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# CODIMENSION ONE ANOSOV FLOWS

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BY

Shigenori Matsumoto

# Preface

In phisics, especially in classical mechanics, the time evolution of a phisical system is often described by an autonomous ordinary differential equation. If the degree of freedom of the system is finite, a state of the system is indicated by a point in a finite dimensional manifold $M$; its time evolution is described by a flow $\phi = \{\phi^t\}$ on $M$. If a state at given time is indicated by a point $x$, then $\phi^t(x)$ indicates the state after the time $t$. For example a state of a point mass moving in $\mathbf{R}^3$ is indicated by (position, velocity), i. e. a point in $\mathbf{R}^6$ and the flow is given by the Newton equation.

Dynamical system is a branch of mathematics which studies the long time behaviour of an orbit of the flow. For example, we are interested in the limit of $\phi^t(x)$ as $t$ tends to the infinity; We aim to decide how many periodic orbits the flow has and what is their natures, etc., etc., $\cdots$.

Instead of considering continuous time, sometimes we are lead to deal with discrete time. For example, in celestial mechanics if we assume that the sun is always fixed and that the earth is rotating in an ellipsoidal orbit around the sun periodically, of period $t_0$, then the equation which describes the motion of the moon is a *time dependent* ordinary differential equation, periodic of period $t_0$. Then it is natural to consider the state at discrete time $0, t_0, 2t_0, 3t_0, 4t_0, \cdots$.

In this case dynamics are described not by a flow but by a diffeomorphism, say $f$. Again a point $x$ of a manifold $N$ indicates a 'state' of a system, and the state after the unit time is indicated by $f(x)$. Here we are assuming that the state is decided only by the previous state. It may seem that this is a rather strong hypothesis and is only applicable to a limited situation. But for example if the succesive states $x_0$ , $x_1$ and $x_2$ determine the next state $x_3$, then instead of the manifold $N$, consider $N \times N \times N$. Then the time change $(x_0, x_1.x_2) \mapsto (x_1.x_2, x_3)$ is given by a diffeomorphism of $N \times N \times N$. In this way by studying diffeomorphisms of higher dimensional manifolds we can deal with fairly wide range of dynamics.

For the most part flows and diffeomorphisms are studied in a parallel way. But always some complications are needed for the case of flows. So we deal with diffeomorphisms first and then come to the flow case.

A diffeomorphism $f$ on a compact manifold $N$ is called *Anosov* if the tangent budle $TN$ splits into the direct sum of two continuous subbundles $E^u$ and $E^s$ both invariant by the derivative $Df$ of $f$ in such a way that $Df$ is expanding in the direction of $E^u$ and is contracting along $E^s$. There is also a concept of Anosov flows. A nonsingular flow on a compact manifold $M$ is called *Anosov* if the normal bundle of the flow has a similar splitting.

Being an Anosov system is a rather strong condition. Therefore it displays striking properties. On the other hand, examples of Anosov *flows* are rather abundant. Especially in dimension three we have many interesting examples.

The purpose of this text is to expose first of all important properties of Anosov systems. For the introductory nature of this preface, let us confine ourself to the case of diffeomorphisms.

The first important result about an Anosov diffeomorphism is that the subbundle $E^u$ and $E^s$ are integrable. A complete proof of this fact needs involvement in a wider range of mathematics, and the proof itself is lengthy, though not uninteresting. So we only give a short geometric proof which assumes that $E^u$ is a $C^1$ subbundle. The foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ defined by $E^u$ and $E^s$ are respectively called *unstable* and *stable foliations.*

A diffeomorphism $f$ is called *structurally stable* if any small perturbation of $f$ (in the $C^1$ topology) has the same topological structure as $f$. In Section 1, among other things we show the following theorem due to Anosov [1].

**Theorem 0.1** *An Anosov diffeomorphism is structurally stable.*

The proof of this theorem is rather simple, once we know the existence of the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$. The structural stability theorem also holds for Anosov flows. This is shown in Section 2.

As Anosov diffeomorphism $f$ is said to be *transitive* if $f$ admits a dense orbit. We shall prove a theorem in Section 1 which include the following.

**Theorem 0.2** *An Anosov diffeomorphism is transitive if and only if one of the foliation $\mathcal{W}^u$ and $\mathcal{W}^s$ has the property that all the leaves are dense.*

So far no example of nontransitive Anosov diffeomorphisms are known.

We have a similar theorem for flows. Examples of nontransitive Anosov flows are known, e. g. on 3-manifolds.

An Anosov diffeomorphism is said to be *of codimension one* if either one of the foliations $\mathcal{W}^u$ or $\mathcal{W}^s$ is of codimension one. There are parallel notions for flows. The later sections are concentrated (except Section 4) to the study of codimension one Anosov *flows.*

In Section 3, we prove the following theorem, originally due to A. Verjovsky [49].

**Theorem 0.3** *A codimension one Anosov flow on a manifold of dimension $\geq 4$ is transitive.*

The proof use some elementary aspect of theory of codimension one foliations, which has been very well developed.

The purpose of later sections is to give a complete proof of the following theorem, formerly known as the Verjovsky conjecture.

**Theorem 0.4** *A codimension one Anosov flow on a manifold with solvable fundamental group is topologically conjugate to the suspension of a hyperbolic automorphism of the torus.*

It was already known by S. Newhouse [34] that *any codimension one Anosov diffeomorphism is topologically conjugate to a hyperbolic automorphism of the torus.* Thus the proof of the Verjovsky conjecture is complete once we show that the flow has a (global) cross section.

A homological condition for a flow to admit a cross section is known by Schwartzman [43]. Section 4 is devoted to the proof of his criterion.

Now in order to show the Verjovsky conjecture, the key step is the study of codimension one foliations. We establish the following theorem in Section 5.

**Theorem 0.5** *Any codimension one foliation $\mathcal{F}$ on a manifold with solvable fundamental group is topologically conjugate to a transversely affine foliation, provided all the leaves are dense and any leaf holonomy is either trivial or has an isolated fixed point.*

Using this theorem, we establish the Verjovsky conjecture in Section 6. A Markov partition for an Anosov flow plays an essential role there. This is prepared in Sections 1 and 2.

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# Contents

# 1 Anosov diffeomorphisms

In 1.1, we study the local behaviour of a diffeomorphism $f$ of a manifold $N$, around a fixed point $x$. We consider the case where the derivative $Df_x$ is *hyperbolic* i. e. has no eigenvalues of mudulus one. In this case the tangent space $T_xN$ splits into the direct sum of two subspaces $E^u_x$ and $E^s_x$, both invariant by $Df_x$. $Df_x$ is expanding along $E^u_x$ and contracting along $E^s_x$. This strong dynamics forces $f$ itself to behave like the derivative $Df_x$ in a small neighbourhood of $x$. Especially there exists a one-to-one immersed copy of $\mathbf{R}^u$, denoted by $W^u_x$, which is invariant by $f$ and tangent to $E^u_x$ at $x$. $W^u_x$ is also characterized as

$$W^u_x = \{y \in N \mid f^{-n}(y) \to x \ (n \to \infty)\}.$$

$W^u_x$ is called the unstable manifold of $x$. Likewise the stable manifold is defined.

Next in 1.2 we consider a diffeomorphism $f$ which is *hyperbolic* on the whole manifold, i. e. admits a splitting of the tangent bundle into the direct sum of the two subbundles $E^u$ and $E^s$, both invariant by $Df$, and $Df$ is expanding along $E^u$ and contracting along $E^s$. Such a diffeomorphism is called an Anosov diffeomorphism.

The main purpose of this section is to study the properties of Anosov diffeomorphisms. We shall expose in 1.3 that the subbundles $E^u$ and $E^s$ is integrable and give birth to a foliation $\mathcal{W}^u$ and $\mathcal{W}^s$. They are called the unstable and stable foliation of $f$. The leaf $W^u_x$ of $\mathcal{W}^u$ through a point $x$ is characterized as

$$W^u_x = \{y \mid d(f^{-n}(x), f^{-n}(y)) \to 0\}.$$

In 1.4 we shall show that an Anosov diffeomorohism $f$ is topologically stable, i. e. any nearby diffeomorphism in the $C^1$ topology is topologically the same as $f$.

$f$ is called transitive if it admits a dense orbit. We shall raise in 1.5 several conditions which are equivalent to the transitiveness. For example, $f$ is transitive if and only if all the leaves of $\mathcal{W}^u$ are dense. This fact will play an important role in a later section.

Finally in 1.6 we expose a Markov partition, which associate Anosov diffeomorphisms to symbolic dynamics.

## 1.1 Local theory of diffeomorphisms

Let $N$ be a smooth manifold of dimension $n$ and let $f$ be a $C^r$ diffeomorphism of $N$ ($r \geq 1$). For a positive integer $n$ we denote by $f^n$ the $n$-ple iterate of $f$, i. e.,

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}.$$

Also , $f^0$ stands for the identity and for $-n$ we let $f^{-n} = (f^{-1})^n$.

A point $x$ of $N$ is called a *periodic point* of period $p > 0$ if $f^p(x) = x$, and a *fixed point* if $p = 1$. The derivative at $x$, $Df_x^p$ is a linear transformation of the tangent space $T_xN$ which approximates $f^p$ around $x$. But for example if $Df_x^p = Id$, then the dynamics of $Df_0^p$ and $f^p$ are quite different in general. We shall consider an opposite extremal case.

**Definition 1.1** A periodic point $x$ of $f : N \to N$ is called *hyperbolic* if the modulus of any eigenvalue of $Df_x^p$ is not 1.

Let $x$ be a hyperbolic fixed point, for simplicity. Let $U$ be a neighbourhood of $x$ and let $h : U \to \overline{U} \subset \mathbf{R}^n$ be a coordinate chart around $x$ such that $h(x) = 0$. Choose a neighbourhood $V$ of $x$ such that $f(V) \subset U$. Then instead of the map $f : V \to U$, we focus our attention on the map $\overline{f} = hfh^{-1}$. $\overline{f}$ is a diffeomorphism from $\overline{V} = h(V)$ into $\mathbf{R}^n$.

Of course 0 is a hyperbolic fixed point of $\overline{f}$. Let $E_0^u$ (resp. $E_0^s$) be the sum of the eigenspaces associated to eigenvalues of moduli greater (resp. less) than one[1]. Then $\mathbf{R}^n$ is the direct sum of $E_0^u$ and $E_0^s$, and the both subspaces are invariant by $D\overline{f}_0$. In what follows we consider the case where $0 < u < n$. For $l > 0$, denote by $E_0^\sigma(l)$ ($\sigma = u, s$) the closed metric ball in $E_0^\sigma$ of radius $l$. Passing to a subset if necessary, one may assume that $\overline{V} = E_0^u(l) \times E_0^s(l)$, that $\overline{f}^{-1}$ is also defined on $\overline{V}$.

Let $W_0^u(l)$ (resp. $W_0^s(l)$) be the set of points $v$ such that $\overline{f}^{-k}(v) \in \overline{V}$ (resp. $\overline{f}^k(v) \in \overline{V}$) for any $k \geq 0$. The following lemma is immediate from the definition.

**Lemma 1.2** *$\overline{f}^{-1}$ (resp. $\overline{f}$) maps $W_0^u(l)$ (resp. $W_0^s(l)$) into itself.*

Now we have the following theorem.

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[1]The letters $u$ and $s$ stand for 'unstable' and 'stable', but at the same time they indicate the dimensions.

**Theorem 1.3** *If $l$ is small enough, $W_0^\sigma(l)$ ($\sigma = u, s$) is a $C^r$ submanifold of $\overline{V}$, tangent to $E_0^\sigma$ at* 0.

Let us give a sketch of the proof. Notice that if $l$ is small enough, the dynamics of $\overline{f}$ is similar to that of $D\overline{f}_0$. $D\overline{f}_0$ streches along $E_0^u$ direction and contracts along $E_0^s$ directin.

We only consider $W_0^u$. It should be the graph of some $C^r$ map from $E_0^u(l)$ to $E_0^s(l)$ whose derivative is very small everywhere. So we will consider a suitable space of maps from $E_0^u(l)$ to $E_0^s(l)$, and construct some transformation on it, in such a way that its fixed point is associated to $W_0^u(l)$. We shall use a fixed point theorem. Thus the space should be complete.

Precisely let $\mathcal{L}$ be the space of Lipschitz maps $w$ from $E_0^u(l)$ to $E_0^s(l)$ with Lipschitz constant $\leq 1$ such that $w(0) = 0$, endowed with the supremum norm. $\mathcal{L}$ is a compact space. For $w \in \mathcal{L}$, let $G(w) \subset \overline{V}$ be its graph.

Now define a transformation

$$\Gamma : \mathcal{L} \to \mathcal{L}$$

by

$$G(\Gamma(w)) = \overline{f}(G(w)) \cap \overline{V}.$$

If $\overline{f}$ is sufficiently near to its derivative $D\overline{f}_0$, then one can show that $\Gamma$ is a contraction[2]. Now for any $w \in \mathcal{L}$, the sequence $w, \Gamma(w), \Gamma^2(w), \cdots$ is a Cauchy sequence. Hence it has a limit, say $w_0$. By the continuity of $\Gamma$, $w_0$ is a fixed point of $\Gamma$. Clearly it is unique. It is not difficult to prove that the graph of $w_0$ coincides with $W_0^u(l)$. This shows that $W_0^u(l)$ is a Lipschitz submanifold.

In order to show that $W_0^u(l)$ is a $C^1$-submanifold, we consider the unit ball $\mathcal{B}$ of the Banach space of the bounded continuous maps from $E_0^u(l)$ to the space of $s$ by $u$ matrices. We construct a suitable transformation of $\mathcal{L} \times \mathcal{B}$ from $\overline{f}$ and its derivative, and seek for a fixed point of the transformation.

Finally the proof that $W_0^u(l)$ is a $C^r$ submanifold needs more complicated argument. See [27] or [29] for details. But since $\overline{f}$ strechs $W_0^u(l)$, it is rather natural to expect that $W_0^u(l)$ is sufficiently smooth.

Now let us return to the hyperbolic fixed point $x \in N$ of $f$.

**Definition 1.4** The *unstable manifold* $W_x^u$ of $x$ is the set of point $y$ such that $f^{-n}(y) \to x$ as $n \to \infty$. Replacing $-n$ by $n$, we also define the stable manifold $W_x^s$.

[2] I. e. a Lipschitz map of Lipschitz constant $< 1$.

**Corollary 1.5 (Unstable Manifold Theorem)** *$W_x^u$ is a one to one immersed copy of $\mathbf{R}^u$, passing through $x$ and tangent to $E_x^u = Dh_0^{-1}(E_0^u)$.*

**Exercise 1.6** *Give a proof of the above corollary by showing that $W^u(x)$ coincides with the union of the increasing sequence $\{f^n(W_x^u(l))\}_{n \geq 0}$, where $W_x^u(l) = h^{-1}(W_0^u(l)$.*

The set $W_x^u$ (resp. $W_x^s$) is called the *unstable* (resp. *stable*) *manifold* of $x$.

## 1.2 Definitions and fundamental properties

Let $N$ be a closed smooth manifold of arbitrary dimension with a fixed Riemannian metric, and let $f$ be a $C^\infty$ diffeomorphism of $N$.

**Definition 1.7** $f$ is called an *Anosov diffeomorphism* if there exists a continuous splitting of the tangent bundle $TN = E^u \oplus E^s$ such that

1. Both $E^u$ and $E^s$ are invariant by the derivative $Df$,

2. There exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n(v)\| \leq C\lambda^n \|v\| \quad v \in E^s \quad n > 0 \tag{1}$$
$$\|Df^{-n}(v)\| \leq C\lambda^n \|v\| \quad v \in E^u \quad n > 0 \tag{2}$$

Of course (1) means that the vectors in $E^s$ shrinks exponentially fast in the positive time direction (to the future), and (2) indicates the expontial decay of vectors in $E^u$ to the past.

Notice that the above definition is independent of the choice of the Riemannian metric.

Clearly we have $0 < u, s < n$. For if, i. g. $u = 0$, then $f^n$ for sufficiently large $n > 0$ is a contraction, and cannot be a diffeomorphism of a compact manifold.

At first sight, Definition 1.7 looks very strong, for it postulates a $Df$-invariant decomposition of the tangent bundle. However this is not the case. There is an equivalent definition in which decomposition of the tangent bundle is not used. In order to expose it, we start with the following easy facts.

If we replace $v$ by $Df^n v$ in (1), we get the following equation which is equivalent to (1).

$$\|Df^{-n}(v)\| \geq C^{-1}\lambda^{-n} \|v\| \quad v \in E^s \quad n > 0. \tag{3}$$

Likewise (2) is equivalent to the following equation.

$$\|Df^n(v)\| \geq C^{-1}\lambda^{-n}\|v\| \quad v \in E^u \quad n > 0. \tag{4}$$

Equation (3) indicates the exponential growth of vectors in $E^s$ to the past. Frequent use will be made of the following convenient lemma.

**Lemma 1.8** *Let $f$ be an Anosov diffeomorphism. Then there exists a Riemannian metric for which we have*

$$\|Df(v)\| < \|v\| \quad \textit{for} \quad v \in E^s \tag{5}$$
$$\|Df(v)\| > \|v\| \quad \textit{for} \quad v \in E^u \tag{6}$$

**Proof** Start with any Riemannian metric, and choose $n > 0$ so that $\|Df^n(v)\| < \|v\|$ for any $v \in E^s$. Define a new metric $|||\cdot|||$ on $E^s$ by

$$|||v||| = \|v\| + \|Df(v)\| + \cdots + \|Df^{n-1}(v)\|.$$

Do the same thing for $E^u$ and take the orthogonal sum of the both. Clearly this new metric suffices for our purpose. □

In fact it is possible to use the condition of Lemma 1.8 for the definition of Anosov diffeomorphisms. However most authors prefer Definition 1.7, which is invariant of the choice of the Riemannian metric.

For a while we use the metric of Lemma 1.8. Let us consider a small cone neighbourhood $C^\sigma$ of $E^\sigma$ in $TN$ ($\sigma = u, s$). Precisely, $C^\sigma$ is the set of vectors which form angles less than some small number with $E^\sigma$. Then clearly the vectors $v \in C^u$ satisfy also the conditon (4), paying the cost of changing the constants $C$ and $\lambda$. Notice that they never satisfy (2). Likewise vectors in $C^s$ satisfy the condition (3). Also notice that $C^u$ are mapped into itself by $Df$, since we have chosen the metric of Lemma 1.8. The following proposition says that this situation is enough for $f$ to be an Anosov diffeomorphism.

**Proposition 1.9** *A diffeomorphism $f$ is Anosov if and only if the following condition is satisfied; There exist a continuous splitting $TN = E^u \oplus E^s$ and a cone neighbourhood $C^\sigma$ of $E^\sigma$ ($\sigma = u, s$) meeting at zero vectors such that*

1. *$Df$ maps the closure of $C^u$ into $C^u$, and $Df^{-1}$ maps the closure of $C^s$ into $C^s$.*

2. *We have*

$$\|Df^{-n}(v)\| \geq C^{-1}\lambda^{-n}\|v\| \quad v \in C^s \quad n > 0, \tag{7}$$
$$\|Df^{n}(v)\| \geq C^{-1}\lambda^{-n}\|v\| \quad v \in C^u \quad n > 0. \tag{8}$$

Notice that in the above proposition, the condition of the invariance of the subbundles $E^\sigma$ is repaced by a priori weeker condition of the invariance in one direction of the cone neighbourhoods.

**Exercise 1.10** *Give a proof of Proposition 1.9.*

An important corollary of Proposition 1.9 is the openness of Anosov diffeomorphisms. Let us denote by $Diff^1(N)$ the space of $C^1$ diffeomorphisms, equipped with the $C^1$ topology.

**Corollary 1.11** *The set of Anosov diffeomorphisms $\mathcal{A}$ is an open subset of $Diff^1(N)$.*

Of course, it is not true that $\mathcal{A}$ is nonempty for any manifold $N$. On the contrary, the manifolds which admit Anosov diffeomorphisms are rather rare. But they exist! Let us give examples.

**Example 1.12** Denote by $SL(n, \mathbf{Z})$ the group of unimodular[3] integral matrices. $A \in SL(n, \mathbf{Z})$ is called *hyperbolic* if its arbitrary eigenvalue has modulus different from 1.

Denote by $E_0^u$ (resp. $E_0^s$) the sum of the generalized eigenspace corresponding to the eigenvalues of moduli greater (resp. less) than one. Then clearly $E_0^\sigma$ ($\sigma = u, s$) is invariant by $A$, $\mathbf{R} = E_0^s \oplus E_0^u$, and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|A^n(v)\| \leq C\lambda^n\|v\|, \quad \forall v \in E_0^s, \quad \forall n > 0 \tag{9}$$
$$\|A^{-n}(v)\| \leq C\lambda^n\|v\|, \quad \forall v \in E_0^u, \quad \forall n > 0. \tag{10}$$

Thus $A : \mathbf{R}^n \to \mathbf{R}^n$ is an Anosov diffeomorphism, but it is not so much interesting, since $\mathbf{R}^n$ is an open manifold.

However since $A$ is unimodular and integral, $A$ defines a diffeomorphism, also called $A$, of the $n$- torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. The above spliting $\mathbf{R}^n = E_0^u \oplus E_0^s$ gives rise to a splitting of the tangent bundle of $T^n$, and it is easy to show that $A$ is an Anosov diffeomorphism.

The simplest example of hyperbolic toral automorphisms is $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

[3]The determinant is 1.

## 1.3 Stable Foliation

Anosov diffeomorphisms are always accompanied with two foliations, called stable and unstable foliations. The study of these foliations is very important for the understanding of Anosov diffeomorphisms.

We start this section with the definition of foliations. For general theory of foliations, see [9], [26] and [47]. Let $N$ be a closed manifold of dimension $n$. As usual the letter $r$ which indicates the regularity class may be 0 (continuous), any positive integer, $\infty$, or $\omega$ (real analytic).

**Definition 1.13** A $C^r$ $(r \geq 0)$ *codimension $q$ (dimension $p$) foliation $\mathcal{F}$* of $N$ is a decomposition of $N$ into one-to-one immersed submanifolds of dimension $p = n - q$, called *leaves* with the following properties.

1. There exists an open covering $\{U_i\}$ and for each $i$ a $C^r$ submersion $f_i : U_i \to \mathbf{R}^q$ such that the inverse image of each point is contained in some leaf.

2. For any $U_i$ and $U_j$ which intersect, there exists a $C^r$ diffeomorphism $g_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ such that $f_i = g_{ij} f_j$ on $U_i \cap U_j$.

Precisely, $\mathcal{F}$ denotes the family of leaves. Thus $F \in \mathcal{F}$ means that $F$ is a leaf of $\mathcal{F}$.

Let us raise some examples of foliations.

**Example 1.14** Let $\phi$ be a nonsingular[4] flow $\phi$ on $N$. Then the decomposition of $N$ into orbits of $\phi$ is a dimension 1 foliation. (We just think of the orbits, and forget about the time parametrization.)

**Example 1.15** Let $f : N \to M$ be a locally trivial bundle map. Then the decomposition of $N$ into the fibers of $f$ is a foliation. Such a foliation is called a *bundle foliation.*

**Example 1.16** Consider the $n$-torus $T^n$. A system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = \text{const.} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = \text{const.} \\ \cdots & \\ a_{q1}x_1 + a_{q2}x_2 + \cdots + a_{qn}x_n & = \text{const.} \end{cases}$$

[4] "Nonsingular" means that no orbit is a point

defines a codimension $q$ foliations on $R^n$ by parallel affine subspaces, provided the coefficient vectors are linearly independent. An affine translation maps a leaf onto a leaf. Therefore this system yields a foliation on $T^n$, called a *linear foliation.*

**Exercise 1.17** *For $n = 2$ and $q = 1$, all the leaves are circles if $a_{11}$ and $a_{12}$ are rationally dependent. Also all the leaves are dense in $T^2$ if $a_{11}$ and $a_{12}$ are rationally independent.*

**Exercise 1.18** *Let $V$ be the subspace of $R^n$ defined by*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots \\ a_{q1}x_1 + a_{q2}x_2 + \cdots + a_{qn}x_n &= 0. \end{cases}$$

*Let $r$ be the rank of the abelian group $V \cap \mathbf{Z}^n$. Then all the leaves are diffeomorphic to $T^r \times \mathbf{R}^{q-r}$.*

**Exercise 1.19** *Let $W$ be the linear subspace of $R^n$ generated by vectors*

$$(a_{11}, a_{12}, \cdots, a_{1n}),\ (a_{21}, a_{22}, \cdots, a_{2n}), \cdots,\ (a_{q1}, a_{q2}, \cdots, a_{qn}).$$

*Let $s$ be the rank of $W \cap \mathbf{Z}^n$. Then the closure of an arbitrary leaf is diffeomorphic to $T^{n-s}$.*

**Example 1.20** Consider $D^2 \times \mathbf{R}$. Denote the polar coordinates of $D^2$ by $(r, \theta)$, and the coordinates of $\mathbf{R}$ by $x$. Then the equation

$$x = \frac{1}{1 - r^2} + \text{const.}$$

defines a foliation on the interior of $D^2 \times \mathbf{R}$. Letting the boundary to be a leaf, this gives birth to a foliation on $D^2 \times \mathbf{R}$. A translation along $\mathbf{R}$ maps a leaf onto a leaf. Therefore, it defines a foliation of the solid torus $D^2 \times S^1$, called the *Reeb component.* Now since 3-sphere is obtained by glueing two solid tori along their boundaries, we obtain a codimension one foliation on $S^3$. This is called the *Reeb foliation.*

Now for a diffeomorphism, there is a construction to define a flow from it, called suspension. This is generalized to foliations as follows.

**Example 1.21** Let $M$ be a closed manifold with fundamental group $\Gamma$. $\Gamma$ acts on the universal covering space $\widetilde{M}$ as the deck transformations. Independently, suppose that the group $\Gamma$ acts on a closed manifold $P$. Consider the diagonal action of $\Gamma$ on the product manifold $\widetilde{M} \times P$. This action is free, properly discontinuous, and preserves the trivial foliation $\mathcal{G} = \{\widetilde{M} \times \{*\}\}$. Therefore one obtains a foliation $\mathcal{F}$ on the quotient manifold $N = \widetilde{M} \times_\Gamma P$. Now the canonical projection of $\widetilde{M} \times P$ onto $M$ is of course equivariant with the actions of $\Gamma$, and therefore induces a submersion $f : N \to M$. This is a locally trivial bundle map with fiber diffeomorphic to $P$. Now the foliation $\mathcal{F}$ is transverse to the fibers.

A bundle structure is called a *foliated bundle* if it is equipped with a foliation tranverse to the fibers and of complementary dimension. It is known that conversely any foliated bundle is constructed in the above way.

When $M$ is $S^1$, thus $\Gamma$ is infinitely cyclic, and the action on $P$ is generated by a diffeomorphism $g : P \to P$, the 1-dimensional foliation thus obtained is the so called *suspension* of $g$.

Another typical example is the following. Let $\Gamma$ be a Fuchsian group such that the quotient of the upper half plane $H^2$ by $\Gamma$ is a Riemann surface $\Sigma$ of genus $g > 1$. Consider the action of $\Gamma$ on the circle at infinity $S^1_\infty$. Then the manifold $N = H^2 \times_\Gamma S^1_\infty$ is the unit tangent bundle of $\Sigma$ and the foliation is the so called stable foliation of the geodesic flow. This will be explained in more detail in Section 2.

Consider a $C^r$ dimension $p$ foliation $\mathcal{F}$. At each point $x \in N$, the tangent space of the leaf through $x$ is a subspace of the tangent space $T_xN$. They define a $C^r$ subbundle of $TN$ of fiber dimension $p$, called the *tangent bundle* of $\mathcal{F}$ and denoted by $T\mathcal{F}$.

Conversely it is not always true that a subbundle $E$ of $TN$ is the tangent bundle of a foliation. We call $E$ *integrable* if it is the tangent bundle of a foliation. Recall the following well-known fact.

**Proposition 1.22 (Frobenius)** *A $C^1$ subbundle $E$ is integrable if and only if for any vector fields $X$ and $Y$ tangent to $E$, the Lie bracket $[X, Y]$ is also tangent to $E$.*

Now we have finished preparations in foliation theory. As before, let $f$ be an Anosov diffeomorphism of a compact manifold $N$, with a splitting of the tangent bundle $TN = E^u \oplus E^s$. Let us state an important theorem associating foliations to Anosov diffeomorphisms.

**Theorem 1.23** *$E^\sigma$ is integrable.*

All the complete proofs of this theorem, as far as the author knows, are rather lengthy and need involvements in fields of mathematics wider than intended for this textbook.

The outline of the proof found in [27] is as follows. Consider the Banach manifold $\mathcal{N}$ of all the maps[5] from $N$ into $N$, and a transformation $f_*$ of $\mathcal{N}$ defined by

$$f_*(h) = f \circ h \circ f^{-1} \quad \forall h \in \mathcal{N}.$$

Then the identity $i \in \mathcal{N}$ is a hyperbolic fixed point of $f_*$. Now apply the Banach space version of the unstable manifold theorem (Corollary 1.5) to this situation. Then we get that the unstable manifold $\mathcal{W}_i^u$ of the inclusion is a one-to-one immersed $C^r$ Banach submanifold. For any $x \in N$ define the subset $W_x^u$ of $N$ to be the set of $y$ such that $y = \gamma(x)$ for some $\gamma \in \mathcal{W}_i^u$. Then $W_x^u$ $(\forall x \in N)$ defines a foliation whose tangent bundle is $E^u$.

The proof in [1] is less formal, but is lengthy. This is rather flustrating since the integrability of $E^\sigma$ seems to be a natural phenomenum caused by the streching-contracting property of Anosov diffeomorphisms.

Let us give a shorter proof based upon geometric observations, under the extra condition that *the splitting* $TN = E^u \oplus E^s$ *is* $C^1$. Let us denote by $\Gamma(E^\sigma)$ the space of $C^1$ cross sections of $E^\sigma$, that is, a vector field tangent to $E^\sigma$. For any vector field $X$ of N, denote the decomposition of $X$ by

$$X = X^u + X^s, \quad \text{where} \quad X^\sigma \in E^\sigma.$$

Also denote by $\|X\|$ the supremum norm of $X$. Now let us try to show that $E^u$ is integrable. Take arbitrary two vector fields $X$ and $Y$ tangent to $E^u$. Clearly we have $f_*[X,Y] = [f_*X, f_*Y]$ and $(f_*X)^s = f_*(X)^s$. Therefore for large $n > 0$, we have

$$[X,Y]^s = f_*^n([f_*^{-n}X, f_*^{-n}Y]^s).$$

Now $f_*^{-n}$ shrinks the vectors in $E^u$. Therefore the vector fields $f_*^{-n}X$ and $f_*^{-n}Y$ are very small in norm, and also $f_*^n$ shrinks the vector $[f_*^{-n}X, f_*^{-n}Y]^s$ which is tangent to $E^s$.

It seems that this observation yields a simple proof. But taking a bracket is a kind of differentiation, and it is not true in general that the bracket of two small vector fields is small. So we need a second effort.

Let $X_1, \cdots, X_r$ be a set of vetor fields tangent to $E^u$ such that at any point $x \in N$, the vectors $X_{1x}, \cdots, X_{rx}$ generate $E_x^u$. Using a partition of unity,

[5] not necessarily continuous

one can easily find such vector fields. Also one can show that any vector field tangent to $E^u$ can be represented as a linear combination of $X_i$'s, with function coefficients. However we need a bit more. Denote by $C^1(N)$ the space of $C^1$ functions of $N$. The supremum norm is denoted by $|\phi|$ for $\phi \in C^1(N)$. The following easy observation will be useful.

> *There exists a constant $a > 0$ such that for any $Y \in \Gamma(E^u)$, we have*
>
> $$Y = \sum_i \phi_i X_i, \quad \textit{where} \quad \phi_i \in C^1(N) \quad \textit{and} \quad \sum_i |\phi_i| \le a\|Y\|.$$

Now for any $Y$, $Z \in \Gamma(E^u)$, let us show that $[Y, Z]^s = 0$. Choose $n > 0$ arbitrarily. Then by the above observation we have the following representations.

$$f_*^{-n}Y = \sum_i \phi_i X_i, \text{ where } \phi_i \in C^1(N) \text{ and } \sum_i |\phi_i| \le a\|f_*^{-n}Y\| \le aC\lambda^n\|Y\|,$$

$$f_*^{-n}Z = \sum_j \psi_j X_j, \text{ where } \psi_j \in C^1(N) \text{ and } \sum_j |\psi_j| \le a\|f_*^{-n}Z\| \le aC\lambda^n\|Z\|.$$

That is,

$$\begin{aligned} Y &= \sum_i \overline{\phi_i} f_*^n X_i, \quad \text{where} \quad \overline{\phi_i} = \phi_i \circ f^n, \\ Z &= \sum_j \overline{\psi_j} f_*^n X_j, \quad \text{where} \quad \overline{\psi_j} = \psi_j \circ f^n. \end{aligned}$$

Now we have

$$\begin{aligned} [Y, Z] &= \sum_{i,j} \overline{\phi_i} f_*^n X_i(\overline{\psi_j}) f_*^n X_j - \overline{\psi_j} f_*^n X_j(\overline{\phi_i}) f_*^n X_i \\ &+ \sum_{i,j} \overline{\phi_i}\,\overline{\psi_j} [f_*^n X_i, f_*^n X_j]. \end{aligned}$$

Therefore

$$[Y, Z]^s = \sum_{i,j} \overline{\phi_i}\,\overline{\psi_j} f_*^n [X_i, X_j]^s.$$

Now choose a constant $b > 0$ such that $\|[X_i, X_j]^s\| \le b$ for any $i$ and $j$. Then

$$\|f_*^n [X_i, X_j]^s\| \le bC\lambda^n.$$

We also have

$$\sum_{i,j} |\overline{\phi}_i||\overline{\psi}_j| = \sum_{i,j} |\phi_i||\psi_j| \leq a^2 C^2 \lambda^{2n} \|Y\| \|Z\|.$$

Finally we get

$$\|[Y, Z]^s\| \leq b a^2 C^3 \lambda^{3n} \|Y\| \|Z\|.$$

Since $n > 0$ is arbitrary and the constants $a$ and $b$ are independent of $n$, we get that $[X, Y]^s = 0$, that is, $[Y, Z] \in \Gamma(E^u)$.

**Definition 1.24** The foliation $\mathcal{W}^u$ (resp. $\mathcal{W}^s$) tangent to $E^u$ (resp. $E^s$) is called the *unstable* (resp. *stable*) *foliation* of the Anosov diffeomorphism $f$. The leaf through a point $x$ of the foliation $\mathcal{W}^\sigma$ is denoted by $W^\sigma_x$.

The $Df$-invariance of the bundle $E^\sigma$ and the uniqueness of the foliation tangent to $E^\sigma$ implies the following proposition.

**Proposition 1.25** *The foliation $\mathcal{W}^\sigma$ is $f$-invariant, i. e. $f$ maps a leaf of $\mathcal{W}^\sigma$ onto a leaf. Precisely we have $f(W^\sigma_x) = W^\sigma_{fx}$.*

Let us state a theorem about the smoothness of the foliation $\mathcal{W}^\sigma$. See [A] or [H] for the proof. Notice that even if a foliation consists of smooth leaves, it does not imply that the foliation is smooth. Recall Difinition 1.13. The smoothability of the foliation depends on the local transverse projection.

**Theorem 1.26** *The foliation $\mathcal{W}^\sigma$ is a continuous foliation by $C^r$ leaves. The leaf $W^\sigma_x$ depends continuously on $x$ in $C^r$ topology.*

From now on, we will use the Riemannian metric in Lemma 1.8. By the compactness of $N$, we have

$$\begin{aligned} \|Df^{-1}(v)\| &\leq \lambda \|v\| \quad \text{for} \quad v \in E^u \\ \|Df(v)\| &\leq \lambda \|v\| \quad \text{for} \quad v \in E^s \end{aligned}$$

for some $0 < \lambda < 1$. We also assume that $E^u$ and $E^s$ is orthogonal.

Denote by $d^\sigma$ the distance in a leaf of $\mathcal{W}^\sigma$ induced by the restricted Riemannian metric. We have the following obvious lemma.

**Lemma 1.27** *For any point $y \in W^u_x$, we have*

$$d^u(f^{-1}x, f^{-1}y) \leq \lambda d^u(x, y).$$

*For any point* $y \in W^s_x$, *we have*

$$d^s(fx, fy) \leq \lambda d^s(x, y).$$

$\square$

For $\epsilon > 0$, denote

$$W^\sigma_x(\epsilon) = \{y \in W^\sigma_x \mid d^\sigma(y, x) < \epsilon\}.$$

Then we have

$$f^{-1}(W^u_x(\epsilon)) \subset W^u_{f^{-1}x}(\lambda\epsilon), \quad f(W^s_x(\epsilon)) \subset W^s_{fx}(\lambda\epsilon). \tag{11}$$

The stable and unstable foliation are orthogonal to each other and of complementary dimensions. Therefore they give a local product structure of the manifold. Namely, we have the following obvious lemma.

**Lemma 1.28** *There exists a number* $\epsilon_0$ *such that for any* $0 < \epsilon < \epsilon_0$ *and for any points* $x$ *and* $y$ *of distance less than* $\epsilon$, *the sets* $W^s_x(2\epsilon)$ *and* $W^u_y(2\epsilon)$ *intersect exactly at one point.* $\square$

Choose $2\epsilon < \epsilon_0$. Let $x$ be an arbitrary point of $N$. Then for any point $y$ in $W^u_x(\epsilon)$ and $z$ in $W^s_x(\epsilon)$, we have that $d(y, z) < 2\epsilon$, and hence from the above lemma that $W^s_y(4\epsilon)$ and $W^u_z(4\epsilon)$ intersects at one point, say, $I(y, z)$.

**Definition 1.29** The set

$$R_x(\epsilon) = \{I(y, z) \mid y \in W^u_x(\epsilon),\ z \in W^s_x(\epsilon)\}$$

is called the *rectangle* at $x$ of radius $\epsilon$

The concept of rectangle will be generalized in a later subsection. The restriction of the foliation $\mathcal{W}^\sigma$ to the rectangle $R_x(\epsilon)$ is a trivial foliation. All the leaves are diffeomorphic to $W^\sigma_x(\epsilon)$. $R_x(\epsilon)$ is homeomorphic to $W^u_x(\epsilon) \times W^s_x(\epsilon)$. For any point $y \in R_x(\epsilon)$, the leaf through $y$ of the restriction of $\mathcal{W}^\sigma$ is denoted by $R_x(\epsilon)^\sigma_y$.

We have the following obvious lemma.

**Lemma 1.30** *There exists a number* $\epsilon_1 > 0$ *such that for any* $0 < \epsilon < \epsilon_1$, $x \in N$, $y \in W^s_x(\epsilon)$ *and* $z \in W^u_x(\epsilon)$, *we have*

$$W^u_y(\lambda^{1/2}\epsilon) \subset R_x(\epsilon)^u_y \subset W^u_y(\lambda^{-1/2}\epsilon),$$

$$W^s_z(\lambda^{1/2}\epsilon) \subset R_x(\epsilon)^s_z \subset W^s_z(\lambda^{-1/2}\epsilon).$$

□

From now on in this section we always choose $\epsilon$ such that $\epsilon < \min\{\epsilon_0/2, \epsilon_1\}$. Let us compare $f(R_x(\epsilon))$ with $R_{fx}(\epsilon)$.

By the relation (11) and by the previous lemma, we have the following lemma.

**Proposition 1.31** *For any point* $y \in R_x(\epsilon) \cap f^{-1}(R_{fx}(\epsilon))$, *we have*

$$R_{fx}(\epsilon)^u_{fy} \subset f(R_x(\epsilon)^u_y), \tag{12}$$

$$R_x(\epsilon)^s_y \subset f^{-1}(R_{fx}(\epsilon)^s_{fy}). \tag{13}$$

**Proof** Notice that in order to show (12), it is enough to consider the case where $y \in W^s_x(\epsilon)$. But then a combination of the relation (11) and the previous lemma shows it. □

Now consider the following decreasing sequence of subsets.

$$R_x(\epsilon) \subset R_x(\epsilon) \cap f(R_{f^{-1}x}(\epsilon)) \subset R_x(\epsilon) \cap f(R_{f^{-1}x}(\epsilon) \cap f^2(R_{f^{-2}x}(\epsilon)) \subset \cdots.$$

Then the width along $\mathcal{W}^s$–direction becomes thinner and thinner, and at last the limit becomes nothing but $W^u_x(\epsilon)$. Also we consider the following decreasing sequence.

$$R_x(\epsilon) \subset R_x(\epsilon) \cap f^{-1}(R_{fx}(\epsilon)) \subset R_x(\epsilon) \cap f^{-1}(R_{fx}(\epsilon) \cap f^{-2}(R_{f^2x}(\epsilon)) \subset \cdots.$$

Let us summerize the result in the following proposition.

**Proposition 1.32** *We have*

$$\bigcap_{n=0}^{\infty} f^n(R_{f^{-n}x}(\epsilon)) = W^u_x(\epsilon), \tag{14}$$

$$\bigcap_{n=0}^{\infty} f^{-n}(R_{f^n x}(\epsilon)) = W^s_x(\epsilon), \tag{15}$$

$$\bigcap_{n=-\infty}^{\infty} f^{-n}(R_{f^n x}(\epsilon)) = \{x\}. \tag{16}$$

**Proposition 1.33** *The foliation $\mathcal{W}^u$ is characterized by the dynamics of $f$ as follows.*

$$W_x^u = \{y \in N \mid d(f^{-n}x, f^{-n}y) \to 0 \ \ (n \to \infty)\}.$$

*Likewise*

$$W_x^s = \{y \in N \mid d(f^{n}x, f^{n}y) \to 0 \ \ (n \to \infty)\}.$$

**Proof** Let us treat only the case of $\mathcal{W}^u$. Suppose $y \in W_x^u$. Then by Lemma 1.27, we have that

$$d^u(f^{-n}x, f^{-n}y) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $d^u(f^{-n}x, f^{-n}y) \geq d(f^{-n}x, f^{-n}y)$, we have

$$d(f^{-n}x, f^{-n}y) \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

On the other hand, suppose $y \in N$ satisfy (17) and let us show that $y$ belongs to $W_x^u$. There exists a number $n_0$ such that if $n \geq n_0$, then $f^{-n}y$ belongs to $R_x(\epsilon)$ Notice that by the $f$-invarinace of the foliation $\mathcal{W}^u$ (Proposition 1.25), $y$ belongs to $W_x^u$ if and only if $f^{-n_0}y$ belongs to $W^u_{f^{-n_0}x}$. So let us rename $f^{-n_0}x$ (resp. $f^{-n_0}y$) by $x$ (resp. $y$) for simplicity. Then we have

$$y \in \bigcap_{n=0}^{\infty} f^n(R_{f^{-n}x}(\epsilon))$$

Therefore it follows from Proposition 1.32 that $y \in W_x^u$. □

## 1.4 Structural stability

An Anosov diffeomorphism satisfies remarkable properties. Among others it is *structurally stable*, i. e. any small perturbation of $f$ has the same topological structure as $f$. Precisely

**Theorem 1.34 (Structural Stability Theorem)** *Let $f$ be a $C^1$ Anosov diffeomorphism of a closed manifold $N$. For any $\epsilon > 0$, there exists a meighbourhood calN of $f$ in $\mathrm{Diff}^1(N)$ with the following property; for any $g \in \mathcal{N}$, there exists a homeomorhism $h$ of $N$, $\epsilon$-near to the identity in the $C^0$-topology, such that $g = h^{-1} \circ f \circ h$.*

The purpose of this subsection is to give a proof of the above theorem. The following two concepts, pseudo orbit tracing property and expansiveness, are very useful. For a while let $f$ be a homeomorphism of a metric space $X$, and $\epsilon$ and $\delta$ positive numbers.

**Definition 1.35** A bi-infinite sequence $\{x_n\}_{n=-\infty}^{\infty}$ of points of $X$ is called a *$\delta$-pseudo orbit* of $f$ if we have $d(f(x_n), x_{n+1}) < \delta$ for any $n$.

**Definition 1.36** A bi-infinite sequence $\{x_n\}_{n=-\infty}^{\infty}$ is said to be *$\epsilon$-traced* by a point $y$ if we have $d(f^n y, x_n) < \epsilon$ for any $n$.

**Definition 1.37** $f$ is said to have *pseudo orbit tracing property* ( POTP for short) if for any $\epsilon > 0$ there exists $\delta > 0$ such that any $\delta$-pseudo orbit is $\epsilon$-traced by some point.

Imagin that we are calculating an orbit of a given diffeomorphism $f$ using a computer. Given a data of the initial point $x$, we compute $fx$ and $f^{-1}x$, but with a small error. Next based upon these results, we compute $f^2x$ and $f^{-2}x$ again with a small error. And then we compute $f^3x$ and $f^{-3}x$ with an error, and so forth $\cdots$. What we get this way is not a true orbit, but exactly a pseudo orbit. Usually if we do this, the accumulation of small errors becomes inneglegible when $n$ becomes big, and there is no guarantee that this sequence is very near to an actual orbit. For example just think of the case where $f$ is the identity.

POTP is a property which says that calculation like this is admissible, however great the number $n$ is. In other words, we are allowed to do small mistakes a million times. Wonderful!

We need another concept, expansiveness, which says that any two distinct orbits must be fairly apart from each other at some time.

**Definition 1.38** $f$ is said to be *expansive* (or $e$-expansive for precision) if there exists a number $e > 0$ such that for any distinct points $x$ and $y$ in $X$, there exists an integer $n$ such that $d(f^n x, f^n y) \geq e$. The number $e$ is called an *expansive constant.*

For example if $f$ is $2\epsilon$-expansive, then the point which $\epsilon$-traces a pseudo orbit is unique. In this subsection, we first establish POTP and expansiveness for Anosov diffeomorphisms, and then use them to prove the structural stability theorem.

From now on, $f$ is to be an Anosov diffeomorphism of a closed manifold $N$.

**Lemma 1.39** *$f$ satisfies the POTP.*

**Proof** Given $\epsilon > 0$, we shall find out $\delta > 0$ such that any $\delta$-pseudo orbit is $\epsilon$-traced. First of all, notice that we may consider only the case where the given $\epsilon$ is sufficiently small. Therefore assume that $\epsilon$ is chosen to satisfy the hypothesis of the previous subsection. Especially Proposition 1.31 holds. This proposition treat with two point $x$ and $fx$. But clearly if we change $fx$ with a point $x_1$ very near to $fx$, then the relation of $f(R_x(\epsilon))$ and $R_{x_1}(\epsilon)$ is not so much different with that of $f(R_x(\epsilon))$ and $R_{fx}(\epsilon)$. Precisely, we have the following. (The point $x$ is denoted by $x_0$.)

**Sublemma 1.40** *Given $\epsilon$ small, there exists a $\delta > 0$ with the following property; If $d(fx_0, x_1) < \delta$, then for any point $y \in R_{x_0}(\epsilon) \cap f^{-1}(R_{x_1}(\epsilon))$, we have*

$$R_{x_1}(\epsilon)^u_{fy} \subset f(R_{x_0}(\epsilon)^u_y), \tag{18}$$
$$R_{x_0}(\epsilon)^s_y \subset f^{-1}(R_{x_1}(\epsilon)^s_{fy}). \tag{19}$$

□

Now let $\{x_n\}_{n=-\infty}^{\infty}$ be a $\delta$-pseudo orbit. Then for any $n$, the relation of $f(R_{x_{n-1}}(\epsilon))$ and $R_{x_n}(\epsilon)$ is as in the sublemma. The contracting property of Anosov diffeomorphism clearly shows the following generalization of Proposition 1.32.

$$\bigcap_{n=0}^{\infty} f^n(R_{x_{-n}}(\epsilon)) = R_{x_0}(\epsilon)^u_{y'} \quad \exists y' \in R_{x_0}(\epsilon), \tag{20}$$
$$\bigcap_{n=0}^{\infty} f^{-n}(R_{x_n}(\epsilon)) = R_{x_0}(\epsilon)^s_{y''} \quad \exists y'' \in R_{x_0}(\epsilon), \tag{21}$$
$$\bigcap_{n=-\infty}^{\infty} f^{-n}(R_{x_n}(\epsilon)) = \{y\} \quad \exists y \in R_{x_0}(\epsilon). \tag{22}$$

Relation (22) shows that the point $y$ $\delta$-traces the pseudo orbit $\{x_n\}$. □

**Lemma 1.41** *There exists a $C^1$ neighbourhood $\mathcal{N}$ of $f$ and a constant $e > 0$ such that any diffeomorphism in $\mathcal{N}$ is $e$-expansive.*

**Proof** It is clear by (16) of Proposition 1.32 that $f$ itself is expansive, with expansive constant equal to the size of the rectangls. In order to show the local uiformity of the expansive constant, just notice that the size of the rectangles can be chosen lower semi-continuously upon the diffeomorphism in the $C^1$-topology. (Recall that the set of Anosov diffeomorphisms form an open set by Corollary 1.11.) □

**Proof of Theorem 1.34** Given $\epsilon > 0$, let us define a $C^1$-neighbourhood $\mathcal{N}$ so that any diffeomorphism $g$ in $\mathcal{N}$ can be conjugated to $f$ by a homeomorphism $h$, $\epsilon$–near to the identity. Let $\mathcal{N}_1$ be a neighbourhood of $f$ such that any diffeomorphism $g$ in $\mathcal{N}_1$ is $2\epsilon$-expansive. Now since the diffeomorphism $f$ has the POTP, there exists $\delta > 0$ such that any $\delta$-pseudo orbit is $\epsilon$-traced. Let $\mathcal{N}_2$ be a $C^0$-neighbourhood of $f$ consisting of those diffeomorphism $g$ such that $d(fx, gx) < \delta$ for any $x \in N$.

Let us show that $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$ satisfies the condition of the theorem. Let $g$ be an arbitrary diffeomorphism in $\mathcal{N}$. Then for any point $x \in N$, the bi-infinite sequence $\{g^n x\}$ is a $\delta$-pseudo orbit for $f$. Therefore it is $\epsilon$-traced by a point, say $h(x)$. Notice that such a point is unique since $f$ is $2\epsilon$-expansive. The uniqueness also shows the equality $f \circ h = h \circ g$. Clearly the map $h : N \to N$ is $\epsilon$-near to the identity.

Let us show the continuity of $h$. (The diffeomorphism $g$ is already fixed.) Suppose $y$ is very near to $x$. Then for some large $N > 0$, $g^n y$ and $g^n x$ are mutually near for $-N \le n \le N$. Now $h(x)$ belongs to $\cap_{n=-N}^{N} f^{-n}(R_{g^n x}(\epsilon))$ and $h(y)$ to $\cap_{n=-N}^{N} f^{-n}(R_{g^n y}(\epsilon))$. We also have that the diameter of $\cap_{n=-N}^{N} f^{-n}(R_{g^n x}(\epsilon))$ $\cup \cap_{n=-N}^{N} f^{-n}(R_g^n y(\epsilon))$ is very small.

Finally we must show that $h$ is injective. Suppose $h(x) = h(y)$. That is, two sequence $\{g^n x\}$ and $\{g^n y\}$ are $\epsilon$-traced by the same point. Thus we have $d(g^n x, g^n y) < 2\epsilon$. Since $g$ is also $2\epsilon$-expansive, we have $x = y$. □

## 1.5 Transitiveness

For a while let $f$ be a homeomorphism of a compact metric space $X$.

**Definition 1.42** A point $x \in X$ is called a *nonwandering point* of $f$ if for any neighbourhood $U$ of $x$, there exists a nonzero integer $n$ such that $U \cap f^n(U) \ne \emptyset$. The set of all the nonwandering points is denoted by $\Omega(f)$ and is called the *nonwandering set.*

For example a periodic point is a wandering point.

**Proposition 1.43** *The nonwandering set $\Omega(f)$ is a nonempty closed set, invariant by $f$, i. e. $f(\Omega(f)) = \Omega(f)$.*

**Proof** The $f$-invariance and the closednedd is obvious. Let us prove that $\Omega(f)$ is nonempty. Choose any point $x \in X$ and consider the set

$$\omega(x) = \{y \mid y = \lim_{k\to\infty} f^{n_k}(x) \text{ for some } n_k \to \infty\}.$$

$\omega(x)$ is a closed $f$-invariant set, and is nonempty since $X$ is compact. $\omega(x)$ is called an $\omega$-limit set. It is easy to show that $\omega(x) \subset \Omega(f)$. □

The dynamics of $f$ outside $\Omega(f)$ is rather simple. Any point cannot come back very near to itself. On the other hand the dynamics inside $\Omega(f)$ can be extremely complicated.

Now let $f$ be an Anosov diffeomorphism of a closed manifold $N$. The following lemma yields an example of nonwandering points.

**Lemma 1.44** *Let $p$ be a periodic point and let $x$ be a point of the intersection of $W_p^u$ and $W_p^s$. Then $x$ is a nonwandering point.*

**Proof** Just for simplicity, let us assume that $p$ is a fixed point. For arbitrarily small $\epsilon > 0$, consider the rectangle $R_x(\epsilon)$. By the iterates $f^n$ $(n > 0)$, The unstable leaf $W_x^u(\epsilon)$ will be streched to be a large ball in unstable leaves. On the other hand, we have $f^n x \to p$ as $n \to \infty$, because $x$ lies in $W_p^s$. This shows that $f^n(W_x^u(\epsilon))$ approaches $W_p^u$. In particular, for any large $n$ the set $f^n(W_x^u(\epsilon))$ intersects the rectangle $R_x(\epsilon)$, showing that $x$ is a nonwandering point. □

The point $x$ in the lemma is called a *homoclinic point.* Notice that the orbit of a homoclinic point does not come back near to itself, although it is a nonwandering point.

**Theorem 1.45** *$\Omega(f)$ coincides with the closure of the set of all the periodic points.*

**Proof** All that needs proof is that the periodic points are dense in $\Omega(f)$. Take an arbitrary point $x \in \Omega(f)$ and a small positive number $\epsilon$. Let us show that there exists a periodic point in the rectangle $R_x(\epsilon)$. Choose $\delta$ as in the previous section (to satisfy Sublemma 1.40). Then there is a point $y$, $\delta$-near to $x$, such that $f^n y$ is also $\delta$-near to $y$. Then clearly we have for $z \in R_x(\epsilon) \cap f^{-n}(R_x(\epsilon))$,

$$\begin{aligned} R_x(\epsilon)^u_{f^n z} &\subset f^n(R_x(\epsilon)^u_z), \\ R_x(\epsilon)^s_z &\subset f^{-n}(R_x(\epsilon)^s_{f^n z}). \end{aligned}$$

The quotient $Q$ of $R_x(\epsilon)$ by the unstable foliation is homeomorphic to an open ball in $\mathbf{R}^s$ and by the above property, we get a map from $Q$ into itself. This map is a contraction and have a fixed point. Let us denote by $L$ the leaf in $R_x(\epsilon)$ of the unstable foliation which corresponds to this fixed point. Then we have that $f^{-n}$ is a contraction of $L$. Therefore it has a fixed point. □

Since the nonwandering set is nonempty (Proposition 1.43), we get the following corollary of Theorem 1.45.

**Corollary 1.46** *An anosov diffeomorphism admits periodic points.*

The rest of this subsection is devoted to the proof of the following theorem.

**Theorem 1.47** *The following five conditions are equivalent.*

1. *The nonwandering set $\Omega(f)$ coincides with the total manifold $N$.*
2. *The periodic points are dense in $N$.*
3. *There exists a point $x$ such that its orbit $\{f^n x\}$ is dense in $N$.*
4. *All the leaves of the unstable foliation $\mathcal{W}^u$ are dense in $N$.*
5. *All the leaves of the stable foliation $\mathcal{W}^s$ are dense in $N$.*

**Definition 1.48** We call $f$ *transitive* if it satisfies one of the above conditions.

Before proving Theorem 1.47, let us prepare a lemma.

**Lemma 1.49** *Suppose that the periodic points are dense in $N$. Let $X$ be a subset of $N$ which satisfies the following conditions;*

1. *$X$ is a nonempty and closed subset,*
2. *$X$ is a union of leaves of the foliation $\mathcal{W}^\sigma$,*
3. *$X$ is invariant by $f^n$ for some $n > 0$.*

*Then we have $X = N$.*

**Proof** Let us treat only the case of $\sigma = u$. It is sufficient to show that $X$ is open. Take a point $x \in X$ and consider a rectangle $R_x(\epsilon)$ for a given small $\epsilon$. Since the periodic points are dense, we need only to show that any periodic point $p$ in $R_x(\epsilon)$ belongs to $X$. Consider a point

$$y \in R_x(\epsilon)^u_x \cap R_x(\epsilon)^s_p \subset X \cap W^s_p.$$

Let $m$ be a common multiplier of $n$ in the lemma and the period of $p$. Then both $X$ and $W^s_p$ are invariant by $f^m$. Now since $y \in W^s_p$, the sequence $\{f^{im}y\}$ approaches $p$, as $i$ tends to the infinity. Since $f^{im}y$ belongs to $X$, the point $p$ is contained in the closed set $X$. □

**Proof of Theorem 1.47** That conditions 1 and 2 are equivalent is immediate from the previous theorem.

Let us show that condition 2 implies 4. That is, assuming that the periodic points are dense in $N$,we shall show that for any point $x$, $W^u_x$ is dense in $N$. Let $Y = \mathrm{Cl}(W^u_x)$. Clearly $Y$ is a union of unstable leaves. Let $n$ be an arbitrary positive integer and let

$$Z_n = \mathrm{Cl}(\bigcup_{i\geq 0} f^{-in}Y).$$

Clearly one has

$$f^{-n}(Z_n) = \mathrm{Cl}(\bigcup_{i\geq 1} f^{-in}Y) \subset Z_n.$$

Therefore we have a decreasing sequence

$$\cdots \subset f^{-3n}(Z_n) \subset f^{-2n}(Z_n) \subset f^{-n}(Z_n) \subset Z_n.$$

Thus the intersection

$$X_n = \bigcap_{i\geq 0} f^{-in}(Z_n)$$

is a nonempty closed subset which is a union of unstable leaves and is invariant by $f^n$. Therefore by Lemma 1.49, we have that $X_n = N$.

Let $p$ be an arbitrary periodic point of period, say $n$. Then since the set $X_n$ constructed above coincides with $N$, we have that

$$X_n \cap W^s_p(\epsilon) = W^s_p(\epsilon).$$

In particular we have

$$f^{-in}(Y) \cap W^s_p(\epsilon) \neq \emptyset$$

for some $i \geq 0$. But since $f^{in}(W_p^s(\epsilon)) \subset W_p^s(\epsilon)$, we have that $Y \cap W_p^s(\epsilon) \neq \emptyset$. Since $\epsilon$ can be arbitrarily small, this shows that $p$ is contained in the closed set $Y$. Since $p$ is an arbitrary periodic point and since we are assuming that the periodic points are dense in $N$, we have that $Y = N$, as is desired.

Next let us show that condition 4 implies 1. By Corollary 1.46, there exists a periodic point $p$. Assume for simplicity that $p$ is a fixed point. We need only show that the stable leaf $W_p^s$ is dense in $N$. For then, since we are assuming that the unstable leaf $W_p^u$ is also dense, the points of intersection of $W_p^u$ and $W_p^s$ are dense in $N$. But we have already shown that these points are nonwandering (Lemma 1.44). Therefore we will get that the nonwandering set coincides with $N$.

Now assume for contradiction that $R = \mathrm{Cl}(W_p^s)$ is not the whole manifold $N$. Clearly $R$ is a union of stable leaves. Since $p$ is a fixed point, $R$ is invariant by $f$. For a small positive number $\epsilon$ define

$$U = \bigcup_{x \in R} W_x^u(\epsilon).$$

If $\epsilon$ is sufficiently small $U$ is a proper subset of $N$. We have for $n \geq 0$

$$f^{-n}(U) \subset \bigcup_{x \in R} W_x^u(\lambda^n \epsilon),$$

where $0 < \lambda < 1$ is the constant in the previous subsection. Now we get the following properties of $U$.

$$\cdots \subset f^{-2}(U) \subset f^{-1}(U) \subset U. \tag{23}$$

$$\bigcap_{n \geq 0} f^{-n}(U) = R. \tag{24}$$

Consider the complement $U^C$ of $U$ and define

$$A = \bigcap_{n \geq 0} f^n(U^C).$$

Then $A$ is a closed $f$-invarinat set. Clearly $A$ and $R$ do not intersect. Especially there exists a constant $d_0$ such that $d(x, y) \geq d_0$ for any $x \in A$ and $y \in R$.

We shall show that $A$ is a union of unstable leaves. Assume $x \in A$ and let us prove that $W_x^u \subset A$. Choose $z \in W_x^u$. That is, $d(f^{-n}x, f^{-n}z) \to 0$ as $n \to \infty$. Assume for contradiction that $z$ is not contained in $A$. Then clearly we have $f^{-n}z$ tends to $R$ as $n \to \infty$, but $f^{-n}x \in A$. A contradiction. This shows that $z$ is also contained in $A$, that is, $A$ is a union of unstable leaves.

Since $A$ is closed, this contradicts condition 4. The proof that 4 implies 1 is done.

So far we have shown that conditions 1, 2 and 4 are equivalent. By a similar argument one can add condition 5 in the group. Also condition 3 implies 1, since the point $x$ of 3 is a nonwandering point and the nonwandering set is closed.

What is left is to deduce 3 using all the other conditions. We are going to show that for any open set $U$ and $V$, there is an integer $n$, such that $U$ and $f^n(V)$ intersect. In other words, that there exists an orbit which meets both $U$ and $V$. This is a sufficient condition for the existence of dense orbits. (See the exercise below.) Choose any periodic point $p$ in $U$ and $q$ in $V$ (condition 2). Let $k$ be a common multiplier of their periods. Since $W_p^u$ and $W_q^s$ are dense (4 and 5), there is a point of intersection $x$. Then $f^{kn}x$ tends to $q$ as $n \to \infty$, and $f^{-kn}x$ tends to $p$ as $n \to \infty$. That is, the orbit of $x$ intersects both $U$ and $V$. □

**Exercise 1.50** *Let $f$ be a homeomorphism of a compact metric space $X$. Suppose that for any open subsets $U$ and $V$ of $X$ there exists an integer $n$ such that $U$ and $f^n(V)$ intersects. Then there exists a dense orbit of $f$.*

The following is an open problem.

**Problem 1.51** *Does there exist a nontransitive Anosov diffeomorphism?*

It is known that if either of unstable or stable foliation has codimension one, then the Anosov diffeomorphism is transitive. Thus the simplest unknown case is when $u = s = 2$.

## 1.6 Markov partitions

A Markov partition is a partition of a manifold, by means of which an Anosov diffeomorphism is (almost) reduced to symbolic dynamics.

Let us explain first what symbolic dynamics is. Let $\Sigma$ be the set of all the bifinite sequence $\mathbf{i} = \{i_n\}_{n=-\infty}^{\infty}$, where each term $i_n$ is either one of $1, 2, \cdots, r$. $\Sigma$ is equipped with the product topology; Namely the subsets consisting of the sequences which mutually coincide at some fixed finite set of $n$'s form an open basis. A homeomorphism $\sigma$, called *shift map* is defined by shifting a sequence in $\Sigma$ one step to the left, i. e. by

$$\sigma(\mathbf{i})_n = i_{n-1}.$$

A pair $(\Sigma, \sigma)$ is called a *full shift.* Its dynamics is easy and well understood. For example,

**Exercise 1.52** *Show that there exist dense orbits, and that the union of the periodic points is dense.*

Also easy to study is a slight generalization of the full shift, called a subshift of finite type. Let $A = \{a_{ij}\}$ be an $r$ by $r$ matrix whose entries are either 0 or 1. Then $A$ defines a directed graph as follows. The vertices are $1, 2, \cdots, r$. There exists a directed edge from the vertex $i$ to $j$ if and only if $a_{ij} = 1$. Consider a bifinite path in this graph, always moving in the direction of edges. It defines a bifinite sequence of the vertices visited in this order. The totality of these sequences form a subset of $\Sigma$, denoted by $\Sigma_A$.

To put it in another word,

$$\Sigma_A = \{\mathbf{i} \in \Sigma \mid a_{i_n i_{n+1}} = 1, \quad \forall n\}.$$

Now clearly $\Sigma_A$ is kept invariant by the shift map $\sigma$. The pair $(\Sigma_A, \sigma)$ is called a *subshift of finite type.*

**Exercise 1.53** *Show tha $\Sigma_A$ is a closed subset of $\Sigma$.*

A subshift of finite type, or a matrix $A$, is called *reduced* if in the corresponding directed graph, given any two points $i$ and $j$, there exists a directed path starting at $i$ and ending at $j$.

**Exercise 1.54** *Show that $A$ is reduced if and only if some power of $A$ is positive (all the entries are positive).*

**Exercise 1.55** *For a reduced subshift of finite type, show that there exist dense orbits, and that the union of the periodic points is dense.*

Of course one can consider closed $\sigma$-invariant subsets of $\Sigma$ of non-finite type. The analysis of such subshifts is hard and gathers interest of an active school of dynamical systems.

Now let us expose Markov partitions, gadgets which associate subshifts of finite type to Anosov systems. Let $f : N \to N$ be an Anosov diffeomorphism. Fix a point $x$ of $N$. First we need to generalize the concept of rectangles.

Let $Q^\sigma$ be a closed set in $W_x^\sigma$ such that $\mathrm{Int}Q^\sigma$ is connected, contains $x$ and that $\mathrm{Cl}(\mathrm{Int}Q^\sigma) = Q^\sigma$. Assume the diameter of $Q^\sigma$ is less than $\epsilon$ mentioned just after Lemma 1.30.

**Definition 1.56** A *rectangle* is a subset $R = Q^s \times Q^u$ of $N$ consisting of the points of intersection of $W^s_y(2\epsilon)$ and $W^u_z(2\epsilon)$, for any $y \in Q^u$ and $z \in Q^s$.

For any point $y \in R$, denote by $R^\sigma(y)$ the connected component at $y$ of $R \cap W^\sigma_y(2\epsilon)$. Thus the set $R = Q^s \times Q^u$ has a product structure defined by the two product foliations $\{R^s(y)\}$ and $\{R^u(y)\}$.

**Definition 1.57** A finite family of rectangles $\mathcal{R} = \{R_i = Q^s_i \times Q^u_i\}_{i=1}^r$ is called a *Markov partition* if the following conditions are satisfied.

1. $\mathcal{R}$ is a covering of $M$ and the interiors are mutually disjoint.
2. For any $i$ and $j$, $\mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j)$ is connected if it is nonempty.
3. For any point $x$ in $\mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j)$, $R^s_i(x)$ is mapped by $f$ into $R^s_j(f(x))$, and also $R^u_j(f(x))$ is mapped by $f^{-1}$ into $R^u_i(x)$.

The condition 3 is in fact stronger than it appears. For example, it implies that an unstable leaf $R^u(x)$ of $R_i$ is mapped by $f$ exactly onto a union of unstable leaves of some rectangles. This shows that a stable boundary component of $R_i$ is mapped by $f$ into a stable boundary component of some other rectangle.

When the dimension of $N$ is[6] 2, the construction of Markov partition is not so difficult. Choose a fixed point $a$. Consider long intervals $I^u$ and $I^s$ (containing $a$ in its interior) in the unstable and stable leaves through $a$. They have four endpoints. By adjusting these points, we got a partition of $N$ into rectangles. Since $a$ is a fixed point, $I^\sigma$ satisfies

$$f(I^u) \supset I^u, \quad f(I^s) \subset I^s.$$

This property implies that the partition is Markov.

But in general the existence of Markov partition is not so easy. We raise the following theorem due to R. Bowen [6] without proof. See also [40].

**Theorem 1.58** *Any Anosov diffeomorphism on a closed manifold admits a Markov partition of arbitrarily small size.* □

Now given a Markov partition $\mathcal{R} = \{R_i\}_{i=1}^r$, define a square matrix $A = \{a_{ij}\}$ of size $r$ by setting $a_{ij} = 1$ if $\mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j) \neq \emptyset$, and otherwise 0. $A$ is called the *transition matrix.* Call a sequece $\mathbf{i} = \{i_n\}$ *admissible* if it belongs to $\Sigma_A$. Notice that an admissible sequence yields a $2\epsilon$-pseudo orbit. (The size of the rectangles is smaller than $2\epsilon$.) The following theorem gives us the connection between Anosov systems and symbolic dynamics.

[6] Then $f$ is known to be topoigically conjugate to a hyperbolic automorphism of $T^2$.

**Theorem 1.59** *1. For any admissible sequence* $\mathbf{i} = \{i_n\}$*, there exists a unique point* $x \in N$ *such that* $f^n(x) \in R_{i_n}$ *for any* $n$.

*2. Conversely given any point* $x \in N$*, there exists a bounded number of admissible sequence* $\mathbf{i} = \{i_n\}$ *such that* $f^n(x) \in R_{i_n}$ *for any* $n$.

**Proof** Let $\mathbf{i} = \{i_n\}$ be an admissible sequence. Think of the following desreasing sequence

$$R_{i_0} \supset R_{i_0} \cap f^{-1}(R_{i_1}) \supset R_{i_0} \cap f^{-1}(R_{i_1}) \cap f^{-2}(R_{i_2}) \supset \cdots.$$

They are closed subsets of $R_{i_0}$, consisting of stable leaves in $R_{i_0}$ They become thiner and thiner. By the hyperbolicity of $f$, we have that $\cap_{n \geq 0} f^{-n}(R_{i_n})$ is a single stable leaf. Likewise $\cap_{n \leq 0} f^{-n}(R_{i_n})$ is a single unstable leaf. Therefore $\cap_{n \in \mathbf{Z}} f^{-n}(R_{i_n})$ is a single point. This shows (1).

(2) is immediate if the point $x$ lies in $J = N \setminus \cup_{i,n} f^{-n} \partial R_i$. Notice that in this case the admissible sequence realizing $x$ is unique because of 1 in Definition 1.57.

If not, approximate $x$ by points $x_k$ in $J$ in an appropriate way. Then the admissible sequences of $x_k$ converges to an admissible sequence $\mathbf{i} = \{i_n\}$ w. r. t. the topology of $\Sigma$. Then one can show that $f^n(x) \in R_{i_n}$ for any $n$. The details, especially the estimate of the number of the admissible sequences corresponding to $x$, is left to the reader. □

**Exercise 1.60** *Give a complete proof of the part (2). (One can very well assume that* $x$ *itself lies in* $\partial R_{i_0}$*. First consider the case where* $x$ *lies in* $\partial Q^s \times \mathrm{Int} Q^u$*. Approximate* $x$ *by the sequences in the stable leaf. Then consider the the opposite case, and finally the case where* $x$ *lies in* $\partial Q^s \times \partial Q^u$.

**Remark 1.61** *In 1, if the sequence* $\mathbf{i}$ *is cyclic, then the corresponding point* $x$ *is a periodic point. This follows directly from the uniqueness*

To sumerize the above theorem, we obtain the following corollary.

**Corollary 1.62** *Let* $\mathcal{R}$ *be a Markov partition and let* $A$ *be the corresponding transition matrix. Then there is a continuous surjection* $h : \Sigma_A \to N$ *such that* $h \circ \sigma = f \circ h$*. The inverse image of any point of* $N$ *has bounded cardinality.*

**Proof** What is left is to show the continuity of $h$, which is left to the reader.
□

**Proposition 1.63** *The subshift* $(\Sigma_A, \sigma)$ *associated to an Anosov diffeomorphism* $f$ *is reduced if and only if* $f$ *is transitive.*

**Proof** Suppose $f$ is transitive. Then there exists a dense orbit. This shows that for any $i$ and $j$, we have $\mathrm{Int}R_i \cap f^{-n}(\mathrm{Int}R_j) \neq \emptyset$, showing that the transition matrix is reduced.

On the contrary, if $(\Sigma_A, \sigma)$ is reduced, it admits a dense orbit (Exercise 1.55). Thus the diffeomorphism $f$ admits a dense orbit, and therefore it is transitive. □

# 2 Anosov flows

Here is a concept of Anosov system also for flows. It postulates the expansion-contraction decomposition of the normal bundle of the nonsingular vector field. In 2.1, we give the definition and show that the set of Anosov flows form an open set in $C^1$ topology. In 2.2, we study the elementary properties of (global) cross sections of the flow.

2.3 is devoted to the construction of examples of Anosov flows. First the suspension of an Anosov diffeomorphism is an Anosov flow. Another important example is the geodesic flow of a negatively curved manifold. Especially we study in detail the dynamics of the geodesic flow of surfaces with constant negative curvature. They are intimately connected with the Lie group $PSL(2, \mathbf{R})$.

An Anosov flow admits two types of foliations. One is the strong (un)stable foliation and the other the weak (un)stable foliation. Their fundamental properties are exposed in 2.4. In 2.5, the structural stability of Anosov flow is proved. The argument is more or less the same as in the case of diffeomophisms. But the POTP and the expansiveness becomes a bit complicated concept. In 2.6, we study the transitivity of Anosov flows. Finally in 2.7, a Markov partition of an Anosov flow is treated. It will play an important role in the last section in which we prove the Verjovsky conjecture.

## 2.1 Definition of Anosov flows

In this paragraph, we will give definitions and fundamental properties of Anosov flows. Let $M$ be a closed smooth manifold, endowed with some Riemannian metric, and $X$ a $C^r$ vector field on it. The flow induced by $X$ is denoted by $\phi = \{\phi^t\}$. $\phi$ is called *nonsingular* if $X$ is nonvanishing. In this case, we denote by $TX$ the one dimensional subbundle of the tangent bundle $TM$ of $M$, spanned by $X$. As a matter of fact, the derivative $D\phi^t$ preserves $TX$.

**Definition 2.1** A nonsingular flow $\phi$ is called an *Anosov flow* if there exists a continuous splitting $TM = TX \oplus E^u \oplus E^s$ such that

1. Both $E^u$ and $E^s$ are invariant by $D\phi^t$ for any $t$,

2. There exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D\phi^t(v)\| \leq C\lambda^t\|v\|, \quad \forall v \in E^s, \quad \forall t > 0$$
$$\|D\phi^{-t}(v)\| \leq C\lambda^t\|v\|, \quad \forall v \in E^u, \quad \forall t > 0.$$

As before $u$ and $s$ denote the dimension of the subbundles.

Apparently the above definition is stronger than that of the diffeomorphism case, because we postulate the existence of flow-invariant subbundles $E^\sigma$. For example, if we change the vector field by multiplying by a positive function, then the time parametrization of the flow is changed, although the 1 dimensional foliation it defines is the same. Notice that $E^\sigma$ is no longer invariant by the new flow. Thus it is not clear directly from the definition that the new flow is also an Anosov flow.

First of all let us show that the definition above is in fact a condition only for the transverse direction of the flow. The map $D\phi^t : TM \to TM$ induces a bundle map of the quotient budle $TM/TX$. By some abuse, we denote it also by $D\phi^t$. Also the metric induced on $TM/TX$ is denoted by $\|\cdot\|$.

**Proposition 2.2** *A flow $\phi$ is an Anosov flow if and only if there exists a continuous splitting $TM/TX = \widehat{E}^u \oplus \widehat{E}^s$ with the following properties.*

1. *Both $\widehat{E}^u$ and $\widehat{E}^s$ are invariant by $D\phi^t$ for any $t$.*

2. *There exist constants $C > 0$ and $0 < \lambda < 1$ such that*

$$\|D\phi^t(v)\| \leq C\lambda^t\|v\|, \quad \forall v \in \widehat{E}^s, \quad \forall t > 0,$$
$$\|D\phi^{-t}(v)\| \leq C\lambda^t\|v\|, \quad \forall v \in \widehat{E}^u, \quad \forall t > 0.$$

**Proof** Let us denote the canonical projection by $p : TM \to TM/TX$. Assume first that $\phi$ is an Anosov flow, with the splitting $TM = TX \oplus E^u \oplus E^s$. Define the splitting of $TM/TX$ by $\widehat{E}^\sigma = p(E^\sigma)$. Then the conditions of the proposition are obviously satisfied.

Next assume that $\phi$ satisfies the conditions of the proposition. Then $p^{-1}(\widehat{E}^u)$ is invariant by $D\phi^t$. Let us denote the restriction of $D\phi^t$ by

$$D^u\phi^t : p^{-1}(\widehat{E}^u) \to p^{-1}(\widehat{E}^u).$$

Our purpose is to find a $D\phi^t$-invariant subbundle $E^u$ of $p^{-1}(\widehat{E}^u)$ of dimension $u$ and transverse to $TX$. Denote by $F^u$ the orthogonal complement of $TX$ in the bundle $p^{-1}(\widehat{E}^u)$. Consider the homomorphism bundle $Homo(F^u, TX)$. Denote by $\Gamma(Homo(F^u, TX))$ the Banach space of the continuous cross sections of

$Homo(F^u, TX)$. Given $\gamma \in \Gamma(Homo(F^u, TX))$, $\gamma(x)$ is a homomorphism from $F^u_x \to TX_x$ for any point $x \in M$. Thus its graph is a $u$-dimensional subspace of $p^{-1}(\widehat{E}^u)_x$, transverse to $TX$. In this way $\gamma \in \Gamma(Homo(F^u, TX))$ defines a subbundle of $p^{-1}(\widehat{E}^u)$, denoted by $G(\gamma)$. Now define a map

$$\Gamma^u\phi^t : \Gamma(Homo(F^u, TX)) \to \Gamma(Homo(F^u, TX))$$

by

$$G(\Gamma^u\phi^t(\gamma)) = D^u\phi^t(G(\gamma)).$$

Then by the conditions of the propositions, one can show that $\Gamma^u\phi^\tau$ is a contraction for large $\tau > 0$. (The details are left to the readers.) Therefore it has a unique fixed point $\gamma_0$. The commutativity of $\Gamma^u\phi^\tau$ and $\Gamma^u\phi^t$, as well as the uniqueness of $\gamma_0$ shows that $\gamma_0$ is invariant by $\Gamma^u\phi^t$ for any $t$. Now $E^u = G(\gamma_0)$ is the desired subbundle. Likewise one can construct the subbundle $E^s$, showing the proposition. □

As a result of the previous proposition, if we change the time parametrization of an Anosov flow, the new flow is also Anosov. We also get the following proposition by the same argument as in Proposition 1.9. Denote by $\mathcal{X}^1(M)$ the space of the $C^1$ vector fields on $M$, equipped with the $C^1$ topology.

**Proposition 2.3** *The set of Anosov flows is an open subset of* $\mathcal{X}^1(M)$.

Before giving examples of Anosov flows, we prepare some fundamental concepts for flows in the next subsection.

## 2.2 Cross sections and suspensions

In this section, we consider a general nonsingular $C^r$ ($r \geq 1$) flow $\phi$ on a closed manifold $M$.

**Definition 2.4** A codimension one closed submanifold $N$ of $M$ is called a *cross section* if

1. the flow $\phi$ is transverse to $N$ and

2. any orbit intersects $N$.

**Proposition 2.5** *Suppose $N$ is a cross section for $\phi$. Then there exists $T > 0$ such that for any point $x \in M$, the point $\phi^t x$ lies in $N$ for some $0 \leq t \leq T$.*

To show this, let us generalize the concept of $\omega$-limit set for flows. For $x \in M$, the *$\omega$-limit set* $\omega(x)$ is the set of points $y$ with the following property; there exists a sequence of positive numbers $t_n \to \infty$ such that $\phi^{t_n}(x) \to y$.

The following properties of $\omega(x)$ are easy to establish.

**Proposition 2.6** *$\omega(x)$ is a nonempty closed subset of $M$, invariant by the flow $\phi$.* □

**Proof of Proposition 2.5.** First let us show that the forward orbit of any point $x$ intersects the cross section $N$. Choose an arbitrary point $y'$ from $\omega(x)$. By the definition of cross section, there exists a point $y$ in the orbit of $y'$ which lies in $N$. By Propositon 2.6, $y$ is a point of $\omega(x)$. That is, there exists a sequence of positive numbers $t_n$ such that $\phi^{t_n}x$ converges to $y$. Considering a flow box at $y$, one can show that a small orbit through $\phi^{t_n}x$ intersects $N$. Thus the forward orbit of $x$ intersects $N$.

Now for any point $x \in M$, define $\tau(x)$ to be the smallest positive number such that $\phi^{\tau(x)}x$ lies in $N$. Then the transversality of $N$ w. r. t. the flow implies that for any $\epsilon > 0$, there exists a neighbourhood $V$ of $x$ such that $\tau(z) < \tau(x) + \epsilon$ for any point $z \in V$. That is, $\tau : M \to \mathbf{R}_{>0}$ is upper semi-continuous and therefore it has a maximal value. This shows the proposition.

□

The following proposition follows from the transversality of $N$ w. r. t. the flow. The proof is omitted.

**Proposition 2.7** *The restriction of the function $\tau$ to $N$, also denoted by $\tau : N \to \mathbf{R}_{>0}$ is a $C^r$ map.* □

**Definition 2.8** The map $\tau : N \to \mathbf{R}_{>0}$ is called the *return time*, and the map $r : N \to N$ defined by $r(x) = \phi^{\tau(x)}x$ is called the *first return map* of the flow $\phi$.

Clearly $r$ is a $C^r$ diffeomorphism.

Conversely suppose that we are given a $C^r$ map $\tau : N \to \mathbf{R}_{>0}$ and a $C^r$ diffeomorphism $\tau : N \to N$. Then we can construct a flow $\psi$ as follows. First consider a subset $P'$ of $N \times \mathbf{R}$ defined by

$$P' = \{(x, t) \mid 0 \leq t \leq \tau(x)\}.$$

Define an equivalence relation $\sim$ by

$$(x, \tau(x)) \sim (r(x), 0).$$

Define a flow $\psi$ on the quotient manifold $P = P'/\sim$ by the vector field $\partial/\partial t$. This flow is called the suspension of $r$ w, r. t. $\tau$.

Now we state the following proposition without a proof.

**Proposition 2.9** *Let $N$ be a cross section for a flow $\phi$ on a closed manifold $M$, with the return time $\tau$ and the first return map $r$. Let $\psi$ (on $P$) be the suspension of $r$ w. r. t. $\tau$. Then the two flows $\phi$ and $\psi$ are conjugate, i. e. there exists a diffeomorphism $h : M \to P$ such that*

$$h(\phi^t x) = \psi^t h(x).$$

□

## 2.3 Examples of Anosov flows

In this section, we will give classical examples of Anosov flows. There are two types, as shown in what follows.

**Example 2.10** Let $f : N \to N$ be an Anosov diffeomorphism. Then the suspension of $f$ w. r. t. any return time is an Anosov flow.

The proof will be obvious by Proposition 2.2.

Let $V$ be an arbitrary closed Riemannian manifold and $M = T_1 V$ be its unit tangent bundle. A point of $M$ is denoted by $(x, v)$ where $x$ is a point of the manifold $V$ and $v$ is a unit tangent vector at $x$. Let us define a flow $\psi = \{\psi^t\}$ on $M$, called the *geodesic flow* of $V$. For any $(x, v)$, let $\gamma_{x,v}$ be the geodesic curve such that $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$. Now define $\psi$ by

$$\psi^t(x, v) = (\gamma_{x,v}(t), \gamma'_{x,v}(t)).$$

The following fact is well known.

**Example 2.11 (Anosov)** If a closed Riemannian manifold is negatively curved, then the corresponding geodesic flow is Anosov.

For the proof, see [1]. The simplest examples of negatively curved manifolds are surfaces of curvature $-1$. Instead of dealing with general manifolds, we shall expose this particular case in details. Let us consider the upper half plane

$$H = \{z = x + iy \in \mathbf{C} \mid y > 0\},$$

equipped with a metric $g_P$, defined by

$$g_P = \frac{dx^2 + dy^2}{y^2}.$$

$H$ is called the *Poincaré plane.*

**Exercise 2.12** *Show that the Gaussian curvature of $g_P$ is constantly equal to* $-1$.

**Exercise 2.13** *Show that the imaginary axis is a geodesic and that the distance of two points $ai$ and $bi$ $(0 < a < b)$ is equal to* $\log(b/a)$.

Now let us study the group of isometries of $H$. Let $SL(2, \mathbf{R})$ be the Lie group of the 2 by 2 real matrices with determinants 1. Then $SL(2, \mathbf{R})$ acts on $H$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

**Exercise 2.14** *Show that this defines an action on $H$ by isometries.*

This action has the kernel $\{\pm I\}$. The quotient group of $SL(2, \mathbf{R})$ by $\{\pm I\}$ is denoted by $PSL(2, \mathbf{R})$. (An element of $PSL(2, \mathbf{R})$ will be denoted by a matrix of $SL(2, \mathbf{R})$ which represent it.)

**Exercise 2.15**

1. *Show that the isotropy subgroup*[7] *at $i$ is isomorphic to $SO(2)$, by the map defined by taking the derivative at $i$.*
2. *Show that for any two points $z_1$ and $z_2$ of $H$, there exists an element of $PSL(2, \mathbf{R})$ which carries $z_1$ to $i$ and $z_2$ to a point on the imaginary axis.*

Now $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm I\}$ acts on the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

[7] the subgroup of $PSL(2, \mathbf{R})$ consisting of the elements which keep $i$ fixed

By Liouville's theorem, $PSL(2, \mathbf{C})$ coincides with the group of holomorphic automorphisms of $\hat{\mathbf{C}}$. Thus the action is conformal, i. e. the derivative preserves the angle.

In the next exercise, by *circles* in $\hat{\mathbf{C}}$ we mean not only usual metric circles in $\mathbf{C}$, but also straight lines in $\mathbf{C}$ plus $\infty$.

**Exercise 2.16** *Show that an element of $PSL(2, \mathbf{C})$ carries a circle to a circle.*

$PSL(2, \mathbf{R})$ is precisely the subgroup of $PSL(2, \mathbf{C})$ which keeps $H$ invariant.

We usually consider $H$ to be a subset of the Riemann sphere $\hat{\mathbf{C}}$. Thus the boundary of $H$ is isomorphic to a circle $\mathbf{R} \cup \{\infty\}$. We denote it by $S^1_\infty$ and call it the *circle at infinity.*

Also there exists an element $P$ of $PSL(2, \mathbf{C})$ which maps $H$ onto the unit disc $D = \{|z| < 1\}$. The action of $PSL(2, \mathbf{R})$ on $H$ is identified with the action of $P \cdot PSL(2, \mathbf{R}) \cdot P^{-1}$ on $D$. In what follows we rather think of $H$ as a unit disc (especially when we consider a picture).

**Proposition 2.17** *A complete geodesic of $H$ is the intersection with $H$ of a circle meeting $S^1_\infty$ perpendicularly.*

**Proof** Given arbitrary two points $z_1$ and $z_2$ of $H$, there exists an element $A$ of $PSL(2, \mathbf{R})$ which maps them to points $w_1$ and $w_2$ on the imaginary axis (Exercise 2.15).

It suffices to show the proposition for $w_1$ and $w_2$, because first of all $A^{-1}$ maps a geodesic to a geodesic, secondly it maps a circle to a circle, and finally as a transformation of $\hat{\mathbf{C}}$ it is conformal.

But the geodesic joining $w_1$ and $w_2$ is the imaginary axis by Exercise 2.13. This finishes the proof. □

Thus any complete geodesic in $H$ has two endpoints in $S^1_\infty$.

**Proposition 2.18** *The group $PSL(2, \mathbf{R})$ coincides with the group Isom$(H)$ of isometries of $H$.*

**Proof** We have already established in Exercise 2.14 that $PSL(2, \mathbf{R})$ is contained in *Isom*$(H)$. Let us show the converse. Let $h$ be an arbitrary isometry. By 2 of Exercise 2.15, there exists $A \in PSL(2, \mathbf{R})$ such that $A(i) = h(i)$. Then $A^{-1} \circ h$ keeps $i$ fixed. By 1 of Exercise 2.15, there exists an element $B \in PSL(2, \mathbf{R})$ keeping $i$ fixed such that the derivative at $i$ coincides with that of $A^{-1} \circ h$. Now $B^{-1} \circ A^{-1} \circ h$ keeps $i$ fixed and the derivative there is the identity. Since $B^{-1} \circ A^{-1} \circ h$ is an isometry, it must be the identity. Thus $h = AB$ belongs to $PSL(2, \mathbf{R})$. $\square$

The following theorem is well known. The proof is omitted.

**Theorem 2.19** *Any simply connected complete Riemannian 2-manifold of curvature constantly equal to -1 is isometric to the Poincaré plane.*

As a consequence, the univeral covering of any closed surface $\Sigma$ of curvature -1 is isometric to $H$. Therefore $\Sigma$ is isometric to a quotient of $H$ by the action of some subgroup $\Gamma$ of $PSL(2, \mathbf{R})$. Of course $\Gamma$ is isomorphic to the fundamental group of $\Sigma$.

Now let us study the action of a single element of $PSL(2, \mathbf{R})$ on $H \cup S^1_\infty$.

**Definition 2.20** A nontrivial element $A$ of $PSL(2, \mathbf{R})$ is called *hyperbolic, parabolic,* or *elliptic* if $|Tr(A)|$ is[8] greater than, equal to, or smaller than 1.

It is easy to show that any hyperbolic element is conjugate to $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ for some $t > 0$. Now the latter has two fixed points 0 and $\infty$ in $S^1_\infty$ and no others in $H \cup S^1_\infty$. The geodesic joining 0 and $\infty$ (the imaginary axis) is kept invariant and on it we obtain the translation by the distance $t$. Thus any hyperbolic element has the same behaviour, i. e. it has one repelling fixed point and one attracting fixed point on $S^1_\infty$, and is the translation on the geodesic joining them.

Any parabolic element is conjugate to $\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}$. Of course the action of the latter is the horizontal translation, keeping the horizontal lines invariant. It has only one fixed point $\infty$.

A circle in $H$ tangent to $S^1_\infty$ at a point $p$ is called a *horocycle* at $p$. (A horocycle for $p = \infty$ is a horizontal line.) A horocycle bounds a disc, called *horodisc*. The usual anti-clockwise orientation of $H$ yields an orientation of a horodisc. Then a prescribed orientation is obtained for each horocycle from the horodisc it bounds.

Now any parabolic element has a unique fixed point $p$ in the circle at infinity, and is a translation along horocycles at $p$. The translation length varies depending on the horocycles. If it is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then the

[8] The absolute value of the trace is well defined for an element of $PSL(2, \mathbf{R})$.

translation is positive w. r. t. the orientation that we described above. If it is conjugate to $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, then the translation is negative.

Any elliptic element is conjugate to $\begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$ for some $\theta$. The latter has a unique fixed point $i$ and the derivative at $i$ is the rotation by the angle $\theta$. Thus any elliptic element $A$ has a unique fixed point $x$, with the derivative at $x$ the rotation by $\theta$, and keeps invariant the equi-distance circles at $x$. As we mentioned earlier, there exists an element $P$ of $PSL(2,\mathbf{C})$ which maps $H$ to the unit disc $D$. One can choose $P$ so that $x$ is mapped to the origin. The conjugation of $A$ by $P$ is a rigid rotation. Therefore if the angle $\theta$ is rational, some iterate of $A$ is the identity, and if it is irrational, then any orbit is dense in an equi-distance circle. We call $A$ a *rational* or *irrational rotation* accordingly. Now we have finished the study of the action of a single element of $PSL(2,\mathbf{R})$. Next we shall investigate the action of a subgroup.

**Definition 2.21** A subgroup $\Gamma$ of $PSL(2,\mathbf{R})$ is said to act on $H$ *properly discontinuously* if for any compact subset $K$ of $H$, there exist at most finite number of elements $\gamma$ of $\Gamma$ such that $\gamma(K) \cap K \neq \emptyset$.

We have the following proposition.

**Proposition 2.22** *A subgroup $\Gamma$ of $PSL(2,\mathbf{R})$ acts on $H$ properly discontinuously if and only if $\Gamma$ is a discrete subgroup of $PSL(2,\mathbf{R})$.*

**Proof** It is obvious that if $\Gamma$ is not discrete, it cannot act on $H$ properly discontinuously. To show the converse, notice that the map

$$p : PSL(2,\mathbf{R}) \to H$$

defined by $p(A) = A \cdot i$ is a bundle map with fiber $SO(2)$. Thus if $K$ is a compact subset of $H$, then $p^{-1}(K)$ is compact, and therefore if $\Gamma$ is discrete, it has the property that $\gamma(p^{-1}(K)) \cap p^{-1}(K) \neq \emptyset$ for but finitely many $\gamma \in \Gamma$. Then $\gamma(K) \cap K \neq \emptyset$ for but finite $\gamma$'s, showing that the action is properly discontinuos. □

**Definition 2.23** A discrete subgroup of $PSL(2,\mathbf{R})$ is called a *Fuchsian group*. A Fuchsian group $\Gamma$ is called *cocompact* if the quotient space $\Gamma \backslash H$ is compact.[9] [10]

[9]The quotient may not be a smooth manifold. It can be a so called orbifold.

[10]In fact, a Fuchsian group is cocompact if and only if the quotient $\Gamma \backslash PSL(2,\mathbf{R})$ is compact.

**Proposition 2.24** *1. A Fuchsian group acts on $H$ freely if and only if $H$ is torsion free.*

*2. A cocompact Fuchsian group cannot contain a parabolic element.*

**Proof 1.** A Fuchsian group $\Gamma$ cannot contain an irrational rotation, since then $\Gamma$ would not be discrete. Therefore elements of $\Gamma$ which have fixed points in $H$ must be rational rotations. But they are precisely elements of finite orfers. 1 follows from this.
**2.** For simplicity, let us consider only the case where the quotient $\Gamma \setminus H$ is a manifold. Then there exists a positive number $\epsilon$ such that any point of $\Gamma \setminus H$ has an embedded $\epsilon$-disk neighbourhood.[11] Consequently any homotopically nontrivial loop in $\Gamma \setminus H$ must have arc length greater than $2\epsilon$.

Suppose that $\Gamma$ contains a parabolic element $\gamma$. One may very well assume that $\gamma$ coincides with $\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}$. Then on the horocycle $\mathrm{Im}(z) = C$, the translation length is $1/C$. This would yield a loop of arbitrarily small length which is homotopically nontrivial. A contradiction. □

**Definition 2.25** A cocompact Fuchsian group without torsion is called a *surface group.*

Notice that a surface group $\Gamma$ is *purely hyperbolic*, i. e. consisting solely of hyperbolic elements and the identity. Also the action of $\Gamma$ on $H$ is free and properly discontinuous, and therefore the quotient space $\Gamma \setminus H$ is a closed surface. The canonical projection $p : H \to \Gamma \setminus H$ is the universal covering map.

Let us expose a typical way of producing examples of surface groups. Consider a regular octagon in $H$ whose edges are geodesics of the same length. If the octagon is small enough, then it looks like an octagon in the Euclidian plane, and the angle of each vertice is near $3\pi/5$. On the other hand, if it is large enough, then the angle is extremely small. Therefore there exists a regular octagon $P$ whose angle is just $\pi/4$.

Choose such an octagon $P$ and name the edges, in cyclic order, $a_1'$, $b_1$, $a_1$, $b_1'$, $a_2'$, $b_2$, $a_2$ and $b_2'$.

One can show that there exists a unique transformation $\alpha_i$ of $PSL(2, \mathbf{R})$ which sends the edge $a_i$ to $a_i'$ and the interior of $P$ outside $P$. Likewise we have a transformation $\beta_i$ sending $b_i$ to $b_i'$ and the interior of $P$ outside.

[11] Such number $\epsilon$ is called an *injectivity radius.*

Choose the vertex $v$ between the edge $a_1'$ and $b_2'$. Consider the image of $v$ by $\alpha_1^{-1}$. It is a vertex between the edges $a_1$ and $b_1'$. Notice that $\beta_1^{-1}$ is the unique transformation which sends $\alpha_1^{-1}(v)$ to a vertex of $P$ without returnig to $v$. Next consider $\beta_1^{-1}\alpha_1^{-1}(v)$. It is between $b_1$ and $a_1$, and $\alpha_1$ is the only transformation mapping it to a new vertex of $P$. Consider $\alpha_1\beta_1^{-1}\alpha_1^{-1}(v)$. $\cdots\cdots$

This way $v$ travels around the vertices of $P$, and returns to $v$ after visiting all the vertices of $P$. One can assure that

$$\beta_2\alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}(v) = [\beta_2,\alpha_2][\beta_1,\alpha_1](v) = v.$$

In a dual way, one can consider the images of $P$ *around the fixed vertex* $v$. First of all, $\beta_2(P)$ is adjacent to $P$, with common edge $b_2'$. Now what is the other edge of $\beta_2(P)$ which has $v$ as an endpoint? The answer is $\beta_2(a_2')$. Consider $a_2'$. What image of $P$ is on the opposite side of $a_2'$? $\alpha_2(P)$ is. This shows that $\beta_2\alpha_2(P)$ has $v$ as a vertex and is next to $\beta_2(P)$. This way one can show that

$$P,\ \beta_2(P),\ \beta_2\alpha_2(P),\ \beta_2\alpha_2\beta_2^{-1}(P),\cdots\cdots,\beta_2\alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1\alpha_1\beta_1^{-1}(P)$$

gather around $v$ in this order. Finally since the total sum of the angles of $P$ is $2\pi$, we have $[\beta_2,\alpha_2][\beta_1,\alpha_1](P) = P$. That is,

$$[\beta_2,\alpha_2][\beta_1,\alpha_1] = id. \tag{25}$$

Let $\Gamma$ be the subgroup of $PSL(2,\mathbf{R})$ gererated by $\alpha_i$ and $\beta_i$. Poincaré Polygon Theorem asserts that all the relations of $\Gamma$ are consequences of (25), and $P$ is the fundamental domain for $\Gamma$.

The outline of the proof is as follows. Consider an abstract group

$$\widehat{\Gamma} = <\alpha_i,\beta_i \mid [\beta_2,\alpha_2][\beta_1,\alpha_1] = id>$$

and $\widehat{\Gamma}\times P$.

Define an equivalence relation $\sim$ on $\widehat{\Gamma}\times\partial P$ by setting

$$(\gamma,x)\sim(\gamma',x') \text{ if } \gamma^{-1}\gamma' = \alpha_i^{\pm 1},\ \beta_i^{\pm 1} \text{ and } \gamma(x) = \gamma'(x').$$

Let

$$\widehat{H} = \widehat{\Gamma}\times P/\sim.$$

($\widehat{H}$ is a model space of $H$ constructed by the tesselation by $\{\gamma P \mid \gamma\in\Gamma\}$, with $\gamma\times P$ corresponding to $\gamma P$ in $H$.)

We leave to the reader the verification of an important fact that $\widehat{H}$ is a complete surface.

Now the group $\widehat{\Gamma}$ acts on $\widehat{H}$ by $\gamma(\gamma', x) = (\gamma\gamma', x)$ and we have the following equivariant pair of maps

$$(q, Q) : (\widehat{\Gamma}, \widehat{H}) \longrightarrow (\Gamma, H),$$

where $q : \widehat{\Gamma} \to \Gamma$ is the canonical map and $Q(\gamma, x) = \gamma x$. Now clearly $Q$ is a submersion. Furthermore since $\widehat{H}$ is complete and $Q$ is a local isometry, $Q$ is a proper map[12], and thus a covering map. Since $H$ is simply connected, $Q$ is a homeomorphism. From this follows that $q$ is an isomorphism. This finishes the proof of the theorem.

The theorem implies that the group $\Gamma$ is properly discontinuous and it is easy to show that the quotient $\Gamma \setminus H$ is the closed oriented surface of genus 2.

Careful reader may notice that there is no need to start with a regular octagon. In fact, by considering an octagon, with the corresponding edges the same length and the total angle of vertices $2\pi$, we get the same conclusion.

In the same way we can construct a surface group realizing a surface of higher genus.

By taking the derivative of the isometric action of $PSL(2, \mathbf{R})$ on $H$, we obtain an action of $PSL(2, \mathbf{R})$ on the unit tangent bundle $T_1H$. What we have shown in Exercise 2.15 is rephrased as follows.

**Proposition 2.26** *$PSL(2, \mathbf{R})$ acts on $T_1H$ freely and transitively, i. e. for any points $(z, v)$ and $(z', v')$ of $T_1H$, there exists a unique element of $PSL(2, \mathbf{R})$ sending $(z, v)$ to $(z', v')$.*

Now let us consider the geodesic flow $\psi$ on $T_1H$. Choose a base point $(i, v_\infty) \in T_1H$, where $v_\infty$ is the unit tangent vector at $i$ tending toward $\infty$ (upward, parallel to the imaginary axis).

By Exercise 2.13 we have $\psi^t(i, v_\infty) = (e^t i, v_\infty)$. Computation shows

$$\psi^t(i, v_\infty) = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} (i, v_\infty).$$

Clearly the geodesic flow must commute with the action of $PSL(2, \mathbf{R})$, which is the group of isometries. That is, we have for $A \in PSL(2, \mathbf{R})$,

$$\psi^t(A(i, v_\infty)) = A \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} (i, v_\infty).$$

[12]The inverse image of a compact is compact.

Define a map

$$\alpha : PSL(2, \mathbf{R}) \to T_1 H$$

by $\alpha(A) = A(i, v_\infty)$. By Proposition 2.26, $\alpha$ is a diffeomorphism. The group $PSL(2, \mathbf{R})$ acts on itself as the left translations and on $T_1 H$ as mentioned above. The diffeomorphism $\alpha$ is equivariant w. r. t. these actions.

Let us identify $T_1 H$ with $PSL(2, \mathbf{R})$ by the diffeomorphism $\alpha$. Then the flow $\psi$, viewed to be on $PSL(2, \mathbf{R})$, is described as

$$\psi^t(A) = A \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix},$$

i. e. given by the *right* translation by $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$.

The space of left invariant vector fields of $PSL(2, \mathbf{R})$ forms a Lie algebra $sl(2, \mathbf{R})$ of the 2 by 2 traceless matrices. Let us set

$$\begin{aligned} X &= \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \\ U &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ S &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

They form a basis of $sl(2, \mathbf{R})$, and satisfy

$$[X, U] = -U, \quad [X.S] = S, \quad [U, S] = -2X. \tag{26}$$

Recall that a *left* invariant vector field induces a flow by the *right* translation by some element.

Now $X$ induces a flow by the right translation by $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$, i. e. the geodesic flow $\psi$. $U$ induces a flow $v = \{v^s\}$ by the right translation by $\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$, and $S$ a flow $\sigma = \{\sigma^s\}$ by $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$.

From equations (26) follows the following;

$$\psi^t \circ v^s = v^{se^t} \circ \psi^t \tag{27}$$

$$\psi^t \circ \sigma^s = \sigma^{se^{-t}} \circ \psi^t \tag{28}$$

Of course one can deduce these relations directly from computation in $PSL(2, \mathbf{R})$. The equation (27) implies that $\psi^t$ preserves a foliation of $v$ and stretchs it $e^t$ times if $t > 0$. Likewise $\psi^t$ shrinks $\sigma$. (Notice that $[X, U] = -U$ implies that the flow of $X$ *streches* the flow of $U$.)[13]

Therefore if we consider the left invariant metric on $PSL(2, \mathbf{R})$ for which the three tangent vectors $X_A$, $U_A$ and $S_A$ form an orthonormal basis of the tangent space at $A \in PSL(2, \mathbf{R})$, then the flow $\psi$ is an Anosov flow of an open manifold $PSL(2, \mathbf{R})$ w. r. t. this metric.

Now assume $\Gamma$ is a cocompact Fuchsian group. Notice that the flows $\psi$, $v$ and $\sigma$, being given by the right translations, commute with the left action by $\Gamma$. Thus they can be pushed down to flows of the quotient $\Gamma \setminus PSL(2, \mathbf{R})$, which we also denote by the same letters. (Notice also the left invariant metric of $PSL(2, \mathbf{R})$ can be pushed down.) Thus $\psi$ is an Anosov flow on a closed manifold $\Gamma \setminus PSL(2, \mathbf{R})$, with $v$ and $\sigma$ giving unstable and stable directions.

Suppose further that $\Gamma$ is a surface group. (Recall then the quotient $\Sigma = \Gamma \setminus H$ is a closed surface.) The identification $\alpha : PSL(2, \mathbf{R}) \to T_1 H$ induces the following diffeomorphism;

$$\overline{\alpha} : \Gamma \setminus PSL(2, \mathbf{R}) \to T_1 \Sigma.$$

Clearly by this identification the geodesic flow of $\Sigma$ coincides with the flow $\psi$.

When the group $\Gamma$ is a cocompact Fuchsian group, but with a torsion, the quotient $\Gamma \setminus H$ is no longer a manifold, but the quotient $\Gamma \setminus PSL(2, \mathbf{R})$ is still a manifold. The push down $\psi$ is an Anosov flow on it. This is a slight generalization of the geodesic flow of a closed surface.

The plane field spanned by $X$ and $S$ is integrable and defines a foliation $\mathcal{V}^s$. Let us study about this foliation. (The foliation $\mathcal{V}^u$ defined by $X$ and $U$ can be dealt with in the same way. So we shall concentrate on $\mathcal{V}^s$.)

The elements $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ and $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ generate a subgroup

$$Aff = \{ \begin{bmatrix} e^x & y \\ 0 & e^{-x} \end{bmatrix} \mid x, y \in \mathbf{R} \}.$$

Clearly the leaf of $\mathcal{V}^s$ through an element $A$ of $PSL(2, \mathbf{R})$ is $A \cdot Aff$.

**Exercise 2.27** *Show that the subgroup Aff is isomorphic to the group of orientation preserving affine transformations on the real line.*

[13] Recall the definition of Lie derivative.

Also notice the following easy facts.

**Proposition 2.28** *1. Aff is precisely the subgroup which keeps $\infty$ fixed.*

*2. For any two points $z$ and $z'$ of $H$, there exists a unique element of Aff which sends $z$ to $z'$.* □

**Corollary 2.29** *Given a leaf $V$ of the weak stable foliation $\mathcal{V}^s$ of $T_1H$, there exists a point $a$ of $S^1_\infty$ such that $V$ is the set of points $(x, v)$ with the property that $a$ is the terminal point of the geodesic tangent to $(x, v)$.*

**Proof** We only need to consider the case where the leaf $V$ passes through the point $(i, v_\infty)$. (The general case follows from the fact that any other leaf is the image of $V$ by the action of an element of $PSL(2, \mathbf{R})$.) $V$ is exactly the orbit through $(i, v_\infty)$ of the action of *Aff* on $T_1H$. (This is true only for the base vector $(i, v_\infty)$.) Notice that the terminal point of the geodesic tangent to $(i, v_\infty)$ is $\infty$. Therefore it follows from the previous proposition that $V$ is the set of points $(x, v)$ such that the terminal point of the geodesic tangent to $(x, v)$ is $\infty$.

□

By this corollary, we get a one to one correspondence between the leaves of $\mathcal{V}^s$ and the points of $S^1_\infty$.

Suppose $\Gamma$ is a cocompact Fuchsian group. Then the foliation $\mathcal{V}^u$ gives birth to a foliation on the quotient space $\Gamma \setminus PSL(2, \mathbf{R})$. denoted by the same letter. This is the so called weak stable foliation associated to the Anosov flow $\psi$, which we will study in the next subsection.

Let us describe a useful way of studying the dynamics of the foliation $\mathcal{V}^s$. First of all define the mapping

$$\beta : PSL(2, \mathbf{R}) \to S^1_\infty \times H$$

by $\beta(A) = (A \cdot \infty, A \cdot i)$. The Fuchsian group $\Gamma$ acts on $S^1_\infty \times H$ diagonally, and on $PSL(2, \mathbf{R})$ from the left. Clearly $\beta$ is a $\Gamma$-equivariant diffeomorphism.

Now recall that a leaf of $\mathcal{V}^s$ in $PSL(2, \mathbf{R})$ is an orbit of the right action of the subgroup *Aff*, that is, of the form $B \cdot$ *Aff* for some $B \in PSL(2, \mathbf{R})$. The subgroup *Aff* keeps $\infty$ fixed. Therefore $\beta$ maps the leaf $B \cdot$ *Aff* to $\{B \cdot \infty\} \times H$. That is, the foliation $\mathcal{V}^s$ is transformed by $\beta$ to a trivial foliation $\mathcal{F}_0 = \{y \times H \mid y \in S^1_\infty\}$.

Again let $\Gamma$ be a cocompact Fuchsian group. $\beta$ induces a diffeomporphism

$$\overline{\beta} : \Gamma \setminus PSL(2, \mathbf{R}) \to \Gamma \setminus (S^1_\infty \times H).$$

The foliation $\mathcal{V}^u$ on $\Gamma \setminus PSL(2, \mathbf{R})$ is identified with the quotient of the trivial foliation $\mathcal{F}_0$ of $S^1_\infty \times H$ by the diagonal action of $\Gamma$. (If the group $\Gamma$ is a surface group, this gives an example of foliated $S^1$ bundles of Example 1.21.)

The transverse dynamics of the foliation $\mathcal{V}^u$ on $\Gamma \setminus PSL(2, \mathbf{R})$ is just the reflexion of the dynamics of the action of the Fuchsian group $\Gamma$ on $S^1_\infty$.

For example, as is well known, a surface group $\Gamma$ has the property that all the $\Gamma$-orbits are dense in $S^1_\infty$. This immediately implies that all the leaves of $\mathcal{V}^s$ are dense.

Now let us study a bit more on a picture of the flows $\psi$ and $\sigma$ on the Poincaré plane $H$.

We say that $(x, v) \in T_1H$ is *positively asymptotic to* a horocycle $C$ if $x$ is on $C$ and $v$ is perpendicular to $C$ pointing inwards. A unit vector $(x, v)$ is positively asymtotic to a unique horocycle. If $(x, v)$ and $(y, w)$ is positively asymtotic to the same horocycle, then

$$d(\psi^t(x, v), \psi^t(y, w)) \to 0 \quad (t \to \infty).$$

Especially the two geodesics tangent to $(x, v)$ and $(y.w)$ have the same terminal point.

**Proposition 2.30** *The flow $\sigma^s$ sends a vector $(x, v)$ to a vector $(y, w)$ positively asymptotic to the same horocycle, with the arc length[14] of the segment of the horocycle from $x$ to $y$ equal to $|s|$, and with the direction from $x$ to $y$ positive (resp. negative) if $s$ is positive (resp. negative).*

**Proof** Recall that on $PSL(2, \mathbf{R})$, the flow $\sigma$ is given by the right translation of the matrix $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, and the identification $\alpha : PSL(2, \mathbf{R}) \to T_1H$ by

$$\alpha(A) = A(i, v_\infty).$$

Thus on $T_1H$, the flow $\sigma$ is given by

$$\sigma^s(x, v) = P \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot (i, v_\infty),$$

[14]w. r. t. the metric $g_P$

for a matrix $P$ such that $P(i, v_\infty) = (x, v)$.

Since the flow $\sigma$ commutes with the action of $PSL(2.\mathbf{R})$, it suffices to show the lemma for a single prescribed vector $(x, v)$. So let us take $(x, v) = (i, v_\infty)$. Then

$$\sigma^s(i, v_\infty) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} (i, v_\infty) = (i + s, v_\infty).$$

The arc length of the segment from $i$ to $i+s$, of the horocycle $\{\mathrm{Im}(z) = 1\}$ is $|s|$ and the direction from $i$ to $i + s$ is just as claimed in the proposition. This completes the proof. □

The flow $\sigma$ (and also $v$) is classically called the *horocycle flow*. The picture of the flow $v$ is similar. We need a concept *negatively asymptotic*. Details are left to the reader.

## 2.4 Foliations associated with Anosov flows

In this paragraph, $\phi$ is to be a $C^r$ $(r \geq 1)$ Anosov flow on a closed manifold $M$. The $\phi$-orbit through a point $x \in M$, $\{\phi^t(x) \mid t \in \mathbf{R}\}$ is denoted by $O(x)$.

First of all we expose the integrability result, without proof.

**Theorem 2.31** *The bundle $E^\sigma$ is integrable. The induced foliation is a continuous foliation by $C^r$ leaves. The leaves are diffeomorphic to $\mathbf{R}^\sigma$.*

*For $\sigma = u$, two points $x$ and $y$ belong to the same leaf if and only if*

$$d(\phi^{-t}x, \phi^{-t}y) \to 0 \quad as \quad t \to \infty.$$

*Likewise for $\sigma = s$, $x$ and $y$ belong to the same leaf if and only if*

$$d(\phi^{t}x, \phi^{t}y) \to 0 \quad as \quad t \to \infty.$$

□

**Definition 2.32** Denote by $\mathcal{W}^\sigma$ the foliation given by $E^\sigma$, and call it the *strong unstable foliation* or *strong stable foliation* of $\phi$ according as $\sigma = u$ or $\sigma = s$.

The leaf of $\mathcal{W}^\sigma$ through a point $x \in M$ is denoted by $W^\sigma_x$.

The following property is immediate from the $\phi^t$-invariance of the splitting $TM = TX \oplus E^u \oplus E^s$.

**Proposition 2.33** *For $t \in \mathbf{R}$, the diffeomorphism $\phi^t$ sends a leaf of $\mathcal{W}^\sigma$ to another leaf.* □

In particular, if $U$ is an small open ball in a leaf of $\mathcal{W}^\sigma$ and if $\epsilon > 0$ is small, then the set

$$\{\phi^t(x) \mid x \in U, \ -\epsilon < t < \epsilon\}$$

is a $C^r$ submanifold in $M$ of dimension $\sigma + 1$. Thus we have;

**Corollary 2.34** *The bundle $TX \oplus E^\sigma$ is integrable. The foliation is a continuous foliation by $C^r$-leaves.* □

**Definition 2.35** Denote by $\mathcal{V}^\sigma$ the foliation associated with $TX \oplus E^\sigma$, and call it the *weak unstable foliation* or *weak stable foliation* of $\phi$ respectively.

The leaf of $\mathcal{V}^\sigma$ through a point $x \in M$ is denoted by $V_x^\sigma$.

Notice that the subbundle $E^\sigma$ depends on the time parametrization of the flow. Therefore the strong foliation $\mathcal{W}^\sigma$ also depends. But as we have shown in the beginning of this sectin, the subbundle $TX \oplus E^\sigma$ does not depend on the time parametrizaton. Therefore the weak foliation does not depend.

A leaf $V$ of $\mathcal{V}^\sigma$ is foliated both by $\mathcal{W}^\sigma |_V$ and by the flow $\phi$.

**Lemma 2.36** *Let $V$ be a leaf of $\mathcal{V}^\sigma$. Given any two points $x$ and $y$ of $V$, the $\phi$-orbit $O(x)$ and the $\mathcal{W}^\sigma$-leaf $W_y^\sigma$ intersect.*

**Proof** An arc in $V$ joining $x$ and $y$ can be approximated by a composite

$$\alpha_1\beta_2\alpha_2\beta_2 \cdot \cdots \cdots \alpha_r\beta_r,$$

where $\alpha_i$ is an arc in a $\phi$-orbit, $\beta_i$ in a $\mathcal{W}^\sigma$-leaf. Let $p_i$ (resp. $q_i$) be the initial (resp. terminal) point of $\alpha_i$. (We have $p_1 = x$.)

The proof is by induction on $r$. If $r = 1$, then the proposition is trivial. So assume $r > 1$. By the induction hypothesis, there exists a point of intersection $z$ of $O(p_2)$ and $W_y^\sigma$. Since $p_2$ and $z$ lies on the same orbit, we have $z = \phi^t(p_2)$ for some $t$. Now $\phi^t$ sends $W_{p_2}^\sigma$ to $W_y^\sigma$. By the definition the point $q_1$ lies both on $O(x)$ and on $W_{p_2}^\sigma$. Therefore the point $\phi^t(q_1)$ lies both on $O(x)$ and $W_y^\sigma$. □

**Proposition 2.37** *Let $V$ be a leaf of $\mathcal{V}^\sigma$.*

1. *If $V$ does not contain a periodic orbit, then any $\phi$–orbit and $\mathcal{W}^\sigma$-leaf in $V$ intersect at exactly one point. The flow $\phi$ and the foliation $\mathcal{W}^\sigma|_V$ gives a product structure of $V$. Especially $V$ is diffeomorphic to $\mathbf{R}^{\sigma+1}$.*

2. *If $V$ contains a periodic orbit $O(p)$, then for any point $x \in V$, $W_x^\sigma$ meets $O(p)$ exactly at one point, say $g(x)$. The mapping $g : V \to O(p)$ is a bundle map with fiber diffeomorphic to $\mathbf{R}^\sigma$. Especially $V$ is diffeomorphic to $S^1 \times \mathbf{R}^\sigma$.*

**Proof** We shall prove the proposition only for $\sigma = s$.

**1.** All that needs proof is that any $\phi$–orbit and $\mathcal{V}^s$-leaf, say $W$, in $V$ intersect at a unique point.

Let $z$ and $z'$ be distinct points of intersection. Since they lie on the same $\phi$-orbit, $z' = \phi^t(z)$ for some $t$. Then $\phi^t$ sends $W$ onto itself. By the definition of Anosov flows, $\phi^{mt}|_W \colon W \to W$ is a contraction for some large integer $m$. (We are considering a distance function on $W$ given by the Riemannian metric induced from $M$.) Therefore $\phi^{mt}|_W$ has a fixed point. This gives birth to a periodic orbit of $\phi$ in $V$, contradicting the assumption.

**2.** Again we only need to show that any $\mathcal{W}^s$-leaf $W$ has a unique point of intersection with $O(p)$. Suppose there exist two, say $p$ and $q$. Let $T$ be the period of $p$ (and of course of $q$). Then by Theorem 2.31, we have

$$d(p,q) = d(\phi^{nT}(p), \phi^{nT}(q)) \to 0 \quad (n \to \infty).$$

Thus we have $p = q$. □

## 2.5 Structural Stability

Let $\mathcal{X}^1(M)$ be the space of $C^1$ vextor fields on $M$, equipped with the $C^1$ topology.

**Theorem 2.38** *An Anosov vector field $X$ is structurally stable, i. e. for any $\epsilon > 0$, there exists a neighbourhood $\mathcal{U}$ of $X$ in $\mathcal{X}^1(M)$ such that for any $Y$ in $\mathcal{U}$, there exists a homeomorphism of $M$ to itself, $\epsilon$-near to the identity, sending an orbit of $Y$ to an orbit of $X$.*

The proof of this theorem is considerably complicated, but certainly parallel with the case of diffeomorphisms. Below we simply mention those points which need modifications. Again $\phi$ is to be an Anosov flow on $M$.

**Definition 2.39** Let $\delta$ and $T$ be positive numbers. Let $\{t_i\}$ be a bi-infinite sequence of numbers greater than $T$. A bi-infinite sequence $\{x_i\}$ of points of $M$ is called a *$(\delta, T)$-pseudo orbit with time sequence* $\{t_i\}$ if $d(\phi^{t_i}(x_i), x_{i+1}) < \delta$ for any $i$.

**Definition 2.40** A bifinite sequence $\{x_i\}$ with the time sequence $\{t_i\}$ is said to be *$\epsilon$-traced by $x$,* if there exists an orientation preserving homeomorphism $g$ of $\mathbf{R}$ onto itself such that $g(0) = 0$ and that $d(\phi^{g(t+T_i)}(x), \phi^t(x_i)) < \epsilon$ for any $i$ and $0 \le t \le t_i$ where

$$T_i = \begin{cases} \sum_{j=0}^{i-1} t_j & \text{if } \ i > 0, \\ 0 & \text{if } \ i = 0, \\ -\sum_{j=i}^{-1} t_j & \text{if } \ i < 0. \end{cases}$$

**Definition 2.41** The flow $\phi$ is said to satisfy *Pseudo Orbit Tracing Property* (POTP) if for any $\epsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that any $(\delta, T)$-pseudo orbit is $\epsilon$-traced by some point.

**Lemma 2.42** *An Anosov flow satisfies POTP.*

**Proof** Given $\epsilon$ and $T$ of Definition 2.41, our purpose is to find out $\delta$. Of course we may assume $\epsilon$ is as small as we want. First of all given small $\epsilon$, we shall construct a field of rectangles of size $\epsilon$, just as we have done for diffeomorphisms. But for flows, the rectangles are of codimension one and transverse to the flow.

Let $R(\epsilon)$ be the rectangle in $\mathbf{R}^u \times \mathbf{R}^s$, centered at the origin and of size $\epsilon$. Choose a submersion $\pi : N \times R(\epsilon) \to M$ with the following properties.

1. $\pi(x, 0) = x$.
2. The image $R_x(\epsilon) = \pi(x \times R(\epsilon))$ is transverse to the flow and of size $\epsilon$.
3. The restriction of the foliation $\mathcal{V}^u$ (resp. $\mathcal{V}^s$) to $R_x(\epsilon)$ gives the horizontal (resp. vertical) foliation.

If $\epsilon$ is sufficiently small, then one can construct such a submersion $\pi$. We do not assume that the horizontal or vertical lines in the rectangles lie in a leaf of the *strong* (un)stable foliation. (This is impossible in general.)

For $0 < \epsilon' < \epsilon$, denote by $R_x(\epsilon')$ a subrectangle of $R_x(\epsilon)$ centered at $x$ and of size $\epsilon'$. Choose a small number $\epsilon' > 0$ with the following property; if $y \in R_x(\epsilon')$, then there exists a strictly increasing continuous function $b : [0, 2T] \to \mathbf{R}$ such that $b(0) = 0$ and $\phi^{b(t)}(y)$ lies on $R_{\phi^t(x)}(\epsilon)$ for $0 < t < 2T$. Below we consider a field of rectangles of size $\epsilon'$.

Now choose $\delta > 0$ such that if the center of some rectangle $R_x(\epsilon')$ is mapped by $\phi^t$ for some $t$ with $T < |t| < 2T$, into a point in $R_y(\epsilon')$ which is $\delta$-near to $y$, then the transition map along the orbit has a continuous extension from a part of $R_x(\epsilon')$ onto a part of $R_y(\epsilon')$ with the analogous property as in Sublemma 1.40.

Let $\{x_i\}$ be an arbitrary $(\delta, T)$-pseudo orbit with the time sequence $\{t_i\}$. One may assume that $T < t_i < 2T$ for any $i$. (If some $t_i$ is too large, then pick up some points from the orbit segment $\{\phi^t(x_i) \mid 0 < t < t_i\}$, and add them to the sequence $\{x_i\}$.)

Just as in the diffeomorphism case, one can find a point $x$ in $R_{x_0}(\epsilon')$ such that the orbit $O(x)$ visit every rectangle $R_{x_i}(\epsilon')$ in a right way, at a point, say $z_i$. By the choice of $\epsilon'$, if $0 < t < t_i$, then $\phi^{b_i(t)}(z_i)$ lies on $R_{\phi^t(x_i)}(\epsilon)$ for some continuous fuction $b_i : [0, t_i] \to \mathbf{R}$. The function $g$ of Definition 2.40 can be defined using the functions $b_i$'s in an obvious way. The lemma is proved. The details are left to the reader. □

**Definition 2.43** A flow $\phi$ is said to be *expansive* if there exists a positive constant $e$, called the *expansive constant*, with the following property; given two points $x$ and $y$, if there exists an orientation preserving homeomorphism $g$ of $\mathbf{R}$ onto itself such that $g(0) = 0$ and that $d(\phi^t(x), \phi^{g(t)}(y)) < e$, then $x$ and $y$ lie on the same orbit and the length of the segment of the orbit joining them is smaller than $2e$.

The proof of the following lemma is analogous to the case of diffeomorphisms and is omitted.

**Lemma 2.44** *For any sufficiently small $e > 0$, there exists a $C^1$ neighbourhood $\mathcal{U}$ of $\phi$ such that any flow in $\mathcal{U}$ is expansive with expansive constant $e$.*
□

**Outline of the proof of Structural Stability Theorem** A purturbed flow $\psi$ yields a pseudo orbit of the original flow $\phi$. Precisely, given a point $x \in M$, a sequence $\{\psi^i(x) \mid i \in \mathbf{Z}\}$ is a $(\delta, 1/2)$-pseudo orbit of $\phi$. Therefore by Lemma 2.42, there are a set $S(x)$ of points which $\epsilon$-traces $\{\psi^i(x)\}$. Now by Lemma 2.44, the set $S(x)$ is contained in a segment of the orbit of $\phi$ of length smaller than $2e$. The expansive constant $e$ can be chosen arbitrarily small. Thus in an appropriate flow box $B$ of $\phi$ centered at $x$, the segment $S(x)$ is contained in a single orbit $H(x)$ of $\phi|_B$. Choosing the point $h(x)$ in $H(x)$ nearest to $x$, we get a homeomorphism $h$ from $M$ to itself sending a $\psi$-orbit to a $\phi$-orbit.

## 2.6 Transitiveness

**Definition 2.45** A point $x \in M$ is called a *nonwandering point* of the flow $\phi$, if for any neighbourhood $U$ of $x$ and for any $T > 0$, there exists $t > T$ such that $\phi^t(U) \cap U \neq \emptyset$.

The set of nonwandering points is called a *nonwandering set* and is denoted by $\Omega(\phi)$.

The nonwandering set is a nonempty closed set, invariant by the flow. For example, a periodic point is a nonwandering point.

For an Anosov flow $\phi$, we have the following facts. The proofs are more or less the same as in the case of diffeomorphisms and are left to the readers. (The proof of Theorem 2.48 becomes simpler than the corresponding theorem for diffeomorphisms.) Recall that $V_x^\sigma$ denotes the leaf of the weak (un)stable foliation $\mathcal{V}^\sigma$ passing through $x$.

**Proposition 2.46** *Let $p$ be a periodic point. Then $V_p^u \cap V_p^s$ is contained in the nonwandering set $\Omega(\phi)$.* □

**Theorem 2.47** *The periodic points are dense in the nonwandering set $\Omega(\phi)$.* □

**Theorem 2.48** *The following five conditions are equivalent.*

1. *The nonwandering set $\Omega(\phi)$ coincides with the total manifold $M$.*
2. *The periodic points are dense in $M$.*
3. *There exists a point $x$ such that its orbit $O(x)$ is dense in $M$.*
4. *All the leaves of the weak unstable foliation are dense in $M$.*
5. *All the leaves of the weak stable foliation are dense in $M$.*

□

**Definition 2.49** We call the flow $\phi$ *transitive* if it satisfies the above conditions.

As stated before, no example has been found so far of nontransitive Anosov *diffeomorphisms*. But for flows, there do exist examples of nontransitive Anosov , provided either the dimension of $M$ is 3 or $s \geq 2$ and $u \geq 2$. See Franks-Williams [14] for the construction, which is based upon the examples in [50].

On the contrary if the dimension of $M$ is $\geq 4$, and if $u = 1$ or $s = 1$, then a theorem of Verjovsky states that any Anosov flow is transitive. The next section is devoted to the proof of this fact.

## 2.7 Markov partitions

Let us study a Markov partition associated with an Anosov flow $\phi$ on a closed manifold $M$. Fix a point $x \in M$. Choose a codimension one disk $D^{n-1}$ centered at $x$ and transverse to the flow $\phi$. Let $D^\sigma$ be the leaf through $x$ of the weak foliation $\mathcal{V}^\sigma$, restricted to $D^{n-1}$. Choose a closed subset $Q^\sigma$ in $D^\sigma$ such that $\text{Int}Q^\sigma$ is connected, contains $x$ and that $\text{Cl}(\text{Int}Q^\sigma) = Q^\sigma$. Assume that the size of $Q^\sigma$ is smaller than a sufficiently small number $\epsilon > 0$.

**Definition 2.50** The set $R = Q^u \times Q^s$ in $D^{n-1}$ of the points of intersection of the leaf of $\mathcal{V}^u|_{D^{n-1}}$ through a point of $Q^s$ and the leaf of $\mathcal{V}^s|_{D^{n-1}}$ through a point of $Q^u$ is called a *rectangle.*

Denote by $\text{Int}R$ the interior of $R$ in $D^{n-1}$. For any point $x \in R$, denote by $R^\sigma(x)$ the leaf of $\mathcal{V}^\sigma|_R$ through $x$. Also denote by $\text{Int}R^\sigma(x) = R^\sigma(x) \cap \text{Int}R$.

Let $\mathcal{R} = \{R_i\}_{i=1}^r$ be a finite disjoint family of rectangles. Denote

$$|\mathcal{R}| = \bigcup_i R_i.$$

Suppose further that any forward-orbit intersects $|\mathcal{R}|$. For any point $x \in |\mathcal{R}|$, let $\tau(x)$ be the smallest positive time such that $\tau(x) \in |\mathcal{R}|$. The map

$$\tau : |\mathcal{R}| \to \mathbf{R}$$

is called the *return time map* and the map

$$f : |\mathcal{R}| \to |\mathcal{R}|$$

defined by $f(x) = \phi^{\tau(x)}(x)$ is called the *first return map.* In general $\tau$ and $f$ are not continuous, but $f$ is a bijection.

**Definition 2.51** The disjoint family $\mathcal{R}$ is called a *Markov partition* for $\phi$ if the following conditions are satisfied.

1. There exists a positive number $\tau_1$ with the following property; for any $x \in M$, there exists $t \in [0, \tau_1]$ such that $\phi^t(x) \in |\mathcal{R}|$.

3. For any $i$ and $j$, the set $\mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j)$ is connected if it is nonempty, and $\tau$ (hence also $f$) is continuous on $R_i \cap f^{-1}(R_j)$.

4. If $x \in \mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j)$, then $f$ maps $\mathrm{Int}R_i^s(x)$ into $\mathrm{Int}R_j^s(f(x))$ and $f^{-1}$ maps $\mathrm{Int}R_j^u(f(x))$ into $\mathrm{Int}R_i^u(x)$.

If the flow $\phi$ is the suspension of an Anosov diffeomorphism $f : N \to N$. Then a Markov partition for $f$ (on $N$) gives birth to a Markov partition for the suspended flow.

Now we have the following theorem due to R. Bowen [6].

**Theorem 2.52** *Any Anosov flow admits a Markov partition of arbitrarily small size.* □

Associated with a Markov partition $\mathcal{R} = \{R_i\}_{i=1}^r$, define an $r$ by $r$ matrix $A = \{a_{ij}\}$ by setting $a_{ij} = 1$ if $\mathrm{Int}R_i \cap f^{-1}(\mathrm{Int}R_j) \neq \emptyset$ and 0 otherwise. $A$ is called a *transition matrix.*

A sequence $\mathbf{i} = \{i_n\}_{n=-\infty}^{\infty}$ of letters $1, 2, \cdots, r$ is called *admissible* if for any $n$, $a_{n,n+1} = 1$. Let

$$\tau_0 = \inf\{\tau(x) \mid x \in |\mathcal{R}|\}.$$

The following theorem is a refinement of the pseudo orbit tracing property and the expansiveness of the flow $\phi$. The proof is analogous and is omitted.

**Theorem 2.53** *1. For any admissible sequence $\mathbf{i} = \{i_n\}$ there exists a unique point $x \in |\mathcal{R}|$ with the following property; there exists a time sequence $\{t_n\}_{n=-\infty}^{\infty}$ such that $t_0 = 0$ and $\tau_0 \leq t_{n+1} - t_n \leq \tau_1$ for any $n$ and that $\phi^{t_n}(x) \in R_{i_n}$.*

*2. Conversely given a point $x \in |\mathcal{R}|$, there exists a bounded number of admissible sequence with the same property as above.* □

**Remark 2.54** *In 1 above, if the sequence $\mathbf{i}$ is cyclic, then the point $x$ must be a periodic point. This follows from the uniqueness of $x$.*

# 3 The Verjovsky theorem

We call an Anosov diffeomorphism (resp. flow) codimension one if the stable or unstable (resp. weak stable or weak unstable) foliation is of codimension one. Such systems are considered to be easier to analyse, since foliations of codimension one are simpler. In fact, it is known that any codimension one Anosov diffeomorphism is topologically conjugate to a hyperbolic automorphism of the torus ([34]).

However flows are more flexible than diffeomorphisms, and it seems difficult to obtain a similar classification result. The first milestone to the study of codimension one Anosov flows is the following theorem of A. Verjovsky. *Any codimension one Anosov flow on a closed manifold of dimension* $\geq 4$ *is transitive.* In [49], the theorem is stated without a condition about the dimension of the manifold. But a counter example in dimension 3 is constructed by Franks-Williams [14].

Also in [49], a stronger result is claimed that *the quotient space of the universal covering space by the lift of the (codimension one) weak stable foliation is homeomorphic to the real line.* However the proof contains a serious gap, and this statement is still an open problem. It is open even for transitive Anosov flows on 3-manifolds.

The purpose of this section is to provide a proof of the Verjovsky theorem cited above. After necassary preliminaries are given in the first subsection, the theorem is proved in the next. Another complete proof is found in T. Barbot [4].

## 3.1 Preparations in foliation theory

Let $\mathcal{F}$ be a transversely oriented[15] codimension one $C^0$ foliation with $C^r$ $(r \geq 1)$ leaves on a closed manifold $M$. One can construct a 1-dimensional subbundle of the tangent bundle of $M$ which is complementary to the tangent bundle $T\mathcal{F}$ of the foliation. We call it a *normal bundle* of $\mathcal{F}$. It defines an oriented 1-dimensional foliation which is transverse to $\mathcal{F}$. Choosing a nonsingular vector field tangent to the normal bundle of $\mathcal{F}$, one can construct a flow $\theta = \{\theta^t\}$ transverse to $\mathcal{F}$.

Given a loop $\gamma$ in a leaf of $\mathcal{F}$ and a positive number $a$, define a mapping

[15] i. e. the qotient bundle $TM/T\mathcal{F}$ is oriented.

$\Gamma : [0,1] \times [-a,a] \to M$ by

$$\Gamma(s,t) = \theta^t(\gamma(s)).$$

Since the flow $\theta$ is transverse to $\mathcal{F}$, the mapping $\Gamma$ is also transverse to $\mathcal{F}$, and therefore the induced foliation[16] $\Gamma^*\mathcal{F}$ is a nonsingular foliation on $[0,1] \times [-a,a]$, transverse to the vertical foliation $\{\{s\} \times [-a,a] \mid s \in [0,1]\}$.

Since $\gamma$ is a loop in a leaf of $\mathcal{F}$, $[0,1]\times\{0\}$ is a leaf of $\Gamma^*\mathcal{F}$. Therefore for some small $b > 0$, if $|t| < b$, the leaf through $(0,t)$ reaches a point in $\{1\} \times [-a,a]$, which we denote by $(1, g(t))$. That is, we obtained an orientation preserving homeomorphism $g$ from $(-b,b)$ into $(-a,a)$ which keeps 0 fixed. The germ of this homeomorphism, denoted by $\mathrm{Hol}_\gamma$, is called a *leaf holonomy along* $\gamma$. It is well known and easy to show that if the leaf loop $\gamma$ is homotopic (in the leaf) to another loop $\gamma'$, keeping the base point fixed, then their leaf holonomies coincide. In particular, if $\gamma$ is homotopically trivial, then the leaf holonomy is trivial.

**Definition 3.1** A subset is called *saturated* if it is a union of leaves.

**Definition 3.2** A subset is called a *minimal set*, if it is nonempty, closed and saturated, and is minimal w. r. t. the inclusion among those subsets

The following proposition is a consequence of Zorn's lemma.

**Proposition 3.3** *Any foliation on a compact manifold has a minimal set.*

A closed leaf is a typical example of minimal set. If all the leaves of $\mathcal{F}$ are dense in $M$, then the total manifold is a minimal set.

**Definition 3.4** A minimal set is called *exceptional* if it is neither a single leaf nor the total manifold.

An example of foliation with an exceptional minimal set is produced as follows. Consider the action of $PSL(2,\mathbf{R})$ on $S^1_\infty$. Let $I_i$ and $J_i$ $(i = 1,2)$ be open intervals in $S^1_\infty$ such that their closures are mutually disjoint. Choose a hyperbolic element $\gamma_i$ of $PSL(2,\mathbf{R})$ which sends the exterior of $I_i$ to the interior of $J_i$. Then it is well known that the group $\Gamma$ generated by $\gamma_1$ and $\gamma_2$ is free. The set

$$\Lambda = \bigcap_{\gamma\in\Gamma} \gamma(S^1_\infty - (I_1 \cup J_1 \cup I_2 \cup J_2))$$

[16] A leaf of $\Gamma^*\mathcal{F}$ is a component of the inverse image of a leaf of $\mathcal{F}$.

is nonempty closed and invariant by the action of $\Gamma$. Moreover $\Lambda$ is known to be the unique minimal subset with this property. This implies e. g, that all the $\Gamma$-orbits in $\Lambda$ are dense in $\Lambda$, and that $\Lambda$ is homeomorphic to the Cantor set.

Let $\Sigma$ be an oriented closed surface of genus $\geq 2$ and let $g : \pi_1(\Sigma) \to \Gamma$ be an arbitrary epimorphism. Then $\pi_1(\Sigma)$ acts on $S^1_\infty$ via $g$. Consider the products $\widetilde{\Sigma} \times S^1_\infty$ and the trivial foliation $\mathcal{F}_0 = \{\Sigma \times \{t\}\}$ on it. Then $\mathcal{F}_0$ yields a foliation $\mathcal{F}$ on the quotient of $\widetilde{\Sigma} \times S^1_\infty$ by the diagonal action of $\pi_1(\Sigma)$. $\mathcal{F}$ admits an exceptional minimal set corresponding to $\Lambda$.

As the final preparation, let us study the structure of an open saturated subset $E$. Fix once and for all a Riemannian metric of $M$. It induces a Riemannian metric on $E$. Denote by $d_E$ the corresponding distance function on $E$. (Notice that $d_E$ is different from the restriction of the distance function on $M$.) The metric completion of $E$ w. r. t. $d_E$ is called the *Dippolito completion* and is denoted by $\widehat{E}$. (To get an idea of Dippolito completion, just think of the case where $E$ is the complement of a single closed leaf $F$. Then two points in $E$ near to each other in $M$ may not be near w. r. t. $d_E$, if they lie on the opposite side of $F$. Thus in the completion $\widehat{E}$, two leaves diffeomorphic to $F$ are added.)

The Dippolito completion $\widehat{E}$ is a manifold with boudary $\delta E$, usually noncompact. The canonical projection $p : \widehat{E} \to \mathrm{Cl}(E)$ is defined in an obvious way.

The Dippolito completion admits a foliation $\widehat{\mathcal{F}}$ induced from $\mathcal{F}$. A boundary component is a leaf of $\widehat{\mathcal{F}}$. The vector field $\theta$ on $M$ transverse to $\mathcal{F}$ also yields a vector field $\widehat{\theta}$ on $\widehat{E}$. Denote by $\delta^+ E$ (resp. $\delta^- E$) the set of points of the boudary $\delta E$ at which the flow $\theta$ is pointing outward (resp. inward). Of course we have $\delta E = \delta^+ E \cup \delta^- E$ and $\delta^\pm E$ is a union of leaves. The following useful lemma is a truncated version of a structure theorem of Dippolito [10].

**Lemma 3.5** *Let $\widehat{E}$ be the Dippolito completion of an open saturated subset $E$ and let $V^-$ be a leaf in $\delta^- E$. Then there exists a compact subset $A^-$ of $V^-$ and a number $T > 0$ such that if $x \in V^- - A^-$, then $\widehat{\theta}^t(x) \in \delta^+ E$ for some $t \in (0, T)$.*

**Proof** Let $\{B_i \times I_i\}$ be a finite family of subsets of $M$ whose interiors form a covering of $M$ such that $B_i \times \{t\}$ is a closed ball in a leaf of $\mathcal{F}$ and $\{x\} \times I_i$ is a closed arc in a $\theta$-orbit. Choose a base point $b_i$ from $B_i$ and identify $\{b_i\} \times I_i$ with $I_i$

Consider the intersection $E \cap I_i$ for each $i$. Then except for finite numbers, all the connected components of the intersection are open intervals in $I_i$ which do not meet the boundary points of the interval.

Let $q_i$ be the terminal point of $I_i$ (in the orientation of $\theta$). If $q_i$ is contained in $E$, define $a_i$, if any, to be the point of $I_i \cap \partial E$ next to $q_i$. (If $q_i$ is not contained in $E$ or if $I_i$ is completely contained in $E$, then $a_i$ is not defined.) Set $A' = \cup_i B_i \times \{a_i\}$. Then the lemma holds for the intersection $A^-$ of $V^-$ and the pull back of $A'$ by the canonical projection $p$. □

## 3.2 The proof of the theorem

Again let $\phi$ be an Anosov flow on $M$. $\phi$ is called *codimension one* if either of the weak stable or unstable foliation is of codimension one.

**Theorem 3.6** *A codimension one Anosov flow is transitive if the dimension $n$ of the manifold is $\geq 4$.*

As we have mentioned in the previous section, the assumptions of the theorem are all actually necessary.[17]

The rest of this section is devoted to the proof of this theorem. In order to fix the idea, we assume that the stable foliation $\mathcal{V}^s$ is of codimension one. Passing to a double cover if necessary, one may assume that $\mathcal{V}^s$ is transversely oriented[18].

Then the strong unstable foliation $\mathcal{W}^u$ is 1-dimensional and oriented. Hence there exists a nonsigular flow $\theta$ tangent to $\mathcal{W}^u$. We consider the leaf holonomy of $\mathcal{V}^s$ according to this flow $\theta$.

The leaves of $\mathcal{V}^s$ are homeomorphic either to $\mathbf{R}^{n-1}$ or to $S^1 \times \mathbf{R}^{n-2}$. ($n$ is the dimension of the manifold $M$.) If it is homeomorphic to $\mathbf{R}^{n-1}$, then of course any leaf holonomy is trivial. If it is homeomorphic to $S^1 \times \mathbf{R}^{n-2}$, then it contains a periodic orbit. The homotopy class of the periodic orbit is a generator of the fundamental group of the leaf. The leaf holonomy along it is an expansion, and 0 is an isolated fixed point. This follows of course from the expanding properties of the Anosov flow along the strong unstable foliation $\mathcal{W}^u$.

The proof is by contradiction. Assume that the flow is not transitive. Then by Theorem 2.48, the stable foliation $\mathcal{V}^s$ is not all-leaves-dense. Of course $\mathcal{V}^s$ does not have a compact leaf. Therefore by Proposition 3.3 $\mathcal{V}^s$ must admit an exceptional minimal set.

[17] I. e. there are counter-examples if one of them is dropped.

[18] One is free to pass to a double cover since the transitiveness of the lifted flow implies that of the original flow.

Let $E$ be a connected component of the complement of the minimal set. Then $E$ is an open saturated set. Consider the Dippolito completion $\widehat{E}$. The foliation $\mathcal{V}^s$, and the flows $\phi$ and $\theta$ have lifts to $\widehat{E}$, denoted by $\widehat{\mathcal{V}}^s$, $\widehat{\phi}$ and $\widehat{\theta}$.

Let $V^-$ be a leaf in $\delta^- E$. We divide the argument into two cases.

**Case 1** *$V^-$ is homeomorphic to* $\mathbf{R}^{n-1}$.

Choose a compact subset $A^-$ of $V^-$ in Lemma 3.5. One may assume that $A^-$ is homeomorphic to the closed disk $D^{n-1}$. Let $U^-$ be the subset of the leaf $V^-$ consisting of those points $x$ such that $\widehat{\theta}^t(x)$ lies on $\delta^+ E$ for some positive $t = \tau(x)$. Lemma 3.5 asserts that $V^- - A^-$ is contained in $U^-$. The function

$$\tau : U^- \to \mathbf{R}$$

defined in this way is continuous. Also define a map

$$h : U^- \to \delta^+ E$$

by $h(x) = \widehat{\theta}^{\tau(x)}(x)$.

Since $V^-$ and $\delta^+ E$ are kept invariant by the flow $\widehat{\phi}$ and an orbit of $\widehat{\theta}$ is mapped by $\widehat{\phi}^t$ to another orbit of $\widehat{\theta}$, we get the followings.

1. The subset $U^-$ is invariant by the flow $\widehat{\phi}$.

2. We have $h(\widehat{\phi}^t(x)) = \widehat{\phi}^t(h(x))$.

Since there is no $\widehat{\phi}$-orbit which is completely contained in the compact set $A^-$, the $\widehat{\phi}$-invariance of $U^-$ implies that $U^-$ is the total of $V^-$. Again by Lemma 3.5, the function $\tau$ is bounded from above. But this contradicts the fact that $\widehat{\phi}$ must expand the $\widehat{\theta}$-orbit.

**Case 2** *The leaf $V^-$ is homeomorphic to* $S^1 \times \mathbf{R}^{n-2}$.

$V^-$ contains a unique periodic orbit $O(p)$ of $\widehat{\phi}$. By some abuse, denote a tubular neighbourhood of $O(p)$ in $V^-$ by $S^1 \times D^{n-2}$. $O(p)$ is identified with $S^1 \times \{0\}$. The orientation of $S^1$ is to be the same as the one given by the flow. One may assume

1. The flow $\widehat{\phi}$ is transverse to the boundary $S^1 \times S^{n-3}$, pointing inward.

2. Each fiber $\{t\} \times D^{n-2}$ lies on a strong stable leaf, and in particular is transverse to $\widehat{\phi}$.

3. The set $A^-$ of Lemma 3.5 is contained in $S^1 \times D^{n-2}$.

By the definition of Anosov flow the only $\widehat{\phi}$-orbit completely contained in $S^1 \times D^{n-2}$ is $O(p)$. Now define the subset $U^-$, the maps $\tau$ and $h$ as in Case 1. Then we have $V^- - O(p)$ is cotained in $U^-$. Of course if $U^-$ is the total of $V^-$, then we are done by the same argument as in Case 1. So assume that $U^- = V - O(p)$ .

Since $n \geq 4$, $U^-$ is connected, and therefore the image $h(U^-)$ is contained in one leaf, say $V^+$, of $\delta^+ E$. (This is the only point where we use the assumption that $n \geq 4$.) By the argument which reverses the sign of $\widehat{\theta}$, one may assume that $V^+$ is also homeomorphic to $S^1 \times \mathbf{R}^{n-2}$ and $h(U^-) = V^+ - O(q)$, where $O(q)$ is the unique periodic orbit in $V^+$. Clearly $h$ is a homeomorphism from $U^-$ onto $h(U^-)$.

Take a base point $x^- = (a, b)$ in $S^1 \times S^{n-3}$ and consider a loop $\gamma$ corresponding to $S^1 \times \{b\}$. Let $I$ be the $\widehat{\theta}$-orbit from $x^-$ to $x^+ = h(x^-)$. Then the leaf holonomy $\mathrm{Hol}_\gamma$ along $\gamma$ is defined *on the whole I onto itself.*

As we have seen before $\mathrm{Hol}_\gamma$ must have isolated fixed points and the point $x^-$ is an expanding fixed point of it. For a while let us assume that $\mathrm{Hol}_\gamma$ has no fixed point in the interior of $I$. (We shall consider the general case afterward.) Choose an arbitrary point $y$ in the interior of $I$. Then we have

$$\begin{aligned} \mathrm{Hol}_\gamma^{-n}(y) &\to x^- \quad \text{as} \quad n \to \infty \\ \mathrm{Hol}_\gamma^{n}(y) &\to x^+ \quad \text{as} \quad n \to \infty \end{aligned}$$

Let $V$ be the $\widehat{\mathcal{V}}^s$ leaf passing through $y$. Let

$$U = \{z \in V \mid z = \widehat{\theta}^t(x) \ \exists t > 0, \ x \in U^-\}.$$

For any point $z$ of $U$, the point $x$ in $U^-$ such that $z = \widehat{\theta}^t(x)$ is unique. Define a map $\pi : U \to U^-$ by setting $x = \pi(z)$. Then since all the forward orbits of $\widehat{\theta}$ starting at points in $U^-$ arrive at points in $V^+$, $\pi$ is a covering map. Also $\pi$ preserves the flow $\widehat{\phi}$.

Denote by $\mathbf{R} \times S^{n-3}$ the inverse image of $S^1 \times S^{n-3}$ of $\pi$. The complement of $\mathbf{R} \times S^{n-3}$ in $V$ has two connected component. One is the inverse image of $V^- - S^1 \times D^{n-2}$, which we call the *exterior* of $\mathbf{R} \times S^{n-3}$. The other is called the *interior*. Since $\widehat{\phi}$ is preserved by $p$, the flow $\widehat{\phi}$ is transverse to $\mathbf{R} \times S^{n-3}$, pointing toward the interior.

Let

$$W = \{z \in V \mid z = \widehat{\theta}^t(x) \ \exists t > 0, \ x \in V^-\}.$$

Of course we have $U \subset W \subset V$. The map $\pi$ can be extended to $\pi : W \to V$. (This may not be a covering map.) Denote by $J$ the forward orbit of $\widehat{\theta}$ starting at the point $p = (a, 0)$. ($p$ is a point in the periodic orbit $O(p)$, which we identified with $S^1 \times \{0\}$.)

Since $V \cap I$ has a point $\mathrm{Hol}_\gamma^{-n}(y)$ arbitrarily close to $x^- \in V^-$, $V \cap J$ also admits a point $p'$ of $V \cap J$ sufficiently close to $p$. Then there exists a subset $B_-$ in $V$ which contains $p'$ and is mapped by $\pi$ homeomorphically onto the fiber $\{a\} \times D^{n-2}$ passing through $p$. $B_-$ separates the interior of $\mathbf{R} \times S^{n-3}$ into two parts. One part is near the leaf $V^-$ and is mapped by $\pi$ onto $S^1 \times D^{n-2}$. Denote by $C_-$ the closure of the other part. Since the flow $\widehat{\phi}$ is transverse to the fiber $\{a\} \times D^{n-2}$, it is transverse to $B_-$. Since $p'$ lies on the strong unstable leaf $J$ through $p$, the $\widehat{\phi}$-orbit through $p'$ lies on the weak unstable leaf through $O(p)$. That is, the orbit (in the positive time direction) runs away from $O(p)$. Therefore the flow $\widehat{\phi}$ is pointing toward $C_-$ at $B_-$.

Do the same argument for the other boundary leaf $V^+$. Then one can define the $(n-2)$-dimensional disk $B_+$, and the part $C_+$ which is not near to $V_+$. The flow $\widehat{\theta}$ is transverse to $B_+$ pointing toward $C_+$.

Now consider the intersection $C = C_- \cap C_+$. It is easy to see that its boundary is homeomorphic to the $(n-2)$-dimensional sphere and the complement of $C$ is noncompact. A well known theorem of Schönflies implies[19] that $C$ is homeomorphic to $(n-1)$-dimensional disc. But on its boundary the flow $\widehat{\phi}$ is pointing inwards, contradicting the dynamics of the Anosov flow on the stable leaf which we established in Proposition 2.37.

Last of all if $\mathrm{Hol}_\gamma$ has a fixed point in the interior of $I$, replace $E$ by a smaller portion and the boundary leaf $V^+$ by the leaf corresponding to the fixed point, nearest ot $x^-$ and then do the same argument.

This completes the proof of Theorem 3.6.

[19] If $V$ is homeomorphic to $S^1 \times \mathbf{R}^{n-2}$, consider the universal covering.

# 4 Asymptotic cycles

In this section, we consider a flow of general type. In [43], Schwartzman introduced the concept of asymptotic cycles. He considered a sequence of arcs $s_n$ lying in orbits of the flow such that length$(s_n) \to \infty$ as $n \to \infty$. By joining the endpoints of $s_n$ by arcs of bounded length in the manifold, we obtain a sequence of closed curves which represents 1st homology classes $C_n$. A homology class $C$ is called an Asymptotic cycle if it is the limit of such sequences $C_n$.

Here we generalize the concept, and asign a homology class, also called asymptotic cycle, to any flow-invariant probability measure. The totality of the asymptotic cycles forms a compact convex subset in the 1st homology group.

This has an application for getting a criterion for the flow to admit a global cross section. We shall prove a theorem of Schwartzman [43] that *if the set of the asymptotic cycles lies in the complement of a codimension one subspace, then the flow admits a cross section.* See also [17] and [15].

## 4.1 Definition and some examples

Let $\phi = \{\phi^t\}$ be a $C^r$-flow $(r \geq 0)$ on a *compact* manifold $M$ without boundary. We use the following notations.

**Notation 4.1** Denote by $C(M)$ the space of $\mathbf{R}$–valued continuous functions on $M$.

**Notation 4.2** Denote by $\mathcal{P} = \mathcal{P}(M)$ the space of probability measures on $M$.

An element $\mu$ of $\mathcal{P}$ is considered to be a linear functional

$$\mu : C(M) \to \mathbf{R},$$

such that

$$\begin{aligned} \mu(f) &\geq 0 \text{ for } f \geq 0 \text{ and} \\ \mu(1) &= 1, \end{aligned}$$

where $1 \in C(M)$ denotes the function constantly equal to one.

As is well known, the space $\mathcal{P}$ is compact by the weak-$*$ topology, that is, the topology of the convergence on each element of $C(M)$. Moreover $\mathcal{P}$ is convex with respect to its natural affine structure.

**Notation 4.3** Let $\mathcal{P}_\phi = \mathcal{P}_\phi(M)$ be the space of $\phi$–invariant probability measures $\mu$, that is, $\phi^t_*(\mu) = \mu$ for any $t$.

Of course $\mathcal{P}_\phi$ is a compact and convex subset of $\mathcal{P}$.

The pupose of this section is to define the following mapping

$$A : \mathcal{P}_\phi \to H_1(M, \mathbf{R}).$$

The image $A_\mu$ of $\mu \in \mathcal{P}_\phi$ is the so called asymptotic cycle of $\mu$.

**Notation 4.4** Denote by $[M, S^1]$ the set of the homotopy classes of the continuous mappings from $M$ to $S^1$.

Here we consider $S^1$ to be the quotient group $\mathbf{R}/\mathbf{Z}$. The additive group structure of $S^1$ yields an additive group structure on $[M, S^1]$. Let us denote by $C_{S^1}$ the fundamental cohomology class of $S^1$. Then a mapping $\alpha : [M, S^1] \to H^1(M; \mathbf{R})$ is defined by $\alpha([g]) = g^*(C_{S^1})$. The following proposition is well known.

**Proposition 4.5** *$\alpha$ is an isomorphism of $[M, S^1]$ onto $H^1(M, \mathbf{Z})$.*

Therefore we get that $H_1(M, \mathbf{R})$ is isomorphic to $\mathrm{Hom}([M, S^1], \mathbf{R})$. Thus in order to define $A : \mathcal{P}_\phi \to H_1(M, \mathbf{R})$, we need to define a mapping (denoted again by $A$)

$$A : \mathcal{P}_\phi \times [M, S^1] \to \mathbf{R}.$$

So choose an arbitrary $\mu \in \mathcal{P}_\phi$ and $[g] \in [M, S^1]$, and let us define a number $A_\mu([g])$. Let $g : M \to S^1$ be a representative of the class $[g]$ and let $x$ be an arbitrary point of $M$. Let us consider the mapping

$$\mathbf{R} \ni t \mapsto g(\phi^t x) \in S^1.$$

Let $h_x : \mathbf{R} \to \mathbf{R}$ be any of its lift. For $t \in \mathbf{R}$, define

$$\Delta g(x, t) = h_x(t) - h_x(0).$$

$\Delta g(x, t)$ measures the increase of the argument of $g : M \to S^1$ along the orbit starting at $x$ and ending at $\phi^t x$.

The proof of the following lemma is immediate.

**Proposition 4.6** *1. $\Delta g(x, t)$ is independent of the choice of the lift and is continuous in $x$ and $t$.*

*2. We have*

$$\Delta g(x, s+t) = \Delta g(x, s) + \Delta g(\phi^s x, t).$$

**Definition 4.7** The class $A_\mu \in H_1(M, \mathbf{R})$ defined by

$$A_\mu([g]) = \frac{1}{t}\int_M \Delta g(x,t)d\mu(x)$$

for any $g \in [M, S^1]$ and $t \neq 0$ is called the *asymptotic cycle* of $\mu$.

Roughly speaking, $A_\mu([g])$ measures the $\mu$-average growth of the argument of $g$ along the flow in unit time.

**Proposition 4.8** *$A_\mu([g])$ is a well defined homomorphism, independent of the choice of $g$ in the class and $t \neq 0$.*

**Proof** First of all, let $[g] = [g']$. This means $g' - g : M \to S^1$ is homotopically trivial, that is, has a lift $h : M \to \mathbf{R}$. Thus we have

$$\Delta g'(x,t) - \Delta g(x,t) = h(\phi^t x) - h(x).$$

By the $\phi^t$-invariance of $\mu$, we get $A_\mu([g]) = A_\mu([g'])$.

Also it is immediate from the definition that $A_\mu : [M, S^1] \to \mathbf{R}$ is a homomorphism.

Next, to show the independence of the choice of $t$, let us denote for a while

$$B^t = \int_M \Delta g(x,t)d\mu(x).$$

Then we have

$$\begin{aligned} B^{s+t} &= \int_M \Delta g(x,s+t)d\mu(x) = \int_M \Delta g(x,s)d\mu(x) + \int_M \Delta g(\phi^s x,t)d\mu(x) \\ &= \int_M \Delta g(x,s)d\mu(x) + \int_M \Delta g(x,t)d\mu(x) = B^s + B^t. \end{aligned}$$

The independence of the choice of $t$ follows from this additivity property and the continuity of $B^t$ on $t$. □

**Remark 4.9** *The mapping $A : \mathcal{M}_\phi \to H_1(M;\mathbf{R})$ is continuous and linear. Therefore its image is compact and convex.*

**Proposition 4.10** *If $\mu_n \in \mathcal{P}$ converges to an invariant probability $\mu \in \mathcal{P}_\phi$, then we have*

$$A_\mu([g]) = \lim_{n\to\infty} \frac{1}{t}\int_M \Delta g(x,t) d\mu_n(x).$$

**Proof** This is straightforward by the definition of the convergence in weak-$*$ topology. Recall that $\Delta g(x,t)$ is continuous in $x$.

$\square$

**Definition 4.11** For $p \in M$ and $T \in \mathbf{R}$, define a probability $\mu(p,T) \in \mathcal{P}$ by

$$(\mu(p,T), h) = \frac{1}{T}\int_0^T h(\phi^t p) dt,$$

for any $h \in C(M)$.

**Corollary 4.12** *Assume for some sequence $p_n$ and $T_n \to \infty$, $\mu(p_n, T_n)$ converges to $\mu$. Then $\mu$ is an invariant probability, and we have*

$$A_\mu([g]) = \lim_{n\to\infty} \frac{1}{T_n}\Delta g(p_n, T_n).$$

**Proof** That $\mu$ is an invariant probability is obvious. To show the last part, we have

$$\begin{aligned}
A_\mu([g]) &= \lim_{n\to\infty} \frac{1}{t}\int_M \Delta g(x,t) d\mu(p_n,T_n)(x) \\
&= \lim_{n\to\infty} \frac{1}{t}(\mu(p_n,T_n),\ \Delta g(\cdot,t)) \\
&= \lim_{n\to\infty} \frac{1}{t}\frac{1}{T_n}\int_0^{T_n} \Delta g(\phi^s p_n, t) ds \\
&= \lim_{n\to\infty} \frac{1}{t}\frac{1}{T_n}\int_0^{T_n} (\Delta g(p_n, s+t) - \Delta g(p_n, s)) ds \\
&= \lim_{n\to\infty} \frac{1}{t}\frac{1}{T_n}(\int_{T_n}^{T_n+t} \Delta g(p_n, s) ds - \int_0^t \Delta g(p_n, s) ds).
\end{aligned}$$

Here notice that the function $\Delta g(x,t)$ converges to $\Delta g(x,0) = 0$, as $t \to 0$ uniformly in $x$. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|\Delta g(x,t)| < \epsilon$ if $|t| < \delta$. This, together with the equality

$$\Delta g(x,s) - \Delta g(x,t) = \Delta g(\phi^t x, s-t),$$

implies the following; for any $\epsilon > 0$, there exists $\delta > 0$ (independent of $x$, $s$ and $t$) such that $|\Delta g(x,s) - \Delta g(x,t)| < \epsilon$ if $|s-t| < \delta$.

Therefore if $|t| < \delta$, then the term

$$\frac{1}{t}\left(\int_{T_n}^{T_n+t} \Delta g(p_n, s)ds - \int_0^t \Delta g(p_n, s)ds\right)$$

is $2\epsilon$-near to $\Delta g(p_n, T_n)$. Therefore we get

$$\begin{aligned} A_\mu([g]) &= \lim_{n\to\infty} \frac{1}{T_n}\frac{1}{t}\left(\int_{T_n}^{T_n+t} \Delta g(p_n, s)ds - \int_t^0 \Delta g(p_n, s)ds\right) \\ &= \lim_{n\to\infty} \frac{1}{T_n}\Delta g(p_n, T_n). \end{aligned}$$

□

**Example 4.13** *Consider the n-torus* $T^n = \mathbf{R}^n/\mathbf{Z}^n$. *Denote the coordinates by* $(x_1, x_2, \cdots, x_n)$. *Given* $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{R}^n$, *define a linear vector field*

$$X = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \cdots + \alpha_n \frac{\partial}{\partial x_n}.$$

*The flow* $\phi$ *induced by* $X$ *preserves the standard volume* $\mu$. *Clearly we have*

$$A_\mu = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{R}^n \approx H_1(T^n; \mathbf{R}).$$

**Example 4.14** *Let* $O(p)$ *be the periodic orbit through a point* $p \in M$ *of (the least) period* $T$. *Then of course* $\mu(p,T)$ *is an invariant probability. Now it is easy to check that* $A_{\mu(p,T)} = (1/T)[O(p)] \in H_1(M;\mathbf{R})$.

## 4.2 Cross sections

In this subsection, we assume that the flow $\phi$ is nonsingular. We will give a condition in terms of asymptotic cycles for a flow to admit a cross section. Recall that a cross section is a closed codimension one submanifold transverse to the flow which intersects any orbit of the flow. First we need preparation. Let $g : M \to S^1$ be a continuous map.

**Definition 4.15** If the limit $g'(p) = \lim_{t\to 0}(1/t)\Delta g(p,t)$ exists for any $p \in M$ and forms a continuous map $g' : M \to \mathbf{R}$, we say that $g$ is $C^1$ *along the orbit.*

**Lemma 4.16** *Suppose that* $g$ *is* $C^1$ *along the orbit and that* $g'$ *is everywhere positive. Then the inverse image* $N$ *by* $g$ *of the point* $0 \in S^1$ *is a cross section.*

**Proof** Clearly $N$ is transverse to the flow. Since the function $g'$ is bounded from below, say, by $\epsilon > 0$, we have that for any point $p \in M$, there exists $0 \leq t \leq 1/\epsilon$ such that $\phi^t(p) \in N$. □

The following theorem will play an important role in the last section.

**Theorem 4.17** *Let $\phi$ be a nonsingular flow on $M$. Suppose that there exists numbers $T > 0$ and $\epsilon > 0$ such that*

$$\frac{1}{T}\Delta g(p,T) > \epsilon \quad (\forall p \in M).$$

*Then the flow $\phi$ admits a cross section.*

**Proof** An element of $\mathbf{R}$ and an element of $S^1 = \mathbf{R}/\mathbf{Z}$ can be added in a natural way, the sum being an element of $S^1 = \mathbf{R}/\mathbf{Z}$. Using this, define a function $\overline{g} : M \to S^1$ by

$$\overline{g}(p) = g(p) + \frac{1}{T}\int_0^T \Delta g(p,s)ds.$$

The proof will be complete once we show that $\overline{g}$ is $C^1$ along the orbit and that $\overline{g}' > 0$. Now

$$\begin{aligned}
\Delta\overline{g}(p,t) &= \Delta g(p,t) + \frac{1}{T}\left(\int_0^T \Delta g(\phi^t p, s)ds - \int_0^T \Delta g(p,s)ds\right) \\
&= \Delta g(p,t) + \frac{1}{T}\left(\int_0^T \Delta g(p,s+t)ds - \int_0^T \Delta g(p,t)ds - \int_0^T \Delta g(p,s)ds\right) \\
&= \frac{1}{T}\left(\int_T^{T+t} \Delta g(p,s)ds - \int_0^t \Delta g(p,s)ds\right)
\end{aligned}$$

Therefore we have

$$\lim_{t\to 0}\frac{1}{t}\Delta\overline{g}(p,t) = \frac{1}{T}\Delta\overline{g}(p,T) > \epsilon,$$

as is required. □

**Corollary 4.18 (Schwartzman)** *Let $\phi$ be a nonsingular flow on $M$. Suppose there exists a class $C \in H^1(M;\mathbf{R})$ such that for any $\mu \in \mathcal{M}_\phi$ we have $(A_\mu, C) > 0$. Then there exists a cross section $N$ of the flow $\phi$.*

**Proof** First consider the case when $C$ is an integral point of $H^1(M;\mathbf{R})$. Let $g: M \to S^1$ represent the class $C \in H^1(M;\mathbf{R})$. We shall show that the hypothesis of the previous theorem is satisfied. If not, there would exist a sequence $p_n \in M$ and $T_n \to \infty$ such that

$$\frac{1}{T_n}\Delta g(p_n, T_n) \to b,$$

for some $b \leq 0$. Passing to a subsequence, consider an invariant probability

$$\mu = \lim_{n\to\infty} \frac{1}{T_n}\mu(p_n, T_n).$$

Then we would have $A_\mu([g]) = b$. A contradiction.

Next let us consider the general case. If a class $C$ satisfies the assumption of the theorem, then all the nearby classes must also satisfies the condition. This is because the set $\{A_\mu \mid \mu \in \mathcal{M}_\phi\}$ is compact. Now just choose a rational class and multiply it to obtain an integral class. □

The following proposition will be useful in the last section.

**Proposition 4.19** *Let $\phi$ be a nonsingular flow on $M$ and let $\pi : \widehat{M} \to M$ be a finite covering. Denote by $\widehat{\phi}$ the lift of $\phi$ to $\widehat{M}$. Then $\phi$ admits a cross section if and only if $\widehat{\phi}$ does.*

**Proof** Suppose that $\widehat{\phi}$ admits a cross section. Then it gives birth to an $S^1$ bundle structure $\widehat{g} : \widehat{M} \to S^1$. Thus $\widehat{g}$ is $C^1$ along the orbit and that $\widehat{g}'$ is everywhere positive.

Define a map $g : M \to S^1$ by $g(x) = \widehat{g}(x_1) + \cdots + \widehat{g}(x_r)$, where $\{x_1, \cdots, x_r\}$ is the inverse image of $x$ by $\pi$. Then clearly $g$ is again $C^1$ along the orbit and $g'$ is everywhere positive. Therefore by Lemma 4.16, the flow $\phi$ admits a cross section.

The converse is clear. □

# 5 Codimension one foliations on solvable manifolds

In this section we study an aspect of the interplays between the structure of a foliation and the topology of the underlying manifold. Let $\mathcal{F}$ be a codimension one foliation on a manifold $M$. We consider the following problem; if the manifold $M$ is simple, then is the structure of the foliation $\mathcal{F}$ also simple? In general the answer is no. For example, $S^3$ admits a foliation with nonvanishing Godbillon-Vey class. Such a foliation should be sufficiently complicated.

In fact if we allow a foliation to admit compact leaves, then starting from any foliation, we can construct another foliatin as complicated as we want i. g. by a modification along any disjoint system of closed embedded loops transverse to the foliation. There is no hope of the classification of such a foliation.

Therefore we need to put the condition that the foliation is without compact leaf. Under this condition, classification results can be found for closed 3-manifolds.

The first result is by S. P. Novikov, who showd that a closed 3-manifold with finite fundamental group does not admits a $C^0$ foliation without closed leaves ([35], [45]).

There is a classification theorem of $C^r$ ($r \geq 2$) foliations on closed 3-manifolds with solvable fundamental group due to E. Ghys and V. Sergiescu ([22]), and independently to J. Plante ([36]).

Also if the manifold is the unit tangent bundle of a closed oriented surface of genus $\geq 2$, then such $C^r$ ($r \geq 2$) foliations are known to be unique up to topological conjugacy ([32]). Quite recently E. Ghys ([21]) obtained a remarkable classification result up to $C^r$ conjugacy.

However if the dimension of the manifold is $\geq 4$, then such a classification as in dimension 3 is impossible. For example, P. Schweitzer showed that any closed manifold of dimension $\geq 4$ with Euler number 0 admits a $C^1$ foliation without compact leaves. This settled down a problem asking the validity of Novikov type theorem in higher dimension. (But P. Schwitzer's example is only $C^1$. The existence of such smooth foliations still remains open.) But it strongly suggests that if we want to construct a qualitative theory of codimension one foliations in higher dimension, then we need an extra condition.

Here we assume that the foliation $\mathcal{F}$ on a manifold $M$ is all-leaves-dense and any leaf holonomy is either trivial or has isolated fixed point. We shall show in the last subsection that $\mathcal{F}$ is topologically conjugate to a transversely affine foliation, provided the fundamental group of $M$ is solvable.

Since transversely affine foliations has nice properties, this result will be applied in the next section to the classification of codimension one Anosov foliations on such manifolds.

## 5.1 Transversely affine foliations

Here we give definitions and fundamental properties of transversely affine foliations.

Recall that a codimension one foliation $\mathcal{F}$ on a manifold $M$ is defined by a family of local submersions $f_i : U_i \to \mathbf{R}$ such that $U_i$ is an open covering of $M$ and that $f_i^{-1}(x)$ is contained in a leaf of $\mathcal{F}$ for any $i$ and $x \in \mathbf{R}$, and a family of local homeomorphisms, called transition functions,

$$g_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$

such that $f_i = g_{ij} f_j$ on $U_i \cap U_j$.

**Definition 5.1** A codimension one foliation is called $C^r$ *transversely affine* if the local submersions $f_i$ are $C^r$ and the transition functions $g_{ij}$ are the restrictions of affine transformatins of $\mathbf{R}$.

In the sequal we only consider transversely oriented foliations. Denote by $Aff_+(\mathbf{R})$ the group of orientation preserving affine transformations of the real line.

Let $\tilde{\mathcal{F}}$ be the lift of a transversely affine foliation $\mathcal{F}$ to the universal covering space $\pi : \widetilde{M} \to M$. Assume that the domains $U_i$ of all the local submersions are evenly covered by the covering map $\pi$. Then one can consider all the lifts

$$f_i^\nu = f_i \circ \pi|_{U_i^\nu} : U_i^\nu \to \mathbf{R}$$

of the local submersions $f_i$, where $U_i^\nu$ are arbitrary lifts of $U_i$. They give $\tilde{\mathcal{F}}$ a structure of a transversly affine foliations.

Suppose that $U_i^\nu$ and $U_j^\mu$ intersect. Then instead of the local submersion $f_j^\mu : U_j^\mu \to \mathbf{R}$, one can take

$$g_{ij} \circ f_j^\mu : U_j^\mu \to \mathbf{R}$$

as the local submersion. (This is possible because $g_{ij}$ is defined on the whole of $\mathbf{R}$.) Then the two local submersions $f_i^\nu$ and $g_{ij} \circ f_j^\mu$ coincides on $U_i^\nu \cap U_j^\mu$. Therefore the domain of definition of the submersion $f_i^\nu$ is extended from $U_i^\nu$ to $U_i^\nu \cup U_j^\mu$.

In this way we can extend the domain of $f_i^\nu$ one by one, just as if we are doing an analytic continuation. Since $\widetilde{M}$ is simply connected, this yields a well defined submersion from the whole $\widetilde{M}$, denoted by

$$D : \widetilde{M} \to \mathbf{R}.$$

Of course $D$ depends upon the choice of the base submersion $f_i^\nu$, which we fix once and for all.

Now the fundamental group $\pi_1(M)$ acts on $\widetilde{M}$ as the group of deck transformations. Choose one element $\gamma \in \pi_1(M)$ and consider the submersion

$$D \circ \gamma : \widetilde{M} \to \mathbf{R}.$$

It is easy to show that this is also a submersion obtained in the same way as above, but starting from a different choice of the base submersion (exactly from $f_i^\nu \circ \gamma : \gamma^{-1}(U_i^\nu) \to \mathbf{R}$ instead of $f_i^\nu$).

Therefore there exists a unique element, say $g$ of $\mathit{Aff}_+(\mathbf{R})$, such that $g \circ D = D \circ \gamma$. This way, we get a mapping

$$h : \pi_1(M) \to \mathit{Aff}_+(\mathbf{R}),$$

by setting $h(\gamma) = g$.

The equality $h(\gamma) \circ D = D \circ \gamma$ implies that $h$ is a homomorphism.

**Definition 5.2** The submersion $D$ is called the *developing map* of $\mathcal{F}$ and the homomorphism $h$ is called the *holonomy homomorphism.* The image $\Gamma = h(\pi_1(M)) \subset \mathit{Aff}_+(\mathbf{R})$ is called the *holonomy group.*

Thus we associate to a transversely affine foliation an equivariant pair

$$(h, D) : (\pi_1(M), \widetilde{M}) \to (\mathit{Aff}_+(\mathbf{R}), \mathbf{R}).$$

Conversely we have the following proposition. The easy proof is omitted.

**Proposition 5.3** *Given any equivariant pair*

$$(h, D) : (\pi_1(M), \widetilde{M}) \to (\mathit{Aff}_+(\mathbf{R}), \mathbf{R}).$$

*of a homomorphism and a submersion, there exists a transversely affine foliation $\mathcal{F}$ on $M$ such that $h$ and $D$ are the holonomy homomorphism and the developing map of $\mathcal{F}$.* □

The above proposition is useful for constructing transversely affine foliations. Below, the group $\mathit{Aff}_+(\mathbf{R})$ is identified with the Lie group of real matrices of the form

$$\begin{bmatrix} e^t & x \\ 0 & 1 \end{bmatrix}.$$

**Example 5.4** Consider the simply connected 3-dimesional solvable Lie group $S_3$ which consists of the real matrices of the form

$$\begin{bmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Define a homomorphism $H : S_3 \to \mathit{Aff}_+(\mathbf{R})$ by

$$H \begin{bmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & x \\ 0 & 1 \end{bmatrix}.$$

Notice that if we replace the part $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ in the definition of $S_3$ by $A^t$, where $A$ is a unimodal integral hyperbolic matix, then we obtain a Lie group which is isomorphic to $S_3$. From this, one can easily show that the Lie group $S_3$ admits a cocompact lattice. Let $\iota : \Lambda \to S_3$ be the inclusion map of a cocompact lattice, and define $p : \mathit{Aff}_+(\mathbf{R}) \to \mathbf{R}$ by

$$p \begin{bmatrix} e^t & x \\ 0 & 1 \end{bmatrix} = x.$$

Then the equivariant pair

$$(H \circ \iota, p \circ H) : (\Lambda, S_3) \to (\mathit{Aff}_+(\mathbf{R}), \mathbf{R})$$

defines a transversely affine foliation on the quotient manifold $\Lambda \setminus S_3$.

In fact, the above foliation is nothing but the (un)stable foliation of the suspension flow of a hyperbolic automorphism of $T^2$.

## 5.2 The Haefliger theorem

Throughout this subsection, $\mathcal{F}$ is to be a codimension one transversely oriented continuous foliation on a manifold $M$. Unlike other sections, we do not assume that the manifold is compact. As is mentioned in the beginning of this section, we need an extra condition on the foliation in order to do qualitative study in higher dimension. Here is a nice assumption which enables us to work.

**Assumption 5.5 (IFP)** Any leaf holonomy is either trivial or has an isolated fixed point.

As a matter of fact, real analytic foliations satisfy IFP. Another example of foliation with IFP which is important to us is the stable foliation of an Anosov flow (when it is of codimension one).

Here is a theorem due to A. Haefliger, which plays a crucial role in what follows.

**Theorem 5.6** *A continuous foliation $\mathcal{F}$ with IFP does not admit a null transversal, i. e. a closed curve transverse to $\mathcal{F}$ which is null homotopic.* □

On the other hand, we have the following fact.

**Proposition 5.7** *Any foliation $\mathcal{F}$ admits a transversal, i. e. a closed curve transverse to it, provided the manifold $M$ is compact.*

**Proof** Suppose for a while that there exists a noncompact leaf $F$. Since $F$ is not closed, there exists a point $x$ in $\mathrm{Cl}(F) - F$. Then near $x$, one can find two points $y$ and $z$ of $F$ which can be joined by a path $\alpha$ which is transverse to $\mathcal{F}$. Let $\beta$ be a leaf path joining $y$ and $z$. Then the composite of $\alpha$ and $\beta$ can be modified slightly to become a transversal.

If all the leaves are compact, then by the Reeb stability theorem ([41], [26]), $\mathcal{F}$ yields a bundle structure of $M$ over $S^1$. In this case it is easy to find a transverse. □

Now notice that if the fundamental group of $M$ is finite, then any foliation on it admits a null transversal. This shows the following corollary of Theorem 5.6.

**Corollary 5.8** *Suppose $M$ is a compact manifold with finite fundamental group. Then $M$ does not admit a foliation with IFP. Especially it does not admit a real analytic foliation.* □

Let us provide an outline of Theorem 5.6. we shall give an outline. Suppose that $\mathcal{F}$ is a foliation with IFP and admits a null transversal $g' : S^1 \to M$. Then $g'$ extends to a map $g : D^2 \to M$. Passing to a small perturbation if necessary, one may assume that $g$ is in general position w. r. t. $\mathcal{F}$. Then the induced foliation $g^*\mathcal{F}$ on $D^2$ has finitely many singularities of the following two types. One is a so called center. Around it $g^*\mathcal{F}$ is a foliation by circles. The other is a saddle. Near by, $g^*\mathcal{F}$ is topologically conjugate to the foliation by $\{x^2 - y^2 = c\}$.

Now since $g|_{S^1}$ is transverse to $\mathcal{F}$, $g^*\mathcal{F}$ must be transverse to $S^1$. By the Poincaré-Hopf theorem, the number of centers minus the number of saddles must be one. In particular, there always exits a center. Any leaf near the center do not intersect $S^1$ Let $C$ be the union of leaves which do not intersect $S^1$. Then $C$ is a closed set. Furthermore one can show that $C$ is connected. The boundary $\partial C$ is a union of finite leaves, possibly including singularities. Let $b : S^1 \to \partial C$ be a surjective submersion, moving anti-clockwise on $\partial C$. $b$ gives a leaf curve of $\mathcal{F}$, also denoted by $b$. By some abuse denote by $(-1, 1)$ a curve through the base point of $b$ which is transverse to $\mathcal{F}$. (0 corresponds to the base point.) Suppose the side $(-1, 0]$ corresponds to the part $C$. Then the holonomy of $b$ must be the identity on this side. This can be shown by looking at the figure of $g^*\mathcal{F}$ inside $C$. On the other hand outside $C$, any leaf must reach $S^1$. This implies that there are no fixed point in $(0, 1)$, contradicting IFP. This finishes the outlined proof of Theorem 5.6.

The general position argument for continuous map and continuous foliation is well developed. One can deal with them as if working in the smooth category. See e. g. [45].

From now on let $\mathcal{F}$ be a foliation with IFP. Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal covering space $\widetilde{M}$. Consider the leaf space $\mathcal{X} = \widetilde{\mathcal{M}}/\tilde{\mathcal{F}}$.

**Lemma 5.9** *The leaf space $\mathcal{X}$ is a 1-connected (possibly non-Hausdorff) 1-manifold.*

**Proof** Let $F$ be a leaf of $\tilde{\mathcal{F}}$ and let $I$ be a transverse open arc passing through $F$. Then any leaf $F'$ of $\tilde{\mathcal{F}}$ intersects $I$ at at most one point. For, if it intersects at two points, then a curve in $I$ and a curve in $F'$ joining the two points form a loop. This loop can be modified to be transverse to $\tilde{\mathcal{F}}$. Since $\widetilde{M}$ is simply connected, we get a null transversal, contradicting Theorem 5.6.

That is, $I$ can be considered to be an open neighbourhood of $F \in \mathcal{X}$. Thus any point of $\mathcal{X}$ has a neighbourhood homeomorphic to an open interval. This shows that $\mathcal{X}$ is a 1-manifold. That $\mathcal{X}$ is 1-connected follows from the corresponding properties of $\widetilde{M}$. □

Fundamental properties about 1-connected 1-manifolds induced from foliations are found in [30]. See also [4].

**Definition 5.10** A pair of points $x_1$ and $x_2$ in $\mathcal{X}$ is called a *nonseparating pair* if any neighbourhood of $x_1$ intersects any neighbourhood of $x_2$. The point $x_i$ is called a *nonseparating point.*

The following lemma follows more or less obviously from the 1-connectedness of $\mathcal{X}$. For a detailed proof, see [30].

**Lemma 5.11** *Let $x_1$ and $x_2$ form a nonseparating pair. Then there exist embeddings $f_i : (-\epsilon, \epsilon) \to \mathcal{X}$ such that $f_i(0) = x_i$ $(i = 1, 2)$ with the following properties.*

1. $f_1((-\epsilon, 0)) = f_2((-\epsilon, 0))$.
2. $f_1([0, \epsilon))$ *and* $f_2([0, \epsilon))$ *do not intersect.*

Fix once and for all an orientation of $\mathcal{X}$.

**Definition 5.12** The pair $x_i$ is called a *right nonseparating pair* if the embeddings $f_i$ are orientation preserving. Otherwise it is called a *left nonseparating pair.*

**Definition 5.13** The 1-manifold $\mathcal{X}$ is called *of type W* if it admits both right and left separating pairs, *of type I* if it admits none, and *of type V* otherwise.

Now since the fundamental group $\pi_1(M)$ acts on $\widetilde{M}$ in a way to preserve the foliation $\widetilde{\mathcal{F}}$, there is an induced action of $\pi_1(M)$ on $\mathcal{X}$. Since the foliation $\mathcal{F}$ is oriented, the action is orientation preserving.

Given a leaf $F$ of $\mathcal{F}$, consider its inverse image in $\widetilde{M}$. This corresponds to a $\pi_1(M)$-orbit $O$ in $\mathcal{X}$. Properties of $F$ can be translated into properties of $O$. For example $F$ is closed if and only if $O$ is; $F$ is dense if and only if $O$ is.

Also the property IFP is interpreted as a condition of the action. Let $b$ be a closed leaf curve of $\mathcal{F}$ based at a point $x_0$ and let $a$ be a path in $M$ joining the base point of the manifold $M$ to $x_0$. Then the point $x_0$ together with the path $a$ designates a point in $\widetilde{M}$ and hence a point, say $x$, in $\mathcal{X}$. Now let $\gamma$ be an element of $\pi_1(M)$ which is given by the composite loop $aba^{-1}$. Then the action of $\gamma$ keeps $x$ fixed. The germ at $x$ of the action coincides with the leaf holonomy along $b$. Conversely the germ at any fixed point of any element of $\pi_1(M)$ corresponds to the leaf holonomy of some leaf curve.

Denote by $\mathrm{Fix}(\gamma)$ the fixed point set of the action of $\gamma \in \pi_1(M)$ on $\mathcal{X}$. Then IFP implies that if $x \in \mathrm{Fix}(\gamma)$ and if $J$ is a sufficiently small interval neighbourhood of $x$, then either $J$ is contained in $\mathrm{Fix}(\gamma)$ or $\mathrm{Fix}(\gamma) \cap J = \{x\}$.

Notice that $\mathrm{Fix}(\gamma)$ is *not closed* in general. (Imagin the exchange of branches.) Let us summerize the property IFP in terms of the action on $\mathcal{X}$.

**Definition 5.14** Let $S$ be a subset of $\mathcal{X}$. A point $x$ of $S$ is called *isolated in* $S$ if $x$ admits a neighbourhood $J$ such that $S \cap J = \{x\}$.

**Lemma 5.15** *Suppose the foliation $\mathcal{F}$ satisfies IFP and let $\gamma$ be an arbitrary element of $\pi_1(M)$. Then any componet of* $\mathrm{Fix}(\gamma)$ *is either an open set or a point which is isolated in* $\mathrm{Fix}(\gamma)$. □

## 5.3 The developing theorem

Again $\mathcal{F}$ is to be an oritented continuous foliation on a manifold $M$. Throughout this subsection, we will work under the following;

**Assumption 5.16 (IFP)** *Any leaf holonomy is either trivial or has an isolated fixed point.*

**(ALD)** *All the leaves of $\mathcal{F}$ are dense in $M$.*

**(SOL)** *The fundamental group of $M$ is solvable.*

The purpose of this section is to show the following theorem

**Theorem 5.17 (Developing theorem)** *Suppose the foliation $\mathcal{F}$ satisfies IFP, ALD and SOL. Then there exist a homomorphism $h$ from $\pi_1(M)$ into the group of orientation preserving homeomorphisms of* $\mathbf{R}$ *and a $h$-equivariant continuous submersion $D : \widetilde{M} \to \mathbf{R}$ such that the pull back of the point foliation of* $\mathbf{R}$ *by $D$ coincides with the lift of $\mathcal{F}$.*

In fact the condition ALD can be replaced by a weaker condition that $\mathcal{F}$ does not admit a closed leaf. (See [33].) The rest of this subsection is devoted to the proof of Theorem 5.17.

We consider the orientation preserving action of $\pi_1(M)$ on the 1-manifold $\mathcal{X}$. By ALD all the orbits are dense in $\mathcal{X}$.

If $\mathcal{X}$ is of type I, that is, homeomorphic to $\mathbf{R}$, then the canonical projection from $\widetilde{M}$ onto $\mathcal{X}$ serves as the submersion $D$ and there are noting to prove. So we assume that $\mathcal{X}$ *is either type V or W.*

First we shall show that type W case is impossible. A key fact is the following lemma.

**Lemma 5.18** *Let $\mathcal{X}$ be of type W. Suppose there exists a connected open set $P$ of $\mathcal{X}$ such that the boundary of $P$ consists of exactly 4-points, $a$, $b$, $c$ and $d$. Assume there exist elements $f$ and $g$ of $\pi_1(M)$ which satisfy $f(a) = b$ and $g(c) = d$ and send $P$ outside $P$. Then $f$ and $g$ generate a free subgroup of $\pi_1(M)$.*

**Proof** The proof is similar to the Poincaré polygon theorem exposed in section 2. Let $\Gamma$ be the subgroup of $\pi_1(M)$ generated by $f$ and $g$. We shall construct a model of the action of $\Gamma$ on $\mathcal{X}$. For this let $\widehat{\Gamma}$ be the abstract free group generated by $f$ and $g$. There is a canonical epimorphism $p : \widehat{\Gamma} \to \Gamma$.

Now as in the proof of the Poincaré polygon theorem, consider the direct product $\widehat{\Gamma} \times \mathrm{Cl}(P)$ and introduce an equivalence relation $\sim$ in $\widehat{\Gamma} \times \partial P$ by

$$(\gamma, x) \sim (\gamma', x') \quad \text{if} \quad \gamma^{-1}\gamma' = f^{\pm 1},\ g^{\pm 1} \quad \text{and} \quad \gamma(x) = \gamma'(x').$$

Then the quotient $\widehat{P} = \widehat{\Gamma} \times \mathrm{Cl}(P)/\sim$ becomes a 1-manifold. Furthermore one can define in a canonical way a $\widehat{\Gamma}$-action on $\widehat{P}$ and a $p$-equivariant immersion $q : \widehat{P} \to P$. Finally one can show that $q$ is an injection. But this implies that $p$ is an isomorphism, as is desired. A full proof will be found in [30]. □

**Lemma 5.19** *$\mathcal{X}$ cannot be of type W.*

**Proof** Suppose $\mathcal{X}$ is of type W. Choose orientation preserving embeddings $u, v : [0,3] \to \mathcal{X}$ such that $u((1,2)) = v((1,2))$ and that $u([0,1])$, $u([2,3])$, $v([0,1])$ and $v([2,3])$ are mutually disjoint. (By ALD right nonseparating points and left nonseparating points are both dense in $\mathcal{X}$. Thus we can find such embeddings.) By ALD there are four points in the same orbit, $a$ in $u([0,1/2])$, $b$ in $u([5/2,3]$, $c$ in $v([0,1/2])$, $d$ in $v([5/2,3]$. Now let $P$ be the open regeon surrounded by the four points. Since the four points are in the same orbit, one can find elements $f$ and $g$ which satisfy the condition of Lemma 5.18. Thus by Lemma 5.18, there exists a free subgroup on two generators in $\pi_1(M)$. But this is against the hypothesis SOL. The claim is now proved. □

From now on, we assume that $\mathcal{X}$ is of type V. To fix the idea, we assume that all the nonseparating pairs are right nonseparating, that is, there is only one end in the $-\infty$-direction, and there are infinitely many ends in the $\infty$-direction. Denote $x \sim y$ if $x$ and $y$ form a nonseparating pair. Since $\mathcal{X}$ is of type V, this is an equivalence relation.

**Notation 5.20** Denote $x \prec y$ if there exists an orientation preserving embedding $f : [0,1] \to \mathcal{X}$ such that $f(0) = x$ and $f(1) = y$. Denote $x \preceq y$ if either $x \prec y$ or $x = y$.

We shall summerize easy properties of the relation $\prec$ in the following lemma.

**Lemma 5.21** *1. The relation $\preceq$ is a partial order.*

*2. For any point $y$ and $z$, there exists a point $x$ such that $x \prec y$ and $x \prec z$.*

*3. If $x \prec z$ and $y \prec z$, then we have either $x \prec y$, $x = y$ or $y \prec x$.* □

**Notation 5.22** *1. For $x \in \mathcal{X}$, denote $(-\infty, x] = \{y \in \mathcal{X} \mid y \preceq x\}$.*

*2. If $x \prec y$, denote $[x, y] = \{z \mid x \preceq z \preceq y\}$.*

As a matter of fact they are homeomorphic to closed intervals in $\mathbf{R}$.

Let $\Gamma$ be the quotient of $\pi_1(M)$ by the normal subgroup formed by the elements which act on $\mathcal{X}$ trivially. Thus $\Gamma$ acts on $\mathcal{X}$ effectively. Of course $\Gamma$ is also a solvable group.

For any $\gamma \in \Gamma$, denote

$$\mathrm{Fix}^{\sim}(\gamma) = \{x \in \mathcal{X} \mid \gamma(x) \sim x\}.$$

Clearly $\mathrm{Fix}^{\sim}(\gamma)$ is a union of $\sim$ equivalence classes, and contains $\mathrm{Fix}(\gamma)$.

Also we make the following definition.

**Definition 5.23** A subset $F$ of $\mathcal{X}$ is called *discrete*, if the intersection of $F$ with any compact interval in $\mathcal{X}$ is a finite set.

**Lemma 5.24** *Suppose* $\mathrm{Fix}^{\sim}(\gamma) = \emptyset$ *for some $\gamma \in \Gamma$. Then $\gamma$ admits a unique $\gamma$-invariant properly*[20] *embedded copy of the real line* $\mathrm{Axis}(\gamma)$*, called the axis of $\gamma$.*

**Proof** Given any point $x$, there exists a point $z$ such that $z \prec x$ and $z \prec \gamma x$. Then $\gamma z \prec \gamma x$ and $z \prec \gamma x$ hold. Therefore we have either $\gamma z \prec z$ or $z \prec \gamma z$. ($\gamma z = z$ cannot occur since $\mathrm{Fix}^{\sim}(\gamma) = \emptyset$.)

[20] Proper means that the inverse image of a compact set is compact.

Suppose, to fix the idea, that $\gamma z \prec z$. Then

$$\mathrm{Axis}(\gamma) = \bigcup_{i \in \mathbf{Z}} \gamma^i([\gamma z, z])$$

is an embedded copy of the real line, invariant by $\gamma$. Since $\mathrm{Fix}^{\sim}(\gamma) = \emptyset$, the orbit $\{\gamma^i z \mid i \in \mathbf{Z}\}$ is a discrete set. From this follows that $\mathrm{Axis}(\gamma)$ is properly embedded.

The proof of the uniqueness is left to the reader. □

Now let

$$\Gamma > \Gamma_1 > \Gamma_2 > \cdots\cdots > \Gamma_r = \Omega > \{e\}$$

be the descending sequence of the solvable group $\Gamma$. As is well known, each $\Gamma_i$ is a normal subgroup. We are interested in the action of the last abelian subgroup $\Omega$ on $\mathcal{X}$. We shall show that any element of $\Omega$ has a nonempty fixed point set of special feature. This implies that the quotient $\mathcal{X}/\Omega$ is again a 1-connected 1-manifold on which the quotient group $\Gamma/\Omega$ acts.

The first step is to show that any element of $\Omega$ admits a fixed point. This will be done in two steps.

**Lemma 5.25** *For any* $\omega \in \Omega$, $\mathrm{Fix}^{\sim}(\omega) \neq \emptyset$.

**Proof** Suppose that $\mathrm{Fix}^{\sim}(\omega) = \emptyset$ for some $\omega \in \Omega$. Then it admits $\mathrm{Axis}(\omega)$. For any $\gamma \in \Gamma$, the conjugate $\gamma\omega\gamma^{-1}$ also has an axis, and we have

$$\mathrm{Axis}(\gamma\omega\gamma^{-1}) = \gamma\mathrm{Axis}(\omega).$$

On the other hand since $\gamma\omega\gamma^{-1}$ belongs to $\Omega$, $\gamma\omega\gamma^{-1}$ commutes with $\omega$. This implies easily that their axes coincide. That is,

$$\gamma\mathrm{Axis}(\omega) = \mathrm{Axis}(\omega).$$

Since $\gamma$ is an arbitrary element of $\Gamma$, this means that $\mathrm{Axis}(\omega)$ is invariant by $\Gamma$. But this is absurd, since by the assumption ALD, all the $\Gamma$-orbits must be dense in $\mathcal{X}$. □

**Lemma 5.26** *For any* $\omega \in \Omega$, $\mathrm{Fix}(\omega) \neq \emptyset$.

**Proof** Suppose $\mathrm{Fix}(\omega) = \emptyset$ for some $\omega$. By the previous lemma, we have $\mathrm{Fix}^{\sim}(\omega) \neq \emptyset$. Let $C$ be a $\sim$ class contained in $\mathrm{Fix}^{\sim}(\omega)$. Let us show that

$$\mathrm{Fix}^{\sim}(\omega) = C.$$

Choose a point $x$ from $\mathcal{X} - C$. First of all if the point $x$ satisfies $c \prec x$ for some $c$ in $C$. Then $\omega c \prec \omega x$, and $\omega c$ also belongs to $C$. But we have $\omega c \neq c$, since we are assuming that $\mathrm{Fix}(\omega) = \emptyset$. Now this shows that $x \not\sim \omega x$.

Secondly consider the case $x \prec c$ for some (and any) element $c$ of $C$. Then we have either $\omega x \prec x$ or $x \prec \omega x$. In any case $x \not\sim \omega x$.

In the remaining case, $x$ lies on a half line (toward $\infty$ direction) starting at a point $x'$ such that $x' \prec c$. Then the above observation about $x'$ also shows that $x \not\sim \omega x$.

Now we have shown that $\mathrm{Fix}^{\sim}(\omega)$ is a single $\sim$ class $C$. An argument similar to the proof of the previous lemma shows that $C$ is invariant by the action of $\Gamma$. A contradiction. □

The next step is to show that there exists an $\Omega$-invariant discrete subset of $\mathcal{X}$. First we need some preparation.

**Notation 5.27** *For a subset $F$ of $\mathcal{X}$, let*

$$\widehat{F} = \{x \mid y \preceq x \preceq z, \quad \exists y, z \in F\}.$$

The proof of the following lemma is left to the readers.

**Lemma 5.28** *For any subset $F$ of $\mathcal{X}$, the boundary $\partial\widehat{F}$ is discrete.*

**Lemma 5.29** *For any nontrivial element $\gamma$ of $\Gamma$, either* $\mathrm{Fix}(\gamma)$ *is discrete, or* $\widehat{\mathrm{Fix}}(\gamma) \neq \mathcal{X}$.

**Proof** Assume that $\mathrm{Fix}(\gamma)$ is not discrete. That is, there exists a sequence $\{x_n\}$ in $\mathrm{Fix}(\gamma)$ such that $x_n \to x_0$. Presumably $x_0$ may not belong to $\mathrm{Fix}(\gamma)$, but it belongs to $\mathrm{Fix}^{\sim}(\gamma)$. Suppose for contradiction that $\widehat{\mathrm{Fix}}(\gamma) = \mathcal{X}$. Then there exists $y \in \mathrm{Fix}(\gamma)$ such that $x_0 \prec y$. This shows $x_0 \in \mathrm{Fix}(\gamma)$. By Lemma 5.15, IFP implies that $\gamma$ is the identity near $x_0$.

Consider the interval $(-\infty, x_0]$. This is invariant by $\gamma$ since $x_0$ is a fixed point. Since $\gamma$ is the identity near $x_0$, IFP implies that $\gamma$ is the identity on the whole of $(-\infty, x_0]$.

Now choose any point $x \in \mathcal{X}$. Then there exist points $z_1$ and $z_2$ such that $x \in [z_1, z_2]$, $z_1 \in (-\infty, x_0)$ and $z_2 \in \mathrm{Fix}(\gamma)$. (This follows from the assumption $\widehat{\mathrm{Fix}}(\gamma) = \mathcal{X}$.) Of course the interval $[z_1, z_2]$ is invariant by $\gamma$. Notice that $[z_1, z_2]$ intersects $(-\infty, x_0)$ in an open interval. Then by IFP we have $x \in \mathrm{Fix}(\gamma)$. Since $x$ is arbitrary, we get that the action of $\gamma$ on $\mathcal{X}$ is trivial. Since $\Gamma$ acts on $\mathcal{X}$ effectively, $\gamma$ must be the identity. A contradiction. □

**Lemma 5.30** *There exists a nonempty discrete subset $S$ of $\mathcal{X}$ which is invariant by $\Omega$.*

**Proof** Choose an arbitrary nontrivial element $\omega$ in $\Omega$. Consider the case where $\mathrm{Fix}(\omega)$ is discrete. Then it is invariant by $\Omega$ since $\Omega$ is abelian. On the contrary if $\mathrm{Fix}(\omega)$ is nondiscrete, then by Lemmas 5.28, 5.29, the set $\partial\widehat{\mathrm{Fix}}(\omega)$ is a nonempty discrete subset. It is also invariant by $\Omega$. □

Let $S$ be a discrete subset invariant by $\Omega$. Then for any $\gamma$ in $\Gamma$ and for any $\omega$ in $\Omega$, we have

$$\gamma^{-1}\omega\gamma S = S.$$

That is,

$$\omega(\gamma S) = \gamma S.$$

Thus the set $\gamma S$ is invariant by $\Omega$. Let

$$\Gamma = \{\gamma_1, \gamma_2, \cdots\cdots\}$$

and for a positive integer $i$ let

$$S_i = \bigcup_{j=1}^{i} \gamma_j S.$$

This yields an increasing sequence of discrete subsets invariant by $\Omega$. By ALD, we have $\cup_i S_i$ is dense in $\mathcal{X}$.

Using this, we show in the next lemma a special feature of $\mathrm{Fix}(\omega)$.

**Lemma 5.31** *Let $\omega \in \Omega$. If $x \in \mathrm{Fix}(\omega)$ and if $y \prec x$, then $y \in \mathrm{Fix}(\omega)$.*

**Proof** Let $x \in \mathrm{Fix}(\omega)$. The interval $(-\infty, x]$ is invariant by $\omega$. For any large $i$, consider a point $y$ in $S_i \cap (-\infty, x)$. Then $\omega y = y$. For if not, we have either $\omega y \prec y$ or $y \prec \omega y$. Accordingly either $\lim_{n\to-\infty} \omega^n y$ or $\lim_{n\to\infty} \omega^n y$ is a point in $(-\infty, x]$. This contradicts that $S_i$ is discrete.

Thus we have

$$S_i \cap (-\infty, x] \subset \mathrm{Fix}(\omega).$$

Since $\cup_i S_i$ is dense in $\mathcal{X}$, we have $(-\infty, x] \subset \mathrm{Fix}(\omega)$, as is desired. □

The proof of the following corollary is left to the readers.

**Corollary 5.32** *The quotient $\mathcal{X}/\Omega$ is a 1-connected 1-manifold on which the group $\Gamma/\Omega$ acts with the same properties as the action of $\Gamma$ on $\mathcal{X}$.* □

**Proof of Theorem 5.17** If $\mathcal{X}/\Omega$ is of type I, then we are done. If not, we repeat the same argument to the action of $\Gamma/\Omega$ on $\mathcal{X}/\Omega$. But this time the step of the solvable group $\Gamma/\Omega$ is smaller than that of $\Gamma$. We continue the same argument. Then at some stage we obtain a 1-manifold of type I as the quotient of $\mathcal{X}$. For if not, we would get a 1-manifold on which the trivial group acts in such a way that all the orbits are dense. □

## 5.4 Solvable group acting on the line

Here we shall show a theorem, due to J. Plante, which asserts that a certain action of a solvable group on the real line $\mathbf{R}$ is conjugate to an action by affine transformations.

As a corollary, we show in Theorem 5.39 that if a codimension one foliation $\mathcal{F}$ satisfies IFP, ALD and SOL, then $\mathcal{F}$ is topologically an affine foliation.

Let $G$ be a countable solvable group acting on $\mathbf{R}$ orientation preservingly. Throughout this subsection we will work under the following;

**Assumption 5.33** *For any nontrivial element $g \in G$, the fixed point set*

$$\mathrm{Fix}(g) = \{x \in \mathbf{R} \mid g(x) = x\}$$

*is discrete.*

By the word *measure* on $\mathbf{R}$, we always mean a measure which is finite on any compact subset of $\mathbf{R}$.

**Definition 5.34** A measure $\mu$ on $\mathbf{R}$ is called *$G$-quasi-invariant* if there exists a homomorphism $a$ from $G$ into the multiplicative group $\mathbf{R}_{>0}$ such that $g_*\mu = a(g)\mu$ for any $g \in G$.

The following is the main theorem of this subsection which we shall establish later.

**Theorem 5.35 (Plante)** *Under Assumption 5.33, there exists a $G$-quasi-invariant measure.*

**Corollary 5.36** *Assume further that all the $G$-orbits are dense. Then there exists a homeomorphism $k : \mathbf{R} \to \mathbf{R}$ and a homomorphism $\phi$ from $G$ to $Aff_+(\mathbf{R})$ such that*

$$k(gx) = \phi(g)k(x) \quad (\forall g \in G, \quad \forall x \in \mathbf{R}). \tag{29}$$

Before giving the proof, we make some conventions.

**Notation 5.37** For any $a, b \in \mathbf{R}$, $a \diamond b$ denotes a signed interval as follows.

$$a \diamond b = \begin{cases} [a, b) & \text{if} \quad a < b \\ \emptyset & \text{if} \quad a = b \\ -[b, a) & \text{if} \quad a > b. \end{cases}$$

Also for a measure $\mu$, $\mu(a \diamond b)$ means an obvious signed value.

This convention is natural and useful, as we see in the next obvious lemma.

**Lemma 5.38** *For any $a$, $b$, $c \in \mathbf{R}$ and a measure $\mu$, we have*

$$\mu(a \diamond c) = \mu(a \diamond b) + \mu(b \diamond c).$$

□

**Proof of 5.36** Choose any $G$-quasi-invariant measure $\mu$. Fix a base point $x_0 \in \mathbf{R}$ and define a map $k$ by

$$k(x) = \mu(x_0 \diamond x).$$

Since all the orbits of the $G$-action are dense in $\mathbf{R}$, there is no atom in $\mu$. Therefore $k$ is a homeomorphism into $\mathbf{R}$.

Next define a map $b : G \to \mathbf{R}$ by

$$b(g) = \mu(x_0 \diamond g(x_0)).$$

Finally define $\phi$ by

$$\phi(g)(x) = a(g)x + b(g).$$

Here $a$ is, of course, a multiplicative homomorphism associated to the $G$-quasi-invariant measure $\mu$ as in 5.34. Now it is a routine work to establish equation (29) using Lemma 5.38.

Finally let us show that $k$ can be taken to be a surjective homeomorphism. If not the image $k(\mathbf{R})$ is an open interval invariant by the subgroup $\phi(G) \subset \mathit{Aff}_+(\mathbf{R})$. First of all notice that $k(\mathbf{R})$ must be an infinite interval. For if not, the action of $\phi(G)$ on $k(\mathbf{R})$ is trivial, contradicting the assumption that all the $G$-orbits are dense.

For simplicity, let $k(\mathbf{R}) = (0, \infty)$. The action of $\phi(G)$ is by multiplications. Now replace $k$ by $\log \circ k$. Then the multiplications are replaced by the translations.

□

**Theorem 5.39** *Suppose a codimension one transversely oriented $C^0$ foliation $\mathcal{F}$ on a manifold $M$ satisfies IFP, ALD and SOL. Then $\mathcal{F}$ is a $C^0$ affine foliation.*

**Proof** By Theorem 5.17, we have an equivalent pair

$$(h, D) : (\pi_1(M), \widetilde{M}) \longrightarrow (\mathit{Homeo}^+(\mathbf{R}), \mathbf{R}).$$

By IFP the action of the group $G = h(\pi_1(M))$ satisfies Assumption 5.33. Also by ALD all the $G$-orbits are dense in $\mathbf{R}$. Thus the theorem follows from Corollary 5.36.

□

Before giving the proof of Theorem 5.35, we prepare some lemmas.

**Lemma 5.40** *Suppose that a countable abelian group $H$ acts on a compact metric space $X$. Then there is a $H$-invariant probability measure on $X$.*

**Proof** We only treat the case where $H$ is infinitely generated. Let $g_1, g_2, \cdots$ be generators of $H$ and let $\mu$ be an arbitrary probability measure of $X$. Then, since the space of probabilities is compact, the sequence

$$\frac{1}{N}\sum_{i=0}^{N-1}(g_1^i)_*\mu$$

has an accumulation point $\mu_1$. Clearly $\mu_1$ is invariant by $g_1$. Likewise an accumulation point $\mu_2$ of the sequence

$$\frac{1}{N}\sum_{i=0}^{N-1}(g_2^i)_*\mu_1$$

is invariant by both $g_1$ and $g_2$. This way, we get a sequence $\mu_1, \mu_2, \cdots$. Clearly an accumulation point of this sequence is a $H$–invariant probability. □

Suppose a group $H$ acts on $\mathbf{R}$ *with an invariant measure* $\mu$. As preparations for the proof of Theorem 5.35, We study properties of such actions.

**Definition 5.41** For $g \in H$, *the translation number* $b_\mu(g) \in \mathbf{R}$ is defined by

$$b_\mu(g) = \mu(x \diamond g(x)),$$

where $x$ is any point of $\mathbf{R}$.

We already defined a homomorphism $b$ in the proof of 5.36. There the measure $\mu$ was quasi-invariant, and therefore we need to fix a base point $x_0$. In contrast, the above definition is made under the hypothesis that $\mu$ is $H$-invariant. This yields much stronger properties of the mapping $b_\mu$. We shall state two lemmas and leave their proofs to the readers.

**Lemma 5.42** *The definition of* $b_\mu(g)$ *is independent of the choice of the point* $x$*, and the mapping* $b_\mu : H \to \mathbf{R}$ *is a homomorphism.* □

**Lemma 5.43** $b_\mu(g) = 0$ *if and only if* $g$ *has a fixed point.* □

We want to establish a group version of the last lemma. For this, denote by $\mathrm{Fix}(H)$ the set of points in $\mathbf{R}$ which is fixed by all the elements of $H$.

**Lemma 5.44**
1. *The homomorphism* $b_\mu$ *is identically zero if and only if* $\mathrm{Fix}(H)$ *is nonempty.*
2. $K = \mathrm{Ker}(b_\mu)$ *is the maximal subgroup of* $H$ *for which* $\mathrm{Fix}(K)$ *is nonempty.*

**Proof** The if part of (1) is immediate. Conversely assume that $b_\mu$ is identically zero. It suffices to show that $\mathrm{Supp}(\mu)$ is contained in $\mathrm{Fix}(H)$. Let $x \in \mathrm{Supp}(\mu)$ and assume that there exists an element $g \in H$ which does not fix $x$. Assume, to fix the idea, that $x < g(x)$. Then the interval $[g^{-1}(x), g(x))$ is a neighbourhood of $x$ and still we have $\mu([g^{-1}(x), g(x))) = b_\mu(g^2) = 2b_\mu(g) = 0$. A contradiction.

(2) follows immediately from (1). □

**Lemma 5.45** *Let $\mu$ and $\nu$ be arbitrary two $H$-invariant measures.*

1. *The homomorphisms $b_\mu$ and $b_\nu$ are parallel, that is, $b_\mu = \lambda b_\nu$ for some $\lambda > 0$.*

2. *If $b_\mu = b_\nu$ and $b_\mu(H)$ is dense in $\mathbf{R}$, then we have $\mu = \nu$.*

**Proof** First notice that the characterization of $\mathrm{Ker}(b_\mu)$ in the above lemma implies that $K = \mathrm{Ker}(b_\mu) = \mathrm{Ker}(b_\nu)$. The quotient group $H/K$ is an abelian group injected into $\mathbf{R}$ in two ways, by $b_\mu$ and by $b_\nu$. Clearly there is a well defined action of $H/K$ on the set $\mathrm{Fix}(K)$. Since this action is free, the order of $\mathrm{Fix}(K)$ yields, in an obvious way, a total order on $H/K$, invariant by the left mutipications (an Archimedian order). Then both homomorphisms $b_\mu$ and $b_\nu$ are order preserving. It is well known, easy to show, that the two homomorphisms are parallel. This shows (1).

To show (2), let us show first that $\mathrm{Supp}(\mu)$ is contained in the closure $C$ of an arbitrary orbit in $\mathbf{R}$. For if not, there would be an interval $(a, b)$ such that $\mu((a, b)) > 0$ and that $C$ does not intersect $(a, b)$. Then using any point in $C$ as a base point to define $b_\mu$, one would get that the image $b_\mu(H)$ is not dense in $\mathbf{R}$.

Of course, the same property holds for $\mathrm{Supp}(\nu)$, and in particular we have $S = \mathrm{Supp}(\mu) = \mathrm{Supp}(\nu)$ is the unique minimal set of the action. Consider an arbitrary orbit in $S$ (dense in $S$). Comparing the two measures $\mu$ and $\nu$ on $a \diamond b$ for any two points $a$ and $b$ in the orbit, one clearly gets that $\mu = \nu$. □

**Proof of Theorem 5.35** Denote by

$$G = G_0 > G_1 > G_2 > \cdots\cdots > G_n > \{0\}$$

the descending sequence of the solvable group $G$.

**Case 1** *The action of $G_n$ is not free.*

Let $K$ be the set of nontrivial element $k \in G_n$ such that $\mathrm{Fix}(k) \neq \emptyset$. By Asumption 5.33, the set $F = \mathrm{Fix}(k)$ is a discrete set. Also notice that any orbit of $k$ other than a fixed point is nondiscrete. For any other element $k'$ of $K$, $\mathrm{Fix}(k')$ is $k$-invariant since $G_n$ is an abelian group. Since it is discrete, $\mathrm{Fix}(k')$ must be contained in $F$. Symmetry shows that $\mathrm{Fix}(k') = F$. Now the conjagation by any element of $G$ keeps the subset $K$ of $G_n$ invariant, and therefore the set $F$ is invariant by the action of $G$. The union of point mass supported on $F$ is a $G$-invariant measure. This proves the theorem.

**Case 2** *$G_n$ acts on $\mathbf{R}$ freely.*

Choose any nontrivial element $h$ of $G_n$. Then since $G_n$ is abelian, there is a well-defined action of the quotient group $G_n/ < h >$ on the circle $\mathbf{R}/ < h >$. By Lemma 5.40, we have an invariant probability. It gives birth to a measure on $\mathbf{R}$, invariant by $G_n$.

Suppose that $G_i$ be the largest subgroup in the descending sequence which has an invariant measure $\mu$. Assume that $i > 0$, for otherwise there is nothing to prove. Since $G_i$ is a normal subgroup of $G$, the induced measure $g_*\mu$ is again a $G_i$-invariant measure for any element $g \in G$. Using (1) of Lemma 5.45 define a homomorphism $a : G \to \mathbf{R}_{>0}$ by

$$b_{g_*\mu} = a(g)b_\mu.$$

We have

$$b_\mu(g^{-1}hg) = a(g)b_\mu(h) \quad (\forall g \in G, \ \ \forall h \in G_i), \tag{30}$$

because

$$\begin{aligned} b_\mu(g^{-1}hg) &= \mu(x \diamond g^{-1}hgx) = \mu(g^{-1}y \diamond g^{-1}hy) \\ &= g_*\mu(y \diamond hy) = b_{g_*\mu}(h) \\ &= a(g)b_\mu(h). \end{aligned}$$

Also it follows from equation (30) that $K = \mathrm{Ker}(b_\mu)$ is a normal subgroup of $G$.

**Claim** *$a$ is nontrivial.*

Suppose on the contrary that $a$ is trivial. First consider the case where $b_\mu(G_i)$ is isomorphic to $\mathbf{Z}$. Let $F = \mathrm{Fix}(K)$. Then the normality of $K$ implies that $F$ is invariant by $G$. Also by equation (30), we have $b_\mu(g^{-1}hg) = b_\mu(h)$ for $g \in G$ and $h \in G_i$. That is, $g^{-1}hg \equiv h \bmod K$.

Now the action of $G$ on $F$ is through the quotient $G/K$ and it commutes with the action of $G_i/K \approx \mathbf{Z}$. Therefore it induces an action of $G/G_i$ on the quotient space $F/G_i$. If we restrict this action to $G_{i-1}/G_i$, then it admits an invariant probability, since the group is abelian. This way, we get an invariant measure for $G_{i-1}$, contrary to the assumption that $G_i$ is the largest group with this property.

Next suppose that $b_\mu(G_i)$ is dense in $\mathbf{R}$. Then since $a$ is trivial, we have $g_*\mu = \mu$ by Lemma 5.45 (2). That is, there exists a $G$-invariant measure, contrary to the hypothesis. This shows Claim.

Let us finish the proof. We already know that the homomorphism $a$ is nontrivial. This implies by equation (30) that $g_\mu(G_i)$ is dense in $\mathbf{R}$. Thus we have $g_*\mu = a(g)\mu$. That is, $\mu$ is a $G$-quasi-invariant measure. $\square$

# 6 Codimension one Anosov flows on solvable manifolds

An Anosov diffeomorphism is said to be *codimension one* if either the stable or the unstable foliation is of codimension one. The structure of codimension one Anosov diffeomorphisms is well understood. We have the following result due to S. Newhouse [34].

Let $f_i$ be a diffeomorphism on a manifold $N_i$ $(i = 1, 2)$. $f_1$ and $f_2$ are said to be *topologically* (resp. $C^r$) *conjugate* if there exists a homeomorphism (resp. $C^r$ diffeomorphism) $h : N_1 \to N_2$ such that $f_2 \circ h = h \circ f_1$.

**Theorem 6.1** *A codimension one Anosov diffeomorphism on a closed manifold is topologically conjugate to a hyperbolic automorphism on the torus.*

The conclusion *topologically conjugate* is the best possible. We cannot hope $C^1$ conjugacy. To see this, just choose any hyperbolic automorphism $f$ of $T^n$ and consider a small perturbation $f'$ of $f$ which has eigenvalues different from those of $f$ at the fixed point. If the perturbation is small, then $f'$ is an Anosov diffeomorphism. By the structural stability theorem, $f'$ is topologically conjugate to $f$, but not $C^1$ conjugate because of the difference of the eigenvalues. Moreover since any hyperbolic automorphisms which are not mutually conjugate by an algebraic automorphism are not topologically conjugate[21], $f'$ is not $C^1$ conjugate to any hyperbolic automorphism.

Thus Theorem 6.1 is the *final theorem* for codimension one Anosov diffeomorphisms. We want to establish the same kind of theorems for codimension one Anosov *flows*. However the flow has more flexibility than the diffeomorphism. This forces us to add an extra condition about the manifold. We assume that the fundamental group of the underlying manifold is solvable. Then we get the following theorem, as conjectured by A. Verjovsky.

Let $\phi_i$ be a flow on a manifold $M_i$ $(i = 1, 2)$. $\phi_1$ and $\phi_2$ are said to be *topologically conjugate* if there exists a homeomorphism from $N_1$ to $N_2$ which sends an orbit of $\phi_1$ to an orbit of $\phi_2$ in a sense preserving way.

**Theorem 6.2** *Any codimension one Anosov flow on a closed manifold with solvable fundamental group is topologically conjugate to the suspension of a hyperbolic automorphism of the torus.*

[21] They give different outer automorphism of the fundamental group

In dimension three, examples of nonclassical Anosov flows are abundant ([14], [25], [23] [16]). But in higher dimension examples of codimension one Anosov flows are rather rare.

It is quite recent that A. El Kacimi constructed an example of nonsolvable Lie group which admits a codimension one Anosov flow and a cocompact discrete subgroup. This brings forth a codimension one Anosov flow on a compact manifold which is not the suspension of a hyperbolic toral automorphism, thus showing that the condition *solvable* is necessary even in higher dimension.

The purpose of this section is to give a proof of Theorem 6.2, following the arguments of D. Fried [15] and J. Plante [37, 36]. Let $\phi$ be an Anosov flow on a closed manifold $M$ which satisfies the conditions of Theorem 6.2.

By Theorem 6.1, we only need to show the following.

**Claim** *The flow $\phi$ admits a cross section.*

But by Proposition 4.19, the flow $\phi$ admits a cross section if and only if its lift to a finite covering does. Therefore we may pass to a double covering and assume that the codimension one weak stable foliation is transversely oriented.

When the dimension of the manifold is 3, Theorem 6.2 is already established by [2]. In this lecture notes, we only treat the case when the dimension of the manifolds $\geq 4$. Then by the Verjovsky theorem (Theorem 3.6), the weak stable foliation $\mathcal{V}^s$ has the property that all the leaves are dense. On the other hand, $\mathcal{V}^s$ satisfies the condition IFP of Section 5 (any leaf holonomy is either trivial or has an isolated fixed point).

Therefore by Theorem 5.39, the foliation $\mathcal{V}^s$ is topologically conjugate to a transversely affine foliaton. That is, we obtain an equivariant pair of maps

$$(h, D) : (\pi_1(M), \widetilde{M}) \longrightarrow (\mathit{Aff}_+(\mathbf{R}), \mathbf{R}),$$

where $D$ is a topological submersion. Since the foliation $\mathcal{V}^s$ is transversely oriented, the image of $h$ is contained in $\mathit{Aff}_+(\mathbf{R})$, the group of orientation preserving affine transformations.

For any $\delta \in \pi_1(M)$, $h(\delta)$ is an affine transformation of the form;

$$\mathbf{R} \ni x \mapsto a(\delta)x + b(\delta) \in \mathbf{R},$$

where $a(\delta)$ is positive. Define a homomorphism

$$\log h' : \pi_1(M) \to \mathbf{R}$$

by $\log h'(\delta) = \log a(\delta)$.

As is well known in algebraic topology, we have the following identification.

$$Homo(\pi_1(M), \mathbf{Z}) \approx [M, S^1] \approx H^1(M, \mathbf{Z}).$$

We also have

$$Homo(\pi_1(M), \mathbf{R}) \approx H^1(M, \mathbf{R}).$$

Thus the homomorphism $\log h'$ defines a class in $H^1(M, \mathbf{R})$, denoted by $C$. Our first trial is to show that the class $C$ satisfies the condition of Corollary 4.18, i. e. that $(A_\mu, C)$ is positive for any $\phi$-invariant probability measure $\mu$. But we only get the following partial result.

**Lemma 6.3** *Let $\gamma$ be a periodic orbit of $\phi$, with the corresponding $\phi$-invariant probability measure $\mu(\gamma)$. Then we have $(A_{\mu(\gamma)}, C) > 0$.*

**Proof** The homology class $A_{\mu(\gamma)}$ is nothing but $[\gamma]/s$, where $s$ is the period of $\gamma$. But $([\gamma], C) = \log h'(\gamma_0)$, where $\gamma_0$ is an element of $\pi_1(M)$ freely homotopic to $\gamma$. Therefore $([\gamma], C)$ is exactly the logarithm of the *linear part* of the leaf holonomy[22] of the foliation $\mathcal{V}^s$ along the periodic orbit $\gamma$. Now the strong unstable foliation $\mathcal{W}^u$ is one dimensional and transverse to $\mathcal{V}^s$. Therefore the expanding property of the flow $\phi$ along $\mathcal{W}^u$ shows that $([\gamma], C)$ is positive. □

The above mentioned expanding property of the flow $\phi$ along $\mathcal{W}^u$ strongly suggests that $(A_\mu, C)$ is positive for any $\phi$-invariant measure $\mu$. However it is rather difficult to show it in a direct manner. So we need to make use of a Markov partition, and consider an approximation of $C$ by a rational class in $H^1(M, \mathbf{Q})$.

Let $\mathcal{R} = \{R_i\}_{i=1}^r$ be a Markov partition for $\phi$. As before we denote $|\mathcal{R}| = \bigcup_i R_i$, and by $f : |\mathcal{R}| \to |\mathcal{R}|$ the first return map, with the return time map $\tau : |\mathcal{R}| \to \mathbf{R}$. There exist positive numbers $\tau_0$ and $\tau_1$ such that $\tau_0 \le \tau(x) \le \tau_1$ for any $x \in |\mathcal{R}|$.

An admissible sequence $\mathbf{i} = \{i_n\}$ is called *cyclic* if there exists a positive integer $q$ such that $i_{n+q} = i_n$ for any $n$. Then we denote

$$\mathbf{i} = ((i_0, i_1, \cdots, i_{q-1})).$$

A cyclic sequence $((i_0, i_1, \cdots, i_{q-1}))$ is called *minimal* if all the indices $i_\nu$ are distinct. The following easy fact will play an important role.

*There are but finite number of minimal cyclic admissible sequences.*

22 considered to be a transverse affine foliation by the topological conjugacy

Recall that an admissible sequence corresponds to a $\phi$-orbit; and especially a cyclic admissible sequence to a periodic orbit. Let

$$\Gamma = \{\gamma_k \mid 1 \le k \le m\}$$

be the family of all the periodic orbits associated with the minimal cyclic admissible sequences. Let $s_k$ be the period of $\gamma_k$.

By Lemma 6.3, we have $(A_{\mu(\gamma_k)}, C) > 0$ for any $1 \le k \le m$. Let $C' \in H^1(M, \mathbf{Q})$ be an approximation of $C$ such that $(A_{\mu(\gamma_k)}, C') > 0$ for any $1 \le k \le m$. Then muntiplying $C'$ by an appropriate integer, one obtains an integral class.

Let $g : M \to S^1$ be a mapping which represents this integral class. Then we have

$$(A_{\mu(\gamma_k)}, [g]) > \eta, \quad 1 \le \forall k \le m, \ \exists \eta > 0. \tag{31}$$

Our purpose is to show that $g$ satisfies the condition of Theorem 4.17. In fact, we will show the following stronger result.

**Claim** *For any sufficiently large $T > 0$, there exists $\epsilon > 0$ such that*

$$\frac{1}{T}\Delta g(x, T) > \epsilon, \quad \forall x \in M.$$

Now it is clear that we only need to show the claim for $x \in |\mathcal{R}|$. Let $T_0$ be a large number to be decided later. For a point $x \in |\mathcal{R}|$, let $\{i_n\}$ be an admissible sequence associated to it. That is, there exists a time sequence $\{t_n\}$ with

$$t_0 = 0, \quad \tau_0 \le t_{n+1} - t_n \le \tau_1, \quad x_n = \phi^{t_n}(x) \in R_{i_n}.$$

We are interested in the sequences $\{x_n\}_{n=0}^N$ and $\{t_n\}_{n=0}^N$ with $t_N \ge T_0$. Clearly it suffices to show the claim only for $T = t_N$.

Now changing the function $g : M \to S^1$ in its homotopy class, one may assume that $g$ is constant on each rectangle $R_i$. For any $0 \le n \le N$, the return time map $\tau$ and the first return map $f$ is continuous on $R_{i_n} \cap f^{-1}(R_{i_{n+1}})$. Therefore the number $\Delta_n g = \Delta g(x, \tau(x))$ is constant for any $x \in R_{i_n} \cap f^{-1}(R_{i_{n+1}})$. Clearly we have $|\Delta_n g| \le L$ for some constant $L$ determined only by $\mathcal{R}$ and $g$. We have

$$\Delta g(x, t_N) = \sum_{n=0}^{N-1} \Delta_n g. \tag{32}$$

Now from the sequence $\mathbf{i} = \{i_0, i_1, i_2 \cdots\cdots i_N\}$, one can choose a number $n_1$ and $q_1 > 0$ such that $i_{n_1} = i_{n_1+q_1}$ and that $i_{n_1}, i_{n_1+1}, \cdots i_{n_1+q_1-1}$ are all distinct. Let $\mathbf{j}_1 = ((i_{n_1}, \cdots, i_{n_1+q_1-1}))$ be the corresponding minimal cyclic admissible sequence. Then $\mathbf{j}_1$ corresponds to a periodic orbit $\gamma_{k_1}$ (of period $s_{k_1}$) in the family $\Gamma$. Then we have

$$\sum_{\nu=n_1}^{n_1+q_1-1} \Delta_\nu g = ([\gamma_{k_1}], [g]) \tag{33}$$

$$\frac{\tau_0}{\tau_1} \le \frac{t_{n_1+q_1} - t_{n_1}}{s_{k_1}} \le \frac{\tau_1}{\tau_0} \tag{34}$$

Now let

$$\mathbf{i}' = \{i_0, \cdots, i_{n_1-1}, i_{n_1+q_1}, \cdots, i_N\}.$$

$\mathbf{i}'$ is also admissible. Again choose numbers $n_2$ and $q_2$ and construct a minimal cyclic admissible sequence $\mathbf{j}_2$ from $\mathbf{i}'$, which corresponds to a periodic orbit $\gamma_{k_2}$ of period $s_{k_2}$. Proceed in this way, until we cannot do any more. Then we get periodic orbits $\gamma_{k_1}, \cdots, \gamma_{k_l}$, and what is left is an admissible sequence of length less than $r$. ($r$ is the number of rectangles in the Markov partition $\mathcal{R}$.)

Thus by (32) and (33), we get

$$\Delta g(x, t_N) = \sum_{j=1}^{l} ([\gamma_{k_j}], [g]) \ + E_1,$$

where $|E_1| \le rL$.

We also have by (34)

$$\frac{\tau_0}{\tau_1} \le \frac{t_N - E_2}{\sum_{j=1}^{l} s_{k_j}} \le \frac{\tau_1}{\tau_0}, \tag{35}$$

where $0 \le E_2 \le r\tau_1$.

Notice that $A_{\mu(\gamma_k)} = [\gamma_k]/s_k$. Thus by (31), we have

$$\Delta g(x, t_N) \ge \eta \sum_{j=1}^{l} s_k \ + E_1. \tag{36}$$

By (35) and (36), we get

$$\frac{1}{t_N} \Delta g(x, t_N) \ge \frac{\eta \sum_{j=1}^{l} s_{k_j} - |E_1|}{(\tau_1/\tau_0) \sum_{j=1}^{l} s_{k_j} + E_2}$$

Now it is easy to deduce

$$\frac{1}{t_N} \Delta g(x, t_N) \ge \frac{\tau_0 \eta}{\tau_1} - \frac{C}{t_N},$$

where $C = C(L, r, \tau_0, \tau_1)$ does not depend on the choice of $x$ or $t_N$. Therefore if we choose $T_0$ such that

$$T_0 \geq \frac{2\tau_1 C}{\tau_0 \eta},$$

then we get

$$\frac{1}{t_N}\Delta g(x, t_N) \geq \frac{\tau_0 \eta}{2\tau_1}$$

if $t_N \geq T_0$. The proof is now complete.

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