수 학 강 의 록 제 29 권

GOTTLIEB GROUPS, GROUP ACTIONS, FIXED POINTS AND RATIONAL HOMOTOPY

JOHN OPREA

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
held at the Seoul National University

John Oprea, Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115, U.S.A.

펴낸날 : 1995년 9월 30일
지은이 : John Oprea
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터〔TEL : 02 - 880 - 6562〕

Preface

These notes are based on a series of talks given at Seoul National University in July 1995. The talks (and these notes) were meant to be an introduction to the mix of classical and rational homotopy ideas and techniques which surround the evaluation map. In particular, the Gottlieb groups are introduced and used to obtain information about spaces in terms of the fibrations for which they are fibres. Also, the beautiful relationship between rational Gottlieb groups and rational Lusternik-Schnirelmann category is discussed as well as applications of rational methods to transformation groups and fixed point theory.

It is my pleasure to thank Seoul National University for its support during my stay in Korea. It is an even greater pleasure to thank my host and friend Doobeum Lee for his generous hospitality during my stay.

Contents

§1 Introduction

There are many connections among the subjects of the title: Gottlieb groups, group actions, rational homotopy theory and fixed point theory. As we shall see below, one of the main theorems about Gottlieb groups follows from Nielsen-Wecken fixed point theory. Conversely, much of what may be computed about the Nielsen number in fixed point theory follows from the Jiang condition — a condition on the Gottlieb group. Also, the advent of rational homotopy theory in the 1970's allowed for new interpretations and calculations of Gottlieb groups as well as applications to fixed point theory. In this paper, I shall attempt to describe some of the connections alluded to above. This is very much my personal slant on things, so I will sometimes write in the first person to avoid imbuing my own views with the authority of *'we'*.

The main goal of fibration theory is to understand the extent to which a given fibration differs from the product (or trivial) fibration. The first information which is apparent is that obtained by noting that a fibration $F \to E \to B$ has associated to it a long exact sequence of homotopy groups

$$\ldots \to \pi_{i+1}(B) \xrightarrow{\partial} \pi_i(F) \to \pi_i(E) \to \pi_i(B) \xrightarrow{\partial} \ldots .$$

Furthermore, because $\pi_i(X \times Y) \cong \pi_i(X) \times \pi_i(Y)$, it is observed that the *connecting homomorphism* ∂ provides a simple measure of the non-triviality of the fibration. Some questions which arise are then:

(1) Must the connecting homomorphism for each fibration be studied separately or is it possible to codify connecting homomorphisms in terms of — what? — the fibre perhaps?

(2) What hypotheses on the connecting map itself lead to strong conclusions about the structure of associated fibrations?

(3) Is the connecting homomorphism related in some way to other homotopical or geometric structures or invariants?

As an example of the type of structure results I'm referring to, let me give a theorem which appeared in a paper having to do with symplectic geometry [GLSW]. The authors wanted to analyze situations where a symplectic form on the (compact) fibre of a bundle would extend to the total space. The following criterion was used to rule out certain cases.

Theorem 1.1. *Let $F \to E \to B$ be a fibration of simply connected spaces. If*

$H^2(E;\mathbb{Q}) \to H^2(F;\mathbb{Q})$ is not surjective, then $H^{2k}(F;\mathbb{Q}) \neq 0$ for all $k \geq 1$ and, therefore, F is not compact.

Proof. The proof given in [GLSW] is similar in spirit to Gottlieb's proof of [G4 Theorem 1], which itself makes use of Weingram's theorem [W]. Gottlieb's result is more general than the result here, however, so there is a simpler proof. First, localize the spaces to make homotopy and homology rational. Denote the localization of a space X by X_0. Consider the commutative diagram induced by Hurewicz maps

$$\begin{array}{ccccccc} \pi_2(\Omega B_0) & \to & \pi_2(F_0) & \to & \pi_2(E_0) & \twoheadrightarrow & \pi_2(B_0) \\ \downarrow & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H_2(\Omega B;\mathbb{Q}) & \to & H_2(F;\mathbb{Q}) & \to & H_2(E;\mathbb{Q}) & \to & H_2(B;\mathbb{Q}) \end{array}$$

where $\pi_2(\Omega B_0)$ is identified with $\pi_3(B_0)$ and the isomorphisms follow from the Hurewicz theorem and the assumption of simple connectivity. By duality, the non-surjectivity of $H^2(E;\mathbb{Q}) \to H^2(F;\mathbb{Q})$ translates into an assumption of non-injectivity of $H_2(F;\mathbb{Q}) \to H_2(E;\mathbb{Q})$. That is, there exists $\alpha \in H_2(F;\mathbb{Q})$ which maps to zero in $H_2(E;\mathbb{Q})$. But then, by the commutativity of the diagram, there exists $\bar{\alpha} \in \pi_2(F_0)$ which has Hurewicz image $h(\bar{\alpha}) = \alpha$ and which maps to zero in $\pi_2(E_0)$. The exactness of the homotopy sequence of the fibration then gives a $\bar{\beta} \in \pi_2(\Omega B_0)$ which maps to $\bar{\alpha}$. Therefore, the commutativity of the left square shows that $H_2(\Omega B;\mathbb{Q}) \to H_2(F;\mathbb{Q})$ is non-trivial. Duality then says that $H^2(F;\mathbb{Q}) \to H^2(\Omega B;\mathbb{Q})$ is non-trivial as well. Hopf's theorem on H-spaces (see [Wh] for example) implies that the rational cohomology algebra of ΩB is a (graded) polynomial algebra on even dimensional generators tensor an (graded) exterior algebra on odd dimensional generators, so the image of $H^2(F;\mathbb{Q})$ in $H^2(\Omega B;\mathbb{Q})$ generates a sub-polynomial algebra over $\mathbb{Q}$. This then means, of course, that no power of an image element can vanish and, hence, no power of any preimage can either. □

I have belabored the proof of this result because it contains many of the ingredients I shall use below. In particular, I want to point out the interplay between the Hurewicz map h and the connecting homomorphism ∂. In fact, such an affiliation was used to great effect by H. Cartan [C] many years before to analyze the real cohomology of homogeneous spaces. For a modern minimal model approach to this subject, see [HT].

Now, a hypothesis on the connecting homomorphism which is particularly powerful is that of surjectivity. In fact, we have

Theorem 1.2. *Suppose $F \to E \to B$ is a fibration of rational spaces. If the connecting homomorphism is surjective in each degree, then F has the homotopy type of a product of $K(\mathbb{Q}, n)$'s.*

Proof. Recall that a *rational space* is of the homotopy type of a simply connected CW complex whose integral homology in each degree is a finite dimensional rational vector space. Such spaces result from localization at 0 for example. Under the 'rational' assumption (using Hopf's theorem on H-spaces), the loop space ΩB has the homotopy type of a product $\prod K(\mathbb{Q}, n_i)$. Because $\partial\colon \pi_*(\Omega B) \to \pi_*(F)$ is a surjection of rational (graded) vector spaces, there is a splitting and a subspace $V \subseteq \pi_*(\Omega B)$ such that $\partial|_V$ is an isomorphism. But, since ΩB is a product, V may be realized by a subproduct $K \subset \Omega B$ with $\pi_*(K) \cong V$ and, consequently, $\partial\colon \pi_*(K) \xrightarrow{\cong} \pi_*(F)$. The Whitehead theorem then shows that F has the homotopy type of K. □

This result seems to be rediscovered every so often. Israel Berstein knew it in the 1960's and H. Haslam [Ha] and Steve Halperin [H1], proved it, in completely different ways, in the 1970's. Examples where the hypothesis of the theorem is satisfied include:

(1) Stiefel-Grassmann fibrations for $k \leq n/2$,

$$U(k) \to V_{k,n}(\mathbb{C}) \to G_{k,n}(\mathbb{C}),$$

since the inclusion of the fibre is nullhomotopic.

(2) Compact fibrations $F \to \Sigma X \to B$, since the inclusion of the fibre is rationally nullhomotopic (see [G3] and [O2]).

Now, in order to understand better the influence of the connecting homomorphism on the structure of fibrations, as well as on fixed point theory and invariants of rational homotopy, I must recall the definitions and notation which serve to codify information about the connecting homomorphism.

§2 Gottlieb Groups Remembered

Throughout this paper, I will always take spaces to be of the homotopy type of CW complexes for which the function space exponential law holds.

Definition 2.1 [G2]. The n^{th} Gottlieb group of a space X, denoted $G_n(X)$, is the subgroup of $\pi_n(X)$ consisting of elements $\alpha \in \pi_n(X)$ which have associated maps $A\colon S^n \times X \to X$ making the following diagram homotopy commutative.

$$(\dagger) \qquad \begin{array}{ccc} S^n \times X & \xrightarrow{A} & X \\ i \uparrow & \nearrow & \alpha \vee 1_X \\ S^n \vee X & & \end{array}$$

Note that the conventions I have assumed about spaces allow us to take the diagram to be strictly commutative when convenient. The relation between this definition and the connecting homomorphism is enunciated by

Theorem 2.2 [G5]. *$G_n(X)$ is equal to*

(1) *Image*$(\mathrm{ev}_\#\colon \pi_n(X^X, 1_X) \to \pi_n(X))$, *where* $\mathrm{ev}\colon X^X \to X$ *is evaluation* $\mathrm{ev}(f) = f(x_0)$ *at a specified basepoint* $x_0 \in X$.

(2) $\bigcup$ *Image*$(\partial_\#\colon \pi_n(\Omega B) \to \pi_n(X))$, *where the union is taken over all fibrations* $X \to E \to B$.

Proof Sketch. (i) The exponential law shows that elements of $\pi_n(X^X, 1_X)$, represented by maps $S^n \to X^X$, correspond to associated maps $S^n \times X \to X$ as in $\dagger$ above. Furthermore, evaluation ev on $\pi_n(X^X, 1_X)$ likewise corresponds to restriction to S^n of the associated map $S^n \times X \to X$.
(ii) The second part follows from the existence of a classifying space for fibrations with fibre X, $\mathrm{Baut}\, X$. Here, $\mathrm{aut}\, X$ denotes the monoid of self equivalences of X, so

$$(\ddagger) \qquad \pi_{i+1}(\mathrm{Baut}\, X) \cong \pi_i(\mathrm{aut}\, X) \cong \pi_i(X^X, 1_X).$$

Every fibration $X \to E \to B$ is a pullback of a universal fibration $X \to \mathrm{Baut}_* X \to \mathrm{Baut}\, X$ via a classifying map $B \to \mathrm{Baut}\, X$. The pullback gives a ladder of exact homotopy sequences which, in particular, provides a commutative square (using $\ddagger$)

$$\begin{array}{ccc} \pi_i(X^X, 1_X) & \to & \pi_i(X) \\ \uparrow & & \uparrow\cong \\ \pi_{i+1}(B) & \xrightarrow{\partial} & \pi_i(X). \end{array}$$

Therefore, any connecting homomorphism ∂ factors through the universal one — and the universal connecting homomorphism is precisely the evaluation map. $\square$

A basic property of Gottlieb groups is that they are annihilators under Whitehead product. Recall that the *Whitehead product* of elements $\alpha \in \pi_k(X)$ and $\beta \in \pi_\ell(X)$ is the element $[\alpha, \beta] \in \pi_{k+\ell-1}(X)$ represented by a map $S^{k+\ell-1} \to X$ determined by taking a composition with the loop commutator map

$$S^{k-1} \times S^{\ell-1} \xrightarrow{\Omega\alpha\times\Omega\beta} \Omega X \times \Omega X \xrightarrow{xyx^{-1}y^{-1}} \Omega X$$

and noting that, since the restriction to $S^{k-1} \vee S^{\ell-1}$ is trivial, the mapping factors through the smash product

$$S^{k-1} \times S^{\ell-1} / S^{k-1} \vee S^{\ell-1} = S^{k+\ell-2}.$$

Finally, the isomorphism $\pi_{k+\ell-2}(\Omega X) \cong \pi_{k+\ell-1}(X)$ provides the element $[\alpha, \beta]$. A standard result about Whitehead products (see [Wh] for example) asserts that $[\alpha, \beta] = 0$ if and only if there exists a map $f\colon S^k \times S^\ell \to X$ such that $f|_S^k = \alpha$ and $f|_S^\ell = \beta$. Elements of Gottlieb groups then have the following remarkable property.

Proposition 2.3. *If $\alpha \in G_n(X)$, then $[\alpha, \beta] = 0$ for all $\beta \in \pi_i(X)$ for all i.*

Proof. Choose a fibration $X \to E \to B$ with $\partial(\hat{\alpha}) = \alpha$ for some $\hat{\alpha} \in \pi_n(\Omega B)$. Recall that every fibration has associated to it a *holonomy* $c\colon \Omega B \times X \to X$ which globalizes the action of the fundamental group of the base on the homology of the fibre. The holonomy obeys the relations: $c|_{\Omega B} = \partial$ and $c|_X = 1_X$. The composition

$$S^k \times S^\ell \xrightarrow{\hat{\alpha}\times\beta} \Omega B \times X \xrightarrow{c} X$$

then gives $c \circ (\hat{\alpha} \times \beta)|_{S^k} = \partial(\hat{\alpha}) = \alpha$ and $c \circ (\hat{\alpha} \times \beta)|_{S^\ell} = \beta$, so the Whitehead product $[\alpha, \beta]$ vanishes. $\square$

Remark 2.4. I have chosen to emphasize the connecting homomorphism in the proof above because this viewpoint fits well with what follows. It is easy also, however, to apply the definition of the Gottlieb group directly to prove the result.

With Proposition 2.3 in mind, define $P_n(X) \subseteq \pi_n(X)$ to be the subgroup of $\pi_n(X)$ consisting of elements whose Whitehead products with all other elements vanish. Then, clearly,

$$G_n(X) \subseteq P_n(X)$$

for all n. In particular, it is well-known that, for $\alpha, \beta \in \pi_1(X)$, the Whitehead product is the commutator, $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, so that $\alpha \in G_1(X)$ must commute with every $\beta \in \pi_1(X)$. In other words, $\alpha \in \mathcal{Z}\pi_1(X)$ (i.e. the center of $\pi_1(X)$). More generally, the Whitehead product of a fundamental group element measures the deviation of the action of that element on higher homotopy from the identity. Therefore, $P(X) = P_1(X)$ consists of elements of the fundamental group which act trivially on all higher homotopy groups. This holds specifically for $G(X) = G_1(X)$. Later I will discuss Gottlieb's question [G1] whether $G(X)$ and $P(X)$ are identical.

A space X is called a *G-space* if $\pi_n(X) = G_n(X)$ for all n. As we have seen, this requires that *all* Whitehead products vanish in $\pi_*(X)$. Of course, H-spaces have this property, so it is natural to ask about their Gottlieb groups. In fact, for an H-space (Y, μ), given $\alpha \in \pi_n(Y)$, there exists an associated map

$$S^n \times Y \xrightarrow{\alpha \times 1_Y} Y \times Y \xrightarrow{\mu} Y$$

defined using the multiplication μ. Therefore, since any homotopy element has an associated map as in †, an H-space is a G-space. *Rationally*, the converse is true by Theorem 1.2, but *integrally* there is

Example 2.5. J. Siegel [S] has given a simple example of a G-space which is not an H-space. Embed S^1 into $SO(3) \times S^1$ by a map $j(e^{i\theta}) = (2\theta, e^{i3\theta})$. Take the homogeneous space

$$T \overset{\text{def}}{=} \frac{SO(3) \times S^1}{j(S^1)}.$$

The usual homogeneous space fibration may be extended to classifying spaces to produce a fibration sequence

$$S^1 \to SO(3) \times S^1 \to T \to BS^1 \to BSO(3) \times BS^1.$$

The connecting homomorphism ∂ then may be identified with the quotient map $SO(3) \times S^1 \to T$ and, since $BS^1 = K(\mathbb{Z}, 2)$, the only possible degree where ∂ might not be surjective is 2. Identifying $\pi_2(BS^1) = \pi_1(S^1)$ and $\pi_2(BSO(3) \times BS^1) = \pi_1(SO(3)) \times \pi_1(S^1)$, the relevant part of the exact homotopy sequence becomes

$$\xrightarrow{\partial} \pi_2(T) \to \pi_1(S^1) = \mathbb{Z} \xrightarrow{j_\#} \pi_1(SO(3)) \times \pi_1(S^1).$$

But j is a 'product map' which induces an analogous homomorphism on fundamental groups. Because $\pi_1(SO(3)) = \mathbb{Z}/2$ and the first factor is multiplied by 2, clearly $j_\#$ is trivial when projected to $\pi_1(SO(3))$. On the second factor, $j_\#$ has degree 3, so it must be the case that $j_\#(1) = (0,3)$ and this is an injection. The exactness of the homotopy sequence then says that $\pi_2(T) \to \pi_1(S^1)$ is trivial and, hence, ∂ is surjective. Thus, T is a G-space.

To show that T is not an H-space, it is sufficient to show that T's cohomology with some coefficients does not support a Hopf algebra structure. The discussion above showed that $\pi_1(T) = \mathbb{Z}/2 \oplus \mathbb{Z}/3$ so that, in particular, $H^1(T;\mathbb{Z}/3) = \mathbb{Z}/3$ with generator x. Now, x has odd degree, so its square is zero. However,

$$H^2(T;\mathbb{Z}/3) = \mathrm{Ext}(H_1(T),\mathbb{Z}/3) \oplus \mathrm{Hom}(H_2(T),\mathbb{Z}/3)$$

and $\mathrm{Ext}(H_1(T),\mathbb{Z}/3) = \mathrm{Ext}(\mathbb{Z}/2 \oplus \mathbb{Z}/3, \mathbb{Z}/3) = \mathbb{Z}/3$, so there is another indecomposable generator β. Now, $\beta^2 = 0$ because β^2 has degree 4 and T is a three dimensional manifold. But, if T were an H-space with multiplication $\mu\colon T \times T \to T$ inducing a Hopf algebra structure on cohomology, then

$$\begin{aligned} \mu^*(\beta^2) &= (\mu^*(\beta))^2 \\ &= (1 \otimes \beta + \beta \otimes 1 + \xi(\alpha \otimes \alpha))^2 \\ &= 2(\beta \otimes \beta) + \ldots \\ &\neq 0 \end{aligned}$$

since the other terms denoted by $\ldots$ and $\beta \otimes \beta$ are linearly independent. This contradiction then shows that a Hopf algebra structure is not possible and, therefore, T cannot be an H-space.

Before I leave this section, I want to concentrate a bit on what is thought to be the most important of the Gottlieb groups, $G(X) = G_1(X)$. Although this evaluation is still reasonable, as we shall see later, the higher groups are playing an ever more important role in topology. I showed above that $G(X) \subseteq \mathcal{Z}\pi_1(X)$ for any X. In case X is a $K(\pi,1)$, more is true. Let $\alpha \in \mathcal{Z}\pi$ and note that, since α is in the center of π, a homomorphism $\phi\colon \mathbb{Z} \times \pi \to \pi$ is defined by $(n,x) \mapsto \alpha^n x$. Any homomorphism of groups is realized by a unique homotopy class of maps on the corresponding Eilenberg-Mac Lane spaces. Hence, the map $\Phi\colon S^1 \times K(\pi,1) \to K(\pi,1)$ exists and, clearly, is an associated map (as in †) for α. Thus, α is in $G(X)$ and $G(X) = \mathcal{Z}\pi$.

Gottlieb was able to characterize $G(X)$ in a very beautiful way by identifying $\pi_1(X)$ with the group of covering transformations of the universal cover $\widetilde{X}$.

Theorem 2.6 [G1]. *$G(X)$ may be identified with the subgroup of covering transformations whose elements are equivariantly homotopic to the identity $1_{\widetilde{X}}$.*

For a proof, see [G1]. Also, recall that an equivariant homotopy, with respect to the covering transformation action, is simply a homotopy which commutes with the action. The main application of Theorem 2.6 is to connect $G(X)$ with another fundamental invariant, the Euler characteristic.

Theorem 2.7 [G1]. *For a space X of the homotopy type of a finite complex, if $\chi(X) \neq 0$, then $G(X) = \{1\}$.*

Proof Sketch. Gottlieb uses Nielsen-Wecken fixed point theory to prove this result. He first identifies fixed point classes associated to a fixed map $f\colon X \to X$ with equivalence classes of lifts $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ (i.e. $\tilde{f}_1 \sim \tilde{f}_2$ if there is a covering transformation γ with $\tilde{f}_2 = \gamma^{-1}\tilde{f}_1\gamma$) such that lifts with no fixed points correspond to the empty fixed point class. Moreover, he shows that this correspondence is preserved under homotopy. This identification allows the transference of fixed point index theory to the equivalence classes of lifts. In particular, the index of a lift $i(\tilde{f})$ is zero if $\tilde{f}$ has no fixed points and the sum of indices over all lifts (i.e. all fixed point classes) is equal to the Lefschetz number. If $f = 1_X$, then every index is zero except for the one associated to the lift $1_{\widetilde{X}}$ since no other lift has fixed points. The Lefschetz number in this case is the Euler characteristic $\chi(X)$, which is non-zero by assumption, so the sum of indices is non-zero as well.

Now, here is where Theorem 2.6 comes in. If $\alpha \in G(X)$ is non-trivial, then there is an equivariant homotopy $H\colon \widetilde{X} \times I \to \widetilde{X}$ with $H_0 = 1_{\widetilde{X}}$ and $H_1 = \alpha$ (considered as a covering transformation). The equivalence relation on lifts then says that the fixed point classes correspond and, hence, the sum of indices should be the same for $1_{\widetilde{X}}$ and α. But, as we saw above, the total index for $1_{\widetilde{X}}$ is $\chi(X) \neq 0$, while the total index for α must be zero because, as a covering transformation, α has no fixed points. This contradiction shows that $\alpha = 1 \in \pi_1(X)$ and $G(X) = \{1\}$. For the details of fixed point index theory as well as another formulation of this proof, see [Br]. □

Corollary 2.8. *If $X = K(\pi, 1)$ is a finite complex with $\chi(X) \neq 0$, then $\mathcal{Z}\pi = \{1\}$.*

Remark 2.9. The Corollary is known as *Gottlieb's Theorem.* Soon after Gottlieb proved his result, Stallings [St] gave a completely algebraic proof in which, among other things, he introduced what is now called the Hattori-Stallings rank. In 1984, Rosset [R] generalized Corollary 2.8 to the following: *For a finite complex*

$X = K(\pi, 1)$, *if* $\chi(X) \neq 0$, *then* $\pi_1(X)$ *contains no non-trivial normal abelian subgroup.* The proof of Rosset's result is algebraic and analytic (even involving rings of operators). In a similar vein, Eckmann tried to apply Rosset's techniques to the non-aspherical case, essentially trying to algebratize Theorem 2.7. He could prove [E]: *For a finite complex* X, *if* $\chi(X) \neq 0$, *then* $\pi_1(X)$ *contains no non-trivial torsionfree normal abelian subgroups which act nilpotently on the homology of the universal cover.* Of course, Theorem 2.6 says that $G(X)$ acts *trivially* on the homology of $\widetilde{X}$, so if we know that $G(X)$ is torsionfree, then Theorem 2.7 is reproved. However, in any other case, Eckmann's result is still weaker than Theorem 2.8. In §7 I shall mention yet another recent proof of Theorem 2.7 which is motivated by fixed point theory and the work of Stallings. This brings up the question of whether a purely homotopical proof of Theorem 2.7 is possible?

§3 Splittings

In Theorems 1.1 and 1.2, the relationship between the connecting homomorphism and the Hurewicz map manifests itself in constraining the structure of fibrations. This relationship, which was known and exploited by Gottlieb (see [G2], [G4] for example), is most easily seen when things are viewed with rational eyes, as in Theorems 1.1 and 1.2. These results may be generalized by the following *rational fibre decomposition theorem*

Theorem 3.1 [O1], [O2]. *Let* $F \to E \to B$ *be a fibration of rational spaces. Then there is a subproduct* $K \subset \Omega B$ *such that* $F \simeq \mathcal{F} \times K$ *and* $H^*(K) \cong$ *Image*$(\partial^* \colon H^*(F) \to H^*(\Omega B))$.

Proof Sketch. Because B is a rational space, ΩB is a product of $K(\mathbb{Q}, n)$'s. For the composition

$$\pi_*(\Omega B) \xrightarrow{\partial_\#} \pi_*(F) \xrightarrow{h} H_*(F),$$

where h is Hurewicz, let K denote the largest subproduct of ΩB upon which $h\partial$ is an isomorphism onto its image. The vector space dual of the image is contained in the vector space dual of $H_*(F)$, $H^*(F)$. Maps $F \to K$ are classified by $H^*(F)$ since K is a product of $K(\mathbb{Q}, m)$'s. Hence, choosing a basis for the dual of Image$(h\partial)$ allows us to construct a map $F \to K$ with a homotopy splitting given by the restriction of the Barratt-Puppe extension (which I also denote by) $\partial \colon \Omega B \to F$ to K. Then, considering the fibration (up to homotopy) $\mathcal{F} \to F \to K$

with section ∂, we see that

$$\pi_n(F) = \pi_n(\mathcal{F}) \oplus \pi_n(K)$$

and the holonomy of the *original* fibration may be composed with the product of appropriate inclusions to yield a map

$$\mathcal{F} \times K \rightarrowtail F \times \Omega B \xrightarrow{c} F$$

which is seen to induce an isomorphism on homotopy groups. Hence, $F \simeq \mathcal{F} \times K$. To see that $H^*(K)$ gives the image of ∂^* in cohomology, it is necessary to use the structure of minimal models in rational homotopy theory. See [O1] for details. □

Remark 3.2.
(1) The space K is called the *Samelson space* of the fibration $F \to E \to B$ because it generalizes work of Samelson on the structure of Lie groups. Furthermore, extending these structure results to a compact transformation group G acting on a manifold M produces a splitting of the above type,

$$H^*(M;\mathbb{R}) \cong A \otimes \Lambda(P)$$

where $\Lambda(P)$ denotes the image of $H^*(M;\mathbb{R})$ under the orbit map $\omega\colon G \to M$, $g \mapsto gx$ for fixed $x \in M$. Of course the orbit map ω may be considered to be the 'connecting homomorphism' of the Borel fibration $M \to MG \to BG$ associated to the action. For this result see [GHV].
(2) Theorem 3.1 provides an immediate proof for Theorem 1.1 by noting that the hypothesis of Theorem 1.1 entails a rational splitting $F \simeq \mathcal{F} \times K(\mathbb{Q}, 2)$.

Say that F is *quasifinite* if $\dim H^*(F;\mathbb{Q}) < \infty$. If F is quasifinite, then K, as a product of $K(\mathbb{Q}, m)$'s cannot have any m even. This follows since $K(\mathbb{Q}, 2k)$ has cohomology a polynomial algebra on one generator. Hence, if F is quasifinite, then K has the rational homotopy type of a product of odd spheres. In particular then, $\chi(K) = 0$. Because Euler characteristic respects products, we have

Proposition 3.3. *If F is rational and quasifinite with $\chi(F) \neq 0$, then $G_*(F) \subset$ Ker(h).*

Proof. Let $\alpha \in G_k(F) \subset \pi_k(F)$ and choose a preimage $\hat{\alpha}$ of α under the evaluation map $\pi_k(F^F, 1_F) \to G_k(F) \subset \pi_k(F)$. By ‡, we may consider $\hat{\alpha} \in \pi_{k+1}(\text{Baut}(F))$

with representative $S^{k+1} \to \mathrm{Baut}(F)$. Pull back the universal fibration over this map to get a fibration

$$F \to E \to S^{k+1}$$

with $\partial(\iota_{k+1}) = \alpha$, where ι_{k+1} denotes the generator of $\pi_k(\Omega S^{k+1})$. Now the rational fibre decomposition theorem may be applied. If $h(\alpha) \neq 0$, then $F \simeq \mathcal{F} \times S_0^k$ where k is odd (since F is quasifinite). But then, $\chi(F) = \chi(\mathcal{F}) \cdot \chi(S^k) = 0$ and this contradicts the hypothesis that the Euler characteristic is non-zero. □

Remark 3.4. Gottlieb proved this and similar results in [G2]. He also knew about certain types of algebraic (i.e. module) splittings of homology and cohomology (with varying coefficients) under certain hypotheses [G4]. The proofs above for results less general than Gottlieb's exemplify the fact that rational homotopy theory often serves as an oasis in homotopy theory where theorems and proofs achieve their cleanest form. In some sense, rational homotopy provides the heuristic reason why an integral homotopical fact is true, but in the translation from the rational world to the integral one, much of the simpler rational structure may be lost. In fact, although Gottlieb actually showed that $G_k(F) \subset \mathrm{Ker}(h_p)$ for all primes p as well as h_0, to the best of my knowledge, the question remains open as to whether $G_k(X) \subset \mathrm{Ker}(h)$ over the integers.

Just as in the remark above, it is possible to give a rational explanation for results like the Transgression Theorem of [CG] in terms of the Samelson space.

Rational Transgression Theorem 3.5 [O2]. *Let $F \to E \to B$ be a fibration of rational spaces with F quasifinite and suppose that $f\colon E \to E$ is a fibre preserving map which induces the identity on B and $g\colon F \to F$. If the Lefschetz number of g, $\Lambda(g)$ is non-zero, then*

$$0 = \partial^*\colon H^*(F) \to H^*(\Omega B).$$

Proof Sketch. The structure of the rational holonomy $\Omega B \times F \to F$ (see [FT]) and the rational fibre decomposition theorem imply that, for non-trivial Samelson space K,

$$\Lambda(g) = \chi(K) \cdot \Lambda(\bar{g})$$

where $\bar{g}\colon \mathcal{F} \xrightarrow{\text{incl}} F \xrightarrow{g} F \xrightarrow{\text{proj}} \mathcal{F}$. But then, if K is non-trivial, then $\chi(K) = 0$ since F is quasifinite and this contradicts the hypothesis that $\Lambda(g) \neq 0$. □

The rational fibre decomposition theorem has an integral analogue in degree 1 which is a homotopical version of a splitting obtained by Conner and Raymond in their work on circle actions. In particular, they showed [CR1]

Conner-Raymond Splitting Theorem 3.6. *If S^1 acts on X so that the orbit map $\omega\colon S^1 \to X$ induces an injection $\omega_*\colon H_1(S^1;\mathbb{Z}) \rightarrowtail H_1(X;\mathbb{Z})$ onto a $\mathbb{Z}$-summand, then*

$$X \cong_{\text{homeo}} (X/S^1) \times S^1$$

and the action is on the second factor by translations.

The crucial ingredient necessary to generalize this result is the fact that the orbit map ω fits into a homotopy commutative diagram

$$\begin{array}{ccc} \Omega BS^1 & \xrightarrow{\partial} & X \\ \phi \uparrow & \nearrow & \omega \\ S^1 & & \end{array}$$

where $S^1 \simeq \Omega BS^1$ and ∂ arises from the Barratt-Puppe extension of the Borel fibration associated to the action

$$\ldots \to \Omega BS^1 \xrightarrow{\partial} X \to XS^1 \to BS^1.$$

The homotopical version of a circle action is simply a map $A\colon S^1 \times X \to X$ associated to a Gottlieb group element $A|_{S^1} = \alpha \in G(X)$. The Conner-Raymond theorem then has the analogue

Theorem 3.7 [O3]. *Let X be a space with $H_1(X;\mathbb{Z})$ finitely generated. If there exists $\alpha \in G(X)$ with Hurewicz image $h(\alpha)$ of infinite order, then there exists a finite cyclic cover of X, $\overline{X}$, with*

$$\overline{X} \simeq Y \times S^1$$

for some Y.

Proof. I shall only give the proof when, as in the Conner-Raymond theorem, $h(\alpha)$ generates a $\mathbb{Z}$-summand, for in this case, X itself splits. So, suppose $h(\alpha)$ generates the $\mathbb{Z}$-summand in $H_1(X;\mathbb{Z}) = \mathbb{Z} \oplus A$ and take the element $\beta \in H^1(X;\mathbb{Z}) = \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z})$ determined by

$$\beta(h(\alpha)) = 1 \in \mathbb{Z} \quad \text{and} \quad \beta(A) = 0.$$

Then β corresponds to a mapping $X \to K(\mathbb{Z},1) = S^1$ by the classification of integral cohomology. This map $X \to S^1$ has a splitting α and, if the homotopy fibre is denoted Y, the same holonomy argument as in Theorem 3.1 gives $X \simeq Y \times S^1$. □

In fact, this result, as well as generalizations and applications may be found in [G6] as well as [O3]. (Also, it appears Thurston knew of this type of theorem years before. See [Mc].) Theorem 3.7 is not only a homotopical version of the Conner-Raymond theorem, but may be used to prove the Conner-Raymond theorem itself.

Proof of Conner-Raymond. The orbit of the circle action is $S^1/\text{Isotropy}$, so the only way that ω_* could map $H_1(S^1;\mathbb{Z})$ *onto* a $\mathbb{Z}$-summand is for the isotropy to be trivial at each point. That is, S^1 must act freely with consequent principal bundle

$$S^1 \to X \to X/S^1.$$

Now, the hypothesis on $\omega = \partial$ translates into saying that $h\partial$ is non-trivial; indeed, is onto a $\mathbb{Z}$-summand. Theorem 3.7 then says that $X \simeq Y \times S^1$ with S^1 modeling the $\mathbb{Z}$-summand. But then X homotopy retracts onto the fibre S^1 in the principal bundle and, consequently,

$$\pi_*(X) \cong \pi_*(X/S^1) \times \pi_*(S^1)$$

compatible with the splitting of homotopy induced by $X \simeq Y \times S^1$. Hence, the composition $Y \to X \to X/S^1$ induces homotopy isomorphisms and, therefore,

$Y \simeq X/S^1$. But then, using the inverse of the displayed homotopy equivalence, there is a homotopy section of the principal bundle $X/S^1 \to X$ which may be made a true section by applying the homotopy lifting property. This implies that the principal bundle is trivial. That is, $X \cong (X/S^1) \times S^1$. □

Theorem 3.7 may be generalized to toral splittings. In fact, since $H_1(X;\mathbb{Z})$ is assumed to be finitely generated, $hG(X)$ is a finitely generated abelian group with a well-defined rank. Define the rank of $h(G(X)$ to be the *h-rank of X*. (The h-rank differs from the Hurewicz rank defined in [G6] by not requiring the $\mathbb{Z}$-factors in $h(G(X))$ to be free summands of $H_1(X;\mathbb{Z})$. In particular, h-rank is greater than or equal to Hurewicz rank and, over $\mathbb{Q}$, they are identical. We then have ([G6], [O3]),

Theorem 3.8. *Let X be a space with $H_1(X;\mathbb{Z})$ finitely generated. If the h-rank of X is s, then there is a finite abelian cover $\overline{X}$ of X with $\overline{X} \simeq Y \times T^s$, where T^s is an s-torus.*

Remark 3.9. Eckmann-Hilton duality provides wedge splittings for cofibres analogous to the product splittings for fibres above. See [O1], [O4] and, for a very general approach, [DG].

I now want to apply the homotopical Conner-Raymond theorem to answer Gottlieb's question of whether $G(X)$ and $P(X)$ are identical. (In fact, Ganea [Ga1] gave an infinite dimensional example X with $G(X) \neq P(X)$ years ago, but the finite dimensional case appeared much harder See §5 for an analysis of Ganea's example.)

Example 3.10. Let $S^3 \to X \to T^4$ be the principal S^3-bundle induced from the Hopf bundle $S^3 \to S^7 \to S^4$ by a degree 1 one map $k\colon T^4 \to S^4$. It is easy to see that X is a compact 7-dimensional manifold which is also a simple space. That is, $\pi_1(X)$ acts trivially on higher homotopy, so $P(X) = \pi_1(X)$.

Now let's look at $G(X)$. Suppose $G(X)$ is non-trivial and let $\alpha \in G(X)$ be a non-zero element. Then Theorem 3.7 implies that there is a finite cyclic cover $\overline{X}$ with $\overline{X} \simeq Y \times S^1$. Now, it is plain that $\overline{X}$ is a simple space and that $\overline{X}$ and X have the same rational homotopy type. Thus, rationally, $X_0 \simeq Y_0 \times S^1_0$. The Postnikov tower for Y_0 is then

$$\begin{array}{ccc} Y_0 & \rightarrow & PK(\mathbb{Q},4) \\ \downarrow & & \downarrow \\ T_0^3 & \rightarrow & K(\mathbb{Q},4) \end{array}$$

and, since $H^4(T^3;\mathbb{Q}) = 0$, $Y_0 \simeq T_0^3 \times S_0^3$. Consequently, $X_0 \simeq T_0^4 \times S_0^3$. This could only happen if the map k were rationally trivial and this contradicts the fact that k has degree 1. Hence, $G(X)$ is trivial and not equal to $P(X)$.

Remark 3.11. The construction in Example 3.10 may be generalized to certain types of principal bundles $G \to X \to T^{k+1}$ for G a simply connected compact Lie group [OP]. An answer to Gottlieb's question, however, might have been found years earlier by adapting a construction of George Cooke [Co]. Here's how.

Let a map θ be defined on the space $Y = (S^n \times S^n) \vee S^{2n}$ in the following manner:

$$\begin{cases} \theta|_{S^n \vee S^n \vee S^{2n}} = 1_{S^n \vee S^n \vee S^{2n}} \\ \theta|_{2n-\text{cell}} \quad \text{wraps with nontrivial degree around } S^{2n}. \end{cases}$$

As Cooke shows, θ induces the identity on homotopy groups, but not on homology groups. Indeed, for the obvious homology basis in degree $2n$, $\theta_*(0,1) = (0,1)$ and $\theta_*(1,0) = (1,1)$ (for a wrapping of degree 1). Therefore, $\theta_*^{2n} \neq id_{H_*}$ for all n. We have just defined a *homotopy action* of $\mathbb{Z}$ on Y. The main point of Cooke's work was to describe conditions under which homotopy actions could be replaced by topological actions having the same homotopical effects. For a homotopy action of $\mathbb{Z}$ the conditions are simple — *any* such homotopy action may be replaced by a homotopically equivalent topological action. The construction is easy: Let M denote the infinite mapping cylinder of θ,

$$M = \frac{Y \times I \times \mathbb{Z}}{\{(y,1,n) \sim (\theta(y),0,n+1)\}}.$$

The inclusion $Y \rightarrowtail M$ is a homotopy equivalence and the shift map $T\colon M \to M$, $T(y,t,n) = (y,t,n-1)$, provides a free, properly discontinuous action of $\mathbb{Z}$ on M which has the same effects on homotopy and homology as θ.

Denote the quotient $M/\mathbb{Z}$ by N and the universal covering by $p\colon M \to N$. Then N *has the homotopy type of a finite complex with* $P(X) = \pi_1(X) = \mathbb{Z}$ *and* $G(X) = \{1\}$. To see this, first note that N is a simple space because its fundamental group $\mathbb{Z}$ acts on higher homotopy via θ. Also, no element of the fundamental group (thought of as a covering transformation) is homotopic to the identity on M since the map induced on $H_*(M)$ is θ_*. By Theorem 2.6, $G(N)$ is trivial.

Therefore, all we must do is show that N is a finite complex. Take the fibration

$$M \xrightarrow{p} N \to S^1$$

and apply Lal's theorem [Lal] to compute the (unreduced) Wall finiteness obstruction,

$$w(N) = p_*(w(M)) \cdot \chi(S^1) = 0.$$

The vanishing of the obstruction implies that N has the homotopy type of a finite complex. I want to put forward a problem I've often given in relation to this construction of Cooke's. Namely, find other examples of maps which induce the identity on homotopy groups, but not on homology groups. In the fashion above, these maps will provide counterexamples to $G(X) = P(X)$.

Simple spaces with trivial $G(X)$ have interesting self-maps from the point of view of fixed point theory. In [P], Pak showed that, for aspherical manifolds with trivial $G(X)$, the representation

$$\psi\colon \pi_0(H(X,x), 1_X) \to \operatorname{Aut} \pi_1(X,x)$$

is faithful (i.e. injective). Here, $H(X,x)$ denotes the group of basepoint preserving self-homeomorphisms of X. This result is saying that based isotopy classes are detectable by their induced homomorphisms on the fundamental group. The question became whether this type of result could be proved for non-aspherical manifolds. The answer turned out to be no as I shall show below. Although I shall use the X constructed in Example 3.10, more general results may be found in [OP] for the principal G-bundles over tori mentioned above.

Let $H(X)$ and $H(X,x)$ denote the spaces of self-homeomorphisms and basepoint-preserving self-homeomorphisms of X respectively. Let $\operatorname{aut}(X)$ and $\operatorname{aut}(X,x)$ denote the spaces of self-homotopy equivalences and basepoint-preserving self equivalences respectively. There is a diagram of evaluation fibrations

$$\begin{array}{ccccc} H(X,x) & \to & H(X) & \to & X \\ \downarrow & & \downarrow & & \| \\ \operatorname{aut}(X,x) & \to & \operatorname{aut}(X) & \to & X \end{array}$$

which induces a mapping of long exact homotopy sequences,

$$
(*)\qquad
\begin{array}{ccccccc}
\pi_1(H(X),1_X) & \xrightarrow{\text{ev}_\#} & \pi_1(X,x) & \to & \pi_0(H(X,x),1_X) & \to & \pi_0(H(X),1_X) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\pi_1(\text{aut}(X),1_X) & \xrightarrow{\text{ev}_\#} & \pi_1(X,x) & \to & \mathcal{E}(X,x) & \to & \mathcal{E}(X)
\end{array}
$$

where we use the notation for the discrete groups of self-homotopy equivalences, $\mathcal{E}(X,x) = \pi_0(\text{aut}(X,x),1_X)$ and $\mathcal{E}(X) = \pi_0(\text{aut}(X),1_X)$. There is a representation

$$\psi'\colon \mathcal{E}(X,x) \to \text{Aut}\,\pi_*(X,x)$$

given by taking induced maps on homotopy. By pre-composing with $\pi_0(H(X,x),1_X) \to \mathcal{E}(X,x)$, we obtain a representation

$$\psi\colon \pi_0(H(X,x),1_X) \to \text{Aut}\,\pi_*(X,x)$$

which is analogous to Pak's representation above. (See [CR2] for a discussion of this representation in the context of aspherical manifolds.) By [McC], the composition of ψ and the connecting map of $(*)$, $\partial\colon \pi_1(X,x) \to \pi_0(H(X,x),1_X)$ is precisely the automorphism of homotopy given by the standard action of the fundamental group. We obtain

Lemma 3.12. . *If X is a simple space, then*

$$\textit{Image}(\partial\colon \pi_1(X,x) \to \pi_0(H(X,x),1_X)) \subset \textit{Ker}\,\psi.$$

Therefore, to obtain homeomorphisms inducing the identity on homotopy groups, it is sufficient to show that the connecting map image above is nontrivial. To do this, it is sufficient to show that $\text{ev}_\#$ is not surjective. But if we take the X of Example 3.10, then $G(X) = \{1\}$, so $\text{ev}_\#\colon \pi_1(\text{aut}(X),1_X) \to \pi_1(X,x)$ is *trivial.* Hence,

$$\text{ev}_\#\colon \pi_1(H(X),1_X) \to \pi_1(X,x)$$

is trivial as well and $\pi_1(X,x) \rightarrowtail \pi_0(H(X,x),1_X)$ is injective. Since $\pi_1(X,x) = \mathbb{Z}^4$, then $\text{rank}(\text{Ker}\,\psi) \geq 4$. Therefore,

Theorem 3.13. *For the space X of Example 3.10, there exist basepoint preserving homeomorphisms which induce the identity on homotopy, but which are not based isotopic to the identity map. Indeed, for the representation $\psi\colon \pi_0(H(X,x),1_X) \to \operatorname{Aut}\pi_*(X,x)$, rank(Ker ψ) ≥ 4.*

This result makes clear that Pak's faithfulness result cannot be generalized to even the simplest non-aspherical manifolds.

§4 Some Rational Homotopy

Localization methods in topology arose in the late 1960's and provided a way to isolate homotopical phenomena according to number theoretic considerations. In particular, as had been apparent in practice for years, the first and simplest step to understanding the homotopical structure of a space is to understand it over the rationals. In 1969 Quillen [Q] made this part of the problem completely algebraic by showing that the rational homotopy category is equivalent to certain algebraic categories. In the 1970's Sullivan [Su] put forth the idea of computing rational homotopy in a fashion akin to computing de Rham cohomology; that is, from forms. Sullivan showed that certain types of forms could be defined on, not just manifolds, but the usual spaces of topology and that these forms carry all the rational information about the space. Moreover, to glean the relevant rational homotopy information, he devised a minimal differential graded algebra model which also contained all rational invariants of the space, but whose algebraic structure was particularly simple.

In this section, I want to describe the rudiments of rational homotopy theory and the connection of rational homotopy with Gottlieb groups. Later I will use the minimal model in discussing fixed points, so I'll introduce that here as well. First, recall that to every (nilpotent) space there is associated a homotopically unique *rationalization* $e\colon X \to X_0$ with the property that, for any rational space Z. there is a bijection of sets of homotopy classes

$$[X_0, Z] \xrightarrow{e^*} [X, Z].$$

For details of localization theory in a general context, see [HMR]. The first question which comes to mind is whether or not the Gottlieb group rationalizes. Consider the following diagram associated to $\alpha = A|_{S^n} \in G_n(X)$:

$$\begin{array}{ccccc}
S^n & \xrightarrow{i} & S^n \times X & \xrightarrow{A} & X \\
j \downarrow & & \downarrow e \times e & & \downarrow e \\
S^n \times X_0 & \xrightarrow{e \times 1} & S_0^n \times X_0 & \xrightarrow{A_0} & X_0.
\end{array}$$

Then we can compute

$$\begin{aligned}
A_0 \circ (e \times 1) \circ j &= e \circ A \circ i \\
&= e \circ A|_{S^n} \\
&= e_\#(\alpha)
\end{aligned}$$

where $e_\#\colon [S^n, X] \to [S^n, X_0]$ is given by $f \mapsto e \circ f$. The mapping $A_0 \circ (e \times 1)$ therefore restricts to 'localization' of α as it should. Also, if we restrict to X in the righthand square, because $A|_X = 1_X$, we have

$$e = e \circ A|_X = A_0|_{X_0} \circ e = e^*(A_0|_{X_0}).$$

Now, $e^*\colon [X_0, X_0] \to [X, X_0]$ is a bijection and, clearly, $e^*(1_{X_0}) = e$. Hence, comparing the two results, $A_0|_{X_0} = 1_{X_0}$. Therefore, $A_0 \circ (e \times 1)$ is an associated map for $e_\#(\alpha)$ and, consequently, $e_\#(\alpha) \in G_n(X_0)$. Hence,

Theorem 4.1. $G_n(X) \otimes \mathbb{Q} \subseteq G_n(X_0)$.

For more on localization and Gottlieb groups, see [L1]. The rational Gottlieb groups are related, somewhat surprisingly I think, to another old invariant of topology, the Lusternik-Schnirelmann (LS) category. Recall that a space X is said to have *category* n if there exist $n+1$ open sets $U_i \subset X$ such that

$$X = \bigcup_{i=1}^{n+1} U_i \quad \text{and} \quad \text{each } U_i \text{ is contractible in } X$$

and n is the least integer for which this is true. The second condition means that there is a homotopy $H\colon U_i \times I \to X$ such that H_0 is the inclusion and H_1 is the constant map. One thing that we can say right away is that category is related to cuplength. Recall that the *cuplength* of X, denoted $\mathrm{cup}(X)$, is the largest integer k so that there exist $x_i \in H^{n_i}(X; R)$, for $i = 1, \cdots, k$ and a nontrivial cup product $0 \neq x_1 x_2 \cdots x_k$. Here R is a ring of coefficients. The following result is well-known and is the basis of many calculations of category.

Proposition 4.2. $cup(X) \leq \text{cat}(X)$.

Proof. Suppose $\text{cat}(X) = k$, choose any $k+1$ elements $x_i \in H^{n_i}(X;R)$, $i = 1\ldots k+1$ with $\deg(x_i) = j_i$ and form the cup product $x_1x_2\cdots x_{k+1}$. Now, because the inclusion $U_i \hookrightarrow X$ is nullhomotopic, the exact cohomology sequence of the pair gives

$$\ldots \to H^{j_i}(X,U_i) \twoheadrightarrow H^{j_i}(X) \xrightarrow{0} H^{j_i}(U_i) \to \ldots,$$

which guarantees that each x_i has a preimage $\bar{x}_i \in H^{j_i}(X,U_i)$ and, therefore, $x_1x_2\cdots x_{k+1}$ has a preimage $\bar{x}_1\bar{x}_2\cdots\bar{x}_{k+1}$. The properties of relative cup products then give (for $N = \sum j_i$)

$$\bar{x}_1\bar{x}_2\cdots\bar{x}_{k+1} \in H^N(X, \cup_{i=1}^{k+1} U_i) = H^N(X,X) = 0.$$

Since $\bar{x}_1\bar{x}_2\cdots\bar{x}_{k+1}$ maps to $x_1x_2\cdots x_{k+1}$, the latter is zero as well. Hence, any cup product of length greater than k is trivial and $\text{cup}(X) \leq \text{cat}(X)$. □

Other important properties of category are given by the following:

(1) Category is an invariant of homotopy type.

(2) If $C_f = Y \cup_f CX$ is a mapping cone, then $\text{cat}(C_f) \leq \text{cat}(Y) + 1$.

(3) If X is a CW-complex, then, by induction on skeleta and property (2), $\text{cat}(X) \leq \dim(X)$.

(4) In fact, (3) may be generalized: If X is $(r-1)$-connected, then $\text{cat}(X) \leq \dim(X)/r$.

The proofs of these properties may be found in [Wh] and the excellent survey [J] for example.

Example 4.3.

1. $\text{cat}(X) = 0$ if and only if X is contractible.
2. $\text{cat}(S^n) = 1$.
3. More generally, $\text{cat}(X) = 1$ if and only if X is a nontrivial co-H space.
4. $\text{cat}(T^n) = n$ (this follows from the proposition and property (3) above).
5. If (M^{2n}, ω) is a simply connected compact symplectic manifold, then $\text{cat}(M) = n = \frac{1}{2}\dim(M)$. First, observe that the volume form is not exact since it represents a nontrivial fundamental class of M. Because $\omega^n/n! = \text{vol}$, the non-degenerate closed 2-form ω cannot be exact either. Hence, ω^n represents a nontrivial cup product of length n in $\mathbb{R}$-cohomology. By property (4) above, $\text{cat}(M) \leq (\dim(M))/2 = n$. Hence, $n \leq \text{cup}(M) \leq \text{cat}(M) \leq \frac{1}{2}\dim(M) = n$ and the result follows.

Now let's see what category has to do with fibrations. The first connection of category with the rational homotopy theory of fibrations appeared in [FH]. Subsequent to this, various simplifications arose allowing minimal model theory to be obviated in the following discussion. In this I shall basically follow [FL].

Lemma 4.4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with i nullhomotopic. Then*

$$\text{cat}(E) \leq \text{cat}(B).$$

Proof. Let $\text{cat}(B) = n$ and take an open cover $\{U_i\}$, $i = 1 \ldots n+1$ realizing the category. Consider the commuative diagram obtained from the homotopy lifting property,

$$\begin{array}{ccc} p^{-1}(U_i) \times I & \xrightarrow{\widetilde{H}} & E \\ p \times 1 \downarrow & & \downarrow p \\ U_i \times I & \xrightarrow{H} & B \end{array}$$

where H is a homotopy contracting U_i in B and $\widetilde{H}$ is the lift of the composition $H \circ (p \times 1)$ with $\widetilde{H}(x,0) = x$. Because $H(u,1) = b_0$, the basepoint of B, commutativity implies that $\widetilde{H}(x,1) \in F = p^{-1}(b_0)$. Now let $L\colon F \times I \to E$ be the nullhomotopy connecting i to the constant map at e_0. That is, $L(x,0) = x \in F$ and $L(x,1) = e_0$, the basepoint of E. Define a homotopy $J\colon p^{-1}(U_i) \times I \to E$ by

$$J(x,t) = \begin{cases} \widetilde{H}(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ L(\widetilde{H}(x,1), 2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $J(x,0) = x \in p^{-1}(U_i)$ and $J(x,1) = e_0$, so $p^{-1}(U_i)$ is contractible in E. Since $\bigcup_{i=1}^{n+1} \{p^{-1}(U_i)\} = E$, then $\text{cat}(E) \leq n = \text{cat}(B)$. □

Now, how does category behave under localization? The answer here is, it behaves well if the space is simply connected and pretty badly otherwise. If X is simply connected, then the Whitehead definition (see [Wh]) of category localizes properly and shows that

$$\text{cat}(X_0) \leq \text{cat}(X).$$

On the other hand, even for a simple space such as the circle, things are bad. Ganea [Ga2] first noticed that, since $S^1_0 \simeq K(\mathbb{Q},1)$ and $\mathbb{Q}$ is not a free group, then S^1_0 is not a co-H space, so, by Example 4.3.3, $\text{cat}(S^1_0) > 1$ (in fact, it is 2) while $\text{cat}(S^1) = 1$. For this reason, we must be careful when we talk about the category of localized spaces.

Theorem 4.5 : The Mapping Theorem [FH], [FL]. *If $f\colon X \to Y$ is a map of simply connected spaces such $f_{\#}\colon \pi_*(X)\otimes\mathbb{Q} \rightarrowtail \pi_*(Y)\otimes\mathbb{Q}$ is an injection, then*

$$\mathrm{cat}(X_0) \leq \mathrm{cat}(Y_0).$$

Proof. Turn $f\colon X \to Y$ into a fibration with homotopy fibre F included into X by $i\colon F \rightarrowtail X$. After rationalizing the spaces (which preserves the fibration), extend the fibration to a Barratt-Puppe sequence

$$\ldots \to \Omega Y_0 \xrightarrow{\partial} F_0 \xrightarrow{i} X_0 \xrightarrow{f} Y_0.$$

Here the maps are rationalized as well, but the subscript is suppressed for notational convenience. Now, because $f_{\#}$ is injective, exactness implies that $i_{\#} = 0$ and, therefore, $\partial_{\#}$ is surjective. But this is precisely the situation of Theorem 1.2, so there is a subproduct $K \subset \Omega Y_0$ such that the restriction $\partial|\colon K \to F_0$ is a homotopy equivalence. Let $\sigma\colon F \to K$ be the homotopy inverse of $\partial|$ (i.e. $\partial\circ\sigma \simeq 1_{F_0}$) and note that we then have

$$i \simeq i \circ 1_{F_0} \simeq i\circ\partial\circ\sigma \simeq *$$

since $i\circ\partial \simeq *$ by the homotopy exactness of the sequence. Therefore, i is nullhomotopic and we are in the situation of Lemma 4.4. Namely,

$$\mathrm{cat}(X_0) \leq \mathrm{cat}(Y_0)$$

which is what we wanted to prove. □

The proof of the mapping theorem exposes several other facts which lead to the link between LS category and the Gottlieb groups. First, for any fibration

$$\ldots \to \Omega B \xrightarrow{\partial} F \xrightarrow{i} E \xrightarrow{p} B$$

if $\partial_{\#} \neq 0$, then, since $\Omega B_0 \simeq \prod K(\mathbb{Q}, n_i)$, there exists a subproduct

$$K = \prod_{i=1}^{s} K(\mathbb{Q}, m_i) \subset \Omega B_0$$

that models Image($\partial_{\#}$). That, is $\partial_{\#}\colon \pi_*(K) \rightarrowtail \pi_*(F_0)$ is injective onto Image($\partial_{\#}$). The mapping theorem then says that

$$(**) \qquad \mathrm{cat}(K) \leq \mathrm{cat}(F_0).$$

Now, Proposition 4.2 says that $\text{cup}(K) \leq \text{cat}(K)$, so we have two cases:

$$\text{cat}(K) = \begin{cases} \infty & \text{if any } m_i \text{ is even} \\ s & \text{if all } m_i \text{ are odd} \end{cases}$$

where s is the number of factors in the product K which, of course, also is $\dim(\text{Image}(\partial_\#))$. Now, if F has finite category (i.e. $\text{cat}(F) < \infty$), then (**) and the estimate of $\text{cat}(K)$ above imply that all the m_i must be odd. Hence, $G_{2j}(F_0) = 0$ for all j. Integrally, this means that even degree Gottlieb groups of finite complexes, say, must be torsion groups. Furthermore, because $\text{cat}(K) = \dim(\text{Image}(\partial_\#))$ for any fibration of simply connected spaces with fibre F, we may create such an entity to model $G_{\text{odd}}(F)$. Namely, pick linearly independent elements $\alpha_1, \ldots, \alpha_r$ in odd degrees $n_1, \ldots, n_r$ respectively which span $G_{\text{odd}}(F_0)$ and form the pullback fibration

$$\begin{array}{ccccccc} \text{aut}(F_0) & \rightarrow & F_0 & \rightarrow & \text{Baut}_*(F_0) & \rightarrow & \text{Baut}(F_0) \\ \uparrow & & \| & & \uparrow & & \uparrow \vee_i B\bar{\alpha}_i \\ \Omega(\vee_i S^{n_i+1}) & \xrightarrow{\partial} & F_0 & \rightarrow & E & \rightarrow & \vee_i S^{n_i+1} \end{array}$$

where $B\bar{\alpha}_i$ is the delooping of chosen preimages of the α_i in $\pi_{n_i}(F_0^{F_0}, 1_{F_0}) \cong \pi_{n_i+1}(\text{Baut}(F_0))$. For this fibration,

$$\dim(G_{\text{odd}}(F_0)) = \dim(\text{Image}(\partial_\#)) = \text{cat}(K) \leq \text{cat}(F_0).$$

To summarize all this, we can write

Theorem 4.6. *If F_0 has finite LS category, then the Gottlieb groups of F obey*

$$\begin{cases} G_{\text{even}}(F_0) = 0 \\ \\ \dim(G_{\text{odd}}(F_0)) \leq \text{cat}(F_0) \end{cases}$$

So we see that there is this intimate connection between Gottlieb groups and LS category. Perhaps, in light of the simple Lemma 4.4, this isn't so surprising after all — or is it? I shall talk more about Gottlieb groups and rational homotopy when I introduce the minimal model. In particular, at that time I will state a tantalizing conjecture of Felix which the reader may ponder.

§5 Group Actions

As the range for connecting homomorphisms, the Gottlieb groups play an interesting role in the subject of group actions. We already have seen the example of the Conner-Raymond splitting theorem, but there is more. The relationship is made apparent by the following diagram of §3 (generalized to G).

$$(***)\qquad \begin{array}{ccc} \Omega BG & \xrightarrow{\partial} & X \\ \simeq\uparrow & \nearrow \omega & \\ G & & \end{array}$$

In other words, the orbit map ω is homotopically modeled by the connecting homomorphism ∂. Therefore, for example, if the action has a fixed point, then $\partial = 0$. The group $G(X)$ is especially important because S^1-actions form the fundamental atoms of compact transformation group theory and, since $\pi_i(S^1) = 0 = H_i(S^1)$ for $i > 1$, the only possible homotopical consequence of the orbit map must reside in $G(X)$. In particular, recently Greg Lupton and I were able to show that the image of the orbit map in $G(X)$ was of infinite order for a circle action on a symplectic manifold X obeying the condition that the symplectic class annihilates the image of Hurewicz [LO1]. Thus, knowing something about the center $\mathcal{Z}\pi_1(X)$ allows some knowledge of the action. For instance, if $\mathcal{Z}\pi_1(X)$ is finite for such a manifold, then *no* non-trivial circle actions are possible. I want to be clear about these interactions, so let me remind you of the connection between the orbit map and ∂.

First, if a compact Lie group, say, acts on a space X, then we would hope to learn about various qualities of the action, and therefore various symmetry properties of X, by studying the orbit space X/G. If G acts freely, then, indeed, this is a good way of initially approaching things, for the quotient map $X \to X/G$ is, in fact, a principal G-bundle. If the action is not free, then X/G may have a quite intricate structure which is not easily amenable to study. In the 1950's, Borel invented a substitute for the orbit space which *is* amenable to homotopical study, the so-called *Borel fibration.* To every G there is associated a universal principal G-bundle

$$G \to EG \to BG,$$

where EG is contactible with free G-action (so $BG = EG/G$), which classifies principal G-bundles over a space X in terms of the homotopy classes of pullback inducing maps $X \to BG$. Given an action $G \times X \to X$, $(g,x) \mapsto gx$, we can form the orbit space

$$XG = \frac{EG \times X}{G}$$

where G acts diagonally EG and X. Because G acts freely on EG, it acts freely on $EG \times X$, so this orbit space is not bad. There are two maps from XG which tell us something about it. First, since elements of XG are equivalence classes $[e,x]$, we can 'project' onto the equivalence class $[x] \in X/G$. It is not hard to show that this map $XG \to X/G$ is a homotopy equivalence *when the action is free.* In fact, although it is harder to show (since it involves the Leray spectral sequence), it is true that, if at each point x the isotropy group of the action defined by $G_x = \{g \in G | gx = x\}$ is finite, then $XG \to X/G$ is a rational homotopy equivalence. Therefore, the Borel space XG at least reduces homotopically to the right object under a free or *almost free* (i.e. finite isotropy) action. Secondly, we can project $[e,x]$ to the equivalence class $[e] \in BG$ to produce what is shown to be a fibration

$$X \to XG \to BG$$

called the *Borel fibration.* A good general reference for the cohomology theory and (rational) homotopy theory of compact group actions is [AP1]. Now we can show

Lemma 5.1. *The diagram* $(***)$ *is homotopy commutative.*

Proof. The Borel fibration and the action $G \times X \xrightarrow{A} X$ may be embedded in the following commutative diagram:

$$\begin{array}{ccccc} G & \rightarrow & G \times X & \xrightarrow{A} & X \\ \downarrow & & \downarrow & & \downarrow \\ EG & \rightarrow & EG \times X & \rightarrow & XG \\ \downarrow & & \downarrow & & \downarrow \\ BG & \xrightarrow{1_{BG}} & BG & \xrightarrow{1_{BG}} & BG. \end{array}$$

The composition of the maps in the top row is the orbit map ω. These fibrations may be extended to Barratt-Puppe sequences which give a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega BG & \xrightarrow{1_{\Omega BG}} & \Omega BG & \xrightarrow{1_{\Omega BG}} & \Omega BG \\ \simeq\downarrow & & \downarrow & & \downarrow \partial \\ G & \rightarrow & G \times X & \xrightarrow{A} & X. \end{array}$$

Since the bottom row is the orbit map and the top row is the identity, then, up to the equivalence $\Omega BG \simeq G$, we see that ω and ∂ are the same. □

Now orbit maps may be studied as connecting homomorphisms of the associated Borel fibrations. For example, the Transgression Theorem of [CG] immediately says that, if the Euler characteristic of X is non-zero, then the orbit map is trivial on cohomology. Various other such results may be found in [G7] and its references. More is even true once we put the fibration theory to work.

Example 5.2. Take the case of a circle action $S^1 \times X \to X$ on a manifold. If $\chi(X) \neq 0$, then the S^1-action on X has a fixed point. There are many ways to see this. In fact, it is true that the Euler characteristic of the fixed set and the space must be the same, so that the fixed set cannot be empty. From the fibration viewpoint though, we take the Borel fibration associated to the action $X \to XS^1 \xrightarrow{p} BS^1$ and consider the *Becker-Gottlieb Euler characteristic transfer for fibrations*

$$H^*(XS^1) \underset{p^*}{\overset{\tau}{\rightleftarrows}} H^*(BS^1)$$

where $\tau(p^*(\alpha)) = \chi(X) \cdot \alpha$ for $\alpha \in H^*(BS^1)$. Since $\chi(X) \neq 0$, then, for rational coefficients say, p^* is injective. But by the fundamental fixed point theorem of Borel (see [Hsi] or [AP1]), this is precisely the condition necessary to ensure a fixed point. Of course, the existence of a fixed point is stronger than simply saying that the orbit map is zero on homology — namely, the orbit map is homotopic to a constant at all points.

Now let's look at some elementary rational homotopy theoretic considerations concerning group actions. We still don't need the minimal model (at least to state the theorems), but we're getting close. The first thing to note is that, according to rational homotopy theorists, there are two types of simply connected spaces of finite LS category: *elliptic and hyperbolic spaces.* An elliptic space X is one for which both

$$\dim H_*(X;\mathbb{Q}) < \infty \quad \text{and} \quad \dim(\pi_*(X) \otimes \mathbb{Q}) < \infty.$$

A hyperbolic space is one with finite dimensional rational homology, but infinite dimensional rational homotopy. In fact, if X is hyperbolic, the sequence

$$\rho_i = \sum_{j<i} \dim(\pi_j(X) \otimes \mathbb{Q})$$

has exponential growth. See [F] for the full story on this dichotomy.

Example 5.3. Spheres, complex projective spaces, Lie groups and, more generally, homogeneous spaces G/H are elliptic spaces.

Definition 5.4. Let X be an elliptic space. The *homotopy Euler characteristic* of X is defined to be

$$\chi_\pi(X) = \sum_{i>0} (-1)^i \dim(\pi_i(X) \otimes \mathbb{Q}).$$

The exact homotopy sequence of a fibration immediately gives

Proposition 5.5. *If $F \to E \to B$ is a fibration, then*

$$\chi_\pi(E) = \chi_\pi(B) + \chi_\pi(F).$$

The fundamental theorem about elliptic spaces and their homotopy Euler characteristics is due to S. Halperin [H2].

Theorem 5.6. *If X is an elliptic space, then $\chi_\pi(X) \le 0$ and $\chi(X) \ge 0$. Moreover, the following are equivalent:*

i $\chi_\pi(X) = 0$,

ii $\chi(X) > 0$,

iii $H^*(X;\mathbb{Q})$ *is a polynomial algebra modulo an ideal generated by a regular sequence.*

Recall that a compact Lie group G has rational cohomology an exterior algebra on a finite set of odd degree generators,

$$H^*(G;\mathbb{Q}) = \Lambda(x_1, \dots, x_s).$$

Because cohomology classifies maps $G \to K(\mathbb{Q}, j)$, we can define a map

$$G \to \prod_{i=1}^{s} K(\mathbb{Q}, |x_i|)$$

which is an isomorphism on rational cohomology. Hence, it is a rational equivalence. The number of factors s in the product is called the *rank* of G. Notice that this is precisely $\dim((\pi_*(G) \otimes \mathbb{Q})$. This leads to

Theorem 5.7 [AH]. *If a compact Lie group acts almost freely on a finite elliptic complex X, then*

$$rank(G) \leq -\chi_\pi(X).$$

Proof. Recall that G acts almost freely if every isotropy group is finite and, in this case, XG has the same rational homotopy type as X/G. In particular, this implies that $\dim H^*(XG;\mathbb{Q}) < \infty$. Now, X is elliptic and $\dim(\pi_*(BG)\otimes\mathbb{Q}) < \infty$ as well (since $\pi_{i+1}(BG) \cong \pi_i(G)$), so the exact homotopy sequence associated to the Borel fibration $X \to XG \to BG$ shows that $\dim(\pi_*(XG)\otimes\mathbb{Q}) < \infty$. Combining this with the finite dimensionality of XG's rational cohomology demonstrates that XG is elliptic.

By Halperin's theorem, $\chi_\pi(XG) \leq 0$. Also, by Proposition 5.5,

$$\chi_\pi(XG) = \chi_\pi(BG) + \chi_\pi(X).$$

Now, we have the following computation for $\chi_\pi(BG)$.

$$\begin{aligned}\chi_\pi(BG) &= \sum_{i>1}(-1)^i \dim(\pi_i(BG)\otimes\mathbb{Q}) \\ &= \sum_{i>0}(-1)^i \dim(\pi_{i-1}(G)\otimes\mathbb{Q}) \\ &= \dim(\pi_*(G)\otimes\mathbb{Q}) \\ &= \mathrm{rank}(G).\end{aligned}$$

Then, rearranging the equation above and substituting for $\chi_\pi(BG)$ gives

$$\mathrm{rank}(G) + \chi_\pi(X) = \chi_\pi(XG) \leq 0,$$

or, rather, $\mathrm{rank}(G) \leq -\chi_\pi(X)$. □

Example 5.8. Let $X = H/K$, a homogeneous space. The formula of Proposition 5.5 applied to the fibration $H/K \to BK \to BH$ shows that $\chi_\pi(H/K) = \mathrm{rank}(K) - \mathrm{rank}(H)$. The theorem above then says that any compact Lie group acting almost freely on H/K must have

$$\mathrm{rank}(G) \leq \mathrm{rank}(H) - \mathrm{rank}(K).$$

In fact, if $\mathrm{rank}(H) = r_H$ and $\mathrm{rank}(K) = r_K$, then for T^{r_H} a maximal torus of H, there is a subtorus T^{r_K} which is a maximal torus for K. The complementary subtorus $T^{r_H - r_K}$ acts almost freely on H/K and this is the largest rank possible by the inequality. Hence, the largest rank of a torus which can act almost freely on H/K is $\mathrm{rank}(H) - \mathrm{rank}(K)$.

The *toral rank* of a space X is the dimension of the largest torus which can act almost freely on the space. Halperin conjectured that, if the toral rank of a (finite) space X is r, then

$$\dim H^*(X;\mathbb{Q}) \geq 2^r.$$

This conjecture is known for various spaces including spaces which, in some way, resemble Kähler manifolds (see [LO1], [A], [AP2]). We also have,

Example 5.9. Let a torus T^r act almost freely on a homogeneous space H/K with $r = \mathrm{rank}(H) - \mathrm{rank}(K)$. Then Halperin's conjecture is true in this case. That is,

$$\dim H^*(H/K;\mathbb{Q}) \geq 2^r.$$

To see this, just consider the fibration $K \to H \to H/K$ and note that the Serre spectral sequence ensures that, for any fibration $F \to E \to B$, $\dim H^*(E;\mathbb{Q}) \leq \dim H^*(B;\mathbb{Q}) \cdot \dim H^*(F;\mathbb{Q})$. Hence,

$$\dim H^*(H;\mathbb{Q}) \leq \dim H^*(K;\mathbb{Q}) \cdot \dim H^*(H/K;\mathbb{Q})$$

and, dividing by $\dim H^*(K;\mathbb{Q})$, we obtain

$$2^r = 2^{r_H - r_K} = \frac{2^{r_H}}{2^{r_K}} = \frac{\dim H^*(H;\mathbb{Q})}{\dim H^*(K;\mathbb{Q})} \leq \dim H^*(H/K;\mathbb{Q}).$$

Remark 5.10. The same types of arguments may be applied to the Samelson space K associated to a fibration [O2]. For example, if F is elliptic, then for any fibration with fibre F,

$$\mathrm{rank}(K) \leq -\chi_\pi(F).$$

Consequently, if F is elliptic and $\chi_\pi(F) = 0$, then the Samelson space is trivial for every fibration in which F is the fibre.

Now let's mix in the Gottlieb group by making a small modification in the proof of [H3 Theorem 1.1]. We shall only consider nice spaces which, in particular, have finite cohomological dimension $cd_{\mathbb{Q}} < \infty$ and for which the rationalized Gottlieb group $G(X) \otimes \mathbb{Q}$ is finite dimensional.

Theorem 5.11 [H3]. *Let X be a connected finite dimensional CW complex with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i > m$. Then*

$$\textit{toral rank}(X) \leq \textit{dim}(G(X) \otimes \mathbb{Q}) - \sum_{i=2}^{m} (-1)^i \textit{dim}(\pi_i(X) \otimes \mathbb{Q}).$$

Proof Sketch. Let T^r act almost freely on X and consider the homomorphism induced by the orbit map on fundamental groups, $\omega_{\#} \colon \pi_1(T^r) \to \pi_1(X)$. We know of course that $\text{Image}(\omega_{\#}) \subset G(X)$, so $\text{Image}(\omega_{\#}) = \oplus_{i=1}^{s} \mathbb{Z} \cdot \alpha_i \oplus \text{torsion}$, where $\alpha_1, \ldots, \alpha_s$ are linearly independent elements in $G(X) \otimes \mathbb{Q}$. Hence, $s \leq \dim(G(X) \otimes \mathbb{Q})$.

There exists a subtorus $T^s \subset T^r$ which realizes $\text{Image}(\omega_{\#})$ and a complementary subtorus $T^{r-s} \subset T^r$ whose fundamental group image is finite. Therefore, the image for this torus can be killed off by taking a finite cover of T^{r-s} — a cover which is also a torus T^{r-s}, but one which has trivial π_1-image. General facts about transformation groups then say that the action of this torus may be lifted to an almost free action on the universal cover of X, $\widetilde{X}$. But then the Allday-Halperin inequality (Theorem 5.7) holds and we get

$$r - s \leq -\sum_{i=2}^{m} (-1)^i \dim(\pi_i(X) \otimes \mathbb{Q})$$

since the homotopy of X and $\widetilde{X}$ agree. Then moving s to the right and bounding it above by $s \leq \dim(G(X) \otimes \mathbb{Q})$ gives

$$\text{toral rank}(X) \leq \dim(G(X) \otimes \mathbb{Q}) - \sum_{i=2}^{m} (-1)^i \dim(\pi_i(X) \otimes \mathbb{Q}).$$

□

Of course we can consider finite group actions as well as connected group actions. In particular, one problem which arose over the years was to determine the Gottlieb group of the orbit space of a finite group acting freely on a (necessarily odd) sphere. In [O5] I developed an obstruction theory for this problem based on a lifting theorem of Gottlieb which we now consider.

Let $\alpha \in G(X)$ and denote by $A \colon S^1 \times X \to X$ an associated map with $A|_{S^1} = \alpha$ and $A|_X = 1_X$. The induced map on cohomology gives,

$$A^*(x) = 1 \otimes x + \lambda \otimes x_A$$

where $x \in H^n(X)$ and λ is a chosen generator of $H^1(S^1)$. Note that $x_A \in H^{n-1}(X)$ and, although x_A depends on λ, we do not denote this. Recall that a fibration $p: E \to B$ is a *principal* $K(\pi,r)$ *fibration* if it is a pullback of the path fibration $K(\pi,r) \to PK(\pi,r+1) \xrightarrow{\rho} K(\pi,r+1)$ by a map $k: B \to K(\pi,r+1)$. If $\iota \in H^{r+1}(K(\pi,r+1);\pi)$ is the characteristic class, then let $k^*(\iota) = \mu \in H^{r+1}(B;\pi)$ and recall that a map $f: Y \to B$ has a lifting $\overline{f}: Y \to E$ if and only if $f^*(\mu) = 0$.

Theorem 5.12 [G2]. *Let $p: E \to B$ be a principal $K(\pi,r)$ fibration and let $A: S^1 \times B \to B$ be a map with $A|_B = 1_B$. Then, there exists a map $\overline{A}: S^1 \times E \to E$ with $\overline{A}|_E = 1_E$ and a commutative diagram*

$$\begin{array}{ccc} S^1 \times E & \xrightarrow{\overline{A}} & E \\ 1_{S^1} \times p \downarrow & & \downarrow p \\ S^1 \times B & \xrightarrow{A} & B \end{array}$$

if and only if $\mu_A = 0 \in H^r(B;\pi)$.

Remark 5.13. This type of lifting result has been generalized several times over the years to include extensions of the notion of Gottlieb group to Gottlieb sets defined by *cyclic maps*. For example, see [HV] or [Hoo]

The obstruction theory of [O5] essentially just proceeds up the Postnikov tower — when a Postnikov tower exists. Also, it is shown in [O5] that, to ensure the existence of a Gottlieb group element in $G(X)$, it is sufficient to check that all obstructions vanish up to $X(n)$ in the tower, where n is the dimension of X. The obstruction theory allows an analysis of Ganea's infinite dimensional example of a space with $G(X) = \{1\}$, but $P(X) = \pi_1(X)$. Compare this with the finite dimensional example in Example 3.10.

Example 5.14 [Ga1]. Construct the Ganea space X as a principal $K(\mathbb{Z}/2,2)$ fibration over $\mathbb{R}P(\infty) = K(\mathbb{Z}/2,1)$ induced by the nontrivial element of

$$H^3(\mathbb{R}P(\infty);\mathbb{Z}/2) \cong \mathbb{Z}/2 = [\mathbb{R}P(\infty), K(\mathbb{Z}/2,3)].$$

As the pullback of the fibration $K(\mathbb{Z}/2.2) \to PK(\mathbb{Z}/2,3) \to K(\mathbb{Z}/2,3)$, X is a simple space. Hence, $P(X) = \pi_1(X) = \mathbb{Z}/2$. Now, the cohomology element inducing the pullback is the cube ι^3 of the polynomial algebra generator ι in degree 1. Let α be the nontrivial element of $\pi_1 X = \mathbb{Z}/2$.

We have $A\colon S^1 \times K(\mathbb{Z}/2,1) \to K(\mathbb{Z}/2,1)$ with $A|S^1$ the generator $\alpha \in \mathbb{Z}/2$ as in the discussion following Example 2.5 and, by Theorem 5.12, we know the obstruction to the existence of a lift $\overline{A}$ is $\iota_A^3 \in H^2(K(\mathbb{Z}/2,1);\mathbb{Z}/2)$. We can compute ι_A^3 by,

$$\begin{aligned} A^*(\iota^3) &= (A^*(\iota))^3 \\ &= (1 \otimes \iota + \lambda \otimes 1)^3 \\ &= 1 \otimes \iota^3 + \lambda \otimes \iota^2. \end{aligned}$$

Hence, $\iota_A^3 = \iota^2 \neq 0$. Therefore a lift of A does not exist and $G(X) = \{1\}$.

Theorem 5.15 [OT], [O5], [B]. *If H is a finite group which acts freely on an odd dimensional sphere, then the Gottlieb group of the orbit space of the action is equal to the center of H.*

My proof of this result in [O5] relies on constructing a Moore-Postnikov tower for the orbit space S^{2n-1}/H *after the first stage.* Then the obstruction theory I have referred to takes over and only one obstruction exists to *any element of the center of H being in $G(X)$*. That obstruction, by the general lifting result Theorem 5.12 above, lies in $H^{2n-1}(H;\mathbb{Z})$. But it is a general fact that, if a finite group H acts freely on a sphere, then all odd dimensional cohomology vanishes (see [O5] for a proof). Hence, $H^{2n-1}(H;\mathbb{Z}) = 0$, the obstruction is trivial and all elements of $\mathcal{Z}H$ are in $G(S^{2n-1}/H)$. Combined with the fact that, generally, $G(X) \subseteq \mathcal{Z}\pi_1$, we see that $G(S^{2n-1}/H) = \mathcal{Z}H$. I should mention that after all this was done, I was sent the paper [OT] which computes this result in a very different way. It might be interesting to compare these approaches. I have been rather vague about the obstruction theory because I want to present some other results about finite groups which will allow a slick proof of Theorem 5.15 in the case of linear actions.

In [L2], George Lang studied the orbit spaces of Lie groups by finite subgroups. I'm going to change his notation a bit to keep in line with the notation of this paper, so let G be a simply connected (compact) Lie group and $\pi \subset G$ a *finite* subgroup. Of course π acts freely on G by left translations, so we have an orbit space G/π which has universal cover G.

Theorem 5.16 [L2]. *With the notation above*

$$\pi \cap \mathcal{Z}G \subseteq G(G/\pi).$$

Proof. Let $\alpha \in \pi \cap \mathcal{Z}G$ thought of as a covering transformation via left translation. By Theorem 2.6, if α is equivariantly homotopic to 1_G, then $\alpha \in G(G/\pi)$. Take a path $\xi\colon I \to G$ with $\xi(0) = \alpha$ and $\xi(1) = e$ (where e is the identity of G). Then it is straightforward to show (using the fact that $\alpha \in \mathcal{Z}G$) that the required equivariant homotopy is obtained as

$$H\colon G \times I \to G \qquad H(g,t) = g \cdot \xi(t).$$

□

Theorem 5.17 [L2]. *If $\mathcal{Z}\pi$ lies in a path component of the centralizer of π in G, $\mathcal{Z}_G\pi$, then*

$$G(G/\pi) = \mathcal{Z}\pi_1(G/\pi) = \mathcal{Z}\pi.$$

Proof. Let $\alpha \in \mathcal{Z}\pi$ and note that the hypothesis provides the existence of a path $\xi\colon I \to \mathcal{Z}_G\pi \subset G$ with $\xi(0) = \alpha$ and $\xi(1) = e$. Then an equivariant homotopy from α to 1_G is given by

$$H\colon G \times I \to G \qquad H(g,t) = \xi(t) \cdot g.$$

This uses the fact that the entire path is contained in π's centralizer in G. Hence, any element of $\mathcal{Z}\pi$ is in $G(G/\pi)$. □

Remark 5.18. In [L2], Lang also makes some interesting calculations of Gottlieb groups for complex projective space and Stiefel manifolds.

Now, how does S. A. Broughton [B] use Lang's results to study $G(S^{2n-1}/H)$? Consider the case where H is a subgroup of the unitary group $U(n)$ acting on $\mathbb{C}^n$ *linearly.* Decompose $\mathbb{C}^n$ into irreducibles $\mathbb{C}^n = \bigoplus_{i=1}^{k} V_i$. Let $\alpha \in ZH$ and note that, by Schur's lemma, because α commutes with every element of H, α acts on each V_i as multiplication by a scalar λ_i. Also note that $|\lambda_i| = 1$ since α is of finite order. Now observe that the k-torus $T^k = S^1 \times \ldots \times S^1$ acts on $\mathbb{C}^n$ unitarily with each element of the j^{th} circle acting by scalar multiplication on V_j (*and thus commuting with* H). Therefore, the connected torus T^k is in the centralizer of H in $U(n)$. Hence we have,

$$ZH \subseteq S^1 \times \ldots \times S^1 \subseteq \mathcal{Z}_{U(n)}H$$

and by Theorem 5.17, $G(S^{2n-1}/H) = \mathcal{Z}H$.

Before I leave this section, I can't resist presenting an example which is not connected with the Gottlieb group, but which is instructive in understanding the Euler characteristic. This type of example must have been known in the 1950's when the Serre spectral sequence proof of the Euler characteristic formula for a fibration $F \to E \to B$

$$\chi(E) = \chi(B) \cdot \chi(F)$$

was given, but I know of no reference to it, so here it is.

Example 5.19. Let $\pi \neq \{1\}$ be a finite group which acts freely on a finite π-CW complex Y with $\chi(Y) \neq 0$. Therefore, $Y \xrightarrow{p} X = Y/\pi$ is a covering map and the following formula holds:

$$\chi(Y) = |\pi|\,\chi(X).$$

Hence, $\chi(X) \neq 0$ as well. The fibrations $E\pi \times \pi \to E\pi \times Y \xrightarrow{*\times p} X$ and $\pi \times Y \to E\pi \times Y \xrightarrow{q\times *} B\pi$ induce fibrations on quotients

$$\begin{array}{ccccc} F & \to & Y\pi & \to & X \\ \\ \mathcal{F} & \to & Y\pi & \to & B\pi. \end{array}$$

The fibres are the π-quotients of the fibres of the inducing fibrations:

$$F = (E\pi \times \pi)/\pi = E\pi \qquad\qquad \mathcal{F} = (\pi \times Y)/\pi = Y.$$

First, since $F = E\pi$ is contractible, $Y\pi \simeq X$, so $\chi(Y\pi) = \chi(X)$. But, secondly, *if the Euler characteristic formula holds in general*, then

$$\chi(Y\pi) = \chi(\mathcal{F}) \cdot \chi(B\pi) = \chi(Y) \cdot 1 = \chi(Y)$$

since, for a finite group, all rational homology vanishes above degree zero and $H_*(B\pi; \mathbb{Q}) \cong H_*(\pi; \mathbb{Q})$. But then

$$\chi(X) = \chi(Y\pi) = \chi(Y) = |\pi|\,\chi(X)$$

which is absurd since we have neither $\chi(X) = 0$ nor $|\pi| = 1$. Hence, for this infinite dimensional example, the Euler charactersitic formula for fibrations cannot hold.

A *DG algebra* is a pair $(\mathcal{A}, d_\mathcal{A})$, where $\mathcal{A}$ is a graded commutative, associative algebra and $d_\mathcal{A}$ is a degree $+1$ differential of $\mathcal{A}$. Any DG algebra $(\mathcal{A}, d_\mathcal{A})$ that we consider here satisfies $H^0(\mathcal{A}, d_\mathcal{A}) = \mathbb{Q}$ and $H^n(\mathcal{A}, d_\mathcal{A})$ is a finite dimensional vector space for each n. We denote the ideal of positive degree elements in an algebra $\mathcal{A}$ by $\mathcal{A}^+$. If V is a vector space, then ΛV denotes the free graded commutative algebra generated by V. If $\{v_1, v_2, \dots\}$ is a basis for V, then we write $V = \langle v_1, v_2, \dots \rangle$ and $\Lambda V = \Lambda(v_1, v_2, \dots)$. A DG algebra is *minimal* if (1) $\mathcal{A} \cong \Lambda V$, as an algebra, for some V and (2) there is a basis $V = \langle v_1, v_2, \dots \rangle$ such that, for each j, $dv_j \in (\Lambda(v_1, \dots, v_{j-1}))^+(\Lambda(v_1, \dots, v_{j-1}))^+$. In particular, differentials of generators are decomposable in a minimal DG algebra. We will write a minimal DG algebra as $(\Lambda V, d)$, or $\Lambda(v_1, v_2, \dots; d)$ if $V = \langle v_1, v_2, \dots \rangle$. Any DG algebra $(\mathcal{A}, d_\mathcal{A})$ has a *minimal model* (i.e. a minimal DG algebra $(\Lambda V, d)$ with a DG homomorphism $\rho : (\Lambda V, d) \to (\mathcal{A}, d_\mathcal{A})$ such that the induced homomorphism on cohomology ρ^* is an isomorphism).

To any space X, with finite-type rational homology, is associated a *minimal model* $(\Lambda V, d)$, where V is a positively graded vector space and ΛV is a freely generated *differential graded (commutative) algebra* (DGA) which is polynomial on even degree generators, exterior on odd degree generators and which has a decomposable differential d. For a *nilpotent space* (e.g. an H-space or a simply connected space), the minimal model $(\Lambda V, d)$ 'models' X in the sense that there are natural isomorphisms $H^*(X; \mathbb{Q}) \cong H^*((\Lambda V, d))$ and $\mathrm{Hom}(\pi_\#(X), \mathbb{Q}) \cong V$. A basic theorem of rational homotopy theory asserts that each nilpotent space X has a minimal model, which contains all the rational homotopy information about the space.

The minimal model of X is constructed from a (non-free) differential graded commutative algebra of rational polynomial forms $A^*(X)$, akin to de Rham forms on a smooth manifold. The construction results in a homomorphism $\rho\colon (\Lambda V, d) \to A^*(X)$ which induces an isomorphism of cohomology. This map ρ enjoys the following lifting property: Given a DGA homomorphism $\phi\colon \mathcal{B} \to A^*(X)$ which induces a cohomology isomorphism, there is a lift $\tilde{\rho}\colon (\Lambda V, d) \to \mathcal{B}$ with $\phi \circ \tilde{\rho} \simeq \rho$. Here, '$\simeq$' denotes DGA-homotopy. This lifting property, together with the fact that a cohomology isomorphism between minimal models is in fact an isomorphism, is sufficient to establish the uniqueness of the minimal model up to isomorphism. Furthermore, a map of spaces $f\colon Y \to Z$ induces (in the usual fashion) a map of forms $f^*: A^*(Z) \to A^*(Y)$. From the lifting property again, we obtain a map of minimal models $(\Lambda V_Z, d_Z) \to (\Lambda V_Y, d_Y)$, unique up to DGA-homotopy.

A minimal DG algebra $\Lambda(V; d)$ is *elliptic* if the graded vector space V and the cohomology $H^*(\Lambda(V; d))$ are finite dimensional. In this case, the *homotopy Euler characteristic* is defined as

$$\chi_\pi(\Lambda) = \dim V^{\text{even}} - \dim V^{\text{odd}}$$

and this, in light of the isomorphism $V \cong \pi_*(X) \otimes \mathbb{Q}$, is the same as our previous definition.

The fact that a space's minimal model contains within it all rational information about the space is precisely what is meant by saying that rational homotopy is *algebraic*. Indeed, all of rational homotopy may be considered from the algebraic point of view. This includes the study of fibrations. The fundamental result about fibrations and minimal models is the

Grivel-Halperin-Thomas Theorem 6.1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with $\pi_1(B) = \{1\}$ (say) and F, E and B of finite type. Let $(\Lambda V, d) \xrightarrow{\phi} A^*(B)$ be a minimal model for B. Then the following diagram is commutative*

$$\begin{array}{ccccc} A^*(B) & \xrightarrow{p^*} & A^*(E) & \xrightarrow{i^*} & A^*(F) \\ \phi \uparrow & & \psi \uparrow & & \theta \uparrow \\ (\Lambda V, d) & \xrightarrow{j} & (\Lambda V \otimes \Lambda W, D) & \rightarrow & (\Lambda W, \bar{D}) \end{array}$$

and ϕ, ψ and θ induce isomorphisms on cohomology. Hence, $\theta: (\Lambda W, \bar{D}) \to A^*(F)$ is a minimal model for the fibre F.

Remark 6.2.

1. The middle DGA $(\Lambda V \otimes \Lambda W, D)$ may not be minimal, but it obeys the rule that, for some well-ordered basis (w_i) of W, $D(w_i) \in \Lambda V \otimes \Lambda W_{<i}$. The bottom row is then called a *Koszul-Sullivan (or K. S.) extension.*
2. The results listed above are found in many sources, including [Su], [GM] and [H4]. The last, in particular, contains a proof of Theorem 6.1.
3. There is a converse to Theorem 6.1 in the sense that any K. S. extension may be realized by a fibration of spaces up to homotopy. This generalizes Sullivan's spatial realization of a DGA.

Example 6.3. Consider an even sphere S^{2n}. To create the minimal model, we need a cocycle in degree $2n$ which maps to the cocycle in $A^*(S^{2n})$ representing the fundamental class of S^{2n}. Call this generator x and take the c.d.g.a. freely generated by it, $(\Lambda(x_{2n}), d=0)$. Now $(\Lambda(x_{2n}), d=0)$ itself maps to $A^*(S^{2n})$ because x freely generates it. We do not yet have a model for S^{2n} since all higher powers of x represent nontrivial cohomology classes in $\Lambda(x)$. Thus these must be killed by the addition of a generator y in degree 4n-1 with $dy = x^2$. It is easy to see that defining d on y in this way kills all extra cohomology. Moreover, we can map y to $A^*(S^{2n})$ by taking it to zero. The minimal model of S^{2n} is then

$$(\Lambda(x_{2n}, y_{4n-1}), dx = 0, dy = x^2).$$

Note that the definition of d on y is precisely the definition of the corresponding k-invariant in the rational Postnikov tower for S^{2n} and that the existence of generators only in degrees $2n$ and $4n-1$ reflects Serre's theorem that even spheres have finite homotopy groups except in those degrees. Similarly, an odd dimensional sphere S^{2n-1} has a minimal model $(\Lambda(x_{2n-1}), d=0)$.

Example 6.4. Consider the extension (where subscripts denote degrees)

$$(\Lambda(e_2), d=0) \to (\Lambda(e_2, x_4, y_7, z_9), D) \to (\Lambda(x_4, y_7, z_9), \bar{D})$$

where

$$D(e) = 0 \quad D(x) = 0 \quad D(y) = x^2 + ae^4 \quad D(z) = e^5.$$

Here, $a \in \mathbb{Q}$ and, for concreteness we take $a = 1$ and $a = 2$. Note that the middle DGA is minimal in this case. There are several interesting things about the minimal DGA $(\Lambda(e_2, x_4, y_7, z_9), D)$. First, it is elliptic. To see this, note that it also arises as the middle of an extension

$$(\Lambda(e_2, z_9), dz = e^5) \to (\Lambda(e_2, x_4, y_7, z_9), D) \to (\Lambda(x_4, y_7), dy = x^2)$$

which models the fibration $S_0^4 \to E \to \mathbb{C}P^5$. Because both base and fibre have finite dimensional rational cohomology, so does E by a Serre spectral sequence argument. Secondly, since $\Lambda = (\Lambda(e_2, x_4, y_7, z_9), D)$ is elliptic, a result of Halperin [H2] tells us that it satisfies Poincaré duality with top dimension 12. General results of Sullivan [Su] and [Ba] then say that there is a 12-dimensional smooth manifold with the rational homotopy type of Λ (for either $a = 1$ or $a = 2$). Now denote the $a = 1$ DGA by Λ_1 with corresponding manifold M_1 and the $a = 2$ DGA by Λ_2 with corresponding manifold M_2 and define a DGA homomorphism $\theta\colon \Lambda \otimes \mathbb{R} \to \Lambda_2 \otimes \mathbb{R}$ by

$$\theta(e) = e \quad \theta(x) = \frac{1}{\sqrt{2}}x \quad \theta(y) = \frac{1}{2}y \quad \theta(z) = z.$$

It is not hard to check that θ commutes with the differential and, therefore is a DGA homomorphism. Indeed, θ is a DGA *isomorphism* since it induces an isomorphism of V to V and the DGA's are freely generated by V. This means that *the real homotopy types of M_1 and M_2 are the same.* These are the homotopy types which might be expected to arise from smooth de Rham forms.

We can try to define an isomorphism θ over $\mathbb{Q}$ by

$$\theta(e) = \lambda e \quad \theta(x) = \sigma x + \rho e^2 \quad \theta(y) = \epsilon y \quad \theta(z) = \tau z + \mu e y$$

where the definition is the most general possible given the degrees of the basis elements. Then, imposing the condition that θ must commute with the differentials gives either $\epsilon = 0$, in which case $\theta(y) = 0$ and θ is not an isomorphism or

$$2 = \left(\frac{\lambda^2}{\sigma}\right)^2 .$$

But λ and σ are in $\mathbb{Q}$, so this a contradiction and no such θ exists over $\mathbb{Q}$. Thus M_1 and M_2 have different *rational* homotopy types. This is a descent phenomenon in homotopy theory analogous to that found in algebra — as might be expected from a completely algebraic version of topology.

Finally, the original extension models the Borel fibration associated to a circle action. To see this, spatially realize Λ by a space Z and use either a Postnikov tower or a homology decomposition to get a space Y of finite type over $\mathbb{Z}$ with $Y \to Z$ a rational equivalence. Hence, $e \in H^2(Z;\mathbb{Q}) \cong H^2(Y;\mathbb{Q})$, so some integral multiple $Ne \in H^2(Y;\mathbb{Z})$. Then use Ne to classify a principal bundle

$$\begin{array}{ccc} X & \to & ES^1 \\ \downarrow & \text{pull} & \downarrow \\ Y & \xrightarrow{Ne} & BS^1 \end{array}$$

so that X inherits a free circle action. Now, denoting the spatial realization of $(\Lambda(x,y,z),\bar{D})$ by $\mathcal{F}$, we can get a mapping of fibrations

$$\begin{array}{ccccc} X_0 & \to & Y_0 & \xrightarrow{Ne} & BS^1_0 \\ \downarrow & & \downarrow\simeq & & \simeq\downarrow \frac{1}{N} \\ \mathcal{F} & \to & Z_0 & \xrightarrow{e} & BS^1_0 \end{array}$$

which induces a homotopy equivalence of fibres $X_0 \to \mathcal{F}$. Therefore, the fibration $X \to Y \to BS^1$ realizes the original K. S. extension and, since X is a principal S^1-bundle over Y, then $Y = X/S^1 = XS^1$ and we obtain the Borel fibration.

A K. S. extension has its own long exact sequence of 'generators' corresponding to the long exact homotopy sequence. This sequence is defined in terms of the twisted differential D and, it turns out that the connecting homomorphism is given by the linear part of D and, thus, measures the deviation of the middle DGA from being minimal.

Example 6.5. Consider the K. S. extension which models the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$,

$$\begin{array}{ccccc} & & A^*(S^3) & \to & A^*(S^1) \\ & \phi \nearrow & \uparrow \psi & & \uparrow \theta \\ (\Lambda(x_2, y_3), dy = x^2) & \to & (\Lambda(x_2, y_3, z_1), D) & \to & (\Lambda(z_1), d = 0) \end{array}$$

where $\phi(x) = 0 = \psi(x)$, $\phi(y) = -\nu = \psi(y)$ (ν a volume form on S^3), $\psi(z) = 0$ and

$$D(x) = 0 \quad D(y) = x^2 \quad D(z) = x.$$

Because $D(z) = x$ is linear, and not decomposable, the middle DGA is not minimal and the DGA connecting homomorphism takes z to x, the dual of the Hopf fibration connecting homomorphism $\pi_2(S^2) \cong \pi_1(S^1)$.

Now that we have seen how the connecting homomorphism of a fibration is modeled, let's look at the rational Gottlieb groups themselves [FH], [F]. For a space X with minimal model $\Lambda = (\Lambda V, d)$, let

$$G_n^{\psi}(X) = \{f\colon V^n \to \mathbb{Q} \mid f \text{ is linear and extends to a } -n\text{-derivation } \theta \text{ of } \Lambda \text{ with } \theta d - (-1)^n d\theta = 0\}.$$

Proposition 6.6. $G_n^{\psi}(X) = G_n(X_0)$.

Proof. The fact that f extends to a derivation is equivalent to having a DGA homomorphism

$$A\colon \Lambda \to H^*(S^n; \mathbb{Q}) \otimes \Lambda$$

and this may be realized at the space level by a map associated to a Gottlieb element. □

Example 6.7. As a somewhat trivial example, recall that, except for the cases $n = 1, 3, 7$ where $G_n(S^n) = \mathbb{Z}$, odd dimensional spheres have $G_{2n+1}(S^{2n+1}) = 2\mathbb{Z}$. Over $\mathbb{Q}$ this means that $G_{2n+1}(S_0^{2n+1}) = \mathbb{Q}$. We can see this in (at least) two ways. First, the minimal model of S^{2n+1} is $\Lambda = \Lambda(z_{2n+1}), d = 0)$ and the proposition bids us look for linear maps $f\colon V^{2n+1} \to \mathbb{Q}$ which extend to derivations of the appropriate sort. But this is immediately satisfied by defining $f(z) = 1$ and extending to $\theta(1) = 0$. Therefore, $G_n^{\psi}(S^{2n+1}) \cong \mathbb{Q}$ generated by f. Secondly, consider the path fibration

$$K(\mathbb{Q}, 2n+1) \to PK \to K(\mathbb{Q}, 2n+2)$$

and model it by a K. S. extension. Because the middle DGA must have trivial cohomology in positive degrees (since PK is contractible), the extension must have the form

$$(\Lambda(z_{2n+2}), d = 0) \to (\Lambda(z_{2n+2}, y_{2n+1}), D) \to (\Lambda(y_{2n+1}), d = 0)$$

where $D(z) = 0$ and $D(y) = z$. Since the latter has a linear part z, the connecting homomorphism of the K. S. extension takes y to z, dual to the isomorphism

$$\partial_{\#}\colon \pi_{2n+2}(K(\mathbb{Q}, 2n+2)) \xrightarrow{\cong} \pi_{2n+1}(K(\mathbb{Q}, 2n+1)).$$

There is a conjecture about rational Gottlieb groups due to Felix which everyone working in the area should keep in mind. The conjecture is an analogue of the following fact: if X is an elliptic space (which we know satisfies Poincaré duality) with top cohomology class in degree n, then there can be no generators of the minimal model of X in degrees greater than $2n-1$.

Conjecture 6.8 [F]. *If X is a finite simply connected CW complex of dimension n, then, for $i > 2n-1$,*

$$G_i(X) \otimes \mathbb{Q} = 0.$$

While the Gottlieb group has always played a role in fixed point theory through the Jiang condition, rational homotopy theory can make its presence felt as well. In [H5], Halperin showed that the Lefschetz number of a self-map of an elliptic space is determined by the induced *homotopy* homomorphism.

Theorem 6.9. *Let X be an elliptic space with minimal model $\Lambda = (\Lambda(V) \otimes \Lambda(W), d)$ where $V = \langle y_1, \dots, y_r \rangle$ with $|y_i| = 2a_i$ even and $W = \langle x_1, \dots, x_q \rangle$ with $|x_i| = 2b_i - 1$ odd. For a self-map $f \colon X \to X$ with minimal model $F \colon \Lambda \to \Lambda$, let $\alpha_1, \dots, \alpha_s$ and $\beta_1, \dots, \beta_t$ be the eigenvalues of V and W respectively which are not equal to one. Then the Lefschetz number of f is given by*

$$L(f) = \frac{\prod_1^t (1 - \beta_i) \cdot \prod_{t+1}^q b_i}{\prod_1^s (1 - \alpha_i) \cdot \prod_{s+1}^r a_i}$$

if $q - t = r - s$ and $L(f) = 0$ otherwise. (Note that α_i and β_i are the eigenvalues of rational homotopy groups.)

Recently, Greg Lupton and I [LO2] have used Halperin's idea to give a new and much simpler proof (and generalization) of a result of Duan [Du]. In the case we look at in [LO2], we can simplify Halperin's proof greatly. Here I shall just use Halperin's formula. A finite H-space plainly is elliptic and has oddly generated minimal model. Furthermore, in the minimal model of an H-space, the differential d is zero. This follows from Hopf's theorem on H-spaces: The cohomology algebra of a connected finite H-space is an exterior algebra on odd generators [Sp, p.269]. Indeed, the exterior cohomology algebra $H^*(X; \mathbb{Q})$ may be mapped cohomologically isomorphically into $A^*(X)$ by taking generators to representing cocycles and then extending freely. The lifting theorem mentioned in the general discussion of minimal models above then provides a map $(\Lambda V, d) \to H^*(X; \mathbb{Q})$ which must also be a cohomology isomorphism and, hence, an isomorphism of DGA's. Duan considered the following situation.

Suppose that (X, μ) is a connected H-space with homotopy unit and that $f_1, f_2 \colon X \to X$ are self-maps. Their *product* is defined as the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f_1 \times f_2} X \times X \xrightarrow{\mu} X.$$

Using this product, form inductively the k^{th} *power* of a self-map f, denoted f^k, by taking $f_1 = f$ and $f_2 = f^{k-1}$. Duan proved

Theorem 6.10. *Let $f \colon X \to X$ be a self-map of a finite, connected H-space. For each $k \geq 2$, the k^{th} power of f has a fixed point.*

Halperin Formula Proof. Because $V^{\text{even}} = 0$, Halperin's formula reduces to its numerator alone. But then, $L(f) \neq 0$ if and only if no eigenvalue is equal to 1. So this is what we must show. Denote the identity map of X by ι. We can write $f^k = \iota^k \circ f$. On homotopy, $(\iota^k)_\# = \times k \colon \pi_\#(X) \to \pi_\#(X)$. Hence $(f^k)_\# = kf_\# \colon \pi_\#(X) \to \pi_\#(X)$ and it follows that, if we denote the eigenvalues of $f_\#$ by $\{\beta_j\}$, then the eigenvalues of $(f^k)_\# \otimes 1$ are $\{k\beta_j\}$. The only way such an eigenvalue $k\beta_j$ can equal 1 is if $\beta_j = \frac{1}{k}$. This is impossible however, because the defining matrix of $f_\#$ may be taken to be integral with characteristic polynomial having integral coefficients. The rational root test says that, in such a case, any rational root would in fact be integral, so that $f_\# \otimes 1$ has no non-integral, rational eigenvalues. Therefore, for $k \geq 2$, $(f^k)_\# \otimes 1$ does not have 1 as an eigenvalue and the result follows. □

§7 The Higher Euler Characteristic

In recent years, classical fixed point theory has been extended to include, for instance, the interesting case of *flows* $\Phi \colon M \times \mathbb{R} \to M$. This is the so-called *one parameter fixed point theory* [GN1], [GN2] and its applications include providing homological invariants for the detection of periodic orbits. It is classical by now that the Gottlieb group enters into fixed point theory in the following way [Br Corollary VIII.E.3, p 142].

Theorem 7.1. *Suppose that X is nice (i.e. a manifold say) and $f \colon X \to X$ is a self-map. Then*

(1) *there exists a self-map $g \colon X \to X$ homotopic to f such that g has exactly $N(f)$ fixed points, where $N(f)$ is the Nielsen number of f;*
(2) *if the Jiang condition, $G(X) = \pi_1(X)$, holds, then $N(f) = 0$ if $L(f) = 0$, so the vanishing of the Lefschetz number is sufficient to ensure that g has no fixed points.*

The second part of the theorem shows that the fixed point theoretic proof of Theorem 2.7 may not be so surprising. There is another recent proof of Theorem 2.7 which also is motivated by fixed point theory. This is due to Geoghegan and Nicas [GN3] and involves their *higher Euler characteristic* $\chi_1(X; R)$. Besides this approach to Gottlieb's Theorem, they show that their higher Euler characteristic also has something to say about the structure of groups whose associated topology is nice.

Theorem 7.2 [GN3]. *Let $X = K(\pi, 1)$ be a finite complex (and suppose π obeys an extra technical hypothesis which for a linear group is satisfied). Then, if $\chi_1(X;\mathbb{Q}) \neq 0$, it follows that $\mathcal{Z}\pi$ is infinite cyclic.*

From Gottlieb's Theorem Corollary 2.8, we immediately see

Corollary 7.3 [GN3]. *With the hypotheses above, if $\chi_1(X;\mathbb{Q}) \neq 0$, then $\chi(X) = 0$.*

The higher Euler characteristic $\chi_1(X;R)$ is most generally defined in terms of Hochschild homology. Here I will use a more straightforward computational definition which Geoghegan and Nicas show is equivalent to the original over $\mathbb{Q}$. While I concentrate on $\chi_1(X;\mathbb{Q})$ here, I should also point out that there is much more interplay between evaluation maps and one-parameter fixed point theory and this is explicated in [GNO].

Recall that, by the exponential law, there is a map $\hat{\alpha} : S^n \to M^M$, $\hat{\alpha}(s)(x) = A(s,x)$ such that evaluation $ev(f) = f(p)$ of a function at a basepoint $m \in M$ entails $ev \circ \hat{\alpha} = \alpha$. Hence, $ev_\#(\hat{\alpha}) = \alpha$ where $ev_\# : \pi_n(M^M, 1_M) \to \pi_n(M)$. Of course, as we have mentioned, a group action $A : S^1 \times M \to M$ provides a Gottlieb element $\alpha \simeq A|_{S^1}$ which may be identified with the homotopy class of the orbit map $a : S^1 \to M$. Also note that it is possible for $\alpha \in G_n(M)$ to be nullhomotopic, but $\hat{\alpha}$ to be essential. *Therefore, our focus is on elements $\hat{\alpha} \in \pi_n(M^M, 1_M)$ and not on their $G_n(M)$-images.*

Following [G8], the associated map A or, equivalently, the element $\hat{\alpha} \in \pi_n(M^M, 1_M)$ may be considered as a clutching map [Sp, p.455] along the equator of S^{n+1} which constructs a fibration

$$M \xrightarrow{i} E \to S^{n+1}$$

with $\alpha = \partial_\#(1) \in Im(\partial_\# : \pi_{n+1}S^{n+1} \to \pi_n M)$. Such a fibration has a Wang sequence associated to it,

$$\ldots \to H^q(E) \xrightarrow{i^*} H^q(M) \xrightarrow{\lambda_{\hat{\alpha}}} H^{q-n}(M) \to H^{q+1}(E) \to \ldots \quad .$$

The map $\lambda_{\hat{\alpha}}$ is a derivation on $H^*(M)$, i.e., satisfies $\lambda_{\hat{\alpha}}(uv) = \lambda_{\hat{\alpha}}(u)v + (-1)^{n|u|}u\lambda_{\hat{\alpha}}(v)$, and is called the Wang derivation. Another way to think of this is as follows. The element $\hat{\alpha} \in \pi_n(M^M, 1_M)$ corresponds to an element in $\pi_{n+1}\,\mathrm{Baut}(M)$ represented by a map $S^{n+1} \to \mathrm{Baut}(M)$. The fibration $M \to E \to S^{n+1}$ above is simply the pullback of the classifying fibration over $\mathrm{Baut}(M)$.

There is a beautiful connection between the Wang sequence and the clutching map A. Namely, for any $u \in H^q(M)$,

$$A^*(u) = 1 \times u + \bar{\sigma} \times \lambda_{\hat{\alpha}}(u),$$

where $\bar{\sigma} \in H^n(S^n)$ is a generator, $\alpha = \mathrm{ev}_\#(\hat{\alpha})$ and $\times$ is the external product. In case $u \in H^n(M)$, we have $\alpha^*(u) = \lambda_{\hat{\alpha}}(u)\bar{\sigma}$, with $\lambda_{\hat{\alpha}}(u) \in H^0(M)$, and the expression above may be re-written

$$\begin{aligned} A^*(u) &= 1 \times u + \bar{\sigma} \times \lambda_{\hat{\alpha}}(u) \\ &= 1 \times u + \alpha^*(u) \times 1 . \end{aligned}$$

We have a basic result, which follows immediately from the exactness of the Wang sequence:

Proposition 7.4. *For $\omega \in H^q(M)$, $\lambda_{\hat{\alpha}}(\omega) = 0$ if and only if there exists $\bar{\omega} \in H^q(E)$ with $i^*\bar{\omega} = \omega$.*

Recall that a fibration $F \to E \to B$ is *totally non-cohomologous to zero* (or *TNCZ*) over a field $\mathbb{F}$ if any of the following three conditions holds:

1. The Serre spectral sequence associated to the fibration collapses to the E_2-term.
2. $H^*(E;\mathbb{F}) \to H^*(F;\mathbb{F})$ is surjective (or, equivalently, $H_*(F;\mathbb{F}) \to H_*(E;\mathbb{F})$ is injective).
3. $H^*(E;\mathbb{F}) \cong H^*(F;\mathbb{F}) \otimes H^*(B;\mathbb{F})$ *as $H^*(B;\mathbb{F})$-modules.*

Therefore we see from Proposition 7.4 that

Corollary 7.5. $\lambda_{\hat{\alpha}} = 0$ *if and only if the Wang fibration associated to $\hat{\alpha}$ is TNCZ.*

Now, recall one of the definitions of the n^{th} *order higher Euler characteristic* of [GN3]. Note that we specialize to coefficients over a field $\mathbb{F}$ which we generally take to be $\mathbb{Z}_p$ for p a prime or ∞. Given $\hat{\alpha} \in \pi_n(M^M, 1_M)$ with Gottlieb image $\alpha \in G_n(M)$, define a homomorphism $\chi_n(M;\mathbb{F}) : \pi_n(M^M, 1_M) \to H_n(M;\mathbb{F})$ by

$$\chi_n(M;\mathbb{F})(\hat{\alpha}) = \sum_{k \geq 0} (-1)^{k+1} \sum_{j_k} \bar{b}_j^k \cap A_*(\sigma \times b_j^k)$$

where $\{b_1^k, \ldots, b_{j_k}^k\}$ is an $\mathbb{F}$-basis for $H_k(M;\mathbb{F})$, $\{\bar{b}_1^k, \ldots, \bar{b}_{j_k}^k\}$ is the Kronecker dual $\mathbb{F}$-basis for $H^k(M;\mathbb{F})$ and σ is a generator for $H_n(S^n;\mathbb{F})$.

There is an intimate connection between the higher Euler characteristic $\chi_n(M;\mathbb{F})$ and the $\lambda_{\hat{\alpha}}$-invariant. Let $\chi(M)$ and h denote the ordinary Euler characteristic and the Hurewicz map mod $\mathbb{F}$ of M respectively. Then

Theorem 7.6.

$$\chi_n(M;\mathbb{F})(\hat{\alpha}) = -\chi(M)\,h(\alpha) + (-1)^n \sum_{k\geq 0}(-1)^{k+1}\sum_{j_k}\lambda_{\hat{\alpha}}(\bar{b}_j^k)\cap b_j^k.$$

Proof. For a basis element $\bar{b}$ with dual b and σ, $\bar{\sigma}$ the respective generators of $H_n(S^n;\mathbb{F})$ and $H^n(S^n;\mathbb{F})$, we compute,

$$\begin{aligned}
\bar{b}\cap A_*(\sigma\times b) &= A_*(A^*(\bar{b})\cap(\sigma\times b)) \\
&= A_*((1\times\bar{b}+\bar{\sigma}\times\lambda_{\hat{\alpha}}(\bar{b}))\cap(\sigma\times b)) \\
&= A_*((1\cap\sigma)\times(\bar{b}\cap b)+(-1)^{(|\bar{\sigma}|)(|b|-|\lambda_{\hat{\alpha}}(\bar{b})|)}(\bar{\sigma}\cap\sigma)\times(\lambda_{\hat{\alpha}}(\bar{b})\cap b)) \\
&= A_*(\sigma\times 1+(-1)^n\langle\bar{\sigma},\sigma\rangle\times(\lambda_{\hat{\alpha}}(\bar{b})\cap b)) \\
&= h(\alpha)+(-1)^n\lambda_{\hat{\alpha}}(\bar{b})\cap b
\end{aligned}$$

since $A_*(\sigma\times 1) = \alpha_*(\sigma) = h(\alpha)$. If we perform this calculation on each b_j^k, we obtain

$$\begin{aligned}
\chi_n(M;\mathbb{F})(\hat{\alpha}) &= \sum_{k\geq 0}(-1)^{k+1}\sum_{j_k} h(\alpha)+(-1)^n\lambda_{\hat{\alpha}}(\bar{b}_j^k)\cap b_j^k \\
&= -\sum_{k\geq 0}(-1)^k\dim H^k(M;\mathbb{F})\cdot h(\alpha)+(-1)^n\sum_{k\geq 0}(-1)^{k+1}\sum_{j_k}\lambda_{\hat{\alpha}}(\bar{b}_j^k)\cap b_j^k \\
&= -\chi(M)\cdot h(\alpha)+(-1)^n\sum_{k\geq 0}(-1)^{k+1}\sum_{j_k}\lambda_{\hat{\alpha}}(\bar{b}_j^k)\cap b_j^k.
\end{aligned}$$

□

Let p denote a prime or ∞ (where $\mathbb{Z}_\infty = \mathbb{Q}$). Denote by h_p the composition of the Hurewicz map $\pi_n(M)\to H_n(M)$ with reduction mod p, $H_n(M)\to H_n(M;\mathbb{Z}_p)$. Then Gottlieb [G2] proved that, if $\chi(M)\neq 0$ and n is odd, then $G_n(M)\subseteq \operatorname{Ker} h_p$. Also, if n is even, then $G_n(M)\subseteq\operatorname{Ker} h$. Therefore, the term $-\chi(M)\,h(\alpha)$ must vanish over any field of coefficients $\mathbb{F}$. Hence, the formula for χ_n becomes

Corollary 7.7.

$$\chi_n(M;\mathbb{F})(\hat{\alpha}) = (-1)^n\sum_{k\geq 0}(-1)^{k+1}\sum_{j_k}\lambda_{\hat{\alpha}}(\bar{b}_j^k)\cap b_j^k.$$

Example 7.8: Spheres.

1. The Gottlieb groups of spheres are: $G_{2n}(S^{2n}) = 0$ and $G_{2n+1}(S^{2n+1}) = 2\mathbb{Z}$ except for the cases $G_1(S^1) = \mathbb{Z}$, $G_3(S^3) = \mathbb{Z}$, $G_7(S^7) = \mathbb{Z}$. If $\hat{\alpha} \in \pi_m({S^m}^{S^m}, 1_{S^m})$ and $\bar{b} \in H^m(S^m; \mathbb{F})$ is a generator, then (as shown in [LO1]),

$$\lambda_{\hat{\alpha}}(\bar{b}) = \langle \bar{b}, h(\alpha) \rangle.$$

Thus, since $\alpha = 0$ for all even spheres, $\chi_{2n}(S^{2n}; \mathbb{F}) = 0$. For odd spheres ($2n + 1 \neq 1, 3, 7$), $\chi_{2n+1}(S^{2n+1}; \mathbb{F})(\hat{\alpha}) = -2\,b$, where α generates $G_{2n+1}(S^{2n+1}) = 2\mathbb{Z}$, b generates $H_{2n+1}(S^{2n+1}; \mathbb{F})$ and $h(\alpha) = 2b$. If $2n + 1 = 1, 3, 7$, then $\chi_{2n+1}(S^{2n+1}; \mathbb{F})(\hat{\alpha}) = -\,b$.

2. $\chi_m(S^m \times S^m; \mathbb{F})(\hat{\alpha}) = 0$, where m is *odd*, $\neq 1, 3, 7$ and $\hat{\alpha}$ represents a generator of one of the $2\mathbb{Z}$ factors in $G_m(S^m \times S^m) \cong G_m(S^m) \times G_m(S^m) \cong 2\mathbb{Z} \times 2\mathbb{Z}$. To see this, let $\{\bar{b}_1^m, \bar{b}_2^m, \bar{b}_1^m \cup \bar{b}_2^m\}$ be the obvious additive basis for cohomology with dual homology basis $\{b_1^m, b_2^m, b_1^m b_2^m\}$. As above, $\lambda_{\hat{\alpha}}(\bar{b}_j^m) = \langle \bar{b}_j^m, h(\alpha) \rangle$ since $h(\alpha)$ and $\bar{b}_j^m$ are both in degree m. Therefore, since we take $h(\alpha) = 2b_1^m$, we have

$$\lambda_{\hat{\alpha}}(\bar{b}_1^m) = 2, \qquad \lambda_{\hat{\alpha}}(\bar{b}_2^m) = 0 \qquad \lambda_{\hat{\alpha}}(\bar{b}_1^m \cup \bar{b}_2^m) = \lambda_{\hat{\alpha}}(\bar{b}_1^m) \cup \bar{b}_2^m = 2\bar{b}_2^m.$$

Taking cap products gives

$$\lambda_{\hat{\alpha}}(\bar{b}_1^m) \cap b_1^m = 2b_1^m, \quad \lambda_{\hat{\alpha}}(\bar{b}_2^m) \cap b_2^m = 0, \quad \lambda_{\hat{\alpha}}(\bar{b}_1^m \cup \bar{b}_2^m) \cap b_1^m b_2^m = 2\bar{b}_2^m \cap b_1^m b_2^m = 2b_1^m.$$

Plugging these calculations into the formula for χ_m gives

$$\chi_m(S^m \times S^m; \mathbb{F})(\hat{\alpha}) = -[2b_1^m + 0 - 2b_1^m] = 0.$$

As a special case, note that $\chi_1(S^1 \times S^1; \mathbb{F}) = 0$ identically. This follows from the same calculation as above and the fact that $G(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} = \pi_1(S^1 \times S^1)$. Similar results apply to $S^3 \times S^3$ and $S^7 \times S^7$.

The cohomological formula for χ_n allows us to extend the higher Euler characteristic to a homomorphism with domain the group of $(-n)$-degree derivations on cohomology;

$$\chi_n(M; \mathbb{F}) : Der_{-n}(H^*(M; \mathbb{F})) \to H_n(M; \mathbb{F})$$

is defined, for $\theta \in Der_{-n}(H^*(M; \mathbb{F}))$, by

$$\chi_n(M; \mathbb{F})(\theta) = (-1)^n \sum_{k \geq 0} (-1)^{k+1} \sum_{j_k} \theta(\bar{b}_j^k) \cap b_j^k.$$

Theorem 7.9. *Suppose $H^*(M;\mathbb{F}) \cong H^*(X;\mathbb{F}) \otimes H^*(Y;\mathbb{F})$ as vector spaces. Let $\theta \in Der_{-n}(H^*(X;\mathbb{F}))$ and extend θ to $\hat{\theta} \in Der_{-n}(H^*(M;\mathbb{F}))$ by taking $\hat{\theta}|_{H^*(Y;\mathbb{F})} = 0$. Then*

$$\chi_n(M;\mathbb{F})(\hat{\theta}) = \chi_n(X;\mathbb{F})(\theta) \cdot \chi(Y).$$

Proof. A basis for $H^*(M)$ may be taken to consist of elements $\bar{b}_j^r \times \bar{\gamma}_i^{k-r}$, where the b's form a basis for $H^*(X)$ and the γ's form a basis for $H^*(Y)$. Then, by the extended definition of θ, $\hat{\theta}(\bar{b}_j^r \times \bar{\gamma}_i^{k-r}) = \theta(\bar{b}_j^r) \times \bar{\gamma}_i^{k-r}$ and

$$\begin{aligned}\hat{\theta}(\bar{b}_j^r \times \bar{\gamma}_i^{k-r}) \cap (b_j^r \times \gamma_i^{k-r}) &= (\theta(\bar{b}_j^r) \times \bar{\gamma}_i^{k-r}) \cap (b_j^r \times \gamma_i^{k-r}) \\ &= (-1)^{r(k-r-(k-r))}(\theta(\bar{b}_j^r) \cap b_j^r) \times (\bar{\gamma}_i^{k-r} \cap \gamma_i^{k-r}) \\ &= \theta(\bar{b}_j^r) \cap b_j^r.\end{aligned}$$

Since this calculation holds for all $\bar{\gamma}_i^{k-r}$, we see that

$$\sum_{j_k(M)} \hat{\theta}(\bar{b}_j^r \times \bar{\gamma}_i^{k-r}) \cap (b_j^r \times \gamma_i^{k-r}) = \sum_{r\geq 0} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \dim H^{k-r}(Y).$$

Then we have

$$\begin{aligned}\chi_n(M;\mathbb{F})(\theta) &= (-1)^n \sum_{k\geq 0}(-1)^{k+1} \sum_{r\geq 0} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \dim H^{k-r}(Y) \\ &= (-1)^n \sum_{k\geq 0}(-1)^{r+1+k-r} \sum_{r\geq 0} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \dim H^{k-r}(Y) \\ &= (-1)^n \sum_{r\geq 0}(-1)^{r+1} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \sum_{k\geq r}(-1)^{k-r} \dim H^{k-r}(Y) \\ &= (-1)^n \sum_{r\geq 0}(-1)^{r+1} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \sum_{k-r\geq 0}(-1)^{k-r} \dim H^{k-r}(Y) \\ &= (-1)^n \sum_{r\geq 0}(-1)^{r+1} \sum_{j_r(X)} \theta(\bar{b}_j^r) \cap b_j^r \cdot \sum_{l\geq 0}(-1)^{l} \dim H^{l}(Y) \\ &= \chi_n(X;\mathbb{F})(\theta) \cdot \chi(Y).\end{aligned}$$

□

Example 7.10. Let $M = X \times Y$ and let $\theta = \lambda_{\hat{\alpha}}$, where $\hat{\alpha} \in \pi_n(X^X, 1_X)$. The extended derivation corresponds to extending the map associated to $\hat{\alpha}$, $A : S^n \times X \to X$, to a map $B : S^n \times X \times Y \to X \times Y$ defined by $B = A \times 1_Y$. This map is associated to an element $\hat{\beta} \in \pi_n(M^M, 1_M)$ and we have $\chi_n(M;\mathbb{F})(\hat{\beta}) = \chi_n(X;\mathbb{F})(\hat{\alpha}) \cdot \chi(Y)$.

Example 7.11. Let $M = S^n \times Y$, where n is an odd number. If $n \neq 1, 3, 7$, then $\chi_n(M; \mathbb{F})(\hat{\alpha}) = -\chi(Y) \cdot h(\alpha) = -2\,\chi(Y) \cdot b$, where α generates $G_n(S^n) = 2\mathbb{Z}$ and b generates $H_n(S^n)$. If $n = 1, 3, 7$, then $\chi_n(M; \mathbb{F})(\hat{\alpha}) = -\chi(Y) \cdot b$, where α generates $G_n(S^n) = \mathbb{Z}$.

The description of $\chi_n(M; \mathbb{F})(\hat{\alpha})$ in terms of $\lambda_{\hat{\alpha}}$ allows us to calculate the former in cases where the latter is known. In particular,

Theorem 7.12. *Let $\hat{\alpha} \in \pi_n(M^M, 1_M)$. If the associated Wang fibration $M \to E \to S^{n+1}$ is totally noncohomologous to zero (TNCZ), then $\chi_n(M; \mathbb{F})(\hat{\alpha}) = 0$.*

We may apply this observation to group actions as follows: suppose $G \times M \to M$ is an action of a compact Lie group on M. Let $S^n \to G$ represent a nontrivial element of $\pi_n(G)$ and form the composition $S^n \to G \to (M^M, 1_M)$. Denote the class of this map by $\hat{\alpha} \in \pi_n(M^M, 1_M)$. Now, $\pi_n(G) \cong \pi_{n+1}(BG)$, so we obtain a pullback of the Borel fibration $M \xrightarrow{j} MG \to BG$ associated to the action,

$$\begin{array}{ccccc} M & \xrightarrow{j} & MG & \to & BG \\ \| & & \uparrow \phi & & \uparrow \\ M & \xrightarrow{i} & E & \to & S^{n+1}. \end{array}$$

The fibration $M \xrightarrow{i} E \to S^{n+1}$ is precisely that associated to $\hat{\alpha}$, so the associated Wang derivation is $\lambda_{\hat{\alpha}}$. By the commutativity of the diagram, if the Borel fibration is TNCZ, then so is the Wang fibration. From Theorem 3.1 we then get

Corollary 7.13. *If the Borel fibration of the action is TNCZ, then $\chi_n(M; \mathbb{F})(\hat{\alpha}) = 0$.*

A particularly important case where a group action has a TNCZ Borel fibration (over $\mathbb{Q}$) is when M is a closed symplectic manifold with a hamiltonian circle action (see [K] for example). Recall that M^{2n} is *symplectic* if there is a closed 2-form ω on M with the property that the wedge product ω^n is a volume form for M. An S^1-action on M is *hamiltonian* if the 1-form $i_X\omega$ is exact, where X is the fundamental vector field associated to the action. Thus,

Corollary 7.14. *If M is a closed symplectic manifold and $\hat{\alpha} \in \pi_1(M^M, 1_M)$ corresponds to a hamiltonian S^1-action, then $\chi_1(M; \mathbb{Q})(\hat{\alpha}) = 0$.*

So, from these results, we see that the higher Euler characteristics may be thought of as obstructions to homotopy properties such as the TNCZ condition, as well as geometric properties such as hamiltonianness of a circle action. Finally, we want to give a different approach to a result of [GN3 Theorem 4.2]. Let

$$S^1 \xrightarrow{\alpha} M \xrightarrow{p} B \xrightarrow{e} BS^1$$

be a principal circle bundle classified by $e : B \to BS^1$. Note that $\alpha = ev_{\#}(\hat{\alpha})$ where $ev_{\#} : \pi_1(M^M, 1_M) \to \pi_1(M)$ is the homomorphism induced by evaluation and $\hat{\alpha} \in \pi_1(M^M, 1_M)$ is associated to the action.

Theorem 7.15 ([GN3 Theorem 4.2]).
1. *If* $e_{\mathbb{F}} \in H^2(B;\mathbb{F})$ *is zero, then* $\chi_1(M;\mathbb{F})(\hat{\alpha}) = -\chi(B) \cdot h(\alpha)$.
2. *If* $e_{\mathbb{F}} \in H^2(B;\mathbb{F})$ *is nonzero, then* $\chi_1(M;\mathbb{F})(\hat{\alpha}) = 0$.

Proof. As usual, we let $\sigma \in H_1(S^1)$ and $\bar{\sigma} \in H^1(S^1)$ be dual generators. All homology and cohomology coefficients are in the field $\mathbb{F}$. Note that $\alpha_*(\sigma) = h(\alpha)$. In the Serre spectral sequence, $d_2(\bar{\sigma}) = e_{\mathbb{F}}$, so $e_{\mathbb{F}} = 0$ implies $\bar{\sigma}$ survives to infinity and produces a nontrivial element in $H^1(M)$ which then maps to $\bar{\sigma}$ via α^*. Hence, α^* is surjective, the fibration is TNCZ and $H^*(M) \cong H^*(S^1) \otimes H^*(B)$ as vector spaces. Now $\lambda_{\hat{\alpha}}$ is a (-1)-degree derivation on $H^*(M)$ which extends the obvious one on $H^*(S^1)$. To see this, note that, because the circle action on M is free, the orbit space B is homotopy equivalent to MS^1, the total space of the Borel fibration associated to the principal circle action. Therefore, we have a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{p} & B & \to & BS^1 \\ \| & & \uparrow \phi & & \uparrow \\ M & \xrightarrow{i} & E & \to & S^2 \end{array}$$

relating the principal bundle to the $\hat{\alpha}$ Wang fibration. From the Wang sequence, we obtain a commutative diagram

$$\begin{array}{ccccc} H^q(E) & \xrightarrow{i^*} & H^q(M) & \xrightarrow{\lambda_{\hat{\alpha}}} & H^{q-1}(M) \\ & \nwarrow \phi^* & \uparrow p^* & & \\ & & H^q(B) & & \end{array}$$

Then $\lambda_{\hat{\alpha}} \circ p^* = \lambda_{\hat{\alpha}} \circ i^* \circ \phi^* = 0$ since $\lambda_{\hat{\alpha}} \circ i^* = 0$ by exactness of the Wang sequence. Hence, $\lambda_{\hat{\alpha}}$ is zero on $\mathrm{Im}(p^*) \cong H^*(B)$ in $H^*(M)$. Of course, the derivation θ on $H^*(S^1)$ defined by $\bar{\sigma} \mapsto 1$ gives $\chi_1(S^1;\mathbb{F})(\theta) = -\sigma$ (see Example 7.8) while the inclusion α_* identifies $-\sigma$ with $-h(\alpha)$. Thus Theorem 7.9 gives

$$\chi_1(M;\mathbb{F})(\hat{\alpha}) = -\chi(B)\cdot h(\alpha).$$

If, on the other hand, $e_{\mathbb{F}} \neq 0$, then consider the Serre sequence associated to the Borel fibration above (with $MS^1 \simeq B$),

$$H^1(B) \xrightarrow{i^*} H^1(M) \xrightarrow{\partial} H^2(BS^1) \xrightarrow{e^*} H^2(B)$$

where $e^*(\iota) = e_{\mathbb{F}}$ and ι is a generator of $H^2(BS^1)$ corresponding to $\bar{\sigma} \in H^1(S^1)$. Now, $e_{\mathbb{F}} \neq 0$ and $H^2(BS^1)$ 1-dimensional imply that $\partial = 0$. Hence i^* must be surjective. The identification of B with MS^1 then shows that the Borel fibration associated to the principal circle action is TNCZ. Therefore Corollary 7.13 implies

$$\chi_1(M;\mathbb{F})(\hat{\alpha}) = 0.$$

□

§8 Final Words

The Gottlieb groups and evaluation map methods continue to find places of application in topology. Besides the work listed above, there is a great deal of interest in other aspects of Gottlieb theory (if I may be so bold) as well. In particular, there is an effort to understand general cyclic maps which has been going on for some time and, more recently, there has been much progress in relativizing Gottlieb groups and understanding the relations between the ordinary and relativized groups [LW], [WL1] [WL2] and [LPW]. I think it would be interesting to see if any of these new invariants have applications akin to the classical Gottlieb groups in fixed point theory, homotopy theory and especially rational homotopy theory?

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