수 학 강 의 록 제 26 권

3 – MANIFOLDS AND THEIR MAPPINGS

DARRYL McCULLOUGH

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

Notes of the Series of Lectures
held at the Seoul National University

Darryl McCullough, Department of Mathematics, Univ. of Oklahoma,
Norman OK 73019−0001, U.S.A.

펴낸날 : 1995년 3월 30 일
지은이 : Darryl McCullough
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터〔TEL : 02-880-6562〕

3-MANIFOLDS AND THEIR MAPPINGS

DARRYL McCULLOUGH

Notes for a series of lectures given at Seoul National University, December 26-29, 1994.

Contents

Preface

These are informal lecture notes distributed as supplementary materials for a series of lectures given at Seoul National University on Dec. 26-29, 1994. Some sections are directly based on material published or to be published by the author, others are original exposition of well-known material. Throughout, the emphasis is on ideas and sketches of proofs, rather than detailed arguments.

The first two chapters are a brief review of some standard algebraic topology, selected for its importance in 3-dimensional topology. The third chapter contains a summary of some of the manifold theory that is heavily used in working with low-dimensional manifolds, such as the isotopy extension theorem, regular and tubular neighborhoods, transversality, and general position.

The next two chapters concern the topological theory of 3-manifolds. The fourth chapter discusses the fundamental theorems — the Kneser-Milnor Factorization Theorem, the Loop Theorem and Sphere Theorem of Papakyriakopoulos — and Waldhausen's theory of mappings of sufficiently large 3-manifolds. The fifth chapter is an exposition of Johannson's version of the Jaco-Shalen-Johannson characteristic decomposition theory of Haken manifolds.

The next three chapters treat recent research in the mappings of compact 3-manifolds. The unifying theme is the study of the mapping class group $\mathcal{H}(M)$ — the group of isotopy classes of (piecewise-linear or differentiable) homeomorphisms of a 3-manifold M — by using the natural homomorphism $\mathcal{H}(M) \to Out(\pi_1(M))$ which takes a mapping class to its induced outer automorphism on the fundamental group. Chapter six deals first with the theory of compression bodies and the associated characteristic decomposition theory, due to Bonahon and McCullough-Miller, and then studies $\mathcal{H}(V) \to Out(\pi_1(V))$ where V is a compression body. Chapter seven focuses on the kernel of $\mathcal{H}(M) \to Out(\pi_1(M))$, giving the Generalized Luft theorem which describes generators for the kernel, and the author's results on infinite generation of the kernel. Chapter eight begins with a general conjectural picture of 3-manifold mapping class groups, and some discussion of the current status of these conjectures. This is followed by a presentation of the author's work on mapping class groups of sufficiently large 3-manifolds.

The ninth chapter describes current research on the finite-index realization problem, which asks when the image of $\mathcal{H}(M) \to Out(\pi_1(M))$ has finite index in $Out(\pi_1(M))$. After some illustrative examples, the finite-index realization theorem is presented. It answers the problem for a large class of 3-manifolds. The final section describes the application of these topological results to obtain information about the deformations of hyperbolic structures on 3-manifolds.

In the tenth and final chapter, we give a list of problems for students. Rather few such lists seem to be available. While ours is not tied directly to the material contained in these notes, we hope that may be of value both to students and lecturers of low-dimensional topology.

The author thanks Seoul National University and GARC for their generous support and hospitality, and especially Sungbok Hong and Hyuk Kim for their personal efforts.

Chapter I. Some algebraic topology used in 3-manifold theory

1. *Homology of spaces*

Let R be a commutative ring with an identity element. Sometimes R will required to be a principal ideal domain.

By a *homology theory* we mean a functor from the category of pairs of spaces and continuous maps to the category of graded R-modules and graded homomorphisms. That is, for each pair (X,A), where A is a subspace of X, there is an R-module $H_*(X,A;R) = \oplus_{q=0}^{\infty} H_q(X,A;R)$, and for each continuous map of pairs $f\colon (X,A) \to (Y,B)$ there are homomorphisms $f_*\colon H_q(X,A;R) \to H_q(Y,B;R)$ for every q, so that $(f \circ g)_* = f_* \circ g_*$. We abbreviate $H_q(X,A;R)$ to $H_q(X,A)$, and $H_q(X,\emptyset)$ to $H_q(X)$. Also, if no ring R is explicitly mentioned, then R is assumed to be $\mathbb{Z}$. For every pair (X,A) and every q there is a homomorphism $\partial\colon H_q(X,A) \to H_{q-1}(A)$, and the following *Eilenberg-Steenrod* axioms must hold:

1. (Homotopy invariance) If $f,g\colon (X,A) \to (Y,B)$ are homotopic, then $f_* = g_*$.
2. (Long exact sequence) There is a long exact sequence

$$\ldots \longrightarrow H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \\ \ldots \longrightarrow H_0(X,A) \longrightarrow 0$$

   where $i\colon (A,\emptyset) \to (X,\emptyset)$ and $j\colon (X,\emptyset) \to (X,A)$ are the inclusion maps.
3. (Excision axiom) If U is an open subset of X whose closure is contained in the interior of A, then the inclusion map $j\colon (X-U, A-U) \to (X,A)$ induces isomorphisms $j_*\colon H_q(X-U,A-U) \to H_q(X,A)$.
4. (Coefficient module) If P is a one point space, then $H_0(P) \cong R$ and $H_q(P) = 0$ for $q \geq 1$.

The module in axiom 4 is called the *coefficients* for the homology theory. There are many ways to define homology groups. (Strictly speaking, one should say homology "modules", but for the common cases $R = \mathbb{Z}$ and $R = \mathbb{Z}/n$, the homology modules are abelian groups, so we often speak of homology groups.) For a fixed coefficient ring R, all the standard definitions produce the same results when X is a simplicial complex and A is a subcomplex, but in more general situations two different homology theories with the same coefficients can assign different homology modules to the same space.

The axioms imply a more general version of the long exact sequence:

THEOREM: *Suppose $B \subseteq A \subseteq X$. Then there are homomorphisms*

$$\partial: H_q(X,A) \to H_{q-1}(A,B)$$

fitting into a long exact sequence

$$\begin{aligned} \ldots \longrightarrow H_q(A,B) \xrightarrow{i_*} H_q(X,B) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} \\ H_{q-1}(A,B) \xrightarrow{i_*} H_{q-1}(X,A) \longrightarrow \ldots \end{aligned}$$

where $i:(A,B) \to (X,B)$ and $j:(X,B) \to (X,A)$ are the inclusion maps.

The axioms also imply the following exact sequence, which is very powerful for computation of homology. It applies in quite general situations, but we will just state it for simplicial complexes.

MAYER-VIETORIS SEQUENCE: *Suppose A and B are subcomplexes of a simplicial complex X, with $A \cup B = X$. Then there are homomorphisms $\partial: H_q(X) \to H_{q-1}(A \cap B)$ fitting into a long exact sequence*

$$\ldots \to H_q(A \cap B) \xrightarrow{(i_*,-j_*)} H_q(A) \oplus H_q(B) \xrightarrow{I_*+J_*} H_q(X) \xrightarrow{\partial} H_{q-1}(A \cap B) \longrightarrow \ldots$$

where $i: A \cap B \to A$, $j: A \cap B \to B$, $I: A \to X$, and $J: B \to X$ are the inclusion maps.

Here are some other consequences of the axioms and the Mayer-Vietoris sequence. Assume that K is a simplicial complex and L is a subcomplex, possibly empty.

1. If K is n-dimensional, or more generally if every simplex of $K - L$ has dimension $\leq n$, then $H_q(K,L) = 0$ for all $q > n$.
2. $H_0(K) \cong \oplus R$ with one summand for each connected component of K.
3. $H_q(S^n) = R$ if $q = 0$ or $q = n$, and $H_q(S^n) = 0$ for all other q.

2. *Cellular homology*

The most concrete way to construct homology groups, and the most useful for computation in low-dimensional topology, is *cellular* homology. It requires that X have a structure as a CW-*complex*. That is, $X = \cup_{q=0}^{\infty} X^{(q)}$ where the 0-*skeleton* $X^{(0)}$ is a countable (possibly finite) discrete set of points, and each $(q+1)$-skeleton $X^{(q+1)}$ is obtained from the q-skeleton $X^{(q)}$ by attaching $(q+1)$-cells. Explicitly, for each q there is a collection $\{e_j \mid j \in J_{q+1}\}$ where

1. Each e_j is a subset of $X^{(q+1)}$ such that if $e'_j = e_j \cap X^{(q)}$, then $e_j - e'_j$ is disjoint from $e_k - e'_k$ if $j, k \in J_{q+1}$ with $j \neq k$.
2. For each $j \in J_{q+1}$, there is an *attaching map* $g_j: (D^{q+1}, \partial D^{q+1}) \to X^{(q+1)}$ such that g_j is a quotient map from D^{q+1} to e_j, and g_j maps $D^{(q+1)} - \partial D^{(q+1)}$ homeomorphically onto $e_j - e'_j$.
3. A subset of X is closed if and only if its intersection with each e_j is closed.

Each $e_j - e'_j$ is called a $(q+1)$-*cell*. When all attaching maps are imbeddings, the CW-complex is called *regular*.

Now for each q let $C_q(X; R)$ be the free R-module with basis the q-cells. We will define the *boundary homomorphism* $\partial: C_{q+1}(X; R) \to C_q(X; R)$. To define $\partial(c_{q+1})$, where c_{q+1} is a fixed $(q+1)$-cell, fix an orientation for D^{q+1}, thus determining an orientation for the q-sphere ∂D^{q+1}, and look at how the attaching map g carries ∂D^{q+1} into $X^{(q)}$. For each q-cell e_k in $X^{(q)}$, fix a point z_k in $e_k - e'_k$. One can use transversality to show that g is homotopic to a map such that for each k, the preimage of z_k is a finite set of points $p_{k,1}, p_{k,2}, \ldots, p_{k,n_k}$, and moreover g takes a neighborhood of each $p_{k,j}$ homeomorphically to a neighborhood of z_k. (By compactness, the preimage of z_k is empty for all but finitely many k). For each j with $1 \leq j \leq n_k$, let $\epsilon_{k,j} = \pm 1$ according to whether g restricted to the neighborhood of $p_{k,j}$ preserves or reverses orientation. Let $\epsilon_k = \sum_{j=1}^{n_k} \epsilon_{k,j}$. Then $\partial c_{q+1} = \sum_{k=1}^{\infty} \epsilon_k e_k$ where almost all $k = 0$. One can prove

1. ∂_q is independent of the homotopy of g used to define it. What happens is that preimage points of z_k appear in ± 1 canceling pairs when f is changed by homotopy.
2. $\partial_q \partial_{q+1} = 0$. The reason is that algebraically, the q-sphere ∂D^{q+1} acts as though it were a regular CW-complex with one q-cell corresponding to each preimage point of a z_k. Since ∂D^{q+1} is a manifold, the boundaries of these q-cells form a collection of $(q-1)$-cells, each appearing a part of the boundary of two q-cells, but with opposite orientations. Consequently, the algebraic sum of the boundaries of these q-cells is 0. Applying ∂_q to $\partial_{q+1}(c_{q+1})$ simply adds up the images of the boundaries of those q-cells, in $C_{q-1}(X; R)$, and the pairs with opposite signs all cancel out, giving 0.

An element of $C_q(X; R)$ is a formal sum $\sum_{k=1}^{n} r_k c_k$, where each c_k is a q-cell; such a sum is called a q-*chain*. Now form a sequence of groups and homomorphisms

$$\ldots \longrightarrow C_{q+1}(X; R) \xrightarrow{\partial_{q+1}} C_q(X; R) \xrightarrow{\partial_q} C_{q-1}(X; R) \longrightarrow$$
$$\cdots \longrightarrow C_1(X; R) \xrightarrow{\partial_1} C_0(X; R) \longrightarrow 0 \;.$$

This is called a *chain complex,* since $\partial_q \partial_{q+1} = 0$ for all q. This means the image of ∂_{q+1} is contained in the kernel of ∂_q for each q. If the image of ∂_{q+1} were equal to the kernel of ∂_q for each q, the sequence would be exact. To measure its deviation from exactness, we define

$$H_q(X;R) = \text{kernel}(\partial_q)/\text{image}(\partial_{q+1}) .$$

Elements of kernel(∂_q) are called *cycles,* and elements of image(∂_{q+1}) are called *boundaries.* An element of $H_q(X;R)$ is a coset $z_q + \partial_{q+1}(C_{q+1}(X;R))$, where $\partial_q z_q = 0$, but is usually written as $[z_q]$. Note that $[z_q] = [z'_q]$ if and only if $z_q = z'_q + \partial_{q+1}(c_{q+1})$ for some $(q+1)$-chain c_{q+1}.

To define f_*, where $f: X \to Y$, we first define $C_q(f): C_q(X;R) \to C_q(Y;R)$. By the Cellular Approximation Theorem (a consequence of transversality), f may be changed by homotopy so that $f(X^{(q)}) \subseteq Y^{(q)}$ for all q. Define $C_q(f)(c_q)$ similarly to the way that ∂c_q was defined. Then $f_*([c_q])$ is defined to be $[C_q(f)(c_q)]$. It is not easy to prove that this is well-defined and satisfies all the Eilenberg-Steenrod axioms, but it can be done. In particular, $H_*(X;R)$ does not depend on the cell structure chosen for X (since the identity map induces an isomorphism on the homologies defined using two different CW-complex structures on X), and f_* depends only on the homotopy class of f.

When A is a subcomplex of X, define the *relative* homology groups $H_q(X,A;R)$ by putting $C_q(X,A;R) = C_q(X;R)/C_q(A;R)$ and noting that ∂_q induces $\partial_q: C_q(X,A;R) \to C_{q-1}(X,A;R)$. Then, $H_q(X,A)$ is defined by the chain complex $C_*(X,A;R)$. The long exact sequence of the second axiom is then a purely algebraic consequence of the existence of short exact sequences $0 \to C_q(A;R) \to C_q(X;R) \to C_q(X,A;R) \to 0$. Note that every element of $H_q(X,A;R)$ is represented by a q-chain whose boundary lies in A.

An important special case of cellular homology is *simplicial homology,* where X is a simplicial complex and each q-simplex is regarded as a q-cell. Because of the large number of simplices needed to triangulate even a fairly simple space, simplicial homology is not very useful for explicit computation, but because all the attaching maps are imbeddings, it is much easier to use in proofs. For example, the definition of ∂_q is much more transparent.

Singular homology is an abstraction of simplicial homology where the simplices are replaced by *singular simplices.* A singular simplex is a map $\sigma: \Delta_q \to X$ where Δ_q is a fixed standard q-simplex. They form a basis for the R-module of singular chains $C_q(X;R)$, which is uncountably generated for most spaces X. This is a computational disadvantage, but note that the singular homology is defined *for any space* X; the rather nice structure of a CW-complex need not be present.

3. *Cohomology of spaces*

Once homology is defined, cohomology can be defined algebraically. This is based on the following fact. If A and B are R-modules, and $\phi: A \to B$ is an R-module homomorphism, then there is an R-module homomorphism $\phi^*: \mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ defined by $\phi^*(\alpha) = \alpha \circ \phi$. Clearly $(\phi \circ \psi)^* = \psi^* \circ \phi^*$, so if we define the *coboundary* by $\delta_q = \partial_q^*$, then $\delta_{q+1}\delta_q = \partial_{q+1}^*\partial_q^* = (\partial_q\partial_{q+1})^* = 0^* = 0$. Therefore, abbreviating $\mathrm{Hom}(C_q(X), R)$ to $C^q(X; R)$, we have a *cochain complex*

$$0 \to C^0(X;R) \xrightarrow{\delta_1} C^1(X;R) \longrightarrow \cdots \longrightarrow C^{q-1}(X;R) \xrightarrow{\delta_q}$$
$$C^q(X;R) \xrightarrow{\delta_{q+1}} C^{q+1}(X;R) \longrightarrow \cdots$$

whose deviation from exactness is measured by the cohomology groups

$$H^q(X;R) = \mathrm{kernel}(\delta_{q+1})/\mathrm{image}(\delta_q)\ .$$

A map $f: X \to Y$ induces $f^*: H^q(Y;R) \to H^q(X;R)$ so that $(f \circ g)^* = g^* \circ f^*$, and there are corresponding versions of the Eilenberg-Steenrod axioms and the Mayer-Vietoris sequence for cohomology.

An important case is when R is a *field,* say $R = F$. Then, it can be proved that $H^q(X;F) \cong \mathrm{Hom}(H_q(X;F), F)$, the dual space of $H_q(X;F)$. Consequently, $H^q(X;F)$ and $H_q(X;F)$ are vector spaces of the same rank, although there is no natural isomorphism between them.

4. *The relation between homology and homotopy*

Throughout this section, the coefficient ring is $R = \mathbb{Z}$. Let $\sigma: (S^n, s_0) \to (X, x_0)$ be a map representing an element $\langle\sigma\rangle$ of $\pi_n(X, x_0)$. Let γ_n be a fixed generator of $H_n(S^n; \mathbb{Z})$. The *Hurewicz homomorphism* $\rho: \pi_n(X, x_0) \to H_n(X)$ is defined by $\rho(\langle\sigma\rangle) = \sigma_*(\gamma_n)$. One can show that this homomorphism is natural, that is, if $f: X \to Y$ is a continuous map, the diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\sigma_\#} & \pi_n(Y) \\ \downarrow \rho & & \downarrow \rho \\ H_n(X) & \xrightarrow{\sigma_*} & H_n(Y) \end{array}$$

commutes. The basic relationship between homotopy groups and homology groups is given by the following theorem, in which the coefficient ring is understood to be $\mathbb{Z}$.

HUREWICZ THEOREM: *Let X be a path-connected space. Then*
(1) $\rho\colon \pi_1(X, x_0) \to H_1(X)$ *is given by abelianization and is surjective.*
(2) *If $n \geq 2$ and $\pi_q(X) = 0$ for $q < n$, then $\rho\colon \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.*

The Hurewicz Theorem has many refinements. There is a relative version relating $\pi_n(X, A)$ and $H_n(X, A)$, and it implies the following.

THEOREM: *Let $f\colon X \to Y$ be a continuous map between path connected spaces. If $f_\#\colon \pi_q(X) \to \pi_q(Y)$ is an isomorphism for $q < n$ and is surjective for $q = n$, then $f_*\colon H_q(X) \to H_q(Y)$ is an isomorphism for $q < n$ and is surjective for $q = n$. The converse holds if X and Y are simply-connected.*

Note that if f induces isomorphisms on all homotopy groups, then it induces isomorphisms on all homology groups, and conversely for simply-connected spaces. When X and Y are CW-complexes (which will be defined in the next section) this condition forces them to be homotopy equivalent:

WHITEHEAD THEOREM: *Let X and Y be connected CW-complexes. If $f\colon X \to Y$ induces isomorphisms on all homotopy groups, then f is a homotopy equivalence.*

A important special case occurs when X is a connected CW-complex and $\pi_q(X) = 0$ for all $q \geq 1$. Taking Y to be a single point, the Whitehead Theorem shows that X is homotopy equivalent to Y and hence X is contractible.

5. *Poincaré Duality*

The most fundamental algebraic result about manifolds is Poincaré Duality. In its crudest form, it can be stated as follows.

POINCARÉ DUALITY: *Let M be a closed n-dimensional manifold which is oriented over the coefficient ring R. Then for each q, $H^q(M; R)$ is isomorphic to $H_{n-q}(M; R)$.*

The way that an $(n-q)$-dimensional homology class determines a homomorphism from $H_q(M; R)$ to R can be described explicitly and geometrically. Among numerous generalizations of Poincaré Duality, a particularly useful one in low-dimensional topology is

LEFSCHETZ DUALITY: *Let M be a compact n-dimensional manifold with boundary which is oriented over the coefficient ring R. Then for each q, $H^q(M;R)$ is isomorphic to $H_{n-q}(M,\partial M;R)$ and $H^q(M,\partial M;R)$ is isomorphic to $H_{n-q}(M;R)$.*

These are special cases of much more general duality theorems. The following is a special case of Theorem VI.2.17 on p. 296 of [Sp].

GENERAL DUALITY THEOREM: *Let M be an n-dimensional manifold which is oriented over the coefficient ring R. Let A and B be compact tame subsets (e. g. finite subcomplexes of some triangulation of M) such that $B \subset A$. Then for each q, $H^q(M-B, M-A;R)$ is isomorphic to $H_{n-q}(A,B;R)$.*

Poincaré Duality is the case when M is compact, $A=M$, and B is empty. The first case of Lefschetz Duality follows when one takes $A=M$, $B=\partial M$, and notes that $M-\partial M$ is homotopy equivalent to M so $H_{n-q}(M,\partial M;R) \cong H^q(M-\partial M;R) \cong H^q(M;R)$. The second case follows by taking $A=M-\partial(M)\times[0,1)$, where $\partial M\times[0,1)$ is an open collar neighborhood of ∂M in M, and $B=\emptyset$. For another application of the General Duality Theorem, if K be a tame knot in S^3, then $H^q(S^3-K;R)\cong H_{3-q}(S^3,K)$. Using long exact sequences this is calculated to be R when $q=0,1$ and 0 otherwise.

The duality theorems have strong "naturality" properties in relation to the induced homomorphisms in homology and cohomology. For example, when $i:\partial M \to M$ and $j:(M,\emptyset)\to(M,\partial M)$ are the inclusion maps, the following diagram is commutative

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^{q-1}(M) & \xrightarrow{i^*} & H^{q-1}(\partial M) & \xrightarrow{\delta} \\
 & \downarrow\cong & & \downarrow\cong & \\
\cdots \longrightarrow & H_{n-q+1}(M,\partial M) & \xrightarrow{\partial} & H_{n-q}(\partial M) & \xrightarrow{i_*}
\end{array}$$

$$\begin{array}{ccccc}
H^q(M,\partial M) & \xrightarrow{j^*} & H^q(M) & \longrightarrow \cdots \\
\downarrow\cong & & \downarrow\cong & \\
H_{n-q}(M) & \xrightarrow{j_*} & H_{n-q}(M,\partial M) & \longrightarrow \cdots
\end{array}$$

where the vertical isomorphisms are given by Poincaré or Lefschetz Duality.

Chapter II. Aspherical complexes

In this chapter, we review asphericity as it is often used in 3-manifold theory.

DEFINITION: A connected space X is called *aspherical* if $\pi_q(X) = 0$ for all $q \neq 1$. When its fundamental group is G, an aspherical space is called a *$K(G,1)$-space,* and a *$K(G,1)$-complex* when it is also a CW-complex.

Note that a connected complex is aspherical if and only if its universal cover is contractible, since then the universal cover is a CW-complex all of whose homotopy groups vanish.

Aspherical complexes arise frequently in low-dimensional topology. Every connected 2-manifold other than the 2-sphere or the projective plane has universal cover whose interior is homeomorphic to $\mathbb{R}^2$, so is aspherical. As we will see in section IV.2, every irreducible orientable 3-manifold with infinite fundamental group is aspherical. However, it is unknown whether the universal cover of such a manifold must always be $\mathbb{R}^3$.

1. *Existence of $K(G,1)$-complexes*

PROPOSITION 1.1: *Let G be any group. Then there exists a connected 2-dimensional CW-complex K_G such that $\pi_1(K_G) \cong G$. If G is finitely presented, then K_G can be selected to be a finite complex.*

PROOF: Let $\langle g_\alpha,\ \alpha \in \mathcal{A} \mid r_\beta,\ \beta \in \mathcal{B}\rangle$ be a presentation for G. Let $K^{(0)}$ consist of a single vertex, and obtain $K^{(1)}$ by attaching one 1-cell for each g_α. This gives a one-point union of circles, whose fundamental group is the free group on the generating set $\{g_\alpha \mid \alpha \in \mathcal{A}\}$. If a 2-cell is attached so that its boundary represents an element r, then van Kampen's theorem shows that the effect on the fundamental group is to quotient out by the normal closure of r. Attaching one 2-cell for each r_β yields a 2-complex with fundamental group G. If the presentation was finite, then only finitely many cells are needed for the construction.

PROPOSITION 1.2: *Let G be a group. Then there exists a $K(G,1)$-complex.*

PROOF: Let $K^{(2)}$ be a 2-complex K_G as constructed in Proposition 1.1. For each nontrivial element of $\pi_2(K^{(2)})$ (actually, it is enough to take a set of generators) attach a 3-cell using that element as the attaching map. Inductively, construct $K^{(n+1)}$ from $K^{(n)}$ by attaching an $(n+1)$-cell for each nontrivial element of $\pi_n(K^{(n)})$, and let K be the union of all $K^{(n)}$ (Usually some of the $K^{(n)}$'s are infinite, and K is infinite-dimensional. In fact, we will see below that whenever G contains torsion elements, K *must* be infinite-dimensional.) To show that $\pi_q(K)=0$ for $q \geq 2$, consider a map $f: S^q \to K$. Since K has the weak topology, the image of f is contained in a finite union of cells of K. Suppose some of these cells have dimension greater than q. Let D be one of these cells, of maximal dimension n. It is the image of a map $(D^n, \partial D^n) \to (K^{(n)}, K^{(n-1)})$ which is an imbedding on the interior of D^n. Let p be the image of 0; using transversality locally near p we may assume that on the preimage of the interior of D, f is transverse to p. Since $n > q$, this means that the image of f is disjoint from p. Therefore f is homotopic to a map that misses the interior of D altogether (compose f with a deformation retraction that pulls $D - \{p\}$ onto $D \cap K^{(n-1)}$). Inductively, we can change f by homotopy so that it maps into $K^{(q)}$. It is homotopic to an attaching map for a $(q+1)$-cell, so can be contracted to a constant map by moving through that cell.

2. *Mappings into $K(G,1)$-complexes*

Since $K(G,1)$-complexes have no higher homotopy, mappings into them are completely controlled by their effect on fundamental groups. As a first step toward making this precise, we have

LEMMA 2.1: *Let X be a connected CW-complex and K a $K(G,1)$-complex. Let x_0 and k_0 be their respective basepoints. If $\phi: \pi_1(X, x_0) \to \pi_1(K, k_0)$ is any homomorphism, then there exists a map $f: (X, x_0) \to (K, k_0)$ with $f_\# = \phi$.*

PROOF: Let T be a maximal tree in $X^{(1)}$, and make $f(T)=k_0$. Each 1-cell a not in T represents an element α in $\pi_1(X,T) \cong \pi_1(X, x_0)$. Its endpoints lie in T so are already mapped to k_0; extend f to a by mapping around a loop representing $\phi(\alpha)$. For each 2-cell b, the boundary (i. e. the restriction of the attaching map to ∂D^2) represents an element β in $\pi_1(X^{(1)}, x_0)$ (well-defined only up to conjugacy since it might not contain the basepoint x_0). Since β is homotopic to a constant map, by moving through b, $\phi(\beta)=1$ so f must carry β to a null homotopic loop in K. Therefore f extends to b. Inductively, for $n \geq 3$, assume that f has been extended to $X^{(n-1)}$. The restriction of f to the boundary of any n-cell e represents an element of $\pi_{n-1}(K)$. Since this group is 0, f extends to a map on e. Therefore f can be inductively extended to all of X.

To describe the general situation, let $[(X,Y),(K,L)]$ denote the set of homotopy classes of maps from X to K that carry Y into L (during all homotopies, as well). This is the set of path components of Maps$((X,Y),(K,L))$. When Y is empty, we can abbreviate $[(X,\emptyset),(K,L)]$ to $[X,K]$.

THEOREM 2.2: *Let X be a connected CW-complex and K a $K(G,1)$-complex. Let x_0 and k_0 be their respective basepoints. Then*

(a) *Sending the homotopy class $\langle f \rangle$ to the induced automorphism $f_\#$ defines a bijection from $[(X,x_0),(K,k_0)]$ to $\mathrm{Hom}(\pi_1(X,x_0),\pi_1(K,k_0))$.*

(b) *Let $\mathrm{OHom}(\pi_1(X,x_0),\pi_1(K,k_0))$ be the set of equivalence classes of homomorphisms from $\pi_1(X,x_0)$ to $\pi_1(K,k_0)$, where $\phi_1 \sim \phi_2$ where there is an inner automorphism μ of $\pi_1(K,k_0)$ such that $\mu\phi_1 = \phi_2$. Then sending the homotopy class $\langle f \rangle$ to the induced automorphism $f_\#$ defines a bijection from $[X,K]$ to $\mathrm{OHom}(\pi_1(X,x_0),\pi_1(K,k_0))$.*

PROOF: By Lemma 2.1, the correspondence is surjective. To prove part (a), it remains to show that if f_0 and f_1 are maps from $(X,x_0) \to (K,k_0)$ inducing the same homomorphism on fundamental groups, then f_0 is homotopic to f_1 preserving x_0. Define $H: X \times \{0,1\} \to K$ by $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. Since $f_0(x_0) = k_0 = f_1(x_0)$, H extends to $x_0 \times I$ by $H(x_0,t) = k_0$. Let T be a maximal tree in the 1-skeleton of X. Since T is contractible, any two maps of T into a path-connected space are homotopic, so H can be extended to $T \times I$. Each 1-simplex s not in T represents an element σ of $\pi_1(X,x_0)$. Since $(f_0)_\#(\sigma) = (f_1)_\#(\sigma)$, H can be extended over s. (There are a few details to worry about here. One way to see it is to note that f_0 and f_1 are homotopic keeping x_0 fixed to maps f_0' and f_1' which both carry T to k_0; then, the restrictions of f_0' and f_1' to σ are actually loops based at k_0 which are homotopic relative to their endpoints, since they represent the same element of $\pi_1(K,k_0)$. A homotopy relative to their endpoints would then define H on $s \times I$.) Once H is defined on $X \times \{0,1\} \cup X^{(1)} \times I$, it is defined on every 2-cell of $X \times I$, and then it extends inductively over all higher-dimensional cells since $\pi_q(K) = 0$ for $q \geq 2$.

For part (b), any element of $\mathrm{OHom}(\pi_1(X,x_0),\pi_1(K,k_0))$ is represented by a homomorphism so, by part (a), is induced by a map. To see that this map is unique up to homotopy, first note that every map $f: X \to K$ is homotopic to a map taking x_0 to k_0; apply the homotopy extension property to extend a map from $X \times \{0\} \cup \{x_0\} \times I$ to K defined to be f on $X \times \{0\}$ and defined on $x_0 \times I$ by using a path from $f(x_0)$ to k_0. Similarly, by the homotopy extension property, there is a homotopy starting at the identity map of K which moves the basepoint k_0 around any specified loop λ. The ending map f_λ induces conjugation by λ on $\pi_1(K,k_0)$. Therefore any f with $f(x_0) = k_0$ is homotopic to $f_\lambda f$, which induces $\mu(\lambda) f_\#$ from $\pi_1(X,x_0)$ to $\pi_1(K,k_0)$. Therefore if f and g induces the same element of $\mathrm{OHom}(\pi_1(X,x_0),\pi_1(K,k_0))$, they are homotopic to maps which take x_0 to

k_0 and induce the same homomorphism from $\pi_1(X, x_0)$ to $\pi_1(K, k_0)$. By part (a), these are homotopic.

3. *Homotopy equivalences of $K(G,1)$-complexes*

Notice that $[(X,Y),(X,Y)]$ becomes a groupoid under composition (the homotopy class of the identity is the identity element) and its invertible elements form a group $\mathcal{E}(X,Y)$ called the *group of homotopy equivalences* of (X,Y). In particular, $\mathcal{E}(X,x_0)$ is called the *group of basepoint-preserving homotopy equivalences* of X and $\mathcal{E}(X)$ is the *group of homotopy equivalences* of X. This group has been studied by various authors. Some notable facts are:

1. If X is a simply-connected finite complex, then $\mathcal{E}(X)$ is finitely presented [Sul1].
2. $\mathcal{E}(S^1 \vee S^2 \vee S^3)$ is not finitely generated [F-K].
3. There exist finite 2-complexes with $\mathcal{E}(X)$ not finitely generated, and compact (nonirreducible) 3-manifolds with $\mathcal{E}(M)$ not finitely generated [McC7].

When X is a $K(G,1)$-complex Theorem 2.2 will describe its homotopy equivalences. First we observe

LEMMA 3.1: *Let $f\colon X_1 \to X_2$ be a map between connected aspherical complexes. If f induces an isomorphism on fundamental groups, then f is a homotopy equivalence.*

PROOF: By Theorem 2.2, there exists a map $g\colon X_1 \to X_2$ that induces the inverse isomorphism $f_{\#}^{-1}$. Then, gf and fg induce the identity (outer) automorphisms of $\pi_1(X_1)$ and $\pi_1(X_2)$. Again by Theorem 2.2, this implies that they are homotopic to the identity maps of X_1 and X_2 respectively.

Lemma 3.1 and Theorem 2.2 now yield our description of $\mathcal{E}(X)$.

COROLLARY 3.2: *Let X be a connected aspherical CW-complex with basepoint x_0. Then sending the homotopy class $\langle f \rangle$ to the induced automorphism $f_{\#}$ defines isomorphisms from $\mathcal{E}(X, x_0)$ to $Aut(\pi_1(X, x_0))$ and from $\mathcal{E}(X)$ to $Out(\pi_1(X, x_0))$, the group of outer automorphisms of $\pi_1(X, x_0)$.*

Another important consequence of Lemma 3.1 and Theorem 2.2 is

COROLLARY 3.3: *Two connected aspherical CW-complexes are homotopy equivalent if and only if their fundamental groups are isomorphic.*

4. *Homology and cohomology of groups*

One can define the homology and cohomology of a group G by letting $H_*(G;R) \cong H_*(K;R)$ and $H^*(G;R) \cong H^*(K;R)$, where K is any $K(G,1)$-complex. This is well-defined by Corollary 3.3, which shows that the homology and cohomology of any two $K(G,1)$-complexes are the same. This definition agrees with the standard group-theoretic definition. To see this, consider the cellular chains of the universal cover $\widetilde{K}$ of K.

$$\cdots \to C_n(\widetilde{K},R) \to \cdots \to C_1(\widetilde{K},R) \to C_0(\widetilde{K},R) \to 0 \; .$$

Since $\widetilde{K}$ is contractible, this chain complex has the homology of a point; that is, H_0 is R and all H_q are 0 for $q > 0$. Therefore one has an exact sequence

$$\cdots \to C_n(\widetilde{K},R) \to \cdots \to C_1(\widetilde{K},R) \to C_0(\widetilde{K},R) \to R \to 0$$

which is a resolution of the trivial RG-module R. The RG-modules $C_n(\widetilde{K},R)$ are free RG-modules. To see this, let $\mathcal{C}_n$ denote the set of n-cells of K and for each $e \in \mathcal{C}_n$ choose a single lift of e to an n-cell $\tilde{e}$ in $\widetilde{K}$. This selects exactly one n-cell in each $\pi_1(K)$-orbit on the set of n-cells of $\widetilde{K}$. Then, each of the other n-cells in the orbit is identified as $g \cdot \tilde{e}$ for a uniquely determined g, so an n-chain can be uniquely written as a sum $\sum_{e \in \mathcal{C}_n} \sum_{i=1}^{N_e} r_{e,i} g_{e,i} \tilde{e}$, where almost all $r_{e,i} = 0$, and this shows the isomorphism from $C_n(\widetilde{K},R)$ to $\oplus_{e \in \mathcal{C}_n} RG$.

According to the group theoretic definition, one can calculate $H_*(G;R)$ as follows: (1) form any free resolution of the trivial RG-module R (2) tensor it over RG with R (3) take the homology of the resulting chain complex. In our context, we can carry this out as follows. Form the chain complex

$$\cdots \to C_n(\widetilde{K},R) \otimes_{RG} R \to \cdots \to C_1(\widetilde{K},R) \otimes_{RG} R \to C_0(\widetilde{K},R) \otimes_{RG} R \to 0 \; .$$

Now, identify $C_n(\widetilde{K},R) \otimes_{RG} R$ with $C_n(K,R)$. This is accomplished by the homomorphism induced by the bilinear map $(g\tilde{e}, r) \mapsto re$. It is surjective since $(\tilde{e}, r) \mapsto e$ and injective since $g\tilde{e} \otimes r = \tilde{e} \otimes r$ in $C_n(\widetilde{K},R) \otimes_{RG} R$ (because the RG-action on R is trivial). So the chain complex is actually

$$\cdots \to C_n(K,R) \to \cdots \to C_1(K,R) \to C_0(K,R) \to 0 \; .$$

After checking that the homomorphisms are still the usual boundary maps, we conclude that the group theoretic homology of G is the same as the cellular homology of K. The proof for cohomology is similar.

This can be extended to the homology and cohomology with coefficients in nontrivial RG-modules, but then one must use the homology and cohomology groups of K with *twisted coefficients,* which are difficult to calculate except in special situations.

As applications of these facts, we have

PROPOSITION 4.1: *If G contains torsion, then any $K(G,1)$-complex is infinite-dimensional.*

PROOF: Let K be any $K(G,1)$-complex. Suppose g is a torsion element of G, of order n, and let G_1 be the subgroup of G generated by g. Let K_1 be the covering space of K corresponding to the subgroup G_1. Then K_1 is also aspherical so $H_q(K_1;\mathbb{Z}) \cong H_q(\mathbb{Z}/n;\mathbb{Z})$ for all n. Now $H_q(\mathbb{Z}/n;\mathbb{Z}) \cong \mathbb{Z}/n$ for all odd q. This can be calculated fairly easily from the algebraic definition (and can also be calculated topologically by constructing an explicit $K(\mathbb{Z}/n,1)$ which is the quotient of a free action of $\mathbb{Z}/n$ on the contractible complex $S^\infty = \cup_{k=1}^{\infty} S^k$). Since the homology is nonvanishing in arbitrarily large dimensions, K_1 and therefore K cannot be finite-dimensional.

Chapter III. Isotopy and general position

In this chapter we review some of the general topology of manifolds that is useful in low-dimensional manifolds.

1. *Gugenheim's Theorems*

The following theorem was proven by V. K. A. M. Gugenheim [Gu].

THEOREM: *Suppose M is piecewise-linearly homeomorphic to D^n or S^n. Then any orientation-preserving piecewise-linear homeomorphism from M to itself is piecewise-linear isotopic to the identity.*

An immediate consequence is

COROLLARY: *Suppose M is piecewise-linearly homeomorphic to D^n or S^n. Then any two orientation-reversing piecewise-linear homeomorphisms from M to itself are piecewise-linearly isotopic.*

PROOF: Suppose f and g are orientation-reversing homeomorphisms from M to M. Then fg^{-1} is orientation-preserving, so is isotopic to the identity, and therefore f is isotopic to g (for if h_t is an isotopy with $h_0 = fg^{-1}$ and $h_1 = 1_M$, then $h_t g$ is an isotopy from f to g.)

Another consequence is

COROLLARY: *Suppose M is piecewise-linearly homeomorphic to D^n. If h is a homeomorphism of M such that $h|_{\partial M}$ is the identity, then h is isotopic to the identity relative to ∂M.*

PROOF: (We will just verify that h is continuously isotopic to the identity.) h must be orientation-preserving so by Gugenheim's Theorem it is isotopic to 1_M. We need to find an isotopy from h to 1_M that is relative to ∂M. The idea is to extend what this isotopy is doing on the boundary to an isotopy on all of D^n by "coning," then correct the original isotopy by composing with the extended one. First assume M is actually equal to D^n. Give D^n radial coordinates (θ, t) where $\theta \in S^{n-1}$ and $0 \le t \le 1$. If $f: \partial D^n \to \partial D^n$ is a homeomorphism, define $C(f): D^n \to D^n$ by $C(f)(\theta,t) = (f(\theta), t)$. Note that $C(1_{\partial D^n}) = 1_{D^n}$. If h_t is the isotopy from h to 1_{D^n}, then $h_t \circ C\big((h_t|_{\partial D^n})^{-1}\big)$ is an isotopy from h to 1_{D^n} relative to ∂D^n. Now, assume M is just homeomorphic to D^n. Let $k: M \to D^n$ be a homeomorphism. Since $k^{-1}hk$ is a homeomorphism of D^n that is the identity on the boundary, there is an isotopy $(k^{-1}hk)_t$ from $k^{-1}hk$ to 1_{D^n} relative to ∂D^n. Then $k(k^{-1}hk)_t k^{-1}$ is an isotopy from h to 1_M relative to ∂M.

Another theorem from [Gu] is the following.

THEOREM: *Let M be a piecewise-linear n-manifold, and let C_1 and C_2 be n-cells imbedded in the interior of M. Let X be a closed subset of M such that C_1 and C_2 are contained in the same component of $M - X$. Then there is a piecewise-linear isotopy j_t of M such that $j_0 = 1_M$, $j_t|_X = 1_X$ for $0 \leq t \leq 1$, and $j_1(C_1) = C_2$.*

COROLLARY: *Let M be an orientable piecewise-linear n-manifold, and let i_0 and i_1 be two imbeddings from D^n into the interior of M. Let X be a closed subset of M such that $i_0(D)$ and $i_1(D)$ are contained in the same component of $M - X$. Then i_0 and i_1 are isotopic if and only if they are both orientation-preserving or both orientation-reversing.*

PROOF: We will need to use the fact that an orientation of M determines a generator of $H_n(M, M - \{x\}; \mathbb{Z})$ for every $x \in M$, and some properties of this. Suppose they are both orientation-preserving or both orientation-reversing. If they are both orientation-reversing, reverse the orientation of M so that we may assume they are both orientation-preserving. By Gugenheim's Theorem, there is an isotopy j_t with j_0 equal to the identity so that $j_1(i_1(D^n)) = i_0(D^n)$. Letting γ be the generator of $H_n(D^n, D^n - \{0\})$ determined by the orientation of D^n, the composite

$$H_n(D^n, D^n - \{0\}) \xrightarrow{(i_1)_*} H_n(M, M - \{i_1(0)\}) \xrightarrow{(j_1)_*} H_n(M, M - \{i_0(0)\}) \xrightarrow{(i_0)_*^{-1}} H_n(D^n, D^n - \{0\})$$

carries γ to γ, since i_1, j_1, and i_0 are both orientation-preserving. Therefore the composite $i_0^{-1} j_1 i_1$ is orientation-preserving. By Gugenheim's first Theorem, there is an isotopy k_t of D^n with $k_0 = 1_{D^n}$ and $k_1 = i_0^{-1} j_1 i_1$. Then $(j_t)^{-1} i_0 k_t$ is an isotopy from i_0 to i_1.

Now suppose i_0 is orientation-preserving and i_1 is orientation-reversing, and that i_0 and i_1 are isotopic. By the Isotopy Extension Theorem (see section 2 below), there is an isotopy j_t of M such that $j_0 = 1_M$ and $j_1 i_1 = i_0$. Let $x_0 = i_0(0)$ and $x_1 = i_1(0)$. Letting γ be the generator of $H_n(D^n, D^n - \{0\})$ determined by the orientation of D^n, the composite

$$H_n(D^n, D^n - \{0\}) \xrightarrow{(i_1)_*} H_n(M, M - \{i_1(0)\}) \xrightarrow{(j_1)_*} H_n(M, M - \{i_0(0)\}) \xrightarrow{(i_0)_*^{-1}} H_n(D^n, D^n - \{0\})$$

sends γ to $-\gamma$. But this composite is the induced homomorphism of $i_0^{-1} j_1 i_1 = 1_{D^n}$, a contradiction.

For nonorientable manifolds, an isotopy of an oriented n-cell that moves it around an orientation-reversing loop reverses the orientation of the n-cell. Using this fact, the previous corollary implies

COROLLARY: *Let M be a piecewise-linear n-manifold, and let i_0 and i_1 be two imbeddings from D^n into the interior of M. Let X be a closed subset of M such that $i_1(D)$ and $i_2(D)$ are contained a nonorientable component of $M - X$. Then i_0 and i_1 are isotopic.*

Some of these results are proven in the smooth category in [Pa1].

2. *Isotopy extension*

In low-dimensional topology, one frequently alters ("moves") submanifolds of M so that they are somehow improved. The tool that allows one to do this is the powerful Palais Fibering Theorem, from [Pa2].

Recall that a map $p: E \to B$ is a *fibration* if it has the *homotopy lifting property.* This means that if $f: K \times I \to B$ and $g: K \times \{0\} \to E$ are maps such that $pg(x,0) = f(x,0)$, then there exists a map $F: K \times I \to E$ such that $pF = f$.

PALAIS FIBERING THEOREM: *Let U be an open subset of a manifold M, and let V be a compact submanifold (possibly with boundary) of M. Let $Diff_{V,U}(M)$ be the space of diffeomorphisms of M which carry V into U, and let $Imb(V, U)$ be the space of imbeddings of V in U. Then the map $Diff_{V,U}(M) \to Imb(V, U)$ defined by restricting diffeomorphisms of M to imbeddings of V is a fibration.*

Note that in the special case when $U = M$, $Diff_{V,U}(M) = Diff(M)$. The Palais Fibering Theorem implies the next useful result.

ISOTOPY EXTENSION THEOREM: *Let V be a compact submanifold of M, and let j_t be an isotopy of imbeddings of V in M such that j_0 is the inclusion. If U is any neighborhood of $j(V \times I)$, then there exists an isotopy J_t of M such that for each t, $J_t \mid_V = j_t$, and $J_t|_{M-U}$ is the identity map of $M - U$.*

PROOF: Let N_0 be a regular neighborhood of $j(V \times I)$ in U, let N_1 be a regular neighborhood of N_0 in U, and let N be the closure of $N_1 - N_0$. Then N is a compact submanifold of M. Define an isotopy of imbeddings f_t of $V \cup N$ into M by letting $f_t = j_t$ on V and the inclusion on N. Regard f_t as a map from I into $Imb(V \cup N, U)$. Define $g: \{0\} \to Diff_{V,U}(M)$ by $g(0) = 1_M$. By the Palais Fibering Theorem, there exists $F: I \to Diff_{V,U}(M)$ so that the restriction of F_t to $V \cup N$ equals f_t. Define J_t to be F_t on N_1 and the identity on $M - N_0$. Then J_t also restricts to f_t, and hence restricts to j_t on V, and J_t is the identity outside of N_0 and hence outside of U.

In low-dimensional topology, the Isotopy Extension Theorem is often applied in the following kind of situation. One has a diffeomorphism h of M, and a submanifold V. One examines $h(V)$, and describes a way to "move" it to a simpler position; that is, one specifies an isotopy j_t of $h(V)$ starting at the inclusion and ending at a simpler submanifold W. The Isotopy Extension Theorem tells us that $h = 1_M \circ h$ can be changed by isotopy to a new diffeomorphism $h' = J_1 \circ h$ with $h'(V) = W$. If K is a closed subset of M such $j_t(V)$ is disjoint from K for all t, then we may assume that each J_t fixes K and hence that $h|_K = h'|_K$.

3. Regular neighborhoods

Regular neighborhoods were invented by J. H. C. Whitehead [Whi]. To define them, we first introduce the idea of a simplicial collapse. Let K be a simplicial complex. Suppose σ is an open simplex of K and τ is a face of σ that is not a face of any other simplex in K. The subcomplex $K_1 = K - \{\sigma, \tau\}$ is said to be obtained from K by an *elementary collapse.* Note that $|K_1|$ is a deformation retract of $|K|$. If there is a sequence of elementary collapses starting at K and ending with L, we say that K *collapses* to L and write $K \searrow L$. Then, $|L|$ is a deformation retract of $|K|$.

Let M be a triangulated manifold, and let $X \subseteq M$ be a polyhedron which is a subcomplex in some subdivision of M (for example, when X is the image of a PL imbedding into M). A *regular neighborhood* of X is a PL n-submanifold N of M so that for some subcomplexes $L \subseteq K$ of a subdivision of the triangulation of M, $(|K|, |L|) = (N, X)$, and $K \searrow L$. This definition does not require that X lie in the topological interior of N, but in practice this is often the case.

Examples (with X contained in the topological interior of N):

1. A regular neighborhood of a point in an n-manifold is an n-ball.
2. If N is a regular neighborhood of ∂M, then (N, M) is homeomorphic to $(\partial M \times [0,1], M \times \{0\})$.
3. If F is a 2-manifold properly imbedded in a 3-manifold M (*properly* means that $F \cap \partial M = \partial F$) then N is homeomorphic to an I-bundle over N.

Regular neighborhoods always exist. In fact, there is a simple description of one:

THEOREM (EXISTENCE OF REGULAR NEIGHBORHOODS): *Let $X = |L|$ where L is a subcomplex of a triangulation T of M. Let K be the collection of all the closed simplices in the second barycentric subdivision T'' of T that have a face in L. Then $|K|$ is a regular neighborhood of X.*

Note that if X lies in the manifold interior of M, then it lies in the topological interior of $|K|$, while if X intersects ∂M, $|K| \cap \partial M$ is a regular neighborhood of $X \cap \partial M$.

Regular neighborhoods are unique in the following sense:

THEOREM (UNIQUENESS OF REGULAR NEIGHBORHOODS): *Let M be a PL n-manifold, X a compact polyhedron PL imbedded in M, and let N_1 and N_2 be two regular neighborhoods of X. Then:*

1. *There is a PL homeomorphism $h: N_1 \to N_2$.*
2. *If X is contained in the topological interiors of N_1 and N_2, then one can choose h so that $h|_X = 1_X$.*
3. *If $N_i \cap \partial M$ is a regular neighborhood of $X \cap \partial M$ for $i = 1, 2$, then there is a PL isotopy $j: M \times I \to M$ such that $j_0 = 1_M$ and $j_1(N_1) = N_2$.*
4. *If X lies in the topological interior of N_i for $i = 1, 2$, then the isotopy j in 3. can be chosen so that $j_t|_X = 1_X$ for $0 \leq t \leq 1$.*

An interesting fact is

THEOREM: *Let N be a regular neighborhood of X in M, such that X lies in the topological interior of N, and let W be the frontier of N in M. Then there is a PL homeomorphism from $(N - X, W)$ to $(W \times [0, 1), W \times \{0\})$.*

4. *Tubular neighborhoods*

In the smooth category, the analogous structure to a regular neighborhood is a *tubular neighborhood.* Let W be a k-dimensional smooth submanifold of a smooth n-manifold M, $k < n$, with $W \cap \partial M = \partial W$. A (closed) tubular neighborhood of W is a closed neighborhood N of W so that (N, W) is diffeomorphic to (E, Z) where E is a bundle over W with fiber an $(n-k)$-dimensional disc, and Z is the zero section consisting of zero in each fiber. The existence (and uniqueness, up to isotopy in M) of tubular neighborhoods is proven in any differential topology book. Here are some useful observations:

1. The boundary of N is a bundle over W with fiber an $(n - k - 1)$-dimensional sphere. $N - W$ is diffeomorphic to $\partial N \times [0, 1)$.
2. When $k = n - 1$, the fiber of N is a 1-disc, which is an interval. The mapping taking the topological boundary $\mathrm{Bd}(N)$ to W by taking the

two endpoints of the fiber to the point of W that meets that fiber is a 2-fold covering. If $\mathrm{Bd}(N)$ is not connected, then $N = W \times [-1,1]$ with W corresponding to $W \times \{0\}$, so N is a *product neighborhood* and W is 2-sided in M. If it is connected, then $N - W = \mathrm{Bd}(N) \times [0,1)$ is connected so W is one-sided in M.

3. In particular, if $k = n - 1$ and W is simply-connected, then W has no connected 2-fold coverings, so the tubular neighborhood is always a product. That is, *a simply-connected codimension-1 submanifold is always 2-sided.*

5. *Transversality*

If two paths $\alpha, \beta: [0,1] \to \mathbb{R}^2$ cross at right angles, then any two paths α' and β' sufficiently close to α and β must also cross. The situation is quite different when $\alpha, \beta: [0,1] \to \mathbb{R}^3$. In this case, we may change α to α' by a homotopy that move points less than ϵ, so that $\alpha'([0,1])$ does not meet $\beta([0,1])$. Roughly speaking, given two maps $\alpha: X_1 \to M^n$ and $\beta: X_2 \to M^n$, where X_1 is a p-dimensional polyhedron and X_2 is a q-dimensional polyhedron, it seems intuitively obvious that α and β should be ϵ-close to maps whose images intersect in a polyhedron of dimension $p+q-n$. However, it is not easy to pin down these phenomena precisely. Two concepts involved are *transversality* and *general position.*

In the smooth category, when one is dealing with submanifolds, transversality is relatively easy to describe. Let M and N be a smooth manifolds of dimensions m and n respectively, and let L be k-dimensional submanifold of N. A smooth map $f: M \to N$ is said to be *transverse* to L if for any point $p \in M$ with $f(p) \in N$, the subspaces $T_{f(p)}L$ and $f_*(T_pM)$ span the tangent space $T_{f(p)}M$. This captures the idea that $f(M)$ and L cut across each other as much as possible.

1. If $m + k < n$, then f is transverse to L only when $f(M) \cap L$ is empty.
2. The implicit function theorem implies that if f is transverse to L, then $f^{-1}(L)$ is a smooth submanifold of M, of codimension k (that is, the dimension of $f^{-1}(L)$ is $m - k$. In particular, if M is compact, then $f^{-1}(L)$ is a compact submanifold of M.
3. (for those familiar with bundle theory) If f is transverse to L, then the normal bundle of $f^{-1}(L)$ in M is the pullback of the normal bundle of L in N. Thus, for example, if L is a point the normal bundle of $f^{-1}(L)$ must be a product bundle.

Here is a transversality theorem proven by R. Thom (see for example [B-J]).

TRANSVERSALITY THEOREM: *Let $f: M \to N$ be a differentiable map, and let L be a differentiable submanifold of N. Then f can be arbitrarily closely approximated by maps $g: M \to N$ transverse to L. If A is a closed subset of M, one can choose g so that $g|_A = f|_A$.*

Every continuous maps can be approximated by transverse maps, by combining the transversality theorem with the following theorem.

SMOOTH APPROXIMATION THEOREM: *Let $f: M \to N$ be a continuous map which is differentiable on an open neighborhood U of the closed set A. Then, arbitrarily close to f, there exists a differentiable map g with $g|_A = f|_A$.*

These theorems are very powerful and can be used to prove many facts usually proven by algebraic methods. Here is one example.

BROUWER NO RETRACTION THEOREM: *There is no retraction from D^n to S^{n-1}.*

PROOF: Suppose there exists a retraction $r: D^n \to S^{n-1}$. Let $S^{n-1} \times [0,1]$ be a smooth closed collar of S^{n-1}. Regarding $D^n - S \times [0,1)$ as a smaller D^n, we can use r to retract D^n to $S^{n-1} \times [0,1]$ and follow this by projection to $S^{n-1} \times \{0\}$, to obtain a retraction r' which is differentiable on $U = S^{n-1} \times [0,1)$. By the smooth approximation theorem, this can be approximated (keeping it the identity on S^{n-1}) by a smooth map. Let $x \in S^{n-1}$. By the transversality theorem, r' can be approximated by a smooth retraction transverse to x. The preimage of this retraction is a compact 1-submanifold in D^n, which consists of arcs and circles. In particular, there must be an even number of endpoints. But x is the only endpoint, a contradiction.

6. *General position*

Smooth transversality captures the idea that $f(M)$ and L cross as much as possible, but suffers from some shortcomings.

1. It is limited to submanifolds and does not adapt easily to the case when M or L are just polyhedra.
2. Even when M and L are manifolds, transverse maps still allow "non-generic" things to happen. For example, when $N = \mathbb{R}^2$, $L = \mathbb{R}$ (the x-axis), and $M = \mathbb{R}$, $f(M)$ could have many lines cutting through origin at different angles. Thus f could be transverse to L, but not "in general position" because by slight changes of f we could have only two lines crossing locally at each intersection point of $f(M)$ and L.
3. It does not clarify what a "self-transverse" map $f: M \to N$ should mean.

Unfortunately, general position is rather difficult to define, even when restricted to manifolds of dimension at most 3 (see pp. 8-13 of [He]). Hempel's approach is to consider a map $f: X \to Y$ and define the *singular set*

$$S(f) = \text{closure of } \{x \in X \mid \#(f^{-1}(f(x)) > 1\} ,$$

where $\#(f^{-1}(f(x))$ denotes the number of points in $f^{-1}(f(x))$. $S(f)$ is the disjoint union of sets $S_i(f)$ defined by

$$S_i(f) = \{x \in X \mid \#(f^{-1}(f(x)) = i\} .$$

For example, let $f: D^2 \to D^2$ by $f(z) = z^2$ (complex multiplication). Then $S(f) = D^2$, $S_1(f) = \{0\}$ and $S_2(f) = D^2 - \{0\}$. Now let $\Sigma_i(f) = f(S_i(f))$. Points of $\Sigma_1(f)$ are called *branch points,* points of $\Sigma_2(f)$ are called *double points,* and points of $\Sigma_3(f)$ are called *triple points.*

Hempel defines *general position* roughly as follows (see p. 10 of his book for the full definition). Let $f: |K| \to M$ be a map from a k-dimensional polyhedron $|K|$ into an n-dimensional manifold M, where $k < n \leq 3$. (For two maps $f: V \to M$ and $g: W \to M$, use $V \cup W$ as $|K|$ and $f \cup g$ as f.) Then f is in general position when $\dim(S_1(f)) \leq n - 3$ and for $i \geq 2$, $\dim(S_i(f)) \leq ik - (i-1)n$ (hence $S_i(f) = \emptyset$ for $i > n$), and on $|K| - S_1(f)$ f is an immersion. (There are several additional technical conditions.)

Hempel's main theorem asserts that any map is ϵ-close to a map in general position and this can be achieved with control:

General Position Theorem: *Suppose K is a finite complex of dimension $k < n \leq 3$, A, B, and C are subcomplexes of K with $K = A \cup B \cup C$, and $A \cap C = \emptyset$. Given an n-manifold M, a PL map $g: |K| \to M$ with $g|_{|B|}$ in general position (and $g(B) \subset \text{int}(M)$) and $\epsilon > 0$ there exists a PL map $f: |K| \to M$ such that*

(a) *$d(f(x), g(x)) < \epsilon$ for all $x \in |K|$.*
(b) *$f|_{|A \cup B|} = g|_{|A \cup B|}$.*
(c) *$f|_{|B \cup C|}$ is in general position with respect to some subdivision of $B \cup C$.*
(d) *If L is a subcomplex of K such that $g|_{|L|}$ is an imbedding, then $f|_{|L|}$ is an imbedding.*

Note that (d) implies that if g is an immersion, then so is f.

We remark that when $|K|$ is compact, given g there is an ϵ such that any map ϵ-close to g is homotopic to g, so one can assume that the map f obtained in the General Position Theorem is homotopic to g.

Here are some special cases of importance.

(1) $|K|$ is a 1-complex, and M is a 3-manifold ($k = 1$, $n = 3$). Then, $\dim(S_2(f)) \leq 2 - 3 = -1$, so $S_2(f)$ and $S_3(f)$ are empty. Therefore $S(f)$ is also empty so a general position map is an imbedding.

(2) $|K|$ is a 1-complex, and M is a 2-manifold ($k=1$, $n=2$). In this case, $S_1(f)$ is empty, $S_2(f)$ consists of pairs of double points, and $S_3(f)$ is empty.

(3) $|K|$ is a 2-manifold (or 2-complex), and M is a 3-manifold. $k = 2$, $n = 3$. First, $\dim(S_3(f)) \leq 3 \cdot 2 - 2 \cdot 3 = 0$ so there can be isolated triple points. $\dim(S_2(f)) \leq 1$ and consists of lines of double points. $\dim(S_1(f)) \leq 0$ and branch points occur at the end pairs of double lines which are identified in the image. For example, when the map $D^2 \to D^2$ sending z to z^2 is regarded as a map from D^2 to $\mathbb{R}^3$ and put in general position, there will be a branch point and two lines of double points from ∂D^2 to the branch point. Their image is an arc ending at the branch point. Note that f cannot be an immersion at a point of $S_1(f)$.

(4) $g: F_1 \cup F_2 \to M$, where F_1 and F_2 are 2-manifolds, M is a 3-manifold, and $g|_{F_i}$ is an imbedding for each i. Then $f|_{F_i}$ will be imbeddings as well, so $S_3(f) = \emptyset$ and, since f is an immersion, $S_1(f) = \emptyset$. The only singularities are arcs and circles of double points.

Chapter IV. The fundamental theorems of 3-manifold theory

In the remainder of these notes, all maps and in particular all imbeddings will be piecewise-linear or smooth. Allowing wild imbeddings would take us into a different subject.

1. *The Kneser-Milnor Factorization Theorem*

Recall that the connected sum $M^n \# N^n$ of two n-dimensional manifolds is obtained by removing the interior of two imbedded n-balls from the interiors of M and N and identifying the resulting punctured manifolds along the $(n-1)$-sphere boundary components created when the balls were removed. Note that $M \# S^n$ is always homeomorphic to M, and $M \# D^n$ is the manifold that results from removing the interior of an n-cell from M. The connected sum depends upon the choice of imbeddings, but isotopic imbeddings give homeomorphic sums. In particular, in a connected manifold there are at most two isotopy classes of imbedded n-balls (see section III.1) and at most two different homeomorphism types can result as the connected sum of M and N. Similarly a boundary connected sum of two manifolds with nonempty boundary is obtained by identifying two $(n-1)$-cells in their boundaries. A manifold M is called *prime* if M is homeomorphic to $M_1 \# M_2$ only when exactly one of $M_1 \# M_2$ is the n-sphere. Also, the operation of connected sum is commutative and associative.

A fundamental structure theorem in 3-manifold theory is the following.

KNESER-MILNOR FACTORIZATION: *Let M be a compact 3-manifold. Then $M = S^3 \# M_1 \# \cdots M_r \# R_1 \cdots \# R_s \# B_1 \# \cdots \# B_t$, where $r, s, t \geq 0$, and*

1. *each M_i is prime and is not one of the two S^2-bundles over S^1,*
2. *each R_j is one of the two S^2-bundles over S^1, and*
3. *each B_k is a 3-ball.*

The numbers r, s, and t are uniquely determined, and the M_i are determined up to homeomorphism and the order in which they appear.

Note that t is exactly the number of 2-sphere boundary components of M. A more precise statement about the unique determination of the R_j can be made, but we will not concern ourselves with that here.

The proof of the Kneser-Milnor theorem is difficult. A nice exposition is given in [He]. The key step is showing that the process of factoring M into summands must terminate. The idea of the argument is to take a collection of disjoint imbedded nonparallel 2-spheres which separates M into summands, and to analyze how it meets the skelta of a triangulation of M (thus, the

Kneser-Milnor theorem rests on the theorem that all 3-manifolds can be triangulated, also a difficult result). By moving the collection around by isotopy into a simple position relative to the simplices in the triangulation, one can obtain an upper bound for the number of spheres in a collection. That is, in any larger collection two of the spheres must be parallel, so they correspond to a trivial connected summand of S^3.

One of the most fundamental concepts in 3-manifold theory is that of irreducibility. A 3-manifold is called *irreducible* if whenever S is an imbedded 2-sphere in M, then there is a 3-ball B imbedded in M whose boundary is S. In particular, every imbedded 2-sphere in M must separate M, so an S^2-bundle over S^1 is not irreducible. However, it is not very difficult to show that a prime manifold must be either irreducible or an S^2-bundle over S^1. Thus, *the manifolds M_i in the Kneser-Milnor factorization of M are irreducible.*

2. *The Loop Theorem and the Sphere Theorem*

The following theorem enables homotopy theoretic information about M to be translated into topological information (and ultimately, into geometric information). Here is a fairly general version (but not the most general).

LOOP THEOREM: *Let M be a 3-manifold and F a connected 2-manifold contained in ∂M. If N is a normal subgroup of $\pi_1(F)$ which does not contain the kernel of the homomorphism $\pi_1(F) \to \pi_1(M)$ induced by inclusion, then there is an imbedding $g\colon (D^2, \partial D^2) \to (M, F)$ such that the element of $\pi_1(F)$ represented by the restriction of g to ∂D^2 is not an element of N.*

An important special case is when $N = \{1\}$. Then, using the fact that an imbedded loop in a 2-manifold is contractible only when it bounds a 2-disc in the 2-manifold, we have the following geometric version of the Loop Theorem.

LOOP THEOREM: *Let M be a 3-manifold and F a connected 2-manifold contained in ∂M. Suppose the kernel of the homomorphism $\pi_1(F) \to \pi_1(M)$ induced by inclusion is nontrivial. Then there is an imbedded disc $(D^2, \partial D^2) \subset (M, F)$ such that ∂D^2 does not bound a disc in F.*

The Loop Theorem is a generalization of a lemma which Max Dehn tried to prove in 1910. Dehn was an outstanding mathematician, but neither he nor anyone else could prove the lemma until 1957, when C. D. Papakyriakopoulos published a proof. The statement above is due to John Stallings, who published a beautiful proof of it in 1960 [St]. Papakyriakopoulos also proved another very important result:

Sphere Theorem: *Let M be an orientable 3-manifold and let N be a proper subgroup of $\pi_2(M)$ which is invariant under the action of $\pi_1(M)$ on $\pi_2(M)$. Then there exists an imbedding $g: S^2 \to M$ which represents an element of $\pi_2(M) - N$.*

Taking N to be the trivial subgroup, we get a simpler geometric statement:

Sphere Theorem: *Let M be an orientable 3-manifold with $\pi_2(M) \neq 0$. Then there exists an imbedded 2-sphere $S \subset M$ which is not contractible to a point in M.*

Again, we obtain a very strong topological conclusion from minimal homotopy theoretic information. The above statements are false for nonorientable manifolds, as the example $\mathbb{RP}^2 \times S^1$ shows, but the following version of the Sphere Theorem due to D. Epstein shows what happens:

Sphere Theorem: *Let M be a 3-manifold and let N be a proper subgroup of $\pi_2(M)$ which is invariant under the action of $\pi_1(M)$ on $\pi_2(M)$. Then there exists a map $g: S^2 \to M$ which represents an element of $\pi_2(M) - N$, satisfying one of the following:*

(a) *g is an imbedding, or*
(b) *g is a two-fold covering from S^2 to $g(S^2)$, which is a projective plane two-sidedly imbedded in M.*

We will now derive some well-known and important consequences of the Sphere Theorem. The first lemma is an interesting fact in its own right, and is also a nice illustration of how the basic theorems of algebraic topology can be applied effectively in 3-manifold topology.

Lemma 1: *Let Σ be a closed simply-connected 3-manifold. Then Σ is homotopy equivalent to the 3-sphere S^3.*

Proof: Since $\pi_1(\Sigma) = 0$, $H_1(\Sigma) = 0$ by the Hurewicz Theorem. Since Σ is simply-connected, it is orientable. By Poincaré Duality, $H_2(\Sigma) \cong H^1(\Sigma) \cong \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}) \cong 0$, and $H_3(\Sigma) \cong H^0(\Sigma) \cong \mathbb{Z}$. For $q \geq 4$, $H_q(\Sigma) = 0$ since Σ is 3-dimensional. By the Hurewicz Theorem, the Hurewicz homomorphism $\rho: \pi_3(\Sigma) \to H_3(\Sigma)$ is an isomorphism. Let $\gamma: S^3 \to \Sigma$ represent a generator of $\pi_3(\Sigma) \cong \mathbb{Z}$, and note that $\rho(\langle 1_{S^3} \rangle)$ is a generator of $H_3(S^3)$. Using properties of the Hurewicz homomorphism, we have

$$\gamma_* \rho(\langle 1_{S^3} \rangle) = \rho \gamma_\#(\langle 1_{S^3} \rangle) = \rho(\langle \gamma \circ 1_{S^3} \rangle) = \rho(\langle \gamma \rangle) .$$

Since $\langle \gamma \rangle$ is a generator of $\pi_3(\Sigma)$ and ρ is an isomorphism, this shows that γ_* is an isomorphism on homology. So $\gamma: S^3 \to \Sigma$ induces an isomorphism on all homology groups. Since S^3 and Σ are simply-connected, the Whitehead Theorem shows that γ induces isomorphisms on all homotopy groups, and hence is a homotopy equivalence.

We remark that the famous *Poincaré Conjecture* asserts that a 3-manifold which is homotopy equivalent to S^3 must actually be *homeomorphic* to S^3. This has resisted the efforts of many outstanding mathematicians, although it has been proven to be true in all other dimensions! That is, for all $q \neq 3$ it is known that a closed manifold which is homotopy equivalent to S^q must be homeomorphic to S^q. This was long known for $q \leq 2$, and was proven for $q \geq 5$ by S. Smale in the 1960's and for $n=4$ by M. Freedman in the 1980's.

LEMMA 2: *Let M be an irreducible orientable 3-manifold. Then $\pi_2(M)=0$.*

PROOF: Suppose $\pi_2(M) \neq 0$. By the Sphere Theorem, there exists an imbedded 2-sphere in M which is not contractible. In particular, S cannot bound a 3-ball in M, so M is not irreducible.

THEOREM: *Let M be a compact orientable irreducible 3-manifold.*

(a) *If $\pi_1(M)$ is finite, then either $M = D^3$ or the universal cover of M is homotopy equivalent to S^3.*

(b) *If $\pi_1(M)$ is infinite, then the universal cover of M is contractible. Consequently $\pi_q(M)=0$ for all $q \geq 0$.*

PROOF: Let $\widetilde{M}$ denote the universal cover of M. For (a), we have $H_1(\widetilde{M})=0$ so by duality theorems $H_1(\partial\widetilde{M})=0$ and therefore $\partial\widetilde{M}$ consists of 2-spheres. So ∂M consists of 2-spheres (there cannot be projective planes since M is orientable, and no other 2-manifolds are covered by the 2-sphere). If ∂M is nonempty, then M must be the 3-ball by irreducibility. If ∂M is empty, then $\widetilde{M}$ is closed and simply-connected, so by lemma 1 it is homotopy equivalent to S^3. For (b), we again have $H_1(\widetilde{M})=0$. By Lemma 2, we have $\pi_2(M) \cong 0$ and hence $H_2(\widetilde{M}) \cong \pi_2(\widetilde{M}) \cong 0$, using the Hurewicz Theorem. Finally, since $\pi_1(M)$ is infinite, $\widetilde{M}$ is noncompact so $H_3(\widetilde{M})=0$. Since M is 3-dimensional, $H_q(\widetilde{M})=0$ for all $q \geq 4$. Thus $\widetilde{M}$ has the homology of a point and is simply-connected, so it is contractible.

As an immediate corollary, we obtain some important information about $\pi_1(M)$.

COROLLARY: *Let M be an irreducible orientable 3-manifold. If $\pi_1(M)$ is infinite, then it is torsionfree.*

PROOF: By Theorem 1(b), such an M is aspherical. By Proposition I.4.1, $\pi_1(M)$ must be torsionfree.

If M is nonorientable, this is false, as shown by the irreducible 3-manifold $\mathbb{RP}^2 \times S^1$. But if M is irreducible, contains no two-sided projective planes, and has infinite fundamental group, then M is aspherical and $\pi_1(M)$ is torsionfree.

Theorem 1 shows that irreducible 3-manifolds with infinite fundamental group are *aspherical*, and as a consequence *their fundamental groups have a profound effect on their topological structure.* As we will see in the next section, in many cases the fundamental group determines the manifold up to homeomorphism. Moreover, the structure of the fundamental group is strongly reflected in the topological structure of M, as is clarified by the Jaco-Shalen-Johannson characteristic submanifold theory (chapter V), and the characteristic compression body (chapter VI). Finally, Thurston's celebrated theory shows that under suitable hypotheses M admits a hyperbolic geometric structure whenever the fundamental group of M does not prohibit the existence of such a structure.

3. *Waldhausen's theorem and the finite-index realization problem*

One of the most significant milestones in the theory of 3-manifolds was Waldhausen's paper [Wa]. To state his main result, we need a definition. A 3-manifold M is called *sufficiently large* when it contains a 2-sided imbedded surface F, not a 2-sphere or projective plane, such that the homomorphism $\pi_1(F) \to \pi_1(M)$ is injective.

WALDHAUSEN'S THEOREM: *Let M and N be compact orientable irreducible 3-manifolds, with N sufficiently large. If $f\colon (M, \partial M) \to (N, \partial N)$ is a map such that*

1. *The restriction of f to ∂M is a homeomorphism onto ∂N.*
2. *$f_{\#}\colon \pi_1(M) \to \pi_1(N)$ is an isomorphism.*

Then f is homotopic, without changing f on ∂M, to a homeomorphism.

(In the absence of assumption 1, much can still be said. See [K-M1] and [K-M2]).

Consider the case when M and N are closed, and N is sufficiently large. By the Theorem in section IV.2 above, the hypotheses imply that M and N are aspherical, so any isomorphism from $\pi_1(M)$ to $\pi_1(N)$ is induced by a homotopy equivalence. Then, Waldhausen's theorem guarantees that f is homotopic to a homeomorphism. So one consequence of Waldhausen's theorem is the following statement.

WALDHAUSEN'S THEOREM: *Let M and N be closed orientable irreducible 3-manifolds, with N sufficiently large. If $\pi_1(M) \cong \pi_1(N)$, then M is homeomorphic to N.*

That is, a closed orientable irreducible sufficiently large 3-manifold is determined up to homeomorphism by its fundamental group. But Waldhausen showed even more: if two homeomorphisms from M to N induce the same isomorphism from $\pi_1(M)$ to $\pi_1(N)$, then by asphericity they are homotopic, and Waldhausen showed that they are actually isotopic. We can combine these results using homotopy equivalences. Since M must be aspherical the natural homomorphism from the group $\mathcal{E}(M)$ of homotopy classes of homotopy equivalences from M to M to $Out(\pi_1(M))$ is an isomorphism (see section II.3). Thus we have a sequence

$$\mathcal{H}(M) \to \mathcal{E}(M) \to Out(\pi_1(M))$$

in which the second arrow is an isomorphism, and Waldhausen's results say that the first arrow is also an isomorphism. That is, every homotopy equivalence is homotopic to a homeomorphism, and if two homeomorphisms are homotopic, then they are isotopic. (Far reaching generalizations of this for parameterized families of homeomorphisms were proved by Hatcher [Hat2].)

When M has boundary, complications start to appear. The correct analogue of Waldhausen's Theorem involves the group $\mathcal{E}(M, \partial M)$ of homotopy classes of homotopy equivalences of pairs from $(M, \partial M)$ to $(M, \partial M)$. Since irreducible 3-manifolds with boundary components other than 2-spheres must be sufficiently large (see [He, p. 63]), we can state this version as follows.

WALDHAUSEN'S THEOREM FOR MANIFOLDS WITH BOUNDARY: *Let M be a compact orientable irreducible 3-manifold with nonempty boundary. Then the natural homomorphism $\mathcal{H}(M) \to \mathcal{E}(M, \partial M)$ is an isomorphism.*

This was proven in [Wa] under the assumption that ∂M is incompressible. When ∂M is compressible, it was proven to be surjective by Evans [Ev] and injective by Laudenbach [Lau].

In the setup above, we have a sequence

$$\mathcal{H}(M) \to \mathcal{E}(M, \partial M) \to \mathcal{E}(M) \to Out(\pi_1(M)) ,$$

where now the last arrow is an isomorphism by asphericity and Waldhausen's Theorem says the first arrow is an isomorphism. Thus $\mathcal{H}(M) \to Out(\pi_1(M))$ is essentially the same as $\mathcal{E}(M, \partial M) \to \mathcal{E}(M)$.

Notice, then, that the deviation of $\mathcal{H}(M) \to Out(\pi_1(M))$ from an isomorphism is entirely related to phenomena involving ∂M and homotopy equivalences of M. The kernel corresponds to homotopies between boundary preserving homotopy equivalences that do not preserve the boundary during the homotopy. Elements of $Out(\pi_1(M))$ that are not in the image of $\mathcal{H}(M) \to Out(\pi_1(M))$ correspond to homotopy equivalences which are not homotopic to maps which preserve ∂M.

When ∂M is incompressible, an argument of Waldhausen [Wa, pp. 82-83] shows that $\mathcal{E}(M, \partial M) \to \mathcal{E}(M)$ is injective, unless M is an I-bundle, in which case the kernel is the subgroup of order 2 generated by reflection in the I-fibers. When the boundary is compressible, we will see in chapters VI and VII that the kernel can be large and complicated.

Even when the boundary is incompressible, $\mathcal{E}(M, \partial M) \to \mathcal{E}(M)$ typically fails to be surjective. Examples 2 and 3 in section IX.1 show some of the ways this can happen, and example 1 there shows another phenomenon that prevents surjectivity when there is compressible boundary. To examine this lack of surjectivity more closely, in chapter IX we will study the following question:

FINITE-INDEX REALIZATION PROBLEM: *For which compact orientable irreducible 3-manifolds M does the image of the homomorphism $\mathcal{H}(M) \to Out(\pi_1(M))$ have finite index?*

As we will see in section IX.3, when M is a hyperbolic 3-manifold the answer to the finite-index realization problem gives information about the deformation spaces of hyperbolic structures on M.

Chapter V. Characteristic decomposition of Haken manifolds

When a compact, orientable, irreducible, sufficiently large 3-manifold has incompressible boundary (possibly empty boundary) it is said to be *Haken.* A remarkable structure theory for Haken manifolds was developed independently by W. Jaco and P. Shalen [J-S] and K. Johannson [Joh]. Johannson's formulation of the theory is designed for applications to the study of mappings between Haken 3-manifolds. In this chapter, we will give an exposition of his theory.

In section 1, we give an introduction to the concepts of boundary patterns and admissible maps, which provide the notation and technical underpinning for his formulation of the characteristic submanifold. Fibered 3-manifolds— I-bundles and Seifert fibered spaces— play a key role, and in section 2, we collect results about them. In section 3, we discuss the characteristic submanifold and list many of its properties. Some other useful results are collected in section 4.

In this chapter, we assume that the reader has a basic understanding of Seifert-fibered 3-manifolds and 3-manifolds that are I-bundles.

1. *Boundary patterns and admissible maps*

The following definitions are due to Johannson. A *boundary pattern* $\underline{\underline{m}}$ for a compact n-manifold M is a finite set of compact, connected $(n-1)$-manifolds in ∂M, such that the intersection of any i of them is empty or consists of $(n-i)$-manifolds. Thus when $n = 3$, the components of the intersections of pairs of elements of the boundary pattern are arcs or circles, and if three elements meet, their intersection consists of a finite collection of points at which three intersection arcs meet.

On a 2-manifold, a boundary pattern is simply a collection of arcs and circles in the boundary, which are disjoint except that two arcs may meet in an endpoint, or in both endpoints. In particular, an *i-faced disc* is a 2-disc whose boundary pattern has i elements, such that every point in the boundary lies in some element of the boundary pattern. A 4-faced disc is called a *square.* The i-faced discs with $i \leq 4$ play an important role in Johannson's theory.

The symbol $|\underline{\underline{m}}|$ will mean the set of points of ∂M that lie in some element of $\underline{\underline{m}}$. It is important in arguments to distinguish between elements of $\underline{\underline{m}}$, which are surfaces in ∂M, and the points of M which lie in these surfaces, and we will always be precise in this distinction. The elements of $\underline{\underline{m}}$ are called *bound sides,* and the closures of the components of $\partial M - |\underline{\underline{m}}|$ are called *free sides.* When $\partial M = |\underline{\underline{m}}|$, $\underline{\underline{m}}$ is said to be *complete.* The *completion* of $\underline{\underline{m}}$ is the complete boundary pattern $\overline{\underline{\underline{m}}}$ which is the union of $\underline{\underline{m}}$ and the collection of free sides.

Boundary patterns on 3-manifolds arise naturally in various ways. Here are some examples.

Example 1: trivial (but important) examples

For any manifold M, one has the empty boundary pattern $\emptyset$. Its completion $\overline{\emptyset}$ is the set of boundary components of M.

Example 2: proper boundary patterns

Let M be a compact 3-manifold and let F_1 be a 2-sided properly imbedded 2-manifold in M. Cut M along F_1 to produce a 3-manifold M_1 with two disjoint copies of F_1 in its boundary. If we want to remember the points that came from points in F_1, we form a boundary pattern $\underline{\underline{m}}_1$ consisting of these two surfaces. Now consider a 2-sided properly imbedded 2-manifold F_2 in M_1; by an arbitrarily small isotopy we may position F_2 so that it meets the elements of $\underline{\underline{m}}_1$ in circles and properly imbedded arcs. Cutting M_1 along F_2 yields a manifold M_2, containing two copies of F_2 in its boundary, together with the surfaces obtained from the elements of $\underline{\underline{m}}_1$ by cutting them along their intersections with F_2. These 2-manifolds in ∂M_2 meet pairwise along circles and arcs, and any three are disjoint, so they form a boundary pattern $\underline{\underline{m}}_2$ for M_2. A 2-sided properly imbedded 2-manifold F_3 in M_2 may be moved by an arbitrarily small isotopy so that it meets the elements in $\underline{\underline{m}}_2$ in circles and arcs, and when M_2 is cut open to obtain M_3, the resulting boundary pattern $\underline{\underline{m}}_3$ may now have three elements meeting at a single point, where a portion of ∂F_3 cuts across an intersection arc or circle of two elements of $\underline{\underline{m}}_2$. Now any F_4 in M_3 may be positioned, by an arbitrarily small isotopy, to avoid the points that are intersections of three elements of $\underline{\underline{m}}_3$. Then, the resulting collection of surfaces in ∂M_4 (and all collections for manifolds resulting from further iterations of this process) will have at most three elements meeting at a point, and consequently will be a boundary pattern.

In general, suppose $(M, \underline{\underline{m}})$ is a 3-manifold with boundary pattern, and F is a 2-sided surface properly imbedded in M, so that ∂F meets the boundaries of the elements of $\underline{\underline{m}}$ transversely, and meets the intersection of each pair of elements transversely, if at all. Let M' be the 3-manifold obtained from M by removing the interior of a small product neighborhood F. The *proper boundary pattern* on M' is the boundary pattern consisting of the two copies of F in ∂M, together with the components of the intersections of the elements of $\underline{\underline{m}}$ with M'.

Example 3: fibered manifolds

There are many nonisotopic I-bundle structures on a 3-dimensional orientable handlebody V. If B is any compact connected 2-manifold whose Euler characteristic equals that of V, then V is the total space of an I-bundle over B, which is twisted if and only if B is nonorientable. These can be distinguished in a natural way using boundary patterns. Assume that V carries a fixed structure as an I-bundle. Each component of the associated ∂I-bundle is a 2-manifold in ∂V, called a *lid* of the I-bundle. There are two lids when the bundle is a product, and one when it is twisted. Let $\underline{\underline{b}}$ be a boundary pattern on B. The preimages of the elements of $\underline{\underline{b}}$ form a collection of squares and annuli in ∂V, called the *sides* of the I-bundle. The lid or lids, together with the sides, if any, form a boundary pattern $\underline{\underline{v}}$ on V. When V carries this boundary pattern, the fibering is called an *admissible I-fibering* of V as an I-bundle over $(B, \underline{\underline{b}})$.

A Seifert fibering on a 3-manifold $(V, \underline{\underline{v}})$ with boundary pattern is called an *admissible Seifert fibering* when the elements of $\underline{\underline{v}}$ are the preimages of the components of a boundary pattern of the orbit surface. Consequently the elements of $\underline{\underline{v}}$ must be tori or fibered annuli.

Example 4: product boundary patterns

If $(M, \underline{\underline{m}})$ is a manifold with boundary pattern, and N is a manifold, then the product boundary pattern on $(M, \underline{\underline{m}}) \times N$ is

$$\{F \times N \mid F \in \underline{\underline{m}}\} \cup \{M \times W \mid W \text{ is a component of } \partial N\} .$$

More generally, if $p\colon E \to (M, \underline{\underline{m}})$ is a locally trivial N-bundle, then there is a boundary pattern

$$\{p^{-1}(F) \mid F \in \underline{\underline{m}}\} \cup \{\text{components of the associated } \partial N\text{-bundle}\} .$$

When $(M, \underline{\underline{m}})$ is a 2-manifold and $N = I$ or $N = S^1$, these agree with the boundary patterns described in example 3.

Example 5: the dual cell construction

Let M be a compact 3-manifold and let F be a compact 2-manifold in ∂M. Fix a triangulation T of F. Take the first barycentric subdivision $T^{(1)}$ of T, and let $\underline{\underline{m}}$ consist of the discs which are the closed stars of the vertices of T in $T^{(1)}$.

Example 6: pared 3-manifolds

When the interior of a Haken 3-manifold M admits a hyperbolic structure, there is a boundary pattern for M called a *pared* structure, so that every parabolic element is homotopic into an element of the boundary pattern. For pared manifolds, the characteristic submanifold has a very restricted structure (see [C-M]).

Maps which respect boundary pattern structures are called admissible. Precisely, a map f from $(M, \underline{\underline{m}})$ to $(N, \underline{\underline{n}})$ is called *admissible* when $\underline{\underline{m}}$ is the disjoint union

$$\underline{\underline{m}} = \coprod_{G \in \underline{\underline{n}}} \{ \text{ components of } f^{-1}(G) \}.$$

Notice that the requirement that the union be disjoint implies that for each element F of $\underline{\underline{m}}$, there is exactly one element of $\underline{\underline{n}}$ that contains the entire image of F. Thus, two neighboring bound sides F_1 and F_2 must be mapped to neighboring bound sides G_1 and G_2, in such a way that $F_1 \cap F_2$ consists of some components of the preimage of $G_1 \cap G_2$ in $F_1 \cup F_2$.

An admissible map $f\colon (M, \underline{\underline{m}}) \to (N, \underline{\underline{n}})$ is called an *admissible homotopy equivalence* if there is an admissible map $g\colon (N, \underline{\underline{n}}) \to (M, \underline{\underline{m}})$ such that gf and fg are admissibly homotopic to the identity maps. When the elements of $\underline{\underline{m}}$ and $\underline{\underline{n}}$ are pairwise disjoint, this simply says that $f\colon (M, |\underline{\underline{m}}|) \to (N, |\underline{\underline{n}}|)$ is a homotopy equivalence of pairs.

Suppose $(X, \underline{\underline{x}})$ is an admissibly-imbedded codimension-zero submanifold of $(M, \underline{\underline{m}})$, which is admissibly imbedded in $(M, \overline{\underline{\underline{m}}})$. The latter assumption guarantees that $X \cap \partial M = |\underline{\underline{x}}|$, and that an element of $\underline{\underline{x}}$ which does not meet any other element of $\underline{\underline{x}}$ must be imbedded in the manifold interior of an element of $\underline{\underline{m}}$. Let $\underline{\underline{x}}''$ denote the collection of components of the frontier of X in M. To *split M along X* means to construct the manifold with boundary pattern $(\overline{M - X}, \widetilde{\underline{\underline{m}}} \cup \underline{\underline{x}}'')$, where the elements of $\widetilde{\underline{\underline{m}}}$ are the closures of the components of $F - (X \cap F)$ for $F \in \underline{\underline{m}}$. The boundary pattern $\widetilde{\underline{\underline{m}}} \cup \underline{\underline{x}}''$ is called the *proper boundary pattern* on $\overline{M - V}$. In particular, if X is a small product neighborhood of a 2-sided surface $(F, \underline{\underline{f}})$ admissibly imbedded in $(M, \underline{\underline{m}})$, then the proper boundary pattern on $\overline{M - X}$ is the one described in example 2 above.

Recall that an *i-faced disc* is a 2-disc whose boundary pattern is complete and has i components. Observe that each element of $\underline{\underline{m}}$ is incompressible if and only if whenever D is an admissibly imbedded 1-faced disc in $(M, \underline{\underline{m}})$, the boundary of D bounds a disc in $|\underline{\underline{m}}|$ which is contained in a single element of $\underline{\underline{m}}$. For most of Johannson's theory, a somewhat stronger condition is needed. The boundary pattern $\underline{\underline{m}}$ is called *useful* when the boundary of every admissibly imbedded i-faced disc in $(M, \underline{\underline{m}})$ with $i \leq 3$ bounds a disc D in ∂M such that $D \cap (\cup_{F \in \underline{\underline{m}}} \partial F)$ is the cone on $\partial D \cap (\cup_{F \in \underline{\underline{m}}} \partial F)$. Notice that $\overline{\emptyset}$ is a useful boundary pattern on M if and only ∂M is incompressible. For another example, the product of a 4-faced disc with S^1 yields a useful boundary pattern on the solid torus, but the product of a 3-faced disc with S^1 does not.

Some confirmation that usefulness is the natural generalization of incompressibility in the context of 3-manifolds with boundary patterns comes from the following generalization of the Loop Theorem of Papakyriakopoulos. It is given as proposition 2.1 in [Joh]. In its statement, J denotes $\cup_{F \in \underline{\underline{m}}} \partial F$.

LOOP THEOREM: *Let $(M, \underline{\underline{m}})$ be a 3-manifold with boundary pattern. The following are equivalent.*

(1) *The boundary of any i-faced disc, $1 \leq i \leq 3$, admissibly imbedded in $(M, \underline{\underline{m}})$, bounds a disc D_0 in ∂M so that $D_0 \cap J$ is the cone on $\partial D_0 \cap J$.*
(2) *For any admissible map $f: (D, \underline{\underline{d}}) \to (M, \underline{\underline{m}})$, where $(D, \underline{\underline{d}})$ is an i-faced disc, there exists a map $g: D \to \partial M$ so that*
   (i) *$g(D) \subset \partial M$ and $g|_{\partial D} = f|_{\partial D}$, and*
   (ii) *$g^{-1}(J)$ is the cone on $g^{-1}(J) \cap \partial D$.*

Usually, a map of a closed 2-manifold into a 3-manifold is considered to be essential when it is injective on fundamental groups. In the context of manifolds with boundary patterns, this concept becomes the following. An admissible map $h: (K, \underline{\underline{k}}) \to (X, \underline{\underline{x}})$, where K is an arc or a circle and $(X, \underline{\underline{x}})$ is a 2- or 3-manifold is called *inessential* if it is admissibly homotopic to a constant map (the constant map might not be admissible, but all the other maps in the homotopy must be admissible), otherwise it is called *essential.* A map $f: (X, \underline{\underline{x}}) \to (Y, \underline{\underline{y}})$ between 2- or 3-manifolds (not necessarily of the same dimension) is called *essential* if for any essential path or loop $h: (K, \underline{\underline{k}}) \to (X, \underline{\underline{x}})$, the composition $fh: (K, \underline{\underline{k}}) \to (Y, \underline{\underline{y}})$ is essential. Notice that when $\underline{\underline{x}}$ is empty, this simply says that f is injective on fundamental groups.

The Loop Theorem of Papakyriakopoulos is often used in processes where the preimage of a surface under a map between 3-manifolds is being simplified. To carry out these processes in 3-manifolds with boundary patterns, the following formulation is needed. It is given as lemma 4.2 of [Joh]:

COMPRESSION LEMMA: *Let $(M, \underline{\underline{m}})$ be a 3-manifold with boundary pattern, and let $(F, \underline{\underline{f}})$ be an admissibly imbedded surface in M with $F \cap \partial M = \partial F$.*

Assume that $\underline{\underline{m}}$ is useful and no component of F is an admissible i-faced disc with $1 \le i \le 3$. Then $(F, \underline{\underline{f}})$ is inessential in $(M, \underline{\underline{m}})$ if and only if there is an admissibly imbedded disc $(D, \underline{\underline{d}})$ in $(M, \underline{\underline{m}})$ such that $(D, \overline{\underline{\underline{d}}})$ is an i-faced disc, $1 \le i \le 3$, and $D \cap F$ is a side of $(D, \overline{\underline{\underline{d}}})$ which is an essential curve in F.

The Compression Lemma is used in the proof in [C-M] of the following result.

HOMOTOPY INVARIANCE OF USEFULNESS: *Let $(M, \underline{\underline{m}})$ and $(N, \underline{\underline{n}})$ be compact orientable irreducible 3-manifolds with boundary pattern, which are admissibly homotopy equivalent.*
(a) *If $\underline{\underline{m}}$ is useful, then so is $\underline{\underline{n}}$.*
(b) *If $\overline{\underline{\underline{m}}}$ is useful, then so is $\overline{\underline{\underline{n}}}$.*

2. *Fibered 3-manifolds*

In example 3 of section 1 above, we defined admissible fiberings of I-bundles and Seifert-fibered manifolds. In this subsection, we will present some results on fibered manifolds.

Assume that $(M, \underline{\underline{m}})$ has a fixed structure as an I-bundle or Seifert-fibered space, with projection map $p: M \to F$. The following definition is from 5.3 of [Joh]. Let G be a manifold. A map $g: G \to M$ is called *vertical* if its image is a union of nonexceptional fibers. It is called *horizontal* if $g^{-1}(\partial M) = \partial G$ and pg is a branched covering map. Branch points occur only if $(M, \underline{\underline{m}})$ is Seifert-fibered, and then they lie over the exceptional points of the orbit surface.

The following definition is from 5.1 of [Joh].

Exceptional Fibered Manifolds: An irreducible 3-manifold $(M, \underline{\underline{m}})$ with an admissible fibering as an I-bundle or Seifert fibered space is called an *exceptional fibered manifold* if it is one of the following.

(EF1) The I-bundle over an i-faced disc, $1 \le i \le 3$.
(EF2) The S^1-bundle over an i-faced disc, $i = 2, 3$, or a Seifert fibered space over a 1-faced disc with at most one exceptional fiber.
(EF3) The I-bundle over the 2-sphere or projective plane.
(EF4) A Seifert fibered space with the 2-sphere as orbit surface and at most three exceptional fibers.
(EF5) A Seifert fibered space with the projective plane as orbit surface and at most one exceptional fiber.

Certain other manifolds are frequently exceptional cases because they admit a horizontal square, annulus, or torus. These are:

(EIB) A manifold with boundary pattern which can be admissibly fibered as an I-bundle over the square, annulus, Möbius band, torus, or Klein bottle.

(ESF) A closed 3-manifold which can be obtained by gluing two I-bundles over the torus or Klein bottle together along their boundaries.

Other than in some of the exceptional cases, an essential imbedded annulus or torus is isotopic to one which is vertical. The precise statement is proposition 5.7 of [Joh]:

ESSENTIAL ANNULUS AND TORUS THEOREM: *Let $(M, \underline{\underline{m}})$ be an I-bundle or Seifert fibered space with a fixed admissible fibration, but not one of the exceptional fibered manifolds (EF1)-(EF5). Suppose that T is an essential square or annulus imbedded in $(M, \underline{\underline{m}})$. Then either*

(i) *there exists an admissible isotopy which makes T vertical, or*

(ii) *$(M, \underline{\underline{m}})$ is one of the exceptions (EIB) and has an admissible I-fibering which makes T vertical.*

Suppose T is an essential torus imbedded in $(M, \underline{\underline{m}})$. Then either

(iii) *there exists an admissible isotopy which makes T vertical, or*

(iv) *$(M, \underline{\underline{m}})$ is the I-bundle over the torus or Klein bottle, or*

(v) *M is one of the exceptions (ESF).*

For annuli which are not imbedded, proposition 5.10 of [Joh] gives a verticalization result:

ESSENTIAL SINGULAR ANNULUS THEOREM: *Let $(M, \underline{\underline{m}})$ be an I-bundle or Seifert fibered space with a fixed admissible fibration, but not one of the exceptional fibered manifolds (EF1)-(EF2). Let $f: T \to M$ be an essential singular square or annulus in $(M, \underline{\underline{m}})$. Then either*

(i) *there exists an admissible homotopy which makes f vertical, or*

(ii) *$(M, \underline{\underline{m}})$ is one of the exceptions (EIB). Moreover, if k is any side of T which is mapped by f into a lid of $(M, \underline{\underline{m}})$, then f is admissibly homotopic to a vertical map by a homotopy which is constant on k.*

In most cases, the fibering of a fibered manifold is unique up to isotopy. The exceptions are determined in corollary 5.9 of [Joh]:

UNIQUE FIBERING THEOREM: *Suppose each of $(M_1, \underline{\underline{m}}_1)$ and $(M_2, \underline{\underline{m}}_2)$ is an I-bundle or Seifert fibered space with a fixed admissible fibration, but neither is a solid torus with $\underline{\underline{m}}_i = \{\partial M_i\}$, or one of the exceptional fibered manifolds (EF3)-(EF5), (EIB), or (ESF). Then every admissible homeomorphism $h: (M_1, \underline{\underline{m}}_1) \to (M_2, \underline{\underline{m}}_2)$ is admissibly isotopic to a fiber-preserving homeomorphism. Moreover,*

(a) *The conclusion holds if M is one of the exceptions (EIB) and h and h^{-1} map lids to lids.*

(b) *If M_1 is an I-bundle and $h\colon M_1 \to M_1$ is the identity on one lid, then the isotopy may be chosen to be constant on this lid.*

A more difficult issue is when a map between two different fibered manifolds is homotopic to a fiber-preserving one. This is determined in proposition 28.4 of [Joh]:

FIBER-PRESERVING MAP THEOREM: *Suppose that each of $(M_1, \underline{\underline{m_1}})$ and $(M_2, \underline{\underline{m_2}})$ is an I-bundle or Seifert fibered space with a fixed admissible fibration, but neither is one of the exceptional manifolds (EF1)-(EF5), (EIB), or (ESF). If $(M_2, \underline{\underline{m_2}})$ is an I-bundle, assume that M_1 is neither a ball or a solid torus. Then every essential map $f\colon (M_1, \underline{\underline{m_1}}) \to (M_2, \underline{\underline{m_2}})$ is admissibly homotopic to a fiber-preserving map.*

In [C-M], the exceptional cases (EIB) arise frequently, and an interesting variant of the Fiber-preserving Map Theorem is needed. Its proof is a fairly straightforward modification of the proof of proposition 28.4 of [Joh].

FIBER-PRESERVING SELF-MAP THEOREM: *Let $(V, \underline{\underline{v}})$ be an I-bundle or Seifert fibered space with a fixed admissible fibration, such that $\overline{\underline{\underline{v}}}$ is useful. Let $f\colon (V, \underline{\underline{v}}) \to (V, \underline{\underline{v}})$ be an essential map. Assume that*

(i) *$(V, \underline{\underline{v}})$ is not one of the exceptional fibered manifolds (EF1)-(EF5), and*

(ii) *$(V, \underline{\underline{v}})$ is not one of the exceptions (ESF), and*

(iii) *if $(V, \underline{\underline{v}})$ is an I-bundle, then f takes lids to lids, and*

(iv) *if V is $S^1 \times S^1 \times I$ or the I-bundle over the Klein bottle and all elements of $\underline{\underline{v}}$ are boundary components, then $(V, \underline{\underline{v}})$ is I-fibered.*

Then f is admissibly homotopic to a fiber-preserving map.

3. *The characteristic submanifold*

According to sections 8 and 9 of [Joh], an irreducible 3-manifold $(M, \underline{\underline{m}})$ with useful boundary pattern contains an admissibly and essentially imbedded fibered 3-manifold $(V, \underline{\underline{v}})$, also admissibly imbedded in $(M, \overline{\underline{\underline{m}}})$, called a *characteristic submanifold* of $(M, \underline{\underline{m}})$. By definition 8.2 of [Joh], V is *full,* which means that the union of V with any of the complementary components of M is not an essential fibered manifold. By corollary 10.9 of [Joh], $(V, \underline{\underline{v}})$ is unique up to admissible isotopy.

Corollary 10.10 of [Joh] gives two geometric characterizations of the characteristic submanifold, one of which is the following.

Engulfing Theorem: *Let* $(M, \underline{\underline{m}})$ *be a Haken 3-manifold with useful boundary pattern, and let* V *be a full essential fibered submanifold of* $(M, \underline{\underline{m}})$. *Then* V *is a characteristic submanifold of* M *if and only if every* I*-bundle or Seifert-fibered space essentially imbedded in* $(M, \underline{\underline{m}})$ *is admissibly isotopic into* V.

Notice that this implies that every essential imbedded square, annulus, or torus imbedded in M is admissibly isotopic into V, since such a surface can be thickened to a fibered manifold.

One of the strongest properties of the characteristic submanifold is theorem 12.5 of [Joh]:

Enclosing Theorem: *Let* $(M, \underline{\underline{m}})$ *be a Haken 3-manifold with useful boundary pattern, and let* V *be its characteristic submanifold. Then every essential singular square, annulus, or torus in* $(M, \underline{\underline{m}})$ *is admissibly homotopic into* V.

A more general version of enclosing is given in proposition 13.1 of [Joh]:

Extended Enclosing Theorem: *Let* $(M, \underline{\underline{m}})$ *be a Haken 3-manifold with useful boundary pattern, and let* V *be its characteristic submanifold. Let* $(X, \underline{\underline{x}})$ *be an* I*-bundle or Seifert fiber space whose complete boundary pattern is useful. Suppose that* $(X, \underline{\underline{x}})$ *is not one of the exceptional cases (EF1)-(EF5). Then every essential map* $f: (X, \underline{\underline{x}}) \to (M, \underline{\underline{m}})$ *is admissibly homotopic into* V.

A Haken 3-manifold $(M, \underline{\underline{m}})$ whose completed boundary pattern $\overline{\underline{\underline{m}}}$ is useful is called *simple* if every component of the characteristic submanifold of $(M, \overline{\underline{\underline{m}}})$ is a regular neighborhood of a side of $(M, \overline{\underline{\underline{m}}})$. Suppose that $(M, \underline{\underline{m}})$ is a Haken 3-manifold and $\overline{\underline{\underline{m}}}$ is useful. Let $(M', \underline{\underline{m}}')$ be the 3-manifold obtained from $(M, \underline{\underline{m}})$ by splitting at the characteristic submanifold of $(M, \overline{\underline{\underline{m}}})$. According to remark 3 on p. 159 of [Joh], $(M', \underline{\underline{m}}')$ is simple. The manifold $(M', \underline{\underline{m}}')$ is involved in one of the main results of [Joh], given as theorem 24.2:

Classification Theorem: *Let* $(M_1, \underline{\underline{m}}_1)$ *and* $(M_2, \underline{\underline{m}}_2)$ *be compact irreducible 3-manifolds with boundary patterns whose completions are useful and nonempty. Let* V_1 *and* V_2 *denote the characteristic submanifolds of* $(M_1, \overline{\underline{\underline{m}}}_1)$ *and* $(M_2, \overline{\underline{\underline{m}}}_2)$ *respectively. Let* $\underline{\underline{v}}_1$, $\underline{\underline{v}}_2$, $\underline{\underline{w}}_1$, *and* $\underline{\underline{w}}_2$ *denote the proper boundary patterns of* V_1, V_2, $\overline{M_1 - V_1}$, *and* $\overline{M_1 - V_2}$ *respectively. Then every admissible homotopy equivalence* $f: (M_1, \underline{\underline{m}}_1) \to (M_2, \underline{\underline{m}}_2)$ *can be changed by admissible homotopy so that* $f|_{V_1}: (V_1, \underline{\underline{v}}_1) \to (V_2, \underline{\underline{v}}_2)$ *is an admissible homotopy equivalence and* $f|_{\overline{M_1 - V_1}}: (\overline{M_1 - V_1}, \underline{\underline{w}}_1) \to (\overline{M_2 - V_2}, \underline{\underline{w}}_2)$ *is an admissible homeomorphism.*

The following result is essentially corollary 18.2 of [Joh] and the remark following it. There, only the conclusion that $(H_t')^{-1}(\Sigma)=\Sigma$ is given. However, this conclusion implies that H_t carries the frontier $\Sigma \cap \overline{M-\Sigma}$ to itself. Since the frontier is bicollared, one may alter H to ensure that $(H_t')^{-1}(\Sigma \cap \overline{M-\Sigma}) = \Sigma \cap \overline{M-\Sigma}$, and then one has the conclusion we give here.

HOMOTOPY SPLITTING THEOREM: *Let $(M, \underline{\underline{m}})$ be an irreducible 3-manifold whose completed boundary pattern $\overline{\underline{\underline{m}}}$ is useful and nonempty, and let Σ be the characteristic submanifold of $(M, \overline{\underline{\underline{m}}})$. Suppose $H\colon (M \times I, \underline{\underline{m}} \times I) \to (M, \underline{\underline{m}})$ is an admissible homotopy such that $H_0^{-1}(\Sigma) = \Sigma$ and $H_1^{-1}(\Sigma) = \Sigma$. Then H is admissibly homotopic, relative to $M \times \partial I$, to H' such that $(H_t')^{-1}(\Sigma)=\Sigma$ and $(H_t')^{-1}(\overline{M-\Sigma})=\overline{M-\Sigma}$ for all $t \in I$.*

4. *Some additional results*

In this section we give a few other results from [Joh] which do not fit conveniently under the previous sections.

Two fundamental results in the theory of mappings of low-dimensional manifolds are Baer's Theorem, and its generalization to dimension 3 due to Waldhausen. These extend to the context of manifolds with boundary patterns. Baer's Theorem becomes proposition 3.3 of [Joh]:

BAER'S THEOREM: *Let $(F, \underline{\underline{f}})$ and $(G, \underline{\underline{g}})$ be connected surfaces with complete boundary patterns. Suppose that $(F, \underline{\underline{f}})$ is not a 1-sided disc or the 2-sphere, and that G is not the projective plane. Then any essential map $f\colon (F, \underline{\underline{f}}) \to (G, \underline{\underline{g}})$ is admissibly homotopic to a covering map. If the restriction of f to ∂F is a local homeomorphism, then the homotopy may be chosen to be constant on ∂F.*

In some statements of the classical Baer's theorem, the case when F is an annulus and f is map carrying ∂F into a single boundary circle of G is allowed as a possibility in the conclusion. In the version of Baer's theorem that we have stated here, such a map would be inessential, since it carries an arc connecting the boundary circles of the annulus to a path which is admissibly homotopic into ∂G, and therefore is excluded by the hypothesis.

Waldhausen's Theorem (see section IV.3) extends to manifolds with complete and useful boundary patterns, as given in proposition 3.4 of [Joh]:

WALDHAUSEN'S THEOREM: *Let $(M, \underline{\underline{m}})$ and $(N, \underline{\underline{n}})$ be connected 3-manifolds with complete and useful boundary patterns. Suppose that M has nonempty boundary and $(M, \underline{\underline{m}})$ is not a 3-ball with one or two sides. Then* *any essential map* $f\colon (M,\underline{\underline{m}}) \to (N,\underline{\underline{n}})$ *is admissibly homotopic to a covering map. If the restriction of* f *to* ∂M *is a local homeomorphism, then the homotopy may be chosen to be constant on* ∂M.

One of the important technical tools in the proof of Waldhausen's Theorem is that homotopic imbedded incompressible surfaces in Haken manifolds are isotopic. In the context of manifolds with boundary patterns, this becomes the following:

PARALLEL SURFACES THEOREM: *Let* M *be an irreducible 3-manifold with complete boundary pattern, and let* $(F,\underline{\underline{f}})$ *and* $(G,\underline{\underline{g}})$ *be connected essential surfaces in* $(M,\underline{\underline{m}})$, *with* $F\cap\partial M=\partial F$ *and* $G\cap\partial M=\partial G$. *Assume that* $(G,\underline{\underline{g}})$ *is admissibly homotopic into* $(F,\underline{\underline{f}})$. *Then* $(G,\underline{\underline{g}})$ *is admissibly isotopic into* $(F,\underline{\underline{f}})$. *Moreover, if* F *and* G *are disjoint, then* $(G,\underline{\underline{g}})$ *is admissibly parallel to* $(F,\underline{\underline{f}})$.

Recall that a Haken 3-manifold $(M,\underline{\underline{m}})$ whose completed boundary pattern $\underline{\underline{\overline{m}}}$ is useful is called *simple* if every component of the characteristic submanifold of $(M,\underline{\underline{\overline{m}}})$ is a regular neighborhood of a side of $(M,\underline{\underline{\overline{m}}})$. Proposition 27.1 of [Joh] gives important information about the mapping class group in this case:

FINITE MAPPING CLASS GROUP THEOREM: *Let* $(M,\underline{\underline{m}})$ *be a simple 3-manifold with complete and useful boundary pattern. Then* $\mathcal{H}(M,\underline{\underline{m}})$ *is finite.*

Chapter VI. The characteristic compression body

1. *Existence and uniqueness*

Compression bodies were defined by Bonahon [Bo] and McCullough and Miller [M-M]. For simplicity we will work only with orientable manifolds, but the theory we will discuss adapts easily to the nonorientable case. A theory of relative compression bodies adapted to Johannson's theory of boundary patterns has been developed in [C-M].

For us, a *compression body* means either a handlebody or a connected irreducible 3-manifold which can be constructed as follows. Start with a collection $\{F_i \mid 1 \leq i \leq m\}$ of closed connected 2-manifolds, none of which is simply-connected. Form a connected irreducible manifold V from $\cup_{i=1}^{m} F_i \times I$ by attaching k 1-handles to $\cup_{i=1}^{m} F_i \times \{1\}$. We denote $F_i \times \{0\}$ by F_i. The fundamental group of V is a free product $\pi_1(F_1) * \cdots * \pi_1(F_m) * H$ where H is a free group of rank $k+1-m$. One component of ∂V is a connected surface F which consists of the intersection of ∂V with $\cup_{i=1}^{m} F_i \times \{1\}$ together with its intersection with the 1-handles. The induced homomorphism $\pi_1(F) \to \pi_1(V)$ is surjective.

In the proof of Theorem 1 below, we will use the following result, which is Theorem 10.5 in [He] simplified to the irreducible orientable case.

FINITE INDEX THEOREM: *Let M be a compact orientable irreducible 3-manifold, and F ($\neq D^2$ or S^2) a compact, connected, incompressible surface in ∂M. If the index of $\pi_1(F)$ in $\pi_1(M)$ is finite, then either*

(i) *$\pi_1(M) \cong \mathbb{Z}$, F is an annulus, and M is a solid torus, or*

(ii) *$\pi_1(F) = \pi_1(M)$ and $M = F \times I$ with $F = F \times \{0\}$, or*

(iii) *$\pi_1(F)$ has index 2 in $\pi_1(M)$ and M is a twisted I-bundle over a compact surface N with F as the associated 0-sphere bundle.*

Our first result is the Existence and Uniqueness Theorem for compression body neighborhoods of compressible boundary components.

THEOREM 1: *Let M be a compact orientable irreducible 3-manifold. Let F be a compressible boundary component of M. Then F has a neighborhood V, with incompressible frontier, satisfying the following.*

(i) *V is a compression body.*

(ii) *Each F_i is either a component of ∂M or is contained in the interior of M.*

(iii) *If M_0 is any component of $\overline{M - V}$ such that $M_0 \cap V$ is connected, then $\pi_1(M_0 \cap V) \to \pi_1(M_0)$ is not surjective.*

The neighborhood V is unique up to admissible ambient isotopy.

A compression body neighborhood of F which satisfies the conditions in Proposition 1 is called a *normally imbedded* compression body neighborhood of F.

PROOF OF THEOREM 1: By inductive application of the Loop Theorem there exists a sequence $D_1, \ldots, D_k$ of disjoint compressing discs with boundary in F so that the frontier of a small regular neighborhood N of $F \cup (\cup_{i=1}^{k} D_i)$ is incompressible in $\overline{M - N}$. If any component of the frontier is a 2-sphere, then it bounds a 3-ball in $\overline{M - N}$; adding the union of such balls to N results in a manifold V with incompressible frontier in M. Therefore V is a compression body satisfying conditions (i) and (ii).

Let M_0 be a component of $\overline{M - V}$ such that $M_0 \cap V$ is connected, and suppose $\pi_1(M_0 \cap V) \to \pi_1(M_0)$ is surjective. Since $M_0 \cap V$ is incompressible, the Finite Index Theorem shows that M_0 is homeomorphic to $(M_0 \cap V) \times I$ with $M_0 \cap V = (M_0 \cap V) \times \{1\}$. Therefore we may add M_0 to V, obtaining a compression body. Repeating this for all such M_0 will make V satisfy condition (iii), while retaining the other conditions.

For the uniqueness, suppose V_1 and V_2 are two normally imbedded compression body neighborhoods of F. Each of the discs called D_i in the construction of V_1 can be deformed isotopically into V_2, since the frontier of V_2 is incompressible, and then the rest of V_1 can be deformed into V_2, so we may assume that V_1 lies in the topological interior of V_2. But if G is any component of the frontier of V_1, contained in a component W of $\overline{V_2 - V_1}$, the condition that $\pi_1(F) \to \pi_1(V_2)$ is surjective implies that $\pi_1(G) \to \pi_1(W)$ is surjective. Since G is incompressible, the Finite Index Theorem implies that W is a product. Therefore there is an isotopy that expands V_1 onto V_2.

¿From the Existence and Uniqueness Theorem, we can easily deduce the following characterization of compression bodies.

COROLLARY 2: *Let W be a compact orientable irreducible 3-manifold. Then W is a compression body if and only there exists a compressible boundary component F of W such that $\pi_1(F) \to \pi_1(W)$ is surjective.*

PROOF: If W is a handlebody, then it is a compression body. Otherwise, let F be a compressible boundary component and let V be a normally imbedded compression body neighborhood of F. Let $V_1, \ldots, V_r$ be the components of $\overline{W - V}$. Since $\pi_1(F) \to \pi_1(W)$ is surjective, V_1 cannot meet V in more than one component of the frontier of V. By condition (iii) in Proposition 1, $\pi_1(V_1 \cap V) \to \pi_1(V_1)$ cannot be surjective. But then, $\pi_1(F) \to \pi_1(W)$ could not have been surjective. We conclude that $\overline{W - V}$ is empty, so $V = W$.

2. *The kernel of* $\mathcal{H}(V) \to Out(\pi_1(V))$

We now begin our examination of the homeomorphisms of compression bodies. As is common in dimensions 2 and 3, we compare the topological automorphisms to the algebraic automorphisms of the fundamental group by examining the natural homomorphism $\mathcal{H}(V) \to Out(\pi_1(V))$ which takes a mapping class $\langle f \rangle$ to its induced outer automorphism $f_\#$ on $\pi_1(V)$. For compression bodies, $Out(\pi_1(V))$ is understandable because $\pi_1(V)$ is a free product

$$\pi_1(V) = \pi_1(F_1) * \cdots * \pi_1(F_m) * H$$

where H is a free group of rank $k + 1 - m$. Many years ago, D. I. Fuks-Rabinovitch [F-R1], [F-R2], following earlier work of J. Nielsen, obtained a presentation for the automorphism group of a free product $G_1 * \cdots G_m * H_1 * \cdots * H_n$, where the factors are indecomposable, each H_i is infinite cyclic, and each G_i is not infinite cyclic. Gilbert [Gil] has given a modern proof of the Fuks-Rabinovitch presentation using methods due to J. McCool. The Fuks-Rabinovitch presentation has six kinds of generators and 48 kinds of relations. A complete list, correcting some typographical errors in the original Fuks-Rabinovitch paper, is given in [M-M]. Also in [M-M], homeomorphisms realizing the generators are constructed. (Actually not all generators can be realized, but the realizable subgroup has finite index in $Out(\pi_1(V))$, as we show in the next section.) To illustrate the most important construction used in realizing the generators, we will discuss one type of generator and its realizing homeomorphism. For studying $\mathcal{H}(V)$, it is convenient to regard V as constructed by attaching the $F_i \times I$ to a 3-ball B, and each of the m 1-handles to that same ball. The center of B is the basepoint x_0. See Fig. 1.

One of the Fuks-Rabinovitch generators is $\rho_{i,j}(\gamma)$. Here, γ is an element of $\pi_1(F_i) = G_i$ and j indicates the j^{th} infinite cyclic group H_j generated by α_j. The automorphism $\rho_{i,j}(\gamma)$ is defined by $\rho_{i,j}(\gamma)(\alpha_j) = \gamma\alpha_j$ and $\rho_{i,j}(\gamma)(x) = x$ for $x \in G_i$ or $x \in H_k$ with $k \neq j$. There is a corresponding homeomorphism $R_{i,j}(\gamma)$ which induces $\rho_{i,j}(\gamma)$, illustrated in Fig. 1. It is constructed as follows. Cut V apart along the right (rather than the left) attaching disc D for the i^{th} 1-handle. Take an isotopy on the cut open manifold which moves the copy of D on the 3-ball around a loop which represents γ^{-1}, then reattach the copies of D. Notice that a loop representing α_j is moved to a loop representing $\alpha_j\gamma$. These loops should be visualized as being in the interior of V.

(Actually, there are many nonisotopic choices for $R_{i,j}(\gamma)$, since different choices of the "sliding path" around which the copy of D moves will yield nonisotopic homeomorphisms. These are all homotopic, since V is aspherical and the homeomorphisms all induce the same automorphism on $\pi_1(V)$. All of these homeomorphisms differ by products of "twist homeomorphisms," discussed below. Here is the idea. Let $\Sigma = \partial B -$ attaching discs. A change in sliding path has the effect of composing with a homeomorphism which only moves points on B. Since Dehn twists generate the group of orientation-preserving mapping classes of Σ, and each simple closed curve in Σ bounds a disc in B, the twists about these discs generated the mapping class group of B relative to the attaching discs.)

There are some other kinds of generating automorphisms, such as performing an automorphism of some G_i. For purposes of understanding the general idea of the program for studying homeomorphisms of V, we need not worry about those. The rigorous development of the theory given in [M-M] requires careful attention to all the Fuks-Rabinovitch generators.

A very important kind of homeomorphism is the "twist homeomorphism." Let D be a properly imbedded 2-disc in a 3-manifold M and let $D \times I$ be a product region with $D = D \times \{0\}$, such that $D \times I \cap \partial M = \partial D \times I$. Regarding D as the unit disc in the complex plane, we define $t_D: M \to M$ by $t_D(x) = x$ if $x \notin D \times I$ and $t_D(r\exp(i\theta), s) = (r\exp(i(\theta + 2\pi s)), s)$. On ∂M, this is a Dehn twist about ∂D. Notice that t_D induces the identity outer automorphism and is homotopic to the identity. This is because $(M - (D \times I)) \cup (\{0\} \times I)$ is a deformation retract of M and t_D restricts to the identity map on this subspace.

The following theorem was proven by Shin'ichi Suzuki [Suz] for the case when V is a handlebody, and for compression bodies in [M-M].

THEOREM 3: *The homeomorphisms realizing the Fuks-Rabinovitch generators, together with the twist homeomorphisms, generate $\mathcal{H}(V)$.*

Theorem 3 is proven by the following approach. Take a homeomorphism h of V. Examine the images of the attaching discs (i. e. the discs $B \cap (\overline{V - B})$ in Figure 1) under h. The generators given in Theorem 3 can be applied to simplify these images until one obtains a product g of generators so that gh fixes all the attaching discs. Applying homeomorphisms in each $F_i \times I$ (these are among the Fuks-Rabinovitch homeomorphisms) and twist homeomorphisms in the 3-ball B, one obtains another product of generators g_1 so that $g_1 gh$ is isotopic to the identity. So h is isotopic to $g^{-1}g_1^{-1}$.

We are now set up to understand the kernel of $\mathcal{H}(V) \to Out(\pi_1(V))$. The following theorem was proven by E. Luft [Lu] in 1978 for handlebodies, and in [M-M] for compression bodies. Let $\mathcal{H}_+(M)$ denote the group of orientation-preserving mapping classes of M, a subgroup of index at most 2 in $\mathcal{H}(M)$.

THEOREM 4: *Let V be a compression body. The kernel of $\mathcal{H}_+(V) \to Out(\pi_1(V))$ is the subgroup generated by twist homeomorphisms.*

Here is the idea of the proof. Let $\mathcal{T}(V)$ be the subgroup of $\mathcal{H}(V)$ generated by twist homeomorphisms. Note that for any homeomorphism h, one has $ht_Dh^{-1} = t_{h(D)}$. Consequently, $\mathcal{T}(V)$ is a normal subgroup of $\mathcal{H}(V)$. Let $\mathcal{K}(V)$ be the kernel of $\mathcal{H}(V) \to Out(\pi_1(V))$. Since $\mathcal{T}(V) \subseteq \mathcal{K}(V)$, there is a natural homomorphism $\Phi\colon \mathcal{H}(V)/\mathcal{T}(V) \to \mathcal{H}(V)/\mathcal{K}(V)$. By Theorem 5 (proved below independently of Theorem 4), $\mathcal{H}(V)/\mathcal{K}(V)$ is a subgroup of finite index in $Out(\pi_1(V))$, and for purposes of explaining the idea we will pretend that it is the entire group $Out(\pi_1(V))$. (In [M-M], the concept of "uniform homeomorphisms" on a certain family of compression bodies must be defined to overcome the fact that $\mathcal{H}(V)/\mathcal{K}(V)$ is not all of $Out(\pi_1(V))$.) To define Φ^{-1}, send each Fuks-Rabinovitch generator to the coset of the "obvious" isotopy class of homeomorphism that induces it; for example, send $\rho_{i,j}(\gamma)$ to $R_{i,j}(\gamma)$. As noted above, $R_{i,j}(\gamma)$ is not unique up to isotopy, but since it is well-defined up to twist homeomorphisms, its coset in $\mathcal{H}(V)/\mathcal{T}(V)$ is well-defined. Once we show that Φ^{-1} is well-defined, Theorem 3 will show that it is surjective, and by construction it is the inverse of Φ. To prove that Φ^{-1} is well-defined, we must show that any product of Fuks-Rabinovitch generators that equals the identity outer automorphism is sent by Φ^{-1} to a product of twist homeomorphisms. It suffices to check this for each of the 48 kinds of relations given in [F-R2] (and listed in [M-M]). The details of these checks are given in [M-M]. A simple type of example is a relation such as number 28, which says that $\rho_{i,j}(\gamma)\rho_{k,\ell}(\delta) = \rho_{k,\ell}(\delta)\rho_{i,j}(\gamma)$ when i, j, k, and ℓ are distinct. One can construct $R_{i,j}(\gamma)$ and $R_{k,\ell}(\delta)$ so that they have disjoint support, and consequently they actually commute as homeomorphisms, verifying that $\Phi^{-1}(R_{i,j}(\gamma)R_{k,\ell}(\delta)R_{i,j}(\gamma)^{-1}R_{k,\ell}(\delta)^{-1}$ is in $\mathcal{T}(V)$. More difficult are cases such as relation 30, which says that

$$\rho_{i,j}(\alpha_i)\rho_{k,i}(x) = \rho_{k,i}(x)\rho_{i,j}(\alpha_i)\rho_{k,j}(x^{-1}) \ .$$

The required check that $R_{i,j}(\alpha_i)R_{k,i}(x)R_{k,j}(x)R_{i,j}(\alpha_i)^{-1}R_{k,i}(x)^{-1}$ is isotopic to a product of twist homeomorphisms requires more complicated geometric arguments.

3. *The image of $\mathcal{H}(V) \to Out(\pi_1(V))$*

The next result is also from [M-M]. Here, we sketch a simplified argument. It requires some knowledge of [M-M], but avoids some of the technicalities of [M-M] (specifically, the concept of uniform homeomorphisms needed to give precise calculations of the index of the image of $\mathcal{H}(V)$ in $Out(\pi_1(V))$).

THEOREM 5: *Let V be an orientable compression body. Then the image of $\mathcal{H}(V)$ in $Out(\pi_1(V))$ has finite index.*

PROOF: Since V is aspherical, any ϕ in $Out(\pi_1(V))$ may be induced by a homotopy equivalence f. We will first show that $Out(\pi_1(V))$ contains a subgroup $Out_1(\pi_1(V))$ of finite index such that if ϕ lies in $Out_1(\pi_1(V))$, then f is homotopic to a homotopy equivalence that preserves each F_i.

Let T_i be the union of the attaching discs for the 1-handles of V that lie in $F_i \times \{1\}$.

For each F_i, the fundamental group is indecomposable and not infinite cyclic, so by the Kurosh subgroup theorem $\pi_1(F_i)$ is conjugate to some $\pi_1(F_j)$ where F_j is also closed. Using asphericity, f may be deformed relative to A so that it takes F_i to F_j. By passing to a subgroup $Out_1(\pi_1(V))$ of finite index in $Out(\pi_1(V))$, we may assume that f preserves each F_i. Since the restriction f_i of f to F_i induces an injection on fundamental groups, and restricts to a homeomorphism on the boundary of F_i, Baer's theorem implies that f_i is properly homotopic to a homeomorphism. From now on, we will assume that each f_i is a homeomorphism. Passing to a subgroup $Out_2(\pi_1(V))$ of finite index in $Out_1(\pi_1(V))$, we may assume that each f_i is orientation-preserving.

To complete the proof of Theorem 5, consider $\phi \in Out_2(\pi_1(V))$ realized by a map f as above. Then ϕ must carry each $\pi_1(F_i)$ to a conjugate of itself. The arguments of Fuks-Rabinovitch [F-R1], [F-R2] show that ϕ can be expressed as a product of certain kinds of generating automorphisms. (Those papers actually deal with a free product in which each factor is indecomposable, but this is used only to know that the automorphism takes each free factor to a conjugate of itself, a condition which we have ensured by our construction of f.)

As discussed in section 2 above, using the "slide homeomorphism" construction of [M-M] one can realize many of these generators by homeomorphisms of V that fix $\partial V - F$. Specifically, the generators called $\rho_{i,j}(x)$, $\lambda_{i,j}(x)$, $\mu_{i,j}(x)$, σ_i, and $\omega_{i,j}$ in [M-M] can be realized by such homeomorphisms. Using the relations that hold between the generators (as given in [M-M], [F-R2], or [Gil]), we can write ϕ as $\phi_1\phi_2$, where ϕ_1 is a product of the generators listed above and $\phi_2 = \prod_{i=1}^{m} \varphi_i$ is a product of "factor automorphisms," that is, automorphisms of $\pi_1(V)$ that arise by applying an automorphism φ_i of $\pi_1(F_i)$ to each element of $\pi_1(F_i)$ appearing in any word of $\pi_1(V)$, while fixing the elements of $\pi_1(F_j)$ for $j \neq i$. Choose a homeomorphism g of V, which is the identity on $\partial V - F$ and induces ϕ_1. Then $g^{-1}f$ induces ϕ_2. Since g is the identity on $\partial V - F$, each φ_i must be induced by f_i. Since f_i is orientation-preserving, it can be extended to a homeomorphism

h_i of $\pi_1(V)$ which is the identity on A and on all F_j for $j \neq i$, and induces φ_i on $\pi_1(V)$. (To construct h_i, extend f_i to a homeomorphism of $F_i \times I$ which is the identity on the T_i, and extend using the identity on the rest of V. When $\pi_1(F_i)$ is nonabelian, the extension of f_i to $F_i \times I$ must be chosen to preserve arcs connecting a basepoint in F_i with the basepoint of V, to ensure that h_i induces φ_i). Let h be the composition $\prod_{i=1}^{m} h_i$; then $h^{-1}g^{-1}f$ is the identity on $\partial V - F$ (so gh is admissible) and induces the identity automorphism on $\pi_1(V)$ (so gh realizes ϕ). This completes the proof of Theorem 5.

Chapter VII. The twist group

1. *The Generalized Luft Theorem*

Recall that the Sphere Theorem shows that if an orientable 3-manifold M is irreducible and has infinite fundamental group, then it is aspherical (see section IV.2). Therefore any map from M to M that induces the identity automorphism is homotopic to the identity map (see section II.2). Said differently, this means that the kernel of the natural homomorphism $\mathcal{H}(M) \to Out(\pi_1(M))$ consists of the mapping classes whose representatives are homotopic to the identity.

Recall that in chapter V we defined twist homeomorphisms. For a 3-manifold M let $\mathcal{T}(M)$ be the subgroup of $\mathcal{H}(M)$ generated by twist homeomorphisms. Since for any homeomorphism h, one has $ht_D h^{-1} = t_{h(D)}$, $\mathcal{T}(M)$ is a normal subgroup of $\mathcal{H}(M)$. Also, twist homeomorphisms lie in the kernel of $\mathcal{H}(M) \to Out(\pi_1(M))$. When the boundary of D is not contractible in ∂M, t_D cannot be isotopic to the identity. For an isotopy to the identity would restrict on ∂M to an isotopy from the Dehn twist about ∂D to the identity. But this Dehn twist induces a nontrivial outer automorphism on $\pi_1(\partial M)$.

There is another obvious example of a homeomorphism which is homotopic but not isotopic to the identity. Suppose M happens to be an I-bundle over a 2-manifold. Then the homeomorphism which is reflection across the midpoint of each I-fiber is homotopic to the identity (just move points within each I-fiber) but not isotopic to the identity, because it is orientation-reversing. (For nonorientable manifolds, this reflection is sometimes isotopic to the identity. When M is a nonorientable I-bundle over the torus, the isotopy looks like an S^1-action on the 0-section. M is the mapping cylinder of an involution which is isotopic to the identity.)

For irreducible orientable 3-manifolds with boundary, these are the only two phenomena. The following result is from [M-M].

GENERALIZED LUFT THEOREM: *Let M be a compact orientable irreducible 3-manifold with nonempty boundary, and basepoint x_0 in the interior of M. Suppose $h\colon (M, x_0) \to (M, x_0)$ is a homeomorphism with $h_\#$ equal to the identity automorphism on $\pi_1(M, x_0)$.*

(a) *If h is orientation-preserving, then h is isotopic relative to x_0 to a product of twist homeomorphisms.*

(b) *If h is orientation-reversing, then M is an I-bundle over a compact 2-manifold. In particular, if M has a compressible boundary component, then M is a handlebody.*

We will sketch part (a). The hard part is the case of a compression body, which we already discussed in chapter V. For part (a), we assume that h is orientation-preserving. One inducts on the number of compressible boundary components. If this number is 0, so that ∂M is incompressible, then h is homotopic to the identity. Since M has incompressible boundary, an argument of Waldhausen [Wa, pp. 82-83] shows that the homotopy can be made to preserve ∂M. Then, a theorem of Laudenbach [Lau, p. 46] shows that the homotopy can be deformed to an isotopy. When there are compressible boundary components, take a normally imbedded compression body neighborhood of one, say V that is a neighborhood of F. One can show that $h(F)=F$ and since V is unique up to isotopy, one may change h by isotopy so that $h(V)=V$. Since $\pi_1(V)$ is a subgroup of h, h induces the identity outer automorphism so on V, h is isotopic to a product of twist homeomorphism, and by induction it is isotopic to a product of twist homeomorphisms on the rest of M.

(There is actually a significant complication when one carries out the details of this proof. When one goes to fit the isotopies on V and $\overline{M-V}$ together, the conclusion is only that h is isotopic to a product of twist homeomorphisms and Dehn twists about torus components of the frontier of V. A fundamental group argument is needed to show that these Dehn twists are trivial. This is not so easy, because a component W of $\overline{M-V}$ may be Seifert-fibered and the trace of the Dehn twist on a torus component of $W \cap V$ may be central in $\pi_1(W)$. In this case, the central element is the trace of an S^1-action on W and one can use this action to produce an isotopy that makes the Dehn twists in the tori of $W \cap V$ trivial.)

Letting $\mathcal{H}_+(M)$ denote the subgroup of $\mathcal{H}(M)$ consisting of orientation-preserving classes, we get an immediate corollary of the Generalized Luft Theorem.

COROLLARY: *Let M be a compact orientable irreducible 3-manifold with nonempty boundary. Then $\mathcal{T}(M)$ is the kernel of $\mathcal{H}_+(M) \to Out(\pi_1(M))$.*

For 3-manifolds which are not irreducible, one can give generators for the kernel of $\mathcal{H}_+(M) \to Out(\pi_1(M))$. This is done in [McC5].

2. *Infinite generation of the twist group*

Because of the Generalized Luft Theorem, one would like to know more about the twist group. Little is known about its actual structure, but some information is obtained in [McC2]. Say that ∂M is *almost incompressible* if in each component F of ∂M, there is at most one simple closed curve (up to isotopy) that bounds a disc in M but does not bound a disc or Möbius band in F. For example, the solid torus has almost incompressible boundary, but the genus 2 handlebody does not.

THEOREM: *Let M be a compact 3-manifold. Then $\mathcal{T}(M)$ is finitely generated if and only if ∂M is incompressible.*

To show the key ideas in the proof, we will prove that $\mathcal{T}(V)$ is not finitely generated when V is the genus 2 orientable handlebody. Let a and b be free generators of $\pi_1(V)$. Think of them as the core circles of the two 1-handles of V. Define $\pi_1(V) \to \mathbb{Z}$ by sending a to 0 and b to 1. The kernel of this homomorphism is the normal closure of a, and the covering space of V corresponding to the kernel is an infinite genus handlebody $\widetilde{V}$ as shown in Figure 2. In Figure 2, we examine the effect of a t_D where D is a certain disc in V. Clearly, the lift $\widetilde{t_D}$ of t_D to $\widetilde{V}$ is given by simultaneous twists about the preimage discs of D. Let α be the circle in $\partial\widetilde{V}$ shown in Figure 2. The effect of $\widetilde{t_D}$ on α is shown in Figure 2. Notice that in homology $H_1(\widetilde{V};\mathbb{Z})$ we have

$$\widetilde{t_{D*}}(\alpha) = \alpha + \text{loops that bound in } \widetilde{V} \ .$$

We say that the *length* of this particular t_D is 3, because the image of α has elements in a portion of $\widetilde{V}$ that is 3 handles long. The length of any element of $\mathcal{T}(V)$ is the number of handles over which the lift spreads α (this is easy to define algebraically).

Here is the key point: Any $\widetilde{t_E}$, where E is a disc in V, fixes all elements of $H_1(\widetilde{V};\mathbb{Z})$ that bound in $\widetilde{V}$, because they must have zero homological intersection with the boundary of every preimage disc of E in $\widetilde{V}$. This is immediate from the general homological formula for Dehn twists $(t_C)_*(x) = x + (x \cdot C)C$, where C is a circle in $\partial\widetilde{V}$ and C is any circle in $\partial\widetilde{V}$.

Denoting length by L, the key point shows that if D and E are any two discs in V, then $L(t_D^{\pm 1} t_E^{\pm 1}) \leq \max\{L(t_D), L(t_E)\}$. That is, the product of the lifts of two twists of V can spread α out no more that either of the two lifts does already. Explicitly,

$$\begin{aligned}\widetilde{t_{D*}}\widetilde{t_{E*}}(\alpha) &= \widetilde{t_{D*}}(\alpha + \text{bounding loops from } \widetilde{t_{E*}}(\alpha)) \\ &= \alpha + \text{bounding loops from } \widetilde{t_{D*}}(\alpha) \\ &\qquad + \text{bounding loops from } \widetilde{t_{E*}}(\alpha)\end{aligned}$$

Therefore, if $\mathcal{T}$ were finitely generated, the length of any element of $\mathcal{T}(V)$ would be at most the maximum length of a generator. To show that there is no bound, we use the discs D_n shown in Figure 3, which illustrates that $L(t_{D_n})=2n+1$. This completes the proof.

In [McC2], this idea is developed (using compression bodies) for general 3-manifolds. When the manifolds are not irreducible, the compression body neighborhoods are no longer unique up to isotopy, but this is irrelevant for examining the twist group.

Also in [McC2], the almost incompressible cases are analyzed. Basically, each boundary component which is almost incompressible but not incompressible contributes one generator to $\mathcal{T}(M)$, and these commute. But there is a certain phenomenon which occurs in the nonorientable case when there are solid Klein bottle irreducible summands, which allows for order 4 elements in $\mathcal{T}(M)$. Here is the general result.

THEOREM: *Let M be a compact connected 3-manifold with almost incompressible boundary.*

(a) *If M is a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbb{Z}/2$.*
(b) *If M is $\mathbb{P}^2$-irreducible and not a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbb{Z}^k$ for some k, and any such group is the twist group of some compact $\mathbb{P}^2$-irreducible 3-manifold.*
(c) *If M is irreducible, or more generally no proper summand of M is a solid Klein bottle, then $\mathcal{T}(M) \cong \mathbb{Z}^k \times (\mathbb{Z}/2)^\ell$ for some k and ℓ, and any such group is the twist group of some compact irreducible 3-manifold.*
(d) *$\mathcal{T}(M) \cong \mathbb{Z}^k \times (\mathbb{Z}/2)^\ell \times (\mathbb{Z}/4)^m$ for some k, ℓ, and m, and any such group is the twist group of some compact 3-manifold.*

Chapter VIII. Mapping class groups of 3-manifolds

1. *A conjectural picture of 3-manifold mapping class groups*

Generalizing the construction of twist homeomorphisms, we define a *Dehn homeomorphism* as follows. Let $(F^{n-1} \times I, \partial F^{n-1} \times I) \subset (M^n, \partial M^n)$, where F is a connected codimension-1 submanifold, and $F \times I \cap \partial M = \partial F \times I$. Let $\langle \phi_t \rangle$ be an element of $\pi_1(Homeo(F), 1_F)$. That is, for $0 \leq t \leq 1$ there is a continuous family of homeomorphisms of F such that $\phi_0 = \phi_1 = 1_F$, the identity map of F. Define $h \in Homeo(M)$ by

$$h = \begin{cases} h(x,t) = (\phi_t(x), t) & \text{if } (x,t) \in F \times I \\ h(m) = m & \text{if } m \notin F \times I \end{cases}$$

We note that when $\pi_1(Homeo(F))$ is trivial, a Dehn homeomorphism must be isotopic to the identity.

Define $\mathcal{D}(M)$ to be the subgroup of $\mathcal{H}(M)$ generated by Dehn homeomorphisms. Note that when M is orientable, $\mathcal{D}(M) \subseteq \mathcal{H}_+(M)$. We call $\mathcal{D}(M)$ the *Dehn subgroup* of $\mathcal{H}(M)$. When M is compact and 2-dimensional, a Dehn homeomorphism with F an arc is isotopic to the identity, and a Dehn homeomorphism with F a circle is usually called a *Dehn twist.* Dehn and Lickorish proved that when M is an orientable 2-manifold, $\mathcal{D}(M) = \mathcal{H}_+(M)$.

¿From now on, we assume that M is compact and 3-dimensional. Also, for technical simplicity, we will often make three more assumptions:

1. M is orientable.
2. M does not have any 2-sphere boundary components (i. e. no prime summand of M is a 3-ball).
3. M does not contain any fake 3-cell. (A fake 3-cell is a manifold homotopy equivalent to D^3 but no homeomorphic to D^3. The Poincaré Conjecture is equivalent to the assertion that fake 3-cells do not exist.)

The following table lists $\pi_1(Homeo(F))$ for connected 2-manifolds, and the names of the corresponding Dehn homeomorphisms of 3-manifolds.

F	$\pi_1(Homeo(F))$	Dehn homeomorphism
$S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$	Dehn twist about a torus
$S^1 \times I$	$\mathbb{Z}$	Dehn twist about an annulus
D^2	$\mathbb{Z}$	twist
S^2	$\mathbb{Z}/2$	rotation about a sphere
$\mathbb{RP}^2$	$\mathbb{Z}/2$	rotation about a projective plane
Klein bottle	$\mathbb{Z}$	Dehn twist about a Klein bottle
Möbius band	$\mathbb{Z}$	Dehn twist about a Möbius band
$\chi(F) < 0$	0	

This brings us to our first conjecture about 3-manifold mapping class groups.

DEHN SUBGROUP CONJECTURE: Let M be a compact 3-manifold. Then $\mathcal{D}(M)$ has finite index in $\mathcal{H}(M)$.

The following theorem from [McC1] reduces the Dehn Subgroup Conjecture to the case of irreducible manifolds, as long as M is orientable.

THEOREM: *Let M be an orientable 3-manifold. Let M_i be the irreducible prime summands of M, $1 \leq i \leq n$. If $\mathcal{D}(M_i)$ has finite index in $\mathcal{H}(M_i)$ for each irreducible prime summand of M, then $\mathcal{D}(M)$ has finite index in $\mathcal{H}(M)$.*

For Haken manifolds, Johannson proved the Dehn Subgroup Conjecture (Corollary 27.6 in [Joh]), and this was extended to all compact orientable irreducible sufficiently large 3-manifolds in [M-M].

Denote by $\mathcal{D}_{>0}(M)$ the subgroup of $\mathcal{D}(M)$ generated by Dehn homeomorphisms using D^2, S^2, and $\mathbb{RP}^2$. The subscript indicates that the surfaces F in the definition of Dehn homeomorphism have positive Euler characteristic.

By an argument similar to the proof of Lemma 1.1 of [McC5]), one can prove the following extension of the Generalized Luft Theorem.

THEOREM: *Let M be a compact 3-manifold.*

1. *If ∂M is incompressible, then $\mathcal{D}_{>0}(M)$ is a finite abelian group.*
2. *If ∂M is almost incompressible, then $\mathcal{D}_{>0}(M)$ is a finitely generated abelian group.*
3. *If ∂M is not almost incompressible, then $\mathcal{D}_{>0}(M)$ is infinitely generated and nonabelian.*

We have a companion conjecture to the Dehn Subgroup Conjecture.

KERNEL CONJECTURE: $\mathcal{D}_{>0}(M)$ has finite index in the kernel of $\mathcal{H}(M) \to Out(\pi_1(M))$.

In general, $\mathcal{D}_{>0}(M)$ need not equal the kernel, as shown by the example of reflection in the fibers of an I-bundle. The conjecture is verified in many cases, such as those covered by the Generalized Luft Theorem in chapter VII. The following theorem from [McC5] reduces the Kernel Conjecture to the case of irreducible manifolds, as long as M is orientable, does not have 2-sphere boundary components, and does not contain counterexamples to the Poincaré Conjecture.

THEOREM: *Suppose M is an orientable 3-manifold with no 2-sphere boundary components and containing no fake 3-cell. Let M_i be the irreducible* *prime summands of M, $1 \leq i \leq n$. Then the kernel of the $\mathcal{H}_+(M) \to Out(\pi_1(M))$ is generated by Dehn homeomorphisms about the connected sum 2-spheres, and homeomorphisms supported on one of the M_i which induce the identity automorphism on $\pi_1(M_i)$.*

This shows that under these hypotheses, if the Kernel Conjecture holds for each M_i, then it holds for M. The main case in which the Kernel Conjecture is unknown is that of irreducible aspherical 3-manifolds which are not sufficiently large. However, even for these the conjecture is known in many cases by recent work of Gabai [Ga1], [Ga2].

Let $Out_\partial(\pi_1(M))$ be the subgroup of $Out(\pi_1(M))$ consisting of the automorphisms ϕ such that for every boundary component F of M, there exists a boundary component G so that $\phi(i_\#(\pi_1(F)))$ is conjugate in $\pi_1(M)$ to $j_\#(\pi_1(G))$, where $i: F \to M$ and $j: G \to M$ are the inclusions. This subgroup must contain the image of $\mathcal{H}(M) \to Out(\pi_1(M))$.

IMAGE CONJECTURE: The homomorphism $\mathcal{H}(M) \to Out_{\partial M}(\pi_1(M))$ has image of finite index.

In general, the image is not all of $Out_{\partial M}(\pi_1(M))$. The following result from [McC5] reduces the Image Conjecture to the case of irreducible manifolds.

THEOREM: *Let M be a compact 3-manifold with irreducible prime summands M_i, $1 \leq i \leq n$. If the image of $\mathcal{H}(M_i) \to Out(\pi_1(M_i))$ has finite index for each i, then the image of $\mathcal{H}(M) \to Out(\pi_1(M))$ has image of finite index.*

We can combine the Kernel Conjecture and the Image Conjecture into a single statement as follows. Call a sequence of groups $1 \to A \to B \to C \to 1$ *almost exact* if A has finite index in the kernel of $B \to C$, and the image of B has finite index in C.

ALMOST EXACTNESS CONJECTURE: Let M be a compact 3-manifold. Then the sequence

$$1 \to \mathcal{D}_{>0}(M) \to \mathcal{H}(M) \to Out_\partial(\pi_1(M)) \to 1$$

is almost exact.

Now assume that M is compact, orientable, and irreducible, and let S^3 be the standard unit sphere in $\mathbb{R}^4$. From section IV.2 we know that if $\pi_1(M)$ is finite, then either M is the 3-ball or the universal cover of M is homotopy equivalent to S^3. The following conjecture is a special case of Thurston's Geometrization Conjecture.

POSITIVE CURVATURE CONJECTURE: Let M be a closed orientable irreducible 3-manifold with finite fundamental group. Then M is homeomorphic to S^3/G, where G is a group of isometries.

That is, the universal cover of M can be identified with S^3 in such a way that the covering transformations of M act as isometries. This is equivalent to the assertion that M admits a Riemannian metric of constant positive curvature. When M is of the form S^3/G, there is a conjectural description of the entire homeomorphism group. Let M have the metric of constant curvature 1 induced from S^3.

GENERALIZED SMALE CONJECTURE: Let $M = S^3/G$, where G is a finite group of isometries. Then the homeomorphism group of M deformation retracts to the group of isometries $Isom(M)$.

Consequently the group of path components if $Homeo(M)$ is the same as the group of path components of $Isom(M)$, and is easily computed. This shows that $\mathcal{H}(M)$ is finite, and verifies the Almost Exactness Conjecture for M. The Smale Conjecture, proven by Hatcher [Hat1], is the case when $M = S^3$. The Generalized Smale Conjecture has been proven for almost all the 3-manifolds with finite fundamental group which contain a one-sided Klein bottle [Iv1], [Iv2], [M-R].

Even when $\pi_1(M)$ is infinite, we expect finite mapping class group when M is not sufficiently large.

FINITENESS CONJECTURE: Let M be a closed orientable irreducible 3-manifold which is not sufficiently large. Then $\mathcal{H}(M)$ is finite.

This has been proven by Gabai for many aspherical but not sufficiently large manifolds [Ga1], [Ga2]. Notice that it is implied by the Dehn Subgroup Conjecture.

Our final conjecture is known for many classes of 3-manifolds, and by a result in [H-M] is reduced to the irreducible case.

FINITE PRESENTATION CONJECTURE: Let M be a compact 3-manifold. Then $\mathcal{H}(M)$ is finitely presented.

In the next section we will discuss work that proves the Finite Presentation Conjecture, and much stronger group-theoretic finiteness properties, for Haken manifolds.

2. *Mapping class groups of Haken 3-manifolds*

In this section, we discuss results from [McC4]. See [McC3] for another summary of [McC4].

When M is Haken, the analysis of $\mathcal{H}(M)$ starts from Johannson's characteristic submanifold theory. As we saw in section V.2, M has a characteristic decomposition consisting of Seifert-fibered and I-bundle components $\Sigma_1, \ldots, \Sigma_r$ and simple components $S_1, \ldots, S_s$. Let $KHomeo(M)$ denote the space of homeomorphisms h of M such that $h(\cup\Sigma_i) = \cup\Sigma_i$. Combining various results of Johannson, Laudenbach, and Hatcher, we have the following result.

THEOREM: *If M is not a torus bundle over S^1 such that the trace of the monodromy homeomorphism is at least 3, then $\pi_0(KHomeo(M)) \to \pi_0(Homeo(M))$ is bijective.*

The surjectivity is the uniqueness of the characteristic submanifold up to isotopy. The injectivity uses a theorem of Laudenbach, later generalized by Hatcher, which shows that isotopic homeomorphisms that preserve an incompressible surface are usually isotopic through homeomorphisms that preserve the incompressible surface.

(When M is a torus bundle over S^1 such that the trace of the monodromy homeomorphism is at least 3, $\pi_0(KHomeo(M)) \to \pi_0(Homeo(M))$ need not be injective. For these manifolds, it is shown in [McC4] that $\mathcal{H}(M)$ is finite. For the remainder of this section, we will assume that M is not one of these manifolds.)

Let $\mathcal{K}(M)$ denote the subgroup of finite index in $\pi_0(KHomeo(M))$ consisting of the classes that take each Σ_j to Σ_j, and each S_k to S_k. From the above theorem, we know that $\mathcal{K}(M) \to \mathcal{H}(M)$ is injective and has image of finite index.

By definition, the restriction of an element of $\mathcal{K}(M)$ to each S_k is well-defined up to isotopy. For S_k that are not the product of an torus with an interval, the Finite Mapping Class Group Theorem of Johannson shows that $\mathcal{H}(S_k)$ is finite. For S_k that are the product of a torus with an interval, it is proven in [McC4] that the image of $\mathcal{K}(M)$ in $\mathcal{H}(S_k)$ is still finite. Therefore the subgroup $\mathcal{K}_0(M)$ consisting of elements whose restriction to each S_k is isotopic to the identity has finite index in $\mathcal{K}(M)$.

The kernel of the homomorphism $\mathcal{K}_0(M) \to \prod \mathcal{H}(\Sigma_i)$ induced by restriction is the abelian subgroup $\mathcal{F}(M)$ of $\mathcal{K}(M)$ generated by Dehn homeomorphisms determined by the annuli and tori that are the components of the frontier of the characteristic submanifold of M. The image is exactly the classes $\mathcal{G}(\Sigma_i)$ that are isotopic to the identity on the frontier of Σ_i, and preserve certain boundary patterns which we suppress from our notation here. Thus we have our *fundamental exact sequence*

$$1 \to \mathcal{F}(M) \to \mathcal{K}_0(M) \to \prod \mathcal{G}(\Sigma_i) \to 1 \ .$$

Consider a Seifert-fibered component Σ_i, with orbit surface F_i. Apart from a few exceptional cases which can be handled explicitly, the orientation-preserving mapping class group $\mathcal{H}_+(\Sigma_i)$ is isomorphic to the group of orientation-preserving fiber-preserving mapping classes $\mathcal{H}^f_+(\Sigma_i)$. From Propositions 25.2 and 25.3 of [Joh], excepting some more cases, there is an exact sequence

$$1 \longrightarrow H_1(F_i, \partial F_i) \longrightarrow \mathcal{H}^f_+(\Sigma_i) \longrightarrow \mathcal{H}^*(F_i) \longrightarrow 1$$

in which $\mathcal{H}^*(F_i)$ is a subgroup of finite index in $\mathcal{H}(F_i')$, where F_i' is the result of removing from F_i the points which correspond to exceptional orbits of Σ_i. The kernel $H_1(F_i, \partial F_i)$ is isomorphic to the group of "vertical" mapping classes that map each fiber to itself. Since $\mathcal{H}^*(F_i)$ is finitely presented, so is $\mathcal{H}^f_+(\Sigma_i)$. With a more careful analysis, one can prove that the groups $\mathcal{G}(\Sigma_i)$ are finitely presented.

When Σ_i is an I-bundle component, $\mathcal{H}(\Sigma_i)$ is essentially the same as the mapping class group of its base surface, and enjoys the same group-theoretic finiteness properties. In particular it is finitely presented. So from the fundamental exact sequence, $\mathcal{H}(M)$ is finitely presented.

In [McC4], stronger group-theoretic finiteness properties of $\mathcal{H}(M)$ are proven. In particular, there are two "finiteness" properties that a group Γ may enjoy:

(1) Γ is of type FL; that is, there is a finite resolution of the trivial Γ-module $\mathbb{Z}$ by finitely generated free $\mathbb{Z}\Gamma$-modules.

(2) Γ is a duality group (over $\mathbb{Z}$); that is, there is a (right) $\mathbb{Z}\Gamma$-module C such that for some nonegative integer n there are natural isomorphisms $H^k(\Gamma; A) \cong H_{n-k}(\Gamma; C \otimes_{\mathbb{Z}} A)$ for all k and all $\mathbb{Z}\Gamma$-modules A.

In (2), n is called the *dimension* of the duality group. A good reference for (1) and for (2) see [B-E] or [Bi]).

When Γ is finitely presented, properties (1) and (2) have topological interpretations. For example, (1) is equivalent to the assertion that there exists a finite $K(G,1)$-complex. Both properties easily imply that the cohomological dimension of Γ is finite. For duality groups the cohomological dimension is equal to the dimension as a duality group.

We say that Γ is of type VFL (respectively, a virtual duality group) if there is a subgroup of finite index in Γ which is of type FL (respectively, which is a duality group).

It has long been known that the mapping class groups of 2-manifolds (of finite type) are finitely presented; this was first proved by McCool [McC]. Work of other authors, notably Harvey [Har3], [Har4] later showed that they are of type VFL, and Harer [Har1] (see also [Har2]) proved that they are virtual duality groups. One of the main results of [McC4] is the following analogue of those results.

THEOREM: *Let M be a Haken 3-manifold. Then the mapping class group $\mathcal{H}(M)$ a virtual duality group.*

This implies that $\mathcal{H}(M)$ is of type VFL. To show the main ideas, we will just sketch the proof that $\mathcal{H}(M)$ is of type VFL when M is a Haken 3-manifold. First consider a Seifert-fibered manifold Σ, fibered over F. Work of Harer [Har1] (extended in [McC4] to nonorientable 2-manifolds) shows that $\mathcal{H}^*(F)$ is of type VFL. In [McC4], it is proven that $\mathcal{H}(\Sigma)$ contains a torsionfree subgroup of finite index, and the intersection of the vertical mapping classes $H_1(F, \partial F)$ with this subgroup is a finitely generated free abelian group. So there is a subgroup of finite index in $\mathcal{H}(\Sigma_i)$ which is an extension of a finitely generated free abelian group by an FL group. Such an extension must be FL. This proves the Theorem for the Seifert fibered case. For I-bundles, the Theorem follows from the 2-dimensional version.

For the general case of incompressible boundary, we use the fundamental exact sequence

$$1 \to \mathcal{F}(M) \to \mathcal{K}_0(M) \to \prod \mathcal{G}(\Sigma_i) \to 1 \ .$$

Somewhat surprisingly, $\mathcal{F}(M)$ can contain torsion, and some effort is required to find a subgroup of finite index in $\mathcal{H}(M)$ that avoids this torsion. Then we have an subgroup of finite index in $\mathcal{K}_0(M)$ which is an extension of a finitely generated free abelian group by an FL group, so $\mathcal{K}_0(M)$ and hence $\mathcal{H}(M)$ are VFL groups.

3. *Mapping class groups of sufficiently large 3-manifolds*

The paper [McC4] also contains some results on manifolds with compressible boundary. For a product-with-handles V, there is a simplicial complex L in which the vertices are the isotopy classes of essential compressing discs in V, and a collection of vertices spans a simplex if and only if the isotopy classes can be represented by a collection of discs in V which are pairwise disjoint. It is proved in [McC4] that L is a finite-dimensional contractible complex admitting a simplicial action of $\mathcal{H}(V)$ with finite quotient. The result of cutting a product-with-handles along a set of compressing discs is a collection of products-with-handles of lower complexity; this enables the stabilizers of simplices in L to be analyzed inductively, obtaining enough information to establish that $\mathcal{H}(V)$ is finitely-presented and virtually of type FL. The compression body case can be applied to extend the theorem above to manifolds with compressible boundary, with a weaker conclusion:

THEOREM: *Let M be a compact orientable irreducible 3-manifold with nonempty boundary. Then M is a finitely presented group of type VFL.*

The proof is based on induction on the number of compressible boundary components, with the induction starting from the Haken case.

For the cases when the boundary of M is compressible it is unknown in general whether $\mathcal{H}(M)$ is a virtual duality group. However, in [McC4] some very special facts about the genus 2 orientable handlebody V_2 are used to prove that $\mathcal{H}(V_2)$ is a virtual duality group of dimension 3.

Chapter IX. The finite-index realization problem

In chapter IV, we stated the following problem:

FINITE-INDEX REALIZATION PROBLEM: *For which compact orientable irreducible 3-manifolds M does the image of the homomorphism $\mathcal{H}(M) \to Out(\pi_1(M))$ have finite index?*

In this chapter we will discuss the solution of this problem given in [C-M]. In section 3, we will show that when M is a hyperbolic 3-manifold the answer to the finite-index realization problem gives information about the deformation spaces of hyperbolic structures on M.

We will denote the image of $\mathcal{H}(M) \to Out(\pi_1(M))$ by $\mathcal{R}(M)$.

1. *Motivating examples*

Before stating the Finite-index Realization Theorem, we motivate it by showing some examples. They illustrate phenomena which affect whether $\mathcal{R}(M)$ has infinite index in $Out(\pi_1(M))$. We are particularly interested in examples related to hyperbolic 3-manifolds. A compact, orientable, irreducible 3-manifold is called *hyperbolizable* when its interior admits a complete hyperbolic structure.

The first two examples illustrate the two basic types of phenomena which cause $\mathcal{R}(M)$ to have infinite index in $Out(\pi_1(M))$ when M is hyperbolizable. The first can occur only when M has a compressible boundary component, and the second only when M has a torus boundary component.

Example 1: When M has a compressible boundary component but is not a compression body the following construction often yields an infinite collection of distinct cosets of $\mathcal{R}(M)$ in $Out(\pi_1(M))$. Let C be a simple closed curve in M with its basepoint in a 1-handle of M, but which does not pass over the 1-handle. A homotopy equivalence of M can be constructed by taking a map which is the identity off of the 1-handle and wraps the 1-handle around C (and then over the original 1-handle). Below, we describe a specific example and give the resulting automorphism explicitly.

Let S be a surface of genus two and let L be the 3-manifold obtained by taking the boundary connected sum of two copies of $S \times I$. Form M_1 by gluing two copies of L together along an incompressible boundary component. Then $\pi_1(M_1) \cong \pi_1(S) * \pi_1(S) * \pi_1(S)$ and has a presentation

$$\langle a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2, e_1, f_1, e_2, f_2 \mid [a_1, b_1] = [a_2, b_2], \\ [c_1, d_1] = [c_2, d_2], [e_1, f_1] = [e_2, f_2] \rangle$$

where $\{a_1, b_1, a_2, b_2\}$ and $\{e_1, f_1, e_2, f_2\}$ generate the fundamental groups of the two incompressible boundary components of M_1 and $\{c_1, d_1, c_2, d_2\}$ generates the fundamental group of the surface we glued along to form M_1. Define an automorphism ϕ which fixes $a_1, b_1, a_2, b_2, c_1, d_1, c_2$, and d_2 and acts on the remaining generators by

$$e_1 \mapsto a_1 e_1 a_1^{-1}, f_1 \mapsto a_1 f_1 a_1^{-1}, e_2 \mapsto a_1 e_2 a_1^{-1}, f_2 \mapsto a_1 f_2 a_1^{-1}.$$

Then no nonzero power ϕ^k of ϕ is realizable by a homeomorphism of M_1, since ϕ^k takes the peripheral element $c_1 e_2$ to the nonperipheral element $c_1 a_1^k e_1 a_1^{-k}$. This automorphism is induced by a homotopy equivalence which is the identity off of the 1-handle in the second copy of L (the copy whose fundamental group is generated by $\{c_1, d_1, c_2, d_2, e_1, f_1, e_2, f_2\}$), and which sends this handle around a loop representing a_1. Using Klein combination, one can construct a (convex cocompact) hyperbolic structure on M_1.

Example 2: The following example illustrates the phenomenon called doubly accidental parabolics. Again we will first describe the general strategy and then give a specific example (due to Thurston). Begin with a submanifold V homeomorphic to $T^2 \times I$ which intersects ∂M in a torus $(T^2 \times \{0\})$ and two annuli in $T^2 \times \{1\}$ which are not isotopic in ∂M. The homotopy equivalence is the identity off of a regular neighborhood of one component A of $\overline{(T^2 \times \{1\}) - \partial M_2}$, and wraps a collar neighborhood of A once around $T^2 \times \{1\}$. Arcs in ∂M which cross one of the annuli in $T^2 \times \{1\}$ are carried to arcs in M which travel around $T^2 \times \{0\}$, and loops which cross these annuli in an essential way can be carried to nonperipheral loops.

Let S be a surface of genus two and α a separating curve on S. Let K be the 2-complex (embedded in 3-space) formed by attaching a torus T to S along the curve α (where α is glued to the longitude of T.) Let M_2 be the manifold obtained by taking a regular neighborhood (in $\mathbb{R}^3$) of K. Notice that M_2 is homeomorphic to $S \times I$ with a tubular neighborhood of $\alpha \times \{\frac{1}{2}\}$ removed. Let ϕ be an automorphism of $\pi_1(K)$ constructed by fixing T and every point on S except an annulus with one boundary component being α. Then take this annulus and wrap it around the meridian of T. A presentation for $\pi_1(M_2)$ is

$$\langle a_1, b_1, a_2, b_2, c \mid [a_1, b_1] = [a_2, b_2], [[a_1, b_1], c] = 1 \rangle \ .$$

In this presentation the automorphism takes the form

$$a_1 \mapsto a_1, b_1 \mapsto b_1, c \mapsto c, a_2 \mapsto c a_2 c^{-1}, b_2 \mapsto c b_2 c^{-1} \ .$$

Notice that no nonzero power ϕ^k of ϕ is realizable by a homeomorphism, since it takes the peripheral element a_1a_2 to the non-peripheral element $a_1c^ka_2c^{-k}$. The characteristic submanifold of M_2 has three components. Two of the components are product I-bundles over the punctured tori which are the components of the complement in S of a regular neighborhood of α. The other component is topologically $T^2 \times I$. One of its boundary tori is the boundary torus of M_2, and the other meets each of the other two boundary components of M_2 in an annulus whose center circle is isotopic to α. A geometrically finite hyperbolic 3-manifold whose conformal extension is homeomorphic to $M_2 - P$ is explicitly constructed in Kerckhoff-Thurston [K-T]. The copies of α in the two nontoral boundary components of M_2 correspond to an accidental parabolic element which appears in a toral cusp and also as two nonhomotopic loops in the conformal boundary of the hyperbolic manifold.

The next two examples illustrate phenomena related to the presence of more complicated Seifert fibered spaces in the characteristic submanifold, which can occur only in nonhyperbolizable examples. Roughly speaking, if Σ has Seifert-fibered components which are complicated and meet the boundary of M, then $\mathcal{R}(M)$ will have infinite index in $Out(\pi_1(M))$, while if all Seifert-fibered components that meet the boundary are uncomplicated, such as the solid torus, the index can be finite. However, in the borderline cases the way in which the components meet the boundary can affect the index, as illustrated in examples 3 and 4. In both examples, Σ is the product of a disc with two holes and the circle, but in example 3 the index is infinite and in example 4 it is finite.

Example 3: Let F be the disk minus two holes, with boundary circles C_1, C_2, and C_3, and let $\Sigma = F \times S^1$. Let S be a compact hyperbolizable 3-manifold whose boundary is a single torus, and form M_3 by identifying the torus boundary component of S with the boundary torus $C_1 \times S^1$ of Σ. A homotopy equivalence of M_3 is constructed as follows. Start with a properly imbedded arc γ in F whose endpoints lie in C_2 and C_3. Let $\gamma \times [-1,1]$ be a product neighborhood of $\gamma = \gamma \times \{0\}$ with $\gamma \times [-1,1] \cap \partial F = \partial\gamma \times I$. Let α be a loop in F, based at a point $x_0 \times \{-1/2\} \in \gamma \times \{-1/2\}$, disjoint from $\gamma \times (-1/2, 1/2)$, and freely homotopic to C_1. Define a homotopy equivalence h_0 of F as follows. It will fix all points outside $\gamma \times [-1,1]$ (in particular, it is the identity on C_1). Map each $\gamma \times \{t\}$ to $\gamma \times \{t\}$ in such a way that for $-1/2 \le t \le 1/2$, $\gamma \times \{t\}$ collapses to $x_0 \times \{t\}$, then map the arc $x_0 \times [-1/2, 1/2]$ around the path product of α and $x_0 \times [-1/2, 1/2]$. Define a homotopy equivalence h of M_3 by taking the product of h_0 and the identity on the S^1-factor on Σ, and by taking the identity on S. The peripheral loop in M_3 represented by C_2 is carried by h^k to the element represented by $C_2C_1^k$ in $\pi_1(F) \times \mathbb{Z} = \pi_1(\Sigma) \subset \pi_1(M_3)$. For nonzero k, this loop is not homotopic into ∂M_3 so h^k is not homotopic to a homeomorphism. Therefore $\mathcal{R}(M_3)$ has infinite index in $Out(\pi_1(M_3))$.

Example 4: Form M_4 from the manifold M_3 in example 3 by attaching another copy of S along $C_2 \times S^1$. We will show that $\mathcal{R}(M_4) = Out(\pi_1(M_4))$. Any outer automorphism of $\pi_1(M_4)$ can be induced by a homotopy equivalence. By Johannson's Classification Theorem (stated in section V.2), such a homotopy equivalence is homotopic to a map f which carries S to S by a homeomorphism and carries Σ to Σ. Any homotopy equivalence of Σ which is an orientation-preserving or orientation-reversing homeomorphism on $(C_2 \cup C_3) \times S^1$ is homotopic to a homeomorphism by a homotopy which is constant on $(C_2 \cup C_3) \times S^1$. (This is not too hard to prove, using the fact that any map of the disc with two holes which is an orientation-preserving or orientation-reversing homeomorphism on two boundary circles is homotopic to a homeomorphism.) Therefore this homotopy equivalence is homotopic to a homeomorphism.

2. *The finite-index realization theorem*

To state the finite-index realization theorem, we must define a kind of 3-manifold which is in some sense close to being a compression body. We call a 3-manifold M *small* when it satisfies one of the following:

(i) M is obtained from a product I-bundle over a closed surface by gluing a 1-handle to one boundary component and a twisted I-bundle (with boundary homeomorphic to the closed surface) to the other, or

(ii) M is obtained from the boundary connected sum of two product I-bundles over closed surfaces by gluing a twisted I-bundle to one or both of the incompressible boundary components, or

(iii) M is obtained from the boundary connected sum of two product I-bundles over homeomorphic closed surfaces by gluing the two incompressible boundary components.

FINITE-INDEX REALIZATION THEOREM:: *Let M be a compact, orientable, irreducible 3-manifold with non-empty boundary.*

1. *If ∂M is compressible, then $\mathcal{R}(M)$ has finite index in $Out(\pi_1(M))$ if and only if M is either small or a compression body.*

2. *If ∂M is incompressible, then $\mathcal{R}(M)$ has finite index in M if and only if every Seifert fibered component V of the characteristic submanifold that intersects ∂M satisfies one of the following:*
   (i) *V is a solid torus, or*
   (ii) *V is an S^1-bundle over the Möbius band or annulus and no component of $V \cap \partial M$ is an annulus, or*
   (iii) *V is fibered over the annulus with one exceptional fiber, and no component of $V \cap \partial M$ is an annulus, or*
   (iv) *V is fibered over the disc with two holes with no exceptional fibers, and $V \cap \partial M$ is one of the boundary tori of V, or*
   (v) *$V = M$, and either V is fibered over the disc with two exceptional fibers, or V is fibered over the Möbius band with one exceptional fiber, or V is fibered over the punctured torus with no exceptional fibers.*

The proof of the finite-index realization theorem is technically involved. First we discuss the case when M has incompressible boundary. The cited theorems of Johannson are stated in section V.2. Start with a homotopy equivalence f inducing an automorphism in $Out(\pi_1(M))$.

(1) Using Johannson's Classification Theorem, deform f so that it preserves the characteristic submanifold Σ.

(2) Using Johannson's Homotopy Splitting Theorem, show that the restrictions of f to the components of Σ and the components of $\overline{M - \Sigma}$ are well-defined (by passing to a subgroup of finite index in $Out(\pi_1(M))$, one may assume that f preserves each such component).

(3) By Johannson's Finite Mapping Class Group Theorem, there are up to isotopy only finitely many restrictions possible on the components of $\overline{M - \Sigma}$, so that by passing to another subgroup of finite index, one may assume f is the identity on $\overline{M - \Sigma}$.

(4) Study the restrictions of f to the components of Σ and characterize topologically the possible components for which the subgroup of their groups of homotopy equivalences consisting of the homotopy classes realizable by homeomorphisms has finite index.

In each step, one must use Johannson's theory of boundary patterns to retain exact control on the boundary. Step (4) breaks into two major cases according to whether the component of Σ is Seifert-fibered or is an I-bundle. For the borderline cases, a lengthy case-by-case analysis must be carried out.

Now we will discuss the case when a component of ∂M is compressible. Let F be a boundary component which is compressible in M. Let V be a normally imbedded compression body neighborhood of F. From section VI.1, such a neighborhood exists and is unique up to admissible isotopy. From section VI.3, $\mathcal{R}(M)$ has finite index in $Out(\pi_1(M))$ when M is a compression body. The two remaining steps are

(1) Show that when M is small, $\mathcal{R}(M)$ has finite index in $Out(\pi_1(M))$, and
(2) Show that when M is not a compression body and not small, $\mathcal{R}(M)$ has in finite index in $Out(\pi_1(M))$.

We sketch the argument for item (1) for the case of a small manifold of Type I. Let M be obtained from $F_1 \times I$ by attaching a 1-handle $D \times I$ to $F_1 \times \{1\}$. Let D denote $D \times \{1/2\}$ and let F_1 denote $F_1 \times \{0\}$. We will show that $\mathcal{R}(M) = Out(\pi_1(M))$ by showing that any homotopy equivalence f from M to M is homotopic to a homeomorphism. Since F_1 is incompressible, one can change f by admissible homotopy so that the image of F_1 is disjoint from D, and hence so that the image of F_1 lies in $F_1 \times I \subset M$. By a further homotopy, we may assume f maps F_1 to F_1. Since $f_\#$ is injective on fundamental groups, Baer's Theorem (see proposition 3.3 of [Joh]) shows that the restriction of f to F_1 is homotopic to a covering map. By an algebraic argument, $f_\#$ must carry $\pi_1(F_1)$ onto $\pi_1(F_1)$, so we may assume that f restricts to a homeomorphism on F_1. Composing f with a homeomorphism which restricts to f^{-1} on F_1 (note that if $h \circ f$ is homotopic to a homeomorphism k, then f is homotopic to the homeomorphism $h^{-1}k$, so we may freely change f by homeomorphisms), we may assume that f is the identity on F_1, and hence on $F_1 \times I$. Now $\pi_1(M) \cong \pi_1(F_1) * \mathbb{Z}$; letting ω be the generator of $\mathbb{Z}$, we now have that $f_\#(x) = x$ for $x \in \pi_1(F_1)$ and $f_\#(\omega) = z_1 \omega^{\pm 1} z_2$. Since $f_\#$ is surjective, algebraic considerations show that $z_1, z_2 \in \pi_1(F_1)$. But for any automorphism of this form, there is a known homeomorphism from [M-M] that induces such an automorphism (a "spin" of the 1-handle carries ω to ω^{-1}, and "slides" of the ends of the 1-handle around loops in $F_1 \times I$ add z_1 and z_2. Since M is aspherical, f is homotopic to this homeomorphism.

For other types of small manifolds, $\mathcal{R}(M)$ need not equal $Out(\pi_1(M))$.

For step (2), one uses the construction of Example 1 to show the index is infinite. There are several cases, according to the nature of $\overline{M - V}$, and in the more difficult cases one must carefully analyze the effects on $\pi_1(M)$.

3. *Deformations of hyperbolic structures*

Recall that we say a compact, orientable, irreducible 3-manifold M is *hyperbolizable* when its interior admits a (complete) hyperbolic structure.

Thurston has shown that if M has nonempty boundary, then M is hyperbolizable if and if and only if every incompressible torus in M is homotopic into ∂M. To understand the different hyperbolic structures on M and manifolds homotopy equivalent to M, one may consider the space $D(\pi_1(M), Isom_+(\mathbb{H}^3))$ of discrete faithful representations of $\pi_1(M)$ into the group $Isom_+(\mathbb{H}^3)$ of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. Each element of $D(\pi_1(M), Isom_+(\mathbb{H}^3))$ gives rise to a hyperbolic 3-manifold $N(\rho) = \mathbb{H}^3/\rho(\pi_1(M))$ which is homotopy equivalent to M. Moreover, the identification of $\pi_1(M)$ with the fundamental group $\rho(\pi_1(M))$ gives rise to a homotopy equivalence $f_\rho: M \to N(\rho)$ which is well-defined up to homotopy. This homotopy equivalence is sometimes called a *marking* of $N(\rho)$.

Since two conjugate representations into $Isom_+(\mathbb{H}^3)$ give rise to isometric hyperbolic 3-manifolds with the same marking, it is also natural to consider the space $AH(\pi_1(M)) = D(\pi_1(M), Isom_+(\mathbb{H}^3))/Isom_+(\mathbb{H}^3)$ where $Isom_+(\mathbb{H}^3)$ acts by conjugation. This is the space $AH(\pi_1(M))$ of marked hyperbolic 3-manifolds homotopy equivalent to M. An obvious question is the following.

QUESTION: *For which compact, orientable, irreducible 3-manifolds M does $AH(\pi_1(M))$ have finitely many components?*

The Ahlfors-Bers quasiconformal deformation theory of Kleinian groups, together with work of Marden, Maskit and Kra, allows one to describe an explicit parameterization of the space of geometrically finite hyperbolic structures on a compact 3-manifold M with a fixed parabolic locus. For simplicity here (although not in [C-M]) we will assume that the parabolic locus is empty.

Let M be a compact, orientable, hyperbolizable (hence irreducible) 3-manifold and let $\rho \in AH(\pi_1(M))$. Denote by $\Omega(\rho)$ the maximal open subset of $\widehat{\mathbb{C}}$ on which $\rho(\pi_1(M))$ acts discontinuously, and let $\widehat{N} = (\mathbb{H}^3 \cup \Omega(\rho))/\rho(\pi_1(M))$. We call $\widehat{N}$ the *conformal extension* of N. When $\widehat{N}$ is compact, N is said to be *convex cocompact.*

By $AH(M)$ we mean the subset of $AH(\pi_1(M))$ consisting of marked hyperbolic 3-manifolds which are actually homeomorphic, not just homotopy equivalent, to *int*(M). Let $CC(M) \subset AH(M)$ denote the space of marked convex cocompact hyperbolic 3-manifolds homeomorphic to *int*(M).

An element $\rho_1 \in CC(M)$ is said to be *quasiconformally conjugate* to ρ if there exists a quasiconformal map $\phi: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\phi \circ \rho(\gamma) \circ \phi^{-1} = \rho_1(\gamma)$ for all $\gamma \in \pi_1(M)$. If $\rho \in CC(M)$, then Bers [Be] (along with Maskit [Mas] and Kra [Kra]) showed how to use the Ahlfors-Bers measurable Riemann mapping theorem ([A-B]) to provide a parameterization of the set $QC(\rho)$

of Kleinian groups which are quasiconformally conjugate to ρ. Let $\mathcal{T}(\partial M)$ denote the Teichmüller space of all (marked) hyperbolic structures on ∂M and let $\mathrm{Mod}_0(M)$ denote the set of isotopy classes of homeomorphisms of ∂M which extend to homeomorphisms of M which are homotopic to the identity. We recall that $\mathcal{T}(\partial M)$ is diffeomorphic to a finite-dimensional Euclidean space and that $\mathrm{Mod}_0(M)$ acts properly discontinuously and freely on $\mathcal{T}(\partial M)$. The work of Bers, Maskit and Kra can be summarized in the following theorem.

QUASICONFORMAL PARAMETERIZATION THEOREM: (Ahlfors, Bers, Kra, Maskit) *Let M be a compact 3-manifold with boundary and $\rho \in CC(M)$. Then $QC(\rho)$ may be identified with $\mathcal{T}(\partial M)/Mod_0(M)$.*

Although we will not use this in the remainder, we note that the results presented in chapter VII show that $\mathrm{Mod}_0(M)$ is finitely generated if and only if the boundary of M is almost incompressible.

Marden's isomorphism theorem [Mar] provides a completely topological characterization of when two convex cocompact hyperbolic 3-manifolds are quasiconformally conjugate.

MARDEN'S ISOMORPHISM THEOREM: *Let ρ and ρ' be elements of $CC(M)$. Then $\rho' \in QC(\rho)$ if and only if there exists a homeomorphism $H\colon \widehat{N}(\rho) \to \widehat{N}(\rho')$ such that $H_* = \rho' \circ \rho^{-1}$.*

Another result of Marden's, later referred to as Marden's Stability Theorem, asserts that $QC(\rho)$ is an open subset of $AH(\pi_1(M))$ when ρ is convex cocompact.

In [C-M], these results are combined to give a complete parameterization of $CC(M)$. For the convex cocompact case, this specializes to the following statement.

PARAMETERIZATION THEOREM: *If M is a hyperbolizable compact 3-manifold with no toroidal boundary components, then $CC(M)$ is homeomorphic to*

$$(\mathcal{T}(\partial M)/Mod_0(M)) \times (Out(\pi_1(M))/\mathcal{R}(M))$$

Hence the number of components of $CC(M)$ is precisely the index of $\mathcal{R}(M)$ in $Out(\pi_1(M))$. The finite-index realization theorem of section IX.2 answers this, at least in terms of the characteristic submanifold of M. In [C-M], the characteristic submanifolds of hyperbolizable manifolds are studied. In the convex-cocompact case, the characteristic submanifold consists entirely of I-bundles and solid tori. Hence the finite-index realization theorem immediately some information related to the Question stated above:

THEOREM: *Let M be a hyperbolizable, compact 3-manifold with no toroidal boundary components.*

1. *If M has compressible boundary, then $CC(M)$ has finitely many components if and only if M is either small or a compression body.*
2. *If M has incompressible boundary, then $CC(M)$ has finitely many components.*

The analogue of part 2 is no longer true when the parabolic locus is nonempty. The corresponding deformation space has infinitely many components precisely when the characteristic submanifold of M (with a boundary pattern determined by the parabolic locus) contains a certain type of structure, illustrated in Example 2 above. Geometrically, it corresponds to the presence of an accidental parabolic element of $\pi_1(M)$ which is homotopic into a toral cusp and homotopic to at least two loops in ∂M which are nonhomotopic in ∂M. These are called *doubly accidental parabolics.*

It is also natural to consider the space $CC(\pi_1(M)) \subset AH(\pi_1(M))$ consisting of all convex cocompact elements of $AH(\pi_1(M))$. If we let $\mathcal{A}(M)$ denote the set of all compact, irreducible 3-manifolds homotopy equivalent to M, then

$$CC(\pi_1(M)) = \cup_{M_i \in \mathcal{A}(M)} CC(M_i).$$

It is a theorem of Swarup [Sw] and Johannson [Joh1] that $\mathcal{A}(M)$ is always finite. Using this, together with an analysis of the homotopy types of compression bodies and small manifolds, one obtains

COROLLARY: *Let M be a hyperbolizable, compact 3-manifold with no toroidal boundary components.*

1. *If M has compressible boundary then $CC(\pi_1(M))$ has finitely many components if and only if $\pi_1(M)$ is either a free group or a free product of two groups one of which is the fundamental group of a closed surface (orientable or non-orientable) and the other is either infinite cyclic or the fundamental group of a closed surface.*
2. *If M has incompressible boundary, then $CC(\pi_1(M))$ has finitely many components.*

The algebraic condition in part 1 says that $\pi_1(M)$ is the fundamental group of either a handlebody or a small manifold.

Chapter X. Elementary and intermediate problem list

These problems are not particularly related to the previous material, but since there are not very many collections of problems in low-dimensional topology, we hope that this list will be useful to students and instructors.

The problems are graded into three levels of difficulty. The "A" problems are relatively short and easy, the "B" problems are of moderate to challenging difficulty, and the "C" problems are even more difficult.

1. *"A" Problems*

A1. Let X be a space. Prove that every point in X has a neighborhood homeomorphic to $\mathbb{R}^n$ if and only if every point has a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$.

A2. Write explicitly the stereographic projection homeomorphism from S^3 to $\mathbb{R}^3 \cup \{\infty\}$.

A3. Let F be the 2-sphere with n crosscaps. Describe explicitly the orientable double covering of F. Do the same for the surface of genus g with n crosscaps.

A4. Prove that $\pi_1(M) \cong \pi_1(M - \partial M)$.

A5. Let M and N be compact n-manifolds. Prove that $M\#N$ is orientable if and only if both M and N are orientable.

A6. Let X be a CW-complex. Prove that if all homotopy groups of X are zero, then X is contractible.

A7. Explain why $M\#S^n = M$ and $M\#D^n = M - \{\text{open disc}\}$.

A8. Let M be obtained by removing an open 3-ball from $\mathbb{RP}^3$. Show that the universal cover of M is $S^2 \times I$.

A9. Let M be a manifold of dimension at least 3. Let D be an n-ball imbedded in M, so that ∂D has a product neighborhood, and let p be point in the interior of D. Prove that the inclusions $M - \{p\} \to M - \text{int}(D)$ and $M - \text{int}(D) \to M$ induce isomorphisms on fundamental groups.

A10. Let X be the 1-point union of two circles. Describe (with pictures) the universal cover of X.

A11. Let $p: (E, e_0) \to (X, x_0)$ be a covering map between path-connected spaces with good local structure (assume they are manifolds, if you like). Let Y be a path-connected space and y_0 a point in Y. Suppose that $f_1, f_2: Y \to E$ are continuous maps with $pf_1 = pf_2$ and $f_1(y_0) = f_2(y_0)$. Prove that $f_1 = f_2$.

A12. Let D be a closed n-ball imbedded in an n-manifold M (as a subcomplex in some simplicial structure), where $n \geq 3$. Show that $\pi_1(M - \text{int}D^n) \cong \pi_1(M)$.

A13. Use van Kampen's theorem to obtain a presentation for the fundamental group of the closed nonorientable surface with n-crosscaps.

A14. Use van Kampen's theorem to calculate the fundamental group of the lens space $L(p,q)$.

A15. Construct an infinite-sheeted covering space $E \to B$ whose group of covering transformations is $\mathbb{Z}/2$.

A16. Recall that a space X is said to be contractible if the identity map of X is homotopic to a constant map. Prove that X is contractible if and only if it is homotopy equivalent to a 1-point space.

A17. Draw an explicit CW-complex structure on the punctured torus $T = S^1 \times S^1 - \text{int}(D^2)$, and use it to calculate $H_*(T;R)$ and $H_*(T,\partial T;R)$.

A18. Prove that $H^0(X,A;R) \cong \text{Hom}(H_0(X,A;R),R)$ if R is commutative.

A19. Let $\phi: V \to W$ be a homomorphism of finite-dimensional vector spaces over a field K and let $\phi^*: \text{Hom}(W,K) \to \text{Hom}(V,K)$ be the induced map on the dual spaces. Prove that the images of ϕ and ϕ^* have the same dimension.

A20. Let M be an n-manifold, let F be a component of ∂M, and let D_1 and D_2 be two $(n-1)$-cells imbedded in F. Prove there is an isotopy j_t of M so that j_0 is the identity, j_t is the identity outside a collar neighborhood of F, and $j_1(D_1) = D_2$.

A21. Let $D_1, E_1, D_2, E_2, \ldots, D_k, E_k$ be PL imbedded n-cells in the interior of an n-manifold M. Assume that $D_i \cap D_j$ and $E_i \cap E_j$ are empty for $i \neq j$. Prove there is an isotopy j_t of M so that j_0 is the identity and $j_1(D_i) = E_i$ for $1 \leq i \leq k$.

A22. Prove that $\mathbb{RP}^2 \times S^1$ is irreducible.

2. *"B" Problems*

B1. Let $A \subseteq X$ and $B \subseteq Y$, and let $h: A \to B$ and $H: Y \to Y$ be homeomorphisms. Let $M(h) = (X \cup Y)/x \sim h(x)$ for $x \in A$. Prove that $M(h) = M(Hh)$.

B2. Show that the quotient space obtained from the 2-sphere by identifying antipodal points is the projective plane.

B3. Let F be a manifold and let f and g be homeomorphisms from F to F. Define $M(f)$ to be $F \times I/(x,0) \sim (f(x),1)$ and $M(f,g)$ to be $(F \times [0,1] \cup F \times [2,3])/(f(x),1) \sim (x,2), (g(x),3) \sim (x,0)$. Show

(a) $M(gfg^{-1}) = M(f)$.
(b) If f is isotopic to g, then $M(f) = M(g)$.
(c) $M(f, g) = M(fg)$. Generalize.
(d) If f^n is isotopic to the identity map 1_F, then $F \times S^1$ is an n-fold covering space of $M(f)$.

B4. Let M and N be closed n-manifolds. Describe the universal cover of $M \# N$.

B5. Let M be a closed 3-manifold. Assume Specker's Theorem: $\pi_2(M)$ is free abelian. Prove that the universal cover of M is homotopy equivalent to either S^3, $\mathbb{R}^3$, or a 1-point union of a collection of 2-spheres.

B6. Let M be a closed 3-manifold. Prove that there exist four 3-cells B_1, B_2, B_3, and B_4 imbedded in M with pairwise disjoint interiors so that $M = \cup_{i=1}^4 B_i$.

B7. Let M be a homology 3-sphere (that is, a 3-manifold such that $H_*(M) \cong H_*(S^3)$), and let C be a tamely imbedded circle in M. Prove that $H_*(M - C) \cong H_*(S^1)$.

B8. Let M be a compact 3-manifold with $\partial M = S^1 \times S^1$. Prove that M cannot be simply connected.

B9. Let $n \geq 1$. Prove that $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; R) \cong R$. More generally, prove that if M is an n-manifold and $x \in M$, then $H_q(M, M - \{x\}; R) \cong R$ if $q = n$ and 0 if $q \neq n$.

B10. Show that the universal cover of $\mathbb{RP}^3 \# \mathbb{RP}^3$ is $S^2 \times \mathbb{R}$.

B11. Let X be the 1-point union of two circles and k 2-spheres. Describe $\pi_2(X)$ as a $\pi_1(X)$-module.

B12. Let M be the manifold obtained from $S^2 \times I$ by identifying $(x, 0)$ with $(\alpha(x), 1)$, where α is the antipodal map. Describe $\pi_2(M)$ as a $\pi_1(M)$-module.

B13. Let X be the cell complex obtained by attaching a 2-cell D^2 to S^1 using a degree n map from ∂D^2 to S^1. Describe $\pi_2(X)$ as a $\pi_1(X)$-module.

B14. Let $p: (E, e_0) \to (X, x_0)$ be a covering map between path-connected spaces with good local structure (assume they are manifolds, if you like). Let $e_0, e_1 \in p^{-1}(x_0)$. Let $\widetilde{\gamma}$ be a path in E from e_0 to e_1 and let γ be the loop $p\widetilde{\gamma}$. Prove that $p_\#(\pi_1(E, e_1)) = [\gamma]^{-1} p_\#(\pi_1(E, e_0))[\gamma]$.

B15. Let $p: (E, e_0) \to (X, x_0)$ be a covering map between path-connected spaces with good local structure (assume they are manifolds, if you like). Let γ_1 and γ_2 be loops in X based at x_0 and let $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$

be their lifts starting at e_0. Prove that $\widetilde{\gamma_1}(1) = \widetilde{\gamma_2}(1)$ if and only if the cosets $p_\#(\pi_1(E, e_0))[\gamma_1]$ and $p_\#(\pi_1(E, e_0))[\gamma_2]$ are equal. Deduce that lifting defines a bijection between $p^{-1}(x_0)$ and the cosets of $p_\#(\pi_1(E, e_0))$ in $\pi_1(X, x_0)$.

B16. Let F be a manifold and let f and g be homeomorphisms from F to F. Define $M(f)$ to be $F \times I/(x,0) \sim (f(x),1)$. Show that $\pi_1(M(f)) \cong (\pi_1(F) * \mathbb{Z})/(\gamma = tf_\#(\gamma)t^{-1})$, where t generates the infinite cyclic free factor. Define a homomorphism from $\pi_1(M(f))$ to $\mathbb{Z}$ by sending t to 1 and γ to 0 for all $\gamma \in \pi_1(F)$. Describe the covering space corresponding to the kernel of this homomorphism.

B17. Let F be a connected 2-manifold in the boundary of a 3-manifold M. Prove that the kernel of $\pi_1(F) \to \pi_1(M)$ contains no orientation-reversing elements.

B18. Let $E \to F$ be an odd-sheeted covering, where F is a nonorientable surface. Prove that E is nonorientable.

B19. Let X be a CW-complex. Prove that X is contractible if and only if every map from a finite complex into X is homotopic to a constant map.

B20. For $n \geq 1$ let $K_n = S^1 \times [n-1, n]$. Let $p: S^1 \to S^1$ be a 2-fold covering map, and form X from $\cup_{n=1}^{\infty} K_n$ by identifying (x, n) in K_n with $(p(x), n)$ in K_{n+1} for all $n \geq 1$. Calculate $\pi_1(X)$.

B21. Regard D^2 as $\{(x,y) \mid x^2 + y^2 \leq 1\}$. Let $\phi: D^2 \to D^2$ be rotation about the origin through an angle of $2\pi/n$. Let E be a small disc centered at $(1/2, 0)$, small enough so that E, $\phi(E), \ldots, \phi^{n-1}(E)$ are disjoint. Let D_n be the disc with n holes $D - \cup_{i=0}^{n-1} \phi^i(\mathrm{int}(E))$, and let $X_n = D_n \times I/(x,0) \sim (\phi(x),1)$. Calculate $\pi_1(X)$, and find a 2-generator presentation for this group.

B22. For the manifold X_n defined in the previous problem, find the free abelian subgroups corresponding to the boundary tori. Find an identification of the boundary tori so that the resulting closed 3-manifold has nontrivial center.

B23. For the manifold X_n used in the previous two problems, prove these is a homeomorphism from X_n to X_n which interchanges its boundary components.

B24. Prove that $H_0(X; R) \cong \oplus R$ with one summand for each path component of X. What about $H_0(X, A; R)$?

B25. Prove that the reduced homology group $\widetilde{H}_0(X; R)$ is isomorphic to $\oplus R$ with one fewer summand than the number of component of X.

B26. Prove that $H^1(X,A;R) \cong \mathrm{Hom}(H_1(X,A;R),R)$ if R is a principal ideal domain.

B27. Let M be a compact simply-connected 3-manifold with nonempty connected boundary. Assuming the fact that $H_n(M;\mathbb{Z})=0$ when M is a connected nonclosed n-manifold, prove that $\partial M = S^2$ and M is contractible.

B28. Let Σ be a closed 3-manifold, let D be a 3-ball in Σ, and let $\Sigma_0 = \Sigma - \mathrm{int}(D)$. Assuming the fact that $H_n(M;\mathbb{Z}) = 0$ when M is a connected nonclosed n-manifold, prove that $\Sigma \simeq S^3$ if and only if $\Sigma_0 \simeq D^3$.

B29. Let $W = D^3 \# D^3 \# D^3$. Let S_1, S_2, and S_3 be the components of ∂W. Let $h\colon S_1 \to S_2$ be a homeomorphism, and define $M(h)$ to be the quotient space of W obtained by identifying x with $h(x)$ for all $x \in S_1$. Up to homeomorphism, find all manifolds of the form $M(h)$.

B30. Let M be a compact orientable irreducible 3-manifold with nontrivial free fundamental group. Prove that M has nonempty boundary

B31. Let M be a compact orientable irreducible 3-manifold with nontrivial free fundamental group. Prove that M is a handlebody. (Note: From the previous problem, M has nonempty boundary. If F is a component of ∂M that is not a 2-sphere, then $\pi_1(F)$ is not free, since F is closed. Since subgroups of free groups are free, $\pi_1(F) \to \pi_1(M)$ cannot be injective.)

B32. Let X be a finite connected 1-dimensional CW-complex. Prove that $\pi_1(X)$ is free of rank $1 - \chi(X)$.

B33. Suppose F is free of rank r and F_1 is a subgroup of index k. Prove that F_1 is free of rank $k(r-1)+1$.

B34. Suppose M^3 is a 3-manifold and F is a connected surface properly imbedded in M (that is, $F \cap \partial M = \partial F$). Suppose that F is two-sided (that is, there is an imbedding $F \times [-1,1] \to M$ such that the image of $F \times \{0\}$ is F). Prove that if F does not separate M, then $H_1(M)$ is infinite.

B35. Let M_1 and M_2 be two connected n-manifolds with nonempty boundaries, and let $D_1 \subset \partial M_1$ and $D_2 \subset \partial M_2$ be $(n-1)$-cells. Define the boundary-connected sum $M_1 \natural M_2$ to be the manifold resulting from $M_1 \cup M_2$ by identifying D_1 and D_2 by a homeomorphism. Investigate how well-defined $M_1 \natural M_2$ is.

B36. Let F be the disc with two holes and denote its boundary circles by C_1, C_2, and C_3. Suppose $f\colon F \times S^1 \to F \times S^1$ is a map which restricts to an orientation-preserving or orientation-reversing homeomorphism on $(C_1 \cup C_2) \times S^1$. Prove that f is homotopic, without changing it on $(C_1 \cup C_2) \times S^1$, to a homeomorphism of $F \times S^1$.

B37. Let X_m denote an m-times punctured 3-cell (that is, the connected sum of $m+1$ 3-cells). Let X_{-1} denote S^3. Show that $X_k \natural X_\ell = X_{k+\ell}$, $X_k \# X_\ell = X_{k+\ell+1}$, and any manifold obtained from $X_k \cup X_\ell$ by identifying a boundary component of X_k and a boundary component of X_ℓ is homeomorphic to $X_{k+\ell-1}$.

B38. Let F be a closed 2-manifold imbedded in S^3. Use the Transversality Theorem to prove that F is 2-sided.

B39. Suppose $\widetilde{M}$ is a covering space of a 3-manifold M. Prove that if $\widetilde{M}$ is irreducible, then M is irreducible. (Remark: The following converse is also true, but is very difficult to prove: If M is irreducible and contains no 2-sided projective planes, then every covering space of M is irreducible.)

B40. Let M be a compact 3-manifold with free fundamental group. Prove that $M = H_1 \# \cdots \# H_r \# M_1 \# \cdots \# M_k \# \Sigma_1 \# \cdots \# \Sigma_\ell$ where each H_i is a handlebody, each M_i is a 2-sphere bundle over the circle, and each Σ_i is an irreducible homotopy 3-sphere.

B41. Without using the Loop Theorem, prove that no loop in the boundary of a 3-manifold that reverses the local orientation in ∂M can be contracted in M.

B42. Let M be a compact simply-connected 3-manifold with nonempty boundary. Prove there is a homotopy equivalence of pairs $(M, \partial M) \simeq (N, \partial N)$ where N is a connected sum of 3-balls.

B43. Suppose M is a compact 3-manifold with incompressible boundary, and $\pi_1(M) = A_1 * A_2$. Show that $M = M_1 \# M_2$ where $\pi_1(M_i) \cong A_i$.

B44. Give an example of a closed orientable irreducible 3-manifold M such that $\pi_1(M) \cong A *_C B$, with $C \neq A$ and $C \neq B$, with C not isomorphic to the fundamental group of a 2-manifold.

B45. Let F be a closed orientable 2-manifold of genus g. Find all n such that $\pi_1(F)$ contains a subgroup isomorphic to the fundamental group of a closed orientable 2-manifold of genus n.

B46. Let F be a closed 2-manifold, not a 2-sphere. Using the cohomology of groups, prove that $\pi_1(F)$ is not free.

B47. Let F be a closed 2-manifold, not a 2-sphere. Using only elementary homology and covering space theory, prove that $\pi_1(F)$ is not free.

3. *"C" Problems*

C1. Draw pictures showing that $\mathbb{RP}^3$ is homeomorphic to the lens space $L(2,1)$.

C2. Let F be a connected manifold and let f be a homeomorphism from F to F. Define $M(f)$ to be $F \times I/(x,0) \sim (f(x),1)$. Suppose that $F \times S^1$ is an n-fold covering space. Prove that f^n must be isotopic to the identity of F if $M(f)$ is 2-dimensional. Prove or give a counterexample when $M(f)$ has dimension 3. Prove or give a counterexample when $M(f)$ has dimension at least 4.

C3. Let M and N be nonsimply connected closed n-manifolds each of whose universal covers is either S^n or $\mathbb{R}^n$. Prove that the universal cover of $M \# N$ is homeomorphic to $S^n - C$, where C is either a set consisting of two points or C is a Cantor set.

C4. Let M be a closed 3-manifold. Prove that the universal cover of M is homotopy equivalent to either S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, or $S^3 - C$, where C is a Cantor set. Deduce Specker's Theorem: $\pi_2(M)$ is free abelian.

C5. Let M be a closed 3-manifold with fundamental group $\mathbb{Z}/2$. Prove that M is homotopy equivalent to $\mathbb{RP}^3$.

C6. Let G be a finite group, and let $n \geq 2$. Prove that there exists a closed n-manifold on which G acts effectively.

C7. Let $p: \partial D^2 \to S^1$ be a 3-fold covering map, and let X be the quotient space formed from $D^2 \cup S^1$ by identifying x with $p(x)$ for all $x \in \partial D^2$. Describe $\pi_2(X)$ as a $\mathbb{Z}$-module and as a $\pi_1(X)$-module.

C8. Let G be a finite group. Find an infinite-sheeted covering space $E \to B$ whose group of covering transformations is G.

C9. For $n \geq 1$ let $K_n = S^1 \times [n-1, n]$. Let $p: S^1 \to S^1$ be a 2-fold covering map, and form X from $\cup_{n=1}^{\infty} K_n$ by identifying (x,n) in K_n with $(p(x), n)$ in K_{n+1} for all $n \geq 1$. Describe the universal cover of X and the action of $\pi_1(X)$ as covering transformations.

C10. Construct a 3-manifold with fundamental group the rational numbers.

C11. Let M be a compact orientable irreducible 3-manifold with $\pi_1(M) \cong A *_C$, or with $\pi_1(M) \cong A *_C B$ with $C \neq A$ and $C \neq B$. Prove that M contains a 2-sided incompressible surface $F \neq S^2$.

C12. Prove or give a counterexample: if $f: X \to Y$ is a map between connected CW-complexes, and f induces isomorphisms on fundamental groups and on all integral homology groups, then f is a homotopy equivalence.

C13. Let M be a compact 3-manifold orientable over the field F. Let $i: \partial M \to M$ be the inclusion, and let $i_*: H_1(\partial M; F) \to H_1(M; F)$ be the induced map on homology. Prove that the kernel of i_* and the image of i_* have the same dimension. Deduce that there is no compact 3-manifold M with $\partial M = \mathbb{RP}^2$.

C14. Use the Lefschetz fixed point formula to prove that there is no compact 3-manifold M with $\partial M = \mathbb{RP}^2$.

C15. Determine the homeomorphism classes of compact 3-manifolds obtained from D^3 by identifying finitely many pairs of disjoint discs in its boundary. Determine the homeomorphism classes obtained from D^3 by identifying finitely many pairs of disjoint surfaces in its boundary.

C16. Consider compact orientable irreducible 3-manifolds M which have a boundary component $F \neq S^2$. Take as known the fact that if $\pi_1(F) \to \pi_1(M)$ is an isomorphism, then $M = F \times I$. Determine the homeomorphism classes of compact orientable irreducible 3-manifolds M having a boundary component $F \neq S^2$ such that $\pi_1(F) \to \pi_1(M)$ is surjective.

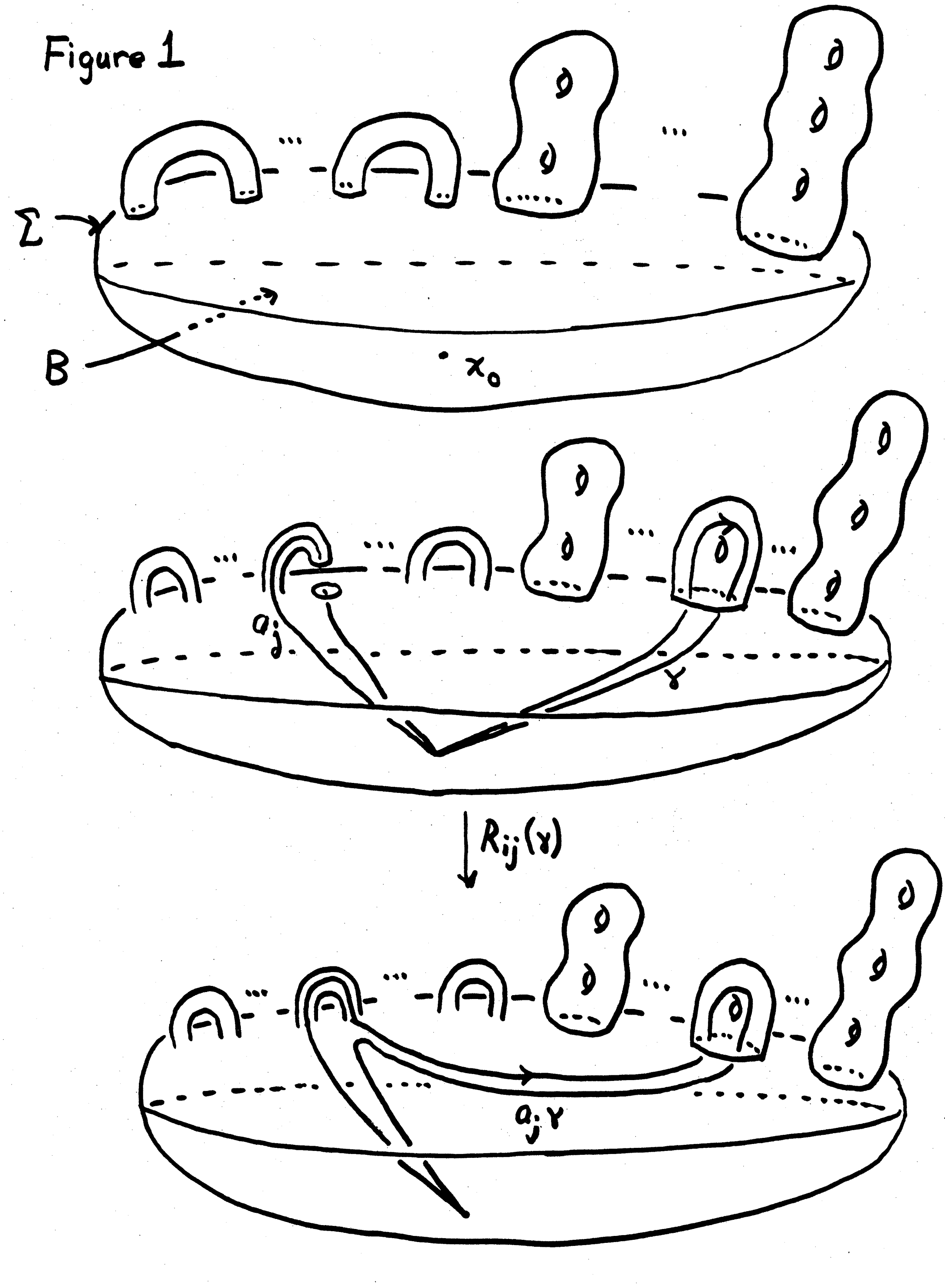
C17. Let M be a closed irreducible 3-manifold such that $\pi_1(M)$ contains an infinite abelian group of finite index. Prove that M has a covering space which is homeomorphic to $S^1 \times S^1 \times S^1$.

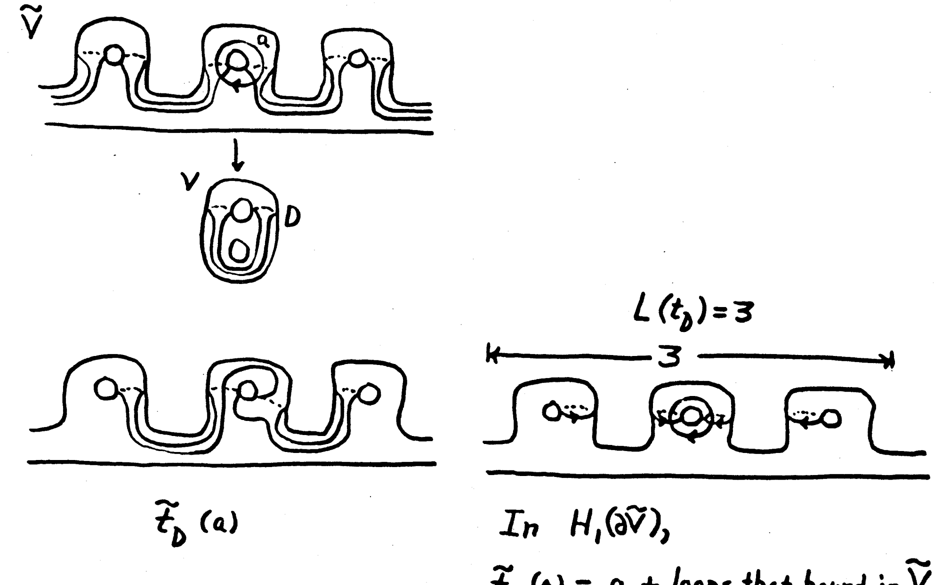
C18. Let M be a compact 3-manifold and suppose that $\pi_1(M)$ contains an element of order 2. Prove that either $M = M_1 \# M_2$, where $\pi_1(M_1)$ is finite, or M contains a 2-sided projective plane.

C19. Let N be a compact orientable 3-manifold. Prove that if some nonzero element of $H_2(\widetilde{N}; \mathbb{Z}/2)$ is fixed by the action of infinitely many elements of $\pi_1(N)$, then N is homotopy equivalent to either $S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$.

C20. For what compact 3-manifolds N is $\pi_2(N)$ finitely generated as an abelian group?

Figure 1

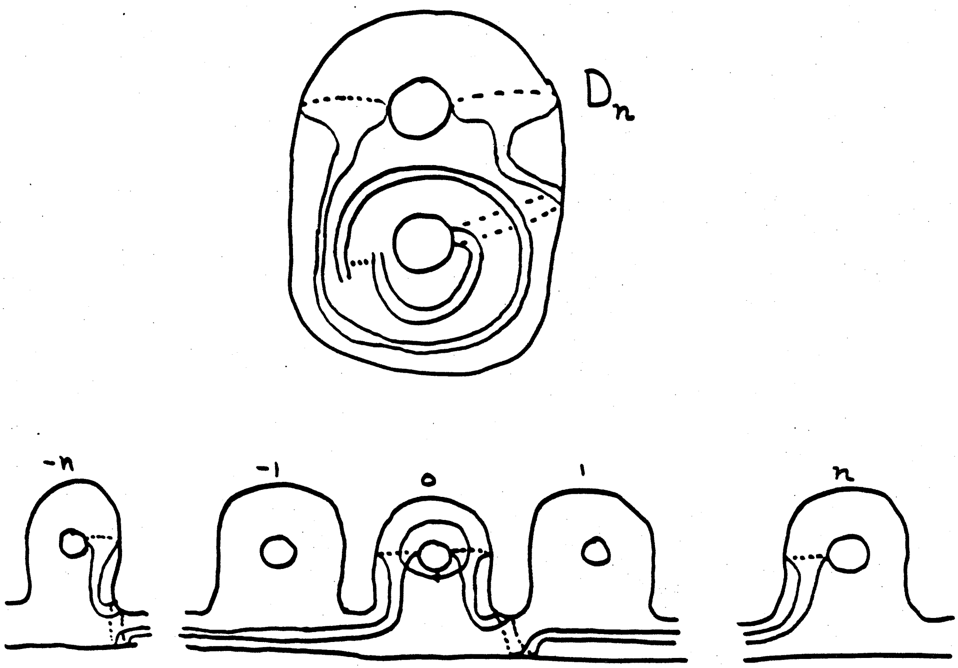




$\tilde{V}$
a
V
D
$L(t_D)=3$
3
$\tilde{t}_D$ (a)
In $H_1(\partial\tilde{V})$,
$\tilde{t}_D$ (a) = a + loops that bound in $\tilde{V}$

Figure 2

Figure 3



$$L(t_{D_n}) = 2n+1$$

References

[Be] L. Bers, Space of Kleinian groups, *Maryland Conference in Several Complex Variables,* Springer-Verlag Lecture Notes in Mathematics 155 (1970), 9–34.

[Bi] R. Bieri, *Homological Dimension of Discrete Groups,* Queen Mary College Mathematics Notes (1976).

[B-E] R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré Duality, *Invent. Math.* 20 (1973), 103-124.

[Bo] F. Bonahon, Cobordism of automorphisms of surfaces, *Ann. Norm. École Norm. Sup.* (4) 16 (1983), 237-270.

[B-J] T. Bröcker and K. Jänich, *Introduction to Differential Topology,* Cambridge University Press, 1982.

[C-M] R. Canary and D. McCullough, Characteristic structure in hyperbolic 3-manifolds, in preparation.

[Ev] B. Evans, Boundary respecting maps of 3-manifolds, *Pacific J. Math* 42 (1972), 639-655.

[Gil] N. Gilbert, Presentations of the automorphism group of a free product, *Proc. London Math. Soc.* (3) 54 (1987), 115-140.

[Gu] V. K. A. M. Gugenheim, Piecewise linear isotopy, *Proc. London. Math. Soc.* 31 (1953), 29-53.

[F-K] D. Frank and D. Kahn, Finite complexes with infinitely-generated groups of self-equivalences, *Topology* 16 (1977), 189-192.

[F-R1] D. I. Fuks-Rabinovitch, On the automorphism group of free products I, *Math. Sb.* 8 (50) (1940), 265-276.

[F-R2] D. I. Fuks-Rabinovitch, On the automorphism group of free products II, *Math. Sb.* 9 (51) (1941), 183-220.

[Ga1] D. Gabai, Homotopy hyperbolic 3-manifolds are virtually hyperbolic, *J. Amer. Math. Soc.* 7 (1994), 193-198.

[Ga2] D. Gabai, On the geometric and topological rigidity of hyperbolic 3-manifolds, preprint.

[Har1] J. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, *Invent. Math.* 84 (1986), 157-176.

[Har2] J. Harer, The cohomology of the moduli space of curves, in *Theory of Moduli,* ed. E. Sernesi, Springer-Verlag Lecture Notes in Mathematics Vol. 1337 (1988), 138-221.

[Har3] W. Harvey, Geometric structure of surface mapping class groups, in *Homological Group Theory,* ed. C. T. C. Wall, London Math. Soc. Lecture Note Series No. 36 (1979), 255-269.

[Har4] W. Harvey, Boundary structure of the modular group, in *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference,* ed. I. Kra and B. Maskit, Annals of Math. Study No. 97 (1981), 245-251.

[Hat1] A. Hatcher, A proof of the Smale Conjecture, $\mathit{Diff}(S^3) \cong O(4)$, *Annals of Math.* 117 (1983), 553-607.

[Hat2] A. Hatcher, Homeomorphisms of sufficiently large $\mathbb{P}^2$-irreducible 3-manifolds, *Topology* 15 (1976), 343-347.

[He] J. Hempel, *3-manifolds,* Annals of Math. Study No. 86, Princeton University Press (1976).

[H-M] Finite presentation of 3-manifold mapping class groups (with Allen Hatcher), *Groups of Self-Equivalences and Related Topics,* ed. R. Piccinini, Springer-Verlag Lecture Notes in Mathematics Vol. 1425 (1990), 48–57.

[Iv1] N. Ivanov, Homotopy of spaces of automorphisms of some three-dimensional manifolds, *Soviet Math. Dokl.* 20 (1979), 47-50.

[Iv2] N. Ivanov, Homotopy of spaces of diffeomorphisms of some three-dimensional manifolds, *J. Soviet Math.* 26 (1984), 1646-1664.

[J-S] W. Jaco and P. Shalen, Seifert fibered spaces in 3-manifolds, *Mem. Amer. Math. Soc.* 220 (1979).

[Joh] K. Johannson, *Homotopy Equivalences of 3-Manifolds,* Springer-Verlag Lecture Notes in Mathematics Vol. 761 (1979).

[Joh1] K. Johannson, On exotic homotopy equivalences of 3-manifolds, *Geometric Topology* (ed. J. C. Cantrell), Academic Press (1979), 101-112.

[K-M1] J. Kalliongis and D. McCullough, π_1-injective mappings of compact 3-manifolds, *Proc. London Math. Soc.* (3), 52 (1986), 173-192.

[K-M2] J. Kalliongis and D. McCullough, Maps inducing isomorphisms on fundamental groups of compact 3-manifolds, *J. London Math. Soc.* (2) 35 (1987), 177-192.

[K-T] S.P. Kerckhoff and W.P. Thurston, Non-continuity of the action of the mapping class group at Bers' boundary of Teichmuller space, *Invent. Math.* 100 (1990), 25–47.

[Kra] I. Kra, On spaces of Kleinian groups, *Comment. Math. Helv.* 47 (1972), 53-69.

[Lau] F. Laudenbach, Topologie de la dimension trois. Homotopie et isotopie, *Astérisque* 12 (1974), 1-152.

[Lu] E. Luft, Actions of the homeotopy group of an orientable 3-dimensional handlebody, *Math. Ann.* 234 (1978), 279-292.

[Mar] A. Marden, The geometry of finitely generated Kleinian groups, *Ann. of Math.* 99 (1974), 383–462.

[Mas] B. Maskit, Self-maps of Kleinian groups, *Amer. J. Math.* 93 (1971), 840-856.

[McC] J. McCool, A presentation for the automorphism group of a free group of finite rank, *J. London Math. Soc.* 8 (1974), 259-266.

[McC1] D. McCullough, Mappings of reducible 3-manifolds, in *Proceeding of the Semester in Geometric and Algebraic Topology of the Banach International Mathematical Center,* ed. H. Torunczyk, Banach Center Publications, Warsaw (1986), 61-76.

[McC2] D. McCullough, Twist groups of compact 3-manifolds, *Topology* 24 (1985), 461-474.

[McC3] D. McCullough, Virtual cohomological dimension of mapping class groups of 3-manifolds, *Bull. Amer. Math. Soc.* 18 (1988), 27-30.

[McC4] D. McCullough, Virtually geometrically finite mapping class groups of 3-manifolds, *J. Differential Geometry* 33 (1991), 1-65.

[McC5] D. McCullough, Topological and algebraic automorphisms of 3-manifolds, *Groups of Self-Equivalences and Related Topics,* ed. R. Piccinini, Springer-Verlag Lecture Notes in Mathematics Volume 1425 (1990), 102-113.

[McC6] Finite aspherical complexes with infinitely-generated groups of self-homotopy-equivalences, *Proc. Amer. Math. Soc.* 80, (1980), 337–340.

[McC7] Compact 3-manifolds with infinitely generated groups of self-homotopy-equivalences, *Proc. Amer. Math. Soc.* 91 (1984), 625–629.

[M-M] D. McCullough and A. Miller, Homeomorphisms of 3-manifolds with compressible boundary, *Mem. Amer. Math. Soc.* 344 (1986), 1-100.

[M-R] D. McCullough and J. H. Rubinstein, The Generalized Smale Conjecture for 3-manifolds with genus 2 one-sided Heegaard splittings, preprint.

[Pa1] R. Palais, Extending diffeomorphisms, *Proc. Amer. Math. Soc.* 11 (1960), 274-277.

[Pa2] R. Palais, Local triviality of the restriction map for embeddings, *Commentarii Math. Helv.* 34 (1960), 305-312.

[Sp] E. Spanier, *Algebraic Topology,* McGraw-Hill (1966).

[St] J. Stallings, On the loop theorem, *Annals of Math.* 72 (1960), 12-19.

[Sul1] D. Sullivan, Infinitesimal computations in topology, *Inst. Hautes Études Sci. Publ. Math.* 47 (1977), 269-331.

[Suz] S. Suzuki, On homeomorphisms of a 3-dimensional handlebody, *Can. J. Math.* 29 (1977), 111-124.

[Sw] G.A. Swarup, Two finiteness properties in 3-manifolds, *Bull. London Math. Soc.* 12 (1980), 296–302.

[Wa] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, *Annals of Math.* 87 (1968), 56-88.

[Whi] J. H. C. Whitehead, Simplicial spaces, nuclei, and m-groups, *Proc. London Math. Soc.* 45 (1939), 243-327.

Lecture Notes Series

1. M.-H. Kim (ed.), Topics in algebra, algebraic geometry and number theory, 1992
2. J. Tomiyama, The interplay between topological dynamics and theory of C*-algebras, 1992 ; 2nd Printing, 1994
3. S. K. Kim, S. G. Lee and D. P. Chi (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part I, 1993
   H. Kim, C. Kang and C. S. Bae (ed.), Proceedings of the lst GARC Symposium on pure and applied mathematics, Part II, 1993
4. T. P. Branson, The functional determinant, 1993
5. S. S.-T. Yau, Complex hyperface singularites with application in complex geometry, algebraic geometry and Lie algebra, 1993
6. P. Li, Lecture notes on geometric analysis, 1993
7. S.-H. Kye, Notes on operator algebras, 1993
8. K. Shiohama, An introduction to the geometry of Alexandrov spaces, 1993
9. J. M. Kim (ed.), Topics in algebra, algebraic geometry and number theory II, 1993
10. O. K. Yoon and H.-J. Kim, Introduction to differentiable maniolds, 1993
11. P. J. McKenna, Topological methods for asymmetric boundary value problems, 1993
12. P. B. Gilkey, Applications of spectral geometry to geometry and topology, 1993
13. K.-T. Kim, Geometry of bounded domains and the scaling techniques in several complex variables, 1993
14. L. Volevich, The Cauchy problem for convolution equations, 1994
15. L. Elden and H. S. Park, Numerical linear algebra algorithms on vector and parallel computers, 1993
16. H. J. Choe, Degenerate elliptic and parabolic equations and variational inequalities, 1993
17. S. K. Kim and H. J. Choe (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part I, The first Korea-Japan conference of partial differential equations, 1993
    J. S. Bae and S. G. Lee (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part II, 1993
    D. P. Chi, H. Kim and C.-H. Kang (ed.), Proceedings of the second GARC Symposium on pure and applied mathematics, Part III, 1993
18. H.-J. Kim (ed.), Proceedings of GARC Workshop on geometry and topology ' 93, 1993
19. S. Wassermann, Exat C*-algebras and related topics, 1994
20. S.-H. Kye, Notes on abstract harmonic analysis, 1994
21. K. T. Hahn, Bloch-Besov spaces and the boundary behavior of their functions, 1994
22. H. C. Myung, Non-unital composition algebras, 1994
23. P. B. Dubovskii, Mathematical theory of coagulation, 1994
24. J. C. Migliore, An introduction to deficiency modules and Liaison theory for subschemes of projective space, 1994
25. I. V. Dolgachev, Introduction to geometric invariant theory, 1994
26. D. McCullough, 3-Manifolds and their mappings, 1995