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ANTIPODAL POINTS AND FIXED POINTS

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PREFACE

The central theme of this monograph involves two related concepts — antipodal points and fixed points — which lie on the borderline of topology and functional analysis. The articles themselves provide a survey of the far-reaching consequences of three classical results.

A celebrated 1912 theorem of L. E. J. Brouwer asserts that every continuous mapping of the unit ball B^n of $\mathbb{R}^n$ into itself has a fixed point. (This fact was known to P. Bohl and perhaps to others even earlier, but Brouwer's explicit statement is the one widely recognized.) A second fundamental result is the contraction mapping principle due to S. Banach, which asserts that each mapping f of a complete metric space M into itself having Lipschitz constant strictly less than 1 has a unique fixed point $\bar{x}$, and moreover $\lim_{n\to\infty} f^n(x) = \bar{x}$ for each $x \in M$. (This fact was known to others as well, but Banach's explicit 1922 statement is the one widely recognized.) Finally, in 1933, K. Borsuk proved that every continuous map of $S^n \to S^n$ (the unit sphere in $\mathbb{R}^n$) satisfying $f(-x) = -f(x)$ is essential. An important consequence of this is the fact that if $f : B^n \to \mathbb{R}^n$ is continuous and satisfies $-f(x) = f(-x)$ for $x \in S^n$, then f has a fixed point. Borsuk also proved that if $f : S^n \to \mathbb{R}^n$ is continuous, then f sends at least one pair of antipodal points into the same point. This result, widely known as the Borsuk-Ulam theorem, was earlier conjectured by S. Ulam. A third result in Borsuk's 1933 paper is an important application to Lusternik-Schneirelmann category.

The first paper in this volume consists of a survey of the Borsuk-Ulam theorem and its various extensions, showing, among other things, how it can be extended to compact Lie groups, Stiefel manifolds, and to multi-valued maps. The advanced techniques of algebraic topology play a fundamental role in this development.

Fixed point theory has always played a central role in the problems of functional analysis, and topology has been involved deeply in both the study of fixed point theory and more directly to problems in analysis in a wide variety of ways. The remaining papers in the volume deal explicitly with two different approaches to fixed point theory. The first of these deals largely with the class of nonexpansive mappings, a limiting case of the class of strict contractions in which the Lipschitz constant k is assumed to be equal to 1. This approach can be thought of as an extensive outgrowth of Banach's theorem. The existence of fixed points for nonexpansive mappings is assured under hypotheses involving both topological and geometric assumptions on the underlying Banach space. The final paper in the volume surveys the large amount of fixed point theory which finds its original inspiration in the profound theorem of Brouwer.

The material presented here includes the texts of some of the main lectures delivered at the First and Second International Miniconferences on Topology and Nonlinear Analysis, which were held at the GARC-RIM-SNU in Seoul, Korea.

The first paper, by Professor Jan Jaworowski, was presented at the First Conference on May 29, 1992.

The second paper, by Professor W. A. Kirk, was delivered at the Second Conference on June 21, 1994.

The third paper, by Professor Sehie Park, originated with the memorial lecture on the occasion of the opening of the GARC on March 21, 1991, and expanded versions of the lecture given as invited talks at the First Miniconference in 1992, Korea University in 1992, Memorial University of Newfoundland in 1992, Busan National University of Technology in 1993, and the Institute of Mathematics, Academia Sinica, Taipei, in 1994. Moreover, while he was visiting Taiwan in February 1995, it was delivered at the colloquium talks in National Changhua University of Education, National Tsinghua University, and Tamkang University. Finally, it was also given at Kyungpook National University, Taegu, in April 1995 by invitation of the TGRC.

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BORSUK–ULAM TYPE THEOREMS FOR ORTHOGONAL GROUPS

JAN JAWOROWSKI

0. INTRODUCTION

One of the most fruitful classical results in topology is the Borsuk-Ulam theorem. It was conjectured by Ulam and proved by Borsuk in 1932 [2]. Two of its classical versions are as follows :

Theorem I. *Let* $f : S^n \to \mathbb{R}^k$ *be a map and let* $A_f = \{x \in S^n \mid f(x) = f(-x)\}$. *Then, if* $k \leq n$, $A_f \neq \emptyset$.

Theorem II. *Let* $f : S^n \to S^n$ *be a map such that* $f(x) \neq f(-x)$ *for each* $x \in X$. *Then* f *is surjective.*

These two versions of the Borsuk-Ulam theorem are closely ralated. They have proved to be a source of a very wide range of ideas and generalizations. We refer the reader to an article by H. Steinlein [26] which lists 457 publications concerned with the Borsuk-Ulam theorem. In this paper we will be mainly concerned with the version given by Theorem I. Generally speaking, Theorem I says that the set A_f, which may be called "degeneracy set" of f with respect to the antipodal map on S^n, is non-empty. The antipodal map is a $\mathbb{Z}_2$-symmetry on S^n. We will show, in particular, how Theorem I can be extended to more general situations, to maps of spaces other than spheres and to symmetries or group actions of groups more general than $\mathbb{Z}_2$. We will also describe "continuous", or "parametrized", versions of the Borsuk-Ulam theorem, for bundles of spaces over a base space. At the end of this paper, we outline recent results of the author on maps of bundles whose fibres are Stiefel manifolds. We show how the size of a set corresponding to A_f can be described by certain polynomials on the Stiefel-Whitney classes of the bundle.

1. NOTATION

We will assume that all the spaces in this paper are paracompact. In what follows we will be using the Alexander-Cech-Spanier cohomology ("ACS cohomology"). The reason for using the ACS cohomology theory is that it has a continuity property which, roughly speaking, can be stated as follows : If a cohomology class of a space X vanishes on a closed subset A of X, then it also vanishes on a neighborhood of A. Throughout of this paper, unless otherwise stated, we will be using the group $\mathbb{Z}_2$ for the coefficient group.

If a group G acts on a space X, the orbit space of the action will be denoted by X/G or by $\overline{X}$.

2. THE INDEX

The classical Borsuk-Ulam theorem is about maps of a sphere with the antipodal action. In 1954, C. T. Yang [27] defined a tool which can be used to estimate the "size" of a space with a free $\mathbb{Z}_2$-action in terms of an integer; this is what is now known as the (cohomology) index of the space. Suppose X is a paracompact space with a free $\mathbb{Z}_2$-action, i.e., with a free involution $T : X \to X$. Associated with the action is its *characteristic class*, i.e., the first Stiefel-Whitney class $u(T) = w_1$ of the free involution. The characteristic class w_1 belongs to the 1-dimensional cohomology group of the orbit space $H^1(X/\mathbb{Z}_2)$ of the action. The Yang index of a space with a free $\mathbb{Z}_2$-action is $\text{Ind}(X,T) := \text{Sup}\{n | u^n(T) \neq 0\}$. The index carries information about the size of X in the sense that if $\text{Ind}(X,T) \geq n$, then $H^n(X/\mathbb{Z}_2) \neq 0$, and thus the covering dimension of $X/\mathbb{Z}_2$ (and hence also of X) is at least n.

In the standard example of an n-sphere S^n with the antipodal involution, the orbit space $S^n/\mathbb{Z}_2$ is the real projective n-space P^n. The classifying space for $G = \mathbb{Z}_2$ is $B\mathbb{Z}_2$, the infinite real projective space P^∞. The cohomology ring of P^∞ is a polynomial ring $\mathbb{Z}_2[u]$ on one generator $u \in H^1(P^\infty)$; the cohomology ring of P^n is obtained from $H^1(P^\infty)$ by introducing the relation $u^n = 0$; and thus $\text{Ind}(S^n, T) = n$. Yang's theorem gives the following estimate for the size of A_f :

Theorem (Yang). *Let X be a space with a free involution $T : X \to X$ and let $f : X \to \mathbb{R}^k$. Then the index of $A_f = \{x \in X | fx = f(Tx)\}$ is at least $n - k$.*

This implies, as before, that the covering dimension of X is at least $n - k$.

The concept of index as an integer associated to a group action on a space can be extended to free actions of other compact Lie groups, like $G = S^1$ and $G = S^3$, in an analogous way, due to the fact that the cohomology structure of the classifying spaces for these groups is similar to that of $B\mathbb{Z}_2$. This was done in [17] and [18]. However, in order to obtain results corresponding to Yang's theorem, one has to give an appropriate definition of the coincidence set A_f. We will discuss such a definition in the next section.

Various extensions of the concept of index were defined and used by Fadell and Husseini ([7], [8]). In [9] and [10] Fadell and Husseini defined a general concept of index for an arbitrary compact Lie group acting on a space. This author defined the concept of index independently in [19] and [20]. In [23] Neza Mramor-Kosta studied the concept of index in this and other settings, such as for actions of cyclic groups and for infinite dimensional representations and bundles of Banach spaces.

The concept of index can be described as follows: Suppose that G is a compact Lie group on a (paracompact) space X. The Borel G-cohomology $H^*_G X$ of X (with coefficients in $\mathbb{Z}_2$, of ACS type) is defined as follows. Let EG be the universal space for X and $X_G := (EG \times X)/G$, where G acts on $EG \times X$ by $g(e, x) = (ge, gx)$. Then $H^*_G X := H^*(X_G)$. If $(\cdot)$ is a one-point space, then $H^*_G(\cdot)$ can be identified with $H^*(BG)$, the ordinary cohomology of the classifying space of G. If G acts freely on X then $(EG \times X)/G$ is the bundle with fibre X associatd to the principal bundle $EG \to X/G$ and $X_G = X/G$, the orbit space of X.

Definiton 2.1. *The G-index*, $\mathrm{Ind}^G X$, *of* X *is defined to be the kernel of the* G*-cohomology map* $c^* : H^*_G(\cdot) = H^*(BG) \to H^*_G X$ *induced by the constant map* $c : X \to (\cdot)$ *of* X *into a one-point space.*

If G acts freely on X, the G-cohomology $H^*_G X$ of X can be identified with $H^*(X/G)$, the cohomology of the orbit space X/G and the constant map $c : X \to (\cdot)$ can be replaced by a classifying map $X/G \to BG$. In particular, in the classical case when $G = \mathbb{Z}_2$ is acting freely on X, the generator of $\mathbb{Z}_2$ represents a free involution on X. In this case, $BG = P^\infty, H^*_G X = H^*(P^\infty) \cong \mathbb{Z}_2[u]$ is a polynomial algebra on one generator $u \in H^1(P^\infty)$; its image under $c^* : H^*_G(\cdot) = H^*(BG) \to H^*_G X$ is the characteristic class of the involution ; and the index can be identified with an integer, as described above.

The following proposition follows immediately from the definition of the index. It expresses its naturality property :

Proposition 2.2. *Let* X *and* Y *be* G*-spaces and let* $f : X \to Y$ *be an equivariant map. Then* $\mathrm{Ind}^G Y \leq \mathrm{Ind}^G X$.

3. THE AVERAGE OF A MAP AND THE BORSUK-ULAM THEOREM FOR A COMPACT LIE GROUP

In theorems of the Borsuk-Ulam type for a general compact Lie group G we usually consider a map $f : X \to W$ of X to a representation space W for G, just as it was done for $G = \mathbb{Z}_2$; and we try to estimate the size of the set A_f, where the G-symmetry becomes degenerate under f. In order to be able to state a right generalization of theorems of Borsuk, Ulam and Yang, we need an appropriate analog of A_f. The set A_f may be defined in various ways depending on the context. By using an invariant measure on G, we can average the map f to obtain an equivariant map $\mathrm{Av}f : X \to W$ and define A_f to be the set of zeros of $f, A_f = (\mathrm{Av}f)^{-1}(0)$. More genrally, for any invariant subspace W_0 of W, we can set $A_f(W_0) := (\mathrm{Av}f)^{-1}(W_0)$ (compare [19], [20]). If $G = \mathbb{Z}_2$ acts on $W = \mathbb{R}^k$ through the antipodal map and $f : X \to \mathbb{R}^k$, then $(\mathrm{Av}f)^{-1}(0)$ corresponds to the set A_f defined in Theorem I. The classical Borsuk-Ulam theorem asserts that for any map $f : S^n \to \mathbb{R}^k$ there is a point in S^n where the average of f (with respect to the antipodal action on the source space and on the target space) is zero.

The following theorem is a general principle of which the theorems of Borsuk, Ulam and Yang are special cases. We refer the reader to [9], [10], [19] and [20]. It is worth noting that the proof we give is analogous to those used in [15, p.113], [16, p.160], [17, p.161] and [18, p.148].

Theorem 3.1 (The Index Theorem). *Let X be a G-space, let W be a representation space for G and let $f : X \to W$ be a map. Then*

$$\mathrm{Ind}^G(A_f(W_0)) \cdot (\mathrm{Ind}^G(W - W_0)) \subset \mathrm{Ind}^G X.$$

Proof. As before, given a space Y, let $c = c_Y : Y \to (\cdot)$ be the constant map of Y into a one-point space. Let $a \in \mathrm{Ind}^G(A_f(W_0))$; that is, $c^*_{A_f}(a) = 0$. Consider $c^*_X(a) \in H^*_G X$. Thus $(c^*_X(a))|A_f = 0$. By the continuity of H^*_G, there exists a neighborhood N of A_f such that $(c^*_X(a))|N = 0$. By the exactness of the G-cohomology sequence of the pair (X, N), $c^*_X(a) = j^*(a')$, where $a' \in H^*_G(X, N)$ and $j : X \to (X, N)$ is the inclusion. Let $b \in \mathrm{Ind}^G(W - W_0)$. Since the restriction of $\mathrm{Av}f$ defines an equivariant map $X - A_f \to W - W_0$, we have by Proposition 2.2 that $\mathrm{Ind}^G(W - W_0) \subset \mathrm{Ind}^G(X - A_f)$. Hence $c^*_X(b) \in \mathrm{Ind}^G(X - A_f)$; that is, $(c^*_X(b))|(X - A_f) = 0$. By the exactness of the G-cohomology sequence of the pair $(X, X - A_f)$, $c^*_X(b) = k^*b'$, where $b' \in H^*_G(X, X - A_f)$ and $k : X \to (X, X - A_f)$ is the inclusion. It follows that $c^*_X(ab) = c^*_X(a)c^*_X(b) = (j^*a')(k^*b') = 0$. Therefore $ab \in \mathrm{Ind}^G X$.

The inclusion of ideals in Theorem 3.1 puts a lower bound on the size of the degeneracy set $A_f(W_0)$. In the classical case of $G = \mathbb{Z}_2$ (as also in the case when $G = S^1$ or S^3 and if the action is free), the inclusion reduces to an inequality for integers, just as in Yang's theorem.

4. Stiefel manifolds and free actions of $O(m)$

A generalization of the Borsuk-Ulam theorem to maps of Stiefel manifolds was proved by the author in [19] and [20]. It may be viewed as extending the Borsuk-Ulam theorem from $G = \mathbb{Z}_2(= O(1))$ to $G = O(m)$. Before we state it (in section 6) we will recall a construction of Stiefel manifolds and Grassmann manifolds. An excellent account of Grassmann manifolds can be found in the lecture notes of Milnor and Stasheff "Characteristic Classes" [22] which will serve as a principal source of reference here.

As usual, $\mathbb{R}^n$ denotes the euclidean n-space and $\mathbb{R}^\infty$ is the infinite union $\mathbb{R}^\infty = \mathbb{R}^0 \cup \mathbb{R}^1 \cup \ldots$ with the inductive topology induced by the inclusions $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \ldots$. The set of orthonormal m-frames of vectors in $\mathbb{R}^{m+n}$ is the Stiefel manifold $V_m(\mathbb{R}^{m+n})$; it has a natural topology as a subset of $\mathbb{R}^{(m+n)m}$. It is easy to see that $V_m(\mathbb{R}^{m+n})$ is a compact manifold. A point v of $V_m(\mathbb{R}^{m+n})$ can be thought of as an $(m+n) \times n$ matrix.

The orthogonal group $O(m+n)$ acts on $V_m(\mathbb{R}^{m+n})$ on the right through its action on $\mathbb{R}^{m+n}$: if $v \in V_m(\mathbb{R}^{m+n})$ and $A \in O(m+n)$ then $v \cdot A$ is just the product of matrices. Two m-frames, v and w are in one orbit of this action if and only if they span the same m-plane. Thus the orbit space $V_m(\mathbb{R}^{m+n})/O(m+n)$ of this action can be identified with the the space of m-planes in $\mathbb{R}^n$. This space, denoted by $G_m(\mathbb{R}^{m+n})$, with the identification topology defined by the orbit map, is the Grassmann manifold of m-planes in $\mathbb{R}^n$.

In the special case of $m = 1, V_1(\mathbb{R}^{1+n}) \cong S^n$, and $G_1(\mathbb{R}^{1+n})$ is the (real) projective space P^n.

These constructions can be extended to the case when $n = \infty$. We will write $V_m := V_m(\mathbb{R}^\infty)$, and $G_m := G_m(\mathbb{R}^\infty)$. V_m and G_m are Hausdorff spaces. The identification topology in these spaces coincides with the topology given by the inclusions $V_0 \subset V_1 \subset \ldots$ and $G_0 \subset G_1 \subset \ldots$, respectively. In other words, $V_m = \varinjlim_n V_m(\mathbb{R}^{m+n})$ and $G_m = \varinjlim_n G_m(\mathbb{R}^{m+n})$. The orbit map $V_m \to G_m$ can also be described as the map which assigns to each m-frame $v \in V_m$ the m-plane in $\mathbb{R}^\infty$ spanned by v.

The following argument (used in [22]) shows that $G_m(\mathbb{R}^{m+n})$ is an mn-dimensional manifold. Let $A \in G_m(\mathbb{R}^{m+n})$ be a fixed m-plane in $\mathbb{R}^{m+n}$, let $A^\perp$ be the orthogonal complement of A in $\mathbb{R}^{m+n}$ and let $U := \{X \in G_m(\mathbb{R}^{m+n}) \,|\, X \cap A = 0\}$. Then an m-plane $X \in U$ can be viewed as the graph of a linear homomorphism $A \to A^\perp$. This shows that $U \cong \mathrm{Hom}(A, A^\perp) \cong \mathbb{R}^{mn}$.

5. The Cohomology of Grassmannians

In this section we are going to discuss some results on the topology of Grassmann manifolds. We will recall some facts given in [22], but we will also discuss new results about the mod 2 cohomology of finite dimensional Grassmann manifolds.

There is an important m-dimensional vector space bundle is over G_m: it is the canonical bundle η^m. Its total space is a subset of $G_m \times \mathbb{R}^\infty$ consisting of pairs (Y, v) where $Y \in G_m$ and v is a vector in Y. The bundle projection sends (Y, v) to it second coordinate, v. By restricting this bundle to $G_m(\mathbb{R}^{m+n})$ we obtain the canonical m-plane bundle over $G_m(\mathbb{R}^{m+n})$. The canonical m-plane bundle η^m is a universal bundle for all $\mathbb{R}^m$-bundles over paracompact spaces.

Every $\mathbb{R}^m$ -bundle over a paracompact base space B has its Stiefel-Whitney classes. They are homogeneous elements of the cohomology ring H^*B with coefficients in $\mathbb{Z}_2$. Throughout of this paper the coefficient group $\mathbb{Z}_2$ for the cohomology will be suppressed from the notation.

The Stiefel-Whitney classes for the canonical bundle η^n over G_m are universal for all $\mathbb{R}^m$- bundles. They will be denoted by $w_0 = 1, w_1, \ldots, w_m$. Thus $w_i \in H^i G_m, i = 0 \ldots, m$.

Definiton 5.1. We define *the height* of a monomial $w_1^{r_1} \ldots w_m^{r_m}$ to be the sum of its exponents, $\mathrm{height}(w_1^{r_1} \ldots w_m^{r_m}) := r_1 + r_2 + \ldots + r_m$.

There are at lest two descriptions of the cohomology ring H^*G_m of Grassmannians. By using a spectral sequence construction, Borel [1] showed that H^*G_m is isomorphic to the polynomial algebra $\mathbb{Z}_2[w_1, \ldots, w_m]$ on the universal Stiefel-Whitney classes $w_1, \ldots, w_m$. On the other hand, Ehresmann [5] constructed an explicit and, in a sense, minimal cell decomposition of the Grassmannians using ideas dating back to Schubert [25].

The cell decomposition of Grassmann manifolds due to Ehresmann (and extended to G_m) can be described as follows (see [22]). Given an m-plane $Y \in G_m$, let us consider the sequence of the dimensions of the intersections of Y with the subspaces $\mathbb{R}^n \subset \mathbb{R}^\infty : 0 \le \dim(Y \cap \mathbb{R}^0) \le \dim(Y \cap \mathbb{R}^1) \le \ldots$. It can be seen that the terms of this sequence can increase by at most 1 ; and that the number of such increases must be m. Let $\sigma_i(Y), i = 1, \ldots, m$, denote the indices where such increases occur. That is: $\dim(Y\cap\mathbb{R}^{\sigma_i(Y)}-1)=i-1$, and $\dim(Y\cap\mathbb{R}^{\sigma_i(Y)})=i$. Rather than with $\sigma_i(Y)$ it is more convenient to work with the integers $s_i(Y):=\sigma_i(Y)-1$. The integers $s_i(Y), i=1,...,m$, satisfy the inequalities $0\leq s_1(Y)\leq\cdots\leq s_m(Y)$.

A Schubert symbol of length m is a sequence $s=(s_1,\ldots,s_m)$ of integers such that $s_1\leq\ldots,\leq s_m$. It can be shown (see [22], p.76) that for each Schubert symbol $s=(s_1,\ldots,s_m)$ the set $e(s)=\{Y\in G_m\,|\,s_i(Y)=s_i, i=1,\ldots,m\}\subset G_m$ is topologically an open cell of dimension $\dim e(s)=s_1+\cdots+s_m$. These cells form a CW-decomposition of G_m. It follows that the number of d-dimensional cells in G_m is equal to the number of partitions of d into m non-negative numbers, $d=s_1+\cdots+s_m$. We will denote this number by $p_m(d)$. By restricting the Grassmannian to a finite dimensional space $\mathbb{R}^{m+n}$ we see that the number of d-dimensional cells in $G_m(\mathbb{R}^{m+n})$ is equal to the number of partitions of d into m non-negative numbers, $d=s_1+\cdots+s_m$, such that $s_i\leq n$ for $i=1,\ldots,m$. This number will be denoted by $p_m^n(d)$.

A counting argument shows that $p_m(d)$ is equal to the number of monomials $w_1^{r_1}\ldots w_m^{r_m}$ of a total degree $d=r_1+2r_2+\cdots+mr_m$. Similarly, the number of monomials $w_1^{r_1}\ldots w_m^{r_m}$ of a total degree $d=r_1+2r_2+\cdots+mr_m$ and height $h=r_1+r_2+\cdots+r_m$ is equal to $p_m^n(d)$. In fact, a bijection the monomials and the partitions

$$\lambda:\text{monomials}\leftrightarrow\text{Schubert symbols}$$

is given by $w_1^{r_1}\ldots w_m^{r_m}\mapsto(r_m, r_m+r_{m-1},\ldots,r_m+r_{m-1}+\cdots+r_1)$.

Under this bijection the monomials of height $\leq n$ correspond to Schubert symbols $s=(s_1,\ldots,s_m)$ such that $s_m\leq n$ which in turn correspond to cells of $G_m(\mathbb{R}^{m+n})$.

It is shown in [22], p.83-84, that the universal Stiefel-Whitney classes $w_1,\ldots,w_m$ are algebraically independent. This fact can be used to determine the rank of the cohomology group H^dG_m in each dimension d:

$$p_m(d)\leq\text{Rank}H^dG_m=\text{Rank}H_dG_m\leq\text{Rank}Z_dG_m\leq\text{Rank}C_dG_m=p_m(d).$$

Here Z_d and C_d are the cycle groups and the boundary groups, respectively; and the last equality follows from the fact that C_dG_m has the d-cells of G_m as its basis.

Thus $\text{Rank}H^dG_m=p_m(d)$. This argument also shows that each d-cell of G_m is at the same time a cycle and a homology class (mod $\mathbb{Z}_2$). Thus in each dimension d the monomials $w_1^{r_1}\ldots w_m^{r_m}$ of a total degree d form an additive basis for H^dG_m. Since, under the bijection λ, the monomials of height $\leq n$

correspond to the cells of $G_m(\mathbb{R}^{m+n})$ it follows that Rank $H^dG_m(\mathbb{R}^{m+n}) = p^n_m(d)$. In the Borel description the cohomology ring $H^*G_m(\mathbb{R}^{m+n})$ is isomorphic to the quotient ring of $H^*G_m(\mathbb{R}^{m+n}) \cong \mathbb{Z}_2[w_1, \ldots, w_m]$ by the ideal generated by the relations $(1 + w_1 + \cdots + w_m)(1 + \overline{w}_1 + \cdots + \overline{w}_m) = 1$, where $\overline{w}_1, \ldots, \overline{w}_m$ are the dual Stiefel-Whitney classes. These facts, however, do not yet imply that the monomials $w_1^{r_1} \ldots w_m^{r_m}$ of a total degree d and height $\leq n$ form an additive basis for $H^dG_m(\mathbb{R}^{m+n})$. For instance, since $G_m(\mathbb{R}^{m+n})$ is a closed manifold of dimension mn , its cohomology group in dimension mn is generated by one element. In the Ehresmann cell decomposition, $G_m(\mathbb{R}^{m+n})$ has one cell in the top dimension mn, given by the Schubert symbol $(n, \ldots, n)$. This cell corresponds to the monomial w_m^n under the bijection λ. This, however, does not imply that w_m^n is necessarily a generator of $H^{mn}G_m(\mathbb{R}^{m+n})$. The fact that the bijection λ indeed furnish a basis for $H^dG_m(\mathbb{R}^{m+n})$ is true. It was proved by the author in [21]:

Theorem 5.2. *The set of monomials $w_1^{r_1} \ldots w_m^{r_m}$ of a total degree $d = r_1 + 2r_2 + \cdots + mr_m$ and height $h = r_1 + r_2 + \cdots + r_m \leq n$ forms an additive basis for $H^dG_m(\mathbb{R}^{m+n})$.*

Theorem 5.2 will play an important role in sections 8 and 9.

6. Borsuk-Ulam Theorem for Stiefel Manifolds

In this section we will discuss a generalization of the Borsuk-Ulam theorem for $G = \mathbb{Z}_2 = O(m)$ given by the author in [19] and [20]. It proceeds as follows. Let $V_m(\mathbb{R}^{m+n})$ be the Stielel manifold of orthonormal m-frames in $\mathbb{R}^{m+n}$ and let $f : X = V_m(\mathbb{R}^{m+n}) \to \mathbb{R}^{m(m+k)} = W$ be a map. In other words, f assigns to every m-frame in $V_m(\mathbb{R}^{m+n})$ an m-tuple of vectors in $\mathbb{R}^{m+k}$. Let W_0 be the subspace of W consisting of the m-tuples which are not linearly independent; in other words, of the m-tuples which, when represented by $m(m+k)$ matrices, are of rank less than m. The orthogonal group $O(m)$ acts freely on $V_m(\mathbb{R}^{m+n})$ and on $\mathbb{R}^{m(m+k)}$ in a standard way, by the right multiplication : if $A \in O(m)$ and $w \in V_m(\mathbb{R}^{m+n})$, or $w \in \mathbb{R}^{m(m+k)}$, is represented by a matrix, then $A \cdot w = wA^T$ (where A^T is the transpose of A). Then $W - W_0$ is exactly the subset of W where the action is free. If $m = 1$, $V_1(\mathbb{R}^{1+n}) = S^n$ and the map $V_m(\mathbb{R}^{m+n}) \to \mathbb{R}^{m(m+k)}$ is $S^n \to \mathbb{R}^{k+1}$, as in the Borsuk-Ulam theorem (Theorem I). Moreover the set $(\mathrm{Av}f)^{-1}(W_0)$ corresponds to A_f. We will write $A_f = (\mathrm{Av}f)^{-1}(W_0)$.

The size of the set A_f in this case, given by the Index Theorem 3.1, can be described more specifically by using the Stiefel-Whitney classes. Since the action of $O(m)$ on $V_m(\mathbb{R}^{m+n})$ and on $W - W_0$ is free, instead of the

$O(m)$-cohomology we can use the ordinary cohomology of the orbit spaces. The orbit space of the standard action of $O(m)$ on $V_m(\mathbb{R}^{m+n})$ is the (real) Grassmann manifold $G_m(\mathbb{R}^{m+n})$ of m-dimensional subspaces of $\mathbb{R}^{m+n}$. The Gramm-Schmidt orthogonalization process provides a homotopy equivalence $W - W_0 \approx V_m(\mathbb{R}^{m+n})$. Thus $\mathrm{Ind}^{O(m)}X = \mathrm{Ind}^{O(m)}\, V_m(\mathbb{R}^{m+n}) = J(m,n)$, $\mathrm{Ind}^{O(m)}(W - W_0) = \mathrm{Ind}^{O(m)}V_m(\mathbb{R}^{m+k}) = J(m,k)$ and the Index Theorem 3.1 says that

$$(\mathrm{Ind}^G A_f)\cdot(\mathrm{Ind}^G(W - W_0)) \subset \mathrm{Ind}^G X.$$

This last inclusion of ideals contains information about the size of the set $\overline{A_f}$ and A_f: it says that those sets cannot be to small (recall that $\overline{X} = X/G$ denotes also the orbit space of a G-action on a space X). In particular, in some special cases, a more specific information can be obtained, in a way analogous to the Yang results described in Section 3. Thus, for instance, in the case $m = 2$, we obtain the following result (see [20, Corollary 5.3]) :

Theorem 6.1. *If $k < n$ and $f : V_2(\mathbb{R}^{n+2}) \to \mathbb{R}^{2(k+2)}$ is a map then the covering dimension of $\overline{A_f}$ is at least $2n - k - 2$. Furthermore, since the orbit map $A_f \to \overline{A_f}$ is a bundle with fibre $O(2)$, the covering dimension $\dim A_f \geq 2n - k - 1$.*

If $n = 2^s - 1$ (and $m = 2$), this result can be improved (see [20, Corollary 6.2]) :

Theorem 6.2. *If $n = 2^s - 1$, $k < n$ and $f : V_2(\mathbb{R}^{n+2}) \to \mathbb{R}^{2(k+2)}$ is a map then $\dim \overline{A_f} \geq n$ and hence $\dim A_f \geq n + 1$.*

7. Parametrized results for $G = \mathbb{Z}_2$

In 1981 the author initiated his research on the question of finding "continuous" or "parametrized" versions of the Borsuk-Ulam theorem. The goal was to extend some of the existing results from single spaces to bundles of spaces over a base space. The reader is referred to [15], [16] and [18].

Suppose that X is a space over a base space B, that is, a space with a map $p : X \to B$, and suppose that $T : X \to X$ is a fibre preserving free involution, that is, a free involution such that $pT = p$. As in Section 3, let $u(T)$ be the characteristic class of T on X. Then, for any i and r, we can define a map $e_i(T) : H^rB \to H^{r+i}\overline{X}$ by $b \to (\overline{p}{*}b)\cup u^i(T)$ for $b \in H^*B$. Thus $e_i(T) : H^*B \to H^*\overline{X}$ is a homomorphism of degree i. The homomorphism $e_i(T)$ was introduced in [16] and [18].

Suppose that $q : W \to B$ is another space over B and $T' : W \to W$ is a free fibre-preserving involution on W'. Suppose further that $f : X \to W$ is a fibre preserving map. In this case, we define the "degeneracy set" of f to be $A_f = \{x \in X | fTx = T'fx\}$. If f is an equivariant map, then A_f is the set of zeros of f. In general, for an arbitrary f, the averaging construction of section 3 can be applied; and then A_f is the set of zeros of Avf. The set A_f is also a space over B. Again, we are interested in the size of A_f. The following theorem was proved by the author in [15] and [16] with some additional assumptions and improved by Nakaoka in [24].

Theorem 7.1. *Let $p : S \to B$ be an n-sphere bundle with the antipodal fibre-preserving involution $T : X \to X$. Let $q : W \to B$ be a vector space bundle with fibre $\mathbb{R}^k$ and let $f : X \to W$ be a fibre-preserving map. Then the map $e_{n-k}(T) : H^rB \to H^{r+n-k}\overline{A_f}$ is injective.*

Just as in the results described in Section 6, this theorem can be used to obtain estimates on the covering dimension of A_f. Thus, for instance, we have the following corollary.

Corollary 7.2. *If, in Theorem 7.1, B is a closed manifold then* $\dim A_f \geq \dim B + n - k$.

In [4] Dold showed that the size of A_f can be described by a polynomial whose coefficients are the Stiefel-Whitney classes of the bundle $p : S \to B$. In the next section we will show how Dold's construction can be extended to maps of bundles of Stiefel manifolds.

8. BUNDLES OF STIEFEL MANIFOLDS

Let $E \to B$ be a (real) vector space bundle of a fibre dimension $m + n$; and let $p : V_m(E) \to B$ be the associated Stiefel manifold bundle. The fibre of $V_m(E)$ is the space of orthonormal m-frames in a fibre of $E \to B$ (which is $\mathbb{R}^{m+n}$). As in Section 4, the orthogonal group $O(m)$ acts freely on $V_m(E)$ by the right multiplication : if $A \in O(m)$ and $w \in V_m(E)$ is represented by a matrix, then $A \cdot w = wA^T$ (where A^T is the transpose of A).

The orbit space $\overline{V}_m(E)$ of the $O(m)$-action on $V_m(E)$ is the total space of the orbit bundle $\bar{p} : \overline{V}_m(E) \to B$. The fibre of $\bar{p}$ is a Grassmann manifold $G_m(\mathbb{R}^{m+n})$ whose cohomology structure was described in Section 5.

Let $w_1, \cdots, w_m$ be the universal Stiefel-Whitney classes, $w_i \in H^iBO(m) = H^i(G_m)$ (as in Section 5, G_m is the infinite Grassmannian). If X is any free $O(m)$-space with the orbit space $\overline{X}$, we will denote by $w_i|\overline{X}$ the Stiefel-Whitney classes of $X \to \overline{X}$. Thus $w_i|\overline{X} \in H^i(\overline{X})$.

Let $v_i := w_i|V_m(E) \in H^i\overline{V}_m(E)$. By Theorem 5.2, the monomials $v_1^{r_1} \cdots v_m^{r_m}$ of height $\leq n$, when restricted to each fibre, form a basis for the cohomology of the fibre. $H^*\overline{V}_m(E)$ is an (H^*B)-module with the action $b \cdot y := (\bar{p} * b) \cup y$ for $b \in H^*B$ and $y \in \overline{V}_m(E)$. We can now apply the Leray-Hirsch theorem to the fibre bundle $\bar{p} : \overline{V}_m(E) \to B$ and obtain the following theorem.

Theorem 8.1. *The monomials $v_1^{r_1} \cdots v_m^{r_m}$ of height $\leq n$ form an H^*B-basis for $H^*\overline{V}_m(E)$.*

Note that the number of monomials of a fixed height n is equal to the number of ordered sequences $r_1, \cdots, r_m$ such that $r_1 + \cdots + r_m = n$.

9. The Stiefel-Whitney Ideal

Definition 9.1. Consider the polynomial algebra $H^*B[x_1, \cdots, x_m]$. Let $e : H^*B[x_1, \cdots, x_m] \to H^*\overline{V}_m(E)$ be the evaluation map defined by the substitution $x_i \to v_i$, $i = 1, \cdots, m$ (compare [4, (1.13)]). By Theorem 8.1, for each polynomial $p(x_1, \cdots, x_m) \in H^*B[x_1, \cdots, x_m]$ of height $n+1$, $e(p(x_1, \cdots, x_m)) = p(v_1, \cdots, v_m)$ can be written uniquely as a linear combination of monomials of height $\leq n$ with coefficients in H^*B. Suppose then that $p(v_1, \cdots, v_m) = b_1 c_1(v_1, \cdots, v_m) + \cdots + b_s c_s(v_1, \cdots, v_m)$. Then the polynomial $p(x_1, \cdots, x_m) + b_1 c_1(x_1, \cdots, x_m) + \cdots + b_s c_s(x_1, \cdots, x_m)$ in $H^*B[x_1, \cdots, x_m]$ will be called *the Stiefel-Whitney polynomial* (corresponding to $p(x_1, \cdots, x_m)$). The ideal in $H^*B[x_1, \cdots, x_m]$ generated by the Stiefel-Whitney polynomials will be called *the Stiefel-Whitney ideal of the bundle E* and it will be denoted by $W(E)$.

The following theorem will be proved in a forthcoming paper ; its proof is purely algebraic.

Theorem 9.2. *The evaluation map $e : H^*B[x_1, \cdots, x_m] \to H^*\overline{V}_m(E)$ is surjective and its kernel is the Stiefel-Whitney ideal $W(E)$ of the bundle E.*

Remark 9.3. The construction of the Stiefel-Whitney ideal can be carried out in the "universal" case. Let $EO(m+n) \to BO(m+n)$ be a universal bundle for $G = O(m+n)$ and let $\phi : B \to BO(m+n)$ be a classifying map for $E \to B$. The naturality of the Stiefel-Whitney ideal implies that the induced map $H^*BO(m+n)[x_1, \cdots, x_m] \to H^*B[x_1, \cdots, x_m]$ maps the Stiefel-Whitney ideal $W(EO(m+n))$ into the Stiefel-Whitney ideal $W(E)$.

Remark 9.4. Suppose that X is a free $O(m)$-space and $X \to B$ is an $O(m)$-locally trivial bundle such that there is a fibre-preserving equivariant homotopy equivalence $X \approx V_m(E)$ over B. Then we can define the

Stiefel-Whitney ideal $W(X)$ of X to be the kernel of the composite map $H^*B[x_1, \cdots, x_m] \to H^*\overline{V}_m(E) \cong H^*\overline{X}$.

We can now state and prove a theorem on fibre preserving maps of bundles of Stiefel manifolds corresponding to the Borsuk-Ulam theorem (Theorem I), to the Yang theorem and to Theorems 3.1, 6.1, 6.2 and 7.1. It is also analogous to the results of Fadell and Husseini [8] and [9]. It describes the index of the "degeneracy set" A_f in terms of the Stiefel-Whintey ideal. A study of the "degeneracy" set A_f for bundles of the Stiefel manifolds in terms of index (and a more general setting of $G = U(m)$ and $G = Sp(m)$) was carried out by N. Mramor-Kosta in [23].

We will use the following notation analogous to that of Section 4. As before, let $E \to B$ be a vector space bundle of a fibre dimension $m+n$ and let $p : V_m(E) \to B$ be the associated Stiefel manifold bundle with the standard action of $O(m)$. Let $E' \to B$ be a vector space bundle of a fibre dimension $m(m+k)$ and also with the standard action of $O(m)$. Suppose $f : V_m(E) \to E'$ is a fibre preserving map. Let E'_0 be the subspace of E' consisting of those elements of E' which (when represented by $m(m+k)$ matrices), are of rank less than m ; i.e., $E' - E'_0$ is exactly the part of E' where the action is free. Let $A_f = (\mathrm{Av} f)^{-1}(E'_0)$. We will write e_f for the composite map

$$e_f : H^*B[x_1, \cdots, x_m] \quad \to \quad H^*\overline{V}_m(E) \quad \to \quad H^*\overline{A}_f,$$

where the second map is induced by the restriction $\overline{A}_f \to \overline{V}_m(E)$.

Theorem 9.5. $(\mathrm{Ker}\ e_f) \cdot W(E' - E'_0) \subset W(E)$.

Note that $E' - E'_0$ equivariantly deformation retracts to $V_m(E)$ (by a Gramm-Schmidt orthogonalization process, as in Section 6) and thus $W(E' - E'_0)$ is defined as in Remark 9.4. The proof of Theorem 9.5 is analogous to that of Theorem 3.1.

Theorem 9.5 imposes a lower bound on the size of A_f in a way analogous to Theorem 7.1 and Corollary 7.2. Each polynomial in $H^*B[x_1, \cdots, x_m]$ defines, through the evaluation map, a homomorphism $H^*B \to \overline{V}_m(E)$; and, by a restriction, a homomorphism $H^*B \to \overline{V}_m(\overline{A}_f)$. A consequence of the inclusion in Theorem 9.5 is that the kernel of e_f cannot be "too small". This implies that for "many" polynomials, the homomorphism $H^*B \to H^*\overline{V}_m(\overline{A}_f)$ is injective, and thus, in the corresonding dimensions, the ACS cohomology is non-trivial. As in Section 7, this implies lower bound estimates on $\dim \overline{A}_f$. But the fibres of the orbit map $A_f \to \overline{A}_f$ are $O(m)$ and thus $\dim A_f = \dim \overline{A}_f + \dim O(m) = \dim \overline{A}_f + (1/2)(m-1)(m-2)$.

10. Parametrized Borsuk-Ulam theorems for multi-valued maps

In this section we will outline some of the results obtained in a recent joint paper of this author with M. Izydorek [13] on multi-valued acyclic maps of sphere bundles into vector space bundles. We return to the case of free actions of $G = \mathbb{Z}_2$. By combining methods used by Dold in [4] with techniques applicable to multi-valued maps we show that some of Dold's results can be extended to such maps. Methods for extending single-valued map results to acyclic multi-valued maps were invented by Eilenberg and Montgomery [6] who applied them to multi-valued fixed point theorems. Such methods are based on the Vietoris mapping theorem. This author used them in [14] to prove a multi-valued version of the Borsuk-Ulam theorem. Subsequently they were extended and refined in various ways by Górniewicz [11] and others. We will show how they can be used to extend parametrized Borsuk-Ulam theorems of Dold [4] which we mentioned here at the end of section 7. We will also indicate how our results can be proved in the relative case, for pairs of spaces rather than for single space only. This allows us to obtain positive results for bundles over manifolds with boundary; for instance, over a closed interval.

Definition 10.1. Let X and Y be spaces and let f be a multi-valued map from X to Y, i.e., a function which assigns to each $x \in X$ a non-empty subset $f(x)$ of Y. We say that f is *upper semicontinous* if each $f(x)$ is compact and if the following condition hols : For every open subset V of Y containing $f(x)$ there exists an open subset U of X containing x such that for each $x' \in U$, $f(x') \subset V$.

For instance, if X and Y are compact then f is upper semicontinuous iff its graph is closed in $X \times Y$.

Definition 10.2. A multi-valued map f from X to Y is said to be $\mathbb{Z}_2$-*admissible* (briefly, *admissible*) if there exists a space Γ and two single valued continuous maps $\alpha : \Gamma \to X$ and $\beta : \Gamma \to Y$ such that :

(i) α is a *Vietoris map*, i.e., it is surjective, proper and each set $\alpha^{-1}(x)$ is $\mathbb{Z}_2$-acyclic.

(ii) For each $x \in X$ the set $\beta(\alpha^{-1}(x))$ is contained in $f(x)$.

We will say that the pair (α, β) is a "selected pair" for f.

For instance, if each $f(x)$ is acyclic (and if f is upper semicontinuous) then f is admissible.

As before, we assume that the spaces considered here are paracompact and we use the Cech cohomology theory H^* with coefficients mod 2 (the coefficient group $\mathbb{Z}_2$ will be suppressed from the notation).

Let $p : E \to B$, $p' : E' \to B$ be vector bundles over the same space B. Let $SE \subset E$ be the sphere bundle of E and let $f : SE \to E'$ be an acyclic fibre preserving map ($p' \circ f = p$, i.e., for each $x \in SE$, $f(x)$ is contained in the fibre ${p'}^{-1}(px)$). Let $A_f = \{x \in SE | f(x) \cap f(-x) \neq \emptyset\}$.

By identifying antipodal points in SE we obtain the projective bundle $\bar{p} : \overline{S}E \to B$ of E and 2-sheeted coverings $SE \to \overline{S}E$ and $A_f \to \overline{A}_f$; let $u \in H^1\overline{S}E$ and $u_f \in H^1\overline{A}_f$ be their characteristic classes. Let $(H^*B)[x]$ be the polynomial ring over H^*B in one indeterminate x. Let m, n be the fibre dimensions of E and E', respectively. We use Stiefel-Whitney classes w_jE, $w_jE' \in H^jB$ and Stiefel-Whitney polynomials $\sum_{j=0}^{m}(w_jE)x^{m-j}$ and $\sum_{j=0}^{m}(w_jE')x^{m-j}$. Since $H^*(\overline{S}E)$ and $H^*(\overline{A}_f)$ are (H^*B)-algebras (via $\bar{p}^* : H^*B \to H^*(\overline{S}E)$), we can substitute u and u_f for the indeterminate x and obtain a homomorphism of (H^*B)-algebras

$$e : (H^*B)[x] \to H^*(\overline{S}E) \to H^*(\overline{A}_f), \quad x \to u.$$

Theorem 10.3. *If $q(x) \in (H^*B)[x]$ is such that $q(u_f) \neq 0$ then $q(x)w'(x) = w(x)q'(x)$ for some polynomial $q'(x) \in (H^*B)[x]$.*

Corollary 10.4. *If m, n are the fibre dimensions of E, E', respectively, then $q(u_f) = 0$ for all polynomials $q(x)$ whose degree with respect to t is smaller than $m - n$. In other words, the H^*B-homomorphism*

$$\bigoplus_{i=0}^{m-n-1} (H^*B)x^i \longrightarrow H^*(\overline{A}_f), \quad x^i \to u_f^i$$

is monomorphic. In particular, if $m > n$ then cohom.dim.$(\overline{A}_f) \geq$ cohom.dim. $(B) + m - n - 1$, *where* cohom.dim. *denotes the cohomological dimension.*

As a special case of 10.4 we obtain the following theorem which was proved in [16] for singule-valued maps.

Corollary 10.5. *Let $p : S \to B$ be an n-sphere bundle with the antipodal involution, let $p' : E' \to B$ be an $\mathbb{R}^k$-bundle and let f be an admissible multi-valued fibre preserving map from S to E' over B. Then there is an injective map*

$$H^j(B) \to H^{j+n-k}(\overline{A}_f).$$

In particular, if $k = n$, this is a map induced by the projection $\overline{A}_f \to B$; i.e.,

$$(\bar{p}|\overline{A}_f)^* : H^jB \to H^j(\overline{A}_f)$$

is injective for all $j \geq 0$.

Corollary 10.6. *If B is a closed manifold and f is an admissible multi-valued fibre preserving map of an n-sphere bundle $p : S \to B$ with the antipodal involution to an $\mathbb{R}^k$-bundle $p' : E' \to B$ then* $\dim \overline{A}_f \geq \dim B + n - k$*, where* dim *denotes the covering dimension.*

To prove Theorem 10.3, we will first prove a lemma which is a version of Theorem 1.3 from [4] adapted to our situation. We show that Dold's theorem is valid not just for maps $f : SE \to E'$, but also in the following, more general setting.

Suppose that X is any space with a free involution $a : X \to X$ and $v : X \to SE$ is an equivariant Vietoris map. Let $g : X \to E'$ be a single-valued map which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & E' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p'} \\ SE & \xrightarrow[p]{} & B \end{array}$$

commutative, $p'g = pv$. Set

$$X_g = \{x \in X \mid gx = g(ax)\}.$$

Just as $H^*(\overline{S}E)$ and $H^*(\overline{A}_f)$ were (H^*B)-algebras via $\bar{p}^* : (H^*B) \to H^*(\overline{S}E)$, $H^*(\overline{X})$ and $H^*(\overline{X}_g)$ are (H^*B)-algebras via the homomorphism $\bar{v}^* \circ \bar{p}^* : H^*B \to H^*\overline{X}$. Thus the characteristic class $u_g \in H_1(\overline{X}_g)$ of the involution $a|X_g : X_g \to X_g$ can be substituted for t to any polynomial $q(x) \in (H^*B)[x]$.

Theorem (1.3) of [4] corresponds to the following lemma in our setting.

Lemma 10.7. *If $q(x) \in (H^*B)[x]$ vanishes on $\overline{X}_g$, $q(u_g) = 0$, then there is a polynomial $q'(x) \in (H^*B)[x]$ such that*

$$q(x)w'(x) = w(x)q'(x).$$

Proof. It is well known (and easily seen) that if X and Y are free $\mathbb{Z}_2$-spaces and $X \to Y$ is an equivariant Vietoris map then the induced map $\overline{X} \to \overline{Y}$ of the orbit spaces is also a Vietoris map. Thus, in our case, $\bar{v} : \overline{X} \to \overline{S}E$ is a Vietoris map.

We will not repeat Dold's argument step by step ; we will only note that his proof can be adapted to our setting due to the fact that the homomorphism

$\bar{v}^*$ induced by the Vietoris map $\bar{v}$ is an isomorphism. Thus, as far as the cohomology is concerned, the arrows $SE \leftarrow X \rightarrow E'$ work just as well as a single arrow $SE \rightarrow E'$.

Proof of Theorem 10.3. Given an admissible multi-valued map f from SE to E', choose a space Γ and (single-valued) maps α and β such that (α, β) is a "selected pair" for f (see Definition 10.2). Let $X = \{(\gamma, \gamma') \in \Gamma \times \Gamma \mid \alpha(\gamma) = -\alpha(\gamma')\}$. Consider the following commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \beta\ } & E' \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ SE & \xrightarrow[\ p\ ]{} & B \end{array}$$

Here π is the first projection, $(\gamma, \gamma') \rightarrow \gamma$, and $v = \alpha_0 \pi$. Then v is a Vietoris map since for each $x \in SE$, $v^{-1}(x) = \alpha^{-1}(x) \times \alpha^{-1}(-x)$ and $\alpha^{-1}(x)$ and $\alpha^{-1}(-x)$ are acyclic.

The space X admits a free involution $(\gamma, \gamma') \rightarrow (\gamma', \gamma)$ and $v : X \rightarrow SE$ becomes then an equivariant map. Let $g = \beta \circ \pi : X \rightarrow E'$. Notice that if $h(\gamma', \gamma)$ for some $(\gamma, \gamma') \in X$ then $f(\alpha(\gamma)) \cap f(\alpha(\gamma')) \neq \emptyset$. Thus $v(X_g) \subset A_f$ and by the naturality of characteristic classes with respect to equivariant maps,

$$(\bar{v}|\overline{X}_g)^*(u_f) = u_g \in H^1\overline{X}_g.$$

Thus $q(u_g) = q((\bar{v}|\overline{X}_g)^*(u_f)) = (\bar{v}|\overline{X}_g)^*(q(u_f)) = 0$.

Sometimes it is useful to have relative versions of the results discussed above. Thus (continuing with the notation used in Section 1) suppose that B_0 is a closed subset of B, $S(E_0) = \pi^{-1}(B_0)$ and $\overline{A}_{0f} = \overline{A}_f \cap \overline{S(E_0)}$. We work with the polynomial ring $(H^*B)[t]$. Then substitution of $u \in H^1(\overline{S}E)$ for x yields a homomorphism of $H^*(B, B_0)$-algebras

$$e : (H^*B)[x] \rightarrow H^*(\overline{S}E) \rightarrow H^*\left(\overline{A}_f, \overline{A}_{0f}\right).$$

The results of [4], §1, remain valid in this relative case. Consequently, we obtain relative versions of Theorem 10.3 also :

Theorem 10.8. *If $q(x) \in H^*(B, B_0)[x]$ is such that $q(u_f) = 0$ then $q(x)w'(x)$ $= w(x)q'(x)$ for some polynomial $q'(x) \in H^*(B, B_0)[x]$.*

Corollary 10.9. *If m, n are fibre dimensions of E, E', respectively, then $q(u_f) = 0$ for all polynomials $q(x)$ whose degree with respect to x is smaller than $m - n$. In other words, the $H^*(B, B_0)$-homomorphism*

$$\bigoplus_{i=0}^{m-n-1} H^*(B, B_0)x^i \longrightarrow H^*(\overline{A}_f, \overline{A}_{0f}), \quad x^i \to u_f^i$$

is monomorphic. In particular, if $m > n$ then $\dim(\overline{A}_f) \geq \dim(B) + m - n - 1$, where dim *denotes the covering dimension.*

Corollary 10.10. *Let $p : (S, S_0) \to (B, B_0)$ be an n-sphere bundle over (B, B_0) with the antipodal involution, let $p' : (E', E'_0) \to (B, B_0)$ be an $\mathbb{R}^k$-bundle and let f be an admissible multi-valued fibre preserving map from S to E'* over *(B, B_0). Then there is an injective map*

$$H^j(B, B_0) \to H^{j+n-k}(\overline{A}_f, \overline{A}_{of}).$$

In particular, if $k = n$, this is map induced by the projection $\overline{A}_f \to B$; i.e.,

$$(\bar{p}|(\overline{A}_f, \overline{A}_{0f}))^* : H^j(B, B_0) \to H^j(\overline{A}_f, \overline{A}_{0f})$$

is injective for all $j \geq 0$.

Corollary 10.11. *If B is a closed manifold with boundary B_0 and f is an admissible multi-valued fibre preserving map of an n-sphere bundle $p : (S, S_0) \to (B, B_0)$ then* $\dim \overline{A}_f \geq \dim B + n - k$.

If B is an interval $0 \leq s \leq 1$, then this implies that there exists a continuum $C \subset A_f$ joining $\pi^{-1}(0)$ with $\pi^{-1}(1)$.

Remark 10.12. When using the Eilenberg-Montgomery technique for multi-valued mappings, some multi-valued mapping theorems may be reducible to the corresponding single-valued cases. Generally speaking, this is the case when the multi-valued map in question has a single-valued cross-section (or a "selector"). To have an example showing the significance of a generalization to multi-valued maps, one has to construct a map with acyclic (for instance, convex) values which would not be "reducible" to a single-valued map, such as of a multi-valued map without a single valued cross-section (or "selector"). Examples of this kind can easily be given ; in fact, they exist already in the classical case of the Borsuk-Ulam theorem, for single spaces over a point (rather than for bundles of spaces).

One sees easily that f is defined in a consistent way and is upper-semicontinuous (its graph is closed). The map is admissible ; its values are acyclic (even convex) : they are all single points except of one value which is a closed interval. In this case the conclusion of the (multi-valued) Brsuk-Ulam theorem is, of course, valid ; but that conclusion cannot be obtained directly from Dold's results because the map has no single-valued corss-section.

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HISTORY AND METHODS OF METRIC FIXED POINT THEORY

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1. INTRODUCTION

Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. It is a sub-branch of the functional analytic theory in which geometric conditions on the mappings and/or underlying spaces play a crucial role. Although it has a purely metric facet, it is also a major branch of nonlinear functional analysis with close ties to Banach space geometry.

For convenience we take an admittedly narrow view of the subject here. We discuss only single-valued mappings, and *primarily* only the nonexpansive mappings. In particular we shall not touch upon results which bridge the topological and metric theories, such as the study of condensing mappings, nor upon the degree-theoretic techniques which are often useful in applications. Also, this survey is not intended to be completely self-contained. We include some proofs, either because they might not be readily available elsewhere or to illustrate the methods involved. Many of the details are found in the books by Aksoy and Khamsi [1] or Goebel and Kirk [43], but for other details one will need to look to the original sources. Several of the results we discuss have emerged subsequent to the publication [1], [43].

For a chronological and methodological perspective, we list below (by name) a few of the more well-known fixed point theorems of functional analysis.

(a) The Zermelo-Bourbaki-Kneser Theorem (1908-1955)
(b) The Brouwer Theorem (1912)
(c) Banach's Contraction Mapping Principle (1922)
(d) The Schauder Theorem (1930)
(e) The Leray-Schauder Theorem (1934)
(f) The Schauder-Tychonoff Theorem (1935)

(g) The Markov-Kakutani Theorem (1936)
(h) Tarski's Theorem (1955)
(i) The Browder-Göhde-Kirk Theorem (1965)
(j) The Ryll-Nardzewski Theorem (1966)
(k) Sadovskii's Theorem (1967)
(l) Caristi's Theorem (1976)
(m) Maurey's Theorems (1981)

Most of these theorems are well-known to specialists in fixed point theory. One might roughly characterize them as follows: (a) and (h) are set-theoretic; (b), (d), (e), and (f) are more topological in nature; the linear structure of the space plays a large role in (g) and (j); (c), (i), (l), and (m) are primarily metric in nature; and (k) provides an example of a result which bridges the metric and topological theories. In our consideration of the metric theory, since (c) is well understood, we shall concentrate on the theory as it pertains to (i), (l), and (m), the rather surprising connections (i) and (l) have with (a), and on a number of more recent developments. We refer to Zeidler [88] for a thorough discussion of the remaining theorems listed as well as numerous other fixed point theorems.

The methods of *metric* fixed point theory are typical of those of functional analysis in general. However we shall make sharper distinctions than usual in the underlying foundational aspects of the theory. Specifically, we shall use (ZF) to refer to the basic (six) axioms of Zermelo and Fraenkel; (ZFDC) adds the so-called Axiom of Dependent Choices (the principle of inductive definition of sequences); and (ZFC) adds the full Axiom of Choice. In Section 2 we are not concerned with the methods involved, but in Sections 3 and 4 we concentrate respectively on that part of the theory which can be developed within (ZF) and within (ZFDC). We note in particular that some of the fundamental existence theorems (theorems of Caristi, Kirk, Soardi) can be proved wholly within (ZF). At the same time much of the remaining theory, at least in the separable case, can be proved within (ZFDC), a fact which many concerned with foundations find quite acceptable. However many results, such as the deep theorems of Maurey, seem to require (ZFC). Most mathematicians do not see a problem with these methods either, although some suggest that more awareness might be advisable. The techniques of (ZFC) include standard applications of the full Axiom of Choice via Zorn's Lemma, Tychonoff's Theorem, and transfinite induction, as well as nonstandard methods including Banach space ultrapowers, Banach Limits, ultranets, etc. These methods have led to some of the most interesting results in the theory. An excellent discussion of the use of nonstandard methods in metric fixed point theory (including the results of Maurey) is found in Aksoy and Khamsi [1].

2. Nonexpansive mappings: Basic theory

The central questions of metric fixed point theory, especially as related to nonexpansive mappings, usually involve the study of the following topics.

(I) Conditions which imply existence of fixed points.
(II) The structure of the fixed point sets.
(III) Asymptotic regularity.
(IV) The approximation of fixed points.
(V) Applications.

Here we take up, in order, some of the central results in each of the above categories.

2.1. Existence of fixed points. We begin with the study of nonexpansive mappings in a Banach space setting. If X is a Banach space and $D \subseteq X$, then a mapping $T : D \to X$ is said to be *nonexpansive* if for each $x, y \in D$,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

The study of the existence of fixed points for nonexpansive mappings has generally fallen into three categories. We shall say that a Banach space has FPP if each of its nonempty bounded closed convex subsets has the fixed point property for nonexpansive self-mappings (which we denote f.p.p.); wc-FPP if each of its weakly compact convex subsets has f.p.p.; and B- FPP if its unit ball (hence any ball) has f.p.p. This latter category is primarily relevant to dual spaces where the unit ball is always compact in its weak* topology relative to any predual. The classical nonreflexive space ℓ^1 provides an example of a space which has B-FPP but not FPP (Karlovitz [51], Lim [64]). Also c_0 provides an example of a space which has wc-FPP but neither FPP nor B-FPP (Maurey [68]).

Clearly one of the central goals of the theory should be to characterize those Banach spaces which have FPP. It is known that essentially all classical reflexive spaces, and in particular all uniformly convex spaces, have FPP, hence wc-FPP, via a geometric property they share called normal structure. As our point of departure, we shall state and prove the original 1965 fixed point theorem of Kirk [58]. It is an examination of the *proof* of this theorem which provides the basis of much that follows in the next section of this report.

Definition 1. A Banach space X is said to have *normal structure* if any bounded convex subset K of X which contains more than one point contains a point x_0 such that

$$\sup\{\|x_0 - x\| : x \in K\} < \operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\}.$$

Such a point x_0 is called a *nondiametral point* of K.

In what follows we shall use the symbol $B(x;r)$ to denote the *closed ball* centered at $x \in K$ with radius $r > 0$. Thus:

$$B(x;r) = \{y \in K : \|x - y\| \leq r\}.$$

Also we need some additional notation.

$$\begin{aligned} \operatorname{diam}(K) &= \sup\{\|u - v\| : u, v \in K\}; \\ r_x(K) &= \sup\{\|x - v\| : v \in K\}, \qquad (x \in K); \\ r(K) &= \inf\{r_x(K) : x \in K\}. \end{aligned}$$

If X is reflexive and if K is a bounded closed and convex subset of X then it readily follows from the weak compactness of K that the set

$$C(K) := \{z \in K : r_z(K) = r(K)\},$$

called the *Chebyshev center* of K, is a *nonempty* closed and convex subset of K.

Theorem 1. *Let X be a reflexive Banach space which has normal structure. Then X has FPP.*

Proof. Let K be a nonempty bounded closed and convex subset of X, and suppose $T : K \to K$ is nonexpansive. Suppose $\Im$ denotes the collection of all nonempty closed convex T-invariant subsets of K. Then if $\Im$ is ordered by set inclusion, it follows from the weak compactness of the members of K (X is reflexive) that every descending chain in $\Im$ has a lower bound—namely the intersection of its members. Thus by Zorn's Lemma, $\Im$ has a minimal element, say K_0.

Obviously $\overline{\operatorname{conv}} T(K_0)$ is nonempty, closed, convex, and T-invariant; thus by minimality it cannot be a proper subset of K_0, so

$$K_0 = \overline{\operatorname{conv}} T(K_0).$$

Let $u \in C(K_0)$; thus $r_u(K_0) = r(K_0)$. Since $\|T(u) - T(v)\| \leq \|u - v\| \leq r(K_0)$ for all $v \in K_0$, it follows that $T(K_0) \subseteq B(T(u); r(K_0))$. Consequently,

$$K_0 = \overline{\text{conv}}T(K_0) \subseteq B(T(u); r(K_0))$$

showing that $r_{T(u)}(K_0) = r(K_0)$; thus $T(u) \in C(K_0)$. We conclude that $C(K_0)$ is T-invariant. The minimality of K_0 implies that $K_0 = C(K_0)$ and in view of normal structure this in turn implies that K_0 consists of a single point which is fixed under T.

Complications in the general study of FPP were noted early. A major obstacle is the obvious fact that fixed point properties for nonexpansive mappings are not invariant under renormings. There are other hindrances as well. It has been known virtually from the outset that FPP for a Banach space depends strongly on 'nice' geometrical properties of the space. On the other hand, two closed convex subsets $K_1, K_2 \subseteq X$ may have f.p.p. yet $K_1 \cap K_2$ *may fail to have* f.p.p.! Indeed, even much more can be said. Goebel and Kuczumow [44] have shown how to construct a descending sequence $\{K_n\}$ of nonempty bounded closed convex subsets of ℓ^1 which has the property that n is odd, K_n has f.p.p., if n is even K_n fails to have f.p.p., and in fact the sequence $\{K_n\}$ may be constructed so that $\bigcap K_n$ falls into either category. The space ℓ^1 provides the setting for another interesting example. It is possible to construct a family $\{K_\epsilon\}$ ($\epsilon > 0$) of bounded closed convex sets in ℓ^1 each of which has f.p.p., but which converges as $\epsilon \to 0$ in the Hausdorff metric to a nonempty bounded closed convex K_0 which fails to have f.p.p.

The above examples illustrate why the problem of classifying Banach spaces which have FPP or sets which have f.p.p. might be extremely difficult. However, Theorem 1 raises the obvious question of precisely how are reflexivity, normal structure, and FPP related.

Karlovitz ([50], [51]) first noted that even in reflexive spaces normal structure is not essential for FPP. An example is provided by the James's spaces X_β, $\beta \geq 0$, defined by:

$$X_\beta = \{x \in \ell^2 : \|x\|_\beta = \max\{\|x\|_{\ell^2}, \beta\|x\|_\infty\}\}.$$

R. C. James observed that while X_β is reflexive (since it is isomorphic to ℓ^2), it fails to have normal structure if $\beta = \sqrt{2}$. In fact, X_β has normal structure $\Leftrightarrow \beta < \sqrt{2}$. Even more is known. The concept of asymptotic normal structure was introduced by Baillon and Schöneberg in 1981 [5]. A Banach space X has *asymptotic normal structure* if each nonempty bounded closed and convex subset K of X which contains more than one point has the property: If $\{x_n\} \subseteq K$ satisfies $\|x_n - x_{n+1}\| \to 0$ then there exists $x \in K$ such that

$$\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}(K).$$

In [5] Baillon and Schöneberg observe that X_β has asymptotic normal structure $\Leftrightarrow \beta < 2$, and they prove the following:

Theorem 2. *In a reflexive Banach space, asymptotic normal structure $\Rightarrow$ FPP.*

In the same paper Baillon and Schöneberg went on to show that even X_2 has FPP, thus showing that asymptotic normal structure is not a necessary condition for FPP. (Surprisingly, P. K. Lin proved in 1985 ([65]) that X_β has FPP for all $\beta > 0$.)

It is actually shown in [5] that in an arbitrary Banach space asymptotic normal structure implies wc-FPP. There has been an interesting further development regarding wc-FPP. In [47] A. Jiménez-Melado and E. Lloréns Fuster introduced a generalization of uniform convexity called orthogonal convexity and proved that weakly compact convex subsets of orthogonally convex spaces have the fixed point property for nonexpansive mappings. (See also [48].)

Orthogonal convexity is defined as follows: For points x, y of a Banach space X and $\lambda > 0$, let

$$M_\lambda(x,y) = \left\{ z \in X : \max\{\|z-x\|, \|z-y\|\} \le \frac{1}{2}(1+\lambda)\|x-y\| \right\}.$$

If A is a bounded subset of X, let $|A| = \sup\{\|x\| : x \in A\}$, and for a bounded sequence $\{x_n\}$ in X and $\lambda > 0$, let

$$D(\{x_n\}) = \limsup_{i\to\infty} \ (\limsup_{j\to\infty} \|x_i - x_j\|);$$
$$A_\lambda(\{x_n\}) = \limsup_{i\to\infty} \ (\limsup_{j\to\infty} |M_\lambda(x_i, x_j)|).$$

A Banach space is said to be *orthogonally convex* if for each sequence $\{x_n\}$ in X which converges weakly to 0 and for which $D(\{x_n\}) > 0$, there exists $\lambda > 0$ such that $A_\lambda(\{x_n\}) < D(\{x_n\})$. It is shown in [47] that every uniformly convex space is orthogonally convex. Other examples given in [47] include Banach spaces with the Schur property (hence ℓ^1), c_0, c, and James's space J.

In 1971 it was observed by Day-James-Swaminathan [25] that every separable space has an equivalent norm which has normal structure (also see van Dulst [28]). *Thus every separable reflexive space has an equivalent norm which has FPP.* (It appears to be an open question whether every reflexive Banach space has an equivalent norm which has normal structure.)

The question of whether reflexivity is essential for FPP remains open, but there is some recent evidence that it might be. First, it is known that the classical nonreflexive spaces c_0 and (as noted above) in ℓ^1 fail to have FPP. Also, Bessaga and Pełczynski have shown that if X is any Banach space with an unconditional basis, then X is non-reflexive $\Leftrightarrow$ X contains a subspace isomorphic to c_0 or ℓ^1. *Thus all classical nonreflexive can be renormed so that they fail to have FPP.*

This raises an obvious question: Can c_0 or ℓ^1 be renormed so that they have FPP? Recall ([46]) that any renorming of ℓ^1 contains almost isometric copies of ℓ^1 suggesting, at least for ℓ^1, that the answer should be no. If indeed the answer is no, then by the Bessaga-Pełczynski result, in any space with an unconditional basis, FPP$\Rightarrow$ reflexivity.

The space L^1 : As we have noted ℓ^1 (hence L^1) fails to have FPP. However, in 1981, Alspach [2] proved much more, namely that L^1 fails to have wc-FPP. At the same time, Maurey [68] proved that all reflexive subspaces of L^1 do have FPP (hence wc-FPP). There has been another recent development. Dowling and Lennard [26] have shown that nonreflexive subspaces of L^1 fail to have FPP. Thus: *A subspace of* L^1 *has* FPP $\Leftrightarrow$*is reflexive.*

But the question remains:

Does FPP $\Rightarrow$ reflexivity?

Of course the reverse implication remains unknown as well. In fact the following question also remains open:

Does superreflexivity $\Rightarrow$ FPP?

Recall that a superreflexive space is one which has the property that every space which is finitely representable in it must itself be reflexive. Superreflexive spaces are also characterized by that fact that they all have equivalent uniformly convex norms (Enflo [33]). The fact that Maurey [68] proved (also in 1981) that superreflexive spaces have FPP for *isometries* suggests that the answer to the above might be yes.

2.2. Structure of the fixed point set. The structure of the fixed point sets of nonexpansive mappings in Banach spaces with FPP is well understood.

Theorem 3 (Bruck [15]). *Let X be a reflexive space, or a separable space, which has FPP, and let $K \subseteq X$ be nonempty bounded closed and convex.*

Then the set of common fixed points of any commutative family of nonexpansive self-mappings of K is a nonempty nonexpansive retract of K.

This raises the obvious question of whether FPP implies the conclusion of Bruck's theorem in general (as it does in the separable case). Of course a positive answer to "FPP $\Rightarrow$ reflexive"would settle this affirmatively as well.

Remark. Under the assumptions of Bruck's theorem, the collection of subsets of K which have f.p.p. includes all the nonexpansive retracts of K.

Proof. Suppose $R : K \to F \subseteq K$ is a nonexpansive retraction, and let $G : F \to F$ be nonexpansive. Then $G \circ R : K \to F$ is nonexpansive, so by FPP there exists $x \in K$ such that $G \circ R(x) = x$. But this implies $x \in F$ and $G(x) = x$. Therefore, $R(x) = x$ is a fixed point of G.

Bruck's proof of the above theorem in the single-mapping case is somewhat involved, relying on a clever use of Tychonoff's Theorem, and the general case is quite difficult. However:

Corollary 1. *Bruck's theorem for finite commutative families follows easily from its validity for a single mapping.*

Proof. Suppose X and K are as in Theorem 3, and suppose T and G are commutative nonexpansive mappings of $K \to K$. Let Fix(T) (etc.) denote the fixed point set of T in K. Then since $T \circ G = G \circ T$, it follows that $G : \text{Fix}(T) \to \text{Fix}(T)$. Since Fix($T$) is by assumption a nonexpansive retract of K, by the above Remark $\text{Fix}(T) \cap \text{Fix}(G) \neq \varnothing$. Let R be a nonexpansive retraction of K onto Fix(T). Then

$$\text{Fix}(T) \cap \text{Fix}(G) = \text{Fix}(G \circ R),$$

and the latter set is also a nonexpansive retract of K. This shows that the common fixed point set of two commuting mappings of $K \to K$ is a nonexpansive retract of K. The general case for a finite family of commuting nonexpansive mappings follows by induction.

We look at the structure of the fixed point sets in a more abstract metric space setting in the next section.

2.3. Asymptotic regularity and approximate fixed points. At the outset we call attention to the survey of Bruck [16].

If K is a subset of a Banach space X, then $f : K \to K$ is said to be *asymptotically regular* (at $x \in K$) if $\|f^n(x) - f^{n+1}(x)\| \to 0$.

In 1976 Ishikawa [45] obtained a surprising result, a special case of which may be stated as follows: Let K be an arbitrary bounded closed convex subset of a Banach space X, $T : K \to K$ nonexpansive, and $\lambda \in (0,1)$. Set $T_\lambda = (1-\lambda)I + \lambda T$. Then for each $x \in K$:

$$\|T_\lambda^n(x) - T_\lambda^{n+1}(x)\| \to 0.$$

Thus by iterating the 'averaged' mapping T_λ one can always reach points which are approximately fixed (but on the other hand, these points may not be near fixed points—indeed, it need not be the case that T even have a fixed point).

In 1978, Edelstein and O'Brien [30] proved that $\{T_\lambda^n(x) - T_\lambda^{n+1}(x)\}$ converges to 0 uniformly for $x \in K$ and, in 1983, Goebel and Kirk [42] proved that this convergence is even uniform for $T \in \Im$, where $\Im$ denotes the collection of all nonexpansive self-mappings of K. Thus:

Theorem 4. *Suppose K is a bounded closed convex subset of a Banach space. Then for each $\epsilon > 0$ there exists a positive integer N such that if $x \in K$ and $T \in \Im$, $n \geq N \Rightarrow$*

$$\|T_\lambda^n(x) - T_\lambda^{n+1}(x)\| \leq \epsilon.$$

Concerning the rate of convergence of $\{T_\lambda^n(x)\}$, Baillon and Bruck [4] have observed that the estimates of [42] or the method of Bruck [16] can be used to establish:

$$\|T_\lambda^n(x) - T_\lambda^{n+1}(x)\| = O\left(\frac{1}{\log n}\right),$$

and they have conjectured (and supported with compelling computational evidence):

$$\|T_\lambda^n(x) - T_\lambda^{n+1}(x)\| = O\left(\frac{1}{\sqrt{n}}\right).$$

As they note, this would be somewhat surprising if true, since it happens that the above is the exact estimate for linear T.

It is interesting to note that the uniform version of Ishikawa's result can be used to say something about the structure of the set of points of a nonexpansive mapping which are 'approximately' fixed. Let K be a bounded closed convex set, and suppose $T : K \to K$ is nonexpansive. For $\epsilon > 0$ set

$$F_\epsilon(T) = \{x \in K : \|x - T(x)\| \leq \epsilon\}.$$

A standard argument shows that these sets are all nonempty. Fix $z \in K$, let $t \in (0,1)$, and consider the contraction mappings $T_t : K \to K$ defined by $T_t(x) = (1-t)z + tT(x)$. If x_t is the (unique) fixed point of T_t, then $\lim_{t\to 1^-} \|x_t - T(x_t)\| = 0$.

Theorem 5 (Bruck [17]). *Suppose K is a bounded closed convex subset of a Banach space X and suppose $T : K \to K$ is nonexpansive. Then $F_\epsilon(T)$ is pathwise connected.*

Proof. Let $f = \frac{1}{2}(I + T)$. Then by Theorem 4 there exists N such that if $n \geq N$, then for each $x \in K$, $\|f^n(x) - f^{n+1}(x)\| \leq \epsilon$. Fix $u, v \in F_\epsilon(T)$, and let $S = \text{seg}[u, v]$. Then the path $f^N(S)$ lies in $F_\epsilon(T)$. Also, for $0 \leq i < N$,

$$\text{seg}[f^i(u), f^{i+1}(u)] \text{ and } \text{seg}[f^i(v), f^{i+1}(v)] \text{ both lie in } F_\epsilon(T).$$

and the union of these segments form a path joining u and v, respectively, to $f^N(u)$ and $f^N(v)$.

There are other interesting results related to approximate fixed points. A convex subset K of a Banach space is said to be *linearly bounded* if the intersection of K with any line in X is bounded. We shall say that a Banach space has the *approximate fixed point property* (AFPP) if any linearly bounded closed convex subset K of X has the a.f.p.p., i.e., if every nonexpansive $T : K \to K$ satisfies $\inf\{\|x - T(x)\| : x \in K\} = 0$.

Theorem 6 (Reich [78], Shafrir [81]). *X has the AFPP $\Leftrightarrow$ X is reflexive.*

In order to characterize those convex subsets of a Banach space which have the a.f.p.p., Shafrir introduced the concept of a directionally bounded set. A *directional curve* in a Banach space X is a curve $\gamma : [0, \infty) \to X$ for which there exists $b \geq 0$ such that

$$t - s - b \leq \|\gamma(s) - \gamma(t)\| \leq t - s.$$

A convex subset K of X is said to be *directionally bounded* if it contains no directional curve. Note that a line is a directional curve with $b = 0$. Thus if K is directionally bounded, then K is linearly bounded. Shafrir proves in [81] that the two concepts are equivalent in X if and only if X is reflexive. The following is also proved in [81] (where it is actually formulated in the more abstract setting of a hyperbolic metric space).

Theorem 7. *A convex subset K of a Banach space has the a.f.p.p. if and only if K is directionally bounded.*

The following analog of the Remark following Theorem 3 also holds for the above theorem.

Remark. If a subset K of a Banach space has the a.f.p.p., then every nonexpansive retract of K has the a.f.p.p.

Proof. Suppose $R : K \to F \subseteq K$ is a nonexpansive retraction, and let $G : F \to F$ be nonexpansive. Then by assumption there exists $\{x_n\} \subseteq K$ such that $\|x_n - G \circ R(x_n)\| \to 0$. But

$$\|R(x_n) - G \circ R(x_n)\| = \|R(x_n) - R \circ G \circ R(x_n)\| \leq \|x_n - G \circ R(x_n)\| \to 0,$$

so $\inf\{\|u - G(u)\| : u \in F\} = 0$.

2.4. Approximation of fixed points. It has been known for some time (see [37]) that even in a uniformly convex setting the iterates of the averaged mapping $f = \frac{1}{2}(I + T)$ of the previous section need not actually converge to a fixed point of T. However in 1971, Kaniel [49] obtained a rather complicated discrete convergence procedure for approximating fixed points of nonexpansive mappings in such spaces. Quite recently, Moloney [69] obtained a refinement of Kaniel's method for constructing such a sequence, a method which in fact applies to asymptotically nonexpansive mappings. We briefly descsibe this result, beginning with the relevant definitions.

The *modulus of convexity* of a Banach space X is the function $\delta_X : [0,2] \to [0,1]$ defined as follows:

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}.$$

It is known that the function δ_X is nondecreasing, and continuous on $[0,2)$. A Banach space is said to be *uniformly convex* if $\delta_X > 0$ whenever $\epsilon > 0$.

We assume that X is a uniformly convex Banach space and $K \subseteq X$ is a given bounded closed and convex subset of X. A mapping $T : K \to K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive real numbers for which $\lim_{n\to\infty} k_n = 1$ and $\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|$ for all $x, y \in K$. In [41], Goebel and Kirk show that such a mapping T always has a fixed point. (There is an extensive literature on asymptotically nonexpansive and related classes of mappings which we do not take up here.) In [69], using some technical lemmas, Moloney constructs an auxiliary mapping $S : K \to K$ which has the properties:

(a) $T(p) = p \Leftrightarrow S(p) = p$;
(b) $\|p - S(x)\| \leq \|p - x\|$;
(c) If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \|S(x_n) - x_n\| = 0$, then $S(x) = x = T(x)$.

Using the mapping S he then constructs a sequence $\{y_n\}$ which always converges strongly to a fixed point of T. However, as in the case of Kaniel's construction, it is not possible to determine how close to the fixed point one is at any step.

The only other (strong) convergence result of note (not requiring some form of strong compactness) seems to be the following:

Theorem 8 (Reich [77]). *Let X be a uniformly smooth Banach space, K a bounded closed convex subset of X, and $T : K \to K$ nonexpansive. For fixed $y \in K$, let $t \in (0,1)$ and let y_t denote the unique fixed point of the contraction mapping*

$$T_t(\cdot) = (1-t)y + tT(\cdot).$$

Then

$$\lim_{t \to 1^-} y_t \text{ exists and is a fixed point of } T.$$

The existence of a fixed point of T in the setting of the above theorem was first proved by Baillon [3]. Later Turett [86] proved that uniformly smooth Banach spaces are actually superreflexive and have normal structure.

A mapping f defined on a subset D of a Banach space X (and taking values in X) is said to be *demiclosed* if it is closed from the weak topology on D to the norm topology on X. Thus, f is demiclosed if for any sequence $\{u_j\}$ in D,

$$\text{weak} - \lim_{n\to\infty} u_j = u \text{ and } \lim_{n\to\infty} \|f(u_j) - w\| = 0 \Rightarrow u \in D \text{ and } f(u) = w.$$

The following theorem has been fundamentally important in the theory of nonexpansive mappings. As with the Kaniel-Moloney constructions discussed above, uniform convexity seems to be the *smoothness* condition essential to its validity.

Theorem 9. *Let X be a uniformly convex Banach space, let K be a closed and convex subset of X, and suppose $T : K \to X$ is nonexpansive. Then the mapping $f = I - T$ is demiclosed on K.*

2.5. Applications. One of the principal applications of the theory of nonexpansive mappings in a functional analytic context has been to the study of monotone and accretive operators.

Accretive operators arise as a very natural generalization of monotone operators. Note that a real-valued function φ of a real variable is monotone increasing provided

$$(s-t)(\varphi(s) - \varphi(t)) \geq 0.$$

This concept extends to Banach spaces as follows. Assume X is a Banach space with dual space X^*, and let $D \subseteq X$.

Definition 2. A mapping $T : D \to X^*$ is said to be *monotone* if for each $u, v \in D$,

$$\langle u - v, T(u) - T(v) \rangle \geq 0.$$

(We use the pairing $\langle x, \xi \rangle$ to denote $\xi(x)$, $x \in X$, $\xi \in X^*$.)

Now let X be a (real) Banach space and define the normalized duality mapping $J : X \to 2^{X^*}$ by setting for $x \in X$,

$$J(x) = \{j \in X^* : \langle x, j \rangle = \|x\|^2 \text{ and } \|j\| = \|x\|\}.$$

Definition 3. A mapping $T : D \to X$ is said to be *accretive* if for each $u, v \in D$ there exists $j \in J(u - v)$ such that

$$\langle T(u) - T(v), j \rangle \geq 0.$$

Note that if X is a Hilbert space then $X = X^*$ and the class of monotone and accretive operators coincide.

A complete characterization of accretive operators was given by Kato in 1967 [53].

Lemma 1. *Let X be a Banach space, $D \subseteq X$, and $T : D \to X$. Then T is accretive if and only if for every $x, y \in D$ and $\lambda \geq 0$,*

$$\|x - y\| \leq \|x - y + \lambda(T(x) - T(y))\|.$$

Thus a mapping T is accretive if and only if for each $\lambda \geq 0$ the mapping $J_\lambda = (I + \lambda T)^{-1}$ is *nonexpansive* on its domain. Using this fact it is possible to extend the definition of accretive mappings to multivalued mappings in a natural way. For a given subset B of X, let

$$|B| = \inf\{\|x\| : x \in B\}.$$

A mapping $A : D \to 2^X$ is said to be *accretive* if for each $x, y \in D$ and $\lambda \geq 0$,

$$\|x - y\| \leq |x - y + \lambda(T(x) - T(y))|$$

Again, the mapping $J_\lambda = (I + \lambda T)^{-1}$ is single-valued and nonexpansive on its domain. If it is the case that the domain of J_λ is all of X for $\lambda > 0$, then A is said to be *m-accretive.*

We now state three results which illustrate the fundamental way in which the theory of nonexpansive and accretive mappings are intertwined. They show respectively the way in which the FPP, B-FPP, and the common fixed point property for commuting families of nonexpansive mappings are each related to basic questions about accretive mappings. The first is a very simple observation about closedness of m-accretive operators which seems to be due to Reich [76] (also see Reich and Torrejón [79]).

Theorem 10. *Let X be a Banach space which has FPP, let $D \subseteq X$ and let $A : D \to X$ be m-accretive. Then if $\{x_n\} \subseteq D$ is bounded and if $y_n \in Ax_n$ satisfies $y_n \to y$, it follows that y is in the range of A.*

Theorem 11 (Kirk-Schöneberg [62]). *Let X be a Banach space which has the B-FPP, let $D \subseteq X$ and let $A : D \to X$ be m-accretive. Suppose*

$$\lim_{\|x\| \to \infty} |Ax| = \infty.$$

Then the range of A is all of X.

Our final result is essentially due to R. H. Martin.

Theorem 12. *Suppose K is a bounded closed convex subset of a Banach space X, and suppose K has the common fixed point property for commuting families of nonexpansive mappings. Suppose $A : K \to X$ is continuous, bounded and accretive, and suppose A satisfies the boundary condition*

$$\lim_{h \to 0+} h^{-1} \operatorname{dist}(x - hA(x), K) = 0.$$

Then 0 is in the range of A.

Proof. By a theorem of R. H. Martin [67] (see also [12, Section 9]), for every $x \in K$ the Cauchy problem

$$x'(t) = -T(x(t));\ x(0) = x$$

has a unique solution on $[0, \infty)$. Put $S(t)x = x(t)$ so that $\{S(t)\}_{t \geq 0}$ is a family of functions of $K \to K$. Then

1) $S(t+s) = S(t) \circ S(s)$ for $s, t \geq 0$; thus the family $\{S(t)\}_{t \geq 0}$ is commutative.
2) $S(t)$ is nonexpansive for $t \geq 0$, i.e., $\|S(t)x - S(t)y\| \leq \|x - y\|$ for each $x, y \in K$.

By the common fixed point property, there exists $x_0 \in K$ such that $x_0 = S(t)x_0$ for each $t \geq 0$. Thus

$$\frac{d}{dt} S(t)x_0 = 0 = x_0'(t) = -A(S(t)x_0) = -A(x_0),$$

proving $0 \in A(K)$.

3. Abstract theory: a set-theoretic approach

There is a set-theoretic basis for much of metric fixed point theory. In this section we discuss Caristi's Theorem, an analog of Theorem 1, a fixed point theorem due to Soardi, and the structure of the fixed point sets of nonexpansive mappings, all in an abstract setting.

3.1. Caristi's Theorem. We begin with two 'equivalent' facts. The first is a special case of Ekeland's celebrated variational principle ([31], [32]), and the second is a well-known theorem due to J. Caristi [23].

(E) (Ekeland, 1974) *Let* (M, d) *be a complete metric space and* $\varphi : M \to \mathbb{R}^+$ *l.s.c. Define:*

$$x \leq y \Leftrightarrow d(x,y) \leq \varphi(x) - \varphi(y), \; x, y \in M.$$

Then $(M \leq)$ *has a maximal element.*

(C) (Caristi, 1976) *Let* M *and* φ *be as above. Suppose* $f : M \to M$ *satisfies:*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \; x \in M.$$

Then f *has a fixed point.*

Proof (E) $\Rightarrow$ (C). With M, φ as above, and f as in Theorem (C), define $x \leq y \Leftrightarrow d(x,y) \leq \varphi(x) - \varphi(y)$, $x, y \in M$. By (E) there exists $\bar{x} \in M$ such that $\bar{x}$ is maximal in $(M, \leq)$. But, $d(\bar{x}, f(\bar{x})) \leq \varphi(\bar{x}) - \varphi(f(\bar{x})) \Rightarrow f(\bar{x}) \geq \bar{x}$. Hence by maximality, $\bar{x} = f(\bar{x})$.

Proof (C) $\Rightarrow$ (E). Assume (E) is false. Then $\forall x \in M \; \exists \; f(x) \in M$ such that $x < f(x)$. It follows that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \; x \in M.$$

By (C) f must have a fixed point $\bar{x}$. But by assumption, $\bar{x} < f(\bar{x})$—a contradiction.

Thus it is easy to see that (E) $\Leftrightarrow$ (C). However the implication (C) $\Rightarrow$ (E) requires at least (ZFDC), an observation due to N. Brunner [18]. On the other hand, Manka [66] has recently shown that (C) holds within (ZF).

Some of the very early proofs of (C) include:

Author	Method	Axioms
Caristi (1976)	Transfinite induction	(ZFC)
C. S. Wong (1976)	Transfinite induction	(ZFC)
Kirk (1976)	Zorn's Lemma	(ZFC)
Browder (1976)		(ZFDC)
Brezis-Browder (1976)		(ZFDC)
Penot (1976)		(ZFDC)
Siegel (1977)		(ZFDC)
Pasicki (1978)		(ZFC)
Brøndsted (1979)		(ZFC)

Since the appearance of (C) and (E) there have been numerous papers devoted to various proofs of these results and to equivalent formulations (e.g., by S. Kasahara [52], S. Park [71], W. Takahashi [85], etc.). We call particular attention the 1986 survey by S. Park [70] on equivalent formulations of Ekeland's variational principle. Many of these proofs require only (ZFDC) (see Section 4).

An extension of a theorem attributed variously to Zermelo, Bourbaki, and Kneser provides the basis for Manka's proof that Caristi's theorem holds in (ZF). In the sequel we shall simply refer to this as Zermelo's Theorem. It should not, however, be confused with his celebrated well-ordering theorem, although apparently the idea is implicit in the proof of that theorem given in [89]. As opposed to that theorem, the following theorem can be proved wholly within (ZF). For a proof, see, e.g., Dunford and Schwartz [29, p.9] or Zeidler [88, p.504].

Theorem 13 (Zermelo [89]). *Let $(E, \leq)$ be a partially ordered set and let $f : E \to E$ satisfy $f(x) \geq x \ \forall x \in E$. Suppose*

every chain in E has a l.u.b.

Then f has a fixed point in E. In fact, given $x \in E$ one can construct $\bar{x} \in E$ such that $\bar{x} \geq x$ and $f(\bar{x}) = \bar{x}$.

In order to prove (C) in (ZF), Manka generalized the above by weakening the chain condition to:

"$\forall a \in E$, every well ordered subset of E with first element a is bounded above, and given such a well ordered set C $\exists$ a function σ which selects an element from the set of all upper bounds of C."

3.2. Nonexpansive mappings—normal structure. We now use Zermelo's Theorem to give another proof of Theorem 1. We restate Theorem 1 in slightly more general form.

Theorem 14. *Suppose K is a nonempty weakly compact convex subset of a Banach space X and suppose K has normal structure. Let $T : K \to K$ be nonexpansive. Then T has a fixed point.*

Note that in saying K has *normal structure* we mean (cf., Definition 1), $r(H) < \text{diam}(H)$ for every convex subset H of K for which $\text{diam}(H) > 0$. The Chebyshev center, $C(H)$, of H is the collection of points of H which serve as centers of balls of minimal radius containing H.

Proof of the Theorem. Closed balls will play a fundamental role in the argument, and we shall restrict ourselves to balls *relative to the underlying domain K*. As before, we use the symbol $B(x; r)$ to denote the *closed ball* centered at $x \in K$ with radius $r > 0$. Thus:

$$B(x; r) = \{y \in K : \|x - y\| \le r\}.$$

Ball intersections will also play a fundamental role. Let

$$\Sigma = \left\{ D : D = \bigcap_{i \in I} B(x_i; r_i) \text{ where } x_i \in K,\ r_i \ge 0 \right\}.$$

Note that since K is bounded, $K \in \Sigma$. As we shall see, it is only elements of Σ that are relevant to our proof.

For $A \subseteq K$, let

$$\text{cov}(A) = \bigcap \{D : D \in \Sigma \text{ and } D \supseteq A\}.$$

Two facts are pertinent.

1. $\text{cov}(A) \in \Sigma$ for each $A \subseteq K$;
2. The members of Σ are all compact in the weak topology.

Step 1. Let

$$\mathcal{M} = \{D \in \Sigma : D \neq \varnothing;\ T : D \to D\}.$$

Define $f_1 : \mathcal{M} \to \mathcal{M}$ by setting

$$f_1(D) = \mathrm{cov}(T(D)).$$

Order $(\mathcal{M}, \leq)$ by $D_1 \leq D_2 \Leftrightarrow D_2 \subseteq D_1$. Then $f_1(D) \geq D\ \forall D \in \mathcal{M}$. Also, every chain in $\mathcal{M}$ has an l.u.b.—namely, the intersection of all its members. By Zermelo's Theorem, given $D \in \mathcal{M}$, there exists $D^* \in \mathcal{M}$ such that

$$f_1(D^*) = D^*.$$

Thus, $D^* = \mathrm{cov}(T(D^*))$.

Step 2. For $D \in \Sigma$, $D \neq \varnothing$, define

$$R(D) = \left\{ r \geq 0 : D \cap \left(\bigcap_{u \in D} B(u; r) \right) \neq \varnothing \right\}.$$

Note that $R(D) \neq \varnothing$ since $\mathrm{diam}(D) = \sup\{\|x - y\| : x, y \in D\} \in R(D)$. Thus $r(D) = \mathrm{g.l.b.} R(D)$ is well defined. Now set

$$C(D) = \left\{ z \in D : z \in \bigcap_{u \in D} B(u; r(D)) \right\}.$$

Assertion. $C(D) \in \Sigma$ and $C(D) \neq \varnothing$.

Proof. $C(D) \in \Sigma$ by definition. Also, by definition, if $r > R(D)$ then

$$C_r(D) := \left\{ z \in D : z \in \bigcap_{u \in D} B(u; r(D)) \right\} \neq \varnothing.$$

We show that $C(D) = \bigcap_{r > r(D)} C_r(D)$ from which the conclusion will follow by weak compactness of the members of Σ.

Obviously, $C(D) \subseteq C_r(D)$ for each $r > r(D)$ since

$$C(D) = D \cap \left(\bigcap_{u \in D} B(u; r(D)) \right) \subseteq D \cap \left(\bigcap_{u \in D} B(u; r) \right) = C_r(D).$$

Thus $C(D) \subseteq \bigcap_{r>r(D)} C_r(D)$. Now suppose there exists $z \in \bigcap_{r>r(D)} C_r(D)$ such that $z \notin C(D)$. Then there exists $u \in D$ such that $\|z-u\| > r(D)$; hence there exists r' satisfying $\|z-u\| > r' > r(D)$. But $\|z-u\| > r'$ implies $z \notin C_{r'}(D)$—a contradiction.

Notice that by normal structure, $\operatorname{diam}(D) > 0 \Rightarrow C(D)$ is a proper subset of D.

Now define $f_2(D) = C(D^*)$ where $C(D^*)$ denotes the Chebyshev center of D^*. Repeating the argument of Step 1, we conclude that f_2 also has a fixed point which, by normal structure, must be a singleton, hence a fixed point of T.

Remarks. Several remarks about the above proof are in order. First, the linear structure of the space does not enter in, so that a much more abstract approach is possible. To describe this approach we introduce the concept of a "convexity structure" (see [72]). A family Σ of subsets of a given set S is called a *convexity structure* if

(i) $\varnothing \in \Sigma$;
(ii) $S \in \Sigma$;
(iii) Σ is closed under arbitrary intersections.

Note that according to this definition the family Σ defined in the proof of Theorem 14 is a convexity structure which, in fact, contains the closed balls of the underlying space.

Now let (M,d) be a bounded metric space and let Σ be a convexity structure in M. We fix the following notation: For $D \in \Sigma$, set

$$\begin{aligned} \operatorname{diam}(D) &= \sup\{d(u,v) : u,v \in D\}; \\ r_x(D) &= \sup\{d(x,v) : v \in D\}, \qquad (x \in M); \\ r(D) &= \inf\{r_x(D) : x \in D\}. \end{aligned}$$

The convexity structure Σ is said to be *normal* if $r(D) > 0$ for each $D \in \Sigma$ for which $\operatorname{diam}(D) > 0$. Also, Σ is said to be *compact* if every family of subsets of Σ which has the finite intersection property has nonempty intersection.

By following precisely the proof just given for Theorem 14, we can obtain the following (see [74], [61], [20]).

Theorem 15. *If (M,d) is a nonempty bounded metric space which possesses a convexity structure which is compact, normal, and contains the closed balls of M, then every nonexpansive mapping $T: M \to M$ has a fixed point.*

The advantage to this abstract formulation is that even in a Banach space setting it frees the underlying topology. The sets in the convexity structure only need consist of sets which are convex and τ-compact or, as we shall see, even τ-countably compact in *some* topology τ for which the norm closed balls are τ-closed. For example τ could be the weak* topology as in the case of $\ell^1 = (c_0)^*$, (Lim [64]) and the Hardy space $H^1(\Delta)$ (Besbes et al. [6]), or τ could be the topology of local convergence in measure in L^1 (Lennard [63]).

3.3. Nonexpansive mappings—uniform relative normal structure.

We now briefly turn to an abstract version of a theorem due to Soardi [84].

A convexity structure Σ in a bounded metric space M is said to be *uniformly relatively normal* if there exists $c \in (0,1)$ such for each $D \in \Sigma$ there exists $z_D \in M$ such that:

(a) $D \subseteq B(z_D; c\,\mathrm{diam}(D))$;

(b) if $D \subseteq B(y; c\,\mathrm{diam}(D))$ for $y \in M$, then $d(z_D, y) \leq c\,\mathrm{diam}(D)$.

Soardi has observed that if X is the complexification of an order complete AM-space with unit, then the order intervals in X form a uniformly relatively normal convexity structure (where $c = 2^{-\frac{1}{2}}$). The same is true if X is an order complete AM-space with unit (where $c = \frac{1}{2}$). Zermelo's theorem may also be used to give a ZF proof of the following abstract version of Soardi's theorem [84]. (For details of the proof, we refer to Büber-Kirk [22]; also Büber [19].)

Theorem 16. *Let M be a nonempty bounded metric space which possesses a convexity structure which is compact, uniformly relatively normal, and contains the closed balls of M. Then every nonexpansive mapping $T : M \to M$ has a fixed point.*

3.4. The structure of the fixed point set of nonexpansive mappings in metric spaces. The fundamental ideas and results of this section are due to Khamsi [55], [57]. This approach will, among other things, show that Theorem 15 extends within (ZF) to finite commutative families of nonexpansive mappings. This theorem also extends to infinite families, but this extension seems to require at least (ZFDC) in the separable case ([21]) and (ZFC) in general.

Again, let (M,d) be a bounded metric space, and let Σ be a compact convexity structure on M. A subset A of M is said to be *admissible* if $A = \mathrm{cov}(A)$, where here we take

$$\mathrm{cov}(A) = \bigcap\{B : B \text{ is a closed ball containing } A\}.$$

We shall let $\varphi(M)$ denote the set of all admissible subsets of M. Then, since Σ contains the closed balls of M it follows that $\varphi(M)$ is itself a compact convexity structure on M, and in fact it turns out that this smaller structure is sufficient for almost all the existence theory for nonexpansive mappings. As before, we say that $\varphi(M)$ is *normal* if $r(A) < \operatorname{diam}(A)$ for each $A \in \varphi(M)$ with $\operatorname{diam}(A) > 0$. ($r(A)$ is defined in the previous section.)

The following is the key to the structure of the fixed point sets of nonexpansive mappings (and to the existence of nonempty common fixed point sets of commuting families of nonexpansive mappings) in a metric space setting. A subset A of M is said to be a 1-*local retract* of M if for each family $\{B_i\}_{i\in I}$ of closed balls centered in A for which

$$\bigcap_{i\in I} B_i \neq \varnothing$$

it is the case that $A \cap (\bigcap_{i\in I} B_i) \neq \varnothing$. It is easy to check that every nonexpansive retract of M is a 1-local retract of M.

We now describe several fundamental properties of 1-local retracts. The proof of the first is routine and the second is immediate.

Proposition 1. *If M is a metric space for which $\varphi(M)$ is compact, then M is complete.*

Proposition 2. *If M is a metric space for which $\varphi(M)$ is compact, and if $\{A_n\}$ is a descending sequence of sets in $\varphi(M)$ for which $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$, then $\bigcap A_n = \{z\}$.*

The following technical proposition collects several additional properties needed in this study.

Proposition 3. *Let M be a metric space and let A be a nonempty subset of M. Then:*

(1) $\operatorname{cov}(A) = \bigcap\{B(x; r_x(A)) : x \in M\}$;
(2) $r_x(A) = r_x(\operatorname{cov}(A))$ *for every* $x \in M$;
(3) $r(\operatorname{cov}(A)) \leq r(A)$;
(4) $\operatorname{diam}(\operatorname{cov}(A)) = \operatorname{diam}(A)$.

Proof.

(1) This is immediate since $B(x; r_x(A))$ is the smallest ball centered at x which contains A.
(2) Since $A \subseteq \operatorname{cov}(A)$ it follows from the definition of r_x that

$$r_x(A) \leq r_x(\operatorname{cov}(A)).$$

On the other hand, (1) implies $r_x(\mathrm{cov}(A)) \le r_x(A)$.

(3) This is immediate from the definition of r and (2).

(4) It obviously suffices to show that $\mathrm{diam}(\mathrm{cov}(A)) \le \mathrm{diam}(A)$. Let $z \in \mathrm{cov}(A)$. Then $z \in B(x; r_x(A))$ for each $x \in M$. In particular, $d(x,z) \le r_x(A) \le \mathrm{diam}(A)$ for each $x \in A$; thus $A \subseteq B(z; \mathrm{diam}(A))$. Hence

$$\mathrm{cov}(A) \subseteq B(z; \mathrm{diam}(A)).$$

It follows that

$$\mathrm{diam}(\mathrm{cov}(A)) \le \mathrm{diam}(A).$$

The following proposition, which shows that in the case of 1-local retracts normality is a hereditary property, is fundamental.

Proposition 4. *Let M be a metric space and suppose $\varphi(M)$ is compact and normal. Suppose N is a given subset of M which is a 1-local retract of M. Then $\varphi(N)$ is compact and normal.*

The key to the proof of the above is the following lemma.

Lemma 2. *Under the assumptions of Proposition 4, $r(\mathrm{cov}(A)) = r(A)$ for each $A \in \varphi(N)$.*

Proof. Clearly we may suppose $\mathrm{diam}(A) > 0$. Since Proposition 3 implies

$$r(\mathrm{cov}(A)) \le r(A)$$

we only need show the reverse inequality. By assumption $A \in \varphi(N)$, so A is of the form

$$A = N \cap \left(\bigcap\{B(x_i; r_i) : x_i \in N\}\right).$$

Also

$$\mathrm{cov}(A) \subseteq \bigcap\{B(x_i; r_i) : x_i \in N\}.$$

Choose $z \in \mathrm{cov}(A)$ and let $r = r_z$. Then

$$z \in S := \left(\bigcap_{x \in A} B(x; r)\right) \cap \left(\bigcap\{B(x_i; r_i) : x_i \in N\}\right).$$

S is a nonempty set belonging to $\varphi(M)$. Since N is a 1-local retract of M and S is the intersection of balls centered in N, $S \cap N \ne \varnothing$. Let $w \in S \cap N$. Then

$$w \in \left(\bigcap\{B(x_i; r_i) : x_i \in N\}\right) \cap N,$$

i.e., $w \in A$. On the other hand, $w \in \bigcap\{B(x;r) : x \in A\}$, so $r_w \leq r$. It follows that

$$r(A) \leq r \leq r_z(\mathrm{cov}(A)).$$

Since z was an arbitrary element of $\mathrm{cov}(A)$ the proof is complete.

Proof of Proposition 4. The definition of a 1-local retract assures that $\varphi(N)$ is compact. To see that $\varphi(N)$ is normal, let $A \in \varphi(N)$ and recall that $\mathrm{diam}(\mathrm{cov}(A)) = \mathrm{diam}(A)$ by Proposition 2. By Lemma 2, $r(\mathrm{cov}(A)) = r(A)$. Since $\varphi(M)$ is normal, $r(\mathrm{cov}(A)) < r(\mathrm{diam}(A))$, i.e., $r(A) < \mathrm{diam}(A)$.

Remark. In Propositions 1–4, one may replace compactness of $\varphi(N)$ with countable compactness.

We now have the following structure theorem.

Theorem 17. *Let M be a bounded metric space for which $\varphi(M)$ is compact and normal, and let $T : M \to M$ be nonexpansive. Then the fixed point set F_T of T is a nonempty nonexpansive retract of M, and moreover, $\varphi(F_T)$ is compact and normal.*

Proof. The fact that $F_T \neq \varnothing$ is immediate from Theorem 15. To see that F_T is a 1-local retract of M let $\{B_i\}$ be a family of closed balls centered in F_T for which

$$S := \bigcap_i B_i \neq \varnothing.$$

Then since T is nonexpansive, $T : S \to S$. Also, since S is admissible, $\varphi(S)$ is compact and normal. Therefore, again by Theorem 15, T has a fixed point in S, i.e., $F_T \cap S \neq \varnothing$. The final assertion of the theorem follows from Proposition 4.

It is even possible to extend Theorem 17 to finite commutative families.

Theorem 18. *Let M be a bounded metric space for which $\varphi(M)$ is compact and normal. Then every finite family Φ of commuting self- mappings of M has a nonempty common fixed point set F_Φ. Moreover, F_Φ is a 1-local retract of M.*

Proof. It suffices to show that $F_\Phi \neq \varnothing$. The fact that F_Φ is a 1-local retract of M can be proved as above. Suppose $\Phi = \{T_1, \ldots, T_n\}$. By the previous theorem, if F_{T_i} denotes the fixed point set of T_i, $i = 1, \ldots, n$, then each of the sets $\varphi(F_{T_i})$ is compact and normal. Since T_1 and T_2 commute, it is immediate that $T_2 : F_{T_1} \to F_{T_1}$. Thus $F_{T_1} \cap F_{T_2} \neq \varnothing$. The conclusion follows by repeating the argument step by step n times.

4. Abstract theory: Consequences of (DC)

We continue here in a constructive vein, but less so than in the previous section. Specifically, we obtain results which may not be possible within (ZF) but which seem to require no more than (ZFDC) (ZF in conjunction with the Axiom of Dependent Choices). Specifically, by (ZFDC) we mean (ZF) with the additional axiom:

(DC). If $A \neq \varnothing$ and if $R : A \to 2^A$, then $\exists\, f : \omega \to A$ such that $\forall n \in \omega$, $f(n+1) \in R(f(n))$.

The above is equivalent to (see [18]):

(M1): If $(M, \leq)$ is a partially ordered set in which each chain is *finite*, then X has a maximal element.

We remark that (ZFDC) seems to be sufficient (and essential) for the development of the foundations of functional analysis in the separable case. See, for example, [36]; also the discussions in [35] and [80]. Regarding the severity of this assumption, A. C. M. van Rooij states in [80]:

> It is a deplorably wide-spread attitude with mathematicians in general and with functional analysts in particular blindly to accept the Axiom of Choice without thinking of its costs. The Axiom enables one to claim the existence of certain objects but only if one manages not to think about what "existence" means. It is a magic key to open a door that is closed to constructivists, but the door leads to a phantom world of things that one cannot touch.
>
> Let me hasten to add that the Countable Axiom of Choice is quite a different matter. It can be understood, it can be made constructive and, best of all, it seems to be everything one needs, at least in Functional Analysis, over $\mathbb{R}$ and $\mathbb{C}$. For instance, it implies the Hahn-Banach Theorem for separable normed spaces. Admittedly it does not yield the Hahn-Banach Theorem for ℓ^∞, but does anyone ever use that?

4.1. Convexity structures. We proceed with the more abstract approach introduced in the previous chapter. Once again we recall that a convexity structure in a set S is a family Σ of subsets of S satisfying (i) $\varnothing \in \Sigma$; (ii) $S \in \Sigma$; (iii) Σ is closed under arbitrary intersections.

Examples of Convexity Structures.

1. Take Σ to be the family of all closed and convex subsets of a given closed convex subset of a Banach space. (This is of course the prototype.)

2. Let B be the unit ball in a Banach space X, and take Σ to be the family of all ball intersections in B :

$$\Sigma = \left\{ \left(\bigcap_{i \in I} B(x_i; r_i) \right) \cap B : x_i \in B,\ r_i \geq 0 \right\}.$$

3. Let (M, d) be a bounded metric space and take Σ to be all ball intersections in M (the admissible sets of Section 3.2).

Remark. A convexity structure is always a subbase for a topology on the underlying set S. By Alexander's subbase theorem (see, e.g., [54, p.139]), S is compact in this topology if the convexity structure is compact. (This fact requires the Axiom of Choice, and it doesn't seem to carry over for countable compactness.)

Definition 4. A convexity structure Σ is said to be [*countably*] *compact* if whenever U is any [countable] subfamily of Σ which has the finite intersection property, $\bigcap U \neq \varnothing$.

Examples of Compact Convexity Structures.

4. The same as Example 1, but with K weakly compact.
5. The same as Example 2, but with B the unit ball in a dual Banach space.
6. The same as Example 3, but with M a hyperconvex metric space.

(Recall that a metric space M is said to be *hyperconvex* if any family $\{B(x_\alpha; r_\alpha)\}$ of closed balls in M satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ has a nonempty intersection.)

7. Let B be the unit ball in $L^1([0,1], \mathbb{R})$ and let Σ be the collection of all closed and convex subsets of B which are compact in the topology of local convergence in measure. (See [63] for the definition.)

For our next example, we consider a non-Archimedean Banach space X. Thus the triangle inequality is replaced with the stronger inequality: $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, $x, y \in X$. A non-Archimedean space X is said to be *spherically complete* if every shrinking collection of balls in X has nonempty intersection. (Note that in such a space, if two balls intersect then one must contain the other.)

8. Let X be a non-Archimedean spherically complete Banach space, and let Σ be the family of all balls in X.

It is interesting to note that in such a setting one has the following 'alternative' theorem. This observation is due to Petalas and Vidalis [75].

Theorem 19. *Suppose X is a spherically complete non-Archimedean normed space and $T : X \to X$ is a nonexpansive map. Then either T has at least one fixed point, or there exists a ball B in X of radius $r > 0$ such that $T : B \to B$ and $\|x - T(x)\| = r$ for each $x \in B$.*

9. This example is also unusual. In establishing his theorem on commutative families of nonexpansive mappings (Theorem 3 of Section 2.2), Bruck actually proved that if X is reflexive or separable and has the FPP, then the family Σ of all nonexpansive retracts of a bounded closed convex subset K of X is a compact convexity structure.

Remark. An interesting question arises in connection with Example 9. Are nonexpansive self-mappings of K continuous relative to the topology Σ generates as a subbase?

4.2. Nonexpansive mappings and countable compactness. We next show that Theorem 14 holds within (ZFDC) if the convexity structure is only assumed to be countably compact. We need the following lemma, which is of interest in itself since it requires no compactness assumption.

Lemma 3. *Let M be a nonempty bounded metric space which possesses a convexity structure Σ containing the closed balls of M. Suppose $T : M \to M$ is nonexpansive and let $\Im = \{D \in \Sigma : D \neq \varnothing \text{ and } T : D \to D\}$. Then for each $D \in \Im$ one can construct a set $D^* \in \Im$ such that $D^* \subseteq D$ and $\operatorname{diam}(D^*) \leq \frac{1}{2}(\operatorname{diam}(D) + r(D))$.*

We state and prove the theorem; the proof of the lemma follows that of Lemma 4.2 of [43]; also see [38].

Theorem 20 ([60, 61]). *Let M be a nonempty bounded metric space which possesses a convexity structure Σ which is normal and countably compact. Then every nonexpansive $T : M \to M$ has at least one fixed point.*

Proof. Let

$$\Im = \{D \in \Sigma : D \neq \varnothing,\ T : D \to D\},$$

and define $\delta : \Im \to \mathbb{R}$ by

$$\delta(D) = \inf\{\operatorname{diam}(F) : F \in \Im,\ F \subseteq D\}.$$

Set $D_1 = \Im$, and with $D_1, \ldots, D_n$ given, select $D_{n+1} \in \Im$ so that $D_{n+1} \subseteq D_n$ and

$$\operatorname{diam}(D_{n+1}) \leq \delta(D_n) + \frac{1}{n}.$$

Let

$$C = \bigcap_{n=1}^{\infty} D_n.$$

Then $C \in \Sigma$, and by countable compactness, $C \neq \varnothing$. Since $T : C \to C$, $C \in \Im$. Also, for each $n \in \mathbb{N}$

$$\operatorname{diam}(C) - \frac{1}{n} \leq \operatorname{diam}(D_{n+1}) - \frac{1}{n} \leq \delta(D_n) \leq \delta(C^*) \leq \frac{1}{2}(\operatorname{diam}(C) + r(C)).$$

Letting $n \to \infty$ we conclude $\operatorname{diam}(C) = r(C)$. By normality of Σ this implies C consists of a single point which is fixed under T.

If the normality assumption is sufficiently strengthened in the above, then no explicit compactness assumption on the convexity structure Σ is needed. A convexity structure Σ is said to be *uniformly normal* if there exists $c \in (0,1)$ such that $r(D) \leq c \operatorname{diam}(D)$ for each $D \in \Sigma$ for which $\operatorname{diam}(D) > 0$.

In a Banach space framework this concept is due to Gillespie and Williams [38] (as is the proof of Lemma 3). It turns out that in a complete metric space a uniformly normal convexity structure is always countably compact. For a proof of this fact see Khamsi [56]. (The proof of Theorem 4.4 of [43] can also be modified to show this.) Thus the following is an immediate consequence of Theorem 20. However, as we show, it also follows immediately from Lemma 3.

Theorem 21. *Let M be a nonempty bounded complete metric space which possesses a convexity structure Σ which is uniformly normal and contains the closed balls of M. Then every nonexpansive $T : M \to M$ has at least one fixed point.*

Proof. Let c be the constant associated with the definition of uniform normality of Σ and let $\mu = \frac{1}{2}(1+c)$. Let $D_1 = M$ and for each n let $D_{n+1} = D_n^*$ where $D_n^* \in \Im$ is the subset of D_n assured by Lemma 3 for which $\operatorname{diam}(D_n^*) \leq \mu \operatorname{diam}(D_n)$. Then $\operatorname{diam}(D_{n+1}) \leq \mu \operatorname{diam}(D_n) \leq \mu^n \operatorname{diam}(D_1)$. Since M is complete and $\lim_{n\to\infty} \operatorname{diam}(D_n) = 0$, it follows that $\bigcap_{n=1}^{\infty} D_n = \{z\}$ for some $z \in M$. Since each of the sets D_n is T-invariant, $T(z) = z$.

4.3. The Goebel-Karlovitz Lemma. There are many situations in the study of metric fixed point theory where the existence of minimal sets seems to be essential. The general problem we take up now is how to prove the existence of a minimal set without using (ZFC) via Zorn's Lemma. Recall that in the initial proof of Theorem 1 we used Zorn's Lemma to obtain a *minimal* nonempty closed convex T-invariant set K_0. We then observed in Section 3 that the existence of such a minimal set was not essential to the proof; a set K for which $K_0 = \overline{\mathrm{conv}}(T(K_0))$ would have done just as well. However the following result, which requires the existence of a minimal set, is fundamental to much of the theory.

Lemma 4 ([40], [51]). *Let X be a Banach space, let $K \subseteq X$, and suppose K is a minimal nonempty weakly compact convex T-invariant set for a nonexpansive mapping T. Then the assumptions $\{x_n\} \subseteq K$ and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$ imply $\lim_{n\to\infty} \|y - x_n\| = \mathrm{diam}(K)$ for each $y \in K$.*

In Section 2.1 we alluded to a fixed point theorem of Baillon and Schöneberg for nonexpansive mappings in spaces having asymptotic normal structure and to three results of Maurey. The proofs of these results are among many which use Goebel-Karlovitz Lemma. Indeed, passing to a minimal set has become a standard approach in the theory (see, e.g., the discussion in Sims [83]).

Minimization. Next we summarize the results of [21]. Basic to those results is the following minimization principle. This result is perhaps known, but it is important for our purposes to note that it can be proved wholly within (ZFDC).

Lemma 5. *Let S be a set, Γ a family of nonempty subsets of S, $\mathcal{B}$ a countable family of subsets of S, and suppose each member of Γ is the intersection of some subfamily of $\mathcal{B}$. Then Γ has a minimal element if each descending sequence in Γ is bounded below (by a member of Γ).*

Proof. Suppose $\mathcal{B} = \{B_1, B_2, \dots\}$, and let $D_0 \in \Gamma$. Let n_1 be the smallest integer for which B_{n_1} contains an element D_1 of Γ which is a proper subset of D_0. Of course if no such integer exists, then D_0 is already minimal in Γ. Having chosen $D_1, \dots, D_k$ and $n_1, \dots, n_k$, let n_{k+1} be the smallest integer strictly larger than n_k for which $B_{n_{k+1}}$ contains a member D_{k+1} which is a proper subset of D_k. Either the process terminates upon reaching a minimal element of Γ or there exists $D \in \Gamma$ such that $D \subseteq \bigcap_{k=1}^{\infty} D_k$. However in the latter case, since $\mathcal{B}$ generates the sets in Γ, $D = \bigcap_{i\in I} B_i$. Also, if $i \in I$ and if $i \neq n_1$, then $i \geq n_2$. Similarly, if $i \neq n_2$, then $i \geq n_3$. By continuing, we see that for some k it must be the case that $i = n_k$ for some k. On the other hand, $D_k \subseteq B_{n_k}$, $k = 1, 2, \dots$. Consequently,

$$\bigcap_{k=1}^{\infty} B_{n_k} \subseteq \bigcap_{i\in I} B_i = D \subseteq \bigcap_{k=1}^{\infty} D_k \subseteq \bigcap_{k=1}^{\infty} B_{n_k}.$$

Thus equality must hold and $D = \bigcap_{k=1}^{\infty} D_k$ is minimal in Γ.

Obviously the lemma implies (and in fact is implicitly equivalent to) the following:

Corollary 2. *Let X be a topological space which is second countable (i.e., which has a countable basis), let Γ be a family of nonempty closed subsets of X, and suppose every descending sequence in Γ is bounded below (by a member of Γ). Then Γ has a minimal element.*

Remark. The corollary is of course known since it is an immediate consequence of the well-known fact that every transfinite descending sequence in a second countable space must be countable. However, this approach requires transfinite induction.

We now assume that (M, d) is a bounded metric space which possesses a countably compact convexity structure Σ, and suppose Σ contains the closed balls of M. Let $D \in \Sigma$ and $p \in M$. Then if $D \neq \varnothing$, the set

$$\{z \in D : d(p, z) = \inf\{d(p, x) : x \in D\}\}$$

is nonempty (and in Σ). Such a set is called a proximinal set in Σ (relative to p). An application of Lemma 5 yields:

Theorem 22. *Suppose (M, d) is a bounded metric space which possesses a countably compact convexity structure Σ which contains the closed balls of M, and suppose the proximinal sets in Σ relative to some point $p \in M$ are separable. Let Γ be any family of nonempty subsets of Σ which is closed under countable intersections. Then Γ has a minimal element.*

In [21] the above theorem is used to show how one may circumvent an application of Zorn's Lemma (in the separable case) in proving the general version of Khamsi's common fixed point theorem for commuting families of nonexpansive mappings ([55], [56]). (Theorem 18 is the finite version of this theorem.) However, there is another application which appears to be new and might be of independent interest.

Theorem 23. *Suppose (M, d) is a bounded metric space which possesses a countably compact convexity structure Σ which contains the closed balls of M, and suppose the proximinal sets of Σ relative to some point $p \in M$ are separable. Then Σ (and the topology it generates as a subbase) is compact.*

We should remark that our proof of the above result does require the Axiom of Choice.

We conclude with an application in Banach spaces. The ball topology b_X on a Banach space X is the coarsest topology relative to which every norm closed ball $B(x;r)$ is b_X-closed. Thus a point $x \in X$ has as a base of b_X-neighborhoods sets of the form

$$V = X \setminus \bigcup_{i-1}^{n} B(x_i; r_i)$$

where $x_1, \ldots, x_n \in X$ and $\|x - x_i\| > r_i$. This topology was introduced by Corson and Lindenstrauss in 1966, and it is studied in depth by Godefroy and Kalton in [39].

Theorem 24. *Suppose the convex subsets of the unit sphere of a Banach space X are separable. Then the following are equivalent.*

(i) *The unit ball B in X is b_X-compact.*

(ii) *For every countable collection $\{B_\alpha, \alpha \in I\}$ of closed balls in X such that $\bigcap_{\alpha\in I} B_\alpha = \varnothing$, there is a finite set $F \subseteq I$ such that $\bigcap_{\alpha\in F} B_\alpha = \varnothing$.*

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EIGHTY YEARS OF THE BROUWER FIXED POINT THEOREM

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0. INTRODUCTION

The Brouwer fixed point theorem is one of the most well-known and useful theorems in topology. Since the theorem and its many extensions are powerful tools in showing the existence of solutions of many problems in pure and applied mathematics, many scholars have been studying its further extensions and applications. The purpose of this article is to survey the developments of the more than eighty years old theorem and related fields in mathematics.

Generalizations of the Brouwer theorem have appeared with related to theory of topological vector spaces in mathematical analysis. The compactness, convexity, single-valuedness, continuity, self-mapness, and finite dimensionality related to the Brouwer theorem are all extended and, moreover, for the case of infinite dimension, it is known that the domain and range of the map may have different topologies. This is why the Brouwer theorem has so many generalizations.

Other directions of its generalizations in topology are studies of spaces having the fixed point property, various degree or index theories, the Lefschetz fixed point theory, the Nielsen fixed point theory, and the fixed point theorems in the Atiyah-Singer index theory which generalizes the Lefschetz theory. However, we will not follow these lines of study.

In closing our introduction, we quote an excellent expression on the current status of fixed point theory as follows:

> "Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis. Moreover, more

or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician."

– M. Hazewinkel, Editor's Preface to the book of Istrăţescu [1981].

1. The Brouwer Fixed Point Theorem

In 1910, the Brouwer theorem appeared.

Theorem [Brouwer, 1912]. *A continuous map from an n-simplex to itself has a fixed point.*

It is clear that, in this theorem, the n-simplex can be replaced by the unit ball $\mathbf{B}^n$ or any compact convex subset of $\mathbf{R}^n$. This theorem appeared as Satz 4 of [1912]. At the end of this paper, it is noted that "Amsterdam, Juli 1910" by Brouwer himself.

Some authors confused that the theorem appeared in Brouwer [1910]. According to Bing [1969], "even before Brouwer's paper [1912] appeared, reference had been made to the Brouwer Fixed Point Theorem. (See Hadamard's reference on page 472 of Tannery [1910].)." In fact, Hadamard gave a proof of the Brouwer theorem using the Kronecker indices in the appendix of Tannery [1910].

According to Freudenthal (the editor of *L.E.J. Brouwer–Collected Works II*, North Holland, Amsterdam, 1976; where the paper [1912] is listed as "1911D".), Hadamard knew the Brouwer theorem (without proof) from a letter of Brouwer (data 4-1-1910).

According to Bing again, Brouwer [1912] himself proved the theorem by showing that homotopic maps of an $(n-1)$-sphere onto itself had the same degree (or rotation of vector fields); hence, there is no retraction of an n-cell onto its boundary; hence each map of an n-cell into itself is not fixed point free.

Alexander [1922] proved a theorem of Brouwer [1910] using the index of a map and applied it to obtain the Brouwer fixed point theorem. Birkhoff and Kellogg [1922] also gave a proof of the theorem of Brouwer [1910] by using classical methods in calculus and determinant theory. The same line of proof of the Brouwer theorem can be found in Dunford and Schwartz [1958].

Knaster, Kuratowski, and Mazurkiewicz [1929] gave a proof of the Brouwer theorem using combinatorial techniques. They used the Sperner lemma

[1928] and showed that the non-retraction theorem holds.

Later there have appeared proofs using algebraic topology, various degree theory, or differential forms. Hirsh [1963] gave a proof of the non-retraction theorem using the method of geometric topology, and Milnor [1978] gave an analytic proof. There are also many other proofs of the Brouwer theorem, and a simple proof using advanced calculus was given by Rogers [1980] and others.

The Brouwer theorem itself gives no information about the location of fixed points. However, there have been developed effective ways to calculate or approximate the fixed points. Such techniques are important in various applications including calculation of economic equilibria. The first such algorithm was the simplicial algorithm proposed by Scarf [1967] and later developed in the so-called homotopy or continuation methods for calculating zeros of function. For details of this topic, see Karmardian [1977], Forster [1980], Zangwill and Garcia [1981], and others.

In the remainder of this section, we discuss certain stories on the forerunners of the Brouwer theorem. The first one is Poincaré [*Sur les courbes définies par les équations différentielles, Ch. XVIII. Distribution des points singuliers*, Oeuvres, t.1, 191-196]. There he used the Kronecker indices to obtain the following consequence:

> *The interior of a surface "sans contact de genre 0 " has always at least a singular point.*

This seems to be an important fact in connection with the Brouwer theorem.

Kaniel [1965] is the first one mentioned that Poincaré is a forerunner of the Brouwer theorem:

Theorem (Kaniel [1965, Theorem 1]). *Let A be a continuous operator defined on a finite dimensional Banach space. If for some R and every $\lambda > 0$*

$$A(u) + \lambda u \neq 0, \quad u \in S_R, \tag{1.1}$$

where S_R is the sphere of radius R around the origin, then there exists a solution to the equation $A(v) = 0, v \in B_R$, where B_R is the ball of radius R around the origin.

Kaniel wrote: "M. Schiffer pointed out that Theorem 1 is not new. It was established by H. Poincaré in 1886 [16] and was rediscovered by P. Bohl in 1904 [1]." Here

[1] Bohl, P., Über die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage. Journal für Math. 127 (1904).

[16] Poincaré, H., Sur les Courbes Définies par les Equations Différentielles, Journal de Math., Vol. II (1886).

For more accurate references, see Bohl [1904] and Poincaré [1886] in the end of this article.

Kaniel's claim was quoted by Reich [1974], which was followed by Bryszewski and Górniewicz [1976]. Istrăţescu [1981] wrote that Poincaré proved Kaniel's Theorem 1 (in an incorrect form), and this was quoted in Editorial comments, the American translation of a Russian Encyclopedia of Mathematics.

However, after throughly reading Poincaré [1986], the present author could not find any fact similar to Kaniel's Theorem 1.

Note that, in Kaniel's Theorem 1, we can see the so-called Leray-Schauder boundary condition, which is not directly related to Leray and Schauder [1934] as we will see in Section 4 of the present article.

Bohl [1904, p.185] proved the following:

Let a domain (G) $-a_i \leq x_i \leq a_i$ $(i = 1, 2, \cdots, n)$ *be given. In this domain let* $f_1, f_2, \cdots, f_n$ *be continuous functions of* x *which do not have a common zero. Then there is a point* $u_1, u_2, \cdots, u_n$ *in the boundary of* G *such that*

$$f_i(u_1, u_2, \cdots, u_n) = N \cdot u_i, \quad N < 0. \quad (i = 1, 2, \cdots, n).$$

The following theorem can be regarded as contained in this theorem:

There do not exist n *continuous functions* $F_1, F_2, \cdots, F_n$*, defined on the domain* (G) $-a_i \leq x_i \leq a_i$ $(i = 1, 2, \cdots, n)$*, which have no common zero and fulfill for the points of the boundary of* (G)

$$F_i = x_i \quad (i = 1, 2, \cdots, n).$$

Hence, Bohl proved for the first time that the boundary of a cube is not a retract of the solid cube, which is equivalent to the Brouwer theorem.

For Bohl's work, R.H. Bing [1969] wrote : "The result is frequently called the Brouwer Fixed Point Theorem although the work of Brouwer [1912] was probably preceded by that of Bohl [1904]. ... In proving the theorem, Bohl considered differentiable maps and used Green's Theorem to show that equivalent integrals did not match if the n-cell had a fixed point free map into itself."

The following is called the non-retraction theorem:

Theorem. *For $n \geq 1, \mathbf{S}^{n-1}$ is not a retract of $\mathbf{B}^n$.*

Smart [1974] wrote: "Bohl [1904] proved a result equivalent to the non-retraction theorem but apparently did not go on to obtain the Brouwer theorem."

On the other hand, Dugundji and Granas [1982, Theorem II (7.2)] claimed that the non-retraction theorem was due to Borsuk and the following to Bohl:

Theorem. *Every continuous $F : \mathbf{B}^{n+1} \to \mathbf{R}^{n+1}$ has at least one of the following properties:*

(a) *F has a fixed point,*
(b) *there is an $x \in S^n$ such that $x = \lambda F x$ for some $0 < \lambda < 1$.*

This follows from Bohl's first theorem: If $f = I - F$ is continuous and fails to have a fixed point, then Bohl's conclusion implies (b).

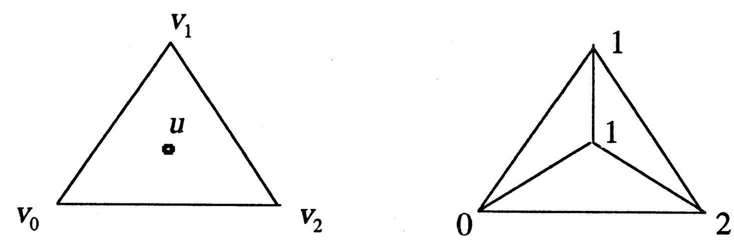
Note that the concept of retraction is due to Borsuk [1931] and that the negation of condition (b) is the so-called Leray-Schauder boundary condition. The above theorem is usually called the Leray-Schauder fixed point theorem.

2. Sperner's Combinatorial Lemma—From 1928

Sperner [1928] gave the following combinational lemma and its applications:

Lemma [Sperner, 1928]. *Let K be a simplicial subdivision of an n-simplex $v_0 v_1 \cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \leq k \leq n, 0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplexes of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

For example, consider a 2-simplex with vertices v_0, v_1, v_2, add a new vertex u as in the figure. Since u lies on $v_0 v_1 v_2$, we may assign to u one of 0, 1 and 2.



If u is assigned 1, then there exists exactly one 2-simplex of the subdivision, whose vertices receive all three integers 0, 1, 2.

Fifty years after the birth of this lemma, at a conference at Southampton, England in 1979, Sperner himself listed early applications of his lemma as follows:

1) Invariance of dimension (Sperner [1928]).
2) Invariance of region (Sperner [1928]).
3) Theorem of verification (Rechtfertigungssatz) in Menger's theory of dimensions (Menger [1928]).
4) Brouwer's fixed point theorem (Knaster, Kuratowski, Mazurkiewicz [1929]).
5) Matrices with elements ≥ 0 (Ky Fan [1958]), theorems of Perron, Frobenius and others.

There have appeared a number of generalizations of the lemma, which was applied to the following:

6) Antipodal theorems (Tucker [1945]; Fan [1952b]): Those are the Lusternik-Schneirelmann theorem on a cover of the n-sphere $\mathbf{S}^n$ consisting of $n+1$ closed subsets and the Borsuk-Ulam theorem on a continuous map $f : \mathbf{S}^n \to \mathbf{R}^n$.
7) Derivation of the Sperner lemma from the Brouwer fixed point theorem (Yoseloff [1974]).
8) Constructive proof of the Fundamental Theorem of Algebra (Kuhn, 1974]).
9) Approximation algorithm to approximate Brouwer fixed point (Scarf [1967]; Kuhn [1969]; Allgower and Keller [1971]; and many others).

In the later years, Sperner unified his own lemma and its extensions due to Tucker and Fan. For the details, see Sperner's articles in Forster [1980].

3. THE KKM THEOREM—FROM 1929

Knaster, Kuratowski, and Mazurkiewicz [1929] obtained the following so-called KKM theorem from the Sperner lemma [1928], and initiated the KKM theory.

Theorem [KKM, 1929]. *Let $F_i \, (0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $v_0 v_1 \cdots v_n$. If the inclusion relation*

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, *then* $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

A special case or dual form of the KKM theorem is already given in Sperner [1922]. The KKM theorem follows from the Sperner lemma and is used to obtain one of the most direct proof of the Brouwer theorem. Therefore, it was conjectured that those three theorems are mutually equivalent. This was clarified by Yoseloff [1974]. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications.

Brouwer

1974 ↙ ↖ 1929

Sperner $\xrightarrow[1929]{}$ KKM

Moreover, many important results in nonlinear functional analysis and other fields are known to be equivalent to those three theorems. Only less than a dozen of those results are shown in text-books such as Aubin [1979, 1982], Aubin and Ekeland [1984], and Zeidler [1986-90] and in surveys such as Gwinner [1981] and others. Further usefulness of those three theorems can also be seen in Nikaido [1970], Zangwill and Garcia [1981], Hildenbrand and Kirman [1976], Ichiishi [1983], and others.

From the KKM theorem, we can deduce the concept of KKM maps as follows: Let E be a vector space and $D \subset E$. A set-valued function (multifunction or map) $G : D \multimap E$ is called a *KKM map* if

$$\operatorname{co} N \subset G(N)$$

holds for each nonempty finite subset N of D.

Granas [1981] gave some examples of KKM maps as follows:

(i) *Variational problems.* Let C be a convex subset of a vector space E and $\phi : C \to \mathbf{R}$ a convex function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : \phi(y) \le \phi(x)\} \quad \text{for } x \in C$$

is a KKM map.

(ii) *Best approximation.* Let C be a convex subset of a vector space E, p a seminorm on E, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : p(fy - y) \le p(fy - x)\} \quad \text{for } x \in C$$

is a KKM map.

(iii) *Variational inequalities.* Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, C a convex subset of H, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : \langle fy, y - x\rangle \leq 0\} \quad \text{for } x \in C$$

is a KKM map.

The study of properties of such KKM maps and their applications is adequately called the KKM theory. See Park [1992c, 1994d]. In the frame of this theory, various fixed point theorems and many other consequences are obtained. See Section 6 of the present article. As the development of this theory, there have appeared many result equivalent to the Brouwer theorem, especially, in nonlinear functional analysis and mathematical economics. For the classical results, see Granas [1981].

Relatively early equivalent forms of the Brouwer theorem are as follows:

1904 Bohl's non-retraction theorem.
1912 Brouwer's fixed point theorem.
1928 Sperner's combinatorial lemma.
1929 The Knaster-Kuratowski-Mazurkiewicz theorem.
1930 Schauder's fixed point theorem.
1934 The Leray-Schauder fixed point theorem.
1935 Tychonoff's fixed point theorem.
1937 von Neumann's intersection theorem.
1941 Intermediate value theorem of Bolzano-Poincaré-Miranda.
1941 Kakutani's fixed point theorem.
1950 Bohnenblust-Karlin's fixed point theorem.
1952 Fan-Glicksberg's fixed point theorem.
1955 Main theorem of mathematical economics on Warlas equilibria of Gale [1955], Nikaido [1956], and Debreu [1959].
1960 Kuhn's cubic Sperner lemma.
1961 Fan's KKM theorem.
1961 Fan's geometric or section property of convex sets.
1966 Fan's theorem on sets with convex sections.
1966 Hartman-Stampacchia's variational inequality.
1967 Browder's variational inequality.
1968 Fan-Browder's fixed point theorem.
1969 Fan's best approximation theorems.
1972 Fan's minimax inequality.
1984 Fan's matching theorems.

Many generalizations of those theorems are also known to be equivalent to the Brouwer theorem.

4. Early Extensions of the Brouwer Theorem—Twenties and Thirties

The Brouwer theorem was extended to continuous selfmaps of compact convex subsets of

(1) certain function spaces, e.g. $L_2[0,1]$ and $\mathbb{C}^n[0,1]$, by Birkhoff and Kellogg [1922];
(2) Banach spaces, by Schauder [1927, 1930]; and
(3) locally convex topological vector spaces, by Tychonoff [1935].

All those results are included in Lefschetz type fixed point theorems, which is in turn contained in the Leray-Schauder theory as extended by Browder and others. For the literature, see van der Walt [1963].

Note that Birkhoff-Kellogg [1922], Schauder [1927], and Tychonoff [1935] applied their results to the existence of solutions of certain differential and integral equations.

There have also appeared extensions for maps which were not selfmaps of compact convex subsets, as follows.

Theorem [Knaster, Kuratowski und Mazurkiewicz, 1929, p.205]. *If $f : \mathbf{B}^n \to \mathbf{R}^n$ is a continuous map such that f maps $\mathbf{S}^{n-1} = \operatorname{Bd}\mathbf{B}^n$ back into $\mathbf{B}^n$, then f has a fixed point.*

This is the origin of the so-called Rothe boundary condition.

Theorem [Schauder, 1930]. *If C is a closed convex subset of a Banach space then every compact continuous map $f : C \to C$ has a fixed point.*

This is a more general version of (2) and especially convenient in applications. Note that this follows from (2) by using Mazur' result [1930] that the convex closure of a compact set in a Banach space is compact. It is later recognized that the closedness of C and the completeness of the space are not necessary.

Theorem [Schauder, 1930]. *If C is a weakly compact convex subset of a separable Banach space, then every weakly continuous map $f : C \to C$ has a fixed point.*

This also follows from (2) by considering the weak topology.

This was also generalized as follows:

Theorem [Krein and Šmulian, 1940]. *Let H be a closed convex subset of a Banach space. If $f : H \to H$ is weakly continuous such that $f(H)$ is separable and the weak closure of $f(H)$ is weaklt compact, then f has a fixed point.*

The KKM fixed point theorem is extended to the following:

Theorem [Rothe, 1938]. *Let V be a closed ball of a Banach space E and $f : V \to E$ a compact continuous map such that $f(\mathrm{Bd}\, V) \subset V$. Then f has a fixed point.*

Altmann [1955] showed that the Rothe condition $f(\mathrm{Bd}\, V) \subset V$ can be replaced by the following:

$$||fx - x||^2 \geq ||fx||^2 - ||x||^2 \quad \text{for all } x \in \mathrm{Bd}\, V.$$

Note that those conditions are all particular to the Leray-Schauder condition.

Applications of theorems of Brouwer, Schauder, Tychonoff, and Rothe appear in many text-books. We list some of them:

Proof of the fundamental theorem of algebra.

Existence of solutions of ordinary differential equations satisfying Lipschitz condition.

Peano's theorem on the existence of solutions of ordinary differential equations.

Alternating current circuits (Periodic solutions of systems of ordinary differential equations).

Solutions of elliptic partial differential equations.

One of the interesting applications of the Brouwer theorem is due to Zeeman [1962], who described a model of brain.

Lomonosov [1963] gave a proof of the existence of invariant subspaces in operator theory; that is, for any continuous linear map f from a Banach space X into itself, there exists a closed subspace X_0 satisfying $f(X_0) \subset X_0$ and $\{0\} \subsetneqq X_0 \subsetneqq X$. He was then a high school boy in Russia and gave a simple proof of this twenty year old problem in a general form using Schauder's theorem.

On the other hand, Kakutani [1943] showed the existence of a fixed-point-free continuous selfmap (even for a homeomorphism) of the unit ball in an infinite dimensional space. Therefore, the compactness in the above theorems on finite dimensional case can not be replaced by bounded closedness or by weak compactness. Moreover, Dugundji [1950] showed that a normed vector space is finite dimensional if and only if every continuous selfmap of its unit ball has a fixed point.

Tychonff's theorem was applied to obtain the following by Markov:

Theorem [Markov, 1936], [Kakutani, 1938]. *Let K be a compact convex subset of a topological vector space E. Let $\mathcal{F}$ be a commuting family of continuous affine maps from K into itself. Then $\mathcal{F}$ has a common fixed point $p \in K$; that is, $fp = p$ for each $f \in \mathcal{F}$.*

Later Kakutani gave a direct proof and several applications.

The Markov-Kakutani theorem was generalized to larger classes of maps by Day [1961] and others.

More early, Schauder raised, as Problem 54 of *The Scottish Book* [Mauldin, ed., 1981], whether a continuous selfmap of a compact convex subset of any topological vector space has a fixed point. If the space is Hausdorff locally convex or if the space has sufficiently many linear functionals, then Schauder's conjecture holds. For some particular spaces, it also holds. However, the problem is not resolved yet for its full generality. For this problem, see Idzik [1988] and his references.

In the mid-thirties, the Leray-Schauder theory [1934] appeared. It assigns a degree to certain maps and establishes properties of the degree which lead to fixed point and domain invariance theorems. This was first done for Banach spaces, and later developed by Leray [1950], Nagumo [1951], Altman [1958a,b] and others for locally convex spaces. When the space is Banach, Granas [1959] obtained a homotopy extension theorem, which yields many of the useful conclusions of the theory while avoiding the more complicated notions of the degree. Moreover, Klee [1960] established the theory without local convexity.

On the other hand, Schaefer [1955a] showed that the problem of solvability of an equation $x = fx$, for a completely continuous map f on a locally convex space E, reduces to finding a priori bounds on all possible solutions for the fanily of equations $x = \lambda fx, \lambda \in (0,1)$. This fact is called the Leray-Schauder alternative by Granas [1993] and its various extensions and modifications have played a basic role in various applications to nonlinear problems. See also Park [1994e].

It is often said that the Leray-Schauder theorem in Section 1 can be obtained in the frame of Leray and Schauder [1934], which seems to be not directly related to the so-called Leray-Schauder boundary condition. This condition seems to be originated from Schaefer [1955b] (see B. Fishel, MR 50#8177) and have been frequently appeared from mid-sixties. It is, the present author guesses, first called the Leray-Schauder condition by Petryshyn [1971]. For the literature on the theory without using degree theory, see forthcoming works of Park [1995].

Independently to the generalizations of the Brouwer theorem, Nikodym [1931] and Mazur and Schauder [1936] initiated the abstract approach to problems in calculus of variations.

Theorem [Mazur-Schauder, 1936]. *Let E be a reflexive Banach space and C a closed convex set in E. Let ϕ be a lower semicontinuous convex and coercive (that is, $|\phi(x)| \to \infty$ as $||x|| \to \infty$) real function on C. If ϕ is bounded from below, then at some $x_0 \in C$ the function ϕ attains its minimum.*

This is a very useful generalization of the classical Bolzano-Weierstrass theorem and was applied to a number of concrete problems in calculus of variations by Mazur and Schauder. However, these results were never published. See Granas [1981]. Later this theorem is generalized to the variational inequality problems in the frame of KKM theory. See Park [1991a,d].

Also independently to the above progress, J. von Neumann [1928] obtained the following minimax theorem, which is one of the fundamental theorems in the theory of games developed by himself:

Theorem [von Neumann, 1928]. *Let $f(x, y)$ be a continuous real-valued function defind for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x \in K} \min_{y \in L} f(x, y) = \min_{y \in L} \max_{x \in K} f(x, y).$$

This is later extended by himself [1937] to the following intersection theorem:

Lemma [von Neumann, 1937]. *Let K and L be two compact convex sets in the Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ in $\mathbf{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics.

According to Debreu (A commentary on the Kakutani fixed point theorem, in *Collected Works of Kakutani*),

> "Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved elementary means."

5. Extensions to Multifunctions and Applications—Forties and Fifties

In order to give simple proofs of von Neumann's Lemma, Kakutani obtained the following generalization of the Brouwer theorem to multifunctions (set-valued functions):

Theorem [Kakutani, 1941]. *If $x \to \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of closed convex subset of S, then there exists an $x_0 \in S$ such tha $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary [Kakutani, 1941]. *Theorem is also valid even if S is an arbitary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary readily implies von Neumann's Lemma. And later Nakaido [1968] noted that those two results are directly equivalent.

According to Debreu (*op. cit.*) again :

> "However the formulation given by Kakutani is far more convenient to use, and his proof is distinctly more appealing.
>
> One of the earliest, and most important, applications of the theorem of Kakutani was made by Nash [1950] in his proof of the existence of an equilibrium for a finite game. It was followed by several hundred applications in the theory of games and in economic theory. In the latter Kakutani's theorem has been more than three decades the main tool for proving the existence of an economic equilibrium (a recent survey by Debreu [1982] quotes some three hundred fifty instances). Other areas of applications were Mathematical Programming, Control Theory and the theory of Differential Equations."

Note that the upper semicontinuity of a multifunction in Kakutani's theorem was also extended. Recall that a multifunction $F : X \multimap Y$, where X and Y are topological spaces, is *upper semicontinuous* (u.s.c.) whenever, for any $x \in X$ and any neighborhood U of Fx, there exists a neighborhood V of x satisfying $F(V) \subset U$.

In the 1950's, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [1950] and to locally convex Hausdorff topological vector spaces by Fan [1952] and Glicksberg [1952]. These extensions were mainly used to extend von Neumann's works in the above. Moreover, these extensions are included in the extensions, due to Eilenberg and Montgomery [1946] and Begle [1950], of Lefschetz's theorem to u.s.c. maps of a compact *lc*-space into the family of its nonempty compact acyclic subsets.

The first remarkble one of generalizations of the von Neumann theorems was Nash's theorem [1951] on equilibrium points of non-cooperative games. The following is formulated by Ky Fan [1966, Theorem 4] :

Theorem [Nash, 1951]. *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\Pi_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any fixed point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \Pi_{j\neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \Pi_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \operatorname*{Max}_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

Further, von Neumann's minimax theorem was extended for arbitrary topological vector spaces as follows:

Theorem [Sion, 1958]. *Let X, Y be a compact convex set in a topological vector space. Let f be a real-valued function defined on $X \times Y$. If*

(1) *for each fixed $x \in X, f(x, y)$ is a lower semicontinuous, quasi-convex function on Y, and*
(2) *for each fixed $y \in Y, f(x, y)$ is an upper semicontinuous, quasi-concave function on X,*

then we have

$$\operatorname*{Min}_{y \in Y} \operatorname*{Max}_{x \in X} f(x, y) = \operatorname*{Max}_{x \in X} \operatorname*{Min}_{y \in Y} f(x, y).$$

Here, f is *lower semicontinuous* whenever the set $\{y \in Y : f(x, y) > p\}$ is open, and *quasi-concave* whenever $\{x \in X : f(x, y) > p\}$ is convex for each $p \in \mathbf{R}$. Moreover, f is *upper semicontinuous* whenever $\{x \in X : f(x, y) < p\}$

is open, and *quasi-convex* whenever $\{y \in Y : f(x,y) < p\}$ is convex for each $p \in \mathbf{R}$.

Sion's proof [1958] was based on the KKM theorem and this seems to be the first application of the theorem after KKM [1929].

As for the Brouwer theorem, in the mid-sixties, there had been developed algorithms on constructive processes approximating effectively to the values of the Kakutani fixed points. For the literature see Section 1.

In closing this section, we quote two stories on the Brouwer and Kakutani theorems.

In the paper entitled "An intuitionist correction of the fixed-point theorem", Brouwer [1952] denied the existence of a fixed point in his earlier theorem [1910], and claimed that there can be only ε-fixed points for each $\varepsilon > 0$. Because the Bolzano-Weierstrass theorem is invalid in the intuitionistic mathematics. Note that his theorem in [1910] implies the Brouwer fixed point theorem as Alexander [1922] showed. Here, we see Brouwer's fate of denying one of his great accomplishments of young days because of his own philosophy.

Comparing the Brouwer and Kakutani theorems, Franklin [1983] quoted a private survey:

> "$\cdots$96% of all mathematicians can state the Brouwer fixed point theorem, but only 5% can prove it. Among mathematical economists, 95% can state it, but only 2% can prove it (and these are all ex-topologists). $\cdots$ while 96% of mathematicians can *state* the Brouwer fixed-point theorem, only 7% can state the Kakutani theorem."

6. Establishment of the KKM Theory — From Sixties to Eighties

A milestone of the history of the KKM theory was erected by Ky Fan [1961]. He extended the KKM theorem to infinite dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma [Fan, 1961]. *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem.

Fan also obtained the following geometric or section property of convex sets, which is equivalent to the preceding Lemma.

Lemma [Fan, 1961]. *Let X be a compact convex set in a topological vector space. Let A be a closed subset of $X \times X$ with the following properties:*

(i) *$(x, x) \in A$ for every $x \in X$.*

(ii) *For any fixed $y \in X$, the set $\{x \in X : (x, y) \notin A\}$ is convex (or empty).*

Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

Fan applied this Lemma to give a simple proof of the Tychonoff theorem.

Moreover, Fan [1964] obtained "a theorem concerning sets with convex sections" and applied it to prove the following results in Fan [1966]:

An intersection theorem (which generalizes the Lemma of von Neumann [1937]).

An analytic formulation (which generalizes the equilibrium theorem of Nash [1951] and the minimax theorem of Sion [1958]).

A theorem on systems of convex inequalities of Fan [1957].

Extremum problems for matrices.

A theorem of Hardy-Littlewood-Pólya concerning doubly stochastic matrices.

A fixed point theorem generalizing Tychonoff [1935] and Iohvidov [1964].

Extensions of monotone sets.

Invariant vector subspaces.

An analogue of Helly's intersection theorem for convex sets.

In the same year, Hartman and Stampacchia [1966] introduced the following variational inequality:

Lemma [Hartman-Stampacchia, 1966]. *Let K be a compact convex subset in $\mathbf{R}^n$ and $f : K \to \mathbf{R}^n$ a cotinuous map. Then there exists $u_0 \in K$ such that*

$$(f(u_0), v - u_0) \geq 0 \quad \text{for } v \in K,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{R}^n$.

Using this result, the authors obtained existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions of Dirichlet boundary value problems associated with certain nonlinear elliptic differential functional equation.

Later the preceding lemma is known to be equivalent to the Brouwer theorem.

The above lemma was extended by Browder [1967] while he was working to extend the theorems of Schauder and Tychonoff motivated by Halpern's work [1965] on fixed point theorems for outward maps:

Theorem [Browder, 1967]. *Let E be a locally convex topological vector space, K a compact convex subset of E, T a continuous mapping of K into E^*. Then there exists an element u_0 of K such that*

$$(T(u_0), u - u_0) \geq 0$$

for all u in K.

Here, E^* is the topological dual of E and $(\,,\,)$ denotes the pairing between elements of E^* and elements of E. This theorem is later extended and improved by Park [1988b] and many others by pointing out that the local convexity is superfluous.

On the other hand, Browder [1968] restated Fan's geometric lemma [1961] in the convenient form of a fixed point theorem by means of the Brouwer theorem and the partition of unity argument. Since then the following is known as the Fan-Browder fixed point theorem:

Theorem [Browder, 1968]. *Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in $K, T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later this is also known to be equivalent to the Brouwer theorem. Browder [1968] applied his theorem to obtain his variational inequality and new fixed point theorems. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to Park [1989a, 1994d].

Motivated by Browder's works [1967, 1968] on fixed point theorems, Fan [1969] deduced the following from his geometric lemma:

Theorem [Fan, 1969]. *Let X be a non-empty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$||y_0 - f(y_0)|| = \operatorname*{Min}_{x \in X} ||x - f(y_0)||.$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.)

Fan also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. Those are known as best approximation theorems and applied to obtain generalizations of the Brouwer theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multifunctions in Fan [1969].

Moreover, Fan [1972] established a minimax inequality from the KKMF theorem:

Theorem [Fan, 1972]. *Let X be a compact convex set in a Hausdorff topological vector space. Let f be a real function defined on $X \times X$ such that:*

(a) For each fixed $x \in X, f(x,y)$ is a lower semicontinuous function of y on X.

(b) For each fixed $y \in X, f(x,y)$ is a quasi-concave function of x on X.

Then the minimax inequality

$$\min_{y\in X}\sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x)$$

holds.

Fan gave applications of his inequality as follows:

A variational inequality (extending Hartman-Stampacchia [1966] and Browder [1967]).

A geometric formulation of the inequality (equivalent to the Fan-Browder theorem).

Separation properties of upper demicontinuous multifunctions, coincidence and fixed point theorems.

Properties of sets with convex sections (Fan [1966]).

A fundamental existence theorem in potential theory.

Furthermore, Fan [1979, 1984] introduced a KKM theorem with a coercivity (or compactness) condition for noncompact convex sets and, from this, extended many of known results to noncompact cases. We list some main results as follows:

Generalizations of the KKM theorem for noncompact cases.

Geometric formulations.

Fixed point and coincidence theorems.

Generalized minimax inequality (extending Allen's variational inequality [1977]).

A matching theorem for open (closed) covers of convex sets.
The 1978 model of the Sperner lemma.
Another matching theorem for closed covers of convex sets.
A generalization of Shapley's KKM theorem (Shapley [1973]).
Results on sets with convex sections.
A new proof of the Brouwer theorem.

While closing a sequence of lectures delivered at the NATO-ASI at Montreal in 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows:

1) Potential theory.
2) Pontrjagin spaces or Bochner spaces in inner product spaces.
3) Operator ideals.
4) Weak compactness of subsets of locally convex topological vector spaces.
5) Function algebras.
6) Harmonic analysis.
7) Variational inequalities.
8) Free boundary value problems.
9) Convex analysis.
10) Mathematical economics.
11) Game theory.
12) Mathematical statistics.

We may add the following fields to this list: nonlinear functional analysis, approximation theory, optimization theory, fixed point theory, and some others.

7. Generalized Fixed Point Theorems on Topological Vector Spaces—From Sixties to Nineties

From Sixties there have appeared many fixed point theorems generalizing the Brouwer or Kakutani theorems for single-valued or multi-valued maps defined on convex subsets of Hausdorff topological vector spaces.

For single-valued continuous maps, Fan [1964] showed that Schauder's conjecture is valid for a topological vector space E on which E^* separates points. Halpern [1965] considered new boundary conditions called outwardness and, later, inwardness; and obtained fixed point theorems for maps satisfying those conditions. For a topological vector space E, a compact convex subset K of E, and a continuous map $f : K \to E$ satisfying certain inwardness or outwardness, generalizations of the Brouwer theorem were due to Halpern [1965], Fan [1969], Reich [1972], Sehgal and Singh [1983], and others whenever E is locally convex; and to Halpern and Bergman [1968], Kaczynski [1983], Roux and Singh [1989], Sehgal, Singh, and Whitfield [1990] whenever E^* separates points of E. In the sequel, a t.v.s. means a Hausdorff topological vector space.

Kakutani's convex-valued u.s.c. multimaps are further extended as follows: For a subset X of a t.v.s. E, a map $F : X \multimap E$ is called

(i) *upper demicontinuous* (u.d.c.) if for each $x \in X$ and open half-space H in E containing Fx, there exists an open neighborhood N of x in X such that $f(N) \subset H$. See Fan [1969].

(ii) *upper hemicontinous* (u.h.c.) if for each $h \in E^*$ and for any real α, the set $\{x \in X : \sup \operatorname{Re} h(Fx) < \alpha\}$ is open in X. See Cornet [1975], Lasry and Robert [1975], and Park [1993d].

(iii) *generalized u.h.c.* if for each $p \in \{\operatorname{Re} h : h \in E^*\}$, the set $\{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X. See Glebov [1969], Cellina [1970], Simons [1986a,b], and Park [1992a,1993d].

For those maps with compact convex domains, the Kakutani theorem was extended by Browder [1968], Fan [1969, 1972], Glebov [1969], Halpern [1970], Cellina [1970], Reich [1972, 1978], Cornet [1975], Lasry and Robert [1975], and Simons [1986a,b] for a locally convex t.v.s. E, and by Granas and Liu [1986], Park [1988b, 1992a, 1993d] and others for a t.v.s. E on which E^* separates points.

In order to assure the existence of a fixed point of maps $f : X \to E$ or $F : X \multimap E$, we need the following:

(i) Certain continuity of the map like as the generalized u.h.c.

(ii) Certain compactness on X — if X is not compact, then certain compactness or coercivity condition suffices for the existence of fixed points.

(iii) Certain boundary conditions. Until mid-sixties, we had only a few of such conditions, for example, Rothe [1937], Altman [1955], or the Leray-Schauder condition.

Halpern [1965] first introduced the outward and, later, inward sets:

Let E be a t.v.s. and $X \subset E$. The inward and outward sets of X at $x \in E, I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For $p \in \{\operatorname{Re} h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| : u \in U, \ v \in V\}.$$

A map $F : X \multimap E$ is said to be

(i) *inward* if $Fx \cap I_X(x) \neq \emptyset$ for each $x \in \text{Bd}\, X$.
outward if $Fx \cap O_X(x) \neq \emptyset$ for each $x \in \text{Bd}\, X$.

(ii) *weakly inward* if $Fx \cap \overline{I}_X(x) \neq \emptyset$ for each $x \in \text{Bd}\, X$.
weakly outward if $Fx \cap \overline{O}_X(x) \neq \emptyset$ for each $x \in \text{Bd}\, X$.

Later Jiang [1988] introduced more general conditions as in Theorem 7.1 below.

For the case the domain X is not compact, for maps $F : X \multimap E$ having certain continuity, boundary conditions, and certain compactness conditions, generalizations of the Kakutani theorem were obtained by Fan [1984], Shih and Tan [1987, 1988], Jiang [1988], Ding and Tan [1992], Park [1992a, 1993d], and others.

All of the generalizations of the Brouwer and Kakutani theorems mentioned above are unified by Park [1992a, 1993d] as follows:

A convex space X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$. See Lassonde [1983].

Let $cc(E)$ denote the set of nonempty closed convex subsets of a t.v.s. E and $kc(E)$ the set of nonempty compact convex subsets of E.

The following is given in Park [1992a, 1993d].

Theorem 7.1. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*
(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in \{\text{Re}\, h : h \in E^*\}$,*

(0) *$p|_X$ is continuous on X;*
(1) *$X_p = \{x \in X : \inf p(Fx) \leq p(x)\}$ is compactly closed in X;*
(2) *$d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \text{Bd}\, X$; and*
(3) *$d_p(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in Fx$.

(II) *Suppose that for each $p \in \{\text{Re}\, h : h \in E^*\}$,*

(0) *$p|_X$ is continuous on X;*
(1)$'$ *$X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X;*

(2)′ $d_p(Fx, \overline{O}_X(x)) = 0$ *for every* $x \in K \cap \mathrm{Bd}\, X$; *and*

(3)′ $d_p(Fx, \overline{O}_L(x)) = 0$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in Fx$. *Further, if* F *is u.h.c., then* $F(X) \supset X$.

The major particular forms of Theorem 7.1 can be adequately summarized by the following enlarged version of the diagrams given in Park [1988b, 1992a].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex Hausdorff topological vector spaces, and IV for topological vector spaces having sufficiently many linear functionals. Moreover, f stands for single-valued maps and F for set-valued maps; and K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex subset of E satisfying certain coercivity conditions with respect to $F : X \multimap E$ with certain boundary conditions.

In fact, Theorem 7.1 contains all of the fixed point theorems in the diagram.

For non-convex-valued multimaps, recently, the author established the fixed point theory for "admissible" maps in very general classes of multifunctions as follows:

A map $F : X \multimap Y$ is *compact* provided $F(X)$ is relatively compact in a topological space Y.

In a t.v.s. E, any convex hulls of its finite subsets will be called *polytopes*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (sngle-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope $P, F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the O'Neill maps $\mathbb{N}$ (with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in a t.v.s., admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, and many others.

We indroduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X, Y) \iff$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is an $\tilde{F} \in \mathfrak{A}_c(K, Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

E	$f : K \longrightarrow K$		$F : K \longrightarrow 2^K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	1950
III	Tychonoff	1935	Fan Glicksberg	1952 1952
IV	Fan	1964	Granas and Liu	1986
	$f : K \longrightarrow E$		$F : K \longrightarrow 2^E$	
I	Knaster, Kuratowski and Mazurkiewicz	1929		
II	Rothe	1937		
III	Halpern Fan Reich Sehgal and Singh	1965 1969 1972 1983	Browder Fan Glebov Halpern Cellina Reich Cornet Lasry and Robert Simons	1968 1969, 1972 1969 1970 1970 1972, 1978 1975 1975 1986
IV	Halpern and Bergman Kaczynski Roux and Singh Sehgal, Singh, and Whitfield	1968 1983 1989 1990	Granas and Liu Park	1986 1988, 1991
			$F : X \longrightarrow 2^E$	
II			Ding and Tan	1992
III			Fan Shih and Tan Jiang	1984 1987, 1988 1988
IV			Park	1992, 1993

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any class belonging to $\mathfrak{A}_c^\kappa$ is called *admissible.* Those classes are all due to the author in his earlier works. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [1991] and $\mathbb{V}_c^\sigma$ due to Park, Singh, and Watson [1994]. Note that $\mathbb{K}_c^\sigma$ contains classes $\mathbb{K}$, the Fan-Browder type maps, $\mathbb{T}$ in Lassonde [1991], and many others. For details, see Park [1993d, 1994d] and Park and Kim [1993g].

Moreover, the approximable maps recently due to Ben-El-Mechaiekh and Idzik [1994] belong to $\mathfrak{A}_c^\kappa$. It was known that any compact-valued closed map defined on a convex subset of a locally convex t.v.s. is approximable whenver its values are all convex, contractible, decomposable, or ∞-proximally connected.

Theorem 7.2. *Let X be a convex subset of a locally convex t.v.s. E. If $T \in \mathfrak{A}_c^\sigma(X, X)$ is compact, then T has a fixed point.*

This is recently due to the author [1993e, 1994d] and has a large number of particular forms due to Schauder [1930], Mazur [1938], Bohnenblust and Karlin [1950], Hukuhara [1950], Singbal [1962], Powers [1970], Rhee [1972], Himmelberg [1972], Ben-El-Mechaiekh, Deguire, and Granas [1982a,b,c], Lassonde [1983, 1990], Ben-El-Mechaiekh *et al.* [1990, 1991], Park [1992b], and Park, Singh, and Watson [1994f].

For non-selfmaps, we have the following in Park [1993d]:

Theorem 7.3. *Let X be a compact convex subset of a t.v.s. E on which E^* separates points. If $F \in \mathfrak{A}_c^\kappa(X, E)$ satisfies $Fx \subset \overline{I}_X(x)$ for each $x \in \mathrm{Bd}\, X$, then F has a fixed point.*

This also has a large number of particular forms.

There have been another way of extending compact maps in certain situations using (generalizations of) the Kuratowski measure of noncompactness. In this direction we have also a very general theorem.

Let E be a t.v.s. and C a lattice with a minimal element, which is denoted by 0. A function $\Phi : 2^E \to C$ is called a *measure of noncompactness* on E provided that the following conditions hold for any $X, Y \in 2^E$:

(1) $\Phi(X) = 0$ if and only if X is relatively compact;
(2) $\Phi(\overline{\mathrm{co}}\, X) = \Phi(X)$, where $\overline{\mathrm{co}}$ denotes the convex closure of X; and
(3) $\Phi(X \cup Y) = \max\{\Phi(X), \Phi(Y)\}$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined either in terms of a family of seminorms or a norm. For details, see Petryshyn and Fitzpatrick [1974a,b].

If Φ is a measure of noncompactnes on $E, D \subset E$, and $T : D \multimap E$, then T is called Φ-*condensing* provided that if $X \subset D$ and $\Phi(X) \leq \Phi(T(X))$, then X is relatively compact; that is, $\Phi(X) = 0$.

Every compact map is Φ-condensing.

The following is recently due to Park [1995]:

Theorem 7.4. *Let D be a closed convex subset of a t.v.s. E on which E^* separates points, and Φ a measure of noncompactness on E. If $T \in \mathfrak{A}_c^\kappa(D, D)$ is Φ-condensing, then T has a fixed point.*

This theorem extends earlier results of Darbo [1955], Sadovskii [1967], Lifšic and Sadovskii [1968], Himmelberg, Porter, and Van Vleck [1969], Daneš [1970], Furi and Vignoli [1970], Nussbaum [1971], Reinermann [1971], Reich [1971, 1972, 1973], Petryshyn and Fitzpatrick [1974a,b], Tarafdar and Výborný [1975], Su and Sehgal [1975], and Ewert [1987].

Let C, D be subsets of a t.v.s. E, $T \in \mathfrak{A}_c(C, D)$, and $\mathcal{M}$ be the class of nonempty compact subsets of D consisting of the functional values of maps in $\mathfrak{A}$. We say that T satisfies the *Schöneberg condition* if

(Sö) $tM \in \mathcal{M}$ for $t \in [0, 1]$ and $M \in \mathcal{M}$

holds. See Schöneberg [1978]. For example, $\mathcal{M}$ can be the class of convex sets for $\mathfrak{A} = \mathbb{K}$, acyclic sets for $\mathfrak{A} = \mathbb{V}$, R_δ sets $\{X = \cap X_i : X_{i+1} \subset X_i, X_i \in \text{AR compact}, i \in N\}$ for $\mathfrak{A} = \mathbb{M}$, and many others.

For $U \subset D$, let $\text{Cl}_D U$ denote the closure of U in D and $\text{Bd}_D U$ the boundary of U in D. On the other hand, $\overline{}$ and Bd will denote the closure and boundary in the whole space E.

Now we give some new fixed point theorems due to Park [1995] for maps satisfying the so-called Leray-Schauder condition:

Theorem 7.5. *Let D be a convex subset of a locally convex t.v.s., $0 \in D, U \subset D$ a neighborhood of D (in D), and $F \in \mathfrak{A}_c(\text{Cl}_D U, D)$ a compact map satisfying* (Sö). *If*

(LS) $Fy \cap \{ry : r > 1\} = \emptyset$ for all $y \in \text{Bd}\, U$,

then the set of fixed points of F in $\text{Cl}_D U$ is nonempty and compact.

This improves, unifies, and extends results of Brouwer [1912], Knaster, Kuratowski, and Mazurkiewicz [1929], Leray and Schauder [1934], Rothe [1937], Eilenberg and Montgomery [1946], Krasnoselskii [1953], Altman [1955], Yamamuro [1963], Shinbrot [1964], Kaniel [1965], Powers [1970], Ma [1972], Potter [1972], Martelli [1973], Furi and Martelli [1974], Su and Sehgal [1975], Fitzpatrick and Petryshyn [1975], Hahn [1976], Reich [1972, 1976, 1979], Górniewicz, Granas, and Kryszewski [1988], Kaczynski and Wu [1992], and Granas [1959, 1976, 1993].

For Φ-condensing maps, we have the following in Park [1995]:

Theorem 7.6. *Let D be a closed convex subset of a t.v.s. E on which E^* separates points, $0 \in D, U \subset D$ a neighborhood of* 0 (in D)*, and $F \in \mathfrak{A}_c(\mathrm{Cl}_D U, D)$ a Φ-condensing map satisfying* (Sö)*. If the condition* (LS) *holds, then the set of fixed points of F in $\mathrm{Cl}_D U$ is nonempty and compact.*

This includes Petryshyn [1971], Reich [1971, 1976, 1979], Gatica and Kirk [1974a,b], Petryshyn and Fitzpatrick [1974a,b, 1975], Su and Sehgal [1975], Martelli [1973, 1975], Lin [1988], and many others.

Those Leray-Schauder type theorems due to Park [1995] are applied to

(i) the so-called Leray-Schauder principles of Schöneberg [1978], Fitzpatrick and Petryshyn [1974], Potter [1972], Browder [1966], and Leray and Schauder [1934];

(ii) the Schaefer type theorems due to Schaefer [1955a,b], Reich [1971, 1972], Šeda [1989], Martelli and Vignoli [1972], Martelli [1975], Górniewicz, Granas, and Kryszewski [1988], and Granas [1993]; and

(iii) the Birkhoff-Kellogg type theorems due to Birkhoff and Kellogg [1922], Yamamuro [1963], Martelli [1975], and Fournier and Martelli [1993].

In the last decade, there have been advancements in the KKM theory also. Recently, Park [1994d] obtained far-reaching generalizations of the KKM theorem, the Fan-Browder theorem, a matching theorem, an analytic alternative, the Ky Fan minimax inequalities, section properties of convex spaces, and other fundamental theorems in the theory from coincidence theorems on composites of admissible maps. These new results extend, improve, and unify main theorems in more than one hundred published works.

On the other hand, the concept of convex sets in a t.v.s. was extended to convex spaces by Lassonde [1983], and further to H-spaces by Horvath [1983, 1984, 1987, 1990, 1991]. A number of other authors also extended the concept of convexity for various purposes. Recently, Park and Kim [1993g, 1995a] unified those concepts and introduced generalized convex spaces or G-convex spaces. For those spaces, the foundations of the KKM theory with respect to admissible maps were established by Park and Kim [1995b], and some general fixed point theorems were obtained by Kim [1995].

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