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INTRODUCTION TO GEOMETRIC INVARIANT THEORY

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by

Igor V. Dolgachev

Dedicated to the memory of Boris Moishezon

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PREFACE

These notes originate in a series of lectures given at the Tokyo Metropolitan University and Seoul National University in the Fall of 1993. These lectures have been extended into a graduate course at the University of Michigan in the Winter of 1994. Almost all of the material in these notes had been actually covered in my course. The main purpose of the notes is to provide a digest to Mumford's book. Their sole novelty is the greater emphasis on dependence of the quotients on linearization of actions and also including toric varieties as examples of torus quotients of open subsets of affine space. We also briefly discuss Nagata's counter-example to Hilbert's Fourteenth Problem. Lack of time (and of interested audience) did not allow me to include such topic as the relationship between geometric invariant theory quotients and symplectic reductions. Only one application to moduli problem is included. This is Mumford's construction of the moduli space of algebraic curves. The more knowledgeable reader will immediately recognize that the contents of these notes represent a small portion of material related to geometric invariant theory. Some compensation for this incompleteness can be found in a bibliography which directs the reader to additional results.

Only the last lecture assumes some advanced knowledge of algebraic geometry; the necessary background for all other lectures is the first two chapters of Shafarevich's book. Because of arithmetical interests of some of my students, I did not want to assume that the ground field is algebraically closed, this led me to use more of the functorial approach to foundations of algebraic geometry.

I am grateful to everyone who attended my lectures in Tokyo, Seoul and Ann Arbor for their patience and critical remarks. I am especially thankful to Sarah-Marie Belcastro and Pierre Giguere for useful suggestions and corrections to preliminary version of these notes. I must also express great gratitude to Professor Uribe for organizing my visit to Tokyo Metropolitan University, and to my former students Jong Keum and Yonggu Kim for inviting me to Seoul National University and for their help in publishing these lecture notes.

Ann Arbor, June 1994 Igor Dolgachev

INTRODUCTION

Geometric invariant theory arises from an attempt to construct the moduli spaces of various algebraic objects. These spaces are algebraic varieties whose points correspond naturally to isomorphism classes of the objects we want to classify. In many situations the construction can be achieved by first finding an algebraic variety X parametrizing representatives of each isomorphism class in such a way that two points correspond to the same class if and only if they belong to the same orbit of a certain algebraic group G acting algebraically on X. In this way the moduli space arises as the orbit space X/G. The problem with this approach is that the latter space may not be defined as an algebraic variety. The reasons for this are quite obvious. Since we expect that the canonical projection $X \to X/G$ is a regular map of algebraic varieties, its fibres, which are the orbits, must be closed subsets of X. However, there is no reason to expect that all orbits are closed. Even they were all closed, the theorem on dimension of fibres of regular maps would tell us that the dimension of an orbit can only increase compared to the dimension of a general one, however the same theorem easily shows that the dimension of the stabilizer group of an orbit can also only jump. Since the dimension of the orbit is equal to the difference between the dimensions of the group and the stabilizer we get a contradiction. A possible solution to this difficulty is leave out some of the objects, i.e., consider a subset U of X, preferably open in the Zariski topology. This subset must be preserved under the action of G, and the orbits in U are good enough to be able to define the quotient. Hopefully, the objects which we have to delete are sufficiently bad to be ignored without much harm (like algebraic varieties with some bad singularities). The orbit space U/G then possesses all natural geometric properties which are included in the definition of a geometric quotient. However, one problem is still left. The quotient space may be a non-complete algebraic variety, leaving us with unsatisfactory feeling that certain objects should be after all not left out from consideration since they represent some natural limits of objects from U. This leads to a compactification of the space X/G which is constructed as a "categorical quotient" of a larger set U'. This consists of a variety $U'//G$ together with a regular map $U' \to U'//G$ which is universal with respect to maps of U' which are constant on orbits.

Geometric invariant theory gives us a recipe for choosing the open sets U and U' in order that both the geometric quotient U/G and the categorical quotient $U//G$ exist. This recipe was proposed by D. Mumford in his epochal book [**Mu1**]. It is based on the following idea due to D. Hilbert. Suppose $X = \mathbf{A}^n$ is affine space and $G = GL(n)$ is the general linear group acting on X by linear change of variables. The set of polynomials which are unchanged under all transformations from G form a finitely generated subalgebra A of the ring of polynomial functions on X. Choose an affine algebraic variety Y whose ring of regular functions is isomorphic to A. Then we have a natural regular map $X \to Y$ which turns out to be the categorical quotient map. In many problems we are interested not in points of X but in points of the associated projective space $\mathbf{P}^{n-1}$ (for example, X is the space of homogeneous forms of some degree d, and $\mathbf{P}^{n-1}$ is the space of hypersurfaces of degree d). Let Z be the closed subspace of $\mathbf{P}^{n-1}$ which is the set of common zeroes of all homogeneous non-constant polynomials from A (in the above example, the corresponding homogeneous forms of degree d are "nullforms" of Hilbert). Then the open subset $U = \mathbf{P}^{n-1} \subset Z$ admits the categorical quotient which is isomorphic to the projective variety with the algebra of projective coordinates isomorphic to A. In the example above, this quotient was taken by Hilbert and his contemporaries as the right moduli space for projective hypersurfaces of degree d. The "bad hypersurfaces" that had to be left out from the consideration are defined by nullforms. The explicit construction of this moduli space is related to finding generators and relations for the algebra of invariants A. This is the subject of Algebraic Invariant Theory (nowadays called Classical Invariant Theory) which was a popular area of mathematics of 19-th century and has been resurrected recently because of its interesting connections to combinatorics and computer computations (see **[Stu]**).

Generalizing Hilbert's idea, Mumford starts from any algebraic variety X and a reductive (e.g. $GL(n)$) algebraic group G acting algebraically on it. Then he chooses a G-equivariant embedding of X into a projective space $\mathbf{P}^N$ and then proceeds as above to define the categorical quotient for some open subset U of $\mathbf{P}^N$. Then he shows that $U \cap X$ admits a categorical quotient isomorphic to the image of $U \cap X$ in $U//G$. There is a more general construction which replaces an embedding into a projective space by a choice of some G-linearized line bundle on X. To obtain a geometric quotient, one should decrease the set U by leaving only points whose orbits are closed in $U \cap X$ and of minimal possible dimension. The points from $U \cap X$ are called semi-stable, and points satisfying the additional property are called stable. In applications, when the variety X is a parametrizing space of some algebraic or geometric objects, the notion of stability often admits a nice algebraic or geometric interpretation (like stable vector bundles or stable degenerate curves). There is a more general aspect of Geometric Invariant Theory which we are not discussing in these lectures. It concerns with the study of algebraic properties of the quotients as well as the properties of the orbits and their closures. We refer the interested reader to **[PV]** for a nice survey of this theory.

Now let us briefly comment on the contents of the notes.

Lectures 1 and 2 introduce the basic notions of algebraic groups and algebraic actions. Although these lectures are self-contained, we want to think that these notions are somewhat familiar to the reader. We use the functorial approach based on the Yoneda Lemma to avoid difficulties in verifying that our constructions are defined over the ground field.

In Lecture 3 we introduce the notion of a G-linearized line bundle and prove that any algebraic action can be linearized. We do not assume that the reader is familiar with the notion of an algebraic vector bundle, and for this reason define everything from scratch. The emphasis on the choice of a G-linearized line bundle for the construction of a quotient is a modern trend in application of the theory to various moduli problems. It turns out that two quotients corresponding to two different choices of linearizations differ from each other by some explicit birational transformation. This allows one to study one by means of another, presumably of simpler geometric or topological structure.

In Lecture 4 we discuss various notions of quotients of an algebraic variety by a group action. Here we introduce the notion of a reductive algebraic group and explain its significance for constructing the quotients.

Lecture 5 is devoted to Hilbert's Fourteenth Problem which asks whether the subalgebra of invariant polynomials under a linear action of an algebraic group is finitely generated. This problem is related to some deep problem in birational geometry of algebraic varieties which we also discuss it here. We prove the Groshans principle here and deduce from this the Weitzenböck theorem on the finite generatedness of the algebra of invariants of the additive group. A counter-example of Nagata to the Hilbert problem is given in this lecture with some proofs left out.

In Lecture 6 we introduce the notion of stability of action and give Mumford's construction of geometric and categorical quotients.

In Lecture 7 we present the main technical tool for verifying the property of stability. It is Hilbert-Mumford's numerical criterion. It consists of replacing the group by any of its one-parameter subgroup and checking the stability for the restricted action. The final form of this criterion can be expressed in terms of some combinatorial data based on the notion of the state polytope of a point.

In Lecture 8 we give the first concrete example of the analysis of stability. This is the case of the action of general linear group in the space of homogeneous polynomials. In some special cases (for example, binary forms or cubic ternary forms) the full description of stable points can be given.

In Lecture 9 another series of examples is discussed. Here we consider ordered sequences of linear subspaces of a fixed projective space and the natural action of the projective linear group on them. Although much is known about the quotients for sequences of points, our knowledge of the moduli space of sequences of subspaces of higher dimension is very limited. We give a tedious analysis of semi-stable orbits in the first non-trivial example: four lines in $\mathbf{P}^3$. The corresponding categorical quotient in this case is isomorphic to the projective plane.

In Lecture 10 we introduce toric varieties as examples of categorical quotients of a subset of affine space by the action of an algebraic torus. This approach to the theory of toric varieties is relatively new and allows one to interpret many properties of toric varieties in terms of geometric invariant theory.

Finally we conclude with Lecture 11 on application of geometric invariant theory to construction of the moduli space of nonsingular projective algebraic curves. As is mentioned in the preface, the material of this lecture involves more algebra-geometrical techniques, and can be omitted by a novice. Again we use the opportunity in this lecture to demonstrate some of the applications of toric geometry; this time to the description of the normalization of the blow-up of monomial ideals in the polynomial rings. Lack of time did not allow me to include other applications of geometric invariant theory. We refer to **[New]** for other elementary introduction into the subject including application to the construction of the moduli space of vector bundles.

Lecture 1. ALGEBRAIC GROUPS

1.0 First let us fix some notation. We shall consider algebraic varieties X over a field k (or algebraic k-varieties). By these we shall mean quasi-projective algebraic k-sets, or, more generally, reduced separated quasi-projective schemes of finite type over k. The field k is not necessarily algebraically closed. For all topological properties of X (e.g., irreducibility) we refer to the variety $X_{\bar{k}}$ obtained by viewing X as a variety over the algebraic closure $\bar{k}$. For any field extension K/k (or any k-algebra K) we shall denote by $X(K)$ the set of K-points of X. Thus if X is affine and $\mathcal{O}(X)$ is its algebra of regular functions (coordinate algebra), we have a natural bijective map $\alpha: X(K) \to Hom_k(\mathcal{O}(X), K)$. If we choose a presentation of $\mathcal{O}(X)$ as the quotient algebra $k[Z_1, \ldots, Z_n]/I$ (or in other words, if we choose an embedding of X into affine space $\mathbf{A}^n$), then

$$X(K) = \{(\alpha_1, \ldots, \alpha_n) \in K^n \mid F(\alpha_1, \ldots, \alpha_n) = 0 \text{for any } F \in I\}.$$

The homomorphism $\mathcal{O}(X) \to K$ defined by a point $x \in X(K)$ is the evaluation at x: $\varphi = P \bmod I \mapsto \varphi(x) := P(\alpha_1, \ldots, \alpha_n)$, where $P \in k[Z_1, \ldots, Z_n]$. By definition the image of a function $\phi \in \mathcal{O}(X)$ under the homomorphism $\mathcal{O}(X) \to K$ defined by a point $g \in X(K)$ will be denoted by $\phi(x)$. If $\phi: K \to K'$ is a homomorphism of k-algebras we have a natural map $X(K) \to X(K')$. It is injective if ϕ is injective. If $K \to K'$ is the inclusion homomorphism, we shall identify the set $X(K)$ with a subset of $X(K')$.

We shall often make no difference between K-points of X and the corresponding homomorphisms $\mathcal{O}(X) \to K$. If X is not necessarily affine, then $x \in X(K)$ is identified with a homomorphism $\mathcal{O}(U) \to K$, where U is any affine open neighborhood of x. For any morphism $f : X \to Y$ we shall denote by $f(K) : X(K) \to Y(K)$ the corresponding map of the sets of K-points. If X, Y are affine, and the map f is given by a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, then for any $x \in X(K), f(K)(x) \in Y(K)$ is given by composing x with f^*.

We denote by pt_k an affine k-variety with $\mathcal{O}(pt_k) = k$. It is defined uniquely up to isomorphism, and $pt_k(K)$ consists of a single element for any K/k. For any $x \in X(k)$ there is a unique regular map

$$pt_k \to X$$

such that the image of the unique element of $pt_k(k)$ is equal to x. If X is affine, this map is defined by the natural homomorphism $x: \mathcal{O}(X) \to k$. We also have the unique (constant) map $X \to pt_k$ which is defined by the natural inclusion $k \hookrightarrow \mathcal{O}(X)$. The canonical isomorphism of the tensor products of k-algebras $\mathcal{O}(X) \otimes_k k \cong \mathcal{O}(X)$ defines a canonical isomorphism $pt_k \times X \cong X$.

1.1 Definition. An *algebraic group* over a field k (or an *algebraic k-group*) is an algebraic variety G over k together with a regular map (*group law*) $\mu: G \times G \to G$ satisfying the usual axioms of a group law:

(i) (associativity) the diagram

$$\begin{array}{ccc} G \times G \times G & \stackrel{\mu \times id}{\longrightarrow} & G \times G \\ \downarrow id \times \mu & & \downarrow \mu \\ G \times G & \stackrel{\mu}{\longrightarrow} & G \end{array}$$

is commutative;

(ii) (the existence of the unit) there exists a point $e \in G(k)$ such that the following diagrams are commutative

$$\begin{array}{ccc} \{pt\} \times G & \stackrel{pt_e \times id}{\longrightarrow} & G \times G \\ pr_2 \searrow & & \swarrow \mu \\ & G & \end{array} , \quad \begin{array}{ccc} G \times \{pt\} & \stackrel{id \times pt_e}{\longrightarrow} & G \times G \\ pr_1 \searrow & & \swarrow \mu \\ & G & \end{array} ;$$

(iii) (the existence of the inverse) there exists a morphism $\beta : G \to G$ such that the following diagrams are commutative:

$$\begin{array}{ccc} G & \stackrel{(id,\beta)}{\longrightarrow} & G \times G \\ \downarrow & & \downarrow \mu \\ \{pt\} & \stackrel{pt_e}{\longrightarrow} & G \end{array} , \quad \begin{array}{ccc} G & \stackrel{(\beta,id)}{\longrightarrow} & G \times G \\ \downarrow & & \downarrow \mu \\ \{pt\} & \stackrel{pt_e}{\longrightarrow} & G \end{array} .$$

1.2 If (G, μ) is an algebraic group, then for any K/k the map

$$\mu(K): G(K) \times G(K) \to G(K)$$

is a group law on the set $G(K)$ with the unit element $e \in G(k) \subset G(K)$ and the inverse operation $x \mapsto x^{-1} := \beta(K)(x)$. If $K \to K'$ is a homomorphism of extensions, then the map $G(K) \to G(K')$ is a homomorphism of groups. This follows from observing that

$$(G \times G)(K) = G(K) \times G(K)$$

and applying this to the diagrams from (1.1). In fact, the same is true for the sets $G(S) := Mor_{Var/k}(S, G)$ of morphisms from any k-variety S to an algebraic k-group G.

1.3 An algebraic group (G, μ) is called *affine* if the variety G is affine. Since morphisms of affine varieties are defined by the homomorphisms of their coordinates algebras, an equivalent definition of an affine algebraic group is obtained by the reversing the arrows in the diagrams from 1.1: An affine algebraic group is an affine algebraic variety G together with a homomorphism of k-algebras ${}^a\mu : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ (called *coaction*) satisfying the following properties:

(i) (associativity) the diagram

$$\begin{array}{ccc} \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) & \stackrel{{}^a\mu \otimes id}{\longleftarrow} & \mathcal{O}(G) \otimes \mathcal{O}(G) \\ \uparrow id \otimes {}^a\mu & & \uparrow {}^a\mu \\ \mathcal{O}(G) \otimes \mathcal{O}(G) & \stackrel{{}^a\mu}{\longleftarrow} & \mathcal{O}(G) \end{array}$$

is commutative;

(ii) (the existence of the unit) there exists a homomorphism of k-algebras $e\colon \mathcal{O}(G) \to k$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}(G)\otimes\mathcal{O}(G) & \overset{e\otimes id}{\longrightarrow} & k\otimes\mathcal{O}(G) \\ {}^a\mu\nwarrow & & \nearrow 1\otimes id \\ & \mathcal{O}(G) & \end{array} \quad , \quad \begin{array}{ccc} \mathcal{O}(G)\otimes\mathcal{O}(G) & \overset{id\otimes e}{\longrightarrow} & \mathcal{O}(G)\otimes k \\ {}^a\mu\nwarrow & & \nearrow id\otimes 1 \\ & \mathcal{O}(G) & \end{array} \quad ;$$

(iii) (the existence of the inverse) there exists a homomorphism of k-algebras ${}^a\beta : \mathcal{O}(G) \to \mathcal{O}(G)$ such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{O}(G)\otimes\mathcal{O}(G) & \overset{id\otimes {}^a\beta}{\longrightarrow} & \mathcal{O}(G)\otimes\mathcal{O}(G) \\ {}^a\mu\uparrow & & \text{mult}\downarrow \\ \mathcal{O}(G) & \overset{e}{\to} k \to & \mathcal{O}(G) \end{array} \quad , \quad \begin{array}{ccc} \mathcal{O}(G)\otimes\mathcal{O}(G) & \overset{{}^a\beta\otimes id}{\longrightarrow} & \mathcal{O}(G)\otimes\mathcal{O}(G) \\ {}^a\mu\uparrow & & \text{mult}\downarrow \\ \mathcal{O}(G) & \overset{e}{\to} k \to & \mathcal{O}(G). \end{array}$$

All tensor products here are over the field k.

1.4 To get the group law on the sets $G(K)$ from the coaction homomorphism ${}^a\mu$ we follow 1.0: a point $g \in G(K)$ is a homomorphism $\mathcal{O}(G) \to K$. Two points $g, g' \in G(K)$ define the map $g \otimes g' \colon \mathcal{O}(G) \otimes \mathcal{O}(G) \to K \otimes_k K$. Composing it with the multiplication mult : $K \otimes_k K \to K$ we get the map $\mathcal{O}(G) \otimes \mathcal{O}(G) \to K$. Finally composing it with the coaction map we get the map: $\mathcal{O}(G) \to K$ which is our product $\mu(K)(g, g') := gg'$. To sum up, gg' is the composition

$$\mathcal{O}(G) \overset{{}^a\mu}{\to} \mathcal{O}(G) \otimes \mathcal{O}(G) \overset{g\otimes g'}{\longrightarrow} K \otimes K \overset{mult}{\longrightarrow} K.$$

1.5 Remark. One can generalize the notion of an affine commutative group by considering any k-algebra A (not even necessarily commutative) equipped with a homomorphism $A \to A \otimes A$ satisfying properties (i)-(iii), where $\mathcal{O}(G)$ is replaced by A. This is called a *Hopf algebra*.

1.6 Remark (for category lovers). Let $\mathcal{C}$ be any category, $\check{\mathcal{C}}$ be the category of presheaves on $\mathcal{C}$ (i.e. contravariant functors from $\mathcal{C}$ to the category of sets **Sets**). Let $h : \mathcal{C} \to \check{\mathcal{C}}$ be the Yoneda functor which assigns to an object $X \in \mathcal{C}$ the presheaf h_X $S \to Mor_{\mathcal{C}}(S, X)$. An object G of $\mathcal{C}$ is called a *group object* if the presheaf h_G is a presheaf of groups, i.e., $h_G : \mathcal{C} \to$ **Sets** factors through the subcategory **Groups** of groups. In other words, G is a group object if each $h_G(S)$ has a structure of a group such that for any morphism $S \to S'$ in $\mathcal{C}$ the natural map $h_G(S') \to h_G(S)$ is a group homomorphism. If $\mathcal{C}$ has a final object $\{pt\}$ and fibred products $X \times_{\{pt\}} Y$, then this definition can be stated in terms of commutative diagrams similar to definition 1.1. Here the unit is given by a morphism $e\colon pt_k \to G$.

An algebraic group is a group object in the category of algebraic varieties over a field k. The final object in this category is the variety pt_k defined in 1.0.

A *group S-scheme* is a group object in the category of S-schemes.

1.7 Examples. 1. pt_k has a unique algebraic group structure. It is called the *trivial group* and is denoted by $\{1\}$.

2. Let G be the affine space $\mathbf{A}_k^n$ with $\mathcal{O}(G) = k[Z_1, \dots, Z_n]$. The group law is given by the the coaction homomorphism:

$$k[Z_1, \dots, Z_n] \to k[Z_1, \dots, Z_n] \otimes k[Z_1, \dots, Z_n],$$

$$Z_i \mapsto Z_i \otimes 1 + 1 \otimes Z_i, \; i = 1, \dots, n.$$

The inversion map

$${}^a\beta \colon k[Z_1, \dots, Z_n] \to k[Z_1, \dots, Z_n]$$

is given by

$$Z_i \mapsto -Z_i, \quad i = 1, \dots, n.$$

The unit element is $e = (0, \dots, 0) \in G(k) = k^n$. The group law on the sets $G(K) = K^n$ is the usual vector addition. This immediately follows from 1.4 (check it!).

This algebraic group is denoted by $\mathbf{G}_{\mathbf{a},k}^n$ and is called the *vector group* of dimension n over the field k. If $n = 1$, this is called the *additive group* over k. It is denoted by $\mathbf{G}_{\mathbf{a},k}$.

3. Let G be the open subvariety of affine space $\mathbf{A}_k^n$ whose complement is the closed subvariety given by the equation $Z_1 \cdot \ldots \cdot Z_n = 0$. It is isomorphic to the closed subvariety of $\mathbf{A}_k^{2n}$ given by the equations $Z_i Z_{n+i} - 1 = 0$, $i = 1, \dots, n$. Its coordinate algebra is equal to the algebra of Laurent polynomials $k[Z, Z^{-1}] := k[Z_1, Z_1^{-1}, \dots, Z_n, Z_n^{-1}]$. We define the group law via the coaction homomorphism:

$$k[Z, Z^{-1}] \to k[Z, Z^{-1}] \otimes k[Z, Z^{-1}],$$

$$Z_i \mapsto Z_i \otimes Z_i, \; i = 1, \dots, n.$$

The inversion map

$${}^a\beta \colon k[Z, Z^{-1}] \to k[Z, Z^{-1}]$$

is given by

$$Z_i \mapsto Z_i^{-1}, \quad i = 1, \dots, n.$$

The unit element is $e = (1, \dots, 1) \in G(k) = (k^*)^n$. The group law on the sets $G(K) = (K^*)^n$ is coordinatewise multiplication. This algebraic group is denoted by $\mathbf{G}_{\mathbf{m},k}^n$ and is called the *algebraic torus* of dimension n over the field k . If $n = 1$, this is called the *multiplicative group* over k and is denoted by $\mathbf{G}_{\mathbf{m},k}$.

We use the name "torus" for two reasons. Assume $k = \mathbf{C}$, the field of complex numbers. Then $G(\mathbf{C}) = (\mathbf{C}^*)^n$. Each copy of $\mathbf{C}^*$ is the image of $\mathbf{C}$ with respect to the exponential map $z \mapsto exp(2\pi i z)$. The kernel of this map is the group of integers. Thus $\mathbf{C}^* \cong \mathbf{C}/\mathbf{Z}$. If we replace $\mathbf{C}$ by $\mathbf{R}$ we obtain a circle. Thus $\mathbf{C}^*$ is a sort of "complex circle", and $(\mathbf{C}^*)^n$ is a "complex torus". On the other hand, as an algebraic variety $\mathbf{G}_{\mathbf{m},k}$ is isomorphic to the affine variety defined by the equation $X^2 + Y^2 - 1 = 0$ (this is true over any field k of characteristic different from 2 and containing the square root of -1). Thus $\mathbf{G}_{\mathbf{m},k}^n \cong (\mathbf{G}_{\mathbf{m},k})^n$ is isomorphic to the product of n "complex circles", i.e. it is again can be viewed as a "complex torus".

4. Let G be the open subvariety of the affine space $\mathbf{A}_k^{n^2}$ whose complement is the closed subvariety given by the equation $det((Z_{ij})_{1 \le i,j \le n}) = 0$. Here we put the variables $Z_1, \dots, Z_{n^2}$

into a square matrix and reindex them. It is isomorphic to a closed subvariety of $\mathbf{A}_k^{n^2+1}$ given by the equation: $det((Z_{ij}))Z_{n^2+1} - 1 = 0$. Its coordinate algebra is isomorphic to $k[Z_{11}, \ldots, Z_{nn}][det((Z_{ij}))^{-1}]$. We define the group law via the coaction homomorphism given by the formula:

$$Z_{ij} \mapsto \sum_{t=1}^{n} Z_{it} \otimes Z_{tj}, \quad i,j = 1, \ldots, n.$$

It is easy to see that the set of K-points of G is identified with the set $GL(n, K)$ of invertible $n \times n$-matrices with entries in K. The group law on the sets $G(K) = GL(n, K)$ is the ordinary matrix multiplication. This algebraic group is denoted by $\mathbf{GL}_k(n)$ and is called the *general linear group* over the field k. Obviously $\mathbf{GL}_k(1) = \mathbf{G}_{\mathbf{m},k}$.

1.8 Algebraic groups over a field k form a category with morphisms taken to be homomorphisms of algebraic groups

Definition. A *homomorphism* $f: G \to G'$ of algebraic groups is a morphism of algebraic varieties such that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ \downarrow f \times f & & \downarrow f \\ G' \times G' & \xrightarrow{\mu'} & G' \end{array}$$

is commutative.

We shall denote the category of algebraic groups over k by $\mathbf{Gr}_k$.

Every algebraic k-group G defines a functor h_G from the category of k-algebras to the category of groups by sending each K to $G(K)$. It is easy to check, by using the Yoneda Lemma (see the beginning of the last Lecture), that any natural transformation (or morphism of functors) from h_G to $h_{G'}$ arises from the unique homomorphism $G \to G'$. In particular, the map $G \to G'$ is an embedding if and only if $h_G \to h_{G'}$ is injective morphism of functors.

The role of subobjects in $\mathbf{Gr}_k$ is played by subgroups:

Definition. A *subgroup* H of an algebraic group G is an algebraic subvariety of G which is an algebraic group such that the canonical inclusion morphism $H \hookrightarrow G$ is a homomorphism of algebraic groups. A subgroup is called *closed* (resp. *open*) if it is a closed (resp. open) subvariety.

1.9 Let $f: X \to Y$ be a morphism of algebraic varieties over a field k. The scheme-theoretical image of f is the closed subvariety Z of X such that f factors through Z, and where Z is minimal with respect to this property. If X is viewed as an algebraic k-set, then Z is just the closure of $f(X(\bar{k}))$. If X is viewed as an algebraic scheme over k and is reduced then Z is the closure of $f(X)$. The image of f is just a subset of Y with the induced topology; it may not be a subvariety of Y.

Theorem. *Let $f: G \to G'$ be a homomorphism of algebraic groups. Then*

*(i) the set-theoretical (or reduced if one uses the language of schemes) fibre of f over $e \in G'(k)$ is a closed subgroup of G (called the **kernel** of f and denoted by* Ker(f)*).*

(ii) the image of f exists (denoted by $\mathrm{Im}(f)$), it coincides with the scheme-theoretical image and is a subgroup of G.

Proof. The first assertion is obvious. Fibres are always closed, and the restriction of the group law to the fibre is obviously a group law. This easily follows from considering the diagrams from (1.1). The second assertion is less trivial. First we consider a subset $H = f(G(\bar{k}))$. It is a subgroup of $G'(\bar{k})$. Let $\bar{H}$ be the Zariski closure of H. It is a subgroup of $G(\bar{k})$. Indeed, for any $x \in H$, the set $x\bar{H}$ is the image of the set $\bar{H}$ under the bijective algebraic map given by the left translation. So its closure $\overline{xH}$ is equal to the image of the closure $x\bar{H}$ which is $\bar{H}$. This gives $H\bar{H} = \bar{H}$. If $y \in \bar{H}$, then $Hy \subset \bar{H}$ implies $\bar{H}y = \overline{Hy} \subset \bar{H}$. Thus $\bar{H}\bar{H} \subset \bar{H}$. Also, $\bar{H}^{-1} := \beta(H)$ equals to $\overline{H^{-1}} = \bar{H}$ because $\beta : G(\bar{k}) \to G(\bar{k})$ is a homeomorphism. Now we use that H contains a dense open subset U of $\bar{H}$. This follows from the Chevalley theorem which asserts that the image of an algebraic set is a constructible subset (a finite union of locally closed subsets). For any $x \in \bar{H}$, $xU^{-1} \cap U \neq \emptyset$ since the intersection of two dense open subsets is always non-empty. This allows us to write x as the product of two elements from H and hence to get $\bar{H} = HH$. Since $\bar{H}$ is a subgroup, we get $\bar{H} = H$.

1.10 A closed subgroup of $\mathbf{GL}_k(n)$ is called a *linear k-group*. A homomorphism $f: G \to GL_k(n)$ is called an n-dimensional linear (rational) representation of G. It is called *faithful* if it is an embedding of algebraic varieties. By (1.9) the image of a faithful linear representation is a linear group. Obviously, a linear group is affine. We shall prove that any affine algebraic k-group is a linear k-group. The idea is the same as for abstract groups. One should consider the representation of G on its space of functions induced by left translations. Suppose G is affine. Then for any $g \in G(K)$ the coaction ${}^a\mu: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ defines a map

$$t_g^*: \mathcal{O}(G) \otimes K \to \mathcal{O}(G) \otimes K.$$

This is obtained by K-linear extension of the map $(g \otimes id) \circ {}^a\mu$. If we view the homomorphism $g : \mathcal{O}(G) \to K$ as evaluation of a function at the point $g \in G(K)$ then for any $x \in G(K)$

$$t_g^*(\phi)(x) = \phi(gx).$$

Lemma. *For any $\phi \in \mathcal{O}(G) \otimes K$ the submodule of $\mathcal{O}(G) \otimes K$ spanned by the functions ${}^g\phi := t_g(\phi), g \in G(K)$, is finitely generated.*

Proof. Let us denote $\mathcal{O}(G) \otimes K$ by $\mathcal{O}(G)_K$. Tensoring the map ${}^a\mu$ with K we have a homomorphism

$${}^a\mu_K : \mathcal{O}(G)_K \to \mathcal{O}(G)_K \otimes \mathcal{O}(G)_K.$$

For any $\phi \in \mathcal{O}(G)_K$, we can write

$${}^a\mu_K(\phi) = \sum_i \phi_i \otimes \varphi_i,$$

where $\phi_i, \varphi_i \in \mathcal{O}(G)_K$. It follows from the definition that

$${}^g\phi = \sum_i \phi_i(g)\varphi_i.$$

This implies that all the translates ${}^g\phi$ belong to the subspace of $\mathcal{O}(G)_K$ spanned by finitely many elements φ_i. This proves the assertion.

Theorem. *Every affine algebraic k-group is isomorphic to a linear k-group.*

Proof. Choose some generators $\phi_1, \ldots, \phi_N$ of the k-algebra $\mathcal{O}(G)$. Let V_K be the linear subspace of $\mathcal{O}(G)$ spanned (over K) by these generators and their translates ${}^g\phi_i, g \in G(K)$. By the previous lemma this is a finite-dimensional G-invariant subspace of $\mathcal{O}(G)_K$. For any $g \in G(K)$ we have a map

$$\rho_K(g): V_K \to V_K, \phi \mapsto {}^{g^{-1}}\phi.$$

It is easy to check that the map $g \mapsto \rho_K(g)$ is a homomorphism from $G(K)$ to the group $GL(V_K)$ of linear automorphisms of the linear K-space V_K. It follows from the proof of the previous lemma that $V_K = V \otimes_k K$ is obtained from $V := V_k$ by extension of scalars. By choosing a basis of n elements in V, for any K/k, we obtain a homomorphism

$$\rho_K: G(K) \to GL(n, K).$$

The set of homomorphisms $\{\rho_K\}$ is compatible in the following way. If $K \to K'$ is a homomorphism of k-algebras, we have the commutative diagrams:

$$\begin{array}{ccc} G(K) & \xrightarrow{\rho_K} & GL(n,K) \\ \downarrow & & \downarrow \\ G(K') & \xrightarrow{\rho'_K} & GL(n,K'). \end{array}$$

In fact we may assume here that K is an arbitrary k-algebra. This shows that we have a homomorphism of algebraic k-groups

$$\rho: G \to \mathbf{GL}_k(n).$$

For any K the homomorphism ρ_K is injective. Indeed, if $g \in \mathrm{Ker}(\rho_K)$, then the left translation map

$$t_g : G(K) \to G(K), x \mapsto gx,$$

induces the identity map on $\mathcal{O}(G)_K$. This implies that t_g is the identity map of $G(K)$, hence $g = e$. Thus the map ρ is an embedding. We finish the assertion by applying the Theorem from (1.9).

Problems.

1. For any abstract finite group G construct an algebraic k-group such that for any field extension K/k its set of K-points is equal to G.
2*. Show that the scheme-theoretical fibre over e of a homomorphism $f: G \to G'$ is a group scheme over k. Give an example when it is not an algebraic group (i.e. the fibre is not reduced).
3. Prove that any algebraic group is a nonsingular algebraic variety.
4. Define the product of algebraic groups and verify that $\mathbf{G_m}_{,k}^n \cong (\mathbf{G}_{\mathbf{m},k})^n, \mathbf{G_a}_{,k}^n \cong (\mathbf{G}_{\mathbf{a},k})^n$.
5. Prove that $Aut_{Gr_k}(\mathbf{G}_{\mathbf{a},k}) \cong k^*$.
6. Let $\mathbf{SL}_k(n)$ be the closed subvariety of $\mathbf{GL}_k(n)$ defined by the equation $det((Z_{ij}) = 1$. Show that $\mathbf{SL}_k(n)$ is an algebraic group over k (*the special linear algebraic group over the field* k), and for any k-algebra K, $\mathbf{SL}_k(n)(K) = SL(n,K) := \{A \in GL(n,K) : det(A) = 1\}$.
7. Show that there are no non-trivial homomorphisms from $\mathbf{G}_{\mathbf{m},k}$ to $\mathbf{G}_{\mathbf{a},k}$ and in the other direction too.
8. Let k'/k be a finite extension of a field k, and let G' be an affine algebraic group over the field k'. Show that there exists an affine algebraic group G over the field k such that for any k-algebra K, $G(K) = G'(k' \otimes_k K)$. Find the coordinate algebra $\mathcal{O}(G)$ when $k' = \mathbf{C}, k = \mathbf{R}, G' = \mathbf{G}_{\mathbf{m},k'}$.

(See also exercises on p.57 and p.63 of **[Hum]**).

Lecture 2. ALGEBRAIC GROUP ACTIONS

2.1 Let G be an algebraic k-group and X be an algebraic k-variety.

Definition. An *algebraic action* of G on X is a morphism of algebraic k-varieties

$$\sigma : G \times X \to X$$

satisfying the following properties:

(i) the diagram

$$\begin{array}{ccc} G \times G \times X & \stackrel{\mu \times id}{\longrightarrow} & G \times X \\ \downarrow id \times \sigma & & \downarrow \sigma \\ G \times X & \stackrel{\sigma}{\longrightarrow} & X \end{array}$$

is commutative;

(ii) the composition

$$X \cong pt_k \times X \stackrel{e \times id}{\longrightarrow} G \times X \stackrel{\sigma}{\longrightarrow} X$$

is the identity morphism.

A pair (X, σ), where $\sigma : G \times X \to X$ is an algebraic action is called a G-variety.

It is clear that for any k-algebra K the action morphism $\sigma : G \times X \to X$ defines the action of the group $G(K)$ on the set $X(K)$. We shall denote it by $(g, x) \mapsto g \cdot x$. The group $G(k)$ acts on all sets $X(K)$ in a compatible way, hence the action defines a homomorphism of (abstract) groups

$$G(k) \to Aut_{Var/k}(X).$$

If G and X are both affine, one can define the action morphism $\sigma : G \times X \to X$ in terms of the *coaction* homomorphism

$$\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X).$$

Its composition with the homomorphism $e \otimes id : \mathcal{O}(G) \otimes \mathcal{O}(X) \to k \otimes \mathcal{O}(X) = \mathcal{O}(X)$ must be the identity.

2.2 For any $g \in G(K)$ the image of a function ϕ under the composition

$$(g \otimes id) \circ \sigma^* : \mathcal{O}(X) \to K \otimes \mathcal{O}(X)$$

is denoted by $g^*(\phi)$. In this way $G(K)$ acts on the K-algebra $K \otimes \mathcal{O}(X)$ by automorphisms of k-algebras. It is clear that for any $g \in G(K), x \in X(K)$,

$$g^*(\phi)(x) = \phi(g \cdot x).$$

We have an analog of the Lemma from Lecture 1, 1.10. Its proof is similar and is left to the reader.

Lemma. *For any $\phi \in \mathcal{O}(X)$ the submodule of $\mathcal{O}(X) \otimes K$ spanned by the functions $g^*(\phi), g \in G(K)$, is spanned by a finite subset from $\mathcal{O}(X)$ (independent of K).*

A function $\phi \in \mathcal{O}(X)$ is called *G-invariant* if

$$\sigma^*(\phi) = 1 \otimes \phi.$$

This of course implies that $\phi(g \cdot x) = \phi(x)$ for any $g \in G(K), x \in X(K)$. If one views a regular function as a morphism $f : X \to \mathbf{A}_k^1$, then the G-invariance of ϕ can be expressed by saying that the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow pr_2 & & \downarrow \phi \\ X & \xrightarrow{\phi} & \mathbf{A}_k^1. \end{array}$$

We shall denote the subset of G-invariant functions by $\mathcal{O}(X)^G$. It is obviously a subalgebra of $\mathcal{O}(X)$.

2.3 Examples. 1. $\sigma : G \times X \to X$ is the second projection map. This action is called *trivial.* If G, X are affine varieties, then the corresponding coaction map is $\phi \mapsto 1 \otimes \phi$.
2. The group law $\mu : G \times G \to G$ is an action of G on itself. This is called the *left translation* action.
3. If $\rho: G \to \mathbf{GL}_k(n)$ is a linear representation, it defines an action of G on the affine space $\mathbf{A}_k^n$ as follows. For any k-algebra K the map $\rho_K : G(K) \to GL(n, K)$ defines the map $\sigma_K : G(K) \times \mathbf{A}_k^n(K) \to \mathbf{A}_k^n(K)$. The set of such maps gives rise to a morphism of functors $h_G \times h_{\mathbf{A}_k^n} = h_{G\times\mathbf{A}_k^n} \to h_{\mathbf{A}_k^n}$. By the Yoneda lemma this defines a morphism $\sigma : G \times \mathbf{A}_k^n \to \mathbf{A}_k^n$.

Now we can generate a lot of concrete examples of linear representations. Here is one which we shall often use.
4. Let $G = \mathbf{G}_{\mathbf{m},k}$, and let $q_1, \ldots, q_n$ be some integers. Define the action of $\mathbf{G}_{\mathbf{m},k}(K) = K^*$ on $K^n = \mathbf{A}_k^n(K)$ by the formula:

$$t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n).$$

This defines a linear action of G on $\mathbf{A}_k^n$. More generally, we may take $G = (\mathbf{G}_{\mathbf{m},k})^r, \mathbf{q}_i \in \mathbf{Z}^r, i = 1, \ldots, n$ and define the action of G on $\mathbf{A}_k^n$ by the formula:

$$\mathbf{t} \cdot (z_1, \ldots, z_n) = (\mathbf{t}^{\mathbf{q}_1} z_1, \ldots, \mathbf{t}^{\mathbf{q}_n} z_n),$$

where for any $\mathbf{t} = (t_1, \ldots, t_r)$ and $\mathbf{q} = (q_1, \ldots, q_r) \in \mathbf{Z}^r$ we set

$$\mathbf{t}^{\mathbf{q}} = t_1^{q_1} \cdot \ldots \cdot t_r^{q_r}.$$

This is called a *diagonal* action of the torus $(\mathbf{G}_{\mathbf{m},k})^r$ on the affine space $\mathbf{A}_k^n$.
5. Let $G = \mathbf{G}_{\mathbf{m},k}$ act on an affine k-variety X. The coaction is given by a homomorphism:

$$\sigma^* : \mathcal{O}(X) \to k[Z, Z^{-1}] \otimes \mathcal{O}(X).$$

For any $\phi \in \mathcal{O}(X)$ we can write:

$$\sigma^*(\phi) = \sum_{i \in \mathbf{Z}} Z^i \otimes \phi_i.$$

where $\phi_i \in \mathcal{O}(X)$.

Let $p_i : \mathcal{O}(X) \to \mathcal{O}(X)$ be the k-linear map $\phi \mapsto \phi_i$, $\mathcal{O}(X)_i := p_i(\mathcal{O}(X))$. We claim that

$$\mathcal{O}(X) = \oplus_{i \in \mathbf{Z}} \mathcal{O}(X)_i, \quad \mathcal{O}(X)_i \cdot \mathcal{O}(X)_j \subset \mathcal{O}(X)_{i+j}.$$

In other words, the action of $\mathbf{G}_{\mathbf{m},k}$ on X defines a $\mathbf{Z}$-grading on the k-algebra $\mathcal{O}(X)$. To check this we use the two axioms of the action. The second axiom tells us that

$$\phi \mapsto \sum_i Z^i \otimes \phi_i \mapsto \sum_i 1 \otimes \phi_i \mapsto \sum_i \phi_i = \phi.$$

This says that $\mathcal{O}(X) = \sum_i \mathcal{O}(X)$. The associativity axiom gives

$$\sum_i Z^i \otimes (\sum_j Z^j \otimes p_j(\phi_i)) = \sum_i Z^i \otimes Z^i \otimes \phi_i.$$

After comparing the coefficients at each Z^i, we find that $\sigma^*(\phi_i) = Z^i \otimes \phi_i$. This shows that the linear maps p_i's are projection operators (i.e., $p_i^2 = p_i$). This immediately implies that the sum is direct. Since σ^* is a homomorphism, we get $\mathcal{O}(X)_i \cdot \mathcal{O}(X)_j \subset \mathcal{O}(X)_{i+j}$.

Conversely, given a $\mathbf{Z}$-grading of $\mathcal{O}(X)$, we define the coaction map σ^* by the formula

$$\sigma^*(\phi) = \sum_i Z^i \otimes \phi_i,$$

where $\phi = \sum_i \phi_i, \phi_i \in \mathcal{O}(X)_i$. It is easy to see that in this way we get a bijection

$$\{\mathbf{G}_{\mathbf{m},k}\text{-actions on } X\} \longleftrightarrow \{\mathbf{Z}\text{-grading of } \mathcal{O}(X)\}.$$

Note that elements of $\mathcal{O}(X)_i$ are characterized by the condition

$$\sigma^*(\phi) = Z^i \otimes \phi.$$

This can be interpreted as follows. Any $g \in G(K)$ defines a homomorphism $\mathcal{O}(G) = k[Z, Z^{-1}] \to K$ which is determined by the image of Z. Let us denote this image by t. Then $\phi \in \mathcal{O}(X)_i$ if and only if for any K/k and $g \in G(K)$

$$g^*(\phi) = t^i \phi.$$

Note that the problem of description of all possible gradings on $\mathcal{O}(X)$ is very difficult even in the simplest case when $X = \mathbf{A}_k^n$ with $\mathcal{O}(X) = k[Z] = k[Z_1, \ldots, Z_n]$. We have described already some actions on this algebra in example 4. In the corresponding grading

$$k[Z]_i = \{P(Z) : P(t^{q_1} Z_1, \ldots, t^{q_n} Z_n) = t^i P(Z_1, \ldots, Z_n),\ \forall K/k, \forall t \in K^*\}.$$

Polynomials from $k[Z]_i$ are called *quasi-homogeneous* polynomials of degree i and weights $q_1, \ldots, q_n$. When $q_1 = \ldots = q_n = 1$, we obtain the standard grading of the ring of polynomials, and the standard notion of a homogeneous polynomial. The problem whether it is true that any grading is obtained from this after applying some automorphism of $k[Z]$ is still open for $n \geq 3$.
6. Let $\mathbf{PGL}_k(n)$ be the algebraic group defined by

$$\mathbf{PGL}_k(n)(K) = PGL(n, K) := GL(n, K)/K^*.$$

As an algebraic variety, $\mathbf{PGL}_k(n)$ is isomorphic to the open subset of the projective space $\mathbf{P}_k^{n^2-1}$ whose complement is the determinantal hypersurface

$$det((T_{ij})_{1 \leq i,j \leq n}) = 0.$$

It is well-known that the complement of a hypersurface in a projective space is an affine algebraic variety (for the proof use the Veronese mapping).

There is a canonical homomorphism $\mathbf{GL}_k(n) \to \mathbf{PGL}_k(n)$ whose kernel is equal to $\mathbf{G}_{\mathbf{m},k}$. Given a linear representation $G \to \mathbf{GL}_k(n)$, by composing, it defines a homomorphism $G \to \mathbf{PGL}_k(n)$. Any such homomorphism is called a *projective representation* of G. It defines an action of G on a projective space $\mathbf{P}_k^{n-1}$. Recall that for any k-algebra K

$$\mathbf{P}_k^{n-1}(K) = \mathbf{P}^{n-1}(K) := \{\text{direct summands of rank 1 in } K^n\}.$$

The action of $PGL(n, K)$ on this set is given via the natural action of $GL(n, K)$ on K^n. Now if X is any quasi-projective subvariety of $\mathbf{P}_k^{n-1}$, G may act on X via its action on $\mathbf{P}_k^{n-1}$ provided that X is G-invariant. This means that the subsets $X(K) \subset \mathbf{P}^{n-1}(K)$ are $G(K)$-invariant subsets. Such an action is called a *projective action.* If the projective representation of G arises from a linear representation (it is not always so), then we say that the action is *linear.* Later on we shall learn how to "linearize" any action.

2.4 A *morphism* between two actions (or between two G-varieties) $\sigma : G \times X \to X$ and $\sigma' : G \times X' \to X'$ is a pair (α, f), where $\alpha : G \to G'$ is a homomorphism of algebraic k-groups, $f : X \to X'$ is a morphism of algebraic k-varieties such that the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \alpha \times f \downarrow & & \downarrow f \\ G' \times X' & \xrightarrow{\sigma'} & X' \end{array}$$

is commutative.

In the case where $G = G'$ and α is the identity, we say that $f : X \to X'$ is a *G-equivariant morphism.* Furthermore, if the action of G on the target space X' is trivial, we say that f is a *G-invariant morphism.* If X is a subvariety of X', and the natural embedding $f : X \to X'$ is G-equivariant, we say that G acts on X via its *induced action*, or the action of G on X is obtained by the *restriction* from the action on X'. Of course this happens if and only if each $X(K)$ is a $G(K)$-invariant subset of $X'(K)$. This can be expressed by saying that X is a G-invariant subvariety of X'.

2.5 Let $\sigma : G \times X \to X$ be an action. It defines a morphism

$$\Psi := (\sigma, pr_2) : G \times X \to X \times X, (g, x) \mapsto (g \cdot x, x).$$

Let $\Delta \subset X \times X$ be the diagonal, and let $\mathcal{S}$ be its pre-image under Ψ. The second projection $pr_2 : G \times X \to X$ induces a morphism $p : \mathcal{S} \to X$. For each point $x \in X(K)$, we have

$$p_K^{-1}(x) = \{(g, x) \in G(K) \times X(K) : g \cdot x = x\}.$$

Under the first projection $(g, x) \mapsto g$ this is mapped bijectively to the stabilizer subgroup $G(K)_x$ of $G(K)$. If $x \in X(k)$ then we can define the fibre $p^{-1}(x)$ as a closed subvariety of $G \times X$ which is isomorphic to a closed subvariety of G. Its reduced structure (or the corresponding algebraic subset of $G(\bar{k})$) is a closed subgroup of G. It is called the *stabilizer subgroup* of the point x and is denoted by G_x. (The scheme-theoretical isotropy subgroup is the scheme-theoretical fibre. It is a group subscheme of G).

Definition. An action $\sigma : G \times X \to X$ is called *free* (resp.*set-theoretically free*) if Ψ is a closed embedding. (resp. all stabilizer subgroups are trivial).

Proposition.
(i) A free action is set-theoretically free.
(ii) $\sigma : G \times X \to X$ is set-theoretically free if and only if the action of the group $G(\bar{k})$ on the set $X(\bar{k})$ is free (i.e. all stabilizer subgroups are trivial).

Proof. (ii) obvious.
(i) Since Ψ is an embedding, for any $x \in X(\bar{k})$, its fibre over a point (x, x) is a point. This implies that $G(\bar{k})_x$ is trivial. By (ii), the action is set-theoretically free.

Example. The following example taken from **[Mu1]** shows that a set-theoretical free action is not free in general. Let $k = \mathbf{C}, G = \mathbf{SL}_k(2)$. Let $V_n = k[Z_1, Z_2]_n$ be the space of homogeneous polynomials of degree n. The group $SL(2, k)$ acts naturally on this space by acting linearly on the variables. Using this we can easily define the algebraic action of the group G on the 7-dimensional affine space X isomorphic to $V_1 \times V_4$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (F(Z_1, Z_2), G(Z_1, Z_2)) = (F(dZ_1 - bZ_2, -cZ_1 + aZ_2), G(dZ_1 - bZ_2, -cZ_1 + aZ_2)).$$

Let Z be a closed subset of $G(k) \times X(k)$ which consists of points

$$(\begin{pmatrix} 0 & -t \\ t^{-1} & 0 \end{pmatrix}, (tZ_1 + Z_2, Z_1^2 Z_2^2)), t \in k^*.$$

The image of Z under the map Ψ is the set of points

$$(tZ_1 - Z_2, Z_1^2 Z_2^2), (tZ_1 + Z_2, Z_1^2 Z_2^2)).$$

In its closure we find the point $((-Z_2, Z_1^2 Z_2^2), ((Z_2, Z_1^2 Z_2^2))$ which does not belong to the image. This shows that Ψ is not closed. Now let us restrict the action to the subvariety X'

formed by the pairs (F, G) such that $F \neq 0, G = F_2^2$, where F_2 is a quadratic polynomial with discriminant 1. Since the stabilizer subgroup of a linear form is conjugate to the group of unipotent matrices

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

and the stabilizer group of F_2^2 is conjugate to the group of orthogonal matrices, we easily see that G acts on X' with trivial stabilizers. Obviously, Z is closed in $G \times X'$. Thus the action is set-theoretically free but not free.

2.6 For any $x \in X(K)$ we have a map $\sigma_x(K) : G(K) \to X(K), g \mapsto g \cdot x$. The image of this map is called the *orbit* of x and is denoted by $G(K) \cdot x$ or $O(x)_K$. If $x \in G(k)$ we have a map of varieties $\sigma_x : G \to X$. Let $O(x)$ be the image of this map (considered as a subset of the algebraic k-set $X(\bar{k})$ or as a subset of the scheme X).

Proposition. *$O(x)$ is open in its closure $\overline{O(x)}$. In particular, it is an algebraic k-subvariety of X (or a locally closed subscheme of X). If G is irreducible, $O(x)$ is an irreducible set of dimension equal to* $\dim G - \dim G_x$.

Proof. Let $\overline{O(x)}$ be the scheme-theoretical image of the map σ_x (considered as a closed k-subset of $X(\bar{k})$ or as a closed reduced subscheme of X). By Chevalley' theorem the image $O(x)$ is a constructible subset in $\overline{O(x)}$ hence contains a dense open subset of the latter. However, the group $G(\bar{k})$ acts transitively on the set $O(x)$. Hence this set is the union of open subsets of $\overline{O(x)}$, hence is open in it. This shows that $O(x)$ is a locally closed subset of X, hence is an algebraic k-set. If G is irreducible, $O(x)$ is irreducible since the image of an irreducible set is irreducible. The assertion about the dimension follows from the theorem on dimension of fibres. In fact, all the fibres of the map σ_x have the same dimension. In fact if $y = g \cdot x = \sigma_x(g) \in O(x)$, then $\sigma_x^{-1}(y) = gG_xg^{-1} \cong G_x$.

An orbit $O(x)$ is called *closed* if it is a closed subset of X. The set of closed orbits is always non-empty. In fact, if $O(x)$ is not closed we take a point in $y \in \overline{O(x)} \setminus O(x)$ and consider its orbit $O(y)$. Its dimension is strictly less than $\dim O(x) = \dim \overline{O(x)}$. If $O(y)$ is not closed, we continue this argument until either we find a closed orbit, or we come to an orbit of dimension 0; however, any subset of dimension 0 is closed.

2.7 Definition. An action $\sigma : G \times X \to X$ is called *transitive* if the map $\Psi : G \times X \to X \times X$ is surjective. We say that X is a *homogeneous space* over G. If furthermore Ψ is an isomorphism, we say that X is a prinicpal homogeneous space over G (or a G-torsor).

Proposition. *A group G is a principal homogeneous space over itself with respect to the left translations. If X is a principal homogeneous space over G and $X(k) \neq \emptyset$, then there exists a G-equivariant isomorphism $G \to X$.*

Proof. The first assertion is obvious. The map $\beta \times id : G \times G \to G \times G$ is the inverse of the map $\Psi = (\mu, id)$. Assume $\Psi : G \times X \to X \times X$ is a k-isomorphism. Consider the second projection of the target and the source of the map. Then Ψ commutes with these projections. Thus for any point $x \in X(\bar{k})$ it defines an isomorphism of the fibres as varieties over $\bar{k}$. If $x \in X(k)$ then these fibres are defined over k and Ψ induces an isomorphism. But the fibre of $pr_2 : G \times X \to X$ over such a point x is isomorphic to G, and the fibre over x of $pr_2 : X \times X \to X$ is isomorphic to X. Let $\alpha : G \to X$ be this isomorphism. It is easy to see that it is G-equivariant with respect to the left translation action of G on itself. Moreover we get $\alpha(e) = x$ and can define the group law μ' on X by setting $\mu' = \alpha \circ \mu \circ (\alpha^{-1} \times \alpha^{-1})$. The unit element is x, and the inverse map is $\alpha \circ \beta \circ \alpha^{-1}$. Now α becomes an isomorphism of algebraic groups.

We say that X is a *non-trivial* principal homogeneous space if it is not isomorphic (as a G-variety) to G.

Remark. Assume k is perfect. Let $\bar{k}_s$ be the algebraic closure of k. One can show that there is a bijective correspondence between the set of isomorphism classes of G-torsors and the set of 1-cohomology $H^1(Gal(\bar{k}_s/k), G(\bar{k}_s))$ (see for example **[Ser]**).

2.8 An example of a homogeneous space is the orbit $O(x)$ of any point $x \in X(k)$ under an action $\sigma : G \times X \to X$. Conversely, any homogeneous space is the orbit of any of its k-points. If $x \in X(k)$ then $X(K)$ can be identified with the set of right cosets $gG_x(K)$ of $G(K)$ with respect to the subgroup $G_x(K)$. This can be expressed by saying that $X = G/G_x$ is the algebraic variety with its functor of K-points equal to $K \to G(K)/G_x(K)$. In fact for any closed subgroup H of an affine k-group G one can define a homogeneous space X and a point $x \in X(k)$ such that $G_x = H$. This homogeneous space is denoted by G/H. Here is its construction.

Let $I \subset \mathcal{O}(G)$ be the ideal of regular functions on G which vanish on H. Let $\phi_1, \ldots, \phi_N$ be its generators. By the Lemma from Lecture 1 (1.10), the k-subspace V spanned by the ϕ_i and its g-translates, $g \in G(k)$ is finite-dimensional. Let $W = V \cap I$ considered as a k-subspace of V. Let X be the Grassmann variety parametrizing subspaces of dimension $n = \dim W$ in V. Recall that by considering the exterior power $\bigwedge^n V$ we can consider X as a closed subvariety of the projective space $\mathbf{P}^d_k$, where $d = \binom{\dim V}{n} - 1$. The group $G(K)$ acts on X by sending a subspace L of $V_K = V \otimes K$ to $g(L)$. This defines an algebraic action of G on X. We can view W as a point $x \in X(k)$. Let $H' \subset G$ be its stabilizer subgroup. We have to show that $H' = H$. It is enough to show that $H'(\bar{k}) = H(\bar{k})$. If $g \in H(\bar{k})$ then for any $h \in H(\bar{k})$, $gh \in H(\bar{k})$ and

$$^g\phi(h) = \phi(gh) = 0.$$

This shows that $g(I) \subset I_{\bar{k}} = I \otimes \bar{k}$ hence $g(W_{\bar{k}}) = W_{\bar{k}}$, i.e. $H(\bar{k}) \subset H'(\bar{k})$. Conversely, if $g \in H'(\bar{k})$, then $g(W) \subset W_{\bar{k}}$, and, in particular, for any $\phi_i \in W$,

$$\phi_i(g) = {}^g\phi_i(e) = 0.$$

This shows that the generators of I vanish at g, and hence $g \in H(\bar{k})$.

We shall show later that the homogeneous space G/H is a geometric quotient of G with respect to the left translation action of H on G.

Problems.

1. Let H be a closed subgroup of an affine algebraic group G. We say that H is normal if for any $g \in G(K), gH(K)g^{-1} \subset H(K)$. Prove that the homogeneous space G/H for a normal H is an affine algebraic group. [Hint: Consider some linear representation of G].
2. Classify all possible actions of $\mathbf{G}_{\mathbf{a},k}$ on $\mathbf{A}^1_k$.
3. Let $\mathbf{SO}_k(n)$ be the subgroup of $\mathbf{SL}_k(n)$ defined by $\mathbf{SO}_k(n)(K) = \{A \in SL(n,K) : {}^tAA = I_n\}$. Construct a non-trivial torsor over the group $\mathbf{SO}_{\mathbf{R}}(2)$.
4. Show that any torsor of the group $\mathbf{G}_{\mathbf{a},k}$ is trivial if $\mathrm{char}(k) = 0$.
5. An action $\sigma : G \times X \to X$ is called *closed* if all orbits are closed. Show that the action is closed if all stabilizer subgroups are of the same dimension.
6. An action is called *proper* if Ψ is proper. Show that all stabilizer subgroups for a proper action of an affine group are finite. Show that the converse is not true.
7. A G-variety is called an *almost homogeneous* if it contains an open orbit whose complement is a finite set of points. Give an example of a projective almost homogeneous G-variety which is not a homogeneous space.
8. Let $f : X \to Y$ be a G-equivariant morphism of homogeneous spaces over an algebraic group G. Show that f is an open map (i.e. the image of an open set is an open set).

Lecture 3. LINEARIZATIONS OF ACTIONS

Here we shall show, as has been promised, that any algebraic action can be induced from a linear action on a projective space. First we need to remind the reader of the general notions involving line bundles and linear systems.

3.1 A *line bundle* over an algebraic k-variety X is a data consisting of a morphism of varieties $\pi : L \to X$ together with an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X and a set $\beta_{\mathcal{U}}$ of isomorphisms $\beta_i, i \in I$, over U_i

$$\beta_i : \pi^{-1}(U_i) \to U_i \times \mathbf{A}_k^1,$$

where the product is considered to be over U_i by means of its first projection. This data must satisfy the following properties:
(i) for any $i, j \in I$, the automorphism of $(U_i \cap U_j) \times \mathbf{A}_k^1$ defined by the composition $(\beta_i | \pi^{-1}(U_i \cap U_j) \times \mathbf{A}_k^1) \circ (\beta_j^{-1} | (U_i \cap U_j) \times \mathbf{A}_k^1)$ is given by an invertible function $g_{ij} \in \mathcal{O}(U_i \cap U_j)^*$ so that for any $x \in (U_i \cap U_j)(K)$ and any $z \in \pi^{-1}(x)$

$$z_i = g_{ij}(x) z_j,$$

where $\beta_i(z) = (x, z_i), \beta_j(z) = (x, z_j)$.
(ii) the functions g_{ij} must satisfy the following conditions:
(a) $g_{ii} = 1$ for any $i \in I$,
(b) $g_{ij} = g_{ji}^{-1}$, for any $i, j \in I$,
(c) $g_{ij} = g_{ik} \cdot g_{kj}$ for any $i, j, k \in I$ after restriction to $U_i \cap U_j \cap U_k$.

The functions g_{ij} are called the *transition functions* of $\pi : L \to X$ with respect to the open cover $\mathcal{U}$. The variety L is called the *total space* of the line bundle, the variety X is called the *base*, and the morphism π is called the *projection.* The cover $\mathcal{U}$ is called the *trivializing cover*, and the isomorphisms β_i are *trivializing isomorphisms.*

Formally speaking we have to denote a line bundle by $(L, X, \pi, \mathcal{U}, \beta_{\mathcal{U}})$, however if no confusion arises we shall denote it by just L.

If $\mathcal{W}$ is an open subcover of an open cover $\mathcal{U}$ we can replace $(L, X, \pi, \mathcal{U}, \beta_{\mathcal{U}})$ by $(L, X, \pi, \mathcal{W}, \beta_{\mathcal{W}})$, where $\beta_{\mathcal{W}}$ is obtained by the restrictions. The corresponding line bundle is said to be *obtained by restriction of the cover.* Its transition functions are obviously obtained by restrictions.

The subvarieties $U_i \times \{0\} \subset U_i \times \mathbf{A}_k^1$ are glued together to form a closed subvariety of L which is called the *zero section.* Under the projection $\pi : L \to X$ it is mapped isomorphically onto X.

An *isomorphism* of line bundles is an isomorphism of their total spaces which commutes with the projections and sends the zero section to the zero section. It is clear that the fibres of the projection $\pi : L \to X$ over any geometric point $x \in X(K)$ (i.e when K is a field) are isomorphic to $\mathbf{A}^1_K$. An isomorphism of line bundles induces a linear isomorphism of the fibres.

3.2 Proposition. *Two line bundles $(L, X, \pi, \mathcal{U}, \beta_{\mathcal{U}})$ and $(L', X, \pi', \mathcal{U}', \beta_{\mathcal{U}'})$ are isomorphic if and only if there exists an open subcover $\mathcal{W} = \{W_i\}_{i\in I}$ of $\mathcal{U}$ and $\mathcal{U}'$, and invertible functions $\phi_i \in \mathcal{O}(W_i)^*$ such that the transition functions g_{ij} and g'_{ij} of the bundles L and L', after restriction to the cover $\mathcal{W}$, satisfy*

$$g_{ij} = \phi_i g'_{ij} \phi_j^{-1} \quad (*)$$

Proof. The condition is sufficient. In fact we define an isomorphism $f : L \to L'$ by the set of isomorphisms $f_i : L_i = \pi^{-1}(W_i) \to L'_i = \pi'^{-1}(W_i)$ by setting:

$$f_i = \beta_i^{-1} \circ \phi_i \circ \beta_i,$$

where ϕ_i is identified with an automorphism of $W_i \times \mathbf{A}^1_k$. The condition $(*)$ shows that the maps f_i and f_j coincide on $\pi^{-1}(W_i \cap W_j)$.

Now let us show that the condition is necessary. Let g_{ij} and g'_{ij} be the transition functions of L and L' with respect to some covers $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{V_j\}_{j\in J}$. We can choose a common subcover $\mathcal{W} = \{W_s\}_{s\in S}$ and restrict the transition functions to it, to assume that $\mathcal{U} = \mathcal{U}'$. Then the composition $\beta' \circ (f|\pi^{-1}(U_i)) \circ \beta_i^{-1}$ is an automorphism of $U_i \times \mathbf{A}^1_k$ which sends the zero section to the zero section. This immediately implies that it is given by a function $f_i \in \mathcal{O}(U_i)^*$. Comparing these two functions on the intersections $U_i \cap U_j$ we immediately see that condition $(*)$ must be satisfied.

3.3 One can construct a line bundle by starting from any open cover $\mathcal{U}$ and invertible functions g_{ij} on $U_i \cap U_j$ satisfying properties (ii) from 3.1. We do it by the gluing construction. We consider the disjoint union of varieties $U_i \times \mathbf{A}^1_k$ and define an equivalence relation by making points $(x, z) \in U_i \times \mathbf{A}^1_k$ and $(x', z') \in U_j \times \mathbf{A}^1_k$ equivalent if and only if $x = x'$ and $z = g_{ij}(x)z'$. The axioms of an equivalence relation are equivalent to the conditions (a),(b), and (c) of (ii) in 3.1, so the set of equivalence classes is equal to the set of points of some algebraic variety L. It comes with its natural projection π to X, and isomorphisms $\beta_i : \pi^{-1}(U_i) \to U_i \times \mathbf{A}^1_k$ which satisfy the definition of a line bundle, and has the functions g_{ij} as its transition functions.

Using the gluing construction one defines the following operations over line bundles.
(i) The *tensor product* $L \otimes L'$. Let $(L, X, \pi, \mathcal{U}, \beta_{\mathcal{U}}), (L', X, \pi', \mathcal{U}, \beta'_{\mathcal{U}})$ be two line bundles with the same trivializing cover $\mathcal{U} = \{U_i\}_{i\in I}$. We define $L \otimes L'$ by gluing the varieties $U_i \times \mathbf{A}^1_k$ by using the transition functions $g_{ij}g'_{ij}$.
(ii) The *inverse* line bundle L^{-1}. Again we use the gluing by means of the transition functions g_{ij}^{-1}. One easily checks that this structure makes the set of line bundles with the same open cover an abelian group. The zero element is the *trivial bundle* $(X \times \mathbf{A}^1_k, X, pr_2, \mathcal{U}, \{id\})$. It is easy to see that we can extend these operations to isomorphism classes of line bundles. The resulting group is called the *Picard group* of X and is denoted by $\mathrm{Pic}(X)$.

(iii) The *pull-back* or *inverse image* $f^*(L)$. Let $f : X \to Y$ be a morphism of varieties and let $(L, Y, \pi, \mathcal{U}, \beta_{\mathcal{U}})$ be a line bundle over Y. We define $f^*(L)$ of L by gluing the products $f^{-1}(U_i) \times \mathbf{A}_k^1$ by using the transition functions $f^*(g_{ij}) \in \mathcal{O}(f^{-1}(U_i \cap U_j)) = \mathcal{O}(f^{-1}(U_i) \cap f^{-1}(U_j))$. The total space of the obtained bundle is equal to the fibred product $X \times_Y L$, and the canonical projection is the first projection of the product. When f is the identity map of a subvariety X of Y, we say that $f^*(L)$ is the *restriction* of L to X and denote it by $L|X$.

Observe that the inverse image operation defines a homomorphism of groups:

$$f^* : Pic(Y) \to Pic(X).$$

3.4 One can naturally generalize the notion of a line bundle as follows. Note that the transition functions can be thought as morphisms $g_{ij} : U_i \cap U_j \to \mathbf{G}_{\mathbf{m},k} = \mathbf{GL}_k(1)$. Let G be any algebraic k-group and $\rho : G \to \mathbf{GL}_k(n)$ be its linear representation. Let $\mathcal{U}$ be an open cover of X as above and $g_{ij} : U_i \cap U_j \to G$ be a collection of morphisms satisfying:

$$g_{ii} = id, g_{ij} = g_{ji}, g_{ij}g_{jk} = g_{ik}.$$

We use the functions $\rho \circ g_{ij} : U_i \cap U_j \to \mathbf{GL}_k(n)$ to define the gluing of the varieties $U_i \times \mathbf{A}_k^n$. The resulting variety E comes with a projection $\pi : E \to X$ whose fibres are n-dimensional affine spaces. Over each open set U_i we have an isomorphism $\beta_i : U_i \times \mathbf{A}_k^n \to \pi^{-1}(U_i)$. If $w \in \pi^{-1}(U_i \cap U_j)$, and $w = \beta_i((x, z_i)) = \beta_j((x, z_j))$, then

$$z_j = \rho(g_{ij}(x))z_j.$$

Here the multiplication is the matrix multiplication. The object $(E, X, \pi, \mathcal{U}, \{\beta_i\}, G, \rho)$ is called a *G-bundle* associated to the representation ρ. If $G = \mathbf{GL}_k(n)$ and $\rho = id$, it is called a *vector bundle of rank n.* Thus a line bundle is a vector bundle of rank 1. Instead of gluing $U_i \times \mathbf{A}_k^n$ by using the transition functions g_{ij} one can glue the varieties $U_i \times G$. For this construction we don't need a linear representation of G. The resulting object is called a *principal G-bundle.* It is clear that we have a bijective correspondence between principal $\mathbf{GL}_k(n)$-bundles and vector bundles of rank n over X.

3.5 Remark. For readers familiar with the language of sheaves one can interpret the previous notions as follows. A set of transition functions defines a Čech 1-cocycle of the sheaf $\mathcal{O}_X^*$ with respect to the chosen open cover. The group of line bundles with the same cover is the group of 1-cocyles $Z^1(\mathcal{U}, \mathcal{O}_X^*)$. Two cocycles define isomorphic line bundles if and only if they are mapped to the same element in the cohomology group $H^1(X, \mathcal{O}_X^*)$. Thus

$$Pic(X) \cong H^1(X, \mathcal{O}_X^*).$$

There are different objects which are classified by the latter cohomology group. These are isomorphism classes of invertible sheaves on X, and isomorphism classes of Cartier divisors on X. We refer to **[Har]** for the corresponding definitions and the relationships. In the more general situation from the previous Remark, one can introduce the sheaf of groups $\mathcal{O}_X(G)$ and the set $H^1(X, \mathcal{O}_X(G))$ which is bijectively equivalent to the set of isomorphism classes of principal G-bundles over X.

3.6 A *section* of a line bundle $(L, X, \pi, \mathcal{U}, \beta_{\mathcal{U}})$ is a morphism $s : X \to L$ such that $\pi \circ s = id$. Let $s_i = s|U_i$; then $\beta_i \circ s_i : U_i \to U_i \times \mathbf{A}^1_k$ are the sections of the trivial line bundle over U_i. We can write $\beta_i \circ s_i(z) = (z, \phi_i(z))$ for some functions $\phi_i \in \mathcal{O}(U_i)$. These functions must satisfy

$$\phi_i = g_{ij}\phi_j$$

when restricted to $U_i \cap U_j$. Conversely, a set of functions ϕ_i satisfying the previous condition defines a section.

We denote the set of sections of a line bundle L by $\Gamma(X, L)$. It has a natural structure of a vector space over the field k. The corresponding operations are obtained from addition and scalar multiplication of the functions ϕ_i. The *zero-section* is defined by the zero functions φ_i. (One defines the *sheaf of sections* $\mathcal{O}_X(L)$ by setting for any open $U \subset X$, $\mathcal{O}_X(L)(U) = \Gamma(U, L|U)$ with obvious restriction maps. It is an invertible sheaf of $\mathcal{O}_X$-modules).

For any $s \in \Gamma(X, L)$ the subset

$$X_s = \{x \in X : s(x) \neq 0\}$$

is an open subset of X. Note that $s(x) \neq 0$ means the image of x under s does not belong to the zero section.

The following result follows from some fundamental results in the theory of coherent sheaves on algebraic varieties. We state it without proof (see **[Har]**, p.228).

Theorem. *Assume X is a projective variety over a field k. Then $\Gamma(X, L)$ is a finite-dimensional linear space over k.*

Example. Let $X = \mathbf{P}^n_k$. We define the line bundle $\mathcal{O}_{\mathbf{P}^n_k}(1)$ by the transition functions

$$g_{ij} = T_j/T_i,$$

where we choose the trivializing cover to be the standard cover $(\mathbf{P}^n_k)_i = \{T_i \neq 0\}, i = 0, \ldots, n$. It is easy to see that its sections can be identified with the space of linear homogeneous polynomials $k[T_0, \ldots, T_n]_1$. The restriction of such a section $F(T_0, \ldots, T_n)$ to $(\mathbf{P}^n_k)_i$ is the inhomogeneous polynomial $\varphi_i = F/T_i$. We see that $\varphi_i = (T_j/T_i)\varphi_j$ so that everything agrees. By definition, for any integer m

$$\mathcal{O}_{\mathbf{P}^n_k}(m) = \mathcal{O}_{\mathbf{P}^n_k}(1)^{\otimes m},$$

where as in arithmetic the negative m-th tensor power means the $(-m)$-th tensor power of the inverse. If $m = 0$ we get the trivial bundle which is denoted by $\mathcal{O}_{\mathbf{P}^n_k}$. One immediately checks that

$$\Gamma(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(m)) \cong \begin{cases} k[T_0, \ldots, T_n]_m & \text{if } m \geq 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

We refer to **[Har]** for the proof that every line bundle on $\mathbf{P}^n_k$ is isomorphic to a bundle $\mathcal{O}_{\mathbf{P}^n_k}(m)$. In particular

$$\text{Pic}(\mathbf{P}^n_k) \cong \mathbf{Z}.$$

One can give the following description of the total space of the line bundle $\mathcal{O}_{\mathbf{P}_k^n}(-1)$. Let L be the blow-up of $\mathbf{A}_k^{n+1}$ at the origin. Recall that this is a closed subvariety of $\mathbf{A}_k^{n+1} \times \mathbf{P}_k^n$ given by the equations $Z_jT_i - Z_iT_j = 0$, where $(Z_1, \ldots, Z_{n+1})$ are coordinate functions on the affine space and $(T_0, \ldots, T_n)$ are projective coordinates in $\mathbf{P}_k^n$. Let $\pi : L \to \mathbf{P}_k^n$ be the second projection. Then

$$\mathcal{O}(\pi^{-1}((\mathbf{P}_k^n)_i)) = k[Z_i, T_0/T_i, \ldots, T_n/T_i] \cong \mathcal{O}((\mathbf{P}_k^n)_i)[Z_i].$$

This shows that $\pi^{-1}((\mathbf{P}_k^n)_i) \cong (\mathbf{P}_k^n)_i \times \mathbf{A}_k^1$. The transition functions are equal to $Z_i/Z_j = T_i/T_j$, i.e.,the inverse of the transition functions for $\mathcal{O}_{\mathbf{P}_k^n}(1)$.

3.7 In algebraic geometry line bundles are used to define mappings of algebraic varieties to projective space. Recall the construction. Let W be a finite-dimensional subspace of $\Gamma(X, L)$. Choose a basis $s_0, \ldots, s_n$ of W and let $X' = \bigcup_{i=0}^{n} X_{s_i}$. Let $\mathbf{P}_k^n$ be the n-dimensional space with its standard affine cover $U_i = \{T_i \neq 0\}, i = 0, \ldots, n$. We define a map $f : X' \to \mathbf{P}_k^n$ by gluing together the maps $f_i : X_{s_i} \to U_i$ corresponding to the homomorphism of rings $\mathcal{O}(U_i) \to \mathcal{O}(X_{s_i})$ defined by mapping each function T_j/T_i to s_j/s_i. Note that for any two sections s, s' of L the ratio s'/s is a rational function on X defined on each open set U_j from the trivializing cover by the ratio φ_j'/φ_j, where φ_j and φ_j' represent s and s' on U_j. If $s(x) \neq 0$ this ratio

is regular at x. We leave to the reader to verify that the maps f_i are compatible on the intersections of their domains. Set-theoretically f is described by the formula:

$$f(x) = (s_0(x), \ldots, s_n(x)) \in \mathbf{P}_k^n(K) = K^{n+1}/K^*, \quad \forall x \in X(K), \forall \text{ field extensions } K/k.$$

Note that although $s(x)$ has no meaning, the right-hand side makes sense since we are considering $(n+1)$-tuples modulo non-zero scalar multiples. If we choose a function φ_i to represent locally a section s over U_i, then $\varphi_i(x)$ is defined up to a scalar factor equal to $g_{ij}(x)$, which is the same for all sections s. The constructed map is called the map given by the *linear system* $(W, (s_0, \ldots, s_n))$. If $W = \Gamma(X, L)$ we say that the linear system is *complete*. It is clear that $X' = X$ if and only if for any point $x \in X$ there exists a section $s \in W$ such that $s(x) \neq 0$. In this case we say that the linear system is *base-point-free*. Any point in $X \setminus X'$ is a *base-point* of W. Obviously, this definition does not depend on the choice of a basis in W.

Since $f^*(T_j/T_i) =$ the restriction of the functions $s_j/s_i = g_{ij}$ to X', we get

$$f^*(\mathcal{O}_{\mathbf{P}_k^n}(1)) = L|X'.$$

Definition. A line bundle L is called *very ample* if the map defined by some of its linear system is an embedding. L is called *ample* if $L^{\otimes m}$ is very ample for some positive m-th tensor power of L.

Example. Obviously $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is very ample on $\mathbf{P}_k^n$. The map defined by its complete linear system is a linear projective automorphism of $\mathbf{P}_k^n$. If we choose for the basis of $\Gamma(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(1))$ the set $(T_0, \ldots, T_n)$, then the map is the identity. The map defined by a complete linear system of $\mathcal{O}_{\mathbf{P}_k^n}(d), d > 0$, is a Veronese map $v_{n,d} : \mathbf{P}_k^n \to \mathbf{P}^{\binom{n+d}{d}-1}$. This shows that if $f : X \to \mathbf{P}_k^n$ is the map defined by L (i.e. by a complete linear system), then the composition:

$$X \xrightarrow{f} \mathbf{P}_k^n \xrightarrow{v_{n,d}} \mathbf{P}^{\binom{n+d}{d}-1}$$

is given by $L^{\otimes d}$.

3.8 Let $\sigma : G \times X \to X$ be an algebraic action of an algebraic group G on an algebraic variety X.

Definition. A *G-linearization* on L is an action $\bar{\sigma} : G \times L \to L$ such that

(i) the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\bar{\sigma}} & L \\ \downarrow 1 \times \pi & & \downarrow \pi \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

is commutative.

(ii) the zero section of L is G-stable.

A G-linearized line bundle (or, for brevity, a *line G-bundle*) over a G-variety X is a pair $(L, \bar{\sigma})$ consisting of a line bundle L over X and its linearization. A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles.

It follows from the definition that for any $g \in G(k), x \in X(k)$ the induced map of the fibres:

$$\bar{\sigma}(g) : L_x \to L_{g \cdot x}$$

is a linear isomorphism. If k is algebraically closed this condition is equivalent to (ii).

Let $\bar{\sigma} : G \times L \to L$ be a G-linearization. By definition of the fibred product we have a unique homomorphism $G \times L = pr_2^*(L) \to \sigma^*(L)$. Here $pr_2 : G \times X \to X$ is the second projection. For any field extension K/k, the fibre of $pr_2^*(L)$ over $(g, x) \in G(K) \times X(K)$ is L_x. The fibre of $\sigma^*(L)$ over the same point is equal to $L_{g \cdot x}$. Since $\bar{\sigma}$ is an action we obtain that the corresponding map of the fibres is an isomorphism. By property (ii), it must be a linear isomorphism. Thus the map

$$\Phi : pr_2^*(L) \to \sigma^*(L)$$

is an isomorphism of vector bundles. One can translate the axioms of action for $\bar{\sigma}$ into the following cocycle condition. Let $p_{23} : G \times G \times X \to G \times X$ be the projection to the product of the second and the third factors. Together with $\mu \times id_X$ and $id_G \times \sigma$ we have three maps from $G \times G \times X$ to $G \times X$. Note that $p_2 \circ (\mu \times id_X) = p_2 \circ p_{23}$ hence we can identify the line bundles $(p_2 \circ p_{23})^*(L)$ and $(p_2 \circ (\mu \times id_X))^*(L)$. Similarly we can identify $[\sigma \circ (\mu \times id_X)]^*(L)$ with $[\sigma \circ (id_G \times \sigma)]^*(L)$ and $(\sigma \circ p_{23})^*(L)$ with $[p_2 \circ (id_G \times \sigma)]^*(L)$. With these identifications the cocycle condition says that the following isomorphisms of line bundles on $G \times G \times X$ are equal:

$$(\mu \times id_X)^*(\Phi) : [p_2 \circ (\mu \times id_X)]^*(L) \to [\sigma \circ (\mu \times id_X)]^*(L) = [\sigma \circ (id_G \times \sigma)]^*(L),$$

$$(id_G \times \sigma)^*(\Phi) \circ p_{23}^*(\Phi) : (p_2 \circ p_{23})^*(L) \to (\sigma \circ p_{23})^*(L) =$$

$$= [p_2 \circ (id_G \times \sigma)]^*(L) \to [\sigma \circ (id_G \times \sigma)]^*(L).$$

Conversely, given an isomorphism of line bundles Φ as above, composing it with the projection $\sigma^*(L) \to L$ we get a map $\bar{\sigma} : G \times L \to L$ for which the diagram from (i) is commutative. The cocycle condition ensures that this map defines an action. Also condition (ii) holds because Φ is an isomorphism of line bundles.

Using the definition of linearization by means of an isomorphism Φ it is easy to define a structure of an abelian group on the set of line G-bundles with the same trivializing cover. If $\Phi : pr_2^*(L) \to \sigma^*(L)$ and $\Phi : pr_2^*(L') \to \sigma^*(L')$ are two line G-bundles, we define their tensor product as the line bundle $L \otimes L'$ with the G-linearization given by the isomorphism:

$$\Phi \otimes \Phi' : pr_2^*(L \otimes L') = pr_2^*(L) \otimes pr_2^*(L') \to \sigma^*(L \otimes L) = \sigma^*(L) \otimes \sigma^*(L')).$$

Here we use the obvious property of the inverse image

$$f^*(L \otimes L') = f^*(L) \otimes f^*(L').$$

The zero element in this group is the trivial line bundle $X \times \mathbf{A}_k^1$ whose linearization is given by the product $\sigma \times id : G \times X \times \mathbf{A}_k^1 \to X \times \mathbf{A}_k^1$. This is called the *trivial linearization* . One checks that this again satisfies the cocycle condition. The structure of the abelian group which we have just defined induces an abelian group structure on the set of isomorphism classes of line G-bundles. We denote this group by $Pic^G(X)$. It comes with the natural homomorphism

$$\alpha : Pic^G(X) \to Pic(X)$$

which is defined by forgetting the linearization.

3.9 Let us now describe the kernel of the homomorphism α. Observe first that if $f : L \to L'$ is an isomorphism of line bundles and $\Phi : pr_2^*(L) \to \sigma^*(L)$ is a G-linearization on L, then we can define a G-linearization on L' by setting $\Phi' = \sigma^*(f)^{-1} \circ \Phi \circ pr_2^*(f)$. Thus if $\alpha((L, \bar{\sigma}))$ is isomorphic to the trivial bundle, we can replace it by an isomorphic G-bundle to assume that L is trivial. This shows that $Ker(\alpha)$ consists of isomorphism classes of linearizations on the trivial line bundle.

Assume that X and G are affine and $L = X \times \mathbf{A}_k^1$ is the trivial line bundle on X. To define a G-linearization on L we have to define a homomorphism

$$\bar{\sigma}^* : \mathcal{O}(L) = \mathcal{O}(X \times \mathbf{A}_k^1) = \mathcal{O}(X)[Z] \to \mathcal{O}(G) \otimes \mathcal{O}(X)[Z].$$

It follows from the definition that under this map

$$\bar{\sigma}^*(Z) = \Psi \otimes Z,$$

where $\Psi \in (\mathcal{O}(G) \otimes \mathcal{O}(X))^* = \mathcal{O}(G \times X)^*$. It follows from the definition of the action that

$$(\mu \otimes id)^*(\Psi) = p_{23}^*(\Psi)(id \otimes \sigma)^*(\Psi).$$

This can be translated into the following property of the function Ψ:

$$\forall x \in X(K), g, g' \in X(K), \Psi(g'g, x) = \Psi(g, x)\Psi(g', gx).$$

The map $\bar{\sigma}^*$ is completely determined by Ψ. The action is defined by the formula

$$\bar{\sigma}(g, (x, z)) = (g \cdot x, \Psi(g, x)z),$$

where $(x, z) \in L(K)$ is defined by the homomorphism of $\mathcal{O}(L)$ which send Z to z and sends any $\phi \in \mathcal{O}(X)$ to $\phi(x)$. We denote by $Z^1_{alg}(G, \mathcal{O}(X)^*)$ the group of such functions on $G \times X$ (with respect to multiplication). For any K/k we have the group

$$Z^1_{alg}(G, \mathcal{O}(X)^*)(K) = \{\Psi_K : G(K) \times X(K) \to K^* \mid \Psi_K(g'g, x) = \Psi(g, x)\Psi(g', gx)\}.$$

Assume that Ψ and Ψ' define isomorphic G-bundles. By definition, there exists an isomorphism of the k-algebras $\mathcal{O}(L) = \mathcal{O}(X)[Z] \to \mathcal{O}(L) = \mathcal{O}(X)[Z]$ which sends Z to ϕZ for some $\phi \in \mathcal{O}(X)^*$ (we use the condition of linearity). It is immediately checked that one must have

$$\sigma^*(\phi)\Psi = \Psi'(1 \otimes \phi).$$

This can be interpreted as follows. Under $\bar{\sigma}$, a point $(x, z) \in L$ is mapped to $(g \cdot x, \Psi(g, x)z)$. Under the isomorphism of G-bundles it is mapped to $(g \cdot x, \phi(g \cdot x)\Psi(g, x)z)$. The same point is mapped under the isomorphism to $(x, \phi(x)z)$, and then under $\bar{\sigma}'$ to $(g \cdot x, \Psi'(g, x)\phi(x)z)$. Conversely, if we define Ψ' by the formula:

$$\Psi' = \sigma^*(\phi)\Psi(1 \otimes \phi^{-1})$$

we obtain an isomorphic G-bundle. When $\Psi = 1$, we get the trivial G-linearization. The group acts by the formula $(x, z) \to (g \cdot x, z)$. The functions Ψ of the form $\sigma^*(\phi)(1 \otimes \phi^{-1})$ define G-linearizations which are isomorphic to trivial G-linearizations. They form a subgroup of $Z^1_{alg}(G, \mathcal{O}(X)^*)$ denoted by $B^1_{alg}(G, \mathcal{O}(X)^*)$. The factor group

$$H^1_{alg}(G, \mathcal{O}(X)^*) := Z^1_{alg}(G, \mathcal{O}(X)^*)/B^1_{alg}(G, \mathcal{O}(X)^*)$$

consists of isomorphism classes of G-linearizations on the trivial line bundle L. Note the special case when for any integral k-algebra K

$$(\mathcal{O}(X) \otimes_k K)^* = K^* \otimes 1.$$

This happens, for example, when X is affine space, or when X is connected and projective. Then for any $\phi \in \mathcal{O}(X)^* = k^*$ we have $\sigma^*(\phi) = 1 \otimes \phi$, hence the group $B^1_{alg}(G, \mathcal{O}(X)^*)$ is trivial, and $H^1_{alg}(G, \mathcal{O}(X)^*) = Z^1_{alg}(G, \mathcal{O}(X)^*)$. Also, if we assume additionally that G is irreducible, $\Psi \in (\mathcal{O}(G) \otimes \mathcal{O}(X))^* = \mathcal{O}(G)^* \otimes 1$ and hence can be identified with an element of $\mathcal{O}(G)^*$. This element defines a homomorphism $k[Z, Z^{-1}] \to \mathcal{O}(G)^*$, hence a map $f : G \to \mathbf{G}_{\mathbf{m},k}$. The property of Ψ guarantees that this is a homomorphism of algebraic groups. Any such homomorphism is called a *rational character* of G. The set of such characters is an abelian group denoted by $\mathcal{X}(G)$. Thus we obtain a canonical isomorphism:

$$H^1_{alg}(G, \mathcal{O}(X)^*) \cong \mathcal{X}(G).$$

We have

$$H^1(G, \mathcal{O}(X)^*)(K) = Hom(G(K), K^*).$$

Suppose now that L is still the trivial line bundle, and that G is affine but X is not. Then we take any open affine cover $\{U_i\}_{i\in I}$ of X, and set, for any $i \in I$, $V_i = \sigma^{-1}(U_i)$. The action $\sigma : G \times X \to X$ can be given by the maps $\sigma_i : V_i \to U_i$. The linearization $\bar{\sigma} : G \times L \to L$ is given by the maps $\bar{\sigma}_i : V_i \times \mathbf{A}^1_k \to U_i \times \mathbf{A}^1_k$, or equivalently, by the homomorphisms $\mathcal{O}(U_i)[Z] \to \mathcal{O}(V_i)[Z]$. As in the affine case this is defined by a function $\Psi_i \in \mathcal{O}(V_i)^*$. These functions are glued to form a global function $\Psi \in \mathcal{O}(G \times X)^*$. Now we can repeat all the formula from above to obtain:

Theorem. *Let G be an affine algebraic group and let X be a G-variety. The set of linearizations on the trivial line bundle $X \times \mathbf{A}^1_k$ is equal to the group $Z^1_{alg}(G, \mathcal{O}(X)^*)$ of functions $\Psi \in \mathcal{O}(G \times X)^*$ satisfying the condition:*

$$\Psi(g'g, x) = \Psi(g, x)\Psi(g', gx), x \in X(K), g, g' \in X(K).$$

The trivial linearizations form a subgroup $B^1_{alg}(G, \mathcal{O}(X)^*)$ isomorphic to the group of functions Ψ of the form $\sigma^*(\phi)pr_2^*(\phi^{-1})$ where $\phi \in \mathcal{O}(X)^*$. The factor group $H^1_{alg}(G, \mathcal{O}(X)^*) = Z^1_{alg}(G, \mathcal{O}(X)^*)/B^1_{alg}(G, \mathcal{O}(X)^*)$ is isomorphic to the group of isomorphism classes of linearizations of the trivial line bundle. In the special case where for any integral finitely generated k-algebra $K, (\mathcal{O}(X) \otimes K)^* = 1 \otimes K^*$, we have $B^1_{alg}(G, \mathcal{O}(X)^*) = \{1\}$, and

$$H^1_{alg}(G, \mathcal{O}(X)^*) \cong \mathcal{X}(G) = Hom(G, \mathbf{G}_{\mathbf{m},k}).$$

3.10 Remark. According to a theorem of Rosenlicht (see **[KKV]**) for any two irreducible algebraic varieties X and Y over an algebraically closed field k, the natural homomorphism

$$\mathcal{O}(X)^* \otimes \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$$

is surjective.

3.11 Now let us study the image of the forgetful homomorphism α. This consists of isomorphism classes of line bundles on X which admit some G-linearization. We start with the following lemma.

Lemma 1. *Let G be an irreducible affine algebraic group, X be an algebraic G-variety. A line bundle L over X admits a G-linearization if and only if there exists an isomorphism of line bundles $\Phi : pr_2^*(L) \to \sigma^*(L)$.*

Proof. We already know that this condition is necessary, so we show that it is sufficient. Assume that such an isomorphism exists. The problem is that it may not satisfy the cocycle condition. If Φ is given, we restrict it to $e \times X$ to obtain an isomorphism $\Phi_e : pr_2^*(L)|e \times X \to$

$\sigma^*(L)|e\times X$. The maps σ and pr_2 coincide on $e\times X$, hence there is a canonical isomorphism of these restrictions. Observe that any two isomorphisms between line bundles differ by an automorphism of one of the bundles. Also an automorphism of a line bundle is given by a global invertible regular function. So by choosing an appropriate function from $\mathcal{O}(X)^*$, and lifting it to $G\times X$, we change Φ to assume that Φ_e is the canonical isomorphism. We use again that the isomorphism $p_{23}^*(\Phi)(id_G\times\sigma)^*(\Phi)$ and $(\mu\times id_X)^*(\Phi)$ differ by a function $\varphi\in\mathcal{O}(G\times G\times X)^*$. We have

$$\Phi(g'g,x)=\Phi(g,x)\Phi(g',g\cdot x)\varphi(g',g,x),$$

where we use for simplicity the argument notation for Φ,φ. We have $\Phi(e,x)\equiv 1$. Thus $\varphi(e,g,x)=\varphi(g',e,x)\equiv 1$. We want to show that $\varphi\equiv 1$. Replacing the field k by its algebraic closure we may assume that k is algebraically closed. By applying Rosenlicht's Theorem (see Remark 3.10) we can write $\varphi(g',g,x)=\varphi_1\otimes\varphi_2\otimes\varphi_3$ for some $\varphi_1,\varphi_2\in\mathcal{O}(G)^*$, and $\varphi_3\in\mathcal{O}(X)^*$. Thus

$$\begin{aligned}\varphi(g',g,x)&=(\varphi_1(g')\varphi_2(g)\varphi_3(x))(\varphi_1(e)\varphi_2(e)\varphi_3(x))=\\&=(\varphi_1(e)\varphi_2(g)\varphi_3(x))(\varphi_1(g')\varphi_2(e)\varphi_3(x))\equiv 1.\end{aligned}$$

Lemma 2. *Assume k is algebraically closed, X is normal (for example, nonsingular) and G is an affine irreducible algebraic group. Let $x_0\in X(k)$. For every line bundle L on $G\times X$ we have*

$$L\cong p_1^*(L|G\times x_0)\otimes p_2^*(L|e\times X).$$

Proof. We only sketch the proof referring to **[KKV]** for the details. First of all we may assume that X is nonsingular. In fact, if we replace X by its subset of nonsingular points X' then, since X is normal, the restriction homomorphism $\mathrm{Pic}(X)\to\mathrm{Pic}(X')$ is injective. So we may replace X by X' to assume that X is nonsingular. Next we use the following fact about algebraic groups (which we shall explain later when we state a structure theorem for algebraic groups). The fact is that, if k is algebraically closed, G contains an open Zariski subset U isomorphic to the product of varieties $\mathbf{A}_k^1$ and $\mathbf{A}_k^1\setminus\{0\}$. We also use the fact that the homomorphisms $pr_2^*: Pic(X)\to Pic(\mathbf{A}_k^1\times X)$ and $pr_2^*: Pic(X)\to Pic(\mathbf{A}_k^1\setminus\{0\}\times X)$ are isomorphisms (see **[Har]**, Chapter II, Proposition 6.6). These facts together imply that the line bundle

$$M=L\otimes(p_1^*(L|G\times x_0)\otimes p_2^*(L|e\times X))^{-1}$$

is trivial when restricted to $U\times X$. This shows that its transition functions can be chosen to be trivial on $U\times X$ (after we replace L by an isomorphic line bundle). Using the relationship between Weil divisors and line bundles one can show that this implies that $M\cong p_1^*(M|G\times x_0)$. Since $M|G\times x_0$ is trivial , M is trivial proving the lemma.

Define now a homomorphism $\delta: Pic(X)\to Pic(G)$ by

$$\delta(L)=(p_2^*(L)\otimes\sigma^*(L^{-1}))|G\times x_0,$$

where x_0 is a chosen point in $X(k)$. Suppose $\delta(L)$ is trivial. By the previous lemma applied to $M=p_2^*(L)\otimes\sigma^*(L^{-1})$ we obtain that $M=p_2^*(M|e\times X)$. But the restriction of σ and p_2 to $e\times X$ is equal. This implies that M is trivial, hence there exists an isomorphism $\Phi: p_2^*(L)\to\sigma^*(L)$. By Lemma 1, L admits a G-linearization. This proves

Theorem. *Assume k is algebraically closed. Let G be an irreducible affine algebraic group and X be a G-variety. Then the following sequence of groups is exact*

$$\{1\} \to \mathrm{Ker}(\alpha) \to Pic^G(X) \xrightarrow{\alpha} Pic(X) \xrightarrow{\delta} Pic(G).$$

Corollary. *Under the assumption of the theorem, the image of $Pic^G(X)$ in $Pic(X)$ is of finite index. In particular, for any line bundle L on X there exists a number n such that $L^{\otimes n}$ admits a G-linearization.*

Proof. Use the fact that for any affine algebraic k-group G the Picard group $Pic(G)$ is finite (see **[KKV]**, p.74).

Remark. For any extension of fields K/k we denote by X_K the variety obtained from X by extension of the ground field. If we assume k is perfect and $\mathcal{O}(X_{\bar{k}})^* = \bar{k}^*$ (for example if $X_{\bar{k}}$ is complete and connected), the assertion of the Corollary remains true. To see this, we have to modify the assertion of Lemma 2 by replacing L by some positive tensor power $L^{\otimes n}$. We modify the proof as follows. Replacing k by a finite Galois extension k', we may assume that G contains an open subset isomorphic to the product of an affine space and an algebraic torus. This follows from the fact that every unipotent algebraic group over a perfect field is isomorphic to an affine space as an algebraic variety (see **[DG]**, p. 536). Then we use that the kernel of the homomorphism $Pic(X \times G) \to Pic(X'_k \times G'_k)$ is isomorphic to the Galois cohomology group

$$H^1(Gal(k'/k), \mathcal{O}(X_{k'} \times G_{k'})^*)) \cong H^1(Gal(k'/k), \mathcal{O}(G_{k'})^*)) = \mathrm{Ker}(Pic(G) \to Pic(G_{k'})).$$

Since $Pic(G)$ is finite, $R := Ker(Pic(X \times G) \to Pic(X_{k'} \times G_{k'}))$ is finite. Thus replacing k by k', and repeating the argument, we obtain that M is of finite order in $Pic(G \times X)$. So replacing L by some tensor power $L^{\otimes n}$, hence replacing M by $M^{\otimes n}$, we obtain that M is trivial. In fact, if we assume additionally that $\mathcal{O}(G_{\bar{k}})^* = \bar{k}^*$ (this happens if G is semi-simple), then the kernel R is isomorphic to the group $H^1(Gal(k'/k), k'^*))$ which is trivial by Hilbert's "Theorem 90". In this case we don't need to raise L to a power, so the assertion of the Theorem is also true.

3.12 Next we shall apply the previous Corollary to prove that any algebraic action can be linearized. Let L be a G-linearized line bundle and $V = \Gamma(X, L)$ its space of sections, and let G be an affine algebraic group. We shall identify V with the subspace of $\mathcal{O}(L^{-1})$ of regular functions whose restriction to the fibres is linear (see Problem 7). By means of the inverse linearization on L^{-1} there is a natural homomorphism

$$\mathcal{O}(L^{-1}) \to \mathcal{O}(G) \otimes \mathcal{O}(L^{-1})$$

which satisfies the axioms of the co-action. By means of this homomorphism $G(K)$ acts linearly on $\mathcal{O}(L_K^{-1})$ preserving the subspace $V_K = \Gamma(X, L)_K$. This defines a linear representation:

$$\rho_K : G(K) \to GL(V_K).$$

Arguing as in Lecture 1, we can show that any finite-dimensional subspace W of V is contained in a finite-dimensional subspace W' such that each W'_K is an invariant subspace with respect to the representation ρ_K. By choosing a basis $(s_0,\dots,s_n)$ in W' we obtain a linear representation

$$\rho: G \to \mathbf{GL}_k(n+1).$$

Now assume that the rational map defined by the linear system W is an embedding. Then the rational map defined by W'

$$f: X \to \mathbf{P}_k^n,\ x \to (s_0(x),\dots,s_n(x)),$$

is also an embedding. For example, X is projective, L is very ample and W is the complete linear system. It is obviously G-equivariant. Thus we have linearized our action of G on X.

Theorem. *Assume k is algebraically closed. Let X be a quasi-projective normal algebraic variety, acted on by an irreducible algebraic group G. Then there exists a G-equivariant embedding $X \hookrightarrow \mathbf{P}_k^n$, where G acts on $\mathbf{P}_k^n$ via its linear representation $G \to \mathbf{GL}_k(n+1)$.*

Remark. If $\mathcal{O}(X_{\bar{k}}) = \bar{k}^*$, using the previous Remark we may assume only that k is perfect. Of course we can get a stronger result if we impose some conditions on G.

Example. Let $G = \mathbf{PGL}_k(n+1)$ act on $X = \mathbf{P}_k^n$ in the natural way. Let us see that the vector bundle $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is not G-linearizable but $\mathcal{O}_{\mathbf{P}_k^n}(n+1)$ is. We view G as an open subset of the projective space $\mathbf{P}_k^N (N = n^2+2n = \dim G)$ whose complement is the determinant hypersurface $det((T_{ij})) = 0$. The action $\sigma: G\times X \to X$ is the restriction to $G \times X$ of the rational map $\sigma: \mathbf{P}_k^N \times \mathbf{P}_k^n \dashrightarrow \mathbf{P}_k^n$ given by the formula

$$\sigma^*(T_i) = \sum_j T_{ij} \otimes T_j.$$

Note that this map is undefined at a point $((a_{ij}), a_0,\dots,a_n) \in \mathbf{P}_k^N(K)\times\mathbf{P}_k^n(K)$ if and only if $\sum_j a_{ij}a_j = 0, i = 0,\dots,n$. But this is possible only if $det((a_{ij})) = 0$, i.e if this point does not belong to $G(K)\times\mathbf{P}_k^n(K)$. By 3.11, Lemma 1, the line bundle $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is G-linearizable if and only if there exists an isomorphism $\sigma^*(\mathcal{O}_{\mathbf{P}_k^n}(1))|G\times\mathbf{P}_k^N \cong pr_2^*(\mathcal{O}_{\mathbf{P}_k^n}(1))|G\times\mathbf{P}_k^N$. It is easy to see that $\sigma^*(\mathcal{O}_{\mathbf{P}_k^n}(1)) \cong pr_1^*(\mathcal{O}_{\mathbf{P}_k^N}(1))\otimes pr_2^*(\mathcal{O}_{\mathbf{P}_k^n}(1))$. Thus $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is G-linearizable if and only if $pr_1^*(\mathcal{O}_{\mathbf{P}_k^N})|G\times\mathbf{P}_k^n = pr_1^*(\mathcal{O}_{\mathbf{P}_k^N}|G)$ is trivial. Obviously $p_1^*: Pic(\mathbf{P}_k^N) \cong \mathbf{Z} \to Pic(\mathbf{P}_k^N\times\mathbf{P}_k^n)$ is injective. Thus we obtain that $\mathcal{O}_{\mathbf{P}_k^n}(1)$ is G-linearizable if and only if $\mathcal{O}_{\mathbf{P}_k^N}|G \cong \mathcal{O}_G$. This is impossible. In fact, it is true that for any hypersurface H of degree d in a projective space $\mathbf{P}_k^N$, the Picard group of the open subvariety $U = \mathbf{P}_k^N \setminus H$ is a cyclic group of order d generated by $\mathbf{P}_k^N|U$. Let us show only that $\mathbf{P}_k^N|U$ is not trivial. This is enough for our purpose. Suppose the contrary is true. If we choose the standard trivializing cover $\mathbf{P}_k^N = U_0\cup\ldots\cup U_N$ of $\mathbf{P}_k^N$ we obtain that $g_{ij} = T_j/T_i = \phi_j/\phi_i$ for some functions $\phi_i \in \mathcal{O}(G\cap U_i)^*$. But $\mathcal{O}(U_i\cap G) = (k[T_1/T_i,\dots,T_n/T/T_i][F/T_i^d])$, where $F=0$ is the equation of the hypersurface H. This immediately implies that $\mathcal{O}(U_i\cap G)^*$ consists of rational functions of the form $c_i(F/T_i^d)^{m_i}$, where $c_i \in k^*$ and m_i is an integer. Thus we get $T_j/T_i = c_j(F/T_j^d)^{m_j}/c_i(F/T_i^d)^{m_i}$. This is possible only if $d = 1, m_j = m_i = 1, c_j = c_i$. A similar argument shows that $\mathcal{O}_{\mathbf{P}_k^N}(d)|U$ is trivial. It remains to observe that in our situation $d = n+1$.

Problems.

1. Prove the total space of the line bundle $\mathcal{O}_{\mathbf{P}^n_k}(1)$ is isomorphic to the open subset of $\mathbf{P}^{n+1}_k$ whose complement is one point.
2. Prove that every line bundle on an affine space is isomorphic to the trivial line bundle.
3. Let L be a line bundle over an algebraic group. Show that the complement L^* to the zero section of L has a structure of an algebraic group such that the projection map $\pi : L^* \to G$ is a homomorphism of groups with kernel isomorphic to $\mathbf{G}_{\mathbf{m},k}$.
4. Assume G is irreducible and k is algebraically closed. Show that $H^1_{alg}(G, \mathcal{O}(X)^*)$ is a homomorphic image of the group $\mathcal{X}(G)$.
5. Use Rosenlicht's Theorem from Remark 3.10 to show that any invertible regular function $f \in \mathcal{O}(G)^*$ on an irreducible affine algebraic group G with value 1 at $e \in G(k)$ defines a rational character of G.
6. Let L be a line bundle. Show that a section of a line bundle $L^{\otimes n}$ can be canonically identified with a regular function on the total space of the inverse line bundle L^{-1} whose restriction to any fibre is a homogeneous polynomial of degree n.
7. Let L be a G-bundle. Using the identification from the previous problem show that the representation of $G(K)$ on $\Gamma(X, L)$ is given by the formula

$$\rho(g)(s)(x) = g \cdot s(g^{-1} \cdot x),$$

for any $g \in G(K), x \in X(K)$.
8. Let $\mathbf{G}_{\mathbf{m},k}$ act on an affine algebraic variety X defining the corresponding grading of $\mathcal{O}(X)$. Let M be a projective module of rank 1 over $\mathcal{O}(X)$ and L be the associated line bundle on X. Show that there is a natural bijective correspondence between G-linearizations on L and structures of a $\mathcal{O}(X)$-graded module on M.
9. Relate the function $\Psi \in \mathcal{O}(G \times X)^*$ from section 3.9 with the isomorphism $\Phi : pr^*(L) \to \sigma^*(L)$ defining a G-linearization. Then compare the two cocycle conditions.

Lecture 4. QUOTIENTS

4.1 Let (X, R) be a set together with an equivalence relation $R \subset X \times X$. A morphism of such pairs $(X, R) \to (Y, R')$ is a map $f : X \to Y$ such that $(f, f)(R) \subset R'$. The quotient set X/R is defined as a morphism $p : (X, R) \to (Y, \Delta_Y)$ such that for every morphism $g : X \to Z$ with the property $(g, g)(R) \subset \Delta_Z$ there exists a unique morphism $\bar{f} : Y \to Z$ with $\bar{f} \circ p = g$. Here $\Delta_X \subset X \times X$ denotes the diagonal. The quotient Y is defined uniquely up to a bijection and is denoted by X/R. By construction of X/R we have

$$R = X \times_{X/R} X = (p, p)^{-1}(\Delta_{X/R}) = \{(x, x') \in X \times X : p(x) = p(x')\}.$$

This equality expresses the property that the fibres of p are equivalence classes.

More generally, if $\mathcal{C}$ is any category with products, we define an equivalence relation on an object X as a subobject $R \subset X \times X$ (or more generally just a morphism $R \to X \times X$) satisfying the obvious axioms (expressed by means of commutative diagrams). Then we repeat the preceding definitions word by word to arrive at the definition of a quotient object X/R and the canonical morphism $p : X \to X/R$. By definition there is a canonical morphism

$$R \to X \times_{X/R} X. \quad (*)$$

There is no reason to expect that in general the morphism $(*)$ will be an isomorphism or an epimorphism.

Let $\sigma : G \times X \to X$ be an algebraic action, and $\Psi : G \times X \to X \times X$ be the corresponding morphism (σ, pr_2). This morphism should be thought as an equivalence relation on X defined by the action. A G-equivariant morphism of G-varieties corresponds to a morphism of sets with an equivalence relation. The definition of a G-invariant morphism $f : X \to Y$ can be rephrased by saying that the map Ψ factors through the natural morphism $X \times_Y X \to X \times X$. This corresponds to the property $(f, f)(R) \subset \Delta$. This suggests the following definition:

Definition. A *categorical quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ such that for any G-invariant morphism $g : X \to Z$ there exists a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. A categorical quotient is called a *geometric quotient* if the image of the morphism Ψ equals $X \times_Y X$. We shall denote the categorical quotient (resp. geometric quotient) by $p : X \to X//G$ (resp. $p : X \to X/G$). It is defined uniquely up to isomorphism.

A different approach to defining a geometric quotient is as follows. We know how to define a geometric quotient as a set. We next discuss topological spaces. We put the structure of a topological space on X/G so that the canonical projection $p : X \to X/G$ is continuous. The weakest topology on X/G for which this should be true is the topology in which a subset $U \subset X/R$ is open if and only if $p^{-1}(U)$ is open. Then we examine ringed spaces, whose definition is given in terms of choosing a class of functions on X (e.g. regular functions, smooth functions, analytic functions). If $\phi \in \mathcal{O}(U)$ is a function on U, then the composition $p^*(\phi) = \phi \circ p$ must be a function on $p^{-1}(U)$. It is obviously a G-invariant function. Using this remark we can define the structure of a ringed space on X/R by setting $\mathcal{O}(U) = \mathcal{O}(p^{-1}(U))^G$. This makes $p : X \to X/R$ a categorical quotient in the category of ringed spaces. Finally, we want that the fibres of p to be orbits. This is condition $(*)$.

Definition. A *good geometric quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ satisfying the following properties:

(i) p is surjective;
(ii) for any open subset U of Y, the pre-image $p^{-1}(U)$ is open if and only if U is open;
(iii) for any open subset U of Y, the natural homomorphism $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions.
(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Proposition 1. *A good geometric quotient is a categorical quotient.*

Proof. Let $q : X \to Z$ be a G-invariant morphism. Pick any open affine cover $\{V_i\}_{i \in I}$ of Z. For any V_i the pre-image $q^{-1}(V_i)$ will be an open G-invariant subset of X. Then we have the obvious inclusion $q^{-1}(V_i) \subset p^{-1}(U_i)$, where $U_i = p(q^{-1}(V_i))$. Comparing the fibres over points $y \in Y(\bar{k})$ and using property (iv), we conclude that the equality takes place. By property (ii), U_i is open in Y. Since p is surjective we get an open cover $\{U_i\}_{i \in I}$ of Y. The map $q^{-1}(V_i) \to V_i$ is defined by a homomorphism

$$\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i)) = \mathcal{O}(p^{-1}(U_i)).$$

Since q is a G-invariant morphism, the image of α_i is contained in $\mathcal{O}(p^{-1}(U_i))^G = \mathcal{O}(U_i)$. This defines a unique homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$ hence a unique map $\bar{q}_i : U_i \to V_i$. It is immediately checked that the maps $\bar{p}_i$ agree on the intersections $U_i \cap U_j$ hence define a unique map $\bar{q} : Y \to Z$ satisfying $q = \bar{q} \circ p$.

Proposition 2. *Let $p : X \to Y$ be a G-equivariant morphism satisfying the following properties:*

(i) for any open subset U of Y, the natural homomorphism $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions.
(ii) if W is a closed G-invariant subset of X then $p(W)$ is a closed subset of Y;
(iii) if W_1, W_2 are closed invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $p(W_1) \cap p(W_2) = \emptyset$. Under these conditions p is a categorical quotient. It is a good geometric quotient if additionally
(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Proof. This is similar to the previous proof. With its notation, let $W_i = X \setminus q^{-1}(V_i)$. This is a closed G-invariant subset of X, hence, by (ii), $U_i = Y \setminus p(W_i)$ is an open subset of Y. Clearly, $p^{-1}(U_i) \subset q^{-1}(V_i)$. Since $\cap_i W_i = \emptyset$, by (iii) we have $\cap_i p(W_i) = \emptyset$, hence $Y = \cup_i U_i$. Now composing the homomorphisms $\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i))^G$ with the restriction homomorphism $\mathcal{O}(q^{-1}(V_i))^G \to \mathcal{O}(p^{-1}(U_i))^G = \mathcal{O}(U_i)$ we get a homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$. This defines a map $U_i \to V_i$ and we finish as before. Now if we assume additionally (iv), then the only property from Proposition 1 to be verified is that p is surjective. But this is obvious because of the uniqueness of the categorical quotient.

Corollary. *Under the assumptions from the previous Proposition, the map $p : X \to Y$ satisfies the following properties:*
(i) two points $x, x' \in X(\bar{k})$ have the same image in Y if and only if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$;
(ii) for each $y \in Y(\bar{k})$ the fibre $p^{-1}(y)$ contains a unique closed orbit.

Proof. In fact, the closures of orbits are closed G-invariant subsets in X. So if $\overline{G \cdot x} \cap \overline{G \cdot x'} = \emptyset$, $p(\overline{G \cdot x}) \cap p(\overline{G \cdot x'}) = \emptyset$. But both sets contain the point $p(x) = p(x')$. Conversely, if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$ and $p(x) \neq p(x')$, we get that $G \cdot x$ and $G \cdot x'$ lie in different fibres. Since the fibres are closed subsets, $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ lie in different fibres. Hence they are disjoint. This contradiction proves (i). To prove (ii) we notice that by (i) two closed orbits in the same fibre must have non-empty intersection. But this is absurd. Since each fibre contains at least one closed orbit, we are done.

Definition. A categorical quotient satisfying properties (i), (ii) and (iii) from Proposition 2 is called a *good categorical quotient.*

Remarks. 1. Note that condition (ii) in the definition of a good geometric quotient is satisfied if we require
(ii)$'$ for any closed G-invariant subset Z of X, the image $p(Z)$ is closed.
Also, together with condition (iii) this implies the surjectivity of the factor map p. In fact, condition (iii) ensures that the map p is dominant, i.e. its scheme-theoretical image is dense in Y. But by (ii)$'$, the image of p must be closed.
2. Condition (iv) is equivalent to
(iv)$'$ for any $y \in Y(\bar{k})$, the fibre $p^{-1}(y)$ is equal to the orbit of any of its points.
3. For any K/k we have a natural map $\Psi_K : G(K) \times X(K) \to X(K) \times_{Y(K)} X(K)$ which is not surjective in general. For any $x \in X(K)$

$$\Psi_K(G(K) \times \{x\}) = G(K) \cdot x \times \{x\} \subset p_K^{-1}(p_K(x)) \times \{x\}.$$

4. Suppose X is an irreducible G-variety over a field of characteristic 0, and $p : X \to Y$ a surjective G-invariant morphism such that its fibres over any point $y \in Y(\bar{k})$ are the orbits. Then $p : X \to Y$ is a geometric quotient. The proof is rather technical and we omit it (see [**Mu1**], Proposition 0.2).
5. The definition of a categorical and geometric quotients are obviously "local" in the following sense. If $p : X \to Y$ is a G-equivariant morphism, and $\{U_i\}$ is an open cover of Y with the property that each $p_i : p^{-1}(U_i) \to U_i$ is a categorical (resp. geometric) quotient, then p is a categorical (resp. geometric) quotient.
6. A good geometric quotient is a good categorical quotient. In fact, we have to verify only conditions (ii) and (iii) of Proposition 2. Since each closed G invariant subset W of X must be the union of fibres, (iii) follows immediately. Also $X \setminus W = f^{-1}(U)$ for some open subset U of Y. By definition of a good geometric quotient, U is open, hence $p(W) = Y \setminus U$ is closed. This checks (ii).

4.2 Examples. 1. Let G be a finite group considered as an algebraic group over a field k, for simplicity assumed to be algebraically closed. Then the geometric quotient X/G always exists. In fact, assume first that X is affine. We shall use some standard properties of integral extensions of rings (see **[Bou]**, Commutative Algebra, Chapter V, §2). The inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ is a finite integral extension of finitely-generated k-algebras. Let Y be an affine algebraic variety with $\mathcal{O}(Y) = \mathcal{O}(X)^G$. By theorems on lifting of ideals in integral extensions, the map $p : X \to Y = X/G$ satisfies properties (ii) and (iii) from Proposition 2. Also, since taking invariants commutes with localizations, property (i) holds also. Now let us show that p is a geometric quotient. Since all orbits of G are finite subsets of X, the image of $\Psi : G \times X \to X \times X$ is closed. It is obviously contained in $X \times_Y X$ which is closed in $X \times X$. Now the group G acts transitively on the set of irreducible components of $X \times_Y X$. In fact we may assume for simplicity that X is irreducible. Then the field of rational functions K' of X is a Galois extension of the field of rational functions $K = K'^G$ of Y. The irreducible components of $X \times_Y X$ correspond to minimal prime ideals in the algebra $K' \otimes_K K'$ which is an integral extension of K' with the Galois group isomorphic to G. Thus G acts transitively on the set of minimal prime ideals (see **[Bou]**, Chapter V,§2). By comparing the dimension of $X \times_Y X$ and $\Psi(G \times X)$ we see that $\Psi : G \times X \to X \times_Y X$ is surjective.

Now let $X \subset \mathbf{P}_k^n$ be quasi-projective but not necessarily affine. Let $\bar{X}$ be the closure of X. Let $O \subset X$ be an orbit and let F be a homogeneous polynomial vanishing on $\bar{X} \setminus X$ but not vanishing at any point of O. Thus O is contained in an affine subset $U = \bar{X} \setminus V(F)$. Recall that the complement to a hypersurface in a projective space is an open affine subset. This implies that U, being closed in an affine set, is affine. Let $U(O) = \cap_{g \in G}(g \cdot U)$. This is an open G-invariant affine subset of X containing O. By letting O vary, we get an open affine G-invariant covering $\{U_i\}$ of X. We already know that each quotient $p_i : U_i \to U_i/G = V_i$ exists. We shall glue the V_i's together to obtain the geometric quotient $p : X \to X/G$. To do this we observe first that $U_i \cap U_j$ is affine and $U_i \cap U_j/G$ is open in V_i and V_j. This follows from the considering the affine case. Thus we can glue all V_i together along the open subset $V_{ij} = U_i \cap U_j/G$. The resulting algebraic variety Y is separated. In fact we use that in the affine situation

$$(X_1 \times X_2)/(G_1 \times G_2) \cong X_1/G_1 \times X_2/G_2,$$

where $G_1 \times G_2$ acts on $X_1 \times X_2$ by the Cartesian product of the actions. Thus the image of $\Delta_X \cap (U_i \cap U_j)$ in $(U_i \times U_j)/(G \times G) \cong U_i/G \times U_j/G$ is closed, and, as is easy to see, coincides with $\Delta_Y \cap (V_i \times V_j)$. This checks that Δ_Y is closed. It remains to prove that X/G is quasi-projective. We shall do this later. Note that, if X is not a quasi-projective algebraic variety, X/G may not exist in the category of algebraic varieties even in the simplest case when G is of order 2. The first example of such action was constructed by M. Nagata **[Na1]** and later a simpler construction was given by H. Hironaka **[Hir]**. However, if we assume that each orbit is contained in a G-invariant open affine subset, the previous construcion works and X/G exists.

2. Let $G = \mathbf{G}_{\mathrm{m},k}$ act on an affine algebraic variety X, and let $A := \mathcal{O}(X) = \oplus_{i \in \mathbf{Z}} A_i$ be the corresponding grading. Assume that $A_i = \{0\}$ for $i < 0$ and $A_0 = k$. Such an action is called a *good* $\mathbf{G}_{\mathrm{m},k}$*-action*. Let us see that $X \to pt_k$ is the categorical quotient.

In fact, the inclusion $A_0 \hookrightarrow A$ induces an isomorphism $A_0 \cong A^G$. Obviously, condition (ii) of Proposition 2 is satisfied too. Now let x_0 be the point from $X(k)$ corresponding to the maximal ideal $m = \oplus_{i>0} A_i$. Obviously x_0 is a closed G-invariant subset (closed G-invariant subsets correspond to homogeneous ideals). The subset $X' = X \setminus \{x_0\}$ is an open G-invariant subset. Let us show that the geometric quotient $X' \to X'/G$ exists. We shall assume first that A is generated by homogeneous elements of degree 1. We choose homogeneous generators $f_0, \dots, f_n \in A_1$ of the k-algebra A. The kernel of the canonical surjection $k[T_0, \dots, T_n] \to A, T_i \mapsto f_i$, is a homogeneous ideal in $k[T_0, \dots, T_n]$. It defines a projective subvariety of $\mathbf{P}_k^n$ which we take for $Y = X'/G$. The standard open cover of $\mathbf{P}_k^n$ defines an open cover $\{U_0, \dots, U_n\}$ of Y. We have $\mathcal{O}(U_i) = A_{(f_i)} = \{\frac{a}{f_i^d}, a \in A_d\}$. The open subsets $D(f_i), i = 1, \dots, n$ cover X', and $\mathcal{O}(D(f_i)) = A_{f_i}$. The subsets $D(f_i)$ are G-invariant, and the induced grading of A_{f_i} is given by $(A_{f_i})_m = \{\frac{a}{f_i^d}, a \in A_{m+d}\}$. In particular we see that $\mathcal{O}(U_i) = \mathcal{O}(D(f_i))^G$. The map $p : X' \to Y$ is given by the maps $D(f_i) \to U_i$ which are defined by the homomorphisms $\mathcal{O}(U_i) \to \mathcal{O}(D(f_i))$. Thus condition (i) of Proposition 2 is satisfied. A closed G-invariant subset of X' is given by a homogeneous ideal in A. Its image in Y is closed, since its intersection with each U_i is given by the dehomogenization of this ideal with respect to the variable T_i. This checks condition (iii). Finally $(A_{f_i})_d = f_i^d A_{(f_i)}$ which gives an isomorphism of $A_{(f_i)}$-algebras $A_{f_i} \cong A_{(f_i)}[Z, Z^{-1}]$. This gives that $X \times_Y X$ is covered by open sets $V_i = D(f_i) \times_{U_i} D(f_i)$ with

$$\mathcal{O}(V_i) \cong A_{(f_i)}[Z, Z^{-1}] \otimes_{A_{(f_i)}} A_{f_i} \cong A_{f_i}[Z, Z^{-1}].$$

It is already clear from this that the fibres of $D(f_i) \to U_i$ over any $x \in X(k)$ are isomorphic to $\mathbf{G}_{\mathrm{m},k}$. We leave to the reader to see that Ψ induces an isomorphism $G \times D(f_i) \to D(f_i) \times_{U_i} D(f_i)$.

Now if A is generated by homogeneous elements $f_i, i = 0, \dots, n$, of arbitrary positive degrees d_i, we construct Y by gluing together the affine varieties U_i corresponding to the algebras $A_{(f_i)}$. We use that

$$A_{(f_i f_j)} \cong (A_{(f_i)})_{f_j^{d_i}/f_i^{d_j}} \cong (A_{(f_j)})_{f_i^{d_j}/f_j^{d_i}}.$$

to identify $U_i \cap U_j$ with the quotients of $D(f_i f_j) = D(f_i) \cap D(f_j)$. This gives a categorical quotient variety denoted by $Proj(A)$. In fact (see **[Bou]**, Chap. III, §1), there exists a positive integer e such that the subalgebra

$$A^{(e)} = \oplus_i A_{ei}$$

is generated by elements of degree e. If we replace X by the variety $\bar{X}$ with $\mathcal{O}(\bar{X}) \cong A^{(e)}$, and define the action of $\mathbf{G}_{\mathrm{m},k}$ on $\bar{X}$ via the grading of $\mathcal{O}(\bar{X})$ by setting $\mathcal{O}(\bar{X})_i = A_{ie}$, we will see that $X'/G \cong \bar{X}'/G$ as algebraic varieties. This follows easily by using natural isomorphisms

$$A^{(e)}_{(f^e)} \cong A_{(f)} \quad \text{for any homogeneous element } f \in A.$$

Since $\mathcal{O}(\bar{X})$ is generated by homogeneous elements of degree 1, $\bar{X}'/G$ is a projective variety. So X'/G is a projective variety. Also observe that, if we consider the homomorphism of groups $\alpha : \mathbf{G}_{\mathbf{m},k} \to \mathbf{G}_{\mathbf{m},k}$ given by the homomorphism of k-algebras $Z \to Z^e$, then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{G}_{\mathbf{m},k} \times X & \longrightarrow & X \\ \alpha \times \varphi \downarrow & & \varphi \downarrow . \\ \mathbf{G}_{\mathbf{m},k} \times \bar{X} & \longrightarrow & \bar{X} \end{array}$$

Here $\varphi : X \to \bar{X}$ is given by the inclusion of the rings $A^{(e)} \hookrightarrow A$. This shows that $\Psi(G \times X')$ and $X' \times_{X/G} X'$ are both mapped onto $\Psi(G \times \bar{X}') = \bar{X}' \times_{\bar{X}'/G} \bar{X}'$ under the map $\varphi \times \varphi$. Using the fact that the map $\varphi \times \varphi$ is a finite morphism, we obtain (by reducing to the case when X is irreducible) that this implies that $\Psi(G \times X') = X' \times_{X/G} X'$. Hence $X' \to X'/G$ is a geometric quotient.

Of course a special case of this example is the case when $X = \mathbf{A}_k^n$ and the action of G is the standard one: $t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n)$. The geometric quotient X'/G is the projective space $\mathbf{P}_k^{n-1}$.

3. Let H be a closed subgroup of an affine algebraic group G and G/H be the homogeneous space we constructed in Lecture 2. The canonical projection $G \to G/H$ is a good geometric quotient. We omit the proof, referring the reader to **[Hum]**, IV,12, where all conditions of the definition are verified.

4.3 Geometric invariant theory suggests a method for constructing quotients. Unfortunately it applies only to a certain special class of algebraic groups. Let us recall some general facts about the structure of affine algebraic groups (see **[Hum]**).

We will be transferring the usual terminology of the theory of groups to algebraic groups. First of all it is time to introduce the notion of an invariant subgroup and the corresponding factor group. The first notion is easily defined by considering the adjoint action $Adj(g,x) = g \cdot x \cdot g^{-1}$ of G on itself. This is defined as the composition of the morphisms

$$G \times G \overset{(1,\beta)\times id_G}{\longrightarrow} G \times G \times G \overset{s}{\to} G \times G \times G \overset{\sigma \times id_G}{\longrightarrow} G \times G \overset{\sigma}{\longrightarrow} G,$$

where s switches the second and the third factor. Next we verify that for any invariant closed subgroup H the homogeneous space G/H has a structure of an affine algebraic group such that the map $G \to G/H$ is a homomorphism of algebraic groups which is universal with respect to homomorphisms $f : G \to G'$ with $H \subset Ker(f)$. We skip this construction but observe that the universal property follows easily from the fact that $G \to G/H$ is a geometric quotient. Our second remark (which should have been made much earlier) is that the properties of irreducibility and connectedness are equivalent when the variety is an algebraic group. This is easy to see by observing that any homogeneous space is a nonsingular variety. We denote the connected component of G containing the unity $e \in G(k)$ by G°. This is a closed invariant subgroup of G. The factor group G/G° is a finite algebraic group over k (not necessary constant).

An algebraic group T is called a *torus* if, considered as a $\bar{k}$-variety, it is isomorphic to $(\mathbf{G}_{\mathbf{m},\bar{k}})^n$. A torus is called *split* if this isomorphism is defined over the ground field k. Every torus can be split after a finite separable extension of k. An algebraic group is called *solvable* if it admits a composition series of closed normal subgroups whose successive quotients are abelian groups. An algebraic group is called *unipotent* if there exists a faithful linear representation $\rho : G \to \mathbf{GL}_k(n)$ such that the image is contained in the subgroup of upper-triangular matrices with diagonal entries equal to 1. A unipotent group is solvable, it contains a composition series with factors isomorphic to $\mathbf{G}_{\mathbf{a}k}$ or (if k is not perfect of $\mathrm{char}(k) = p > 0$) to the constant group $(\mathbf{Z}/p)_k$, associated to the group $\mathbf{Z}/p$. It is isomorphic to the semi-direct product of these groups. Each algebraic group G contains a maximal connected solvable invariant subgroup. It is called the *radical* of G and is denoted by $Rad(G)$. A group G is called *reductive* if its radical is a torus. A group G is called *semi-simple* if its radical is trivial. A maximal solvable subgroup B of a semi-simple group G is called a *Borel subgroup*. It has the property that for any subgroup P containing B the homogeneous space G/P is projective. Any two Borel subgroups are conjugate to each other. A Borel subgroup contains a maximal connected abelian subgroup isomorphic to a torus. It is called a *maximal torus*. All maximal tori are conjugate. Their common dimension is called the *rank* of G.

Each Borel subgroup is isomorphic to a semi-direct product (defined in the natural way) of its maximal unipotent subgroup U and a maximal torus contained in B. There exist two Borel subgroups B^+ and B^- such that $T = B^+ \cap B^-$ is a maximal torus in both of them. Let U^+ and U^- be the corresponding maximal unipotent subgroups of B^+ and B^-. Then the multiplication map $U^- \times T \times U^+ \to G$ is an isomorphism onto an open Zariski subset (called a *big cell*). Over a perfect field k any unipotent group is isomorphic to an affine space (as an algebraic variety). Thus after some finite extension k'/k, the group $G_{k'}$ contains an open subset isomorphic to the product of affine lines and a split torus. This is the fact we have used in Lecture 3.

There is a complete classification of semi-simple groups over an algebraically closed field k. Examples of simple groups are the classical groups

$$SL_k(n+1)(\text{type } A_n), O_k(2n+1)(\text{type } B_n), Sp_k(2n)(\text{type } C_n), O_k(2n)(\text{type } D_n).$$

There are also some exceptional groups of type F_4, G_2, E_6, E_7, E_8. Every simple algebraic group is isogenous to one of these groups (i.e. there exists a surjective homomorphism from one to another with a finite kernel). Here the subscript indicates the rank of the corresponding group.

4.4 Examples. 1. Let $G = \mathbf{GL}_k(n)$. Its radical is the subgroup of scalar matrices, and is isomorphic to $\mathbf{G}_{\mathbf{m},k}$. The quotient groups $\mathbf{SL}_k(n)$ is simple. Any Borel subgroup is conjugate to the subgroup $UT_k(n)$ of upper-triangular matrices. Its maximal unipotent subgroup is the group of upper-triangular matrices with diagonal entries equal to 1 (upper unipotent matrices). The maximal torus in this subgroup is the subgroup $D_k(n)$ of diagonal matrices. It is isomorphic to $(\mathbf{G}_{\mathbf{m},k})^{n-1}$. Thus $\mathbf{SL}_k(n)$ is of rank $n-1$. If $LT_k(n)$ denote the subgroup of lower triangular matrices, then $UT_k(n) \cap LT_k(n) = D_k(n)$. The LU-decomposition tells us that there exists an open Zariski subset of $\mathbf{SL}_k(n)(\bar{k})$ such that each matrix A from this set can be written as a product $A = LU$, where $L \in LT_k(n)(\bar{k}), U \in UT_k(n)(\bar{k})$. This decomposition of A can be easily converted to decomposition into the product of a lower unipotent matrix, a diagonal matrix and an upper unipotent matrix.
2. Let k_0 be a field of characteristic $p > 0$, and $k = k_0(t)$ be the field of rational functions with coefficients from k_0. Let G be the subgroup of $(\mathbf{G}_{\mathbf{a},k})^2$ defined by the equation

$Z_1^p + Z_1 = tZ_2^p$ in $\mathbf{A}_k^2$. For any K/k we have $G(K) = \{(a,b) \in K^2 : a^p + a = tb^p\}$. It is immediately verified that this is a subgroup of the vector group K^2. Since a subgroup of a unipotent group is obviously unipotent, G is a one-dimensional connected unipotent group. The projection to the second factor defines a surjective homomorphism $p : G \to \mathbf{G}_{\mathbf{m},k}$ of algebraic k-groups, with kernel $(\mathbf{Z}/p)_k$. However it is not isomorphic (even as a variety!) to $\mathbf{G}_{\mathbf{a},k}$. In fact one checks easily that the projective curve $X : T_1^p + T_1 T_0^{p-1} - tT_2^p = 0$ in $\mathbf{P}_k^2$ has no singular k-points and contains G as an open subset. If G were isomorphic to $\mathbf{A}_k^1$ we could extend this isomorphism to an isomorphism of projective curves $X \cong \mathbf{P}_k^1$. But this is absurd since X has a singular $\bar{k}$-point $(0, \sqrt[p]{t}, 1)$. However if we extend k by adding a p-th root of t, the variety X becomes isomorphic to $\mathbf{A}_k^1$. Note that the restriction of the homomorphism p to the open subset $\mathbf{G}_{\mathbf{a}k} \setminus \{0\}$ has all fibres isomorphic to $\mathbf{G}_{\mathbf{a}k}$, so it is a group scheme over $\mathbf{G}_{\mathbf{a}k} \setminus \{0\}$.

4.5 A reductive group over a field of characteristic 0 satisfies the following property of *linear reducibility*:

For every rational linear representation $\rho : G \to GL_k(n)$ and a G-invariant linear subspace V, there exists a G-invariant linear subspace V' such that $V \cap V' = \{0\}$ and $\mathbf{A}_k^n \cong V \times V'$.

In particular, if $v \in k^n \setminus \{0\}$ is $G(k)$-invariant, there exists a G-invariant linear function on k^n which does not vanish at v. To see this we write $k^n = kv \oplus V'$ for some $G(k)$-invariant subspace V' and consider the projection map to kv. Unfortunately, this property does not hold in general if k is of positive characteristic. However if we replace the words "linear function" with "homogeneous polynomial function" this property holds in any characteristic. This is a famous theorem of W. Haboush [**Hab**]. In fact, this is the main property which we shall need to construct quotients.

Definition. An affine algebraic group G is called *geometrically reductive* if for any rational linear representation $\rho : G \to GL_k(n)$ and a non-zero vector $v \in k^n$ there exists a non-constant G-invariant homogeneous polynomial function $f \in \mathcal{O}(\mathbf{A}_k^n)$ which does not vanish at v.

So by Haboush's Theorem any reductive group is geometrically reductive. Conversely, any geometrically reductive group must be reductive. (see [**MN**]). Note that in the case of positive characteristic a group G satisfying the property of *linear reducibility* has G° isomorphic to a torus and the order of G/G° is coprime to the characteristic.

4.6 We shall often use the following lemma.

Key Lemma. *Let X be an affine G-variety, and let Z_1 and Z_2 be two closed G-invariant k-subsets with $Z_1(\bar{k}) \cap Z_2(\bar{k}) = \emptyset$. Assume G is geometrically reductive. Then there exists a G-invariant function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z_1) = 0, \phi(Z_2) = 1$.*

Proof. First choose some $\varphi \in \mathcal{O}(X)$, not necessary G-invariant, such that $\varphi(Z_1) = 0, \varphi(Z_2) = 1$. This is easy. Since the sum of the ideals defining Z_1 and Z_2 is the unit ideal, we can find a function $\alpha \in I(Z_1)$ and a function $\beta \in I(Z_2)$ such that $1 = \alpha + \beta$. Then we take $\varphi = \alpha$. Applying the Lemma from Lecture 2, 2.2, we can find a G-invariant finite-dimensional vector k-subspace V of $\mathcal{O}(X)$ such that for each K/k, the space V_K is spanned by translates $g^*(\varphi), g \in G(K)$. Let $\varphi_1, \ldots, \varphi_n$ be its basis. Consider a map

$f : X \to \mathbf{A}_k^n$ defined by these functions. Clearly, $f(Z_1) = (0, \ldots, 0), f(Z_2) = (1, \ldots, 1)$. The group G acts linearly on the affine space defining a linear representation. By definition of geometrically reductive groups, we can find a non-constant G-invariant homogeneous polynomial $F \in k[Z_1, \ldots, Z_n]$ such that $F(1, \ldots, 1) \neq 0$. Then $\phi = f^*(F) = F(\varphi_1, \ldots, \varphi_n)$ satisfies the assertion of the lemma.

We shall apply this lemma to prove the main result of this Lecture:

Theorem (M.Nagata). *Let G be a geometrically reductive group acting on an affine variety X. Then the subalgebra $\mathcal{O}(X)^G$ is finitely generated over k, and if Y denotes an affine algebraic variety with $\mathcal{O}(Y) = \mathcal{O}(X)^G$, then the map $p : X \to Y$ induced by the inclusion of the k-algebras is a good categorical quotient.*

Proof. We postpone the proof of that $\mathcal{O}(X)^G$ is finitely generated until the next Lecture. To show that p is a categorical quotient, let us apply Proposition 2. First of all, property (i) easily follows from the fact that taking invariants commutes with localizations. More precisely, if $f \in \mathcal{O}(X)^G$, then $(\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f$. This is easy and we skip the proof. Let Z be a closed G-invariant subset of X. Suppose $p(Z)$ is not closed. Let $y \in \overline{p(Z)} \setminus p(Z)$. Then $W_1 = Z$ and $W_2 = p^{-1}(y)$ are two closed G-invariant subsets of X with empty intersection. By the Key Lemma, there exists a function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z) = 0, \phi(p^{-1}(y)) = 1$. Since $\phi = p^*(\varphi)$ for some $\varphi \in \mathcal{O}(Y)$, we obtain $\varphi(p(Z)) = 0, \varphi(y) = 1$. But this is absurd since y belongs to the closure of $p(Z)$. This verifies condition (ii). Now let Z_1 and Z_2 be two disjoint G-invariant closed subsets of X. As above we find a function $\varphi \in \mathcal{O}(Y)$ with $\varphi(p(W_1)) = 0, \varphi(W_2) = 1$. This obviously implies that $p(Z_1) \cap p(Z_2) \neq \emptyset$. This checks (iii).

4.7 Example. Let $G = \mathbf{GL}_k(N)$ act on itself by the adjoint action, i.e., $g \cdot x = gxg^{-1}$. For each matrix $g \in GL(n, K)$ we consider the characteristic polynomial

$$det(g - tI_n) = (-t)^n + c_1(g)(-t)^{n-1} + \ldots + c_n(g).$$

Define the maps $c_K : GL(n, K) \to K^n$ by the formula $c_K(g) = (c_1(g), \ldots, c_n(g))$. As is easy to see these maps define a G-equivariant morphism

$$c : GL_k(n) \to \mathbf{A}_k^n.$$

We claim that this a categorical quotient. To check this it is enough to verify that $\mathcal{O}(G)^G = k[c_1, \ldots, c_n)] \cong k[Z_1, \ldots, Z_n]$. Obviously $k[c_1, \ldots, c_n] \subset \mathcal{O}(G)^G$. Let $\phi \in \mathcal{O}(G)^G$. Let U be the subset of $GL(n, \bar{k})$ which consists of diagonalizable matrices with distinct eigenvalues. This is an open dense Zariski subset. The restriction of ϕ to U is determined by the restriction of ϕ to the subset $D(n)$ of diagonal matrices. Each such function is a symmetric polynomial function in the diagonal elements. By the theorem on elementary symmetric functions, it is a polynomial in $c_1|D(n), \ldots, c_n|D(n)$. Thus there exists a polynomial $F(Z_1, \ldots, Z_n)$ such that $\phi - F(c_1, \ldots, c_n) = 0$ when restricted to U. This implies that $\phi = F(c_1, \ldots, c_n)$.

4.8 The algebra of invariants $\mathcal{O}(X)^G$, where G is a reductive algebraic group, and X is an affine algebraic variety, inherits many algebraic properties of $\mathcal{O}(X)$. We shall not go into this interesting area of algebraic invariant theory, however we mention the following simple but important result.

Proposition. *Let G be a reductive algebraic group acting on a normal algebraic variety X. Then the categorical quotient $X//G$ is a normal algebraic variety.*

Proof. Let K be the field of rational functions on X. It is clear that the field L of rational functions on $X//G$ is contained in the subfiled K^G of G-invariant rational functions. We have to check that the ring $\mathcal{O}(X)^G$ is integrally closed in L. Suppose $x \in L$ satisfies a monic equation

$$x^n + a_1 x^{n-1} + \ldots + a_0 = 0$$

with coefficients a_i from $\mathcal{O}(X)^G$. Since X is normal, $x \in \mathcal{O}(X) \cap K^G = \mathcal{O}(X)^G$ and the assertion is verified.

Remark. One should not think that the field of rational functions of $X//G$ is equal to the field of G-invariant rational functions on X. This is not true in general. However this is true in the following cases:

(i) G is a finite group;
(ii) $\mathcal{O}(X)$ is a unique factorization domain and $\mathcal{X}(G) = \{1\}$;
(iii) the connected component of the identity of G is a solvable linear group.

Problems.

1. Let $\mathbf{G}_{\mathbf{a},k}$ act on $\mathbf{A}_k^2$ by the formula $t \cdot (z_1, z_2) = (z_1, z_2 + tz_1)$. Consider the map $\mathbf{A}_k^2 \to \mathbf{A}_k^1$, $(z_1, z_2) \mapsto z_1$. Is it a categorical quotient? If it is, is it a geometric quotient?
2. Let $\mathbf{G}_{\mathbf{m},k}$ act on $\mathbf{A}_k^n$ by the formula $t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n)$ for some positive integers $q_1, \ldots, q_n$ coprime to char(k). Show that the geometric quotient $\mathbf{A}_k^n \setminus \{0\}/\mathbf{G}_{\mathbf{m},k}$ constructed in Example 2 is isomorphic to a quotient of $\mathbf{P}^{n-1}$ by a finite group.
3. Let $A = \oplus_{i \in \mathbf{Z}} A_i$ be a graded finitely generated k-algebra, and $A^{(e)} = \oplus_{i \in \mathbf{Z}} A_{ei}$. Show that, if e is coprime to char(k), $A^{(e)} = A^G$, where G is a cyclic group of order e.
4. Show that $\mathbf{G}_{\mathbf{a},k}$ is not geometrically reductive.
5. In the notation of Nagata's Theorem show that for any open subset U of Y, the restriction map $p^{-1}(U) \to U$ is a categorical quotient with respect to the induced action of G.
6. Describe the orbits and the fibres of the categorical quotient from Example 4.7 when $n = 2$.
7. Let G act on an irreducible affine variety X and let $f : X \to Y$ be a G-invariant morphism to a normal affine variety. Assume that codim$(Y \setminus f(X)) \geq 2$ and there exists an open subset U of Y such that for all $y \in U$ the fibre f^{-1} contains a dense orbit. Show that $Y \cong X//G$.
8. Give an example of an irreducible affine G-variety X such that the field of fractions of $\mathcal{O}(X)^G$ is not equal to the field of G-invariant rational functions on X.
9. Prove the assertions (i) and (ii) from Remark 4.8.
10. Give an example of a torus which is not split over the ground field.

Lecture 5. HILBERT'S FOURTEENTH PROBLEM

5.1 Here we shall prove the assertion in Nagata's Theorem that the ring of invariants of a geometrically reductive group is finitely generated. We shall also give a counter-example of Nagata for a group which is not geometrically reductive. These results are all related to one of the Hilbert's Fourteenth Problems. The precise statement of this problem is as follows:

Problem. *Let k be a field, $k(t_1,\dots,t_n)$ be its purely transcendental extension, and let K/k be a field extension contained in $k(t_1,\dots,t_n)$. Is the k-algebra $K\cap k[t_1,\dots,t_n]$ finitely generated?*

Hilbert himself gave a positive answer to this question in the situation when K is the field of rational functions invariant with respect to a linear action of $G = \mathbf{SL}_k(n)$ in $k[t_1,\dots,t_n]$. The subalgebra $K\cap k[t_1,\dots,t_n]$ is of course the subalgebra of invariant polynomials $k[t_1,\dots,t_n]^G$. A special case of his problem asks whether the same is true for an arbitrary group G acting linearly in the ring of polynomials. A first counter-example was given by M. Nagata in 1959 **[Na2]**. We shall briefly explain it in this lecture. Let us first give a geometric interpretation of Hilbert's Problem 14 due to O. Zariski.

For any subfield $K\subset k(t_1,\dots,t_n)$ we can find an irreducible algebraic variety X over k with the field of rational functions $k(X)$ isomorphic to K. The inclusion of the fields gives rise to a rational map

$$f:\mathbf{P}_k^n \dashrightarrow X.$$

Let $Z\subset \mathbf{P}_k^n\times X$ be the closure of the graph of the regular map of some open subset of $\mathbf{P}_k^n$ defined by f. Let H be the hyperplane at infinity in $\mathbf{P}_k^n$ and $D = pr_2(pr_1^{-1}(H))$. This is a closed subset of X. By blowing up, if necessary, we may assume that D is the union of codimension 1 irreducible subvarieties D_i and $pr_1(pr_2^{-1}(D_i))$ is contained in H. Thus for any rational function $\phi\in k(X), f^*(\phi)$ is regular on $\mathbf{P}_k^n\setminus H$ if and only if ϕ has poles only along the D_i's. But, after identifying $k(X)$ with K (by means of f^*) and $\mathcal{O}(\mathbf{P}_k^n\setminus H)$ with $k[t_1,\dots,t_n]$, this implies that $K\cap k[t_1,\dots,t_n]$ is isomorphic to the ring $R(D)$ of rational functions on X with poles only along the D_i's. So the problem is reduced to the problem of finite-generatedness of the algebras $R(D)$ where D is any *positive divisor* (union of codimension 1 irreducible subvarieties) on an algebraic variety X. Assume moreover that X is nonsingular. Then each positive divisor can be given locally by an equation $\phi_U = 0$, where ϕ_U on X is regular on some open subset U of X. These functions must satisfy $\phi_U = g_{UV}\phi_V$ on $U\cap V$ for some $g_{UV}\in\mathcal{O}(U\cap V)^*$. This leads us to a line bundle $L(D)$ defined by the transition functions g_{UV}. Rational functions R with poles along D must satisfy $a_U = R\phi_U^n\in\mathcal{O}(U)$ for some $n\geq 0$. This implies that the functions a_U satisfy

$a_U = g^n_{UV} a_V$; hence they form a section of the line bundle $L(D)^{\otimes n}$. This shows that the algebra $R(D)$ is equal to the union of linear subspaces $\Gamma(X, L(D)^{\otimes n})$ of the field $k(X)$. Let

$$R^*(D) = \oplus_{n\geq 0}\Gamma(X, L(D)^{\otimes n}).$$

Recall that we can view $\Gamma(X, L(D)^{\otimes n})$ as the space of regular functions on the line bundle $L(D)^{-1}$ whose restriction to fibres are homogeneous polynomials of degree n. This allows one to consider the algebra $R^*(D)$ as the algebra $\mathcal{O}(L(D)^{-1})$. Let P be the variety obtained from $L(D)^{-1}$ by adding the point at infinity in each fibre of L^{-1}. More precisely, let $\mathcal{O}_X$ be the trivial line bundle. Then the variety P can be constructed as the quotient of the rank 2 vector bundle $L(D)^{-1} \oplus \mathcal{O}_X \setminus \{\text{zero section}\}$ by the group $\mathbf{G}_{m,k}$ acting diagonally on fibres. Here the direct sum means that the transition functions of the vector bundle are chosen to be diagonal matrices

$$\begin{pmatrix} g_{UV} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we obtain that $R^*(D)$ is equal to the ring $R(S)$ where S is the divisor at infinity in P. This leads to the following:

Problem (Zariski). *Let X be a nonsingular algebraic variety and D be a positive divisor on X. Is the algebra $R^*(D)$ finitely generated?*

We shall see later that Nagata's counterexample to the Hilbert problem is of the form $R^*(D)$. It turns out that the algebras $R^*(D)$ are often non-finitely generated. However if we impose conditions on D (for example, that the complete linear system defined by $L(D)$ has no base points) then it is finitely generated. One of the fundamental questions in algebraic geometry is the question of finite generatedness of the ring $R^*(D)$, where D is the canonical divisor of X. This is closely related to the theory of minimal models of algebraic varieties (see **[Mor]**).

5.2 Let us now turn our attention to the algebras of invariants. First we consider a simple situation. Let G be a reductive group over a field of characteristic zero acting linearly in $\mathbf{A}^n_k$. Let $A = \mathcal{O}(\mathbf{A}^n_k) = k[Z_1, \ldots, Z_n]$. It is clear that G leaves each subspace $A_d = k[Z_1, \ldots, Z_n]_d$ invariant and

$$A^G = \oplus_{d\geq 0} A^G_d.$$

The linear representation of G in the space A_d is completely reducible (here we use that $\mathrm{char}(k) = 0$). Let $r_d : A_d \to A^G_d$ be the G-invariant projection operator. The sum of these operators defines a projection

$$r : A \to A^G.$$

Using the uniqueness of the operators r_d it is easy to see that it is a homomorphism of A^G-modules, i.e., $r(ab) = ar(b)$ for any $a \in A^G, b \in A$.

Now we consider the ideal I in A generated by homogeneous polynomials of positive degree from A^G. By Hilbert's Basis Theorem (which he proved exactly for this purpose!),

I is generated by a finite set of elements $f_1, \ldots, f_N$. We may assume that each f_i is a homogeneous polynomial of some degree d_i from A^G. Let us show that the polynomials f_i generate A^G as a k-algebra. For any $f \in A_d^G$ we can write

$$f = \sum_{i=1}^{N} a_i f_i, \quad a_i \in A_{d-d_i}.$$

After applying r, we get

$$f = r(f) = \sum_{i=1}^{N} r(a_i f_i) = \sum_{i=1}^{N} r(a_i) f_i.$$

By induction on the degree of f we may assume that the $r(a_i)$ are all polynomials in the f_i's; hence f is a polynomial in the f_i as well.

5.3 To move from $k[Z_1, \ldots, Z_n]^G$ to A^G, where A is any finitely generated algebra, we need the following result:

Lemma 1. *Let $\rho : G \to \mathbf{GL}_k(n)$ be a linear representation of a reductive algebraic group G over a field k of characteristic zero, and let X be a G-invariant closed algebraic subvariety in $\mathbf{A}_k^n$ defined by an ideal $I \subset \mathcal{O}(\mathbf{A}_k^n) = k[Z_1, \ldots, Z_n]$. Then*

$$\mathcal{O}(X)^G \cong k[Z_1, \ldots, Z_n]^G / (I \cap k[Z_1, \ldots, Z_n]^G).$$

Before we prove this lemma, we introduce some terminology. We say that an affine algebraic group G *acts rationally* on a k-algebra A if there is given a coaction map $\sigma^* : A \to \mathcal{O}(G) \otimes A$ in the sense of Lecture 1. When $A = \mathcal{O}(X)$ for some affine algebraic variety (e.g. A is finitely generated k-algebra) this is equivalent to an action of G on X. We say that an ideal $I \subset A$ is G-invariant if $\sigma^*(I) \subset \mathcal{O}(G) \otimes I$. Again if A is finitely generated this is the same as saying that the affine subvariety of X defined by I is G-invariant. Clearly the action of G on A induces an action of G on A/I. So Lemma 1 follows from the following more general result:

Lemma 1′. *Let a geometrically reductive algebraic group G act rationally on a k-algebra A leaving an ideal I invariant. For any $a \in (A/I)^G$ there exists $d > 0$ such that $a^d \in A^G/(I \cap A^G)$. If G is reductive and char$(k) = 0$ then d can be chosen to be 1, i.e., $(A/I)^G \subset A^G/(I \cap A^G)$.*

Here as usual $A^G = \{a \in A : \sigma^*(a) = 1 \otimes a\}$ is the subalgebra of G-invariant elements

Proof. Let $\bar{a}$ be a non-zero element from $(A/I)^G$, and let a be its representative in A. Let V be a finite-dimensional G-invariant subspace of A spanned by the translates of a. If

$$\sigma^*(a) = \sum_i \alpha_i \otimes a_i,$$

then V is spanned by the a_i's. We denote by $\bar{\sigma}^*$ the induced co-action map $A/I \to \mathcal{O}(G) \otimes A/I$. We have

$$\bar{\sigma}^*(\bar{a}) = \sum_i \alpha_i \otimes \bar{a}_i = 1 \otimes \bar{a}.$$

This shows that $W := V \cap I$ is a G-invariant subspace of codimension 1 in V and we can write any element of V in the form

$$v = \lambda a + w$$

for some $\lambda \in k$ and $w \in W$. Let $l : V \to k$ be the linear map $v \mapsto \lambda$. Since

$$\sigma^*(\lambda a + w) = \lambda(\sigma^*(a) - 1 \otimes a) + \lambda(1 \otimes a) + \sigma^*(w) \in \lambda(1 \otimes a) + \mathcal{O}(G) \otimes W$$

the map l is G-invariant. Consider it as an element of the dual space V^*, where the group G acts on V^* naturally and l is a G-invariant element. Choose a basis $(v_1, \ldots, v_n)$ of V with $v_1 = a$, and $v_i \in W$ for $i \geq 2$. Then we can identify V^* with $\mathbf{A}_k^n$, by using the dual basis so that $l = (1, 0, \ldots, 0)$. By definition of geometric reductivity, we can find a G-invariant homogeneous polynomial $F(Z_1, \ldots, Z_n)$ of degree d such that $F(1, 0, \ldots, 0) \neq 0$. We may assume that $F = Z_1^d + \ldots$. Now we can identify v_i with Z_i, hence $F(v_1, \ldots, v_n) - a^d \in (v_2, \ldots, v_n) \subset I \cap A^G$. Since $F(v_1, \ldots, v_n) \in A^G$, we are done.

5.4 Thus we have proved that $\mathcal{O}(X)^G$ is finitely generated in the case that G is a reductive group over a field of characteristic 0. Let us now treat the general case of a geometrically reductive group.

Theorem (Nagata). *Let A be a finitely generated k-algebra and let G be a geometrically reductive group acting rationally on A. Then the subalgebra A^G is finitely generated.*

Proof. We would like to assume that $A = S/I$, where S is a polynomial algebra on which G acts linearly, inducing the action of G on A. To do so, we choose a set of generators for A, and then consider the linear action of G on the vector space V spanned by the translates of the generators. If $(f_1, \ldots, f_N)$ is a basis of V, then we consider the surjection $S = k[Z_1, \ldots, Z_n] \to A, Z_i \mapsto f_i$,. Since V contains the set of generators of A, this is a surjective G-equivariant homomorphism.

We will use Lemma 1$'$ again; it implies that the algebra $(A/I)^G$ is integral over $A^G/(I \cap A^G)$. Here we consider the first algebra as an algebra over the second one by means of the canonical injective homomorphism $A^G/(I \cap A^G) \to (A/I)^G$ induced by the projection $A \to A/I$. In particular, we obtain

$$(A/I)^G \quad \text{is finitely generated} \Rightarrow A^G/(I \cap A^G) \quad \text{is finitely generated.}$$

The converse is true if $(A/I)^G$ is a domain and its field of fractions is finitely generated over the field of fractions of $A^G/(I \cap A^G)$.

Assume $A^G = (S/I)^G$ is not finitely generated. Let $\mathcal{N}$ be the set of all ideals $\Im$ in S with the property that $(S/\Im)$ is not finitely generated. This set is nonempty, so we can find a maximal element Q. We get a contradiction if we show that $(S/Q)^G$ is finitely generated. Thus we may assume that our ideal $I = Q$ and hence satisfies the property that for any ideal J' in S strictly containing I, the algebra $(S/J')^G$ is finitely generated. Equivalently, for any nonzero ideal J in $A = S/I, (A/J)^G$ is finitely generated.

Suppose for a moment that I is a homogeneous ideal. Then A inherits the standard grading of S, and since G preserves the grading of S, S^G and A^G are graded k-algebras. Let J be a non-zero ideal of A. Since $(A/J)^G$ is finitely generated, by above, $A^G/(J \cap A^G)$ is finitely generated. If the ideal $J \cap A^G$ is finitely generated, we add its set of generators to the set of representatives in A^G of generators of the algebra $A^G/J \cap A^G$ to obtain that the ideal $(A^G)_+ = \oplus_{d>0}(A^G)_d$ is finitely generated. By using the same inductive argument as in 5.2, we obtain that A^G is finitely generated k-algebra. So we find the contradiction as soon as we find such J. If A/I has no zero divisors this is really easy. One can take J to be the principal ideal fA, where $f \in (A^G)_+$. Then $fA \cap A^G = fA^G$ since, for any $x \in A$, $g^*(fx) - fx = f(g^*(x) - x) = 0$ for all $g \in G(K)$ implies $x \in A^G$.

So we may assume that any $f \in (A^G)_+$ is a zero divisor. Then the annulator ideal $R := \{a \in A : fa = 0\}$ is non-zero. The algebras $A^G/(fA \cap A^G)$ and $A^G/(R \cap A^G)$ are finitely generated (because $(A/fA)^G$ and $(A/R)^G$ are). Let B be the subring of A^G generated by representatives of generators of both algebras. It is mapped surjectively to both the algebras $A^G/(fA \cap A^G)$ and $A^G/(R \cap A^G)$. Let $c_1, \ldots, c_n$ be representatives in A of generators of $(A/R)^G$ as a $B/(R \cap B)$-module. Since $g^*(c_i) - c_i \in R$ for all $g \in G(K)$, we get $f(g^*(c_i) - c_i) = 0$, i.e., $fc_i \in A^G$. Let us show that $A^G = B[fc_1, \ldots, fc_n]$. Then we will be done (in the homogeneous case). If $a \in A^G$, we can find $b \in B$ such that $a - b \in fA$ (since B is mapped surjectively to $A^G/fA \cap A^G$). Then $a - b = fr$ is G-invariant implies that r is G-invariant modulo I. Thus there is an element $c \in \Sigma_i Bc_i$ such that $r - c \in I$. This implies $a = b + fr = b + fc \in B[fc_1, \ldots, fc_n]$ as we wanted.

Let us no longer assume that I is a homogeneous ideal. Again we choose Q as above (not necessarily homogeneous). If $A^G = (S/Q)^G$ contains a zero-divisor f we apply the previous argument to get a contradiction. Otherwise, A^G is a domain integral over $S^G/(Q \cap S^G)$. By the previous case S^G is finitely generated (as G acts linearly on it); hence, $S^G/(Q \cap S^G)$ is finitely generated. Now A^G is finitely generated provided we verify that its field of fractions is finitely generated over k. If A were a domain this is obvious (a subfield of a finitely generated field is finitely generated). In the general case we use the total ring of fractions of A, the localization A_S with respect to the set S of non-zero-divisors. For any maximal ideal m of A_S we have $m \cap A^G = 0$ since A^G is a domain. This shows that the field of fractions of A^G is a subfield of A_S/m. But the latter is a finitely generated field equal to the field of fractions of $A/m \cap A$. The proof is now complete.

5.5 Let us discuss the case of not necessarily geometrically reductive groups. We shall give later an example of Nagata which shows that A^G is not finitely generated for a particular non-reductive group G. Notice that according to a result of Vladimir Popov [**Pop**], if A^G is finitely generated for any finitely generated algebra A, then G must be geometrically reductive.

This shows that for any non-reductive G there exists a finitely generated k-algebra on which G acts rationally such that A^G is not finitely generated. Unfortunately Popov's proof does not give any explicit example of such A.

Since any affine algebraic group H is a closed subgroup of a reductive group G, we may ask how the rings A^G and A^H are related.

Lemma (Groshans Principle). *Let an algebraic group G act rationally on a finitely generated k-algebra A. Then*

$$A^H \cong (\mathcal{O}(G)^H \otimes A)^G.$$

Proof. Let X be an affine variety with $\mathcal{O}(X) = A$. Consider the action of $G \times H$ on $G \times X$, defined by the formula $(g,h)\cdot(g',x) = (gg'h^{-1}, gx)$. Then

$$\mathcal{O}(G \times X)^{G\times H} \cong (\mathcal{O}(G)^H \otimes \mathcal{O}(X))^G.$$

On the other hand, the projection $G \times X \to X$ is a morphism of the $G \times H$-variety $G \times X$ to the H-variety X with respect to the projection homomorphism $G \times H \to H$. This defines a morphism of categorical quotients

$$(G \times X)//G \times H \to X//H.$$

Its inverse is defined by the natural inclusion map $X \hookrightarrow G \times X, x \mapsto (e,x)$. Thus

$$\mathcal{O}(X)^H = \mathcal{O}(X//H) \cong \mathcal{O}(G \times X//(G\times H)) \cong \mathcal{O}(G \times X)^{G\times H} \cong (\mathcal{O}(G)^H \otimes \mathcal{O}(X))^G.$$

Corollary. *Assume $\mathcal{O}(G)^H = \mathcal{O}(G/H)$ is finitely generated. Then A^H is finitely generated*

Example. Let H be a Borel subgroup of a connected reductive group G. Then G/H is projective and hence $\mathcal{O}(G/H) = k$ is finitely generated. Hence A^H is finitely generated.

Of course to apply the Groshans Principle we have to verify first that the action of H is obtained by restriction of an action of a reductive group G containing H as a closed subgroup. Let us give an application of this principle by proving the following classical result:

Weitzenböck's Theorem. *Let $\rho : \mathbf{G}_{\mathbf{a},k} \to \mathbf{GL}_k(n)$ be a linear representation of the additive group. Then $\mathcal{O}(\mathbf{A}_k^n)^{\mathbf{G}_{\mathbf{a},k}}$ is finitely generated.*

Proof. We assume $\mathrm{char}(k) = 0$, referring for the general case to **[Fau]**. Using a Lie algebra argument one can show that the image of $\mathbf{G}_{\mathbf{a},k}$ is contained in a maximal unipotent subgroup of $\mathbf{GL}_k(n)$ (see **[Bor]**, Chapter I,§4). Using the Jordan decomposition it easily implies that ρ can be extended to a linear representaion $\rho' : \mathbf{SL}_k(2) \to \mathbf{GL}_k(n)$, where $\mathbf{G}_{\mathbf{a},k}$ is considered as the subgroup U of upper-triangular unipotent matrices in $\mathbf{SL}_k(2)$. Applying Groshans's principle to the pair $(G,H) = (\mathbf{SL}_k(2), \mathbf{G}_{\mathbf{a},k})$ acting on $\mathbf{A}_k^n$, we see that it is enough to check that $\mathcal{O}(\mathbf{SL}_k(2)/U)$ is finitely generated. Since, for any K/k, $SL(2,K)$ acts transitively on $K^2 \setminus \{0\}$ with stabilizer of $(1,0)$ isomorphic to $U(K)$, we obtain that $\mathbf{SL}_k/U$ is naturally isomorphic to $\mathbf{A}_k^2 \setminus \{0\}$. Hence $\mathcal{O}(\mathbf{SL}_k(2)/U) \cong \mathcal{O}(\mathbf{A}_k^2)$ is finitely generated.

5.6 Now we are ready to present Nagata's counter-example to the 14th Hilbert Problem. The group G here is isomorphic to the product of $n-3$ copies of the additive group and the torus $T = (\mathbf{G}_{m,k})^{n-1}$. We shall specify the number n later. Our group G will act linearly on the space $\mathbf{A}_k^{2n}$ as follows. Let G' be the subgroup of $\mathbf{SL}_k(2n)$ with $G'(K)$ equal to the set of matrices of the form:

$$\begin{pmatrix} c_1 & a_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & c_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & c_2 & a_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & c_2 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & 0 & c_n & a_n \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & c_n \end{pmatrix}.$$

Here $a_i \in K, c_i \in K^*, c_1 \dots c_n = 1$. The group G' acts on $\mathbf{A}_k^{2n}$ via the natural action of $\mathbf{SL}_k(2n)$ on $\mathbf{A}_k^{2n}$. If we separate the variables to denote any vector $(x_1, \dots, x_{2n}) \in K^{2n}$ by $(x_1, y_1, \dots, x_n, y_n)$, then $g \in G'(K)$ is given by the matrix as above acts by the formula:

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (c_1x_1 + a_1y_1, c_1y_1, \dots, c_nx_n + a_ny_n, c_ny_n).$$

We realize G as the subgroup of G' of matrices where the non-diagonal elements $(a_1, \dots, a_n)$ belong to the linear subspace defined by a system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0, i = 1, 2, 3.$$

So the assertion is that the ring of invariants $k[X_1, \dots, X_n, Y_1, \dots, Y_n]^G$ is not finitely generated under some assumption on n and on k which we shall explain later.

Lemma 1. *Assume k is infinite and a_{ij} are algebraically independent over the prime field contained in k. Then*

$$k(X_1, \dots, X_n, Y_1, \dots, Y_n)^G = k(T, Z_1, Z_2, Z_3),$$

where

$$T = Y_1 \dots Y_n, \ Z_i = \sum_{j=1}^{n} a_{ij}(\frac{X_jT}{Y_j}), i = 1, 2, 3.$$

Proof. Under the action of g, defined by the matrix from above, we have

$$g^*(\frac{X_j}{Y_j}) = \frac{X_j}{Y_j} + a_j, \ g^*(T) = T$$

and, since $\sum_{j=1}^{n} a_{ij}a_j = 0$, we obtain that $g^*(Z_i) = Z_i, i = 1, 2, 3$. This checks that the right-hand-side is contained in the left-hand-side. Since a_{ij} are algebraically independent over the prime field, we can express $X_iT/Y_i, i = 1, 2, 3$, linearly through Z_1, Z_2, Z_3 to obtain

$$k(X_1, \dots, X_n, Y_1, \dots, Y_n) = k(Z_1, Z_2, Z_3, X_4, \dots, X_n, Y_1, \dots, Y_n) =$$

$$= k(T, Z_1, Z_2, Z_3, X_4, \ldots, X_n, Y_1, \ldots, Y_{n-1}).$$

Let H be the subgroup of G given by the equations $x_5 = \ldots = x_n = 0, c_i = 1, i = 1, \ldots, n$. Obviously it is isomorphic to $\mathbf{G}_{\mathbf{a},k}$. We see that

$$k(X_1, \ldots, X_n, Y_1, \ldots, Y_n)^G \subset k(T, Z_1, Z_2, Z_3, X_4, \ldots, X_n, Y_1, \ldots, Y_{n-1})^H =$$

$$= k(t, Z_1, Z_2, Z_3, X_5, \ldots, X_n, Y_1, \ldots, Y_{n-1}).$$

Continuing in this way, we eliminate $X_5, \ldots, X_n$ to obtain

$$k(X_1, \ldots, X_n, Y_1, \ldots, Y_n)^G \subset k(T, Z_1, Z_2, Z_3, Y_1, \ldots, Y_{n-1}).$$

Now we throw in the torus part T which acts on Y_i by multiplying them by c_i. It is clear that any T-invariant rational function in $Y_1, \ldots, Y_{n-1}$ with coefficients from the field $k(T, Z_1, Z_2, Z_3)$ must be a constant. This proves the lemma.

Consider now the polynomial ring $k[Z_1, Z_2, Z_3]$ and view any column (a_{1j}, a_{2j}, a_{3j}) of the matrix (a_{ij}) as the homogeneous coordinates of a point P_i in the projective plane $\mathbf{P}^2_k$. Let $R(m)$ be the ideal in $k[Z_1, Z_2, Z_3]$ generated by homogeneous polynomials F such that each P_i is a point of multiplicity $\geq m$ on the curve $F = 0$.

Lemma 2.

$$k[X_1, Y_1, \ldots, X_n, Y_n]^G = k(T, Z_1, Z_2, Z_3) \cap k[X_1, Y_1, \ldots, X_n, Y_n] =$$

$$= \{\sum_{m \in \mathbf{Z}} F(Z_1, Z_2, Z_3) T^{-m} : F \in R(m)\}.$$

Proof. We only sketch the proof referring for the details to the original paper of Nagata [**Na2**]. Let $V_i = X_i T / Y_i, i = 1, \ldots, n$. Since a_{ij} are algebraically independent over k, we have

$$k[V_1, \ldots, V_n] = k[Z_1, Z_2, Z_3, V_4, \ldots, V_n].$$

This implies that

$$k[Z_1, Z_2, Z_3, X_4, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] = k[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}].$$

The intersection of the first ring with $k(Z_1, Z_2, Z_3, Y_1, \ldots, Y_n)$ is equal to

$$k[Z_1, Z_2, Z_3, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}].$$

The intersection of the latter ring with $k(T, Z_1, Z_2, Z_3)$ is equal to $k[T, T^{-1}, Z_1, Z_2, Z_3]$. Consider the difference

$$z_i = a_{3i} Z_1 - a_{1i} Z_3 =, z_i' = a_{3i} Z_2 - a_{2i} Z_3.$$

As is easily seen they are both divisible by Y_i in the polynomial ring $k[X, Y] = k[X_1, \ldots, X_n, Y_1, \ldots, Y]$. Since z_i and z_i' generate (after dehomogenization) the maximal ideal of the point P_i, we see that for any polynomial $F \in R(m), FT^{-m} \in k[X, Y]$. We skip the proof of the converse.

Lemma 3. *For any homogeneous ideal $I \subset k[Z_1, Z_2, Z_3]$ let $\deg(I)$ denote the smallest positive integer d such that $I \cap k[Z_1, Z_2, Z_3]_d \neq \{0\}$. Assume that n is chosen to be such that $\deg(R(m)) > m\sqrt{n}$ for all $m > 0$. Then for any natural number m there exists a natural number N such that $R(m)^N \neq R(mn)$.*

Proof. Let $R(m)_d = k[Z_1, Z_2, Z_3]_d \cap R(m)$ be the space of homogeneous polynomials of degree d from $R(m)$. The expected dimension of this space is $(d+2)(d+1)/2 - n(m+1)m/2$. We use the fact the condition that P_i is a point of multiplicity $\geq m$ is expressed by vanishing of all derivatives of the dehomogenized polynomial up to the order $m-1$. Thus we see we see that $\lim_{m\to\infty} \deg(R(m))/m \leq \sqrt{n}$. In view of our assumption we must have $\lim_{m\to\infty} \deg(R(m))/m = \sqrt{n}$. Since again by assumption $\deg(R(m))/m > \sqrt{n}$ we see that for sufficiently large N, $\deg R(mN) < \deg R(m)^N = N\deg(R(m))$. This implies that $R(mN)$ is strictly larger than $R(m)^N$.

Lemma 4. *The assumptions of the previous lemma are satisfied when $n = s^2$ where $s \geq 4$ and the coordinates of points P_i generate a field of sufficiently high transcendence degree over k.*

Proof. We omit the proof of this Lemma.

Let us show now that these four lemmas imply the assertion. Assume the algebra $k[X, Y]^G$ is generated by finitely many polynomials $P_i(X, Y)$. We can write them in the form $P_i = \sum_m F_{i,m} T^{-m}$ as in Lemma 2. Let $r = max_{i,m}\{degF_{i,m}\}$. By lemma 3, we can find $F \in R(rN)$ for suffiently large N such that $F \notin R(r)^N$. Obviously $P = FT^{-rN}$ can not be expressed as a polynomial in F_i's. This contradiction proves the assertion.

Problems.

1. Show that the algebra $k[X,Y]^G$ from the counter-example of Nagata is isomorphic to the algebra $R^*(D)$ where D is the exceptional divisor on the surface S obtained by blowing up a finite set of points in $\mathbf{P}_k^2$.
2. Prove that the algebra $R^*(D)$ is finitely generated if the the line bundle $L(D)$ is ample.
3. Give an example of a homogeneous space G/H such that $\mathcal{O}(G/H)$ is not finitely generated.
4. Let H be a closed reductive subgroup of an affine algebraic group G which acts on G by left translations. Show that the homogeneous space G/H is affine.

Lecture 6. STABILITY

¿From now on we will assume that G is a *reductive* algebraic group acting on an irreducible algebraic variety X. In this lecture we shall explain a general construction of geometric and categorical quotients suggested by D. Mumford. The idea is to cover X by open affine G-invariant sets U_i and then to construct the categorical quotient $X//G$ by gluing together the quotients $U_i//G$. The latter quotients are defined by Nagata's theorem. Unfortunately, such a cover does not exist in general. However we find such a cover for some open subset of X, so we can define only a "partial" quotient $U//G$. The construction of U depends on a parameter, a choice of a G-linearized line bundle L.

6.1 Definition. *Let L be a G-linearized line bundle on X. We set*

(i) $X^{ss}(L) = \{x \in X(\bar{k}) : \exists s \in \Gamma(X, L^{\otimes n})^G$ *for some* $n \geq 0$, *such that the set* $X_s := \{y \in X : s(y) \neq 0\}$ *is affine and contains* $x\}$. *A point from this set is called semi-stable with respect to* L.

(ii) $X^s(L) = \{x \in X(\bar{k}) : \exists s \in \Gamma(X, L^{\otimes n})^G$ *for some* $n \geq 0$, *such that the set* X_s *is affine, contains* x *and the action of* G *in* X_s *is closed* $\}$. *A point from this set is called stable with respect to* L.

(iii) $X^s(L)_{(0)} = \{x \in X^s(L) : G_x$ *is a finite group* $\}$. *A point from this set is called properly stable.*

(iv) $X^{sss}(L)) = X^{ss}(L) \setminus X^s(L)$. *A point from this set is called strictly semi-stable with respect to* L.

(v) $X^{us}(L) = X \setminus X^{ss}(L)$. *A point of this set is called unstable with respect to* L.

Remarks. 1. Obviously the subsets $X^{ss}(L)$ and $X^s(L)$ are open and G–invariant (but could be empty).

2. If L is ample and X is projective, the sets X_s are always affine, so this condition in the definition of semi-stable points can be dropped. In fact, for any $n > 0$, $X_{s^n} = X_s$ so we may assume that L is very ample. Let $f : X \to \mathbf{P}_k^N$ be a closed embedding defined by some complete linear system associated to L. Then X_s is equal to the pre-image of an affine open subset in $\mathbf{P}_k^N$ which is the complement of a hyperplane. Because a closed subset of an affine set is affine, we obtain the assertion.

3. The restriction of L to $X^{ss}(L)$ is ample. This follows from the following criterion of ampleness: L is ample on a variety X if and only if there exists an affine open cover of X formed by the sets X_s, where s is a global section of some tensor power of L. We refer for the proof to **[Har]**, p.155.

4. The definition of the sets $X^{ss}(L), X^s(L)$, and so on does not change if we replace L by $L^{\otimes n}$ (as a G-linearized line bundle).

5. Assume L is ample. If $x \in X^{ss}(L)$ and the orbit $G \cdot x$ is closed in $X^{ss}(L)$ and G_x is finite, then $x \in X^s(L)_{(0)}$. In fact let $x \in X_s$ be semi-stable. Then the set $Z = \{y \in X_s : \dim G_y > 0\}$ is closed in X_s and does not contain $G \cdot x$. As G is reductive, there exists a function $\phi \in \mathcal{O}(X_s)^G$ such that $\phi(x) \neq 0, \phi(Z) = 0$. One can show that there exists some number $r > 0$ such that $\phi s^{\otimes r}$ extends to a section s' of some tensor power of L (see **[Har]**, Chapter II, 5.14). Since X is irreducible, this section must be G-invariant. Thus $x \in X_{s'} \subset X_s$ and each point in $X_{s'}$ has a 0-dimensional stabilizer. This implies that the orbits of all points in $X_{s'}$ are closed in $X_{s'}$. This checks that x is stable. In fact, it is properly stable.

6*. Let $i : Y \hookrightarrow X$ be a closed G-invariant embedding, and $L_Y = i^*(L)$ where L is a an ample G-linearized line bundle on X. Assume that X is projective and G is linearly reductive, e.g. with char$(k) = 0$. Then for any $y \in Y(\bar{k})$

$$y \in Y^{ss}(i^*(L)) \Leftrightarrow i(y) \in X^{ss}(L),$$

$$y \in Y^{s}(i^*(L)) \Leftrightarrow i(y) \in X^{s}(L),$$

$$y \in Y^{s}(i^*(L))_{(0)} \Leftrightarrow i(y) \in X^{s}(L)_{(0)}.$$

First we use that the canonical map

$$\Gamma(i^*) : \Gamma(X, L^{\otimes N})^G \to \Gamma(Y, i^*(L)^{\otimes N})^G$$

is surjective for sufficiently large N. To see this we consider the invertible sheaf $\mathcal{L}$ of sections of L and the exact sequence

$$0 \to \mathcal{I}_Y \otimes \mathcal{L}^{\otimes N} \to \mathcal{L}^{\otimes N} \to i^*(\mathcal{L})^{\otimes N} \to 0,$$

where $\mathcal{I}_Y$ is the sheaf of ideals of Y. Applying the exact sequence of cohomology and using the fact that $H^1(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes N}) = 0$ for sufficiently large N since L is ample (see **[Har]**, p. 229), we get the exact sequence

$$0 \to \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes N}) \to \Gamma(X, \mathcal{L}^{\otimes N}) \to \Gamma(Y, i^*(\mathcal{L})^{\otimes N}) \to 0.$$

Since G is linearly reductive, we obtain an exact sequence

$$0 \to \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes N})^G \to \Gamma(X, \mathcal{L}^{\otimes N})^G \to \Gamma(Y, i^*(\mathcal{L})^{\otimes N})^G \to 0.$$

¿From this, our result follows quickly. For any $s \in \Gamma(X, L^{\otimes N})^G, s(i(y)) \neq 0$ implies $\Gamma(i^*)(s)(y) \neq 0$. Conversely if $s' \in \Gamma(X, i^*(L)^{\otimes N})^G$ with $s'(y) \neq 0$, then we find some $s \in \Gamma(X, L^{\otimes N})^G$ with $\Gamma(i^*)(s) = s'$. Obviously $s(i(y)) \neq 0$. This proves the assertion for the sets of semi-stable points. The remaining assertions are now obvious.

6.2 Theorem. *There exists a good categorical quotient $\pi : X^{ss}(L) \to X^{ss}(L)//G$. There is an open subset U in $X//G$ such that $X^s(L) = \pi^{-1}(U)$ and $\pi|X^s(L) : X^s(L) \to U$ is a geometric quotient of $X^s(L)$ by G. Moreover there exists an ample line bundle M on $X^{ss}(L)//G$ such that $\pi^*(M) = L^{\otimes n}$, restricted to $X^{ss}(L)$, for some $n \geq 0$. In particular, $X^{ss}(L)//G$ is a quasi-projective variety.*

Proof. As any open subset of X is quasi-compact in the Zariski topology we can find a finite set $\{s_1, \ldots, s_r\}$ of invariant sections of some tensor power of L such that $X^{ss}(L)$ is covered by the sets X_{s_i}. Obviously we may assume that all s_i belong to $\Gamma(X, L^{\otimes N})^G$ for some sufficiently large N. Let $U_i = X_{s_i}, i = 1, \ldots, r$. For every U_i, we consider the ring $\mathcal{O}(U_i)^G$ of G–invariant regular functions and let $\pi_i : U_i \to Y_i := U_i//G$ with $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$ as constructed in Nagata's theorem. For each i, j we can consider s_i/s_j as a regular G-invariant function on U_j. Let $\phi_{ij} \in \mathcal{O}(Y_j)$ be the corresponding regular function on the quotient. Consider the principal open subset $D(\phi_{ij}) \subset Y_i$. Obviously

$$\pi_i^{-1}(D(\phi_{ij})) = \pi_i^{-1}(D(\phi_{ji})) = U_i \cap U_j.$$

This easily implies that the both sets $D(\phi_{ij})$ and $D(\phi_{ji})$ are categorical quotients of $U_i \cap U_j$. By the uniqueness of categorical quotient there is an isomorphism $\alpha_{ij} : D(\phi_{ij}) \to D(\phi_{ji})$. It is easy to see that the set of isomorphisms $\{\alpha_{ij}\}$ satisfies the conditions of gluing, so we can patch together the quotients Y_i and the maps π_i to obtain a morphism $\pi : X^{ss}(L) \to Y$, where $Y = X^{ss}//G$. To show that Y is separated it is enough to observe that it admits an affine open cover by the sets Y_i which satisfies the following properties: $Y_i \cap Y_j \cong U_i \cap U_j//G$ are affine and $\mathcal{O}(Y_i \cap Y_j)$ is generated by restrictions of functions from $\mathcal{O}(Y_i)$ and $\mathcal{O}(Y_j)$. The latter property follows from the fact that $\mathcal{O}(U_i \cap U_j)$ is generated by restrictions of functions from $\mathcal{O}(U_i)$ and $\mathcal{O}(U_j)$.

In fact, the separatedness also follows from the assertion that Y is quasi-projective. So let us concentrate on proving the latter. Note that the cover $\{U_i\}_{i=1,\ldots,r}$ of $X^{ss}(L)$ is a trivializing cover for the line bundle L' obtained by restriction of L to $X^{ss}(L)$. In fact, by Remark 3, L' is ample hence we may assume that some tensor power $L^{\otimes tN}$ is very ample. This implies that $L'^{\otimes tN}$ is equal to the line bundle $f^*(\mathcal{O}_{\mathbf{P}_k^n}(1))$ for some embedding $f : X^{ss}(L) \to \mathbf{P}_k^n$. The section $s_i^{\otimes t}$ of $L'^{\otimes tN}$ is equal to the section $f^*(h)$ where h is a section of $\mathcal{O}_{\mathbf{P}_k^n}(1)$. Thus the open subset U_i is equal to $f^{-1}(V_i)$ where V_i is an open subset of $\mathbf{P}_k^n$ isomorphic to affine space. This shows that L' restricted to U_i is equal to $(f|U_i)^*(\mathcal{O}_{\mathbf{P}_k^n}(1)|V_i)$. However, $\mathcal{O}_{\mathbf{P}_k^n}(1)|V$ is isomorphic to the trivial line bundle since any line bundle over affine space is isomorphic to the trivial bundle. By fixing some of the trivializing isomorphisms we can identify the functions $(s_i/s_j)|U_i \cap U_j$ with the transition functions g_{ij} of L'. As we have shown before, $s_i/s_j = \pi^*(\phi_{ij})$ for some functions $\phi_{ij} \in \mathcal{O}(Y_j)$. We use transition functions $h_{ij} = \phi_{ij}|Y_i \cap Y_j$ to define a line bundle M on Y. Obviously $\pi^*(M) \cong L'$. Let us show that M is ample. First we define its sections t_j by setting $t_j|Y_i = \phi_{ij}$ for a fixed j and variable i. As for any i_1, i_2

$$\phi_{i_2j} = \phi_{i_1j}\phi_{i_2i_1},$$

$t_j|Y_{i_1} \cap Y_{i_2}$ differ by the transition function of M; hence t_j is in fact a section of M. Clearly $\pi^*(t_j) = s_j$ and $Y_{t_j} = Y_j$. Again as above since all Y_j are affine, we obtain that M

is ample. Since $\pi : X^{ss}(L) \to Y$ is obtained by gluing together good categorical quotients, the morphism π is a good categorical quotient.

It remains to show that the restriction of π to $X^s(L)$ is a geometric quotient. By definition $X^s(L)$ is covered by affine open G-invariant sets where G acts with closed orbits. Since π is a good categorical quotient, for any $x \in X^s(L)$ the fibre $\pi^{-1}(\pi(x))$ consists of one orbit. Thus $\pi|X^s(L)$ is a good geometric quotient.

In the case when L is ample and X is projective, the following construction of the categorical quotient $X^{ss}(L)//G$ is equivalent to the previous one.

Proposition. *Assume X is projective and L is ample. Let*

$$R = \bigoplus_{n\geq 0} \Gamma(X, L^{\otimes n}).$$

Then

$$X^{ss}(L)//G \cong \mathrm{Proj}(R^G).$$

In particular, the quotient $X^{ss}(L)//G$ is a projective variety.

Proof. First of all, we observe that by Nagata's theorem, the algebra R^G is finitely-generated. It has also the natural grading induced by the grading of R. The reader should go back to Lecture 4 to recall the definition of Proj(A) for any finitely generated graded k-algebra A. Replacing L by $L^{\otimes d}$, we may assume that R^G is generated by elements $s_0, \ldots, s_n$ of degree 1. Let $Y = \mathrm{Proj}(R^G)$ be the projective subvariety of $\mathbf{P}_k^n$ corresponding to the homogeneous ideal I equal to the kernel of some homogeneous surjection $k[T_0, \ldots, T_n] \to R^G, T_i \mapsto s_i$. The elements s_i generate the ideal $m = R_+^G$ generated by homogeneous elements of positive degree. Thus the affine open sets $U_i = X_{s_i}$ cover $X^{ss}(L)$. On the other hand the open sets $Y_i = Y \cap \{T_i \neq 0\}$ form an open cover of Y with the property that $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$. The maps $U_i \to Y_i$ define a morphism $X^{ss}(L) \to Y$ which coincides with the categorical quotient defined in the proof of the previous Theorem.

Remark. Note that the morphism $X^{ss}(L) \to X^{ss}(L)//G$ is affine, i.e., the pre-image of an affine open set is affine. There is also the following converse of the previous theorem. Let U be a G–invariant open subset of X such that the geometric quotient $\pi : U \to U/G$ exists and is an affine map. Assume U/G is quasi-projective. Then there exists a G–linearized line bundle L such that $U \subseteq X^s(L)$. We refer for the proof to **[Mu1]**, p. 41.

6.3 Examples. 1. Let X be the affine space $\mathbf{A}_k^n$ and G the multiplicative group $\mathbf{G}_{\mathbf{m},k}$. Let it act on X by the formula

$$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n), t \in \mathbf{G}_{\mathbf{m},k}(K), z_i \in K.$$

Let $L = X \times \mathbf{A}_k$ be the trivial bundle on X. By Lecture 3, its G-linearization is defined by the formula:

$$t \cdot (z, v) = (t \cdot z, \phi_t v),$$

where $t \mapsto \phi_t$ is a homomorphism $\chi : \mathbf{G}_{\mathbf{m},k} \to \mathbf{G}_{\mathbf{m},k}$. It is easy to see that any such homomorphism is given by a formula: $t \to t^\alpha$ for some integer α. In fact $\chi^* : k[T, T^{-1}] \to$

$k[T, T^{-1}]$ is defined by the image of T, and the condition that this map is a homomorphism implies that the image is a power of T. So let L_α denote the G-linearized line bundle which is trivial (as a line bundle) and the linearization is given by the formula:

$$t \cdot (z, v) = (t \cdot z, t^\alpha v).$$

A section $s : X \to L_\alpha$ of L_α is given by the formula

$$s(z) = (z, F(z))$$

for some polynomial $F(Z) \in k[Z] = \mathcal{O}(\mathbf{A}_k^n)$. The group G acts on the space of sections by the formula $s \mapsto {}^t s$, where

$${}^t s(z) = (z, t^\alpha \cdot F(t^{-1} \cdot z)).$$

Thus $s \in \Gamma(X, L_\alpha^{\otimes r})^G$ if and only if

$$F(t \cdot z) = t^{r\alpha} \cdot F(z) \quad \text{for all } z \in K^n, t \in K^*.$$

A polynomial which satisfies this property must be homogeneous of degree $r\alpha$. Since there are no non-zero homogeneous polynomials of negative degree, we get $X^{ss}(L_\alpha) = \emptyset$ if $\alpha < 0$. In this case $X^{ss}(L)//G = \emptyset$. If $\alpha = 0$, F must be a constant, hence, since X is affine, $X^{ss}(L_\alpha) = X$. Since $\mathcal{O}(X)^G = k$, we get $X^{ss}(L_\alpha)//G = pt_k$.

If $\alpha > 0$,

$$\bigoplus_{r \geq 0} \Gamma(X, L_\alpha^{\otimes r})^G = \bigoplus_{r \geq 0} k[Z]_{r\alpha} = k[Z]^{(\alpha)},$$

where $k[Z]$ is equipped with the standard grading. Since all monomials Z_i^α belong to this ring, $X^{us}(L_\alpha) = \{0\}$, hence $X^{ss}(L) = \mathbf{A}_k^n \setminus \{0\}$. Since all orbits of G in this set are closed, and G acts with trivial stabilizers, we obtain $X^{ss}(L_\alpha) = X^s(L_\alpha)_{(0)}$. To see what the quotient is, we may assume that $\alpha = 1$ (see Remark 4 in 6.1). Then, in the notation of the proof of the Theorem, we may take Z_i to be the section s_i. Then the sets X_{s_i} coincide with the sets $U_i = D(Z_i) = \mathbf{A}_k^n \setminus V(Z_i)$. The ring $\mathcal{O}(U_i)^G$ is equal to the homogeneous localization $k[Z]_{(Z_i)}$ so that each Y_i is isomorphic to the affine space $\mathbf{A}_k^{n-1}$. In this way we easily see that

$$X^{ss}(L_1)//G = X^s(L)/G = \mathbf{P}_k^{n-1}.$$

Now if X is any closed subvariety of $\mathbf{A}_k^n$ given by a homogeneous ideal I, then the action of $\mathbf{G}_{\mathbf{m},k}$ on $\mathbf{A}_k^n$ induces an action on X. Let $\bar{L}_1$ be the restriction of the G-linearized bundle L_1 to X. Then by Remark 6 from 6.1, $X^{ss}(\bar{L}_1) = X' = X \setminus \{0\}$. We leave it to the reader to verify that the construction of $X^{ss}(\bar{L}_1)//\mathbf{G}_{\mathbf{m},k}$ coincides with the one described in Example 2 of Lecture 4.

2. Let G be again $\mathbf{G}_{\mathbf{m},k}$ and $X = \mathbf{A}_k^4$ with the action given by the formula:

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4).$$

As in in the previous example, each G-linearized line bundle is isomorphic to the trivial line bundle with the G-linearization defined by an integer α. We have

$$\Gamma(X, L_\alpha^{\otimes r})^G = k[Z]_{r\alpha}.$$

However this time the grading in $k[Z_1, \ldots, Z_4]$ is weighted with weights $(1, 1, -1, -1)$.

Assume $\alpha = 0$. Then for any $r > 0, 1 \in \Gamma(X, L_0^{\otimes r})^G = \Gamma(X, L_0)^G$. Hence $X = X^{ss}(L)$, and

$$\mathcal{O}(X)^G = k[Z]_0 = k[Z_1Z_3, Z_1Z_4, Z_2Z_3, Z_2Z_4] \subset k[Z].$$

We have a canonical surjection

$$k[T_1, T_2, T_3, T_4] \to \mathcal{O}(X)^G,\ T_1 \to Z_1Z_3, T_2 \mapsto Z_1Z_4,\ T_3 \mapsto Z_2Z_3,\ T_4 \mapsto Z_2Z_4.$$

This shows that

$$\mathcal{O}(X)^G \cong k[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3).$$

Thus $X^{ss}(L)//\mathbf{G}_{\mathbf{m},k}$ is isomorphic to the closed subvariety Y_0 of $\mathbf{A}_k^4$ given by the equation

$$T_1T_4 - T_2T_3 = 0.$$

This is a quadric cone. It has one singular point at the origin.

Assume $\alpha > 0$. Again, without loss of generality we may take $\alpha = 1$. It is easy to see that

$$\bigoplus_{r>0} k[Z]_r = k[Z]_{>0} = Z_1k[Z]_{\geq 0} + Z_2k[Z]_{\geq 0}.$$

Thus

$$X^{ss}(L_1) = \mathbf{A}_k^4 \setminus V(Z_1, Z_2).$$

This set is covered by the open subsets $U_1 = D(Z_1)$ and $U_2 = D(Z_2)$. We have

$$\mathcal{O}(U_1)^G = k[Z]_{(Z_1)} = k[Z]_0[Z_2/Z_1], \mathcal{O}(U_2)^G = k[Z]_{(Z_2)} = k[Z]_0[Z_1/Z_2].$$

We claim that $X^{ss}(L_1)/G$ is isomorphic to a closed subvariety Y' of $\mathbf{A}_k^4 \times \mathbf{P}_k^1$ given by the equations

$$T_1Z_2 - T_3Z_1 = 0,\ T_2Z_2 - T_4Z_1 = 0,\ T_1T_4 - T_2T_3 = 0.$$

Here we use (Z_1, Z_2) for homogeneous coordinates in $\mathbf{P}_k^1$. In fact, this variety is covered by two affine open sets Y_i' given by $Z_i \neq 0, i = 1, 2$. It is easy to see that $\mathcal{O}(Y_i') \cong \mathcal{O}(U_i)^G$. We also verify that these two sets are glued together as they should be according to our construction of the categorical quotient. Thus we obtain an isomorphism $Y' \cong Y_+ := X^{ss}(L_1)//\mathbf{G}_{\mathbf{m},k}$. In fact, we have $X^{ss}(L_1) = X^s(L_1)$ so that Y_+ is a geometric quotient. Note that we have a canonical morphism

$$f_+ : Y_+ \to Y_0$$

which is given by the inclusion of the rings $k[Z]_0 \subset \mathcal{O}(U_i)^G$. Geometrically it is induced by the projection $\mathbf{A}_k^4 \times \mathbf{P}_k^1 \to \mathbf{A}_k^4$. Over the open subset $Y_0 \setminus \{0\}$ this morphism is an isomorphism. In fact, $Y_0 \setminus \{0\}$ is covered by the open subsets $U_i = Y_0 \cap D(T_i), i = 1, \ldots, 4$. The pre-image $\bar{U}_1 = f_+^{-1}(U_1)$ is contained in the open subset where $Z_1 \neq 0$. Since $Z_2/Z_1 = T_3/T_1$, we see that f_+ induces an isomorphism $\mathcal{O}(U_1) \to \mathcal{O}(\bar{U}_1)$. We treat the other pieces U_i similarly. Over the origin, the fibre of f_+ is isomorphic to $\mathbf{P}_k^1$. Also, we can immediately check that Y_+ is a nonsingular variety. Thus $f_+ : Y_+ \to Y_0$ is a resolution of singularities of Y_0. It is called *small* because the exceptional set is of codimension > 1. The reader, familiar with the notion of the blowing up, will recognize Y_+ as the variety obtained by blowing up along the closed subvariety of Y_0 defined by the equations $T_1 = T_3 = 0$.

Assume $\alpha < 0$. Similar arguments show that $Y_- = X^s(L_{-1})/\mathbf{G}_{\mathbf{m},k}$ is isomorphic to the closed subvariety of $\mathbf{A}_k^4 \times \mathbf{P}_k^1$ given by the equation

$$T_1Z_4 - T_2Z_3 = 0, T_3Z_4 - T_4Z_3 = 0, T_1T_4 - T_2T_3 = 0.$$

We have a morphism

$$f_- : Y_- \to Y_0$$

which is an isomorphism over $Y_0 \setminus \{0\}$ and the fibre over $\{0\}$ is isomorphic to $\mathbf{P}_k^1$. The diagram

$$\begin{array}{ccc} Y_+ & & Y_- \\ f_+ \searrow & & \swarrow f_- \\ & Y_0 & \end{array}$$

represents a type of birational transformations between algebraic varieties which is called nowadays a "flip". Note that Y_+ is not isomorphic to Y_-; they are isomorphic outside the fibres $f_\pm^{-1}(0) \cong \mathbf{P}_k^1$.

3. Let $\rho : G \to GL_k(n+1)$ be a linear representation of a reductive algebraic group G. Consider the corresponding action of G in $\mathbf{P}_k^n$. We know from Lecture 3 that the line bundle $L = \mathcal{O}_{\mathbf{P}_k^n}(n+1)$ admits a canonical $GL(n+1)$-linearization. This defines a G-linearization on L. For any $x \in \mathbf{P}_k^n(\bar{k})$ let $x^* = (a_0, \ldots, a_n)$ be a vector in $\mathbf{A}_k^{n+1}(\bar{k}) = \bar{k}^{n+1}$ lying over x (i.e., its coordinates are homogeneous coordinates of x). We claim

(i) $x \in X^{ss}(L) \Leftrightarrow$ the closure $\overline{G \cdot x^*}$ of its orbit does not contain 0;
(ii) $x \in X^s(L)_{(0)} \Leftrightarrow$ the orbit $G \cdot x^*$ is closed in $\mathbf{A}_k^{n+1}$ and G_{x^*} is finite;
(iii) $x \in X^{us}(L) \Leftrightarrow$ the closure $\overline{G \cdot x^*}$ of its orbit contains 0.

Let us check (i). Assume $0 \notin \overline{G \cdot x^*}$. The sets $\{0\}$ and $\overline{G \cdot x^*}$ are two disjoint closed G-invariant subsets in $\mathbf{A}_k^n$. Since G is reductive, we can find a G-invariant polynomial F such that $F(x^*) = 1, F(0) = 0$. Write F as a sum of homogeneous polynomials $F = F_0 + F_1 + \ldots + F_r$. We get that $F_0 = 0$, and that one of the F_i's does not vanish at x^*. But then F_i^{n+1} defines a section $s \in \Gamma(X, L^{\otimes i})^G$ with $s(x) \neq 0$. This shows that $x \in X^{ss}(L)$. Conversely if $s \in \Gamma(X, L^{\otimes i})^G$ with $s(x) \neq 0$ and F is the corresponding G-invariant homogeneous polynomial of degree $i(n+1)$, then $F(x^*) \neq 0$. For any point $v \in \overline{G \cdot x^*}, F(v) = F(x^*) \neq 0$. Hence $0 \notin \overline{G \cdot x^*}$.

Let us check (ii). Let $x \in X^s(L)_{(0)}$ and $x \in X_s$ be as in the definition of stable points. Then s corresponds to a homogeneous polynomial F of positive degree d and $F(x^*) = a \neq 0$. If $v \in \overline{G \cdot x^*}$, then $F(v) = a$; hence, $\overline{G \cdot x^*} \subset \{F = a\}$. Under the canonical projection $\pi : \mathbf{A}_k^{n+1} \setminus \{0\} \to X = \mathbf{P}_k^n$ the image of the set $Z_a = \{F = a\}$ is equal to X_s and the restriction of π to Z_a defines a finite map $Z_a \to X_s$ of degree d. In fact, we can view this map as the linear projection map of the hypersurface $F(Z_0, \ldots, Z_n, Z_{n+1}) - aZ_{n+1}^d \subset \mathbf{P}_k^{n+1}$. We have

$$\pi(G \cdot x^*) = \pi(\overline{G \cdot x^*}) = G \cdot x.$$

If $G \cdot x^*$ is not closed, the image of a closed orbit $G \cdot y^*$ contained in $\overline{G \cdot x^*} \setminus G \cdot x^*$ is a closed non-empty G-invariant subset of $G \cdot x$. Hence it must be equal to $G \cdot x$. But then dim $G_{y^*} > 0$ implies dim $G_x > 0$ contradicting the assumption. Thus $G \cdot x^*$ is closed. Obviously the condition that G_x is finite implies G_{x^*} is finite. Conversely if $G \cdot x^*$ is closed, and by a similar argument we find that its image in X_s is equal to $G \cdot x$ and is closed. Again, G_{x^*} is finite implies that G_x is finite. Lastly, statement (iii) follows from (i).

Problems.

1. Let X be a homogeneous space with respect to an action of an affine algebraic group G. Assume X is not affine. Show that for any $L \in \mathrm{Pic}^G(X)$ the set $X^{ss}(L)$ is empty.
2. A G-linearized line bundle is called *G-effective* if $X^{ss}(L) \neq \emptyset$. Show that $L \otimes L'$ is G-effective if both L and L' are G-effective.
3*. Let $\mathbf{G}_{\mathrm{m},k}$ act on an affine algebraic variety X and $\mathcal{O}(X) = \sum_{i \in \mathbf{Z}} \mathcal{O}(X)_i$ be the corresponding grading. Define $A_0 = \mathcal{O}(X)_0, A_{\geq 0} = \oplus_{i \geq 0} \mathcal{O}(X)_i, A_{\leq 0} = \oplus_{i \leq 0} \mathcal{O}(X)_i, A_{>0} = \oplus_{i>0} \mathcal{O}(X)_i, A_{<0} = \oplus_{i<0} \mathcal{O}(X)_i$. Let $L \in Pic^G(X)$, which is trivial as a line bundle. Show that there are only three possibilities (up to isomorphism): $X^{ss}(L) = X, X \setminus V(I_+), X \setminus V(I_-)$, where I_+ (resp. I_-) is the ideal in $\mathcal{O}(X)$ generated by A_+ (resp. A_-). Show that in the first case $X^{ss}(L)//\mathbf{G}_{\mathrm{m},k}$ is isomorphic to $\mathrm{Spec}(A_0)$, in the second (resp. the third) case $X^{ss}(L)//\mathbf{G}_{\mathrm{m},k}$ is isomorphic to $\mathrm{Proj}(A_{\geq 0})$ (resp. $\mathrm{Proj}(A_{\leq 0})$.
4. In Example 2 from 6.3 show that the fibred product $\tilde{Y} = Y_+ \times_{Y_0} Y_-$ is a nonsingular variety. Its projection to Y_0 is an isomorphism outside the origin, and the pre-image E of the origin is isomorphic to $\mathbf{P}^1_k \times \mathbf{P}^1_k$. Show that the restrictions of the projections from $\tilde{Y}$ to $Y_\pm$ to E coincide with the two projection maps $\mathbf{P}^1_k \times \mathbf{P}^1_k \to \mathbf{P}^1_k$.
5. Let G be a finite group acting algebraically on X. Show that for any $L \in \mathrm{Pic}^G(X)$, $X^{ss}(L) = X^s(L)$. Also $X^s(L) = X$ if L is ample. Show that the assumption of ampleness is essential (even for the trivial group!).
6. Let $G = \mathbf{SL}_k(n)$ act on the affine space of $(n \times n)$-matrices $M(n,k)$ by conjugation. Consider the corresponding action of G in the projective space $X = \mathbf{P}(M(n,k))$. Find the sets $X^{ss}(L), X^s(L), X^s(L)_{(0)}$, where $L \in Pic^G(X)$. Recall that $\mathrm{Pic}(X) \cong \mathbf{Z}$.

Lecture 7. NUMERICAL CRITERION OF STABILITY

In this lecture we prove a numerical criterion of stability due to David Hilbert and David Mumford. It is stated in terms of the restriction of the action to one-parameter subgroups of G.

7.1 Definition. A one-parameter subgroup of G is a homomorphism of algebraic groups

$$\lambda : \mathbf{G}_{\mathbf{m},k} \to G.$$

The set of one parameter subgroups of G is denoted by $\mathcal{X}_*(G)$.

Examples. 1. Let $G = \mathbf{G}^n_{\mathbf{m},k}$ be a n-dimensional torus. A one-parameter subgroup $\lambda : \mathbf{G}_{\mathbf{m},k} \to G$ is given by the homomorphism $\lambda^* : \mathcal{O}(G) \cong k[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to \mathcal{O}(\mathbf{G}_{\mathbf{m},k}) \cong k[T, T^{-1}]$. This homomorphism is determined by its values on $T_1, \dots, T_n$ which as easily seen must be powers of T. Thus λ is determined by n integers $(m_1, \dots, m_n)$ such that $\lambda^*(T_i) = T^{m_i}$. Since G is commutative, the set $\mathcal{X}_*(G)$ has a natural structure of an abelian group. The map $\lambda \mapsto (m_1, \dots, m_n)$ defines a natural isomorphism

$$\mathcal{X}_*(\mathbf{G}^n_{\mathbf{m},k}) \cong \mathbf{Z}^n.$$

Note that we also have a natural isomorphism

$$\mathcal{X}(\mathbf{G}^n_{\mathbf{m},k}) \cong \mathbf{Z}^n,$$

where $\mathcal{X}(G) = Hom(G, \mathbf{G}_{\mathbf{m},k})$ is the group of characters of G. The composition map:

$$\mathcal{X}_*(\mathbf{G}^n_{\mathbf{m},k}) \times \mathcal{X}(\mathbf{G}^n_{\mathbf{m},k}) \to \mathcal{X}(\mathbf{G}_{\mathbf{m},k}), (\lambda, \chi) \to \chi \circ \lambda,$$

corresponds to the natural dot-product bilinear pairing:

$$\mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}.$$

We shall denote the value of this pairing by $\langle \lambda, \chi \rangle$.

2. Let $\lambda \in \mathcal{X}_*(\mathbf{GL}_k(n))$. Consider the natural linear action of $\mathbf{GL}_k(n)$ on $\mathbf{A}^n_k$ and let $\mathbf{G}_{\mathbf{m},k} \times \mathbf{A}^n_k \to \mathbf{A}^n_k$ be the action of $\mathbf{G}_{\mathbf{m},k}$ obtained by the composition of this action with the map $(\lambda \times id) : \mathbf{G}_{\mathbf{m},k} \times \mathbf{A}^n_k \to \mathbf{GL}_k(n) \times \mathbf{A}^n_k$. Let

$$Z_i \to \sum_{j=1}^{n} Z_{ij} \otimes Z_j$$

be the coaction for $\mathbf{GL}_k(n)$ and

$$Z_i \to \sum_{j=1}^{n} \lambda^*(Z_{ij}) \otimes Z_j$$

be the corresponding coaction map for $\mathbf{G}_{\mathbf{m},k}$. As always a $\mathbf{G}_{\mathbf{m},k}$-action defines (and is defined by) a grading of the algebra of functions on the variety. In our case we have a grading of $k[Z_1, \ldots, Z_n]$. The above formula shows that the subspace of linear polynomials V is invariant with respect to this grading. Furthermore, we can write

$$V = \bigoplus_{i=1}^{n} V_i,$$

where

$$V_i = \{\phi \in V : \phi(\lambda(t) \cdot x) = t^i \phi(x) \quad \text{for all } t \in K^*, x \in K^n\}.$$

Let $(\phi_1, \ldots, \phi_n)$ be a basis in V such that each function ϕ_i belongs to some subspace V_{m_i}. Let $g \in GL(n,k) = \mathbf{GL}_k(n)(k)$ send each coordinate function Z_i to ϕ_i. In these new coordinates, we have for any $x = (x_1, \ldots, x_n) \in \mathbf{A}_k(K) = K^n$,

$$\lambda(t) \cdot (x_1, \ldots, x_n) = (t^{m_1} x_1, \ldots, t^{m_n} x_n).$$

In other words, the one-parameter subgroup

$$g \circ \lambda \circ g^{-1} : \mathbf{G}_{\mathbf{m},k} \to \mathbf{GL}_k(n), t \mapsto g\lambda(t)g^{-1}$$

is defined by the formula

$$t \mapsto \begin{pmatrix} t^{m_1} & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & t^{m_2} & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 & t^{m_n} \end{pmatrix}.$$

What we have proved here is that any linear representation of $\mathbf{G}_{\mathbf{m},k}$ is *diagonalizable.* Using the same idea one can prove that any linear representation $\rho : \mathbf{G}^r_{\mathbf{m},k} \to \mathbf{GL}^n_k$ is diagonalizable, i.e., there exists $g \in GL(n,k)$ such that

$$g\rho((t_1, \ldots, t_r))g^{-1} = \begin{pmatrix} \mathbf{t}^{\mathbf{m}_1} & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & \mathbf{t}^{\mathbf{m}_2} & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 & \mathbf{t}^{\mathbf{m}_n} \end{pmatrix},$$

where we use the vector notation

$$\mathbf{t^m} = t_1^{m_1} \dots t_r^{m_r}$$

for any $(t_1, \dots, t_r) \in (K^*)^r, (m_1, \dots, m_r) \in \mathbf{Z}^r$. This fact is used in proving that all maximal tori in a reductive algebraic group are conjugate.

7.2 The idea of the stability criterion is as follows. Suppose G acts on a projective variety $X \subset \mathbf{P}_k^n$ via its linear representation $\rho : G \to \mathbf{GL}_k(n+1)$. This can be achieved by taking a very ample G-linearized line bundle L on X. As in Example 3 from 6.3, we denote by x^* a representative of a point $x \in X(\bar{k}) \in \bar{k}^{n+1}$. By this example $x \in X^{ss}(L)$ if and only if $0 \notin \overline{G \cdot x}$. By 7.1 we may choose coordinates in the affine space $\mathbf{A}_k^{n+1}$ such that for any $x^* = (x_0, \dots, x_n) \in \mathbf{A}_k^{n+1}(K)$,

$$\lambda(t) \cdot x^* = (t^{m_0} x_0, \dots, t^{m_n} x_n).$$

Suppose, for $x_i \neq 0$, that the m_i are strictly positive. Then the map:

$$\lambda_{x^*} : \mathbf{A}_k^1 \setminus \{0\} \to \mathbf{A}_k^{n+1}, t \to \lambda(t) \cdot x^*$$

can be extended to a regular map $\mathbf{A}_k^1 \to \mathbf{A}_k^{n+1}$ by sending the origin of $\mathbf{A}_k^1$ to the origin of $\mathbf{A}_k^{n+1}$. It is clear that the latter belongs to the closure of the orbit of x^*, hence our point x is unstable (Example 3 from Lecture 6). Similarly, if all m_i are negative, we change λ to λ^{-1} defined by the formula $\lambda^{-1}(t) = \lambda(t^{-1})$ to reach the same conclusion. Let us set

$$\mu(x, \lambda) := \min_i \{m_i : x_i \neq 0\}.$$

So we can restate the previous remark by saying that if there exists $\lambda \in \mathcal{X}_*(G)$ such that $\mu(x, \lambda) > 0$ or $\mu(x, \lambda^{-1}) > 0$, then x is unstable. In other words, we have a necessary condition for semi-stability:

$$\text{if } x \in X^{ss}(L), \text{ then for all } \lambda \in \mathcal{X}_*(G), \mu(x, \lambda) \leq 0.$$

Assume the previous condition is satisfied and $\mu(x, \lambda) = 0$ for some λ. Let us show that x is not properly stable. Assume the contrary. In the previous notation, let $I = \{i : x_i \neq 0, m_i > 0\}$, and let $y = (y_0, \dots, y_n)$, where $y_i = x_i$ if $i \neq I$, and $y_i = 0$ if $i \in I$. Obviously, y belongs to the closure of the orbit of x under the action of the subgroup $\lambda(\mathbf{G}_{\mathbf{m},k})$. By definition of stability, y must be in the orbit. However, obviously $\lambda(\mathbf{G}_{\mathbf{m},k})$ fixes y, so that y cannot be properly stable. Thus we obtain a necessary condition for properly stable points:

$$\text{if } x \in X^s(L)_{(0)}, \text{ then for any } \lambda \in \mathcal{X}_*(G), \mu(x, \lambda) < 0.$$

We have to show first that the numbers $\mu(x, \lambda)$ are independent of a choice of coordinates in $\mathbf{A}_k^{n+1}$, and, more importantly, that the previous condition is sufficient for semi-stability. Let us prove first the independence. Let x^* be as above. For any $t \in \bar{k}^*$ the corresponding point $\lambda(t) \cdot x$ is equal to the point

$$(t^{m'_0} x_0, \dots, t^{m'_n} x_n),$$

where $m_i' = m_i - \mu(x,\lambda)$ if $x_i \neq 0$ and anything otherwise. Thus when we let t go to 0, we obtain a point in X with coordinates $y = (y_0, \ldots, y_n)$, where $y_i \neq 0$ if and only if $x_i \neq 0$ and $m_i = \mu(x,\lambda)$. The precise meaning of "let t go to 0" is the following. For any one-parameter subgroup $\lambda : \mathbf{G}_{\mathbf{m},k} \to G$ and a point $x \in X(k)$ we have a map

$$\lambda_x : \mathbf{A}_k^1 \setminus \{0\} \to X, t \to \lambda(t) \cdot x.$$

Since X is projective this map can be extended to a unique regular map

$$\bar{\lambda}_x : \mathbf{P}_k^1 \to X.$$

We set

$$\lim_{t \to 0} \lambda(t) \cdot x := \bar{\lambda}_x(0),$$

$$\lim_{t \to \infty} \lambda(t) \cdot x := \bar{\lambda}_x(\infty).$$

Obviously

$$\lim_{t \to \infty} \lambda(t) \cdot x = \lim_{t \to 0} \lambda^{-1}(t) \cdot x.$$

So our point y is equal to $\lim_{t \to 0} \lambda(t) \cdot x$. Now it is clear that for any $t \in \bar{k}$

$$\lambda(t) \cdot y = y,$$

that is, y is a fixed point for the subgroup $\lambda(\mathbf{G}_{\mathbf{m},k})$ of G. Also the definition of y is independent of any coordinates. Furthermore, for any vector y^* over y,

$$\lambda(t) \cdot y^* = t^{\mu(x,\lambda)} y^* \quad (*)$$

This can be interpreted as follows. Restrict the action of G on X to the action of $\mathbf{G}_{\mathbf{m},k}$ defined by λ. Then L has a natural $\mathbf{G}_{\mathbf{m},k}$-linearization and, since y is a fixed point, $\mathbf{G}_{\mathbf{m},k}$ acts on its fibre L_y defining a linear representation

$$\rho_y : \mathbf{G}_{\mathbf{m},k} \to \mathbf{GL}_k(1) = \mathbf{G}_{\mathbf{m},k}.$$

¿From Lecture 3 we know the geometric interpretation of the total space of the line bundle $\mathcal{O}_{\mathbf{P}_k^n}(-1)$. It follows from this that the fibre of the canonical projection $\mathbf{A}_k^{n+1} \setminus \{0\} \to \mathbf{P}_k^n$ over a point $x \in X$ can be identified with $\mathcal{O}_{\mathbf{P}_k^n}(-1)_x \setminus \{0\}$. Thus from (*) we get that $\mathbf{G}_{\mathbf{m},k}$ acts on the fibre L_y^{-1} by the character $t \mapsto t^{\mu(x,\lambda)}$ hence it acts on the fibre L_y by the character $t \mapsto t^{-\mu(x,\lambda)}$. This gives us a coordinate-free definition of $\mu(x,\lambda)$. In fact, this allows one to define the number $\mu^L(x,\lambda)$ for any G-linearized line bundle L as follows. Let $y = \lim_{t \to 0} \lambda(t) \cdot x$. Then $\lambda(\mathbf{G}_{\mathbf{m},k}) \subset G_y$ and, as above, there is a representation of $\mathbf{G}_{\mathbf{m},k}$ in the fibre L_y. It is given by an integer which is taken to be $-\mu^L(x,\lambda)$.

7.3 Now we are ready to state the main result of this Lecture.

Theorem. *Let G be a reductive group acting on a projective algebraic variety X. Let L be an ample G-linearized line bundle on X and $x \in X(k)$. Then*

$$x \in X^{ss}(L) \Leftrightarrow \mu^L(x,\lambda) \leq 0 \quad \text{for all } \lambda \in \mathcal{X}_*(G),$$

$$x \in X^{s}(L)_{(0)} \Leftrightarrow \mu^L(x,\lambda) < 0 \quad \text{for all } \lambda \in \mathcal{X}_*(G).$$

First of all, replacing L by sufficiently high tensor power, we can place ourselves in the following situation. G acts in a projective space $\mathbf{P}^n_k$ by means of a linear representation $\rho : G \to \mathbf{GL}_k(n)$, X is G-invariant closed subvariety of $\mathbf{P}^n_k$. We have to prove the following:

Let $x \in X$ and $x \notin X^s(L)_0$. Then there exists $\lambda \in \mathbf{X}_*(G)$ such that $\mu^L(x,\lambda) \geq 0$. Moreover, if $x \in X^{us}(L)$ then there exists $\lambda \in \mathbf{X}_*(G)$ such that $\mu^L(x,\lambda) > 0$.

To simplify the notation we drop L in $\mu^L(x,\lambda)$ remembering that $L = i^*(\mathcal{O}_{\mathbf{P}^n_k}(1)^{\otimes n+1})$.

We shall need the following fact:

Lemma (Cartan-Iwahori-Matsumoto). *Let $R = k[[T]]$ be the ring of formal power series with coefficients in k, and let $K = k((T))$ be its fraction field. For any reductive algebraic group G, any element of the set of double cosets $G(R)\backslash G(K)/G(R)$ can be represented by a one-parameter subgroup $\lambda : \mathbf{G}_{\mathbf{m},k} \to G$ in the following sense. One considers λ as a $k(T)$-point of G and identifies $k(T)$ with the subfield of $k((T))$ by considering the Laurent expansion of rational functions at the origin of $\mathbf{A}^1_k$.*

Proof. We shall do it only for the case $G = \mathbf{SL}_k(n)$ or $\mathbf{GL}_k(n)$, referring to the original paper of Iwahori and Matsumoto for the general case (see **[IM]**).

A K-point of G is a matrix A with entries in K. We can write it as a matrix $T^r\bar{A}$, where $\bar{A} \in GL(n,R)$. As R is a PID, we can reduce the matrix $\bar{A}$ to the diagonal form to be able to write

$$A = \bar{C}_1\bar{D}\bar{C}_2,$$

where $\bar{C}_i \in G(R)$, and $\bar{D}$ is a diagonal matrix $\text{diag}[T^{r_1},\ldots,T^{r_n}]$. Now we can define a one-parameter subgroup of G by

$$\lambda(t) = \text{diag}[t^{r_1},\ldots,t^{r_n}].$$

Then λ represents the double coset of the point $A \in G(K)$ as asserted.

Proof of the Theorem. Suppose x is not stable (a fortiori, not properly stable). Then the map $a : G \to V = \mathbf{A}^{n+1}_k, g \mapsto g \cdot x^*$, is not proper. In fact, otherwise the orbit $g \cdot x^*$ is closed, and hence x is semi-stable. By the Valuative Criterion of properness (**[Har]**, p. 101), there exists a R-point of V such that, viewed as a K-point of V, it has a pre-image under $a_K : G(K) \to V(K)$ but it does not arise from any R-point of G. In other words, there exists an element $g \in G(K) \setminus G(R)$ such that $g \cdot x^* \in V(R) = R^{n+1}$. By the previous Lemma we can write $g = g_1[\lambda]g_2$, where $g_1, g_2 \in G(R)$, and $[\lambda] \in G(K)$ which arise from a one-parameter subgroup $\lambda \in \mathcal{X}_*(G)$. Let $\bar{g}_2$ be the image of g_2 under the "reduction" homomorphism $G(R) \to G(k)$ corresponding to the natural homomorphism $R \to k, \sum_i a_iT^i \to a_0$. We can write:

$$\bar{g}_2^{-1}g_1^{-1}g = (\bar{g}_2^{-1}[\lambda]\bar{g}_2)\bar{g}_2^{-1}g_2.$$

The expression in the bracket is a K-point of G defined by a one-parameter subgroup $\lambda' = \bar{g}_2^{-1}\lambda\bar{g}_2$ of G. Choose a basis $(e_0,\ldots,e_n)$ in k^{n+1} such that the action of $\lambda'(\mathbf{G}_{\mathbf{m},k})$ is diagonalized. That is, we may assume that

$$\lambda'(t)\cdot e_i = t^{r_i}e_i, i=0,\ldots,n.$$

This is equivalent to

$$[\lambda']\cdot e_i = T^{r_i}e_i, i=0,\ldots,n.$$

Thus, if we write $x^* = x_0^*e_0+\ldots+x_n^*e_n$, we obtain

$$(\bar{g}_2^{-1}g_1^{-1}g\cdot x^*)_i = ([\lambda']\cdot(\bar{g}_2^{-1}g_2\cdot x^*))_i = T^{r_i}(\bar{g}_2^{-1}g_2\cdot x^*)_i.$$

Since $g\cdot x^*\in R^{n+1}$, this tells us that

$$(\bar{g}_2^{-1}g_2\cdot x^*)_i = T^{-r_i}(\bar{g}_2^{-1}g_1^{-1}g\cdot x^*)_i \in T^{-r_i}R. \quad (*)$$

This implies that $r_i\geq 0$ if $x_i^*\neq 0$. In fact the element $\bar{g}_2^{-1}g_2$ is reduced to the identity modulo (T), hence $(\bar{g}_2^{-1}g_2\cdot x^*)_i$ modulo (T) is a constant equal to x_i^*. On the other hand it is equal to $T^{-r_i}a_i$ modulo (T) for some $a_i\in R$. This of course implies that $r_i\geq 0$ if $x_i^*\neq 0$.

Recalling our definition of $\mu(x,\lambda')$ we see that $\mu(x,\lambda')\geq 0$. If x is unstable, we can additionally observe that the point $g\cdot x^*\in R^{n+1}$ is reduced to the zero modulo (T) because $0\in\overline{G\cdot x^*}$. This implies that the right-hand-side of (*) belongs to $T^{-r_i+1}R$ and hence we get $r_i>0$ if $x_i^*\neq 0$. This proves the theorem.

7.4 Assume $G=\mathbf{G}_{\mathbf{m},k}^r$ is a torus acting linearly on $X\subset\mathbf{P}_k^n$. By 7.1 we know that we can find a basis $(X_0\ldots,X_n)$ in $V=\mathcal{O}(\mathbf{A}_k^{n+1})_1$ such that the linear action of G in $\mathbf{A}_k^{n+1}$ is given by the coaction formula:

$$a^*(X_i) = \mathbf{T}^{\mathbf{m}_i}\otimes X_i,$$

where $\mathbf{T}^{\mathbf{m}_i} = T_1^{m_{i1}}\ldots T_t^{m_{ir}}\in\mathcal{O}(G)$. For any $t=(t_1,\ldots,t_r)\in G(K)=(K^*)^r$ and any $v\in\mathbf{A}_k^{n+1}(K)$ we have

$$X_i(t\cdot v) = \mathbf{t}^{\mathbf{m}_i}X_i(v).$$

The map $t\mapsto\mathbf{t}^{\mathbf{m}_i}$ is a character of G. Let us identify the group $\mathcal{X}(G)$ with the group of monomial functions in $\mathcal{O}(G)$, and also with the group $\mathbf{Z}^{\mathbf{r}}$ of monomial exponents. For any character $\chi\in\mathcal{X}(G)$ let

$$V_\chi = \{\phi\in V : a^*(\phi)=\chi\otimes\phi\} = \{\phi\in V : \phi(t\cdot x)$$

$$=\chi(t)\phi(x),\ \forall K/k, \forall t\in G(K), \forall x\in\mathbf{A}_k^{n+1}(K)\}.$$

Then we obtain that $X_i\in V_{\mathbf{m}_i}$ and

$$V = \bigoplus_{\chi\in\mathcal{X}(G)} V_\chi.$$

The latter decomposition is coordinate-free. For any $v \in \mathbf{A}_k^{n+1}(K)$, let $v_i = X_i(v)$. In other words, if we use the X_i's to identify $\mathbf{A}_k^{n+1}(K)$ with K^{n+1}, then $v = (v_0, \ldots, v_n)$. It follows from above the group G acts on a vector v by the formula

$$t \cdot v = (\mathbf{t}^{\mathbf{m}_0} v_0, \ldots, \mathbf{t}^{\mathbf{m}_n} v_n).$$

Let W_χ be the linear subspace of $\mathbf{A}_k^{n+1}$ defined by the ideal generated by the functions from $V_{\chi'}, \chi' \neq \chi$. Then, for any $v \in W_\chi(K)$,

$$t \cdot v = \chi(t) v, \ \forall t \in G(K).$$

Considering W_i as a linear subspace of $\mathbf{P}_k^n$, we see that it consists of fixed points of G. The vector space $\mathbf{A}_k^{n+1}(K)$ decomposes into the direct sum of the subspaces W_χ, and if we write $v = \sum_\chi v_\chi \in \mathbf{A}_k^{n+1}(K)$, where $v_\chi \in W_\chi(K)$, we obtain for any $t \in G(k)$

$$t \cdot v = \sum_\chi \chi(t) v_\chi.$$

This gives a coordinate-free description of the action of G in $\mathbf{A}_k^{n+1}$.

Now for any $\lambda \in \mathcal{X}_*(G)$

$$\lambda(t) \cdot v = \sum_\chi t^{\langle \lambda, \chi \rangle} v_\chi.$$

We define for any $x \in \mathbf{P}_k^n(k)$, the *state set* of x:

$$st(x) = \{\chi \in \mathcal{X}(G) : x_\chi^* \neq 0\}$$

and the *state polytope* of x

$$\overline{st(x)} = \text{convex hull of st}(x) \text{ in } \mathcal{X}(G) \otimes \mathbf{R} \cong \mathbf{R}^n.$$

It follows from 7.2 that

$$\mu^L(x, \lambda) = \min_{\chi \in st(x)} \langle \lambda, \chi \rangle.$$

This can be restated in the following way:

Theorem. *Let G be a torus and L be an ample G-linearized line bundle over a projective G-variety X. Then*

$$x \in X^{ss}(L) \Leftrightarrow 0 \in \overline{st(x)},$$
$$x \in X^s(L)_0 \Leftrightarrow 0 \in \text{interior}\{\overline{st(x)}\}.$$

Proof. We use a well-known fact (the supporting hyperplane lemma) from the theory of convex sets. Let Δ be a closed convex subset of $\mathbf{R}^n$. For any point $a \in \mathbf{R}^n \setminus \text{interior}(\Delta)$ (resp. $a \in \mathbf{R}^n \setminus \Delta$ there exists an affine function $\phi : \mathbf{R}^n \to \mathbf{R}$ such that $\phi(a) \leq 0$ (resp. $\phi(a) < 0$), and $\phi(\Delta) \subset \mathbf{R}_{\geq 0}$. Moreover, the proof of this fact shows that one can choose ϕ

with integral coefficients if Δ is a convex hull of the set of points with integral coordinates. We refer for the proofs to any text-book on convex sets (see for example [**Brø**]). The result follows.

7.5 Now let G be any reductive group acting linearly on a projective variety $X \subset \mathbf{P}^n_k$, L be some positive tensor power of $\mathcal{O}_{\mathbf{P}^n_k}(1)$. We know that any one-parameter subgroup of G has its image in a maximal torus T of G, hence can be considered as a one-parameter subgroup of T. Thus if we restrict the action of G to T, we obtain from the numerical criterion that any $x \in X^{ss}(L)$ must belong to the subset $X^{ss}_T(L_T)$, where the subscript T indicates the restriction of the action (and the linearization) to T. Now, applying Theorem 7.3, we obtain

$$X^{ss}(L) = \bigcap_{T \in MT(G)} X^{ss}_T(L_T)$$

$$X^{s}(L)_0 = \bigcap_{T \in MT(G)} X^{s}_T(L_T),$$

where $MT(G)$ is the set of maximal tori in G. Let us fix one maximal torus T. Then for any other maximal torus T', we can find $g \in G(k)$ such that $gT'g^{-1} = T$. By Example 2 from the previous lecture x is semi-stable (resp. properly stable) with respect to $\lambda(\mathbf{G}_{\mathbf{m},k})$ if and only if $0 \notin \overline{\lambda(\mathbf{G}_{\mathbf{m},k}) \cdot x^*}$ (resp. $\overline{\lambda(\mathbf{G}_{\mathbf{m},k}) \cdot x^*}$ is closed and the stabilizer of x^* in $\lambda(\mathbf{G}_{\mathbf{m},k})$ is finite). From this it immediately follows that this property is satisfied if and only if $g \cdot x$ is semi-stable (resp. properly stable) with respect to $g\lambda g^{-1}(\mathbf{G}_{\mathbf{m},k})$. This implies

$$x \in X^{ss}_{T'}(L_{T'}) \Leftrightarrow g \cdot x \in X^{ss}_T(L_T),$$

and similar assertion for properly stable points. Putting this together we obtain

Theorem. *Let T be a maximal torus in G. Then*

$$x \in X^{ss}(L) \Leftrightarrow \forall g \in G(k), g \cdot x \in X^{ss}_T(L_T),$$

$$x \in X^{s}(L)_0 \Leftrightarrow \forall g \in G(k), g \cdot x \in X^{s}_T(L_T)_0.$$

Together with the Theorem from the previous section we get an explicit criterion for checking stability. We shall demonstrate how it works in the next lecture.

Problems.

1. An algebraic group G is called *diagonalizable* if $\mathcal{O}(G)$ is generated as k-algebra by the characters $\phi : G \to \mathbf{G}_{\mathbf{m},k}$ considered as regular functions on G. Prove that a torus is a diagonalizable group and every connected diagonalizable group is isomorphic to a torus. Give examples of non-connected diagonalizable groups.
2. Check the following properties of the function $\mu^L(x, \lambda)$:
(i) $\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g)$ for any $g \in G(k), \lambda \in \mathcal{X}_*(G)$;
(ii) for any $x \in X, \lambda \in \mathcal{X}_*(G)$, the map $Pic^G(X) \to \mathbf{Z}$ defined by the formula $L \mapsto \mu^L(x, \lambda)$ is a homomorphism of groups;
(iii) if $f : X \to Y$ is a G-equivariant morphism of G-varieties, and $L \in Pic^G(Y)$, then $\mu^{f^*(L)}(x, \lambda) = \mu^L(f(x), \lambda)$;
(iv) $\mu^L(x, \lambda) = \mu^L(\lim_{t\to 0} \lambda(t) \cdot x, \lambda)$.
3. Prove that G acts properly on $X^s(L)_{(0)}$ (i.e., the map $\Psi : G \times X^s(L)_{(0)} \to X^s(L)_{(0)} \times X^s(L)_{(0)}$ is proper).
4. Let T be an r-dimensional torus acting linearly in a projective space $\mathbf{P}^n_k$. Show that $Pic^T(\mathbf{P}^n_k) \cong \mathbf{Z}^{r+1}$ and the set of $L \in Pic^T(\mathbf{P}^n_k)$ such that $(\mathbf{P}^n_k)^{ss}(L) \neq \emptyset$ is a finitely generated semigroup of $\mathbf{Z}^{r+1}$.
5. In the notation of Problem 6 from Lecture 6, using the numerical criterion of stability, find the sets $X^{ss}(L)$ and $X^{ss}(L)_{(0)}$.

Lecture 8. EXAMPLE: PROJECTIVE HYPERSURFACES

Let $G = \mathbf{SL}_k(n+1)$ act linearly in $\mathbf{A}_k^{n+1}$ in the natural way (as a subgroup of $\mathbf{GL}_k(n+1)$). This action defines an action of G in the subspace $k[Z_0, \ldots, Z_n]_d \subset \mathcal{O}(\mathbf{A}_k^{n+1})$ of homogeneous polynomials of degree $d > 0$. We view the latter as the space of k-points of the affine space $\mathbf{A}_k^N$, where $N = \binom{n+d}{d}$. The k-points of the associated projective space $\mathbf{P}_k^{N-1}$ can be interpreted as hypersurfaces of degree d in $\mathbf{P}_k^n$. For this reason we shall denote this projective space by $\mathrm{Hyp}_n(d)$.

In this lecture we shall try to describe the sets of semi-stable and stable points for this action. Note that since $Pic(\mathbf{P}_k^{N-1}) \cong \mathbf{Z}$ and $\mathcal{X}(G) = \{1\}$ there is no choice for a non-trivial linearization; we take $L = \mathcal{O}_{\mathbf{P}_k^{N-1}}(1)$. We shall assume that k is algebraically closed.

8.1 We begin with the simplest non-trivial case where $n = 1$. The elements of the space $k[Z_0, Z_1]_d$ are binary forms of degree d. The corresponding hypersurfaces can be viewed as finite subsets of points in $\mathbf{P}_k^1$ taken with some multiplicities (or, equivalently, as effective divisors on $\mathbf{P}_k^1$ or closed subschemes of $\mathbf{P}_k^1$). If

$$F = \sum_{i=0}^{d} a_i Z_0^{d-i} Z_1^i \in K[Z_0, Z_1]_d,$$

and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, K)$$

then the action of G on $k[Z_0, Z_1]_d$ is defined by the formula:

$$F \mapsto (g^{-1})^*(F) = \sum_{i=0}^{d} a_i (dZ_0 - bZ_1)^{d-i} (-cZ_0 + aZ_1)^i.$$

Let T be the maximal torus of G which consists of diagonal matrices and equal to the image of the one-parameter group

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let us first investigate the stability with respect to T. For this we shall follow the last section of the previous Lecture. We have to compute the state set of the point $H \in \mathrm{Hyp}_1(d)$. We have

$$K[Z_0, Z_1]_d = \bigoplus_{i=0}^{d} K Z_0^{d-i} Z_1^i$$

and

$$\lambda(t)^*(Z_0^{d-i}Z_1^i) = t^{d-2i}Z_0^{d-i}Z_1^i.$$

Let

$$S = \{-d, -d+2, \ldots, d-2, d\} \subset \mathbf{Z} = \mathcal{X}(T).$$

For any point $H \in \mathrm{Hyp}_1(d)$ given by the equation $F = 0$, its state set $st(H)$ (with respect to the action of T) is a subset of S. Let α_{min} (resp. α_{max}) be the smallest (resp. largest) element of this set.

By the theorem from 7.4 of Lecture 7, we know that H is semi-stable (resp. properly stable) with respect to T if and only if

$$\alpha_{min} \leq 0 \leq \alpha_{max} \quad (\text{resp. } \alpha_{min} < 0 < \alpha_{max}). \ (*)$$

Obviously, $\alpha_{min} = -d+2i$, where i is the maximum power of Z_0 which divides F. Similarly, $\alpha_{max} = d - 2i$, where i is the maximum power of Z_1 which divides F. This can be interpreted as follows:

H is semi-stable (resp. properly stable) with respect to T if and only if the points $(0,1)$ and $(1,0)$ are zeroes of H of multiplicity $\leq d/2$ (resp. $< d/2$).

¿From this we easily deduce

Theorem. $\mathrm{Hyp}_1(d)^{ss}$ *(resp.* $\mathrm{Hyp}_1(d)_0^s$*) is equal to the set of hypersurfaces with no roots of multiplicity* $> d/2$ *(resp.* $\geq d/2$*).*

Proof. Suppose H is semi-stable and has a root $(z_0, z_1) \in \mathbf{P}^1_k(k)$ of multiplicity $> d/2$. Let $g \in G(k)$ take this point to the point $(1,0)$. Then $H' = g \cdot H$ has the point $(1,0)$ as a root of multiplicity $> d/2$. This shows that H' is unstable with respect to T. Hence H is unstable with respect to G contradicting the assumption. Conversely, assume H has no roots of multiplicity $> d/2$ and is unstable. Then there exists a maximal torus T' with respect to which H is unstable. Let $gT'g^{-1} = T$ for some $g \in G(k)$. Then $g \cdot H$ is unstable with respect to T. But then it has one of the points $(1,0)$ or $(0,1)$ as a root of multiplicity $> d/2$. Thus H has $g^{-1} \cdot (1,0)$ or $g^{-1} \cdot (0,1)$ as a root of multiplicity $< d/2$.

A similar argument proves the assertion about proper stability.

Corollary. *Assume d is odd. Then*

$$\mathrm{Hyp}_1(d)^{ss} = \mathrm{Hyp}_1(d)_0^s.$$

Assume d is even and let $H \in \mathrm{Hyp}_1(d)^{ss} \setminus \mathrm{Hyp}_1(d)_0^s$. This means that H has a root of multiplicity $d/2$ but no roots of multiplicity greater than $d/2$. Consider the fibre of the projection $\mathrm{Hyp}_1(d)^{ss} \to \mathrm{Hyp}_1(d)^{ss}//G$ containing H. Since our categorical quotient is good, the fibre contains a unique closed orbit. H belongs to this orbit if and only if its stabilizer is of positive dimension. Any group element stabilizing H stabilizes its set of roots. It is easy to see that any subset of $\mathbf{P}^1_k(k)$ consisting of more than 2 points has a finite stabilizer. Thus, H must have only two roots. Since one of these roots is of multiplicity $d/2$, the other one is also of multiplicity $d/2$. Since any two-point sets on $\mathbf{P}^1_k(k)$ are projectively equivalent, this tells us that

$$\mathrm{Hyp}_1(d)^{ss} \setminus \mathrm{Hyp}_1(d)_0^{\overline{s}} G \cdot H_0,$$

where H_0 is given by the equation $(Z_0 Z_1)^{d/2} = 0$. In particular, $\mathrm{Hyp}_1(d)^{ss}//G$ is obtained from $\mathrm{Hyp}_1(d)_0^s/G$ by adding one point.

The variety $C_1^d := \mathrm{Hyp}_1(d)^{ss}//G$ is an irreducible normal projective variety of dimension $d-3$. A much deeper result is that it is a rational variety. This was proven only very recently by F. Bogomolov and P. Katsylo (see **[Bog]**).

8.2 Let us consider some cases with small d.

If $d = 1$ we have $\mathrm{Hyp}_1(1)^{ss} = \emptyset$. If $d = 2$ we have $\mathrm{Hyp}_1(2)_0^s = \emptyset$ and

$$\mathrm{Hyp}_1(2)^{ss} = \mathrm{Hyp}_1(2)^s$$

consists of subsets of two distinct points in $\mathbf{P}^1_k$. There is only one orbit of such subsets.

The set $\mathrm{Hyp}_1(3)^{ss}$ consists of three distinct points in $\mathbf{P}^1_k$. By a projective transformation they can be reduced to the points $\{0, 1, \infty\}$. So the variety C_1^d is again the point variety pt_k.

The set $\mathrm{Hyp}_1(4)_0^s$ consists of subsets of four distinct points in $\mathbf{P}^1_k$ and the set $\mathrm{Hyp}_1(4)^{ss}$ consists of closed subsets $V(F)$ where F has at most double roots. Since $\mathrm{Hyp}_1(4)_0^s$ is an open Zariski subset of the projective space $\mathbf{P}^4_k$, and the fibres of the projection $\mathrm{Hyp}_1(4)_0^s \to \mathrm{Hyp}_1(4)_0^s/G$ are of dimension $3(= \dim \mathbf{SL}_k(2))$, we obtain that C_1^4 is a normal, hence nonsingular, curve. Since it is obviously unirational, it must be isomorphic to $\mathbf{P}^1_k$. The image of the set of semi-stable but not properly stable points is one point. If we consider the map

$$\pi : \mathrm{Hyp}_1(4)^{ss} \to C_1^4 \cong \mathbf{P}^1_k$$

as a rational function on $\mathrm{Hyp}_1(4)_0^s$ then we can find its explicit expression as a rational function $R(a_0, \ldots, a_4)$ in the coordinates of a binary form. Unfortunately, this is not easy to explain and is the subject of the Classical Invariant Theory, which we have no intention of discussing here (for a good introduction to this theory we can refer the reader to **[Stu]**). We only give the answer (see **[Sal]**, p.189)

$$R(a_0, \ldots, a_4) = \frac{A^3}{4A^3 - B^2},$$

where

$$A = a_0^2 - 3a_1 a_3 + 12 a_0 a_4, B = 27 a_1^2 a_4 + 27 a_0 a_3^2 + 2 a_2^3 - 72 a_0 a_2 a_4 - 9 a_1 a_2 a_3.$$

Here the denominator is equal to the discriminant of the form so it vanishes when the form has a multiple root. In particular, the function R maps the semi-stable but not properly stable orbits to the point ∞.

Consider the special case when $F = T_0(T_1^3 + aT_0^2T_1 + bT_0^3)$. If char$k \neq 3$ then each orbit contains a representative of such form. Then

$$R = \frac{a^3}{4a^3 + 27b^2}.$$

The denominator is the discriminant of the cubic polynomial $x^3 + ax + b$. The reader familiar with the theory of elliptic curves will immediately recognize this function. It is the j-invariant of elliptic curves given in the Weierstrass form:

$$y^2 = x^3 + ax + b.$$

This coincidence is not accidental. The equation above describes an elliptic curve as a double cover of $\mathbf{P}_k^1$ branched over four points (which are the infinity point and the three roots of the equation $x^3+ax+b = 0$). In other words, they are the zeroes of the binary form $T_0(T_1^3 + aT_0^2T_1 + bT_0^3)$. Two elliptic curves are isomorphic if and only if the corresponding sets of four points on $\mathbf{P}_k^1$ are in the same orbit with respect to our action of $\mathbf{SL}_k(2)$.

8.3 Let n be arbitrary. Recall that a hypersurface $V(F) \in \mathrm{Hyp}_n(d)$ is a nonsingular variety if the equations

$$F = 0, \partial F/\partial T_i = 0, i = 0, \ldots, n$$

have no common zeroes. If char(k) does not divide d, we can write

$$dF = \sum_{i=0}^{n} T_i \partial F/\partial T_i$$

so that the first equation can be eliminated. Let D be the resultant of the polynomials $\partial F/\partial T_i$. It is a homogeneous polynomial of degree $(n+1)(d-1)^n$ in the coefficients of the form F. It is called the *discriminant* of F. Its value at F is equal to zero if and only if $V(F)$ is singular. Since the latter property is independent of a choice of coordinates, the hypersurface $V(D) \subset \mathrm{Hyp}_n(d)$ is invariant with respect to the action of $G = \mathbf{SL}_k(n+1)$. This means that for any $g \in G(K)$ we have $g^*(D) = \phi(g)D$ for some $\phi(g) \in K^*$. One immediately verifies that the function $g \mapsto \phi(g)$ is a character of $\mathbf{SL}_k(n+1)$. Since the latter is a simple group, it group of characters is trivial. This implies that $\phi(g) = 1$ for all g, hence D is an invariant polynomial. Since it is not identically zero, we obtain:

Theorem. *Assume* char(k) *is prime to* d. *Any nonsingular hypersurface is a semi-stable point of* $\mathrm{Hyp}_n(d)$.

If $d \geq n+1$, one can replace "semi-stable" with "properly stable". This follows from the fact, that under these assumptions, the group of automorphisms of a nonsingular hypersurface is finite.

Assume $d = 2$ and char$(k) \neq 2$. Then $\mathrm{Hyp}_n(2)$ is the space of quadrics. The space $k[T_0, \ldots, T_n]_2$ is the space of quadratic forms $F = \sum_{i,j=0}^n a_{ij}T_iT_j$, or, equivalently, the space of symmetric matrices

$$B = (b_{ij})_{i,j=0,\ldots,n}, \quad b_{ii} = 2a_{ii}, b_{ij} = b_{ji} = a_{ij}, i \neq j.$$

A quadric $V(F)$ is nonsingular if and only if the rank of the corresponding matrix is equal to $n+1$. The determinant function on $k[T_0, \dots, T_n]_2$ is the resultant R from above. Thus all nonsingular quadrics are semi-stable. We know that by a linear change of variables every quadratic form can be reduced to the sum of squares $X_0^2 + \dots + X_r^2$, where the number r is equal to the rank of the matrix B as above. In our situation we are allowed to use only linear transformations with determinant 1 but since we are considering homogeneous forms only up to a multiplicative factor, the result is the same. So we have exactly n orbits for the action of $\mathbf{SL}_k(n+1)$ on $\mathrm{Hyp}_n(2)$. Only one of them is open, and its complement is equal to the set of zeroes of the discriminant. By Hilbert's Nullstellensatz, any invariant polynomial must be a power of the discriminant. Thus only non-degenerate quadrics are semi-stable points. The stabilizer of the quadratic form $T_0^n + \dots + T_n^2$ is the special orthogonal group $SO_k(n+1)$. Since it is of positive dimension (if $n > 0$), there are no properly stable points. Since the orbit of non-degenerate forms is closed in the set of semi-stable point, all semi-stable points are stable.

8.4 Let $n = 2$ and $d = 3$. Every homogeneous form of degree 3 in three variables (*a ternary cubic*) can be written in the form:

$$F = a_{300}T_0^3 + a_{210}T_0^2T_1 + a_{201}T_0^2T_2 + a_{120}T_0T_1^2 + a_{111}T_0T_1T_2 +$$
$$+ a_{102}T_0T_2^2 + a_{030}T_1^3 + a_{021}T_1^2T_2 + a_{012}T_1T_2^2 + a_{003}T_2^3.$$

Let T be the diagonal maximal torus in $\mathbf{SL}_k(3)$. Its group of K-points consists of diagonal matrices $\mathrm{diag}[t_1, t_2, t_1^{-1}t_2^{-1}]$.

For any degree 3 monomial $T_0^iT_1^jT_2^k$ we have

$$\lambda(t) \cdot T_0^iT_1^jT_2^k = t_1^{i-k}t_2^{j-k}T_0^iT_1^jT_2^k.$$

Thus the set of characters which can enter in the state set of some F is the set of following 10 lattice points in the real plane, with coordinates $(i-k, j-k)$:

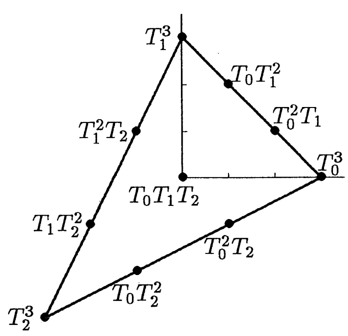


Fig.1

Suppose $V(F)$ is unstable with respect to T. Then the origin lies outside of the state polytope of F with respect to T which is equal to the convex hull of the points corresponding to the monomials entering in the equation of F with non-zero coefficients. After permuting the coordinates we may assume that

$$F = a_{300}T_0^3 + a_{210}T_0^2T_1 + a_{201}T_0^2T_2 + a_{120}T_0T_1^2 + a_{030}T_1^3.$$

Computing the partial derivatives we find

$$\begin{aligned}\partial F/\partial T_0 &= 3a_{300}T_0^2 + 2a_{210}T_0T_1 + a_{120}T_1^2 + 2a_{201}T_0T_2,\\ \partial F/\partial T_1 &= a_{210}T_0^2 + 2a_{120}T_0T_1 + 3a_{030}T_1^2,\\ \partial F/\partial T_2 &= a_{201}T_0^2.\end{aligned}$$

This tells us that the point $(0,0,1)$ is a singular point of $V(F)$. In the inhomogeneous coordinates $X = T_0/T_2, Y = T_1/T_2$ the equation of $V(F)$ is

$$F = a_{201}X^2 + a_{300}X^3 + a_{210}X^2Y + a_{120}XY^2 + a_{030}Y^3.$$

¿From this we see that the singular point is not an ordinary double point.

If $a_{201} = 0$, the curve is reducible. It is the union of three lines passing through the point $(0,0,1)$. Some of these lines may coincide if F (considered as a binary cubic form) has multiple roots.

Assume $a_{201} \neq 0$. Replacing T_2 by $a_{201}^{-1}(T_2 - a_{300}T_0 - a_{210}T_1)$, we may assume that $a_{201} = 1, a_{300} = a_{210} = 0$ so that

$$F = X^2 + a_{120}XY^2 + a_{030}Y^3.$$

If $a_{030} = 0$, the curve is reducible. If $a_{120} \neq 0$, it is the union of a line $T_0 = 0$ and the conic $T_0 + T_1^2 = 0$. They are tangent at the point $(0,0,1)$. If $a_{120} = 0$ the curve is the union of two lines $T_0 = 0$ and $T_2 = 0$, the first one is taken with multiplicity 2.

If $a_{030} \neq 0$, the curve is irreducible. By scaling we may assume that $a_{030} = 1$. If $\mathrm{char}(k) \neq 3$, we replace Y by $Y + \frac{1}{3}a_{120}X$ to kill a_{120}, and then change T_2 again to kill the coefficients appearing at X^2Y and X^3. Thus we reduce our equation to the standard equation of a cuspidal cubic:

$$F = X^2 + Y^3 = 0$$

or in projective coordinates:

$$F = T_0^2T_2 + T_1^3 = 0.$$

If $\mathrm{char}(k) = 3$ we cannot get rid of the term XY^2. By scaling the coordinates we reduce the equation to the form:

$$F = T_0^2T_2 + \epsilon T_0T_1^2 + T_1^3 = 0,$$

where $\epsilon = 0$ or 1. Observe that the curve with $\epsilon = 1$ differs from the curve with $\epsilon = 0$ by the property that it has no inflection tangent at any nonsingular point.

Now let us find semi-stable but not properly stable points. This corresponds to 0 being on the boundary of the state polytope. Again the picture shows that, up to permutation of coordinates, the equation of the curve can be taken in the form:

$$F = a_{300}T_0^3 + a_{210}T_0^2T_1 + a_{201}T_0^2T_2 + a_{120}T_0T_1^2 + a_{111}T_0T_1T_2 + a_{030}T_1^3 + a_{021}T_1^2T_2.$$

Computing the partials we find

$$\begin{aligned}\partial F/\partial T_0 &= 3a_{300}T_0^2 + 2a_{210}T_0T_1 + 2a_{201}T_0T_2 + a_{120}T_1^2 + a_{111}T_1T_2,\\ \partial F/\partial T_1 &= a_{210}T_0^2 + 2a_{120}T_0T_1 + 3a_{030}T_1^2 + 2a_{021}T_1T_2 + a_{111}T_0T_2,\\ \partial F/\partial T_2 &= a_{201}T_0^2 + a_{021}T_1^2 + a_{111}T_0T_1.\end{aligned}$$

We see that the point $(0,0,1)$ is singular. In affine coordinates $X = T_0/T_2, y = T_1/T_2$ we can write

$$F = a_{201}X^2 + a_{111}XY + a_{021}Y^2 + a_{300}X^3 + a_{210}X^2Y + a_{120}XY^2 + a_{030}Y^3.$$

We claim that the binary quadratic form $a_{201}X^2 + a_{111}XY + a_{021}Y^2$ is non-degenerate. In fact, if it is degenerate, a linear transformations of the variables T_0 and T_1 allows one to replace the equation to assume that $a_{201}X^2 + a_{111}XY + a_{021}Y^2 = X^2$. Then $a_{111} = a_{021} = 0$ and we are in the previous case (i.e., $V(F)$ is unstable with respect to some torus conjugate to the diagonal torus by means of the transformation we have just used).

Note that we can do everything in arbitrary characteristic. In fact, if we view a quadratic binary form as a set of two points on $\mathbf{P}_k^1$ we can always reduce them, by a linear change of variables, to 0 and ∞ or to the double 0. This corresponds to the reduction of the quadratic form to either XY or X^2.

Now we can apply a linear change of the unknowns T_0 and T_1 to assume that

$$F = XY + a_{300}X^3 + a_{210}X^2Y + a_{120}XY^2 + a_{030}Y^3.$$

Replacing T_2 by $T_2 - a_{210}T_0 - a_{120}T_1$ we may assume that $a_{210} = a_{120} = 0$ so that

$$F = XY + a_{300}X^3 + a_{030}Y^3.$$

If one (but not both) of the coefficients a_{030} and a_{300} vanishes, the curve is the union of a line and a conic. They intersect each other at 2 distinct points.

If $a_{030} = a_{300} = 0$, we get the union of three non-concurrent lines $T_0T_1T_2 = 0$.

Assume now that a_{030} and a_{300} are not zero. Applying the linear transformation

$$(T_0, T_1, T_2) \to (a_{030}^{\frac{1}{3}}T_0, a_{300}^{\frac{1}{3}}T_1, a_{030}^{-\frac{1}{3}}a_{300}^{-\frac{1}{3}}T_2)$$

we reduce the equation to the form:

$$XY + X^3 + Y^3 = 0.$$

If $\mathrm{char}(k) \neq 3$ it can be reduced to the familiar equation of a nodal cubic:

$$Y^2 + X^2 + X^3 = 0.$$

We leave this as an exercise for the reader.

Now we can finish the analysis of stability. Assume $V(F)$ is unstable. Then it is unstable with respect to some maximal torus. By changing the coordinates we may assume that it is the diagonal torus. Then we deduce from above that $V(F)$ must be isomorphic to one of the following curves

(us1) irreducible cuspidal curve;
(us2) the union of an irreducible conic and its tangent line;
(us3) the union of three concurrent lines;
(us4) the union of two lines, one of them is double;
(us5) one triple line.

Assume $V(F)$ is semi-stable but not properly stable. Then $V(F)$ must be isomorphic to one of the following curves:

(sss1) irreducible nodal curve;
(sss2) the union of an irreducible conic and a line intersecting it at two distinct points;
(sss3) the union of three non-concurrent lines.

As all the above curves are singular, all non-singular curves must be properly stable. In particular, their group of projective automorphisms is finite. Since we listed all possible singular curves there are no more properly stable points. Let us show that curves of types (sss) are not stable. Consider the quotient map

$$\pi : \mathrm{Hyp}_2(3)^{ss} \to \mathrm{Hyp}_2(3)^{ss}//\mathbf{SL}_k(3).$$

The dimension of those fibres containing properly stable curves is equal to $\dim \mathbf{SL}_k(3) = 8$. Note that in the process of the previous analysis, we have found that curves of type (sssi), i = 1,2,3, each form a single orbit represented by the curves

$$T_0T_1T_2 + T_0^3 + T_1^3 = 0, T_0T_1T_2 + T_1^3 = 0, T_0T_1T_2 = 0,$$

respectively. Moreover the curves of type (sss2) and (sss3) have stabilizer of positive dimension. In fact the torus $\lambda(\mathbf{G}_{\mathbf{m},k})$, where $\lambda(t) = (t, 1, t^{-1})$ stabilizes the second curve, and the maximal diagonal torus stabilizes the third curve. This shows that the orbits of curves of type (sss2) and (sss3) are of dimension ≤ 7. Thus they lie in the closure of some orbit of dimension 8. It cannot be a properly stable orbit, hence the only possible case is that it is the orbit of curves of type (sss1). Hence this orbit is nether closed nor stable.

Since $\mathrm{Hyp}_2(3)$ is of dimension 9, we obtain $\dim \mathrm{Hyp}_2(3)^{ss}//\mathbf{SL}_k(3) = 1$. It is a normal projective unirational curve, hence we find that

$$\mathrm{Hyp}_2(3)^{ss}//\mathbf{SL}_k(3) \cong \mathbf{P}_k^1.$$

Since there is only one closed semi-stable but not stable orbit, namely the set of three non-concurrent lines, we obtain

$$\mathrm{Hyp}_2(3)^s/\mathbf{SL}_k(3) = \mathrm{Hyp}_2(3)_0^s//\mathbf{SL}_k(3) \cong \mathbf{A}_k^1.$$

Thus projective isomorphism classes of nonsingular plane cubics are parametrized by the affine line. It is easy to see that the orbit of the curve $T_0T_1T_2 = 0$ is of dimension 6. In the same fibre we find two other orbits: nodal irreducible cubics (of dimension 8) and of curves of type (sss2) (of dimension 7). The second orbit lies in the closure of the first one, and the closed orbit lies in the closure of the second one.

If $\text{char}(k) \neq 3$, we have 5 unstable orbits: irreducible cuspidal cubics (of dimension 8), curves of type (us 2) (of dimension 7), of type (us3) (of dimension 5), of type (us4) (of dimension 4), and of type (us5) (of dimension 2). It is easy to see that the orbit of type (us$i$) lies in the closure of the orbit of type (us$i-1$).

If $\text{char}(k) = 3$ we have two unstable orbits of type (us1), and four other unstable orbits lying in the closure of the previous two orbits.

Again as in 8.2, one may ask for the explicit formula for the quotient map. It can be given by the following rational function J in the coefficients a_{ijk} (see **[Sa2]**, pp.189-192 or **[Stu]**, pp.167-173) :

$$J = \frac{S^3}{T^2 + 64S^3},$$

where

$$\begin{gathered} S = abcm - (bca_2a_3 + cab_1b_3 + abc_1c_2) - m(ab_3c_2 + bc_1a_3 + ca_2b_1) - \\ -m^4 + 2m^2(b_1c_1 + c_2a_2 + a_3b_3) + \\ +(ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2) - 3m(a_2b_3c_1 + a_3b_1c_2) - \\ -(b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) + (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2), \end{gathered}$$

$$\begin{gathered} T = a^2b^2c^2 - 6abc(ab_3c_2 + bc_1a_3 + ca_2b_1) - 20abcm^3 + 12abcm(b_1c_1 + c_2a_2 + a_3b_3) + \\ +6abc(a_2b_2c_1 + a_3b_1c_2) + \\ +4(a^2bc_2^3 + a^2cb_3^2 + a^2cb_3^3 + b^2ca_3^3 + b^2ac_1^3 + c^2ab_1^3 + c^2ba_2^3) + \\ +36m^2(bca_2a_3 + cab_1b_3 + abc_1c_2) - \\ -24m(bcb_1a_3^2 + bcc_1a_2^2 + cac_2b_1^2 + caa_2b_3^2 + aba_3c_2^2 + abb_3c_1^2) - 3(a^2b_3^2a_3^2 + b^2c_1^2a_3^2 + c^2a_2^2b_1^2) - \\ -12(bcc_2a_3a_2^2 + bcb_3a_2a_3^2 + cac_1b_3b_1^2 + caa_3b_1b_3^2 + abb_1c_2c_1^2) - \\ -12m^3(ab_3c_2 + bc_1a_3 + ca_2b_1) + \\ +12m^2(ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2) - \\ -60m(ab_1b_3c_1c_2 + bc_1c_2a_2a_3 + ca_2a_3b_1b_3) + \\ +12m(aa_2b_3c_2^2 + aa_3c_2b_3^2 + bb_3c_1a_3^2 + bb_1a_3c_1^2 + cc_1a_2b_1^2 + cc_2b_1a_2^2) + \\ +6(ab_3c_2 + bc_1a_3 + ca_2b_1)(a_2b_3c_1 + a_3b_1c_2) + \\ +24(ab_1b_3^2c_1^2 + ac_1c_2^2b_1^2 + bc_2c_1^2a_2^2 + ba_2a_3^2c_2^2 + ca_3a_2^2b_3^2 + cb_3b_1^2a_3^2) - \end{gathered}$$

$$-12(aa_2b_1c_2^3 + aa_3c_1b_3^3 + bb_3c_2a_3^3 + bb_1a_2c_1^3 + cc_1a_3b_1^3 + cc_2b_3a_2^3) - 6b_1c_1c_2a_2a_3b_3-$$
$$-8m^6 + 24m^4(b_1c_1 + c_2a_2 + a_3b_3) - 36m^3(a_2b_3c_1 + a_3b_1c_2) - 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2)+$$
$$+36m(a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + c_2a_2 + a_3b_3) + 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3)-$$
$$-12(b_1^2c_1^2c_2a_2 + b_1^2c_1^2a_3b_3 + c_2^2a_2^2a_3b_3 + c_2^2a_2^2b_1c_1 + a_3^2b_3^2b_1c_1 + a_3^2b_3^2)-$$
$$-12m^2(b_1c_1c_2a_2 + c_2a_2a_3b_3 + a_3b_3b_1c_1) - 24m^2(b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2)+$$
$$+18(bcb_1c_1a_2a_3 + cac_2a_2b_3b_1 + aba_3b_3c_1c_2).$$

Here we use the following dictionary between our notation of coefficients and Salmon's:

$$(a_{300}, a_{210}, a_{201}, a_{120}, a_{111}, a_{102}, a_{030}, a_{021}, a_{012}, a_{003}) = (a, a_2, a_3, 3b_1, 6m, 3c_1, b, 3b_3, 3c_2, c).$$

The denominator $D = T^2 + 64S^3$ is the *discriminant* of the cubic, which is the resultant polynomial R we used in 8.3. It vanishes on the closed subset of singular cubics. In particular, the function J sends semi-stable but not properly stable orbits to the point ∞. It is known that each plane nonsingular cubic over an algebraically closed field of characteristic different from 2 and 3 can be reduced to the Weierstrass equation

$$T_1^2T_0 - T_2^3 - aT_2T_0^2 - bT_0^3 = 0.$$

In this special case the value of the function J is equal to

$$J = \frac{a^3}{4a^3 + 27b^2}.$$

This is the j-invariant of the elliptic curve. Note that we came to the same function by studying the orbits of binary quartics.

8.5 The following are the other values of (d, n), where the analysis of stability has been worked out:

$(d, n) = (2, 4), (2, 5), (3, 3)$ **[Mu1]**, $(2, 6)$ **[Sh1]**, $(3, 4)$ **[Sh2]**.

8.6 Let us cite the following description of the algorithm for finding unstable plane curves ("nullforms") given by David Hilbert in his 1897 course on invariant theory. (see **[Hil]**, Lecture XLV, July 27, 1897):

"Draw an arbitrary straight line through the center M, and then determine all those vertices which lie on this line or on the side of A off the line. The coefficients $a_{n_1n_2n_3}$ in the ternary form of order n, corresponding to the vertices n_1, n_2, n_3, are to be set equal to zero, while the other coefficients are left arbitrary.

Here, A is an arbitrary but fixed vertex."

Problems.

1. Let $(a_i, b_i), i = 1, 2, 3, 4$, be four distinct roots of a binary quartic F. Let $[ij]$ denote the determinant of the matrix $\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$. The expression $r = [12][34]/[13][24]$ is called the *cross-ratio* of the four points. Prove that two binary quartics define the same orbit in $\mathrm{Hyp}_1(4)$ if and only if the corresponding cross-ratio coincide after some permutation of the roots.
2. Let H be the subgroup of $G = \mathbf{SL}_k(2)$ generated by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Show that the homogeneous space G/H is isomorphic to the complement of the quartic $V(D)$ in $\mathbf{P}_k^3$, where D is the discriminant of binary cubic forms.
3. Show that there are exactly two orbits in $\mathrm{Hyp}_1(4)^s$ with non-trivial stabilizer. Show that the closures of these orbits in $\mathrm{Hyp}_2(4)$ are given by the equations $A = 0$ and $B = 0$, where A, B are the polynomials of degree 2 and 3 defined in 8.3.
4. Show that $\mathrm{Hyp}_1(4)^{us}$ is isomorphic to a surface of degree 6 in $\mathbf{P}_k^4$. Its singular set is isomorphic to a Veronese curve of degree 4.
5. Find all projective automorphisms of a nonsingular cubic curve (may assume that $\mathrm{char}(k) \neq 2, 3$).
6. Find all projective automorphisms of an irreducible cuspidal cubic.
7. Make the analysis of stability in the case $(d, n) = (3, 3)$ and compare the result with the answer in **[Mu1]**.
8. Prove that nonsingular quadrics are semi-stable in all characteristics.

Lecture 9. EXAMPLE: CONFIGURATIONS OF LINEAR SUBSPACES

Let $G(r+1,n+1)$ denote the Grassmann variety of $(r+1)$-dimensional linear subspaces in the affine space $\mathbf{A}_k^{n+1}$ (or, equivalently, of r-dimensional linear projective subspaces in $\mathbf{P}_k^n$). The group $G = \mathbf{SL}_k(n+1)$ acts naturally on $G(r+1,n+1)$ via its linear representation in $\mathbf{A}_k^{n+1}$. In this lecture we shall investigate the stability of the diagonal action of G on the variety

$$X = G(r+1,n+1)^m$$

of ordered m-tuples of linear r-dimensional subspaces in $\mathbf{P}_k^n$.

9.1 First we have to describe possible linearizations of this action. Recall that a K-point of the variety $G(r+1,k+1)$ is a direct summand projective submodule $\bar{W}$ of K^{n+1} of rank $r+1$. If K is a field, $\bar{W}$ is given in K^{n+1} by a system of homogeneous linear equations which we can take for the definition of the corresponding linear projective subspace W of $\mathbf{P}_K^n$. We assign to $\bar{W}$ its exterior power $\bigwedge^{r+1}(\bar{W})$ to obtain a point of $\mathbf{P}_k^N(K)$. This defines a closed embedding of $G(r+1,n+1)$ into the projective space $\mathbf{P}_k^N$, $N = \binom{n+1}{r+1} - 1$, called the *Plücker embedding.* When $\bar{W}$ is free and has the ordered set $(v_0, \ldots, v_r)$ for its basis, $\bigwedge^{r+1}(\bar{W})$ is free of rank 1 with basis

$$v_0 \wedge \ldots \wedge v_r = \sum_{0 \le i_0 < \ldots < i_r \le n} p_{i_0,\ldots,i_r} e_{i_0} \wedge \ldots e_{i_r},$$

where $e_0, \ldots, e_n$ is the standard basis of K^{n+1}. The coefficients $p_{i_0,\ldots,i_r}$ can be taken as projective coordinates of the point $W \in G(r+1,n+1)(K)$ in the Plücker embedding (the *Plücker coordinates*).

It will be convenient to represent a free $\bar{W}$ by a matrix of size $(r+1) \times (n+1)$ with entries in K and rows equal to the basis vectors $v_0, \ldots, v_r$

$$A = \begin{pmatrix} a_{00} & a_{01} & \ldots & \ldots & a_{0n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{r0} & a_{r1} & \ldots & \ldots & a_{rn} \end{pmatrix}.$$

In this way the Plücker coordinates $p_{i_0,\ldots,i_r}$ are the maximal minors $A_{i_0,\ldots,i_r}$ of this matrix formed by the columns $A_{i_0}, \ldots, A_{i_r}$. Note that two different matrices A, A' correspond to the same subspace if and only if there exists an invertible matrix $C \in GL(r+1,K)$ such that $A' = CA$. Also note that the matrix A is of maximal rank since its rows are linearly independent. Thus we can view K-points of $G(r+1,n+1)(K)$ corresponding to free K-modules as orbits of $GL(r+1,K)$ in the open subset of $Mat_{r+1,n+1}(K)$ consisting of matrices of rank $r+1$. To make $G(r+1,n+1)$ a quotient variety we consider the trivial line bundle L on the space $\mathbf{A}_k^{(r+1)(n+1)}$ and $\mathbf{GL}_k(r+1)$-linearize it by the formula

$$C \cdot (A,t) = (CA, det(C)t),$$

where $C \in GL(r+1,K), A \in \mathbf{A}_k^{(r+1)(n+1)}(K) = Mat_{r+1,n+1}(K), t \in \mathbf{A}_k^1(K) = K$. Then the functions $P_{i_0,\dots,i_r} : A \mapsto A_{i_0,\dots,i_r}$ are invariant sections of this bundle. This shows that matrices $A \in Mat_{(r+1)(n+1)}(\bar{k})$ of maximal rank are semi-stable points with respect to L. In fact they are properly stable points since the group $GL(n+1,\bar{k})$ acts freely on the open subset $\{P_{i_0,\dots,i_r} \neq 0\}$. Thus ve can view $G(r+1,n+1)$ as an open subvariety of the geometric quotient $(\mathbf{A}_k^{(r+1)(n+1)})^s(L)/\mathbf{GL}_k(r+1)$. Since we know that the Grassmann variety is a projective variety we obtain that it is equal to the whole quotient:

$$G(r+1,n+1) \cong (\mathbf{A}_k^{(r+1)(n+1)})^s(L)/\mathbf{GL}_k(r+1).$$

Another way to see $G(r+1,n+1)$ as a geometric quotient is to use that it is a homogeneous space with respect to the action of $\mathbf{GL}_k(n+1)$. The stabilizer group of the subspace spanned by the first $r+1$ unit vectors $e_0,\dots,e_r$ is the subgroup P of block-matrices of the form:

$$\begin{pmatrix} A_{r+1,r+1} & B_{r+1,n-r} \\ 0_{n-r,r+1} & C_{n-r,n-r} \end{pmatrix},$$

where the subscripts indicate the sizes of the blocks.

Let $\mathcal{O}_{G(r+1,n+1)}(1)$ denote the line bundle over $G(r+1,n+1)$ obtained as the pull-back of the bundle $\mathcal{O}_{\mathbf{P}_k^N}(1)$ with respect to the Plücker embedding. It is easy to see that this bundle is isomorphic to the line bundle obtained from the $\mathbf{GL}_k(r+1)$-linearized bundle L on $\mathbf{A}_k^{(r+1)(n+1)}$ by the construction described in the proof of Theorem 6.2 from Lecture 6. Since the Plücker embedding is obviously $\mathbf{SL}_k(n+1)$-equivariant, we get a canonical $\mathbf{SL}_k(n+1)$-linearization on $\mathcal{O}_{G(r+1,n+1)}(1)$. It can be proven that $\mathcal{O}_{G(r+1,n+1)}(1)$ generates $Pic(G(r+1,n+1))$ (this follows from computation of the Picard groups of homogeneous spaces). Since $\mathbf{SL}_k(n+1)$ is simple, and hence has no non-trivial characters, we obtain

$$Pic^{\mathbf{SL}_k(n+1)}(G(r+1,n+1)) \cong \mathbf{Z}$$

and $\mathcal{O}_{G(r+1,n+1)}(1)$ can be taken as a generator. Note that $\mathcal{O}_{G(r+1,n+1)}(1)$ is obviously ample, and its complete linear system defines a Plücker embedding. We shall use the notation $Z_{i_0,\dots,i_r}$ to denote the projective coordinates in $\mathbf{P}_k^N$ (we order them lexicographically). The value of these coordinates at any free module $L \in G(r+1,n+1)(K)$ is equal to the Plücker coordinates $p_{i_0,\dots,i_r}$ of L. Since $G(r+1,n+1)$ is not contained in a linear subspace of $\mathbf{P}_k^N$, the restriction map

$$\Gamma(\mathbf{P}_k^N, \mathcal{O}_{\mathbf{P}_k^N}(1)) \to \Gamma(G(r+1,n+1), \mathcal{O}_{G(r+1,n+1)}(1))$$

is injective. One can also show that it is surjective.

Let $X = G(r+1,n+1)^m$. For any vector $\mathbf{k} = (k_1,\dots,k_m) \in \mathbf{Z}^m$ we set

$$L_{\mathbf{k}} = \otimes_{i=1}^m p_i^*(\mathcal{O}_{G(r+1,n+1)}(1)^{\otimes k_i})$$

where $p_i : X \to G(r+1,n+1)$ is the i-th projection. One can prove that every line bundle over X is isomorphic to $L_{\mathbf{k}}$ for some $\mathbf{k}$. Since each projection is a $\mathbf{SL}_k(n+1)$-equivariant morphism, each $L_{\mathbf{k}}$ admits a canonical $\mathbf{SL}_k(n+1)$-linearization. Thus

$$Pic^{\mathbf{SL}_k(n+1)}(X) \cong \mathbf{Z}^m.$$

Also, $L_{\mathbf{k}}$ is ample if and only if all k_i are positive. In fact, if some tensor power of $L_{\mathbf{k}}$ defines a closed embedding $X \to \mathbf{P}_k^M$ then the restriction of $L_{\mathbf{k}}$ to any factor $G(r+1,n+1)$ is an ample line bundle. It is obvious that this restriction is isomorphic to $\mathcal{O}_{G(r+1,n+1)}(1)^{\otimes k_i}$, which is ample if and only if $k_i > 0$. Conversely, any $L_{\mathbf{k}}$ with positive $\mathbf{k}$ is very ample. It defines a projective embedding of X which is equal to the composition

$$X \to (\mathbf{P}_k^N)^m \to \prod_{i=1}^m \mathbf{P}_k^{\binom{N+k_i}{N}-1} \to \mathbf{P}_k^{\Pi_{i=1}^m \binom{N+k_i}{N}-1},$$

where the first map is the product of the Plücker embeddings, the second map is the product of the Veronese embeddings, and the last map is the Segre map.

9.2 Now we are ready to describe semi-stable and stable sequences $\mathcal{W} = (W_1, \ldots, W_m)$ of $\bar{k}$-points of X. We shall replace k with $\bar{k}$ to assume that k is algebraically closed.

Theorem. *Let* $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbf{Z}^m$, *with all* $k_i > 0$. *Then* $\mathcal{W} \in X^{ss}(L_{\mathbf{k}})$ *(resp.* $\in X_0^s(L_{\mathbf{k}})$*) if and only if for any proper subspace* W *of* $\mathbf{P}_k^n$

$$(n+1)\sum_{j=1}^m k_j[\dim(W_j \cap W) + 1] \le (r+1)(\dim W + 1)\sum_{j=1}^m k_j$$

(resp. there is strict inequality).

Proof. Let T be the maximal diagonal torus in $\mathbf{SL}_k(n+1)$. Each one-parameter subgroup of T is defined by $\lambda(t) = diag[t^{q_0}, \ldots, t^{q_n}]$, where $q_0 + \ldots + q_n = 0$. By permuting coordinates we may assume that

$$q_0 \ge q_1 \ge \cdots \ge q_n \quad (*).$$

Suppose $\mathcal{W}$ is semi-stable. Let $\bar{E}_s, s = 0, \ldots n$ be the linear space spanned by the unit vectors $e_0, \ldots, e_s$ and E_s the corresponding projective subspace. For any $W \in G(r+1, n+1)(k)$ and any integer $j, 0 \le j \le r$, there is a unique integer ν_j for which

$$\dim(W \cap E_{\nu_j}) = j, \dim(W \cap E_{\nu_j - 1}) = j - 1.$$

To see this we list the numbers $a_s = \dim(W \cap E_s), s = 0, \ldots, n$, and observe that $0 \le a_s - a_{s-1} \le 1$ and $a_n = r$, as each E_{s-1} is a hyperplane in E_s and $E_n = \mathbf{P}_k^n$. Then we see that each j occurs among these numbers and define ν_j to be the first s when $a_s = j$.

With this notation we obtain that W can be representable by a matrix A of the form

$$\begin{pmatrix} a_{00} & \dots & a_{0\nu_0} & 0 & \dots & \dots & \dots & \dots & 0 \\ a_{10} & \dots & \dots & a_{1\nu_1} & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ a_{r0} & \dots & \dots & \dots & \dots & a_{r\nu_r} & 0 & \dots & 0 \end{pmatrix}$$

where $a_{j\nu_j} \neq 0$ for all j. It is clear from viewing the maximal minors of this matrix that $p_{i_0,\dots,i_r}(W) = 0$ if $i_j > \nu_j$ for any value of j and $p_{\nu_0\dots\nu_r}(W) \neq 0$.

Now we notice that the projective coordinates of $\mathcal{W} = (W_1, \dots, W_m)$ in the embedding defined by the line bundle $L_{\mathbf{k}}$ are equal to the product of m monomials of degree k_i in Plücker coordinates of W_i. Since for each λ as in $(*)$ we have

$$p_{i_0,\dots,i_r}(\lambda(t)\cdot W) = t^{q_{i_0}+\dots+q_{i_r}} p_{i_0,\dots,i_r}(W),$$

we easily find that

$$\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) = \sum_{i=1}^{m} k_i(\sum_{j=0}^{r} q_{\nu_j^{(i)}}).$$

Using that $\dim(W_i \cap E_j) - \dim(W_i \cap E_{j-1}) = 0$ if $j \neq \nu_j^{(i)}$, we can rewrite the previous sum as follows

$$\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) = \sum_{i=1}^{m} k_i \left[\sum_{j=0}^{n} q_j(\dim(W_i \cap E_j) - \dim(W_i \cap E_{j-1}))\right] =$$

$$= \sum_{i=1}^{m} k_i \left[(r+1)q_n + \sum_{j=0}^{n-1}(\dim(W_i \cap E_j)+1)(q_j - q_{j+1})\right] =$$

$$= \left[\sum_{i=1}^{m} k_i\right](r+1)q_n + \sum_{j=0}^{n-1}\left[\sum_{i=1}^{m} k_i(\dim(W_i \cap E_j)+1)(q_j - q_{j+1})\right].$$

Consider the following special one-parameter subgroups λ_s given by

$$q_0 = \dots = q_s = n - s, q_{s+1} = \dots = q_n = -(s+1), 0 \leq s \leq n-1.$$

Plugging in these values of q_j, we find

$$\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda_s) = -(\sum_{i=1}^{m} k_i)(r+1)(s+1) + (n+1)(\sum_{i=1}^{m} k_i(\dim(W_i \cap E_s)+1). \quad (**)$$

If $\mathcal{W}$ is semi-stable (resp. properly stable) this number must be non-positive (resp. negative). Since any s-dimensional linear subspace of $\mathbf{P}_k^n$ is projectively equivalent to E_s, we obtain the necessary condition for semi-stability or proper stability from the Theorem. It is also sufficient. In fact, if it were satisfied but $\mathcal{W}$ were not semi-stable, we could find some $\lambda \in \mathcal{X}_*(\mathbf{SL}_k(n+1))$ such that $\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) > 0$. By choosing an appropriate coordinates, we may assume that $\lambda \in \mathcal{X}_*(T)$ and satisfies $(*)$. Then we use the easy fact that each λ satisfying $(*)$ can be written as a positive linear combination of λ_s. From this we deduce that $\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda_s) > 0$ for some s. Then the above computations show that $(**)$ does not hold for some s contradicting our assumption.

Corollary. *Assume the numbers* $\sum_{i=1}^{m} k_i$ *and* $n+1$ *are coprime. Then*

$$X^{ss}(L_{\mathbf{k}}) = X_0^s(L_{\mathbf{k}}).$$

9.3 Let us consider a special (but important) case when $r = 0$, i.e., $G(r+1, n+1) = \mathbf{P}_k^n$, so that all W_i are points. In this case $X = (\mathbf{P}_k^n)^m, L_{\mathbf{k}} = \otimes_{i=1}^m p_i^*(\mathcal{O}_{\mathbf{P}_k^n}(k_i))$. We get

Theorem. *Let* $\mathcal{P} = (p_1, \ldots, p_m) \in (\mathbf{P}_k^n(k))^m$. *Then*

$$\mathcal{P} \in ((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}})(\text{ resp. } \in ((\mathbf{P}_k^n)^m)_0^s(L_{\mathbf{k}}))$$

if and only if for every proper linear subspace W *of* $\mathbf{P}_k^n$

$$\sum_{i, p_i \in W} k_i \leq \frac{\dim W + 1}{n+1} (\sum_{i=1}^m k_i)$$

(resp. $<$*).*

In particular, if all $k_i = 1$, this condition can be rewritten in the form

$$\#\{i : p_i \in W\} \leq \frac{\dim W + 1}{n+1} m \quad (\text{resp. } <).$$

Corollary.

$$((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}}) \neq \emptyset \Leftrightarrow (n+1)\max_i\{k_i\} \leq \sum_{i=1}^m k_i.$$

$$((\mathbf{P}_k^n)^m)_0^s(L_{\mathbf{k}}) \neq \emptyset \Leftrightarrow (n+1)\max_i\{k_i\} < \sum_{i=1}^m k_i.$$

Proof. Let

$$((\mathbf{P}_k^n)^m)^{gen} := \{\mathcal{P} = (p_1, \ldots, p_m) : \text{each subset of } n+1 \text{ points } p_i \text{ spans } \mathbf{P}_k^n\}.$$

This is obviously an open non-empty subset of $(\mathbf{P}_k^n)^m$. We know that $((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}})$ is an open subset. So if it is not empty it has non-empty intersection with $((\mathbf{P}_k^n)^m)^{gen}$. If we take a point $\mathcal{P} = (p_1, \ldots, p_m)$ in the intersection, we obtain, since no two points p_i coincide, $(n+1)k_i \leq \sum_{i=1}^m k_i$ for each $i = 1, \ldots, m$. Conversely, if this condition is satisfied then each point $\mathcal{P} \in ((\mathbf{P}_k^n)^m)^{gen}$ is semi-stable with respect to $L_{\mathbf{k}}$. In fact, each subspace W of dimension s contains at most $s+1$ points p_i. Hence

$$\sum_{i, p_i \in W} k_i \leq (\dim W + 1)\max_i\{k_i\} \leq \frac{\dim W + 1}{n+1} (\sum_{i=1}^m k_i).$$

This proves the assertion about semi-stability. Similarly we prove the assertion about proper stability.

Remark. Let

$$\Delta_{n,m} = \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m : \sum_{i=1}^{m} x_i = n+1, 0 \leq x_i \leq 1, i = 1, \ldots, m\}.$$

This is called a $(m-1)$-dimensional *hypersimplex* of type n. Note that the ordinary simplex corresponds to the case when $n = 0$. One can restate the previous corollary in the following form. Consider the cone over $\Delta_{n,m}$ in $\mathbf{R}^{m+1}$

$$C\Delta_{n,m} = \{(x, \lambda) \in \mathbf{R}^m \times \mathbf{R}_+ : x \in \lambda\Delta_{n,m}\}.$$

We have the injective map

$$Pic^{\mathbf{SL}_k(n+1)}((\mathbf{P}_k^n)^m) \to \mathbf{R}^{m+1}, L_{\mathbf{k}} \mapsto (k_1, \ldots, k_m, (n+1)^{-1} \sum_{i=1}^{m} k_i),$$

which allows us to identify $Pic^{\mathbf{SL}_k}((\mathbf{P}_k^n)^m)$ with a subset of $\mathbf{R}^{m+1}$. We have

$$Pic^{\mathbf{SL}_k(n+1)}((\mathbf{P}_k^n)^m) \cap C\Delta_{n,m} = \{\, L \in Pic^{\mathbf{SL}_k(n+1)}((\mathbf{P}_k^n)^m) : ((\mathbf{P}_k^n)^m)^{ss}(L) \neq \emptyset\}.$$

In fact if the first m coordinates of a point $x \in \mathbf{R}^{m+1}$ from the left-hand-side are all positive, this follows immediately from the previous Corollary. Suppose some of the first coordinates of x are equal to zero, say the first t coordinates. Then $L_{\mathbf{k}} = p^*(L'_{\mathbf{k}})$, where $p : (\mathbf{P}_k^n)^m \to (\mathbf{P}_k^n)^{m-t}$ is the projection to the last $m-t$ factors, and $\mathbf{k}' = (k_{t+1}, \ldots, k_m)$. By applying the Corollary to $L'_{\mathbf{k}}$, we obtain that $((\mathbf{P}_k^n)^{m-t})^{ss}(L'_{\mathbf{k}}) \neq \emptyset$. It is easy to see that

$$(\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}}) = p^{-1}(((\mathbf{P}_k^n)^{m-t})^{ss}(L'_{\mathbf{k}})),$$

and we have a commutative diagram

$$\begin{array}{ccc} ((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}}) & \xrightarrow{p} & ((\mathbf{P}_k^n)^{m-t})^{ss}(L'_{\mathbf{k}}) \\ \downarrow & & \downarrow \\ ((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}})//\mathbf{SL}_k(n+1) & \xrightarrow{\bar{p}} & ((\mathbf{P}_k^n)^{m-t})^{ss}(L'_{\mathbf{k}})//\mathbf{SL}_k(n+1) \end{array}$$

where the vertical arrows are quotient maps and the map $\bar{p}$ is an isomorphism.

Observe that $\mathcal{P} \in ((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}}) \setminus ((\mathbf{P}_k^n)^m)_0^s(L_{\mathbf{k}})$ if and only if there exists a subspace W of dimension $d, 0 \leq d \leq n-1$ such that

$$(n+1) \sum_{i, p_i \in W} k_i = (\mathrm{dim} W + 1) \sum_{i=1}^{m} k_i.$$

This is equivalent to the condition that $L_{\mathbf{k}}$ belongs to the hyperplane

$$H_{I,d} := \{(x_1, \ldots, x_m, \lambda) \in \mathbf{R}^m : \sum_{i \in I} x_i = \lambda d\},$$

where I is a non-empty subset of $\{1,\ldots,m\}$. Let C be a connected component of $C\Delta_{n,m}\setminus\bigcup_{I,d}H_{I,d}$ (called a *chamber*). Then for any $L_{\mathbf{k}}\in C$ we have $((\mathbf{P}^n_k)^m)^{ss}(L_{\mathbf{k}})=((\mathbf{P}^n_k)^m)^s_0(L_{\mathbf{k}})$.

One can show that any two line bundles from the same chamber have the same set of semi-stable points. We refer the reader to **[DH]** for more general and precise results on this subject.

9.4 Examples. 1. Let us take $r=0,n=1,\mathbf{k}=(1,\ldots,1)$, i.e., consider ordered sequences of m points in $\mathbf{P}^1_k$. The condition of semi-stability tells us that

$$\mathcal{P}=(p_1,\ldots,p_m)\in((\mathbf{P}^1_k)^m)^{ss}(L_{\mathbf{k}})\Leftrightarrow \text{each } p_i \text{ is repeated in } p \text{ at most } m/2 \text{ times.}$$

Note that this condition is similar to one we obtained in the previous lecture for binary homogeneous forms. This is not accidental. The permutation group Σ_m acts naturally on $(\mathbf{P}^1_k)^m$ by permuting the factors. The geometric quotient $(\mathbf{P}^1_k)^m/\Sigma_m$ consists of unordered m-tuples of points and can be easily identified with the space $\mathrm{Hyp}_1(m)$. It is easy to see that the notion of semi-stability for ordered and unordered point sets (with respect to $L_{(1,\ldots,1)}$) coincide.

Let us set

$$P^m_n(\mathbf{k}):=((\mathbf{P}^n_k)^m)^{ss}(L_{\mathbf{k}})//\mathbf{SL}_k(n+1).$$

If $\mathbf{k}=(1,\ldots,1)$ we denote it by P^m_n. It follows from the construction of the quotient that

$$P^m_n(\mathbf{k})=\mathrm{Proj}(\oplus_{d\geq 0}\Gamma((\mathbf{P}^n_k)^m,L_{\mathbf{k}}^{\otimes d})^{\mathbf{SL}_k})=\mathrm{Proj}(\oplus_{d\geq 0}S^d(V(k_1)\otimes\ldots\otimes V(k_m))^{\mathbf{SL}_k}),$$

where we denote by $V(i)$ the space $\Gamma(\mathbf{P}^n_k,\mathcal{O}_{\mathbf{P}^n_k}(i))=k[Z_0,\ldots,Z_n]_i$, and $S^d(V)$ is the d-th symmetric power of a linear space V. The classical theory of invariants teaches us how to compute the graded algebra

$$R^m_n(\mathbf{k}):=\oplus_{d\geq 0}S^d(V(k_1)\otimes\ldots\otimes V(k_m))^{\mathbf{SL}_k}.$$

Let $\mathcal{P}=(p_1,\ldots,p_m)\in(\mathbf{P}^n_k)^m$ and

$$A(\mathcal{P})=[p_1^*,\ldots,p_m^*]$$

be the matrix of size $(n+1)\times m$ whose columns are the vectors of projective coordinates of the points p_i. Let $M_{i_1,\ldots,i_{n+1}}$ be the maximal minor of this matrix composed of columns $p^*_{i_1},\ldots,p^*_{i_{n+1}}$. Since $M_{i_1,\ldots,i_{n+1}}$ is a multi-linear function in the vectors p_i^*, we can view it as a section of $L_{1,\ldots,1}$ over $(\mathbf{P}^n_k)^{n+1}$. Denote this section by $[i_1,\ldots,i_{n+1}]$. Now suppose we have w increasing sequences $(i_1,\ldots,i_{n+1}),(j_1,\ldots,j_{n+1}),\ldots,(q_1,\ldots,q_{n+1})$ of $n+1$ integers between 1 and m such that each number $s\in\{1,\ldots,m\}$ occurs exactly dk_s times. This of course implies that

$$d(k_1+\ldots+k_m)=(n+1)w.$$

Then, by lifting sections $[i_1,\ldots,i_{n+1}],\ldots,[q_1,\ldots,q_{n+1}]$ of $L_{1,\ldots,1}$ to sections of its inverse image with respect to the projection $(\mathbf{P}^n_k)^m\to(\mathbf{P}^n_k)^{n+1}$ and then tensoring them, we can consider the product

$$[i_1,\ldots,i_{n+1}][j_1,\ldots,j_{n+1}]\cdots[q_1,\ldots,q_{n+1}]$$

as a homogeneous element of degree d of the algebra $R_n^m(\mathbf{k})$.

The First Fundamental Theorem of Invariant Theory asserts that these functions generate the whole algebra $R_n^m(\mathbf{k})$. Also the straightening algorithm (see **[Stu]**, p. 82) allows one to express each function as a linear combination of functions satisfying the condition that $i_\alpha \leq j_\alpha \leq \ldots \leq q_\alpha$ for all $\alpha = 1, \ldots, n+1$. Consider for example the simplest case when $n = 1, m = 4, \mathbf{k} = (1,1,1,1)$. Then the piece of degree 1 of (R_1^4) is spanned by two functions [12][34] and [13][24]. The value of the ratio $r = [12][34]/[13][24]$ on 4 ordered points (p_1, p_2, p_3, p_4) with the coordinate matrix

$$A = \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \end{pmatrix}$$

is equal to

$$r(p_1, p_2, p_3, p_4) = \frac{(b_1 a_0 - a_1 b_0)(c_0 d_1 - c_1 d_0)}{(a_0 c_1 - a_1 c_0)(b_0 d_1 - b_1 d_0)}.$$

This is called the *cross-ratio* of the four points in the projective line. If we choose coordinates in the form $(1, x_i), i = 1, \ldots, 4$, assuming that none of the points is the infinity point, we obtain

$$r(p_1, p_2, p_3, p_4) = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}.$$

This is the familiar expression for the cross-ratio of four ordered points in the complex plane (see **[Alf]**).

If $p = (0, 1, \infty, x) = ((1,0), (1,1), (0,1), (1,x)$ we get

$$r(0, 1, \infty, x) = 1 - x.$$

This implies that two distinct ordered quadruples of points in $\mathbf{P}_k^1$ are projectively equivalent if and only if their cross-ratios coincide.

Note that the cross-ratio of four distinct points never takes the values $0, 1, \infty$. The quadruples (p_1, p_2, p_3, p_4) go to 0 if $p_1 = p_2$ or $p_3 = p_4$. The only closed orbit in the fibre over 0 consists of configurations with $p_1 = p_2, p_3 = p_4$.

We refer the reader for more details about the computations of the algebras R_n^m and the geometry of the spaces P_n^m to **[DO]**.

2. Let us take n and r the same as before but change $\mathbf{k}$ such that $k_1 + \ldots + k_{m-1} < k_m$. Then the sequence $(p_1, \ldots, p_1, p_m)$ is semi-stable.

3. Let us take $r = 0, n = 2, \mathbf{k} = (1, \ldots, 1)$.

$$(p_1, \ldots, p_m) \in ((\mathbf{P}_k^2)^m)^{ss}(L_{\mathbf{k}}) \Leftrightarrow \leq m/3 \text{ repetitions}, \leq 2m/3 \text{ points on a line}.$$

Semi-stability coincides with proper stability when 3 does not divide m.

4. Let us take $r = 1, n = 3, \mathbf{k} = (1, \ldots, 1)$. Then we are dealing with sequences $(l_1, \ldots, l_m)$ of lines in $\mathbf{P}_k^3$. Let us apply the criterion of semi-stability, first taking W to be a point, then a line, and finally a plane. In the first case we obtain

$$\#\{i : W \in W_i\} \leq m/2,$$

that is, no more than $m/2$ lines intersect at one point.

Taking W to be a line, we obtain

$$2\#\{i : W = W_i\} + \#\{i : W_i \neq W, W \cap W_i \neq \emptyset\} \leq m,$$

in particular, no more than $m/2$ lines coincide and no more than $m - 2t$ lines W_i intersect a line W_j which is repeated t times.

Finally, taking W to be a plane, we get

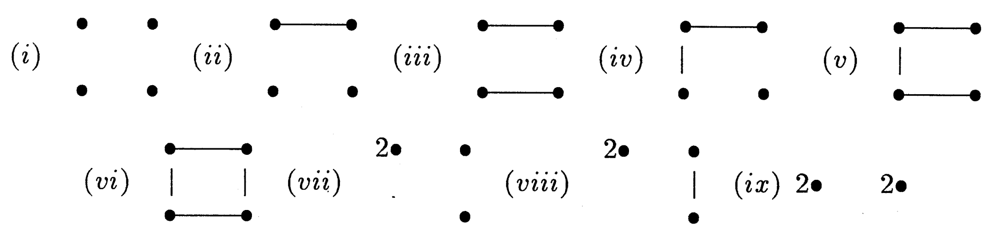
$$2\#\{i : W_i \subset W\} + \#\{i : W_i \not\subset W, W \cap W_i \neq \emptyset\} \leq 3m/2,$$

that is, no more than $m/2$ lines are coplanar.

For example, there are no semi-stable points if $m = 1$. If $m = 2$, a pair of lines is semi-stable if and only if it is a pair of skew lines. It easy to see that by a projective transformation a pair of skew lines is reduced to the two lines given by the equations $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$. Thus we have one orbit (stable but not properly stable). Similarly, if $m = 3$ we get one stable orbit represented by the lines $x_0 = x_1 = 0, x_2 = x_3 = 0$, and $x_0 + x_1 = x_2 + x_3 = 0$.

9.5 Let us concentrate on the case $m = 4$. First there are no properly stable quadruples of lines. This follows from the fact that there is always a line intersecting 4 lines in $\mathbf{P}^3_k$ (called a *transversal line*). To see this, we argue as follows: The assertion is obvious if two lines lie in a plane π. We take the line in π which joins the intersection points of π with the other lines. Assume now that the lines are skew. Choose 3 points on each of the first 3 lines l_1, l_2, l_3. Then there exists a quadric passing through these points (since the space of quadrics in $\mathbf{P}^3_k$ is of dimension 9). By Bezout's Theorem this quadric Q contains the lines l_1, l_2, l_3. Since these lines are skew, the quadric is nonsingular and the three lines belong to the same ruling. The fourth line l_4 intersects Q at two points. Let l be the line on Q from another ruling which passes through one of these two points (which may coincide). Then it intersects all the lines l_i. Note that one of the following three possible cases may occur: there are either exactly two transversals, one transversal, or infinitely many. For example when the lines are skew, we have two transversals if one of the lines intersects the quadric through the remaining lines at two distinct points (resp. at one point, resp. is contained in the quadric).

Using Example 4 in 9.4 we can easily list all possible "topological" types of semi-stable quadruples of lines. These types are described by the following incidence graphs:



Here a dot indicates a line, two dots are connected by an edge if the two lines intersect. The number 2 indicates that the line is repeated in the sequence.

Recall that $\Gamma(X, L_{(1,1,1,1)})^{\mathbf{SL}_k(4)}$ contains the subspace spanned by tensor products

$$Z^{(1)}_{i_1j_1} \otimes Z^{(2)}_{i_2j_2} \otimes Z^{(3)}_{i_3j_3} \otimes Z^{(4)}_{i_4j_4}. \quad (***)$$

We know that each line W is given by a decomposable 2-vector $v_1 \wedge v_2$. For any pair of lines W_a, W_b among 4 lines $W_1, \ldots, W_4$ let $W_a \wedge W_b$ denote the wedge product of the corresponding tensors. If we fix a basis in the space we can identify this with a number. In the matrix representation of lines this is just the determinant of the 4×4-matrix which is obtained by putting the first matrix in the first two rows, and putting the second matrix in the last two rows. If (a, b) and (c, d) are complementary subsets of $\{1, 2, 3, 4\}$, the product $(W_a \wedge W_b)(W_c \wedge W_d)$ is a linear combination of monomials

$$p^{(1)}_{i_1j_1} p^{(2)}_{i_2j_2} p^{(3)}_{i_3j_3} p^{(4)}_{i_4j_4},$$

where $p^{(s)}_{i_sj_s}$ are the Plücker coordinates of the line W_s. Let $[ab][cd] \in \Gamma(X, L_{(1,1,1,1)})^{\mathbf{SL}_k(4)}$ denote the same linear combination but with each $p^{(1)}_{i_1j_1} p^{(2)}_{i_2j_2} p^{(3)}_{i_3j_3} p^{(4)}_{i_4j_4}$ replaced with the monomial $(***)$. We shall continue to denote a quadruple of lines (W_1, W_2, W_3, W_4) by $\mathcal{W}$. We have

$$[ab][cd](\mathcal{W}) = (W_a \wedge W_b)(W_c \wedge W_d).$$

Observe that this expression does not depend (up to a multiplicative factor) on the choice of a basis in each subspace as soon as we employ the same basis in the whole space. Thus we have three different sections: [12][34], [13][24] and [14][23]. They form a base-point-free linear system which defines the $\mathbf{SL}_k(4)$-equivariant map

$$f : X \to \mathbf{P}^2_k, \ \mathcal{W} \mapsto ((W_1 \wedge W_2)(W_3 \wedge W_4), (W_1 \wedge W_3)(W_2 \wedge W_4), (W_1 \wedge W_4)(W_2 \wedge W_3)).$$

To see that there are no base-points we observe that $[ab](\mathcal{W}) = 0$ if and only if the lines W_a and W_b intersect. It is easy to see that [12][34], [13][24] and [14][23] vanish simultaneously on $\mathcal{W}$ if and only if one line intersects the remaining lines or three lines are coplanar. Both of these conditions are excluded by semi-stability. Since the map f is equivariant, it factors through a map

$$\bar{f} : X//\mathbf{SL}_k(4) \to \mathbf{P}^2_k.$$

We shall show that this map is an isomorphism by describing the fibres of the map f. This will also give us the classification of all semi-stable orbits.

Let us start with values of f at orbits of type (i). Let Q be the quadric containing the first three lines. Since all nonsingular quadrics are projectively isomorphic we can assume that Q is given by the equation

$$Z_0Z_2 - Z_1Z_3 = 0.$$

It contains two line rulings defined by the equations

$$\lambda Z_0 - \mu Z_3 = \lambda Z_1 - \mu Z_2 = 0;$$

$$\lambda' Z_0 - \mu' Z_1 = \lambda' Z_3 - \mu' Z_2 = 0.$$

The two rulings define an isomorphism $\mathbf{P}_k^1 \times \mathbf{P}_k^1 \cong Q$ which can be given by the formula

$$((\lambda, \mu), (\lambda', \mu')) \mapsto (\mu\mu', \lambda'\mu, \lambda\lambda', \lambda\mu').$$

The fibres of the first projection $\mathbf{P}_k^1 \times \mathbf{P}_k^1 \to \mathbf{P}_k^1$ are the lines from the first ruling and the fibres of the second projections are the lines from the second ruling. Using this isomorphism it is easy to see that the group of projective automorphisms of the quadric Q is generated by $\mathrm{Aut}(\mathbf{P}_k^1) \times \mathrm{Aut}(\mathbf{P}_k^1)$ and the involution which interchanges the factors. Explicitly, the automorphisms of the first factor $(\lambda, \mu) \mapsto (a\lambda + b\mu, c\lambda + d\mu)$ act on Q by the formulas

$$(Z_0, Z_1, Z_2, Z_3) \mapsto (cZ_3 + dZ_0, cZ_2 + dZ_1, aZ_2 + bZ_1, aZ_3 + bZ_0),$$

and similarly for the second factor

$$(Z_0, Z_1, Z_2, Z_3) \mapsto (cZ_1 + dZ_0, aZ_1 + bZ_0, aZ_2 + bZ_3, cZ_2 + dZ_3).$$

Without loss of generality we may assume that the first three lines W_1, W_2, W_3 belong to the first ruling. So applying an automorphism of the first factor we can assume that these lines are the pre-images of the points $0, 1$ and ∞. Their wedge representations are

$$W_1 = e_0 \wedge e_1, W_2 = (e_0 + e_3) \wedge (e_1 + e_2), W_3 = e_2 \wedge e_3.$$

Now consider the intersection points of the fourth line W_4 with Q. These two points lie on two lines from the first ruling (which may coincide) and hence, applying automorphisms of the second factor we can reduce the equation of W_4 to the form

$$W_4 = (ae_0 + e_3) \wedge (be_1 + e_2 + \alpha e_0).$$

Here $a, b \neq 0, 1$. Also $a \neq b, \alpha = 0$ if W_4 intersects Q at two different points, $a = b, \alpha = 1$ if W_4 is tangent to Q, and $a = b, \alpha = 0$, if W_4 coincides with a line of the first ruling.

We find

$$([12][34], [13][24], [14][23])(\mathcal{W}) = (-ab, -1 - ab + a + b, -1).$$

By a linear change of the coordinates in $\mathbf{P}_k^2$ we may assume that the image is equal to the point $((1-a)(1-b), a+b-2, 1)$. If we identify $\mathbf{P}_k^2$ with $\mathrm{Hyp}_1(2)$, by assigning to a point (a_0, a_1, a_2) the zeroes of the binary form $a_0 t_0^2 + a_1 t_0 t_1 + a_2 t_1^2$, we see that our image corresponds to the binary form with two roots equal to $1-a$ and $1-b$. ¿From Example 1 in 9.4 we infer that the value $f(\mathcal{W})$ is equal to the unordered pair of two cross-ratios of four ordered points $W_1 \cap l, \ldots, W_4 \cap l$, where l is a transversal line. Note that we have two transversals if W_4 intersects Q at two points, one if it is tangent to Q and infinitely many (lines from the second ruling) if W_4 is contained in Q. In the latter case the cross-ratio is independent of the choice of a transversal line.

Let X^{reg} denote the set of quadruples $\mathcal{W}$ of skew lines with two common transversals. Since each orbit in X^{reg} is completely determined by its image under f, we obtain that all orbits in X^{reg} are stable. Also since every fibre of $f : X^{reg} \to \mathbf{P}_k^2$ consists of one orbit, we obtain that the map $\bar{f}$ defines an isomorphism

$$X^{reg}/\mathbf{SL}_k(4) \to U^{reg} \subset \mathbf{P}_k^2,$$

where U^{reg} consists of binary quadrics $a_0t_0^2+a_1t_0t_1+a_2t_1^2$ with no roots equal to $(1,0),(1,1)$, $(0,1)$ and no double roots. Thus the closed set $\mathbf{P}_k^2 \setminus U^{reg}$ is equal to the union $C \cup T$, where C is the conic

$$a_1^2 - 4a_0a_2 = 0$$

and T is the union of its three tangent lines

$$l_{12} : a_0 = 0, l_{13} : a_0 + a_1 + a_2 = 0, l_{14} : a_2 = 0.$$

Note that we have

$$(a_0, a_1, a_2) = (-[13][24], [13][24] - [12][34] + [14][23], -[14][23]).$$

This we take for the new formulas for our map f. The stabilizer of any quadruple $\mathcal{W}$ from X^{reg} is isomorphic to $\mathbf{G}_{\mathbf{m},k}$. It consists of all automorphisms of the second factor of $Q = \mathbf{P}_k^1 \times \mathbf{P}_k^1$ which leave the set $W_4 \cap Q$ invariant. This confirms our earlier observation that there are no properly stable points in X. If $\mathcal{W}$ has one transversal or infinitely many, then

$$f(\mathcal{W}) \in C \quad \text{but } f(\mathcal{W}) \notin T.$$

Here we use the obvious observation that the pre-image of any point from T has at least one pair of intersecting lines. The fibre of f over each point of $C \setminus (C \cap T)$ consists of two orbits, of dimensions 14 and 12. The closed orbit consists of quadruples with infinitely many transversals. Its stabilizer is isomorphic to $\mathbf{SL}_k(2)$.

Now let us find the fibres over points from T. We shall examine, case by case, all other possible topological types $(ii) - (ix)$ of semi-stable quadruples.

Assume $\mathcal{W}$ is of type (ii) or (iv) or (vii). In all of these cases we have three skew lines. By choosing a quadric through these lines, we may assume, as in the previous case that

$$W_1 = e_0 \wedge e_1, W_2 = (e_0 + e_3) \wedge (e_1 + e_2), W_3 = e_2 \wedge e_3.$$

Now W_4 intersects Q at two lines from the first ruling, and one or two of them are from $\mathcal{W}$. Without loss of generality we may assume that W_4 intersects W_3. Applying automorphisms of the second factor of $\mathbf{P}_k^1 \times \mathbf{P}_k^1$, we may assume that $W_4 \cap W_3 = e_3$ and the second point of intersection of W_4 with Q is $e_2 + ae_1 + be_0$. Here

(i) $a = b = 0$ if $W_4 = W_3$,
(ii) $a = 0, b = 1$ if W_4 is tangent to Q,
(iii) $a = 1, b = 0$ if W_4 intersects W_2, and
(iv) $a \neq 0, 1, b = 0$ if W_4 intersects Q transversally and intersects only one W_3.

We find

$$f(\mathcal{W}) = (1-a, a-2, 1) \in l_{13}.$$

If $a \neq 0,1$, the image does not lie in $C \cup l_{12} \cup l_{14}$. If $a = 1$, $f(\mathcal{W}) = l_{13} \cap l_{13}$. If $a = 0$, $f(\mathcal{W}) = l_{13} \cap C$. None of these orbits is closed. For example, if $a = b = 0$, i.e.,

$$W_1 = e_0 \wedge e_1, W_2 = (e_0 + e_3) \wedge (e_1 + e_2), W_3 = W_4 = e_2 \wedge e_3,$$

we can apply the one-parameter subgroup $\lambda(t) = \operatorname{diag}(1, t^{-1}, t^{-1}, t^2)$ to get

$$\lim_{t \to 0} \lambda(t) \cdot \mathcal{W} = (W_1, e_0 \wedge (e_1 + e_2), W_3, W_3).$$

This is a quadruple of topological type (viii).

Now let us consider the remaining types $(iii), (v), (vi), (viii)$ and (ix). Without loss of generality we may assume that W_1 intersects W_2 and W_3 intersects W_4. Assume first that the plane containing W_1 and W_2 does not contain the point $W_3 \cap W_4$. After applying a linear transformation we can write

$$W_1 = e_0 \wedge e_1, W_2 = e_0 \wedge e_2, W_3 = e_3 \wedge (a_2 e_2 + a_1 e_1 + a_0 e_0), W_4 = e_3 \wedge (b_2 e_2 + b_1 e_1 + b_0 e_0).$$

Here a_2, b_2 cannot be both zero, as otherwise we have three coplanar lines. So we may assume that $a_2 = 1$. Replacing e_2 by $e_2 + a_0 e_0$, we may further assume that $a_0 = 0$. We find that

$$f(\mathcal{W}) = (b_1, -b_1 - b_2 a_1, b_2 a_1) \in l_{13}.$$

Suppose first that $b_2, b_1 \neq 0$. By scaling, we may further assume that $b_2 = b_1 = 1$, and $b_0 = 0$ or 1. This gives us

$$W_1 = e_0 \wedge e_1, W_2 = e_0 \wedge e_2, W_3 = e_3 \wedge (e_2 + a e_1), W_4 = e_3 \wedge (e_2 + e_1 + b e_0).$$

If $a \neq 0,1$, the image of $\mathcal{W}$ is not in the union of the conic C and the two other lines l_{12}, l_{14}. If additionally $b = 0$ the planes $\langle W_1, W_2 \rangle$ and $\langle W_3, W_4 \rangle$ intersect along the line l spanned by e_1 and e_2. Furthermore the parameter a is determined by the cross-ratio of the four points $(W_1 \cap l, W_2 \cap l, W_3 \cap l, W_4 \cap l)$. These orbits are closed and of dimension 13. If $b_0 \neq 0$, the line $\langle W_1, W_2 \rangle \cap \langle W_3, W_4 \rangle$ moves with the parameter a. We leave to the reader to check that these orbits are not closed and of dimension 14.

If $a = b = 1$, we have

$$f(\mathcal{W}) \in C \cap l_{13}.$$

In this case the planes $\langle W_1, W_2 \rangle$ and $\langle W_3, W_4 \rangle$ intersect along the line spanned by $e_1 + e_2$ and e_0. This line passes through the point $W_1 \cap W_2$. This orbit is not closed and of dimension 13.

Assume that $b_1 = 0$. The case $a_1 = 0$ is reduced to this by reordering the lines. We get

$$f(\mathcal{W}) = (0, -1, 1) \in l_{13} \cap l_{12}$$

and

$$W_1 = e_0 \wedge e_1, W_2 = e_0 \wedge e_2, W_3 = e_3 \wedge (e_2 + a_1 e_1), W_4 = e_3 \wedge (e_2 + b_0 e_0).$$

Here $a_1 \neq 0$ as otherwise W_2 intersects the remaining lines. So we may assume $a_1 = 1$. Observe that W_2 intersects now W_4 so that $\mathcal{W}$ is of type (v). If further $b_0 = 0$, W_1 intersects W_3 so that $\mathcal{W}$ is of type (vi). The latter orbit is closed and is of dimension 13; it lies in the closure of the former orbit.

Finally we consider the case when the intersection point of the lines W_3 and W_4 lies in the plane $\langle W_1, W_2 \rangle$. We can write

$$W_1 = e_0 \wedge e_1, W_2 = e_0 \wedge e_2, W_3 = e_3 \wedge (a_2 e_2 + a_1 e_1 + a_0 e_0),$$

$$W_4 = (a_2 e_2 + a_1 e_1 + a_0 e_0) \wedge (b_2 e_2 + b_1 e_1 + b_0 e_0).$$

Here $a_2, a_1, b_2 \neq 0$ as otherwise either one of the lines W_1, W_2 intersects the remaining lines or W_1, W_2, W_4 lie in the same plane. Thus replacing $a_2 e_2 + a_0 e_0$ by e_2 we may assume that $a_2 = a_1 = b_2 = 1, a_0 = 0$. Furthermore, replacing e_3 by $e_3 + b_2(e_0 + e_1)$ we may assume that $b_2 = 0$. So

$$W_1 = e_0 \wedge e_1, W_2 = e_0 \wedge e_2, W_3 = e_3 \wedge (e_2 + e_1), W_4 = (e_2 + e_1) \wedge (e_3 + b e_1 + a e_0).$$

We find that

$$f(\mathcal{W}) = (1, -2, 1) \in C \cap l_{13}.$$

Assume $b \neq 0$. Applying the transformation

$$(e_0, e_1, e_2, e_3) \to (b^2 e_0, b^{-1} e_1 - abe_0, b^{-1} e_2 + abe_0, e_3),$$

we eliminate the coefficient b. If $a = 0$, $W_3 = W_4$ and $\mathcal{W}$ is of type $(viii)$. By changing e_1 to $e_1 + e_2$ and then applying the one-parameter subgroup $\operatorname{diag}(1, t, 1, t^{-1})$ we find

$$\lim_{t \to 0} \lambda(t) \cdot \mathcal{W} = (W_1, W_1, W_3, W_3).$$

If $a \neq 0$, we have the orbit representing $\mathcal{W}$ with the peculiar property that the line $\langle W_1, W_2 \rangle \cap \langle W_1, W_2 \rangle$ contains the points $W_1 \cap W_2$ and $W_3 \cap W_4$. The closure of this orbit contains the orbits with $a = 0$.

This finishes the computations. We obtain from the previous analysis that the fibre of f over each point of $\mathbf{P}^2_k$ contains a unique closed orbit. Hence it is the quotient map and we obtain

$$X^{ss} // \mathbf{SL}_k(4) \cong \mathbf{P}^2_k.$$

It is convenient to collect all the information about semi-stable orbits in the following table. We shall use the following notations:

$$C^\circ = C \setminus (C \cap T); P = \{l_{12} \cap l_{13}, l_{12} \cap l_{23}, l_{13} \cap l_{23}\}; \; T^\circ = T \setminus (C \cap T) \setminus P,$$

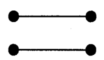
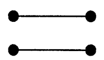
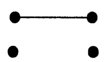
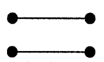
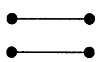
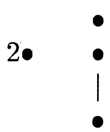
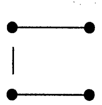
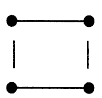
as well as the following abbreviations:

s = stable; nsc= non-stable, closed; nsnc = non-stable, non-closed;

t = the number of transversal lines to W.

dim = dimension of the orbit.

The last column gives the equations of a representative of the orbit.

Type	dim	f($\mathcal{W}$)	t	stability	canonical form
	14	$\in U^{reg}$	2	s	$e_0 \wedge e_1, (e_1+e_2) \wedge (e_0+e_3), e_2 \wedge e_3,$ $(ae_0+e_3) \wedge (be_1+e_2), a \neq b, a, b \neq 0, 1$
	14	$\in C^{\circ}$	1	nsnc	$e_0 \wedge e_1, (e_1+e_2) \wedge (e_0+e_3), e_2 \wedge e_3,$ $(ae_0+e_3) \wedge (ae_1+e_2+e_0), a \neq 0, 1$
	12	$\in C^{\circ}$	∞	nsc	$e_0 \wedge e_1, (e_1+e_2) \wedge (e_0+e_3), e_2 \wedge e_3,$ $(ae_0+e_3) \wedge (ae_1+e_2), a \neq 0, 1$
	14	$\in T^{\circ}$	2	nsnc	$e_0 \wedge e_1, (e_1+e_2) \wedge (e_0+e_3),$ $e_2 \wedge e_3, e_3 \wedge (ae_1+e_2)(a \neq 0, 1)$
	14	$\in T^{\circ}$	1	nsnc	$e_0 \wedge e_1, e_0 \wedge e_2, (e_2+ae_1) \wedge e_3,$ $e_3 \wedge (e_1+e_2+e_0)(a \neq 0, 1)$
	13	$\in T^{\circ}$	1	nsc	$e_0 \wedge e_1, e_0 \wedge e_2, e_3 \wedge (e_1+e_2)$ $(e_2+ae_1) \wedge e_3, (a \neq 0, 1)$
	14	$\in T \cap C$	1	nsnc	$e_0 \wedge e_1, (e_1+e_2) \wedge (e_0+e_3),$ $e_2 \wedge e_3, e_3 \wedge (e_0+e_2)$
	13	$\in T \cap C$	1	nsnc	$e_0 \wedge e_1, e_0 \wedge e_2, e_3 \wedge (e_1+e_2),$ $e_3 \wedge (e_2+e_1+e_0)$
	12	$\in T \cap C$	1	nsnc	$e_0 \wedge e_1, e_0 \wedge e_2, e_3 \wedge (e_1+e_2),$ $(e_1+e_2) \wedge (e_0+e_3)$
	12	$\in T \cap C$	∞	nsnc	$e_0 \wedge e_1, (e_0+e_3) \wedge (e_1+e_2), e_2 \wedge e_3, e_2 \wedge e_3$
	12	$\in T \cap C$	∞	nsnc	$e_0 \wedge e_1, e_0 \wedge (e_1+e_2), e_2 \wedge e_3, e_2 \wedge e_3$
	11	$\in T \cap C$	∞^2	nsc	$e_0 \wedge e_1, e_0 \wedge e_1, e_2 \wedge e_3, e_2 \wedge e_3$
	14	$\in P$	1	nsnc	$e_0 \wedge e_1, e_0 \wedge e_2, (e_2+e_1) \wedge e_3, (e_0+e_2) \wedge e_3$
	13	$\in P$	1	nsc	$e_0 \wedge e_1, e_0 \wedge e_2, (e_2+e_1) \wedge e_3, e_2 \wedge e_3$

9.6 As far as I know, nothing is known about the quotient spaces for sequences of more than four lines in $\mathbf{P}_k^3$. For example, we do not know whether the quotients are rational. As the following theorem shows the spaces P_n^m of configurations of points are rational varieties.

Theorem. *Assume* $((\mathbf{P}_k^n)^m(L_\mathbf{k}))^{ss} \neq \emptyset$. *Then* $P_n^m(\mathbf{k})$ *is a rational variety of dimension* $n(m-n-2)$.

Proof. Consider the subset U^{reg} of ordered point sets in which any subset of $n+1$ points spans $\mathbf{P}_k^n$. Let Z be its closed subset consisting of point sets $\mathcal{P} = (p_1, \ldots, p_m)$ with

$$p_1^* = (1, \ldots, 0), p_2^* = (0, 1, 0, \ldots, 0), \ldots, p_{n+1}^* = (0, \ldots, 0, 1), p_{n+2}^* = (1, \ldots, 1).$$

We have the map

$$\sigma : G \times Z \to U^{reg}$$

defined by the action of G on X. For any $\mathcal{P} \in U^{reg}$ the projective coordinates of the first $n+1$ points are linearly independent. So we can find an element $g \in \mathbf{SL}_k(n+1)$ which transforms these points to the unit vectors $e_1, \ldots, e_{n+1}$. Now the coordinates of the point p_{n+2} are non-zero since otherwise we find $n+1$ points among $p_1, \ldots, p_{n+2}$ lying in a hyperplane. Since the action of the diagonal matrices does not change the projective coordinates of the first $n+1$ points, we can use them to normalize the coordinates of the point p_{n+2}. This shows that the map σ is surjective. It is also injective. In fact, if $g \cdot \mathcal{P} = g' \cdot \mathcal{P}'$ we get $g'^{-1} g \cdot \mathcal{P} = \mathcal{P}'$. But the latter implies that $g'^{-1}g$ fixes the vectors $e_1, \ldots, e_{n+1}, e_1 + \ldots + e_{n+1}$ up to multiplicative factors. This gives that $g'^{-1}g = 1$, i.e., $g' = g$ and hence $\mathcal{P} = \mathcal{P}'$. Note that σ is obviously G-invariant when we consider the left translation of G on the left factor and the trivial action on the right factor. Passing to the quotients we obtain

$$U^{reg}/G \cong (G \times Z)/G \cong Z \cong (\mathbf{P}_k^n)^{m-n-2}.$$

Here we mean the quotient with respect to any $L_\mathbf{k}$ for which $((\mathbf{P}_k^n)^m(L_\mathbf{k}))^{ss} \neq \emptyset$. It follows easily from the proof of Corollary 9.3 that U^{reg} is an open subset of $((\mathbf{P}_k^n)^m)_0^s(L_\mathbf{k})$. Thus U^{reg}/G is an open subset of $P_n^m(\mathbf{k})$ and we are done.

Problems.

1. Prove that the orbit of $\mathcal{P} = (p_1, \ldots, p_m)$ in $((\mathbf{P}_k^n)^m)^{ss}(L_{\mathbf{k}})$ is closed but not properly stable if and only if there exists a partition of $\{1, \ldots, m\}$ into subsets $J_s, s = 1, \ldots, r$, such that for any s one can find a proper subspace W_s of $\mathbf{P}_k^n$ with $\sum_{i \in J_s, p_i \in W_s} k_i = \frac{\dim W+1}{n+1}(\sum_{i=1}^m k_i)$.
2. Show that the projection $\mathbf{P}_k^m \to \mathbf{P}_k^{m-1}$ induces the map $P_n^m \to P_n^{m-1}$. Describe the fibres of the map $P_1^5 \to P_1^4$ and show that P_1^5 is isomorphic to the blow-up of 4 points in $\mathbf{P}_k^2$.
3. Draw a picture of the hypersimplex $\Delta_{1,4}$ and describe the chambers of the cone $C\Delta_{1,4}$.
4. Consider the action of the permutation group Σ_4 on P_1^4 and show that the kernel of this action is isomorphic to the group $(\mathbf{Z}/2\mathbf{Z})^2$. Find the orbits whose stabilizers are of order strictly larger than 4. Compute the corresponding cross-ratios.
5. Prove that the algebra R_1^4 of bracket polynomials is isomorphic to the algebra of polynomials in two variables (without using the fact that $P_1^4 \cong \mathbf{P}_k^1$).
6. Find the equation (in terms of functions $[ij]$) of the closure of the locus of quadruples of lines in $\mathbf{P}_k^3$ which have only one transversal line.
7. For each semi-stable orbit of quadruples of lines in $\mathbf{P}_k^3$ find the semi-stable orbits lying in its closure.
8. Prove that $P_n^m(L_{\mathbf{k}})$ is isomorphic to a categorical quotient of some open subset of the Grassmannian $G(n+1, m)$ with respect to the action of the torus $\mathbf{G}_{\mathbf{m},k}^m$ via its standard action in $\mathbf{A}_k^m$.
9. Prove that the closure of the locus of those $(W_1, \ldots, W_5) \in G(2,4)^5$ which admit a common transversal line is of codimension 1. Find its equation in terms of the functions $[ij]$.
10. Study the stability of ordered point sets on a nonsingular quadric in $\mathbf{P}_k^3$ with respect to the action of its automorphism group. Construct the categorical quotient space in the case of four points.

Lecture 10. TORIC VARIETIES

In this lecture we shall consider an interesting class of algebraic varieties which arise as categorical quotients of some open subsets of the affine space. These varieties are generalizations of the projective spaces and admit a very explicit description in terms of some combinatorial data of convex geometry. In algebraic geometry they are often used as natural ambient spaces for imbedding of algebraic varieties and for compactifying moduli spaces. They have served as a powerful tool for proving some of the fundamental conjectures in the combinatorics of convex polyhedra.

10.1 Let $T = \mathbf{G}^r_{\mathbf{m},k}$ act linearly on $\mathbf{A}^N_k$ by the formula

$$(t_1, \dots, t_r) \cdot (z_1, \dots, z_N) = (\mathbf{t}^{\mathbf{a}_1} z_1, \dots, \mathbf{t}^{\mathbf{a}_m} z_N),$$

where

$$\mathbf{a}_j = (a_{1j}, \dots, a_{rj}) \in \mathbf{Z}^r, \ \mathbf{t}^{\mathbf{a}_j} = t_1^{a_{1j}} \dots t_r^{a_{rj}}.$$

The vectors $\mathbf{a}_j$ can be viewed as characters of T under the identification of the group $\mathcal{X}(T)$ with $\mathbf{Z}^r$. Since $Pic(\mathbf{A}^N_k)$ is trivial and $\mathcal{O}(\mathbf{A}^N_k)^* = k^*$ we have a natural isomorphism (see Lecture 4)

$$\mathrm{Pic}^T(\mathbf{A}^N_k) \cong \mathcal{X}(T) \cong \mathbf{Z}^r.$$

Let us fix $\mathbf{a} = (\alpha_1, \dots, \alpha_r) \in \mathbf{Z}^r$ and denote by $L_{\mathbf{a}}$ the corresponding linearized line bundle. It is the trivial line bundle $\mathbf{A}^N_k \times \mathbf{A}^1_k$ with the linearization defined by the formula

$$t \cdot (z, w) = (t \cdot z, \mathbf{t}^{\mathbf{a}} w).$$

We identify its sections with polynomials $F \in k[Z_1, \dots, Z_N]$. A polynomial F defines an invariant section of some non-negative tensor power $L_{\mathbf{a}}^{\otimes d}$ if

$$F(\mathbf{t}^{\mathbf{a}_1} Z_1, \dots, \mathbf{t}^{\mathbf{a}_m} Z_N) = \mathbf{t}^{d\mathbf{a}} F(Z_1, \dots, Z_N).$$

This is equivalent to F being a linear combination of monomials $Z^{\mathbf{m}}$, where the vector of exponents $\mathbf{m}$ is a solution of the system of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \dots & \dots & a_{rn} \end{pmatrix} \bullet \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix} = \begin{pmatrix} da_1 \\ \vdots \\ da_r \end{pmatrix}.$$

We write it in the matrix form

$$A \bullet \mathbf{m} = d\mathbf{a}.$$

Let S' be the set of non-negative integral solutions of the equivalent system

$$(A| - \mathbf{a}) \bullet \begin{pmatrix} \mathbf{m} \\ d \end{pmatrix} = 0,$$

where the matrix of coefficients is obtained from A by augmenting it with one more column formed by the vector $-\mathbf{a}$. Note that the set of all real non-negative solutions of this system is a convex cone spanned by a finite set of vectors with rational coordinates (*a rational convex polyhedral cone*). It is obtained by intersecting the positive octant in $\mathbf{R}^{n+1}$ with the linear subspace given by the nullspace of the matrix $(A|-\mathbf{a})$. The set S' is the set of integral points inside of this cone.

Lemma (P.Gordan). *Let C be a rational polyhedral convex cone in $\mathbf{R}^n$. Then $C \cap \mathbf{Z}^n$ is a finitely generated submonoid of $\mathbf{Z}^n$.*

Proof. Let C be spanned by vectors $v_1, \ldots, v_k$ which we may assume to be integral. The set

$$K = \{\sum_i x_i v_i \in \mathbf{R}^n : 0 \le x_i \le 1\}$$

is compact and hence its intersection with $\mathbf{Z}^n$ is finite. Let $\{w_1, \ldots, w_N\}$ be this intersection. This obviously includes the vectors v_i. We claim that this set generates the monoid $\mathcal{M} = C \cap \mathbf{Z}^n$. In fact we can write each $m \in \mathcal{M}$ in the form $m = \sum_i (x_i + m_i) v_i$, where m_i is a non-negative integer and $0 \le x_i \le 1$. Thus $m = (\sum_i x_i v_i) + (\sum_i m_i v_i)$ is the sum of some vector w_j and a positive linear combination of vectors v_i. This proves the assertion.

For any commutative monoid $\mathcal{M}$ we denote by $k[\mathcal{M}]$ the monoid algebra. It is a free abelian group generated by elements of $\mathcal{M}$ with the multiplication law given on the generators by the monoid multiplication. If $\mathcal{M} = \mathbf{Z}^n$ we can identify $k[\mathcal{M}]$ with the algebra of Laurent polynomials $k[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ by assigning to each $\mathbf{m} = (m_1, \ldots, m_n)$ the monomial $Z^{\mathbf{m}}$. If $\mathcal{M}$ is a submonoid of $\mathbf{Z}^n$ we identify $k[\mathcal{M}]$ with the subalgebra of $k[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ which is generated by monomials $Z^{\mathbf{m}}, \mathbf{m} \in \mathcal{M}$.

Now we can easily make the natural isomorphism

$$\oplus_{d \ge 0} \Gamma(\mathbf{A}_k^N, L_{\mathbf{a}}^{\otimes d})^T \cong k[S],$$

where S is the projection of S' to $\mathbf{Z}^N, (\mathbf{m}, d) \mapsto \mathbf{m}$. Obviously S is isomorphic to S', and hence, by Gordan's Lemma, $k[S]$ is a finitely-generated algebra.

Let $\mathcal{A}$ be the ideal $\oplus_{d>0} k[S]_d$. This is a monomial ideal, i.e., it can be generated by monomials.

Let $Z^{\mathbf{m}_1}, \ldots Z^{\mathbf{m}_s}$ be a minimal set of monomial generators of the ideal $\mathcal{A}$. For each $\mathbf{m}_j = (m_{1j}, \ldots, m_{Nj})$, let $I_j := \{i : m_{ij} \ne 0\}$. For each subset I of $\{1, \ldots, N\}$, let $Z_I = \prod_{i \in I} Z_i$. Obviously, the open sets $D(Z^{\mathbf{m}_j}) = \mathbf{A}_k^N \setminus \{Z^{\mathbf{m}_j} = 0\}$ and $D(Z_{I_j}) = \mathbf{A}_k^N \setminus \{Z_{I_j} = 0\}$ coincide. By definition of semi-stability

$$(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}}) = \mathbf{A}_k^N \setminus \{Z_{I_1} = \ldots Z_{I_s} = 0\} = \bigcup_{j=1}^{s} D(Z_{I_j}).$$

For any $j = 1, \ldots, s$, let

$$R_j = \mathcal{O}(D(Z_{I_j}))^T = \{\frac{F(Z)}{(Z_{I_j})^p} : p \ge 0, F(Z) \in (Z_{I_j})^p k[M]\},$$

where

$$M = \{\mathbf{m} \in \mathbf{Z}^N : A \bullet \mathbf{m} = 0\}.$$

Obviously, the algebra $k[M]$ is the subalgebra of T-invariant Laurent polynomials in $Z_1, \ldots, Z_N$.

We know that the categorical quotient is obtained by gluing together the affine algebraic varieties X_j defined by $\mathcal{O}(X_j) \cong R_j$. We shall now describe these rings and their gluing in terms of certain combinatorial structures.

10.2 Let $\mathbf{Z}^N \to \mathbf{Z}^r$ be the map given by the matrix A. Its kernel is the group M. By restricting linear functions on $\mathbf{Z}^N$ to M, we obtain the homomorphism of the dual abelian groups

$$(\mathbf{Z}^N)^* \to N := M^*.$$

Let $e_1^*, \ldots, e_N^*$ be the dual basis of the standard basis of $\mathbf{Z}^N$, and let $\bar{e}_1^*, \ldots, \bar{e}_N^*$ be the images of these vectors in M^*. For each I_j let σ_j be the convex cone in the linear space

$$N_{\mathbf{R}} := N \otimes \mathbf{R} \cong \mathbf{R}^n.$$

spanned by the vectors $\bar{e}_i^*, i \notin I_j$. For any convex cone σ in a real vector space V the subset

$$\check{\sigma} = \{x \in V^* : \langle x, y \rangle \geq 0, \ \forall y \in \sigma\}.$$

is a rational polyhedral convex cone in the dual space.

Lemma 1. $R_j \cong k[\check{\sigma}_j \cap M]$.

Proof. Obviously R_j is isomorphic to $k[\mathcal{M}]$, where

$$\mathcal{M} = \{\mathbf{m} \in M : \mathbf{m} + p\sum_{i \in I_j} e_i \in \mathbf{Z}^N_{\geq 0} \text{for some } p \geq 0\} =$$

$$= \{\mathbf{m} = (m_1, \ldots, m_N) : m_i \geq 0, \forall i \notin I_j\}.$$

Here, as usual, we denote by e_i the unit vectors in $\mathbf{R}^N$. For each $i \notin I_j$,

$$\bar{e}_i^*(\mathbf{m} + p\sum_{i \in I_j} e_i) = \bar{e}_i^*(\mathbf{m}) = m_i \geq 0 \Leftrightarrow \mathbf{m} \in \mathcal{M}.$$

On the other hand

$$\mathbf{m} \in \check{\sigma}_j \Leftrightarrow \bar{e}_i^*(\mathbf{m}) \geq 0, \forall i \notin I_j.$$

Lemma 2. *Let Σ be the set of convex cones $\sigma_j, j = 1, \ldots, s$. For any $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face in both σ and σ'.*

Proof. Let $I = I_a, J = I_b$. We want to show that $\sigma_a \cap \sigma_b$ is a common face of σ_a and σ_b. Recall that a face of a convex set σ is the intersection of σ with a hyperplane such that σ lies in one of the two halfspaces defined by the hyperplane. We know that

$\mathcal{O}(D(Z_I Z_J))^T$ is equal to the localization $\mathcal{O}(D(Z_I))^T_{Z^{\mathbf{c}}}$, where $\mathbf{c} = (c_1, \ldots, c_N) \in M$ and $c_i = 0$ for $i \notin I \cup J$. Considering $\mathbf{c}$ as a linear function on M^* we have

$$\mathbf{c}(\bar{e}_i^*) = e_i^*(\mathbf{c}) = 0 \quad \text{for } i \notin I \cup J.$$

This shows that $\mathbf{c}$ is identically zero on $\sigma_a \cap \sigma_b$. On the other hand, it follows from Lemma 1 that $\mathbf{c}$ is non-negative on σ_a and on σ_b. This proves the assertion.

Definition. A finite collection $\Sigma = \{\sigma_i\}_{i \in I}$ of rational convex polyhedral cones in $\mathbf{R}^n$ such that $\sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j is called a *fan.*

In a coordinate-free approach one replaces the space $\mathbf{R}^n$ by any finite-dimensional real linear space V, then chooses a lattice N in V, i.e. a finitely generated abelian subgroup of the additive group of V with $N \otimes \mathbf{R} = V$, and considers N-rational convex polyhedral cones, i.e., cones spanned by a finite subset of N. Then a N-*fan* Σ is a finite collection of N-rational polyhedral cones in V satisfying the property from the above definition. A version of this definition includes in the fan all faces of all cones $\sigma \in \Sigma$.

Let $M = N^*$ be the dual lattice in the dual space V^*, by Gordan's Lemma for each $\sigma \in \Sigma$ the algebra $A_\sigma = k[\check{\sigma} \cap M]$ is finitely generated. Let $X_\sigma = \mathrm{Spec}(A_\sigma)$ be the affine variety with $\mathcal{O}(X_\sigma)$ isomorphic to $k[\check{\sigma} \cap M]$. Since for any $\sigma, \sigma' \in \Sigma, \sigma \cap \sigma'$ is a face in both cones, we obtain that $k[(\sigma \cap \sigma')\check{} \cap M]$ is a localization of each algebra A_σ and A'_σ. This shows that $\mathrm{Spec}(k[(\sigma \cap \sigma')\check{} \cap M])$ is isomorphic to an open subset of X_σ and X'_σ. This allows us to glue together the varieties X_σ to obtain a separated (abstract) algebraic variety. It is denoted by X_Σ and is called the *toric variety* associated to the fan Σ. It is not always a quasi-projective algebraic variety.

By definition X_Σ has a cover by open affine subsets U_σ isomorphic to X_σ. Since each algebra A_σ is a subalgebra of $k[M] \cong k[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ we obtain a morphism $T = (\mathbf{G}_{\mathrm{m},k})^n \to X_\Sigma$. It is easy to see that this morphism is T-equivariant if one considers the action of T on itself by left translations and on X_Σ by means of $\mathbf{Z}^r$-gradings of each algebra A_σ. If each cone $\sigma \in \Sigma$ does not contain a linear subspace, the morphism $T \to X_\Sigma$ is an isomorphism onto an open orbit. In general X_Σ always contains an open orbit isomorphic to a factor group of T. All toric varieties X_Σ are normal and, of course, rational.

Keeping our old notations we obtain

Theorem. *Let $(\mathbf{Z}^N)^* \to M^*$ be the transpose of the identity map $M \to \mathbf{Z}^N$ and let N be its image. Let Σ be the N-fan formed by the cones $\sigma_j, j = 1, \ldots, s$. Then*

$$(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})//T \cong X_\Sigma.$$

10.3 Recall that a cone in a linear space V is called simplicial if it is spanned by a part of a basis of V. A fan is called *simplicial* if each $\sigma \in \Sigma$ is simplicial. The geometric significance of this property is given by the following result, the proof of which can be found in any book on toric varieties (for example in **[Fu2]**).

Lemma. *A fan Σ is simplicial if and only if each affine open subset $U_\sigma, \sigma \in \Sigma$, is isomorphic to the product of a torus and the quotient of an affine space by a finite abelian group.*

In our situation, we have

Proposition. *Let X_Σ be the toric variety $(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})//T$. Then Σ is simplicial if and only if*

$$(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}}) = (\mathbf{A}_k^N)^{s}(L_{\mathbf{a}}).$$

Proof. Assume some $\sigma \in \Sigma$ is not simplicial. We have to show that there exists a semi-stable but not stable point. Let $\bar{e}_i^*, i \notin I$, be the spanning vectors of σ. Since σ is not simplicial, $\sum_{i\notin I} n_i \bar{e}_i^* = 0$ for some integers n_i not all of which are zero. This implies that $\sum_{i\notin I} n_i e_i^*$ belongs to the annihilator $M^\perp$ of M in $(\mathbf{Z}^N)^*$. If we identify $(\mathbf{Z}^N)^*$ with $\mathbf{Z}^N$, then $M^\perp$ is isomorphic to the submodule spanned by the rows $\bar{A}_i$ of the matrix A. Thus we can write

$$\sum_{i\notin I} n_i e_i = b_1\bar{A}_1 + \ldots + b_r\bar{A}_r$$

for some $\mathbf{b} = (b_1,\ldots,b_r) \in \mathbf{Z}^r$. This implies that $\mathbf{b}\bullet\mathbf{a}_j = 0$ for $j \in I$, where a_j are the columns of the matrix A.

Let us consider a one-parameter subgroup $\lambda_0 \in \mathcal{X}_*(T)$ defined by

$$\lambda_0(t) = (t^{b_1},\ldots,t^{b_r}).$$

For any $t \in K^*$ and $z \in \mathbf{A}_k^N(K)$ we have

$$\lambda_0(t)\cdot z = (t^{\mathbf{b}\bullet\mathbf{a}_1}z_1,\ldots,t^{\mathbf{b}\bullet\mathbf{a}_N}z_N). \quad (*)$$

Take a point $p = (z_1,\ldots,z_N)$, where $z_j = 1$ if $j \in I$ and 0 otherwise. Since $Z_I(p) \neq 0$, we see that $p \in (\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})$. On the other hand, for all $t \in k, \lambda_0(t)\cdot p = p$. This shows that the stabilizer subgroup T_p of the point z contains the group $\lambda(\mathbf{G}_{\mathbf{m},k})$. Obviously we may assume that the set $(\mathbf{A}_k^N)^s(L_{\mathbf{a}})$ is not empty. Because it is an open subset of the affine space we can find a stable point z' with all non-zero coordinates. It follows from formula $(*)$ that $\lambda(\mathbf{G}_{\mathbf{m},k}) \subset G_{z'}$ for some $\lambda \in \mathcal{X}_*(T) = \mathbf{Z}^r$ if and only if $\lambda\cdot\mathbf{a}_j = 0$ for all $j = 1,\ldots,N$. It is easy to see that each connected subgroup of a torus is a torus, and hence it is generated by one-parameter subgroups. Thus the connected component of the stabilizer of each point $z \in \mathbf{A}_k^N$ contains the subtorus T' generated by $\lambda(\mathbf{G}_{\mathbf{m},k})$ where $\lambda\cdot A = 0$. For stable points the connected component of the stabilizer is exactly T'. However we have found that G_p contains a subgroup λ_0 for which $\lambda_0\cdot A \neq 0$.It is not in the left kernel of A. Hence $\dim G_p > \dim G'_z$, and hence p cannot be stable point.

Conversely, assume that there exists a semi-stable but not stable point. Arguing as above, we find a one-parameter subgroup λ_0 such that $\lambda_0\cdot A \neq 0$ but $\lambda_0\cdot\mathbf{a}_j = 0$ for all $j \in I$ where $\sigma_I \in \Sigma$. Then $(b_1,\ldots,b_N) = \lambda_0\cdot A$ has not all coordinates b_j zero for $j \notin I$ and $b_j = 0$ for all $j \in I$. This gives $\sum_{j\notin I} b_j\bar{e}_j^* = 0$, hence σ_I is not simplicial.

10.4 Since every line bundle on an affine variety is ample, we obtain that the toric varieties $X_\Sigma = (\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})//T$ are always quasi-projective. Let us find out when they are projective.

Definition. A fan Σ in a linear space V is called *complete* if

$$V = \bigcup_{\sigma\in\Sigma}\sigma.$$

For the proof of the following basic result we refer to **[Fu2]**.

Lemma. *A fan Σ is complete if and only if the toric variety X_Σ is complete.*

Theorem. *Assume that $L_{\mathbf{a}}$ is not a trivial linearized bundle (i.e., $\mathbf{a} \neq 0$) and $(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})$ is not empty. The toric variety $(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})//T$ is projective if and only if 0 is not contained in the convex hull of the character vectors $\mathbf{a}_j, j = 1, \ldots, N$.*

Proof. Suppose 0 is not in the convex hull of the vectors $\mathbf{a}_j$. This is equivalent to the existence of a vector $\lambda \in \mathbf{Z}^r$ such that $\lambda \bullet \mathbf{a}_j > 0$ for all $j = 1, \ldots, N$. This is a well-known fact from the theory of convex sets. Formula $(*)$ from above shows that this is equivalent to the existence of a one-parameter subgroup λ of T which acts on $\mathbf{A}_k^N$ by the formula:

$$\lambda(t) \cdot z = (t^{q_1} z_1, \ldots, t^{q_N} z_N),$$

where all q_i are positive integers. So suppose such a λ exists. Let $T' = \lambda(\mathbf{G}_{\mathbf{m},k})$. Note that $\mathbf{A}_k^N \setminus \{0\}$ is the set of semi-stable points with respect to $L_{\mathbf{a}}$ for the action of T' on $\mathbf{A}_k^N$. Then $Y := (\mathbf{A}_k^N \setminus \{0\})//T'$ is a projective variety (see Lecture 4, Examples 4.2). The line bundle $L_{\mathbf{a}}$ descends to an ample line bundle L' on Y (equal to $\mathcal{O}_Y(\lambda \bullet \mathbf{a})$). Since T is commutative, it acts on Y via the quotient torus T/T' and the linearization of $L_{\mathbf{a}}$ descends to a linearization of T on L'. We have

$$\Gamma(Y, L'^{\otimes d})^T \cong \Gamma(\mathbf{A}_k^N, L_{\mathbf{a}}^{\otimes d})^T.$$

This easily implies that

$$(\mathbf{A}_k^N)^{ss}(L_{\mathbf{a}})//T \cong Y^{ss}(L')//T.$$

Now by the Main Theorem of Lecture 6 the latter quotient is projective.

Conversely, assume that X_Σ is projective. Then by the previous lemma, the fan Σ is complete. In particular this implies that the convex hull of the vectors $\bar{e}_i^*$ is equal to the whole space $N \otimes \mathbf{R}$. Thus we can write $0 = \sum_i b_i \bar{e}_i^*$ where $b_i \geq 0$ but not all are zero. This is equivalent to $\sum_i b_i e_i^* \in M^\perp$. This implies that there is a linear combination of the rows of the matrix A which is a non-negative non-zero vector. This of course is equivalent to the existence of a non-zero vector $\lambda \in \mathbf{Z}^r$ with $\lambda \bullet a_j \geq 0$ for $j = 1, \ldots, N$. The theorem is proven.

10.5 Examples. 1. Let $\mathbf{G}_{\mathbf{m},k}$ act on $\mathbf{A}_k^{n+1}$ by the formula:

$$t \cdot (z_0, \ldots, z_n) = (tz_0, \ldots, tz_n),$$

We have

$$A = (1 \;\; \ldots \;\; 1),$$

$$M = \{(m_0, \ldots, m_n) \in \mathbf{Z}^{n+1} : \sum_{i=1}^{n} m_i = 0\}.$$

It is easy to see that the basis of M consists of vectors $v_i = e_i - e_{i+1}, i = 1, \ldots, n$. If we choose the dual basis $(v_1^*, \ldots, v_n^*)$ of $N = M^*$, the vectors $\bar{e}_i^*$ are equal to

$$\bar{e}_1^* = v_1^*, \bar{e}_2^* = -v_1^* + v_2^*, \ldots, \bar{e}_n^* = -v_{n-1}^* + v_n^*, \bar{e}_{n+1}^* = -v_n^*.$$

We can take for a new basis of M^* the vectors $\bar{e}_i^*, i = 2, \ldots, n+1$, because

$$\bar{e}_1^* = -(\bar{e}_2^* + \ldots + \bar{e}_{n+1}^*).$$

Let us linearize the action by taking the line bundle L_a, where $a = 1$. Then we have an isomorphism of graded rings

$$\oplus_{d \geq 0} \Gamma(\mathbf{A}_k^{n+1}, L_1^{\otimes d})^{\mathbf{G}_{\mathbf{m},k}} = k[Z_0, \ldots, Z_n].$$

Obviously the minimal generators of the ideal $\mathcal{A}$ are the unknowns Z_i. Thus the cones of our fan Σ are

$$\sigma_j = \operatorname{span}\{\bar{e}_1^*, \ldots, \bar{e}_{j-1}^*, \bar{e}_{j+1}^*, \ldots, \bar{e}_{n+1}^*\}, j = 1, \ldots, n+1.$$

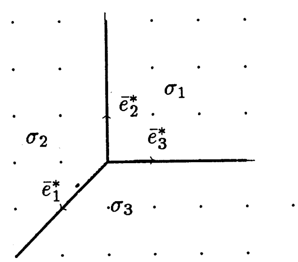


Fig.1

This is the fan defining the projective space $\mathbf{P}_k^n$ (see **[Fu2]**). Let us see the corresponding gluing. We can take for a basis of M the dual basis of $(\bar{e}_2^*, \ldots, \bar{e}_{n+1}^*)$ which is the set of vectors

$$e_2 - e_1, \ldots, e_{n+1} - e_1.$$

We easily find

$$k[\check{\sigma}_1 \cap M] = k[\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}], \ \ldots, \ k[\check{\sigma}_{n+1} \cap M] = k[\frac{Z_0}{Z_n}, \ldots, \frac{Z_{n-1}}{Z_n}].$$

These are the coordinate rings of the standard open subsets of $\mathbf{P}_k^n$. The gluing is achieved by identifying all the coordinate rings with subrings of the ring if the Laurent polynomials $k[Z_0^{\pm 1}, \ldots, Z_n^{\pm 1}]$.

2. Consider the action of $\mathbf{G}_{\mathbf{m},k}$ on $\mathbf{A}_k^4$ by the formula

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4),$$

We have

$$A = (1 \ \ 1 \ \ -1 \ \ -1),$$

$$M = \{(m_1, m_2, m_3, m_4) \in \mathbf{Z}^4 : m_1 + m_2 - m_3 - m_4 = 0\}.$$

Let us choose the following basis of M

$$v_1 = -e_1 + e_2, v_2 = e_1 + e_3, v_3 = e_1 + e_4.$$

We can express the vectors $\bar{e}_i^*$ in terms of the dual basis $(v_1^*, \ldots, v_n^*)$ of $N = M^*$ as follows

$$\bar{e}_1^* = -v_1^* + v_2^* + v_3^*, \bar{e}_2^* = v_1^*, \bar{e}_3^* = v_2^*, \bar{e}_4^* = v_3^*.$$

Choose $L = L_1$, then the monoid of solutions of the equation

$$m_1 + m_2 - m_3 - m_4 - d = 0, m_i \geq 0, d > 0,$$

is spanned by the vectors $(1,0,0,0,1), (0,1,0,0,1)$. This means that the unknowns Z_1, Z_2 are the minimal generators of the ideal $\mathcal{A}$. Thus the fan Σ consists of two cones

$$\sigma_1 = \text{span}\{\bar{e}_2^*, \bar{e}_3^*, \bar{e}_4^*\}, \ \sigma_2 = \text{span}\{\bar{e}_1^*, \bar{e}_3^*, \bar{e}_4^*\}.$$

The dual cones are

$$\check{\sigma}_1 = \text{span}\{-e_1 + e_2, e_1 + e_3, e_1 + e_4\}, \ \check{\sigma}_2 = \text{span}\{-e_2 + e_1, e_2 + e_3, e_2 + e_4\}.$$

The quotient X_Σ is obtained by gluing together two nonsingular algebraic varieties with the coordinate algebras

$$k[\check{\sigma}_1 \cap M] \cong k[Z_1Z_3, Z_1Z_4][\frac{Z_2}{Z_1}], \ \ k[\check{\sigma}_2 \cap M] \cong k[Z_2Z_3, Z_2Z_4][\frac{Z_1}{Z_2}].$$

Similarly if we take $L = L_{-1}$ we get that the fan Σ consists of two cones

$$\sigma_1 = \text{span}\{\bar{e}_1^*, \bar{e}_2^*, \bar{e}_4^*\}, \ \sigma_2 = \text{span}\{\bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*\}.$$

The quotient X_Σ is obtained by gluing together two nonsingular algebraic varieties with the coordinate algebras

$$k[\check{\sigma}_1 \cap M] \cong k[Z_1Z_3, Z_2Z_3][\frac{Z_4}{Z_3}], \ \ k[\check{\sigma}_2 \cap M] \cong k[Z_1Z_4, Z_2Z_4][\frac{Z_3}{Z_4}].$$

If we now change the linearization by taking $L = L_0$ we get $L = L_0^{\otimes d} = L_0$ for all $d \geq 0$, hence $\mathcal{A}$ is generated by 1. Then we have only one cone spanned by the four vectors $\bar{e}_i^*$. The toric quotient is isomorphic to the affine variety with the coordinate algebra

$$k[\check{\sigma} \cap M] \cong k[Z_1Z_3, Z_1Z4, Z_2Z_3, Z_2Z_4] \cong k[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3).$$

One should compare this with our previous computation of this quotient in Lecture 6. We see here a general phenomena: two toric varieties X_Σ and X'_Σ whose fans have the same set of one-dimensional edges of its cones (called the *1-skeleton* of a fan) differ by a special birational modification. We refer the interested reader for more details to **[Rei]**.

10.6 One can go in the opposite direction by identifying any toric variety X_Σ with a categorical quotient of some open subset of an affine space. We state without proof the following result of D. Cox **[Cox]**:

Theorem. *Let X_Σ be a toric variety determined by a $\mathbf{Z}^n$-fan Σ. To each one-dimensional edge of the 1-skeleton of Σ assign a variable Z_i and consider the polynomial algebra $k[Z_1, \ldots, Z_N]$ generated by these variables. For each cone $\sigma \in \Sigma$ let $Z_{I(\sigma)} \in k[Z_1, \ldots, Z_N]$ where $I(\sigma) \subset \{1, \ldots, N\}$ is the complementary set of the 1-skeleton of σ. Let $U = \mathbf{A}_k^N \setminus V(\{Z_{I(\sigma)}\}_{\sigma \in \Sigma})$. Let $\bar{e}_i^*$ be the primitive vectors of the lattice $\mathbf{Z}^n$ which span one-dimensional edges of the cones from Σ. Let B be the $(n \times N)$-matrix whose columns are the vectors $\bar{e}_i^*$, and let A be a $(r \times N)$-matrix whose rows form a basis of the group of integral solutions of the equation $B \cdot x = 0$. Then*

(i)

$$X_\Sigma \cong U//T,$$

with the action of $T = (\mathbf{G}_{\mathbf{m},k})^r$ given by the formula

$$t \cdot (z_1, \ldots, z_N) = (t^{\mathbf{a}_1} z_1, \ldots, t^{\mathbf{a}_N} z_N),$$

where $\mathbf{a}_j$ are the columns of A.

(ii) X_Σ is simplicial if and only if $U//T = U/T$.

Remark. Note that applying this construction to the toric varieties X_Σ obtained as the quotients $(\mathbf{A}_k^N)^{ss}(L_\mathbf{a})//T$ we get $U = (\mathbf{A}_k^N)^{ss}(L_\mathbf{a})$ and the action is isomorphic to the one we started with. However, in general, $U \neq (\mathbf{A}_k^N)^{ss}(L_\mathbf{a})$ for any $\mathbf{a} \in \mathbf{Z}^r$. One reason for this is that our quotients are always quas-projective and there are examples of non-quasi-projective toric varieties. Another reason is simpler. The fans we are getting from our quotient constructions are "full" in the following sense. One cannot extend it to a larger fan with the same 1-skeleton.

The torus T which acts on U has a very nice interpretation. Its character group $\mathcal{X}(T)$ is naturally isomorphic to the group $\mathrm{Cl}(X_\Sigma)$ of classes of Weil divisors on X_Σ.

Example. Let Σ consists of the following four cones in $\mathbf{R}^2$

$$\sigma_1 = \mathrm{span}\{e_1, e_2\}, \sigma_2 = \mathrm{span}\{e_1, -e_2\}, \sigma_3 = \mathrm{span}\{-e_1, -e_2\}, \sigma_4 = \mathrm{span}\{-e_1, e_2\}.$$

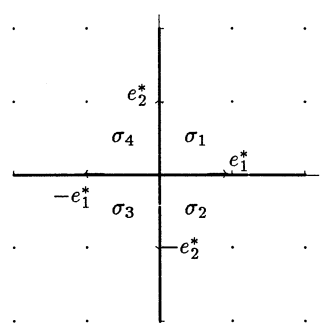


Fig.2

We have

$$U = \mathbf{A}_k^4 \setminus \{Z_3Z_4 = Z_1Z_2 = Z_2Z_3 = Z_1Z_4 = 0\},$$

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

hence the action is given by

$$(t_1, t_2) \cdot (z_1, z_2, z_3, z_4) = (t_1z_1, t_2z_2, t_1z_3, t_2z_4).$$

The variety X_Σ is obtained by gluing four affine planes with coordinate rings

$$k[Z_1, Z_2], k[Z_1Z_2^{-1}], k[Z_1^{-1}, Z_2^{-1}], k[Z_1^{-1}, Z_2].$$

It is easy to see that X_Σ is isomorphic to the product $\mathbf{P}_k^1 \times \mathbf{P}_k^1$. This also is seen from observing that

$$U/T = (\mathbf{A}_k^2 \setminus \{Z_1 = Z_3 = 0\})/\mathbf{G}_{\mathbf{m},k} \times (\mathbf{A}_k^2 \setminus \{Z_2 = Z_4 = 0\})/\mathbf{G}_{\mathbf{m},k} = \mathbf{P}_k^1 \times \mathbf{P}_k^1.$$

Problems.

1. Consider the action $t \cdot (z_1, z_2, z_3) = (tz_1, t^{-1}z_2, tz_3)$ and take $L = L_1$. Show that the quotient X_Σ is isomorphic to the blowing-up of $\mathbf{A}_k^2$ at the origin. Draw the corresponding fan.
2. Let $T = (\mathbf{G}_{\mathbf{m},k})^4$ act on $\mathbf{A}_k^6$ by the formula

$$t \cdot z = (t_1 t_2 t_3 t_4^{-1} z_1, t_3 t_4 t_1^{-1} z_2, t_1 z_3, t_2 z_4, t_3 z_5, t_4 z_6).$$

Take $L = L_{\mathbf{a}}$, where $\mathbf{a} = (1,1,1,1,1,1)$. Show that the quotient is isomorphic to the blowing-up of the projective plane at three points. Draw the picture of the fan.
3. Take a fan Σ in $\mathbf{R}^3$ formed by three two-dimensional cones spanned by the unit vectors e_1, e_2, e_3. Using Cox's theorem represent the toric variety X_Σ as a geometric quotient.
4. A toric variety X_Σ is nonsingular if and only if each $\sigma \in \Sigma$ is spanned by a part of a basis of the lattice N. Show that $U/T = X_\Sigma$ is nonsingular if and only if stabilizer of each point of U is equal to the same subgroup of T.
5. Let Σ be a N-fan and Σ' be a N'-fan. Show that the cones $\sigma \times \sigma', \sigma \in \Sigma, \sigma' \in \Sigma'$, form a $(N \oplus N')$-fan. Denoting this fan by $\Sigma \times \Sigma'$, show that $X_{\Sigma \times \Sigma'} \cong X_\Sigma \times X_{\Sigma'}$.
6. Let $\mathbf{G}_{\mathbf{m},k}$ act on $\mathbf{A}_k^n$ by the formula $t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n)$, where $q_1, \ldots, q_n$ are positive integers. Show that the geometric quotient $\mathbf{A}_k^1 \setminus \{0\}/\mathbf{G}_{\mathbf{m},k}$ exists (it is denoted by $\mathbf{P}(q_1, \ldots, q_n)$ and is called the weighted projective space). Show that it is a toric variety and find the corresponding fan.
7. Let Σ be a N-fan, X_Σ be the associated toric variety. Identify the lattice N with the group of one-parameter subgroups of the torus T acting on X_Σ. Let T' be the dense orbit of T in X_Σ. Show for any $\lambda \in N, z \in T'$, $\lim_{t \to 0} \lambda(t) \cdot z$ exists in X_Σ if and only if $\lambda \in \sigma$ for some $\sigma \in X_\Sigma$.

Lecture 11. MODULI SPACE OF CURVES

In this last lecture we shall discuss how the methods of geometric invariant theory are used to construct the moduli space of nonsingular projective curves. Similar methods are used to construct other moduli spaces arising in algebraic geometry. We will use the language of schemes, and, in particular, some facts about cohomology of schemes. All the necessary background can be found in the first four chapters of **[Har]**.

11.1 Roughly speaking, a moduli space is an algebraic variety whose points are in a one-to-one correspondence with the set of isomorphism classes of algebra-geometric objects (certain classes of algebraic varieties, vector bundles on a fixed variety, and so on). This correspondence must be in some sense canonical, or natural. The formalization of these ideas leads to the concept of a representable functor.

Let $\mathcal{C}$ be a category with its set of objects $\mathrm{Ob}(\mathcal{C})$ and sets of morphisms $\mathrm{Mor}_{\mathcal{C}}(S, S')$. For any $X \in \mathrm{Ob}(\mathcal{C})$ one defines the (covariant) functor

$$h_X : \check{\mathcal{C}} \to \mathbf{Sets},$$

by setting

$$h_X(S) = \mathrm{Mor}_{\mathcal{C}}(S, X), \ \forall S \in \mathrm{Ob}(\mathcal{C}),$$
$$h_X(\varphi) : h_X(S) \to h_X(S'), \ \phi \mapsto \phi \circ \varphi, \ \forall \varphi \in \mathrm{Mor}_{\mathcal{C}}(S', S).$$

Recall that $\check{\mathcal{C}}$ denotes the dual category, which has the same set of objects as $\mathcal{C}$ but the morphisms are defined by reversing the arrows (i.e., $\mathrm{Mor}_{\mathcal{C}}(S, S') = \mathrm{Mor}_{\check{\mathcal{C}}}(S', S)$). With this trick one can consider only covariant functors, the contravariant functors become covariant functors on the dual category.

By assigning to each object $X \in \mathrm{Ob}(\mathcal{C})$ the functor h_X, and to each morphism $\varphi : X \to Y$ the morphism of functors $h(\varphi) : h_X \to h_Y$ defined by composing any $\phi \in h_X(S)$ with φ on the right, we obtain a functor

$$h : \mathcal{C} \to \mathrm{Funct}(\check{\mathcal{C}}, \mathbf{Sets})$$

from the category $\mathcal{C}$ to the category of contravariant functors from $\mathcal{C}$ to **Sets** (where morphisms are morphisms of functors, also called natural transformations of functors). The next fundamental lemma says that this functor allows one to consider $\mathcal{C}$ as a full subcategory of $\mathrm{Funct}(\check{\mathcal{C}}, \mathbf{Sets})$.

Lemma (Yoneda). *For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map*

$$h_{X,Y} : \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathrm{Funct}(\check{\mathcal{C}}, \mathbf{Sets})}(h_X, h_Y), \ \varphi \mapsto h(\varphi),$$

is bijective.

Proof. Let us construct the inverse of the map $h_{X,Y}$. Suppose we are given a morphism of functors $\alpha : h_X \to h_Y$. Taking $S = X$, we obtain a morphism

$$\phi = \alpha(id_X) \in h_Y(X) = \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

Let us show that $h_{X,Y}(\phi) = \alpha$. By definition, for any $S \in \mathrm{Ob}(\mathcal{C})$, $h_{X,Y}(\phi)(S)$ is the map $h_X(S) \to h_Y(S)$ which is defined by composing morphisms $S \to X$ with ϕ. Let $(S \xrightarrow{f} X) \in h_X(S)$. By definition of morphisms of functors we have a commutative diagram

$$\begin{array}{ccc} h_X(X) & \xrightarrow{h_X(f)} & h_X(S) \\ \alpha(X)\downarrow & & \downarrow \alpha(S) \\ h_Y(X) & \xrightarrow{h_Y(f)} & h_Y(S) \end{array}.$$

If we write $f = f \circ id_X$, we obtain that $h_X(f)(id_X) = f$, hence

$$\alpha(S)(f) = h_Y(f)(\alpha(X)(id_X)) = h_Y(f)(\phi) = \phi \circ f.$$

This verifies that the map $\alpha \mapsto \phi$ is the left inverse of $h_{X,Y}$. We leave it to the reader to verify that it is also the right inverse.

Definition. A contravariant functor $F : \mathcal{C} \to \mathbf{Sets}$ is called *representable* if there is an isomorphism of functors $F \cong h_X$ for some $X \in \mathrm{Ob}(\mathcal{C})$. The object X is defined uniquely (up to isomorphism) by this property. It is called the *representing object* of F. The element $u_F \in F(X)$ corresponding to the identity morphism in $h_X(X)$ is called the *universal element* of F.

Let F be a representable functor, and let X be its representing object. It follows from the proof of the Yoneda Lemma that for any $S \in \mathrm{Ob}(\mathcal{C})$ and any $a \in F(S)$, there exists a unique morphism $\varphi : S \to X$ such that

$$a = F(\varphi)(u_F).$$

Examples. 1. Let $\mathcal{C} = (\mathbf{Sets})\check{}$. Consider the functor $F : \mathcal{C} \to \mathbf{Sets}$ whose value at any set S is equal to its n-th Cartesian power S^n. Then this functor is representable by the set $[1, n] = \{1, \dots, n\}$. The universal element is the element $(1, 2, \dots, n) \in [1, n]^n$.
2. Let I be an ideal in the polynomial ring $k[Z_1, \dots, Z_n]$, and let $\mathcal{C}$ be the dual category of the category of commutative k-algebras. Consider the functor $F : \mathcal{C} \to \mathbf{Sets}$ which assigns to an algebra A the set

$$Sol(I, A) = \{(a_1, \dots, a_n) \in A^n : F(a_1, \dots, a_n) = 0, \forall F \in I\}.$$

This functor is representable by the factor-algebra $k[Z_1, \dots, Z_n]/I$.
3. Let $\mathcal{C} = Sch$ be the category of schemes (over $\mathbf{Z}$). For any scheme S let $F(S)$ be the set whose elements are isomorphism classes of locally free sheaves $\mathcal{E}$ of rank r together with a surjection $\phi : \mathcal{O}_S^n \to \mathcal{E}$ with fixed n. If $f : S' \to S$ is a morphism of schemes, we define the map $F(S) \to F(S')$ by using the operations of the inverse transform f^* of a sheaf and of a homomorphism of sheaves. This functor is representable by the Grassmann variety

$G(n-r,n)$. Recall that for any commutative ring A, the set $G(n-r,n)(A)$ consists of submodules M of A^n which are projective modules of rank $n-r$ and are direct summands of A^n. Replacing M by the factor module A^n/M, we obtain an equivalent description of the set $G(n-r,n)(A)$ as the set of surjections $A^n \to N$, where N is a projective A-module of rank r. Thus for any affine scheme $S = \mathrm{Spec}(A)$ any element $\phi : \mathcal{O}_S^n \to \mathcal{E}$ of $F(S)$ defines a point from $G(n-r,n)(S)$. If S is not affine, we choose an open affine covering of S and construct a morphism $S \to G(n-r,n)$ whose restriction to each affine subset $U = \mathrm{Spec}(A)$ is the point of $G(n-r,n)(A)$ defined by the restriction of $\phi : \mathcal{O}_S^n \to \mathcal{E}$ to U. The universal object for this functor is a locally free sheaf $\mathcal{Q}$ over $G(n-r,n)$ together with a surjection $\mathcal{O}_{G(n-r,n)}^n \to \mathcal{Q}$. It is called the *universal quotient bundle*. For any morphism $f : S \to G(n-r,n)$ the inverse transform $f^*(\mathcal{O}_{G(n-r,n)}^n) = \mathcal{O}_S^n \to f^*(\mathcal{Q})$ is an element of the set $F(S)$, and any element from this set is obtained in this way from a unique morphism $f : S \to G(n-r,n)$.

Note the special case when $r = 1$. The Grassmannian becomes the projective space $\mathbf{P}^{n-1}$, and the universal quotient bundle becomes the invertible sheaf $\mathcal{O}_{\mathbf{P}^{n-1}}(1)$. The surjection $\mathcal{O}_{\mathbf{P}^{n-1}}^n \to \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ is given by a choice of homogeneous coordinates $T_0, \ldots, T_{n-1}$ which can be considered as a basis in the space of global sections of $\mathcal{O}_{\mathbf{P}^{n-1}}(1)$. We obtain the standard description of a morphism $f : S \to \mathbf{P}^{n-1}$. It is given by an invertible sheaf L on S together with its n global sections $s_0, \ldots, s_{n-1}$ which generate L at each point $s \in S$.

11.2 Let us now consider the category Sch/k of schemes over a field k. In this case a representing object of a representable functor F is called a *fine moduli scheme* of F.

For example, consider the functor

$$M_g(S) = \{\text{families of curves of genus } g \text{ over } S\}/\text{modulo isomorphism over } S.$$

Here a *family of curves* of genus g over a scheme S is defined as a proper smooth morphism of relative dimension 1 with connected geometric fibres $f : X \to S$. By definition, its fibre over each point $s \in S$ is a geometrically connected smooth complete curve of genus g over the residue field $k(s)$ of S. If $\phi : S' \to S$ is a morphism of k-schemes, we define the map $M_g(S) \to M_g(S')$ by using the operation of the base change:

$$M_g(\phi)(X \to S) = X \times_S S' \to S'.$$

Suppose this functor is representable by a scheme $\mathbf{M}_g$ over k. Then we have a universal family $\pi : \mathcal{X}_g \to \mathbf{M}_g$ with the following property: for any family of curves $f : X \to S$ there exists a unique morphism $g : S \to \mathbf{M}_g$ such that $(X \to S) \cong (\mathcal{X}_g \times_{\mathbf{M}_g} S \to S)$. By taking $S = \mathrm{Spec}(k)$ we obtain a bijection

$$\{\text{isomorphism classes of curves of genus } g \text{ over } k\} \to \mathbf{M}_g(k).$$

Also, it follows from the definition of functors that the unique map $S \to \mathbf{M}_g$ defined by a family $f : X \to S$ is given by

$$s \to \text{isomorphism class of the fibre } X_s.$$

Thus we have found a natural parametrization of the former set by the points of an algebraic variety (or a scheme).

Unfortunately, life is not that easy and the functor M_g is not representable for any g. The reason is simple. For example, if k is not algebraically closed we observe that for any extension of fields k'/k we have an obvious injection $\mathbf{M}_g(k) \hookrightarrow \mathbf{M}_g(k')$. On the other hand, $M_g(\mathrm{Spec}(k)) \to M_g(\mathrm{Spec}(k'))$ may not inject, as there could be curves not isomorphic over k but isomorphic over the extension k'. Over any algebraically closed field one can give an example of a family $f : X \to S$ of algebraic curves where all geometric fibres are isomorphic to the same curve C, but the family is not isomorphic to the trivial family $C \times S \to S$. Both families must define the same map to $\mathbf{M}_g$ and hence must be isomorphic. The easiest case of this example can be given in the case $g = 0$. This is a ruled surface not isomorphic to the quadric $\mathbf{P}^1_k \times \mathbf{P}^1_k$. For example, let X be the blow-up of one point x in the projective plane $\mathbf{P}^2_k$. It is easy to see that the linear projection $p : \mathbf{P}^2_k \setminus \{x\} \to \mathbf{P}^1_k$ extends to a morphism $f : X \to \mathbf{P}^1_k$ with all fibres isomorphic to $\mathbf{P}^1_k$. This surface is not isomorphic to the quadric. See details in **[Har]**.

Thus, our functor does not have the fine moduli scheme. A weaker notion is the following:

Definition. A *coarse moduli scheme* of a functor $F : (Sch)^{\check{}} \to \mathbf{Sets}$ is a scheme X such that there is a morphism of functors

$$\Phi : F \to h_X$$

satisfying the properties

(i) for any scheme Y and a morphism of functors $\Phi' : F \to h_Y$ there exists a unique morphism $\phi : X \to Y$ such that $h(\phi) \circ \Phi = \Phi'$;

(ii) for any algebraically closed extension Ω/k, the map

$$\phi(\Omega) : F(Spec(\Omega)) \to X(\Omega)$$

is bijective.

Using geometric invariant theory we shall show that the coarse moduli scheme for the functor M_g exists and is an algebraic variety.

11.3 The idea for constructing a coarse moduli scheme for curves (and some other objects) is the following. First we embed all curves into a projective space in such a way that two curves are isomorphic if and only if they are projectively isomorphic. Then we find a variety X which parametrizes all projective curves arising in this way and show that there is a functor $h_X \to M_g$ satisfying the following property: there exists an action of the projective linear group G on X such that for any algebraically closed extension Ω/k, $X(\Omega) \to M_g(Spec(\Omega))$ is the quotient map $X(\Omega) \to X(\Omega)/G(\Omega)$. There is a distinguished element $(\pi : \mathcal{X} \to X) \in M_g(X)$. For any morphism of functors $M_g \to h_N$ the image of π defines a G-invariant morphism $X \to N$. This allows one to construct a morphism $M_g \to h_{X/G}$ and prove that X/G is the coarse moduli scheme. The variety X which we will be looking for is a subvariety of the *Hilbert scheme* of curves of genus g.

Roughly speaking, a Hilbert scheme is an algebraic variety parametrizing subvarieties of a projective space of given degree and dimension. In the case when the varieties are hypersurfaces of some degree d in a projective space $\mathbf{P}_k^n$, the Hilbert scheme is just the projective space $\mathrm{Hyp}_d(n)$ of dimension $\binom{n+d}{d}$. In the general case, this problem has no obvious solution. Making it even harder, let us consider the more general problem of classifying all closed subchemes of a given projective algebraic variety X over k.

Let us introduce the following functor on the category of k-schemes with values in the category of sets:

$$Hilb_{X/k}(S) = \{\text{closed subschemes } Z \text{ of } X \times_k S \text{ which are flat over } S\}.$$

Recall that a morphism $Z \to S$ is called *flat* if for any point $s \in S$ and any point $z \in Z$ over s the local ring $\mathcal{O}_{Z,z}$ is a flat $\mathcal{O}_{S,s}$-module. The reason for this condition will be clear later. Choose an ample invertible sheaf $\mathcal{L}$ on X. For any coherent sheaf $\mathcal{F}$ on X we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$. The following result can be found in **[Har]**, Chapter III, Theorem 9.9.

Lemma. *Let $\mathcal{F}$ be a coherent sheaf over X.*

(i) There exists a polynomial $P_{\mathcal{F}}^L(t) \in \mathbf{Q}[t]$ such that for all $n \geq 0$

$$P_{\mathcal{F}}^L(n) = \chi(X, \mathcal{F}(n)) := \sum_{i=0}^{\dim_k X} (-1)^i \dim_k H^i(X, \mathcal{F}(n)).$$

(ii) If n is large enough,

$$P_{\mathcal{F}}^L(n) = \dim_k \Gamma(X, \mathcal{F}(n)).$$

(iii) If $f : X \to S$ is a flat morphism, then for any fibre X_s over a closed point $s \in S$, $P_{\mathcal{O}_{X_s}}^L(t)$ is independent of s.

Definition. The polynomial $P_{\mathcal{F}}^L(t)$ is called the *Hilbert polynomial* of $\mathcal{F}$ with respect to L. If $\mathcal{F} = \mathcal{O}_Z$, where Z is a closed subscheme of X (where we consider $\mathcal{O}_Z$ as a coherent sheaf on X with support on Z), the Hilbert polynomial is denoted by $P_Z^L(t)$ and is called the *Hilbert polynomial* of Z with respect to L.

Using property (iii) of the previous lemma, we may split the values $\mathrm{Hilb}_{X/k}(S)$ of the functor $\mathrm{Hilb}_{X/k}$ into the disjoint subsets $\mathrm{Hilb}_{X/k}^P(S)$, where for any $P \in \mathbf{Q}[t]$

$$\mathrm{Hilb}_{X/k}^P(S) = \{Z \in \mathrm{Hilb}_{X/k}(S) : \text{for any closed } s \in S, P_{Z_s}^L(t) = P\}.$$

Theorem. *(A. Grothendieck). Let X be a projective variety with a fixed ample line bundle L on it. The functor $\mathrm{Hilb}_{X/k}^P$ is representable by a projective algebraic variety. It is denoted by $\mathbf{Hilb}_{X/k}^P$ and is called the Hilbert scheme of X (with respect to P).*

Proof. We only sketch the main idea of the proof referring the reader for the details to **[Gro]**. Applying Theorem 8.8 from **[Har]**, Chapter III, one shows that for any $Z \subset X \times S$ from $Hilb_{X/k}^P(S)$, there exists a number N such that for all $n \geq N$ the sheaf $(pr_2)_*(\mathcal{O}_Z \otimes pr_1^*(\mathcal{O}_X(n)))$ is locally free on S of rank equal to $P(n)$ and the natural homomorphism

$(pr_2)_*(pr_1^*(\mathcal{O}_X(n))) \to (pr_2)_*(\mathcal{O}_Z(n))$ is surjective. This defines a surjective map of linear spaces

$$\Gamma(X, \mathcal{O}_X(n)) \to \Gamma(Z, \mathcal{O}_Z(n)), \quad n \geq N.$$

The hard part of the proof consists of proving that N can be chosen independently of S and is determined only by the Hilbert polynomial (in the case $X = \mathbf{P}_k^n$ one can find the proof in Lecture 14 of [**Mu2**]). This allows one to define a map of sets

$$\mathrm{Hilb}_{X/k}^P(S) \to \mathrm{Mor}_{Sch/k}(S, G(c-p, c)),$$

where $c = \dim_k \Gamma(X, \mathcal{O}_X(N)), p = P(N)$. By varying S, we obtain a morphism of functors

$$\mathrm{Hilb}_{X/k}^P \to h_{G(c-p,c)}.$$

Finally one proves that this morphism can be represented by a closed immersion of schemes

$$\mathbf{Hilb}_{X/k}^P \hookrightarrow G(c-p, c).$$

11.4 We shall use Hilbert schemes only in the case when X is a projective space $\mathbf{P}_k^n$. In this case there is another construction for parametrizing closed subvarieties Z of $\mathbf{P}_k^n$. It is based on the notion of the *Cayley form* of Z. We shall assume for simplicity that the ground field k is algebraically closed.

Let Z be a closed subvariety of $\mathbf{P}_k^n$ of dimension r and degree d. Let $\check{\mathbf{P}}_k^n$ denote the dual projective space parametrizing hyperplanes H in $\mathbf{P}_k^n$ (which can be considered as the Grassmann variety $G(n, n+1)$). Consider the closed subvariety T of $\mathbf{P}_k^n \times (\check{\mathbf{P}}_k^n)^{r+1}$ with

$$T(k) = \{(x, H_1, \ldots, H_{r+1}) : x \in \bigcap_{i=1}^{r+1} H_i\}$$

We leave it to the reader to write T as the zero set of a $r+1$-multilinear form in projective coordinates in $\mathbf{P}_k^n$ and $\check{\mathbf{P}}_k^n$. Let $p_1 : T \to \mathbf{P}_k^n$ and $p_2 : T \to (\check{\mathbf{P}}_k^n)^{r+1}$ be the projections. For any reduced subvariety Z of dimension r and degree d in $\mathbf{P}_k^n$ we set

$$\mathrm{cay}(Z) = p_2(p_1^{-1}(Z) \cap T).$$

This is a closed subvariety of $(\check{\mathbf{P}}_k^n)^{r+1}$ with

$$\mathrm{cay}(Z)(k) = \{(H_1, \ldots, H_{r+1}) : Z \cap H_1 \ldots \cap H_{r+1} \neq \emptyset\}.$$

Let us see that $\mathrm{cay}(Z)$ is a hypersurface in $(\check{\mathbf{P}}_k^n)^{r+1}$ of multi-degree $(d, \ldots, d)$. For this it suffices to show that the intersection of $\mathrm{cay}(Z)$ with the general fibre of each projection $(\check{\mathbf{P}}_k^n)^{r+1} \to (\check{\mathbf{P}}_k^n)^r$ is a hypersurface of degree d. Without loss of generality we may assume that the projection is onto the product of the first r factors. Fix r "general" hyperplanes $H_1, \ldots, H_r$; then the intersection $Z \cap H_1 \cap \ldots \cap H_r$ consists of d points. Hence

$$\{H \in \check{\mathbf{P}}_k^n(k) : (Z \cap H_1 \cap \ldots \cap H_r) \cap H \neq \emptyset\} = \text{the union of } d \text{ hyperplanes}.$$

This proves the claim.

Let $\mathrm{Div}^{(d,\ldots,d)}((\check{\mathbf{P}}_k^n)^{r+1})$ denote the projective space (of dimension $\binom{n+d}{d}^{r+1}-1$) of hypersurfaces of multi-degree $(d,\ldots,d)$ in $(\check{\mathbf{P}}_k^n)^{r+1}$. We have constructed a map

$$\mathrm{cay}:\{\text{closed subvarieties of } \mathbf{P}_k^n \text{ of dimension } r \text{ and degree } d\}\to \mathrm{Div}^{(d,\ldots,d)}((\check{\mathbf{P}}_k^n)^{r+1}).$$

The value of this map on any Z is called the *Cayley form* of Z.

A closely related notion is the *Chow form* of Z. Consider the natural rational map

$$(\check{\mathbf{P}}_k^n)^{r+1} \dashrightarrow G(n-r,n+1),\quad (H_1,\ldots,H_{r+1})\to H_1\cap\ldots\cap H_{r+1}.$$

It is defined on the open subset U which consists of hyperplanes $(H_1,\ldots,H_{r+1})$ which intersect along a subspace of dimension $n-r-1$. Then the image of $\mathrm{cay}(Z)\cap U$ consists of all codimension $r+1$ linear subspaces of $\mathbf{P}_k^n$ which intersect Z. This is a hypersurface in $G(n-r,n+1)$ which is called the Chow form of Z and denoted by $\mathrm{chow}(Z)$. If $\mathcal{O}_{G(n-r,n+1)}(1)$ is the line bundle defining the Plücker embedding of $G(n-r,n+1)$, then $\mathrm{chow}(Z)$ is given by a global section of $\mathcal{O}_{G(n-r,n+1)}(d)$.

Examples. 1. Let Z be a hypersurface of degree d in $\mathbf{P}_k^n$. Then it coincides with its Chow form.

2. Let L be a linear subspace in $\mathbf{P}_k^n$ of dimension r. We shall identify $\mathbf{P}_k^n(k)$ with the projective space $P(E)$ associated to a linear space E. Then $\check{\mathbf{P}}_k^n(k)$ can be identified with the projective space $P(E^*)$ associated to the dual space E^*. Let $v_1,\ldots,v_{r+1}$ be a basis of the linear subspace $\bar{L}$ defining L. For any $\phi\in E^*$ we denote by $V(\phi)$ the hyperplane defined by the equation $\phi=0$. Then, using some standard facts from multi-linear algebra, we obtain

$$L\cap V(\phi_1)\cap\ldots V(\phi_{r+1})\neq\emptyset \Leftrightarrow \langle v_1\wedge\ldots\wedge v_{r+1},\phi_1\wedge\ldots\wedge\phi_{r+1}\rangle=0.$$

Here we denote by $\langle,\rangle$ the canonical pairing between the spaces $\bigwedge^{r+1}(V)$ and $\bigwedge^{r+1}(V^*)$. If we choose a basis $e_0,\ldots,e_n$ of V and its dual basis $e_0^*,\ldots,e_n^*$ in V^*, then the previous condition can be written in the form

$$\sum_{0<i_1<\ldots<i_{r+1}<n} p_{i_1,\ldots,i_{r+1}}a_{i_1}^{(1)}\ldots a_{i_{r+1}}^{(r+1)}=0,$$

where $p_{i_1,\ldots,i_{r+1}}$ are the Plücker coordinates of L, and $\phi_j=\sum_i a_i^{(j)}e_i^*, j=1,\ldots,r+1$. This expression is an $(r+1)$-multilinear linear form defining the equation of the Cayley form of L in $(\check{\mathbf{P}}_k^n)^{r+1}$. If we view $a_{i_1}^{(1)}\ldots a_{i_{r+1}}^{(r+1)}$ as the Plücker coordinates of the linear subspace of V^* which is orthogonal to the subspace $\phi_1=\ldots=\phi_{r+1}=0$ of V, then the above expression is the equation of the image of the Chow form of L under the map $G(n-r,n+1)\to G(r+1,n+1)$ given by $W\to W^{\perp}$.

Now we want to define a morphism

$$\Phi:\mathbf{Hilb}_{\mathbf{P}_k^n}^P\to \mathrm{Div}^{(d,\ldots,d)}(\check{\mathbf{P}}_k^n)^{r+1}$$

such that its value at a closed subvariety of degree d and dimension r with Hilbert polynomial P is equal to $\mathrm{cay}(Z)$. The problem here is that we do not know the values of this map on non-reduced closed subschemes. For any closed subscheme Z we define the *cycle* of Z as a formal linear combination

$$\mathrm{cyc}(Z) = \sum_i m_i Z_i,$$

where Z_i is an irreducible component of Z and m_i is equal to the length of the local ring $\mathcal{O}_{Z_i,\eta_i}$ of Z_i at its generic point η_i. One extends the notion of the Cayley form to any cycle of irreducible reduced varieties by setting

$$\mathrm{cay}(\sum_i m_i Z_i) = \sum_i m_i \mathrm{cay}(Z_i)$$

where the latter sum is considered as a divisor in $(\check{\mathbf{P}}_k^n)^{r+1}$. It is shown in **[Mu1]**, Chapter 5, 4.6, that there is a morphism $\Phi : \mathbf{Hilb}_{\mathbf{P}_k^n}^P \to \mathrm{Div}^{(d,\ldots,d)}(\check{\mathbf{P}}_k^n)^{r+1}$ with the property

$$\Phi(Z) = \mathrm{cay}(cyc(Z))$$

for any $Z \in \mathbf{Hilb}_{\mathbf{P}_k^n}^P(k)$. Note that the degree d and the dimension r of Z can be read off the Hilbert polynomial P as $P(t) = (d/r!)t^r +$ terms of lower order. Also it is known that

$$d = \deg(Z) := \sum_i m_i \deg(Z_i).$$

11.5 Recall from **[Har]**, p.180, that for any smooth variety X of dimension n there is an invertible sheaf ω_X whose global sections are regular differential n-forms on X. The inverse sheaf $\omega_X^{-1}$is equal to the maximal exterior power of the tangent bundle T_X of X. If X is a projective connected curve of genus g, the degree of ω_X is equal to $2g-2$. Thus, if $g > 1$, the ν-canonical complete linear system $|\omega_X^{\otimes \nu}|, \nu \geq 3$, defines a closed embedding $X \hookrightarrow \mathbf{P}_k^{(2\nu-1)(g-1)-1}$ (see **[Har]**, p.308). This fact allows one to embed all nonsingular curves of genus $g \geq 2$ into the same projective space, say $\mathbf{P}_k^{5g-6}$, as curves of degree $6(g-1)$. Since any isomorphism $f : X \to X'$ of curves defines an isomorphism $\omega_X \cong f^*(\omega'_X)$, we obtain that $X \cong X'$ if and only if their images in $\mathbf{P}_k^{5g-6}$ are projectively equivalent. This suggests a construction of a coarse moduli scheme of curves of genus $g \geq 2$ as a geometric quotient of some subset of the Hilbert scheme $\mathbf{Hilb}_{\mathbf{P}_k^N}^P$ where

$$N = 5g-6, \ P(t) = dt + 1 - g, \ d = 6g-6.$$

Let us fulfill this program.

First let us define the appropriate subset of the Hilbert scheme. We assume that $g \geq 2$ leaving the case $g = 1$ to the reader (see Problem 2).

Lemma. *There is a unique subscheme $\mathbf{H}_g$ of the Hilbert scheme $\mathbf{Hilb}^P_{\mathbf{P}^N_k}$ such that any morphism $f : S \to \mathbf{Hilb}^P_{\mathbf{P}^N_k}$ factors though $\mathbf{H}_g$ if and only if*

(i) the S-subscheme $Z \subset S \times \mathbf{P}^N_k$ defined by f is a family of curves of genus g;

(ii) the embedding $Z_s \hookrightarrow \mathbf{P}^N_k$ is given by the 3-canonical linear system;

(iii) the restriction of the morphism $\Phi : \mathbf{Hilb}^P_{\mathbf{P}^N_k} \to \mathrm{Div}^{(d,d)}((\check{\mathbf{P}}^N_k)^2)$ to $\mathbf{H}_g$ is a closed embedding.

Proof. Let $\mathcal{Z} \to \mathbf{Hilb}^P_{\mathbf{P}^N_k} \times \mathbf{P}^N_k$ be the universal object for the functor $\mathrm{Hilb}^P_{\mathbf{P}^N_k}$ and $p : \mathcal{Z} \to \mathbf{Hilb}^P_{\mathbf{P}^N_k}$ be the first projection. Since the set of points $s \in S$ over which a morphism $X \to S$ is smooth is an open subset of S, we can find a maximal open subset U of $\mathbf{Hilb}^P_{\mathbf{P}^N_k}$ over which the morphism p is smooth. Since the number of connected components of a geometric fibre $\mathcal{Z}_s$ of p is determined by the dimension of the space $\dim_k\Gamma(\mathcal{Z}_s, \mathcal{O}_{\mathcal{Z}_s})$, and the latter is an upper semi-continuous function for a flat morphism ([**Har**], p. 288), we obtain that there exists an open subset U' of U such that all geometric fibres over U' are connected nonsingular curves. Their genus is determined by the Hilbert polynomial.

To achieve the second property we have to use a more powerful technique. It is based on the notion of the *relative Picard scheme* (see [**Mu2**]). For each morphism $f : X \to S$ of schemes we consider the functor $Pic_{X/S}$ on the category of S-schemes which assigns to any $S' \to S$ the Picard group $Pic(X \times_S S')$ of isomorphism classes of invertible sheaves on the fibred product $X \times_S S'$. For any morphism $\phi : S'' \to S'$ of S-schemes, the operation of the inverse image of a coherent sheaves defines a homomorphism of groups $Pic_{X/S}(S') \to Pic_{X/S}(S'')$. In the case where f is a family of curves this functor is representable by the S-scheme $\mathbf{Pic}_{X/S}$. It contains a closed subscheme $\mathbf{Pic}^0_{X/S}$ which represents the subfunctor $Pic^0_{X/S}$ whose value on S' is the subgroup of $Pic(X \times_S S')$ which consists of isomorphism classes of invertible sheafs whose restriction to each fibre of the map $X \times_S S' \to S'$ is of degree zero. The scheme $\mathbf{Pic}^0_{X/S}$ is a proper group scheme over S (an *abelian scheme*), its geometric fibres are abelian varieties, the jacobian varieties of the geometric fibres of f. Let us take for f the restriction $\mathcal{Z}' \to U'$ of $p : \mathcal{Z} \to \mathbf{Hilb}^P_{\mathbf{P}^N_k}$ over U'. Let $\omega_{\mathcal{Z}'/U'}$ be the relative canonical sheaf of f. Its restriction to each fibre is the canonical sheaf of the fibre. Consider the invertible sheaf $\mathcal{L}$ on $\mathcal{Z}'$ equal to the tensor product $\omega^{\otimes 3}_{\mathcal{Z}'/U'} \otimes q^*(\mathcal{O}_{\mathbf{P}^N_k}(-1))$, where $q : \mathcal{Z}' \to \mathbf{P}^N_k$ is induced by the second projection $\mathcal{Z}' \subset \mathbf{Hilb}^P_{\mathbf{P}^N_k} \times \mathbf{P}^N_k \to \mathbf{P}^N_k$. Since both $\omega^{\otimes 3}_{\mathcal{Z}'/U'}$ and $q^*(\mathcal{O}_{\mathbf{P}^N_k}(1))$ restrict to an invertible sheaf of degree d on each fibre of f (which is a curve of degree d in $\mathbf{P}^N_k$), we see that $\mathcal{L}$ belongs to the group $Pic^0_{\mathcal{Z}'/U'}(U') \subset Pic(\mathcal{Z}' \times_{U'} U') = Pic(\mathcal{Z}')$. Hence it defines a morphism

$$\varphi : U' \to \mathbf{Pic}^0_{\mathcal{Z}'/U'}.$$

If s is a closed point of U' and $i : s \hookrightarrow U'$ is the closed embedding, we can consider s as a U'-scheme. Then $Pic^0_{\mathcal{Z}'/U'}(s)$ is equal to the group $Pic^0(\mathcal{Z}' \times_{U'} s) = Pic^0(\mathcal{Z}'_s)$ of isomorphism classes of invertible sheaves of degree 0 on the curve $p^{-1}(s) = \mathcal{Z}'_s \subset \mathbf{P}^N_k$ represented by the point s of the Hilbert scheme. By definition of representable functors, the map φ sends the point s to the isomorphism class of the sheaf $\mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}'_s}$. Let $\mathbf{0}$ be the zero section of the group scheme $\mathbf{Pic}^0_{\mathcal{Z}'/U'}$. Its closed points correspond to the isomorphism classes of the structure sheaves on the fibres of $\mathcal{Z}' \to U'$. Thus if we set

$$\mathbf{H}_g = \varphi^{-1}(\mathbf{0}),$$

we obtain the needed subset of $\mathbf{Hilb}^P_{\mathbf{P}^N_k}$. The last property from (iii) follows from the fact that the Cayley form determines uniquely any nonsingular curve.

Theorem. *The geometric quotient* $\mathcal{M}_g = \mathbf{H}_g/PGL(n+1)$ *is a coarse moduli scheme of* M_g.

Proof. Let $p : \mathbf{H}_g \to \mathcal{M}_g$ be a geometric quotient $\mathbf{H}_g/PGL(n+1)$. Let us construct a morphism of functors $M_g \to h_{\mathcal{M}_g}$. Let S be a scheme over the field k and $f : X \to S$ be a family of curves of genus g. By **[Har]**, Chapter III, Corollary 12.9, the sheaf $\mathcal{E} = f_*(\omega_{X/S}^{\otimes 3})$ is a locally free sheaf on S of rank $N + 1 = 5g - 5$. For any closed point $s \in S$ we have

$$\mathcal{E} \otimes_{\mathcal{O}_{S,s}} k(s) \cong H^0(X_s, \omega_{X_s}^{\otimes 3}).$$

Let $\{U_i\}_{i \in I}$ be a trivializing affine open cover for $\mathcal{E}$. If we choose a basis $\sigma_0, \ldots, \sigma_N$ of the free $\mathcal{O}_S(U_i)$-module $M = \mathcal{O}_S(U_i) \otimes_k H^0(X_s, \omega_{X_s}^{\otimes 3})$, we will be able to define a morphism:

$$\phi_i : X_i := f^{-1}(U_i) \to U_i \times \mathbf{P}^N_k$$

by sending a point $x \in X_s$ to the point $(s, (\sigma_0(s)(x), \ldots, \sigma_N(s)(x)))$. This morphism is a closed embedding and its image $Z_i \subset U_i \times \mathbf{P}^N_k$ satisfies properties (i) and (ii) from the previous Lemma. Let $\mathcal{Z}_g \to \mathbf{H}_g$ be the restriction to $\mathbf{H}_g$ of the universal family over the Hilbert scheme. Applying the Lemma, we obtain a unique morphism $U_i \to \mathbf{H}_g$ such that the $\mathcal{Z}_g \times_{\mathbf{H}_g} U_i \cong X_i$. Now we use that the restriction of the maps ϕ_i and ϕ_j to $f^{-1}(U_i \cap U_j)$ differ by a projective automorphism of $\mathbf{P}^N_k$ defined by the transition function g_{ij} of $\mathcal{E}$. This shows that the compositions

$$\bar{\phi} : U_i \to \mathbf{H}_g \to \mathcal{M}_g$$

agree on $U_i \cap U_j$. Thus, starting from an element of $M_g(S)$, we have constructed a morphism $S \to \mathcal{M}_g$. It remains to verify the two properties from the definition of a coarse moduli scheme.

Let us start with the property of universality. Suppose we have a morphism of functors $M_g \to h_N$. By taking $S = \mathbf{H}_g$, and taking $\mathcal{Z}_g \to \mathbf{H}_g$ as an element from $M_g(S)$, we obtain an element of $h_N(S)$ which is a morphism $\mathbf{H}_g \to N$. The group $\mathbf{PGL}_k(N+1)$ acts on $\mathbf{H}_g$ via projective transformations of $\mathbf{P}^N_k$ which transform the universal family $\mathcal{Z}_g \to \mathbf{H}_g$ to an isomorphic family of curves, i.e., it defines the same element of $M_g(S)$. Hence it defines the same image in h_N, and hence the map $\mathbf{H}_g \to N$ factors through a unique morphism $\mathcal{M}_g \to N$, making the commutative diagram

$$\begin{array}{ccc} M_g & \longrightarrow & h_N \\ & \searrow \quad \nearrow & \\ & h_{\mathcal{M}_g} & \end{array}.$$

This checks the universality. Now if we take $S = \mathrm{Spec}(\Omega)$, where Ω is any algebraically closed extension of k, we get a map $M_g(\Omega) \to \mathbf{H}_g(\Omega)$ which sends a nonsingular projective connected curve Γ of genus g over Ω to the projective isomorphism class of the curve $\phi(\Gamma) \subset \mathbf{P}_k^N$. But we have explained already that $\Gamma \cong \Gamma'$ if and only if their images in $\mathbf{P}_k^N$ under the map defined by the tri-canonical linear system are projectively isomorphic. This shows that our map $M_g(\Omega) \to \mathbf{H}_g(\Omega)/\mathbf{PGL}_k(N+1)$ is bijective. This verifies the second property of a coarse moduli scheme.

11.6 It remains to prove that a geometric quotient $\mathbf{H}_g/PGL(n+1)$ exists.

Let us consider the closed embedding

$$\Phi : \mathbf{H}_g \hookrightarrow \mathcal{D}_g := \mathrm{Div}^{(d,d)}((\check{\mathbf{P}}_k^N)^2)$$

defined in Lemma 11.5. It is obviously G-equivariant, where the action of G on $\mathcal{D}_g$ is induced by the diagonal action of G on $(\check{\mathbf{P}}_k^N)^2$ via its dual projective representation. It is enough to show that the image of $\mathbf{H}_g$ in the projective space $\mathcal{D}_g$ is contained in the set of properly stable points with respect to $\mathcal{O}_{\mathcal{D}_g}(1)$ and the action of $\mathbf{SL}_k(N+1)$. We will now show this. Recall that geometric points of $\Phi(\mathbf{H}_g)$ are the Cayley forms of nonsingular connected projective curves Γ in $\mathbf{P}_k^N$ of genus g and degree $d = 6(g-1)$, embedded by the tri-canonical linear system. For any $g \in SL(N+1,k)$, $g \cdot \Gamma$ is the image of Γ under the projective transformation defined by g. To check that Γ is properly stable we apply the numerical criterion of stability. We use the following

Lemma. *Let G act on a projective variety X, and let L be a G-linearized ample line bundle. For any $\lambda \in \mathcal{X}_*(G)$, and $x \in X(k)$,*

$$\mu^L(x,\lambda) = \mu^L(\lim_{t\to 0} \lambda(t)\cdot x, \lambda).$$

Proof. Linearizing the action we may assume that $x = (x_0, \ldots, x_n) \in \mathbf{P}_k^n$ and $\lambda(t)$ acts on x by the formula $\lambda(t)\cdot x = (t^{r_0}x_0, \ldots, t^{r_n}x_n)$. We know that

$$\mu^L(x,\lambda) = \min\{r_i : x_i \neq 0\}.$$

Without loss of generality we may assume that this minimum is equal to r_0. Then

$$\lambda(t)\cdot x = (x_0, t^{r_0 - r_1}x_1, \ldots, t^{r_0 - r_n}x_n)$$

and

$$\lim_{t\to 0} \lambda(t)\cdot x = y := (x_0, a_1x_1, \ldots, a_nx_n),$$

where $a_i = 1$ if $r_i = r_0$ and 0 otherwise. Now $\lambda(t)$ acts on y by the formula

$$\lambda(t)\cdot y = (t^{r_0}x_0, t^{r_1}a_1x_1, \ldots, t^{r_n}a_nx_n),$$

and the assertion follows immediately.

We need to find the limit

$$\gamma(\lambda) = \lim_{t \to 0} \lambda(t) \cdot \mathrm{cay}(\Gamma).$$

We have already observed that it is enough to verify that $\mu^L(x, \lambda) < 0$ for all one-parameter subgroups λ which are given by the diagonal matrices $diag(t^{r_0}, \ldots, t^{r_N})$, where

$$r_0 \geq r_1 \geq \ldots \geq r_N, \sum_i r_i = 0.$$

We can write such a one-parameter subgroup as a non-negative rational linear combination of the subgroups $\lambda_s, s = 0, \ldots, N-1$, with

$$r_0^{(s)} = \ldots = r_s^{(s)} = N - s, r_{s+1}^{(s)} = \ldots, r_N^{(s)} = -s - 1.$$

If we write

$$\lambda = \sum_{s=0}^{N-1} a_s \lambda_s,$$

we easily get

$$r_i - r_{i+1} = a_i(N+1), i = 0, \ldots, N-1.$$

In particular, the sequence $r_0 \geq r_1 \geq \ldots \geq r_N$ is strictly decreasing if and only if all a_i are nonzero. Let $\chi_1, \ldots, \chi_r$ be the characters of the maximal torus T which correspond to non-zero coordinates of x (see Lecture 7). Let $S(\lambda) = \{s : a_s \neq 0\}$. Then

$$\mu(x, \lambda) = \min_i \{\langle \lambda, \chi_i \rangle\} \leq \min_i \max_{s \in S(\lambda)} \{\langle \lambda_s, \chi_i \rangle\}(\sum_{s=0}^{N-1} a_s) \leq \min_i \max_s \{\langle \lambda_s, \chi_i \rangle\}(\sum_{s=0}^{N-1} a_s).$$

This shows that it is enough to verify that $\mu(x, \lambda) < 0$ for one-parameter subgroups with $S(\lambda) = \{0, \ldots, N\}$, i.e. , satisfying

$$r_0 > r_1 > \ldots > r_N, \sum_i r_i = 0.$$

In fact we can forget about any subset of λ's such that the differences $r_i - r_{i+1}$ satisfy some linear equations.

11.7 Let $x = (x_0, \ldots, x_N)$ be a point of $\mathbf{P}_k^N$ with $x_N \neq 0$. We have

$$\lambda(t) \cdot x = (t^{r_0} x_0, \ldots, t^{r_N} x_N) = (t^{r_0 - r_N} x_0, \ldots, t^{r_{N-1} - r_N} x_{N-1}, x_N).$$

When t goes to 0 we obtain

$$\lim_{t \to 0} \lambda(t) \cdot x = (0, \ldots, 0, 1).$$

This shows that all points of Γ not lying in the hyperplane $x_N = 0$ "specialize" to the point $P_N = (0, \ldots, 0, 1)$. Let us see that the whole curve Γ specializes to a cycle of lines taken with some multiplicities. Consider the map

$$f : \mathbf{G}_{\mathbf{m},k} \times \Gamma \to \mathbf{P}_k^N$$

which is obtained as the composition of $\lambda \times \mathrm{id}_\Gamma : \mathbf{G}_{\mathbf{m},k} \times \Gamma \to \mathbf{SL}_k(N+1) \times \mathbf{P}_k^N$ and the action morphism $\sigma : \mathbf{SL}_k(N+1) \times \mathbf{P}_k^N \to \mathbf{P}_k^N$. Let $Q_1, \ldots, Q_\nu$ be the set of points of Γ with the last coordinate equal to 0. Choose a local parameter η_j of Γ at the point $Q_j, i = 1, \ldots, \nu$. Let $X_0, \ldots, X_N$ be projective coordinates in $\mathbf{P}_k^N$. Suppose some $X_k(Q_j) \neq 0$. Then $X_i/X_k \in \mathcal{O}_{\Gamma, Q_j}$ for each i and hence we can write

$$X_i = \epsilon_i^{(j)} \eta_j^{s_i^{(j)}}, i = 0, \ldots, N,$$

where $\epsilon_i^{(j)}$ is an invertible element in the local ring $\mathcal{O}_{\Gamma, Q_j}$ and $s_i^{(j)}$ are non-negative integers, one of them equal to 0. Now the morphism $f : \mathbf{G}_{\mathbf{m},k} \times \Gamma \to \mathbf{P}_k^N$ defines a rational map

$$f : \mathbf{A}_k^1 \times \Gamma \dashrightarrow \mathbf{P}_k^N$$

given near the point $0 \times Q_i$ by the formula

$$X_i = \epsilon_i^{(j)} \eta_j^{s_i^{(j)}} T^{r_i}, i = 0, \ldots, N,$$

where $\mathbf{A}_k^1 = \mathrm{Spec} k[T]$.

Suppose we find a normal algebraic surface V together with two morphisms $\pi : V \to \mathbf{A}_k^1 \times \Gamma$ and $f' : V \to \mathbf{P}_k^N$ satisfying the following properties

(i) π is a proper birational morphism which is an isomorphism outside the points $Q_j, j = 1, \ldots, \nu$;

(ii) the diagram of rational maps

$$\begin{array}{ccc} & V & \\ {}^{\pi}\swarrow & & \searrow^{f'} \\ \mathbf{A}_k^1 \times \Gamma & \overset{f}{\dashrightarrow} & \mathbf{P}_k^N \end{array}.$$

is commutative.

Let $p_1 : \mathbf{A}_k^1 \times \Gamma \to \mathbf{A}_k^1$ be the first projection. Then the composition

$$\pi' = p_1 \circ \pi : V \to \mathbf{A}_k^1$$

is a proper map. For any $t \neq 0$

$$\pi'^{-1}(a) = \lambda(t) \cdot \Gamma \cong \Gamma.$$

On the other hand, over the origin the fibre is equal to the divisor D of zeroes of the function $\pi'^*(T)$. Set-theoretically this fibre is equal to the union $\Gamma' \cup E$, where Γ' is the proper transform of $0 \times \Gamma$ and E is the union of the pre-images of the points Q_j. The map f' blows down Γ' to the point P_N and maps E onto some curve in $\mathbf{P}_k^N$. Let E_α be an irreducible component of E, and let m_α be its multiplicity in the divisor D. Let $\bar{E}_\alpha = f'(E_\alpha)$ and let n_α be the degree $[R(E_\alpha) : R(\bar{E}_\alpha)]$ of the extension of the fields of rational functions defined by the map $E_\alpha \to \bar{E}_\alpha$. Here we set $n_\alpha = 0$ if f' blows down E_α to a point.

Proposition.

$$\lim_{t\to 0}\lambda(t)\cdot \mathrm{cay}(\Gamma) = \sum_{\alpha} m_\alpha n_\alpha \mathrm{cay}(\bar{E}_\alpha).$$

Proof. Since the map $\Phi : \mathbf{H}_g \to \mathcal{D}_g$ is $\mathbf{SL}_k(N+1)$-equivariant, we have

$$\mathrm{cay}(\lambda(t)\cdot\Gamma) = \lambda(t)\cdot \mathrm{cay}(\Gamma).$$

Let $Z_r(X)$ denote the group of algebraic r-cycles on an algebraic variety. This is a free abelian group generated by irreducible r-dimensional closed reduced subvarieties of X. For every proper morphism $\phi : X \to Y$ the image of a closed subset is closed. Let C be an r-dimensional closed subvariety of X. Define

$$f_*(C) = \deg(C/f(C))f(C),$$

where $\deg(C/f(C))$ is equal to $R(C) : R(f(C))$ if $f(C)$ is r-dimensional and zero otherwise. This extends by linearity to a homomorphism

$$f_* : Z_r(X) \to Z_r(Y)$$

which is called the *push-forward homomorphism* of cycles.

Two algebraic r-cycles C and C' are called *rationally equivalent* if there are a finite number of $r+1$-dimensional reduced subvarieties Z_i of X and rational functions $r_i \in R(\bar{Z}_i)$ on their normalizations $\bar{Z}_i$ such that $C - C' = \sum_i (\pi_i)_*(div(r_i))$, where $\pi_i : \bar{Z}_i \to X$ is the composition of the normalization map and the inclusion map $Z_i \to X$. In particular, two linearly equivalent divisors on a normal variety X of dimension n are rationally equivalent $(n-1)$-cycles on X. One can prove that the push-forward of rationally equivalent cycles are rationally equivalent ([**Ful**], p.11). Returning to our situation we observe that the map $(f',\pi') : V \to \mathbf{P}_k^N \times \mathbf{A}_k^1$ is proper. In fact, the composition of this map with the proper projection map $\mathbf{P}_k^N \times \mathbf{A}_k^1 \to \mathbf{A}_k^1$ equals the composition of the two proper maps $\pi : V \to \mathbf{A}_k^1 \times \Gamma$ and the projection $\mathbf{A}_k^1 \times \Gamma \to \mathbf{A}_k^1$. So the properness of (f',π') follows from Corollary 4.8 of Chapter 2 of [**Har**]. We have

$$\sum_{\alpha} m_\alpha n_\alpha(\bar{E}_\alpha \times \{0\}) = f'_*(div(\pi^*(T))),\ \lambda(t)\cdot\Gamma\times\{t\} = f'_*(div(\pi^*(T-t))).$$

Thus the cycles $\sum_\alpha m_\alpha n_\alpha(\bar{E}_\alpha \times \{0\})$ and $\lambda(t)\cdot\Gamma\times\{t\}$ are rationally equivalent in $\mathbf{P}_k^N \times \mathbf{A}_k^1$. If we identify both cycles with cycles in $\mathbf{P}_k^N$, then we get an algebraic family of rationally equivalent 1-cycles in $\mathbf{P}_k^N$ parametrized by $\mathbf{A}_k^1$ (see [**Ful**], Chapter 10). All members of this family have the same degree, and we have a regular map $\beta : \mathbf{A}_k^1 \to \mathcal{D}_g$ such that

$$\beta(0) = \mathrm{cay}(\sum_{\alpha} m_\alpha n_\alpha(\bar{E}_\alpha)),\ \beta(t) = \mathrm{cay}(\lambda(t)\cdot\Gamma), t \neq 0.$$

Now the assertion follows from the definition of $\lim_{t\to 0}\lambda(t)\cdot\mathrm{cay}(\Gamma)$.

11.8 Let us construct the surface V which "resolves" the rational map $f : \mathbf{A}^1_k \times \Gamma - \to \mathbf{P}^N_k$. This surface is obtained by blowing up the ideals $\mathcal{I}_j, j = 1, \ldots, \nu$, generated by the functions $\epsilon_i^{(j)} \eta_j^{s_i^{(j)}} T^{r_i'}$, where $r_i' = r_i - r_N$. To see what happens over each point Q_j, it is enough to consider the local situation, when $\mathbf{A}^1_k \times \Gamma$ is replaced by a regular local ring with local parameters x, y and the ideal $\mathcal{I}_j$ is replaced by the ideal generated by the monomials $x^{r_i'} y^{s_i^{(j)}}, i = 0, \ldots, N$. We arrive at the same situation if we consider the blowing-up of the ideal in the ring of polynomials $k[X,Y]$ generated by the monomials $X^{r_i'} Y^{s_i^{(j)}}, i = 0, \ldots, N$. This is a "torical situation". We need some more constructions from toric geometry.

Let $N \cong \mathbf{Z}^r$ be a lattice, $M = N^*$ be the dual lattice, Σ be a N-fan in the linear space $N_{\mathbf{R}}$, and X_Σ be the toric variety associated to Σ. Let $T = \mathrm{Spec}(k[M])$ be the torus acting on X_Σ. We assume for simplicity that no cones in Σ contains a linear subspace. This means that X_Σ contains an open orbit isomorphic to T so we may identify the field K of rational functions on X_Σ with the field of rational functions on T. Let $\mathcal{F}$ be a coherent subsheaf of the constant sheaf K (a sheaf of *fractional ideals*) which is T-invariant (for example the ideal sheaf of a T-invariant closed subset of X_Σ). Its restriction to each affine piece $X_\sigma, \sigma \in \Sigma$, is a M-graded finitely generated $k[\check{\sigma} \cap M]$-submodule of K. We can take for its generators a set of monomials $Z^{\mathbf{m}}, \mathbf{m} \in G(\mathcal{F})$. Define the function

$$\mathrm{ord}_{\mathcal{F}} : |\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \to \mathbf{R}$$

by setting for all $x \in |\Sigma|$

$$\mathrm{ord}_{\mathcal{F}}(x) = \min_{\mathbf{m} \in G(\mathcal{F})} \langle \mathbf{m}, x \rangle.$$

This function depends only on $\mathcal{F}$ and satisfies the following properties

(i) $\mathrm{ord}_{\mathcal{F}}(cx) = c\,\mathrm{ord}_{\mathcal{F}}(x), \forall c \in \mathbf{R}_{>0}$;

(ii) $\mathrm{ord}_{\mathcal{F}}$ is continuous, piecewise linear;

(iii) $\mathrm{ord}_{\mathcal{F}}(N \cap |\Sigma|) \subset \mathbf{Z}$;

(iv) $\mathrm{ord}_{\mathcal{F}}$ is convex on each $\sigma \in \Sigma$.

Conversely, given a function $f : |\Sigma| \to \mathbf{R}$ satisfying the previous four conditions, one can construct a unique $\mathcal{F}_f$ with $f = \mathrm{ord}_{\mathcal{F}_f}$ as follows. The restriction of $\mathcal{F}$ to each affine piece X_σ is the $k[\check{\sigma} \cap M]$-submodule of K

$$(\mathcal{F}_f)_\sigma = \bigoplus_{\mathbf{m} : \langle \mathbf{m} - f, \sigma \rangle \geq 0} k Z^{\mathbf{m}}.$$

For example, the identically zero function corresponds to the structure sheaf $\mathcal{O}_{X_\Sigma}$. If we want $\mathcal{F}$ to be an invertible sheaf we have to require additionally

(v) $\mathrm{ord}_{\mathcal{F}}$ is linear on each $\sigma \in \Sigma$.

In fact, if this property is satisfied, then for each $\sigma \in \Sigma$ there exists $\mathbf{m}_\sigma \in M$ such that $f(x) = \langle -\mathbf{m}_\sigma, x \rangle$ for any $x \in \sigma$. This implies that

$$(\mathcal{F}_f)_\sigma = \bigoplus_{\mathbf{m} : \langle \mathbf{m} + \mathbf{m}_\sigma, \sigma \rangle \geq 0} k Z^{\mathbf{m}} = Z^{-\mathbf{m}_\sigma} k[\check{\sigma} \cap M]$$

is a free module generated by $Z^{-\mathbf{m}_\sigma}$.

The sheaves $\mathcal{F}_f$ are coherent sheaves of T-invariant *complete* fractional ideals, i.e., each $k[\check{\sigma} \cap M]$- module $\Gamma(X_\sigma, \mathcal{F}(X_\sigma)) \subset K$ is integrally closed in K. In general, $\mathcal{F}_{\mathrm{ord}_{\mathcal{F}}}$ is not equal to $\mathcal{F}$. However it is equal to the normal closure of $\mathcal{F}$.

The operation $\mathcal{F} \to \mathrm{ord}_{\mathcal{F}}$, $f \to \mathcal{F}_f$ allows one to construct the normalization of the blow-up of $\mathcal{F}$. Recall from **[Har]**, p.163, that for any coherent sheaf of ideals $\mathcal{I}$ on a noetherian scheme X, one defines the blowing-up scheme $B_{\mathcal{I}}$ which comes with a projective morphism $\pi : B_{\mathcal{I}} \to X$ satisfying the following properties:

(i) $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{B_{\mathcal{I}}}$ is an invertible sheaf on $B_{\mathcal{I}}$;
(ii) π is an isomorphism outside the closed subscheme defined by $\mathcal{I}$;
(iii) π is universal with respect to the previous properties.

To describe the blowing-up of the sheaf $\mathcal{F}_f$ we have to explain the functorial property of the construction $\Sigma \to X_\Sigma$. Let $\varphi : N' \to N$ be a homomorphism of lattices, $\varphi_{\mathbf{R}} : N'_{\mathbf{R}} \to N_{\mathbf{R}}$ be the corresponding linear map, Σ' be a N'-fan, and Σ be a N-fan. Assume that

$$\forall \sigma' \in \Sigma', \quad \varphi_{\mathbf{R}}(\sigma') \subset \sigma \quad \text{for some } \sigma \in \Sigma.$$

Then we can define the morphism $f(\varphi) : X_{\Sigma'} \to X_\Sigma$ of the toric varieties by gluing together the morphisms of the affine varieties $X_{\sigma'} \to X_\sigma$ corresponding to the natural homomorphism of the rings $k[\check{\sigma} \cap M] \to k[\check{\sigma}' \cap M']$ induced by the transpose map ${}^*\varphi : M = N^* \to M' = N'^*$. We have

$$f(\varphi) \quad \text{is a proper morphism if and only if } \varphi(|\Sigma'|) = |\Sigma|.$$

Now we can state the following result, for the proof of which we refer the reader to **[KSM]**.

Proposition. *Let $\mathcal{I}$ be a coherent sheaf of T-invariant ideals on X_Σ. Let Σ' be obtained from Σ by subdividing each $\sigma \in \Sigma$ into the largest rational convex subcones σ_i such that $\mathrm{ord}_{\mathcal{F}}$ is linear on each σ_i. Then the morphism $X_{\Sigma'} \to X_\Sigma$ is the composition of the blowing-up $B_{\mathcal{I}}$ and the normalization map $X_{\Sigma'} \to B_{\mathcal{I}}$.*

To see better the geometry of the map $f(\varphi)$ and, in particular, of the map $\pi : X_{\Sigma'} \to X_\Sigma$, we use the following description of the orbital decomposition of a toric variety X_Σ. First let $\bar{\Sigma}$ be obtained from Σ by adding to it all faces of all cones $\sigma \in \Sigma$. For each $\tau \in \bar{\Sigma}$ we define the T-orbit O^τ as follows. Let τ be a face of some $\sigma \in \Sigma$. Then $\check{\tau}$ is spanned by $\check{\sigma}$ and the linear function $\mathbf{m} \in M = N^*$ such that $\mathbf{m}$ vanishes on τ. This shows that $k[\check{\tau} \cap M]$ is equal to the localization $k[\check{\sigma} \cap M]_{Z^{\mathbf{m}}}$. Therefore $\mathrm{Spec}(k[\check{\tau} \cap M])$ is an open affine subset of $\mathrm{Spec}(k[\check{\sigma} \cap M])$. Denote by $\tau^\perp$ the linear subspace of $M_{\mathbf{R}}$ of functions which vanish on τ. Then $\tau^\perp \subset \check{\tau}$ and the subspace of $k[\check{\tau} \cap M]$ spanned by monomials $Z^{\mathbf{m}}$ with $\mathbf{m} \notin \tau^\perp$ is an ideal with the quotient ring isomorphic to $k[\tau^\perp \cap M] \cong k[\mathbf{Z}^d]$, where $d = \dim\tau^\perp = r - \dim\tau$. This defines a closed subset of X_σ isomorphic to the torus $(\mathbf{G}_{\mathbf{m},k})^d$ which is a T-orbit. In this way we obtain a one-to-one correspondence $\tau \leftrightarrow O^\tau$ between the set $\bar{\Sigma}$ and the set of T-orbits. It satisfies the properties

(i) $\dim O^\tau = \mathrm{codim}\tau$;
(ii) τ is a face in τ' if and only if $O^{\tau'}$ is contained in the closure of O^τ.

If φ maps a cone $\sigma' \in \bar{\Sigma}'$ into a cone $\sigma \in \bar{\Sigma}$, then

$$f(\varphi)(O^{\sigma'}) \subset O^{\sigma}.$$

Codimension 1 orbits correspond to 1-dimensional cones from $\bar{\Sigma}$. The set of such cones is denoted by $\Sigma^{(1)}$ and is called the *1-skeleton* of Σ. We have already used this set in Lecture 10. For every $\tau \in \Sigma^{(1)}$ the closure of the orbit O^{τ} is a T-invariant Weil divisor on X_{Σ}. We shall denote it by D_{τ}. Let $\mathbf{n}_{\tau}$ be the primitive lattice vector spanning τ. For any $\mathbf{m} \in M$ the monomial $Z^{\mathbf{m}}$ is a T-invariant rational function on X_{Σ}. Its divisor must be a linear combination of the divisors D_{τ}. The explicit formula is

$$\mathrm{div}(Z^{\mathbf{m}}) = \sum_{\tau \in \Sigma^{(1)}} \langle \mathbf{m}, \mathbf{n}_{\tau} \rangle D_{\tau}. \qquad (*)$$

11.9 We shall apply the previous construction to the special case when Σ consists of a single cone σ equal to the first quadrant of $\mathbf{R}^2$. We consider N to be the standard lattice $\mathbf{Z}^2$ so that the dual lattice is $M = \mathbf{Z}^2$ and the pairing between the lattices is the usual dot-product. In this case $\check{\sigma}$ is equal to σ (but drawn in the "dual" picture). Thus

$$X_{\Sigma} = \mathbf{A}_k^2 = \mathrm{Spec}(k[\check{\sigma} \cap \mathbf{Z}^2]) = \mathrm{Spec}(k[X, Y]),$$

where $X = Z^{(1,0)}, Y = Z^{(0,1)}$. Let I be a monomial ideal in $k[X, Y]$. Because we are in the affine case, this corresponds to a sheaf of T-invariant ideals $\mathbf{I}$ on X_{Σ}. Let $\{Z^{\mathbf{m}}\}_{\mathbf{m} \in G(I)}$ be the set of monomial generators of I. Suppose $\mathbf{m} = \mathbf{m}' + \mathbf{a}$, where $\mathbf{a} \in \mathbf{Z}^2_{\geq 0}$. Then for any $x \in \sigma$,

$$\mathbf{m} \cdot x \leq \mathbf{m}' \cdot x.$$

This shows that the function $\mathrm{ord}_{\mathcal{F}}$ is determined only by the minimal elements of the set $G(I)$ with respect to the order $\mathbf{m} \geq \mathbf{m}' \Leftrightarrow \mathbf{m} - \mathbf{m}' \in \mathbf{Z}^2_{\geq 0}$. To find this minimal set, we plot the set $\mathbf{m} \in G(I)$ in $\mathbf{R}^2$, then the set of minimal points will lie on the *Newton polygon* of $G(I)$. This is defined as the union of finite edges of the convex hull of the set

$$\bigcup_{\mathbf{m} \in G(I)} (\mathbf{m} + \mathbf{Z}^2_{\geq 0}).$$

Let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be the points from $G(I)$ lying on the Newton polygon. Set

$$\sigma_i = \{x \in \sigma : \mathrm{ord}_{\mathcal{F}}(x) = \mathbf{m}_i \cdot x, i = 1, \ldots, n\}.$$

It is clear that $\mathrm{ord}_{\mathcal{F}}$ is linear on each σ_i, and the set $\Sigma' = \{\sigma_1, \ldots, \sigma_n\}$ satisfies the property from the previous Proposition. We have

$$\sigma_i \cap \sigma_j = \{x \in \sigma : \mathbf{m}_i \cdot x = \mathbf{m}_j \cdot x \leq \mathbf{m}_k \cdot x, \ k \neq i, j\}.$$

This shows that Σ' is obtained by subdividing σ by inserting 1-dimensional rays which are perpendicular to the edges of the Newton polygon. If a point from $G(I)$ lies on an edge, and differs from a vertex, we may delete it without changing Σ'. Also, observe that for any $\mathbf{m} \in M$, the blowing-up schemes of I and $Z^{\mathbf{m}}I$ are isomorphic. This allows one to translate the Newton polygon by any lattice points without changing the blowing-up. In particular we may always assume that I contains some powers of the variables.

It follows from the previous section that for any 1-dimensional cone $\tau \in \Sigma' \setminus \Sigma$, the divisor D_τ is mapped to the origin of $\mathbf{A}_k^2$. Each such divisor D_τ is equal to the closure of a one-dimensional torus orbit, and the points in the closure correspond to 2-dimensional cones σ and σ' such that $\tau = \sigma \cap \sigma'$. This easily implies that $D_\tau \cong \mathbf{P}_k^1$. The number of such divisors is equal to the number of edges of the Newton polygon.

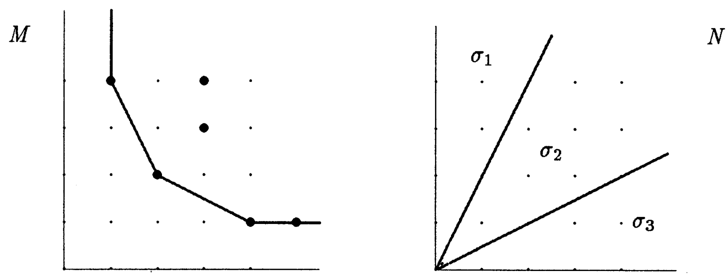


Fig. 1

Examples. 1. Let $I = (X, Y)$.

Fig. 2

The variety X_Σ is obtained by gluing together two affine varieties with the coordinate rings $k[\check{\sigma}_1 \cap M] = k[X^{-1}Y, X]$ and $k[\check{\sigma}_2 \cap M] = k[Y^{-1}X, Y]$. It is isomorphic to the subvariety of $\mathbf{A}_k^2 \times \mathbf{P}_k^1$ defined by the equation

$$T_1 X - T_0 Y = 0.$$

The open set X_{σ_1} is equal to the set where $T_0 \neq 0$, and the second set X_{σ_2} is equal to the set where $T_1 \neq 0$. It is the usual description of the variety obtained by blowing up the maximal ideal of the origin of the affine plane. The T-orbits of $\mathbf{A}_k^2$ are the origin, the two coordinate axes with the origin deleted, and the rest. Now X_Σ has two zero-dimensional orbits which are blown down to the origin. It has three one-dimensional orbits, the orbit $O^{\sigma_1 \cap \sigma_2}$ is blown-down to the origin, and the other two are mapped isomorphically onto the one-dimensional orbits of $\mathbf{A}_k^2$. The blowing-up map $\pi : X_\Sigma \to \mathbf{A}_k^2$ is an isomorphism over the open torus orbit.

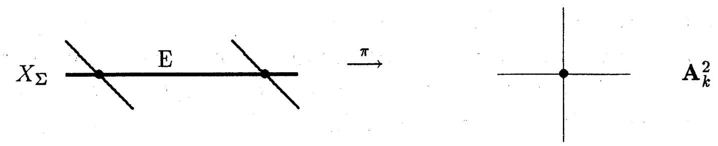


Fig. 3

The curve $E \cong \mathbf{P}_k^1$ which is the closure of the orbit $O^{\sigma_1 \cap \sigma_2}$ is the exceptional curve of the blowing-up. It is equal to the pre-image $\pi^{-1}(0)$ of the origin.

2. Let $I = (X^3, XY, Y^3)$.

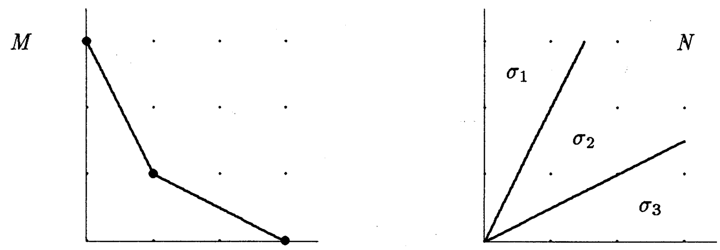


Fig. 4

This time X_Σ consists of three affine pieces

$$X_{\sigma_1} = \mathrm{Spec} k[X^{-2}Y, X],\ X_{\sigma_2} = \mathrm{Spec} k[X^{-1}Y^2, Y, X, X^2Y^{-1}],\ X_{\sigma_3} = \mathrm{Spec} k[Y^{-2}X, Y].$$

The first and the third piece are isomorphic to the affine plane since the cones σ_1 and σ_3 are spanned by a basis of the lattice N. The second piece is isomorphic to the affine cone over the Veronese curve of degree 3 in $\mathbf{P}_k^3$ since

$$k[X^{-1}Y^2, Y, X, X^2Y^{-1}] \cong k[T_0, T_1, T_2, T_3]/(T_0T_3 - T_1T_2, T_0T_2 - T_1^2, T_1T_3 - T_2^2).$$

Thus X_Σ is singular at the 0-dimensional orbit O^{σ_2}. The exceptional curve is equal to the union of the closures of the orbits $O^{\sigma_1 \cap \sigma_2}$, $O^{\sigma_2 \cap \sigma_3}$.

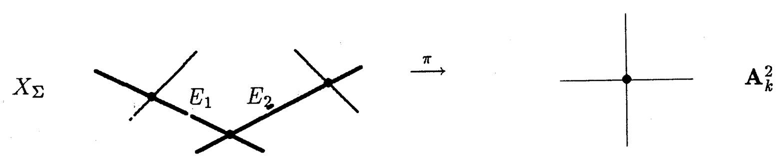


Fig. 5

11.10. Let us return to the situation of section 11.7. Applying the results of the previous sections we obtain the following description of the fibres of the morphism $\pi : V \to \mathbf{A}_k^1 \times \Gamma$

over the points $Q_j, j = 1, \ldots, \nu$. For each $Q = Q_j$ we consider the monomial ideal I generated by the monomials

$$X^{r'_i} Y^{s_i}, i = 0, \ldots, N,$$

where $r'_i = r_i - r_N, i = 1, \ldots, N, s_i = s_i^{(j)}$. Let $Newt$ be the Newton polygon for this set of monomials. Let β be the largest i such that $s_i = 0$. Since $r'_N = 0$, the Newton polygon $Newt$ starts at the point $(0, s_N)$ and ends at $(r'_\beta, 0)$. Let

$$\beta = e(1) < \ldots < e(\delta) = N$$

be the sequence of numbers such that the points

$$(r'_{e(1)}, 0), (r'_{e(2)}, s_{e(2)}), \ldots, (0, s_N)$$

are the consecutive vertices of $Newt$. Let $(p_i, q_i), i = 1, \ldots, \delta$, be the primitive lattice vectors such that

$$(r'_{e(i)} - r'_{e(i+1)})p_i + (s_{e(i)} - s_{e(i+1)})q_i = 0.$$

These vectors span the one-dimensional cones which subdivide the positive quadrant in the torical blowing-up of the ideal I.

Thus we deduce from the previous section that, set-theoretically,

$$\pi^{-1}(Q) = E_1 \cup \ldots \cup E_\delta,$$

where the curves E_1 are all isomorphic to $\mathbf{P}^1_k$, and each E_i intersects only E_{i-1} and E_{i+1}, transversally at one point.

Moreover, the divisor of the function $\pi^*(T)$ on V is equal to

$$\operatorname{div}(\pi^*(T)) = \pi^{-1}(\{0\} \times \Gamma) + \sum_{i=1}^{\delta} m_i E_i,$$

where $\pi^{-1}(\{0\} \times \Gamma)$ denote the proper transform of $\{0\} \times \Gamma$, and m_i is equal to the order of the zero of the function $\pi^*(T)$ at the curve E_i. The latter can be defined by using the toric geometry. By formula $(*)$ from 11.8, we get

$$m_i = (1, 0) \cdot (p_i, q_i) = p_i.$$

Similarly, we have

$$\operatorname{div}(\pi^*(\eta)) = \pi^{-1}(\mathbf{A}^1_k \times Q) + \sum_{i=1}^{\delta} n_i E_i,$$

where

$$n_i = (0, 1) \cdot (p_i, q_i) = q_i.$$

This shows that the rational function $\pi^*(\eta^{p_i}/T^{q_i})$ has order 0 at the curve E_i. Moreover, it is easy to see that this function generates the field of rational functions on E_i (use that the vector $(q_i, -p_i)$ spans $\tau^{\perp} \cap M$, where $\tau = \mathbf{R}(p_i, q_i))$. Now recall that in an affine neighborhood of the point Q, the rational map $\mathbf{A}_k^1 \times \Gamma \dashrightarrow \mathbf{P}_k^N$ can be given by the formulas

$$X_i = \epsilon_i \eta^{s_i} T^{r'_i}, i = 0, \ldots, N.$$

We may assume that no other point (r'_i, s_i) lies on $Newt$. This follows from the remark made at the end of section 11.6. Thus the order of the zero of $f'^*(X_k/X_l)$ at the curve E_i is equal to

$$(s_k - s_l)p_i + (r'_k - r'_l)q_i > 0 \text{ if } \{k, l\} \not\subset \{e(i), e(i+1)\}$$

and zero otherwise. This implies that the curve E_i is mapped by the map $f' : V \to \mathbf{P}_k^N$ to the line $L_{e(i),e(i+1)}$ given by the equations $X_i = 0, i \neq e(i), e(i+1)$. Since

$$f'^*(X_{e(i)}/X_{e(i+1)}) = \pi^*(\eta^{s_{e(i)} - s_{e(i+1)}} T^{r'_{e(i)} - r'_{e(i+1)}}) =$$

$$= \pi^*((\eta^{p_i}/T^{q_i})^{\frac{s_{e(i)} - s_{e(i+1)}}{p_i}}),$$

we obtain that

$$\deg(E_i/f'(E_i)) = \frac{s_{e(i+1)} - s_{e(i)}}{p_i}.$$

Since E_i enters into $\mathrm{div}(\pi^*(T))$ with multiplicity p_i, we get from Proposition 11.7 that the push-forward under the map f' of the cycle $\mathrm{div}(\pi^*(T))$ on V is equal to

$$\sum_{i=1}^{\mu} (s_{e(i+1)} - s_{e(i)}) L_{e(i),e(i+1)}.$$

Collecting together all the points $Q_j, j = 1, \ldots, \nu,$, we get

Theorem. *Let $\Gamma \subset \mathbf{P}_k^N$ be a curve represented by a point of $\mathbf{H}_g$, and let $Q_i, \ldots, Q_\nu$ be its points lying in the hyperplane $X_N = 0$. For each point Q_j, let the inclusion $\Gamma \hookrightarrow \mathbf{P}_k^N$ be given in an affine neighborhood of Q_j by the formula*

$$X_i = \epsilon_i^{(j)} \eta_j^{s_i^{(j)}}, i = 0, \ldots, N,$$

where $\epsilon_i^{(j)}$ is an invertible function at Q_j and η_j is a local parameter at Q_j. Let $\lambda(t) = diag(t^{r_0}, \ldots, t^{r_N})$ be a one-parameter subgroup of $\mathbf{SL}_k(N+1)$ satisfying conditions
(i) $r_0 > r_1 > \ldots > r_N$,
(ii) for each $j = 1, \ldots, \nu$, the Newton polygon $Newt_j$ of the set $B(j)$ of vectors $v_i^{(j)} = (r_i - r_N, s_i^{(j)}), i = 0, \ldots, N$, has no lattice points except its vertices.

For each j, let $\beta_j = e_j(1) < \ldots < e_j(\delta_j) = N$ be the sequence of the indices i such that the vectors $v_i^{(j)}$ are the vertices of $Newt_j$. Then

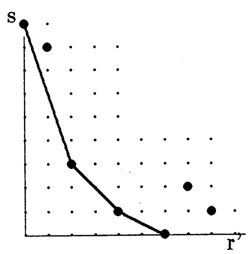
$$\lim_{t \to 0} \lambda(t) \mathrm{cay}(\Gamma) = \sum_{j=1}^{\nu} \left[\sum_{i=1}^{\delta_j - 1} (s_{e_j(i+1)}^{(j)} - s_{e_j(i)}^{(j)}) \mathrm{cay}(L_{e_j(i), e_j(i+1)}) \right],$$

where $L_{e_j(i),e_j(i+1)}$ *is the line given by the equations*

$$X_i = 0, i \neq e_j(i), e_j(i+1).$$

Note that for each j, the lines $L_{e_j(i),e_j(i+1)}$ form a chain of δ_j lines joining the point $P_N = (0,\ldots,0,1)$ with the point $P_{\beta_j} = (0,\ldots,0,\underbrace{1}_{\beta_j^{th}\,position},0,\ldots,0)$

Example.



$$\begin{gathered} X_0 = \eta \\ X_1 = \eta^2 \\ X_2 = 1 \\ X_3 = \eta \\ X_4 = \eta^3 \\ X_5 = \eta^8 \\ X_6 = \eta^9 \\ \lambda(t) = \mathrm{diag}(t^4, t^3, t^2, 1, t^{-2}, t^{-3}, t^{-4}) \end{gathered}$$

Fig. 6

This gives us, assuming that we have only one point Q_j, the limit cycle

$$L_{2,3} + 2L_{3,4} + 6L_{4,6}.$$

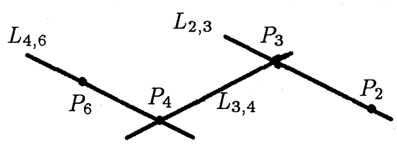


Fig. 7

11.11 Now it is easy to compute $\mu(\lambda, \mathrm{cay}(\Gamma))$. where λ satisfies properties (i) and (ii) from the previous Theorem. ¿From Example 2 in 11.4, we know the equation of the Cayley form of a linear subspace. Applying this to the lines $L^{(j)}_{e_j(i),e_j(i+1)}$, we find

$$\mathrm{cay}(L^{(j)}_{e_j(i),e_j(i+1)}) : A^{(1)}_{e_j(i)} A^{(2)}_{e_j(i+1)} = 0,$$

where we use $(A_0^{(\alpha)}, \ldots, A_N^{(\alpha)}), \alpha = 1, 2$, to denote the homogeneous coordinates in each copy of the space $\check{\mathbf{P}}^N_k$. Since $\lambda(t)$ acts on the coordinate $A_i^{(\alpha)}$ by multiplying it by t^{r_i}, we obtain from Lemma 11.6

$$\mu(\lambda, \mathrm{cay}(\Gamma)) = \sum_{j=1}^{\nu} \left[\sum_{i=1}^{\delta_j - 1} (s^{(j)}_{e_j(i+1)} - s^{(j)}_{e_j(i)})(r_{e_j(i)} + r_{e_j(i+1)}) \right].$$

Now let us observe that the sum corresponding to each j is equal to the twice of the area under the Newton polygon $Newt_j$ plus $2r_N s_N^{(j)}$. If we take any other polygon whose edges join consecutive pairs of points $(r_i, s_i^{(j)})$ which start at $(0, s_N^{(j)})$ and end at $(r'_{\beta_j}, 0)$, we get a larger area. ¿From this we infer that

$$\mu(\lambda, \mathrm{cay}(\Gamma)) = \sum_{j=1}^{\nu} \min_{\beta_j = i_1 < \ldots < i_\delta = N} \left[\sum_{k=1}^{\delta-1} (s_{i_{k+1}}^{(j)} - s_{i_k}^{(j)})(r_{i_{k+1}} + r_{i_k}) \right].$$

Let

$$s'_{k,j} = \min_{i \geq k} s_i^{(j)}, \quad e_k = \sum_{j=1}^{\nu} s'_{k,j}.$$

This has the following interpretation. The first number is the multiplicity of the base point Q_j of the linear system A_k of divisors on Γ cut out by the linear system of hyperplanes in $\mathbf{P}_k^N$ spanned by the hyperplanes $\{X_i = 0\}, i = k, \ldots, N$. The second number is the degree of the divisor of the base points of A_k.

Now we can rewrite $\mu(\lambda, \mathrm{cay}(\Gamma))$ in the form

$$\mu(\lambda, \mathrm{cay}(\Gamma)) = \sum_{j=1}^{\nu} \min_{0 = i_1 < \ldots < i_\delta = N} \left[\sum_{k=1}^{\delta-1} (s'_{i_{k+1},j} - s'_{i_k,j})(r_{i_{k+1}} + r_{i_k}) \right] \leq$$

$$\leq \min_{0 = i_1 < \ldots < i_\delta = N} \sum_{k=1}^{\delta-1} \left[\sum_{j=1}^{\nu} s'_{i_{k+1},j} - \sum_{j=1}^{\nu} s'_{i_k,j} \right] (r_{i_{k+1}} + r_{i_k}) =$$

$$= \min_{0 = i_1 < \ldots < i_\delta = N} \sum_{k=1}^{\delta-1} (e_{i_{k+1}} - e_{i_k})(r_{i_{k+1}} + r_{i_k}).$$

Next we need the following combinatorial lemma.

Lemma. *Suppose the integers $e_0 \leq \ldots \leq e_N$ satisfy*
(i) $e_0 = 0, e_N = N + g$;
(ii) $e_i \leq i$ *for* $i = 0, \ldots, N - g$;
(iii) $e_i \leq i + g$ *for* $i = N - g + 1, \ldots, N$.

Then

$$\min_{0 = i_1 < \ldots < i_\delta = N} \sum_{k=1}^{\delta-1} (e_{i_{k+1}} - e_{i_k})(r_{i_{k+1}} + r_{i_k}) < 0.$$

Proof. Suppose the minimum is achieved for some sequence $i_1 < \ldots < i_\delta$. If $i_k \leq N - g$ and $e_{i_{k+1}} < i_{k+1}$ we can add 1 to both $e_{i_k}, e_{i_{k+1}}$ with the difference in sum equal to

$$(r_{i_k} + r_{i_{k-1}}) - (r_{i_{k+2}} + r_{i_{k+1}}) = (r_{i_k} - r_{i_{k+2}}) + (r_{i_{k-1}} - r_{i_{k+1}}) \geq 0.$$

Similarly we consider the case when $i_k \geq N-g+1$. This shows that the minimum only increases if we replace e_i by its upper bound i or $g+i$. Consider the following $g+1$ sequences $i_1 < \ldots < i_\delta$:

$$(0,1,\ldots,N),(0,1,\ldots,N-g-1,N-g+2,\ldots,N),(0,1,\ldots,N-g-2,N-g+3,\ldots,N),\ldots$$

$$\ldots,(0,1,\ldots,N-2g+1,N),(0,N).$$

Let $\Sigma_i, i=0,\ldots,g$, be the corresponding partial sums. We find

$\Sigma_0 = -(r_0+r_N)+g(r_{N-g+1}+r_{N-g})$;

$\Sigma_1 = -(r_0+r_N)+(g+2)(r_{N-g+2}+r_{N-g-1})-2(r_{N-g+1}+r_{N-g})$;

$\Sigma_i = -(r_0+r_N)+(g+2i)(r_{N-g+i+1}+r_{N-g-i})-2(r_{N-g-i+1}+\ldots+r_{N-g+i}); i = 2,\ldots,g-1$

$\Sigma_g = (N+g)(r_0+r_N)$.

After we multiply Σ_i by $\frac{1}{(g+2i)(g+2i+2)} = \frac{1/2}{g+2i} - \frac{1/2}{g+2i+2}, i=0,\ldots,g-1$, and Σ_g by $\frac{1}{3g(N+g)}$, and add up, we get

$$\frac{1}{3g}(r_{N-2g+1}+r_{N-2g+2}+\ldots+r_{N-1}+r_N) \quad (**)$$

As $N = 5(g-1) > 2g, r_0 \geq \ldots \geq r_N, r_0+\ldots+r_N = 0$, we obtain

$$(r_{N-2g+1}+\ldots+r_N) \leq 2gr_{N-2g} \leq \frac{2g(r_0+\ldots+r_{N-2g})}{N-2g+1} \leq -\frac{2g(r_{N-2g+1}+\ldots+r_N)}{N-2g+1}.$$

This shows that $(**)$ must be strictly negative. Thus for at least one sequence from above the sum is negative, and hence the assertion of the lemma is verified.

11.12 Now we are ready to prove the main theorem.

Theorem. *Let $\Gamma \subset \mathbf{P}_k^N$ be a curve represented by a point of $\mathbf{H}_g$. Then it is properly stable with respect to the natural action of $\mathbf{SL}_k(N)$ and the linearization defined by the embedding* cay : $\mathbf{H}_g \hookrightarrow \mathrm{Div}^{(d,d)}(\check{\mathbf{P}}_k^N)^2)$. *Here $g \geq 2, N = 5g-6, d = 6(g-1)$.*

Proof. As we have explained earlier, it is enough to verify that the numbers e_i (defined as the degrees of the base locus of the linear system A_i cut out on Γ by hyperplanes $a_iX_i+\ldots+a_NX_N = 0$) satisfy the assumption of the previous Lemma. Let $L = \mathcal{O}_\Gamma(H)$ be the line bundle corresponding to a hyperplane section H of Γ. The linear system A_i corresponds to a subspace of the projective space associated to the linear space

$$H^0(\Gamma, \mathcal{O}_\Gamma(H-D_i))) \subset H^0(\Gamma, L),$$

where

$$D_i = \sum_{j=1}^{\nu} s'_{i,j}Q_j$$

is the divisor of the base points. Since $\dim A_i = N - i$, applying the Riemann-Roch theorem, we obtain

$$N - i + 1 \leq \dim_k H^0(\Gamma, \mathcal{O}_\Gamma(H - D_i)) = N - e_i + 1 + \dim_k H^1(\Gamma, \mathcal{O}_\Gamma(H - D_i)).$$

By Serre duality, $H^1(\Gamma, \mathcal{O}_\Gamma(H - D_i)) \cong H^0(\Gamma, \mathcal{O}_\Gamma(K_\Gamma - (H - D_i)))$ where K_Γ is the canonical divisor. As $H - D_i$ is linearly equivalent to a positive divisor,

$$\dim_k H^1(\Gamma, \mathcal{O}_\Gamma(H - D_i)) \leq \dim_k H^1(\Gamma, \mathcal{O}_\Gamma(K_\Gamma))) = g.$$

This gives $e_i \leq i + g$ for all i.

Moreover, if $i \leq N - g, \dim A_i \geq g$, hence $\dim_k H^1(\Gamma, \mathcal{O}_\Gamma(H - D_i)) \geq g + 1$, and

$$H^1(\Gamma, \mathcal{O}_\Gamma(H - D_i)) = 0.$$

This gives $e_i \leq i$ for $i \leq N - g$. This verifies the conditions of the Lemma and proves the Theorem.

So we have proved the existence of the coarse moduli scheme for nonsingular projective curves of genus $g \geq 2$.

11.13. Finally let us comment on the properties of the coarse moduli scheme $\mathcal{M}_g$ of curves of genus $g \geq 2$. Again, for lack of appropriate techniques we are not able to give complete proofs.

Theorem. *dim* $\mathbf{H}_g = (5g - 5)^2 + 3(g - 1) - 1$.

Proof. First we use the description of the Zariski tangent space $T(\mathbf{Hilb}_{X/k})_Z$ of the Hilbert scheme $\mathbf{Hilb}_{X/k}$ at the k-point represented by a subscheme $Z \subset X$. We assume that X and Z are smooth. Let Θ_Z be the tangent sheaf of Z and Θ_X be the tangent sheaf of X. The quotient sheaf $\mathcal{N}_{Z/X} = (\Theta_X \otimes \mathcal{O}_Z)/\Theta_Z$ is called the *normal sheaf* of Z in X. We have (see **[Mu2]**, Lecture 22):

$$T(\mathbf{Hilb}_{X/k})_Z \cong H^0(Z, \mathcal{N}_{Z/X}).$$

Also, it is known that $\mathbf{Hilb}_{X/k}$ is smooth at Z if

$$H^1(Z, \mathcal{N}_{Z/X}) = 0.$$

In our case we can compute everything very easily because Z is a nonsingular curve. The tangent sheaf of the projective space $\mathbf{P}_k^N$ is determined by the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}_k^N} \to \mathcal{O}_{\mathbf{P}_k^N}^{\oplus N+1} \to \Theta_{\mathbf{P}_k^N} \to 0.$$

Applying the Riemann-Roch Theorem, we obtain

$$\chi(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) = \dim_k H^0(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) - \dim_k H^1(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) = \chi(Z, \Theta_{\mathbf{P}_k^N} \otimes \mathcal{O}_Z) - \chi(Z, \Theta_Z) =$$

$$= (N+1)\chi(Z, \mathcal{O}_Z(1)) - \chi(Z, \mathcal{O}_Z) - \chi(Z, \Theta_Z) =$$

$$= (N+1)(d+1-g) - (1-g) - ((2-2g)+1-g) = (5g-5)^2 + 4g - 4.$$

The degree of $\mathcal{N}_{Z/\mathbf{P}_k^N}$ is equal to $\deg(\Theta_{\mathbf{P}_k^N} \otimes \mathcal{O}_Z) - \deg(\Theta_Z) = d(N+1) - (2-2g)$. By Serre duality,

$$H^1(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) \cong H^0(Z, \mathcal{H}om(\mathcal{N}_{Z/\mathbf{P}_k^N}, \mathcal{O}_Z) \otimes \omega_Z).$$

The degree of the sheaf $\mathcal{H}om(\mathcal{N}_{Z/\mathbf{P}_k^N}, \mathcal{O}_Z) \otimes \omega_Z$ is equal to $(N-1)(2g-2) - d(N+1) + 2 - 2g$. Since $d = 6(g-1)$ and $N = 5(g-1)$, this number is negative, and hence

$$H^1(Z, Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) = 0, \dim_k H^0(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) = \chi(Z, \mathcal{N}_{Z/\mathbf{P}_k^N}) = (5g-5)^2 + 4g - 4.$$

Thus we get that $\mathbf{Hilb}_{\mathbf{P}_k^N}$ is smooth of dimension $(5g-5)^2 + 4g - 4$ at any point of $\mathbf{H}_g$. It follows from the construction of $\mathbf{H}_g$ that it is obtained as the intersection of two sections in the Picard scheme $\mathbf{Pic}^0_{Z'/U'}$. Since the fibres of $\mathbf{Pic}^0_{Z'/U'}$ are Jacobian varieties of nonsingular curves of genus g, their dimension is equal to g. Thus the codimension of the zero section is equal to g. This shows that

$$\dim \mathbf{H}_g = \dim \mathbf{Hilb}_{\mathbf{P}_k^N} - g = (5g-5)^2 + 3g - 4.$$

Corollary (Riemann). *Let $\mathcal{M}_g$ be the coarse moduli scheme of curves of genus $g \geq 2$. Then*

$$\dim \mathcal{M}_g = 3g - 3$$

Proof. We know that

$$\mathcal{M}_g \cong \mathbf{H}_g / \mathbf{PGL}_k(N+1),$$

and all points of $\mathbf{H}_g$ are properly stable. This implies that all orbits are of dimension equal to $\dim \mathbf{PGL}_k(N+1) = (N+1)^2 - 1 = (5g-5)2 - 1$. Together with the previous Theorem this gives the dimension of $\mathcal{M}_g$.

As a geometric invariant theory quotient of a normal irreducible variety is normal and irreducible, we obtain the remaining assertions.

Remarks. 1. Using more of the deformation theory one can show that $\mathbf{H}_g$ is smooth. Using the proof of the previous theorem, this fact implies that the two sections of the relative Picard scheme intersect transversally. Also one can show that $\mathbf{H}_g$ is irreducible and hence $\mathcal{M}_g$ is irreducible. There is a transcendental proof over the field of complex numbers which uses the Teichmüller theory. There is also an algebraic proof, going back to Riemann, which uses representation of curves as ramified covers of the projective line with ordinary ramification points. Unfortunately, it works only if characteristic is zero or sufficiently large. The first purely geometric proof applied to any characteristic was given by P. Deligne and D. Mumford (see [**DM**]).
2. Since $\mathbf{H}_g$ is smooth and $\mathbf{PGL}_k(N+1)$ acts with finite stabilizers, the singularities of $\mathcal{M}_g$ are rather mild. They are locally isomorphic to the quotients of the affine space $\mathbf{A}_k^{3g-3}$

by a linear action of a finite group. This follows from Luna's Slice Theorem (see **[Lun]**). Unfortunately, for lack of time, we could not discuss this important result of geometric invariant theory .

3. Since the constructions of the Hilbert scheme and its subvariety $\mathbf{H}_g$ do not use any particular ground field and can be done over $\mathbf{Z}$, the moduli space $\mathcal{M}_g$ also can be defined as a scheme over $\mathbf{Z}$.

4. The variety $\mathcal{M}_g$ is known to be rational for $g \leq 6$, unirational for $g \leq 15$, and not unirational for $g \geq 19$.

5. One can construct a natural compactification $\bar{\mathcal{M}}_g$ of $\mathcal{M}_g$ by allowing the inclusion of certain singular curves. This can be done using essentially the same methods but with substantially more technical difficulties. Instead of nonsingular curves of genus g one considers stable curves C with $\dim_k H^1(C, \mathcal{O}_C) = g$. A stable curve is defined as a connected algebraic curve whose singularities are at most ordinary double points, and each smooth irreducible component of genus 0 intersects at least three other components (this ensures that the group of automorphisms of the curve is finite). Let $P(t) = 2\nu(g-1)t + 1 - g, N = P(1)$, and let U_ν be set of semi-stable points of $\mathbf{Hilb}_{\mathbf{P}_k^N}^{P(t)}$ with respect to the action of $G = \mathbf{PGL}_k(N+1)$ and the linearized line bundle coming from the embedding of the Hilbert scheme into the Grassmannian. It is proved that, for sufficiently large ν, the closed subscheme H_ν of U_ν corresponding to the curves embedded by the ν-canonical linear system parametrizes stable curves of genus g. This shows that the quotient H_ν/G is a closed subvariety of a projective variety $U_\nu//G$. Since G acts on H_ν with finite stabilizers, all points of H_ν are properly stable, and the quotient $H_\nu//G$ is in fact a geometric quotient. This is a coarse moduli scheme for the functor of families of stable curves. We refer for the details and for further references to **[Gie]**.

Problems.

1. Prove that the group of automorphisms of a nonsingular projective curve of genus $g \geq 2$ is finite.
2. Consider the functor $E : Sch/k \to \mathbf{Sets}$ by setting $E(S) = \{$family of curves of genus 1 with a section$\}$/modulo isomorphism. Show that the functor E admits a coarse moduli scheme isomorphic to the affine line.
3. Show that the Hilbert scheme of 0-dimensional closed subschemes of a nonsingular curve X is isomorphic to the disjoint sum of symmetric products $X^{(n)} := X^n/S_n$.
4. Compute the Chow and the Cayley forms of a Veronese curve of degree 3 (the image of the Veronese map $v_3 : \mathbf{P}^1_k \to \mathbf{P}^3_k$).
5. Find the relationship between the equation of the Cayley form of a closed reduced subvariety Z of dimension r in $\mathbf{P}^n_k$ and the equation of the hypersurface in $\mathbf{P}^{r+1}_k$ obtained by projection of Z from a general point in $\mathbf{P}^n_k$.
6. Using toric geometry describe the normalization of the blowing-up of the ideal in $k[Z_1, Z_2, Z_3]$ generated by the monomials $Z_1^3, Z_1 Z_2 Z_3, Z_2^2, Z_3^3$.

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