수 학 강 의 록 제 24 권

AN INTRODUCTION TO DEFICIENCY MODULES AND LIAISON THEORY FOR SUBSCHEMES OF PROJECTIVE SPACE

JUAN C. MIGLIORE

서 울 대 학 교
수 학 연 구 소 · 대 역 해 석 학 연 구 센 터

• Notes of the Series of Lectures
held at Seoul National University

Juan C. Migliore, Department of Mathematics, University of Notre Dame,
Notre Dame, IN 46556, USA

펴낸날 : 1994년 11월 15일
지은이 : Juan C. Migliore
펴낸곳 : 서울대학교 수학연구소 · 대역해석학연구센터[TEL : 02-880-6562]

To Mom and Dad
and
Michelle

Preface

This monograph is an extremely expanded version of a series of talks that I gave at Seoul National University in the Fall of 1992, at the invitation of Professor Chang-Ho Keem. It is based on a course that I have taught at Notre Dame, and I hope to use it in the next few years when I teach the course again.

There are really two principal subjects, as indicated in the title: deficiency modules and Liaison Theory, for subschemes of projective space. Deficiency modules are in some sense the main subject, though, since our discussion of Liaison Theory relies heavily on them.

Both deficiency modules and Liaison Theory are used very extensively in Algebraic Geometry in one form or another. However, there does not seem to be a good general introduction to either subject in the literature, from the point of view in which we are interested, so this was one of the motivations for this monograph. It is not entirely self-contained, but I have tried to at least state the necessary background results, and I have included a very extensive list of references (books, research papers and expository papers) which I hope will help the reader look up any material which he or she would like to pursue further.

Of course, it is impossible to completely cover such a broad subject. I have chosen topics which are most interesting to me, and I apologize for any omissions or oversights. When the proofs are reasonable in length I have tried to include them, or at least give the reader an idea of how the proof goes. In some cases I have had to simply state results with no proof. Only a few of the results and observations given here are new (e.g. Proposition 4.2.15, Remark 3.3.1, Remark 5.3.13), and I have tried to give careful credit where it is due.

The set of deficiency modules of a subscheme V of $\mathbb{P}^n$ is a collection of graded modules over the polynomial ring. Under reasonable assumptions these modules have finite length. The number of modules associated to V is just the dimension of V. One can get a lot of information about V from these modules, and this monograph is an attempt to describe some of the techniques used and some of the information that can be obtained from these modules.

Liaison is an equivalence relation among subschemes of given dimension in a projective space. Roughly, two schemes are said to be directly linked if their union is a complete intersection, and this notion generates the equivalence relation of Liaison. It is not at all obvious, a priori, that there is any connection between Liaison and deficiency modules; but in fact, there is a strong one. Especially in the context of Liaison Theory (but also sometimes more generally), deficiency modules are also sometimes known as *Hartshorne-Rao modules.* However, they actually have been important in the literature even before Rao's work.

Chapter 1 gives the background needed in the following chapters. It also defines the deficiency modules, gives important facts about them, and gives a number of examples.

Chapter 2 gives some applications of deficiency modules, and in particular of submodules of the deficiency module. The term "deficiency module" comes because these modules measure the failure of our scheme V to be arithmetically Cohen-Macaulay. We show that submodules of the (first) deficiency module also measure various types of deficiency. Among the other applications are a generalization of Dubreil's theorem, and a discussion of when the property of being arithmetically Cohen-Macaulay is "lifted" from the general hyperplane (or hypersurface) section of V up to V.

In studying graded modules over the polynomial ring, it is important to know something about the module structure. One way of looking at it is to ask what effect "multiplying" by a general linear form has on the module. The extreme case is where multiplication by *any* linear form is zero. One would naturally expect that such extreme behavior in the deficiency modules of V would be reflected in many ways in the scheme V. Chapter 3 explores the case where V is a curve with this property, i.e. a so-called *Buchsbaum* curve. In fact, Buchsbaum curves recur often in these notes since they provide interesting examples of many things that we will discuss. Chapter 3 also discusses Liaison Addition, which is an important construction originally due to Schwartau. It is used to construct Buchsbaum curves, and a special case ("basic double linkage") is used in Liaison Theory.

Chapter 4 begins the study of Liaison Theory. Many things can be said about Liaison Theory for subschemes of projective space in general (arbitrary codimension), and I have tried to give a good overview of these in this chapter. For instance, we see why it is often more useful to look at *even liaison* (i.e. restricting to an even number of links).

Many of the more powerful results in Liaison Theory are known at the moment only in codimension two. This is the topic of Chapter 5. We begin with Rao's result parameterizing the even liaison classes. Next we turn to the problem of describing the structure of an even liaison class. Finally, in the last part of Chapter 5 we give a number of applications of Liaison Theory, to give an idea of the breadth of possible ways in which it can and has been used.

There are many mathematicians whom I must acknowledge. I am very grateful to Joe Harris for introducing me to Liaison Theory and for patiently guiding me through my first steps in the field; to Phil Schwartau for teaching me his more algebraic way of looking at it (and for writing such a wonderful thesis); to Giorgio Bolondi and Tony Geramita for their friendship and for the many years in which we have collaborated– my most enjoyable moments as a mathematician have come in working with them, and much of that work is described in the pages that follow. I would also like to thank Kyung-Hye Cho, Tony Geramita, Heath Martin, Scott Nollet, Yves Pitteloud, Phil Schwartau and especially Chris Peterson for their careful reading of the manuscript and their helpful suggestions. I am grateful to the Mathematics Department and the Global Analysis Research Center of Seoul National University for their hospitality and support. Most of all, I would like to thank my friend Chang-Ho Keem for inviting me to Korea to give these talks, and for his great kindness and hospitality while I was there. Without the opportunity that he provided, this monograph would never have been written.

Finally, and most importantly, I must thank my parents and my wife, Michelle, for their love and encouragement through the various stages of my career. This monograph is dedicated to them.

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Chapter 1

Background

Throughout this monograph, $k = \overline{k}$ shall always denote an algebraically closed field. We will occasionally also require that it have characteristic zero, but we will always make it clear when we are making this assumption. We shall denote by S the homogeneous polynomial ring $k[X_0, \cdots, X_n]$ and we let $\mathbb{P}^n = \mathbb{P}^n_k = \text{Proj } S$. Since S is a graded ring, it is the direct sum of its homogeneous components: $S = \bigoplus_{d \geq 0} S_d$, where S_d is the vector space of homogeneous polynomials of degree d. We denote by $\mathfrak{m}$ the maximal ideal $\mathfrak{m} = (X_0, \cdots, X_n) \subset S$.

We assume some knowledge of sheaves and sheaf cohomology. For a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules we will sometimes use the notation $H^i_*(\mathcal{F})$ for the graded S-module

$$\bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t)).$$

(See also page 6.) As usual, we use lower case for dimension:

$$h^i(\mathcal{F}) = \dim_k(H^i(\mathbb{P}^n, \mathcal{F})).$$

The main purpose of this chapter is to give some of the necessary background for the material in the subsequent chapters, and to define the deficiency modules.

1.1 Finitely Generated Graded S-Modules

Let $M = \bigoplus_{j \in \mathbb{Z}} M_j$ be a finitely generated graded S-module. Notice that $M_j = 0$ for $j \leq j_0$ for some j_0. Also, $\dim_k M_j < \infty$ for all j. How does one describe the module structure of M? (It's not enough to know the dimensions of all the components, although there are many things that can be said just from this information– see [10].) In particular, how does "multiplication by a homogeneous polynomial" work? Given $F \in S_d$ we get a homomorphism $\phi_{i,F} : M_i \to M_{i+d}$ for each i. We need to know all possible $\phi_{i,F}$.

Any homogeneous polynomial F can be written as a sum of products of linear forms (in fact, a sum of products of variables). Hence, because of the module structure, it is enough simply to be able to describe the situation when $d = 1$; if we know how all $n+1$ variables act on M then we know how any homogeneous polynomial acts on M. In this case ($d = 1$), for each i there is a homomorphism $\phi_i : S_1 \to \text{Hom}\,(M_i, M_{i+1})$ taking $L \mapsto \phi_{i,L}$. The collection of all ϕ_i determines the module structure. (Notice that the module structure also forces some compatibility conditions on the various ϕ_i.) We first describe how each of the ϕ_i can be viewed as a matrix of linear forms (in the dual variables).

Choose a basis for M_i and one for M_{i+1}. In terms of these bases, write $\phi_i(X_0) = A_0,\ldots,\phi_i(X_n) = A_n$ where the A_i are matrices of scalars. Let $L = \alpha_0 X_0 + \cdots + \alpha_n X_n$ be a linear form (i.e. an element of S_1). So $\phi_i(L) = \alpha_0 A_0 + \cdots + \alpha_n A_n$. Hence ϕ_i may be viewed as a $(\dim M_{i+1}) \times (\dim M_i)$ matrix of linear forms in the variables α_i.

Notice that the ϕ_i's are not isomorphism invariants of the graded module (since they depend on the choice of bases), but the degeneracy loci they determine are. That is, for any r the scheme $W_{i,r}$ defined by the vanishing of the $(r+1) \times (r+1)$ minors of the matrix ϕ_i is an isomorphism invariant. In particular, if $m = \text{rk}\,(\phi_{i,L})$ for the generic $L \in S_1$ then $\{\, L' \in S_1 \mid \text{rk}\,(\phi_{i,L'}) < m \,\}$ is an isomorphism invariant. The projectivization V_i of this set is a subvariety of the dual projective space $(\mathbb{P}^n)^* = \mathbb{P}(S_1)$. If M' is another graded S-module with associated subvarieties $V_i' \subset (\mathbb{P}^n)^*$, and if $M \cong M'$, then for each i, $V_i = V_i'$. Note that $W_{i,m-1}$ is supported on V_i. "Most of the time," $V_i = W_{i,m-1}$.

There is an expected codimension and degree for $W_{i,r}$:

Lemma 1.1.1 *Let ϕ be a $q \times p$ matrix of linear forms, and let W_r be the subscheme of $\mathbb{P}^n$ defined by the vanishing of the $(r+1) \times (r+1)$ minors of ϕ. Then the expected codimension of W_r is $(p-r)(q-r)$. If W_r is not empty and has the expected codimension then its degree is given by*

$$deg\ W_r = \prod_{i=0}^{p-r-1} \left[\binom{q+i}{r} \Big/ \binom{r+i}{r} \right].$$

For example, if ϕ is a square matrix then $p = q$. We expect that $p = q = r+1$ and W_r has codimension (1)(1), and that in fact W_r is the hypersurface given by the vanishing of det ϕ, hence of degree $\prod_0^0 \left[\binom{q}{q-1} / \binom{r}{r} \right] = q$. Lemma 1.1.1 is a special case of Porteous' formula. See [6] page 86 and [91] Lemma 1.4 for more details.

We will also eventually be interested in the *dual module* of M. There are two kinds of dual modules that we could define: $M^{\vee k} = \text{Hom}_k(M, k)$ and $M^{\vee S} = \text{Hom}_S(M, S)$. (This notation is borrowed from [122].) When it is clear from context which dual is being used, we will sometimes abuse notation and write $M^\vee$ for the dual module. (It will often be the case that our module M has finite length; in this case we leave it as an exercise to check that $M^{\vee S} = 0$.) $M^{\vee k}$ has the following structure as a graded S-module: $(M^{\vee k})_i = M^*_{-i}$ $(= \text{Hom}_k(M_{-i}, k)$, the dual vector space) for all i, and $\phi_i^\vee = {}^t\phi_{-i-1}$. For a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$, we denote by $\mathcal{F}^\vee$ the dual sheaf $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n})$.

Another operation on graded modules which will be important is shifting: we define the module $M(d)$ to be given by $M(d)_i = M_{d+i}$ (a shift to the *left* by d), with the obvious re-indexing of the ϕ_i.

Another way to view the module structure is the following. For each i we have a homomorphism $\mu_i : M_i \otimes S_1 \to M_{i+1}$. (This takes $m \otimes L \mapsto \phi_i(L)(m)$.) This point of view allows us to define the notion of *minimal generators* of M. We will always assume that such generators are homogeneous. Indeed, in each component M_{i+1} choose a basis for the image of μ_i and extend it with vectors $m_1, \cdots, m_t$ to a basis for M_{i+1}. Note that in the first non-zero component of M, $\{ m_1, \cdots, m_t \}$ is a basis of the whole component. (Before this, μ_i is zero.) The set of all such (homogeneous) elements m_i of M is a *minimal generating set* of M. The number of elements in degree d of a minimal generating set is well-defined, and we denote it by $\nu_d(M)$. We denote by $\nu(M)$ the sum of the $\nu_d(M)$, which is the number of elements in a minimal generating set of M.

(Equivalently, a minimal generating set is a set of homogeneous elements of M whose residues in the k-vector space $M/\mathfrak{m}M$ form a k-basis.)

Extending this idea is the notion of minimal free resolutions. (This brief discussion will follow [122].) Let $\{m_1, \cdots, m_a\}$ be a minimal generating set for our finitely generated graded module M, and assume that the generators are in degree $\alpha_1, \cdots, \alpha_a$ respectively. Then we obtain a surjective degree zero homomorphism

$$\bigoplus_{i=1}^{a} S(-\alpha_i) \xrightarrow{f_0} M \to 0$$

in the natural way. Note that the map f_0 is not uniquely determined, but the free module $F_0 = \bigoplus S(-\alpha_i)$ is. Let K_1 be the kernel of f_0. It can be shown that K_1 is invariant up to isomorphism. (That is, if we chose a different f_0' with corresponding kernel K_1' then $K_1 \cong K_1'$.)

K_1 is again finitely generated since F_0 is Noetherian, so one can do the same procedure with K_1 instead of M, and get a free module F_1 surjecting to K_1:

$$\begin{array}{ccccc} F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & M \to 0 \\ & \searrow \nearrow & & & \\ & K_1 & & & \\ & \nearrow \searrow & & & \\ 0 & & 0 & & \end{array}$$

The kernel K_i obtained in the ith step is called the *ith syzygy module* of M. K_1 gives the relations among the minimal generators of M, K_2 gives the relations among the relations, etc. One can continue in this manner, and the Hilbert Syzygy Theorem (cf. for instance [118]) guarantees that after at most $n+1$ steps the kernel K_{n+1} will be free. (Recall that $n+1$ is the dimension of the ring S. See also the discussion of the Auslander-Buchsbaum theorem on page 11.) Therefore we have a long exact sequence

$$0 \to F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

which is the *minimal free resolution* of M. (Note that we stop as soon as the kernel is free, so some of the F_i above may be zero.) The simplest example is the minimal free resolution of a complete intersection; see Example 1.4.1. If $F_i = \bigoplus \beta_{i,j} S(-j)$, the $\beta_{i,j}$ are called the *graded Betti numbers* of the resolution. (Some authors use the notation $F_i = \bigoplus S(-j)^{\beta_{i,j}}$.)

An exact sequence of the above form (where all but the rightmost module M are free) but not necessarily obtained in the given way is called simply a *free resolution* of M. In general one may desire to know if a given free resolution is minimal. A useful criterion is the following: it is minimal if and only if after choosing bases for the F_i and representing the homomorphisms f_i by the corresponding matrices, the matrices f_i have no entries which are non-zero scalars (i.e. the entries all lie in $\mathfrak{m}$).

A useful way of producing free resolutions is the so-called *mapping cone* procedure (cf. [82] Chapter II, section 4). The basic idea, in our context, is this: given a short exact sequence of finitely generated graded S-modules

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$$

and free resolutions

$$0 \to F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M_1 \to 0$$

and

$$0 \to G_{n+1} \xrightarrow{g_{n+1}} G_n \xrightarrow{g_n} \cdots \xrightarrow{g_1} G_0 \xrightarrow{g_0} M_2 \to 0$$

then a free resolution for M_3 is given by

$$0 \to F_{n+1} \to F_n \oplus G_{n+1} \to \cdots \to F_1 \oplus G_2 \to F_0 \oplus G_1 \to G_0 \to M_3 \to 0.$$

(We leave it as an exercise to determine the maps of this free resolution, in terms of the ones for M_1 and M_2.) Note that this is not necessarily minimal, even if the resolutions of M_1 and M_2 are. For instance the length might be too long. But even if the length is not an obstacle, it could violate the criterion for minimality mentioned above. It depends on the map α in the short exact sequence of modules.

An important construction for us will be the sheafification of a graded module (see Hartshorne [60]). It associates to any graded S-module M a quasi-coherent sheaf $\tilde{M}$. If M is finitely generated then $\tilde{M}$ is in fact coherent. Here are some useful examples:

Example 1.1.2 (a) If M is a graded module of finite length then $\tilde{M} = 0$.
(b) If Y is a closed subscheme of $\mathbb{P}^n$ and I is a homogeneous ideal which defines Y scheme-theoretically (e.g. the saturated ideal of Y– see below for the definition) then $\tilde{I} = \mathcal{I}_Y$, the ideal sheaf of Y in $\mathbb{P}^n$. Also, $\widetilde{S/I_Y} \cong \mathcal{O}_Y$, the structure sheaf of Y. □

Sheafification also preserves exactness. For instance, given a short exact sequence of graded modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we get a short exact sequence of sheaves

$$0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0.$$

For example, the exact sequence $0 \to I_Y \to S \to S/I_Y \to 0$ yields the standard exact sequence of sheaves

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0. \tag{1.1}$$

Furthermore, if we begin with a minimal free resolution

$$\begin{array}{ccccccccccccc}
0 & \to & F_{n+1} & \xrightarrow{f_{n+1}} & F_n & & \xrightarrow{f_n} & \cdots & \xrightarrow{f_1} & & F_0 & \xrightarrow{f_0} & M \to 0 \\
 & & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & \\
 & & & & & K_n & & & & K_1 & & & \\
 & & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & \\
 & & & & & 0 \qquad 0 & & & & 0 \qquad 0 & & &
\end{array}$$

we can sheafify and get a corresponding diagram for the sheaves. If M is the coordinate ring S/I of a locally Cohen-Macaulay, equidimensional subscheme of $\mathbb{P}^n$ of dimension r then $F_0 = S$, $K_1 = I$ and K_i is locally free for $i \geq n - r$. (See the next section.)

In the other direction, one can obtain from a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{F}$ a graded S-module

$$H^0_*(\mathcal{F}) \stackrel{\text{def}}{=} \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(d)).$$

Note, though, that while $\widetilde{H^0_*(\mathcal{F})} = \mathcal{F}$ the reverse is not true: $H^0_*(\tilde{M}) \neq M$ in general. (It is generally bigger than M.) Furthermore, given a short exact sequence of sheaves one of course does not necessarily obtain a short exact sequence of cohomology modules. One gets a long exact sequence involving higher cohomology. Specifically, one can obtain graded modules by applying the higher cohomology functor to $\mathcal{F}$: we define

$$H^i_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(d)).$$

These objects fit into the long exact cohomology sequence arising from the short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

yielding

$$\begin{array}{ccccccc} 0 \to & H^0_*(\mathcal{F}_1) & \to & H^0_*(\mathcal{F}_2) & \to & H^0_*(\mathcal{F}_3) & \to \\ & H^1_*(\mathcal{F}_1) & \to & H^1_*(\mathcal{F}_2) & \to & H^1_*(\mathcal{F}_3) & \to \cdots \end{array}$$

A special case of the above discussion concerns the relation between homogeneous ideals in S and closed subschemes of $\mathbb{P}^n$. Let I be a homogeneous ideal in S. Then I determines a closed subscheme Y of $\mathbb{P}^n$. However, there are infinitely many ideals which define Y, and we define "the" ideal of Y to be the largest such. Indeed, there is a bijective correspondence between closed subschemes of $\mathbb{P}^n$ and *saturated* ideals, defined as follows. Let I be a homogeneous ideal. Then the *saturation* of I is

$$\overline{I} = \{ \ F \in S \mid \forall\, i \ (0 \le i \le n) \text{ there is an } r \text{ such that } X_i^r \cdot F \in I \ \}.$$

For any ideal I, $H^0_*(\tilde{I}) = \overline{I}$ (compare with above). If I defines a closed subscheme Y then $\overline{I} = H^0_*(\mathcal{I}_Y) \supseteq I$. The Hilbert function of Y is the function $\mathbb{Z} \to \mathbb{N}$ defined by $H(Y,t) = \dim_k(S/\overline{I})_t$. For large values of t, this function is a polynomial of degree equal to the dimension of the scheme Y. This polynomial is called the *Hilbert polynomial* of Y. We refer to [60] for details.

Example 1.1.3 (a) If I is the ideal in $S = k[X_0, X_1, X_2, X_3]$ given by

$$I = (X_0^2, X_0X_1, X_0X_2, X_0X_3, X_1^2, X_1X_2, X_1X_3) = (X_0, X_1)_2$$

then $\overline{I} = (X_0, X_1)$ and I defines a line.
(b) In $\mathbb{P}^3$, let V be the four points $[0,0,0,1], [0,0,1,0], [0,1,0,0]$, $[1,0,0,0]$. The saturated ideal of V has six generators, all in degree 2. In particular, dim $(I_V)_2 = 6$. Choose four general elements F_1, F_2, F_3, F_4 of this vector space, spanning an ideal I. Then F_1, F_2, F_3 cut out the eight points of a complete intersection, four of which are the points of V, and F_4 picks out these points and avoids the other four. Hence one can see from a geometrical point of view that I defines the scheme V but is not saturated. □

An important tool is the notion of *Castelnuovo-Mumford regularity.* We first recall

Definition 1.1.4 Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then $\mathcal{F}$ is said to be *m-regular* if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$. □

Note that by a theorem of Serre ([60] Theorem III.5.2), $\mathcal{F}$ is m-regular for some m.

Theorem 1.1.5 (Castelnuovo-Mumford [104]) *Let $\mathcal{F}$ be an m-regular coherent sheaf on $\mathbb{P}^n$. Then*

(1) $H^i(\mathbb{P}^n, \mathcal{F}(k)) = 0$ whenever $i > 0$ and $k + i \geq m$.

(2) $H^0(\mathbb{P}^n, \mathcal{F}(k))$ is spanned by

$$H^0(\mathbb{P}^n, \mathcal{F}(k-1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(1)) \quad \text{if } k > m$$

In particular, the S-module $H^0_*(\mathcal{F})$ defined above is generated in degree $\leq m$.

(3) $\mathcal{F}(k)$ is generated as an $\mathcal{O}_{\mathbb{P}^n}$-module by its global sections, for all $k \geq m$.

Note that part (1) says that if $\mathcal{F}$ is m-regular then it is $(m+1)$-regular. Hence it makes sense to define the *regularity* of $\mathcal{F}$, sometimes called the *Castelnuovo-Mumford regularity*, by

$$reg(\mathcal{F}) = \min\{m | \mathcal{F} \text{ is } m\text{-regular}\}.$$

Remark 1.1.6 It is worth noting that there is a strong connection between the regularity and minimal free resolutions. Let $M = H^0_*(\mathcal{F})$ and assume that M is finitely generated as an S-module. Consider the *minimal* free resolution

$$\begin{array}{ccccccccccc} 0 \to \cdots & \longrightarrow & \bigoplus_j S(-a_{i,j}) & \longrightarrow & \cdots & \longrightarrow & \bigoplus_j S(-a_{0,j}) \to M \to 0 \\ & \searrow \nearrow & & \searrow \nearrow & & \nearrow & \\ & K_{i+1} & & K_i & & K_1 & \\ & \nearrow \searrow & & \nearrow \searrow & \nearrow & & \\ 0 & & 0 \quad 0 & & 0 & & \end{array}$$

with the corresponding short exact sequences, as described starting on page 4. Then one can check, by sheafifying and carefully following the cohomology of these short exact sequences from the resolution, that

$$reg(\mathcal{F}) = \max_{i,j}\{a_{i,j} - i\}.$$

The proof is left as an exercise. It is not trivial, but apart from a careful study of the indices, the main fact that one needs is that $H^n(\mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $k \geq -n$. $\square$

If M is a graded S-module of finite length, we define the *diameter* of M, diam M, to be the number of components from the first non-zero one to the last (inclusive). So, for example, a module which is one-dimensional in degrees 1 and 5 but zero elsewhere would have diameter 5.

An interesting invariant of a graded S-module of finite length is the least integer k such that all homogeneous polynomials of degree k annihilate M. This has been called the *Buchsbaum index* of the module M [54]. See also Remark 1.4.8, Remark 2.2.8 and Remark 3.1.4. It was observed in [99], and is not hard to show, that if $k = 2$ then M decomposes as a direct sum of modules of diameter $k = 2$. This is not true for $k \geq 3$.

1.2 The Deficiency Modules $(M^i)(V)$

In this section we introduce the deficiency modules of a subscheme of projective space, and then in the subsequent sections we give examples and first results. The deficiency modules have been very important in the literature, both in the theory of Liaison and in various papers that have appeared recently. They are the central topic of these notes. We will discuss many of these applications later. In the context of Liaison, especially for curves in $\mathbb{P}^3$, they are often referred to as the *Hartshorne-Rao module(s)* (or simply the *Rao module(s)*) of the scheme. (See Chapters 4 and 5 for more details about the role these modules play in Liaison Theory.) However, this is somewhat inappropriate away from curves in $\mathbb{P}^3$, since these modules were also heavily studied by others around the same time (or even earlier); for instance, Schenzel [120] and Evans and Griffith [46] played an important role in the theory. In the important paper [80], Lazarsfeld and Rao use the term "deficiency module" even for curves in $\mathbb{P}^3$. We maintain their terminology here.

Given a closed r-dimensional subscheme V of $\mathbb{P}^n$, we define the *deficiency modules*

$$(M^i)(V) = H^i_*(\mathcal{I}_V) \text{ for } 1 \leq i \leq r.$$

The reason for this name is that the collection of these modules measures the failure of V to be arithmetically Cohen-Macaulay (see below). The first such module (i.e. taking $i = 1$) in particular measures the failure of V to be projectively normal, i.e. the failure of the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_V(d))$$

to be surjective for all d. This would be true even if dim $V = 0$, although we have not defined the deficiency module for this case– we have modules for $1 \leq i \leq \dim V$. We will see some other ways in which this module (and certain submodules) measures the failure of V (and its ideal) to satisfy certain properties. Some of these results will in fact be true also for the case dim $V = 0$.

The module $(M^i)(V)$ can also be expressed in the following way:

$$(M^i)(V) \cong \left[\mathrm{Ext}_S^{n-i+1}(S/I_V, S)(-n-1)\right]^{\vee k} \text{ for } 1 \leq i \leq r = \dim V.$$

We leave the details to the reader. A proof for the case $n = 3$ can be found on pages 48–50 of [122], and the general case is proved similarly.

A special case of the above is when V is a curve. In this case we simply write $M(V)$ for the deficiency module.

If V has dimension $r \geq 1$, we will now describe how the collection of r deficiency modules measures the failure of V to be arithmetically Cohen-Macaulay. First recall the

Definition 1.2.1 Let I be the saturated ideal of a closed subscheme V of $\mathbb{P}^n$. Then V is *arithmetically Cohen-Macaulay (aCM)* if and only if dim S/I = depth S/I (where "dim" is the Krull dimension). In this case S/I is said to be a *Cohen-Macaulay ring.* □

From the point of view of Ext, another equivalent formulation of the aCM property is that the projective dimension (i.e. the length of a minimal free resolution of S/I) is equal to the codimension of V. (This fact also holds for dimension 0.) In general, the projective dimension is greater than or equal to the codimension, and is related to the codimension, via the depth of S/I, by the Auslander-Buchsbaum theorem (special case): pd $S/I +$ depth $S/I =$ depth S; see for instance [130] Theorem 4.4.15. In the case where S/I is a Cohen-Macaulay ring (i.e. I defines an aCM scheme), depth $S -$ depth $S/I = n + 1 -$ dim S/I is exactly the codimension; in general pd $S/I =$ depth $S -$ depth S/I is larger than the codimension since dim $S/I \geq$ depth S/I.

If dim $V = 0$, V is automatically aCM. For dim $V = r \geq 1$, the property of being aCM is equivalent to the condition that $(M^i)(V) = 0$ for $1 \leq i \leq r$. (This can be shown from the previous paragraph and the connection between the $(M^i)(V)$ and Ext above; it can also be seen directly from Definition 1.2.1, using the point of view of hyperplane sections described in the next section.) Hence we may view the $(M^i)(V)$ as measuring the failure of V to be arithmetically Cohen-Macaulay, as indicated above.

If V is aCM then the rank of the last free module in a minimal free resolution of I_V (or, equivalently of S/I_V) is called the *Cohen-Macaulay type* of V. In particular, if the Cohen-Macaulay type is 1 then V is said to be *arithmetically Gorenstein.* If V is aCM and has codimension two then the minimal free resolution for I_V is a short exact sequence:

$$\begin{array}{ccccccccccc} 0 & \to & \bigoplus_{j=1}^{r-1} S(-b_j) & \xrightarrow{A} & \bigoplus_{i=1}^{r} S(-a_i) & & \xrightarrow{B} & & S & \to & S/I_V & \to & 0 \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & I_V & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array}$$

Here $B = (F_1, \cdots, F_r)$ where the F_i are the minimal generators of I_V, and A is called the *Hilbert-Burch matrix* of I_V.

Now suppose that V is locally Cohen-Macaulay and equidimensional of dimension r. Then a minimal free resolution of S/I_V has the form

$$
\begin{array}{ccccccccccccccccc}
0 & \rightarrow & F_n & \xrightarrow{f_n} & F_{n-1} & \xrightarrow{f_{n-1}} & & \cdots & & \xrightarrow{f_1} & F_1 & \xrightarrow{f_0} & S & \xrightarrow{f_0} & S/I_V & \rightarrow & 0 \\
 & & & & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & & & \\
 & & & & & K_{n-1} & & & & K_2 & & & I & & & & \\
 & & & & \nearrow & & \searrow & & \nearrow & \searrow & & \nearrow & \searrow & & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

It then follows from the local version of the Auslander-Buchsbaum theorem that K_i is *locally* free for $i \geq n - r$.

A related and important function of the deficiency modules is that they determine whether the scheme is locally Cohen-Macaulay and equidimensional:

Theorem 1.2.2 *Assume that dim $V = r \geq 1$. Then V is locally CM and equidimensional if and only if the modules $(M^i)(V)$ have finite length for $1 \leq i \leq r$.*

This theorem can be found in [122] Theorem 9 or [64] (37.4). As a very special case, notice that if a scheme V is arithmetically Cohen-Macaulay then it is locally Cohen-Macaulay, but the converse of course does not hold. Notice also that $(M^i)(V)_j = 0$ automatically for all $j \gg 0$. (This is a theorem of Serre; see for instance [60] Theorem III.5.2 (b)). Hence V will fail to be locally Cohen-Macaulay and equidimensional if and only if one of its deficiency modules is non-zero for infinitely many components *in negative degree*.

These modules are also related to the question of connectedness. For example, we have

Theorem 1.2.3 *Let V be a closed subscheme of $\mathbb{P}^n$.*

(a) If V is reduced and connected then $(M^1)(V)_0 = 0$.

(b) If V is reduced then dim $(M^1)(V)_0 =$ (number of connected components of V) -1.

(c) If dim $(M^1)(V)_0 = 0$ (e.g. if V is arithmetically Cohen-Macaulay) then V is connected.

Proof:

(a) is Lemma 4.4 of [42]. Now consider the exact sequence

$$0 \to H^0(\mathcal{I}_V) \to H^0(\mathcal{O}_{\mathbb{P}^n}) \to H^0(\mathcal{O}_V) \to (M^1)(V)_0 \to 0.$$

If V is reduced and connected then from (a) we get that $h^0(\mathcal{O}_V) = 1$. So applying this to each connected component of V and using this exact sequence again gives (b), and similarly (c) since in any case $h^0(\mathcal{O}_V) \geq 1$. □

We will also prove later that every arithmetically Buchsbaum curve in $\mathbb{P}^3$ is connected, except for two skew lines. (See Definition 1.4.7 and Corollary 3.1.3.) This implies that every Buchsbaum scheme of higher dimension is connected, since the Buchsbaum property is preserved under hyperplane sections.

In the case of curves in $\mathbb{P}^3$, a useful connection between the deficiency (= Hartshorne-Rao) module $M(C)$ and the ideal of the curve I_C is the following theorem of Rao ([113] Theorem 2.5):

Theorem 1.2.4 *Let C be a curve in $\mathbb{P}^3$ and let $M(C)$ be its deficiency module. Let $M(C)$ have a minimal free resolution*

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \to L_2 \to L_1 \to L_0 \to M(C) \to 0.$$

Then I_C has a minimal free resolution of the form

$$0 \to L_4 \xrightarrow{(\sigma_4,0)} L_3 \oplus \bigoplus_1^r S(-l_i) \to \bigoplus_1^m S(-e_i) \to I_C \to 0$$

for some integers e_i, l_i, r, m.

Under certain circumstances (for instance if C is *minimal* in its even liaison class with respect to degree), then it can furthermore be shown that the direct summand $\bigoplus_1^r S(-l_i)$ does not occur. This is a special case of the work of Martin-Deschamps and Perrin [87], about which we will say more, especially in Chapter 5.

Finally, it is natural to ask what modules of finite length can be the deficiency module of a curve? More generally, what collection of r modules of finite length can be the deficiency modules of a closed subscheme of dimension r? The answer, due to Evans and Griffith [46], is "all" if you allow for a sufficiently large rightward shift of the modules (making the same shift for each module). Rao [113] also proved this for curves in $\mathbb{P}^3$ in a different way, and showed that the curve can be taken to be smooth. Rao's approach motivated similar constructions in [87] (again curves in $\mathbb{P}^3$) and in [19] (for surfaces in $\mathbb{P}^4$).

The question remains whether, given a collection of graded S-modules of finite length, it is necessarily the collection of deficiency modules of a scheme of appropriate dimension (without shifting). The answer is "no." The point (which is an important ingredient in the structure theorem for codimension two even liaison classes, as we will see) is that there is a "leftmost" shift of the modules for which they form the deficiency modules of some scheme. We will see in Chapter 3, with Basic Double Linkage, that once the collection of modules actually is the set of deficiency modules of a scheme then any rightward shift is also the collection of deficiency modules of some scheme. This is not true for leftward shifts, thanks to the following

Proposition 1.2.5 ([25]) *Given a collection $\{M_1, \cdots, M_r\}$ of graded S-modules of finite length ($1 \leq r \leq n-2$), there is a scheme X of dimension r with the following properties:*

(a) There is an integer d such that $(M^i)(X) = M_i(d)$ for all $1 \leq i \leq r$.

(b) If Y is a scheme of dimension r with $(M^i)(Y) = M_i(e)$ for $1 \leq i \leq r$ then $d \geq e$.

Note that the integers d and e above are the same for all i; that is, the modules in the collection are all shifted together.

(Recall that $M(d)$ is a shift d places to the left, for d positive.) We will not prove this proposition here (but see Remark 1.3.3 (c) where we prove it in the case of curves); however, the main idea for the general case is to reduce to the case of a curve, by taking hyperplane sections.

In the context of curves, this says that once the module is known (up to shift) then there is a lower bound on the degree in which it can start.

In Remark 1.3.3 (c) we give the details. It is given purely in terms of the dimensions of the components of the module. Sharp examples can be given (e.g. a double line in $\mathbb{P}^4$; cf. [92]), but it is not sharp for every module. Martin-Deschamps and Perrin have given a more detailed analysis [87] of the leftmost shift attained by any given module, for curves in $\mathbb{P}^3$.

Another interesting problem is to bound the module (in various senses) in terms of the degree and (arithmetic) genus of the curve. For curves in $\mathbb{P}^3$, some useful results can be found in the paper of Martin-Deschamps and Perrin [89]. Here they give explicit bounds on the dimensions of the components of the deficiency module in terms of the degree d and (arithmetic) genus g of the curve $C \subset \mathbb{P}^3$. They do not assume that the curve is reduced or irreducible. In particular, they give a lower bound r_a for the degree in which the deficiency module may start, and an upper bound r_o for the degree in which it may end, and they give a bound for the dimension of any component. Specifically, they show

$$r_a \geq g + 1 - \frac{(d-2)(d-3)}{2} \qquad r_o \leq \frac{d(d-3)}{2} - g$$

and if C is not a plane curve then

$$\forall n \in \mathbb{Z},\ h^1(\mathcal{I}_C(n)) \leq \frac{(d-2)(d-3)}{2} - g.$$

Again, sharp examples exist ([89] gives an example; double lines [92] provide another), but these bounds are not sharp in general.

In the case of integral curves in $\mathbb{P}^n$, $r_a \geq 1$ (cf. [42] and Theorem 1.2.3) and $r_o \leq d - n$ (cf. [56]). (In fact, in [56] it is shown that $r_o = d - n$ if and only if C is smooth and rational, and either $d = n + 1$ or else $d > n + 1$ and C has a $(d + 2 - n)$-secant line.)

1.3 Hyperplane and Hypersurface sections

Given a subscheme $V \subset \mathbb{P}^n$ with defining (saturated) ideal I_V, and given a general homogeneous polynomial F of degree d, we would like to define the subscheme $Z = V \cap F$ cut out by F. We can view it as a subscheme of $\mathbb{P}^n$, or as a subscheme of F (thought of as a hypersurface). By abuse of notation, we will use the letter F for both the hypersurface and the polynomial. If $d = 1$ we can also view $Z = V \cap F$ as a subscheme of $\mathbb{P}^{n-1}$.

Here "define $V \cap F$" means to give the *saturated* ideal, either in S or in $S/(F)$. The natural "guess" is to form the ideal $I_V + (F) \subset S$, or $\frac{I_V+(F)}{(F)} \subset S/(F)$. These aren't quite right: in general we need to take the saturation of these ideals. (In Chapter 2 we will discuss "how far" these ideals are from already being saturated, and show that the failure is measured by a submodule of the first deficiency module.)

So we define the hypersurface section Z of V by F in $\mathbb{P}^n$ to be the scheme with saturated ideal

$$I_Z = \overline{I_V + (F)}, \tag{3.1}$$

and in the hypersurface F we have the saturated ideal

$$I_{Z|F} = \overline{\left(\frac{I_V + (F)}{(F)}\right)} \tag{3.2}$$

in the ring $S/(F)$. If $d = 1$, this is called the *hyperplane section* of V by F.

How are these related to each other and to I_V? Note that

$$\frac{I_V + (F)}{(F)} \cong \frac{I_V}{I_V \cap (F)} = \frac{I_V}{F \cdot I_V}.$$

(The isomorphism is standard (cf. [7] p. 19). The equality is by generality of F: suppose that

$$I_V = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_r$$

is the primary decomposition of I_V in S. Since I_V is saturated, $\mathfrak{m}$ is not an associated prime. Choose F so that it is not in any associated prime $\mathcal{P}_i$. Now, if $FG \in I_V \cap (F)$, it is in each primary ideal $\mathcal{Q}_i$. But no power of F is in $\mathcal{Q}_i$, so $G \in \mathcal{Q}_i$ for each i, hence $G \in I_V$.)

As a result, we get two useful exact sequences of sheaves:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\times F} \mathcal{I}_Z \to \mathcal{I}_{Z|F} \to 0 \tag{3.3}$$

(from sheafifying $0 \to (F) \to I_V + (F) \to \frac{I_V+(F)}{(F)} \to 0$) and

$$0 \to \mathcal{I}_V(-d) \xrightarrow{\times F} \mathcal{I}_V \to \mathcal{I}_{Z|F} \to 0 \tag{3.4}$$

(from sheafifying $0 \to I_V(-d) \xrightarrow{\times F} I_V \to \frac{I_V}{F \cdot I_V} \to 0$).

Remark 1.3.1 (a) Notice that (3.3) implies

$$H^i_*(\mathcal{I}_Z) \cong H^i_*(\mathcal{I}_{Z|F}) \text{ for } 1 \le i \le n-2. \tag{3.5}$$

We will use this fact several times in this chapter and the next.
(b) From (3.3) we also have the short exact sequence

$$0 \to S(-d) \xrightarrow{\times F} I_Z \to I_{Z|F} \to 0.$$

Hence one can verify that there is a one-to-one correspondence between the minimal generators of $I_{Z|F}$ and those of I_Z other than (possibly) F itself.
(c) We now claim that F is a minimal generator of I_Z. Indeed, first let V_1 be an irreducible component of V, of degree d_1, corresponding to a primary ideal $\mathcal{Q}_1$ with associated prime $\mathcal{P}_1$. Let $Z_1 \subset Z$ be the hypersurface section of V_1 cut out by F. By assumption, dim $Z_1 =$ dim $V_1 - 1$ and deg $Z_1 = dd_1$.

Now suppose that $F = \sum X_i F_i$ where $F_i \in I_Z$. If $F_i \notin \mathcal{P}_1$ then F_i cuts V_1 in a subscheme Z' of dimension dim Z_1 and degree (deg $F_i)(d_1) < dd_1$. But on the other hand, $F_i \in I_Z \subset I_{Z_1}$ so $Z_1 \subset Z'$ (exercise). But then deg $Z_1 \le$ deg Z', contradicting the degree calculations above. Thus $F_i \in \mathcal{P}_1$.

The same is true of all the other components of V. That is, F_i vanishes on the support of each component of V. This holds for all i, so F also vanishes on the support of each component of V. This contradicts the generality of F. □

Now that we have the notion of a hyperplane or hypersurface section, it is natural to ask what properties are passed from a closed subscheme V of $\mathbb{P}^n$ to its (general) hyperplane or hypersurface section, and conversely what properties of the general hyperplane or hypersurface section force a conclusion about V. In some sense the most interesting aspect of this question is the case when V is a curve, and we will discuss this more in Chapter 2. However, now we give a result for higher dimension. This is also a special case of a result in [72].

Theorem 1.3.2 *Let V be a locally Cohen-Macaulay, equidimensional closed subscheme of $\mathbb{P}^n$ and let F be a general homogeneous polynomial of degree d cutting out on V a scheme $Z \subset V \subset \mathbb{P}^n$. Assume that dim $V \ge 2$. Then V is arithmetically Cohen-Macaulay if and only if Z is.*

Proof:

Consider the exact sequence in cohomology obtained from (3.4):

$$\cdots \to H^i(\mathcal{I}_V(t-d)) \xrightarrow{\times F} H^i(\mathcal{I}_V(t)) \to H^i(\mathcal{I}_{Z|F}(t))$$

$$\to H^{i+1}(\mathcal{I}_V(t-d)) \xrightarrow{\times F} H^{i+1}(\mathcal{I}_V(t)) \to \cdots$$

where $1 \leq i \leq (\dim V)-1$. Notice that Z has dimension equal to $(\dim V)-1$. If V is aCM then $(M^i)(V) = 0$ for all $1 \leq i \leq \dim V$ so $H^i_*(\mathcal{I}_{Z|F}) = 0$ for all $1 \leq i \leq \dim V - 1$. But then by (3.5) $H^i_*(\mathcal{I}_Z) = 0$ for $1 \leq i \leq \dim Z$, so Z is aCM.

Conversely, if Z is aCM then $H^i(\mathcal{I}_V(t-d)) \xrightarrow{\times F} H^i(\mathcal{I}_V(t))$ is surjective for all t and $H^{i+1}(\mathcal{I}_V(t-d)) \xrightarrow{\times F} H^{i+1}(\mathcal{I}_V(t))$ is injective for all t. This is impossible unless V is aCM, since $(M^i)(V)$ has finite length by Theorem 1.2.2. □

It is interesting to note that Chang has shown in [34] that at least in the case where V has codimension two and $d = 1$, this theorem does not hold if "aCM" is replaced by "Buchsbaum" (defined in §4 of this chapter). Indeed, she gives an example of a non-Buchsbaum 3-fold whose general hyperplane section is Buchsbaum. (This is the only direction in which such an example is possible– as we will see, the general hyperplane or hypersurface section of a Buchsbaum scheme is again Buchsbaum.)

Note also that without the assumption that V is locally Cohen-Macaulay and equidimensional, Theorem 1.3.2 would not be true. For instance, V could be the union of an aCM surface and a point.

Remark 1.3.3 (a) Notice that if V is aCM then in particular we have

$$I_V / [I_V(-d)] \cong I_{Z|F} \cong I_Z/(F).$$

The second isomorphism comes from (3.3) and is true regardless of whether or not V is aCM. The first isomorphism comes from (3.4), and it implies that the generators of $I_{Z|F}$ are in bijective (degree preserving) correspondence with those of I_V, and also that properties such as regular sequences are preserved (in general). Of course one has to be careful: for instance, if V is a twisted cubic in $\mathbb{P}^3$, there is a regular sequence of two quadrics Q_1 and Q_2 containing V, and the complete intersection of these two quadrics is the union of V and a line λ. Suppose $d = 1$. For a general F, the images of Q_1 and Q_2 in $I_{Z|F}$ form a regular sequence in $S/(F)$. However, if F is a plane which contains λ then these images have a common component (namely λ itself) and so do not form a regular sequence.

(b) Many strong results are known about the *general* hyperplane section of an *integral* curve, especially from the point of view of the Hilbert function or the resolution of the points of the hyperplane section. The main results say that the general hyperplane section is a set of points such that two subsets of these points, of the same cardinality, are indistinguishable numerically– they have the same Hilbert function (i.e. they have the *Uniform Position Property*), and in fact the same graded Betti numbers (i.e. they have the *Uniform Resolution Property*). See for instance [41], [51], [57], [58], [59], [84], [119] among others. See also §5.3.

(c) We will now prove Proposition 1.2.5 for the case of curves (for simplicity). This was proved in [91] and independently in [122] by a different method. Specifically, we claim first that for any $d \leq 0$, $\dim M(C)_{d-1} \leq \dim M(C)_d$. (But see (d) for a stronger statement.)

Let $C \subset \mathbb{P}^n$ be a locally Cohen-Macaulay equidimensional curve. Let H be a hyperplane not containing any component of C. Following convention, we will denote by $\mathcal{I}_{C\cap H}$ the ideal sheaf of the points of the hyperplane section in the projective space $H = \mathbb{P}^{n-1}$. (Above this was denoted by $\mathcal{I}_{Z|F}$.) Then for any $d \leq 0$, $H^0(\mathcal{I}_{C\cap H}(d)) = 0$ so we have the exact sequence (from (3.4))

$$0 \to H^1(\mathcal{I}_C(d-1)) \to H^1(\mathcal{I}_C(d)) \to \cdots$$

It follows that $h^1(\mathcal{I}_C(d-1)) \leq h^1(\mathcal{I}_C(d))$. On the other hand, $M(C)$ has finite length so it has a last non-zero component. What the above argument shows is that this last component cannot be in negative degree. Hence there is a leftmost possible shift for which $M(C)$ is the deficiency module of some curve. (Of course C itself may not achieve this bound.) This idea was explored further in [91] and in [25]. Furthermore, it was shown in [11] (in the generality of codimension two subschemes of projective space), [87] (for curves in $\mathbb{P}^3$) and [27] (for codimension two subschemes of a smooth Gorenstein variety), that the schemes for which this bound is achieved are very special from the point of view of Liaison. They are the so-called *minimal* schemes of the even liaison class. We will discuss this much more in Chapter 5.

(d) In fact, it was shown in [91] that if $M(C)$ has components in negative degree then we must have $\dim M(C)_i < \dim M(C)_{i+1}$ for all $i \leq -1$. $\square$

1.4 Examples

In this section we give a number of examples to illustrate the ideas in the preceding three sections.

Example 1.4.1 *Complete intersections*

If V is a subscheme of $\mathbb{P}^n$ of codimension r then the saturated ideal of V clearly has at least r generators. If the number of generators is equal to the codimension then V is said to be a *complete intersection*. From the algebraic point of view, V is a complete intersection if and only if the saturated ideal $I_V = (F_1, \cdots, F_r)$ where $(F_1, \cdots, F_r)$ is a regular sequence. (In fact, it's not hard to show that a regular sequence always defines a saturated ideal.)

The minimal free resolution of a complete intersection, known as the Koszul resolution, is particularly simple. It basically says that the only relations are the "obvious" ones, $F_iF_j - F_jF_i = 0$, and similarly for the second and higher syzygies (see page 4). Formally, let $d_i = \deg F_i$. Then the minimal free resolution is given by

$$\cdots \xrightarrow{\phi_3} \bigwedge^2 \Big(\bigoplus_{1 \leq i \leq r} S(-d_i) \Big) \xrightarrow{\phi_2} \bigoplus_{1 \leq i \leq r} S(-d_i) \xrightarrow{\phi_1} I_V \rightarrow 0$$

where

$$\begin{aligned} \phi_2[(f_1, \cdots, f_r) \wedge (g_1, \cdots, g_r)] &= [\phi_1(f_1, \cdots, f_r)](g_1, \cdots, g_r) \\ &\quad - [\phi_1(g_1, \cdots, g_r)](f_1, \cdots, f_r). \end{aligned}$$

If we use the standard bases for $\bigwedge^2(\bigoplus_{1 \leq i \leq r} S(-d_i))$ and $\bigoplus_{1 \leq i \leq r} S(-d_i)$ we see that ϕ_2 is represented by an $r \times \binom{r}{2}$ matrix, each of whose entries is either zero or an F_i (up to ± 1). Similarly, the ith free module occurring in this resolution ($1 \leq i \leq r$) has rank $\binom{r}{i}$ and the matrices all have entries which are either zero or an F_i (up to ± 1). Notice that in particular V is arithmetically Gorenstein (see page 11).

In particular, the projective dimension of I_V is equal to the codimension, r, so V is aCM. Hence the deficiency modules of V are zero (unless dim $V = 0$, in which case there are no deficiency modules, by definition).

The degree of a complete intersection is simply the product of the degrees of the defining polynomials (by Bezout's theorem). In the case of a curve, the arithmetic genus g is

$$\frac{1}{2}(\prod d_i)(\sum d_i - (n+1)) + 1.$$

To see this, one can for instance take the general case first. Take each hypersurface section to be smooth, and so one can use the adjunction formula ([55] p. 147) to compute $2g-2$, and hence g. But the arithmetic genus depends only on the Hilbert polynomial, which is determined by the graded Betti numbers of the resolution. These depend only on the degrees d_i, and not on smoothness. Hence this value of g is correct for any such complete intersection curve.

A special case of a complete intersection is a hypersurface (assuming that it is locally Cohen-Macaulay and equidimensional) since in this case the ideal is principal (hence isomorphic to $S(-d_1)$– this was used in §3 to get (3.3)) and the codimension is 1.

In codimension two, say $I_V = (F_1, F_2)$ is a regular sequence (i.e. F_1 and F_2 have no common component). Then the Koszul resolution for I_V is given by

$$0 \to S(-d_1-d_2) \xrightarrow{\begin{bmatrix} F_2 \\ -F_1 \end{bmatrix}} S(-d_1) \oplus S(-d_2) \xrightarrow{[F_1 \ F_2]} I_V \to 0.$$

The case of higher codimension proceeds analogously. □

We now pass to the case of curves and compute some deficiency modules.

Example 1.4.2 Let C be a set of two skew lines λ_1 and λ_2 in $\mathbb{P}^n$. Consider the exact sequence

$$0 \to H^0(\mathcal{I}_C(i)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(i)) \to H^0(\mathcal{O}_C(i)) \to M(C)_i \to 0.$$

For $i < 0$ we get that $M(C)_i = 0$ and for $i = 0$ we get dim $M(C)_0 = 1$. On the other hand, for $i \geq 1$ we claim that C imposes $2i+2$ independent conditions on hypersurfaces of degree i. Indeed, choose $i+1$ distinct points on each of the two lines λ_1 and λ_2. Clearly a hypersurface of degree i contains C if and only if it contains all of these $2i+2$ points. But excluding any one of these points, it is a simple exercise to find a union of i hyperplanes which contains all of the remaining points.

Hence $h^0(\mathcal{I}_C(i)) = \binom{i+n}{n} - (2i+2)$ while of course $h^0(\mathcal{O}_{\mathbb{P}^n}(i)) = \binom{i+n}{n}$ and $h^0(\mathcal{O}_C(i)) = 2i+2$. Therefore $M(C)_i = 0$ for all $i \neq 0$. In particular, $M(C)$ is non-zero but it is annihilated by the maximal ideal $\mathfrak{m}$ of S. Because of this, C is an example of a so-called *Buchsbaum* curve, which we will discuss shortly (and in Chapter 3). □

Example 1.4.3 Let C be a smooth rational quartic curve in $\mathbb{P}^3$. Consider again the exact sequence

$$0 \to H^0(\mathcal{I}_C(i)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(i)) \to H^0(\mathcal{O}_C(i)) \to M(C)_i \to 0.$$

By Riemann-Roch, $h^0(\mathcal{O}_C(i)) = 4i+1$ for $i \geq 0$. On the other hand, $h^0(\mathcal{I}_C(i)) = 0$ for $i \leq 1$ so this exact sequence gives $M(C)_i = 0$ for $i \leq 0$ and $\dim M(C)_1 = 1$.

Now let H be a general hyperplane and consider the exact sequence

$$0 \to H^0(\mathcal{I}_C(i-1)) \to H^0(\mathcal{I}_C(i)) \to H^0(\mathcal{I}_{C\cap H}(i)) \to M(C)_{i-1} \to$$
$$\to M(C)_i \to H^1(\mathcal{I}_{C\cap H}(i)) \to H^2(\mathcal{I}_C(i-1)) \to \cdots$$

where $\mathcal{I}_{C\cap H}$ is the ideal sheaf of the hyperplane section in the hyperplane H. (This is the sequence (3.4) of §3 of Chapter 1.) Observe that $C \cap H$ is a set of four points in the plane with no three on a line. It is in fact a complete intersection, so the cohomology of $\mathcal{I}_{C\cap H}$ is not hard to compute (from the Koszul resolution, which is a short exact sequence since the codimension is two). In particular, $h^1(\mathcal{I}_{C\cap H}(i)) = 0$ for $i \geq 2$ (which gives us $h^0(\mathcal{I}_{C\cap H}(i)) = \binom{i+2}{2} - 4$ for $i \geq 2$). Furthermore, C has degree four and arithmetic genus 0 so it is not a complete intersection– from the formula in Example 1.4.1 we know that the complete intersection of two quadrics in $\mathbb{P}^3$ would have arithmetic genus 1. Therefore, since C is smooth, it can lie on only one quadric and $h^0(\mathcal{I}_C(2)) = 1$. Then setting $i = 2$ in the above exact sequence gives $M(C)_2 = 0$, and proceeding to $i = 3, 4, \cdots$ gives that $M(C)$ is zero in every degree other than 1, and it is one-dimensional in degree 1.

Therefore a rational quartic has the same module as does a set of two skew lines, but shifted. This will be very relevant when we discuss Liaison Theory. Notice, incidentally, that there are deeper (but quicker) reasons why $M(C)_i = 0$ for $i \geq 2$. One could apply the theory of Liaison, which we will discuss later. Alternatively, one could use the main theorem of [56] (which says, in the special case of a reduced, irreducible curve in $\mathbb{P}^3$, that $M(C)_i = 0$ for $i \geq \deg C - 2$), or the fact [66] that a general rational curve C in $\mathbb{P}^n$ of degree d has *maximal rank*, i.e. that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(i)) \to H^0(\mathcal{O}_C(i))$ has maximal rank for all i. (Using the exact sequence (1.1) and taking cohomology, one sees that this is equivalent to the condition that $h^0(\mathcal{I}_C(i)) \cdot h^1(\mathcal{I}_C(i)) = 0$ for all i.) □

Example 1.4.4 Let C be the disjoint union in $\mathbb{P}^3$ of a line λ and a plane curve Y of degree d (so $I_C = I_\lambda \cap I_Y$). A special case is the disjoint union of two lines, which we discussed from a geometric point of view above. Now we show how to derive the deficiency module in a more algebraic way. We have an exact sequence

$$0 \to I_C \to I_Y \oplus I_\lambda \to I_Y + I_\lambda \to 0$$

and we can sheafify it and take cohomology to obtain

$$\begin{array}{ccccccccccc} 0 & \to & I_C & \to & I_Y \oplus I_\lambda & & \longrightarrow & & S & \to & M(C) & \to & 0 \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & I_Y + I_\lambda & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array}$$

(Note that $H^0_*(\widetilde{I_Y + I_\lambda}) = S$ since Y and λ are disjoint– use the Nullstellensatz. Also, in particular λ and Y are complete intersections so as we have seen, $H^1_*(\mathcal{I}_Y \oplus \mathcal{I}_\lambda) = 0$.) Notice that the ideal $I_Y + I_\lambda$ has three (independent) generators in degree 1. Hence $M(C) \cong k[x]/(F)$ where $F \in k[x]$ has degree d. In particular,

$$\dim M(C)_i = \begin{cases} 1 & \text{if } 0 \leq i \leq d-1; \\ 0 & \text{otherwise.} \end{cases}$$

Consider for instance the case $d = 2$. Here the module is one-dimensional in each of degrees 0 and 1, and zero otherwise. From the point of view of

§1, what is the module structure? We have a homomorphism $\phi_0 : S_1 \to Hom(M(C)_0, M(C)_1)$ which we view as a 1×1 matrix whose entry is a linear form in the dual variables. Either ϕ_0 is the zero homomorphism, or else the degeneracy locus V_0 is a plane in the dual projective space. We now check that the latter is in fact the correct conclusion, and we determine from a geometric point of view the relation between this plane and the curve C.

Let H be a general hyperplane determined by a linear form L, and consider multiplication by L on the deficiency module $M(C)$. Consider the exact sequence (3.4) from §3 (where now for convenience we denote by $\mathcal{I}_{C\cap H}$ the sheaf $\mathcal{I}_{Z|F}$):

$$0 \to H^0(\mathcal{I}_C) \to H^0(\mathcal{I}_C(1)) \to H^0(\mathcal{I}_{C\cap H}(1)) \to M(C)_0 \xrightarrow{\phi_0(L)} M(C)_1 \to \cdots .$$

As long as H meets the curve in finitely many points, we have that rank $\phi_0(L) = 0$ if and only if $h^0(\mathcal{I}_{C\cap H}(1)) = 1$ if and only if the three points of intersection of C with H are collinear, i.e. if and only if H passes through the point P of intersection of λ with the plane of Y. So the degeneracy locus V_0 is the plane in the dual projective space which is dual to the point P. Notice that initially we only considered planes meeting C properly. However, since the degeneracy locus is closed we get in this case that any linear form L vanishing on a component of C also gives a rank 0 homomorphism on $M(C)$, simply because it also vanishes at P. See also Examples 4.4.1 and 4.4.2.

One can also check that this plane determines the module structure of $M(C)$ (although in general the degeneracy loci are not enough). A similar analysis can be done with the case $d \geq 3$ as well, with the same conclusions about V_0. □

Example 1.4.5 Suppose, in the last example, that we had let λ and Y meet. If they lie in the same plane then C is a plane curve, hence a hypersurface and so arithmetically Cohen-Macaulay. The only other possibility is that they do not lie in the same plane and meet in exactly one point P. We claim that this curve is also aCM.

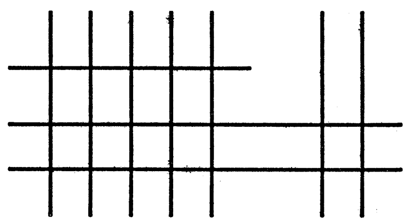
In this case $H^0_*(\widetilde{I_Y + I_\lambda}) = I_Y + I_\lambda = I_P$. One can see this since $I_Y \subset I_P$ and $I_\lambda \subset I_P$ so $I_Y + I_\lambda \subset I_P$. But on the other hand, I_P is generated by three independent linear forms and $I_Y + I_\lambda$ contains three independent linear forms, so they agree in degree 1 (we already have one inclusion) and so $I_P \subset I_Y + I_\lambda$.

Therefore we get the exact diagram

$$\begin{array}{ccccccccccc}
0 & \to & I_C & \to & I_Y \oplus I_\lambda & & \longrightarrow & & I_P & \to & M(C) & \to & 0 \\
 & & & & & \searrow & & \nearrow & & & & & \\
 & & & & & & I_P & & & & & & \\
 & & & & & \nearrow & & \searrow & & & & & \\
 & & & & 0 & & & & 0 & & & &
\end{array}$$

so $M(C) = 0$ and C is aCM as claimed. □

Example 1.4.6 The simplest case in Example 1.4.4 is the disjoint union of two lines in $\mathbb{P}^3$, which we already discussed, and the second simplest is the disjoint union of a line and a conic. The latter has deficiency module which has dimension 1 in each of two components, and the module structure is non-trivial. (The multiplication from the first nonzero component to the second, induced by a general linear form, is an isomorphism.) For comparison, we give an example of a curve with the same module dimensions as this one, but different structure.



Let C be the union of lines in the above picture. It is understood that lines which do not intersect in the picture in fact do not intersect. One can verify from geometry that three "horizontal" lines and the five leftmost "vertical" lines lie on a smooth quadric surface, and that the last two lines do not lie on this surface. Then one can give elementary but tedious arguments to show that this curve has deficiency module which is one-dimensional in degrees 2 and 3, and 0 elsewhere. (This curve was constructed using Liaison Addition, which we will describe in Chapter 3. These dimensions come for free as a result of that theorem.) Also, it is not hard to check that the curve lies on no quadric or cubic surfaces, by geometrical considerations. However, the general hyperplane section of this curve does lie on a cubic plane curve. Then from the exact sequence

$$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_C(3)) \to H^0(\mathcal{I}_{C\cap H}(3)) \to M(C)_2 \xrightarrow{\phi_0(L)} M(C)_3 \to \cdots$$

we get that the general linear form, and hence arbitrary linear form, induces the zero homomorphism on the deficiency module. □

The curve in Example 1.4.6 is the simplest non-trivial example of an arithmetically Buchsbaum curve. Specifically:

Definition 1.4.7 A curve $C \subset \mathbb{P}^n$ is *arithmetically Buchsbaum* (or simply *Buchsbaum*) if its deficiency module is annihilated by the maximal ideal $(X_0, \cdots, X_n)$ of S. □

By "simplest non-trivial example" we mean that the curve of Example 1.4.6 is a curve of least degree which is Buchsbaum and which has a deficiency module which is non-zero in at least two degrees. (Obviously if the module is non-zero in at most one degree then the curve must be Buchsbaum for "trivial" reasons.) The minimality of this curve is not obvious: it is a consequence of the Lazarsfeld-Rao property which will be described later. See also [22], [23] and [49]; some relevant facts from these papers will be discussed in Chapter 3. Of course this curve is not the unique curve with this minimality property, since there is some flexibility in the lines chosen for the components of the curve.

Note that if C is Buchsbaum then its deficiency module, *as a graded S-module*, is a direct sum of twists of copies of the field k; that is, the module structure is just that of a k-vector space. So, for example, the resolution of such a module is the direct sum of twists of copies of the resolution of k, and so is easy to write down. We will use this fact in the proof of Corollary 2.2.6.

There is also a notion of arithmetically Buchsbaum subschemes of higher dimension, which is not quite what one would expect. The analogous definition, extending the one just given, is an inductive one: a scheme V of dimension $r \geq 2$ is arithmetically Buchsbaum if all of its deficiency modules (in the range $1 \leq i \leq r = \dim V$) are annihilated by the maximal ideal, *and* if its general hyperplane section is arithmetically Buchsbaum. (See [126] for an excellent overview of the theory of Buchsbaum rings, and see [32] and [33] for a description of the arithmetically Buchsbaum schemes in codimension two.) A scheme for which the deficiency modules are all annihilated by the maximal ideal, without the assumption about the general hyperplane section, is said to be *quasi-Buchsbaum*. This is not sufficient for the scheme to be Buchsbaum, since examples exist of schemes whose deficiency modules are annihilated by the maximal ideal, but for which the deficiency modules of the general hyperplane section are *not* annihilated by the maximal ideal (cf. for instance [103]).

One can check, though, that an r-dimensional scheme $V \subset \mathbb{P}^n$ is Buchsbaum if and only if its deficiency modules are all annihilated by the maximal ideal, *and* the same is true for the deficiency modules of the scheme $V \cap \Lambda$, obtained by intersecting V with a general linear space Λ of any dimension k satisfying $n - r + 1 \leq k \leq n - 1$.

In these notes, for the most part the only Buchsbaum schemes that we will consider are Buchsbaum curves.

Buchsbaum curves can be viewed in many ways as a generalization of the aCM curves. Certainly from the module point of view it is clear that an aCM curve is Buchsbaum. There are also some other ways that come up, especially for curves in $\mathbb{P}^3$, which we will discuss in Chapter 3 once we prove a striking result of Amasaki (Corollary 2.2.6) relating the k-dimension of the deficiency module and the least degree of a surface containing the curve. We will also discuss a surprising fact about Buchsbaum curves in the last chapter: every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure (i.e. a union of lines with at most double points). This will be an application of the Lazarsfeld-Rao property and Liaison Addition. (This is a partial answer to a classical question: does every smooth curve in $\mathbb{P}^3$ specialize to a stick figure?)

Remark 1.4.8 As mentioned at the end of §1, it is interesting to study the Buchsbaum index of a graded module M of finite length, i.e. the least degree k such that all forms of degree k annihilate M. If M is the deficiency module of a curve C, we also say that C is *k-Buchsbaum*. As a special case, a Buchsbaum curve is 1-Buchsbaum (and an arithmetically Cohen-Macaulay curve is 0-Buchsbaum). Notice that every curve is k-Buchsbaum for some k. k-Buchsbaum curves (and generalizations to higher dimensions) have been studied extensively. See for instance [8], [9], [29], [43], [54], [67], [68], [69], [99]. (This is not intended to be a complete list.) □

Example 1.4.9 One can use the computer program "Macaulay" [14] to compute the deficiency module of a subscheme of projective space. For example, the following script computes (not necessarily in the most efficient way) the deficiency module of a curve in projective space, using scripts already written by D. Eisenbud and M. Stillman. The name of the script is def_mod.

```
incr-set prlevel 1
if #0=2 START
HERE:
incr-set prlevel -1
;;; Usage:
;;;  <def_mod   I   def
;;;
;;;   Computes the deficiency module of I.
;;;   This assumes that I is an ideal that defines a variety
;;;   of dimension one. The script returns the deficiency
;;;   module in def. The deficiency module is the first
;;;   cohomology module of the ideal sheaf.
;;;
incr-set prlevel 1
jump END
ERONE:
shout echo The ideal needs to define a one dimensional variety
kill @zz @I @i @tt
jump HERE
ERTWO:
shout echo The first entry needs to be an ideal
kill @zz
jump HERE
;;; Parameters and output values: I is an ideal.
;;;                    def is a presentation for a module.
;;;
;
```

```
START:
nrows #1 @zz
if (@zz<1) ERTWO
if (@zz>1) ERTWO
std #1 @I

nvars @I @i
codim @I @zz
int @tt @i-@zz-1
if (@tt>1) ERONE
if (@tt<1) ERONE

int @j @i-1
<ext(-,R) @j @I @C
std @C @CC
<ext(-,R) @i @CC @D
std @D @DD
copy @DD #2
std #2 #2
<prune #2 #2
std #2 #2

kill @zz @I @tt @i @j @C @CC @D @DD
END:
incr-set prlevel -1
```

A quick way to read the dimensions of the components of the module is to use hilb, as illustrated by the following simple Macaulay session (compare with Example 1.4.4):

```
% <ring 4 wxyz R

% <ideal line w x

% <ideal plane_quintic y z5
```

```
% intersect line plane_quintic curve
; 0.[126k]1.2..3..4..5..6..
; computation complete after degree 6

% std curve curve
; 23.4.5.6.7.
; computation complete after degree 7

% <def_mod curve def

% hilb def

;        1 t  0
;       -3 t  1
;        3 t  2
;       -1 t  3
;       -1 t  5
;        3 t  6
;       -3 t  7
;        1 t  8

;        1 t  0
;        1 t  1
;        1 t  2
;        1 t  3
;        1 t  4

; codimension = 4
; degree      = 5
; genus       = 6
```

Here, `line` is the ideal of a line, `plane_quintic` is the ideal of a (nonreduced) plane quintic disjoint from `line`, `curve` is their union and `def` is the deficiency module. To read off the dimensions and degrees of the nonzero components simply look at the part of the printout above `codimension`: this module is one-dimensional in each of degrees 0,1,2,3 and 4.

Many extremely useful scripts have been produced by Eisenbud, Stillman and others, and are available with the program "Macaulay." □

Chapter 2

Submodules of the Deficiency Module

In this chapter we will define certain submodules of the first deficiency module of a closed subscheme Z of $\mathbb{P}^n$, and see how they give a more refined measure of the failure of the ideal of Z to have certain nice properties. We then describe some applications of these submodules, from the literature.

2.1 Measuring Deficiency

Recall that we defined the first deficiency module of a closed subscheme V of $\mathbb{P}^n$ to be $H^1_*(\mathcal{I}_V)$. If dim $V = 1$ then this is unambiguously called *the* deficiency module of V. Since this chapter deals only with this particular module (and submodules thereof), we denote it simply by $M(V)$. In section I.2 we saw one reason for the name "deficiency module": it measures the failure of V to be projectively normal. We now define a submodule of this module, which will be one of the main objects of study in this chapter, and we will see other reasons for the name. For now we do not require that V be locally Cohen-Macaulay or equidimensional, and we allow dim $V = 0$ (although in this case $M(V)$ will not have finite length).

Recall that if M is a graded S-module, $A \subset M$ a submodule and $J \subset S$ an ideal then by definition

$$[A :_M J] = \{\ m \in M \mid F \cdot m \in A \text{ for all } F \in J\ \}.$$

This is a submodule of M. If $M = S$ then A is an ideal and we often write $[A : J]$ for $[A :_S J]$. this is called an *ideal quotient.* Now the submodule referred to above is given by

Definition 2.1.1 Let $F \in S_d$ be a homogeneous polynomial of degree d. Then we define $K_F = [0 :_{M(V)} (F)]$. □

In the special case where V is a curve in $\mathbb{P}^n$ and deg $F = 1$, K_F is a key component in a formula for the arithmetic genus of V; see page 112.

Proposition 2.1.2 *Let $F \in S_d$ be a general homogeneous polynomial of degree d. Then*

(a) K_F has finite length.

(b) If $J = \dfrac{I_V}{F \cdot I_V} \cong \dfrac{I_V + (F)}{(F)}$ *then* $K_F(-d) \cong \dfrac{I_{Z|F}}{J} \cong \dfrac{I_Z}{I_V + (F)}$.

Proof: Let Z be the subscheme cut out on V by F and consider the exact sequence in cohomology obtained from (3.4) of Chapter 1

$$\begin{array}{ccccccccccc}
0 & \to & I_V(-d) & \xrightarrow{\times F} & I_V & \to & I_{Z|F} & \longrightarrow & M(V)(-d) & \xrightarrow{\times F} & M(V) \\
 & & & & & & & \searrow \quad \nearrow & & & \\
 & & & & & & & K_F(-d) & & & \\
 & & & & & & & \nearrow \quad \searrow & & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

Since $I_{Z|F}$ is an ideal, $(K_F)_i$ is zero for $i \leq 0$, and it is zero for $i \gg 0$ since this is true of $M(V)_i$ (regardless of whether or not V is locally Cohen-Macaulay or equidimensional). Hence K_F has finite length. This proves (a).

The first isomorphism of (b) also follows from the above exact sequence (see page 16). For the second isomorphism consider the exact sequence

$$0 \to I_V \cap (F) \to I_V \oplus (F) \to I_V + (F) \to 0$$

which, after sheafification and taking cohomology, gives

$$
\begin{array}{ccccccccc}
0 \to I_V(-d) & \xrightarrow{\begin{bmatrix} F \\ F \end{bmatrix}} & I_V \oplus (F) & \longrightarrow & & I_Z & \longrightarrow & & M(-d) \xrightarrow{\times F} M \\
& & & \searrow & \nearrow & & \searrow & \nearrow & \\
& & & & I_V + (F) & & K_F(-d) & & \\
& & & \nearrow & \searrow & & \nearrow & \searrow & \\
& & & 0 & & 0 & & 0 &
\end{array}
$$

where $M = M(V)$ (note that $(F) \cong S(-d)$ so its higher cohomology vanishes) from which (b) follows immediately. □

The last two isomorphisms of this proposition say that the submodule K_F measures the failure of both "denominators" to be saturated. If V is aCM (for example) then the "denominators" are automatically saturated. In the case of a curve V in $\mathbb{P}^n$ they are saturated if and only if V is aCM. We now consider a submodule of K_F, which was introduced in [97].

Definition 2.1.3 Let $F_1, F_2 \in S$ be general homogeneous polynomials. Let $A = (F_1, F_2)$. Then $K_A = [0 :_{M(V)} A] = K_{F_1} \cap K_{F_2}$. □

It turns out that K_A also measures a "deficiency." For two ideals I and J it is always true that $IJ \subset I \cap J$. Then it is natural to ask when we have equality, and furthermore whether there is a natural measure of the failure of equality to hold. In the special case where A is a codimension two complete intersection, we have:

Proposition 2.1.4 ([97]) *Let deg $F_i = d_i$ $(i = 1, 2)$ and assume that A is sufficiently general (that is, A is the ideal of a complete intersection meeting V in codimension two (or disjoint from V)). Let $R = S/(F_1)$ and $J = I_V/(F_1 \cdot I_V)$ as above. Then*

$$K_A \cong \frac{[J :_R F_2]}{J}(d_1) \cong \frac{I_V \cap A}{I_V \cdot A}(d_1 + d_2)$$

The first part of the proposition is a technical result which will be used in the next section. The second one gives an answer to the question posed above: the module K_A measures the failure of $I_V \cdot A$ to be saturated, i.e. to equal the intersection. (Recall that the intersection of two saturated ideals is again saturated.) The proof of this proposition is somewhat technical, but to a large extent it is a diagram chase using the commutative diagram

$$\begin{array}{ccccccccc}
\xrightarrow{\times F_1} & (I_V)_{m+d_1} & \xrightarrow{r} & (I_{Z|F_1})_{m+d_1} & \to & M_m & \xrightarrow{\times F_1} & M_{m+d_1} \\
& \downarrow \times F_2 & & \downarrow \times \overline{F_2} & & \downarrow \times F_2 & & \\
\xrightarrow{\times F_1} & (I_V)_{m+d_1+d_2} & \xrightarrow{r} & (I_{Z|F_1})_{m+d_1+d_2} & \to & M_{m+d_2} & \xrightarrow{\times F_1} & M_{m+d_1+d_2}
\end{array}$$

where $M = M(V)$, $\overline{F_2}$ is the image of F_2 in $S/(F_1)$ and r is the restriction of I_V to the ideal $I_{Z|F_1}$ of the hypersurface section Z in $S/(F_1)$. See [97] for details.

Remark 2.1.5 (a) In [97] a generalization of this result is also given in the case where dim $V \leq 1$: it is shown that if A is the ideal of a codimension two aCM subscheme which is disjoint from V then $\frac{I_V \cap A}{I_V \cdot A}$ is isomorphic to a naturally defined submodule of a certain direct sum of copies of shifts of $M(V)$. This is obtained using the minimal free resolution of A and the homomorphism induced on the direct sum of copies of M(V) (with shifts) by the Hilbert-Burch matrix of A (see §2 of Chapter 1).

It would be tempting to conjecture that a more general version of Proposition 2.1.4 holds: if A is the ideal of an aCM subscheme which is disjoint from V and $K_A = [0 :_{M(V)} A]$ then $K_A \cong \frac{I_V \cap A}{I_V \cdot A}$ (with some shift). Unfortunately this is not true. For example, let A be the ideal of a twisted cubic curve Y in $\mathbb{P}^3$ (which is aCM) and let V be a smooth rational quartic curve in $\mathbb{P}^3$ disjoint from Y. Then from Example 1.4.3 we know that $M(V)_i = 0$ for $i \neq 1$ and dim $M(V)_1 = 1$. Therefore $K_A \cong k$, occurring in degree 1. On the other hand, we can check that $\frac{I_V \cap A}{I_V \cdot A}$ is at least two dimensional in degree 4. Indeed, knowing that $M(V)_4 = 0$ allows us to use the first exact sequence in Example 1.4.3 to compute that $h^0(\mathcal{I}_V(4)) = 35 - 17 = 18$. Then Y imposes at most $(4)(3) + 1 = 13$ conditions on this linear system, so dim $(I_V \cap A)_4 \geq 5$. But dim $(I_V \cdot A)_4 = (1)(3) = 3$ (compute the number of quadrics in I_V and in A) so dim $\left(\frac{I_V \cap A}{I_V \cdot A}\right)_4 \geq 2$ (in fact $= 2$) so this cannot be a submodule of $M(V)$ with any shift.

However, Proposition 2.1.4 has been generalized in a different way in [86]. If A is the ideal of *any* subscheme of $\mathbb{P}^n$ which is disjoint from V then $\frac{I_V \cap A}{I_V \cdot A}$

fits in a certain long exact sequence, from which many things can be deduced. In the case where A is aCM of codimension two, we recover the result from [97] mentioned above. However, this is a very special case in the context of [86].

(b) It is natural to ask, for two homogeneous ideals I and J of S, when is it true that $IJ = I \cap J$? From a theorem of Serre ([123] p. 143) one can deduce the following. Let $I = I_V$ and $J = I_W$ be the saturated ideals of two schemes V and W. Assume that V and W are disjoint. Then $IJ = I \cap J$ if and only if dim $V +$ dim $W = n - 1$ and both V and W are arithmetically Cohen-Macaulay. (A new proof of this result can be found in [86].) However, one can see from Proposition 2.1.4 that this is not the only case in which $IJ = I \cap J$. For example, if W is a complete intersection surface in $\mathbb{P}^4$ and V is a surface with $M(V) = 0$ but $H^2_*(\mathcal{I}_V) \neq 0$, and if V and W meet in a finite number of points, then it is still true that $IJ = I \cap J$, even though V and W are not disjoint and V is not even arithmetically Cohen-Macaulay. $\square$

2.2 Generalizing Dubreil's Theorem

The goal of this section is to show how the submodule K_A introduced in the last section can be used to generalize a classical result of Dubreil. A special case of this result is due to Amasaki (see below). In order to state Dubreil's theorem we first make

Definition 2.2.1 Let I be a homogeneous ideal of S. Then

(a) $\alpha(I) = \min \{ i \in \mathbb{Z} \mid I_i \neq 0 \}$.

(b) $\nu(I) =$ number of minimal generators of I.

$\square$

Dubreil's theorem, in its simplest form, is the following (cf. [38]).

Theorem 2.2.2 (Dubreil) *Let $R = k[x, y]$ and let $I \subset R$ be a homogeneous ideal. Then $\nu(I) \leq \alpha(I) + 1$.* $\square$

The standard application of Dubreil's theorem is the

Corollary 2.2.3 *Let $V \subset \mathbb{P}^n$ be a codimension two aCM subscheme with defining saturated ideal I_V. Then $\nu(I_V) \leq \alpha(I_V) + 1$.*

Proof:

Since V is aCM, so is its general hyperplane section $V \cap H$ (as long as $\dim V \geq 1$), and hence the intersection of V with a general linear subvariety (by taking a sequence of hyperplane sections). Suppose first that $n \geq 3$. Let L be a defining linear form for the hyperplane H. Then from the exact sequence

$$0 \to I_V(-1) \xrightarrow{\times L} I_V \to I_{V \cap H} \to 0$$

(where the last ideal is in the ring of the hyperplane $S/(L)$; that is, a polynomial ring in one fewer variable) we see that $\nu(I_V)$ and $\alpha(I_V)$ are preserved under hyperplane sections. So without loss of generality assume that V is a finite set of points in $\mathbb{P}^2$, $S = k[X_0, X_1, X_2]$ and let L be a general linear form (hence a non-zerodivisor in S/I_V). Then $\nu(I_V) = \nu(J)$ and $\alpha(I_V) = \alpha(J)$ where

$$J = \frac{I_V}{L \cdot I_V} \cong \frac{I_V}{I_V \cap (L)} \cong \frac{I_V + (L)}{(L)} \subset k[x, y]$$

so Dubreil's theorem applies to J. (The reason that we need a non-zerodivisor is that this guarantees the first isomorphism above. Otherwise J is not an ideal in $k[x, y]$ so Dubreil's theorem does not apply.) □

The following generalization of this result can be found implicitly in [1] and explicitly, with a different proof, in [50]. We remark that for a Buchsbaum curve C, the dimension $N = \dim_k M(C)$ is called the *Buchsbaum type* of C.

Theorem 2.2.4 (Amasaki) *Let C be a Buchsbaum curve in $\mathbb{P}^3$ (cf. Definition 1.4.7). Let $N = \dim_k M(C)$. Then $\nu(I_C) \leq \alpha(I_C) + 1 + N$.* □

Note that formally this implies the corollary above, in the case of curves in $\mathbb{P}^3$, since aCM curves are trivially Buchsbaum. In this section we will prove a generalization from [97] of Theorem 2.2.4.

From now on, assume that V is a closed subscheme of $\mathbb{P}^3$, but of any codimension ≥ 2 and not necessarily locally Cohen-Macaulay or equidimensional. Assume that $A = (L_1, L_2)$ where deg $L_i = 1$ $(i = 1, 2)$ and the L_i are chosen generically.

Let $J = \frac{I_V}{L_1 \cdot I_V} \cong \frac{I_V + (L_1)}{(L_1)}$ as above. Let $\overline{J} = \frac{J + (L_2)}{(L_2)} \subset k[x, y]$. As noted above, at this stage we have to be careful. However, what we can say is that

$$\overline{J} \cong \frac{J}{(L_2) \cap J} \cong \frac{J}{L_2 \cdot [J : L_2]}. \tag{2.1}$$

Theorem 2.2.5 *Let $V \subset \mathbb{P}^3$ be a closed subscheme of dimension ≤ 1. Let $L_1, L_2 \in S_1$ be generically chosen linear forms and let $A = (L_1, L_2)$. Then $\nu(I_V) \leq \alpha(I_V) + 1 + \nu(K_A)$.*

Proof:

Note that $J \supset L_2 \cdot [J : L_2] \supset L_2 J$. Then

$$\begin{aligned}
\nu(I_V) &= \nu(J) \\
&= \nu\left(\tfrac{J}{L_2 J}\right) \\
&\leq \nu\left(\tfrac{J}{L_2 \cdot [J:L_2]}\right) + \nu\left(\tfrac{L_2 \cdot [J:L_2]}{L_2 J}\right) \\
&= \nu(\overline{J}) + \nu\left(\tfrac{[J:L_2]}{J}\right) && \text{(by (2.1))} \\
&= \nu(\overline{J}) + \nu(K_A) && \text{(by Prop. 2.1.4)} \\
&\leq \alpha(I_V) + 1 + \nu(K_A) && \text{(by Dubreil)} \qquad \square
\end{aligned}$$

This generalizes Amasaki's theorem above. This result has in turn been generalized in [86] to arbitrary codimension in $\mathbb{P}^n$, using Koszul homology, but the statement requires a good deal of explanation and notation and we will not give it here. We now show, following [50], how to deduce from Theorem 2.2.4 Amasaki's important theorem concerning the least degree of a surface containing a Buchsbaum curve in $\mathbb{P}^3$. (Amasaki's proof is along different lines.)

Corollary 2.2.6 (Amasaki) *Let $C \subset \mathbb{P}^3$ be a Buchsbaum curve and let $N = dim_k M(C)$ be the Buchsbaum type. Then $\alpha(I_C) \geq 2N$.*

Proof:

We first recall a result of [30]. That is, $\nu(I_C) \geq 3N+1$. Assuming this, the result follows immediately from Theorem 2.2.4 above: $3N+1 \leq \nu(I_C) \leq \alpha(I_C)+1+N$.

To prove that $\nu(I_C) \geq 3N+1$, the idea of [30] is to use Rao's result, Theorem 1.2.4. Indeed, note that as an S-module, k has a minimal free resolution

$$0 \to S(-4) \to 4S(-3) \to 6S(-2) \to 4S(-1) \to S \to k \to 0$$

(it is isomorphic to $S/(X_0, X_1, X_2, X_3)$, a complete intersection, so this is just the Koszul resolution). But since C is Buchsbaum, $M(C)$ is isomorphic to a direct sum, with twists, of copies of k. (See page 26.) Hence the minimal free resolution of $M(C)$ is just a direct sum, with twists, of this resolution. That is, a minimal free resolution of $M(C)$ has the form

$$0 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to M(C) \to 0$$

where rk $F_i = N \cdot \binom{4}{i}$.

Now, Theorem 1.2.4 says that a minimal free resolution of I_C has the form

$$0 \to F_4 \to F_3 \oplus \bigoplus_1^r S(-l_i) \to \bigoplus_1^m S(-e_i) \to I_C \to 0$$

where now $m = \nu(I_C) = \text{rk } F_3 + r - \text{rk } F_4 + 1 \geq \text{rk } F_3 - \text{rk } F_4 + 1 = 3N+1$. □

C. Peterson [110] has used Theorem 2.2.5 to study the deficiency module of powers of certain ideals of curves in $\mathbb{P}^3$. In particular he has shown that the deficiency modules of these curves grow very quickly:

Corollary 2.2.7 (Peterson) *Let C be a reduced, locally Cohen-Macaulay curve in $\mathbb{P}^3$ which is not a complete intersection, and such that I_C^n is saturated with no embedded components for $n \gg 0$. Let C_n be the curve defined by I_C^n. Then $\nu(K_A(C_n)) > sn^2$ for some $s > 0$ and all $n \gg 0$.*

Remark 2.2.8 (a) What can we say about r in the resolution above (without recourse to [87])? Observe that if C lies on a surface of minimal degree $2N$ then $3N+1 \leq \nu(I_C) \leq \alpha(I_C)+1+N = 3N+1$ so $\nu(I_C) = 3N+1$. Therefore in this case $r = 0$. This was observed in [50].

(b) Since Theorem 2.2.5 is much more general than Theorem 2.2.4, and Theorem 1.2.4 holds for any (locally CM equidimensional) curve in $\mathbb{P}^3$, the approach described above will work, in principal, for any curve in $\mathbb{P}^3$. However, to get a result as "clean" as Corollary 2.2.6 it is necessary for the invariants in the lower and the upper bounds of $\nu(I_C)$ to be combined in a nice way. This was easy for the Buchsbaum case but not so easy in general.

One case in which a reasonably nice answer was obtained (in [97]) is in the 2-Buchsbaum case. In general a curve C is said to be *k-Buchsbaum* if $M(C)$ is annihilated by all forms of degree k, but not by all forms of degree $k-1$ (see also [43], [99]). In particular, consider $k=2$. A simple example of such a curve is any non-Buchsbaum curve whose deficiency module is non-zero in exactly two components, in consecutive degrees (i.e. *diameter* two). Suppose C is such a curve, and suppose that the non-zero components of the module have dimensions a and b respectively. By liaison, which we will discuss later, without loss of generality we may assume that $a \leq b$. Then it can be shown that $\alpha(I_C) \geq 2b-a$. Furthermore, for any choice of a and b, sharp examples can be constructed. (There is one exception, which we have already seen. If $a=b=1$ then of course $2b-a=1$ cannot be sharp since a plane curve is always aCM. The disjoint union of a line and a conic is an example of a curve with these dimensions.)

A complete answer can be given in the 2-Buchsbaum case in general, similar to but slightly more complicated than the Buchsbaum case. It is based on this special case of diameter two, and the fact (which is an interesting exercise) that the module of a 2-Buchsbaum curve decomposes as a direct sum of modules of diameter ≤ 2. See [97] for details.

(c) Note that Theorem 2.2.5 holds for any curve in $\mathbb{P}^3$, not necessarily locally Cohen-Macaulay or equidimensional. Furthermore, it even holds for a finite set of points in $\mathbb{P}^3$, which has codimension three.

(d) A natural question is whether the converse of Dubreil's theorem holds: if C is a curve in $\mathbb{P}^3$ with $\nu(I_C) \leq \alpha(I_C)+1$, then does it follow that C is aCM? The answer is "no." A simple example is the following. Let C_1 and C_2 each be the complete intersection of two quadric surfaces in $\mathbb{P}^3$, and assume that C_1 and C_2 are disjoint. Let $C = C_1 \cup C_2$. Note that C_1 and C_2 are each aCM, but C is not (since it is not connected; cf. Theorem 1.2.3). But it follows immediately from Proposition 2.1.4 (letting $I_V = I_{C_1}$ and $A = I_{C_2}$, and noting that $K_A = 0$) that $I_C = I_{C_1} \cdot I_{C_2}$; hence $\nu(I_C) = 4 < 4+1 = \alpha(I_C)+1$.

Many other counterexamples exist, even with smooth irreducible curves. For instance, one can also check that if C is a smooth curve of degree 7 and genus 4, not lying on a quadric surface, then I_C has 4 minimal generators, and this is $\alpha(I_C) + 1$. But C is not aCM.

(e) Another natural question is whether this result as stated for subschemes of $\mathbb{P}^3$ would also hold for curves in $\mathbb{P}^4$. The answer is "no." For example, take V to be a set of two skew lines in $\mathbb{P}^4$. Here $\nu(I_V) = 5$ but $\alpha(I_V) = 1$ and $\nu(K_A) = \dim_k M(V) = 1$. □

2.3 Lifting the Cohen-Macaulay Property

For technical reasons, in this section we will assume that the characteristic of the base field is zero. (This is mainly used in the proof of a lemma which leads to Proposition 2.3.2– cf. [124], [72], [78], [98]). We saw in Theorem 1.3.2 that for dimension ≥ 2, V is aCM if and only if any hypersurface section Z of V, not containing a component of V, is aCM. Of course in general this is not true for curves, since every finite set of points is aCM while not every curve is. So one would naturally like to know if there are any conditions on the hypersurface section Z which *force* the curve to be aCM. The "cleanest" result is the following (but 2.3.2 is more general):

Theorem 2.3.1 ([98]) *Let $V \subset \mathbb{P}^n$ be a non-degenerate, locally Cohen-Macaulay, equidimensional curve. Let $F \in S_d$ be general, cutting out a zero-scheme Z. Assume that Z is a complete intersection. Then V is a complete intersection, unless*

(a) $n = 3, d = 1, V$ lies on a quadric and has even degree

(b) $n = 3, d = 1$, deg $V = 4$ and V does not lie on a quadric

(c) $n > 3$, $d = 1$, V is a double line.

It's clear that (a), (b) and (c) are exceptions; what is quite surprising is that they are the *only* exceptions. In particular, as soon as $d \geq 2$ then there are *no* exceptions. An interesting application of this result can be found in Proposition 4.2.15.

[98] is just one of the more recent of a series of papers on this general subject. A very special case of this theorem was proved in [49], and the more general question was asked for curves in $\mathbb{P}^3$ and $d = 1$. This question was answered by Strano in [124]. Then Strano's student Re proved the analogous result for curves in $\mathbb{P}^n$ and $d = 1$ in [116]. This was re-proved in [72], and some generalizations were proved for Gorenstein curves, but always assuming $d = 1$. To our knowledge [98] is the first work in this direction for $d \geq 2$. More recently, some results were obtained in [100] for higher Cohen-Macaulay type.

[98] is based on a "translation" of the preparatory results of [72] to allow for higher d, and it follows the same approach. Interestingly, it uses in a heavy way the module K_F described in §1.

Theorem 2.3.1 is a consequence of the following result from [98] (analogous to one in [72]), which gives a nice condition on the hypersurface section that forces the curve to be aCM. The notation is as follows. Let $V \subset \mathbb{P}^n$ be a non-degenerate, locally Cohen-Macaulay equidimensional curve, and let $F \in S_d$ be a general homogeneous polynomial of degree d cutting out on V a zeroscheme Z. Consider the minimal free resolution of I_Z (in $\mathbb{P}^n$):

$$0 \to F_n \to \cdots \to F_1 \to I_Z \to 0$$

and write $F_n = \bigoplus_{i=1}^{b_n} S(-m_i)$. Let K_F be the kernel of the multiplication map on $M(V)$ induced by F, as in Definition 2.1.1 and let $K = K_F(-d)$ (K is the kernel of the map induced by $F : M(V)(-d) \to M(V)$). Let $b = \min\{\, j \mid K_j \neq 0 \,\}$. Equivalently, b is the least degree in which $I_{Z|F}$ contains an element (necessarily a minimal generator) which does not lift to I_V (apply the cohomology functor to the exact sequence (3.4) of Chapter 1– see for instance the proof of Proposition 2.1.2). We stress this last interpretation since it will be used several times in the proof of 2.3.1.

Proposition 2.3.2 *Assume that V is not aCM, so that $K \neq 0$. Then $b+n \geq \min\{m_i\}$ (where m_i are defined in the paragraph above).* □

The proof of this result is based on the approach of [72]; see [98] for details. The key technical step is the extension to higher degree d of their "Socle Lemma," which we now recall. For any graded S-module N we define the *initial degree*

$$i(N) \;=\; \inf\,\{t \in \mathbb{Z} \mid N_t \neq 0\}.$$

When N is an ideal in S, we sometimes use $\alpha(N)$ for $i(N)$; see for instance §2.2.

Lemma 2.3.3 *Let M be a non-zero, finitely generated, graded S-module. For $d \geq 1$ let $F \in S_d$ be a general homogeneous polynomial of degree d. Let*

$$0 \to K \to M(-d) \xrightarrow{F} M \to \mathfrak{C} \to 0$$

be exact. If $K \neq 0$ then let $b = i(K) = \min\{ j \mid K_j \neq 0 \}$ as above. Then

$$b \geq i([0 :_{\mathfrak{C}} \mathfrak{m}^d]) + d.$$

We refer the reader to [98] for the proofs of Lemma 2.3.3 and its corollary, Proposition 2.3.2. See Example 4.4.3 for an illustration of another way in which Lemma 2.3.3 can be applied.

Proof of 2.3.1:

Say Z is the complete intersection of hypersurfaces (in $\mathbb{P}^n$) of degrees $a_1, \cdots, a_n$ with $a_1 \leq \cdots \leq a_n$. Since Z is a complete intersection, the minimal free resolution of I_Z is the Koszul resolution, and the last free part of this resolution has rank one and is twisted by $\sum a_i$ (see Example 1.4.1). (So this is $\min\{m_i\}$ in 2.3.2.) Suppose V is not a complete intersection. Then V is not arithmetically Cohen-Macaulay. (If it were, the Cohen-Macaulay type would be preserved in passing to Z so Z would not be a complete intersection.) Hence by Proposition 2.3.2 we have $b + n \geq \sum a_i$. Notice that one of the a_i is d (by Remark 1.3.1(c)) and one is b (by the discussion preceding 2.3.2, and Remark 1.3.1(b)). We will denote by $\sum^\bullet a_i$ the sum $a_1 + \cdots + a_n - b$, so we have $\sum^\bullet a_i \leq n$.

Since I_Z has n minimal generators, we have two possibilities: either $\sum^\bullet a_i = n - 1$ or $\sum^\bullet a_i = n$. In the first case all the a_i other than b are 1. But b is the least degree in which $I_{Z|F}$ has a generator which does not lift to V, so here $b = 1$ as well since V is non-degenerate, and so deg $Z = 1$. It follows that deg $V = 1$. Impossible (V is non-degenerate).

Now suppose that $\sum^\bullet a_i = n$. Then $n-2$ of the a_i are 1, one is 2 and one is b. Furthermore, one of these (other than b) is d. If $n \geq 4$ then at least two of the a_i are 1. At most one of these corresponds to F (if $d = 1$) so at least one of the generators of $I_{Z|F}$ has degree 1. Hence by definition $b = 1$ as well, and so deg $Z = 2$. Then the non-degeneracy of V means in particular that deg $V > 1$, so this forces $d = 1$, deg $V = 2$, V non-reduced (i.e. a double line).

The last possibility is $n = 3$. Then Z is the complete intersection of surfaces of degree 1,2 and b. If $b = 1$ then the same reasoning as the last paragraph gives $d = 1$ and V is a double line. So assume that $b \geq 2$. We claim that the definition of b then forces $d = 1$. Indeed, d is either 1 or 2; if it were 2 then $I_{Z|F}$ has a generator of degree 1, and since $b \geq 2$ this means that this generator must lift to I_V, contradicting the non-degeneracy of V.

So Z is a zero-scheme in $\mathbb{P}^2$ which is the complete intersection of a conic (not necessarily reduced) and a plane curve of degree b. If $b = 2$ then deg Z = deg $V = 4$ and V may or may not lie on a quadric. If $b > 2$ then by the definition of b, V lies on a quadric and clearly has even degree. □

A similar, but not as complete, analysis can be done for the case where Z is merely assumed to be arithmetically Gorenstein (i.e. having Cohen-Macaulay type 1). The following theorem summarizes much of what is known in this case (beyond Theorem 2.3.1):

Theorem 2.3.4 *Let $X \subset \mathbb{P}^n$ be a curve and let Z be its general degree d hypersurface section. Assume that Z is arithmetically Gorenstein.*

(a) ([72]) If $d = 1$ and X is reduced and connected, not lying on a quadric hypersurface, then X is arithmetically Gorenstein.

(b) ([131]) If $d = 1$ and X is integral and non-degenerate, and if X is not arithmetically Gorenstein, then Z is contained in a rational normal curve in the hyperplane and deg $X \equiv 2 \pmod{n-1}$. Conversely, given any integer $m \geq n+1$ such that $m \equiv 2 \pmod{n-1}$, there exists a smooth irreducible curve $X \subset \mathbb{P}^n$ of degree m which is not arithmetically Gorenstein but whose general hyperplane section is arithmetically Gorenstein.

(c) ([98]) Assume that X is reduced, not lying on a quadric hypersurface. If $n \geq 5$ or $d \geq 2$ then X is arithmetically Gorenstein.

(d) ([131]) Assume that X is reduced, irreducible and non-degenerate, and $d \geq 2$. Then X is arithmetically Gorenstein.

The proof is done via Proposition 2.3.2 and an analysis of the minimal free resolution of an arithmetically Gorenstein ideal.

Chapter 3

Buchsbaum Curves and Liaison Addition

One of the goals of these lecture notes is to show how much information one can get about a scheme from knowledge of its deficiency modules. This is especially true for curves in $\mathbb{P}^3$. (For instance see Rao's theorem, Theorem 1.2.4.) As an illustration of this connection, in this chapter we consider the case of Buchsbaum curves. Recall that a Buchsbaum curve is one whose deficiency module structure is trivial, i.e. it is annihilated by all linear forms.

In this chapter we will first apply some of the results and techniques of the previous chapters to describe some aspects of the theory of Buchsbaum curves. In the first section, most of the results in fact hold for Buchsbaum curves in any projective space. We will then describe in §2 a useful construction known as Liaison Addition. This was introduced by P. Schwartau in his Ph.D. thesis (Brandeis University, 1982) and generalized in [52]. The basic goal of this construction is to construct (reducible) schemes from given ones, so that the various deficiency modules of the new scheme are all direct sums of the corresponding modules for the component schemes (with twists). This has been applied in many ways, but for the moment we will focus on one application in §3: constructing "nice" Buchsbaum curves. An important observation (which we will use later) will be that the curves we produce with this construction are "the best possible" from the point of view of the general results obtained in §1. A special case of Liaison Addition, Basic Double Linkage, is very important in Liaison Theory. We will introduce this notion in this chapter (Remark 3.2.4), and discuss it more carefully in Chapters 4 and 5.

3.1 Buchsbaum Curves

One of the themes that will emerge in this section is that there are several ways in which the Buchsbaum property can be viewed as a generalization of the aCM property. In fact, the definition we are using was not the original one. The notion of a Buchsbaum ring was first introduced by J. Stückrad and W. Vogel after a negative answer by Vogel to a question of D. Buchsbaum as to whether, for an ideal generated by a system of parameters, there is a constant value (depending only on the ring and not on the system of parameters) for the difference between the length and multiplicity of the ideal. (The answer is "yes" for a CM ring; in fact, the difference is zero.) For a very complete and useful description of Buchsbaum rings, beginning with this point of view and developing all the cohomology theory, we refer the reader to [126]. For a classification of Buchsbaum subschemes of $\mathbb{P}^n$ see [32], [33].

The first result, which is classical for aCM schemes, is the following:

Proposition 3.1.1 ([49]) *Let $C \subset \mathbb{P}^n$ be a Buchsbaum curve and let H be any hyperplane which contains no component of C. Then the Hilbert function of the hyperplane section $C \cap H$ is independent of the choice of H.*

Proof:

Consider the exact sequence

$$0 \to (I_C)_{i-1} \to (I_C)_i \to (I_{C\cap H})_i \to M(C)_{i-1} \xrightarrow{\times 0} M(C)_i.$$

The dimension of the third term (equivalently, the Hilbert function of $C \cap H$) depends only on the dimension of the first, second and fourth terms. But none of these depends on H. (Of course this result holds for Buchsbaum schemes of higher dimension, and in fact all that is needed is that the first deficiency module be annihilated by the maximal ideal, so it holds even more generally.) □

Probing slightly deeper, we have

Proposition 3.1.2 ([49]) *Let $C \subset \mathbb{P}^n$ be a Buchsbaum curve. Let H be a general hyperplane, let $C \cap H$ be the hyperplane section and let $I_{C\cap H}$ be its ideal in the hyperplane H. Let $\alpha = \alpha(I_C)$. Then*

(a) $\alpha - 1 \leq \alpha(I_{C\cap H}) \leq \alpha$.

(b) $\alpha - 1 = \alpha(I_{C\cap H})$ *if and only if* $M(C)_{\alpha-2} \neq 0$. *In this case,*

$$h^0(\mathcal{I}_{C\cap H}(\alpha - 1)) = \dim M(C)_{\alpha-2}.$$

(c) $M(C)_i = 0$ *for all* $i \leq \alpha - 3$.

(d) C *is locally Cohen-Macaulay and equidimensional.*

Proof:

Let L be a linear form defining the hyperplane H and let L' be another linear form which is not a scalar multiple of L. By abuse of notation we will also denote by L' the restriction of L' to the hyperplane H. As usual, for any integer d we denote by $\phi_d(L)$ the homomorphism $M(C)_d \to M(C)_{d+1}$ induced by L, which in our situation is always the zero homomorphism. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to (I_C)_{d-1} & \xrightarrow{\times L} & (I_C)_d & \xrightarrow{r_d} & H^0(\mathcal{I}_{C\cap H}(d)) & \to & M(C)_{d-1} & \xrightarrow{\phi_{d-1}(L)} & 0 \\ & & \downarrow{\times L'} & & \downarrow{\times L'} & & \downarrow{\phi_{d-1}(L')} & & \\ 0 \to (I_C)_d & \xrightarrow{\times L} & (I_C)_{d+1} & \xrightarrow{r_{d+1}} & H^0(\mathcal{I}_{C\cap H}(d+1)) & \to & M(C)_d & \xrightarrow{\phi_d(L)} & 0 \end{array}$$

The second inequality of (a) is always true, regardless of whether C is Buchsbaum or not. Then (a), (b) and (c) follow from the following observation. Let $F \in H^0(\mathcal{I}_{C\cap H}(d))$. Then $L'F \in \text{im } r_{d+1}$. This follows from commutativity of the diagram, exactness and the fact that $\phi_{d-1}(L')$ is the zero homomorphism. We leave the remaining details to the reader. As for (d), this is also immediate since (c) guarantees that $M(C)_i = 0$ for $i \leq -1$ (for example), and as we noted in §2 of Chapter 1, $M(C)_i = 0$ for $i \gg 0$ automatically. Then apply Theorem 1.2.2 □

This Proposition is already interesting in that it shows that for any Buchsbaum curve the initial degree of the ideal of the hyperplane section differs by at most 1 from that of the ideal of the curve. For an aCM curve this difference is necessarily zero, while on the other hand for any $r \geq 0$ one can find a curve (of large degree and of course not Buchsbaum) for which this difference is r. So this is another sense in which the Buchsbaum property generalizes the aCM property. In general, it is a very interesting problem to study the difference between these initial degrees (for the general hyperplane section) and to relate that to properties of the curve (for instance its degree). Work along these lines has been done by Laudal [79], Strano [125], Mezzetti [90], etc. The work in §3 of the last chapter can also be viewed as a contribution to this problem: in that case the difference was forced to be zero (since the curve was forced to be aCM).

This Proposition also has some striking consequences, which give information about the *liaison classes* of Buchsbaum curves in $\mathbb{P}^3$, but which can be stated simply in terms of the deficiency module, so we present them here.

Corollary 3.1.3 ([49]) *Let $C \subset \mathbb{P}^n$ be Buchsbaum and as usual let $N = \dim_k M(C)$. Then*

- *(a) If $C \subset \mathbb{P}^3$ then $M(C)_i = 0$ for $i \leq 2N-3$. That is, the left-most component of $M(C)$ occurs in degree $\geq 2N-2$.*
- *(b) If $C \subset \mathbb{P}^3$ and if C is not the disjoint union of two lines then C is connected.*
- *(c) If* diam $M(C) \geq 3$ *then C does not have maximal rank.*

Proof:

We remark that (b) and (c) were also proved in [44] in the case of integral curves in $\mathbb{P}^3$. (b) first appeared in [93] but with a complicated proof. The proof in [49] was very quick but used some results involving liaison. The proof given here is in the same spirit but uses the ideas of §3 of Chapter 2.

Now, (a) follows easily from part (c) of Proposition 3.1.2 and from Amasaki's theorem (Corollary 2.2.6). As for (b), suppose C is a disconnected Buchsbaum curve. Then in particular $\dim_k M(C)_0 \geq 1$ (see Theorem 1.2.3). Now let H be a general hyperplane and consider the usual exact sequence

$$0 \to H^0(\mathcal{I}_C) \to H^0(\mathcal{I}_C(1)) \to H^0(\mathcal{I}_{C\cap H}(1)) \to M(C)_0 \to 0$$

(recall that C is Buchsbaum). Since C is locally Cohen-Macaulay and equidimensional but not aCM, $h^0(\mathcal{I}_C(1)) = 0$ (otherwise C would be a hypersurface in $\mathbb{P}^2$, hence aCM). We conclude that $h^0(\mathcal{I}_{C\cap H}(1)) = \dim M(C)_0 \geq 1$. By Proposition 3.1.2(c), $\alpha = 2$ so by Amasaki's theorem $N = 1$ and $M(C)$ occurs in degree 0. In particular $h^0(\mathcal{I}_{C\cap H}(1)) = 1$.

$C \cap H$ is a set of points in $\mathbb{P}^2$ lying on a line; hence it is a hypersurface in that line, and so it is a complete intersection in H and thus also in $\mathbb{P}^3$. Then use the argument in §3 of Chapter 2: $b = 1$ and $C \cap H$ is a complete intersection of surfaces of degrees $a_1 = 1, a_2 = 1$ and $a_3 \geq 2$. Proposition 2.3.2 then says that $1 + 3 \geq 1 + 1 + a_3$, so $a_3 = 2$. Therefore deg C = deg $(C \cap H) = 2$ and in order for C to be disconnected it must consist of two skew lines.

For (c), recall that for C to have maximal rank means that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_C(d))$$

has maximal rank, i.e. is either injective or surjective, for all d. Equivalently, for each d either $h^0(\mathcal{I}_C(d)) = 0$ or $h^1(\mathcal{I}_C(d)) = 0$ (or both). Then the result follows from the definition of α and from (c) of Proposition 3.1.2. □

Remark 3.1.4 An analog, for k-Buchsbaum curves, of many of the results in this section can be found in [99]. □

Remark 3.1.5 (a) Part (c) of this corollary is stated for higher projective space, while parts (a) and (b) are stated for $\mathbb{P}^3$ only. It is natural to ask whether they are true in higher projective space. Note that both of them use Amasaki's theorem in the proof, so one suspects right away that there will be problems. And indeed, both are false in higher projective space. For (a), a counterexample can be found in [52] Proposition 2.7. Here a Buchsbaum curve C_r is constructed in $\mathbb{P}^4$ using Liaison Addition (see the next section), where $\dim_k M(C_r) = r$ and the module begins (and in fact is concentrated) in degree $r - 1$. As for (b), consider for instance three general skew lines in $\mathbb{P}^5$. The module is 2-dimensional and concentrated in degree 0, hence C is Buchsbaum. However, using a similar argument it should be possible to classify the disconnected Buchsbaum curves in $\mathbb{P}^n$.

(b) One of the main ideas of the structure of an even liaison class of codimension two subschemes of $\mathbb{P}^n$, which we will discuss in §2 of Chapter 5, is that the schemes whose modules have the leftmost possible shift are very special. We have already discussed possible shifts in Chapter 1. Notice that part (a) of Corollary 3.1.3 gives the leftmost possible shift for a Buchsbaum curve in $\mathbb{P}^3$ (with specified Buchsbaum type N, i.e. with the dimension N of the deficiency module specified). One of the consequences of the structure theorem (Theorem 5.2.1) is that any Buchsbaum curves which are extremal with respect to this bound have the same degree. For example, if the module has dimension k and is supported in one degree then the leftmost possible shift would be when the non-zero component occurs in degree $2k-2$. In this case, the curve necessarily has degree $2k^2$ (cf. [22]). (If $k=1$ then two skew lines give an example of an extremal curve.) □

In Chapter 2 we discussed Dubreil's theorem bounding the number of minimal generators of the ideal of an aCM curve in $\mathbb{P}^3$ (and generalizing this result). A related problem is to bound the *degrees* of the minimal generators of the ideal of the curve, and a common technique is to relate this to the generators of the ideal of the general hyperplane section.

It is already clear that these two cannot be related without some extra information (for instance something about the module). For example, if C is a double line (i.e. a non-reduced scheme of degree two supported on a line) then its general hyperplane section has degree 2 and in fact is a complete intersection, so its ideal is generated in degrees ≤ 2. However, the ideal of C can have a minimal generator of arbitrarily large degree (and correspondingly the deficiency module can get arbitrarily large). See [92] for details.

Now we will prove a result for Buchsbaum curves which generalizes a standard result for aCM curves. We need the following lemma.

Lemma 3.1.6 *Let V be a zeroscheme in $\mathbb{P}^n$ with saturated ideal I_V. Let $t = \min\ \{ i \in \mathbb{Z} \mid h^1(\mathcal{I}_V(i)) = 0 \}$. Then I_V is generated in degree $\leq t+1$.*

Proof: Let L be a general linear form; in particular, L does not vanish at any of the points on which V is supported. Thus the saturation of $I_V + (L)$ is all of S by the Nullstellensatz, so there is no hyperplane section, and using the notation of §3 of Chapter 1 we have $\mathcal{I}_{Z|L} \cong \mathcal{O}_{\mathbb{P}^{n-1}}$. Hence the exact sequence (3.4) of that section becomes

$$0 \to \mathcal{I}_V(-1) \xrightarrow{\times L} \mathcal{I}_V \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0.$$

Twisting by $d \geq t+1$ and letting $R = k[X_0, \cdots, X_{n-1}]$, we get

$$0 \to (I_V)_{d-1} \xrightarrow{\times L} (I_V)_d \xrightarrow{r_d} R_d \to 0.$$

Now, let $F \in (I_V)_d$ where $d \geq t+2$. We want to show that F is not a minimal generator. But $r_d(F) \in R_d$ can be written as $\sum X_i G_i$ where $G_i \in R_{d-1}$. Since r_{d-1} is surjective (since now $d \geq t+2$) we get $r_d(F) = \sum X_i r_{d-1}(F_i) = r_d(\sum X_i F_i)$ for some $F_i \in (I_V)_{d-1}$. Then by exactness, $F - \sum X_i F_i = LG$ for some $G \in (I_V)_{d-1}$ so F is not a minimal generator. □

Corollary 3.1.7 ([49]) *Let $C \subset \mathbb{P}^n$ be a Buchsbaum curve and for a general hyperplane H let $t = \min \{ i \mid h^1(\mathcal{I}_{C \cap H}(i)) = 0 \}$. Then the saturated ideal I_C of C is generated in degree $\leq t+1$.*

Proof:
Consider the exact sequence

$$0 \to (I_C)_{t-1} \to (I_C)_t \to (I_{C \cap H})_t \to M(C)_{t-1} \xrightarrow{\times 0} M(C)_t \to 0$$

(where the last 0 is from the definition of t). This guarantees that $M(C)_t = 0$, and since $h^1(\mathcal{I}_{C \cap H}(i)) = 0$ for all $i \geq t$, we also have that $M(C)_i = 0$ for all $i \geq t$. With this information and using the above exact sequence (twisted by 1), the same argument as in Lemma 3.1.6 gives the result. □

Note that Lemma 3.1.6 and Corollary 3.1.7 could also be proved using Theorem 1.1.5.

3.2 Liaison Addition

Liaison Addition was introduced in the Ph.D. thesis of Philip Schwartau [122] in 1982. His work was motivated by the following naive question: Let C_1 and C_2 be curves in $\mathbb{P}^3$ with deficiency modules M_1 and M_2 respectively. Does there exist a curve C with $M(C) \cong M_1 \oplus M_2$? The following example shows that this is too ambitious:

Example 3.2.1 Let C_1 and C_2 each be a set of two skew lines in $\mathbb{P}^3$, so as we saw in Example 1.4.2, M_1 and M_2 are both one-dimensional k-vector spaces, and as graded modules they are non-zero only in degree zero. Hence $M_1 \oplus M_2 \cong k^2$, also occurring in degree zero. Now suppose that there were to exist such a curve C. Let H be a general plane and consider the exact sequence

$$0 \to H^0(\mathcal{I}_C) \to H^0(\mathcal{I}_C(1)) \to H^0(\mathcal{I}_{C\cap H}(1)) \to k^2 \to 0$$

(where $\mathcal{I}_{C\cap H}$ is the ideal sheaf of the hyperplane section in the hyperplane H). We know from Theorem 1.2.2 that C is locally Cohen-Macaulay and equidimensional. On the other hand, if C lay on a plane it would be a hypersurface in that plane and hence (as noted in Example 1.4.1) a complete intersection, which contradicts the fact that C is not aCM. Therefore $h^0(\mathcal{I}_C(1)) = 0$. Then exactness above gives that $h^0(\mathcal{I}_{C\cap H}(1)) = 2$. But $C \cap H$ is a set of points in the plane, and $h^0(\mathcal{I}_{C\cap H}(1)) = 2$ implies that $C \cap H$ consists of exactly one point. Hence deg $C = 1$, and because C is locally Cohen-Macaulay and equidimensional this means that it is a line. Again this contradicts the fact that C is not aCM. □

Example 3.2.2 Schwartau remarks that however the question is phrased, simply taking the union of the given curves cannot work. Indeed, if C_1 and C_2 are disjoint lines then both are aCM (hence have trivial deficiency module) while their union is not. □

Schwartau discovered the correct way to rephrase the question for curves in $\mathbb{P}^3$, and in fact his theorem is stated in the more general context of "adding" two codimension two subschemes of projective space. This theorem was generalized in [52] to allow higher codimension and a greater number of "added" schemes (rather than only two). Here we will not give this theorem in its full generality, but rather give a simplified version which will suit our purposes. We refer the reader to [52] for the stronger version of the theorem.

We will now state the theorem. Note that we use the cohomology notation $H^i_*(\mathcal{I}_V)$ rather than the notation $(M^i)(V)$. This is because technically $(M^i)(V)$ is only defined for $1 \leq i \leq \dim V$, while in this theorem for some V we may need i to be larger.

Theorem 3.2.3 **([52])** *Let $V_1, \cdots, V_r$ be closed subschemes of $\mathbb{P}^n$, with $2 \leq r \leq n$. (We allow the possibility that $V_i = \emptyset$, in which case $I_{V_i} = S$.) Assume* *that codim $V_i \geq r$ for all i. By making general choices (possibly of large degree), we may choose homogeneous polynomials*

$$F_i \in \bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} I_{V_j} \tag{1.1}$$

for $1 \leq i \leq r$, such that $(F_1, \cdots, F_r)$ forms a regular sequence and hence gives the saturated ideal of a complete intersection scheme V of codimension r. (This is possible because of our assumption that codim $V_i \geq r$ for all i). Let deg $F_i = d_i$. Let $I = F_1 I_{V_1} + \cdots + F_r I_{V_r}$ and let Z be the closed subscheme of $\mathbb{P}^n$ defined by I. We denote by $H(Z,t)$ the Hilbert function of Z (see page 7). Then

(a) *As sets, $Z = V_1 \cup \cdots \cup V_r \cup V$.*

(b) *For each $1 \leq j \leq n - r = \dim V$, $H^j_*(\mathcal{I}_Z) \cong H^j_*(\mathcal{I}_{V_1})(-d_1) \oplus \cdots \oplus H^j_*(\mathcal{I}_{V_r})(-d_r)$.*

(c) *I is saturated ($I = I_Z$).*

(d) *$H(Z,t) = H(V,t) + H(V_1, t - d_1) + \cdots + H(V_r, t - d_r)$.*

Proof:

If $P \in V_i$ for some i or if $P \in V$ then clearly from the way I is defined, every $F \in I$ vanishes at P. So as sets, $Z \supseteq V_1 \cup \cdots \cup V_r \cup V$. Now let P be any point of $\mathbb{P}^n$ *not* on $V_1 \cup \cdots \cup V_r \cup V$. In particular there is some F_i not vanishing at P (since $P \notin V$), and since (for that i) $P \notin V_i$, we have some $G \in I_{V_i}$ not vanishing at P. So $GF_i \in I$ does not vanish at P, hence $P \notin Z$. This proves (a).

From Example 1.4.1 we know that the resolution for I_V ends with

$$\cdots \xrightarrow{\phi_3} \bigoplus_{1 \leq i < j \leq r} S(-d_i - d_j) \xrightarrow{\phi_2} \bigoplus_{1 \leq i \leq r} S(-d_i) \xrightarrow{\phi_1} I_V \to 0 \tag{1.2}$$

where $\phi_1 = (F_1, \cdots, F_r)$ and ϕ_2 is the matrix of relations: it is an $r \times \binom{r}{2}$ matrix of homogeneous polynomials, and each column C_t consists of exactly two non-zero entries (each being one of the F_i, up to ± 1) so that the matrix product $\phi_1 \cdot C_t$ is of the form $F_i F_j - F_j F_i = 0$. In particular, for $1 \leq i \leq r$

each entry of the ith row of ϕ_2 is an element of I_{V_i} (possibly 0), because of the assumption (1.1). (What we are saying is that F_i never occurs in the ith row, but all the other generators do.)

The significance of this observation is that the image of ϕ_2 is actually contained in $\bigoplus_{1\le i\le r} I_{V_i}(-d_i)$. Therefore we also have a long exact sequence

$$\cdots \xrightarrow{\phi_3} \bigoplus_{1\le i<j\le r} S(-d_i-d_j) \xrightarrow{\phi_2} \bigoplus_{1\le i\le r} I_{V_i}(-d_i) \xrightarrow{\phi_1} I \to 0. \tag{1.3}$$

If we let K be the cokernel of ϕ_3, exactness gives that K is also the kernel of ϕ_1 (in either (1.2) or (1.3)). Therefore we have two short exact sequences

$$0 \to K \to \bigoplus_{1\le i\le r} S(-d_i) \xrightarrow{\phi_1} I_V \to 0 \tag{1.4}$$

and

$$0 \to K \to \bigoplus_{1\le i\le r} I_{V_i}(-d_i) \xrightarrow{\phi_1} I \to 0. \tag{1.5}$$

Let $\mathcal{K}$ be the sheafification of K and sheafify these two exact sequences. Note that the sheafification of I is $\mathcal{I}_Z$. The idea is to use (1.4) to get information about K, and then apply this to (1.5). Since ϕ_1 is surjective in (1.4) and I_V is saturated, we get that $H^1_*(\mathcal{K}) = 0$. Similarly we get that $H^0_*(\mathcal{K}) = K$ and that $H^j_*(\mathcal{K}) = 0$ for $2 \le j \le n-r+1$ (using the fact that V has dimension $n-r$ and is aCM).

Now apply this information to (1.5) (sheafifying and taking cohomology). (b) follows immediately from the vanishing of the higher cohomology of $\mathcal{K}$. As for (c), this follows from the facts that $H^1_*(\mathcal{K}) = 0$, I_{V_i} is saturated for each i, and $H^0_*(\mathcal{K}) = K$. (The point is that when you sheafify the short exact sequence and then take cohomology, you again get a short exact sequence and the first two terms have not changed; hence $I = H^0_*(\mathcal{I}_Z) = I_Z$.) $\square$

Remark 3.2.4 (a) As we noted above, this theorem is a special case of a more general result in [52]. (One can allow V to be much more general than simply a complete intersection, and one can "add" more than r schemes. But this version seems to be the simplest to apply in any case.) The thesis of Schwartau proved this theorem in the special case where $r = 2$, codim $V_i = 2$ and V_i are not trivial. This thesis was not published elsewhere.

On the other hand, [52] was written with the advantage of hindsight: knowing the work of [122] made it somewhat natural to guess what ideal would most likely produce the desired scheme (once one decided in which direction it should be generalized). As a result, the main theorem in [52] is not only much more general but the proof is much simpler. The approach of [122] is more constructive.

(b) As mentioned in the theorem, some of the V_i can be trivial. In the case where V_1 is non-trivial and all the rest of the V_i are trivial, the resulting scheme Z is called a *basic double link* of V_1. So in this case we have $F_1 \in S$, a general homogeneous polynomial of any degree $d_1 \geq 1$, and $F_2, \cdots, F_r \in I_{V_1}$, such that $(F_1, F_2, \cdots, F_r)$ form a regular sequence and hence define a complete intersection V.

This was introduced for curves in $\mathbb{P}^3$ by Lazarsfeld and Rao [80], and extended to the current generality in [25] without recourse to the above theorem. The connection to Liaison Addition was made in [52]. Basic double linkage in codimension two plays a crucial role in the theory of Liaison, and especially in the structure theorem for an even liaison class (cf. [80], [11], [87], etc.). This will be discussed later. For now notice three things:

(1) As sets, $Z = V_1 \cup V$.

(2) $I_Z = F_1 I_{V_1} + (F_2, \cdots, F_r)$.

(3) $(M^i)(Z) \cong (M^i)(V_1)(-d_1)$.

(c) In the next section we will see one important application of Liaison Addition: the construction of Buchsbaum curves. This theorem can also be used to create quick and simple examples of saturated ideals of schemes with embedded components. For example (from [52]), consider the case of curves in $\mathbb{P}^3$ and $r = 2$. It is fairly easy to show using part (b) of the theorem that Z is locally Cohen-Macaulay, equidimensional and one-dimensional if and only if V_1 and V_2 are (but allowing V_2 to possibly be trivial as well). (Use Theorem 1.2.2.)

A simple example is the following. Let V_1 be a point, say with ideal (X_1, X_2, X_3) and let V_2 be trivial. Form a basic double link taking F_1 and F_2 to be linear forms in I_{V_1}: say $F_1 = X_1, F_2 = X_2$. The ideal

$I_Z = X_1(X_1, X_2, X_3) + (X_2) = (X_1^2, X_1X_3, X_2)$ is the saturated ideal of a line Z with an embedded point (at V_1). □

3.3 Constructing Buchsbaum Curves in $\mathbb{P}^3$

In this section we first show how to use Liaison Addition to construct Buchsbaum curves. We will focus on curves in $\mathbb{P}^3$ for simplicity (and because this is the most useful for us) but the same ideas work for curves in $\mathbb{P}^n$ thanks to the generality of the construction described in §2. We will follow [26] for the most part, because we would like to show that the curves we construct are "nice" from a number of viewpoints used in that paper (and we will use these properties later when we use Liaison techniques to show that every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure, which is one of the main results of [26]). However, similar ideas for constructing appropriate Buchsbaum curves in $\mathbb{P}^3$ via Liaison Addition were used in [122], [22], [23] and [24]. The construction of Buchsbaum curves in $\mathbb{P}^4$ with some "nice" properties was done in [52].

Let $N = \dim_k M(C)$ where C is a Buchsbaum curve. Let I_C be its saturated ideal. For the curves we construct, the bounds of Theorem 2.2.4, Corollary 2.2.6 and Corollary 3.1.3 will be sharp. That is, our curves will satisfy

(a) $\nu(I_C) = \alpha(I_C) + N + 1$ (Theorem 2.2.4).

(b) $\alpha(I_C) = 2N$ (Corollary 2.2.6).

(c) The leftmost nonzero component of $M(C)$ occurs in degree $2N - 2$ (Corollary 3.1.3).

The curves we construct in this section will also be *hyperplanar stick figures* (cf. [26]). This means that each curve we construct is

(d) a reduced union of lines with no three meeting in a point (i.e. a stick figure) and

(e) contained in a (reduced) union of $2N$ hyperplanes such that the intersection of any three of the hyperplanes is a point and the intersection of any two of them is not a component of C (i.e. hyperplanar).

(We will use the fact that our curves are hyperplanar in §5.3.4 when we show that any Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure.)

The basic idea here is that we will use sets of two skew lines in $\mathbb{P}^3$ as "building blocks" to construct Buchsbaum curves with larger deficiency modules. We can use Liaison Addition to do this. Recall that a set of two skew lines has deficiency module which is just one-dimensional (in degree zero)– cf. Example 1.4.2. But any graded S-module, whose multiplication by linear forms is trivial, is isomorphic (as an S-module) to a direct sum (with twists) of copies of this one-dimensional vector space (see page 26), so Liaison Addition lends itself naturally to this problem.

Suppose we are trying to construct a Buchsbaum curve C whose deficiency module has components of dimension $n_1 > 0, n_2 \geq 0, \cdots, n_{r-1} \geq 0, n_r > 0$ (where these are the dimensions of all the components of the module, from the first non-zero one to the last). The proof works by induction on $N = n_1 + \cdots + n_r$. Note that we would like the first component to occur in degree $2N - 2$.

For $N = 1$, C is the disjoint union of two lines. It is trivial to verify that all the conditions (a)-(e) are verified. So assume that $N > 1$. For convenience we will assume that $n_1 > 1$. The case $n_1 = 1$ is similar and is left to the reader. (The main difference is that the choice of F_1 below will be slightly different.)

Suppose that C_1 is a curve whose module components have dimensions $n_1 - 1, n_2, \cdots, n_r$ and which satisfies all the conditions above (with now N replaced by $N-1$). Let $F_2 \in I_{C_1}$ be the union of $2(N-1)$ planes guaranteed by (e). The singular locus $Sing\ F_2$ is a union of lines having no component in common with C_1. Let C_2 be a disjoint union of two lines, also disjoint from C_1 and from $Sing\ F_2$.

Note that $Sing\ C_1$, $(C_1 \cap Sing\ F_2)$ and $\{$ triple points of F_2 $\}$ are all finite sets disjoint from C_2. Let F_1 be a union of two planes which contains C_2 but avoids these three finite sets, and such that the line $Sing\ F_1$ avoids Y_1, $Sing\ F_2$ and $C_2 \cap F_2$. (Choosing one plane first gives an additional finite number of points for the second plane to avoid. Notice that the fact that $Sing\ F_1$ avoids $C_2 \cap F_2$ implies that F_2 avoids $C_2 \cap Sing\ F_1$, which is the analogue of the condition that F_1 avoids $C_1 \cap Sing\ F_2$ which we required in the choice of F_1.)

Let C be the result of applying Liaison Addition to C_1 and C_2 using $F_2 \in I_{C_1}$ and $F_1 \in I_{C_2}$, so $I_C = F_1 I_{C_1} + F_2 I_{C_2}$. Clearly $\alpha(I_C) = 2N$, and by Liaison Addition $M(C) \cong M(C_1)(-2) \oplus M(C_2)(-2N+2)$. Then by the inductive hypothesis, one can check that (b) and (c) hold for C, and so by Remark 2.2.8(a) we also have (a) above.

We now check that C is a stick figure. By the generality of the choices made, C is a reduced union of lines. (We need the fact that F_1 contains no component of C_1 or of $Sing\ F_2$ and F_2 contains no component of C_2 or of $Sing\ F_1$, which follow from the above set-up.) As for the absence of triple points, it is somewhat tedious to check all the details but it follows from the fact that both C_1 and C_2 are hyperplanar with F_2 and F_1 the corresponding unions of planes, and these are sufficiently general so that they meet well. C is the union of C_1, C_2 and the complete intersection of F_1 and F_2, and the two conditions of (e) are exactly what are needed to avoid triple points. (The first condition avoids triple points lying on a plane and the second avoids triple points that are not on a plane.) We refer to [26] for details.

To see that C is hyperplanar we need to exhibit a reduced union Σ of $2N$ planes containing C such that the two conditions in (e) hold. One would be tempted to take for Σ the union of F_1 and F_2, but since C contains in particular the complete intersection of F_1 and F_2, the second condition of (e) fails. However, by the way things were chosen, F_2 does almost all of the job (it contains C_1 and the complete intersection of F_1 and F_2). One can check that taking for Σ the union of F_2 and a general union of two planes containing C_2, this will do the trick. Again there are several details to check, for which we refer the reader to [26].

As mentioned above, the case $n_1 = 1$ is very similar. Here, instead of taking F_1 to be a union of two planes containing C_2, we take it to be a union of two planes containing C_2 and an appropriate number of general planes. (The number is chosen in order to construct the right module with Liaison Addition.)

Remark 3.3.1 (a) One can also use Liaison Addition to construct Buchsbaum curves in the other extreme: for any module dimensions $n_1, \cdots, n_r$ as above, one can construct a Buchsbaum curve whose deficiency module has these dimensions, sharp with respect to conditions (a), (b) and (c), but which is supported on a line. The idea is that rather than using two skew lines as the "building blocks" one can use a double line of arithmetic genus -1 (and at each step use the same double line rather than a general one). Such a double line has one-dimensional deficiency module, so the same ideas apply.

(b) Similar constructions (for subschemes which are reduced unions of codimension two linear varieties with "good" properties) are possible in higher dimensions as well. For example, for surfaces in $\mathbb{P}^4$ some work in this direction was done in [26].

(c) One can check that the degree of the curve C constructed in this section (in particular satisfying condition (c) on page 57) is

$$2N^2 - N + n_1 + 3n_2 + \cdots + (2r-1)n_r.$$

This can be shown directly using induction, or it can be shown using Corollary 2.18 (b) of [24].

(d) See Example 1.4.6 for the case $r = 2$ and $n_1 = n_2 = 1$. □

Chapter 4

Introduction to Liaison Theory in Arbitrary Codimension

In this chapter we introduce Liaison Theory. While many of the important results in the theory are currently known only in codimension two, the definitions and first results hold much more generally. In this chapter we will always assume the context of subschemes of arbitrary codimension in any projective space. (It should be noted that even this is a restriction– a great deal of work has been done in a more general algebraic context. We refer the interested reader to [71], from which he or she can "get a foot in the door" and branch off to other references. However, more in the spirit of the ideas in these notes, we refer the reader also to [120] and to [27], and to the recent paper [62]. In particular, liaison can be studied by looking more generally at residuals in a Gorenstein scheme, not simply in a complete intersection.)

There do not seem to be many introductory references for Liaison Theory. One useful source, albeit only for curves in $\mathbb{P}^3$, is the lecture notes [115] (in Italian). In this context, [87] is also a must-read. See also [121] and [126] for a discussion of Liaison Theory and its connections to Buchsbaum rings. The paper [62] develops the theory from the point of view of generalized divisors on a Gorenstein scheme. (See Example 4.1.6 (4.) below.)

Historically, Liaison began in the last century as a tool to study curves in projective space. The idea was to start with a curve, say in $\mathbb{P}^3$, and look at its residual in a complete intersection. Since complete intersections are in some sense the simplest curves, it turns out that a lot of information can be carried over from a curve to its residual (or vice versa). So it was hoped that using this process of taking residuals, one could always pass to a "simpler" curve and so it would be easier to get information about the original curve.

This idea in fact works nicely for aCM curves in $\mathbb{P}^3$. This was essentially shown by Apéry and Gaeta in the 1940's, and it was proved rigorously with modern machinery by Peskine and Szpiro [109] in 1974. However, Joe Harris conjectured that this notion would not work in the non-aCM case even for curves in $\mathbb{P}^3$ ([59] p. 80). This was proved by Lazarsfeld and Rao in 1982– the idea is that the "general smooth curve" is already the simplest curve in its liaison class, in many senses. This is a special case of (and indeed it inspired!) the structure theorem for codimension two even liaison classes (cf. [11], [87] [107]) which we will discuss in the next chapter. Work of Huneke and Ulrich [71] can also be interpreted as showing that this idea also fails even for aCM curves in higher codimension.

Liaison has been used very extensively in the literature as a means of studying curves (or indeed higher dimensional varieties) and of producing interesting examples. Furthermore, beginning with the paper of Peskine and Szpiro, Liaison has attracted a great deal of attention as a subject in and of itself, rather than simply a tool to study projective varieties. We will discuss several of these results in these chapters. As we will see, the deficiency modules play an important role also in this field, thanks especially to work of Rao [113], [114].

4.1 Definitions and First Examples

As we indicated above, liaison involves studying properties that are preserved when the union of two schemes is a complete intersection. Actually, this point of view is a bit too naive: when the two schemes have no common component then there are no problems, but otherwise "union" is too weak and we need to take a more algebraic approach. In any case we begin with the weaker notion:

Definition 4.1.1 Let V_1, V_2 be subschemes of $\mathbb{P}^n$ such that no component of V_1 is contained in any component of V_2 and conversely. Then V_1 is *(geometrically) directly linked* to V_2 by a complete intersection X if $V_1 \cup V_2 = X$.

From the point of view of the saturated ideals, this says that $I_{V_1} \cap I_{V_2} = I_X$. V_1 is said to be *residual* to V_2 in the complete intersection X. □

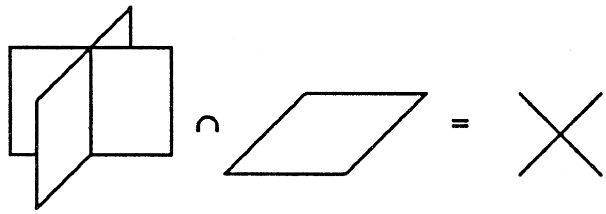


Figure 4.1: Geometric Link

Notice that deg V_1 + deg V_2 = deg X. (See Corollary 4.2.10 for a more general statement.) The simplest example of two curves that are geometrically directly linked is the union of two lines, given as the intersection of a pair of planes with another plane (see Figure 4.1). The problem comes when we try to extend this notion to the case where the curves may have common components. For example, if the second surface (the plane) contains the line of intersection of the two planes comprising the first surface, this is still a perfectly good complete intersection (see Figure 4.2).

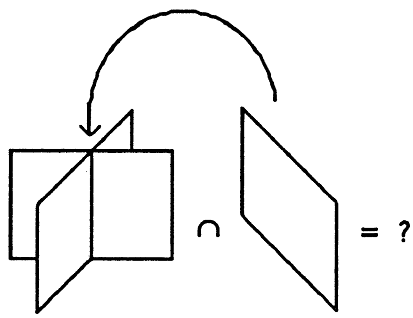


Figure 4.2: Algebraic Link

The only natural way to interpret this would be to say that the line of intersection is linked to itself (since the complete intersection still has degree two), but using only unions of course we do not have the equality in the definition. The solution is to use ideal quotients.

Definition 4.1.2 Let V_1, V_2 be subschemes of $\mathbb{P}^n$ of codimension r and let $F_1, \cdots, F_r \in I_{V_1} \cap I_{V_2}$ define a complete intersection scheme X of codimension r. Then V_1 is *(algebraically) directly linked* to V_2 by the complete intersection X ($V_1 \overset{X}{\sim} V_2$) if and only if $[I_X : I_{V_1}] \cong I_{V_2}$ and $[I_X : I_{V_2}] \cong I_{V_1}$. V_1 is *residual* to V_2 in the complete intersection X. □

We can check that this definition makes sense in the example described above in Figures 4.1 and 4.2:

Example 4.1.3 Let $n = 3$ and let $S = k[X_0, \cdots, X_3]$. Let $I_X = (X_0X_1, X_2)$ and $I_{V_1} = (X_0, X_2)$. Then $I_{V_2} = [I_X : I_{V_1}] = (X_1, X_2)$. (So the first case is ok.)

Now let $I_X = (X_0X_1, X_0 + X_1)$ and $I_{V_1} = (X_0, X_1)$. Then $I_{V_2} = [I_X : I_{V_1}] = (X_0, X_1) = I_{V_1}$ (so it is "self-linked"). □

Remark 4.1.4 As one would expect from the way we have set things up, one can show that if V_1 and V_2 are geometrically linked then they are algebraically linked. Indeed, assume that they are geometrically linked. This means that $I_{V_1} \cap I_{V_2} = I_X$ and that neither I_{V_1} nor I_{V_2} is contained in any associated prime of the other. We wish to show that $[I_X : I_{V_1}] = I_{V_2}$ (and vice versa). The inclusion $\supseteq$ is left as a quick exercise. For the inclusion $\subseteq$, say $F \in [I_X : I_{V_1}]$, so $F \cdot I_{V_1} \subset I_X$. Choose $G \in I_{V_1}$ such that G is not in any associated prime of I_{V_2} (i.e. G vanishes on no component of I_{V_2}). Then

$$FG \in I_X = \underbrace{Q_1 \cap \cdots \cap Q_r}_{I_{V_2}} \cap \underbrace{Q'_1 \cap \cdots \cap Q'_s}_{I_{V_1}}$$

In particular $FG \in Q_i$ for all i, but no power of G is in Q_i. Therefore by the definition of "primary," $F \in Q_i$ for all i so $F \in I_{V_2}$.

Conversely, if V_1 and V_2 have no common components and are algebraically directly linked then they are geometrically directly linked. See Proposition 4.2.2 for the proof.

From now on we will assume that our links are algebraic links, and will write $V_1 \overset{X}{\sim} V_2$ without further comment. □

Note that the definition of direct linkage is symmetric but not usually reflexive (only in the case of "self-linkage") or transitive. So we need to extend it in order to get an equivalence relation:

Definition 4.1.5 *Liaison* is the equivalence relation generated by direct linkage. That is, we say that V_1 is *linked* to V_2, $V_1 \sim V_2$, if there is a sequence of schemes $W_1, \cdots, W_k$ and a sequence of complete intersections $X_1, \cdots, X_{k+1}$ such that

$$V_1 \stackrel{X_1}{\sim} W_1 \stackrel{X_2}{\sim} \cdots \stackrel{X_k}{\sim} W_k \stackrel{X_{k+1}}{\sim} V_2.$$

The equivalence classes generated by this procedure are called *liaison classes* (or *linkage classes*). If $k+1$ is even, we say that V_1 and V_2 are *evenly linked.* Notice that even linkage also generates an equivalence relation, and the equivalence classes are called *even liaison classes.* An even liaison class will usually be denoted by $\mathcal{L}$. □

Now that we have defined our equivalence classes, there are many natural questions that arise. We will discuss many of the answers in these lectures. We will see that by far, the most complete picture is in the case of codimension two.

Questions

1. Find connections between directly linked schemes (degree, genus, and especially more subtle connections). Find properties that are preserved. For example, we'll see that the property of being aCM is preserved, as is the property of being Buchsbaum. Also the dimension is preserved. As a consequence, find invariants of a liaison class or of an even liaison class.

2. Is this a trivial equivalence relation? From the above, we can already deduce that the answer is "no." Indeed, since the property of being aCM is preserved and the property of being Buchsbaum is preserved, we see that there have to be at least three classes, since there exist non-aCM Buchsbaum curves and there exist non-Buchsbaum curves in $\mathbb{P}^n$. This was already known to Gaeta in the 40's. (In fact, in codimension two the aCM subschemes *form* one liaison class. In higher codimension there are infinitely many aCM liaison classes.)

3. Parameterize the (even) liaison classes– give necessary and sufficient conditions for two subschemes to be in the same (even) liaison class.

4. Describe any one even liaison class. (We will see below that the even liaison class of a given scheme is the "right" object to study, rather than

the entire liaison class.) Is there a standard way to describe "distinguished" elements in that class? What *structure* does the class have? In particular, is there a structure common to all even liaison classes? What consequences does this structure have?

5. Can liaison be studied in greater generality? For instance, can we talk about linkage of subschemes of something other than projective space? We will not talk about this much, but the answer is emphatically "Yes!" In this context important work has been done by Huneke, Kustin, Miller and Ulrich, among others. (For instance, see [71], where one can find further references. Also, [27] can be viewed as an attempt to bridge some of these ideas.) On the other hand, Hartshorne [62] has recently developed a more general theory of divisors. This approach allows him to give a new definition of linkage, which is equivalent to that of algebraic linkage (Definition 4.1.2) but more in the spirit of geometric linkage. See also Example 4.1.6 (4.) below. Another direction is to ask whether we can perform links using something other than complete intersections. It turns out, thanks largely to Schenzel [120], that if we replace "complete intersection" by "arithmetically Gorenstein" then many of the same results hold. (Again, see also [27].) On the other hand, Walter [129] has shown that if we replace "arithmetically Gorenstein" with "aCM" then that is too much– in that case there is only one equivalence class for any given codimension in projective space. Similar results have recently been obtained by Martin [85]. And finally, liaison has been generalized to the notion of *residual intersections* (where even the dimension is not preserved); this was introduced by Artin and Nagata [3] and studied for instance by Huneke and Ulrich in [76].

Example 4.1.6 1. We saw that in $\mathbb{P}^3$ we can directly link a line to another line (provided that the lines either meet in one point or coincide). In fact, if one considers $\mathbb{P}^3$ as a complete intersection linear subvariety of $\mathbb{P}^n$, one can show that the same holds in $\mathbb{P}^n$ (but now the codimension is $n-1$ so the number of generators of the complete intersection performing the link is $n-1$ rather than two).

2. Any curve of type (n,n) on a smooth quadric in $\mathbb{P}^3$ is a complete intersection, so a twisted cubic is directly linked to a line, two skew lines are directly linked to two skew lines (in the opposite ruling), etc. Furthermore,

one can directly link two skew lines to a double line (i.e. a scheme of degree 2 supported on a line) in the opposite ruling by the same reasoning.

3. Schwartau [122] gives a simple proof that any two complete intersections of the same codimension are linked. The proof follows from the following lemma: If $I_{X_1} = (F_1, \cdots, F_{d-1}, F)$, $I_{X_2} = (F_1, \cdots, F_{d-1}, G)$ and $I_X = (F_1, \cdots, F_{d-1}, FG)$ then $X_1 \overset{X}{\sim} X_2$. So now starting with arbitrary complete intersections Y_1 and Y_2 one can produce a sequence of links from one to the other in d steps: just apply the lemma to change one generator at a time.

In particular, any two hypersurfaces are linked. (For example, any two plane curves are linked. Note that a plane curve is still a complete intersection even if it is embedded in a higher projective space, so even in this context any two plane curves are linked.)

As remarked earlier, we will see in Chapter 5 that for codimension two, being in the linkage class of a complete intersection is equivalent to being aCM. In higher codimension, this is not true. A great deal of work has been done on subschemes of projective space that are *licci*; that is, subschemes that are in the linkage class of a complete intersection. See for instance [71], [73], [74], [75], [101], [127], [128].

4. If two curves on a smooth surface F in $\mathbb{P}^3$ are linearly equivalent then they are evenly linked. Here is an intuitive proof. Say C is linearly equivalent to C' on F. Then $C - C'$ is the divisor of a rational function $\frac{G_1}{G_2}$, so there is some divisor D on F such that G_1 cuts out $C + D$ (i.e. C is linked to D) and G_2 cuts out $C' + D$ (i.e. D is linked to C'). Hence C is evenly linked (in two steps) to C'. More generally, we may replace F by a complete intersection subscheme of $\mathbb{P}^n$ and obtain a similar result for schemes of any dimension and codimension (cf. [109] Exemple 2.4).

Using his notion of generalized divisors on Gorenstein schemes, Hartshorne [62] has recently shown that the above idea using linear equivalence is in a sense the central idea behind liaison, and does not require the hypothesis of smoothness used above. He derives from this approach all of the basic theorems of liaison found in the first three sections of this chapter.

5. There are two very useful programs for making computations in Commutative Algebra, which in particular are very nice sources of examples for

Liaison theory. They are Macaulay [14] and CoCoA [53]. □

Example 4.1.7 We mentioned above that liaison has also been used extensively as a tool for constructing interesting varieties. Here is a very short list of examples. (They will be described in much better detail in §5.3.) Note that they are all in the context of codimension two.

1. In [58] Harris gives sharp bounds for the genus of an integral curve lying on an irreducible surface S of degree k in $\mathbb{P}^3$ (in terms of k and the degree d of the curve). The curves which achieve his bounds are exactly the integral curves residual to a plane curve in a complete intersection of S with a surface of degree $\left[\frac{d-1}{k}\right] + 1$.

2. In [84], Maggioni and Ragusa use liaison to produce smooth aCM curves in $\mathbb{P}^3$ for every Hilbert function of "decreasing type."

3. In [15], Beltrametti, Schneider and Sommese use liaison to construct smooth aCM threefolds of degree 9 and 10 in $\mathbb{P}^5$ and to show that they are the unique examples with the given invariants. In particular, for degree 10 every smooth threefold in $\mathbb{P}^5$ is aCM. On the other hand, in [102] Miró-Roig uses liaison to construct smooth non-aCM threefolds in $\mathbb{P}^5$ of degree $10n, n > 1$, completing the work begun by Banica [13] of determining the degrees of non-aCM smooth threefolds of $\mathbb{P}^5$. Beltrametti, Schneider and Sommese also do a similar analysis of the case of degree 11 in [16]. See also [40] for an overview of work on this problem and that of surfaces in $\mathbb{P}^4$.

4. In [26], the *structure* of an even liaison class (see Question 4 above) is used to show that every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure (see §5.3 for more details). □

4.2 Relations Between Linked Schemes

In this section we will give some answers to the first question raised in the last section. That is, we will find some connections between linked schemes. The main goal here is to provide some basic tools, which will be used later. We shall see in the next section that there are some subtle invariants of a liaison class, but for now we begin with some more natural observations. We will assume some knowledge about primary decomposition and associated primes– see for instance [7] for details.

Lemma 4.2.1 *Let I_{V_1} be a saturated ideal and $I_X \subset I_{V_1}$ a complete intersection of the same dimension. Consider $J = [I_X : I_{V_1}]$. Then J is saturated.*

Proof:

This follows from the fact that I_X is saturated; we leave the details as an exercise. □

Our next observation is that the dimension is preserved under liaison. In fact, a scheme cannot participate in a link unless it is "unmixed" (i.e. its associated primes all have the same height):

Proposition 4.2.2 *Assume that $V_1 \overset{X}{\sim} V_2$. Then*

(a) As sets, $V_1 \cup V_2 = X$.

(b) V_1 and V_2 are equidimensional of the same dimension, and have no embedded components.

(c) If V_1 and V_2 have no common component then they are geometrically linked.

Proof:

We follow the proof in [122]. Recall that by definition, $I_X \subset I_{V_1} \cap I_{V_2}$, $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$. For part (a) have to show that the radicals $\sqrt{I_{V_1} \cap I_{V_2}} = \sqrt{I_X}$.

From the above facts it follows that $I_{V_1} \cdot I_{V_2} \subseteq I_X \subseteq I_{V_1} \cap I_{V_2}$. Then (a) will follow once we prove that in fact $\sqrt{I_{V_1} \cdot I_{V_2}} = \sqrt{I_{V_1} \cap I_{V_2}}$. This is left as an exercise.

For (b), we will show that every associated prime of either I_{V_1} or I_{V_2} is an associated prime of I_X. Then we will be done, since all associated primes of I_X have the same height: its deficiency modules are zero so it is in particular locally Cohen-Macaulay and equidimensional, by Theorem 1.2.2, and this implies that all associated primes have the same height.

Let $\mathcal{P}$ be an associated prime of I_{V_1}, so $\mathcal{P} = [I_{V_1} : x]$ for some homogeneous $x \in S, x \notin I_{V_1}$. In particular, $x\mathcal{P} \subset I_{V_1}$. But $I_{V_1} = [I_X : I_{V_2}]$, so

$$x\mathcal{P} \cdot I_{V_2} \subset I_X \tag{4.1}$$

Now, $x \notin I_{V_1}$ so $xI_{V_2} \not\subset I_X$. Thus there exists $y \in I_{V_2}$ with $xy \notin I_X$. But by (4.1), $xy\mathcal{P} \subset I_X$. Therefore, $\mathcal{P} \subseteq [I_X : xy]$.

We now claim that in fact $\mathcal{P} = [I_X : xy]$. Once we prove this we will be done since the associated primes of I_X are exactly the prime ideals among all ideals of the form $[I_X : z]$ with $z \in S, z \notin I_X$.

To prove the claim consider the set

$$\{ \text{ ideals } [I_X : z] \mid [I_X : z] \supseteq [I_X : xy] \}.$$

Let $\mathcal{Q}$ be a maximal element of this set (possibly $[I_X : xy]$ itself). In particular $\mathcal{Q}$ is an associated prime of I_X. Now, $\mathcal{P} \subset \mathcal{Q}$ and both are prime ideals. If they are not equal then height $\mathcal{P} <$ height $\mathcal{Q}$; that is, codim $V(\mathcal{P}) <$ codim $V(\mathcal{Q})$ (where for an ideal I, $V(I)$ is the vanishing locus of I). This says that V_1 has a component of dimension strictly larger than any component of X (which all have the same dimension). But $X = V_1 \cup V_2$ so this is impossible. This proves (b).

To prove (c), we have to show that $I_X = I_{V_1} \cap I_{V_2}$. The inclusion $\subseteq$ is part of the definition of algebraic linkage, so we have only to prove the reverse inclusion.

Since V_1, V_2 and X are all equidimensional of the same dimension by (b), the associated primes all have the same height, and correspond to components of V_1, V_2 and/or X of maximal dimension. Hence it follows from (a) (independently of the hypothesis of (c)) that any associated prime of I_X is an associated prime of either I_{V_1} or I_{V_2}.

We have by hypothesis that $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$. Let $F \in I_{V_1} \cap I_{V_2}$. By choosing polynomials of large degree, we can choose $G_1 \in I_{V_1}$ such that no power of G_1 is in any associated prime of I_{V_2} (since I_{V_1} and I_{V_2} do not share any associated primes). (It helps to think geometrically.) Similarly we can choose $G_2 \in I_{V_2}$ such that no power of G_2 is in any associated prime of I_{V_1}.

Let $F \in I_{V_1} \cap I_{V_2}$. We want to show that $F \in I_X$. If we write a primary decomposition $I_X = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_t$, then we need to show that $F \in \mathcal{Q}_i$ for all

i. Suppose that for some i, $F \notin \mathcal{Q}_i$. Let $\mathcal{P}_i$ be the corresponding associated prime. We have seen that $\mathcal{P}_i$ is then an associated prime of either I_{V_1} or I_{V_2} but not both (by hypothesis). Suppose, without loss of generality, that $\mathcal{P}_i$ is an associated prime of I_{V_1}. Then $F \cdot G_2 \in I_X$ by definition of the ideal quotient (since $F \in I_{V_1}$ and $[I_X : I_{V_2}] = I_{V_1}$). So $F \cdot G_2 \in \mathcal{Q}_i$. But no power of G_2 is in $\mathcal{P}_i$, so $F \in \mathcal{Q}_i$. □

Remark 4.2.3 Our definition of algebraic linkage appears, at first, to not be quite the same as that used by Peskine and Szpiro [109]; however, the definitions are equivalent. The main point is that there is a natural isomorphism

$$\frac{[I_X : I_{V_1}]}{I_X} \cong \mathrm{Hom}_S(S/I_{V_1}, S/I_X).$$

(Similarly reversing the roles of V_1 and V_2.) The definition of [109] uses the latter formulation. This equivalence is left to the reader as an exercise. Notice that another way of saying this is that I_{V_2} is the annihilator of I_{V_1} in S/I_X. □

A useful application of liaison is to produce new schemes from given ones. To do this, one typically starts with V_1 and chooses an arbitrary complete intersection X containing V_1 and looks for a residual V_2 using the ideal quotient. One needs to know that in fact this construction yields $V_2 \overset{X}{\sim} V_1$.

Proposition 4.2.4 *Let V_1 be a closed subscheme of $\mathbb{P}^n$ with saturated ideal I_{V_1}. Let $I_X \subset I_{V_1}$ be a complete intersection of the same dimension. Consider $J = [I_X : I_{V_1}]$ (which is automatically saturated). Let V_2 be the scheme defined by J, so $J = I_{V_2}$. Then*

(a) $V_1 \cup V_2 = X$ as sets.

(b) V_2 has no embedded components and is equidimensional of the same dimension as X.

(c) If V_1 has embedded components or is not equidimensional then V_1 is not linked to V_2.

(d) ([109] Proposition 2.1) If V_1 has no embedded components and is equidimensional then $V_1 \overset{X}{\sim} V_2$.

Proof:

For (a), note that $[I_X : I_{V_1}] = I_{V_2}$ implies in particular that $I_{V_1}I_{V_2} \subseteq I_X$. It also implies that $I_X \subset I_{V_2}$, so we have that $I_X \subset I_{V_1} \cap I_{V_2}$. Then the same proof as in Proposition 4.2.2 (a) works.

For (b), note that in Proposition 4.2.2 (b) what we really proved was that if $I_{V_1} = [I_X : I_{V_2}]$ and if $\mathcal{P}$ is an associated prime of I_{V_1} then $\mathcal{P}$ is an associated prime of I_X. Here we have exactly the same situation but with V_1 and V_2 interchanged. Hence we conclude that every associated prime of I_{V_2} is an associated prime of I_X so we have (b).

For (c) and (d) we need to consider $[I_X : [I_X : I_{V_1}]]$ and check if this is equal to I_{V_1}. Then the proof of (b) gives that $[I_X : [I_X : I_{V_1}]]$ has no embedded components and is equidimensional, so we immediately get (c).

(d) is somewhat deeper, although notice that it is immediate if V_1 and V_2 have no common component (all components have the same codimension by (b) and by hypothesis) since then one can show that they are geometrically linked. For the general case, by Remark 4.2.3, we need to show that $\text{Ann}(\text{Ann}(I_{V_1})) = I_{V_1}$ in S/I_X. Since X is a complete intersection, S/I_X is in particular a Gorenstein ring. Then (d) follows from some basic facts about Gorenstein rings. We omit the details (but see the following example). □

Example 4.2.5 We will show that (d) above is not necessarily true if X is not arithmetically Gorenstein. This example was produced using the computer program Macaulay [14]. Let $S = k[X_0, \cdots, X_3]$ and let $I_X = (X_0, X_1)^2 = (X_0^2, X_0X_1, X_1^2)$. X has degree 3 and is not arithmetically Gorenstein. Let $I_{V_1} = (X_0^2, X_0X_1, X_1^2, X_0X_2 - X_1X_3)$. V_1 has degree 2. Let $J = [I_X : I_{V_1}]$. One calculates that $J = (X_0, X_1) = I_{V_2}$, which is the ideal of a line V_2 (degree 1) supported on the same set as X and V_1. But $[I_X : I_{V_2}] = I_{V_2}$, so we do not recover I_{V_1}. □

As we have just seen, in a link the only components that matter are those of maximal dimension. So it is natural to ask what happens if V_1 has embedded components or is not equidimensional, X is a complete intersection of the same dimension containing V_1, and we twice apply the ideal quotient with I_X as in Proposition 4.2.4. That is, what can we say about $[I_X : [I_X : I_{V_1}]]$? In fact, it can be checked that the components of I_{V_1} of non-minimal height disappear.

We now describe a very useful exact sequence from [109], relating the ideal sheaves $\mathcal{I}_X, \mathcal{I}_{V_1}$ of X and V_1 and the dualizing sheaf ω_{V_2} of V_2.

Proposition 4.2.6 *Let $V_1 \overset{X}{\sim} V_2$ where $V_1, V_2 \subset \mathbb{P}^n$ of codimension r and X is the complete intersection of hypersurfaces of degree $d_1, \cdots, d_r$. Let $d = \sum d_i$. Then*

$$0 \to \mathcal{I}_X \to \mathcal{I}_{V_1} \to \omega_{V_2}(n+1-d) \to 0 \tag{4.2}$$

Proof:

From Remark 4.2.3, and sheafifying, we get

$$\mathcal{I}_{V_1}/\mathcal{I}_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_X)$$

We need to show that this last term is isomorphic to $\omega_{V_2}(n+1-d)$. This follows immediately from [65] (5.20) by sheafifying. However, we will give another proof which is a modification of a proof in [115]. Consider the exact sequence of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Applying $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, -)$ we get

$$\begin{array}{ccccccc} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_{\mathbb{P}^n}) & \to & \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_X) & \to & \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{I}_X) & \to \\ & \to & \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_{\mathbb{P}^n}) & \to & \cdots . & \end{array}$$

But by [60] III.7.3, $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $0 \le i < r$. Hence

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_X) \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{I}_X).$$

But in fact we can carry this idea even farther. Consider the Koszul resolution for the complete intersection X (see Example 1.4.1):

$$\begin{array}{ccccccccc} 0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\alpha} \mathcal{F}_{r-1} & & \longrightarrow & & \mathcal{F}_{r-2} & & \longrightarrow & & \cdots \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & K_{r-2} & & & & K_{r-3} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ 0 & & & & 0 & & & & 0 \end{array}$$

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \bigoplus \mathcal{O}_{\mathbb{P}^n}(-a_i) & & \longrightarrow & & \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0 \\ \searrow & \nearrow & & \searrow \; \nearrow & & & & \searrow & & \nearrow & \\ & K_2 & & K_1 & & & & & \mathcal{I}_X & & \\ \nearrow & \searrow & & \nearrow \; \searrow & & & & \nearrow & & \searrow & \\ 0 & & 0 & & 0 & & & & & & 0 \end{array}$$

We get that

$$\begin{aligned}
\mathcal{I}_{V_1}/\mathcal{I}_X &\cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},\mathcal{I}_X) \\
&\cong \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},K_1) \\
&\cdots \\
&\cong \mathcal{E}xt^{r-2}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},K_{r-3}) \\
&\cong \mathcal{E}xt^{r-1}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},K_{r-2})
\end{aligned}$$

since the $\mathcal{F}_i$ are free. What is this last term? We use the left-most short exact sequence in the diagram above and get an exact sequence

$$\begin{aligned}
0 \to \mathcal{E}xt^{r-1}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},K_{r-2}) \to \mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},\mathcal{O}_{\mathbb{P}^n}(-d)) &\xrightarrow{\alpha} \\
\mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},F_{r-1}) &\to \cdots
\end{aligned}$$

But we know exactly what α is, since X is a complete intersection– its entries are the generators of I_X. Hence α annihilates every $\mathcal{O}_X$-module. Therefore the image of α is zero and we get

$$\begin{aligned}
\mathcal{I}_{V_1}/\mathcal{I}_X &\cong \mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},\mathcal{O}_{\mathbb{P}^n}(-n-1))(n+1-d) \\
&\cong \mathcal{E}xt^{r}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2},\omega_{\mathbb{P}^n})(n+1-d) \\
&= \omega_{V_2}(n+1-d)
\end{aligned}$$

as desired. □

Remark 4.2.7 The exact sequence in Proposition 4.2.6 gives us that $\mathcal{I}_{V_1}/\mathcal{I}_X \cong \omega_{V_2}(n+1-d)$. But this is the kernel of the natural map $\mathcal{O}_X \to \mathcal{O}_{V_1}$. Thus we also have an exact sequence

$$0 \to \omega_{V_2}(n+1-d) \to \mathcal{O}_X \to \mathcal{O}_{V_1} \to 0.$$

We have a similar exact sequence if we reverse the roles of V_1 and V_2. □

Notice that so far, we have not needed to assume that our schemes are locally Cohen-Macaulay. A useful consequence of Proposition 4.2.6 is that in the locally Cohen-Macaulay case one can produce a locally free resolution of the linked scheme from a locally free resolution of the original scheme plus knowledge of the degrees of the generators of the complete intersection. This is by way of the Mapping Cone procedure.

Proposition 4.2.8 ([109]) *Let* $V_1 \overset{X}{\sim} V_2$ *where* $V_1, V_2 \subset \mathbb{P}^n$ *of codimension* r *and* X *is the complete intersection of hypersurfaces of degree* $d_1, \cdots, d_r$. *Let* $d = \sum d_i$. *Assume that* V_1 *is locally Cohen-Macaulay, and suppose that we are given a locally free resolution for* $\mathcal{I}_{V_1}$ *of the form*

$$\begin{array}{ccccccc} 0 \to \mathcal{G} \to \mathcal{F}_{r-1} \to \cdots \to \mathcal{F}_1 & & \longrightarrow & & \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{V_1} \to 0 \\ & \searrow & & \nearrow & \\ & & \mathcal{I}_{V_1} & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

Then there is a locally free resolution for $\mathcal{I}_{V_2}$ *of the form*

$$0 \to \mathcal{F}_1^\vee(-d) \to \left(\Lambda^1(\oplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i))\right)^\vee(-d) \oplus \mathcal{F}_2^\vee(-d)$$

$$\to \left(\Lambda^2(\oplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i))\right)^\vee(-d) \oplus \mathcal{F}_3^\vee(-d) \to \cdots$$

$$\to \left(\Lambda^{r-1}(\oplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i))\right)^\vee(-d) \oplus \mathcal{G}^\vee(-d) \to \mathcal{I}_{V_2} \to 0.$$

Proof:

Such a locally free resolution for $\mathcal{I}_{V_1}$ can be obtained, for example, as in §2 of Chapter 1. (All that is needed for this construction is that all the sheaves $\mathcal{G}$ and $\mathcal{F}_i$ be locally free, but in particular we can arrange that the $\mathcal{F}_i$ be free.)

By Proposition 4.2.6 we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_{V_1} \to \omega_{V_2}(n+1-d) \to 0.$$

The Koszul resolution for $\mathcal{I}_X$ (cf. Example 1.4.1) and the locally free resolution for $\mathcal{I}_{V_1}$ combine with this short exact sequence to form a commutative diagram

$$
\begin{array}{cccccc}
 & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \\
 & \mathcal{O}_{\mathbb{P}^n}(-d) & \rightarrow & \mathcal{G} & & \\
 & \downarrow & & \downarrow & & \\
 & \vdots & & \vdots & & \\
 & \downarrow & & \downarrow & & \\
 & \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i) & \rightarrow & \mathcal{F}_1 & & \\
 & \downarrow & & \downarrow & & \\
0 \rightarrow & \mathcal{I}_X & \rightarrow & \mathcal{I}_{V_1} & \rightarrow \omega_{V_2}(n+1-d) \rightarrow 0 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}
$$

Then the mapping cone of this diagram gives a locally free resolution of ω_{V_2} (where we write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n}$):

$$0 \rightarrow \mathcal{O}(-d) \rightarrow \mathcal{G} \oplus \bigoplus_i \mathcal{O}(d_i - d) \rightarrow \cdots \rightarrow \bigoplus_i \mathcal{O}(-d_i) \oplus \mathcal{F}_2 \rightarrow \mathcal{F}_1$$

$$\rightarrow \omega_{V_2}(n+1-d) \rightarrow 0.$$

Now we apply $\mathcal{H}om_{\mathcal{O}}(-, \mathcal{O})$, and get

$$0 \rightarrow \mathcal{F}_1^\vee \rightarrow \bigoplus_i \mathcal{O}(d_i) \oplus \mathcal{F}_2^\vee \rightarrow \cdots \rightarrow \mathcal{G}^\vee \oplus \bigoplus_i \mathcal{O}(d - d_i) \rightarrow \mathcal{O}(d)$$

$$\rightarrow \mathcal{E}xt^r_{\mathcal{O}}(\omega_{V_2}(n+1-d), \mathcal{O}) \rightarrow 0.$$

The Proposition will be proved once we show

(a) this is exact, i.e. $\mathcal{E}xt^i_{\mathcal{O}}(\omega_{V_2}, \mathcal{O}) = 0$ for $0 \leq i \leq r-1$;

(b) $\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^n}}(\omega_{V_2}(n+1-d), \mathcal{O}_{\mathbb{P}^n}) = \mathcal{O}_{V_2}(d)$.

To see this, consider a locally free resolution for $\mathcal{O}_{V_2}$:

$$0 \rightarrow \mathcal{G}_r \rightarrow \cdots \rightarrow \mathcal{G}_1 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{V_2} \rightarrow 0.$$

Applying $\mathcal{H}om(-, \mathcal{O})$ and using [60] Lemma III.7.3, we get a locally free resolution

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{G}_1^\vee \rightarrow \cdots \rightarrow \mathcal{G}_r^\vee \rightarrow \omega_{V_2}(n+1) \rightarrow 0.$$

Applying $\mathcal{H}om(-,\mathcal{O})$ again we get

$$0 \to \mathcal{G}_r \to \cdots \to \mathcal{G}_1 \to \mathcal{O} \to \mathcal{E}xt^r_{\mathcal{O}}(\omega_{V_2}(n+1),\mathcal{O}) \to 0.$$

This is not exact a priori, but we know it is by comparison with the original resolution. Therefore we get $\mathcal{E}xt^i_{\mathcal{O}}(\omega_{V_2},\mathcal{O}) = 0$ for $0 \leq i \leq r-1$ and we have shown (a).

For (b), we again compare the two resolutions. We get that

$$\begin{aligned} \mathcal{E}xt^r_{\mathcal{O}}(\omega_{V_2}(n+1-d),\mathcal{O}) &= \mathcal{E}xt^r_{\mathcal{O}}(\omega_{V_2}(n+1),\mathcal{O})(d) \\ &= \mathcal{O}_{V_2}(d) \end{aligned}$$

as desired. □

As an immediate corollary (considering the projective dimension and using the local version of the Auslander-Buchsbaum theorem– see page 11), we get the important fact that the property of being locally Cohen-Macaulay is preserved under liaison:

Corollary 4.2.9 *Let $V_1 \overset{X}{\sim} V_2$ where $V_1, V_2 \subset \mathbb{P}^n$ and X is a complete intersection. Then V_1 is locally Cohen-Macaulay if and only if V_2 is locally Cohen-Macaulay.*

These results are very useful in general. We will give an important consequence of Proposition 4.2.8, namely the Hartshorne-Schenzel Theorem, in the next section. For now we give some more elementary consequences of Proposition 4.2.6.

Corollary 4.2.10 *If $V_1 \overset{X}{\sim} V_2$, and if one (hence both) of V_1 and V_2 are locally Cohen-Macaulay, then $\deg V_1 + \deg V_2 = \deg X$.*

Proof:

This is clear in the case of geometric linkage. We will sketch the proof in the case of algebraic linkage. For convenience say that dim V_1 = dim V_2 = dim $X = p$.

Note that for $t \gg 0$,

$$\begin{aligned} h^0(\mathcal{O}_{V_1}(t)) &= P(V_1,t) \qquad \text{(the Hilbert polynomial of } V_1) \\ &= \frac{\deg V_1}{p!} t^p + \text{(lower terms)}. \end{aligned}$$

Hence for $t \gg 0$, since $h^1(\mathcal{I}_{V_1}(t)) = 0$, we have

$$h^0(\mathcal{I}_{V_1}(t)) = \binom{t+n}{n} - \left[\frac{\deg\, V_1}{p!}t^p + (\text{lower terms})\right].$$

Similarly,

$$h^0(\mathcal{I}_X(t)) = \binom{t+n}{n} - \left[\frac{\deg\, X}{p!}t^p + (\text{lower terms})\right].$$

Now, for $t \gg 0$ use [60] pp. 230, 243 and 244 to deduce that

$$\begin{aligned} P(V_2, -t) &= h^0(\mathcal{O}_{V_2}(-t)) - h^1(\mathcal{O}_{V_2}(-t)) + \cdots + (-1)^p h^p(\mathcal{O}_{V_2}(-t)) \\ &= (-1)^p h^p(\mathcal{O}_{V_2}(-t)) \\ &= (-1)^p h^0(\omega_{V_2}(t)). \end{aligned}$$

That is, for $t \gg 0$ we have

$$\begin{aligned} h^0(\omega_{V_2}(t)) &= (-1)^p \left[\frac{\deg\, V_2}{p!}(-t)^p + (\text{lower terms})\right] \\ &= \left[\frac{\deg\, V_2}{p!}t^p + (\text{lower terms})\right]. \end{aligned}$$

Finally, use the exact sequence of Proposition 4.2.6, twist by $t \gg 0$, take cohomology, compute dimensions using the above facts, and compare leading terms. □.

One can also relate the Hilbert polynomials of the linked schemes. (See for instance [62], Proposition 4.7.) In the case of curves, a nice formula emerges. Let g_1 and g_2 be the arithmetic genera of C_1 and C_2 respectively.

Corollary 4.2.11 *If* $C_1 \overset{X}{\sim} C_2$ *then*

$$g_2 - g_1 = \frac{1}{2}(d - n - 1)(deg\ C_2 - deg\ C_1).$$

Proof:

The proof is similar to that of the last corollary. Notice that for curves, being locally Cohen-Macaulay is equivalent to having no embedded points, and this is guaranteed by the fact that the curves are linked (Proposition 4.2.2). The only other fact that is needed is that the arithmetic genus of the complete intersection X is

$$\frac{1}{2}\left(\prod d_i\right)\left(\sum d_i - (n+1)\right) + 1$$

(cf. Example 1.4.1). Then compute cohomology dimensions in the exact sequence of Proposition 4.2.6 but keep track of the "lower terms." □

Corollary 4.2.12 *Let C_1 and C_2 be curves. If $C_1 \overset{X}{\sim} C_2$, and if deg $C_1 =$ deg C_2, then $g_1 = g_2$.*

It is an amusing exercise to study the converse of Corollary 4.2.12 and see what it would take to find a counterexample. For instance, suppose $n = 3$.

Another useful fact is that linkage is preserved under hyperplane or hypersurface section (see §3 of Chapter 1). This is intuitively obvious, but there is something to prove.

Proposition 4.2.13 *Let $V_1 \overset{X}{\sim} V_2$ in $\mathbb{P}^n$, where $2 \leq \operatorname{codim} X = r < n$. Let F be a general hypersurface of degree d. Then $(V_1 \cap F) \overset{X\cap F}{\sim} (V_2 \cap F)$ in $\mathbb{P}^n$.*

Proof:

Notice that $X \cap F$ is a complete intersection: if $I_X = (F_1, \cdots, F_r)$ then the saturated ideal $I_{X\cap F} = (F_1, \cdots, F_r, F) = I_X + (F)$. By Proposition 4.2.4 we need to show that the saturated ideals of the hypersurface sections satisfy

(a) $I_{X\cap F} \subset I_{V_1\cap F} \cap I_{V_2\cap F}$, and
(b) $[I_{X\cap F} : I_{V_1\cap F}] = I_{V_2\cap F}$.

The proof of (a) is quick:

$$\begin{aligned} I_{X\cap F} &= I_X + (F) \\ &\subset I_{V_1} + (F) \\ &\subset \overline{I_{V_1} + (F)} \\ &= I_{V_1\cap F} \end{aligned}$$

and similarly for V_2.

For (b), notice that both sides of the desired equality represent saturated, unmixed ideals of schemes of the same codimension. We first show that these schemes have the same degree; after that it suffices to show one of the two inclusions to get equality.

We know that deg V_1 + deg V_2 = deg X. Hence the left-hand side of the desired equality is the saturated ideal of a scheme of degree (deg X) $\cdot d$ − (deg V_1) $\cdot d$ = (deg V_2) $\cdot d$, which is the same as the degree of the scheme represented by the right-hand side.

Now we just show that $[I_{X \cap F} : I_{V_1 \cap F}] \supset I_{V_2 \cap F}$. We leave it to the reader to verify the fact that if I and J are ideals with $I = \bar{I}$ then $[I : J] = [I : \bar{J}]$. Having this fact, we only need to show that $[I_X + (F) : I_{V_1} + (F)] \supset I_{V_2 \cap F}$. But clearly $[I_X + (F) : I_{V_1} + (F)] \supset I_{V_2} + (F)$, and the left-hand side is saturated, so we are done. □

We can use this last result to remove the hypothesis of being locally Cohen-Macaulay from Corollary 4.2.10:

Corollary 4.2.14 *If $V_1 \overset{X}{\sim} V_2$ then deg V_1 + deg V_2 = deg X.*

Proof:
If the schemes are zero-dimensional then Corollary 4.2.10 applies. Otherwise take enough hyperplane sections to reduce to this case, and note that Proposition 4.2.13 guarantees that linkage is preserved, while Bezout's theorem guarantees that the degrees are preserved. □

A natural question is whether the converse of Proposition 4.2.13 is true. The answer is affirmative, at least in the case of geometric linkage. This is an interesting application of Theorem 2.3.1.

Proposition 4.2.15 *Let V_1 and V_2 be locally Cohen-Macaulay equidimensional subschemes of $\mathbb{P}^n$. Assume that for a general homogeneous polynomial F of degree d, $V_1 \cap F$ is directly linked to $V_2 \cap F$ and that $V_1 \cap F$ and $V_2 \cap F$ have no common components. Then V_1 is directly linked to V_2, unless one of the following holds:*

(i) $d = 1$, $n = 3$ and $V_1 \cup V_2$ lies on a quadric surface;

(ii) $d = 1$, $n = 3$ and deg V_1 + deg V_2 = 4.

Proof:

Since $V_1 \cap F$ and $V_2 \cap F$ have no common component and are directly linked, they are geometrically linked (by Proposition 4.2.2). That is, their union is a complete intersection. But since they have no common component, we also have that V_1 and V_2 have no common component (and no component of one is contained in a component of the other since the components are all of the same dimension). So the scheme-theoretic union $V_1 \cup V_2$ has the property that its general degree d hypersurface section is a complete intersection. Then by Theorem 2.3.1 (for curves) or Theorem 1.3.2 (for higher dimension), $V_1 \cup V_2$ is a complete intersection, so since they have no common component they are directly linked. □

Something similar should be true for the case of algebraic linkage, although not quite as strong (i.e. the list of exceptions should be slightly bigger). This is still an open problem. The following example illustrates some of the possible considerations and obstacles.

Example 4.2.16 In $\mathbb{P}^3$, let V_1 be a double line (i.e. a scheme of degree 2 supported on a line– cf. [92]) and let V_2 be a scheme of degree 4 supported on the same line as V_1. The general hyperplane section of V_1 is a complete intersection. Assume that the general hyperplane section of V_2 is a complete intersection, say (x^2, y^2). By considering links of the form $(x^2, y^2 \cdot \ell)$ (where ℓ is a linear form) one can verify that the general hyperplane section of V_2 is directly linked to the general hyperplane section of V_1. However, by changing the linkage class of V_1 if necessary (cf. [92]) we have that V_1 is not directly linked to V_2. □

4.3 The Hartshorne-Schenzel Theorem

In this section we give a more subtle relation between linked schemes, in the form of a necessary condition for two schemes to be linked. That is, we will show the invariance (up to shifts and duals and re-indexing) of the deficiency modules. The fact that the dimensions of the components (suitably re-indexed) is preserved was known already by Gaeta [48], but the invariance of the module structure of course requires knowledge of schemes.

Throughout this section we will assume that the schemes we consider are locally Cohen-Macaulay, so that we obtain a locally free sheaf from the resolution of the ideal sheaf. This assumption has recently been removed by Hartshorne in the case of even liaison; he proved Corollary 4.3.3 below without this hypothesis by using reflexive sheaves and his notion of generalized divisors ([62] Proposition 4.5). Notice that Theorem 4.3.1 itself cannot be extended to the non locally Cohen-Macaulay case since for any such V, $(M^i)(V)$ will then fail to have finite length, but it is always zero in large degree, so the conclusion of the theorem cannot hold. So in a sense, Corollary 4.3.3 is more basic than Theorem 4.3.1. See also Example 4.1.6 (4.).

The theorem was proved for curves in $\mathbb{P}^3$ by Hartshorne (cf. [113]), for curves in $\mathbb{P}^n$ by Chiarli [36], and for arbitrary dimension by Schenzel [120]) and subsequently by Migliore [94]. A different proof has recently been given by Hartshorne [62]. We follow the proof given in [94]. We will use the following notation:

$$(M^i)^\vee(V) := \left[(M^i)(V)\right]^{\vee k} = \operatorname{Hom}_k((M^i)(V), k)$$

(see page 3).

Theorem 4.3.1 *Let $V_1, V_2 \subset \mathbb{P}^n$ have dimension r and assume that $V_1 \overset{X}{\sim} V_2$ where $I_X = (F_1, \cdots, F_{n-r})$ and deg $F_i = d_i$. Let $d = \sum d_i$. Then*

$$(M^{r-i+1})(V_2) \cong (M^i)^\vee(V_1)(n+1-d) \quad \textit{for } 1 \leq i \leq r.$$

Proof:

Consider a locally free resolution for $\mathcal{I}_{V_1}$ as in Proposition 4.2.8

$$\begin{array}{ccccccccccc}
0 \to \mathcal{G}_{n-r} \to \mathcal{F}_{n-r-1} & & \longrightarrow & & \cdots \to \mathcal{F}_2 & & \longrightarrow & & \mathcal{F}_1 \to \mathcal{I}_V \to 0 \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & \mathcal{G}_{n-r-1} & & & & \mathcal{G}_2 & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
0 & & & & 0 \quad 0 & & & & 0
\end{array} \tag{4.3}$$

where $\mathcal{G}_{n-r}$ is locally free and the $\mathcal{F}_i$ are free. Then by Proposition 4.2.8 we have a locally free resolution

$$
\begin{array}{ccccccc}
0 \to \mathcal{F}_1^\vee(-d) \to \bigoplus_i \mathcal{O}(d_i - d) \oplus \mathcal{F}_2^\vee(-d) & \to & \cdots & & & & \\
 & \searrow & & \nearrow & & & \\
 & & \mathcal{K}_2 & & & & \\
 & \nearrow & & \searrow & & & \\
0 & & & & 0 & &
\end{array}
\tag{4.4}
$$

$$
\begin{array}{ccccc}
\cdots & \to & \mathcal{G}_{n-r}^\vee(-d) \oplus \bigoplus_i \mathcal{O}(-d_i) \to \mathcal{I}_{V_2} \to 0. & & \\
 & \searrow & & \nearrow & \\
 & & \mathcal{K}_{n-r-1} & & \\
 & \nearrow & & \searrow & \\
0 & & & & 0
\end{array}
$$

From the short exact sequences obtained from this latter resolution, beginning from the left, one sees that

$$0 = H_*^{n-r-2}(\mathcal{K}_2) \cong \cdots \cong H_*^2(\mathcal{K}_{n-r-2}) \cong H_*^1(\mathcal{K}_{n-r-1})$$

$$\vdots$$

$$0 = H_*^{n-2}(\mathcal{K}_2) \cong \cdots \cong H_*^{r+2}(\mathcal{K}_{n-r-2}) \cong H_*^{r+1}(\mathcal{K}_{n-r-1})$$

Therefore we have

$$
\begin{array}{rcll}
(M^{r-i+1})(V_2) & = & H_*^{r-i+1}(\mathcal{I}_{V_2}) & \\
 & \cong & H_*^{r-i+1}(\mathcal{G}_{n-r}^\vee(-d)) & \text{(by (4.4))} \\
 & \cong & H_*^{n-r+i-1}(\mathcal{G}_{n-r}(d-n-1))^* & \text{(by Serre duality)} \\
 & \cong & H_*^{n-r+i-2}(\mathcal{G}_{n-r-1}(d-n-1))^* & \text{(by (4.3))} \\
 & \vdots & & \\
 & \cong & H_*^{i+1}(\mathcal{G}_2(d-n-1))^* & \text{(by (4.3))} \\
 & \cong & H_*^{i}(\mathcal{I}_{V_1}(d-n-1))^* & \text{(by (4.3))} \\
 & = & (M^i)^\vee(V_1)(n+1-d) & \square
\end{array}
$$

Remark 4.3.2 With this theorem we can now answer some of the questions raised in §1 of this chapter. First, the equivalence relation of liaison is not trivial, at least in dimension > 0. Indeed, this theorem says that the deficiency modules are invariant, up to duals and shifts and re-indexing, in a liaison class. But as we remarked in §2 of Chapter 1, a theorem of Evans and

Griffith [46] states that any collection of modules can be realized, up to shift, as the deficiency modules of a scheme of appropriate dimension. (Huneke and Ulrich have shown that it is not trivial in dimension 0 either, apart from sets of points in $\mathbb{P}^2$. However, this does not follow from Theorem 4.3.1. Their approach is to study the linkage class of a complete intersection, and show that not every set of points is in this class. In fact, they study arithmetically Cohen-Macaulay schemes of any dimension from this point of view.)

This already says that there are at least as many liaison classes (roughly) as there are collections of graded S-modules. However, the question remains as to whether to each collection of modules (up to shifts and duals) there is associated just one liaison class, or many. We will discuss this later. (This is related to the problem of finding sufficient conditions for two schemes to be linked.)

Another immediate consequence is that the property of being arithmetically Cohen-Macaulay is invariant in a liaison class, since this property depends only on the collection of modules being zero.

We can also show that the property of being arithmetically Buchsbaum is preserved under liaison. For curves it is an immediate consequence of the definition (Definition 1.4.7) since it depends only the fact that the deficiency module is annihilated by the maximal ideal $\mathfrak{m} = (X_0, \cdots, X_n)$, and this does not change with shifts or duals. For higher dimension, the definition of being arithmetically Buchsbaum is given after Definition 1.4.7. What is different in higher dimension is that we require that $\mathfrak{m}$ should annihilate the deficiency modules not only of the original scheme V, but also of the general hyperplane section $V \cap H_1$, the general hyperplane section $V \cap H_1 \cap H_2$, etc. That is, we require that the scheme obtained by intersecting V with a general linear space of dimension $\geq n - \dim V + 1$ also have its deficiency module(s) annihilated by $\mathfrak{m}$. But then the fact that the Buchsbaum property is preserved under liaison follows from Proposition 4.2.13.

A useful application of Theorem 4.3.1 is to show that two schemes are *not* linked, by showing that their collection of modules are not the same even after dualizing and shifting. We will apply this idea in a geometric way in the next section. □

We have been a little careless in saying that Theorem 4.3.1 implies that the deficiency modules are invariant up to shifts and duals and re-indexing.

For curves there is no problem, but for higher dimension one has to be a little careful. It is very important to realize that the twist $n+1-d$ is the same for each module. Hence the "positioning" of the modules $(M^i)(V)$ with respect to each other stays the same in even liaison and "flips" in odd liaison. In particular we have

Corollary 4.3.3 *Let V_1 and V_2 be evenly linked schemes of dimension r. Then there is an integer p such that $(M^i)(V_1)(p) \cong (M^i)(V_2)$ for each $1 \leq i \leq r$.*

The point of this corollary is that the same p is used for each i. That is, in even liaison the *configuration* of modules is preserved up to shift. This will play an important role in the structure theorem in the next chapter. We will discuss it further below. This is our first indication that even liaison is considerably simpler than odd liaison, since dual modules do not have to be considered. It will turn out that an even liaison class has a "nice" structure, while a liaison class does not (apart from being a union of two even liaison classes in general).

Remark 4.3.4 One of the questions raised in §1 was whether we could perform links using something other than complete intersections and still get a good linkage theory. We remarked that Schenzel [120] has shown that most of the theory works if we replace "complete intersection" by "arithmetically Gorenstein." (See also [27].) One of the results is that in fact Theorem 4.3.1 continues to hold in that context.

On the other hand, as we mentioned in §1, Walter has shown that if we replace "complete intersection" by "arithmetically Cohen-Macaulay" then there is just one equivalence class (for fixed dimension). It is easy to see that at the very least we cannot expect the module to be invariant in this case: let V_1 be a set of two skew lines in $\mathbb{P}^3$ and let V_2 be a line meeting both components of V_1. Then $V_1 \cup V_2$ is arithmetically Cohen-Macaulay, but V_2 has a trivial deficiency module while V_1 has a one-dimensional module (cf. Example 1.4.2). □

We can now begin to describe the structure of an even liaison class, although the main theorem will come in Chapter 5 (Theorem 5.2.1). The first ingredient is the notion of Basic Double Linkage (see Remark 3.2.4), and we will now see why it is so called. (We will change the notation slightly to be consistent with that of the current chapter: r shall now denote the dimension rather than the codimension.)

Recall that we start with a scheme $V_1 \neq \emptyset$ of dimension r and we choose general homogeneous polynomials $F_1 \in S$ of degree d_1 and $F_2, \cdots, F_{n-r} \in I_{V_1}$ of degree $d_2, \cdots, d_{n-r}$ respectively, such that $(F_1, F_2, \cdots, F_{n-r})$ form a regular sequence. We then consider the scheme Z with defining (saturated) ideal $I_Z = F_1 I_{V_1} + (F_2, \cdots, F_{n-r})$.

Recall also that for $1 \leq i \leq r$ we have $(M^i)(Z) \cong (M^i)(V_1)(-d_1)$. The key observation now is that we in fact have something stronger: Z is linked to V_1 in two steps. (This was observed in [80] for codimension two and in [25] and [52] in general.) The basic idea is to choose a general homogeneous polynomial $G \in I_{V_1}$ of sufficiently large degree so that $(G, F_2, \cdots, F_{n-r})$ forms a regular sequence, and hence so does $(GF_1, F_2, \cdots, F_{n-r})$. Then one can check that if one links V_1 to a scheme W using the complete intersection $(G, F_2, \cdots, F_{n-r})$, and then links W using the complete intersection $(GF_1, F_2, \cdots, F_{n-r})$, the resulting scheme is exactly Z.

Now fix an even liaison class $\mathcal{L}$ of dimension r subschemes of $\mathbb{P}^n$. We have associated to $\mathcal{L}$ a configuration of modules $\{(M_1), \cdots, (M_r)\}$, unique up to shift. By Proposition 1.2.5, a sufficiently large leftward shift of this configuration of modules (i.e. $\{(M_1)(d), (M_2)(d), \cdots, (M_r)(d)\}$ for $d \gg 0$) is not the configuration of modules of any scheme $V \in \mathcal{L}$, and so there is a leftmost shift (i.e. maximum d) for which there exists some $V \in \mathcal{L}$ with $(M^i)(V) = M_i(d)$ for some d.

Definition 4.3.5 We will denote by $\mathcal{L}^0$ the set of schemes $V \in \mathcal{L}$ whose configuration of modules coincides with this leftmost one. The elements of $\mathcal{L}^0$ are the *minimal* elements of the even liaison class $\mathcal{L}$ and the corresponding configuration of modules is said to be in the *minimal shift*. Let $V_0 \in \mathcal{L}^0$. We will denote by $\mathcal{L}^h$ the set of schemes $V \in \mathcal{L}$ satisfying $M^i(V) \cong M^i(V_0)(-h)$ for $1 \leq i \leq r$. By Basic Double Linkage, $\mathcal{L}^h \neq \emptyset$ for all $h \geq 0$. □

As a consequence, we have partitioned $\mathcal{L} = \mathcal{L}^0 \cup \mathcal{L}^1 \cup \mathcal{L}^2 \cdots$ according to the shift of the configuration of modules. The key to answering question 4 of §1 (i.e. to describe the structure of an even liaison class) is to study the relations between the elements of these subsets. This structure has been called the *Lazarsfeld-Rao Property* (cf. [25], [11]). This is known to hold for codimension two (cf. [11], [87]), and it will be described carefully in Chapter 5. However, the structure can be phrased in arbitrary codimension, even if it is unknown to hold in codimension ≥ 3, so we will end this section with a description of that property.

Definition 4.3.6 [25] Let $\mathcal{L}$ be an even liaison class of dimension r subschemes of $\mathbb{P}^n$. We say that $\mathcal{L}$ has the *Lazarsfeld-Rao Property* if the following conditions hold:

(a) If V_1, $V_2 \in \mathcal{L}^0$ then there is a deformation from one to the other through subschemes all in $\mathcal{L}^0$; (in particular, all subschemes in the deformation are in the same even liaison class).

(b) Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ $(h \geq 1)$, there exists a sequence of subschemes $V_0, V_1, \cdots, V_t$ such that for all i, $1 \leq i \leq t$, V_i is a basic double link of V_{i-1} and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.

This is also sometimes referred to as the *LR-Property*. □

4.4 Geometric Invariants of a Liaison Class

In the last section we saw that the graded modules $(M^i)(V)$ $(1 \leq i \leq \dim V)$ are invariant, up to shifts and duals (resp. shifts), in the liaison class (resp. even liaison class) of V. This implies that the degeneracy loci V_k, described in §1 of Chapter 1, are invariant (after suitable re-indexing). These loci essentially parameterize the set of linear forms that have "unusual" rank when viewed as homomorphisms from a component $(M^i)(V)_k$ to the next component $(M^i)(V)_{k+1}$.

In this section we will show how this fact can be used to study the liaison class of V when V is a curve (locally Cohen-Macaulay and equidimensional, as always), and hence $i = 1$. Our main references are [91] and [94]. However, we remark that it would be interesting to understand the connection between the V_k and V in the case where $\dim V$ is larger and especially when i is larger.

The general philosophy that emerges, which is illustrated in several examples below, is that the points of V_k correspond to those hyperplanes meeting V in "unusual" ways, either by containing a component of the curve or by having unusual postulation for the hyperplane section. This often yields useful necessary conditions for curves to be evenly linked, as we will see.

We remark that this general description should still be true for the higher cohomology modules, but it needs to be made precise. (The meaning of "unusual" may be entirely different for larger i.)

We will use the letter C to denote our curves rather than V, to avoid confusion with the degeneracy loci V_k. We now illustrate the ideas with several examples.

Example 4.4.1 Let $C \subset \mathbb{P}^3$ be the disjoint union of a line λ and a plane curve Y of degree $d \geq 1$. This situation was analyzed in Example 1.4.4. It was shown that $M(C)_i$ is one-dimensional for $0 \leq i \leq d-1$, and 0 elsewhere. Furthermore, the degeneracy locus $V_0 \subset (\mathbb{P}^3)^*$ is just the plane dual to the point P of intersection of λ with the plane of Y. Since this is an isomorphism invariant, we have that any curve evenly linked to C has the same degeneracy locus in the first non-trivial module multiplication. Furthermore, by Remark 1.3.3 (d), if we denote by $\mathcal{L}$ the even liaison class of C, then $C \in \mathcal{L}^0$.

We now describe a simple link that can be done with C. (We will use the same names for surfaces and for polynomials defining those surfaces.) Let $F_1 = \Lambda_1 \cup \Lambda_2$ be a surface of degree 2 consisting of the plane Λ_1 of Y and a plane Λ_2 containing λ. Let F_2 be a surface of degree $d+1$ containing C but not containing any component in common with F_1. (F_1, F_2) is a complete intersection, and hence gives a residual curve C'.

What is C'? Looking first at the intersection of F_2 with Λ_1 we see that the residual to Y is a line λ' on Λ_1 passing through P (since Y avoids P but both Λ_1 and F_2 contain P). The residual to λ in the complete intersection of F_2 with Λ_2 is a plane curve Y' of degree d. Hence $C' = Y' \cup \lambda'$. If Y' and λ' were to meet, then by Example 1.4.5 C' would be aCM. But we know that C is not aCM and that the property of being or not being aCM is preserved under liaison. Therefore C' is also the disjoint union of a line and a plane curve of degree d, with P again the point of intersection of λ' and the plane of Y'.

We now ask when two such curves can be in the same liaison class. The following can be shown from the above discussion (the details are left as an exercise). Let C be the disjoint union of a line λ and a plane curve Y of degree d, with P the point of intersection of λ with the plane of Y. Let C', λ', Y', P' and d' be similarly defined. Then C and C' are in the same liaison class if and only if $d = d'$ and $P = P'$.

(To show that the conditions $d = d'$ and $P = P'$ imply that C and C' are linked, for now one has to proceed by brute force, constructing a sequence of links in the manner outlined. However, once we have Rao's theorem in the next chapter, this will follow from the observation that these two conditions force $M(C)$ and $M(C')$ to be isomorphic.) □

Example 4.4.2 Let C be a nondegenerate set of d skew lines in $\mathbb{P}^n$, with $2d > n + 1$. Assume that the general hyperplane section $Z = C \cap H$ of C is nondegenerate in the hyperplane H. We first compute that $\dim M(C)_0 = d - 1$ and $\dim M(C)_1 = 2d - (n + 1)$. (These come from the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0,$$

twisting by either 0 or 1, and taking cohomology.)

Now, for a linear form L defining a hyperplane H which does not contain any component of C, there is an exact sequence

$$0 \to H^0(\mathcal{I}_{C\cap H}(1)) \to M(C)_0 \xrightarrow{\phi_0(L)} M(C)_1 \to \cdots.$$

(This is from the exact sequence in cohomology associated to the short exact sequence (3.4) of §1 of Chapter 1. The first 0 is because C is non-degenerate.) The hypothesis that the general hyperplane section is a set of points which is nondegenerate in H guarantees that $\phi_0(L)$ is injective for general L. In particular, we must have that $\dim M(C)_0 \leq \dim M(C)_1$, i.e. $n \leq d$.

In fact, clearly if L does not vanish on any component of C then L can be viewed as a point of V_0 if and only if the points of $H \cap C$ are degenerate in H. (For example, if C is a curve in $\mathbb{P}^3$ then L is a point of V_0 if and only if H contains a d-secant line of C.)

On the other hand, we still have to consider the linear forms L which do vanish on a component λ of C. Let Y be the remaining set of lines. Then for any k we have the following two exact sequences of sheaves:

$$0 \to \mathcal{I}_C(k) \to \mathcal{I}_Y(k) \to \mathcal{O}_\lambda(k) \to 0$$

and

$$0 \to \mathcal{I}_Y(k) \xrightarrow{\times L} \mathcal{I}_C(k+1) \to \mathcal{I}_{C\cap H}(k+1) \to 0.$$

Here $C \cap H$ is one-dimensional. Notice that the composition

$$0 \to \mathcal{I}_C(k) \to \mathcal{I}_Y(k) \xrightarrow{\times L} \mathcal{I}_C(k+1)$$

induces the map $\phi_k(L)$ on $M(C)$. But now taking $k = 0$ we get

$$0 \to H^0(\mathcal{O}_\lambda) \to M(C)_0 \to M(Y)_0$$

and

$$0 \to H^0(\mathcal{I}_{C\cap H}(1)) \to M(Y)_0 \to M(C)_1.$$

Since $h^0(\mathcal{O}_\lambda) = 1$, we get that $\phi_0 : M(C)_0 \to M(C)_1$ is not injective. Combining this with the fact that the general linear form induces an injection, we get that L corresponds to a point of V_0.

The conclusion, then, is that L (up to scalar multiples) corresponds to a point of V_0 if and only if either L vanishes on a component of C or else the points of $H \cap C$ are degenerate in H (and for general L this does not occur). Note that if the points of $H \cap C$ are degenerate in H, then any linear form L' vanishing on the span of $H \cap C$ will also correspond to a point of V_0. Hence in the dual projective space $(\mathbb{P}^n)^*$, V_0 is a union (possibly infinite) of linear subvarieties: it contains the codimension two linear subvarieties dual to the lines of C, as well as the linear spaces dual to the spans of the degenerate sets $H \cap C$. C can often be recovered from V_0 in this way. (See Example 4.4.4.) $\square$

Example 4.4.3 Suppose that, as in the previous example, we take C to be a nondegenerate set of d skew lines in $\mathbb{P}^n$ with $2d > n+1$, but now we assume that the general hyperplane section is *degenerate* in the hyperplane. We will show that C must lie on a quadric hypersurface.

We use the "Socle Lemma" of [72] (see Lemma 2.3.3 and take $d = 1$) and apply it to $M(C)$. Since $h^0(\mathcal{I}_{C\cap H}(1)) \neq 0$ by hypothesis, this means that $b = 1$. Hence Lemma 2.3.3 guarantees that $i([0 :_{\mathfrak{C}} \mathfrak{m}]) \leq 0$. But $M(C)$ begins in degree 0, and in this degree $M(C) = \mathfrak{C}$. Hence $[0 :_{M(C)} \mathfrak{m}]$ is nonzero in degree 0.

In particular, for any two linear forms L_1, L_2, if we let $A = (L_1, L_2)$ then there is some element in $M(C)_0$ annihilated by A. That is, the submodule K_A (cf. Definition 2.1.3) is nonzero in degree 0. This implies, by Proposition 2.1.4, that $I_C \cap A$ is non-zero in degree 2. That is, even the union of C with a general linear subspace of codimension two lies on a quadric hypersurface. $\square$

Example 4.4.4 Let C now be a set of d skew lines in $\mathbb{P}^3$. We give a brief summary of what happens in this case; see [91] for more details. Since we know that $M(C) \cong k$ for $d = 2$, we will assume that $d \geq 3$ so that there is nontrivial module structure to study. We have seen in Example 4.4.2 that the degeneracy locus V_0 identifies the hyperplanes in $\mathbb{P}^3$ which contain either a d-secant line or else a component of C.

There are two possibilities (from the point of view of what we want to discuss): either C lies on a quadric surface or it does not. If C lies on a quadric surface Q then clearly it must be smooth in order to contain skew lines. One checks that the hyperplanes tangent to Q are precisely the hyperplanes that we are looking for. One can thus recover Q from the module, but no more. And indeed, C is linked via Q and an appropriate union of d hyperplanes to a set of d skew lines in the other ruling of Q. It follows (since dim $M(C)_0 =$ dim $M(C')_0 = d-1$) that any set of skew lines in the liaison class (either even or odd) of C also has d components, and also lies on Q, but there is no uniqueness– any such set of lines is in the liaison class.

In fact, it can be shown that if C is a set of $d \geq 3$ skew lines lying on a quadric surface Q, and if C' is set of d' skew lines, then C is evenly linked to C' if and only if $d = d'$ and C' also lies on the same ruling of Q. They are oddly linked if and only if $d = d'$ and C' lies on the other ruling of Q.

On the other hand, if C does not lie on a quadric surface then by Example 4.4.3, the general hyperplane section does not consist of collinear points. That is, the general hyperplane section gives an injection of $M(C)_0 \rightarrow M(C)_1$. Now, since C does not lie on a quadric surface, it must consist of at least four lines. And for the same reason, one can check that it has at most two d-secants. (If it had three d-secants, Bezout's theorem would force C to lie on the quadric surface spanned by the three d-secants.) Then V_0 consists of a finite set of lines in $(\mathbb{P}^3)^*$, and so from the configuration of lines one can recover C from $M(C)$. That is, C is the only set of skew lines in its even liaison class. (See [80] for another example, more algebraic, of how to recover a curve from the deficiency module, under certain circumstances.)

However, there may be a set of skew lines (at most one) oddly linked to C. A simple example is provided by a so-called "double-six" configuration on a cubic surface (cf. [55]; see also Example 4.4.5), in which a set of six skew lines is directly linked to another set of six skew lines. See [91] for more details. □

Example 4.4.5 Let C be a general rational sextic curve in $\mathbb{P}^3$. We will briefly describe the lovely geometry relating the geometry of C to the geometry of the degeneracy locus V_1; for details see [91]. The problem we would like to solve is to determine which other curves C' of arithmetic genus 0 are in the liaison class of C.

Because C has maximal rank (cf. [12]), C lies on a unique cubic surface S. Since C is general, S is smooth. Hence we know all the divisors on S, and in particular there are 27 lines (cf. [60], [55]). The geometry of S will play an important role.

One immediate observation, made in Example 4.1.6, is that any curve which is linearly equivalent to C on S is evenly linked to C. Also, any curve linked to C by S and a quartic surface has degree 6 as well, and hence also has arithmetic genus 0 (cf. Corollary 4.2.11). We will see that these are the only possibilities.

On the other hand, maximal rank (together with the Riemann-Roch theorem) also allows us to compute that the deficiency module $M(C)$ is 3-dimensional in degrees 1 and 2, and zero elsewhere. This shows, in particular, that if C' is in the liaison class of C with arithmetic genus 0 then C' also has degree 6. (One can also use the Lazarsfeld-Rao Property to deduce this.)

It is not hard to show that the general hyperplane section of C consists of 6 points not on a conic. (For example, the general hyperplane section does not have any collinear points, so if it lay on a conic it would be a smooth conic rather than a union of two lines. But then it would be a complete intersection, and so by Strano's theorem [124] (see Theorem 2.3.1) C would have to be a complete intersection.) Hence the degeneracy locus V_1 consists of those linear forms whose corresponding hyperplanes intersect C in six points on a (possibly singular) conic.

By Porteous' formula (cf. Lemma 1.1.1) we see that V_1 is a cubic hypersurface in $(\mathbb{P}^3)^*$. (This is also clear since it is obtained by taking the determinant of a 3×3 matrix of linear forms, but it is nice to see Porteous' formula at work.) We would like to identify this cubic surface and to describe its relationship to the cubic surface S.

We can only give the basic idea of the answer here. There is a special type of configuration of lines on S called a "double-six" , and there are only 36 such configurations. These configurations consist of two separate sets of 6 skew lines, meeting in a special way. Of these 36, there is exactly one with the property that each of one of the sets of 6 skew lines is a 4-secant to C, and each of the other 6 lines is disjoint from C. But then it is clear that the lines dual to each of these lines all lie on V_1. (One has to check that a plane containing any of these 12 lines meets C in 6 points which lie on a conic in that plane: in the former case the conics are all unions of lines, and in the latter case they are generically smooth. We leave the verification as an exercise, or see [91] for details.)

But these 12 dual lines also form a double-six configuration in $(\mathbb{P}^3)^*$, and hence V_1 is the unique cubic surface containing this configuration. Knowing that this is V_1 limits the possible linear systems of curves of degree 6 and arithmetic genus 0 on cubic surfaces in $\mathbb{P}^3$ (by looking at all the double-sixes on V_1 and going back to their duals in $\mathbb{P}^3$). The last step is to consider a certain line bundle on V_1 and use it to "weed out" the sextic curves which do not lie on S. $\square$

The examples in this section serve as a useful illustration of the fact that the curves in an even liaison class which are *minimal* (cf. Remark 1.3.3 (c)) really do fall into nice families, including the possibility that they are unique. This will be part of the Lazarsfeld-Rao property described in Chapter 5.

Chapter 5

Liaison Theory in Codimension Two

As we have mentioned before, the strongest results to date on Liaison Theory are for the case where the schemes have codimension two. Until recently, it was also necessary to restrict to schemes that are locally Cohen-Macaulay and equidimensional. In a recent preprint, Nollet [107] has modified the proofs of these theorems to remove the hypothesis of locally Cohen-Macaulay, now assuming only that all components have the same dimension. This is an important contribution, but for simplicity in this chapter we maintain this hypothesis. The main results which are known in this case, but which are missing in the case of arbitrary codimension, are a sufficient condition for two subschemes to be linked (or evenly linked), and a structure theorem describing an even liaison class. These will be discussed in this chapter. See also [95] for an expository description of some of the main results in codimension two.

Liaison Theory (as a subject rather than simply a tool) began with Apéry and Gaeta in the 1940's (cf. [4], [5], [47], [48]). In these papers, it was shown (essentially) that a smooth curve C in $\mathbb{P}^3$ is in the linkage class of a complete intersection if and only if it is arithmetically Cohen-Macaulay (i.e. its deficiency module is zero). This result was extended to arbitrary codimension two subschemes of projective space by Peskine and Szpiro [109], and they put the whole theory of Liaison into the framework of modern scheme theory. Some of their work was described in the last chapter.

After the landmark paper of Peskine and Szpiro, the next major contribution to the theory of Liaison was by Rao in his two papers [113] and [114]. The latter paper contains the main result of the former as a special case, but the former can be viewed as the starting point of the burst of activity for codimension two subschemes that has occurred in Liaison Theory in the last decade and a half.

Rao's main result is to give a necessary and sufficient condition for two codimension two subschemes of projective space to be evenly linked. (He also observes that his result holds in a more general context.) In the last chapter we have seen a necessary condition: the Hartshorne-Schenzel theorem. In the case of curves in $\mathbb{P}^3$ he notes that this is also sufficient. However, for $\mathbb{P}^n$ it is necessary to approach the problem from the point of view of locally free sheaves, as we will see. The main result is given in terms of stable equivalence classes of locally free sheaves.

In this chapter we will also give the structure theorem of [11] (in $\mathbb{P}^n$) and [87] (in $\mathbb{P}^3$), the so-called Lazarsfeld-Rao Property, which describes an even liaison class and how the elements of the class are related to each other. This can also be done for the more general context mentioned by Rao (cf. [27]). Passing to curves in $\mathbb{P}^3$, we have even stronger results from [87] (and, as mentioned above, we have Rao's converse to the Hartshorne-Schenzel theorem).

Finally, in the last section we give some applications of liaison. For example, we will describe how it has been used in the classification of surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$. We will also show how the Lazarsfeld-Rao Property has been used to prove that every Buchsbaum curve specializes to a stick figure, proving a special case of the classical question of whether every smooth curve specializes to a stick figure.

Throughout this chapter, we will assume that all schemes in question are locally Cohen-Macaulay and equidimensional.

5.1 Rao's Results

If $V \subset \mathbb{P}^n$ has codimension two (and is locally Cohen-Macaulay and equidimensional, as usual), then we have seen that there is a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_V \to 0. \tag{5.1}$$

This is obtained, as in §2 of Chapter 1, from a minimal free resolution of the saturated homogeneous ideal I_V. (The integers a_i are the degrees of the minimal generators of I_V.) It follows that $\mathcal{G}$ is locally free of rank $m-1$, and that $H^1_*(\mathcal{G}) = 0$. (The latter fact follows because we began with a minimal free resolution, so the map $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_V$ is already surjective on global sections.) The fact that $\mathcal{G}$ is locally free, especially, was important to prove Proposition 4.2.8. It follows from Proposition 4.2.8 that if V and V' are evenly linked, and if $\mathcal{I}_V$ has the locally free resolution above, then $\mathcal{I}_{V'}$ has a locally free resolution

$$0 \to \mathcal{G}' \to \mathcal{F}' \to \mathcal{I}_{V'} \to 0$$

where $\mathcal{F}'$ is free and $\mathcal{G}' = \mathcal{G}(c) \oplus \mathcal{A}$, with $\mathcal{A}$ free and $c \in \mathbb{Z}$. Note that here the direct summands of $\mathcal{F}'$ do not necessarily correspond to a minimal generating set for $I_{V'}$.

For many purposes, for instance Rao's proof of injectivity and surjectivity of the correspondence in Theorem 5.1.3 below, and the proof in [11] of the Lazarsfeld-Rao Property, it is convenient to have a short exact sequence of the form

$$0 \to (\text{free}) \to \mathcal{K} \to \mathcal{I}_V \to 0 \tag{5.2}$$

where $\mathcal{K}$ is locally free. This is obtained using the above ideas as follows. Choose any complete intersection X containing V, and let W be the residual. Suppose that I_X is generated by forms of degrees b_1 and b_2, and let $b = b_1 + b_2$. If W has a locally free resolution

$$0 \to \mathcal{E} \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_W \to 0$$

as above, where $\mathcal{E}$ is locally free, then Proposition 4.2.8 gives that V has a locally free resolution

$$0 \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(a_i - b) \to \mathcal{E}^\vee(-b) \oplus \mathcal{O}_{\mathbb{P}^n}(-b_2) \oplus \mathcal{O}_{\mathbb{P}^n}(-b_1) \to \mathcal{I}_V \to 0.$$

Then simply take $\mathcal{K} = \mathcal{E}^\vee(-b) \oplus \mathcal{O}_{\mathbb{P}^n}(-b_2) \oplus \mathcal{O}_{\mathbb{P}^n}(-b_1)$.

Remark 5.1.1 As Rao observes, if the generators of I_X are taken from a minimal generating set of I_W then the summands $\mathcal{O}_{\mathbb{P}^n}(-b_2) \oplus \mathcal{O}_{\mathbb{P}^n}(-b_1)$ of $\mathcal{K}$ are redundant and can be canceled. This idea can be used, for example, to show that any arithmetically Cohen-Macaulay codimension two subscheme V of $\mathbb{P}^n$ is in the linkage class of a complete intersection (i.e. licci). We briefly describe the idea.

Note first that if V is arithmetically Cohen-Macaulay then $\mathcal{G}$ is in fact free of rank $m-1$, not just locally free. When we perform a link, the ideal I_X of the complete intersection X that we choose has two generators, and it can happen that 0, 1 or both of these generators are part of a minimal generating set of I_V. (In particular, for any V, arithmetically Cohen-Macaulay or not, it is always possible to find a complete intersection X satisfying any of these three possibilities.)

If we link V by a complete intersection X, both of whose generators are minimal generators of I_V, one can check that the canceling referred to at the beginning of this remark gives us that the residual to V in the link has one fewer minimal generator $(m-1)$ than does V. (Use the fact that $\mathcal{G}$ is free.) So one can proceed by induction. This is essentially the approach of Gaeta and Peskine-Szpiro. However, this result also follows from Rao's more general theorem below. □

Definition 5.1.2 Two locally free sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\mathbb{P}^n$ are *stably equivalent* if there exist free sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ and an integer c such that $\mathcal{G}_1 \oplus \mathcal{L}_1 \cong \mathcal{G}_2(c) \oplus \mathcal{L}_2$. □

Rao's main result on liaison in codimension two (cf. [114]) is the following:

Theorem 5.1.3 (Rao) *In $\mathbb{P}^n$, $n \geq 2$, the even liaison classes in codimension two are in bijective correspondence with the stable equivalence classes of locally free sheaves $\mathcal{G}$ on $\mathbb{P}^n$ with $H^1(\mathbb{P}^n, \mathcal{G}(p)) = 0$ for all $p \in \mathbb{Z}$.*

The correspondence is given as above: to the locally Cohen-Macaulay, equidimensional scheme V is associated the locally free sheaf $\mathcal{G}$ obtained in the exact sequence (5.1). The fact that this correspondence is well-defined is confirmed by the observation above using Proposition 4.2.8, which says that if V and V' are evenly linked then the locally free sheaves one obtains are stably equivalent.

We will only give the basic idea of Rao's proof, and refer the reader to [114] for the details. To show injectivity, suppose that V_1 and V_2 are codimension two subschemes of $\mathbb{P}^n$ which yield locally free sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$, as in (5.1), which are stably equivalent. Rao shows that then V_1 and V_2 are evenly linked. His idea is that we can use the above type of arguments to obtain two locally free resolutions

$$0 \to \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-a_j) \xrightarrow{\alpha} \mathcal{K}(a) \to \mathcal{I}_{V_1} \to 0$$

$$0 \to \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-b_j) \xrightarrow{\beta} \mathcal{K}(b) \to \mathcal{I}_{V_2} \to 0$$

The maps α and β have the form $(s_1, \cdots, s_r)$ and $(t_1, \cdots, t_r)$, where the s_i and t_i are global sections of various twists of $\mathcal{K}$. He then shows how, by a specific series of pairs of links, we may begin with the exact sequence involving α and (essentially) end up with the exact sequence involving β. That is, he shows how a prescribed even number of links will take us from V_1 to V_2.

For surjectivity, Rao begins with a locally free sheaf $\mathcal{G}$ as in the theorem, and considers the dual sheaf $\mathcal{G}^\vee$. His goal is to set up an exact sequence of the form (5.1) with the locally free sheaf $\mathcal{G}$. He will do this by first finding one of the form (5.2). By considering global sections of $\mathcal{G}^\vee(p)$ for $p \gg 0$, he obtains a locally Cohen-Macaulay scheme Y with resolution

$$0 \to (\text{free}) \to \mathcal{G}^\vee(c) \to \mathcal{I}_Y \to 0$$

for some $c \in \mathbb{Z}$. Then any direct link $Y \sim V$ gives a codimension two locally Cohen-Macaulay equidimensional scheme V with resolution of the desired form. This completes our description of Rao's proof of the bijection.

Remark 5.1.4 This theorem of course implies the Hartshorne-Schenzel theorem (Theorem 4.3.1) for codimension two even liaison. In fact, the cohomology of the sheaf $\mathcal{G}$ gives us back the deficiency modules: $H_*^{i+1}(\mathbb{P}^n, \mathcal{G}) \cong (M^i)(V)$ for $1 \le i \le n-2$, and this cohomology is invariant under stable equivalence (up to shifts).

This theorem also implies that codimension two arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ comprise an entire liaison class. Indeed, the arithmetically Cohen-Macaulay codimension two subschemes of $\mathbb{P}^n$ are exactly those for which the sheaf $\mathcal{G}$ in (5.1) is free, and any two free sheaves are obviously stably equivalent.

As a corollary, among zeroschemes in $\mathbb{P}^2$ there is just one liaison class. □

Thanks to the classification, due to Horrocks [70], of stable equivalence classes of vector bundles $\mathcal{G}$ on $\mathbb{P}^n$ with $H^1_*(\mathcal{G}) = 0$, we have a very nice restatement of Rao's result for the case of curves in $\mathbb{P}^3$. This result actually preceded Rao's main result described above (cf. [113]).

Theorem 5.1.5 (Rao) *Let C and C' be curves in $\mathbb{P}^3$ with deficiency modules $M(C)$ and $M(C')$. Then C is evenly linked to C' if and only if $M(C)$ is isomorphic to some shift of $M(C')$.*

That is, in the language of Theorem 5.1.3, the even liaison classes of (locally Cohen-Macaulay, equidimensional) curves in $\mathbb{P}^3$ are in bijective correspondence with the isomorphism classes of graded S-modules of finite length, after identifying those that differ only by shift. Hence this provides the converse to the Hartshorne-Schenzel theorem for curves in $\mathbb{P}^3$.

Of course it then follows immediately from the Hartshorne-Schenzel theorem and its corollary (Theorem 4.3.1, Corollary 4.3.3) that C is oddly linked to C' if and only if $M(C)$ is isomorphic to some shift of $M(C')^{\vee k}$.

Rao also mentions in [114] that the Horrocks classification also yields a (somewhat more complicated) classification of the even liaison classes of surfaces in $\mathbb{P}^4$ in terms of graded modules:

Corollary 5.1.6 *For graded S-modules M_1, M_2 of finite length, consider equivalence classes of triples (M_1, M_2, η), where $\eta \in Ext^2_S(M_1^{\vee k}, M_2^{\vee k})$ and $(M_1, M_2, \eta) \sim (N_1, N_2, \zeta)$ iff we have $M_1 \xrightarrow{\phi_1} N_1(a)$ and $M_2 \xrightarrow{\phi_2} N_2(a)$ isomorphisms for some integer a which maps η to ζ. Then the even liaison classes of surfaces in $\mathbb{P}^4$ are in bijective correspondence with the equivalence classes of such triples.*

This shows that the converse to the Hartshorne-Schenzel theorem is false for surfaces in $\mathbb{P}^4$. Indeed, we need not only the preservation of the modules $(M^1)(V)$ and $(M^2)(V)$ up to shift, but also an element of $\mathrm{Ext}^2_S(M_1^{\vee k}, M_2^{\vee k})$. Bolondi [19] has used this classification to give a construction of the minimal elements in any even liaison class of surfaces in $\mathbb{P}^4$, much as Martin-Deschamps and Perrin [87] did for curves in $\mathbb{P}^3$.

It is also worth noting that under Liaison Addition (cf. §2 of Chapter 3), the behavior of stable equivalence classes of vector bundles in the case of codimension two subschemes of $\mathbb{P}^n$ works as one would expect (and so we really do have an "addition" in the context of liaison, even in codimension two). Specifically, we have

Theorem 5.1.7 ([25]) *Let V_1 and V_2 be codimension two subschemes of $\mathbb{P}^n$, and choose $F_1 \in I_{V_2}$ and $F_2 \in I_{V_1}$ of degrees d_1 and d_2 respectively, such that F_1 and F_2 form a regular sequence, defining a complete intersection X. Let Z be the scheme defined by the saturated ideal $F_1 \cdot I_{V_1} + F_2 \cdot I_{V_2}$. Assume that we have locally free resolutions*

$$0 \to \mathcal{G}_1 \to \mathcal{F}_1 \to \mathcal{I}_{V_1} \to 0$$

$$0 \to \mathcal{G}_2 \to \mathcal{F}_2 \to \mathcal{I}_{V_2} \to 0$$

where the $\mathcal{G}_i$ are locally free, the $\mathcal{F}_i$ are free and $H^1_*(\mathcal{G}_i) = 0$ for $i = 1, 2$. Then

(1) Z has a locally free resolution

$$0 \to \mathcal{G}_1(-d_1) \oplus \mathcal{G}_2(-d_2) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_1 - d_2) \to \mathcal{F}_1(-d_1) \oplus \mathcal{F}_2(-d_2) \to \mathcal{I}_Z \to 0.$$

(2) The vector bundle corresponding to Z is in the same stable equivalence class of vector bundles as the direct sum of the vector bundles corresponding to V_1 and V_2, suitably shifted.

(3) There is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d_1 - d_2) \to \mathcal{I}_{V_1}(-d_1) \oplus \mathcal{I}_{V_2}(-d_2) \to \mathcal{I}_Z \to 0$$

Proof:
(1) and (3) can be proved easily using the methods of [52] described in §2 of Chapter 3 (and in fact an analogous statement can be obtained in higher codimension). We omit the details. It was originally proved in [25] in a different way. (2) follows immediately from (1). □

Rao also remarks that his approach in [114] works in greater generality: one can replace $\mathbb{P}^n$ by any complete, connected arithmetically Gorenstein variety P (cf. page 11) of dimension at least two, with a very ample line bundle $\mathcal{L}$, and doing linkage using only complete intersections coming from powers of $\mathcal{L}$. That is, if we view P as embedded in $\mathbb{P}^n$, these complete intersections on P are the restrictions to P of codimension two complete intersections in $\mathbb{P}^n$ meeting P in a subsheme of codimension two in P. (So, for instance, if P is a smooth quadric surface in $\mathbb{P}^3$ then linking is done only with divisors of type (a,a). For example, let Z_1 and Z_2 each consist of one point on P. If Z_1 and Z_2 are on the same ruling then they are not directly linked, even though they are the intersection of a divisor of type $(0,1)$ with one of type $(2,0)$. On the other hand, if Z_1 and Z_2 are not on the same ruling then they are directly linked by a complete intersection of two divisors of type $(1,1)$. This is because the latter is the intersection of P with a line not lying on P, while the former is not.)

In higher codimension there is no known analogue to Rao's converse of the Hartshorne-Schenzel theorem for curves in $\mathbb{P}^3$, or to his restatement of the problem in terms of stable equivalence classes of locally free sheaves. Any such result would be a very welcome addition to the theory.

5.2 The Structure of an Even Liaison Class

As we mentioned on page 65, there are several natural questions to answer about the equivalence relation of Even Liaison. The first two (connections between linked schemes and whether all schemes are linked) were answered in Chapter 4. The third and fourth are solved only in codimension two. The third, to parameterize the even liaison classes, was answered by Rao, and his work was described in §1 of this chapter. In this section we consider the fourth question, to describe the structure of an even liaison class. This structure has been called the Lazarsfeld-Rao Property, and was introduced in full generality in Definition 4.3.6. In this section we will first recall the set-up, but in the context of codimension two in projective space. (See the last part of §3 of Chapter 4 for the set-up in arbitrary codimension.) We will then outline the proof of the structure theorem from [11], and we will give full proofs of those parts where the details are not too involved.

Recall, from Definition 4.3.5 and the discussion following it, that any non-arithmetically Cohen-Macaulay even liaison class $\mathcal{L}$ may be partitioned $\mathcal{L} = \mathcal{L}^0 \cup \mathcal{L}^1 \cup \cdots \cup \mathcal{L}^h \cdots$ according to the shift of the deficiency modules $(M^1), \cdots, (M^{n-2})$. At the heart of the Lazarsfeld-Rao Property is the set of *minimal* elements of the even liaison class, i.e. the elements of $\mathcal{L}^0$. The rest of $\mathcal{L}$ will be "built up" from these elements.

The first step is Basic Double Linkage. We now record the definition and important properties in the case of codimension two (but see Remark 3.2.4 and page 86 for more details about these facts in the general case of arbitrary codimension). Let V_1 be a codimension two subscheme of $\mathbb{P}^n$, and choose $F_2 \in I_{V_1}$ of degree d_2. Let F_1 be a form of degree d_1 such that F_1 and F_2 form a regular sequence, defining a complete intersection X. Then the ideal $F_1 \cdot I_{V_1} + (F_2)$ is the *saturated* ideal of a scheme Z, with the following properties:

(1) As sets, $Z = V_1 \cup X$ (in particular codim $Z = 2$).

(2) For all $1 \leq i \leq n-2$, $(M^i)(Z) \cong (M^i)(V_1)(-d_1)$.

(3) Z is linked to V_1 in two steps. In particular, if $V_1 \in \mathcal{L}^h$ then $Z \in \mathcal{L}^{h+d_1}$.

(4) If V_1 has a locally free resolution

$$0 \to \mathcal{G} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_{V_1} \to 0$$

where $H^1_*(\mathcal{G}) = 0$ (cf. §1), then $\mathcal{I}_Z$ has a locally free resolution

$$0 \to \mathcal{G}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_1-d_2) \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-d_1-a_i) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2) \to \mathcal{I}_Z \to 0.$$

The only fact which we have not already seen is (4). This can be proved either by applying the mapping cone procedure (Proposition 4.2.8) twice and canceling redundant terms (as was done in Remark 5.1.1), or by applying Theorem 5.1.7 and using the fact that Basic Double Linkage is really an application of Liaison Addition, taking one of the schemes to be trivial. Either (1) or (3) of Theorem 5.1.7 can then be used to give a proof. For instance, it follows by taking a mapping cone (see page 5) of the diagram

$$
\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 & 0 & & \mathcal{G}(-d_1) & \oplus & 0 & \\
 & \downarrow & & & \downarrow & & \\
 & \mathcal{O}_{\mathbb{P}^n}(-d_1-d_2) & \to & \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-d_1-a_i) & \oplus & \mathcal{O}_{\mathbb{P}^n}(-d_2) & \\
 & \downarrow & & & \downarrow & & \\
0 \to & \mathcal{O}_{\mathbb{P}^n}(-d_1-d_2) & \to & \mathcal{I}_{V_1}(-d_1) & \oplus & \mathcal{O}_{\mathbb{P}^n}(-d_2) & \to \mathcal{I}_Z \to 0 \\
 & \downarrow & & & \downarrow & & \\
 & 0 & & & 0 & &
\end{array}
$$

We remark that for curves in $\mathbb{P}^3$, the fact that Z is evenly linked to V_1 would follow from (2) by Rao's theorem (Theorem 5.1.5). But in higher dimension or codimension we do not have these results, and in any case we have the further information that the linking can be done in two steps.

Recall from Definition 4.3.6 that an even liaison class $\mathcal{L}$ of codimension-two subschemes of $\mathbb{P}^n$ has the *Lazarsfeld-Rao Property* (or simply the *LR-Property*) if the following conditions hold:

(a) If $V_1, V_2 \in \mathcal{L}^0$ then there is a deformation from one to the other through subschemes all in $\mathcal{L}^0$; (in particular, all subschemes in the deformation are in the same even liaison class).

(b) Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ ($h \geq 1$), there exists a sequence of subschemes $V_0, V_1, \cdots, V_t$ such that for all i, $1 \leq i \leq t$, V_i is a basic double link of V_{i-1} and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.

The main theorem which we will describe in this section is the following:

Theorem 5.2.1 ([11]) *Every even liaison class of non-arithmetically Cohen-Macaulay codimension two subschemes of* $\mathbb{P}^n$ *has the Lazarsfeld-Rao Property.*

Remark 5.2.2 In this remark we give the history of work on this problem. The LR-Property was first proved for special even liaison classes of curves in

$\mathbb{P}^3$ by Lazarsfeld and Rao [80]. Their main theorem was that if C is a curve in $\mathbb{P}^3$ with index of speciality $e = e(C) = \max\{t \mid h^1(\mathcal{O}_C(t)) \neq 0\}$, and if C lies on no surface of degree $e+3$, then C is minimal in its even liaison class, and this even liaison class has the property described above. (Furthermore, if C lies on no surface of degree $e+4$ then $\mathcal{L}^0 = \{C\}$; that is, C is the only curve in $\mathcal{L}^0$.) This paper inspired the main structure theorem, both in [11] and in [87].

Lazarsfeld and Rao's paper was motivated by the conjecture of Harris mentioned on page 62. That is, the *general* curve of fixed degree and genus cannot be "linked down" to another curve of smaller degree and/or genus. Indeed, they recall that when $d \geq 2g-1$ the family of all smooth irreducible curves in $\mathbb{P}^3$ of degree d and genus g is irreducible. (Note also that if a smooth curve X of degree d and genus g satisfies $d \geq 2g-1$ then $e(X) \leq 0$.) Then they show that for a fixed integer $g \geq 0$ there exists a constant $C(g) \geq 2g-1$ such that a sufficiently general curve X of genus g and degree $d \geq C(g)$ lies on no surface of degree $\sqrt{5d}$ or less. Hence their structure theorem applies to the even liaison class of X.

The next case proved was in [24], where it was shown that the even liaison class of every *arithmetically Buchsbaum* curve (see Definition 1.4.7) in $\mathbb{P}^3$ has the LR-Property. If the deficiency module $M(C)$ of an arithmetically Buchsbaum curve C has diameter one then the minimal curves all fail to lie on surfaces of degree $e(C)+3$ (cf. [22]), so the theorem of [80] applies. In fact, the LR-Property was important in obtaining the results in [22]. But if the diameter is greater than one then no curve in the liaison class satisfies the hypothesis of Lazarsfeld and Rao [23]. Hence [24] gave a large class of curves not covered by [80].

The proof of the main result of [24] used the paper of Bolondi [20], which provided the approach to get the desired deformations. (This paper in turn was based on the original paper of Lazarsfeld and Rao.) It was inspired by the papers [1], [22], [23], [49] and [50], which provided technical tools as well as very compelling evidence that the LR-Property should also hold at least for Buchsbaum curves.

The paper [25] suggested a general approach to the problem and gave several useful technical results. It also proved the first case of the LR-Property in codimension two in any projective space, but it was a very limited case.

The main structure theorem for codimension two subschemes of $\mathbb{P}^n$ (Theorem 5.2.1 above) was proved in [11]. The proof will be described below. A similar proof was subsequently given in [27] for even liaison classes of codimension two subschemes of an arithmetically Gorenstein variety in $\mathbb{P}^n$. (See also page 100.)

The main structure theorem was given independently for curves in $\mathbb{P}^3$ by Martin-Deschamps and Perrin [87]. Although their results hold only for curves in $\mathbb{P}^3$, they give a much more detailed description of the structure and the connections between the curves and the deficiency module (such as an improvement of Theorem 1.2.4 of Rao, and an analysis of the connections with the locally free sheaves mentioned in §1 of this chapter). They also show how to construct the minimal elements of the even liaison class. A similar analysis was subsequently done by Bolondi [19] for surfaces in $\mathbb{P}^4$.

It should also be noted that in a more algebraic context, many results in this direction have been obtained by Huneke and Ulrich. See especially [71], Section 6. □

There are two main technical tools used in the proof of Theorem 5.2.1 found in [11]. The first tool is a sufficient condition for the existence of a deformation of the desired form. Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$. It turns out [25] that if two elements of $\mathcal{L}$ are in the same shift $\mathcal{L}^h$ and have the same Hilbert function, then such a deformation exists. This result (essentially) was known for aCM subschemes of codimension two in $\mathbb{P}^n$ [45], and proved for curves in $\mathbb{P}^3$ by Bolondi [20], inspired by the paper [80]. The proof in [25] followed the same idea, and we will now describe it.

Proposition 5.2.3 ([25]) *Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$, and let $V_1, V_2 \in \mathcal{L}^h$. Assume that V_1 and V_2 have the same Hilbert function (i.e. $h^0(\mathbb{P}^n, \mathcal{I}_{V_1}(t)) = h^0(\mathbb{P}^n, \mathcal{I}_{V_2}(t))$ for all t). Then*

(a) it also holds that $h^{n-1}(\mathbb{P}^n, \mathcal{I}_{V_1}(t)) = h^{n-1}(\mathbb{P}^n, \mathcal{I}_{V_2}(t))$ for all t;

(b) there exists an irreducible flat family $\{V_s\}_{s \in S}$ of codimension two subschemes of $\mathbb{P}^n$ to which both V_1 and V_2 belong; and

(c) S can be chosen so that for all $s \in S$, $V_s \in \mathcal{L}^h$, and so that furthermore V_s has the same Hilbert function as that of V_1 and V_2.

Proof:

Consider a locally free resolution

$$0 \to \mathcal{G} \xrightarrow{f} \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_{V_1} \to 0 \tag{5.3}$$

where $H^1_*(\mathcal{G}) = 0$ (cf. §1). Since V_2 is evenly linked to V_1, a repeated use (an even number of times!) of the mapping cone procedure (cf. Proposition 4.2.8 and §1 page 96) gives that V_2 has a locally free resolution

$$0 \to \mathcal{G}(d) \oplus \mathcal{A} \xrightarrow{g} \mathcal{B} \to \mathcal{I}_{V_2} \to 0 \tag{5.4}$$

where $\mathcal{A}$ and $\mathcal{B}$ are direct sums of line bundles. Since V_1 and V_2 are in the same shift, it follows by taking cohomology that $d = 0$.

By adding a trivial addendum $\mathcal{A}$ to the first two terms in the locally free resolution (5.3), we get

$$0 \to \mathcal{G} \oplus \mathcal{A} \xrightarrow{f \oplus I_A} \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \oplus \mathcal{A} \to \mathcal{I}_{V_1} \to 0. \tag{5.5}$$

We rewrite (5.4) and (5.5) as

$$0 \to \mathcal{E} \xrightarrow{u} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-h_i) \to \mathcal{I}_{V_2} \to 0,$$

$$0 \to \mathcal{E} \xrightarrow{v} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-k_i) \to \mathcal{I}_{V_1} \to 0,$$

where $\mathcal{E} = \mathcal{G} \oplus \mathcal{A}$ (and hence $H^1_*(\mathcal{E}) = 0$) and $r = rk\mathcal{E} + 1$. Because V_1 and V_2 are assumed to have the same Hilbert function, a calculation (taking cohomology) shows that $h_i = k_i$ for all i.

Hence we have $u, v \in \operatorname{Hom}(\mathcal{E}, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-h_i))$. Now, following [80], we can produce the desired deformation. Given $s \in k$ (the base field), let

$$w_s = su + (1-s)v \in \operatorname{Hom}\left(\mathcal{E}, \bigoplus \mathcal{O}_{\mathbb{P}^n}(-h_i)\right).$$

For general $s \in k$, the cokernel of w_s is the ideal sheaf $\mathcal{I}_{V_s}$ of a codimension two subscheme of $\mathbb{P}^n$, and these subschemes fit together in an irreducible flat family (see also [80] and [77]). By Theorem 5.1.3, V_s is in the same even liaison class as V_1 and V_2. By taking cohomology one can check that they are all in the same shift and that they all have the same Hilbert function. □

The second technical tool, in some sense the heart of the proof of Theorem 5.2.1, is the following key lemma.

Lemma 5.2.4 *Let $\mathcal{E}$ be a rank $(r+1)$ vector bundle on $\mathbb{P}^n$ and let*

$$\phi : \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{E}$$

$$\psi : \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-b_i) \to \mathcal{E},$$

$a_1 \leq a_2 \leq \cdots \leq a_r, b_1 \leq b_2 \leq \cdots \leq b_r$, be morphisms whose degeneracy loci have codimension two. Then there exists a morphism

$$\zeta : \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-c_i) \to \mathcal{E}$$

with $c_i = \min\{a_i, b_i\}$ $\forall i$, whose degeneracy locus has codimension two.

The proof is rather involved, and we refer the reader to [11] for details.

With these two tools, there are essentially three steps in the proof of Theorem 5.2.1:

(1) Take care of the minimal elements (i.e. prove condition (a) of the LR-Property);

(2) Relate a locally free resolution for a minimal element $V_0 \in \mathcal{L}^0$ to one for an arbitrary element $V \in \mathcal{L}^h$; and

(3) Find the right sequence of basic double links to complete condition (b) of the LR-Property.

The first step is accomplished by the following:

Proposition 5.2.5 *If V_1, $V_2 \in \mathcal{L}^0$ then there is an irreducible flat family $\{V_s\}_{s \in S}$ of codimension two subschemes of $\mathbb{P}^n$ to which both V_1 and V_2 belong.*

Proof:

By Proposition 5.2.3 it is enough to show that V_1 and V_2 have the same Hilbert function. Consider a locally free resolution

$$0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{I}_{V_1} \to 0 \tag{5.6}$$

with $\mathcal{P}$ a direct sum of line bundles and $\mathcal{F}$ a vector bundle with $H_*^{n-1}(\mathcal{F}) = 0$ (see (5.2) of §1). Since V_2 is evenly linked to V_1, by repeatedly using the mapping cone procedure (Proposition 4.2.8), one obtains a locally free resolution

$$0 \to \mathcal{P} \oplus \mathcal{B} \to \mathcal{F} \oplus \mathcal{A} \to \mathcal{I}_{V_2} \to 0 \tag{5.7}$$

where $\mathcal{A}$ and $\mathcal{B}$ are direct sums of line bundles.

On the other hand, we can trivially add $\mathcal{A}$ to sequence (5.6) to get

$$0 \to \mathcal{P} \oplus \mathcal{A} \to \mathcal{F} \oplus \mathcal{A} \to \mathcal{I}_{V_2} \to 0. \tag{5.8}$$

Clearly we will be done if we can show that $\mathcal{P} \oplus \mathcal{A} \cong \mathcal{P} \oplus \mathcal{B}$ (just take cohomology on (5.7) and (5.8)).

Suppose we write $\mathcal{P} \oplus \mathcal{A} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-a_i)$ and $\mathcal{P} \oplus \mathcal{B} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-b_i)$. If $a_i \neq b_i$ for some i then by Lemma 5.2.4 there exists a morphism

$$\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-c_i) \to \mathcal{F} \oplus \mathcal{A}$$

with $c_i = \min\{a_i, b_i\}$ for all i, whose degeneracy locus Z has codimension two. Hence its ideal sheaf has a locally free resolution

$$0 \to \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-c_i) \to \mathcal{F} \oplus \mathcal{A} \to \mathcal{I}_Z(\delta) \to 0.$$

By considering Chern classes, one can check that $\sum_{i=1}^r a_i = \sum_{i=1}^r b_i$, and that hence $\delta = \sum_{i=1}^r (c_i - a_i) < 0$. But this means that the deficiency modules of Z are shifted to the left of those of V_1, contradicting the hypothesis that $V_1 \in \mathcal{L}^0$. □

A similar proof takes care of step (2) of the proof. Specifically, one proves

Proposition 5.2.6 *Let $V_0 \in \mathcal{L}^0$ have the locally free resolution*

$$0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{I}_{V_0} \to 0$$

(with $\mathcal{P}$ and $\mathcal{F}$ as above) and let $V \in \mathcal{L}^h$ have the locally free resolution

$$0 \to \mathcal{P} \oplus \mathcal{B} \to \mathcal{F} \oplus \mathcal{A} \to \mathcal{I}_V(\delta) \to 0$$

(with $\mathcal{A}$ and $\mathcal{B}$ direct sums of line bundles). If $\mathcal{P} \oplus \mathcal{A} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-a_i)$ and $\mathcal{P} \oplus \mathcal{B} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-b_i)$ with $a_1 \leq \cdots \leq a_r$ and $b_1 \leq \cdots \leq b_r$, then $b_i \geq a_i$ for all i.

We omit the details of the proof.

It remains only to complete step (3) of the proof. Again we omit the details, but the proof is similar to that given originally by Lazarsfeld and Rao in [80] and we will give the basic idea. We consider locally free resolutions of the type we have been using above. If $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ then we have the two resolutions mentioned in Proposition 5.2.6, with $b_i \geq a_i$ for all i.

We have seen that whenever we get a short exact sequence of this form, with the first two terms of the sequence the same, it follows that the corresponding schemes are in the same shift ($\delta = 0$) and have the same Hilbert function, and so there is the desired deformation. So we have to focus on those i for which $a_i < b_i$, and perform basic double links to produce new schemes for which "fewer" of these degrees are different. The idea of the proof is to start with the maximum of such i (essentially) and perform a basic double link starting from V_0, using $d_1 = b_i - a_i$ and $d_2 = b_i$, to produce a scheme V_1. (See the set-up for basic double links on page 102. V_1 here plays the role of Z in that set-up.) This is not quite correct, because of a technicality, but it gives the right idea. One then uses fact (4) in the description of basic double links, on page 102, to produce the appropriate locally free resolution for V_1. One compares this to the locally free resolution of the scheme V (suitably altered), and finds that "fewer" of the degrees differ. So after a finite number of steps the degrees are all the same, and we are done.

This completes our description of the proof of Theorem 5.2.1.

Notice that it is possible to construct (with, say, the computer program Macaulay [14]) a minimal element in the even liaison class of a given scheme

V. From V one obtains the vector bundle $\mathcal{F}$, and then one finds the "smallest" set of rk $\mathcal{F} - 1$ sections of $\mathcal{F}$ (chosen generally) which drop rank in codimension two. As in Lemma 5.2.4 and the discussion after it, this will be the minimal element. Peterson has implemented this idea for curves in $\mathbb{P}^3$.

5.3 Applications

In this section we will describe a number of way in which Liaison Theory has been applied in the literature. The topics chosen here are by no means close to being a complete list. Furthermore, because of space limitations we will obviously not be able to give more than an overview of the techniques and ideas involved in these topics. We just want to give a flavor of some of the ways in which Liaison Theory can and has been used.

The first topic deals with trying to find curves in $\mathbb{P}^3$ having certain nice properties, related to the Hilbert function of the general hyperplane section. Liaison is used to find these curves. We will describe work from [58] and [84].

The second topic passes to surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$. We would like to use Liaison to find smooth examples, working toward a classification theorem. We describe work of several authors.

The next application involves some simple consequences of the Lazarsfeld-Rao Property, giving some ways in which properties of the whole even liaison class (say of a curve in $\mathbb{P}^3$) can be given just from knowledge of a minimal element. In particular, we will show how in principle we can give a complete list of all possible pairs (d, g) (degrees and genera) of Buchsbaum curves in $\mathbb{P}^3$.

We then show how the Lazarsfeld-Rao Property can give information about how codimension two subschemes of $\mathbb{P}^n$ can specialize to "nice" subschemes. In particular, we describe the proof from [26] that every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure. This generalizes the corresponding fact for arithmetically Cohen-Macaulay curves, due to Gaeta. To do this, we will also have to use some of the results and ideas from Chapter 3.

Finally, we give some connections between low rank vector bundles on $\mathbb{P}^n$ and subschemes of $\mathbb{P}^n$ defined by a small number of equations. Our discussion is based on Chapter 3 of [110].

5.3.1 Smooth Curves in $\mathbb{P}^3$

We would first like to describe two problems concerning curves in $\mathbb{P}^3$. These problems both turn out to involve the Hilbert function in a key way, and they both use Liaison to find the desired curves. Furthermore, in both cases although the problem is given in general, for the solution it turns out to be enough to consider arithmetically Cohen-Macaulay curves.

The first problem is to find a bound for the (arithmetic) genus of a nondegenerate, integral curve lying on an irreducible surface of given degree k. This was solved in [58]. The second problem is to describe all possible Hilbert functions for the general hyperplane section of an integral curve. This was solved in [84]. (See [57] and [119] for similar results, and [51] for a discussion of the relations between these approaches.)

Let $C \subset \mathbb{P}^3$ be an integral curve of degree d and arithmetic genus g. Let L be a general linear form defining a hyperplane $H \subset \mathbb{P}^3$. Let K_L be the submodule of the deficiency module $M(C)$ annihilated by L (see Definition 2.1.1). Let $\Gamma = C \cap H$ be the hyperplane section of C by H (where Γ is viewed as a finite set of points in $\mathbb{P}^2$). Let $I_\Gamma = (F_1, F_2, \cdots, F_r)$ be the saturated ideal of Γ (see page 7) in the ring $R = k[X_0, X_1, X_2]$, and set $d_i = \deg F_i \leq \deg F_{i+1} = d_{i+1}$. Let $H(\Gamma, t) = \dim_k(R/I_\Gamma)_t$ be the Hilbert function of Γ. Let $\Delta H(\Gamma, t) = H(\Gamma, t) - H(\gamma, t-1)$ be the first difference of $H(\Gamma, t)$.

Here are some important properties of Γ and of $\Delta H(\Gamma, t)$ (see [60] for basic facts about Hilbert functions):

(1) ([58]) Γ has the *Uniform Position Property* (U.P.P.). That is, any two subsets of Γ with the same cardinality have the same Hilbert function. Most of the properties described here are a consequence of U.P.P. (In fact, the general hyperplane section of an integral curve in any projective space $\mathbb{P}^r$ is a set of points with U.P.P., provided that the field has characteristic zero.)

(2) $\Delta H(\Gamma, t) = 0$ for $t < 0$; $\Delta H(\Gamma, 0) = 1$; $\Delta H(\Gamma, t) = 0$ for $t \gg 0$ (since Γ is a zeroscheme, so eventually the Hilbert function of Γ is equal to the Hilbert polynomial, which is the constant d).

(3) $\sum_t \Delta H(\Gamma, t) = d = \deg \Gamma = \deg C$ (use (2)).

(4) $\Delta H(\Gamma, t) = t + 1$ for $0 \leq t \leq d_1 - 1$.

(5) $\Delta H(\Gamma, t) = d_1$ for $d_1 - 1 \leq t \leq d_2 - 1$.

(6) $\Delta H(\Gamma, d_2) < d_1$.

(7) $\Delta H(\Gamma, t)$ is of *decreasing type.* That is, for $t \geq d_2$ we have either $\Delta H(\Gamma, t) < \Delta H(\Gamma, t-1)$ or $\Delta H(\Gamma, t) = 0$. (This is a consequence of U.P.P.)

(8) Let $l \gg 0$. Then $g = \sum_{i=1}^{l}[d - H(\Gamma, i)] - \dim K_L$. (This is an elementary calculation; see for instance [96].) Harris [58] shows that this is equal to $\sum_{i=0}^{l}[(i-1)\Delta H(\Gamma, i)] + 1 - \dim K_L$. (For this fact C can be in any projective space.)

Remark 5.3.1 The "decreasing type" description of the possible Hilbert functions for Γ given in (7) is central to both problems that we will discuss in this section. Any zeroscheme in $\mathbb{P}^2$ will have a first difference function that is non-increasing in the range $t \geq d_2$; U.P.P. forces it to be strictly decreasing until it reaches zero. Indeed, it follows from work of Davis [37] that zeroschemes which are not of decreasing type can be decomposed in a very nice way, which violates U.P.P. See also [18] for a more geometric interpretation of this phenomenon, viewed (in any projective space) in the context of Macaulay's growth condition [81]. □

Now suppose we are interested in finding an upper bound for the genus of C, in terms of its degree d and the fact that C lies on an irreducible surface of degree k. From (8) above,

$$g \leq \sum_{i=1}^{l}[d - H(\Gamma, i)],$$

with equality if and only if C is arithmetically Cohen-Macaulay.

Now, we would like to determine the largest possible value for the right-hand side of the inequality, given the degree d. Clearly the idea is to have the growth of $H(\Gamma, i)$ be as slow as possible, subject to the requirement that it have decreasing type. With no condition on the surfaces on which C lies, this happens when the first difference function is

$$\Delta H(\Gamma,t)=\begin{cases}0, & \text{if } t\le -1;\\ 1, & \text{if } t=0;\\ 2, & \text{if } 1\le t\le \lceil\frac{d}{2}\rceil-1\\ \delta, & \text{if } t=\lceil\frac{d}{2}\rceil\\ 0, & \text{if } t>\lceil\frac{d}{2}\rceil\end{cases}$$

where $\delta = 0$ or 1, depending on whether d is odd or even, respectively. Castelnuovo's bound ([31] or [60] pp. 351–352) follows from this:

$$g\le\begin{cases}\left(\frac{d}{2}-1\right)^2, & \text{if } d \text{ is even;}\\ \left(\frac{d-1}{2}\right)\left(\frac{d-3}{2}\right), & \text{if } d \text{ is odd}\end{cases}$$

and this bound is achieved if and only if C is arithmetically Cohen-Macaulay and lies on a quadric surface.

Harris' idea was that Castelnuovo's bound above, and the extremal curves obtained, really come about by liaison. The curves are either the complete intersection of a quadric and a surface of degree $\frac{d}{2}$ (if d is even), or else residual to a line in the complete intersection of a quadric and a surface of degree $\frac{d+1}{2}$ (if d is odd). In either case this gives both the bound and produces the sharp examples.

Now we impose the condition that C lie on an irreducible surface of degree k. Harris analyzed the possible Hilbert functions that could arise, as above, once he realized that they had to be of decreasing type. He broke the argument into two cases, depending on whether $d > k(k-1)$ or $d \le k(k-1)$. His conclusion in either case was that *the greatest genus of a curve lying on an irreducible surface S of degree k is that of a curve residual to a plane curve.* However, the particular complete intersection required for the link depends on whether $d > k(k-1)$ or $d \le k(k-1)$. (Only in the former case is S necessarily one of the surfaces involved in the complete intersection.) Furthermore, not only can some extremal curves be obtained in this way by liaison, but in fact any extremal curve *must* be residual to a plane curve.

Specifically, Harris proves the following. Let

$$\pi(d,l)=\frac{d^2}{2l}+\frac{1}{2}d(l-4)+1-\frac{\epsilon}{2}\left(l-\epsilon-1+\frac{\epsilon}{l}\right)$$

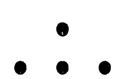
where $0 \leq \epsilon \leq l-1, -\epsilon \equiv d (\text{mod } l)$. Then the genus g of an irreducible curve C lying on an irreducible surface S of degree k satisfies

$$g \leq \begin{cases} \pi(d,k), & \text{if } d > k(k-1); \\ \pi\left(d, \left\lfloor \frac{d-1}{k} \right\rfloor + 1\right), & \text{if } d \leq k(k-1). \end{cases}$$

Harris' work, then, described "extremal" kinds of Hilbert functions of decreasing type, showed that they must actually occur as the Hilbert function of the general hyperplane section of an integral, arithmetically Cohen-Macaulay curve, and showed that these curves are extremal with respect to his bound on the genus. But what about the other Hilbert functions of decreasing type? In [59], Harris and Eisenbud ask what may be the Hilbert function of a set of points in $\mathbb{P}^{r-1}$ with U.P.P., and whether every such function actually occurs as the Hilbert function of the general hyperplane section of some integral curve in $\mathbb{P}^r$.

In the case $r = 3$, we describe the answer given in [84]. As mentioned above, it follows from work of Davis that if a zeroscheme in $\mathbb{P}^2$ has Hilbert function which is not of decreasing type, then it cannot have U.P.P. In [84], Maggioni and Ragusa show that *any Hilbert function of decreasing type occurs as the Hilbert function of the general hyperplane section of some smooth, arithmetically Cohen-Macaulay curve in* $\mathbb{P}^3$. Since such a hyperplane section automatically has U.P.P., it follows immediately that both questions of Harris and Eisenbud have affirmative answers for $r = 3$. (In the case $r \geq 4$, neither question has yet been answered, although some progress has been made. See for instance [58], [105], [117], [18].)

An important fact used by Maggioni and Ragusa is that if Z is a set of points in $\mathbb{P}^2$ with U.P.P., then Z lies in a complete intersection of type (d_1, d_2) (defined on page 111). This is not true in general if you remove the condition that Z have U.P.P. A simple counterexample without U.P.P. is a set of points Z with the following configuration:



(that is, three points on a line and one point off the line). Z clearly does not have U.P.P. since one subset of three points lies on a line while the other subsets of three points do not have this property. Here, $d_1 = d_2 = 2$, but the points do not lie on a complete intersection of type $(2,2)$. The smallest complete intersection containing Z is one of type $(2,3)$.

Notice that the values of d_1 and d_2 can be read off directly from the Hilbert function. (See the properties of Hilbert functions of points in $\mathbb{P}^2$ listed above.) For instance, if Z is a zeroscheme in $\mathbb{P}^2$ with Hilbert function

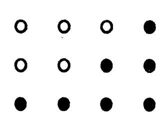
i	$\cdots$	-1	0	1	2	3	4	5	6	7	$\cdots$
$\Delta H(Z,i)$	$\cdots$	0	1	2	3	3	3	3	1	0	$\cdots$

then $d_1 = 3$ and $d_2 = 6$.

Maggioni and Ragusa use a theorem from [39] which says (in our situation) how the Hilbert function of a set of points in $\mathbb{P}^2$ behaves under liaison. Specifically, say Z is a zeroscheme in $\mathbb{P}^2$ linked to a zeroscheme Z' by a complete intersection X of type (a_1, a_2). (Notice that given a_1 and a_2, we know precisely what the Hilbert function of X is.) Let $N = a_1 + a_2 - 2$. Then

$$\Delta H(Z', i) = \Delta H(X, N - i) - \Delta H(Z, N - i).$$

For example, the following configuration of points is a complete intersection of type $(3,4)$ linking the "open points" Z to the "solid" points Z':



(Think of the complete intersection X as consisting of three "horizontal" lines and four "vertical" lines.) The theorem of [39] says that to compute the Hilbert function of Z' we subtract that of Z from that of X and then "read backwards." In this case, we have that Z lies on a unique conic, so one quickly computes that $\Delta H(Z,i)$ is given by

$$\Delta H(Z,i) = \begin{cases} 0 & \text{if } i \leq -1; \\ 1 & \text{if } i = 0; \\ 2 & \text{if } i = 1; \\ 2 & \text{if } i = 2; \\ 0 & \text{if } i \geq 3 \end{cases}$$

Then from

i	$\cdots$	-1	0	1	2	3	4	5	6	$\cdots$
$\Delta H(X,i)$	$\cdots$	0	1	2	3	3	2	1	0	$\cdots$
$-\ \Delta H(Z,i)$	$\cdots$	0	1	2	2	0	0	0	0	$\cdots$
$\Delta H(Z',5-i)$	$\cdots$	0	0	0	1	3	2	1	0	$\cdots$

we get (by reading backwards) that $\Delta H(Z',i)$ is given by

$$\Delta H(Z',i) = \begin{cases} 0 & \text{if } i \leq -1; \\ 1 & \text{if } i = 0; \\ 2 & \text{if } i = 1; \\ 3 & \text{if } i = 2; \\ 1 & \text{if } i = 3; \\ 0 & \text{if } i \geq 4. \end{cases}$$

Now, the idea of Maggioni and Ragusa goes as follows. Start with a Hilbert function of decreasing type. We want to produce a smooth arithmetically Cohen-Macaulay curve C in $\mathbb{P}^3$ whose general hyperplane section has the given Hilbert function. We reason backwards. If such a smooth curve C were to exist, its general hyperplane section Z would have U.P.P. Hence d_1 and d_2 can be read from the Hilbert function, and a complete intersection of type (d_1, d_2) would link Z to another set of points Z'. The Hilbert function of Z' would be given by the theorem of [39] above. Let us call this new Hilbert function H' (which formally depends only on the given Hilbert function). The complete intersection linking Z to Z' would lift to one containing C, and the residual curve C' would have a general hyperplane section with Hilbert function H'. (Z' may not be sufficiently "general," but in any case the Hilbert function does not vary with the hyperplane for arithmetically Cohen-Macaulay curves– see for instance Proposition 3.1.1.)

Now, Maggioni and Ragusa produce a stick figure C', using a simple construction, whose general hyperplane section has Hilbert function H'. They then show that C' lies on a smooth surface S of degree d_1 (where d_1 still refers to the Hilbert function of C). Finally, they consider the linear system $d_2H - C'$ on S and show that the general element C of this linear system is smooth. But C is thus directly linked to C' by a complete intersection of type (d_1, d_2), and so we have produced the desired curve.

Remark 5.3.2 As these two applications show, it is of interest to have results which tell us when we can use liaison to produce smooth curves (or higher dimensional varieties). A very general result (holding for any dimension) is the theorem of Peskine and Szpiro mentioned below (Theorem 5.3.3). For the case of curves in $\mathbb{P}^3$ there is a result of Nollet [108] which shows how one can produce smooth curves starting from "very" non-reduced curves. Specifically, let $Y \subset \mathbb{P}^3$ be a smooth curve with homogeneous ideal I_Y, and let W be the curve whose homogenous ideal is the saturation of I_Y^d, for some integer $d > 1$. Let n be an integer such that $\mathcal{I}_W(n)$ is generated by global sections. If $m > n$ then for a general pair of surfaces F and G of degree m containing W, we have

$$F \cap G = \bar{W} \cup C$$

where $W \subset \bar{W}$, $Supp(\bar{W}) = Y$ and C is a smooth, irreducible curve. In other words, W itself is linked to the union of a smooth curve C and a curve supported on Y, so it is easy to recover the smooth curve C. Nollet also gives formulas for the degrees and arithmetic genera of C and $\bar{W}$. Furthermore, he shows that if $m \gg 0$ then (F, G) is actually the lowest degree complete intersection containing C. (See also Example 5.3.4.) □

5.3.2 Smooth Surfaces in $\mathbb{P}^4$ and Threefolds in $\mathbb{P}^5$

A (special case of a) conjecture of Hartshorne [61] states that any smooth codimension two subvariety of $\mathbb{P}^n$, for $n \geq 6$, must be a complete intersection. Curves in $\mathbb{P}^3$ are rather well known (for instance, it is known what pairs (d, g) of degrees and genera can occur). Hence it is of interest to study surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$, to see what smooth varieties exist. An excellent reference for this subject is the paper [40].

The main problem in this area is to classify surfaces in $\mathbb{P}^4$ or threefolds in $\mathbb{P}^5$, including existence and uniqueness results. According to [17] (page 325), "Adjunction Theory is a major tool to get maximal lists of all possible cases. Liaison is the major tool used to get existence results." The classification is generally done according to the degree.

The idea, then, is to start with known varieties, and link to get new (smooth) varieties. The main fact which is used to produce these varieties is a theorem of Peskine and Szpiro ([109] Proposition 4.1) which gives a necessary condition for the residual variety to be smooth. (More precisely, it says that the singular locus of the residual is "small.") We give a special case of this theorem, quoted from [40] (Theorem 2.1), which deals with the situation in which we are interested:

Theorem 5.3.3 *Let $X \subset \mathbb{P}^n$, $n \leq 5$, be a local complete intersection of codimension two. Let m be a twist such that $\mathcal{I}_X(m)$ is globally generated. Then for every pair $d_1, d_2 \geq m$ there exist forms $F_i \in H^0(\mathcal{I}_X(d_i))$, $i = 1, 2$, such that the corresponding hypersurfaces V_1 and V_2 intersect properly and link X to a variety X'. Furthermore, X' is a local complete intersection with no component in common with X, and X' is nonsingular outside a set of positive codimension in Sing X.*

Example 5.3.4 Note that if X is smooth, or if X is a local complete intersection with a zero-dimensional singular locus, then X' is smooth and has no component in common with X. These conclusions cannot be expected to hold in general without hypotheses of this sort, even for curves in $\mathbb{P}^3$. For example, consider the ideal $(X_1, X_2)^3$. This is a saturated ideal defining a scheme X of degree 6 supported on a line, λ (see below). Then we claim that any complete intersection containing X will link X to a residual X' which is non-reduced and has a common component with X, namely the line λ. Indeed, any element of I_X vanishes to order three on the line λ, so any complete intersection will vanish to order at least 9 on λ, and thus X' has a component of degree at least 3 supported on λ. (By the way, why is $(X_1, X_2)^3$ a saturated ideal defining a curve of degree 6? One can check that

$$\begin{aligned} [(X_1^2, X_2^2) : (X_1, X_2)] &= (X_1, X_2)^2 \\ [(X_1^3, X_2^3) : (X_1, X_2)^2] &= (X_1, X_2)^3. \end{aligned}$$

Now use the fact that ideal quotients of this sort are saturated (Lemma 4.2.1), the fact that the sum of the degrees of linked schemes is equal to the degree of the complete intersection linking them (Corollary 4.2.10), and the fact that the ideal (X_1^k, X_2^k) defines a complete intersection of degree k^2 for all k. This same argument works to show that for any complete intersection Y in $\mathbb{P}^3$, I_Y^k is saturated for all k.) □

Example 5.3.5 As an example of Theorem 5.3.3 in action, we mention the paper [102]. The object of [102] is to complete the list of the degrees for which there exist smooth threefolds in $\mathbb{P}^5$ which are not arithmetically Cohen-Macaulay. (The Hilbert-Burch matrices, hence the degrees, of smooth arithmetically Cohen-Macaulay codimension two subvarieties of $\mathbb{P}^n$ are known–for instance cf. [35] and [27].) It was shown by Banica [13] that for any odd integer $d \geq 7$ and any even integer $d = 2k > 8$ with $k = 5s+1, 5s+2, 5s+3$ or $5s+4$ there exist smooth threefolds in $\mathbb{P}^5$ of degree d which are not arithmetically Cohen-Macaulay. (In smaller degrees they are all arithmetically Cohen-Macaulay.)

Hence it remained to consider multiples of 10. It was shown by Beltrametti, Schneider and Sommese [15] that any smooth threefold of degree 10 is arithmetically Cohen-Macaulay. Miró-Roig's idea was to use Liaison (and Theorem 5.3.3 in particular) to produce smooth, non-arithmetically Cohen-Macaulay examples for degrees $d = 10n, n > 1$.

Her starting point is the smooth non-arithmetically Cohen-Macaulay threefold $Y \subset \mathbb{P}^5$ of degree 12 having a locally free resolution

$$0 \to \mathcal{O} \oplus \mathcal{O}(1)^3 \to \Omega(3) \to \mathcal{I}_Y(6) \to 0$$

where Ω is the cotangent bundle of $\mathbb{P}^n$ (cf. [13]). By studying the cohomology of Ω, and using Theorem 1.1.5, one obtains that $\mathcal{I}_Y(6)$ is globally generated. Hence Theorem 5.3.3 gives that a general complete intersection of type $(6,7)$ yields a residual X which is smooth. Note that X has degree 30, is not arithmetically Cohen-Macaulay (since the property of being arithmetically Cohen-Macaulay is preserved under liaison), and one can check that $\mathcal{I}_X(8)$ is globally generated.

From X, Miró-Roig produces her smooth threefolds of degree $10n, n \geq 5$, by linking with general hypersurfaces of degree 10 and $n+3$. For degrees 20 and 40 she uses the same idea, but starts with different Y. We omit the details. □

The classification of smooth surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$ is not yet complete. A very nice table summarizing the state of affairs to date is given in §7 of [40]. It covers surfaces in $\mathbb{P}^4$ up to degree 15 and threefolds in $\mathbb{P}^5$ up to degree 18. That paper, together with the book [17], give most of the necessary background, as well as a very extensive set of references, from which the reader can learn more about this fascinating subject, using many different approaches (in addition to Liaison and Adjunction Theory). Another useful source is the paper [111].

5.3.3 Possible Degrees and Genera in a Codimension Two Even Liaison Class

In this section and the next, we describe some consequences of the Lazarsfeld-Rao Property (Theorem 5.2.1). In this section we look at some simple consequences, and in the next section we consider a slightly more surprising consequence.

Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$. The Lazarsfeld-Rao Property says, in effect, that up to deformation, all of $\mathcal{L}$ is described once you know all possible basic double links that can be performed. (See the discussion beginning on page 102. We will use the notation from that discussion without comment.) And essentially, all you need to know to do this is the Hilbert function of the minimal elements. We will give some idea of the extent to which this can be done.

As a first observation, notice that all minimal elements of $\mathcal{L}$ (i.e. all elements of $\mathcal{L}^0$) have the same Hilbert function (see the proof of Proposition 5.2.5). In particular, they all have the same degree and arithmetic genus, and in any given degree the size of the corresponding components of the saturated ideals is the same (so they lie on the same number of hypersurfaces of given degree).

Let us assume we know the Hilbert function of any minimal element V_1 of $\mathcal{L}$. Assume that we have performed a basic double link, using $F_2 \in I_{V_1}$ of degree d_2 and a general element $F_1 \in S_{d_1}$, arriving at a scheme $Z \in \mathcal{L}^{d_1}$. Then from (4) on page 102, we can compute the cohomology of the vector bundle $\mathcal{G}$, and hence the Hilbert function of Z. In particular we know the degree of Z (namely $\deg Z = \deg V_1 + d_1 d_2$) and the arithmetic genus of Z. But we also know all the possible basic double links that can be performed on Z. So with only the initial information of the Hilbert function of V_1, we can (in principle) describe all the possible sequences of basic double links that can be performed starting with V_1. Hence we can describe all the possible degrees and arithmetic genera of elements of the even liaison class $\mathcal{L}$.

In practice, it is not so easy to carry out the above process from scratch.

However, there are several useful facts from [25] §5 which make the job much easier. Before stating these facts, we introduce the following notation for basic double links in codimension two (see page 102): Let V_1 be a codimension two subscheme of $\mathbb{P}^n$ and $F_2 \in I_{V_1}$ a form of degree d_2. Choose $F_1 \in S_{d_1}$ such that F_1 and F_2 form a regular sequence. Let Z be the scheme obtained by performing the corresponding Basic Double Linkage. Then we write

$$V_1 : (d_2, d_1) \to Z$$

(by Proposition 5.2.3, up to deformation it is enough to know the degrees d_1 and d_2).

The first fact, which can be proved fairly quickly using Proposition 5.2.3, is the following:

Lemma 5.3.6 *Let V_1 be a codimension two subscheme of $\mathbb{P}^n$. Consider the basic double links*

$$V_1 : (a, b + c) \to Z$$

and

$$V_1 : (a, b) \to Y_1 : (a, c) \to Z'.$$

Then Z and Z' are in the same shift of the same even liaison class, and there is an irreducible flat family of codimension two subschemes of $\mathbb{P}^n$ to which both belong.

In particular, the basic double link

$$V_1 : (a, b) \to Z$$

is equivalent (up to deformation) to the sequence of basic double links

$$V_1 : (a, 1) \to Z_1 : (a, 1) \to \cdots \to Z_{b-1} : (a, 1) \to Z_b.$$

Hence any sequence of basic double links is equivalent (up to deformation) to one (longer, in general) where the polynomials playing the role of F_1 all have degree 1.

The next fact can also be proved using similar methods. (The proof can be found in the proof of Lemma 5.2 of [25].)

Lemma 5.3.7 *Assume that $a > b$ and that there exists a sequence of basic double links*

$$V_1 : (a,1) \to Y : (b,1) \to Z.$$

(It is not always the case that such a sequence exists, even if the first basic double link exists.) Then the sequence

$$V_1 : (a-1,1) \to Y' : (b+1,1) \to Z'$$

also exists, Z and Z' are in the same shift of the same even liaison class, and there is an irreducible flat family of codimension two subschemes of $\mathbb{P}^n$ to which both belong.

Using these facts, the following very useful theorem is proved in [25]. It severely restricts the possible sequences of basic double links that one has to consider in the above program of describing the entire even liaison class. It will also play a crucial role in the next section.

Theorem 5.3.8 ([25] Corollary 5.3) *Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$. Let $X \in \mathcal{L}^0$ and let $X' \in \mathcal{L}^h$. Let*

$$X = X_0 : (b_1, f_1) \to X_1 : (b_2, f_2) \to \cdots \to X_{p-1} : (b_p, f_p) \to X_p$$

be a sequence of basic double links, where $X_p \in \mathcal{L}^h$ has the same Hilbert function as X'. Let $s = \alpha(I_X)$ (see Definition 2.2.1). Then there exists another sequence of basic double links

$$X = Y_0 : (s,b) \to Y_1 : (g_2, 1) \to \cdots \to Y_{r-1} : (g_r, 1) \to Y_r$$

where

(1) $b \geq 0, s < g_2 < g_3 < \cdots < g_r$ *and* $b + r - 1 = h$*;*

(2) $\deg X' = \deg X + sb + g_2 + \cdots + g_r$*;*

(3) X_p and Y_r have the same Hilbert function (as that of X') and are in $\mathcal{L}^h$ (hence we have the desired deformation).

Moreover, the sequence $(b; g_2, \cdots, g_r)$ is uniquely determined by X'.

Example 5.3.9 We illustrate the ideas above by giving a complete list of all possible degrees and arithmetic genera of Buchsbaum (non-arithmetically Cohen-Macaulay) curves in $\mathbb{P}^3$ of degree ≤ 10. Recall the notation from §3.3: if C is a Buchsbaum curve, we assume that the module has components of dimension $n_1 > 0, n_2 \geq 0 \cdots, n_{r-1} \geq 0, n_r > 0$ respectively, and we let $N = \dim_k M(C) = n_1 + \cdots + n_r$. From Remark 3.3.1 (c), the following are the only three possibilities for the dimensions of the components of $M(C)$:

(a) $r = 1, n_1 = 1$ (in this case, the minimal curve C_1 has degree 2 and arithmetic genus -1);

(b) $r = 1, n_1 = 2$ (in this case, the minimal curve C_2 has degree 8 and arithmetic genus 5);

(c) $r = 2, n_1 = 1, n_2 = 1$ (in this case, the minimal curve C_3 has degree 10 and arithmetic genus 10).

The genera in the last two cases can be computed using Theorem 3.2.3 (d). Since basic double links are a special case of Liaison Addition, it is also possible to use Theorem 3.2.3 (d) to compute the arithmetic genus of the curve obtained from a given curve by performing a basic double link. (There are other ways to do this, for instance using the fact that a basic double link really is a sequence of two links (cf. page 86), and then applying Corollary 4.2.11 twice.) One obtains

Lemma *Let $C : (a, b) \to Y$, where C and Y are curves in $\mathbb{P}^3$. Let $g(C)$ and $g(Y)$ denote the genera of C and Y respectively. Then*

$$g(Y) = b \cdot \deg C + \frac{1}{2}ab(a + b - 4) + g(C).$$

Now, to know which Buchsbaum curves of degree ≤ 10 exist, it is enough to know what sequences of basic double links can be performed on the above three curves and still obtain a resulting curve of degree ≤ 10, keeping in mind Theorem 5.3.8. Clearly no such basic double link can be done to C_2 or C_3 since their degrees are already too large. (Recall that their homogeneous ideals start in degree 4, by Corollary 2.2.6, so the result of the smallest basic double links on C_2 and C_3 would have degrees 12 and 14, respectively.)

The result is the following list of possibilities. We modify slightly the notation of Theorem 5.3.8 for basic double links, denoting a typical sequence by $X : (b; g_2, \cdots, g_r)$ $(b \geq 0, s < g_2 < \cdots < g_r)$. For example,

$$\begin{aligned} C_1 : (3;0) &\iff C_1 : (2,3) \to Z \\ C_1 : (0;4) &\iff C_1 : (4,1) \to Z \\ C_1 : (3;4,5) &\iff C_1 : (2,3) \to Y_1 : (4,1) \to Y_2 : (5,1) \to Z \end{aligned}$$

So, for instance, $C_1 : (0;4)$ is the union of a plane curve of degree 4 and two skew lines, each meeting the plane curve once, while $C_1 : (3;0)$ is the union of a complete intersection of type $(2,3)$ and two skew lines, each meeting the complete intersection in 3 points. Then Figure 5.1 gives all the possible pairs (d, g) that can occur, and it gives all the possible sequences of basic double links (in view of Theorem 5.3.8). Notice that this does not tell us which of the corresponding families contain *integral* curves. (See [2], [88] and [106].) □

As a final illustration of how the whole even liaison class $\mathcal{L}$ can be described from knowledge of the minimal elements, we consider curves in $\mathbb{P}^3$ and the problem of finding maximal rank curves in $\mathcal{L}$. (See page 23 for the definition. The main idea behind the notion of "maximal rank" is that the deficiency module must end before the ideal can begin.)

Of course since all the elements of $\mathcal{L}^0$ have the same Hilbert function and the same deficiency module, $\mathcal{L}^0$ contains some maximal rank curves if and only if every element of $\mathcal{L}^0$ has maximal rank.

To see whether there exist curves in $\mathcal{L}$ of maximal rank, it is enough to consider only the minimal curves and any curves obtained from them by sequences of basic double links (because of the Lazarsfeld-Rao Property). Now, using the facts about basic double links starting on page 102, it is not hard to check the following: if we perform the basic double link $X : (a, b) \to Z$, then in passing from X to Z the deficiency module moves to the right exactly b places, while the beginning of the ideal moves to the right at most b places (and exactly b places if a is sufficiently large, namely $a = \deg F_2 \geq (\deg F_1) + \alpha(I_X) = b + \alpha(I_X)$).

Hence it follows that *$\mathcal{L}$ contains maximal rank curves in every shift (of infinitely many degrees in each shift) if and only if the minimal elements of $\mathcal{L}$ have maximal rank.* (This was first observed in [22].)

degree	genus	sequence of BDL's
2	-1	C_1
3	–	(does not exist)
4	0	$C_1 : (1; 0)$
5	1	$C_1 : (0; 3)$
6	3	$C_1 : (2; 0)$
		$C_1 : (0; 4)$
7	4	$C_1 : (1; 3)$
	6	$C_1 : (0; 5)$
8	5	C_2
	6	$C_1 : (1; 4)$
		$C_1 : (0; 2, 4)$
	8	$C_1 : (3; 0)$
	10	$C_1 : (0; 6)$
9	8	$C_1 : (0; 3, 4)$
	9	$C_1 : (2; 3)$
		$C_1 : (1; 5)$
	15	$C_1 : (0; 7)$
10	10	C_3
	11	$C_1 : (2; 4)$
		$C_1 : (0; 3, 5)$
	13	$C_1 : (1; 6)$
	15	$C_1 : (4; 0)$
	21	$C_1 : (0; 8)$

Figure 5.1: Buchsbaum (non-aCM) curves in $\mathbb{P}^3$ of degree ≤ 10

5.3.4 Stick Figures

It is a classical problem whether every smooth curve in $\mathbb{P}^3$ specializes to a stick figure. Recall that a stick figure is a union of lines such that at most two of the lines meet in any point. See [63] for background on stick figures. It was shown by Gaeta [48] that every arithmetically Cohen-Macaulay curve specializes to a stick figure. (See also [28] for an analogous statement in higher dimension, still codimension two.)

In this section we describe the approach taken in [26] to prove that every *Buchsbaum* curve in $\mathbb{P}^3$ specializes to a stick figure. The idea is to show how the results on basic double links from the last section can be used, in conjunction with the results on Buchsbaum curves in Chapter 3 and the Lazarsfeld-Rao Property, to extend an idea of Bolondi [21].

Bolondi's idea, briefly, is the following. Let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$ and assume that $\mathcal{L}^0$ contains a stick figure C_0. Assume furthermore that in the initial degree of the ideal of this curve (i.e. $\alpha(I_{C_0})$) there is a polynomial consisting of a product of linear forms; that is, a union of planes. Hence in any degree d for which $(I_{C_0})_d$ is not zero, there is a union of d planes containing C_0. Then given any curve C in $\mathcal{L}$, we can choose our sequence of basic double links (guaranteed by the Lazarsfeld-Rao Property) so that each basic double link $C_i : (a, b) \to C_{i+1}$ uses polynomials F_1 and F_2 (see page 102) which are unions of planes (by abuse of notation), and F_1 can be chosen generally enough so that the intersection of F_1 and F_2 is a reduced union of lines. Hence C specializes to a reduced union of lines.

The problem is that this union of lines may be forced to have more than two of the lines meeting in a point. A simple example is to let C_0 be the degree 9 curve consisting of four general lines on a plane L_1, four general lines on a plane L_2, and the line of intersection of L_1 and L_2. Then L_1L_2 is the only surface of degree 2 containing C_0, so the basic double link $C_0 : (2,1) \to C_1$ results in a curve C_1 with a triple point.

The first step in resolving this problem, used already by Bolondi, is to assume that the minimal element is *hyperplanar*, i.e. that the reduced union of planes of minimal degree on which it lies (by assumption) has the further properties that any three planes meet in a point and no component of C lies on the intersection of any two of the planes. (See also page 57.) *We saw in §3.3 that for any Buchsbaum even liaison class $\mathcal{L}$ we can construct a* *hyperplanar stick figure in $\mathcal{L}^0$.* (This is needed for Theorem 5.3.11.)

It is slightly more subtle to see why this is not enough. Consider the following example.

Example 5.3.10 In this example we will use in a heavy way the fact that a basic double link is the union of a given curve and a certain complete intersection (see page 102). Let C_0 be a set of two skew lines and consider the sequence of basic double links

$$C_0 : (20,1) \to C_1 : (15,1) \to C_2 : (4,1) \to C_3.$$

C_1 is the union of C_0 and 20 lines on a plane L_1. C_2 is the union of C_1 and 15 lines on a plane L_2. However, because $15 < 20$, the surface F_2 of degree 15 (containing C_1) used in this latter basic double link contains L_1 as a component. Hence *the line of intersection of L_1 and L_2 is a component of C_2*. Next, *any surface of degree 4 containing C_2 contains both L_1 and L_2 as components.* Hence C_3 contains a triple point (coming from the two facts written in italics). □

The idea used in [26] is that this type of obstruction can be avoided by using Theorem 5.3.8, which allows us to assume that the sequence of basic double links is done in strictly increasing degree. One checks that this is the only thing that can go wrong, and so we have

Theorem 5.3.11 **([26])** *Let $\mathcal{L}$ be an even liaison class of curves in $\mathbb{P}^3$ and assume that there is a hyperplanar stick figure $C_0 \in \mathcal{L}^0$. Then every curve in $\mathcal{L}$ specializes to a stick figure.*

As a corollary of Theorem 5.3.11 we have the promised result for Buchsbaum curves (thanks to the construction in §3.3):

Corollary 5.3.12 **([26])** *Every Buchsbaum curve in $\mathbb{P}^3$ specializes to a stick figure.*

In [26] there is a more general theorem, generalizing the notion of a stick figure to codimension two subschemes of projective space. This result is applied in a similar way to certain Buchsbaum surfaces in $\mathbb{P}^4$, although the result is not as strong as the one above for curves.

Remark 5.3.13 Thanks to Remark 3.3.1, we can use the same techniques to prove a result in the opposite extreme: Every Buchsbaum curve in $\mathbb{P}^3$ specializes to a curve supported on a line (even staying in the same liaison class throughout the deformation). It is not known whether every curve has this property. □

5.3.5 Low Rank Vector Bundles and Schemes Defined by a Small Number of Equations

In this section we give a very brief description of how Liaison Theory can be applied to the study of vector bundles of low rank on $\mathbb{P}^n$, and to the study of schemes defined by a small number of equations. We refer to Peterson's thesis [110], Chapter 3, for a much fuller discussion of this topic; almost all of this section is based on that discussion, and is just intended to give a flavor of what can be done.

How many equations "should" it take to define a scheme? That is, given a scheme V, what is the ideal with the smallest number of generators whose saturation is I_V? Consider for instance curves in $\mathbb{P}^3$ (locally Cohen-Macaulay and equidimensional, as always). If C is a complete intersection then of course its saturated homogeneous ideal is generated by two elements (and in particular C is defined scheme-theoretically by two elements). It is also possible that the saturated ideal of C may be generated by three elements: for instance, a twisted cubic is such a curve. These are called *almost complete intersections.*

On the other hand, a result of Peskine and Szpiro says that as long as $C \subset \mathbb{P}^3$ is at least a local complete intersection then C is defined scheme-theoretically by four equations (see below). An intuitive argument for this is as follows. Choose two general polynomials F_1 and $F_2 \in I_C$ of sufficiently large degree, defining a complete intersection X which links C to some curve C' having no component in common with C. At "most" (but not all) points of C, F_1 and F_2 are enough to cut out C locally. Choose a third general polynomial $F_3 \in I_C$. This is enough to cut out C at the remaining points. However, (F_1, F_2, F_3) vanishes not only on C but also at the points of intersection of F_3 with C'; hence we need a fourth polynomial to eliminate those points.

The interesting question is to describe those curves in $\mathbb{P}^3$ which are cut out scheme-theoretically by three polynomials. Such a curve is called a *quasi-complete intersection.* Among these, of course, are the almost complete intersections.

In general, any codimension d subscheme of $\mathbb{P}^n$ whose saturated homogeneous ideal is generated by $d+1$ elements is called an *almost complete intersection.* If V is defined scheme-theoretically by $d+1$ equations (i.e. if there is an ideal with $d+1$ generators whose saturation is I_V) then V is said to be a *quasi-complete intersection.* The observation of Peskine and Szpiro referred to above actually says that if V is a local complete intersection in $\mathbb{P}^n$ then V can in any case be defined scheme-theoretically by $n+1$ equations ([109] pp. 301–302).

A rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is said to be of *low rank* if $r < n$. Liaison provides a connection between low rank vector bundles and codimension two schemes which can be defined by a (not necessarily saturated) ideal with a small number of generators, namely n or fewer.

In one direction, given a scheme V which is defined scheme-theoretically by m equations then we have an exact sequence

$$0 \to \mathcal{E} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_V \to 0. \tag{5.9}$$

The surjection comes from the fact that V is defined scheme-theoretically by m equations. Since we do not assume that these m polynomials define a saturated ideal, we do not have that $H^1_*(\mathcal{E}) = 0$ (compare with the sequence (5.1) and the discussion immediately following it). However, since V is locally Cohen-Macaulay and equidimensional, $\mathcal{E}$ is at least locally free. (We do not distinguish between locally free sheaves and vector bundles. See [60] Exercise II.5.18.) If $m \leq n$, then $r = \text{rk } \mathcal{E} = m - 1 < n$.

In the other direction, given a vector bundle $\mathcal{E}$ of rank $m-1$, for large d we have an exact sequence

$$0 \to \mathcal{O}^{m-2} \to \mathcal{E}(d) \to \mathcal{I}_{V'}(\delta) \to 0$$

where δ can be computed from a Chern class calculation, and V' has codimension two. Then link V' by a complete intersection (F_1, F_2), where deg

$F_i = a_i$, $i = 1, 2$, and let V be the residual. This gives, after twisting by $-\delta$ and applying Proposition 4.2.8, the exact sequence

$$0 \to \mathcal{E}^\vee(\delta - d - a_1 - a_2) \to \mathcal{O}(\delta - a_1 - a_2)^{m-2} \oplus \mathcal{O}(-a_2) \oplus \mathcal{O}(-a_1) \to \mathcal{I}_V \to 0$$

(where we write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n}$). Hence V is defined scheme-theoretically by m equations.

It is natural to ask if there is any connection between a set of minimal generators for the saturated ideal of a scheme V (or the number $\nu(I_V)$ of such generators) and a set of polynomials which defines V scheme-theoretically. In this regard we have the following useful theorem of Portelli and Spangher:

Theorem 5.3.14 **([112])** *Let I be the homogeneous ideal of a scheme V in $\mathbb{P}^n$. Let $s = \nu(I_V)$. Let t denote the minimum number of elements required to generate V scheme-theoretically. Then there exists a minimal system $g_1, \cdots, g_s$ of homogeneous generators of I such that $g_1, \cdots, g_t$ define V scheme-theoretically.*

Peterson uses this result to prove the following theorem:

Theorem 5.3.15 **([110])** *Let V be an equidimensional, locally Cohen-Macaulay codimension two subscheme of $\mathbb{P}^n$ with homogeneous ideal I_V and ideal sheaf $\mathcal{I}_V$. If V is a quasi-complete intersection then*

$$\nu(I_V) - \nu\left((M^{n-2})^\vee(V)\right) = 3.$$

(See page 82 for notation).

We now show how Peterson uses this theorem to reprove a result from [30] about quasi-complete intersection Buchsbaum curves in $\mathbb{P}^3$; this result brings together several of the topics which we have seen in these chapters.

Corollary 5.3.16 **([30])** *Let C be a Buchsbaum, non-aCM curve in $\mathbb{P}^3$. Let $N = dim_k M(C)$. If C is a quasi-complete intersection then $\nu(I_C) = 4$ and $N = 1$.*

Proof:

Notice that since $M(C)$ and $M^\vee(C)$ have trivial module structure, we have $\nu(M^\vee(C)) = N$. Recall also the result of Bresinsky, Schenzel and Vogel [30] that $\nu(I_C) \geq 3N + 1$ (see page 39 for the proof). Hence Theorem 5.3.15 gives $3 = \nu(I_C) - N \geq 2N + 1$. Since in any case $N > 0$, we get the desired result. □

Finally, it is observed in [30] and in [110] that the connection (described above) between vector bundles of low rank and schemes defined scheme-theoretically by a small number of equations easily gives the following corollary. *Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^3$ with $N = \sum_{n \in \mathbb{Z}} h^1(\mathbb{P}^3, \mathcal{E}(n))$. If $N > 1$ then no section of $\mathcal{E}$ can have a Buchsbaum curve as zero scheme.*

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