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MATHEMATICAL THEORY OF COAGULATION

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P. B. DUBOVSKIĬ

Dedicated to my father on occasion of his seventy-fifth birthday

PREFACE

This Lecture Notes are based partially on research which I carried on at the Pohang University of Science and Technology (Postech) during 1993-1994 years. The work became a reality due to the financial and moral support of the Global Analysis Research Center (GARC).

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Contents

0.1 Introduction

The science of disperse systems is the study of solid or liquid particles suspended in a medium (usually in a gas). Such systems are created during various natural processes, such as smoke particles from forest fires, sand and dust storms, blowing snow, condensation of water vapor in the atmosphere, volcanic dust, spores and seeds from plant life, and meteoritic dust. Several industrial operations produce aerosols either as an intentional part of the operation or as an undesirable byproduct. Some of the deliberate aerosols are fluidized catalysts, emulsions of various sorts, the spray drying of viscous liquids, and the atomization of liquid fuels and so on. The military is also very concerned with the science of aerosols in order that the propagation of radioactive clouds from atomic explosions and the production and propagation of smoke screens can be better understood.

Once a disperse system is produced it is usually unstable because it will change with time. The most important process corresponding to the evolution of disperse systems, is the process of coagulation (merging) of particles in a disperse system. The theory of coagulation of aerosols deals with the process of adhesion or coalescence of aerosol particles when they come in contact with one another. The aim of this theory is the description of the particle size distribution as a function of time and space as the disperse system undergoes changes due to various physical influences.

The major part of the current study will deal with the coagulation processes as it affects particle size distribution. Since condensation, evaporation, fragmentation, precipitation, and coagulation are all an integral part of the evolution of the particle size spectrum, these other processes will also be considered in some of the mathematical models discussed in this notes.

We suppose that the number of particles per unit volume is sufficiently small so that the probability of triple, quadruple, or more frequent encounters is negligible; thus binary collisions are assumed to occur simultaneously with the chance splitting of single particles in two. Using this basic assumption, Smoluchowski [63] obtained the following infinite set of nonlinear differential equations:

$$\frac{\partial c_i(t)}{\partial t} = \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j}c_{i-j}(t)c_j(t) - c_i(t)\sum_{j=1}^{\infty} K_{i,j}c_j(t) \tag{0.1}$$

with the initial conditions

$$c_i(0) = c_i^{(0)} \geq 0. \tag{0.2}$$

Applications of (0.1) can be found in many problems including chemistry (e.g. reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), engineering (behaviour of a fuel mixture in engines), astrophysics (formation of stars and planets), meteorology (coagulation of drops in atmospheric clouds).

The constant coefficients $K_{i,j}$ are to be non-negative and symmetric, i.e. $K_{i,j} \geq 0$ and $K_{i,j} = K_{j,i}$ for all $i,j \geq 1$. The complexity of the system (0.1), (0.2) is determined by the form of $K_{i,j}$'s as functions of the indices i,j. The function $K_{i,j}$ is called a coagulation kernel; it describes intensity of interaction between particles of mass i and j and is supposed to be known. The unknown non-negative function $c_i(t)$ is the concentration of particles with mass i, $i \geq 1$.

Müller [56] rewrote equation (0.1) in terms of an integrodifferential equation for the time evolution of the particle mass density function. Let $c(x,t)dx$ be the average number of particles per unit volume at time t whose masses lie between x and $x + dx$. All other averages are referred to a unit volume, too. The function $K(x,y)$ (coagulation kernel) is introduced by assuming that the average number of coalescences between particles of mass x to $x + dx$ and those of mass y to $y + dy$, is $c(x,t)c(y,t)K(x,y)dxdydt$ during the time interval $(t, t + dt)$. The quantity $c(x,t)$ is the particle mass density function. Then the model (0.1), (0.2) converts into

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy -$$

$$-c(x,t)\int_0^\infty K(x,y)c(y,t)dy, \quad x,t \geq 0, \tag{0.3}$$

$$c(x,0) = c_0(x) \geq 0, \quad x \geq 0. \tag{0.4}$$

The first integral in (0.3) expresses the fact that a particle of mass x can only come into existence if two particles with masses $x-y$ and y collide. The second integral says that each particle of mass x disappears from the interval x to $x+dx$ after colliding with a particle of mass y . The coagulation kernel $K(x,y)$, or the collision frequency factor, is dependent upon the physics of collisions. It can be both bounded or unbounded function. For instance, when coagulation is controlled by Brownian diffusion, then

$$K(x,y) = \text{const} \cdot (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3}).$$

If we consider the gravitational coalescence , which refers to the fusion of particles of different sizes colliding due to the difference in their rates of fall in a gravity field, then the coagulation kernel can be presented as

$$K(x,y) = \text{const} \cdot (x^{1/3} + y^{1/3})^2 |x^{2/3} - y^{2/3}|.$$

Golovin [41] considered the simplified kernel

$$K(x,y) = \text{const} \cdot (x+y).$$

Martynov and Bakanov [48] considered

$$K(x,y) = \beta_1 + \beta_2(x+y) + \beta_3 xy \qquad \text{where } \beta_i = \text{const}.$$

Other kernels and more extensive discussion can be found in Drake's review [18] or in Voloschuk [77].

Melzak [54] extended the model concerned for the case when particles undergo breakdown. He proposed the following equation:

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy - c(x,t)\int_0^\infty K(x,y)c(y,t)dy +$$

$$+ \int_x^\infty \Psi(y,x)c(y,t)dy - \frac{c(x,t)}{x}\int_0^x y\Psi(x,y)dy. \tag{0.5}$$

The particle-mass distribution varies as a result of two processes, coagulation and fragmentation. The breakdown function $\Psi(x,y) \geq 0$ enters through the assumption that $c(x,t)\Psi(x,y)dxdydt$ is the average number of particles of mass y to $y+dy$ created from the breakdown of particles of mass x to $x+dx$, during the time interval $(t, t+dt)$. Therefore $\Psi(x,y) = 0$ if $x < y$. The third integral in (0.5) describes production of particles x in the process of breakdown of particles y $(x \leq y < \infty)$; the fourth one reflects disappearance of particles x due to their breakdown into particles y $(0 \leq y \leq x)$.

Friedlander [31] considered the case of fragmentation which allows splitting of particles into two other particles. This way gives us the coagulation-fragmentation equation

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy - c(x,t)\int_0^\infty K(x,y)c(y,t)dy +$$

$$+\int_0^\infty F(x,y)c(x+y,t)dy - \frac{1}{2}c(x,t)\int_0^x F(x-y,y)dy. \qquad (0.6)$$

Equation (0.6) can be obtained from Melzak's model (0.5). Really, if it is assumed that each particle can only be split into two sub-particles then $\Psi(x,y) = \Psi(x, x-y)$ and equation (0.5) may be rewritten in the form (0.6) where the fragmentation kernel F becomes $F(x-y,y) = \Psi(x,y)$. The physical motivation of introducing the fragmentation kernel F similar to the coagulation kernel K. It is important to note that the fragmentation kernel is symmetric function unlike the breakdown function $\Psi(x,y)$. All the functions concerned are non-negative.

The discrete case for (0.6) is

$$\frac{dc_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j}c_{i-j}(t)c_j(t) - c_i(t)\sum_{j=1}^{\infty} K_{i,j}c_j(t) +$$

$$+\sum_{j=1}^{\infty} F_{i,j}c_{i+j}(t) - \frac{1}{2}c_i(t)\sum_{j=1}^{i-1} F_{i-j,j}. \qquad (0.7)$$

If we suppose that the particles can coagulate to form large particles or fragment to form smaller ones by addition or loss the smallest particle with mass 1 respectively, then we come to the Becker–Döring cluster equations

$$\frac{dc_i(t)}{dt} = J_{i-1}(c) - J_i(c), 2 \le i < \infty \tag{0.8}$$

$$\frac{dc_1(t)}{dt} = J_1(c) - \sum_{i=1}^{\infty} J_i(c), \tag{0.9}$$

where

$$J_i(c) = a_i c_1 c_i - b_{i+1} c_{i+1}, \tag{0.10}$$

The equations (0.8)–(0.10) can be obtained if in (0.7) we put

$$K_{i,j} = \begin{cases} a_i, & j = 1, \quad i \ge 2 \\ 2a_1, & i = j = 1 \\ 0, & i \ge 2 \text{ and } j \ge 2; \end{cases}$$

$$F_{i,j} = \begin{cases} b_{i+1}, & j = 1, \quad i \ge 2 \\ 2b_2, & i = j = 1 \\ 0, & i \ge 2 \text{ and } j \ge 2. \end{cases}$$

The original model was proposed by Becker and Döring [9] but it is the modified form given by Penrose and Lebowitz [59] which will be studied here.

The spatially inhomogeneous model of coagulation was presented by Berry [10] and Levin and Sedunov [45]:

$$\frac{\partial c(x,z,t)}{\partial t} + \frac{\partial}{\partial x}(r(x,z,t)c(x,z,t)) + \mathrm{div}_z(v(x,z,t)c(x,z,t)) =$$

$$= \frac{1}{2}\int_0^x K(x-y,y)c(x-y,z,t)c(y,z,t)dy -$$

$$-c(x,z,t)\int_0^\infty K(x,y)c(y,z,t)dy. \tag{0.11}$$

In (0.11) the space variable $z \in R^3$, the known vector-function $v(x,z,t) \in R^3$ is the velocity of space motion of particles with mass x . The scalar function $r(x,z,t) \in R^1$ is equal to rate of growth of particles due to condensation or evaporation processes (e.g. condensation of vapour on water drops in atmospheric clouds). In evaporation processes the function $r(x,z,t)$ is negative. In physically real situations we often have $r \sim x^\alpha$, $\alpha > 0$, $0 < x_0 \le x \le \overline{x}$ where x_0 is a critical mass of a particle which splits regions of its stable and unstable state; $\overline{x}$ is a conventional boundary of satiation, after which the function $r(x)$ may be considered as bounded. The physical meanings of functions $v(x,z,t)$ and $r(x,z,t)$ are confirmed by mathematical reasonings. Namely, the characteristic equations to (0.11) yield

$$\frac{dx}{dt} = r(x,z,t), \qquad \frac{dz}{dt} = v(x,z,t).$$

So, the second term on the left-hand side of (0.11) represents grows by condensation or decay by evaporation. The third one represents the change in c due to motion in the physical space. Hence, if there is only motion in the direction of the gravity field, then the third term accounts for the effect of settling or sedimentation. We could add to the right-hand side of (0.11) a source term for new particles. This source term may account for droplet formation due to water vapor condensing on atmospheric condensation nuclei. Also, we could consider fragmentation and efflux terms in (0.11). The last one is mathematically expressed by adding $a(x,z,t)c(x,z,t)$, $a \ge 0$ to the left-hand side of (0.11) and describes absorbtion phenomena (see, e.g., [32, 33, 66, 79]).

In this notes we treat the general coagulation models from mathematical point of view. Usually the results and methods being discussed hold both for discrete and continuous models and we shall not usually specify this fact. We shall emphasize if results for discrete case cannot be transformed for continuous one or vice versa.

0.2 Main properties of the coagulation equation

The main observation is that the equations (0.3), (0.5) and (0.6) possess the mass conservation law. Actually, from definition of the distribution function c we conclude that the total mass of particles per unit of volume is expressed by the first moment of $c(x,t)$:

$$N_1(t) = \int_0^\infty xc(x,t)dx. \tag{0.12}$$

We multiply the equations by x and assume that all integrals in (0.3), (0.5) and (0.6) exist. Then after integration we obtain the mass conservation law

$$\frac{dN_1(t)}{dt} = 0. \tag{0.13}$$

To derive (0.13) we have used the change of variables $x' = x - y$, $y' = y$ in the first integral convolution-like term in the right-hand side of equations. The Jacobian of such transform is equal to 1. In this place we have employed the symmetry property of a coagulation kernel, too. From physical point of view the mass conservation law is very natural. The solutions to the coagulation equation for whih the equality (0.13) holds for all $t \geq 0$, we call mass conserving solutions.

For pure coagulation equation (without fragmentation and other processes) the coagulation must lead to dissipation of total amount of particles which is expressed by the zero moment of solution:

$$N_0(t) = \int_0^\infty c(x,t)dx. \tag{0.14}$$

The mathematical treatment of equation (0.3) confirms that the dissipation law is valid:

$$\frac{dN_0(t)}{dt} \leq 0. \tag{0.15}$$

The same results hold for corresponding discrete equations.

We should pay attention that for unbounded coagulation kernels the values of the right-hand side of equation (0.1) may belong to another functional space than its domain. This brings us the main difficulties in studying the coagulation equation with unbounded kernels.

It is worth pointing out that the coagulation equation is similar to Boltzmann equation of gas kinetics, but unlike the Boltzmann equation, the coagulation one considers nonelastic collisions of particles. Therefore for the coagulation equation we do not possess the conservation of the second moment of solutions which physically expresses the energy conservation law. The Boltzmann equation conserves zero, first and second moments of solutions. As we have seen, the coagulation–fragmentation one conserves only the first moment.

Another important observation is that the initial value problem being considered possesses the property of immediate spreading of perturbations. In fact, let us restrict ourselves to the simplest case of equation (0.1) with the constant coagulation kernel $K_{i,j} \equiv 1$ and the initial data

$$c_i(0) = (1, 0, 0, \dots). \tag{0.16}$$

We introduce the generating function

$$G(z,t) = \sum_{i=1}^{\infty} z^i c_i(t).$$

Then we obtain the ordinary differential equation

$$\frac{dG}{dt} = \frac{1}{2}G^2 - \frac{1}{1+t/2}G,$$

whence

$$G(z,t) = \sum_{i=1}^{\infty} z^i \frac{(t/2)^{i-1}}{(1+t/2)^{i+1}}.$$

Consequently,

$$c_i(t) = \frac{(t/2)^{i-1}}{(1+t/2)^{i+1}} > 0 \quad \text{for all } t > 0,\ i \geq 1. \tag{0.17}$$

This example demonstrates the property of immediate "scooping out" of the initial distribution (0.16) and immediate spreading of perturbations. This property demonstrates common properties of coagulation equation and parabolic equations. Another conclusion from this example is that there is no sense to restrict initial data to, e.g., compactly supported ones. The minor natural initial data which should be considered to study the coagulation equation, are of exponential type

$$c_i(0) \leq e^{-\lambda i}. \tag{0.18}$$

Really, for small $t > 0$ we obtain from (0.17) that

$$c_i(t) \sim t^{i-1} = \left(e^{\ln t}\right)^{i-1}.$$

Other results connected with estimates of area of solutions' positivity, are discussed in next chapters.

In chapter 1 we demonstrate the problems which can arise in the coagulation equation and cause the surprising breakdown of the mass conservation law. This justifies the next chapters where general theory for the coagulation–fragmentation equation is constructed. In chapter 2 we prove existence of solutions for approximated (truncated) problems. In chapter 3 we pass to limit in the sequence of approximated solutions to demonstrate the existence for the problem with unbounded kernels. In chapter 4 we discuss uniqueness of solutions. Chapter 5 is devoted to some properties of solutions: their boundedness, positivity and asymptotics for large x. In chapters 6–9 we study asymptotic properties of solutions for large t and their convergence to equilibria. Chapters 10 and 11 are devoted to the space inhomogeneous case, which, as we shall see, brings in many difficulties.

Chapter 1. COAGULATION EQUATION WITH MULTIPLICATIVE KERNELS AND INFRINGEMENT OF MASS CONSERVATION LAW

In this chapter we are concerned with the pure coagulation equation (0.3) which reflects main properties of different coagulation models. It turns out that there are coagulation kernels such that the mass conservation law (0.13) undergoes the breakdown in a finite time even if initial data are very "good" (e.g., smooth and have a compact support). To demonstrate why such situation can happen, we multiply (0.3) by x and integrate over $[0, m]$. Then we obtain

$$\frac{d}{dt}\int_0^m xc(x,t)dx = \frac{1}{2}\int_0^m \int_0^{m-x} (x+y)K(x,y)c(x,t)c(y,t)dydx -$$

$$-\int_0^m \int_0^\infty xK(x,y)c(x,t)c(y,t)dydx =$$

$$= -\int_0^m \int_{m-x}^\infty xK(x,y)c(x,t)c(y,t)dydx \leq 0. \tag{1.1}$$

Passing to limit $m \to \infty$ yields

$$\frac{d}{dt}\int_0^\infty xc(x,t)dx \leq 0. \tag{1.2}$$

If the integral

$$\int_0^\infty \int_0^\infty xK(x,y)c(x,t)c(y,t)dydx \tag{1.3}$$

is bounded then passing to limit in (1.1) gives us zero and we obtain the mass conservation law. Otherwise

$$\frac{d}{dt}\int_0^\infty xc(x,t)dx < 0 \tag{1.4}$$

and the breakdown of the mass conservation law can happen. In this chapter we demonstrate that there may be cases when (1.4) is valid yielding the paradoxical infringement of the mass conservation law.

In the first section we explicitely derive the solution for discrete case of coagulation. Then we discuss some developments for continuous case.

1.1 Discrete coagulation equation with $K_{i,j} = i\,j$

In the case concerned the coagulation equation (0.3) takes the following form

$$\frac{dc_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1}(i-j)jc_{i-j}(t)c_j(t) - ic_i(t)\sum_{j=1}^{\infty} jc_j(t). \tag{1.5}$$

We consider the simplest initial conditions

$$c_i(0) = (1, 0, 0, \dots). \tag{1.6}$$

For the total number of particles expressed by the zero moment N_0 we obtain from (1.5)

$$\frac{dN_0(t)}{dt} = -\frac{1}{2}N_1^2(t). \tag{1.7}$$

Hence, if the mass conservation law holds for all $t \geq 0$ (i.e. $N_1(t) \equiv$const) then after the finite time equal to $2N_0(0)$ $(N_1 = 1)$ the solution to (1.7) become negative and we come to the dilemma: either there exists no non-negative solution to (1.5) or the mass conservation law fails. Let us analyse why the mass conservation law can fail. With this aim we multiply (1.5) by i and summarize from 1 to n :

$$\frac{d}{dt}\sum_{i=1}^{n} ic_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} i(i-j)jc_{i-j}c_j - \sum_{i=1}^{n} i^2 c_i \sum_{j=1}^{\infty} jc_j =$$

$$= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}(i^2 j + ij^2)c_i c_j - \sum_{i=1}^{n} i^2 c_i \sum_{j=1}^{\infty} jc_j. \tag{1.8}$$

Consequently, the right-hand side of (1.8) tends to zero as $n \to \infty$ (yielding the mass conservation law) if the second moment of solution $N_2(t)$ is bounded. If it becomes unbounded then passing to limit in the right-hand side of (1.8) brings us indeterminancy $\infty - \infty$ and, generally speaking, the mass conservation law fails. Really, if mass is indeed conserved, $N_1(t) = 1$, one may solve (1.5) by solving the simpler system

$$\frac{dc_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1}(i-j)jc_{i-j}(t)c_j(t) - ic_i(t), \tag{1.9}$$

recursively. This yields the formula

$$c_i(t) = \frac{i^{i-2}}{i!} t^{i-1} \exp(-it), \quad i \geq 1. \tag{1.10}$$

However, the desired conservation of mass of (1.10) breaks down for $t > 1$, and, hence, (1.10) is no longer a valid solution past the critical "gelation" time $t = 1$.

Therefore we consider the time-dependent first moment

$$N_1(t) = \sum_{i=1}^{\infty} i c_i(t)$$

so that (1.5) becomes

$$\frac{dc_i(t)}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} (i-j) j c_{i-j}(t) c_j(t) - N_1(t) i c_i(t) \tag{1.11}$$

or, alternatively, with

$$\phi_i(t) = \exp\left(i \int_0^t N_1(s) ds \right) c_i(t)$$

we see

$$\frac{d\phi_i(t)}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} (i-j) j \phi_{i-j}(t) \phi_j(t) \tag{1.12}$$

with (1.6) implying

$$\phi_1(0) = 1, \qquad \phi_i(0) = 0, \quad 2 \leq i < \infty. \tag{1.13}$$

Now solve (1.12), (1.13) recursively to obtain

$$\phi_i(t) = \frac{i^{i-3}}{(i-1)!} t^{i-1}; \tag{1.14}$$

hence,

$$c_i(t) = \frac{i^{i-3} t^{i-1}}{(i-1)!} \exp\left(-i \int_0^t N_1(s) ds \right). \tag{1.15}$$

But, by definition, $N_1(t) = \sum_{i=1}^{\infty} ic_i(t)$ so that $N_1(t)$ must satisfy the equation

$$N_1(t) = \sum_{i=1}^{\infty} \frac{i^{i-2}t^{i-1}}{(i-1)!} \exp\left(-i \int_0^t N_1(s)ds\right)$$

or, equivalently,

$$tN_1(t) = \sum_{i=1}^{\infty} \frac{i^{i-1}\left(t \exp\left(-\int_0^t N_1(s)ds\right)\right)^i}{i!}. \tag{1.16}$$

Next note the relevant identity

$$\sum_{i=1}^{\infty} \frac{i^{i-1}\left(xe^{-x}\right)^i}{i!} = x, \quad 0 \leq x \leq 1. \tag{1.17}$$

For the moment assume

$$t \exp\left(-\int_0^t N_1(s)ds\right) \leq e^{-1}, \quad \text{for all} \quad t > 0, \tag{1.18}$$

and let $x(t)$ be that value x, $0 \leq x(t) \leq 1$, which satisfies the equation

$$x(t)e^{-x(t)} = t \exp\left(-\int_0^t N_1(s)ds\right). \tag{1.19}$$

As the graph of xe^{-x} is monotone increasing for $0 \leq x \leq 1$ and monotone decreasing for $x \geq 1$ with a maximum e^{-1} at $x = 1$, (1.18) will imply a solution of (1.19) and $x(t)$ can always be uniquely found. Thus (1.16), (1.17) imply

$$tN_1(t) = x(t). \tag{1.20}$$

Equations (1.19), (1.20) provide two equations in the two unknowns $x(t)$, $N_1(t)$ for all $t > 0$. Now substitute the $x(t) = tN_1(t)$ from (1.20) into (1.19) to obtain

$$N_1(t) \exp\left(-tN_1(t)\right) = \exp\left(-\int_0^t N_1(s)ds\right), \quad t > 0, \tag{1.21}$$

and, hence,

$$\ln N_1(t) - tN_1(t) = -\int_0^t N_1(s)ds. \tag{1.22}$$

At points of differentiability of $N_1(t)$, differentiate (1.22) to see that $N_1(t)$ satisfies

$$\left(\frac{1}{N_1(t)} - t\right)\frac{dN_1(t)}{dt} = 0, \tag{1.23}$$

where recall of the initial data is given by

$$N_1(0) = 1. \tag{1.24}$$

For $0 \le t < 1$, the solution of (1.23), (1.24) is $N_1(t) = 1$ and (1.18) is satisfied. For $t > 1$, a choice must be made between a function satisfying $\frac{d}{dt}N_1(t) = 0$ (which, for continuous $N_1(t)$, would mean $N_1(t) = 1$) and $N_1(t) = 1/t$. If $N_1(t) = 1$ is chosen, then (1.20) becomes $t = x(t)$, i.e., $x(t) > 1$ which contradicts the definition of $x(t)$. This necessitates the choice $N_1(t) = 1/t$ which yields $x(t) = 1$ and also satisfies (1.18). Enforcing continuity at $t = 1$ yields

$$N_1(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 1/t, & t > 1, \end{cases} \tag{1.25}$$

as the solution of (1.21). Finally, substitute (1.25) into (1.20). This yields the solution

$$c_i(t) = \begin{cases} i^{i-3}t^{i-1}\exp(-it)/(i-1)!, & 0 \le t \le 1, \\ i^{i-3}e^{-i}/(i-1)!\dfrac{1}{t}, & t > 1, \end{cases} \qquad i \ge 1. \tag{1.26}$$

We should note in conclusion that actually the problem concerned became one of looking for a fixed point of the nonlinear equation (1.16). Of course, the ability to solve (1.16) in explicit form is, to some extent, luck. Similar explicit result does not hold in the continuous coagulation model.

1.2 Coagulation kernel $K(x,y) = xy$

In this section we are concerned with continuous coagulation equation

$$c_t(x,t) = \frac{1}{2}k\int_0^x (x-y)yc(x-y,t)c(y,t)dy - kxc(x,t)\int_0^\infty yc(y,t)dy. \quad (1.27)$$

We will discuss the case of general continuous initial mass spectra $c(x,t)$. Here some properties of the solution differ strongly from those in the discrete case, in particular long time properties. Note that we may set $k = N_1(0) = N_2(0) = 1$ without loss of generality by choosing proper units for x, t and $c(x,t)$: $N_2(0)/N_1(0)$, $1/kN_2(0)$, and $N_1(0)^3/N_2(0)^2$, respectively. For the kernel xy we make a preliminary investigation of the moment equations (1.27):

$$\frac{dN_r(t)}{dt} = \frac{1}{2}\int_0^\infty\int_0^\infty [(x+y)^r - x^r - y^r]\, xyc(x,t)c(y,t)dxdy \quad (1.28)$$

where N_r are moments of the solution:

$$N_r(t) = \int_0^\infty x^r c(x,t)dx. \quad (1.29)$$

These can be derived from the coagulation equation if one assumes that orders of integration can be freely interchanged.

For $r = 1$ it follows that the total mass $N_1(t)$ provided that $N_2(t) < \infty$. This condition is necessary to give a well-defined meaning to (1.28), as the right-hand side equals $N_1(N_2 - N_2)$. When $c(x,t)$ is such that $N_2 = \infty$, the determination of moments requires more care, as discussed below.

For $r = 0$ one finds

$$\frac{dN_0}{dt} = -\frac{1}{2}N_1^2. \quad (1.30)$$

As long as $N_1(t) = 1$, the general solution of (1.30) is

$$N_0(t) = N_0(0) - t/2. \quad (1.31)$$

Note the unphysical prediction by (1.31) that $N_0(t)$ becomes negative for $t > t_0 = 2/N_0(0)$.

For $r = 2$ one finds $\frac{d}{dt}N_2 = N_2^2$, provided $N_3(t) < \infty$. This yields

$$N_2(t) = \frac{1}{1-t}, \tag{1.32}$$

where $N_2(t)$ approaches infty within a finite time $t_c = 1/N_2(0) = 1$. (Recall that units are chosen such that $N_1(0) = N_2(0) = 1$.) For $r = 3$ one finds

$$N_3(t) = N_3(0)(1-t)^{-3}, \tag{1.33}$$

provided $N_4(t) < \infty$.

In order to find $N_1(t)$ we consider the Laplace transform of the mass distribution $xc(x,t)$, defined as

$$f(p,t) = \int_0^\infty x e^{-px} c(x,t)dx. \tag{1.34}$$

After multiplying (1.27) by x and taking the Laplace transform we obtain

$$f_t + f_p \cdot (f - N_1(t)) = 0. \tag{1.35}$$

The equation (1.35) is to be solved subject to the initial condition:

$$f(p,0) = f_0(p), \qquad (f_0(0) = -f_0'(0) = 1). \tag{1.36}$$

The characteristic equation to (1.35) is

$$\frac{dp}{dt} = f - N_1(t). \tag{1.37}$$

Since the right-hand side of (1.35) is equal to zero then along characteristic curves $f \equiv$const and $f(p,t) = f_0(p_0)$ where p_0 is the starting point of the characteristics passing through (p,t). Then from (1.37) we conclude

$$f = f_0\left(p - ft + \int_0^t N_1(s)ds\right) = f(p,t). \tag{1.38}$$

The solution (1.38) still contains the unknown mass of particles, $N_1(t) = f(0,t)$, which may be determined self-consistently by putting $p = 0$ to yield the functional equation

$$N_1 = f_0\left(\int_0^t N_1(s)ds - tN_1\right). \tag{1.39}$$

It can be solved by differentiating (1.39) with respect to time:

$$\frac{dN_1}{dt} = -t\frac{dN_1}{dt} \cdot f_0'\left(\int_0^t N_1(s)ds - tN_1\right). \tag{1.40}$$

We assume that the function f_0' has no singularities in the complex half-plane $Re\ p \geq 0$, i.e. the initial function $c_0(x)$ has the bounded first moment. Then from (1.40) we have a constant solution

$$N_1^{(1)}(t) = N_1^{(1})(0) = 1, \tag{1.41}$$

and a time-dependent solution, parametrically given by

$$N_1^{(2)} = f_0(\xi), \qquad t^{-1} = -f_0'(\xi), \quad \xi \geq 0. \tag{1.42}$$

In fact, to obtain (1.42) we denote in (1.40)

$$\xi = \int_0^t N_1(s)ds - tN_1$$

and use (1.39). Since the function $N_1(t)$ cannot increase (see (1.2)), then $\xi \geq 0$. In this important place we have the similarity with (1.23). Both functions f_0 and $-f_0'$ are positive, monotonically decreasing, equal to 1 at $\xi = 0$, and tend to zero as $\xi \to \infty$. Therefore the monotonically decreasing solution $N_1^{(2)}(t)$ appears at $t = 1$ (not earlier: $\xi \geq 0$!) and replaces the constant solution (1.41). The replacement takes place because, as we have seen, the constant solution (1.41) cannot be valid for all $t > 0$. The replacement cannot occur after $t = 1$ since the function $N_1(t)$ must be continuous but the moment $t = 1$ is the only moment when $N_1^{(2)} = N_1^{(1)} = 1$. Therefore we obtain the expression which generalizes (1.25):

$$N_1(t) = \begin{cases} 1, & t \leq 1, \\ N_1^{(2)}(t), & t \geq 1. \end{cases} \tag{1.43}$$

There occurs a phase transition (gelation) at the gel point $t_c = 1$. In the sol phase ($t < 1$) $N_1(t)$ is constant; in the gel phase ($t > 1$) $N_1(t)$ decreases to zero as time progresses. The loss of mass, starting at $t = 1$, is associated with the formation of an infinite cluster (gel, superparticle). It is a loss to infinity due to the cascading growth of larger and larger particles (clusters), where the process accelerates, as the clusters grow larger, since the rate is given by $K(x, y) = xy$. The mass deficit, $1 - N_1(t)$, is called the gel fraction, which is only nonvanishing past the gel point $t_c = 1$.

We need only to show that there exists a solution to (1.27) with the first moment expressed by (1.43). With this aim we rewrite (1.27) in the form

$$d_t(x,t) = \frac{1}{2}x \int_0^x d(x-y,t)d(y,t)dy - xd(x,t)N_1(t) \qquad (1.44)$$

where $N_1(t)$ is defined in (1.43) and $d(x,t) = xc(x,t)$. So,

$$d(x,0) = xc_0(x). \qquad (1.45)$$

Let $D(p,t)$ be the Laplace transform to $d(x,t)$. Then we obtain similarly to (1.38):

$$D - D_0\left(p - Dt + \int_0^t N_1(s)ds\right) = 0. \qquad (1.46)$$

If we denote the left-hand side of (1.46) as $F(D,p)$ then $F_D = 1 + tD_0' > 0$ in a vicinity of $p = 0$ for sufficiently small $t > 0$. The implicit function theorem ([80], p. 149) yields the existence of a local in time solution to (1.46). Since D_0 is a Laplace transform of d_0 then there exists the inverse Laplace transform of D and, consequently, the initial value problem (1.44), (1.45) has a local in time solution $d(x,t)$ with bounded zero moment (which corresponds to $D(0,t)$). Its nonnegativity and continuity easily follow from (1.44). Let $D(0,t) = \tilde{N}_1(t)$. Then (1.46) yields

$$\frac{d}{dt}\tilde{N}_1(t) = (N_1(t) - \tilde{N}_1(t) - t\frac{d}{dt}\tilde{N}_1(t))D_0'\left(\int_0^t N_1(s)ds - t\tilde{N}_1(t)\right). \qquad (1.47)$$

From (1.47) and the above reasonings, following after (1.40), we conclude that $\tilde{N}_1(t) = N_1(t)$ indeed. Therefore we can extend the local in time solution of (1.44), (1.45) globally in time provided that $xc_0(x)$ has a bounded first moment. The uniqueness follows from section 4.3. Consequently, the following theorem has been proved.

Theorem 1.1. *Let the initial function* $c_0(x)$ *be continuous, nonnegative and have bounded first moment. Then there exists unique nonnegative continuous solution of the equation (1.27) and a critical time moment* $t_c > 0$ *such that the first moment of the solution is expressed by (1.43).*

Let us consider as an example the monodisperse initial conditions $c_0(x) = \delta(x-1)$ where δ is the Dirac delta–functional. Then $f_0(p) = \exp(-p)$. From (1.42) and (1.43) we deduce for the mass $N_1(t)$ the expression (1.25).

1.3 Remarks

In the section 1.1 we repeat reasonings of Slemrod [62]. The identity (1.17) may be found in the collection of Jolley ([44], p .24, Series 130). The interval of convergence $0 \le x \le 1$ is not noted by Jolley but is easily obtained by ratio test for $0 \le x < 1$ and Stirling's formula at $x = 1$. In fact, this is proven in the paper of McLeod [52]. Detailed derivation of (1.10) and (1.14) may be found in [52], too.

In 1962 McLeod [52] proved the local existence and uniqueness theorem for the problem (1.5), (1.6). He noted also, that the desired conservation of density breaks down. His result is the first one connected with the treatment of unbounded coagulation kernels. For continuous case he succeeded to prove similar existence and uniqueness theorem in [53].

Almost in 20 years after McLeod's result Leyvraz and Tschudi [46] succeeded to prolong McLeod's solution globally in time. They solved (1.5), (1.6) by setting

$$\phi_i(t) = ic_i(t)\exp\left(i\int_0^t N_1(s)ds\right) \quad \text{and} \quad G(z,t) = \sum_{i=1}^{\infty} \phi_i(t)z^i.$$

A straightforward computation shows that the generating function G satisfies the quasilinear partial differential equation

$$\frac{\partial G}{\partial t} = zG\frac{\partial G}{\partial z}, \quad 0 \le z \le 1, \quad t > 0, \tag{1.48}$$

with initial data

$$G(z,0) = z.$$

Equation (1.48) may be integrated via the method of characteristics to obtain $G(z,t)$ from which one may recover $\phi_i(t)$, $i \geq 1$, and finally $c_i(t)$. This way was used by Leyvraz and Tschudi to obtain the solution (1.26).

The approach, similar to Leyvraz and Tschudi's one, with replacing the generating function onto Laplace transform, was employed for continuous case by Ernst, Ziff and Hendriks [28] and Galkin [36]. Ernst *et al* [28] showed that the equation (1.36) can be solved by introducing the inverse function, $p(F,t)$. In fact, using $f_p = (p_f)^{-1}$ and $f_t = -p_t/p_f$ we see that p satisfies $p_t = f - N_1(t)$, with the initial condition $p(f,0) = f_0^{-1}(f)$, the solution of which is given by

$$p = f_0^{-1}(f) + ft - \int_0^t N_1(s)ds$$

From the last equality we immediately obtain (1.38). In section 1.2 we follow mainly to Ernst, Ziff and Hendriks [28].

Using the characteristic method Galkin [36] studied (1.35) with $N_1(t) = f(0,t)$. He showed that the critical time t_c corresponds to the first intersection of characteristic curves. He proved that

$$N_0(t) = C(\pi(t)) - \frac{1}{2}tC_p^2(\pi(t)), \quad N_1(t) = -C_p(\pi(t)),$$

where the function $C(p)$ is the Laplace transform of $c_0(x)$ and the nonnegative function $\pi(t)$ is the starting point of the characteristic curve which at the time t intersects with the straight line $p = 0$. That function is defined by

$$\pi(t) = 0, \quad 0 \leq t \leq t_c = C_{pp}^{-1}(0),$$

$$C_{pp}(\pi(t)) = 1/t, \quad t \geq t_c.$$

It is possible to observe that these results coinside with ones presented in section 1.2.

Chapter 2. EXISTENCE FOR KERNELS WITH COMPACT SUPPORT

In this chapter we are concerned with the following general spatially homogeneous coagulation-fragmentation equation

$$\frac{\partial c(x,t)}{\partial t} + a(x,t)c(x,t) - q(x,t) = \frac{1}{2}\int_0^x K(x-y,y,t)c(x-y,t)c(y,t)dy -$$

$$-c(x,t)\int_0^\infty K(x,y,t)c(y,t)dy + \int_0^\infty F(x,y,t)c(x+y,t)dy -$$

$$-\frac{1}{2}c(x,t)\int_0^x F(x-y,y,t)dy. \tag{2.1}$$

As was pointed out in introduction, the non-negative function a describes efflux terms. The function q describes, in turn, sources of particles. The equation (2.1) must be equipped with the initial condition

$$c(x,0) = c_0(x), \quad x \geq 0. \tag{2.2}$$

2.1 Functional spaces

We introduce some functional spaces. Firstly, we fix T, $0 < T \leq \infty$. Let Π_T be the strip

$$\Pi_T = \{(x,t):\ x \in [0,\infty), 0 \leq t < T\}$$

and $\Pi_T(X)$ be the rectangle

$$\Pi_T(X) = \{(x,t):\ 0 \leq x \leq X, 0 \leq t < T\}.$$

We denote by $\Omega_\lambda(T)$ and $\Omega_{0,r}(T)$ the spaces of continuous functions c with bounded norms

$$\|c\|_\lambda^{(T)} = \sup_{0\leq t\leq T}\int_0^\infty \exp(\lambda x)|c(x,t)|dx, \ \lambda > 0$$

and

$$\|c\|_{0,r}^{(T)} = \sup_{0\leq t\leq T}\int_0^\infty (1+x^r)|c(x,t)|dx, \quad r \geq 1.$$

Let

$$\Omega(T) = \bigcup_{\lambda>0} \Omega_\lambda(T).$$

It should be noted that the following inclusions take place

$$\Omega_0 \stackrel{\text{def}}{=} \Omega_{0,0} \supset \Omega_{0,1} \supset \cdots \supset \Omega_{0,r} \supset \cdots \supset \Omega_{\lambda_1} \supset \Omega_{\lambda_2}, \quad 0 < \lambda_1 < \lambda_2 < \infty$$

and

$$\Omega_{0,r} \supset \Omega, \quad r \geq 0.$$

The space $\Omega(T)$ may be equipped with the topology of the inductive limit of topologies in $\Omega_\lambda(T)$, i.e. a set is open in $\Omega(T)$ if its intersection with $\Omega_\lambda(T)$ is open in the topological space $\Omega_\lambda(T)$ for all $\lambda > 0$. We denote C and BC the spaces of continuous and bounded continuous functions correspondingly. Cones of nonnegative functions in the above spaces are denoted using the superscript +, e.g. $\Omega^+_{0,r}(T)$, $\Omega^+_\lambda(T)$.

2.2 Local existence

Theorem 2.1. *Let the functions $K(x,y,t)$ and $F(x,y,t)$ be continuous, nonnegative, symmetric and have a compact support for each moment $0 \leq t < T$. Let*

1^0

$$c_0 \in \Omega^+_{0,r}(0), \quad q \in \Omega^+_{0,r}(T), \quad a \in C^+(\Pi_T), \quad r \geq 1; \tag{2.3}$$

or

2^0

$$c_0 \in \Omega^+_\lambda(0) \quad q \in \Omega^+_\lambda(T), \quad a \in C^+(\Pi_T). \tag{2.4}$$

Then there exists $\tau > 0$ such that the initial value problem (2.1), (2.2) has at least one solution

$$c \in \Omega^+_{0,r}(\tau) \qquad or \qquad c \in \Omega^+_\lambda(\tau) \tag{2.5}$$

correspondingly.

If, in addition, functions c_0 and q are bounded in Π_T, then, in addition to (2.5),

$$c \in BC(\Pi_T). \tag{2.6}$$

Proof. Let us note first that there exists a constant A such that the functions K and F have a compact support in $[0,A]\times[0,A]$. Hence, the solution to (2.1), (2.2) for $x>2A$ takes the values

$$c(x,t)=\exp\left(-\int_0^t a(x,s)ds\right)\left\{c_0(x)+\int_0^t \exp\left(\int_0^s a(x,s_1)ds_1\right)q(x,s)ds\right\}. \tag{2.7}$$

We pick up the modified initial and sources functions

$$\tilde{c}_0(x)=\begin{cases} c_0(x), & x\le 2A,\\ \min\{c_0(2A),\ c_0(x)\}, & x>2A.\end{cases}$$

and

$$\tilde{q}(x,t)=\begin{cases} q(x,t), & x\le 2A,\\ \min\{q(2A,t),\ q(x,t)\}, & x>2A\end{cases}\qquad t\ge 0.$$

The spaces $\tilde{\Omega}_\lambda=\Omega_\lambda\bigcap BC$ and $\tilde{\Omega}_{0,r}=\Omega_{0,r}\bigcap BC$ become Banach spaces if we introduce the following norms

$$|||c|||_\lambda^{(T)}=\sup_{0\le t<T,x\ge 0}|c(x,t)|+\sup_{0\le t<T}\int_0^\infty \exp(\lambda x)|c(x,t)|dx,\ \ \lambda>0$$

and

$$|||c|||_{0,r}^{(T)}=\sup_{0\le t<T,x\ge 0}|c(x,t)|+\sup_{0\le t<T}\int_0^\infty (1+x^r)|c(x,t)|dx,\qquad r\ge 1.$$

Our plan is to prove existence in Banach spaces $\tilde{\Omega}_\lambda$ or $\tilde{\Omega}_{0,r}$ with initial and sources functions $\tilde{c}_0$, $\tilde{q}$, then we use (2.7) to obtain the solution of the original problem with c_0 and q. Those reasonings are caused by continuous functions which are unbounded on $x\in[0,\infty)$.

To prove Theorem 2.1 we rewrite the problem (2.1), (2.2) in the following integral form

$$c(x,t)=\exp\left(-\int_0^t a(x,s)ds\right)\left\{c_0(x)+\int_0^t(\mathbf{S}(c)(x,s)+q(x,s))\cdot\right.$$

$$\cdot \exp\left(\left(\int_0^s a(x,s_1)ds_1\right) ds\right\} \stackrel{\text{def}}{=} Y(c)(x,t). \tag{2.8}$$

where the collision operator $\mathbf{S}(c)$ is expressed by the right-hand side of the equation (2.1). Since the coagulation and fragmentation kernels K, F have a compact support, the efflux function $a(x,t)$ is non-negative and functions c_0, q satisfy (2.3) or (2.4), then the integral operator Y, defined in (2.8), maps the Banach spaces $\tilde{\Omega}_{0,r}(t)$, $\tilde{\Omega}_\lambda(t)$ into itself:

$$Y: \tilde{\Omega}_{0,r}(t) \mapsto \tilde{\Omega}_{0,r}(t), \quad Y: \tilde{\Omega}_\lambda(t) \mapsto \tilde{\Omega}_\lambda(t) \quad \text{for all} \quad 0 \le t \le T.$$

We prove Theorem 2.1 using the contraction mapping theorem. Let us establish first that for small $\tau > 0$ there exists in $\Omega_{0,r}(\tau)$ a closed ball which is invariant relatively to the mapping Y. Really, let $\|c\|_{0,r}^{(\tau)} \le z$. Then (2.8) yields

$$\|Y(c)\|_{0,r}^{(\tau)} \le M \cdot (1 + \tau z + \tau z^2), \tag{2.9}$$

where a constant M depends on c_0, q, K and F, Hence,

$$\|Y(c)\|_{0,r}^{(\tau)} \le z$$

if $M \cdot (1 + \tau z + \tau z^2) \le z$. The last inequality holds if $\tau < 1/M$ and

$$\frac{1 - M\tau - \sqrt{(1 - M\tau)^2 - 4M^2\tau}}{2M\tau} \le z \le \frac{1 - M\tau + \sqrt{(1 - M\tau)^2 - 4M^2\tau}}{2M\tau}. \tag{2.10}$$

Secondly, we need to check whether the mapping Y is contractive. From (2.8) we obtain for a positive constant M_1 that

$$\|Y(c) - Y(d)\|_{0,r}^{(\tau)} \le M_1(z+1)\tau\|c - d\|_{0,r}^{(\tau)} \tag{2.11}$$

provided that

$$\|c\|_{0,r}^{(\tau)} \le z, \qquad \|d\|_{0,r}^{(\tau)} \le z.$$

Hence, the mapping Y is contractive in $\Omega_{0,r}(\tau)$ for $\tau < [M_1(z+1)]^{-1}$. Using this result together with the inequalities (2.10) we conclude that for sufficiently small $\tau > 0$ there exists an invariant ball of radius z. In that ball the mapping Y is a contraction. Consequently, that ball contains a fixed point of Y. Existence of a fixed point of the mapping Y in $\tilde{\Omega}_\lambda(\tau)$ in the case $\mathbf{2}^0$ follows analogiously.

We need else to show that the solution to (2.8) obtained is nonnegative. With this aim we prove the following simple lemma.

Lemma 2.1. *Let a mapping Y be contractive and maps a Banach space X into itself and have there an invariant ball. Let α, $\alpha < 1$ be the contraction constant of Y and let x be its fixed point. Suppose that the operator Y is the limit of a sequence of operators Y_n, i.e. for any $M > 0$*

$$\lim_{n\to\infty} \sup_{\|x\|\le M} \|Y(x) - Y_n(x)\| = 0. \tag{2.12}$$

Let in addition there exist bounded sequence $\{x_n\}$ of fixed points of operators Y_n,

$$\|x_n\| \le M, \quad n \ge 1.$$

Then

$$\|x_n - x\| \to 0, \quad n \to \infty.$$

Proof. From the inequalities

$$\|x_n - x\| = \|Y_n(x_n) - Y(x)\| \le \|Y_n(x_n) - Y(x_n)\| + \|Y(x_n) - Y(x)\| \le$$
$$\le \|Y_n(x_n) - Y(x_n)\| + \alpha\|x_n - x\|$$

we obtain

$$(1-\alpha)\|x_n - x\| \le \sup_{\|x\|\le M} \|Y_n(x) - Y(x)\| \to 0, \quad n \to \infty.$$

This proves the assertion of Lemma 2.1. □

Lemma 2.2. *Let conditions of Theorem 2.1 hold. Then any continuous solution to the initial value problem (2.1), (2.2) is nonnegative.*

Proof. Let initial data c_0 and the function of sources q be strictly positive. Suppose that there exist a point (x_0, t_0) such that $c(x_0, t_0) = 0$ and the point (x_0, t_0) is "the first" point with that property, i.e.

$$c(x,t) > 0 \qquad \text{for all} \quad 0 \le x \le \max\{x_0, A\}, \quad t \in [0, t_0). \tag{2.13}$$

Since the solution is continuous and satisfies (2.8), it must be continuously differentiable in t. Hence, from (2.1) we obtain

$$\frac{\partial c(x_0,t_0)}{\partial t} = \frac{1}{2}\int_0^{x_0} K(x_0 - y, y, t_0)c(x_0 - y, t_0)c(y, t_0)dy +$$

$$+\int_0^A F(x_0,y)c(x_0+y,t_0)dy + q(x_0,t_0) > 0.$$

The positivity of the time derivative proves that there exist points (x,t_0), $x < x_0$ where the function $c(x,t_0)$ is negative. This contradicts to our assumption that the point (x_0,t_0) is "the first". Consequently, the solution is positive provided that initial data and sources are positive.

If initial data and/or sources are not strictly positive then we construct sequences of positive functions $\{c_0^n\}$, $\{q^n\}$ which satisfy the conditions of Theorem 2.1 and converge in $\Omega_{0,r}(\tau)$ to c_0, q correspondingly uniformly with respect to $t \in [0,\tau]$. As we have already proved, those two sequences generate the sequence $\{c^n\}$ of positive continuous solutions to the problem (2.1), (2.2) (generally speaking, those solutions may be unbounded in $C[0,\infty)$). We introduce the family of operators

$$Y_n : \ \Omega_{0,r}(\tau) \mapsto \Omega_{0,r}(\tau)$$

as

$$Y_n(c)(x,t) = \exp\left(-\int_0^t a(x,s)ds\right)\cdot$$

$$\cdot\left\{c_0^n(x) + \int_0^t \exp\left(\int_0^s a(x,s_1)ds_1\right)(\mathbf{S}(c)(x,s) + q^n(x,s))ds\right\}. \qquad (2.14)$$

We note that

$$\sup_{\|c\|_{0,r}^{(\tau)} \le M} \|Y_n(c) - Y(c)\|_{0,r}^{(\tau)} \le \int_0^\infty (1+x^r)|c_0^n(x) - c_0(x)|dx +$$

$$+\tau \sup_{0\le t\le\tau} \int_0^\infty (1+x^r)|q(x,s) - q^n(x,s)|dx \ \to \ 0, \ n\to\infty. \qquad (2.15)$$

Since the mapping Y is contractive in $\Omega_{0,r}(\tau)$ (it follows from the proof of contraction in $\tilde{\Omega}_{0,r}(\tau)$), we can apply Lemma 2.1 and conclude

$$\|c^n - c\|_{0,r}^{(\tau)} \to 0, \ n\to\infty.$$

This proves Lemma 2.2 and Theorem 2.1. □

2.3 Global existence

The aim of this section is to prove the following theorem.

Theorem 2.2. *Let the conditions of Theorem 2.1 hold. Then in the strip* Π_T *exists the solution to the initial value problem (2.1), (2.2) such that*

$$\mathbf{1^0} \qquad c \in \Omega_{0,r}^{+}(T) \qquad or \qquad \mathbf{2^0} \qquad c \in \Omega_{\lambda}^{+}(T)$$

correspondingly. This solution is unique in $\Omega_{0,r}(T)$ *or* $\Omega_{\lambda}(T)$ *respectively. If* $a = q \equiv 0$ *then the solution is mass conserving.*

First, we observe the boundness of all moments

$$N_k(t) = \int_0^\infty x^k c(x,t)dx, \quad 0 \le k \le r. \tag{2.16}$$

Really, the integration (2.1) yields

$$\frac{dN_0(t)}{dt} = -\frac{1}{2}\int_0^\infty \int_0^\infty K(x,y,t)c(x,t)c(y,t)dydx +$$

$$+\frac{1}{2}\int_0^\infty \int_0^x F(x-y,y,t)c(x,t)dydx + \int_0^\infty \left[q(x,t) - a(x,t)c(x,t)\right] dx. \tag{2.17}$$

All the integrals exist due to compactly supported kernels K and F. Hence,

$$N_0(t) \le N_0(0) + \int_0^t \left\{\frac{1}{2}\overline{F}N_1(s) + \int_0^\infty q(x,s)dx\right\} ds. \tag{2.18}$$

Here

$$\overline{K} = \sup\ K, \quad \overline{F} = \sup\ F.$$

Integrating (2.1) with the weight x gives us boundness of the total mass of particles in the system concerned which, as was mentioned, is expressed by the first moment of solution:

$$N_1(t) \le N_1(0) + \int_0^t \int_0^\infty xq(x,s)dx. \tag{2.19}$$

Substituting (2.19) into (2.18) yields boundness of the zero moment N_0. For the k-th moment, $k \geq 2$ we obtain

$$\frac{dN_k(t)}{dt} \leq \frac{1}{2}\overline{K}\int_0^\infty \int_0^\infty \left[(x+y)^k - x^k - y^k\right] c(x,t)c(y,t)dxdy + \int_0^\infty x^k q(x,t)dx. \tag{2.20}$$

Hence, boundness of the k-th moment, $k \geq 2$ is based on boundness of previous ones. Thus, step by step, we obtain boundness of all moments till $N_r(t)$. We are in position now to demonstrate boundness of $c(x,t)$. With this aim we note that

$$c(x,t) \leq c_0(x) + \int_0^t \left\{\frac{1}{2}\overline{K}\int_0^x c(x-y,s)c(y,s)dy + \overline{F}\int_x^\infty c(y,s)dy + q(x,s)\right\} ds. \tag{2.21}$$

We introduce

$$\hat{c}(t) = \sup_{0 \leq x \leq X} c(x,t). \tag{2.22}$$

Substituting (2.18), (2.22) into (2.21) yields

$$\hat{c}(t) \leq \hat{c}(0) + \int_0^t \left\{\frac{1}{2}\overline{K}\hat{c}(s)N_0(s) + \overline{F}N_0(s) + q(x,s)\right\} ds. \tag{2.23}$$

Finally, we arrive to the Gronwall's inequality [Har]

$$\hat{c}(t) \leq M_0 + M_1 \int_0^t \hat{c}(s)ds$$

which proves boundness of $\hat{c}(t)$, $0 \leq t \leq T$:

$$\hat{c}(t) \leq M_0 \exp(M_1 T), \quad 0 \leq t \leq T. \tag{2.24}$$

As usually, M_0, M_1 are positive constants.

We can observe now that for large values x the right-hand side of (2.1) is less than $T \sup q(x,t)$. For small values x we have the majorant estimate (2.24). Therefore the solution to (2.1), (2.2) is bounded in the norm $\|.\|_{0,r}^{(T)}$. Taking into account non-negativity of local solution we prolong it to all $0 \leq t \leq T$. This proves existence.

The mass conservation follows trivially from integration of (2.1) with weight x. Existence of all integrals holds due to compactly supported kernels K, F.

To prove uniqueness in $\Omega_{0,r}(T)$ we assume that there are two solutions c and d with the same initial function c_0. We consider their difference

$$|c(x,t)-d(x,t)| \le \int_0^t \left\{ \frac{1}{2}\int_0^x K(x-y,y,s)|c(x-y,s)-d(x-y,s)|\cdot \right.$$

$$\cdot|c(y,s)+d(y,s)| + |c(x,s)-d(x,s)| \int_0^\infty K(x,y,s)c(y,s)dy +$$

$$+d(x,s)\int_0^\infty K(x,y,s)|c(y,s)-d(y,s)|dy + \int_x^\infty F(y-x,x,s)|c(y,s)-d(y,s)|dy +$$

$$\left. +\frac{1}{2}|c(x,s)-d(x,s)|\int_0^x F(x-y,y,s)dy \right\} ds. \tag{2.25}$$

Integration of (2.25) from 0 to ∞ yields

$$\|c(.,t)-d(.,t)\|_0^{(0)} \le \int_0^t \left\{ \frac{3}{2}\overline{K}\|c(.,s)-d(.,s)\|_0^{(0)} \cdot (\|c(.,s)\|_0^{(0)} + \right.$$

$$\left. +\|d(.,s)\|_0^{(0)}) + \frac{3}{2}\overline{F}A\|c(.,s)-d(.,s)\|_0^{(0)} \right\} ds. \tag{2.26}$$

The interchange of the order of integration is justified by Fubini's theorem [26], and other steps follow from the conditions on coagulation and fragmentation kernels. Since $c, d \in \Omega_{0,r}$ then the value

$$\|c(.,s)\|_0^{(0)} = \int_0^\infty |c(x,s)|ds$$

is bounded uniformly with respect to t, $0 \le t \le T$. Then (2.26) and Gronwall's inequality [42] imply that

$$\|c(.,t)-d(.,t)\|_0^{(0)} = 0 \qquad \text{for all } 0 \le t \le T.$$

From the continuity of $c(x,t)$ and $d(x,t)$ it follows that $c(x,t) = d(x,t)$ for $(x,t) \in \Pi_T$. The proof for case $\mathbf{2^0}$ is the same. This completes the proof of Theorem 2.2. □

Chapter 3. EXISTENCE FOR UNBOUNDED COAGULATION KERNELS

In this chapter we prove existence theorem for the general coagulation-fragmentation equation which can be written as

$$\frac{\partial}{\partial t}c(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy-$$

$$-c(x,t)\int_0^\infty K(x,y)c(y,t)dy - \frac{1}{2}c(x,t)\int_0^x F(x-y,y)dy+$$

$$+\int_0^\infty F(x,y)c(x+y,t)dy, \tag{3.1}$$

$$c(x,0) = c_0(x) \geq 0. \tag{3.2}$$

Theorem 3.1. *Let the functions $K(x,y)$ and $F(x,y)$ be nonnegative and symmetric. Suppose also that*

$$K(x,y) \leq k(1+x+y), \quad k>0 \tag{3.3}$$

and there exist positive constants m, m_1 and b such that

$$\int_0^x F(x-y,y)dy \leq b(1+x^{m_1}), \quad F(x-y',y) \leq b(1+x^m), \tag{3.4}$$

$$0 \leq y \leq y' \leq x, \quad x \in [0,\infty).$$

Let the initial data function satisfy either:

1^0 $c_0 \in \Omega^+_{0,r}(0)$, $r > max\,\{m,1\}$, *and* $r \geq m_1$;

or

2^0 $c_0 \in \Omega^+(0)$.

Then the problem (3.1), (3.2) has, respectively, either:

1^0 *at least one solution in* $\Omega^+_{0,r}(T)$, $0 < T \leq \infty$;

or

2^0 *a solution in* $\Omega^+(T)$, $0 < T < \infty$.

3.1 Uniform boundness of sequence of approximated solutions

When K and F belong to class (3.3), we construct a sequence of continuous kernels $\{K_n, F_n\}_{n=1}^{\infty}$ from the class (3.3) with a compact support for each $n \geq 1$, such that

$$K_n(x,y) = K(x,y), 0 \leq x,y \leq n, n \geq 1, \tag{3.5}$$

$$F_n(x,y) = F(x,y), 0 \leq x,y \leq n, n \geq 1, \tag{3.6}$$

$$K_n(x,y) \leq K(x,y), 0 \leq x,y < \infty, n \geq 1, \tag{3.7}$$

$$F_n(x,y) \leq F(x,y), 0 \leq x,y < \infty, n \geq 1. \tag{3.8}$$

In accordance with Theorem 2.2, the sequence $\{K_n, F_n\}_{n=1}^{\infty}$ generates on Π_T a sequence $\{c_n\}_{n=1}^{\infty}$ of nonnegative continuous solutions to the problem (3.1), (3.2) with the kernels K_n, F_n. These solutions belong to $\Omega_{0,r}^{+}(T)$ or $\Omega^{+}(T)$ respectively.

Let us denote the r-th moment of the functions c_n as

$$N_{r,n}(t) = \int_0^{\infty} x^r c_n(x,t)dx, r \geq 0, n \geq 1.$$

By direct integration of (3.1) with the weight x , we obtain the mass conservation law

$$N_{1,n}(t) = N_1 = \text{const}, n \geq 1, 0 \leq t < T \tag{3.9}$$

All the integrals exist due to the compact support of the kernels. Integrating (3.1) with the weight x^2 and using (3.3), we also obtain

$$\frac{dN_{2,n}(t)}{dt} \leq kN_1^2 + 2kN_1 N_{2,n}(t).$$

Hence, $N_{2,n}(t)$ is bounded on $t \in [0,T)$:

$$N_{2,n}(t) \leq \bar{N}_2, 0 \leq t < T, \; n \geq 1. \tag{3.10}$$

Similarly, step by step, we obtain the uniform boundedness of $N_{r,n}(t)$ with respect to $n \geq 1, \; 0 \leq t < T$. The uniform boundedness of the zero moment $N_{0,n}$ follows via (3.4) from the inequalities

$$\frac{dN_{0,n}}{dt} \leq \frac{1}{2}\int_0^{\infty} c_n(x,t) \int_0^{x} F(x-y,y)dydx \leq \frac{1}{2}b(N_{0,n} + N_{m_1,n})$$

and the condition $m_1 \leq r$. Consequently,

$$N_{k,n}(t) \leq \bar{N}_k = \text{const if } t \in [0,T),\ n \geq 1,\ 0 \leq k \leq r. \tag{3.11}$$

We are now in a position to formulate the following Lemma:

Lemma 3.1. *The sequence $\{c_n\}_{n=1}^{\infty}$ is uniformly bounded on each rectangle $\Pi_T(X)$, $T < \infty$.*

Proof. Since solutions c_n of (3.1),(3.2) with kernels K_n, F_n are nonnegative then by virtue of (3.3),(3.6),(3.8), (3.11), we obtain for $(x,t) \in \Pi_T(X)$:

$$c_n(x,t) \leq \bar{c_0} + \int_0^t \left(\frac{1}{2}k(1+X)c_n * c_n(x,s) + b(\bar{N_0} + \bar{N_r})\right) ds. \tag{3.12}$$

Here $\bar{c}_0 = \sup_{0 \leq x \leq X} c_0(x)$ and $f * g$ is the convolution,

$$f * g(x) = \int_0^x f(x-y)g(y)dy.$$

We define the "upper" function for the integral inequality (3.12) to be

$$g(x,t) = g_0 + \int_0^t \left(\frac{1}{2}k(1+X)g * g(x,s) + g(x,s)\right) ds, \tag{3.13}$$

$$0 \leq t < T, 0 \leq x < \infty,$$

where $g_0 = \max\{\bar{c}_0, b(\bar{N_0} + \bar{N_r})\} = \text{const}$. Taking the Laplace transform of this relation with respect to x, we obtain

$$g(x,t) = g_0 \exp\left(\frac{1}{2}g_0 kx(1+X)(e^t - 1) + t\right), 0 \leq t < T, 0 \leq x < \infty. \tag{3.14}$$

Our next purpose is to prove that the inequality

$$c_n(x,t) \leq g(x,t) \text{ for } (x,t) \in \Pi_T(X)$$

holds for each integer $n \geq 1$.

We introduce the auxiliary function

$$g_\varepsilon(x,t) = g_0 + \varepsilon + \int_0^t \left(\frac{1}{2}k(1+M)g_\varepsilon * g_\varepsilon(x,s) + g_\varepsilon(x,s)\right) ds, \tag{3.15}$$

$$(x, t) \in \Pi_T, \varepsilon > 0.$$

Clearly $c_n(x, 0) < g_\varepsilon(x, 0)$ for $0 \le x \le X$. We assume that, for some $n \ge 1$, there is a set D of points $(x, t) \in \Pi_T(X)$ on which $c_n(x, t) = g_\varepsilon(x, t)$. Since D does not contain points on the coordinate axes, we choose $(x_0, t_0) \in D$ so that the rectangle $Q = [0, x_0) \times [0, t_0)$ contains no points of D. Since g_ε and c_n are continuous, we have $c_n(x, t) < g_\varepsilon(x, t)$ for $(x, t) \in Q$. The values of c_n and g_ε coincide at the point (x_0, t_0). Hence

$$c_n(x_0, t_0) = g_\varepsilon(x_0, t_0) > g_0 + \varepsilon + \int_0^{t_0} \left(\frac{1}{2} k(1 + X) c_n * c_n(x_0, s) + c_n(x_0, s) \right) ds. \tag{3.16}$$

This is proved by using the fact that the values of the arguments of g_ε in the integrand (3.15) are in Q. Combining (3.12) and (3.16) we arrive at the contradiction $c_n(x_0, t_0) > c_n(x_0, t_0)$, which proves that D is empty and

$$c_n(x, t) < g_\varepsilon(x, t), \qquad (x, t) \in \Pi_T(X), \ n \ge 1.$$

Using (3.14) we have the continuity of g_ε as a function of ε. Letting ε tend to zero we find that actually

$$c_n(x, t) \le g(x, t) \quad \text{for} \quad (x, t) \in \Pi_T(X), n \ge 1,$$

and hence the sequence $\{c_n\}_{n=1}^{\infty}$ is bounded uniformly on $\Pi_T(X)$:

$$0 \le c_n(x, t) \le g_0 \exp\left(\frac{1}{2} g_0 k X(1 + X)(e^T - 1) + T \right) = M_1 = \text{const.} \tag{3.17}$$

This proves Lemma 3.1. □

3.2 Compactness of approximated sequence

Lemma 3.2. *The sequence $\{c_n\}_{n=1}^{\infty}$ is relatively compact in the uniform-convergence topology of continuous functions on each rectangle $\Pi_T(X)$, $T < \infty$.*

Proof. Step 1. We show the equicontinuity of $\{c_n\}_{n=1}^{\infty}$ with respect to t. From (3.1) we note that for $0 \le t \le t' \le T, 0 \le x \le X, n \ge 1$ the following inequality takes place

$$|c_n(x, t') - c_n(x, t)| \le \int_t^{t'} \left\{ \frac{1}{2} \int_0^x K_n(x - y, y) c_n(x - y, s) c_n(y, s) dy + \right.$$

$$+c_n(x,s)\int_0^\infty K_n(x,y)c_n(y,s)dy + \int_0^\infty F_n(x,y)c_n(x+y,s)dy+$$

$$+\frac{1}{2}c_n(x,s)\int_0^x F_n(x-y,y)dy\Big\} ds. \tag{3.18}$$

It follows from (3.7),(3.8) and (3.17) that the first and the fourth terms of the integrand in (3.18) are uniformly bounded. The second and the third terms in (3.18) are uniformly bounded by virtue of the uniform boundedness of the sequence $\{c_n\}_{n=1}^\infty$ on $\Pi_T(X)$, equations (3.5)–(3.8) and the inequalities

$$\int_0^\infty K_n(x,y)c_n(y,s)dy \le k(1+X)\bar{N}_0 + kN_1, \tag{3.19}$$

$$\int_0^\infty F_n(x,y)c_n(x+y,s)dy = \int_x^\infty c_n(y,s)F(y-x,x)dy \le b(\bar{N}_0+\bar{N}_r) \tag{3.20}$$

with $0 \le s \le T$, $n \ge 1$. Applying (3.19),(3.20) to (3.18), we finally obtain

$$\sup_{0\le x\le X} |c_n(x,t') - c_n(x,t)| \le M_2|t'-t|, 0 \le t \le t' \le T, n \ge 1. \tag{3.21}$$

The constant M_2 is independent of n; hence $\{c_n\}_{n=1}^\infty$ is equicontinuous with respect to the variable t on $\Pi_T(X)$.

Step 2. We next establish that $\{c_n\}_{n=1}^\infty$ is equicontinuous with respect to x. Let $0 \le x \le x' \le X$; then for each $n \ge 1$ we have

$$|c_n(x',t) - c_n(x,t)| \le |c_0(x') - c_0(x)|$$

$$+\int_0^t \Big\{\frac{1}{2}\int_x^{x'} K_n(x'-y,y)c_n(x'-y,s)c(y,s)dy +$$

$$+\frac{1}{2}\int_0^x |K_n(x'-y,y) - K_n(x-y,y)|c_n(x'-y,s)c_n(y,s)dy+$$

$$+\frac{1}{2}\int_0^x K_n(x-y,y)\cdot|c_n(x'-y,s) - c_n(x-y,s)|c_n(y,s)dy+$$

$$+|c_n(x',s) - c_n(x,s)|\int_0^\infty K_n(x',y)c_n(y,s)dy+ \tag{3.22}$$

$$+c_n(x,s)\int_0^\infty |K_n(x',y)-K_n(x,y)|c_n(y,s)dy+ \tag{3.23}$$

$$+\int_{x'}^\infty c_n(y,s)|F_n(x,y-x')-F_n(x,y-x)|dy+ \tag{3.24}$$

$$+\int_{x'}^\infty c_n(y,s)|F_n(x',y-x')-F_n(x,y-x')|dy+ \tag{3.25}$$

$$+\frac{1}{2}|c_n(x',s)-c_n(x,s)|\int_0^{x'} F_n(x'-y,y)dy+$$

$$+\frac{1}{2}c_n(x,s)\int_x^{x'} F_n(x'-y,y)dy+\int_x^{x'} c_n(y,s)$$

$$F_n(x,y-x)dy++\frac{1}{2}c_n(x,s)\int_0^x |F_n(x'-y,y)-F_n(x-y,y)|dy\Big\}\,ds. \tag{3.26}$$

It follows from (3.5),(3.6) that the kernel sequence $\{K_n,F_n\}_{n=1}^\infty$ we have constructed is equicontinuous on each rectangle $[0,X]\times[0,z], z>0$.

Let us remark that if $\phi(x)$ is nonnegative and measurable and $\psi(x)$ is positive and nondecreasing for $x>0$, then

$$\int_z^\infty \phi(x)dx \le \frac{1}{\psi(z)}\int_0^\infty \phi(x)\psi(x)dx, \quad z>0, \tag{3.27}$$

if the integrals exist and are finite.

Our aim now is to show that if the difference $|x'-x|$ is small enough, then the left-hand side of (3.26) is small also. Fix an arbitrary $\varepsilon>0$ and choose $\delta(\varepsilon), 0<\delta(\varepsilon)<\varepsilon$, such that

$$\sup_{|x'-x|<\delta} |c_0(x')-c_0(x)|<\varepsilon, \tag{3.28}$$

$$\sup_{|x'-x|<\delta} (|K_n(x',y)-K_n(x,y)|+|F_n(x',y)-F_n(x,y)|)<\varepsilon, \tag{3.29}$$

$$\sup_{|x'-x|<\delta} |F_n(x,y-x')-F_n(x,y-x)|<\varepsilon. \tag{3.30}$$

The inequalities (3.29) and (3.30) hold uniformly with respect to $n \geq 1$ and $0 \leq y \leq z$. The rule for choosing the constant $z = z(\varepsilon)$ is given below in expressions (3.34), (3.36). Introduce the modulus of continuity

$$\omega_n(t) = \sup_{|x'-x|<\delta} |c_n(x',t) - c_n(x,t)|, \; 0 \leq x, x' \leq X.$$

Using (3.3),(3.7),(3.8),(3.17), we can easily demonstrate the smallness of terms in (3.26) whose integrals are over finite intervals. To show the smallness of the term at (3.22) we have to use the uniform boundedness of the integral which follows from (3.3),(3.9),(3.11):

$$|c_n(x',s) - c_n(x,s)| \int_0^\infty K_n(x',y)c_n(y,s)dy \leq$$

$$\leq k\omega_n(s)((1+X)\bar{N}_0 + N_1), \quad n \geq 1, \; 0 \leq x, x' \leq X.$$

The summands in the terms (3.23)-(3.25) are more complicated. Let us consider (3.23). Using the partitioning $\int_0^\infty = \int_0^z + \int_z^\infty$, we obtain with (3.3), (3.11),(3.29), that

$$\int_0^\infty |K_n(x',y) - K_n(x,y)|c_n(y,s)dy \leq$$

$$\leq \varepsilon\bar{N}_0 + 2k(1+X)\int_z^\infty c_n(y,s)dy + 2k\int_z^\infty yc_n(y,s)dy. \tag{3.31}$$

Let us use (3.27) with $\phi(x) = c_n(x)$, $\psi(x) = x$ or $\phi(x) = xc_n(x)$, $\psi(x) = x^{r-1}$ in the second and third terms of (3.31) respectively. Also, recall equation (3.10). Then we arrive at the expressions

$$\int_z^\infty c_n(y,s)dy \leq \frac{1}{z}N_1, \tag{3.32}$$

$$\int_z^\infty yc_n(y,s)dy \leq \frac{1}{z^{r-1}}\bar{N}_r. \tag{3.33}$$

If we choose z such that

$$\frac{1}{z}N_1 \leq \varepsilon \quad \text{and} \quad \frac{1}{z^{r-1}}\bar{N}_r \leq \varepsilon \tag{3.34}$$

then from (3.31)

$$\int_0^\infty |K_n(x',y) - K_n(x,y)| c_n(y,s)dy \le \text{const}\cdot\varepsilon. \tag{3.35}$$

The same reasoning should be used to estimate the terms (3.24) and (3.25). For (3.24) we obtain

$$\int_{x'}^\infty c_n(y,s)|F_n(y-x',x) - F_n(y-x,x)|dy \le$$

$$\le \varepsilon\bar{N}_0 + \int_z^\infty c_n(y,s)F_n(y-x',x)dy + \int_z^\infty c_n(y,s)F_n(y-x,x)dy \le$$

$$\le \varepsilon\bar{N}_0 + 2b\int_z^\infty c_n(y,s)(1+y^m)dy \le \varepsilon\bar{N}_0 + 2b\frac{\bar{N}_1}{z} + 2b\frac{\bar{N}_r}{z^{r-m}}.$$

If (3.34) holds and

$$\frac{\bar{N}_0}{z^{r-m}} < \varepsilon \tag{3.36}$$

then

$$\int_{x'}^\infty c_n(y,s)|F_n(x,y-x') - F_n(x,y-x)|dy \le \text{const}\cdot\varepsilon. \tag{3.37}$$

Finally, using (3.17),(3.28),(3.29),(3.30),(3.35) and (3.37) we obtain from the whole inequality (3.26):

$$\omega_n(t) \le M_3\cdot\varepsilon + M_4\int_0^t \omega_n(s)ds, \quad 0 \le t \le T.$$

Here the positive constants M_3 and M_4 are independent of n and ε. Hence by Gronwall's inequality

$$\omega_n(t) \le M_3\varepsilon\exp(M_4T) \stackrel{\text{def}}{=} M_5\cdot\varepsilon. \tag{3.38}$$

We conclude from (3.21) and (3.38) that

$$\sup_{|x'-x|<\delta, |t'-t|<\delta} |c_n(x',t') - c_n(x,t)| \le (M_2+M_5)\varepsilon, \tag{3.39}$$

$$0 \le x, x' \le X, 0 \le t, t' \le T.$$

The assertion of Lemma lm3.2 is then a consequence of (3.17),(3.39) and Arzela's theorem [26]. Lemma 3.2 has now been proved. □

3.3 Proof of Theorem 3.1: Case 1^0

To prove Theorem 3.1 we employ the standard diagonal method. We select a subsequence $\{c_i\}_{i=1}^{\infty}$ from $\{c_n\}_{n=1}^{\infty}$ converging uniformly on each compact set in Π_T to a continuous nonnegative function c. Let us consider an integral $\int_0^z x^k c(x,t)dx,\quad 0 \le k \le r$. Since for all $\varepsilon > 0$ there exists $i \ge 1$ such that

$$\int_0^z x^k c(x,t)dx \le \int_0^z x^k c_i(x,t)dx + \varepsilon \le \bar{N}_k + \varepsilon, \tag{3.40}$$

then

$$\int_0^\infty x^k c(x,t)dx \le \bar{N}_k, \quad 0 \le k \le r \tag{3.41}$$

because in (3.40) both z and ε are arbitrary. Similarly we obtain

$$\int_0^\infty xc(x,t)dx \le N_1. \tag{3.42}$$

The inequality (3.42) can be transformed into an equality giving the mass conservation law: this will be proved below. We should show now that the function $c(x,t)$ is a solution to the initial value problem (3.1),(3.2). To prove this assertion we write the equations (3.1), (3.2) in the integral form for c_n with K_n, F_n and change c_n, K_n, F_n to $c_n - c + c$, $K_n - K + K$, $F_n - F + F$ respectively. Then we obtain

$$(c_i - c)(x,t) + c(x,t) = c_0(x) +$$

$$+ \int_0^t \left\{ \frac{1}{2} \int_0^x (K_i - K)(x-y,y)c_i(x-y,s)c_i(y,s)dy + \right.$$

$$+ \frac{1}{2} \int_0^x K(x-y,y)(c_i(x-y,s) - c(x-y,s))c_i(y,s)dy +$$

$$+ \frac{1}{2} \int_0^x K(x-y,y)(c_i(y,s) - c(y,s))c(x-y,s)dy +$$

$$+ \frac{1}{2} \int_0^x K(x-y,y)c(y,s)c(x-y,s)dy -$$

$$- c_i(x,s) \int_0^\infty (K_i - K)(x,y)c_i(y,s)dy -$$

$$-(c_i - c)(x,s)\int_0^\infty K(x,y)c_i(y,s)dy-$$

$$-c(x,s)\int_0^\infty K(x,y)(c_i - c)(y,s)dy - c(x,s)\int_0^\infty K(x,y)c(y,s)dy+$$

$$+\int_0^\infty (F_i - F)(x,y)c_i(x+y,s)dy + \int_0^\infty F(x,y)(c_i - c)(x+y,s)dy+$$

$$+\int_0^\infty F(x,y)c(x+y,s)dy - \frac{1}{2}c_i(x,s)\int_0^x (F_i - F)(x-y,y)dy-$$

$$-\frac{1}{2}(c_i - c)(x,s)\int_0^x F(x-y,y)dy-$$

$$-\frac{1}{2}c(x,s)\int_0^x F(x-y,y)dy\Bigg\}\, ds. \tag{3.43}$$

Passing to the limit as $i \to \infty$ in (3.43) we can see that the terms with integrals over $[0,\infty)$ tend to zero due to the estimations of their "tails", which may be obtained with (3.27), (3.41), (3.42) taken into account similarly to (3.35),(3.37):

$$\left|\int_z^\infty (K_i - K)(x,y)c_i(y,s)dy\right| \le \frac{2k}{z}(1+x)N_1 + \frac{2k}{z^{r-1}}\bar{N}_r), \tag{3.44}$$

$$\left|\int_z^\infty K(x,y)(c_i - c)(y,s)dy\right| \le \frac{2k}{z}(1+x)N_1 + \frac{2k}{z^{r-1}}\bar{N}_r), \tag{3.45}$$

$$\left|\int_z^\infty (F_i - F)(x,y)c_i(x+y,s)dy\right| \le \frac{2b}{z}N_1 + \frac{2b}{z^{r-m}}\bar{N}_r, \tag{3.46}$$

$$\left|\int_z^\infty F(x,y)(c_i - c)(x+y,s)dy\right| \le \frac{2b}{z}N_1 + \frac{2b}{z^{r-m}}\bar{N}_r. \tag{3.47}$$

Other difference terms in (3.43) can be easily shown to tend to zero. Finally, we find that the function c is a solution of the problem (3.1), (3.2) written in integral form:

$$c(x,t) = c_0(x) + \int_0^t \left\{\frac{1}{2}\int_0^x K(x-y,y)c(x-y,s)c(y,s)dy-\right.$$

$$-c(x,s)\int_0^\infty K(x,y)c(y,s)dy+$$

$$+\int_0^\infty F(x,y)c(x+y,s)dy-\frac{1}{2}c(x,s)\int_0^x F(x-y,y)dy\Big\}\,ds. \tag{3.48}$$

It follows from (3.44)–(3.47) and the continuity of $c(x,t)$ that the right-hand side in (3.1), evaluated at c, is a continuous function on Π_T. Differentiation of (3.38) with respect to t establishes that c is a solution of (3.1),(3.2).

3.4 Proof of Theorem 3.1: Case 2^0.

To prove the second case of Theorem 3.1 it suffices to prove that, similarly to (3.11), the functions c_n belong to $\Omega^+(T)$ uniformly, that is there exists $\lambda > 0$ such that for all $n \geq 1$, $t \in [0,T]$

$$\int_0^\infty \exp(\lambda x)c_n(x,t)dx \ \leq \ \text{const.} \tag{3.49}$$

Actually, in this case the uniform convergence on each compact set to $c(x,t)$ implies that $c \in \Omega_\lambda^+(T)$ with the same λ. Denote

$$\sigma_n(\lambda,t) = \int_0^\infty (\exp(\lambda x)-1)c_n(x,t)dx.$$

Multiplying (3.1) by $\exp(\lambda x)-1$ and taking into account the positivity of $c_n(x,t)$, we obtain

$$\frac{\partial \sigma_n}{\partial t} \leq k(\frac{1}{2}\sigma_n^2 + \sigma_n\frac{\partial}{\partial\lambda}\sigma_n - \sigma_n N_1), \tag{3.50}$$

$$\sigma_n(\lambda,0) = \int_0^\infty (\exp(\lambda x)-1)c_0(x)dx. \tag{3.51}$$

Let us consider the "upper" function $\sigma(\lambda,t)$ which satisfies the following equation:

$$\frac{\partial \sigma}{\partial t} = k(\frac{1}{2}\sigma^2 + \sigma\frac{\partial\sigma}{\partial\lambda} - \sigma N_1), \tag{3.52}$$

$$\sigma(\lambda,0) \stackrel{\text{def}}{=} \sigma_0(\lambda) > \sigma_n(\lambda,0), \lambda > 0;\ \sigma_0(0) = \sigma_n(0,0) = 0. \tag{3.53}$$

If the problem (3.52),(3.53) has a smooth enough solution then

$$\sigma_n(\lambda,t) \le \sigma(\lambda,t) \tag{3.54}$$

for $0 \le \lambda < \bar\lambda,\ \bar\lambda > 0,\quad 0 \le t \le T$ for some $\bar\lambda$. To show this fact we use the substitution

$$\sigma_n = \exp(-kN_1 t)\alpha_n, \quad \sigma = \exp(-kN_1 t)\alpha.$$

Then from (3.50) and (3.52) we have

$$\frac{d}{dt}\alpha_n \le \frac{1}{2}k\exp(-kN_1 t)\alpha_n^2, \tag{3.55}$$

$$\frac{d}{dt}\alpha = \frac{1}{2}k\exp(-kN_1 t)\alpha^2, \tag{3.56}$$

where $\frac{d}{dt}$ in (3.55),(3.56) means differentiation along characteristics of (3.50), (3.52) respectively. Let $(\hat\lambda,\hat t)$ be the first point where $\sigma_n(\lambda,t) = \sigma(\lambda,t)$, i.e. $\sigma_n(\lambda,t) < \sigma(\lambda,t)$ for $0 \le \lambda < \bar\lambda,\ 0 \le t < \hat t$. Due to (3.53) we have $\hat t > 0$. Also, the functions σ_n increase in λ. Then we obtain the following contradiction:

$$\alpha(\hat\lambda,\hat t) = \alpha(\lambda(t),t) + \frac{1}{2}k\int_t^{\hat t}\exp(-kN_1 s)\alpha^2(\lambda(s),s)ds$$

$$> \alpha_n(\lambda_n(t),t) + \frac{1}{2}k\int_t^{\hat t}\exp(-kN_1 s)\alpha_n^2(\lambda_n(s),s)ds = \alpha_n(\hat\lambda,\hat t). \tag{3.57}$$

The first and second integrations in (3.57) are along characteristics of the equations (3.52) and (3.50) respectively. We have used the fact that $\lambda(s) > \lambda_n(s)$. The inequality (3.54) has now been proved.

Our next aim is to show that there exists a solution to (3.52),(3.53), which is bounded in a neighborhood of zero for all $0 \le t \le T$. Firstly, we formulate without proof the following well-known lemma which is fundamental to the characteristics method.

Lemma 3.3. *Let functions $a(z,t,u)$ and $f(t,u)$ be continuous in $R^n \times R^1_+ \times R^1$ and $R^1_+ \times R^1$ respectively and $u(z,t)$ be a solution to the problem*

$$u_t(z,t) + a(z,t,u)u_z(z,t) = f(t,u) \tag{3.58}$$

$$u(z,0) = u_0(z), \qquad z \in R^n,\quad t \in R^1_+.$$

Let the function v be a solution to the simplified problem

$$v_t(t, v_0) = f(t, v) \qquad (3.59)$$
$$v(0, v_0) = v_0 = const.$$

Let $z_0(z,t)$ be the beginning of the characteristics for the problem (3.58) which pass through the point (z,t). Then

$$u(z,t) = v(t, u_0(z_0(z,t))). \qquad (3.60)$$

To study (3.52),(3.53) we consider the following problem:

$$\frac{\partial \sigma}{\partial t} = k(\frac{1}{2}\sigma^2 + \sigma\frac{\partial \sigma}{\partial \lambda} - g(\lambda)\sigma) + \varepsilon, \qquad (3.61)$$

$$\sigma(\lambda, 0) = \sigma_0(\lambda), \quad \lambda \geq 0, t \geq 0. \qquad (3.62)$$

Lemma 3.4. *Let $\sigma_0(\lambda) > 0$ if $\lambda > 0$, $\sigma_0(0) = 0$; $g(\lambda) = G - \delta(\lambda)$, $G = const > 0$; $\delta(\lambda) \to 0$ as $\lambda \to 0$ and $\sigma_0'(0) \leq G$. Let $\sigma_0(\lambda)$ be a holomorphic function in a neighborhood of $\lambda = 0$. Let us fix $T > 0$. Then there exist $\hat{\lambda}(T) > 0$ and $\hat{\varepsilon}(T) > 0$ such that the initial value problem (3.61),(3.62) has for $t \in [0,T]$ a unique solution for $0 \leq \lambda < \hat{\lambda}, 0 \leq \varepsilon < \hat{\varepsilon}$.*

Proof. Firstly, let $\delta(\lambda) = 0$. We consider the auxiluary problem

$$v_t = \frac{1}{2}kv^2 - kGv + \varepsilon, \quad v|_{t=0} = v_0$$

with the solution

$$v(t, v_0) = v_2 + (v_1 - v_2)\left[1 + \left(\frac{v_1 - v_2}{v_0 - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1}.$$

Here v_1 and v_2 are roots of the trinomial $\frac{1}{2}kv^2 - kGv + \varepsilon$. Choosing ε small enough, we have $v_1 >> v_2 \geq 0$. Using Lemma 3.3, we have

$$\sigma(\lambda, t) = v_2 + (v_1 - v_2)\left[1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1}. \qquad (3.63)$$

We investigate the quantity λ_0. Let $\lambda(t)$ be a solution of the characteristic equation of the problem (3.61), (3.62):

$$d\lambda/dt = -k\sigma(\lambda, t).$$

Using (3.63), we obtain

$$\lambda(t) = \lambda_0 - k\int_0^t \{v_2 + (v_1 - v_2) \cdot \left[1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}ks(v_1 - v_2))\right]^{-1}\}ds,$$

whence

$$\lambda = \lambda_0 - kv_1t + 2\log\left(1 + \left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2} - 1\right)\exp(\frac{1}{2}kt(v_1 - v_2))\right) - 2\log\left(\frac{v_1 - v_2}{\sigma_0(\lambda_0) - v_2}\right).$$

By substituting (3.63) into the last expression, we obtain the equality:

$$\lambda_0 = \lambda + kv_1t + 2\log\left(\frac{\sigma - v_2}{v_1 - v_2} + \left(1 - \frac{\sigma - v_2}{v_1 - v_2}\right)\exp(-\frac{1}{2}kt(v_1 - v_2))\right). \tag{3.64}$$

Using (3.63), we introduce for consideration the function

$$S(\sigma, \lambda, t) = \sigma - v_2 - (v_1 - v_2)(\sigma_0(\lambda_0) - v_2) \cdot \left[\sigma_0(\lambda_0) - v_2 + (v_1 - \sigma_0(\lambda_0))\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-1}.$$

From (3.64) we can see that for small σ, v_2 and λ, the value λ_0 is small for all t, $0 \le t \le T$. Consequently, the function S is analytic in the polycircle

$$\left\{(\lambda, \sigma, t) : \ |\lambda| < \hat{\lambda}, \ |\sigma| < \hat{\sigma}, \ |t| < T\right\}$$

for small $\hat{\lambda}$ and $\hat{\sigma}$, because $\sigma_0(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$ and $\sigma_0(0) = 0$. For the derivative we obtain

$$\frac{\partial S(0,0,t)}{\partial \sigma} = 1 - 2(v_1 - v_2)^2 \sigma_0'(\lambda_0^0) \exp(\frac{1}{2}kt(v_1 - v_2))$$
$$\cdot(1 - \exp(-\frac{1}{2}kt(v_1 - v_2)))[v_1 \exp(-\frac{1}{2}kt(v_1 - v_2)) - v_2]^{-1}$$
$$\cdot \left[(\sigma_0(\lambda_0^0) - v_2)(1 - \exp(\frac{1}{2}kt(v_1 - v_2))) + (v_1 - v_2)\exp(\frac{1}{2}kt(v_1 - v_2))\right]^{-2} \tag{3.65}$$

where

$$\lambda_0^0 = \lambda_0|_{\lambda=0,\sigma=0} = kv_1t + 2\log\left(\frac{v_1}{v_1 - v_2}\left(\exp(-\frac{1}{2}kt(v_1 - v_2)) - 1\right) + 1\right)$$

and $0 \leq |t| \leq T$. By analysing this expression for $\frac{\partial S(0,0,t)}{\partial \sigma}$ with the conditions of the lemma taken into account, we conlude that

$$\frac{\partial S(0,0,t)}{\partial \sigma} \neq 0$$

for all $|t| \leq T$. This last assertion is especially descriptive when $\varepsilon = 0$: in this case we have $v_2 = 0$ and $v_1 = 2G$. Then

$$\frac{\partial S(0,0,t)}{\partial \sigma} = 1 - \sigma_0'(0)G^{-1}(1 - \exp(-Gkt)) \neq 0$$

if all the conditions of Lemma 3.4 hold.
Using the implicit function theorem, we establish the existence of a solution to (3.61),(3.62) which is unique and analytic in the polycircle

$$\left\{(\lambda, t):\ |\lambda| < \hat{\lambda},\ |t| < T\right\}$$

for $\hat{\lambda}$ small enough.
If $\delta(\lambda) \neq 0$ then we can easily show (similar to obtaining the inequality (3.54)) that $\sigma < \tilde{\sigma}$ where

$$\tilde{\sigma}_t = k\left(\frac{1}{2}\tilde{\sigma}^2 + \tilde{\sigma}\tilde{\sigma}_\lambda - G_1\tilde{\sigma}\right) + \varepsilon,$$

$$\tilde{\sigma}_0(\lambda) > \sigma_0(\lambda), \ \lambda > 0, \ \tilde{\sigma}_0(0) = 0$$

with

$$G_1 = G - \sup_{0 \le \lambda \le \hat{\lambda}} \delta(\lambda).$$

Then, by repeating the above arguments, Lemma 3.4 can similarly be proved.

Applying Lemma 3.4 to the problem (3.52),(3.53) with $\varepsilon = 0$, $\delta = 0$, $G = N_1$, we obtain that for all $t, \in [0, T]$ and $\lambda \in [0, \hat{\lambda})$:

$$\sigma(\lambda, t) \le \text{const}. \tag{3.66}$$

From (3.54), (3.66) we establish the correlation

$$\int_0^\infty (\exp(\lambda x) - 1)c_n(x,t)dx \le \text{const}, 0 \le \lambda < \hat{\lambda}, \ 0 \le t \le T, \ n \ge 1. \tag{3.67}$$

Consequently, (3.49) follows from (3.67) and (3.11). Hence, $c \in \Omega^+(T)$:

$$\int_0^\infty \exp(\lambda x)c(x,t)dx \le \text{const}, \ \ 0 \le \lambda < \bar{\lambda}, \ \ 0 \le t \le T. \tag{3.68}$$

The proof of Theorem 3.1 is now complete. □

Remark 3.1. *It is worth pointing out that the solution does not belong to $\Omega_\lambda(T)$ even if $c_0 \in \Omega_\lambda(0)$. Actually, for the constant kernels $K \equiv 1$, $F \equiv 0$ we obtain from (3.1):*

$$\frac{d\sigma}{dt} = \frac{1}{2}\sigma(t)^2$$

where

$$\sigma(t) = \int_0^\infty (\exp(\lambda x) - 1)c(x,t)dx.$$

Hence, $\sigma(t) \to \infty$ as $t \to 2/\sigma(0) < \infty$. Consequently, the right "tails" of solutions (i.e. for large values of x) increase in time. This growth is fast enough for the solution to leave $\Omega_\lambda(T)$ within a finite time but it is sufficiently slow to remain inside $\Omega(T)$ for all $T > 0$.

3.5 Mass conservation

Theorem 3.2. *Let the conditions of Theorem 3.1 hold. Let, in addition,*

$$r \geq 2 \text{ and } \int_0^x yF(x-y,y)dy \leq const \cdot (1+x^r) \qquad (3.69)$$

then the mass conservation law holds.

Proof. We are ready now to improve the inequality (3.42) and demonstrate that for all $t \geq 0$ the function $c(x,t)$ yields, similarly to (3.9), the mass conservation law

$$N_1 = \int_0^\infty xc(x,t)dx = \text{const}.$$

This equality holds due to the boundedness of the upper moments of $c(x,t)$ for all $t \geq 0$ (see (3.41)). Actually, by integrating (3.1) with weight x, we obtain

$$\frac{dN_1(t)}{dt} = -\lim_{n\to\infty} \int_0^n \int_{n-x}^\infty (xK(x,y)c(x,t)c(y,t) - xF(x,y)c(x+y,t))dydx.$$

Passing to the limit we obtain zero if the integrals

$$\int_0^\infty \int_0^\infty xK(x,y)c(x,t)c(y,t)dxdy \quad \text{and} \quad \int_0^\infty \int_0^\infty xF(x,y)c(x+y,t)dxdy$$

are bounded. The first integral with the coagulation kernel is bounded due to (3.3) and boundedness of the second moment N_2. For the integral with the fragmentation kernel we appeal to (3.69),(3.27) and (3.41) to see that

$$\int_0^\infty \int_0^\infty xF(x,y)c(x+y,t)dydx = \int_0^\infty c(x,t) \int_0^x yF(x-y,y)dydx \leq \text{const}(\bar{N}_0 + \bar{N}_r).$$

This proves Theorem 3.2 □

Remark 3.2. *If at a critical time $t_c < \infty$ the second moment $N_2(t)$ had become infinite then the formal integration of (3.1) over $[0,\infty)$ with weight x would give us in the coagulation part the indeterminance $\infty - \infty$ which could lead to the infringement of the mass conservation law (see chapter 1).*

3.6 Remarks

In chapter 3 we follow Dubovskiĭ and Stewart [24] and Galkin [34]. The first global existence and uniqueness result was proved in 1957 by Melzak [54]. His Theorem treats bounded kernels and claims:

Theorem 3.3. *Let* c_0 *be a continuous, nonnegative, bounded and integrable function. Let kernels* $K(x,y)$, $\Psi(x,y)$ *be continuous and*

$$0 \leq K(x,y) = K(y,x) \leq const < \infty,$$

$$0 \leq \Psi(x,y) \leq const < \infty,$$

$$\int_0^x y\Psi(x,y)dy \leq x, \quad \int_0^x \Psi(x,y)dy \leq const < \infty.$$

Then the equation (0.5) possesses a unique solution $c(x,t)$, *which is continuous, bounded, nonnegative, integrable in* x *for each* t, *and analytic in* t *for each* x.

Boundness of coagulation kernels K ensures continuity of the collision operator, expressed by the right-hand side of (3.1) and its invariant property, i.e. the operator reflects the space of integrable functions into itself. Unbounded coagulation kernels do not possess such property and it is the main obstacle them to be analysed. Existence for unbounded coagulation kernels from class (3.3) was proved by Galkin [34] and White[78]. They did not take fragmentation into account. Burobin and Galkin [13] considered kernels which belong (3.3) for large arguments and have singular behaviour for $x,y \to 0$. Then Galkin and Dubovskiĭ [39] and Spouge [65,66] proved existence for kernels $K(x,y) = o(x)o(y)$ with several variations. The existence theorem in large for unbounded coagulation kernels with fragmentation ones taken into account, was proved by Ball and Carr [5] and Stewart [69]. In all above references the rate of growth of coagulation kernels is not more than one (i.e. $K(x,y) \sim y,\ y \to \infty,$ for fixed x).

Chapter 4. UNIQUENESS THEOREMS

In the first section of this chapter we prove uniqueness result which naturally links with the existence theorem from previous chapter if initial data belong to the class $\Omega(0)$. Then we demonstrate two other uniqueness results which are valid for solutions from the less restrictive class $\Omega_{0,1}(T)$ of functions with bounded first moment (i.e. bounded total mass of particles).

4.1 Uniqueness theorem in $\Omega(T)$

Theorem 4.1. *Let the case* 2^0 *of the Theorem 3.1 hold and* $m_1 \leq 1$. *Then the solution to the initial value problem (3.1), (3.2) is unique in the class* $\Omega(T)$.

To prove Theorem 4.1 we use the following lemma.

Lemma 4.1. *Let* $v(\lambda, t)$ *be a real continuous function having continuous partial derivatives* v_λ *and* $v_{\lambda\lambda}$ *on*

$$D = \{0 \leq \lambda \leq \lambda_0,\ 0 \leq t \leq T\}.$$

Assume that $\alpha(\lambda), \beta(\lambda, t), \gamma(\lambda, t)$ *and* $\theta(\lambda, t)$ *are real and continuous on* D *, having continuous partial derivatives there in* λ *and that the functions* $v, v_\lambda, \beta, \gamma$ *are nonnegative. Suppose that the following inequalities hold on* D*:*

$$v(\lambda, t) \leq \alpha(\lambda) + \int_0^t (\beta(\lambda, s) v_\lambda(\lambda, s) + \gamma(\lambda, s) v(\lambda, s) + \theta(\lambda, s)) ds, \tag{4.1}$$

$$v_\lambda(\lambda, t) \leq \alpha_\lambda(\lambda) + \int_0^t \frac{\partial}{\partial \lambda} (\beta(\lambda, s) v_\lambda(\lambda, s) + \gamma(\lambda, s) v(\lambda, s) + \theta(\lambda, s)) ds. \tag{4.2}$$

Let $m_0 = \sup_{0 \leq \lambda \leq \lambda_0} \alpha, m_1 = \sup_D \beta,\ m_2 = \sup_D \gamma,\ m_3 = \sup_D \theta$. *Then*

$$v(\lambda, t) \leq m_0 \exp(m_2 t) + (m_3/m_2)(\exp(m_2 t) - 1)$$

in any region $R \subset D$*:*

$$R = \{(\lambda, t) : 0 \leq t \leq t' < T';\ \lambda_1 - m_1 t \leq \lambda \leq \lambda_0 - m_1 t,\quad 0 < \lambda_1 < \lambda_0\}.$$

where $T' = \min\{\lambda_1/m_1), T\}$.

Proof. Let us denote the right-hand side of the inequality (4.1) by $w(\lambda, t)$. By differentiating in t, λ, we obtain from (4.1),(4.2):

$$w_t \le \beta w_\lambda + \gamma w + \theta \le m_1 w_\lambda + \gamma w + \theta.$$

Hence for the derivative along the characteristic $\frac{d\lambda}{dt} = -m_1$ we have

$$\frac{dw}{dt} \le \gamma w + \theta. \tag{4.3}$$

Let us denote $u(t) = M_0 \exp(m_2 t) + (M_3/m_2)(\exp(m_2 t) - 1)$ with $M_0 > m_0$, $M_3 > m_3$. Obviously, $u(0) > w(\lambda, 0)$ for all $\lambda \in [0, \lambda_0]$. Let $(\hat{\lambda}, \hat{t})$ be the first point on a characteristic straight line, where $w = u$. Then at the point $(\hat{\lambda}, \hat{t})$

$$\frac{d(u-w)}{dt} \le 0$$

and consequently

$$w_t - C_1 w_\lambda \ge u_t. \tag{4.4}$$

From $u_t = m_2 u + M_3$ we can easily see that at the point $(\hat{\lambda}, \hat{t})$ the equality $u_t = m_2 w + M_3$ holds. Recalling (4.4), we obtain a contradiction with (4.3):

$$\frac{dw}{dt} = w_t - m_1 w_\lambda \ge m_2 w + M_3 > m_2 w + m_3 \ge \gamma w + \theta.$$

This proves Lemma 4.1 □

Proof of Theorem 4.1. We shall prove the uniqueness of solution $c \in \Omega^+(T)$ in $\Omega(T)$ by contradiction. Suppose that there are two distinct solutions c and g of the initial value problem (3.1),(3.2) in $\Omega(T)$. Using the notation $u = |c - g|$, $\psi = |c + g|$ and conditions (3.3), we find that

$$\begin{aligned} u(x,t) \le \int_0^t \Big\{ \frac{1}{2} k(1+x) \int_0^x u(x-y,s)\psi(y,s)dy + \\ + \frac{1}{2} k u(x,s) \int_0^\infty (1+x+y)\psi(y,s)dy + \frac{1}{2} k\psi(x,s) \int_0^\infty (1+x+y)u(y,s)dy \\ + \int_x^\infty F(y-x,x)u(y,s)dy + \frac{1}{2} u(x,s) \int_0^x F(x-y,y)dy \Big\} ds. \end{aligned} \tag{4.5}$$

Since $c, g \in \Omega(T)$, we have $u, \psi \in \Omega(T)$, and $u, \psi \geq 0$ on Π_T. Let $\hat{\lambda} > 0$ be chosen such that

$$\int_0^\infty \exp(\hat{\lambda}x)u(x,t)dx \leq \text{const} < \infty,$$

$$\int_0^\infty \exp(\hat{\lambda}x)\psi(x,t)dx \leq \text{const} < \infty \tag{4.6}$$

uniformly with respect to t, $0 \leq t \leq T$, and let

$$0 \leq \lambda < \hat{\lambda}. \tag{4.7}$$

Integration of inequality (4.5) with the weight $\exp(\lambda x)$ yields

$$\int_0^\infty \exp(\lambda x)u(x,t)dx \leq \int_0^t \left\{ \int_0^\infty \int_0^\infty \frac{1}{2}k(\exp(\lambda(x+y))+\exp(\lambda x)+\exp(\lambda y)) \right.$$

$$\cdot(1+x+y)u(x,s)\psi(y,s)dxdy + \int_0^\infty \exp(\lambda x)u(x,s)\cdot$$

$$\left. \cdot \left(\int_0^x \exp(\lambda y - \lambda x)F(x-y,y)dy + \frac{1}{2}\int_0^x F(x-y,y)dy \right) dx \right\} ds.$$

Here we have changed the order of integration in the integral, using Fubini's theorem [26]. We strengthen this inequality with (3.4) and $m_1 \leq 1$ taken into account:

$$\int_0^\infty \exp(\lambda x)u(x,t)dx \leq \frac{3}{2}\int_0^t \left\{ \int_0^\infty \int_0^\infty k\exp(\lambda x + \lambda y) \right.$$

$$\left. \cdot(1+x+y)u(x,s)\psi(y,s)dxdy + b\int_0^\infty (1+x)\exp(\lambda x)u(x,s)dx \right\} ds. \tag{4.8}$$

The following inequality can be proved similarly:

$$\int_0^\infty x\exp(\lambda x)u(x,t)dx \leq \frac{3}{2}\int_0^t \left\{ \int_0^\infty \int_0^\infty k\exp(\lambda x + \lambda y)(x+y) \right.$$

$$\left. \cdot(1+x+y)u(x,s)\psi(y,s)dxdy + b\int_0^\infty x(1+x)\exp(\lambda x)u(x,s)dx \right\} ds. \tag{4.9}$$

Let

$$U(\lambda,t)=\int_0^\infty \exp(\lambda x)u(x,t)dx;\quad \Psi(\lambda,t)=\int_0^\infty \exp(\lambda x)\psi(x,t)dx.$$

The functions U and Ψ are analytic in the half-plane $Re(\lambda)<\hat{\lambda}$ for any fixed $t,\ \ 0\le t\le T$. Let λ be on the real axis and satisfy

$$0\le\lambda\le\lambda_0<\hat{\lambda}. \tag{4.10}$$

The inequalities (4.6) then ensure that, for any integer $i\ge 1$,

$$\sup_{0\le t\le T,0\le\lambda\le\lambda_0}\left\{\frac{\partial^i}{\partial\lambda^i}U(\lambda,t),\ \frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right\}<\infty. \tag{4.11}$$

Moreover, since $u(x,t)$ and $\psi(x,t)$ are continuous on Π_T and inequalities (4.6) are satisfied, for a given $\varepsilon>0$ there are corresponding numbers $\delta(\varepsilon)>0$ and $\delta_i(\varepsilon)>0$ such that, if $0\le t,t'\le T$, and $i\ge 1$,

$$\sup_{0\le\lambda\le\lambda_0}\{|U(\lambda,t')-U(\lambda,t)|,|\Psi(\lambda,t')-\Psi(\lambda,t)|\}<\varepsilon,\quad |t'-t|<\delta,$$

$$\sup_{0\le\lambda\le\lambda_0}\left\{\left|\frac{\partial^i}{\partial\lambda^i}U(\lambda,t')-\frac{\partial^i}{\partial\lambda^i}U(\lambda,t)\right|,\left|\frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t')-\frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right|\right\}<\varepsilon,\quad |t'-t|<\delta_i. \tag{4.12}$$

In fact, to show (4.12) it is enough to split the integrals in (4.6) and use the uniform smallness of the "tails" $\int_z^\infty$, which holds due to (4.7), (4.10) and the inequality (3.27) with, for example, $\psi(x)=\exp(\frac{1}{2}(\hat{\lambda}-\lambda_0)x)$. It follows from (4.11),(4.12) that U and Ψ are continuous together with all their partial derivatives with respect to λ in $D=\{0\le\lambda\le\lambda_0,\ 0\le t\le T\}$. The inequalities (4.8),(4.9) imply that

$$U(\lambda,t)\le\frac{3}{2}\int_0^t\{(k\Psi(\lambda,s)+b)U_\lambda(\lambda,s)+(k\Psi(\lambda,s)+k\Psi_\lambda(\lambda,s)+b)U(\lambda,s)\}ds,$$

$$U_\lambda(\lambda,t)\le\frac{3}{2}\int_0^t\frac{\partial}{\partial\lambda}\{(k\Psi+b)U_\lambda+(k\Psi+k\Psi_\lambda+b)U(\lambda,s)\}ds,$$

and U and Ψ are nonnegative in D together with their partial derivatives with respect to λ. We can thus apply Lemma lm4.1 in D. Let

$$c_1 = \frac{3}{2}(k \sup_D \Psi + b), \ c_2 = \frac{3}{2} k \sup_D (\Psi + \Psi_\lambda) + \frac{3}{2} b.$$

Then $U(\lambda, t) = 0$ in the region R defined in Lemma 2.5. Since $u(x,t)$ is continuous, $u(x,t) = 0$ for $0 \le t \le t'$, $0 \le x < \infty$; hence $U(\lambda, t) = 0$ not only in R, but for all $0 \le \lambda \le \lambda_0$, $0 \le t \le t'$. Applying the same reasoning to the interval $[t', 2t']$, we conclude that $u(x,t) = 0$ for $0 \le t \le 2t'$, $0 \le x < \infty$ and, continuing this process, we establish that $u(x,t) = 0$ on Π, that is, $c = g$ on Π_T. This completes the proof of Theorem 4.1 □

4.2 Uniqueness theorem in $\Omega_{0,1}(T)$

Let the coagulation kernels be symmetric and satisfy the following condition. Suppose that for all $x \ge 0$ there exists $X(x) \ge 1$ such that

$$K(x,y) = a(x)y + b(x,y) \quad \text{if} \quad y \ge X(x) \tag{4.13}$$

and there exist positive constants λ, G such that

$$\sup_{0 \le y \le X(x)} K(x,y) + \sup_{y \ge X(x)} b(x,y) + a(x)X(x) \le G \exp(\lambda x), \quad x \ge 0. \tag{4.14}$$

Functions a and b have to be nonnegative.

We shall consider fragmentation kernels which are nonnegative, symmetric and satisfy for positive constants μ and A the following condition:

$$\int_0^x F(x-y,y) \exp(-\mu y) dy \le A, \qquad x \ge 0. \tag{4.15}$$

This class includes bounded and all above-mentioned fragmentation kernels (e.g. $F(x,y) = (x+y)^{-1}$).

The aim of this section is to prove the following theorem.

Theorem 4.2. *The initial value problem (3.1),(3.2) with a coagulation kernel from the class (4.13) and a fragmentation kernel from (4.15) has at most one nonnegative continuous solution among all continuous functions with the same first moment:*

$$\int_0^\infty xc(x,t)dx = \int_0^\infty xd(x,t)dx < \infty, \quad t \geq 0. \tag{4.16}$$

We formulate the following lemma which can be proved similarly to Lemma 4.1.

Lemma 4.2. *Let $v(q,t)$ be a real continuous function having continuous partial derivatives v_q and v_{qq} on*

$$D = \{0 < q_0 \leq q \leq q_1,\ 0 \leq t \leq T\}.$$

Assume that $\alpha(q), \beta(q,t), \gamma(q,t)$ and $\theta(q,t)$ are real continuous functions on D and their first partial derivatives in q are continuous. Let v, v_{qq}, β, γ be nonnegative and $v_q, \alpha_q, \beta_q, \gamma_q, \theta_q$ be nonpositive functions on D. Suppose also, that the following inequalities hold on D:

$$v(q,t) \leq \alpha(q) + \int_0^t (-\beta(q,s)v_q(q,s) + \gamma(q,s)v(q,s) + \theta(q,s))ds; \tag{4.17}$$

$$v_q(q,t) \geq \alpha_q(q) + \int_0^t \frac{\partial}{\partial q}(-\beta(q,s)v_q(q,s) + \gamma(q,s)v(q,s) + \theta(q,s))\, ds. \tag{4.18}$$

Let $c_0 = \sup_{q_0 \leq q \leq q_1} \alpha, c_1 = \sup_D \beta,\ c_2 = \sup_D \gamma,\ c_3 = \sup_D \theta$. Then

$$v(q,t) \leq c_0 \exp(c_2 t) + (c_3/c_2)(\exp(c_2 t) - 1)$$

in any region $R \subset D$:

$$R = \{(q,t) : 0 \leq t \leq T';\ q_0 + c_1 t \leq q \leq q_1 - \varepsilon + c_1 t,\ 0 < \varepsilon < q_1 - q_0, \},$$

$$T' = \min\{T, \varepsilon/c_1\}.$$

Proof of Theorem 4.2. We shall prove by contradiction. Suppose that there are two distinct continuous solutions c and d of the initial value problem

(3.1),(3.2) with the same initial data and first moment. Let us denote $u = c - d$. Then we obtain from (3.1):

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)(c+d)(y,t)dy-$$
$$-u(x,t)\int_0^\infty K(x,y)c(y,t)dy - d(x,t)\int_0^\infty K(x,y)u(y,t)dy-$$
$$-\frac{1}{2}u(x,t)\int_0^x F(x-y,y)dy + \int_0^\infty F(x,y)u(x+y,t)dy. \tag{4.19}$$

Let us write (4.19) in the following integral form

$$u(x,t) = \int_0^t \exp\left(-\int_s^t \left\{\int_0^\infty K(x,y)c(y,\tau)dy + \frac{1}{2}\int_0^x F(x-y,y)dy\right\} d\tau\right)\cdot$$
$$\cdot\left(\frac{1}{2}\int_0^x K(x-y,y)u(x-y,s)(c+d)(y,s)dy-\right.$$
$$\left.-d(x,t)\int_0^\infty K(x,y)u(y,s)dy + \int_0^\infty F(x,y)u(x+y,s)dy\right) ds. \tag{4.20}$$

Utilizing (4.16), we consider the second summand in (4.20) separately:

$$\int_0^\infty K(x,y)u(y,t)dy = \int_0^{X(x)} K(x,y)u(y)dy + \int_{X(x)}^\infty (a(x)y+b(x,y))u(y,t)dy =$$
$$= \int_0^{X(x)} K(x,y)u(y)dy + \int_{X(x)}^\infty b(x,y)u(y,t)dy - a(x)\int_0^{X(x)} yu(y,t)dy.$$

Whence,

$$\left|\int_0^\infty K(x,y)u(y,t)dy\right| \le$$
$$\le \left[\sup_{0\le y\le X(x)} K(x,y) + \sup_{y\ge X(x)} b(x,y) + a(x)X(x)\right]\int_0^\infty |u(y,t)|dy. \tag{4.21}$$

Using (4.21) and (4.14), we obtain from (4.20):

$$|u(x,t)| \le \int_0^t \left\{\frac{1}{2}\int_0^x K(x-y,y)|u(x-y,s)|\cdot|c+d|(y,s)dy+\right.$$
$$\left.+G\exp(\lambda x)|d(x,s)|\int_0^\infty |u(y,s)|dy + \int_0^\infty F(x,y)|u(x+y,s)|dy\right\} ds. \tag{4.22}$$

Let

$$U(q,t) = \int_0^\infty \exp(-qx)|u(x,t)|dx,$$

$$\psi = \max\{|c+d|, |c|, |d|\}, \qquad \Psi(q,t) = \int_0^\infty \exp(-qx)\psi(x,t)dx.$$

Let q be on the real axis. Functions U and Ψ decrease in q, $q \geq 0$. Boundness of the values $U(0,t), \Psi(0,t)$ ensures that all partial derivatives in q of U, Ψ are bounded on $q > 0$. In addition, the functions U and Ψ are continuous with all their derivatives in q for any fixed t, $0 \leq t \leq T$. Since $u(x,t)$ and $\psi(x,t)$ are continuous, then U ans Ψ are continuous together with all their partial derivatives with respect to q for $q > 0$, $0 \leq t \leq T$.

If we choose $q_0 > \max\{\lambda, \mu\}$ and utilize (4.15), then for $0 < q_0 \leq q \leq q_1 < \infty$ the following inequality takes place

$$\int_0^\infty \int_0^\infty F(x,y)\exp(-qx)|u(x+y,t)|dy =$$
$$= \int_0^\infty |u(x,t)| \int_0^x F(x-y,y)\exp(-qy)dydx \leq AU(0,t). \tag{4.23}$$

By integrating (4.22) with weight $\exp(-qx)$ and taking into account (4.23), (4.14), we obtain

$$U(q,t) \leq \int_0^t \{GU(q,s)\Psi(q-\lambda,s) - GU_q(q,s)\Psi(q-\lambda,s)+$$
$$+G\Psi(q-\lambda,s)U(0,s) + AU(0,s)\}ds. \tag{4.24}$$

Our next step is to estimate $U(0,s)$. Let q_2 be the solution to the algebraic equation

$$U(0,t) = U(q_2,t) - q_2 U_q(q_1,t). \tag{4.25}$$

Due to the decreasing of the function $U(q,t)$ with q, the equation (4.25) has the only root $q_2 > q_1$. Hence, for $0 \leq q \leq q_1$, $0 \leq t \leq T$ we obtain

$$U(0,t) \leq U(q,t) - QU_q(q,t), \tag{4.26}$$

where $Q = \sup_{0 \le t \le T} q_2(t)$. By substituting (4.26) into (4.23) we come to the following inequality

$$U(q,t) \le \int_0^t \{V(q,s)U(q,s) - W(q,s)U_q(q,s)\}\, ds, \quad 0 < q_0 \le q \le q_1 \tag{4.27}$$

where functions V and W are positive, continuous and have negative first derivative in q. Similarly, by integrating (4.22) with weight $x\exp(-qx)$, we obtain

$$-U_q(q,t) \le -\int_0^t \frac{\partial}{\partial q} \{V(q,s)U(q,s) - W(q,s)U_q(q,s)\}\, ds, \tag{4.28}$$

if $0 < q_0 \le q \le q_1$. We choose the value of q_1 sufficiently large to take $\varepsilon \ge Tc_1$ and to obtain $T' = T$. Applying Lemma 4.2 to (4.27),(4.28) we obtain $U(q,t) = 0$ in the region R defined in Lemma 4.2. Since $|u(x,t)|$ is continuous, $u(x,t)$=0 for $0 \le t \le T$, $0 \le x < \infty$. Consequently, $c = d$. This completes the proof of the Theorem 4.2 □

4.3 Uniqueness theorem for another class of unbounded kernels

In this section we prove the following theorem.

Theorem 4.3. *If (H1)*

$$K(x,y) \le \phi(x)\phi(y), \quad x,y \ge 0 \tag{4.29}$$

where $\phi(x) \le k\sqrt{1+x}$ *for some constant* k *, and; (H2): for all* $x \ge 0$ *there is a constant* m *such that*

$$\int_0^x \sqrt{1+y}F(x-y,y)dy \le m\sqrt{1+x}, \tag{4.30}$$

then solutions to (3.1), (3.2) are unique in $\Omega_{0,1}(T)$.

Proof. For $\lambda \in R^1$ define $\mathrm{sgn}(\lambda)$ to equal 1,0,-1 whenever $\lambda > 0, \lambda = 0$ or $\lambda < 0$ respectively. Let c and d be two solutions to (3.1) on $[0,T]$, where $c(0) = d(0)$, and set as in section 4.2 $u = c - d$. For $n \geq 1$ define

$$w^n(t) = \int_0^n \sqrt{1+x}|u(x,t)|dx.$$

We write (4.19) in the integral form

$$u(x,t) = \int_0^t \left\{ \frac{1}{2} \int_0^x K(x-y,y)u(x-y,s)(c+d)(y,s)dy - \right.$$

$$-d(x,s) \int_0^\infty K(x,y)u(y,s)dy - u(x,s) \int_0^\infty K(x,y)c(y,s)dy +$$

$$\left. + \int_0^\infty F(x,y)u(x+y,s)dy - \frac{1}{2}u(x,s) \int_0^x F(x-y,y)dy \right\} ds. \qquad (4.31)$$

Multiplying $|u|$ by $\sqrt{1+x}$ and applying Fubini's theorem to (4.31) we obtain for each n and $t \in [0,T]$

$$w^n(t) = \int_0^t \int_0^n \sqrt{1+x}\,\mathrm{sgn}(u(x,s)) \left[\frac{1}{2} \int_0^x (K(x-y,y)u(x-y,s)(c+d)(y,s) - \right.$$

$$-F(x-y,y)u(x,s))dy -$$

$$\left. - \int_0^\infty (K(x,y)\{d(x,s)u(y,s) + u(x,s)c(y,s)\} - F(x,y)u(x+y,s))\,dy \right] dxds. \qquad (4.32)$$

Using the substitution $x' = x - y$, $y' = y$ in the first integral on the right-hand side of (4.32) we find that

$$w^n(t) = \int_0^t \int_0^n \int_0^{n-x} \left[\frac{1}{2}\sqrt{1+x+y}\,\mathrm{sgn}(u(x+y,s)) - \sqrt{1+x}\,\mathrm{sgn}(u(x,s)) \right] \cdot$$

$$\cdot [K(x,y)\{d(x,s)u(y,s) + u(x,s)c(y,s)\} - F(x,y)u(x+y,s)]\,dydxds -$$

$$- \int_0^t \int_0^n \int_{n-x}^\infty \sqrt{1+x}\,\mathrm{sgn}(u(x,s))[K(x,y)\{d(x,s)u(y,s) + u(x,s)c(y,s)\} -$$

$$-F(x,y)u(x+y,s)]dydxds. \qquad (4.33)$$

We note that by interchanging the order of integration (and interchanging the roles of x and y) the symmetry of kernels K, F yields the identity

$$\int_0^n \int_0^{n-x} \sqrt{1+x}\,\mathrm{sgn}(u(x,s))\,[K(x,y)c(x,s)c(y,s) - F(x,y)c(x+y,s)]\,dydx =$$

$$= \int_0^n \int_0^{n-x} \sqrt{1+y}\,\mathrm{sgn}(u(y,s))\,[K(x,y)c(x,s)c(y,s) - F(x,y)c(x+y,s)]\,dydx. \tag{4.34}$$

(4.34) similarly holds for solution d. For $x, y \geq 0$ and $t \in [0,T]$ define f by

$$f(x,y,t) = \sqrt{1+x+y}\,\mathrm{sgn}(u(x+y,t)) - \sqrt{1+x}\,\mathrm{sgn}(u(x,t)) - \sqrt{1+y}\,\mathrm{sgn}(u(y,t)).$$

Using (4.34) show that (4.33) can be rewritten as

$$w^n(t) = \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} f(x,y,s)K(x,y)c(x,s)u(y,s)dydxds +$$

$$+\frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} f(x,y,s)K(x,y)d(y,s)u(x,s)dydxds -$$

$$-\frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} f(x,y,s)F(x,y)u(x+y,s)dydxds -$$

$$\int_0^t \int_0^n \int_{n-x}^{\infty} \sqrt{1+x}\,\mathrm{sgn}(u(x,s))[K(x,y)\{d(x,s)u(y,s) + u(x,s)c(y,s)\} -$$

$$-F(x,y)u(x+y,s)]dydxds = \int_0^t \sum_{i=1}^{4} S_i^n(s)ds, \tag{4.35}$$

where S_i^n, $i = 1, \ldots, 4$, are the corresponding integrands in the preceding line.

We now consider each S_i^n individually. Noting that for all λ, $\gamma \in R^1$ we have $\mathrm{sgn}(\lambda)\mathrm{sgn}(\gamma) = \mathrm{sgn}(\lambda\gamma)$ and $|\lambda| = \lambda\mathrm{sgn}(\lambda)$, we find that

$$f(x,y,s)u(y,s) \leq \left[\sqrt{1+x+y} + \sqrt{1+x} - \sqrt{1+y}\right] |u(y,s)| \leq 2\sqrt{1+x}|u(y,s)|. \tag{4.36}$$

The very important place in these reasonings is the negative sign before $\sqrt{1+y}$ in (4.36) unlike the positive one before $\sqrt{1+x}$. Thus by Hölder's inequality, hypothesis (H1) and (4.36),

$$\int_0^t S_1^n(s)ds \le L_1 \int_0^t w^n(s)ds, \tag{4.37}$$

where

$$L_1 = k^2 \sup_{s\in[0,t]} \|c(.,s)\|_{0,1}.$$

Similarly, there is a constant L_2 such that

$$\int_0^t S_2^n(s)ds \le L_2 \int_0^t w^n(s)ds. \tag{4.38}$$

To consider S_3^n we first observe that

$$-f(x,y,s)u(x+y,s) \le \left[\sqrt{1+x}+\sqrt{1+y}-\sqrt{1+x+y}\right] |u(x+y,s)|. \tag{4.39}$$

By (4.39), Fubini's theorem, hypothesis (H2) and the symmetry of F

$$\int_0^t S_3^n(s)ds \le$$

$$\le \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} \left[\sqrt{1+x}+\sqrt{1+y}-\sqrt{1+x+y}\right] F(x,y)|u(x+y,s)|dydxds$$

$$\frac{1}{2}\int_0^t \int_0^n \int_x^{n} \left[\sqrt{1+x}+\sqrt{1+y-x}-\sqrt{1+y}\right] F(x,y-x)|u(y,s)|dydxds$$

$$= \frac{1}{2}\int_0^t \int_0^n \int_0^{x} \left[\sqrt{1+x-y}+\sqrt{1+y}-\sqrt{1+x}\right] F(x-y,y)|u(x,s)|dydxds$$

$$\le \frac{1}{2}\int_0^t \int_0^n \int_0^{x} \sqrt{1+y}F(x-y,y)|u(x,s)|dydxds \le L_3 \int_0^t w^n(s)ds \tag{4.40}$$

where $L_3 = m/2$. Let χ denote the characteristic function, i.e. for any set E we have $\chi_E(x) = 1$ if $x \in E$ and zero otherwise. Define

$$g_n(x,s) = \chi_{[0,n]}(x)(1+x)c(x,s)\int_{n-x}^{\infty} \sqrt{1+y}\; c(y,s)dy$$

and

$$l_n(s) = \int_0^\infty g_n(x,s)dx.$$

Clearly, for each $s \in [0,t]$

$$|g_n(x,s)| \leq (1+x)c(x,s)\|c(.,s)\|_{0.1}$$

for all n and $g_n(x,s) \to 0$ as $n \to \infty$. Thus by the dominated convergence theorem $l_n(s) \to 0$ as $n \to 0$. Moreover,

$$|l_n(s)| \leq \sup_{s\in[0,t]} \|c(.,s)\|_{0,1}^2,$$

and therefore a further application of the dominated convergence theorem leads to

$$\int_0^t l_n(s)ds \to 0 \qquad \text{as} \qquad n \to \infty. \tag{4.41}$$

Using the symmetry of F, Fubini's theorem and hypothesis (H2) we have for each $s \in [0,t]$

$$\int_0^n \int_{n-x}^\infty \sqrt{1+x}F(x,y)c(x+y,s)dydx = \int_0^n \int_n^\infty \sqrt{1+x}F(x,y-x)c(y,s)dydx$$

$$= \int_n^\infty \int_0^n \sqrt{1+y}F(x-y,y)c(x,s)dydx \leq \int_n^\infty \int_0^x \sqrt{1+y}F(x-y,y)c(x,s)dydx$$

$$\leq m\int_n^\infty \sqrt{1+x}\ c(x,s)dx \to 0 \qquad \text{as} \qquad n \to \infty. \tag{4.42}$$

The right-hand side of (4.42) is always bounded by the constant

$$m \sup_{s\in[0,t]} \|c(.,s)\|_{0,1}$$

and therefore

$$\int_0^t \int_0^n \int_{n-x}^\infty \sqrt{1+x}F(x,y)c(x+y,s)dydxds \to 0 \qquad \text{as} \qquad n \to \infty. \tag{4.43}$$

(4.41) and (4.43) are similarly true for the solution d and consequently

$$\int_0^t S_4^n(s)ds \to 0 \qquad \text{as} \qquad n \to \infty. \tag{4.44}$$

We now set $L = L_1 + L_2 + L_3$. Applying (4.37), (4.38), (4.40) and (4.44) to equation (4.35) and taking limits as $n \to \infty$ gives

$$\int_0^\infty \sqrt{1+x}|u(x,t)|dx = \lim_{n\to\infty} w^n(t) \le \lim_{n\to\infty} L\int_0^t w^n(s)ds + \lim_{n\to\infty}\int_0^t S_4^n(s)ds$$

$$= L\int_0^t\int_0^\infty \sqrt{1+x}|u(x,s)|dxds. \tag{4.45}$$

An application of Gronwall's inequality to (4.45) shows that

$$\int_0^\infty \sqrt{1+x}|u(x,t)|dx = 0$$

for all $t \in [0,T]$ and therefore $c(x,t) = d(x,t)$ which completes the proof of Theorem 4.3 □

4.4 Remarks

The first uniqueness theorem for the problem concerned was proved by Melzak [54] for bounded coagulation and fragmentation kernels with the additional condition

$$\int_0^x F(x-y,y)dy \le \text{const}.$$

Aizenman and Bak [1] demonstrated uniqueness for constant coagulation and fragmentation kernels $K = F = $const.

It was shown by Galkin and Dubovskiĭ [39] that for kernels $K(x,y) \le k(1+x^\alpha y^\alpha)$, $\alpha < 1$, $F = 0$ the problem concerned can have at most one solution in class of functions, integrable with weight $\exp(\lambda x^\alpha)$, $\lambda > 0$.

The results of section 4.1 generalize Galkin's ones (see [34]) where the uniqueness in $\Omega(T)$ is proved for pure coagulation equation (without fragmentation and other processes). A variant of Lemma 4.1 was formulated without proof in [34]. The proof of Lemma 4.1 is based on the proof of Haar's lemma [42].

The condition (4.16) takes place, e.g., for coagulation kernels with linear growth on infinity and bounded fragmentation ones. In this case the equality (4.16) means the mass conservation law. If the kernel K grows faster than linear function then the mass conservation law can be infringed. This phenomenon is discussed in chapter chap1. It takes place for the important multiplicative case $K = xy$. For this case the behaviour of the total mass (which is expressed by the first moment of solution) is well-known. Consequently, the condition (4.16) takes place for the multiplicative coagulation kernel as well as for mass conserving kernels.

The class (4.13) includes many physically reasonable coagulation kernels. Particularly, this class has large intersection with coagulation kernels satisfying (3.3) or (4.28). Also, the class (4.13) includes bounded coagulation kernels considered by Melzak [54], linear kernels [34] and multiplicative ones ($K = (Ax + B)(Ay + B)$) which are concerned in many papers (see below). In addition, the class (4.13) includes the following kernels:

$$K(x,y) = \alpha(x,y) + \beta(y)x + \beta(x)y + \gamma(x,y)$$

where

$$\gamma(x,y) = \begin{cases} g_1(x)x + g_2(x)y + g_3(x)xy, & y \geq x \\ g_1(y)y + g_2(y)x + g_3(y)xy, & y \leq x. \end{cases}$$

Functions α, β and g_i, $i = 1,2,3$ are to be nonnegative and bounded.

Theorem 4.2 is true for the discrete form of the problem (3.1),(3.2) and for case including sources and efflux terms with $a(x,t) \leq \text{const} \cdot (1+x)$.

Existence theorems for pure coagulation equation were proved (see Chapter 3) if the coagulation kernels satisfy the condition $K(x,y) \leq k(1+x+y)$ and solutions have bounded p-th moment, $p > 1$. Theorem 4.2 proves uniqueness and completes the study of correctness for the important case

$$K(x,y) = \alpha(x,y) + \beta(x)y + \beta(y)x + \begin{cases} g_1(x)x + g_2(x)y, & y \geq x \\ g_1(y)y + g_2(y)x, & y \leq x \end{cases}$$

with bounded functions α, β, g_1, g_2.

Theorem 4.2 includes as well the important part of the Spouge's conditions ensuring existence [65,66]. Namely, his conditions on fragmentation, sources and efflux satisfy Theorem 4.2.

In addition, we have the large intersection with Galkin and Dubovskiĭ's [39] conditions on the coagulation kernels ensuring existence. These kernels include many unbounded kernels modelling fast interaction of particles with approximately equal masses ($x \approx y$). The following function $K(x,y) \in$ (4.13) can serve as an example:

$$K(x,y) = \alpha(x,y) + \begin{cases} \exp(\nu(2y-x)), & y \leq x \\ \exp(\nu(2x-y)), & y \geq x \end{cases}, \quad 0 \leq \nu \leq \lambda.$$

The function α is bounded.

For the coagulation kernel $K = xy$ without fragmentation Ernst, Ziff and Hendriks [28] and Galkin [36] found exact behaviour of the first moment of solution (see chapter 1). Consequently, this case conforms to the condition (4.16) of Theorem 4.2, and we have global uniqueness of solution. Uniqueness theorem for such coagulation model was proved by McLeod [53] for short time interval when the mass conservation law takes place (before gel creating). For the Flory-Stockmayer discrete model of polymerization with $K_{i,j} = (Ai + 2)(Aj + 2)$ Ziff and Stell [83] found the value of the first moment of solution for all $t \geq 0$. Consequently, in this case we obtain uniqueness of solution, too.

Recently Bruno, Friedman and Reitich [11] considered a special coagulation model. They succeded to prove uniqueness for bounded coagulation kernels only, though their existence theorem allows to concern unbounded ones. Approach of section 4.2 supplements their results and enables to prove uniqueness of solution for kernels from the class (4.13). We have also application of discrete versions of uniqueness Theorems 4.1 and 4.2 for the Becker-Döring equations (chapter 9).

In section 4.3 we follow results of Ball and Carr [5] and Stewart [70]. As we aware, this uniqueness theorem was the first one which treated the coagulation equation for unbounded kernels with a initial function from the class $\Omega_{0,1}$.

Chapter 5. SOME PROPERTIES OF SOLUTIONS

In this chapter we find some estimates for asymptotic behaviour of solutions to (3.1), (3.2). Existence and uniqueness of solutions are established in chapters 3 and 4.

5.1 Maximum principle

Let G be an open set in a metric space B whose closure $\overline{G}$ is a compact subset. We denote ∂G the boundary of G and let

$$C(T) = G \times (0,T]; \quad \overline{C}(T) = G \times [0,T]; \quad \partial\overline{C}(T) = (\overline{G} \times \{0\}) \bigcup (\partial G \times (0,T]).$$

So, $\partial\overline{C}(T)$ is the parabolic boundary of the cylinder $\overline{C}(T)$.

Theorem 5.1. *Let a continuous real functions $v(x,t)$, $q(t)$ and $a(t)$ be defined in the cylinder $\overline{C}(T)$ Let v have in $C(T)$ a continuous time derivative v_t. Let for each fixed $t \in (0,T]$ the points $\overline{x} \in G$ be the point of maximum, i.e.*

$$v(\overline{x},t) = \max_{x \in \overline{G}} v(x,t).$$

Suppose

$$v_t(\overline{x},t) \le q(t) - a(t)v(\overline{x},t), \quad 0 < t \le T, \quad x \in G. \tag{5.1}$$

Then

$$\max_{\overline{C}(T)} v(x,t) = \max_{\partial\overline{C}(T)} v(x,t).$$

Remark 5.1. *If, in addition, the derivative v_t is continuous in t in $\overline{G} \times (0,T]$ and the inequality (5.1) holds for $\overline{x} \in \overline{G}$ (not only for $\overline{x} \in G$), then*

$$\max_{\overline{C}(T)} v(x,t) = \exp\left(-\int_0^t a(s)ds\right) \max_{x \in \overline{G}} v(x,0) + \int_0^t \exp\left(-\int_s^t a(s_1)ds_1\right) q(s)ds. \tag{5.2}$$

Proof. Suppose that $a = q \equiv 0$. Then (5.1) can be rewritten as

$$v_t(\overline{x},t) \le 0, \quad 0 < t \le T, \quad x \in G. \tag{5.3}$$

Let us replace the correlation (5.3) onto more strong condition

$$v_t(\overline{x}, t) < 0. \tag{5.4}$$

Let us assume that the maximum value of function v is achieved in a point $(x_0, t_0) \in C(T)$. Then from (5.4) we immediately obtain that there exists a point $t_1 \in (0, t_0)$ such that

$$v(x_0, t_1) > v(x_0, t_0), \quad 0 < t_1 < t_0.$$

This contradiction proves that actually $(x_0, t_0) \notin C(T)$.

The final proof is based on the consideration the sequence of functions

$$v_n(x, t) = v(x, t) + \frac{1}{n}(T - t), \quad n \geq 1$$

and passing to limit $n \to \infty$.

If $q \neq 0$ or $a \neq 0$ then we introduce the auxuliary function

$$\tilde{v}(x, t) = \exp\left(\int_0^t a(s)ds\right) v(x, t) - \int_0^t \exp\left(\int_0^s a(s_1)ds_1\right) q(s)ds,$$

which satisfies (5.3), and iterate the above arguments. This proves Theorem 5.1. □

Let us consider a useful generalization of Theorem 5.1 which can be proved similarly.

Theorem 5.3. *Suppose the derivative* v_t *is continuous in* t *in* $\overline{G} \times (0, T]$ *and for* $\overline{x} \in \overline{G}$ *the following inequality holds*

$$v_t(\overline{x}, t) \leq 0 \quad \textit{if} \quad v(\overline{x}, t) \geq \max_{x \in \overline{G}} v(x, 0).$$

Then

$$\max_{\overline{C(T)}} v(x, t) = \max_{x \in \overline{G}} v(x, 0).$$

5.2 Application the maximum principle to the coagulation equation

In this section we are concerned with the coagulation equation with effluxes and sources

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy - c(x,t)\int_0^\infty K(x,y)c(y,t)dy -$$

$$-a(x,t) + q(x,t) \tag{5.5}$$

which is equipped the initial condition

$$c(x,0) = c_0(x). \tag{5.6}$$

We say that a continuous coagulation kernel satisfies the condition (M) if

$$0 \le K(x,y) = K(y,x) \qquad \text{and}$$

$$K(x-y,y) \le K(x,y), \quad 0 \le y \le \frac{1}{2}x, \quad 0 < x < \infty.$$

We should point out that the sufficient condition for validity of (M) is the increasing of the function $K(x,y)$ on x if $x \in [y,\infty)$ for each fixed $y > 0$.

The following coagulation kernels are of interest for describing real physial and chemical processes and can serve as examples of functions which satisfy the conditions (M):

1) 1; 2) $x+y$; 3) $|x-y|$; 4) $(x^{1/3}+y^{1/3})^2 \cdot |x^{1/3}-y^{1/3}|$; 5) $(x^{1/3}+y^{1/3})^3$;

6) $\left[M + \left(\dfrac{x}{y+a}\right)^{1/3} + \left(\dfrac{y}{x+a}\right)^{1/3}\right]^3, \quad a > 0, \quad M \ge 0.$

Theorem 5.2. *Let the kernel K satisfy the condition (M). Then the solution of the initial value problem (5.5), (5.6) satisfies the maximum principle: 1) on each interval $0 \le x \le b$*

$$\max_{0\le x\le b} c(x,t) \le \exp\left(-\int_0^t \min_{0\le x\le b} a(x,s)ds\right) \max_{0\le x\le b} c_0(x) +$$

$$+\int_0^t \exp\left(-\int_s^t \min_{0\le x\le b} a(x,s_1)ds_1\right)\max_{0\le x\le b} q(x,s)ds, \quad t\ge 0; \tag{5.7}$$

2) if $\sup_{0\le x<\infty} c_0(x)<\infty$ *and* $\sup_{0\le x<\infty} q(x,t)<\infty$ *for all* $t\ge 0$ *then*

$$\sup_{0\le x<\infty} c(x,t)\le \exp\left(-\int_0^t \inf_{0\le x<\infty} a(x,s)ds\right)\sup_{0\le x<\infty} c_0(x)+$$

$$+\int_0^t \exp\left(-\int_s^t \inf_{0\le x<\infty} a(x,s_1)ds_1\right)\sup_{0\le x<\infty} q(x,s)ds, \quad t\ge 0. \tag{5.8}$$

Proof. We fix arbitrary $t>0$. Let at the point $\overline{x}\in[0,b]$ the following equality hold

$$c(\overline{x},t)=\max_{0\le x\le b} c(x,t).$$

Then

$$\frac{\partial c(\overline{x},t)}{\partial t}=\int_0^{\overline{x}/2}\left[K(\overline{x}-y,y)c(\overline{x}-y,t)-K(\overline{x},y)c(\overline{x},t)\right]c(y,t)dy-$$

$$-c(\overline{x},t)\int_{\overline{x}/2}^{\infty} K(\overline{x},y)c(y,t)dy-a(\overline{x},t)+q(\overline{x},t).$$

Due to non-negativity c,K and the condition (M) we obtain

$$c_t(\overline{x},t)\le \max_{0\le x\le b} q(x,t)-\min_{0\le x\le b} a(x,t)\cdot c(\overline{x},t).$$

Using Remark 5.1 we come to (5.7). Since

$$\sup_{0\le x<\infty} c(x,t)=\lim_{b\to\infty}\max_{0\le x\le b} c(x,t),$$

then the assertion 2) follows from 1) as $b\to\infty$. This proves Theorem 5.2 □

Let us apply Theorem 5.2 to estimate solutions for sufficiently large x for pure coagulation equation ($a=q\equiv 0$). The estimates do not change in time. Let, as before, c be the solution to the problem (5.1), (5.2) and $f(x)$ be a real function on $[0,\infty)$ which is positive on $x>0$. The transform

$c \mapsto d$ where $d(x,t) = f(x)c(x,t)$, yields the following initial value problem for the function d :

$$\frac{\partial d(x,t)}{\partial t} = f(x)\left\{\frac{1}{2}\int_0^x K_1(x-y,y)d(x-y,t)d(y,t)dy - \right.$$

$$\left. - d(x,t)\int_0^\infty K_1(x,y)d(y,t)dy, \right\} \tag{5.9}$$

$$d(x,0) = d_0(x), \tag{5.10}$$

where $d_0(x) = f(x)c_0(x)$, and the modified coagulation kernel K_1 is

$$K_1(x,y) = \frac{K(x,y)}{f(x)f(y)}, \qquad x, y > 0.$$

If the kernel K satisfies the condition (M) and $\sup_{0\le x<\infty} d_0(x) = M < \infty$, then repeating the reasonings of Theorem th5.2 yields

$$\sup_{0\le x<\infty} d(x,t) \le M, \quad t \ge 0.$$

Consequently, we obtain the following estimate for solutions of the Cauchy problem (5.1), (5.2)

$$c(x,t) \le \frac{M}{f(x)}, \quad 0 < x < \infty, \quad t \ge 0. \tag{5.11}$$

Let us demonstrate how to obtain estimates of the type (5.11) for the interesting and important coagulation kernel $K(x,y) = |x-y|$. We put $f(x) = x^\alpha$ $(\alpha > 0)$. Let us look for the maximum value of the parameter α when the kernel

$$K_1(x,y) = x^{-\alpha}|x-y|y^{-\alpha} \tag{5.12}$$

is an increasing function with respect to x on $0 < y \le x < \infty$ (i.e. when the kernel $K_1(x,y)$ satisfies the condition (M)). That condition is equivalent to the inequality

$$\frac{d}{dx}\frac{x-y}{x^\alpha y^\alpha} \ge 0 \qquad \text{if} \quad 0 < y \le x < \infty.$$

Hence, the inequality $(1-\alpha)x+\alpha y \geq 0$ must be true for all $0 < y \leq x < \infty$. Thus, we should pick up $\alpha_{\max} = 1$. Consequently, the kernel (5.12) satisfies the condition (M) on $0 \leq \alpha \leq 1$ (for $\alpha > 1$ it is wrong). If the initial condition of the problem (5.1), (5.2) is such that

$$\sup_{0\leq x<\infty} x^{\alpha} c_0(x) = M < \infty \tag{5.13}$$

for a $al \in [0,1]$, then the solution of the coagulation equation with the kernel $K(x,y) = |x-y|$ obeys the inequality

$$c(x,t) \leq \frac{M}{x^{\alpha}}, \qquad 0 < x < \infty, \quad t \geq 0. \tag{5.14}$$

Analogiously we can show that the estimate (5.14) is valid for solutions of the initial value problem (5.1), (5.2) with the following coagulation kernels:

$$\begin{array}{llll} \text{a)} & x+y & \text{if} & 0 \leq \alpha \leq \frac{1}{2}; \\ \text{b)} & (x^{1/3}+y^{1/3})^3 & \text{if} & 0 \leq \alpha \leq \frac{1}{2}; \\ \text{c)} & (x^{1/3}+y^{1/3})^2 |x^{1/3}-y^{1/3}| & \text{if} & 0 \leq \alpha \leq \frac{27}{32}. \end{array}$$

For the case c) the upper bound for the parameter α can be more precise: it belongs to the interval $\frac{27}{32} < \alpha_{\max} < 1$.

In conclusion we should emphasize that the estimates (5.11) are uniform with respect to the time variable t and enable to make the judgement about decreasing of solutions as $x \to \infty$.

Let us consider more complicated coagulation-fragmentation equation (0.6). Iterating reasonings of Theorem 5.2 yields

$$\frac{\partial c(\overline{x},t)}{\partial t} = \int_0^{\overline{x}/2} \left[K(\overline{x}-y,y)c(\overline{x}-y,t) - K(\overline{x},y)c(\overline{x},t)\right] c(y,t)dy-$$

$$-\int_{\overline{x}}^{\infty} \left[K(\overline{x},y)c(\overline{x},t) - F(y-\overline{x},\overline{x})\right] c(y,t)dy-$$

$$-c(\overline{x},t)\int_{\overline{x}/2}^{\overline{x}} K(\overline{x},y)c(y,t)dy - c(\overline{x},t)\int_{0}^{\overline{x}} F(\overline{x}-y,y)dy.$$

Let there exists a positive constant m such that the functions K and F obey the following condition (M_1):

$$mK(x,y) \geq F(y-x,x) \quad \text{for all} \quad y \geq x.$$

Also, we suppose that condutions (M) hold. If there exists a time moment t such that $c(\overline{x},t) \geq m$ then we obtain

$$\frac{\partial c(\overline{x},t)}{\partial t} \leq 0.$$

Applying Theorem 5.3 yields the following result

Theorem 5.4. *Let the kernels K, F satisfy the conditions (M) and (M_1). Then the solution of the equation (0.6) satisfies the foolowing estimates:*
1) on each interval $0 \leq x \leq b$

$$\max_{0\leq x\leq b} c(x,t) \leq \max\{m, \max_{0\leq x\leq b} c_0(x)\}, \quad t \geq 0;$$

2) if $\sup_{0\leq x<\infty} c_0(x) < \infty$, then

$$\sup_{0\leq x<\infty} c(x,t) \leq \max\{m, \sup_{0\leq x<\infty} c_0(x)\}, \quad t \geq 0.$$

5.3 Positivity of solutions

In this section we discuss some results connected with positivity of solutions. A domain where a solution is stricktly positive, we call the positivity set of the solution. The totality of the positivity sets gives valuable information about the behaviour of $c(x,t)$, especially for small t. Hence this information will be valuable for numerical work in which the pure coalescence terms are evaluated, where each evaluation is for a small time step.

Let $P = \{x: \quad c_0(x) > 0\}$ and $Z = \{x: \quad c_0(x) = 0\}$, and define the n-th positivity set of the solution $c(x,t)$ as

$$Z^n = \left\{x: \quad \frac{\partial^k c_0(x)}{\partial t^k} = 0, \quad 0 \leq k \leq n-1; \quad \frac{\partial^n c_0(x)}{\partial t^n} > 0\right\}, \quad n \geq 1.$$

We also need to define the following operation between sets. If P and Q are arbitrary sets of non-negative real numbers, then their sum is defined as

$$P+Q=\{r: \quad r=p+q; \quad p\in P,\ q\in Q\}.$$

Thus the following notation is meaningful: $P_1=P,\ P_2=P+P,\dots,P_n=P_{n-1}+P$. We are in position to formulate the following theorem.

Theorem 5.5. *Let us suppose that there exists a unique non-negative continuous solution $c(x,t)$ to (0.3), (0.4) which is analytic in t for each x. Let $K(x,y)$ be strictly positive. Then the positivity sets of the solution $c(x,t)$ are given by*

$$Z^n=Z\bigcap(P_{n+1}-\cup_{i=1}^{n}P_i).$$

The proof of Theorem 5.5 is based on finding the form of nth derivative of $c(x,t)$ with respect to t. This derivative exists due to analyticity of the solution.

Corollary 5.1. $c(x,t)>0$ *for* $t>0$ *and for all* x *if and only if* $Z\subset\cup_{i=2}^{\infty}P_i$.

Example. Suppose that $c_0(x)$ is defined as

$$c_0(x)=\begin{cases}0, & 0\le x\le\frac{1}{2},\\ \frac{3}{2}(x-\frac{1}{2}), & \frac{1}{2}\le x\le 1,\\ \frac{3}{2}(\frac{3}{2}-x), & 1\le x\le\frac{3}{2},\\ 0, & x\ge\frac{3}{2}.\end{cases}$$

Hence, we have

$$P=\left(\frac{1}{2},\frac{3}{2}\right),\qquad Z=\left[0,\frac{1}{2}\right]\bigcup\left[\frac{3}{2},\infty\right).$$

Thus, $P_2 = (1,3)$, $P_3 = (\frac{3}{2}, \frac{9}{2})$, $P_4 = (2,6)\dots$ and $\cup_{i=2}^{\infty} P_i = (1,\infty)$. Therefore $Z \not\subset \cup_{i=2}^{\infty} P_i$ and by Corollary 5.1 there are $x, t > 0$ such that $c(x,t) = 0$. From Theorem 5.3 we have

$$Z^1 = \left[\frac{3}{2}, 3\right), \quad Z^2 = \left[3, \frac{9}{2}\right), \quad Z^3 = \left[\frac{9}{2}, 6\right)$$

and so on. Due to analyticity we can expand $c(x,t)$ in a series

$$c(x,t) = c_0(x) + \sum_{i=1}^{\infty} \frac{\partial^i c_0(x)}{\partial t^i} \frac{t^i}{i!}.$$

Hence the significance of the Z^n's is that coefficients of the above series are positive over Z^i but equal to zero in outer points. There we see that positivity sets contribute information about the evolution of $c(x,t)$, especially for small t.

Let us discuss the problem concerned for the coagulation equation with fragmentation. We consider the discrete equation (0.7). Suppose that both coagulation and fragmentation kernels satisfy the positivity condition

$$K_{1,i} > 0, \qquad F_{1,i} > 0, \qquad i \geq 1. \tag{5.15}$$

Then either the solution to (0.7) is trivial (zero for all arguments) or strictly positive for all $t > 0$. Namely, the following theorem holds.

Theorem 5.6. *Let (5.15) hold and c be a nonnegative continuous solution of (0.7) on $[0,T]$. Suppose that there exists $k \geq 1$ such that $c_k(0) > 0$. Then $c_i(t) > 0$ for all $t \in (0,T]$ and all $i \geq 1$.*

Proof. Suppose for contradiction that $c_i(\tau) = 0$ for some i and some $\tau \in (0,T]$. If $i > 1$ then since

$$\frac{dc_i(t)}{dt} = \vartheta_i(t) - c_i(t)\theta_i(t),$$

where

$$\vartheta_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j} c_{i-j} c_j + \sum_{j=1}^{\infty} F_{i,j} c_{i+j},$$

$$\theta_i(t) = \sum_{j=1}^{\infty} K_{i,j} c_j + \frac{1}{2} \sum_{j=1}^{i-1} F_{i-j,j},$$

we have that

$$0 = c_i(\tau) \exp\left(\int_0^\tau \theta(s)ds\right) = c_i(0) + \int_0^\tau \exp\left(\int_0^t \theta(s)ds\right) \vartheta_i(t)dt.$$

Hence

$$\frac{1}{2} \sum_{j=1}^{i-1} K_{i-j,j} c_{i-j}(t) c_j(t) = 0$$

for all $t \in [0, \tau]$, and thus either $c_{i-1}(\tau) = 0$ or $c_1(\tau) = 0$ for all j, $1 \leq j \leq i-1$. We obtain $c_{i-1}(\tau) = 0$ if $c_1(\tau) \neq 0$. We repeat similar arguments and obtain $c_{i-2}(\tau) = 0$ if $c_1(\tau) \neq 0$ and so on. Finally, we establish $c_1(\tau) = 0$.

For c_1 we have

$$\frac{dc_1(t)}{dt} = -c_1(t)\phi(t) + h(t), \quad t \in (0, T]. \tag{5.16}$$

where

$$\phi(t) = \sum_{j=1}^{\infty} K_{i,j} c_j(t), \quad h(t) = \sum_{j=2}^{\infty} F_{j-1,1} c_j(t). \tag{5.17}$$

It follows easily from (5.16) that

$$c_1(\tau) \exp\left(\int_0^\tau \phi(s)ds\right) = c_1(0) + \int_0^\tau \exp\left(\int_0^t \phi(s)ds\right) h(t)dt.$$

Hence $c_1(0) = 0$ and $h(t) = 0$ for all $t \in (0, \tau)$. Since each c_i is continuous, we obtain from (5.16) $c_i = 0$ for all $i \geq 2$, and thus $c(0) = 0$, a contradiction. Theorem 5.6 has now been proved □

It is interesting to point out that we have essentially used positivity of fragmentation kernel. If, e.g., we consider pure coagulation, then $h(t) = 0$ and we do not obtain the result. Theorem 5.5 claims that the assertion of Theorem 5.6 is not valid for pure coagulation.

5.4 Remarks

In Theorems 5.1, 5.2 we follow Galkin and Tupchiev [40] who proved them for pure coagulation case $a = q \equiv 0$. Theorem 5.5 is due to Melzak [55], Theorem 5.6 generalizes the similar result of Ball, Carr and Penrose [6] which they established for Becker-Döring equations. The proof of Theorem 5.6 iterates verbatim [6]. A number of other results closely connected with the contents of sections 5.1, 5.2 can be found in [35].

Chapter 6. TREND TO EQUILIBRIUM FOR CONSTANT COAGULATION AND FRAGMENTATION KERNELS

In this chapter we examine the convergence to equilibrium of the coagulation–fragmentation equation

$$\frac{\partial}{\partial t}c(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy - \int_0^\infty K(x,y)c(x,t)c(y,t)dy -$$

$$-\frac{1}{2}\int_0^x F(x-y,y)c(x,t)dy + \int_0^\infty F(x,y)c(x+y,t)dy, \tag{6.1}$$

$$c_0(x) = c(x,0) \geq 0 \tag{6.2}$$

for constant (not equal) coagulation and fragmentation kernels. The plan of the charter is as follows. Section 6.1 introduces the notation, definitions and compactness results we shall require. In section 6.2 the invariance principle for lower semicontinuous Lyapunov functionals introduced by Dafermos is extended to the weak L^1 topology. In Section sec6.3 we prove that if the kernels are constants then the equilibrium solution is unique for given initial data. Solutions are shown to be weakly relatively compact in $L^1(0,\infty)$ in section 6.4, via the Dunford-Pettis compactness theorem. This then allows us to apply the extension of the invariance principle derived in section 6.2 to a suitable semicontinuous Lyapunov functional and conclude that solutions tend weakly in $L^1(0,\infty)$ to a unique equilibrium solution whose form is related to the initial data and the relative magnitudes of the kernels K and F.

6.1 Preliminaries

In order to study solutions to (6.1) we introduce the usual Banach space L^1 and recall (see chapter 2)

$$\Omega_{0,1}^{+}(0) = \{f \in C \bigcap L^1(0,\infty) : \|f\|_{0,1}^{(0)} < \infty \ and \ f \geq 0 \ \} \tag{6.3}$$

where $\|f\|_{0,1}^{(0)} = \int_0^\infty (1+x)\,|f(x)|\,dx$. We will further often omit zero indexes and use just $\|f\|_{0,1}$, $\Omega_{0,1}^{+}$. The concept of gauge spaces will be used because the weak convergence required cannot be described by a metric.

Definition 6.1. *For any set Y a map is called a guage (or semimetric) on Y whenever*

(i) $d(x,y) \geq 0$ *for all x and y*

(ii) if $x = y$ *then* $d(x,y) = 0$

(iii) $d(x,y) = d(y,x)$ *for all x and y*

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ *for all x, y and z.*

A gauge has all the properties of a metric except that $d(x,y)$ can be zero for $x \neq y$. A family of gauges on Y, say $\Upsilon = \{d_\alpha : \alpha \in A\}$ for some index set A, is called separating if for each pair of points $x \neq y$ there is a $d_\alpha \in \Upsilon$ such that $d_\alpha(x,y) \neq 0$. The topological space arising from such a family Υ is called a Hausdorff gauge space (see Ash [3]). It is known that $f_n \to f$ weakly in $L^1(0,\infty)$ if and only if for each $\phi \in L^\infty(0,\infty)$

$$\int_0^\infty \phi(x) f_n(x) dx \to \int_0^\infty \phi(x) f(x) dx \text{ as } n \to \infty \tag{6.4}$$

(see Dunford and Schwartz [25], page 289). Defining

$$d_\phi(f,g) = \left| \int_0^\infty \phi(x)\{f(x) - g(x)\} dx \right| \tag{6.5}$$

for $\phi \in L^\infty(0,\infty)$ and $f, g \in L^1(0,\infty)$ it can easily be shown that weak convergence in L^1 is equivalent to convergence in the Hausdorff gauge space $\Upsilon = \{d_\phi : \phi \in L^\infty(0,\infty)\}$, the gauges d_ϕ being defined by (6.5).

The following compactness result, known as the Dunford-Pettis theorem [26], will be used in section 6.4 below:

Theorem 6.1. *Let R be the set of real numbers. For a subset P of functions contained in $L^1(R)$ to be weakly relatively compact it is necessary and sufficient that the following three conditions be fulfilled:*

(i)

$$\sup \left\{ \int_R |f| \, d\mu : f \in P \right\} < \infty \tag{6.6}$$

(ii) given $\epsilon > 0$ there exists a compact set K such that

$$\sup \left\{ \int_{R \setminus K} |f| \, d\mu : f \in P \right\} \leq \epsilon \tag{6.7}$$

(iii) given $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\sup\left\{\int_A |f|\,d\mu : f \in P\right\} \le \epsilon \tag{6.8}$$

whenever $\mu(A) \le \delta$, where $\mu(\cdot)$ denotes Lebesgue measure.

6.2 The Invariance Principle

Let Υ be the gauge space on L^1 introduced in section 6.1 above with the gauges being defined by equation (6.5). Let c be a solution to (6.1) with initial data c_0 and, for emphasis, denote this solution in L^1 by $\hat{c}(c_0,t)$. We introduce the following definitions:

Definition 6.2.
(i) The motion through c_0 is the time map $\hat{c}(c_0,t) : [0,\infty) \to \Omega^+_{0,1}$.
(ii) The positive orbit $O^+(\hat{c})$ of the motion through c_0 is the range of the map in part (i) for $t \ge 0$.
(iii) the ω-limit set of the motion through c_0 is defined by

$$\omega(c_0) = \{f \in L^1 : \hat{c}(c_0,t_n) \to f \text{ weakly in } L^1 \text{ for some sequence of times } t_n \to \infty\}.$$

Theorem 6.2. *If the positive orbit $O^+(\hat{c})$ through c_0 is weakly relatively compact in $L^1(0,\infty)$ then*
(i) $\omega(c_0)$ is nonempty,
(ii) $\hat{c}(c_0,t) \to \omega(c_0)$ weakly in $L^1(0,\infty)$ as $t \to \infty$,
(iii) $\omega(c_0)$ is positively invariant, i.e. for each $y \in \omega(c_0)$, $\hat{c}(y,t) \in \omega(c_0)$ for any $t \ge 0$.

Proof. (i) Let $\{t_n\}$ be a sequence in $[0,\infty)$ with $t_n \to \infty$. Then $\{\hat{c}(c_0,t_n)\}$ is a sequence contained in $O^+(c_0)$. Since $O^+(\hat{c})$ is relatively compact there is a subsequence $\{t_{n_k}\}$ such that $\{\hat{c}(c_0,t_{n_k})\}$ converges weakly to some function $y \in L^1$. By Definition 6.1 (iii) $y \in \omega(c_0)$ and therefore $\omega(c_0) \ne \emptyset$.

(ii) Suppose there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that for some $d_\phi \in \Omega$

$$d_\phi(\hat{c}(c_0, t_n), \omega(c_0)) \not\to 0. \tag{6.9}$$

As in part (i), since $O^+(\hat{c})$ is weakly relatively compact, there is a subsequence and a function $y \in L^1$ such that $\hat{c}(c_0, t_{n_k}) \to y$ weakly, that is, $d_\phi(\hat{c}(c_0, t_{n_k}), \omega(c_0)) \to 0$ for all $d_\phi \in \Upsilon$. This contradicts (6.9) since any subsequence of a weakly convergent sequence must converge to the same limit. Hence $d_\phi(\hat{c}(c_0, t), \omega(c_0)) \to 0$ as $t \to \infty$ for all $d_\phi \in \Upsilon$ and $\hat{c}$ converges weakly in L^1 to $\omega(c_0)$.
(iii) Let $y \in \omega(c_0)$. Then $\hat{c}(c_0, t_n) \to y$ weakly for some sequence $\{t_n\}$. Fix $t \geq 0$. From continuity in time t we have

$$\hat{c}(c_0, t_n + t) = \hat{c}(\hat{c}(c_0, t_n), t) \to \hat{c}(y, t) \tag{6.10}$$

weakly as $t_n \to \infty$. Thus by definition $\hat{c}(y, t) \in \omega(c_0)$ □

Definition 6.3. *A map $V : \Omega^+_{0,1} \to [0, \infty)$ is called a lower semicontinuous Lyapunov functional for the solution c if*
(i)

$$V(\hat{c}(c_0, t)) \leq V(c_0) \text{ for all } c_0 \in \Omega^+_{0,1} \text{ and } t \geq 0 \tag{6.11}$$

(ii) for any weakly convergent sequence $c_n \to c$ in L^1

$$V(c) \leq \liminf_{n\to\infty} V(c_n). \tag{6.12}$$

Theorem 6.3. *(Invariance Principle) Let V be a lower semicontinuous functional for the solution c. Suppose that $\omega(c_0) \neq \emptyset$ for some $c_0 \in \Omega^+_{0,1}$ and let $y \in \omega(c_0)$. Then for all $z \in \omega(c_0)$*

$$V(z) = V(y). \tag{6.13}$$

In particular, for any $y \in \omega(c_0)$ and $t \geq 0$

$$V(\hat{c}(y, t)) = V(y). \tag{6.14}$$

Proof. Let $y, z \in \omega(c_0)$. By Definition 6.2 (iii) there are sequences $\{t_n\}$, $\{s_m\}$ with $t_n \to \infty$, $s_m \to \infty$ such that as $n, m \to \infty$

$$\hat{c}(c_0, t_n) \to y \quad \text{weakly in} \quad L^1 \tag{6.15}$$

$$\hat{c}(c_0, s_m) \to z \quad \text{weakly in} \quad L^1. \tag{6.16}$$

Consider a subsequence $\{s_{m_n}\}$ of $\{s_m\}$ with $s_{m_n} \geq t_n + n$ and set $q_n = s_{m_n} - t_n$. Let $\hat{c}(y,t)$ be a solution to (6.1) with initial data $y \in \Omega^+_{0,1}$. As solutions are continuous in t we have that

$$\hat{c}(c_0, s_{m_n}) = \hat{c}(c_0, q_n + t_n) = \hat{c}(\hat{c}(c_0, t_n), q_n). \tag{6.17}$$

From the triangle inequality for gauges in Definition def6.1(iv) it follows from (6.15) that for each $d_\phi \in \Omega$

$$d_\phi(\hat{c}(y, q_n), z) \leq d_\phi(\hat{c}(y, q_n), \hat{c}(\hat{c}(c_0, t_n), q_n)) + d_\phi(\hat{c}(c_0, s_{m_n}), z). \tag{6.18}$$

Since (6.15) holds

$$d_\phi(\hat{c}(y, q_n), \hat{c}(\hat{c}(c_0, t_n), q_n)) \to 0. \tag{6.19}$$

In the gauge space Υ subsequences of convergent sequences converge to the same limit as the original sequence and hence by (6.16) as $n \to \infty$

$$d_\phi(\hat{c}(c_0, s_{m_n}), z) \to 0. \tag{6.20}$$

Thus by (6.18), (6.19) and (6.20) as $n \to \infty$

$$d_\phi(\hat{c}(y, q_n), z) \to 0 \tag{6.21}$$

and so by the arbitrariness of $\phi \in L^\infty$

$$\hat{c}(y, q_n) \to z \quad \text{weakly in } L^1 \quad \text{as} \quad n \to \infty. \tag{6.22}$$

It now follows from (6.22) and Definition 6.3 that

$$V(z) \leq \liminf_{n \to \infty} V(\hat{c}(y, q_n)) \leq V(y). \tag{6.23}$$

Interchanging the roles of y and z in the above argument shows $V(y) \leq V(z)$ and hence $V(y) = V(z)$.

Now fix $t \geq 0$. By Theorem th6.2 $\omega(c_0)$ is positively invariant and therefore $\hat{c}(y,t) \in \omega(c_0)$ if $y \in \omega(c_0)$. Setting $z = \hat{c}(y,t)$ in equation (6.13) yields (6.12) □

6.3 Uniqueness of Equilibrium

Before examining equilibrium solutions to (6.1) and (6.2) we make some general comments on the time dependent solutions. Multiplying (6.1) by x^i, $i = 0, 1$, and integrating over $[0, \infty)$, gives the moment equations for $N_i(t)$ defined for constant kernels K and F by

$$\frac{d}{dt}N_i(t) = \frac{d}{dt}\int_0^\infty x^i c(x,t)dx$$
$$= \frac{1}{2}\int_0^\infty\int_0^\infty [Kc(x,t)c(y,t) - Fc(x+y,t)] \times \left[(x+y)^i - x^i - y^i\right] dxdy \tag{6.24}$$

provided $N_2(0) < \infty$. This last condition, which we shall always assume, ensures boundness of the second moment $N_2(t)$ for all $t \geq 0$ (see chapter 3). For constant kernels the mass of the system, $N_1(t)$, must remain constant:

$$N_1(t) = \int_0^\infty xc_0(x)dx, \qquad t \geq 0. \tag{6.25}$$

Observing that

$$\int_0^\infty\int_0^\infty c(x+y,t)dydx = \int_0^\infty\int_x^\infty c(y,t)dydx = \int_0^\infty \left(c(y,t)\int_0^y dx\right) dy = N_1, \tag{6.26}$$

the differential equation for $N_0(t)$ (which corresponds to the number of particles in the system) is, by (6.24) and (6.25),

$$\frac{d}{dt}N_0(t) = -\frac{1}{2}KN_0^2(t) + \frac{1}{2}FN_1 \tag{6.27}$$

with $N_0(0) = \int_0^\infty c_0(x)dx$. Solving (6.27) shows that for any initial data $c_0 \in X^+$ we must have

$$\left|1 - \frac{2\sqrt{FN_1}}{N_0(t)\sqrt{K} + \sqrt{FN_1}}\right| = \left|1 - \frac{2\sqrt{FN_1}}{N_0(0)\sqrt{K} + \sqrt{FN_1}}\right| \exp\left(-t\sqrt{KFN_1}\right). \tag{6.28}$$

It follows that if we set $\lambda = F/K$ then

$$N_0(t) \to \sqrt{\lambda N_1} \; as \; t \to \infty. \tag{6.29}$$

Equation (6.28) further shows that if the initial data satisfies $N_0(0) = \sqrt{\lambda N_1}$ then $N_0(t) = N_0(0)$ for all $t \geq 0$. Although weak convergence to equilibrium will be proved in section 6.4 below, we remark that if it is assumed $c(\cdot) \to \overline{c}(\cdot)$ weakly in $L^1(0,\infty)$ as $t \to \infty$ for some equilibrium solution $\overline{c} \in L^1(0,\infty)$, then we necessarily have, from the definition of weak convergence,

$$N_0(t) = \int_0^\infty c(x,t)dx \to \int_0^\infty \overline{c}(x)dx. \tag{6.30}$$

Hence, by (6.29), any equilibrium solution $\overline{c}$ for (6.1) to which solutions may converge must satisfy the following relationship, which involves the initial data in terms of N_1

$$\int_0^\infty \overline{c}(x)dx = \sqrt{\lambda N_1}. \tag{6.31}$$

In order to prove that equilibrium solutions are unique for a given initial data we require the following lemma, which is a special case of that mentioned by Tricomi [74], page 12:

Lemma 6.1. *Let $A(y) \in L^1(0,m)$ for some $m > 0$ with $A(y) > 0$ a.e. and define for $n = 1,2,3,...$*

$$F_0(u) = 1,\ F_1 = \int_0^u A(y)dy,\ F_2(u) = \int_0^u A(y)F_1(y)dy, ...$$

$$...F_n(u) = \int_0^u A(y)F_{n-1}(y)dr.$$

Then

$$F_n(u) = \frac{1}{n!}F_1^n(u) \tag{6.32}$$

Proof. The result is clear for $n = 1$. Assume true for all integers up to $n-1$. Then, since $dF_1/dy = A(y)$ we see that

$$F_n(u) = \int_0^u A(y)F_{n-1}(y)dy = \frac{1}{(n-1)!}\int_0^u A(y)F_1^{n-1}(y)dy = \frac{1}{n!}\int_0^u \frac{dF_1^n}{dy}dy = \frac{1}{n!}F_1^n(u)$$

and the result follows by induction □

We are now in a position to prove that the equilibrium solution for (6.1) is unique.

Theorem 6.4. *Suppose that the kernels K and F are positive constants and set $\lambda = F/K$. Then equilibrium solutions to (6.1) are unique in $L^1(0,\infty)$ for each $c_0 \in X^+$. The unique equilibrium is given by*

$$\overline{c}(x) = \lambda \exp\left(-x\sqrt{\lambda/N_1}\right) \tag{6.33}$$

where N_1 is given in terms of the initial data by (6.25).

Proof. The equilibrium equation is, after rearranging terms and using the condition (6.31),

$$\overline{c}(x) = \frac{\lambda\sqrt{\lambda N_1}}{\sqrt{\lambda N_1} + x\lambda/2} + \frac{1}{\sqrt{\lambda N_1} + x\lambda/2}\int_0^x \overline{c}(y)\left(\frac{\overline{c}(x-y)}{2} - \lambda\right) dy. \tag{6.34}$$

Note that $\overline{c} \equiv 0$ cannot be an equilibrium solution unless $N_1 = 0$. Let f and g be two equilibrium solutions for non-zero initial data and choose $m > 0$. From equation (6.34) we have, by adding and subtracting $g(y)f(x-y)$ to the integrand and using the properties of convolutions,

$$\int_0^m |f(x) - g(x)|\, dx = \frac{1}{\sqrt{\lambda N_1} + x\lambda/2}\int_0^m \left|\int_0^x f(y)\left(\frac{f(x-y)}{2} - \lambda\right) - \right.$$

$$\left. -g(y)\left(\frac{g(x-y)}{2} - \lambda\right) dy \right| dx \leq \frac{1}{2\sqrt{\lambda N_1}}\int_0^m \int_0^x 2\lambda\,|f(y) - g(y)| +$$

$$+ |f(y) - g(y)|\,|f(x-y)| + |f(x-y) - g(x-y)|\,|g(y)|\, dydx \leq$$

$$\leq \frac{1}{2\sqrt{\lambda N_1}}\int_0^m \int_0^x |f(y) - g(y)|\,(2\lambda + |f(x-y)| + |g(x-y)|)\, dydx. \tag{6.35}$$

Making the change of variables $u = x - y$, $y = v$ and changing the order of the integrations, inequality (6.35) becomes

$$\begin{aligned}\int_0^m |f(x) - g(x)|\, dx &\leq \frac{1}{2\sqrt{\lambda N_1}}\int_0^m A(u)\int_0^{m-u} A(v)dvdu \\ &= \frac{1}{2\sqrt{\lambda N_1}}\int_0^m A(u)F_1(m-u)du\end{aligned} \tag{6.36}$$

where

$$A(u) = 2\lambda + |f(u)| + |g(u)| > 0 \tag{6.37}$$

and we have introduced $F_n(u)$ as defined in Lemma 6.1. Employing (6.37) we can re-insert the inequality (6.36) to show that

$$\int_0^m |f(x)-g(x)|\,dx \leq \frac{1}{2\sqrt{\lambda N_1}} \int_0^m A(u) \int_0^{m-u} A(y) \int_0^{m-u-y} |f(z)-g(z)|\,dzdydu$$

$$\leq \frac{1}{2\sqrt{\lambda N_1}} \int_0^m A(u) \int_0^{m-u} A(y) \int_0^{m-u} A(z)dzdydu = \frac{1}{2\sqrt{\lambda N_1}} \int_0^m A(u)F_2(m-u)du. \tag{6.38}$$

In conjunction with Lemma 6.1, repeated insertions of (6.36) demonstrate that for $n = 1, 2, 3, ...$

$$\begin{aligned}\int_0^m |f(x)-g(x)|\,dx &\leq \frac{1}{2\sqrt{\lambda N_1}} \int_0^m A(u)F_n(m-u)du \\ &= \frac{1}{2\sqrt{\lambda N_1}}\frac{1}{n!} \int_0^m A(u)\left(\int_0^{m-u} A(y)dy\right)^n du \\ &\leq \frac{1}{2\sqrt{\lambda N_1}}\frac{1}{n!}\left(\int_0^m A(y)dy\right)^{n+1}.\end{aligned} \tag{6.39}$$

Clearly, the right hand side of (6.39) tends to zero as $n \to \infty$ for any $A(y) \in L^1(0,m)$. Hence by the arbitrariness of m we must have $f(x) = g(x)$ a.e., that is, equilibria are unique. Since (6.31) holds, the unique equilibrium solution is found by inspection to be given by (6.33), as can be verified directly □

6.4 Weak Convergence to Equilibrium

This section begins by showing that all solutions $c(x,t)$ to (6.1) are weakly compact with respect to time t in L^1. This information will then enable us to apply the Invariance Principle to a suitable Lyapunov functional and hence prove that solutions converge weakly to the equilibrium solution (6.31).

Theorem 6.5. *Let $c(x,t)$ be any solution to (6.1) having initial data $c_0 \in \Omega^+_{0,1}$ and bounded second moment $N_2(0)$. Then the positive orbit $O^+(c)$ defined by*

$$O^+(c) = \bigcup_{t\geq 0}\left\{\int_0^\infty c(x,t)dx\right\} \tag{6.40}$$

is weakly relatively compact in $L^1(0,\infty)$.

Proof. We show that the three conditions of Theorem 6.1 (Dunford-Pettis) are satisfied.

(i) By equation (6.29) $\int_0^\infty c(x,t)dx$ converges as $t \to \infty$ and therefore there is a constant $L > 0$ such that

$$\sup\left\{\int_0^\infty c(x,t) : t \geq 0\right\} \leq L < \infty. \tag{6.41}$$

(ii) Choose $\epsilon \geq 0$. Then there exists $m > 0$ such that $N_1/m \leq \epsilon$ and therefore

$$\begin{aligned}\int_m^\infty c(x,t)dx &= \int_m^\infty \frac{x}{x}c(x,t)dx \\ &\leq \frac{1}{m}\int_m^\infty xc(x,t)dx \\ &\leq \frac{1}{m}N_1 \leq \epsilon.\end{aligned} \tag{6.42}$$

This inequality is uniformly true for all $t \geq 0$ and hence for the compact set $[0,m]$

$$\sup\left\{\int_{[0,\infty)\setminus[0,m]} c(x,t)dx : t \geq 0\right\} \leq \epsilon. \tag{6.43}$$

(iii) For any set $A \subseteq R$ define the characteristic function on A by

$$\chi_A(x) = \begin{cases} 1 x \in A \\ 0 x \notin A \end{cases} \tag{6.44}$$

and introduce the function u as

$$u(z,t) = \int \chi_A(x+z)c(x,t)dx, \tag{6.45}$$

$$u(z,0) = \int \chi_A(x+z)c_0(x)dx. \tag{6.46}$$

For convenience we define $c(x,t)$ to be zero whenever $x < 0$, which ensures that the integrals in (6.45) and (6.46) are defined on $(-\infty,\infty)$. Since c_0 is integrable it follows by absolute continuity that for

$$u(z,0) < \epsilon \quad \text{whenever} \quad \mu(A) < \delta_1 \tag{6.47}$$

uniformly in z for some $\delta_1 > 0$ where $\mu(\cdot)$ denotes Lebesque measure. Using the substitutions $X = x - y$, $Y = y$ and the non-negativity of solutions we have, upon reverting to the x and y notation,

$$\frac{\partial}{\partial t}u(z,t) \leq \iint \chi_A(x+y+z)\chi_{[0,x+y]}(y)c(x,t)c(y,t)dydx$$
$$+\int \chi_A(x+z)\int_x^\infty c(y,t)dydx - \int \chi_A(x+z)c(x,t)\int_0^\infty c(y,t)dy$$
$$\leq \frac{1}{2}\int_0^\infty c(y,t)dy \sup_{0\leq z\leq\infty}\int \chi_A(x+z)c(x,t)dx$$
$$+\mu(A)\int_0^\infty c(y,t)dy - u(z,t)\int_0^\infty c(y,t)dy$$
$$= \frac{1}{2}N_0(t)\sup_{0\leq z<\infty} u(z,t) + \mu(A)N_0(t) - N_0(t)u(z,t) \tag{6.48}$$

and hence

$$\frac{\partial}{\partial t}\left(u(z,t)\exp\left(\int_0^t N_0(s)ds\right)\right) \leq \frac{N_0(t)}{2}\exp\left(\int_0^t N_0(s)ds\right)\sup_{0\leq z<\infty} u(z,t)$$
$$+\exp\left(\int_0^t N_0(s)ds\right)N_0(t)\mu(A). \tag{6.49}$$

Integrating (6.49) results in the inequality

$$u(z,t)\exp\left(\int_0^t N_0(s)ds\right) \leq \int_0^t \frac{N_0(s)}{2}\exp\left(\int_0^s N_0(r)dr\right)$$
$$\times\left\{\sup_{0\leq z<\infty} u(z,s) + 2\mu(A)\right\}ds + u(z,0). \tag{6.50}$$

Taking supremums over z on both sides of (6.50) and setting

$$v(t) = \sup_{0\leq z<\infty} u(z,t)\exp\left(\int_0^t N_0(s)ds\right) \tag{6.51}$$

$$w(t) = \int_0^t N_0(s)\mu(A)\exp\left(\int_0^s N_0(r)dr\right)ds + \sup_{0\leq z<\infty} u(z,0) \tag{6.52}$$

inequality (6.50) becomes

$$v(t) \le \int_0^t \frac{N_0(s)}{2} v(s)ds + w(t). \tag{6.53}$$

Applying Gronwall's Inequality to (6.53) gives

$$\begin{aligned} v(t) \le w(0)\exp\left(\int_0^t \frac{N_0(s)}{2}ds\right) + \int_0^t \frac{d}{ds}g(s)\exp\left(\int_s^t \frac{N_0(\tau)}{2}d\tau\right)ds \\ = w(0)\exp\left(\int_0^t \frac{N_0(s)}{2}ds\right) \\ + \int_0^t N_0(s)\mu(A)\exp\left(\int_0^t N_0(r)dr\right)\exp\left(\int_s^t \frac{N_0(\tau)d\tau}{2}d\tau\right)ds. \end{aligned} \tag{6.54}$$

Multiplying the above inequality (6.54) by $\exp\left(-\int_0^t N_0(s)ds\right)$ and using the uniform bound L on solutions from equation (6.41) we arrive at the inequality

$$\sup_{0\le z<\infty} u(z,t) \le w(0) + \mu(A)L\int_0^t \exp\left(-\int_s^t \frac{N_0(\tau)}{2}d\tau\right)ds. \tag{6.55}$$

Since $N_0(t)$ converges to $\sqrt{\lambda N_1}$ by equation (6.29), $N_0(t)$ is bounded. Further, $N_0(t) \ne 0$ for any $t \ge 0$ by (6.28).

Hence there exists $K > 0$ such that

$$N_0(t) \ge K \tag{6.56}$$

uniformly in t. From (6.47) $w(0) < \epsilon$ whenever $\mu(A) < \delta_1$ and it therefore follows from (6.55) and (6.56) that there is a $\delta_2 > 0$ such that

$$\begin{aligned} \sup_{0\le z<\infty} u(z,t) \le \epsilon + \mu(A)L\int_0^t \exp\left(-\frac{K(t-s)}{2}\right)ds \\ = \epsilon + \mu(A)L\frac{2}{K}\left(1-\exp\left(-\frac{Kt}{2}\right)\right) \le \epsilon + \mu(A)L\frac{2}{K} \le Q\epsilon \end{aligned} \tag{6.57}$$

for some constant $Q > 0$ whenever $\mu(A) < \min\{\delta_1, \delta_2\}$. Hence there is a $\delta > 0$ such that

$$\begin{aligned} \int_A c(x,t)dx &= \int \chi_A(x)c(x,t)dx \\ &\le \sup_{0\le z<\infty}\int \chi_A(x+z)c(x,t)dx \\ &= \sup_{0\le z<\infty} u(z,t) \le \epsilon \end{aligned} \tag{6.58}$$

uniformly in t whenever $\mu(A) < \delta$.

From equations (6.41), (6.43) and (6.58) the three conditions of the Dunford - Pettis theorem hold and therefore the positive orbit $O^+(c)$ is weakly relatively compact in $L^1(0,\infty)$, that is, for any sequence of times $\{t_n\}$ there is a subsequence $\{t_{n_k}\}$ such that $c(\cdot, t_{n_k})$ converges in $L^1(0,\infty)$. This proves Theorem 6.5 □

With some additional assumptions imposed on solutions c we can introduce a Lyapunov functional.

Theorem 6.6. *Let $\lambda = F/K$, $c_0 > 0$ and suppose that for a given solution c*

(a) $c(\ln c - 1) \in L^1(0,\infty)$ for all $t \geq 0$,

(b) for each $t \geq 0$ there exists some $\Lambda \in L^1(0,\infty)$ and $\delta > 0$ such that

$$|c(x,t+h)(\ln c(x,t+h) - 1) - c(x,t)(\ln c(x,t) - 1)| \leq |h|\,\Lambda(x)$$

whenever $|h| < \delta$.
Then the functional

$$V(c) = \int_0^\infty c\left(\ln\frac{c}{\lambda} - 1\right) dx \tag{6.59}$$

is a weakly lower semicontinuous Lyapunov functional on $\Omega^+_{0,1}$, that is,
(i) for any solution c to (6.1) satisfying the initial condition (6.2)

$$V(c(\cdot,t)) \leq V(c_0(\cdot)) \text{ for all } t \geq 0, \tag{6.60}$$

and

(ii) if $c_n \to c$ weakly in $L^1(0,\infty)$ then

$$V(c) \leq \liminf_{n\to\infty} V(c_n). \tag{6.61}$$

Remark 6.2. *Conditions of Theorem 6.6 looks very naturally. In fact, If* $c_0(x) \le Ax^{-r}$, $r > 1$, *then (see chapter 3) for all* $t > 0$ $c \in \Omega_{0,r}$ *and we can expect that there exists a positive constant* A_1 *such that* $c(x,t) \le A_1 x^{-r}$. *Combining this fact with positivity of* $c(x,t)$ *and taking into account that*

$$\lim_{c \to 0} c \ln c = 0,$$

the function $c \ln c$ *increases for* $0 < c < e^{-1}$, *we obtain for discrete case of equation (6.1) the condition (a) of Theorem 6.6. For continuous one we need suppose its validity. The condition (b) follows from Lipshitz continuity of* $c(x,t)$ *in* t, *which is proved in Lemma lm3.3.*

Proof the Theorem 6.6. From the positivity of c_0 we easily obtain $c(x,t) > 0$ (cf. chapter 5 and Lemma 2.2) and, hence, conclude that $\ln c(x,t)$ exists for all $x \ge 0$ a.e.

(i) From the right hand side of (6.1) we can assume $\partial c/\partial t$ exists for $x \ge 0$ a.e.. Hence by hypothesis (a)

$$\frac{\partial c}{\partial t} \ln c = \frac{\partial}{\partial t} c(\ln c - 1) i \quad \text{exists for } x \ge 0 \text{ a.e. and } \quad c(\ln c - 1) \in L^1. \tag{6.62}$$

Employing standard results, hypothesis (b) in conjunction with (6.62) then gaurantees that $\frac{\partial}{\partial t} c(\ln c - 1) \in L^1$ and that we can evaluate dV/dt by differentiating under the integral sign. Straightforward calculations show that

$$\begin{aligned} \frac{d}{dt} V(c(\cdot,t)) = -\frac{1}{2} \int_0^\infty \int_0^\infty & [\ln(Kc(x,t)c(y,t)) - \ln(Fc(x+y,t))] \\ & \times [Kc(x,t)c(y,t) - Fc(x+y,t)]\, dxdy \\ & \le 0, \end{aligned} \tag{6.63}$$

this last inequality arising from the fact that $(\ln x - \ln y)(x - y) > 0$ for all $x, y > 0$. Integrating (6.63) between zero and t yields

$$V(c(\cdot,t)) \le V(c(\cdot,0) = V(c_0(\cdot)). \tag{6.64}$$

(ii) V can be written as

$$V(c) = \int_0^\infty c\,(\ln c - 1)\, dx - \ln(\lambda) \int_0^\infty c(x) dx. \tag{6.65}$$

The first integral in (6.65) is known to be *strongly* lower semicontinuous in $\Omega^+_{0,1}$ by Lemma 3.4 in [1]; this integral is also convex and therefore by standard results in convex analysis (see Ekeland and Temam [27], page 11) it must be weakly lower semicontinuous in $\Omega^+_{0,1}$. Since $c_n \to c$ weakly in L^1, $\int_0^\infty c_n dx \to \int_0^\infty c\, dx$. The right hand side of (6.65) is therefore weakly lower semicontinuous. This proves Theorem 6.6 □

We are now in a position to prove the main result of Chapter 6.

Theorem 6.7. *Let K and F be constant kernels with $\lambda = F/K$. Let $c > 0$ be a solution to (6.1) having its initial second moment $N_2(0)$ bounded and satisfying $c(x,0) = c_0(x) \in \Omega^+_{0,1}$. Suppose hypotheses (a) and (b) in Theorem 6.6 also hold. Then*

$$c \to \bar{c} = \lambda \exp\left(-x\sqrt{\lambda/N_1}\right) \tag{6.66}$$

weakly in $L^1(0,\infty)$ as $t \to \infty$ where $\bar{c}$ is the unique equilibrium solution to (6.1).

Proof. By Theorem 6.5 the positive orbit $O^+(c)$ is weakly relatively compact in $L^1(0,\infty)$, that is, to each sequence $\{t_n\}$ there exist a subsequence $\{t_{n_k}\}$ and a function $f \in L^1$ such that

$$c(\cdot, t_{n_k}) \to f(\cdot) \text{ as } t_{n_k} \to \infty \tag{6.67}$$

weakly in $L^1(0,\infty)$. By Theorem 6.2 the ω-limit set $\omega(c_0)$ is nonempty and

$$c(\cdot, t) \to \omega(c_0) \text{ as } t \to \infty \tag{6.68}$$

weakly in L^1. From Theorem 6.6 $V(c)$ is a semicontinuous Lyapunov functional for c. Hence by Theorem 6.3 $V(f)$ is constant for all $f \in \omega(c_0)$. From (6.63) $V(f)$ is constant implies that $Kf(x)f(y) = Ff(x+y)$ which in turn implies from (6.1) that all the elements $f \in \omega(c_0)$ are equilibria. From Theorem 6.4 the equilibrium solution is unique and so $\omega(c_0) = \{\bar{c}\}$ where, by (6.31), $f = \bar{c} = \lambda \exp(-x\sqrt{\lambda/N_1})$. It now follows from (6.68) that $c \to \bar{c}$ weakly in $L^1(0,\infty)$ as $t \to \infty$ □

6.5 Remarks

In this chapter we follow Stewart and Dubovskiĭ [SD] and extend to arbitrary constant kernels the work on convergence to equilibrium via a Lyapunov functional begun by Aizenman and Bak [1] who examined the case $K = F = 2$. Here, we weaken the convergence and prove that for constant kernels K and F (not necessarily equal) any solution to (6.1) with initial condition (6.2) must converge weakly to a unique equilibrium solution which is stated explicitely.

Global existence and uniqueness for solutions to (6.1) with (6.2) when $K = F = 2$ have been proved by Aizenman and Bak [1]. As mentioned in [1], it is possible to normalise the equation when the kernels are constants so that only equal kernels need be considered; but in this chapter we do not use such approach since we show directly the influence of the kernels upon the Lyapunov functional or other properties of solutions and we therefore examine (6.1) with independent kernels K and F. Much of the motivation for this chapter comes from the use of the invariance principle in Ball, Carr and Penrose [6] (see chapter 9) for Becker–Döring system of discrete equations we build upon some results from Aizenman and Bak and extend (to the weak topology) a version of the invariance principle proved by Dafermos [16, 17] for lower semicontinuous Lyapunov functionals. Weak convergence methods have previously been used in the study of (6.1) by Stewart [69].

The concept of gauge spaces was used in Stewart [69]. The Definitions 6.2, 6.3, Theorems 6.2, 6.3 extend the results by Dafermos [16, 17] to the gauge space Υ.

Chapter 7. EXISTENCE AND CONVERGENCE TO EQUILIBRIUM FOR LINEAR COAGULATION AND CONSTANT FRAGMENTATION KERNELS

7.1 Existence and uniqueness of an equilibrium solution

We are concerned with kernels of the form

$$K(x,y) = a + k(x+y) + g\ xy, \quad F = b \tag{7.1}$$

with nonnegative constants $a,\ b,\ k,\ g$. An equilibrium solution $\bar{c}(x)$ to (3.1) has to satisfy the following equation

$$\begin{aligned}\frac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)dy - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)dy\\ -\frac{1}{2}\bar{c}(x)\int_0^x F(x-y,y)dy + \int_0^\infty F(x,y)\bar{c}(x+y)dy = 0.\end{aligned} \tag{7.2}$$

Let

$$N_0 = \int_0^\infty \bar{c}(x)dx, N_1 = \int_0^\infty x\bar{c}(x)dx.$$

Then, integrating (7.2) and taking into account (7.1), we obtain

$$\frac{a}{2}N_0^2 + kN_0N_1 + (gN_1 - b)\frac{N_1}{2} = 0. \tag{7.3}$$

Therefore

$$N_0 = \frac{1}{a}\sqrt{(k^2 - ag)N_1^2 + abN_1} - \frac{kN_1}{a}.$$

From (7.3) we conclude that nonzero nonnegative equilibrium solutions cannot exist if $N_1 > b/g$. If $a = k = 0$ then $N_1 = b/g$. If $g = 0$ then an equilibrium solution may be exists for any $N_1 > 0$. Using (7.1) we may rewrite (7.2) in the form

$$\begin{aligned}\bar{c}(x) = \left(\frac{a}{2}\bar{c} * \bar{c}(x) + \frac{kx}{2}\bar{c} * \bar{c}(x) + \frac{g}{2}(x\bar{c}) * (x\bar{c})(x) + bN_0 - \bar{c} * b(x)\right)\\ \cdot\frac{1}{aN_0 + kN_1 + x(b/2 + kN_0 + gN_1)}.\end{aligned} \tag{7.4}$$

Our aim is to show that $\bar{c}$ as a solution to (7.4) is an equilibrium solution to (3.1). This will be proved at the end of section sec7.3. If we denote the right-hand side of (7.4) as $A(\bar{c})$, we obtain

$$|A\bar{c}_1 - A\bar{c}_2| \leq \frac{1}{aN_0 + kN_1}(\frac{a}{2}|\bar{c}_1 - \bar{c}_2| * |\bar{c}_1 + \bar{c}_2| + \frac{kx}{2}|\bar{c}_1 - \bar{c}_2| * |\bar{c}_1 + \bar{c}_2| + \frac{g}{2}|x\bar{c}_1 - x\bar{c}_2| * |x\bar{c}_1 + x\bar{c}_2| + |\bar{c}_1 - \bar{c}_2| * b).$$

Let us consider the operator A as a mapping of the Banach space $C[0, \alpha]$ onto itself. Then we obtain

$$\|A\bar{c}_1 - A\bar{c}_2\| \leq \|\bar{c}_1 - \bar{c}_2\| \frac{\alpha}{aN_0 + kN_1} \left[\frac{1}{2}\left(a + \alpha k + \alpha^2 g\right) \|\bar{c}_1 + \bar{c}_2\| + b\right].$$

Hence, the operator A is contractive if

$$\|\bar{c}\| < \frac{aN_0 + kN_1 - \alpha b}{\alpha(a + \alpha k + \alpha^2 g)} \stackrel{\text{def}}{=} R_\alpha.$$

To use the contraction mapping theorem [funkan] we have to check if the ball $B(R_\alpha)$ is invariant. We obtain

$$\|A\bar{c}\| \leq \frac{1}{2(aN_0 + kN_1)} \left[(\alpha a + \alpha^2 k + \alpha^3 g)\|\bar{c}\|^2 + 2bN_0 + 2\alpha b\|\bar{c}\|\right].$$

By solving the inequality $\|A\bar{c}\| \leq \|\bar{c}\|$ we may see that the ball $B(R_\alpha)$ remains invariant if

$$\|\bar{c}\| \leq \frac{aN_0 + kN_1 - \alpha b + \sqrt{(aN_0 + kN_1 - \alpha b)^2 - 2\alpha bN_0(a + \alpha k + \alpha^2 g)}}{\alpha(a + \alpha k + \alpha^2 g)}$$

whence we obtain that the square root expression should be nonnegative. This condition with $R_\alpha > 0$ allows us to find a suitable value of α. Now we are in a position to prove the following lemma:

Lemma 7.1. *Let b satisfy the above-mentioned conditions. Then there exists a unique solution to (7.4) on the interval $[0, \alpha]$ which is continuous and belongs to the ball $B(R_\alpha)$.*

Proof. Existence and uniqueness of a continuous solution $\bar{c}$ in the ball $B(R_\alpha)$ follows from the contraction mapping theorem. We now prove global uniqueness only. Suppose that there exists another solution $\bar{e}$ to (7.4). Its continuity follows from its integrability and remark that the operator A maps any integrable function to a continuous one. Let us consider the restriction of $\bar{e}$ to an interval $[0,\varepsilon], \varepsilon < b$. Choosing ε small enough we obtain that the ball $B(R_\varepsilon)$ contents two solutions $\bar{c}$ and $\bar{e}$. (Actually, R_ε tends to infinity as $\varepsilon \to 0$.) This result contradicts the uniqueness in this ball. This proves Lemma 7.1 □

Our next step is to extend the solution obtained to the interval $[0,\infty)$.

Lemma 7.2. *There exists a unique continuous solution to (7.4) for all* $x \geq 0$.

Proof. Let us consider the operator A as a mapping $A : C[\alpha, 2\alpha] \to C[\alpha, 2\alpha]$ and denote by $d(m)$ a solution of (7.4) on $[\alpha, 2\alpha]$. The function $d(x)$ obeys the equality

$$d(x) = \frac{1}{aN_0 + kN_1 + x(b/2 + kN_0 + gN_1)} \left[(a+kx) \int_\alpha^x d(y)\bar{c}(x-y)dy + \right.$$

$$+ \frac{a+kx}{2} \int_{x-\alpha}^{\alpha} \bar{c}(x-y)\bar{c}(y)dy + g \int_\alpha^x y(x-y)d(y)\bar{c}(x-y)dy +$$

$$\left. + \frac{g}{2} \int_{x-\alpha}^{\alpha} y(x-y)\bar{c}(x-y)\bar{c}(y)dy + b\left(N_0 - \int_0^\alpha \bar{c}(x)dx - \int_\alpha^x d(y)dy\right) \right]. \quad (7.5)$$

Here the function $\bar{c}$ is the solution to (7.4) on $[0,\alpha]$. Its existence and uniqueness were proved in Lemma 7.1. By standard results on integral equations, the linear Volterra equation (7.5) has a unique continuous solution $d(x)$ on the interval $[\alpha, 2\alpha]$. Put $\bar{c}(x) = d(x)$ if $\alpha < x \leq 2\alpha$. Obviously, $\bar{c}$ satisfies (7.4) for all $x \in [0, 2\alpha]$. Its continuity follows from the proof of Lemma 7.1. Now we can analogously extend the solution obtained to $[2\alpha, 4\alpha]$ and so on. From uniqueness on $[\alpha, 2\alpha]$ it follows also that the solution constructed has no branch points, otherwise we can choose b on a branch point. This completes the proof of Lemma 7.2 □

Remark 7.1. *It follows from the proof of Lemma 7.1 that the function* $\bar{c}$ *is infinitely many times differentiable.*

Remark 7.2. *The integrability and positivity of $\bar{c}$ are not proved yet. These properties will be discussed later. It is worth pointing out the importance of the non-zero term bN in the numerator of the right-hand side (7.4). If we had replaced bN with $b\int_0^\infty \bar{c}(x)dx$ then the contractions $A\bar{c}$ would tend to the trivial zero steady solution and we would not obtain the nontrivial solution by this approach. If $b = 0$ then by our uniqueness result only the zero continuous equilibrium solution is possible. It is also worth pointing out that the continuity condition is essential, because there are examples of nonzero discontinuous steady solutions for the pure coagulation equation* [23].

7.2 Strong Linear Stability

Equilibrium solutions to (7.4) are denoted by $\bar{c}(x)$. Let us assume that $g = 0$, that is we consider further the kernels

$$K(x,y) = a + k(x+y), \qquad F = b. \tag{7.6}$$

In this case there is no prohibition for equilibria for any $M > 0$, as is pointed out in section sec7.1. In accordance with Theorem th3.1 the initial value problem (3.1),(3.2) has a mass conserving nonnegative solution $c(x,t)$ if the initial function c_0 is continuous and has bounded moments. Therefore $c(x,t)$ can converge as $t \to \infty$ to the equilibrium with the same total mass N_1:

$$\int_0^\infty x c_0(x)dx = \int_0^\infty x\bar{c}(x)dx = N_1. \tag{7.7}$$

This reason forces us to consider the case $g = 0$, otherwise we cannot warrant the mass conservation law. Let us show that

$$N_0(t) \to N_0 \quad \text{as} \quad t \to \infty. \tag{7.8}$$

Here and further $N_0(t)$ denotes the moment of the time-dependent solution unlike N_0. The integration (3.1) yields

$$\frac{dN_0(t)}{dt} = -\frac{a}{2}N_0^2(t) - kN_1N_0(t) + \frac{b}{2}N_1. \tag{7.9}$$

By using (7.3) and solving (7.9) we obtain

$$|N_0(t) - z_1| \cdot |N_0(t) - z_2| = |N_0(0) - z_1| \cdot |N_0(0) - z_2| \exp(-\frac{1}{2}at)$$

where the constants z_1, z_2 are the roots of the quadratic equation

$$\frac{1}{2}az^2 + kN_1 z - \frac{1}{2}bN_1 = 0.$$

Hence, we obtain (7.8). We have obtained also that if $N_0(0)$ satisfies (7.3) then $N_0(t) = N_0$ for all $t \geq 0$. The value of N_1 in (7.3) is defined by the initial distribution c_0 in (7.7).

To examine the general convergence of the solution $c(x,t)$ to $\bar{c}(x)$ where $\bar{c}$ is the solution of (7.4), the function $f = c - \bar{c}$ is introduced. The substitution of $f(x,t)$ into (3.1) using (7.4),(7.6),(7.7) gives us

$$\frac{\partial f}{\partial t} = (a + kx)(\frac{1}{2}f * f + f * \bar{c} - fN_0(t))$$
$$-kfN_1 - \frac{1}{2}bfx - b * f + (b - a\bar{c} - kx\bar{c})(N_0(t) - N_0) \tag{7.10}$$

with $f(x,0) = f_0(x) = c_0(x) - \bar{c}(x)$. Our main aim now is to show that $f(x,t) \to 0$ as $t \to \infty$. If we consider $u(x,t)$ to be a linear perturbation of $f(x,t)$ then (7.10) can be linearised giving

$$u_t = (a + kx)u * \bar{c} - auN_0(t) - kxuN_0 - kuN_1 - \frac{1}{2}bxu - b * u,$$
$$u(x,0) = u_0(x). \tag{7.11}$$

Taking the Laplace transform of (7.11) we come to the partial differential equation for the Laplace transform $U(p,t)$ of $u(x,t)$:

$$U_t + (k\overline{C} - kN_0 - \frac{1}{2}b)U_p = (a\overline{C} - k\overline{C}_p - \frac{b}{p} - aN_0(t) - kN_1)U. \tag{7.12}$$

By the substitution

$$U = \exp(-a\int_0^t N_0(s)ds - kN_1 t)W \tag{7.13}$$

we obtain from (7.12)

$$W_t + (k\overline{C} - kN_0 - \frac{1}{2}b)W_p = (a\overline{C} - k\overline{C}_p - \frac{b}{p})W. \tag{7.14}$$

The characteristic equation for (7.14) is of the form

$$dt = \frac{dp}{k\overline{C}(p) - kN_0 - b/2} = \frac{dW}{(a\overline{C}(p) - k\overline{C}(p) - b/p)W}. \tag{7.15}$$

By solving (7.15) and denoting for a fixed $p_0 \geq 0$

$$I(p) = \int_{p_0}^{p} \frac{dq}{k\overline{C}(q) - kN_0 - b/2},$$

we obtain, with (7.13) taken into account,

$$\begin{aligned} U(p,t) = \exp(-a\int_0^t N_0(s)ds - kN_1 t)U_0(i(p,t)) \\ \cdot \exp\left(\int_p^{i(p,t)} \frac{a\overline{C}(q) - k\overline{C}_p(q) - b/q}{kN_0 + b/2 - k\overline{C}(q)}dq\right) \end{aligned} \tag{7.16}$$

where

$$i(p,t) = I^{-1}(I(p) + t).$$

Here I^{-1} is the inverse function of I. The existence of I^{-1} is warranted by the increasing monotonicity of the function I. For any fixed $t > 0$ the integral $\int_p^{i(p,t)}$ in (7.16) decreases in p for all $p > p_0$ due to the decreasing of $\overline{C}$ and $-\overline{C}_p$. Increasing of both functions I and I^{-1} means that the decreasing of $U_0(i(p,t))$ in p is not less than the decreasing of $U_0(p)$ because $i(p,t) \geq p$. Therefore there exists an inverse Laplace transform $u(x,t)$ of the right-hand side in (7.16) which is the solution of (7.11) and we have for a positive constant G the following estimate:

$$\|u(.,t)\|_C \leq G\exp\left(-a\int_0^t N_0(s)ds - kN_1 t\right)\|u_0\|_C \tag{7.17}$$

where norms are from the space $C[0,B]$ for any fixed $0 < B < \infty$. The constant G depends on B but does not depend on t. Hence $u(x,t) \to 0$ strongly in $C[0,B]$ as $t \to \infty$, that is, the equilibrium solution $\bar{c}$ is (exponentially) strongly asymptotically stable in $C[0,B]$.

Remark 7.3. *We need to consider $B < \infty$ because we do not know at this point whether the function $\bar{c}$ belongs to the space $L^1[0,\infty)$ or $L^\infty[0,\infty)$. We prove these important properties of $\bar{c}$ in the next section.*

Example. For the simple case $k = 0$, $a = b = 1$ with $\overline{C}(p) = (p+\lambda)^{-1}$ we find

$$I(p) = 2(p - p_0), \qquad i(p,t) = p + t/2$$

and (7.16) is replaced by

$$U(p,t) = U_0(p + t/2)\exp(-t/\lambda)\frac{p^2(p+t/2+\lambda)^2}{(p+t/2)^2(p+\lambda)^2}.$$

The inverse Laplace transform gives us

$$u(x,t) = \exp(-\frac{1}{2}xt - t/\lambda)u_0(x) - \lambda t\exp(-\frac{1}{2}xt - t/\lambda)$$
$$\times\left[u_0(x) * (A(t) + xB(t))\exp(-\lambda x) + u_0(x) * (B(t) - A(t))\exp(-\frac{1}{2}xt)\right]$$

where

$$A(t) = (t/2 - \lambda)^{-1} + \frac{1}{2}\lambda t(t/2 - \lambda)^{-3},$$

$$B(t) = -\frac{1}{4}\lambda t(\lambda - t/2)^{-2}$$

provided $t \neq 2\lambda$. If $t = 2\lambda$ then we obtain

$$u(x,2\lambda) = \exp(-\lambda x)\left[u_0(x) - 2\lambda^2 u_0(x) * (x\exp(-\lambda x)) + \frac{1}{6}\lambda^4 u_0(x) * (x^3\exp(-\lambda x))\right].$$

For this example the estimate (7.17) is more descriptive.

7.3 Nonlinear estimates for solutions and convergence

We are now ready to exlpoit the estimate (7.17). Let us denote $u(x,t) = T_t u_0(x)$ where $u(x,t)$ is the solution of equation (7.11) and T_t is the resulting semigroup operator. From the inequality (7.11) we obtain for the usual semigroup norm

$$\|T_t\| = \sup_{\|u_0\|_C \leq 1} \|T_t u_0\|_C \leq G\exp(-a\int_0^t N_0(s)ds - kN_1 t) \leq G\exp(-\nu t), \tag{7.18}$$

$$0 < \nu \leq kN_1 + a \inf_{t>0} t^{-1} \int_0^t N_0(s)ds = kN_1 + a\min\{N_0(0), N_0\}. \tag{7.19}$$

The nonlinear initial value problem (7.10) can now be written in integral form (similar to the case in [D90-2]) as

$$f(x,t) = T_t f_0 + \int_0^t T_{t-s}\left[\frac{1}{2}(a+kx)f * f(.,s) \right. \\ \left. +(b - a\bar{c} - kx\bar{c} - kxf(.,s))(N_0(s) - N_0)\right] ds. \tag{7.20}$$

We now introduce the norm

$$\|f\|_\nu = \sup_{t\geq 0} \exp(\nu t)\|f(.,t)\|_C. \tag{7.21}$$

If the right-hand side of the equation (7.20) is denoted by $D(f(.,t))$ then clearly for any fixed $t \geq 0$ D maps $C[0,B]$ into itself. Expressions (7.18) and (7.20) yield

$$\|D(f(.,t))\|_C \leq G\exp(-\nu t)(\|f_0\|_C + \int_0^t \exp(\nu s)(\frac{1}{2}(a+kB)B\|f(.,s)\|_C^2 \\ + \sup_{x\in[0,B]} |b - a\bar{c} - kx\bar{c}| \cdot |N_0(s) - N_0| + kx\|f(.,s)\|_C|N_0(s) - N_0|)ds). \tag{7.22}$$

Multiplying (7.22) by $\exp(\nu t)$ we establish the correlation

$$\|D(f)\|_\nu \leq G\|f_0\|_C + \frac{GB}{2\nu}(a+kB)\|f\|_\nu^2 + Ga_1 + a_2\|f\|_\nu. \tag{7.23}$$

where

$$a_1 = \sup_{x\in[0,B]} |b - a\bar{c} - kx\bar{c}| \int_0^\infty \exp(\nu s)|N_0(s) - N_0|ds$$

and

$$a_2 = GkB \int_0^\infty |N_0(s) - N_0| ds.$$

From (7.22) it is possible to reveal that if

$$\|f_0\|_C + a_1 \le \frac{\nu(1-a_2)^2}{2G^2 B(a+kB)} \tag{7.24}$$

and

$$a_2 < 1 \tag{7.25}$$

then the mapping D has an invariant ball in $C[0,B]$ with radius η satisfying $\eta_1 \le \eta \le \eta_2$ where η_1 and η_2 are the real positive roots of the quadratic equation

$$\frac{GB}{2\nu}(a+kB)z^2 - (1-a_2)z + G(\|f_0\|_C + a_1) = 0. \tag{7.26}$$

In fact, if $\|f\|_\nu \le \eta$ for some $\eta \in [\eta_1, \eta_2]$, then from (7.23) we obtain

$$\|D(f)\|_\nu \le G\|f_0\|_C + \frac{GB}{2\nu}(\alpha + \delta B)\eta^2 + Ga_1 + a_2\eta \le \eta \tag{7.27}$$

which follows from the facts that $\eta_1 \le \eta_2$ and the conditions (7.23)–(7.25) hold. We now try to find conditions for D to be a contraction in $C[0,B]$. For any f_1 and f_2 it follows from (7.18) and (7.20) that

$$\begin{aligned} \|D(f_1) - D(f_2)\|_C &\le \\ \frac{1}{2}G(a+kB)\int_0^t &\exp(-\nu(t-s))\|(f_1 - f_2) * (f_1 + f_2)\|_C ds \\ +kGB\int_0^t &\exp(-\nu(t-s))|N_0(s) - N_0| \cdot \|f_1 - f_2\|_C ds \\ &\le \frac{BG}{2\nu}(a+kB)\exp(-\nu t)\|f_1 - f_2\|_\nu(\|f_1\|_\nu + \|f_2\|_\nu) \\ &\quad + a_2 \exp(-\nu t)\|f_1 - f_2\|_\nu. \end{aligned} \tag{7.28}$$

If the functions f_1 and f_2 belong to a ball with radius η, that is, $\|f_1\|_\nu \le \eta$ and $\|f_2\|_\nu \le \eta$, then from (7.27) we obtain

$$\|D(f_1) - D(f_2)\|_\nu \le \left(\frac{BG\eta}{\nu}(a+kB) + a_2\right) \cdot \|f_1 - f_2\|_\nu. \tag{7.29}$$

Thus the mapping D is a contraction mapping in the ball with radius

$$\eta < \frac{(1-a_2)\nu}{BG(a+kB)} \stackrel{\text{def}}{=} \eta_0.$$

From equation (7.26) η_1 and η_2 are given by

$$\eta_{1,2} = \frac{(1-a_2)\nu}{BG(a+kB)} \cdot \left(1 \pm \sqrt{1 - \frac{2G^2B(\alpha+\delta B)(\|f_0\|_C + a_1)}{\nu(1-a_2)^2}}\right)$$

and hence the bound of contraction belongs to the closed interval $[\eta_1, \eta_2]$. From standard arguments using (7.26), (7.28) and the contraction mapping theorem ([80]) we see that there exists a solution of the initial value problem (7.10) which is unique in the ball of radius $\|f\|_\nu \leq \eta_0$ and belongs to the ball of radius $\|f\|_\nu \leq \eta_1 < \eta_0$. Moreover, this solution tends to zero not slower than $\exp(-\nu t)$.

From the nonnegativity of $c(x,t)$ as a solution to (3.1),(3.2) and its trend to $\bar{c}(x)$ we can easily see that the function $\bar{c}$ is nonnegative. Using the mass conservation law and the nonnegativity of functions $\bar{c}$ and $c(x,t)$ we now see that $\bar{c}$ is integrable with weight x on all $[0,\infty)$ and, in addition,

$$\int_0^\infty x\bar{c}(x)dx \leq N_1. \tag{7.30}$$

By integrating (7.4) directly, we find that

$$\int_0^\infty \bar{c}(x)dx = N_0$$

otherwise the right-hand side of (7.4) cannot be integrated. Taking (7.3) into account we also obtain that

$$\int_0^\infty x\bar{c}(x)dx = N_1.$$

Therefore the function $\bar{c}$ is indeed the solution to (7.2) with the kernels (7.6). Using Lemma 7.2 we can now prove the following theorem:

Theorem 7.1. *Let the conditions of Theorem 3.1 hold and kernels K, F satisfy (7.6). Then*
(1) there exists a unique nonnegative continuous equilibriium solution to (3.1) with the first moment bounded;
(2) the time-dependent solution tends to this equilibrium in $C[0,B]$ for any $0 < B < \infty$ and in $L^1[0,\infty)$ as $t \to \infty$ if the estimates (7.24) and (7.25) hold. The rate of the convergence is proportional to $\exp(-\nu t)$ where ν is defined in (7.19).

Remark 7.4. *The estimates (7.24) and (7.25) mean smallness of difference between the initial function and the equilibrium.*

Proof of Theorem 7.1. Case (1) and convergence in $C[0,B]$ were proved above. To prove convergence in $L^1[0,\infty)$ it suffices to note that "tails" of the integral of $c(x,t)$ are bounded uniformly in t thanks to (3.42). By increasing the constant B we obtain the desired result, which completes the proof of Theorem 7.1 □

7.4 Remarks

In this chapter we follow Dubovskiĭ and Stewart [24]. Using approach of Ball, Carr and Penrose [6] (see chapter 9), Carr [14] studied the discrete coagulation–fragmentation equation and proved convergence to equilibrium. He assumed that $K_{i,j} \le k(i^\alpha + j^\alpha)$ where $\alpha < \gamma$ and the constant γ is defined by the fragmentation kernel via the inequality

$$\sum_{j=1}^{i/2} j^r F_{i-j,j} \ge C(r) i^{r+\gamma}, \; i \ge 3, r \ge 0.$$

$C(r)$ is a constant. For $F \equiv$const we have $\gamma = 1$ and, hence, the coagulation kernel has less than linear growth (in chapter 7 it is linear). Also, there must exist a sequence Q_i such that

$$K_{i,j} Q_i Q_j = F_{i,j} Q_{i+j}, \quad i,j \ge 1. \tag{7.31}$$

The condition (7.31) immediately gives us an equilibrium solution $c_i = Q_i c_1^i$ so that each pair $K_{i,j} c_i c_j - F_{i,j} c_{i+j}$ yields zero. In chapter 7 kernels do not satisfy (7.31) and therefore the proof of existence of an equilibrium is more difficult. It is worth noting that for constant coagulation and fragmentation kernels (which we treat in chapter 6) such sequence Q_i exists.

Chapter 8. CONVERGENCE TO EQUILIBRIUM FOR COAGULATION EQUATION WITH SOURCES

In this chapter we are concerned with the coagulation equation with sources

$$\frac{\partial c(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)dy-$$

$$-c(x,t)\int_0^\infty K(x,y)c(y,t)dy + q(x), \quad x \geq 0, t > 0, \tag{8.1}$$

$$c(x,0) = c_0(x) \geq 0, \quad x|geq0. \tag{8.2}$$

Solvability and uniqueness of the time-dependent problem (8.1), (8.2) for coagulation kernels (3.3) was proved in chapters 3 and 4. Our aim is to reveal some properties of the equilibrium solution and prove convergence of the time-dependent solution to the stationary one.

8.1 Properties of the stationary solutions

The stationary form of the equation (8.1) is

$$\frac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)dy - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)dy + q(x) = 0, \quad x \geq 0. \tag{8.3}$$

Let $\bar{c}(x)$ be its nonnegative measurable solution for which the integrals in (8.3) are bounded for anu $x \geq 0$. Obviously, for the coagulation kernel $K(x,y)v(x)v(y)$ the solution of (8.3) is $\bar{c}(x)/v(x)$ for any function $v(x) \geq 0$.

Integrating (8.3) with the weight x yields

$$\int_0^\infty\int_0^\infty xK(x,y)\bar{c}(x)\bar{c}(y)dxdy = \infty. \tag{8.4}$$

In fact, otherwise the first and the second summands in (8.3) which become equal to (8.4), yield zero, and we come to the contradiction with the positivity of

$$\int_0^\infty xq(x)dx.$$

From (8.4) we conclude that if $K(x,y) \leq M = \text{const}$ then the first moment of the function $\bar{c}(x)$ is unbounded. From physical point of view this simple result is very natural: long-time influx of particles in the disperse system brings up the infinite total mass. Nevertheless, the total amount of particles expressed by the zero moment of $\bar{c}(x)$, may be bounded. For instance, the boundedness of the zero moment takes place for a constant coagulation kernel (see below). For the kernels which describe weak coagulation (e.g. $K(x,y) = \exp(-x-y)$), the zero moment can be infinite similarly to the first one. We define the moments of the solution as

$$N_\alpha = \int_0^\infty x^\alpha \bar{c}(x)dx.$$

If we restrict ourselves with solutions $\bar{c}(x)$ with bounded zero and unbounded first moments, then the natural question arises: "When the α-th moment of the equilibrium solution becomes unbounded?" The following theorem gives the answer to this question.

Theorem 8.1. *Let symmetric nonnegative continuous coagulation kernel be bounded in $L^\infty(R^2_+)$ and nonzero nonnegative function of sources q have bounded first moment. Let there exist at least one nonnegative measurable solution $\bar{c}$ of (8.3). Then on $\alpha \geq 1/2$ the moments N_α are equal to infinity.*

Remark 8.1. *The hypothesis of solvability of the equation (8.3) is essential. Actually, if $K(x,y) = 0$ on $x > 1$ or $y > 1$ and the sources function q is not equal to zero on $x > 2$, then the equation (8.3) is unsolvable.*

Proof of Theorem 8.1. Multiplying (8.3) by x^α and integrating yields

$$\frac{1}{2}\int_0^\infty \int_0^\infty [(x+y)^\alpha - x^\alpha - y^\alpha]\, K(x,y)\bar{c}(x)\bar{c}(y)dxdy = -Q_\alpha \qquad (8.5)$$

where

$$Q_\alpha = \int_0^\infty x^\alpha q(x)dx > 0. \qquad (8.6)$$

The following inequality holds for all $x, y \geq 0$:

$$(x+y)^\alpha - x^\alpha - y^\alpha \geq (2^\alpha - 2)x^{\alpha/2}y^{\alpha/2}, \qquad \text{if} \quad 0 \leq \alpha \leq 1,\ \alpha \geq 2. \quad (8.7)$$

To prove (8.7) it suffices to note that the minimum of the function

$$\frac{(x+y)^\alpha - x^\alpha - y^\alpha}{x^{\alpha/2} y^{\alpha/2}}$$

is achieved at $x = y$.

We substitute (8.7) into (8.5) and obtain

$$2Q_\alpha \le (2 - 2^\alpha) M N^2_{\alpha/2}, \qquad 0 \le \alpha \le 1. \tag{8.8}$$

Here

$$M = \sup_{0 \le x, y < \infty} K(x,y),$$

If to assume $N_{1/2} < \infty$ then at $\alpha = 1$ we obtain from (8.8) the contradiction $Q_1 \le 0$. This proves the Theorem 8.1 □

Further we consider the constant case $K(x,y) \equiv 1$. The case $K = \text{const}$ can be transformed onto $K = 1$ by change of variables $\tau = Kt$. We put for convenience $Q = Q_0$ where Q_0 is defined in (8.6). It is easily to observe that $N_0 = \sqrt{2Q}$. We substitute this correlation into (8.3):

$$\frac{1}{2} \bar{c} * \bar{c}(x) - \sqrt{2Q}\,\bar{c}(x) + q(x) = 0. \tag{8.9}$$

In (8.9) $\bar{c} * \bar{c}$ means the convolution:

$$\bar{c} * \bar{c}(x) = \int_0^x \bar{c}(x-y)\bar{c}(y)\,dy.$$

We avail ourselves of the Laplace transform and obtain from (8.9)

$$\bar{c}(x) = \sqrt{2Q} \sum_{i=1}^{\infty} \frac{(2i-3)!!\, q^{[i]}(x)}{(2Q)^i i!}, \tag{8.10}$$

where

$$(2i-3)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2i-3), \quad (-1)!! = 1,$$

$$q^{[i]} = q * q * \ldots * q \quad \text{(the convolution is used } i-1 \text{ times).}$$

By definition $q^{[0]} = 1$, $q^{[1]} = q$. The expression (8.10) testifies the non-negativity and uniqueness of the solution to (8.9) and allows explicitely find $\bar{c}(x)$ for simple source functions. For example, if $q(x) = \exp(-ax)$ then

$$\bar{c}(x) = \sqrt{2a}\sum_{i=1}^{\infty}\frac{(2^i-3)!!(ax)^{i-1}}{2^i i!(i-1)!}\mathrm{e}^{-ax} = \sqrt{a/2}\exp\left(-\frac{ax}{2}\right)\left(I_0\left(\frac{ax}{2}\right) - I_1\left(\frac{ax}{2}\right)\right) \tag{8.11}$$

where I_0, I_1 are modified Bessel functions. Also, the equality (8.10) allows to conclude that $N_\alpha < \infty$ for all $0 \le \alpha < 1/2$ provided that

$$q(x) \le M_0 \exp(-ax), \qquad M_0 = \text{const.} \tag{8.12}$$

Really, in this case $Q \le M_0/a$ and from (8.10) we conclude

$$\bar{c}(x) \le \sqrt{2M_0 a}\sum_{i=1}^{\infty}\frac{(2^i-3)!!(ax)^{i-1}}{2^i i!(i-1)!}\mathrm{e}^{-ax} \tag{8.13}$$

By integrating (8.13) with weight x^α, $0 \le \alpha \le 1$ we obtain

$$N_\alpha = \frac{\sqrt{M_0}a^{-(\alpha+1/2)}}{\sqrt{2\pi}} \cdot \sum_{i=1}^{\infty}\frac{\Gamma(i-1/2)\Gamma(i+\alpha)}{\Gamma(i+1)\Gamma(i)}$$

where $\Gamma(i)$ is Euler's gamma-function. We have utilized that

$$\Gamma(i+1/2) = 2^{-i}\sqrt{\pi}(2i-1)!!.$$

Applying the Raabe's test of summation of series ([29], p.273), we find $N_\alpha < \infty$ provided that $\alpha < 1/2$. Consequently, in this case the estimate $\alpha = 1/2$ of Theorem 8.1 is exact, and we come to the following lemma.

Lemma 8.1. *Let the conditions of Theorem 8.1 and (8.12) hold. Then* $N_\alpha < \infty$ *provided that* $\alpha < 1/2$.

If we consider the discrete stationary coagulation equation

$$\frac{1}{2}\sum_{j=1}^{i-1}\bar{c}_{i-j}\bar{c}_j - \bar{c}_i\sum_{j=1}^{\infty}\bar{c}_j + q_i = 0, \quad i \ge 1$$

with sources $q = (Q, 0, 0, \dots, 0, \dots)$, then

$$\bar{c}_i = \sqrt{2Q}\frac{(2i-3)!!}{2^i i!}, \quad i \ge 1.$$

In this case we also have $N_\alpha < \infty$ on $\alpha < 1/2$.

8.2 Convergence to equilibrium

Theorem 8.2. *Let conditions of Theorem 8.1 hold, the coagulation kernel* K *be a constant and the sources function* q *be continuous. Then the solution of the problem (8.1), (8.2) converges to equilibrium as* $t \to \infty$ *in* Ω_0 *and in* $C[a,b]$ *for all* $0 \le a < b < \infty$. *The rate of convergence is proportional to* $\exp\left(-\sqrt{2Q}\, t\right)$.

Proof. As we have already mentioned, we can transform any constant coagulation kernel to the unit one. Hence, let $K = 1$. We denote $f(x,t) = c(x,t) - \bar{c}(x)$. Then for the function f we obtain from (8.1)–(8.3):

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2} f * f(x,t) - f(x,t) \int_0^\infty f(y,t)dy + \bar{c} * f(x) -$$

$$-\bar{c}(x) \int_0^\infty f(y,t)dy - f(x,t) \int_0^\infty \bar{c}(y)dy, \tag{8.14}$$

$$f(x,0) = f_0(x) \stackrel{\text{def}}{=} c_0(x) - \bar{c}(x). \tag{8.15}$$

We denote $F(t) = \int_0^\infty f(x,t)dx$. Integrating the equation (8.1) yields

$$F(t) = \frac{2\sqrt{2Q}}{\left(1 + \frac{2\sqrt{2Q}}{F(0)}\right) \exp(\sqrt{2Q}\, t) - 1}. \tag{8.16}$$

Hence,

$$F(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{8.17}$$

Obviously, $F(t) \equiv 0$ provided that $F(0) = 0$. If $F(0) > 0$ then from (8.16) we obtain

$$0 < F(t) < F(0) \exp(-\sqrt{2q}\, t). \tag{8.18}$$

For $-\sqrt{2Q} \le F(0) < 0$ we have $-\sqrt{2Q} \le F(t) < 0$ and, in addition,

$$|F(t)| \le 2|F(0)| \exp(-\sqrt{2Q}\, t) \le 2\sqrt{2Q} \exp(-\sqrt{2q}\, t). \tag{8.19}$$

Also, we observe

$$\int_0^t |F(s)|ds \le \frac{2F(0)}{F(0) + 2\sqrt{2Q}} \le 2 \tag{8.20}$$

provided that $F(0) < 0$.

With the aim to show that $f(x,t) \to 0$ as $t \to \infty$ we consider the linearised equation (8.14):

$$u_t(x,t) = -F(t)u(x,t) + \bar{c} * u(x,t) - F(t)\bar{c}(x) - \sqrt{2Q}\, u(x,t), \quad u(x,0) = u_0(x). \tag{8.21}$$

Employing the Laplace transform gives the solution to (8.21):

$$u(x,t) = \exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right)\left\{u_0(x) + u_0(x) * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}(x)t^i}{i!} - \right.$$

$$\left. - \int_0^t F(s)\exp\left(\sqrt{2Q}\, s + \int_0^s F(s_1)ds_1\right)\sum_{i=0}^{\infty} \frac{\bar{c}^{[i+1]}(x)(t-s)^i}{i!}ds\right\}. \tag{8.22}$$

Therefore we use the method of variation of constants to look for solution of (8.14), (8.15) in the form

$$f(x,t) = \exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right)\left\{g(x,t) + g(x,t) * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}(x)t^i}{i!} - \right.$$

$$\left. - \int_0^t b(s)\sum_{i=0}^{\infty} \frac{\bar{c}^{[i+1]}(x)(t-s)^i}{i!}ds\right\}, \tag{8.23}$$

where

$$b(t) = F(t)\exp\left(\sqrt{2Q}\, t + \int_0^t F(s)ds\right).$$

From (8.18), (8.19) we conclude

$$b(t) \le F(0)\exp(F(0)/\sqrt{2Q}) \quad \text{if} \quad F(0) > 0; \tag{8.24}$$

$$|b(t)| \le 2|F(0)| \quad \text{if} \quad F(0) \le 0. \tag{8.25}$$

Substituting (8.23) into (8.14) yields

$$g_t + g_t * \sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!} = \frac{1}{2}\exp\left(-\sqrt{2Q}\, t - \int_0^t F(s)ds\right)\cdot$$

$$\cdot\left\{g^{[2]}+g^{[2]}*\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]}+2g^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}-\right.$$

$$-2g*\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds-$$

$$-2g*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds+$$

$$\left.+\left(\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds\right)^{[2]}\right\},\qquad g(x,0)=f_0(x). \tag{8.26}$$

With the aim of the Laplace transform we conclude from (8.26) that the function g satisfies the equation

$$g_t=\frac{1}{2}\exp\left(-\sqrt{2Q}\,t-\int_0^t F(s)ds\right)\cdot\left\{g^{[2]}+g^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}+\right.$$

$$+\left(\bar{c}^{[2]}+\bar{c}^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)*\left(\int_0^t b(s)ds+\sum_{i=1}^{\infty}\frac{(-\bar{c})^{[i]}}{i!}\int_0^t b(s)s^i ds\right)^{[2]}-$$

$$\left.-2g*\int_0^t b(s)\sum_{i=0}^{\infty}\frac{\bar{c}^{[i+1]}(t-s)^i}{i!}ds\right\},\qquad g(x,0)=f_0(x). \tag{8.27}$$

We write (8.27) in the integral form, then estimate g and $-g$ with (8.20), (8.24) and (8.25) taken into account, and finally establish the inequality

$$|g|_t\le\frac{1}{2}\exp\left(2-\sqrt{2Q}\right)\cdot\left\{|g|^{[2]}+|g|^{[2]}*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}+\right.$$

$$\left.+A^2*\left(\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]}+\left(\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right)^{[2]}\right)+2A|g|*\sum_{i=1}^{\infty}\frac{\bar{c}^{[i]}t^i}{i!}\right\}. \tag{8.28}$$

In (8.28) the constant A is equal to one of the upper estimate of $|b(t)|$ in dependence on the sign of $F(0)$.

Let us fix $m > 0$. For any $\varepsilon > 0$ we can find constants M and M_1 such that

$$\sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}(x)t^i}{i!} \le M\mathrm{e}^{\varepsilon t}, \quad 0 \le x \le m, \quad t \ge 0; \tag{8.29}$$

$$1 * \left(\left(\sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!} \right)^{[3]} + \left(\sum_{i=1}^{\infty} \frac{\bar{c}^{[i]}t^i}{i!} \right)^{[2]} \right) \le M_1\mathrm{e}^{\varepsilon t}, \quad 0 \le x \le m, \quad t \ge 0. \tag{8.30}$$

Hence, $|g| < h$ on $0 \le x \le m$, $t > 0$, where the function h satisfies the equation

$$h_t(x,t) = \frac{1}{2}\exp(2 - (\sqrt{2Q} - \varepsilon)t) \cdot \left(h^{[2]} + h^{[2]} * M + A^2 M_1 + 2AM * h \right), \tag{8.31}$$

$$h(x,0) = h_0 = \text{const} > \sup_{0 \le x \le m} |f_0(x)|. \tag{8.32}$$

Let us note that $h(x,t)$ increases in x for all $t > 0$. Actually, since $h_0 = \text{const}$ then from (8.31) $h_t(x,t) > h_t(x_1,t)$ for $x > x_1$, $t \ge 0$. Hence, $h^{[2]}$ increases in x, too, and, consequently,

$$M * h^{[2]}(x,t) \le Mmh^{[2]}(x,t), \quad M * h(x,t) \le Mmh(x,t)$$

for $0 \le x \le m$, $t \ge 0$. We substitute these expressions into (8.31) and establish that $h(x,t) < H(x,t)$ for $0 \le x \le m$, $t \ge 0$, if

$$H_t(x,t) = \frac{1}{2}\exp(2-(\sqrt{2Q}-\varepsilon)t) \cdot \left(H^{[2]}(x,t)(1 + Mm) + H(x,t)(1 + 2AMm) \right),$$

$$H(x,0) = H_0 = \text{const} > \max\{h_0, A^2 M\}.$$

We solve this equation and obtain

$$H(x,t) = H_0 E(t) \exp\left(H_0 x(E(t) - 1)\frac{1 + Mm}{1 + 2AMm} \right) \tag{8.33}$$

where

$$E(t) = \exp\left((1 - \exp(-(\sqrt{2Q} - \varepsilon)t))\frac{\mathrm{e}^2(1 + 2AMm)}{2(\sqrt{2Q} - \varepsilon)} \right) \le$$

$$\leq \exp\left(\frac{e^2(1+2AMm)}{2(\sqrt{2Q}-\varepsilon)}\right) = E_0.$$

Finally, from (8.33) we obtain boundedness of $g(x,t)$:

$$|g(x,t)| \leq H_0 E_0 \exp\left(H_0 m(E_0-1)\frac{1+Mx_0}{1+2AMm}\right) = G,\ 0 \leq x \leq m,\ t \geq 0. \tag{8.34}$$

Now we substitute (8.20), (8.29), (8.30) and (8.34) into (8.23) and conclude that $c(x,t)$ tends to $\bar{c}(x)$ as $t \to \infty$ uniformly with respect to $x \in [0,m]$:

$$|c(x,t)-\bar{c}(x)| \leq \exp(2-\sqrt{2Q}\,t)\left(G+GMme^{\varepsilon t}+AMe^{\varepsilon t}\right) \leq M_2 e^{-(\sqrt{2Q}-\varepsilon)t}, \tag{8.35}$$

$$0 \leq x \leq m,\ t \geq 0.$$

We should emphasize that the constants G and M depend on m and ε. This proves convergence in the space $C[a,b]$ for any $0 \leq a < b < \infty$.

To prove convergence in the space $\Omega_{0,0}$ we note that

$$\int_0^\infty |c(x,t)-\bar{c}(x)|dx = \int_0^m |c(x,t)-\bar{c}(x)|dx + \int_m^\infty |c(x,t)-\bar{c}(x)|dx \leq$$

$$\leq M_2 m e^{-(\sqrt{2Q}-\varepsilon)t} + \int_m^\infty c(x,t)dx + \int_m^\infty \bar{c}(x)dx. \tag{8.36}$$

Let us fix $\epsilon > 0$ and pick up $m \geq N_\alpha/\epsilon$. Then

$$\int_m^\infty \bar{c}(x)dx \leq \epsilon. \tag{8.37}$$

Really, to obtain (8.37) we employ Lemma 8.1 and the inequality (3.27). Since $F(t) \to 0$ $t \to \infty$ (see (8.17)) and (8.35) is valid, then there exists t_0 such that for all $t > t_0$

$$\int_m^\infty c(x,t)dx = \int_0^\infty c(x,t)dx - \int_0^m c(x,t)dx = \int_m^\infty \bar{c}(x)dx + \delta(t) \tag{8.38}$$

where $\delta(t) \leq \epsilon$ for all $t > t_0$. Inserting (8.37) and (8.38) into (8.36) yields

$$\int_0^\infty |c(x,t)-\bar{c}(x)|dx \leq M_2 m e^{-(\sqrt{2Q}-\varepsilon)t} + 3\epsilon. \tag{8.39}$$

(8.35) and (8.39) prove Theorem 8.2 □

8.3 Remarks

As we aware, the existence of a stationary solution for the coagulation equation with sources and convergence to it, was studied by Gajewski [32], where he was concerned with the equation with effluxes which essentially help to construct the results. Without efflux terms both existence of equilibrium and convergence to it were not proved.

Chapter 9. TREND TO EQUILIBRIUM IN THE BECKER–DÖRING CLUSTER EQUATIONS

In this chapter we study the Becker-Döring coagulation–fragmentation model (0.8)–(0.10).

As in chapter 2 we fix T, $0 < T \le \infty$ and introduce the space $\Omega_{1,r}(T)$ of continuous functions c with bounded norm

$$\|c\|_{1,r}^{(T)} = \sup_{0 \le t < T} \sum_{i=1}^{\infty} i^r |c_i(t)|, \quad r \ge 0,$$

and the space $\Omega_\lambda(T) \subset \Omega_{1,r}(T)$ with the bounded norm for $\lambda > 1$:

$$\|c\|_{\lambda}^{(T)} = \sup_{0 \le t < T} \sum_{i=1}^{\infty} \lambda^i |c_i(t)|.$$

To simplify notations we write just $\Omega_{1,r}$ instead of $\Omega_{1,r}(0)$ and X instead of $\Omega_{1,1}(0)$. In this case the space X is the Banach sequence space with norm $\|c\| = \sum_{i=1}^{\infty} i|c_i|$.

Definition 9.1. *We say a sequence* $\{c^{(j)}\}$ *of elements of* X *converges weak* to* $c \in X$ *(symbolically* $c^{(j)} \xrightarrow{*} c$ *) if*

$$(i) \qquad \sup_{j \ge 1} \|c^{(j)}\| < \infty;$$

$$(ii) \qquad c_i^{(j)} \to c_i \quad \textit{as} \quad j \to \infty \quad \textit{for each} \quad i \ge 1.$$

To justify the terminology, note that (cf. [25], p. 354) X can be identified with the dual of the space Y of sequences $y = y_i$ satisfying $\lim_{i \to \infty} i^{-1} y_i = 0$ with norm $\|y\| = \max i^{-1}|y_i|$ and that weak* convergence as defined above is exactly weak* convergence in $X = Y^*$.

For $\rho > 0$ let $B_\rho = \{y \in X : \ \|y\| \le \rho\}$. We make B_ρ^+ into a metric space by giving it the metric

$$d(y,z) = \sum_{i=1}^{\infty} |y_i - z_i|.$$

Clearly a sequence $\{c^{(j)}\} \subset B_\rho^+$ converges weak* to $y \in X^+$ if and only if $y \in B_\rho^+$ and B_ρ^+ is compact when endowed with the metric d; equivalently, any bounded subsequence in X has a weak* convergent subsequence. To avoid confusion B_ρ^+ endowed with the d metric will be written as B_ρ^{d+}.

Let $E \subset X$. A function $\theta : E \mapsto R$ is sequentially weak * continuous if $\theta(y^{(j)}) \overset{*}{\rightharpoonup} \theta(y)$ whenever $y^{(j)}$, $y \in E$ with $y^{(j)} \overset{*}{\rightharpoonup} y$ as $j \to \infty$. For example, the function $\sum_{i=1}^{\infty} g_i y_i$ is well defined for all $y \in X$ if $|g_i| = O(i)$, but is sequentially weak * continuous if and only if $|g_i| = o(i)$.

We will use the following simple lemma.

Lemma 9.1. *If* $y^{(j)} \overset{*}{\rightharpoonup} y$ *in* X *and* $\|y^{(j)}\| \to \|y\|$, *then* $y^{(j)} \to y$ *in* X.

Proof. Define $z_i^{(j)} = |y_i^{(j)}| + |y_i| - |y_i^{(j)} - y_i| \geq 0$. Then $z_i^{(j)} \to 2|y_i|$ as $j \to \infty$ for each i. Since for any m

$$\sum_{i=1}^{\infty} i z_i^{(j)} \geq \sum_{i=1}^{m} i z_i^{(j)},$$

it follows that

$$\liminf_{j \to \infty} \sum_{i=1}^{\infty} i z_i^{(j)} \geq 2 \sum_{i=1}^{\infty} i |y_i|.$$

Hence

$$\limsup_{j \to \infty} \|y^{(j)} - y\| = 2\|y\| - \liminf_{j \to \infty} \sum_{i=1}^{\infty} i z_i^{(j)} \leq 0,$$

which proves the assertion □

9.1 Existence, uniqueness and mass conservation

Combining Theorems 3.1 and 3.2 we obtain the following result for the Becker-Döring equations.

Theorem 9.1. *Assume that* $a_i = O(i)$ *and there exists a positive constant* m *such that*

$$b_i \leq const \cdot i^m.$$

Suppose $c_0 \in \Omega^+_{1,r}(0)$, $r > \max\{m, 1\}$. *Then there exists at least one solution to the problem (0.8)–(0.10)* $c \in \Omega^+_{1,r}(T)$, $0 < T \le \infty$. *If* $r \ge 2$ *and* $b_i \le const \cdot i^r$ *then the solution is mass conserving.*

If we suppose $c_0 \in \Omega^+(0)$ *then there exists a solution* $c \in \Omega^+(T)$ *for all* $T < \infty$.

From the uniqueness Theorems 4.1 and 4.2 we obtain

Theorem 9.2. *(i) Let* $a_i = O(i)$ *and* $b_i = O(i)$. *If* $c_0 \in \Omega^+_{\lambda_0}(0)$ *then there exists* $1 < \lambda < \lambda_0$ $(\lambda = \lambda(T))$ *such that the initial value problem (0.8)–(0.10) has at most one solution in* $\Omega_\lambda(T)$, $T < \infty$;

(ii) Let k, H *are nonnegative constants and*

$$a_i = ki + h_i \ge 0, \quad i \ge I(i), \qquad and \quad h_i \le H, \ i \ge 1$$

and the fragmentation coefficients b_i *are bounded uniformly with respect to* i. *Then the solution to the problem (0.8)–(0.10) is unique in* $\Omega_{1,1}(T)$.

9.2 Equilibria and Lyapunov functions

There are three classes of kinetic coefficients to consider when studying the evolution of (0.8), (0.9). The first class is pure fragmentation in which we assume $a_i = 0$, $b_i > 0$ for all i, in this case the equilibria satisfy $\bar{c}_i = 0$ for $i \ge 2$ and are given by $\bar{c} = (\bar{c}_1, 0, 0, \dots)$. The second class of coefficients to consider is pure coagulation which we assume $a_i > 0$, $b_i = 0$ for all i. In this case the equilibria are obtained by solving the equations $a_i \bar{c}_1 \bar{c}_i = 0$, $i \ge 1$ and thus the equilibrium states are given by $\bar{c} \in \Omega^+_{1,1}$ with $\bar{c}_1 = 0$. Note there are infinitely many equilibria with fixed mass $\rho = \sum_{i=1}^\infty i\bar{c}_i$ for each ρ (it is more convenient for us to write in this chapter ρ instead of usual N_1).

We consider the most interesting case in which we assume

$$a_i > 0, b_i > 0 \ \text{for all} \quad i. \tag{9.1}$$

In this case the equilibria satisfy

$$\frac{\bar{c}_{i+1}}{\bar{c}_i} = \frac{a_i}{b_{i+1}} \bar{c}_1, \qquad i \ge 1, \tag{9.2}$$

and therefore we have the form

$$\bar{c}_i = Q_i \bar{c}_1^i, \qquad i \geq 1 \tag{9.3}$$

where the Q_i are defined by

$$Q_1 = 1, \qquad \frac{Q_{i+1}}{Q_i} = \frac{a_i}{b_{i+1}}, \quad i \geq 1. \tag{9.4}$$

In order for $\bar{c}$ given by (9.3) to be an equilibrium state, $\bar{c}_1$ must be chosen so that $\bar{c} \in X^+$. For $z \geq 0$ define

$$F(z) = \sum_{i=1}^{\infty} i Q_i z^i. \tag{9.5}$$

The radius of convergence z_s of this series is given by

$$z_s^{-1} = \lim_{i \to \infty} \sup Q_i^{1/i}. \tag{9.6}$$

We shall always assume that

$$\lim_{i \to \infty} \sup Q_i^{1/i} < \infty \tag{9.7}$$

so that $0 < z_s \leq \infty$. Note that F is smooth and strictly increasing $0 \leq z \leq z_s$. Define

$$\rho_s = \sup_{0 \leq z < z_s} F(z). \tag{9.8}$$

If $z_s = \infty$, then $\rho_s = \infty$. If $0 < z_s < \infty$, and in the case when $0 < \rho_s < \infty$, we have $\rho_s = F(z_s)$. We thus obtain the following characterization of equilibria.

Lemma 9.2. *Let (9.1) hold.*
(i) Let $\rho < \infty$, $0 \leq \rho \leq \rho_s$. Then there is exactly one equilibrium state $\bar{c}^\rho$ with mass ρ, and it is given by

$$\bar{c}_i^\rho = Q_i z(\rho)^i, \quad i \geq 1, \tag{9.9}$$

where $z(\rho)$ is the unique root of $F(z) = \rho$.
(ii) If $\rho_s < \rho < \infty$, there is no equilibrium state with mass ρ.

Next we need some properties of the function $V(c)$ defined by

$$V(c) = \sum_{i=1}^{\infty} c_i \left(\ln \left(\frac{c_i}{Q_i} \right) - 1 \right), \qquad c \in X^+ \tag{9.10}$$

where the summand is defined to be zero when $c_i = 0$.

Lemma 9.3. *The function*

$$G(c) = \sum_{i=1}^{\infty} c_i(\ln\ c_i - 1) \tag{9.11}$$

*is finite and sequentially weak * continuous on* X^+.

Proof. Let $0 < \varepsilon < \frac{1}{2}$. If $c \in X^+$ and $1 \le m \le n$, then by Hölder's inequality

$$\sum_{i=m}^{n} c_i^{1-\varepsilon} \le \left(\sum_{i=m}^{n} ic_i\right)^{1-\varepsilon} \left(\sum_{i=m}^{n} i^{1-1/\varepsilon}\right)^{\varepsilon}. \tag{9.12}$$

In particular, setting $m = 1$ and using the inequality

$$x|\ln\ x| \le \text{const}(x^{1+\varepsilon} + x^{1-\varepsilon}), \quad x > 0, \tag{9.13}$$

it is easily seen that the series defining G is absolutely convergent. To prove the sequential weak* continuity, let $c^{(j)} \in X^+$ for $j \ge 1$ with $c^{(j)} \overset{*}{\rightarrow} c$ as $j \to \infty$. Then

$$G(c^{(j)}) = \left(\sum_{i=1}^{m-1} + \sum_{i=m}^{\infty}\right) c_i^{(j)}(\ln\ c_i^{(j)} - 1),$$

and by (9.12) the second sum is bounded in absolute value by

$$\text{const}\left(\frac{\|c^{(j)}\|^{1+\varepsilon}}{m^{1+\varepsilon}} + \frac{\|c^{(j)}\|}{m} + \|c^{(j)}\|^{1-\varepsilon}\left(\sum_{i=m}^{\infty} i^{1-1/\varepsilon}\right)^{\varepsilon}\right),$$

and therefore tends to zero as $m \to \infty$ uniformly in j. Since $c_i^{(j)} \to c_i$ for each i we obtain $\lim_{j\to\infty} G(c^{(j)}) = G(c)$ as required. This proves Lemma 9.3 □

Note that it follows either directly from the proof of the Lemma 9.3 or from the sequential weak* continuity that G is bounded above and below on B_ρ^+ for each $\rho \ge 0$.

We note that by (9.10), (9.11),

$$V(c) = G(c) - \sum_{i=1}^{\infty} ic_i \ln(Q_i^{1/i}). \tag{9.14}$$

It thus follows from (9.7) that V is bounded below on B_ρ^+ for every $\rho \geq 0$. In general V may take the value $+\infty$, but if

$$\liminf_{i\to\infty} Q_i^{1/i} > 0, \tag{9.15}$$

then V is bounded above on B_ρ^+ for every $\rho \geq 0$.

For $0 < z < \infty$ we define

$$V_z(c) = V(c) - \ln z \sum_{i=1}^{\infty} ic_i = \sum_{i=1}^{\infty} c_i \left(\ln\left(\frac{c_i}{Q_i z^i}\right) - 1\right). \tag{9.16}$$

Lemma 9.4. *Let* $\rho < \infty$, $0 \leq \rho \leq \rho_s$. *Then*

$$V_z(\bar{c}^\rho) = \int_0^\rho \ln\left(\frac{z(s)}{z}\right) ds. \tag{9.17}$$

Proof. Since

$$V_z(\bar{c}^\rho) = \rho \ln\left(\frac{z(\rho)}{z}\right) - \sum_{i=1}^{\infty} Q_i z(\rho)^i,$$

we have that

$$\frac{d}{d\rho}V_z(\bar{c}^\rho) = \ln\left(\frac{z(\rho)}{z}\right) + \rho\frac{z'(\rho)}{z(\rho)} - F(z(\rho))\frac{z'(\rho)}{z(\rho)} = \ln\left(\frac{z(\rho)}{z}\right), \quad 0 < \rho < \rho_s.$$

The result follows since $z(\rho) \sim \rho$ as $\rho \to 0+$. This proves Lemma 9.4 □

By (9.14), (9.16) and Lemma 9.3 $V_z(.)$ is sequentially weak * continuous on X^+ if and only if the functional

$$c \mapsto \sum_{i=1}^{\infty} c_i \ln(Q_i z^i)$$

is, that is, if and only if $\ln(Q_i z^i) = o(i)$. Since $i^{-1}\ln(Q_i z^i) = \ln(Q_i^{1/i} z)$ then we have proved

Lemma 9.5. $V_z(.)$ *is sequentially weak* continuous on* X^+ *if and only if* $\lim_{i\to\infty} Q_i^{1/i}$ *exists and* $z = z_s$.

We are in position now to prove the important equality related to V.

Theorem 9.3. *Suppose that (9.1), (9.7), (9.15) hold and that* $a_i = O(i/\ln i), b_i = O(i/\ln i)$. *Let* c *be a solution of (0.8), (0.9) on some interval* $[0,T]$ *with* $c_r(0) > 0$ *for some* r. *Then*

$$V(c(t)) + \int_0^t D(c(s))ds = V(c(0)) \qquad \textit{for all} \quad t \in [0,T], \tag{9.18}$$

where

$$D(c) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty}(a_i c_1 c_i - b_{i+1}c_{i+1})(\ln(a_i c_1 c_i) - \ln(b_{i+1}c_{i+1})). \tag{9.19}$$

Proof. For $n = 2, 3, \ldots$ define

$$V^{(n)}(c) = \sum_{i=1}^{n} c_i \left(\ln\left(\frac{c_i}{Q_i}\right) - 1\right),$$

$$D_n(c) \stackrel{\text{def}}{=} \sum_{i=1}^{n}(a_i c_1 c_i - b_{i+1}c_{i+1})(\ln(a_i c_1 c_i) - \ln(b_{i+1}c_{i+1})).$$

By Theorem 5.6 we see that $c_i(t) > 0$ for all $t \in (0,T)$, $i \geq 1$. For a.e. $t \in (0,T)$ we find

$$\frac{d}{dt}V^{(n)}(c(t)) = -D_{n-1}(c) - (\ln c_1)\sum_{i=n}^{\infty} J_i - J_n \ln\left(\frac{c_n}{Q_n}\right) =$$

$$= -D_n(c) - (\ln c_1)\sum_{i=n+1}^{\infty} J_i - J_n \ln\left(\frac{c_{n+1}}{Q_{n+1}}\right). \tag{9.20}$$

For sufficiently large n we have that $\ln c_n < 0$ on $(0,T)$, and hence

$$-J_n \ln c_n \leq -a_n c_1 c_n \ln c_n, \qquad -J_n \ln c_{n+1} \geq b_{n+1}c_{n+1}\ln c_{n+1}. \tag{9.21}$$

Since the solution c is mass conserving then $nc_n \to 0$ as $n \to \infty$ uniformly on $(0,T)$, so that by our hypotheses on a_i, b_i the right-hand side of (9.21) also tend to zero uniformly. Furthemore, by Theorem 5.6 and the definition of a solution,

$$\lim_{n\to\infty}\int_\tau^t (\ln c_1)\sum_{i=n}^{\infty} J_i ds = 0, \quad 0 < \tau < t < T, \tag{9.22}$$

and by (9.7), (9.15) and the correlation

$$n\int_{\tau}^{t} J_n(c(s))ds \to 0, \quad n\to\infty, \tag{9.23}$$

we obtain

$$\lim_{n\to\infty}\int_{\tau}^{t} J_n \ln Q_n ds = \lim_{n\to\infty}\int_{\tau}^{t} J_n \ln Q_{n+1} ds = 0, \quad 0<\tau<t<T. \tag{9.24}$$

Combining (9.20)–(9.24) we deduce that as $n\to\infty$,

$$\int_{0}^{t} D_{n-1}(c(s))ds + o(1) \le V^{(n)}(c(\tau)) - V^{(n)}(c(t)) \le$$

$$\le \int_{\tau}^{t} D_n(c(s))ds + o(1), 0<\tau<t<T. \tag{9.25}$$

Since

$$(x-y)(\ln x - \ln y) > 0 \quad \text{for} \quad x,y>0,\ x\ne y, \tag{9.26}$$

we deduce from (9.25) and the monotone convergence theorem that

$$V(c(t)) + \int_{\tau}^{t} D(c(s))ds = V(c(\tau)).$$

Since $c:\ [0,T)\mapsto X$ is continuous, since, by Lemma 9.1, (9.7) and (9.15), $V:\ X^+\mapsto R^1$ is continuous, the result follows from letting $\tau\to 0+$. This proves Theorem 9.3 □

The following theorem is less restrictive than Theorem 9.3 but then the equality (9.18) is not proved, and it should be replaced by the inequality.

Theorem 9.4. *Let* $a_i = O(i)$, $c_0\in\Omega^+_{1,r}(0)$ *and the conditions of Theorem 9.1 hold. Let (9.1) and (9.7) hold, and suppose further that* $c_0\ne 0$, $V(c_0)<\infty$. *Then there exists a solution* $c\in\Omega^+_{0,r}(T)$ *of (0.8), (0.9) with* $c(0)=c_0$ *satisfying the energy inequality*

$$V(c(t)) + \int_{0}^{t} D(c(s))ds \le V(c_0) \quad \textit{for all} \quad t\in[0,T). \tag{9.27}$$

Proof. For n sufficiently large the approximation solutions c^n defined in the proof of Theorem 3.1 satisfy, by the same argument as in Theorem 5.6, Lemma 2.2 $c_i^n(t) > 0$ for all $t > 0$, $1 \le i \le n$, and hence

$$V(c^n(t)) + \int_0^t D_{n-1}(c^n(s))ds = V(c_0), \quad t \in [0,T). \tag{9.28}$$

Since $D_{n-1}(c^n) \ge D_m(c^n)$ for $n > m$, we have

$$\liminf_{n\to\infty} \int_0^t D_{n-1}(c^n(s))ds \ge \int_0^t D(c(s))ds.$$

Finally, by (9.7), Lemma 9.1, and the fact that (see proof of Theorem 3.1) $c^n(t) \to c(t)$ in $\Omega_{1,r}(T)$,

$$\liminf_{n\to\infty} V(c^n(t)) \ge V(c(t)).$$

The inequality (9.27) follows by passing to the limit in (9.28). This proves Theorem 9.4 □

In the above proof we did not specify that the approximating sequence c^n is actually the proper subsequence of the set of solutions of approximated problems.

9.3 Asymptotic behaviour of solutions

In order to learn the asymptotic behaviour of solutions we use some definitions and facts.

Definition 9.2. *(cf. Definition 6.2). A generalized flow G on a metric space Y is a family of continuous mappings $\phi : [0,\infty) \mapsto Y$ with the properties*

(i) if $\phi \in G$ and $\tau \ge 0$, then $\phi_\tau \in G$, where $\phi_\tau(t) \overset{\text{def}}{=} \phi(t+\tau)$, $t \in [0,\infty)$.

(ii) if $y \in Y$, there exists at least one $\phi \in G$ with $\phi(0) = y$, and

(iii) if $\phi_j \in G$ with $\phi_j(0)$ convergent in Y as $j \to \infty$, then there exist a subsequence ϕ_{j_k} of ϕ_j and an element $\phi \in G$ such that $\phi_{j_k}(t) \to \phi(t)$ in Y uniformly for t in compact intervals of $[0,\infty)$.

A generalized flow G with the property that for each $y \in Y$ there is a unique $\phi \in G$ with $\phi(0) = y$ is called a semigroup; we then write $T(t)y = \phi(t)$, so that the mappings $T(t) : Y \mapsto Y$, $t \geq 0$, satisfy

(i) $T(0) =$ identity,

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$,

(iii) the mapping $(t, y) \mapsto T(t)y$ is continuous from $[0, \infty) \times Y \mapsto Y$.

Given a generalized flow on a metric space Y and some $\phi \in G$ we denote by $O^+(\phi) = \cup_{t \geq 0} \phi(t)$ the *positive orbit* of ϕ and by $\omega(\phi) = \{y \in Y : \phi(t_j) \to y$ for some sequence $t_j \to \infty\}$ the *ω-limit* set of ϕ. A subset $E \subset Y$ is said to be quasi-invariant if given any $y \in E$ and $t \geq 0$ there exists $\psi \in G$ with $\psi(t) = y$ and $O^+(\psi) \subset E$. The following result is classic.

Theorem 9.5. *Let G be a generalized flow on Y, let $\phi \in G$ and suppose that $O^+(\phi)$ is relatively compact. Then $\omega(\phi)$ is nonempty and quasi-invariant, and dist$(\phi(t), \omega(\phi)) \to 0$ as $t \to \infty$.*

Proof. We prove the quasi-invariance, the other assertions being obvious. Let $y \in \omega(\phi)$, so that $\phi(t_j) \to y$ for some sequence $t_j \to \infty$. Let $t \geq 0$ and consider the sequence $\phi(t_j - t)$. Since $O^+(\phi)$ is relatively compact there is a subsequence t_{j_k} such that $\phi(t_{j_k} - t) = \phi_{t_{j_k} - t}(0)$ is convergent. By property (iii) in the definition of a generalized flow there exist a further subsequence, again denoted t_{j_k}, and an element $\psi \in G$ such that $\phi_{t_{j_k} - t}(s) = \phi(t_{j_k} - t + s) \to \psi(s)$ as $k \to \infty$ uniformly for s in compact intervals of $[0, \infty)$. Clearly $O^+(\psi) \subset \omega(\phi)$ and $\psi(t) = y$. Theorem 9.5 has been proved □

A function $\mathbf{V} : Y \mapsto R$ is called a Lyapunov function if $t \mapsto \mathbf{V}(\phi(t))$ is nonincreasing on $[0, \infty)$ for each $\phi \in G$. For generalized flows the simplest form of the "invariance principle" consists of the following immediate consequence of Theorem 9.5. If $\mathbf{V}$ is a continuous Lyapunov function and if $O^+(\phi)$ is relatively compact, then $\omega(\phi)$ consists of complete orbits along which $\mathbf{V}$ has the constant value $\mathbf{V}^\infty = \lim_{t \to \infty} \mathbf{V}(\phi(t))$. This information may determine $\omega(\phi)$.

Theorem 9.6. *Let conditions of Theorem 9.1 hold. Let* G *denote the set of all solutions* c *of (0.8), (0.9) on* $[0,\infty)$. *Then* G *is a generalized flow on* X^+.

Proof. Continuity of solutions is proved in Theorem 9.1. Property (i) in the definition of a generalized flow is obvious from (0.8), (0.9), while property (ii) follows from Theorem 9.1. It thus remains to prove the upper semicontinuity property (iii). Let $c^{(j)}$ be a sequence of solutions of (0.8), (0.9) on $[0,\infty)$ satisfying $c^{(j)}(0) \to c_0$ in $\Omega_{0,r}(0)$ as $j \to \infty$. Repeating the proof of Theorem 3.1 with $c^{(j)}$ playing the role of the approximating solutions, and using the mass conservation and estimates of "tails" obtained with (3.27) taken into account, we obtain a subsequence $c^{(j_k)}$ and a solution c such that $c_i^{(j_k)}(t) \to c_i(t)$ uniformly on $[0,T]$ for every $T > 0$ and $i \geq 1$. Also,

$$\sum_{i=1}^{\infty} ic_i^{(j_k)}(t) = \sum_{i=1}^{\infty} ic_i^{(j_k)}(0) \to \sum_{i=1}^{\infty} ic_i(0) = \sum_{i=1}^{\infty} ic_i(t)$$

as $k \to \infty$, for every $t \geq 0$. Property (iii) follows by Lemma 9.1 □

Theorem 9.7. *Assume* $a_i = o(i)$, $b_i = o(i)$. *For* $\rho > 0$ *let* G_ρ *denote the set of all solutions* c *on* $[0,\infty)$ *with* $c_0 \in B_\rho^{d,+}$. *Then* G_ρ *is a generalized flow on* $B_\rho^{d,+}$.

Proof. We must check property (iii) in the definition of a generalized flow. Let $c^{(j)}$ be a sequence of solution with $c^{(j)}(0) \stackrel{*}{\rightharpoonup} c_0$ as $j \to \infty$. It follows from mass conservation and proof of Theorem 9.1 that the derivative $dc_i^{(j)}(t)/dt$ exists for each $i \geq 1$ and is absolutely bounded independently of j and $t \geq 0$. Hence by the Arzela-Ascoli theorem there exist a diagonal subsequence $c^{(j_k)}$ of $c^{(j)}$ and a function $c:\ [0,\infty) \mapsto X^+$ such that $c_i^{(j_k)}(t) \to c_i(t)$ uniformly for t in compact subsets of $[0,\infty)$ for each i. This implies also that $d(c^{(j_k)}(t), c(t)) \to 0$ uniformly on compact subsets. Clearly c satisfies (0.8), the equation for $i \geq 2$. To pass to the limit in the c_1 equation (0.9) (written as usual in the integral form), we use the sequential weak * continuity of the functions

$$\sum_{i=1}^{\infty} a_i y_i, \quad \sum_{i=1}^{\infty} b_i y_i$$

and the bounded convergence theorem. Thus c is a solution. This proves Theorem 9.7 □

We prove now the final results of this chapter. First, we consider a case in which $z_s = \infty$.

Theorem 9.8. *Assume* $a_i > 0$, $b_i > 0$ *for all* i, $a_i = O(i)$ *and*

$$\lim_{i\to\infty} Q_i^{1/i} = 0. \tag{9.29}$$

Let c *be any solution of (0.8), (0.9) on* $[0,\infty)$ *satisfying* $c_0 \neq 0$, $V(c_0) < \infty$, *and the energy inequality (9.27). Let* $\rho_0 = \sum_{i=1}^{\infty} ic_i(0)$. *Then* $c(t) \to c^{\rho_0}$ *strongly in* X *as* $t \to \infty$, *where* c_0^{ρ} *is the unique equilibrium state with mass* ρ_0 *(given by (9.9)).*

Proof. By Theorem 9.6 the set G of all solutions of (0.8), (0.9) on $[0,\infty)$ is a generalized flow on X^+. By (9.27) $V(c(t)) \le V(c(0))$ for all $t \ge 0$, and by (9.14), Lemma 9.3 it follows that

$$-\sum_{i=1}^{\infty} ic_i(t)\ln(Q_i^{1/i}) \le M < \infty, \quad t \ge 0. \tag{9.30}$$

As is easily shown, (9.29) and (9.30) imply that $O^+(c)$ is relatively compact in X^+. Since V is not continuous on X^+ we cannot apply the invariance principle directly to determine $\omega(c)$. Instead we note by (9.27) that for any n, any $T > 0$ and any sequence $t_j \to \infty$,

$$\lim_{j\to\infty} \int_0^T D_n(c(t_j+s))ds = \lim_{j\to\infty} \int_{t_j}^{t_j+T} D_n(c(s))ds = 0. \tag{9.31}$$

Let $\hat{c} \in \omega(c)$, so that $c(t_j) \to \hat{c}$ in X for some sequence $t_j \to 0$. By the proof of Theorem 9.5 there is a subsequence, again denoted t_j, and a solution d on $[0,\infty)$, such that $d(0) = \hat{c}$ and $c(t_j + \cdot) \to d(\cdot)$ in $C([0,T];X)$. Since $\sum_{i=1}^{\infty} id_i(t) = \rho_0 > 0$, we have by Theorem 5.4 that $d_i(t) > 0$ for all i and all $t > 0$. Thus by (9.31) and Fatou's lemma,

$$\int_0^T D_n(d(s))ds = 0. \tag{9.32}$$

Since n is arbitrary, it follows from (9.32), (9.26) and the continuity of each d_i that

$$d_i(s) = Q_i d_1^i(s), \qquad i \geq 1,\ s \in [0, T].$$

In particular, $\hat{c}_i = Q_i \hat{c}_1^i$ for $i \geq 1$, and since $\sum_{i=1}^{\infty} i\hat{c}_i = \rho_0$, this implies that $\hat{c} \equiv c^{\rho_0}$. Hence $\omega(c) = \{c^{\rho_0}\}$ and the result follows from Theorem 9.5 □

We now discuss the case $0 < z_s < \infty$. This is more difficult because if $\rho_0 = \sum_{i=1}^{\infty} ic_i(0) > \rho_s$, then the positive orbit of c is never relatively compact in X.

Theorem 9.9. *Assume $a_i > 0$, $b_i > 0$ for all i, $a_i = O(i/\ln i), b_i = O(i/\ln i)$, and that $\lim_{i\to\infty} Q_i^{1/i} = 1/z_s$ exists with $0 < z_s < \infty$. Let mass conservation conditions of Theorem 9.1 hold and c be a solution on $[0, \infty)$ with mass ρ_0. Then $c(t) \overset{*}{\rightharpoonup} c^{\rho}$ as $t \to \infty$ for some ρ with $0 \leq \rho \leq \min\{\rho_0, \rho_s\}$.*

Proof. The case $\rho_0 = 0$ being trivial, we suppose $\rho_0 > 0$. By Theorem 9.7 G_{ρ_0} is a generalized flow on $B_{\rho_0}^{d+}$. By Lemma 9.5 V_{z_s} is continuous on $B_{\rho_0}^{d+}$, and by mass conservation and Theorem 9.3

$$V_{z_s}(c(t)) + \int_0^t D(c(s))ds = V_{z_s}(c(0)),\ t \geq 0. \tag{9.33}$$

Since the first moment of c is bounded, $O^+(c)$ is relatively compact in $B_{\rho_0}^{+}$. By the invariance principle $\omega(c)$ is nonempty and consists of solutions $c(\cdot)$ along which V_{z_s} has the constant value $V_{z_s}^{\infty} = \lim_{t\to\infty} V_{z_s}(c(t))$. Applying (9.33) to a nonzero such solution $\hat{c}(\cdot)$ we see that necessarily $\hat{c}_i(t) = Q_i \hat{c}_1^i(t)$, $i \geq 1$, and by mass conservation it follows that $\hat{c}$ is an equilibrium. Hence $\omega(c)$ consists of equilibria c^{ρ} with $0 \leq \rho \leq \min\{\rho_0, \rho_s\}$ and $V_{z_s}(c^{\rho}) = V_{z_s}(\infty)$. But by Lemma 9.4 $V_{z_s}(c^{\rho})$ is stricktly decreasing in ρ, and thus $\omega(c) = \{c^{\rho}\}$ for a unique ρ. The result follows from Theorem 9.5 □

9.4 Remarks

The concept of generalized flow was introduced by Ball [7] to give a natural abstraction for the (not necessarily unique) solution of an autonomous evolution equation. In this chapter we follow to Ball, Carr and Penrose [6]. The existence result in Theorem 9.1 follows from general coagulation equation and, hence, can be improved for more simple Becker-Döring equations (see [6]).

Using the relaxed invariance principle, Slemrod [60] improved results of Theorem 9.9 by replacing the conditions $a_i = O(i/\ln i)$, $b_i = O(i/\ln i)$ onto $a_i = O(i)$, $b_i = O(i)$.

Chapter 10. SPATIALLY INHOMOGENEOUS COAGULATION EQUATION

We are concerned with the following equation

$$\frac{\partial c_i(z,t)}{\partial t} + \mathrm{div}_z(v_i(z,t)c_i(z,t)) + a_i(z,t)c_i(z,t) =$$

$$\frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j}c_{i-j}(z,t)c_j(z,t) - c_i(z,t)\sum_{j=1}^{\infty} K_{i,j}c_j(z,t) + q_i(z,t) \qquad (10.1)$$

with initial data

$$c_i(z,0) = c_i^{(0)}(z) \geq 0. \qquad (10.2)$$

Its physical interpretation is described in Introduction (section 0.1, equation (0.11)). We consider the kernels

$$K_{i,j} \leq kij \qquad (10.3)$$

and prove the local existence, uniqueness and stability theorem for space inhomogeneous problem (10.1), (10.2) with kernels (10.3). We succeed to extend the local existence theorem for all $t > 0$ for sufficiently small initial data and sources. From Chapter 1 we know that for kernels (10.3) the mass conservation law may be broken. We prove here that the mass conservation holds at least locally in time.

Finally, we show that for kernels $K_{i,j} = ij$ the sequence of solutions of regularized equations does not converge to the solution of the original problem and, consequently, we cannot construct this solution via the approximated solutions and simulate it numerically without special tools. This example demonstrates also that our results cannot be improved via the approach of approximated solutions.

10.1 Local Existence

First, we fix $T > 0$ and write (10.1), (10.2) in the following integral form

$$c_i(z,t) = \exp\left(-\int_0^t (a_i + \mathrm{div}_z v_i)(z_i(s),s)ds\right) c_i^{(0)}(z_{i,0}) +$$

$$+\int_0^t \mathbf{S}(c)_i(z_i(s),s)\exp\left(-\int_s^t (a_i+\mathrm{div}_z v_i)(z_i(s),s)ds\right)ds. \qquad (10.4)$$

Here $\mathbf{S}(c)_i$ is the collision operator expressed by the right hand side of (10.1); $z_i(s)$ is the characteristic curve passing through the point $(z,t) \in R^3 \times R^1_+$. It satisfies the characteristics equation $dz_i/dt = v_i$ with $z_{i,0} = z_i(0),\ z_i(t) = z$. Under solution of the initial value problem (10.1), (10.2) we will understand the solution to (10.4). We consider the regularized equations to (10.1) which can be obtain by its truncation:

$$\frac{\partial c_i(z,t)}{\partial t} + \mathrm{div}_z(v_i(z,t)c_i(z,t)) + a_i(z,t)c_i(z,t) =$$

$$\frac{1}{2}\sum_{j=1}^{i-1} K^n_{i-j,j}c_{i-j}(z,t)c_j(z,t) - c_i(z,t)\sum_{j=1}^{n} K^n_{i,j}c_j(z,t) + q_i(z,t). \qquad (10.5)$$

The truncated kernels can be defined as (cf. chapter 3)

$$K^n_{i,j}(z,t) = K_{i,j}(z,t), \quad i+j \le n, \qquad (10.6)$$

$$K^n_{i,j}(z,t) = 0, \quad i+j > n, \quad 1 \le n \le \infty. \qquad (10.7)$$

We impose the following conditions on space transfer and effluxes:

$$a_i \in C,\ v_i \in C^{1,0},\ a_i + \mathrm{div}_z v_i \ge -b = \mathrm{const},\ 0 \le t \le T,\ z \in R^3,\ i \ge 1 \qquad (10.8)$$

and for all $i \ge 1$ and each point $(z,t) \in R^3 \times R^1_+$ there exists a characteristic curve $z_i(s)$ passing through (z,t) on $s = t$.

We introduce the space $\Omega_{1,r}(T),\ r \ge 0$ of continuous functions $c_i(z,t)$, $i \ge 1$ with the bounded norm

$$\|c\|^{(T)}_{1,r} = \sup_{0\le t\le T}\sum_{j=1}^{\infty} j^r \sup_{z\in R^3}|c_j(z,t)| + \sup_{z\in R^3, 0\le t\le T, i\in N}|c_j(z,t)|$$

and the space $\Omega_\lambda(T) \subset \Omega_{1,0}(T),\ \lambda \ge 1$ with the bounded norm

$$\|c\|^{(T)}_{\lambda} = \sup_{0\le t\le T}\sum_{j=1}^{\infty} \lambda^j \sup_{z\in R^3}|c_j(z,t)| + \sup_{z\in R^3, 0\le t\le T, i\in N}|c_j(z,t)|.$$

If $\lambda > 1$ then $\Omega_\lambda(T) \subset \Omega_{1,r}(T),\ r \ge 0$. As before, the cones of nonnegative functions in the above spaces we denote via the superscript "+". We are now in position to formulate the auxiliary lemma.

Lemma 10.1. *Let the conditions (10.3), (10.8) hold, the functions $K_{i,j}(z,t)$ be continuous ($i,j \geq 1$), and*
I $q \in \Omega^+_{1,r}(T)$, $c^{(0)} \in \Omega^+_{1,r}(0)$, $r \geq 1$;
or
II $q \in \Omega^+_\lambda(T)$, $c^{(0)} \in \Omega^+_\lambda(0)$, $\lambda > 1$.

Then for any $n < \infty$ the equation (10.5) written in the integral form (10.4) with the initial function $c^{(0)}$ has the unique solution

I $c \in \Omega^+_{1,r}(T)$; *or*

II $c \in \Omega^+_\lambda(T)$ *correspondingly. In both cases the solution has a continuous derivative along characteristics.*

The proof is similar to the proof of Theorem 2.1. We consider the integral equation $c = R(c)$ where the operator R, expressed by the right hand side of (10.4), maps spaces $\Omega_{1,r}$ and Ω_λ into itself. We utilize the contraction mapping theorem to demonstrate local existence of a continuous solution. Thanks to its continuity we obtain from (10.4) its differentiability along characteristics and then prove its nonnegativity similarly to Lemma 2.2. To extend the solution for all $0 \leq t \leq T$ we take into account that from (10.4)

$$c_i(z,t) \leq \mathrm{e}^{bt} c_i^{(0)}(z_{i,0}) +$$

$$+ \int_0^t \left\{ \frac{1}{2} \sum_{j=1}^{i-1} K^n_{i-j,j} c_{i-j}(z_i(s),s) c_j(z_i(s),s) + q_i(z_i(s),s) \right\} \mathrm{e}^{b(t-s)} ds. \quad (10.9)$$

We use the "upper" function h which satisfies the differential equation

$$\frac{dh_i(t)}{dt} = \frac{1}{2} \overline{K} \sum_{j=1}^{i-1} h_{i-j}(t) h_j(t) + (Q+b) h_i(t) \quad (10.10)$$

and the initial conditions

$$h_i(0) > \sup_z c_i^{(0)}. \quad (10.11)$$

In (10.10) $\overline{K} = \sup_{i,j} K^n_{i,j}$ and the constant Q we choose such that $Qh_i(0) > \sup_{z,t} q_i(z,t)$ for $1 \leq i \leq n$. Using the generating function we solve (10.10), (10.11) explicitly and obtain differentiability and uniform boundedness $h_i(t)$ for all $1 \le i \le n,\ 0 \le t \le T$.

Let the point (z,t) be the first where $c_i(z,t) = h_i(t)$, i.e.

$$h_j(s) > c_j(z_j(s), s), \quad 1 \le j \le i, \ \ 0 \le s < t.$$

Then we obtain from (10.9), (10.10)

$$c_i(z,t) < e^{bt} h_i(0) + \int_0^t \left\{ \frac{1}{2} \sum_{j=1}^{i-1} \overline{K} h_{i-j}(s) h_j(s) + Q h_i(s) \right\} e^{b(t-s)} ds = h_i(t).$$

Hence, we come to the contradiction $c_i(z,t) < h_i(t)$ which demonstrates that the function h really is a majorant for c:

$$h_i(t) > c_i(z,t), \ 1 \le i \le n, \ \ z \in R^3, 0 \le t \le T. \tag{10.12}$$

This estimate of boundedness enables us to extend the local solution for all $0 \le t \le T$ since values c_i for $i > N$ are controlled easily from (10.5). Actually, they don't undergo influence of nonlinear terms. The global uniqueness follows from the local one thanks to the estimate (10.12), too. This proves Lemma 10.1.

Remark 10.1. *If we impose the additional condition*

$$v_i(z,t) \in C^{2,0}, \quad q_i, a_i, K_{i,j} \in C^{1,0}, \quad c^{(0)} \in C^1, \quad i,j \ge 1 \tag{10.13}$$

then the solution c of (10.4) is continuously differentiable in z. Taking into account its differentiability along characteristics we obtain that it has a continuous derivative in t, too. Hence, it satisfies (10.5). To demonstrate this fact we introduce the space $\Omega^1_{1,r}(T)$ with the norm

$$|||c||| = \|c\|^{(T)}_{1,r} + \sup_{z \in R^3, 0 \le t \le T, i \in N} \left| \frac{\partial c_i(z,t)}{\partial z} \right|$$

and then repeat the above arguments.

Lemma 10.2. *Let the conditions (10.3), (10.8) hold,*

$$\sup_{0\le t\le T} q \in \Omega^{+}_{\lambda_0}(0),\ c^{(0)} \in \Omega^{+}_{\lambda_0}(0),\ \lambda_0 > 1$$

and the functions $K_{i,j}(z,t),\ i,j \ge 1$ *be continuous. Let* $1 < \tilde{\lambda} < \lambda_0$. *Then there exists* $\tau > 0$ *such that*

$$\|c^n\|^{(\tau)}_{\lambda} \le M = const, \quad 1 \le \lambda \le \tilde{\lambda}. \tag{10.14}$$

Proof. In accordance with Lemma 10.1 $c^n \in \Omega^{+}_{\lambda_0}(T)$. We make the substitution $d^n_i = ic^n_i$ and similarly to Lemma 10.1 construct the majorant function $h_i(t) > d^n_i(z,t),\ z \in R^3,\ i \ge 1$ which satisfies the equation

$$\frac{dh_i(t)}{dt} = \frac{1}{2}ki\sum_{j=1}^{i-1} h_{i-j}(t)h_j(t) + (Q+b)h_i(t), i \ge 1 \tag{10.15}$$

with the initial condition

$$h^{(0)}_i > \max\{ic^{(0)}_i,\ i\sup_{z,t} q_i(z,t)\},\ h^{(0)} \in \Omega^{+}_{\lambda}(0),\ 1 < \lambda < \lambda_0. \tag{10.16}$$

We introduce the generating function

$$u(\xi,t) = \sum_{i=1}^{\infty} \xi^i h_i(t)$$

and obtain from (10.15), (10.16):

$$u_t = \frac{1}{2}k\xi u u_\xi + (1+b)u,\ u(\xi,0) = \sum_{i=1}^{\infty} \xi^i h^{(0)}_i. \tag{10.17}$$

The initial value problem (10.17) satisfies the conditions of Cauchy-Kovalevskaya theorem [30] and, consequently, has an analytic solution $u(\xi,t)$ in a neighbourhood of the point $\xi = \tilde{\lambda} > 1,\ t = 0$. This proves existence of $h(t),\ 0 \le t \le \tau,\ \tau > 0$. Its nonnegativity follows directly from (10.15), (10.16). As far as the sum

$$\sup_{0\le t\le \tau} \sum_{i=1}^{\infty} h_i(t)$$

is bounded then h_i are bounded uniformly with respect to $i \ge 1$. Consequently, the norm $\|h\|^{(\tau)}_{\lambda}$ is bounded. This proves Lemma 10.2 □

Lemma 10.3. *Let the conditions of Lemma 10.2 hold. Then for each fixed* $i \geq 1$ *the sequence* $\{c_i^n\}_{n=1}^{\infty}$ *is a compact subset of the space of continuous functions on each compacta in* $R^3 \times [0, \tau]$.

Proof. We consider a box

$$\Pi(I, Z, \tau) = \{(i, z, t) : 1 \leq i \leq I,\ |z| \leq Z,\ 0 \leq t \leq \tau\}.$$

From (10.4) we obtain

$$|c_i^n(z', t) - c_i^n(z, t)| \leq e^{bt} \left| c_i^{(0)}(z'_{i,0}) - c_i^{(0)}(z_{i,0}) \right| +$$

$$\left| \exp\left(- \int_0^t (a_i + \mathrm{div}_z v_i)(z'_i(s), s) ds \right) - \right.$$

$$\left. - \exp\left(- \int_0^t (a_i + \mathrm{div}_z v_i)(z_i(s), s) ds \right) \right| c_i^{(0)}(z'_{i,0}) +$$

$$+ \int_0^t |\mathbf{S}(c^n)_i(z'_i(s), s) - \mathbf{S}(c^n)_i(z_i(s), s)| \cdot$$

$$\cdot \exp\left(- \int_s^t (a_i + \mathrm{div}_z v_i)(z'_i(s_1), s_1) ds_1 \right) ds +$$

$$+ \int_0^t |\mathbf{S}(c^n)_i(z_i(s), s)| \left| \exp\left(- \int_s^t (a_i + \mathrm{div}_z v_i)(z'_i(s_1), s_1) ds_1 \right) - \right.$$

$$\left. - \exp\left(- \int_s^t (a_i + \mathrm{div}_z v_i)(z_i(s_1), s_1) ds_1 \right) \right| ds. \tag{10.18}$$

For any $\varepsilon > 0$ we can pick up $\delta > 0$ such that if $|z' - z| < \delta$, $|z'|, |z| \leq Z$ then in $\Pi(I, Z, \tau)$ the following correlations hold

$$\left| \exp\left(- \int_0^t (a_i + \mathrm{div}_z v_i)(z'_i(s), s) ds \right) - \right.$$

$$\left. - \exp\left(- \int_0^t (a_i + \mathrm{div}_z v_i)(z_i(s), s) ds \right) \right| < \varepsilon, \tag{10.19}$$

$$|c_i^{(0)}(z'_{i,0}) - c_i^{(0)}(z_{i,0})| < \varepsilon,\ |q_i(z'_i(s), s) - q_i(z_i(s), s)| < \varepsilon, \tag{10.20}$$

$$|K_{i,j}(z_i'(s),s) - K_{i,j}(z_i(s),s)| < \varepsilon. \tag{10.21}$$

The inequality (10.21) must hold on $i \le I$ and $j \le J$. The value J will be define a bit later in (10.23). We note that from (10.1), (10.3)

$$|\mathbf{S}(c^n)_i(z_i'(s),s) - \mathbf{S}(c^n)_i(z_i(s),s)| \le \frac{1}{2}kI^2 \sum_{j=1}^{i-1} |c_{i-j}^n(z_i'(s),s) - c_{i-j}^n(z_i(s),s)| \cdot$$

$$\cdot |c_j^n(z_i'(s),s) + c_j^n(z_i(s),s)| + |c_i^n(z_i'(s),s) - c_i^n(z_i(s),s)| \cdot$$

$$\sum_{j=1}^{\infty} K_{i,j}^n(z_i'(s),s) c_j^n(z_i'(s),s) +$$

$$+ c_i^n(z_i(s),s) \sum_{j=1}^{\infty} |K_{i,j}^n(z_i'(s),s) - K_{i,j}^n(z_i(s),s)| c_j^n(z_i'(s),s) +$$

$$+ c_i^n(z_i(s),s) \sum_{j=1}^{\infty} K_{i,j}^n(z_i(s),s) |c_j^n(z_i'(s),s) - c_j^n(z_i(s),s)| +$$

$$+ |q_i(z_i'(s),s) - q_i(z_i(s),s)|. \tag{10.22}$$

We split the infinite series in (10.22) in the following fashion

$$\sum_{j=1}^{\infty} = \sum_{j=1}^{J} + \sum_{j=J+1}^{\infty}, \quad \text{where } J \ge I, \ \ \lambda^{J+1} > J+1, \ \ \text{and} \ \ \left(\lambda/\tilde{\lambda}\right)^{J+1} < \varepsilon \tag{10.23}$$

for some λ, $1 < \lambda < \tilde{\lambda}$. Hence, from (10.3), (10.14), (10.21) and (10.23) we obtain

$$\sum_{j=1}^{\infty} K_{i,j}^n(z_i'(s),s) c_j^n(z_i'(s),s) \le kIJ \sum_{j=1}^{J} c_j^n(z_i'(s),s) +$$

$$+ kI \sum_{j=J+1}^{\infty} j c_j^n(z_i'(s),s) \le kIJ \|c^n\|_1^{(\tau)} + kI \|c^n\|_\lambda^{(\tau)} = \text{const}; \tag{10.24}$$

$$\sum_{j=1}^{\infty} |K_{i,j}^n(z_i'(s),s) - K_{i,j}^n(z_i(s),s)| c_j^n(z_i'(s),s) \le \varepsilon \sum_{j=1}^{J} c_j^n(z_i'(s),s) +$$

$$+2kI \sum_{j=J+1}^{\infty} \tilde{\lambda}^j c_j^n(z_i'(s), s) \leq \varepsilon M + 2kI \left(\lambda/\tilde{\lambda}\right)^{J+1} \|c^n\|_{\lambda}^{(\tau)} \leq \varepsilon M(1+2kI). \tag{10.25}$$

Similarly we estimate another infinite series in (10.22):

$$\sum_{j=1}^{\infty} K_{i,j}^n(z_i(s), s)|c_j^n(z_i'(s), s) - c_j^n(z_i(s), s)| \leq kIJ^2\theta^n(s) + 2k\varepsilon IM. \tag{10.26}$$

We introduce the moduli of continuity

$$\eta_i^N = \sup_{|z|\leq Z, |t'-t|<\delta} |c_i^n(z, t') - c_i^n(z, t)|, \quad 1 \leq i \leq I, \quad n \geq 1, \tag{10.27}$$

$$\theta^n(t) = \sup_{|z-z'|<\delta, 1\leq i\leq J} |c_i^n(z', t) - c_i^n(z, t)|, \quad 0 \leq t \leq \tau, \quad n \geq 1 \tag{10.28}$$

and obtain from (10.18) with (10.19), (10.20), (10.24)–(10.26) taken into account:

$$\theta^n(t) \leq \text{const} \cdot \varepsilon + \text{const} \int_0^t \theta^n(s) ds, \; n \geq 1.$$

Consequently,

$$\theta^n(t) < M_1 \varepsilon, \; 0 \leq t \leq \tau, \quad M_1 = \text{const}. \tag{10.29}$$

Similarly we choose $|t' - t| < \delta, \; 0 \leq t', t \leq \tau$ and obtain

$$\eta_i^n \leq \text{const} \cdot \varepsilon + \text{const} \int_0^t \theta^n(s) ds \leq M_2 \varepsilon, \quad 1 \leq i \leq I. \tag{10.30}$$

From (10.23),(10.27), (10.28), (10.29), (10.30) we obtain

$$\sup_{|t'-t|<\delta, |z'-z|<\delta} |c_i^n(z', t') - c_i^n(z, t)| \leq \text{const} \cdot \varepsilon, \tag{10.31}$$

$$1 \leq i \leq I, \; 0 \leq t', t \leq \tau, \; |z'|, |z| \leq Z, \; n \geq 1.$$

Applying the Arzela's theorem to (10.14), (10.31) we conclude the proof of Lemma 10.3 □

We are now in position to formulate the local existence theorem.

Theorem 10.1. *Let conditions of Lemma 10.2 hold. Then for some* $\tau > 0$ *and any* $\tilde{\lambda}$, $1 < \tilde{\lambda} < \lambda_0$ *there exists a solution to (10.4) which is unique in* $\Omega_{\tilde{\lambda}}(\tau)$ *and has continuous dependence on initial data and sources.*

Proof. We construct the sequence of truncated kernels K^n which converges to the original kernel K. In accordance with Lemma 10.1 this sequence generates the sequence $\{c^n\}_{n=1}^{\infty}$ of continuous solutions of (10.4). Using Lemma 10.3 we pick up for $i = 1$ from $\{c^n\}$ a subsequence which converges uniformly in $\Pi(1, Z, \tau)$ to a continuous nonnegative function $c_1(z,t)$. Then we pick up a subsubsequence which converges uniformly in $\Pi(2, 2Z, \tau)$ to $c_1(z,t), c_2(z,t)$ and so on. Finally, we obtain a sequence of solutions to (10.4) with truncated kernels which converges to a continuous for each $i \geq 1$ function

$$\{c_i(z,t)\}_{i=1}^{\infty}, \ (z,t) \in R^3 \times [0,\tau].$$

To demonstrate that the function obtained actually satisfies (10.4) with the original kernel K, we should note that as far as $c^n \in \Omega_{\tilde{\lambda}}^{+}(\tau)$ uniformly with respect to $n \geq 1$ then also

$$c \in \Omega_{\tilde{\lambda}}^{+}(\tau).$$

Moreover, from (10.14) we can see

$$\|c\|_{\lambda}^{(\tau)} \leq M, \quad 1 \leq \lambda \leq \tilde{\lambda}. \tag{10.32}$$

Passing to limit $n \to \infty$ in (10.4) is possible due to estimates of series "tails" (10.24)–(10.26). This proves existence in Theorem 10.1.

To prove uniqueness we consider solutions c and d to (10.4) with initial data $c^{(0)}$, $d^{(0)}$ and sources q^1, q^2 respectively. We denote $u_i = \sup_z |c_i - d_i|$ and obtain from (10.4) with (10.3) taken into account:

$$u_i(t) \leq e^{bt} u_i^{(0)} + \int_0^t \left\{ \frac{1}{2} k \sum_{j=1}^{i-1} (i-j) j u_{i-j}(s) \psi_j(s) + k i u_i(s) \sum_{j=1}^{\infty} j \psi_j(s) + \right.$$

$$\left. + k i \psi_i(s) \sum_{j=1}^{\infty} j u_j(s) + r_i(s) \right\} e^{b(t-s)} ds. \tag{10.33}$$

In (10.33) we put

$$\psi_i(t) = \sup_z |c_i(z,t) + d_i(z,t)|, r_i(t) = \sup_{z \in R^3} |q_i^1(z,t) - q_i^2(z,t)|.$$

Summation of (10.33) with the weight λ^i, $1 < \lambda \leq \tilde{\lambda}$ yields

$$U(\lambda,t)\mathrm{e}^{-bt} \leq U(\lambda,0) + \int_0^t \left\{ \frac{5k}{2}\lambda^2 U_\lambda \Psi_\lambda + R \right\} \mathrm{e}^{-bs} ds \tag{10.34}$$

where

$$U(\lambda,t) = \sum_{i=1}^{\infty} \lambda^i u_i(t), \quad \Psi(\lambda,t) = \sum_{i=1}^{\infty} \lambda^i \psi_i(t).$$

Similarly, using the correlations

$$\sum_{i=1}^{\infty} i\lambda^i u_i = \lambda U_\lambda, \ \sum_{i=1}^{\infty} i^2 \lambda^i u_i = \lambda^2 U_{\lambda\lambda} + \lambda U_\lambda,$$

we obtain

$$U_\lambda(\lambda,t)\mathrm{e}^{-bt} \leq U_\lambda(\lambda,0) + \int_0^t \frac{\partial}{\partial \lambda} \left\{ \frac{5k}{2}\lambda^2 U_\lambda \Psi_\lambda + R \right\} \mathrm{e}^{-bs} ds. \tag{10.35}$$

Applying Lemmas 4.1, 4.2 to (10.34), (10.35) and using (10.32) we conclude that in the domain R defined in Lemma 4.1 with $c_1 = 5k\tilde{\lambda}^2 M$ the following correlation holds

$$\sum_{i=1}^{\infty} \lambda^i u_i(t) \leq c_0 \mathrm{e}^{bt} + c_2 t \mathrm{e}^{bt} \tag{10.36}$$

where

$$c_0 = \|c^{(0)} - d^{(0)}\|_\lambda^{(0)}, \quad c_2 = \|q^1 - q^2\|_\lambda^{(\tau)}.$$

Therefore

$$\|c - d\|_\lambda^{(t)} \leq c_0 \mathrm{e}^{bt} + c_2 t \mathrm{e}^{bt} \tag{10.37}$$

in the domain R. Since the functions $c, d \in \Omega_\lambda(\tau)$ and the constant c_1, which define the domain R, does not change, then we pick up $T' < \tau$ as the initial moment, repeat the above reasonings N times unless $NT' \geq \tau$. Finally, (10.37) yields the continuous dependence of solution on initial data and sources on $0 \leq t \leq \tau$. The uniqueness of the solution is a simple corollary. This proves Theorem 10.1 □

Remark 10.2. *Utilizing Remark 10.1 and the Arzela theorem for the space C^1 of continuously differentiable functions, we obtain existence and uniqueness of the continuously differentiable solution to the original problem (10.1), (10.2).*

Let us stop on the problem of mass conservation for the basic space homogeneous problem with

$$v = q = a \equiv 0. \tag{10.38}$$

Summation (10.1) with the weight i yields

$$\frac{d}{dt}\sum_{i=1}^{\infty} ic_i(t) = -\lim_{n\to\infty}\left\{\sum_{i=1}^{n-1}\sum_{j=n-i}^{\infty} iK_{i,j}c_i(t)c_j(t) + \sum_{j=1}^{\infty} nK_{n,j}c_n(t)c_j(t)\right\}. \tag{10.39}$$

Deriving (10.39) we have used the symmetry of the coagulation kernel $K_{i,j} = K_{j,i}$. Hence, the limit is equal to zero (yielding the mass conservation law) if the double sum

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} iK_{i,j}c_i(t)c_j(t) \tag{10.40}$$

is bounded. Boundedness (10.40) for $0 \le t \le \tau$ follows from (10.3) and (10.14). Thus, we have proved the following corollary.

Cofollary 10.1. *Let for space homogeneous problem the conditions of Theorem 10.1 hold and (10.38) is true. Then the solution obtained is mass conserving.*

10.2 Global Existence for Small Data and Sources

Lemma 10.4. *Let the conditions of Lemma 10.2 hold and, in addition,*

$$a_i(z,t) + div_z v_i(z,t) \ge \delta > 0, \;\; i \ge 1, \;\; z \in R^3, \;\; 0 \le t \le T \le \infty, \tag{10.41}$$

$$c^{(0)} \in \Omega^+_{\lambda_0}(0), \quad q \in \Omega^+_{\lambda_0}(T). \tag{10.42}$$

Then there exist $1 < \lambda < \lambda_0$ and $\Lambda > 0$ such that any continuous nonnegative solution to (10.1), (10.2) obeys the inequality

$$c_i(z,t) \le \Lambda i^{-1}\lambda^{-i}\exp(-\delta t), \;\; i \ge 1, \;\; z \in R^3, \;\; 0 \le t \le T \tag{10.43}$$

provided that the initial data $c^{(0)}$ and sources q are sufficiently small (defined below).

Proof. We use the substitution

$$c_i(z,t) = d_i(z,\tau)e^{-\delta t}i^{-1}, \quad \tau = 1 - e^{-\delta t}, \quad \tau \in [0,1), \quad i \geq 1. \tag{10.44}$$

Similarly to Lemma 10.1 we obtain that

$$d_i(z,\tau) < h_i(\tau), \quad i \geq 1, \quad z \in R^3, \quad 0 \leq \tau < 1 \tag{10.45}$$

where

$$\delta \frac{dh_i}{d\tau} = \frac{1}{2} ki \sum_{j=1}^{i-1} h_{i-j}h_j + Qh_i, \tag{10.46}$$

$$h_i(0) = h_i^{(0)} > \max\{ic_i^{(0)},\ Q^{-1}i \sup_{z,t} q_i(z,t)\}, \quad i \geq 1. \tag{10.47}$$

Due to (10.42) we can pick up λ_1, $1 < \lambda_1 < \lambda_0$ such that $h(0) \in \Omega^+_{\lambda_1}(0)$. We introduce the generating function

$$H(\xi,\tau) = \sum_{i=1}^{\infty} \xi^i h_i(\tau) \tag{10.48}$$

and obtain from (10.46), (10.47):

$$\frac{\partial H}{\partial \tau} = \xi k \delta^{-1} H H_\xi + Q\delta^{-1} H, \tag{10.49}$$

$$H(\xi,0) = H_0(\xi). \tag{10.50}$$

Inserting

$$H = \exp(Q\delta^{-1}\tau)\tilde{H} \tag{10.51}$$

in (10.49), (10.50) yields

$$\tilde{H}_\tau = \xi k \delta^{-1} \exp(Q\delta^{-1}\tau)\tilde{H}\tilde{H}_\xi, \quad \tilde{H}(\xi,0) = H_0(\xi).$$

It is easy to show that

$$\tilde{H}(\xi,\tau) \leq G(\xi,\tau), \ \xi < \lambda_1 \tag{10.52}$$

where

$$G_\tau = \lambda_1 k \delta^{-1} \exp(Q\delta^{-1}\tau) G G_\xi, \quad G(\xi, 0) = H_0(\xi). \tag{10.53}$$

In fact, we can use the characteristics approach and obtain

$$G(\xi,\tau) = H_0(\xi_0^{(G)}), \quad \tilde{H}(\xi,\tau) = H_0(\xi_0^{(H)}), \quad \xi_0^{(G)} > \xi_0^{(H)}.$$

Taking into account the monotonic increasing property of H_0, we establish (10.52).

We solve (10.53) and obtain

$$G(\xi,\tau) = H_0\left(\xi + \lambda_1 k Q^{-1} G(\xi,\tau)\left(e^{Q\tau/\delta} - 1\right)\right).$$

Utilizing the implicit function theorem we establish existence of an analytic solution of the problem (10.53)

$$G(\xi,\tau) \le \overline{G} = \text{const} \tag{10.54}$$

on the intervals $0 \le \tau < 1$, $1 \le \xi < \lambda_1$ provided that the initial data are sufficiently small:

$$H_0' < \frac{Q}{\lambda_1 k\left(\exp(Q/\delta - 1)\right)}. \tag{10.55}$$

Inserting (10.47), (10.48), (10.50) into (10.55) we can see that to satisfy (10.55) it suffices to pick up $1 < \lambda < \lambda_1$ such that

$$\sum_{i=1}^{\infty} i\lambda^{i-1} h_i^{(0)} < \frac{Q}{\lambda_1 k\left(\exp(Q/\delta - 1)\right)}. \tag{10.56}$$

If (see (10.47))

$$Q^{-1} \sup_{z,t} q_i(z,t) \ge c_i^{(0)}, \; i \ge 1, \tag{10.57}$$

then (10.56) transforms into

$$\sum_{i=1}^{\infty} i^2 \lambda^{i-1} q_i < \frac{Q^2}{\lambda_1 k\left(\exp(Q/\delta - 1)\right)} \quad \text{where} \quad q_i = \sup_{z,t} q_i(z,t). \tag{10.58}$$

Since the function $f(x) = x^2/(e^x - 1)$ has its maximum at the point $\bar{x} \in (1,2)$ and $f(\bar{x}) > 0.6$ then to obtain less restrictions on the initial data we should choose $Q = \bar{x}\delta$ in accordance with (10.58). Thus, we obtain the condition on sources:

$$\sum_{i=1}^{\infty} i^2 \lambda^{i-1} q_i < \frac{0.6\delta^2}{\lambda_1 k}.$$

In general, when (10.57) does not hold, the expression (10.56) yields us the condition on initial data and sources via (10.44), (10.47). From (10.44), (10.45), (10.47), (10.51), (10.52), (10.54) we obtain

$$\sum_{i=1}^{\infty} i\lambda^i c_i(z,t) < \overline{G} \exp\left(Q/\delta - \delta t\right).$$

Hence, we come to (10.43) with $\Lambda = \overline{G}e^{Q/\delta}$. This proves Lemma 10.4 □

We are now in position to formulate the following Theorem.

Theorem 10.2. *Let the conditions of Lemma 10.4 hold. Then there exists a global solution to (10.5) which is unique in $\Omega_\lambda(T)$, has continuous dependence on initial data and sources and asymptotically tends to zero with exponential rate as $t \to \infty$.*

If we add the conditions (10.13) then the solution obtained has continuous derivatives and satisfies (10.1), (10.2).

Proof of Theorem 10.2 is based on estimate (10.43) and iterating the proof of Theorem 10.1 with τ replaced by T, $0 < T \leq \infty$. Continuous differentiability follows from Remarks 10.1 and 10.2. This proves Theorem 10.2 □

Now we demonstrate that for the basic space homogeneous case with the kernel $K_{i,j} = ij$ the solutions of truncated problems do not converge to the solution of the original problem, whose existence is proved in chapter 1. Let, as before, c^n be solutions of truncated problems with kernels (10.6), (10.7). We consider the residual

$$c_i(t) - c_i^n(t) =$$

$$= \int_0^t \left\{ \frac{1}{2} \sum_{j=1}^{i-1} (i-j)j(c_{i-j} - c_{i-j}^n)(s)(c_j + c_j^n)(s) - ic_i N_1(s) + ic_i^n(s) N_1(0) \right\} ds. \tag{10.59}$$

In (10.59) $N_1(t)$ is the total mass of particles at time t which is equal to the first moment of the solution. We have used the mass conservation property of the regularized solutions c^n (chapter 2). From Lemma 10.3 and uniqueness we have the uniform convergence $c^n \to c$ on each compact subset. Consequently,

$$c_i^n(t) \to c_i(t) \quad n \to \infty$$

till the mass conservation law holds only. After the critical time which, as well known, is equal to $[N_2(0)]^{-1}$ (chapter 1), the convergence to the solution of the original problem fails, since in (10.59) we obtain the nonzero summand $ic_i(M(s) - M(0))$. Therefore to prove the global existence theorem for general kernels (10.3) for any initial data, we must use completely different approach.

10.3 Remarks

In [39,66] the existence theorem was established for kernels $K_{i,j} \le o(i)o(j)$, $i, j \to \infty$. Existence of a solution to the pure (without sources, effluxes and space transfer) coagulation equation with the important kernel $K_{i,j} = ij$ was proved in [52] (local existence) and in [46] (global existence). Therefore there exists the gap in study of the above problem, which is the absence of an existence theorem for the space homogeneous coagulation equation with kernels

$$o(i)o(j) \le K_{i,j}(z,t) \le kij, \quad k = \text{const}, \; z \in R^3, \; t \ge 0. \tag{10.60}$$

The key reason of this gap is absence of the uniform estimate of the first moment of solutions of the regularized problems (usually with truncated kernels). The above estimate allows to pass to limit in the infinite series in (10.1). In [46, 62] the equation with $K_{i,j} = ij$ is solved directly and this method cannot be used for more general kernels (10.60).

The space inhomogeneous coagulation equation with unbounded coagulation kernels was studied by Burobin [12] for coagulation kernels with linear growth, and Galkin [38] for kernels of a special type. Galkin [37] and Dubovskiĭ [20] succeeded to prove existence and uniqueness theorems for bounded coagulation kernels. In [20] the convergent iteration method is constructed. The unique solvability of the problem with coagulation kernels of linear growth and particle fractionation taken into account was demonstrated by Dubovskiĭ [19].

Chapter 11. EXISTENCE AND UNIQUENESS FOR SPATIALLY INHOMOGENEOUS COAGULATION–ONDENSATION EQUATION WITH UNBOUNDED KERNELS

We are concerned with the space-inhomogeneous Smoluchowski equation with condensation processes taken into account.

$$\frac{\partial}{\partial t}c(x,z,t)+\frac{\partial}{\partial x}(r(x)c(x,z,t))+\operatorname{div}_z(v(x,z)c(x,z,t))=$$
$$=\frac{1}{2}\int_0^x K(x-y,y)c(x-y,z,t)c(y,z,t)dy-$$
$$-c(x,z,t)\int_0^\infty K(x,y)c(y,z,t)dy; \quad x,t\in R^1_+=[0,\infty), \quad z\in R^3. \tag{11.1}$$

The equation (11.1) must be supplemented by an initial distribution

$$c(x,z,0)=c_{01}(x,z)\geq 0, \tag{11.2}$$

and a distribution function of condensation germs

$$c(0,z,t)=c_{02}(z,t). \tag{11.3}$$

The results in the spatially inhomogeneous case are much more poor than in the homogeneous one, it may be explained by the following reasons. The formal integration of (11.1) with weight x yields (the condensation r is assumed to be equal to zero)

$$\int_{-\infty}^{+\infty}\int_0^\infty xc(x,z,t)dxdz=\text{const}. \tag{11.4}$$

In the space homogeneous case we obtain more valuable equality

$$\int_0^\infty xc(x,t)dx=\text{const}. \tag{11.5}$$

The correlations (11.4) and (11.5) express the mass conservation law. The large difference between the conservation laws (11.4) and (11.5) can be seen from the integral form of the problem (11.1),(11.2) (to simplify the exposition we take $r = 0$ and $v = v(x)$):

$$c(x,z,t) = c_0(x, z - v(x)t) + \int_0^t \left\{ \frac{1}{2}\int_0^x K(x-y,y)c(x-y, z-v(x)(t-s),t)c(y, z-v(x)(t-s),t)dy - c(x, z-v(x)(t-s),t)\int_0^\infty K(x,y)c(y, z-v(x)(t-s),s)dy \right\} ds. \tag{11.6}$$

To prove existence of solution to (11.6) we usually build a sequence of solutions $c_n(x,z,t)$ of a regularized (more simple) problem. Such sequence should converge to a function c. As we have seen in chapters 3, 10, the main problem is to prove that the function constructed $c(x,z,t)$ is a solution to the original equation (11.6). Namely, we must demonstrate possibility to pass to the limit as $n \to \infty$ in the equation (11.6) with c replaced by c_n. The most difficult stage is to show the admission to pass to the limit under sign of the integral over the infinite domain

$$\int_0^\infty K(x,y)c_n(y, z-v(x)(t-s),s)dy.$$

Let the coagulation kernel $K(x,y)$ be bounded. Then we ought to demonstrate the uniform smallness of the integral "tails"

$$\int_m^\infty c_n(y, z-v(x)(t-s),s)dy$$

for all $n \geq 1$. The value of m is taken sufficiently large. As long as we have the estimation like (11.5) then the problem can be solved by the following well-known trick (see (3.27):

$$\int_m^\infty c_n(y, z-v(x)(t-s),s)dy \leq \leq \frac{1}{m}\int_0^\infty y c_n(y, z-v(x)(t-s),s)dy \leq \frac{\text{const}}{m} \to 0, \quad m \to \infty.$$

The correlation (11.4) does not give us such convergence and we ought to seek other approaches. It is worth to note that if the space velocity v does not depend on x, then a variant of the strong mass conservation law (11.5) holds and we obtain the desired trend. Namely this fact was used by Burobin in [12] where he considered the case $v(x)$ =const for $x \geq x_0$ which reduces the problem to the space uniform situation with the strong conservation law (11.5).

In this chapter we prove existence and uniqueness theorem for sufficiently small initial data with coagulation kernels admitting linear growth on infinity. These kernels include the considerable class of physically real ones.

11.1 Main Result

Fix $T > 0$ and denote $\Omega_\lambda(T)$ the space of continuous functions in $R^1_+ \times R^3 \times [0, T]$ with the norm

$$\|c\|_\lambda = \sup_{0 \leq t \leq T} \int_0^\infty \exp(\lambda x) \sup_{z \in R^3} |c(x, z, t)| dx.$$

We define $\Omega(T) = \bigcup_{\lambda > 0} \Omega_\lambda(T)$. Let $\Omega^+_\lambda(T)$, $\Omega^+(T)$ be nonnegative cones in corresponding spaces.

Theorem 11.1. *Let the coagulation kernel K be continuous, nonnegative and symmetric function, i.e. $K(x, y) = K(y, x) \geq 0$. Let also*

$$K(x, y) \leq k(x + y) \text{ where } k = \text{const} .$$

Let the function $r(x)$ be nonnegative, bounded with its derivative and have continuous second derivative. Let the following inequality hold

$$div_z v(x, z) + r'(x) \geq \delta > 0. \tag{11.7}$$

Suppose that functions v, c_{01}, c_{02} are continuous and, in addition, c_{01} and c_{02} are nonnegative. We impose the following conditions ensuring smallnes of the initial data:

$$c_{01}(x, z) < A \exp(-ax), \quad a > 0; \tag{11.8}$$

$$\sup_{0 \leq t \leq T} \exp(\delta t) c_{02}(z, t) < A. \tag{11.9}$$

Let $r(x) \geq 0$ *and*

$$R = \max\left\{\sup_{R^1_+} r(x), \sup_{R^1_+} |r'(x)|\right\} < \frac{\delta}{1+a}, \tag{11.10}$$

$$2\sqrt{\frac{kA}{\delta - R(1+a)}} + 2\frac{kAr(0)}{\delta(\delta - R(1+a))} < a. \tag{11.11}$$

Let also

$$c_{01}(0,z) = c_{02}(z,0), \;\; z \in R^3.$$

Then there exists a continuous, differentiable along characteristics of the equation (11.1) nonnegative solution $c \in \Omega^+(T)$*. This solution is unique in* $\Omega(T)$ *the additional condition provided*

$$div_z v(x,z) + r'(x) \leq M(1+x), \quad M = const, \quad x \in R^1_+, \quad z \in R^3.$$

First, we formulate an auxiliary result.

Lemma 11.1. *Let conditions of the Theorem 11.1 hold and the coagulation kernel* K *have a compact support. Then there exists a unique solution* $c \in \Omega^+_a(T)$ *to the problem (11.1)-(11.3).*

The proof of Lemma 11.1 is based upon replacement of the integral with infinite upper limit in the main equation (11.1) to the integral over a compact domain. Due to this replacement the collision operator, which is expressed by the right-hand side of the equation (11.1), maps $\Omega_a(T)$ into itself (see Theorems 2.1, 2.2).

We approximate the original unbounded kernel by a sequence $\{K_n\}_{n=1}^{\infty}$ of kernels with compact supports. Each kernel from this sequence must satisfy the conditions of Theorem 11.1. Recalling Lemma 11.1 we get a sequence $\{c_n\}_{n=1}^{\infty}$ solutions of the problem (11.1)-(11.3) with kernels K_n and the same initial data c_{01} and c_{02}.

11.2 A PRIORI INTEGRAL ESTIMATE

We make the change of variables

$$c_n(x,z,t) = (1-\tau)\hat{c}_n(x,z,\tau), \quad \tau = 1 - \exp(-\delta t), n \geq 1.$$

Then the problem (11.1)-(11.3) takes the following form:

$$\delta\frac{\partial}{\partial\tau}\hat{c}_n(x,z,\tau)+(1-\tau)^{-1}\left(v(x,z),\nabla_z\hat{c}_n(x,z,\tau)\right)+$$
$$+(1-\tau)^{-1}r(x)\frac{\partial}{\partial x}\hat{c}_n(x,z,\tau)=$$
$$=\frac{1}{2}\int_0^x K_n(x-y,y)\hat{c}_n(x-y,z,\tau)\hat{c}_n(y,z,\tau)dy-$$
$$-\hat{c}_n(x,z,\tau)\int_0^\infty K_n(x,y)\hat{c}_n(y,z,\tau)dy-$$
$$-[\mathrm{div}_z v(x,z)+r'(x)-\delta](1-\tau)^{-1}\hat{c}_n(x,z,\tau) \tag{11.12}$$

with the initial and boundary conditions

$$\hat{c}_n(x,z,0)=c_{01}(x,z),\quad \hat{c}_n(0,z,\tau)=(1-\tau)^{-1}c_{02}(z,t). \tag{11.13}$$

Lemma 11.2. *Let the conditions of Theorem 11.1 hold and the continuous function g be a solution of the equation*

$$\delta g_\tau(x,\tau)+\frac{r(x)}{1-\tau}g_x(x,\tau)=\frac{1}{2}kx\int_0^x g(x-y,\tau)g(y,\tau)dy, \tag{11.14}$$
$$g(x,0)=A\exp(-ax),\quad g(0,\tau)=A. \tag{11.15}$$

Then

$$\hat{c}_n(x,z,\tau)<g(x,\tau),\quad x\in R^1_+,\ z\in R^3,\ \tau\in[0,1),\ n\geq 1. \tag{11.16}$$

Proof. Let a point (x_0,z_0,τ_0) be the first point where the functions $\hat{c}_n$ and g are equal:

$$\hat{c}_n(x_0,z_0,\tau_0)=g(x_0,\tau_0),\quad \hat{c}_n(x,z,\tau)<g(x,\tau), \tag{11.17}$$
$$0\leq\tau<\tau_0,\ 0\leq x<x(\tau),\ z=z(\tau).$$

In (11.17) $x(\tau)$, $z(\tau)$ mean the values on the characteristic passing through the point (x_0,z_0,τ_0) with $x(\tau_0)=x_0, z(\tau_0)=z_0$. Such point (x_0,z_0,τ_0) exists thanks the continuity of $\hat{c}_n$, g, positivity of r and due to the expressions (11.8), (11.9) and (11.15). We integrate (11.12), (11.14) along characteristics and obtain

$$\hat{c}_n(x_0, z_0, \tau_0) \leq \frac{1}{2}\delta^{-1} \int_0^{\tau_0} \int_0^{x_0} K_n(x(s)-y, y)\hat{c}_n(x(s)-y, z(s), s)\hat{c}_n(y, z(s), s)dyds$$

$$< \frac{1}{2}\delta^{-1}k \int \int_0^{\tau_0} \int_0^{x_0} g(x(s) - y, s)g(y, s)dyds = g(x_0, \tau_0). \qquad (11.18)$$

The inequality (11.18) yields the contradiction

$$\hat{c}_n(x_0, z_0, \tau_0) < g(x_0, \tau_0)$$

which proves Lemma 11.2 □

Remark 11.1. *Lemma 11.2 demonstrates the nontrivial influence which bring in the condensation term. We are able to replace the problem (11.14), (11.15) by essentially simpler problem*

$$\delta g_\tau(x, \tau) = \frac{1}{2}kx \int_0^x g(x - y, \tau)g(y, \tau)dy, g(x, 0) = A, \ \ g(0, \tau) = A,$$

but in this case the majorant function g *is not integrable. If* $g(x, 0) = A\exp(-ax)$, *then we cannot omit the condensation summand in (11.14) since the characteristic curves in (11.12) increase.*

By integrating (11.14) with the weight $\exp(\lambda x)$, we obtain

$$\delta H_\tau(\lambda, \tau) - (1 - \tau)^{-1}r(0)g(0, \tau) - (1 - \tau)^{-1}\lambda \int_0^\infty \exp(\lambda x)r(x)g(x, \tau)dx -$$

$$-(1 - \tau)^{-1} \int_0^\infty \exp(\lambda x)r'(x)g(x, \tau)dx = kH(\lambda, \tau)H_\lambda(\lambda, \tau), \qquad (11.19)$$

$$H(\lambda, 0) = \frac{A}{a - \lambda}, \quad \lambda \in [0, a), \quad \tau \in [0, 1). \qquad (11.20)$$

In (11.19), (11.20) we have used the notation

$$H(\lambda, \tau) = \int_0^\infty \exp(\lambda x)g(x, \tau)dx.$$

Taking (11.10) into account, we obtain from (11.19):

$$\delta H_\tau - kHH_\lambda \leq \frac{Ar(0)}{1 - \tau} + \frac{1 + a}{1 - \tau}RH. \qquad (11.21)$$

To find an estimate for the function $H(\lambda, \tau)$ we need the following lemma.

Lemma 11.3. *A solution of the differential inequality (11.21) with the initial condition (11.20) obeys for some* $0 < \tilde{\lambda} < a$ *the following correlation*

$$H(\lambda, \tau) < F(\lambda, \tau), \quad \tau \in [0,1), \quad 0 \le \lambda \le \tilde{\lambda},$$

where the function F *is defined as a solution to the majorant equation*

$$\delta F_\tau(\lambda, \tau) - kF(\lambda, \tau)F_\lambda(\lambda, \tau) = \frac{Ar(0)}{1-\tau} + \frac{1+a}{1-\tau} RF(\lambda, \tau), \tag{11.22}$$

$$F(\lambda, 0) = \frac{D}{a-\lambda}, \quad \lambda \in [0, \tilde{\lambda}], \quad \tau \in [0,1], \tag{11.23}$$

$$D > A. \tag{11.24}$$

Proof. We shall prove by contradiction. Consider the family of characteristics of the problem (11.22), (11.23). Define

$$Q(\lambda_0, \tau_0) = \{(\lambda, \tau) : 0 \le \tau \le \tau_0, \quad 0 \le \lambda < \lambda(\tau)\},$$

where $\lambda(\tau)$ is a value of λ on the charateristic curve $\Gamma(\lambda_0, \tau_0)$ which goes through the point (λ_0, τ_0). In addition, we suppose $0 < \lambda(0) < a$. We choose a point (λ_0, τ_0) such that

$$F(\lambda_0, \tau_0) = H(\lambda_0, \tau_0), \quad \text{but } H(\lambda, \tau) < F(\lambda, \tau) \quad \text{if} \quad (\lambda, \tau) \in Q(\lambda_0, \tau_0).$$

We point out that $\tau_0 > 0$ because (11.24) holds. Let us consider the characteristic curve $\Gamma'(\lambda_0, \tau_0)$ of the problem (11.21), (11.20) with the inclusion $\Gamma'(\lambda_0, \tau_0) \in Q(\lambda_0, \tau_0)$ taken into account. Then

$$\begin{aligned} H(\lambda_0, t_0) &\le H(\lambda_0^2, 0) + \int_{\Gamma'(\lambda_0, \tau_0)} \left\{ \frac{R(1+a)}{1-\tau} H(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} d\tau < \\ &< H(\lambda_0^1, 0) + \int_{\Gamma(\lambda_0, \tau_0)} d\tau \left\{ \frac{R(1+a)}{1-\tau} H(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} < \\ &< F(\lambda_0^1, 0) + \int_{\Gamma(\lambda_0, \tau_0)} d\tau \left\{ \frac{R(1+a)}{1-\tau} F(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} = F(\lambda_0, \tau_0). \end{aligned}$$

In the last expression λ_0^1 and λ_0^2 are beginnings of the characteristic curves $\Gamma(\lambda_0, \tau_0)$ and $\Gamma'(\lambda_0, \tau_0)$, and $\lambda_0^1 > \lambda_0^2$. We have used as well that $\lambda < a$ and the function H increases in λ. Finally, $H(\lambda_0, \tau_0) < F(\lambda_0, \tau_0)$. This contradiction with the hypotesis $F(\lambda_0, \tau_0) = H(\lambda_0, \tau_0)$ proves Lemma 11.3 □

Let us consider properties of the function $F(\lambda, \tau)$. We make in (11.22), (11.23) change of variables

$$F(\lambda, \tau) = (1-\tau)^{-\eta} L(\lambda, \tau), \quad \lambda \in [0, \tilde{\lambda}], \tau \in [0,1). \tag{11.25}$$

Hence,

$$\delta L_\tau(\lambda, \tau) - k(1-\tau)^{-\eta} L(\lambda, \tau) L_\lambda(\lambda, \tau) = Ar(0)(1-\tau)^{\eta-1}, \tag{11.26}$$

$$L(\lambda, 0) = \frac{D}{a-\lambda}, \quad \lambda \in [0, \tilde{\lambda}], \quad \tau \in [0,1). \tag{11.27}$$

In the expression (11.25) the notation $\eta = R(1+a)\delta^{-1}$ is introduced. The characteristic equation of the problem (11.26), (11.27) has the form

$$\frac{d\lambda}{d\tau} = -k\delta^{-1}(1-\tau)^{-\eta} L(\lambda, \tau). \tag{11.28}$$

As far as on each characteristic

$$L(\lambda, \tau) = \frac{D}{a-\lambda_0} + \int_0^\tau Ar(0)\delta^{-1}(1-\tau)^{\eta-1} d\tau,$$

then from (11.28) we obtain

$$\frac{d\lambda}{d\tau} = -k\delta^{-1}(1-\tau)^{-\eta} \left(\frac{D}{a-\lambda_0} - Ar(0)\delta^{-1}\eta^{-1}\left[(1-\tau)^\eta - 1\right] \right).$$

Hence,

$$\lambda(\tau) = \frac{Akr(0)\tau}{\delta^2 \eta} + \lambda_0 - \frac{k(1-(1-\tau)^{1-\eta})}{\delta(1-\eta)} \left[\frac{Ar(0)}{\delta\eta} + \frac{D}{a-\lambda_0} \right]. \tag{11.29}$$

Let us ascertain whether the characteristics (11.29) with starting points λ_0^1 and λ_0^2 can intersect. If they intersect then

$$\lambda_0^1 - \lambda_0^2 = \frac{2kD}{\delta(1-\eta)} \left[1 - (1-\tau)^{1-\eta}\right] \frac{\lambda_0^1 - \lambda_0^2}{(a-\lambda_0^1)(a-\lambda_0^2)},$$

whence

$$1-(1-\tau)^{1-\eta}=\delta k^{-1}D^{-1}(a-\lambda_0^1)(a-\lambda_0^2)(1-\eta).$$

Consequently, the characteristic curves of the problem (11.26), (11.27) have no intersection, if

$$D<\frac{\delta a^2}{k}\left(1-R\frac{1+a}{\delta}\right) \tag{11.30}$$

and the initial conditions are sufficiently small:

$$0<\lambda_0^1,\lambda_0^2<a-\sqrt{\frac{kD}{\delta-R(1+a)}}. \tag{11.31}$$

The inequality (11.30) brings us the correlation

$$R<\frac{\delta}{1+a}. \tag{11.32}$$

We should reveal now when the problem (11.26),(11.27) has smooth solution for small $\lambda>0$ for all $\tau\in[0,1)$. This condition holds if characteristics have no intersection and $\lambda(1)>0$. On $\tau=1$ we obtain from (11.29)

$$\lambda_0-\frac{kAr(0)}{\delta^2(1-\eta)}-\frac{kD}{\delta(1-\eta)(a-\lambda_0)}>0.$$

Hence,

$$\frac{kAr(0)}{2\delta^2(1-\eta)}+\frac{a}{2}-C<\lambda_0<\frac{kAr(0)}{2\delta^2(1-\eta)}+\frac{a}{2}+C, \tag{11.33}$$

where

$$C=\sqrt{\left(\frac{kAr(0)}{2\delta^2(1-\eta)}+\frac{a}{2}\right)^2-\frac{k}{\delta(1-\eta)}(Aar(0)\delta^{-1}+D)}. \tag{11.34}$$

To obtain suitable $\lambda_0>0$ we ought to have the concordance between (11.31) and (11.33). Hence, the following correlation have to take place

$$\frac{kAr(0)}{2\delta^2(1-\eta)}+\frac{a}{2}-C<a-\sqrt{\frac{kA}{\delta(1-\eta)}}. \tag{11.35}$$

We have changed D in (11.31) onto A thanks (11.24). Hence, (11.35) holds provided that

$$a > 2\sqrt{\frac{kA}{\delta(1-\eta)} + \frac{kAr(0)}{\delta^2(1-\eta)}} \tag{11.36}$$

It is easy to see that the subradicand expression in (11.34) is positive if the more strong inequality than (11.36) is true:

$$a > 2\sqrt{\frac{kA}{\delta(1-\eta)} + 2\frac{kAr(0)}{\delta^2(1-\eta)}}. \tag{11.37}$$

The inequality (11.37) ensures the correctness of (11.31) and (11.35) and holds thanks the condition (11.11) of the Theorem 11.1. Consequently, for $0 \leq \tau \leq 1$ and small enough $\lambda > 0$ there exists a continuous function $F(\lambda, \tau)$ and the supremum $\sup_{0\leq\tau\leq1} F(\lambda, \tau)$ covers the integrals

$$\sup_{t\in[0,\infty)} \int_0^\infty \exp(\lambda x) \sup_{z\in R^3} c_n(x, z, t)dx$$

uniformly with respect to $n \geq 1$ for suffiiently small $\lambda > 0$. Hence, we have proved the following lemma.

Lemma 11.4 (main). *Let the conditions of Theorem 11.1 hold. Then there exist positive constants* $\tilde{\lambda}$ *and* E *such that*

$$\sup_{0\leq\lambda\leq\tilde{\lambda}} \|c_n\|_\lambda \leq E < \infty, \quad n \geq 1.$$

11.3 Proof of Existence

Lemma 11.5. *Let the conditions of Theorem 11.1 hold. Then the sequence* $\{c_n\}_{n=1}^\infty$ *is uniformly bounded and equicontinuous on each compact in* $R_+^1 \times R^3 \times [0, T]$.

The proof is similar to the proof of Lemma 10.3 □

By standard diagonal process we choose from the sequence $\{c_n\}_{n=1}^\infty$ a subsequence $\{c_{n'}\}_{n'=1}^\infty$, which converges on each compact to a continuous function $c \geq 0$. Such subsequence exists due to Lemma 11.5. As the corollary of the Lemma 11.4 we have as well

$$\sup_{0 \leq \lambda \leq \tilde{\lambda}} \|c\|_\lambda \leq E.$$

Lemma 11.4 allows us to pass to limit in the equation (11.1) written in the integral form. Actually, this lemma ensures that "tails" of integrals $\int_m^\infty$ tend to zero uniformly with respect to n as $m \to \infty$. Consequently, the function c satisfies the integral equation

$$c(x,z,t) = c_{01}(x(0), z(0)) +$$

$$+ \int_{\Gamma(x,z,t)} \left\{ \frac{1}{2} \int_0^{x(s)} K(x(s)-y, y) c(x(s)-y, z(s), s) c(y, z(s), s) dy - \right.$$

$$- \left[r'(x(s)) + \operatorname{div}_z v(x(s), z(s)) \right] c(x(s), z(s), s) -$$

$$\left. -c(x(s), z(s), s) \int_0^\infty K(x(s), y) c(y, z(s), s) dy \right\} ds, \qquad (11.38)$$

where $\Gamma(x,z,t)$ is the part $s \leq t$ of the characteristic going through the point (x,z,t). In (11.38) we assume that the characteristic begins at the coordinate axis $t = 0$. As long as the characteristic begins at the coordinate axis $x = 0$ then the expression (11.38) will be unsufficiently changed. Direct differentiation (11.38) persuades us that the function c has continuous derivatives along characteristics. Existence of a solution $c \in \Omega^+(T)$ has been proved.

11.4 Proof of Uniqueness

We prove uniqueness by contradiction. Suppose that there are two solutions to the problem (11.1)-(11.3) $c_1, c_2 \in \Omega(T)$. We make the substitution

$$c_i = (1+x)^{-1} d_i(x), \quad i = 1, 2$$

and denote

$$u(x,t) = \sup_{z \in R^3} |d_1(x,z,t) - d_2(x,z,t)|, \quad \psi(x,t) = \sup_{z \in R^3} |d_1(x,z,t) + d_2(x,z,t)|.$$

From the equation (11.1) written in the integral form, we obtain

$$u(x_0,t_0) \le \int_{\Gamma(x_0,t_0)} \{\frac{1}{2}k\int_0^{x(s)} (1+x(x_0,t_0,s))u(x(x_0,t_0,s)-y,s)\psi(y,s)dy+$$

$$+k(1+x(x_0,t_0,s))u(x(x_0,t_0,s),s)\int_0^\infty \psi(y,s)dy+$$

$$+k(1+x(x_0,t_0,s))\psi(x(x_0,t_0,s),s)\int_0^\infty u(y,s)dy+$$

$$+M(1+x)u(x(x_0,t_0,s),s)\}ds, \qquad (11.39)$$

$$x_0 \in R^1_+, \quad 0 \le t_0 \le T.$$

Here $\Gamma(x_0,t_0)$ is an ortogonal projection of the curve $\Gamma(x_0,z,t_0)$ on the plane (x,t); $x(x_0,t_0,s)$ is a current value of the variable x on the curve $\Gamma(x_0,t_0)$ which depends on the parameter $s \le t_0$. We should point out that $\Gamma(x,t)$ is an integral curve of the equation

$$\frac{dx}{dt} = r(x). \qquad (11.40)$$

Lemma 11.6. *Let a nonnegative function $u(x,t)$ be a solution to the integral inequality (11.39). Then there exists such continuous differentiable by both arguments function $f(x,t)$ that*

$$u(x,t) \le f(x,t), \quad x \in R^1_+, \quad 0 \le t \le T, \qquad (11.41)$$

$$f(x,0) \equiv f(0,t) \equiv 0 \qquad (11.42)$$

and the function f satisfies the following differential inequality

$$\frac{\partial}{\partial t}f(x,t) + r(x)\frac{\partial}{\partial x}f(x,t) \le \frac{1}{2}k(1+x)\int_0^x f(x-y,t)\psi(y,t)dy+$$

$$+k(1+x)f(x,t)\int_0^\infty \psi(y,t)dy+$$

$$+k(1+x)\psi(x,t)\int_0^\infty f(y,t)dy + M(1+x)f(x,t). \qquad (11.43)$$

Proof. Let us denote the right-hand side of the inequality (11.39) as $f(x_0, t_0)$ and the contents of the braces $\{.\}$ under the main integral in (11.39) as $w(x(s), s)$. Then for derivatives of the function f we obtain

$$\frac{\partial f(x_0, t_0)}{\partial t_0} = w(x_0, t_0) + \int_0^{t_0} w'_x(x(x_0, t_0, s), s) x'_{t_0}(x_0, t_0, s) ds, \qquad (11.44)$$

$$\frac{\partial f(x_0, t_0)}{\partial x_0} = \int_0^{t_0} w'_x(x(x_0, t_0, s), s) x'_{x_0}(x_0, t_0, s) ds. \qquad (11.45)$$

Our nearest goal is to estimate the derivative $x'_{t_0}(x_0, t_0, s)$. Integration of the equation (11.40) yields

$$t_0 - s = Q(x_0) - Q(x(x_0, t_0, s)),$$

where $Q(x)$ is the primitive function to $[r(x)]^{-1}$. Hence,

$$x(x_0, t_0, s) = Q^{-1}(Q(x_0) - t_0 + s). \qquad (11.46)$$

Here the function Q^{-1} is the inverse function to Q. Utilizing the rule for differentiating of the inverse function and taking into account $Q'(x) = [r(x)]^{-1}$, we obtain

$$x'_{t_0}(x_0, t_0, s) = -r(x(x_0, t_0, s)). \qquad (11.47)$$

Similarly from (11.46) we conclude

$$x'_{x_0}(x_0, t_0, s) = \frac{r(x(x_0, t_0, s))}{r(x_0)}. \qquad (11.48)$$

By substituting the expressions (11.47) and (11.48) into (11.44), (11.45) and utilizing the inequality (11.39) and definition of function f, we establish (11.41). This proves Lemma 11.6 □

Proof of Theorem 11.1. Let us integrate (11.43) in x with weight $\exp(\lambda x)$ and $x \exp(\lambda x)$, $0 \le \lambda \le \tilde{\lambda}$. Then for

$$G(\lambda, t) = \int_0^\infty \exp(\lambda x) f(x, t) dx, \quad \Psi(\lambda, t) = \int_0^\infty \exp(\lambda x) \psi(x, t) dx$$

we obtain the following correlations with (11.42) taken into account

$$G_t(\lambda,t) \le \left(\frac{3}{2}k\Psi(\lambda,t)+M\right)G_\lambda(\lambda,t)+\left[\frac{5}{2}k(\Psi+\Psi_\lambda)+R(2+\tilde{\lambda})+M\right]G(\lambda,t),$$

$$G_{\lambda t}(\lambda,t) \le \frac{\partial}{\partial\lambda}\left(\left(\frac{3}{2}k\Psi(\lambda,t)+M\right)G_\lambda(\lambda,t)+\right.$$
$$\left.+\left[\frac{5}{2}k(\Psi+\Psi_\lambda)+R(2+\tilde{\lambda})+M\right]G(\lambda,t)\right).$$

Thanks (11.42) and Lemma 4.1 we obtain

$$\int_0^\infty \exp(\lambda x)f(x,t)=0 \tag{11.49}$$

in the region R defined in Lemma 4.1. Since $f(x,t)$ is continuous, $f(x,t)=0$ for $0\le t\le t'$, $0\le x<\infty$. Consequently, the integral (11.49) is equal to zero not only in R but for all $0\le\lambda\le\tilde{\lambda}$, $0\le t\le t'$. Applying the same reasonings to the interval $[t',2t']$ we conclude that $f(x,t)=0$ for $0\le t\le 2t'$, $0\le x<\infty$, and, continuing this process, we establish that $f(x,t)\equiv 0$. Utilizing (11.41) completes the proof of the Theorem 11.1 □

Remark 11.2. *The above proof of uniqueness in the class $\Omega(T)$ is valid for more general coagulation kernels $K(x,y)\le k(1+x)(1+y)$.*

11.5 Remarks

In Chapters 10, 11 we follow Dubovskiĭ [21] and Chae and Dubovskiĭ [15]. Influence of the condensation processes on the evolution of a coagulating system was studied for the space homogeneous case by Srivastava and Passarelli [67], Gajewski and Zaharias [32,33] and by Dubovskiĭ in [22] (where the generalized solutions in terms of Borel measures were constructed).

In chapters 10, 11 we use ideas of Arsen'ev [2], Maslova *et al* [49, 50, 51], Nishida and Imai [57], Ukai and Asano [75, 76] where it was proved for Boltzmann equation of the kinetic theory of gases, that the problem has a solution in a sufficiently small neighbourhood of an equilibrium state (with a Maxwell distribution function). An important feature of the proof is the establishment of the exponential contraction of the semigroup generated by the linearized Boltzmann equation (Arsen'ev, [2]). Since Smoluchowski's equation has the trivial solution $\equiv 0$ as an equilibrium state, so that the global results of chapters 10, 11 are obtained for sufficiently small initial data.

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