-1)$ [97], analogous to the classical $84(g-1)$-bound for closed Riemann surfaces of genus $g > 1$.

Let M be a closed orientable 3-manifold. We will give the following definition according to B.Zimmermann [101].

Definition 6.1 A closed orientable 3-manifold M is called *maximally symmetric* if M has a Heegaard splitting of genus $g > 1$ and a finite group G of orientation-preserving homeomorphisms of maximal possible order $12(g-1)$ which preserves both handlebodies of the Heegaards splitting (but does not leave invariant a Heegaard splitting of genus 0 or 1).

It was shown in [99], that several of best-known 3-manifolds are maximally symmetric, for example, the 3-sphere, the projective 3-space, the 3-torus, the Poincaré homology 3-sphere, the Seifert-Weber hyperbolic dodecahedral space; also it is proven that an irreducible maximally symmetric 3-manifold are hyperbolic or are Seifert fibred.

In this section we will show that the smallest known hyperbolic 3-manifold $\mathcal{M}_1$ is also maximally symmetric. Really this result is expected, because according to Corollary 6.3 the manifold $\mathcal{M}_1$ is of Heegaard genus two, and according to [58] and [36], the isometry group of $\mathcal{M}_1$ is of order 12.

We will demonstrate this property of $\mathcal{M}_1$ using the following nice criterion from [101].

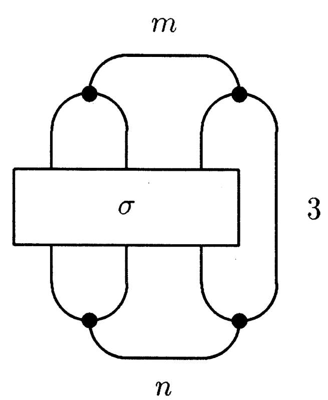

Figure 6.21. The singular set of $\theta(\sigma, m, n)$.

Let us consider an orbifold with underlying space S^3 whose singular set is isomorphic to the spatial graph with four vertices pictured in Figure 6.21, where

σ denotes a 3-strings braid and 3, m, n are branch indices of corresponding edges with $m, n \in \{2, 3, 4, 5\}$ and indices of other edges are equal 2. Following [101], we denote this orbifold by $\theta(\sigma, m, n)$.

Theorem 6.4 ([101]) *The maximally symmetric 3-manifolds (M, G) are exactly the finite regular coverings of the orbifolds $\theta(\sigma, m, n)$.*

We will apply this criterion to the manifold $\mathcal{M}_1$. As it was shown in Corollary 6.1, the manifold $\mathcal{M}_1$ can be obtained as the 2-fold covering of the 3-sphere branched over the knot 9_{49} pictured in Figure 6.10. According to the orbifold theory terminology [77], we say that $\mathcal{M}_1$ covers the π-orbifold $9_{49}(2)$ with the 3-sphere as underlying space and the knot 9_{49} with branch index 2 as its singular set. We will redraw the knot 9_{49} using its presentation in [15, p.265], in the following form pictured in Figure 6.22.

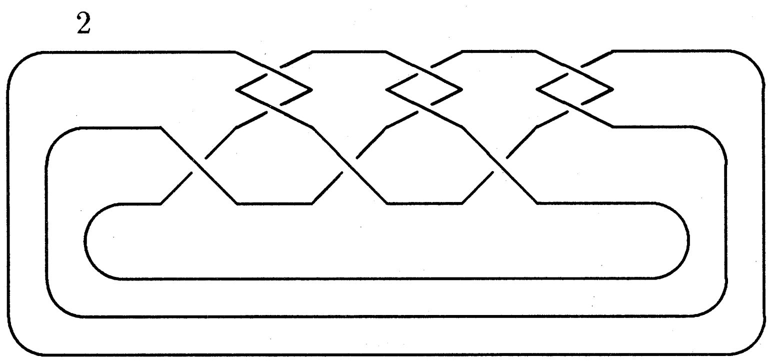

Figure 6.22. The singular set of $9_{49}(2)$.

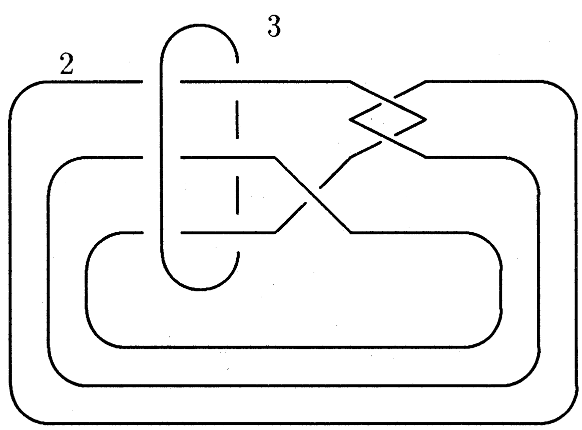

Figure 6.23. The singular set of $7_1^2(2, 3)$.

So it is obvious that the orbifold $9_{49}(2)$ has the symmetry of order 3 and the quotient space under this symmetry action is the orbifold $7_1^2(2, 3)$ whose singular set is the 2-component link 7_1^2 with branch indices 2 and 3, pictured in Figure 6.23.

Using Reidemeister's moves one can redraw the link 7_1^2 in more symmetric form pictured in Figure 6.24 (see also [70, p.416]).

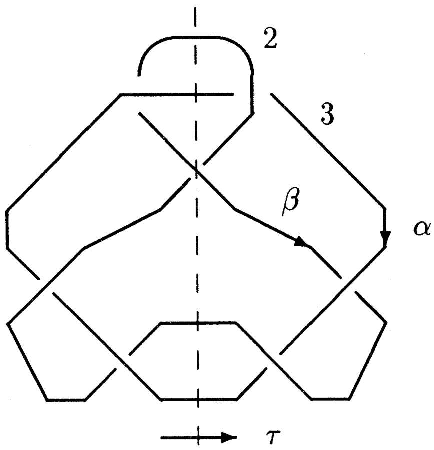

Figure 6.24.

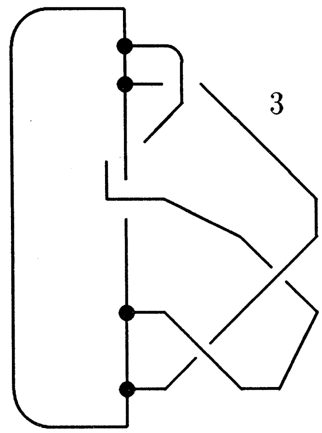

Figure 6.25.

As we see, the singular set of $7_1^2(2,3)$ has an invertable involution τ of order 2 whose fixed axe intersects it in four points. The singular set of the quotient space $7_1^2(2,3)/\tau$ is shown in Figure 6.25. The singular set of $7_1^2(2,3)/\tau$ is a spatial graph with four vertices, that one edge has branch index 3 and branch indices of other edges are equal 2. So, using Reidemeister's moves we can redraw the singular set as in Figure 6.26 and we see that $7_1^2(2,3)/\tau$ is the orbifold $\theta(\sigma, 2, 2)$,

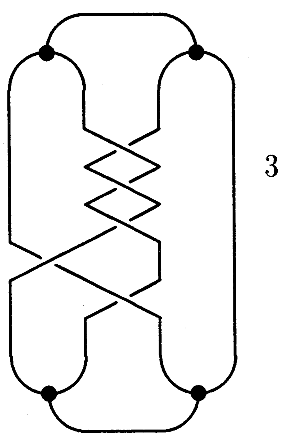

Figure 6.26.

where $\sigma = \sigma_2^3\sigma_1^{-1}\sigma_2$ in standard generators of a braid group. Thus we get

Proposition 6.3 *The smallest known compact hyperbolic manifold $\mathcal{M}_1$ is a regular covering of the orbifold $\theta(\sigma_2^3\sigma_1^{-1}\sigma_2, 2, 2)$.*

We remark, that the singular set of the orbifold $\theta(\sigma_2^3\sigma_1^{-1}\sigma_2, 2, 2)$ is so-called *spatial tetrahedron*, i.e. a spatial graph isomorphic to the 1-skeleton of the tetrahedron, and can be described as the knot 5_2, which is a closure of the braid $\sigma_2^3\sigma_1^{-1}\sigma_2$, with two bridges.

According to Theorem 6.4 we have

Corollary 6.4 *The manifold $\mathcal{M}_1$ is maximally symmetric.*

Because the link 7_1^2 is a 2-bridge link [70], its components are symmetric, and we can exchange indices 2 and 3 in Figure 6.23 and consider 2-fold covering of the orbifold $7_1^2(2,3)$. One can check that in this case we will get the orbifold $5_2(3)$ with the knot 5_2 as singular set and with branch index 3. By the Wirtinger algorithm we see, that the group $\pi(7_1^2(2,3))$ of the orbifold $7_1^2(2,3)$ has the representation

$$\pi(7_1^2(2,3)) = \langle \alpha, \beta \mid \quad \alpha^3 = \beta^2 = 1, \quad \alpha = w\,\alpha\,w^{-1} \rangle,$$

where

$$w = \beta\,\alpha\,\beta\,\alpha\,\beta^{-1}\,\alpha^{-1}\,\beta^{-1}\,\alpha^{-1}\,\beta^{-1}\,\alpha\,\beta\,\alpha\,\beta,$$

and generators α and β correspond to Figure 6.24. Therefore we can consider an epimorphism

$$\varphi : \pi(7_1^2(2,3)) \to \mathbb{Z}_3 \oplus \mathbb{Z}_2 = \langle a \mid a^3 = 1 \rangle \oplus \langle b \mid b^2 = 1 \rangle$$

defined by equalities

$$\varphi(\alpha) = a, \qquad \varphi(\beta) = b.$$

Thus similar to arguments from the proof of Theorem 5.2, we will get the following diagram of coverings:

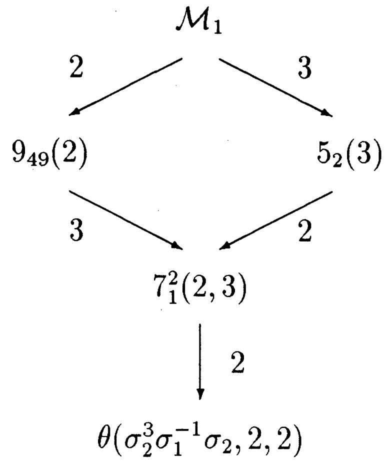

Figure 6.27.

The complete diagram of coverings corresponding to isometries of $\mathcal{M}_1$ is presented in [53].

6.5 The Meyerhoff–Neumann manifold $\mathcal{M}_3$

This section is devoted to an interesting connection between the Fibonacci manifold M_4 and the third smallest known manifold $\mathcal{M}_3$.

We recall that R. Meyerhoff and W. Neumann [56] have obtained the hyperbolic manifold $\mathcal{M}_3 = W(3,-2;6,-1)$ by means of Dehn surgery on the Whitehead link W. It was calculated in [56], that $vol(\mathcal{M}_3)$ approximately up to 10^{-50} equals to the volume of the regular ideal tetrahedron in the Lobachevsky space. They asked if these volumes are strictly equal and if the manifold $\mathcal{M}_3$ is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. In this section we will give affirmative answers on these questions.

Theorem 6.5 ([82]) *The Fibonacci manifold M_4 is a two-fold unbranched covering of the Meyerhoff–Neumann manifold $\mathcal{M}_3$.*

Proof. Let us consider the hyperbolic π–orbifold $Th_4(2)$ whose underlying space is the 3-sphere S^3 and singular set is the Turk's head knot $Th_4 = 8_{18}$ (see Figure 6.28). According to Corollary 5.10 this orbifold is hyperbolic, and following to notations in the proof of Theorem 5.2 we denote by T_4 its fundamental group: $Th_4(2) = \mathbb{H}^3 \,/\, \mathrm{T}_4$.

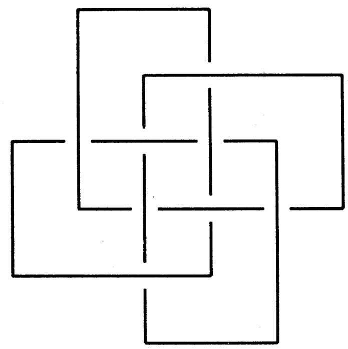

Figure 6.28. The Turk's head knot Th_4.

As we see from Figure 6.28, the orbifold $Th_4(2)$ has the rotation symmetry ρ of the order four such that the singular set remains invariant under ρ action. Let us consider the involution ρ^2. The quotient space $Th_4(2)\,/\,\rho^2$ is a π-orbifold $D(2,2)$. The singular set of the orbifold $D(2,2)$ is the two-component link in Figure 6.29.

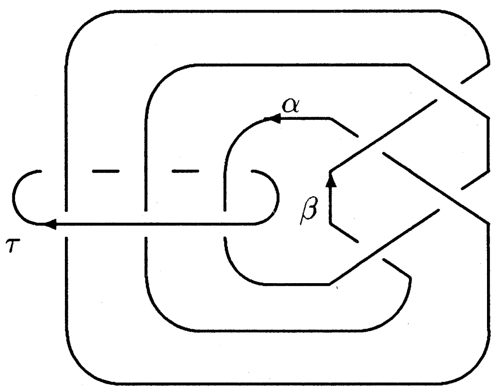

Figure 6.29. The singular set of the orbifold $D(2,2)$.

Using Wirtinger algorithm and results from [26] for the fundamental group of an orbifold, we can find the fundamental group Δ of the orbifold $D(2,2)$:

$$\Delta = \langle \alpha, \beta, \tau \mid (\tau\alpha\tau\beta)^2\, \alpha\, (\beta\tau\alpha\tau)^2 = \tau\beta\tau, \quad (\beta\alpha)^2\, \tau\beta\tau\, (\alpha\beta)^2 = \tau\alpha\tau, \quad \alpha^2 = \beta^2 = \tau^2 = 1 \rangle. \quad (6.1)$$

In this representation generators α, β, τ canonical correspond to arcs with the same labels on the link diagram in Figure 6.29.

By rigidity theorem the involution ρ^2 is isotopic to an isometry of the hyperbolic orbifold $Th_4(2)$. Therefore the group Δ can be realized as a discrete subgroup of the isometry group of the Lobachevsky space $\mathbb{H}^3$. In this case τ is a lifting of the involution ρ^2 on the universal covering. According to Theorem 5.2, the Fibonacci manifold $M_4 = \mathbb{H}^3 \,/\, F(2,8)$ is the two-fold covering of the π-orbifold $Th_4(2) = \mathbb{H}^3 \,/\, \mathrm{T}_4$. Since for M_4, $Th_4(2)$ and $D(2,2) = \mathbb{H}^3 \,/\, \Delta$ we have the covering diagram

$$M_4 \xrightarrow{2} Th_4(2) \xrightarrow{2} D(2,2), \tag{6.2}$$

which implies an embeddings for subgroups:

$$F(2,8) \,\triangleleft\, \mathrm{T}_4 \,\triangleleft\, \Delta, \tag{6.3}$$

where $| \Delta : \mathrm{T}_4 | = 2$ and $| \mathrm{T}_4 : F(2,8) | = 2$.

Let us consider an epimorphism

$$\theta \,:\, \Delta \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \;=\; \left\langle a \mid a^2 = 1 \right\rangle \oplus \left\langle t \mid t^2 = 1 \right\rangle$$

defined by correspondences:

$$\theta(\alpha) \;=\; \theta(\beta) \;=\; a, \quad \theta(\tau) \;=\; t. \tag{6.4}$$

By the construction of the two-fold cover $Th_4(2) \to D(2,2)$, the loop τ from the fundamental group Δ of the orbifold $D(2,2)$ lifts to a trivial loop in the fundamental group T_4 of the orbifold $Th_4(2)$. By the same, loops α and β lift to loops which generate cyclic subgroups of order 2 in the group T_4. Therefore

$$\mathrm{T}_4 \;=\; \theta^{-1}(\mathbb{Z}_2) \;=\; \theta^{-1}\left(\left\langle a \mid a^2 = 1 \right\rangle\right). \tag{6.5}$$

Let us consider the 4–fold cover $M_4 \to D(2,2)$. In this case loops α, β and τ lift to trivial loops in the group $F(2,8)$. Hence

$$F(2,8) \;=\; \theta^{-1}(1) \;=\; \operatorname{Ker}\theta. \tag{6.6}$$

The group

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \;=\; \left\langle a \mid a^2 = 1 \right\rangle \oplus \left\langle t \mid t^2 = 1 \right\rangle$$

contains a cyclic subgroup of order two generated by $d = a + t$. Let us define an epimorphism

$$\lambda \,:\, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2,$$

by the correspondence

$$\lambda(a) \;=\; \lambda(t) \;=\; d. \tag{6.7}$$

Then for an epimorphism $\varphi = \lambda \circ \theta$ such that

$$\varphi : \Delta \longrightarrow \mathbb{Z}_2 = \left\langle d \mid d^2 = 1 \right\rangle . \tag{6.8}$$

we have

$$\varphi(\alpha) = \varphi(\beta) = \varphi(\tau) = d. \tag{6.9}$$

Denote $\Phi = \operatorname{Ker}\varphi$ and consider the orbifold $U = \mathbb{H}^3 / \Phi$. By the construction of the epimorphism φ, the orbifold cover

$$U = \mathbb{H}^3 / \Phi \xrightarrow{2} D(2,2) = \mathbb{H}^3 / \Delta \tag{6.10}$$

is branched over both components of the singular set of the orbifold $D(2,2)$. In this case loops α, β and τ lift to trivial loops in the group Φ. Therefore U is a hyperbolic orbifold and the singular set of U is empty. Hence U is a hyperbolic manifold.

Our next step is to prove that $U = \mathcal{M}_3$. By Theorem 6.2 the manifold $\mathcal{M}_3 = W(3,-2;6,-1)$ can be obtained as the 2-fold covering of S^3 branched over the 2-component link $\mathcal{L}_3 = \mathcal{L}(3,-2;6,-1) = 10^2_{138}$ (see Corollary 6.1), pictured in Figure 6.12. By using the Reidemeister's moves one can see that two-component links in Figure 6.29 and in Figure 6.12 are equivalent. Therefore manifolds U and $\mathcal{M}_3$ are obtained as 2-fold coverings of the three-sphere branched over the same link. Hence, manifolds U and $\mathcal{M}_3$ are homeomorphic and moreover, by Mostow rigidity theorem, they are isometric. Thus $\mathcal{M}_3 = \mathbb{H}^3 / \Phi$. Since $\varphi = \lambda \circ \theta$ we get the inclusion for fundamental groups

$$\Phi = \operatorname{Ker}\varphi > F(2,8) = \operatorname{Ker}\theta, \tag{6.11}$$

and the covering diagram for manifolds

$$M_4 = \mathbb{H}^3 / F(2,8) \xrightarrow{2} N = \mathbb{H}^3 / \Phi. \tag{6.12}$$

Comparing covering diagrams (6.2), (6.10) and (6.12) we get the following diagram in Figure 6.30 :

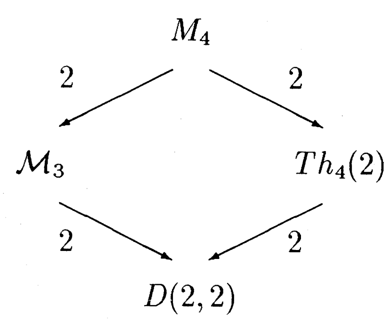

Figure 6.30.

Groups $F(2,8)$ and Φ are fundamental groups of hyperbolic manifolds M_4 and $\mathcal{M}_3$ respectively. Hence these groups are torsion-free. Therefore we can conclude that the cover (6.12) induced by (6.11) is unbranched, and the theorem is proved. □

From Corollary 5.7 and the covering diagram for manifolds M_4 and $\mathcal{M}_3$, we have affirmative answers on above questions.

Theorem 6.6 ([80], [82]) *The Meyerhoff–Neumann hyperbolic manifold* $\mathcal{M}_3 = W(3,-2;6,-1)$ *is arithmetic over the field* $\mathbb{Q}(\sqrt{-3})$ *and its volume is exactly equal to the volume of the regular ideal tetrahedron in* $\mathbb{H}^3$.

Proof. It was shown in Corollary 5.7, that the volume of the hyperbolic Fibonacci manifold M_4 is strictly equal to the double volume of the regular ideal tetrahedron in $\mathbb{H}^3$. Thus by Theorem 6.5 we get $vol(\mathcal{M}_3) = \frac{1}{2}vol(M_4) =$ the volume of the regular ideal tetrahedron $= 2\Lambda(\pi/6) = 1.0149\ldots$. Moreover, by (6.11) fundamental groups Φ and $F(2,8)$ of manifolds $\mathcal{M}_3$ and M_4 are commensurable. It was proved in [28] that the Fibonacci manifold M_4 is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. Therefore the same is true for the manifold $\mathcal{M}_3$. □

We remark that Theorem 6.6 was proven by A. Reid in [67] by arithmetic methods, using theory of quaternion algebras. Moreover, manifolds $\mathcal{M}_3$ and M_4 arise in [67] as first examples of non-Haken hyperbolic 3-manifolds which are covered by a manifold that fibers over the circle.

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