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BLOCH - BESOV SPACES AND THE BOUNDARY BEHAVIOR OF THEIR FUNCTIONS

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PREFACE

This lecture note is an outcome of a series of lectures given by the author at Seoul National University in September, 1993, as part of the research developement program of the Global Analysis Research Center.

It is intended to introduce to the readers some of the recent results of various spaces of holomorphic functions and Mobius invariant harmonic functions of several complex variables with special emphasis on the Bloch-Besov functions in the unit ball B. There has been considerable research activity on this area in the past decade. The main purpose of this lecture note is to provide to the readers a cohesive treatment of the subject. It is hoped that this lecture note will not only provide a useful source of information on the subject, but also motivate the readers for further research of this important field.

In the first chapter we introduce some basic notation and preliminary results which will be used throughout this lecture note. In the second chapter we review standard properties of the reproducing kernel under a rather general setting, and consider an important special case of weighted Bergman kernel and Bergman spaces on the unit ball.

In the third chapter we discuss the basic properties of the Bergman metric and related Bergman geometry in the view point of analysis. In constructing the Bergman metric we take a non-standard approach by embedding a domain into an infinite dimensional projective space by a holomorphic mapping. We also give an inequality between the Bergman metric and the Carathéodory metric.

The notions of Bloch and Besov spaces of M-harmonic (Mobius invariant harmonic) functions are being introduced in Chapter 4. In the first section of

Chapter 5, several characterizations of Besov p-spaces with weights are given, and more interesting properties of such spaces are studied in Section 2. In section 3, it is demonstrated that there are three values of weights of Besov spaces for which the corresponding spaces are most interesting. Certain characteristic properties of such spaces are briefly discussed in Chapter 6.

In Chapter 7, the boundary behavior of M-harmonic fuctions and M-subharmonic functions are discussed. It is shown that an M-harmonic function in a Besov p-spaces of weight n has admissible boundary limits almost everywhere on the boundary of the unit ball when $p > 2n$. A similar result is not true when the weight is larger than n. In fact, there is a function in such a space which has no radial limits on a set of positive measure. On the other hand, it is shown that the M-harmonic Besov functions with weights less than n/τ, where $\tau > 1$ is the order of tangency of the approach region with the boundary of the unit ball, have tangential limits of order τ almost everywhere on the boundary, and are Lipschitz continuous on the closure of the ball if the weighs are negative.

Finally, the author would like to thank the faculty members of the Department of Mathematics at Seoul National University for their warm hospitality during his stay. He is particularly indebted to Professors Sang Moon Kim, Chung-Hyuk Kang, Dong-Pyo Chi, and Jongsik Kim for their effort in inviting him to the GARC.

The author is deeply saddend to learn the passing of Professor Jongsik Kim, the Director of Global Analysis Research Center at Seoul National University. It is a great loss to all of us who knew him personally and a loss to mathematics in general.

CHAPTER I INTRODUCTION

In this section we introduce some basic notation and preliminary results which will be used throughout this lecture notes. The notation of this notes follows closely those in [44], many of the proofs of the preliminary facts can be found there.

1. Notation.

Throughout these lecture notes, $\mathbb{C}$ will denote the set of complex numbers and $\mathbb{C}^n$ the cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by

$$< z, w >= \sum_{j=1}^{\infty} z_j \bar{w}_j$$

and the norm by $|z| = \sqrt{< z, z >}$. The standard orthonormal basis in $\mathbb{C}^n$ are denoted by $e_1, \cdots, e_n$, where $e_j = (0, \cdots, 1, \cdots, 0)$ with 1 at the jth place for $j = 1, \cdots, n$. For a $\epsilon \mathbb{C}^n, r > 0$, let

$$B(a, r) = \{z \epsilon \mathbb{C}^n; |z - a| < r\}.$$

For the sake of simplicity, the unit ball $B(0, 1)$ will be denoted by either B or B_n. The boundary ∂B of B is the unit sphere $S = \{z : |z| = 1\}$. When $n = 1$, the unit disc in $\mathbb{C}$ will be denoted by $U = B_1$ and the boundary by T. For $n > 1$, the cartesian product U^n of n copies of the disc U is called the unit polydisc in $\mathbb{C}^n$. The torus $T^n = \{z : |z_j| = 1, j = 1, \cdots, n\}$ is called the distinguished boundary of U^n, which is a proper subset of the topological boundary ∂U^n. The polydisc of radii $r = (r_1, \cdots, r_n)$ with the center $a = (a_1, \cdots, a_n)$ is the set

$$U^n(a, r) = \{z \epsilon \mathbb{C}^n : |z_j - a_j| < r_j, \quad j = 1, \cdots, n\}$$

Since our discussion of functions of several complex variables requires multi-index notation, we introduce the following standard conventions. Let $\mathbb{N} = \{1, 2, \cdots\}$ denote the set of natural numbers. A multi-index α is an ordered n-tuple : $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_j \epsilon \mathbb{N}, \ j = 1, \cdots, n$. For a multi-index α and $z \epsilon \mathbb{C}^n$, set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$
$$\alpha! = \alpha_1! \cdots \alpha_n!,$$
$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

As in the case of $n = 1$, for $z = (z_1, \cdots, z_n)$, we write $z_j = x_j + iy_j$, $j = 1, \cdots, n$, and define the differential operators :

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

If no confusion arises we will use the notation $\partial_j = \frac{\partial}{\partial z_j}$ and $\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j}$. Moreover, for all multi-indices α, we will write

$$\partial^\alpha f = \frac{\partial^\alpha f}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

2. Definition and Preliminary Results.

Let Ω be an open set in $\mathbb{C}^n, k \epsilon \mathbb{N}$, we denote by $C^k(\Omega)$ the space of real or complex valued functions on Ω which have continuous derivatives of order α for al multi-indices α with $|\alpha| \leqq k$. $C_c^k(\Omega)$ denotes those functions in $C^k(\Omega)$ with compact suport. Similarly, we define $C^\infty(\Omega)$ and $C_c^\infty(\Omega)$.

Definition 2.1. Let Ω be an open set of $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if f is holomorphic in each variable separately i.e., for each a $\epsilon\Omega$ and each $i = 1, \cdots, n$, the function

$$\lambda \to f(a + \lambda e_i)$$

is holomorphic in an open neighborhood of 0 in $\mathbb{C}$. The set of holomorphic functions on Ω will be denoted by $\mathrm{H}(\Omega)$.

The space $\mathrm{H}(\Omega)$ is an algebra with respect to pointwise addition and product, and also a Frechet space with the topology of uniform convergence on Ω.

It is a classical result, due to Hartogs [28], that if f is holomorphic in each variable separately as defined above, then f is continuous in Ω.

As a result, we obtain the following Cauchy integral formula [38].

Theorem 2.1 (The Cauchy Integral Formula for Polydiscs). *If f is a holomorphic function in an open set $\Omega \subset \mathbb{C}^n$, for each a $\epsilon\Omega$ and n-tuple r of positive numbers with $\overline{U(a,r)} \subset \Omega$, we have*

(2.1)

$$f(z) = \int_{|w_1 - a_1| = r_1} \cdots \int_{|w_n - a_n| = r_n} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \cdots dw_n.$$

for all $z \in U^n(a, r)$.

Among many consequences of the Cauchy integral formula, we list several properties that are pertinent here :

(i) If $f \in H(\Omega)$, *where* Ω *is an open set in* $\mathbb{C}^n$, *for every* $a \in \Omega$ f *has a power series expansion*

$$f(z) = \sum_{\alpha} a_\alpha (z-a)^\alpha \tag{2.2a}$$

which converges absolutely and uniformly in every polydisc centered at a and whose closure is contained in Ω, *where the sum runs over all multi-indices* α, *and* a_α *is given by*

$$a_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial {z_1}^{\alpha_1} \cdots \partial {z_n}^{\alpha_n}} = \frac{\partial^\alpha f(a)}{\alpha!}. \tag{2.2b}$$

(ii) The <u>*identity theorem*</u> *holds : Let* Ω *be a domain (connected open set) and* $f \in H(\Omega)$. *If* $f(z) = 0$ *in some non-empty open subset of* Ω, *then* $f \equiv 0$ *on* Ω.

(iii) The <u>*maximum modulus principle*</u> *holds : If* $f \in H(\Omega)$, Ω *is connected and* $|f|$ *has a local maximum in* Ω, *then f is constant.*

(iv) For each compact subset K *of* Ω, *there exists a constant* C *which depends only on* K, α *such that for all* $f \in H(\Omega)$

$$\sup_{z \in K} |\partial^\alpha f(z)| \leqq C \sup_{z \in K} |f(z)|. \tag{2.3}$$

Consequently, if $f_n \in H(\Omega)$ *and* $f_n \to f$ *uniformly on compact subsets of* Ω, *then* $\partial^\alpha f_n \to \partial^\alpha f$ *uniformly on compact subsets of* Ω *for all multiindices* α.

(v) Suppose $f \in C^1(\Omega)$. *Then* $f \in H(\Omega)$ *if and only if f satisfies the* <u>*Cauchy-Riemann equations*</u> : $\bar{\partial}_j f = 0$ *for* $j = 1, \cdots, n$.

(vi) If $f \in C^1(\Omega)$, *then the* <u>*(real) gradient*</u> *of f in complex form is given by*

$$df(z) = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \bar{z}_1}, \cdots, \frac{\partial f}{\partial \bar{z}_n}\right) = (\partial f, \bar{\partial} f). \tag{2.4a}$$

If f is holomorphic in Ω, *it reduces to*

$$\nabla f(z) = \partial f(z) = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right). \tag{2.4b}$$

For a function $f \epsilon C^\infty(B)$ and $m = 1, 2, \cdots$, we define

(2.4c)

$$\partial^m f(z) = (\partial^\alpha f(z))_{|\alpha|=m}, \quad \bar\partial^m f(z) = \left(\bar\partial^\alpha f(z)\right)_{|\alpha|=m},$$
$$d^m f(z) = \left(\partial^\alpha \bar\partial^\beta f(z)\right)_{|\alpha|+|\beta|=m},$$

where $\partial^\alpha f(z) = \frac{\partial^{|\alpha|} f(z)}{\alpha z_\alpha}$, $\bar\partial^\alpha f(z) = \frac{\partial^{|\alpha|} f(z)}{\alpha \bar z_\alpha}$, *and α and β are multiindicies.*
Further we define

(2.4d)

$$|\partial^m f(z)| = \sum_{|\alpha|=m} \left|\frac{\partial^{|\alpha|} f(z)}{\partial z_\alpha}\right|, \quad |\bar\partial^m f(z)| = \sum_{|\alpha|=m} \left|\frac{\partial^{|\alpha|} f(z)}{\partial \bar z_\alpha}\right|,$$
$$|d^m f(z)| = \sum_{|\alpha|+|\beta|=m} \left|\frac{\partial^m f(z)}{\partial z_\alpha \partial \bar z_\beta}\right|.$$

(vii) If $w = \varphi(z) = (\varphi_1(z), \cdots, \varphi_k(z))$ is a C^1 mapping of a domain $\Omega_1 \subset \mathbb{C}^n$ into a domain $\Omega_2 \subset \mathbb{C}^k$, and $f \in C^1(\Omega_2)$, then for $g(x) = f(\varphi(z))$, the following complex forms of the chain rule hold :

$$\frac{\partial g}{\partial z_j} = \sum_{i=1}^k \left(\frac{\partial f}{\partial w_i}\frac{\partial w_i}{\partial z_j} + \frac{\partial f}{\partial \bar w_i}\frac{\partial \bar w_i}{\partial z_j}\right) \tag{2.5a}$$

$$\frac{\partial g}{\partial \bar z_j} = \sum_{i=1}^k \left(\frac{\partial f}{\partial w_i}\frac{\partial w_i}{\partial \bar z_j} + \frac{\partial f}{\partial \bar w_i}\frac{\partial \bar w_i}{\partial \bar z_j}\right). \tag{2.5b}$$

Definition 2.2. Let Ω be an open set in $\mathbb{C}^n$. A mapping

$$\varphi = (\varphi_1, \cdots, \varphi_k) : \Omega \to \mathbb{C}^k$$

is called holomorphic if each component function $w_i = \varphi_i(z)$ $(i = 1, \cdots, k)$ is holomorphic in Ω. Namely, the mapping $\varphi \epsilon C^1(\Omega)$ is holomorphic if and only if it satisfies :

$$\frac{\partial \varphi_i}{\partial \bar z_j} = 0 \quad (i = 1, \cdots, k; j = 1, \cdots, n).$$

The following theorem is an immediate consequence of the chain rules (2.5a) and (2.5b).

Theorem 2.2. *Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^k$ be open sets. The mapping*

$$\varphi = (\varphi_1, \cdots, \varphi_k) : \Omega \to D$$

is holomorphic if and only if for every $f \epsilon H(D), f \circ \varphi \epsilon H(\Omega)$. Moreover,

$$\nabla (f \circ \varphi)(z) = \varphi'(z) \cdot \nabla f(z),$$

where $\varphi'(z) = \frac{d\varphi}{dz}(z) = (\partial \varphi_i / \partial z_j)$ $(i = 1, \cdots k; j = 1, \cdots, n)$ denotes the $k \times n$ (holomorphic) Jacobian matrix of φ.

Definition 2.3. The mapping $\varphi : \Omega \to D$ is said to be non-singular at $z^\circ \epsilon \Omega$ if the rank of φ' is maximal at z°, that is,

$$\text{rank}\varphi'(z^\circ) = \min(n, k).$$

In the case where $n = k$, the (holomorphic) Jacobian determinant of φ will be denoted by

$$(J_\varphi)(z) = \det\varphi'(z).$$

The set of all holomorphic mappings $\varphi : \Omega \to D$ will be denoted by $Hol(\Omega, D)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^k$ be open sets. If $\varphi \epsilon$ Hol (Ω, D) and $\psi \epsilon$ Hol $(D, \mathcal{C}^\ell)$, then $\psi \circ \varphi \epsilon$ Hol $(\Omega, \mathcal{C}^\ell)$ and*

$$(\psi \circ \varphi)'(z) = \psi'(w) \cdot \varphi'(z), \quad w = \varphi(z).$$

If $n = k = \ell$, then

$$J(\psi \circ \varphi)(z) = (J\psi)(w)(J\varphi)(z).$$

That $\varphi \epsilon$ Hol (Ω, D) is bijective is enough to guarantee that the inverse mapping φ^{-1} is also holomorphic and that $J_\varphi(z) \neq 0$.

3. Automorphism Groups

Definition 3.1. Let Ω and D be domains in $\mathbb{C}^n$. A mapping

$$\varphi : \Omega \to D$$

is said to be biholomorphic if it is bijective, and both φ and φ^{-1} are holomorphic.

Definition 3.2. Let Ω be a domain in $\mathbb{C}^n$. The group of all biholomorphic mappings of Ω onto itself under composition is called the (holomorphic) automorphism group of Ω and denoted by Aut (Ω).

A domain Ω is homogeneous if the group Aut (Ω) is transitive, i.e., any two points in Ω can be mapped each other by a member of Aut(Ω).

Theorem 3.1 (Cartan's Theorem I) [38]. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $z^\circ \epsilon \Omega$. If $\varphi \epsilon$ Hol (Ω, Ω) such that*

$$\varphi(z^\circ) = z^\circ \quad \text{and} \quad \varphi'(z^\circ) = I = \textit{identity map},$$

then $\varphi(z) = z$ for all $z \epsilon \Omega$.

This theorem is often called Cartan's uniqueness theorem which is useful in computing automorphism groups of various circular domains.

Definition 3.3. A domain $\Omega \subset \mathbb{C}^n$ is called circular about 0 $\epsilon\Omega$ if $e^{i\theta} z \epsilon \Omega$ whenever $z \epsilon \Omega$ and $\theta \epsilon \mathbb{R}$, complete circular if $\lambda z \epsilon \Omega$ whenever $\lambda \epsilon \mathbb{C}$, $|\lambda| < 1$, and Reinhardt circular if $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \epsilon \Omega$ whenever $z \epsilon \Omega$ and $\theta_i \epsilon \mathbb{R}$ for $i = 1, \cdots, n$

Theorem 3.2 (Cartan's Theorem II) [38]. *Let Ω and D be bounded circular domains in $\mathbb{C}^n$ that contains the origin 0. Every biholomorphic mapping $\varphi : \Omega \to D$ such that $\varphi(0) = 0$ reduces to a linear map in $\mathbb{C}^n$.*

In the case of the unit disc $U \subset \mathbb{C}$, the group Aut (U) is well-known. It consists of the maps

$$\varphi_a(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}, \quad z \epsilon U,$$

where a ϵU and $\theta \epsilon \mathbb{R}$. These mappings are known as the Moebius transformations of U. It is easily cheked that the mapping φ_a satisfies :

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \text{and} \quad \varphi_a^{-1} = \varphi_a.$$

We now describe the group Aut B when $n > 1$. Let a ϵ B and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by

$$\text{(3.1a)} \qquad P_a z = \frac{< z, a >}{< a, a >} a, \quad \text{if } a \neq 0, \quad \text{and } P_0 = 0, \quad \text{if } a = 0.$$

Let $Q_a = I - P_a$. *Define* φ_a *on* B *by*

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - < z, a >}. \tag{3.1b}$$

Notice that $\varphi_0(z) = -z$. *It is not hard to see that* φ_a *is actually an automorphism of* B, *i.e.,* $\varphi_a \epsilon$ *Aut* B.

The automorphisms φ_a have several important properties that will be used repeatedly in this lecture. The following theorem contains some of those properties :

Theorem 3.3 [44]. *Let* $a \epsilon B$. *Then*

(i) $\varphi_a(0) = a, \varphi_a(a) = 0$, and $\varphi_a(\varphi_a(z)) = z$
(ii) $\varphi'(0) = -(1-|a|^2)P_a - \sqrt{1-|a|^2} Q_a$
(iii) For all z, w $\epsilon \bar{B}$, we have

$$1 - < \varphi_a(z), \varphi_a(w) > = \frac{(1-|a|^2)(1 - < z, w >)}{(1 - < z, a >)(1 - < a, w >)}, \tag{3.2a}$$

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1 - < z, a > |^2}. \tag{3.2b}$$

An important consequence of (i) of Theorem 3.3 is that the map $\varphi_b \circ \varphi_a$ *carries a to b. Therefore, Aut (B) acts transitively on B. It is also important to notice that the map :* $a \to \varphi_a$ *is continuous from B into Aut B. Hence, if we identify* φ_a *with a, then* $\varphi_{a_n} \to \varphi_a$ *in Aut B if and only if* $a_n \to a$ *in B. Finally, if* $\psi \epsilon$ *Aut B and* $a = \psi^{-1}(0)$, *then there exists a unique* $U \epsilon U_n$, *the group of unitary transformations of* $\mathbb{C}^n$, *such that* $\psi = U\varphi_a$. *This result follows from the fact that* $\psi \circ \varphi_a$ *is an automorphism of B which fixes 0, and thus by Cartan's theorem is linear, and hence must be unitary.*

Theorem 3.4. *Let* ψ *be a holomorphic mapping from an open set* Ω *into an open set D in* $\mathbb{C}^n$. *Then the determinant* $J_{\mathbb{R}}\psi$ *of the real Jacobian matrix of* ψ *satisfies the following identity :*

$$J_{\mathbb{R}}\psi(z) = |J\psi(z)|^2 = |\det \psi'(z)|^2. \tag{3.3a}$$

Moreover, if in particular $\psi \epsilon$ Aut (B) and $a = \psi^{-1}(0)$, then

$$(3.3b) \qquad |J\psi(z)|^2 = \left[\frac{1-|a|^2}{|1-<z,a>|^2}\right]^{n+1} = \left[\frac{1-|\psi(z)|^2}{1-|z|^2}\right]^{n+1}.$$

4. Integral Formulas on B

Let ν be the Lebesque measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. If V denotes the usual euclidean volume measure in $\mathbb{C}^n$, then $c_n d\nu = dV$, where $c_n = V(B_n) = \pi^n/n!$. Let σ be the rotation invariant surface measure on S normalized by $\sigma(S) = 1$. The following integration formulas, the proofs of which may be found in [44], will be used throughout this lecture.

Theorem 4.1.

$$(4.1a) \qquad \int_{\mathbb{C}^n} f d\nu = 2n \int_0^\infty r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr$$

$$(4.1b) \qquad \int_S f d\sigma = (1/2\pi) \int_S \int_0^{2\pi} f(e^{i\theta}\zeta) d\theta d\sigma(\zeta)$$

$$(4.1c) \qquad \int_S f d\sigma = (1/2\pi) \int_{B_{n-1}} \int_0^{2\pi} f(\zeta', e^{i\theta}\zeta_n) d\theta d\nu(\zeta')$$

$$(4.1d) \qquad \int_S f d\sigma = \int_U f(U\eta) dU.$$

Here $\zeta' = (\zeta_1, \cdots, \zeta_{n-1}) \epsilon \mathbb{C}^{n-1}$ and dU denotes the Haar measure on the group $U = U(n)$ of the unitary transformations of $\mathbb{C}^n$.

A linear transformation $U \epsilon \mathcal{U}$ if and only if $< Uz, Uw > = < z, w >$ for all $z, w \epsilon \Omega$. The group $\mathcal{U}$ is a compact subgroup of $0(n)$, the orthogonal group of $\mathbb{C}^n$.

In addition to the above integral formulas, the following formulas hold :

Theorem 4.2 [44]. *If f is a function of a single variable, then for $n > 1$, and $\zeta \epsilon S$, we have*

$$\int_S f(<\zeta,\eta>)d\sigma(\zeta) = \frac{n-1}{\pi}\int_U (1-r^2)^{n-1} f(re^{i\theta})rdrd\theta. \tag{4.2}$$

If α, β are multi-indices, then the following hold :

$$\int_S \zeta^\alpha \bar{\zeta}^\beta d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}, \quad \text{if } \alpha = \beta, \quad = 0, \quad \text{if } \alpha \neq \beta. \tag{4.3a}$$

$$\int_B z^\alpha \bar{z}^\beta dV(z) = \frac{n!\alpha!}{(n+|\alpha|)!}, \quad \text{if } \alpha = \beta, \quad = 0, \quad \text{if } \alpha \neq \beta. \tag{4.3b}$$

If λ denotes the measure defined on B by

$$d\lambda(z) = (1-|z|^2)^{-n-1}d\nu(z), \tag{4.4a}$$

then

$$\int_B f d\lambda = \int_B (f \circ \psi)(z) d\lambda(z) \tag{4.4b}$$

for every $f \epsilon L^1(B,\lambda)$ and every $\psi \epsilon$ Aut B. The measure λ is invariant under Aut B.

For a ϵ B and $r\epsilon(0,1)$, we define : $E(a,r) = \varphi_a(rB)$, $rB = B(0,r)$. Then $z\epsilon E(a,r)$ if and only if $|\varphi_a(z)| < r$. A little manupulation shows that $E(a,r)$ is given by an ellipsoid of the form :

$$E(a,r) = \left\{ z\epsilon B : \frac{|P_z - c|^2}{\rho^2} + \frac{|Qz|^2}{\rho} < r^2 \right\}, \tag{4.5a}$$

where $P = P_a$, $Q = Q_a$, and

$$c = \frac{(1-r^2)a}{1-r^2|a|^2}, \quad \rho = \frac{1-|a|^2}{1-r^2|a|^2}. \tag{4.5b}$$

This means that the intersection of $E(a,r)$ with the complex line [a] generated by a is a disc with center c and radius $r\rho$, whereas it's intersection with the $(n-1)$ dimensional subspace perpendicular to [a] is a ball of radius $r\sqrt{\rho}$. An easy computation shows that

$$\lambda(E(a,r)) = \frac{r^{2n}}{(1-r^2)^n}, \tag{4.5c}$$

and that it is independent of a. It is also clear that the ratio between $\nu(E(a,r))$ and $\nu(rB)$ is given by ρ^{n+1}. Therefore, we obtain

$$(J_R\varphi_a)(0) = \lim_{r\to 0} \frac{\nu(E(a,r))}{\nu(rB)} = (1-|a|^2)^{n+1}. \tag{4.5d}$$

Chapter II Reproducing Kernel and Weighted Bergman Spaces

In this chapter we discuss some elementary but basic properties of reproducing kernel under a rather general setting, and then consider an important special case of weighted Bergman kernel and Bergman spaces on the open unit ball B more in details. See also [11], [6], [32].

1. Reproducing Kernel

Let H be a Hilbert space of functions on some set Ω such that the point evaluations $f \to f(z)$ are continuous linear functional on H for all $z \epsilon \Omega$. Then, by the Riesz representation theorem, for each $z \epsilon \Omega$, there exists a unique function $K_z \epsilon H$ such that

(1.1a) $$f(z) = (f, K_z), \quad f \epsilon H \quad \text{and} \quad z \epsilon \Omega.$$

We define the reproducing kernel as the function $K(z, w) = K_w(z)$ of z and $w \epsilon \Omega$. Clearly,

(1.1b) $$K(z, w) = K_w(z) = (K_w, K_z).$$

Consequently,

(1.1c) $$K(w, z) = \overline{K(z, w)}$$

(1.1d) $$K(z, z) = \|K_z\|^2 \geqq 0$$

(1.1e) $$|K(z, w)|^2 \leqq K(z, z)K(w, w)$$

(1.1f) $$|f(z)| \leqq \|f\|\|K_z\| = K(z, z)^{1/2}\|f\|$$

(1.1g) $K(z, z) = 0$ if and only if $f(z) = 0$ for every $f \epsilon H$.

Furthermore, we have

Proposition 1.1. *Suppose that $\{\varphi_\alpha\}$ is an orthonormal basis for the Hilbert space H. Then for every compact subset K of Ω,*

$$K(z,w) = K_w(z) = \sum_{\alpha=1}^{\infty} \varphi_\alpha(z)\overline{\varphi_\alpha(w)} \tag{1.1h}$$

converges absolutely and uniformly on $K \times K$. In particular, $K(z,w)$ is independent of the choice of the orthonormal basis $\{\varphi_\alpha\}$.

Proof. By the Riesz-Fisher theorem, combined with (1.1f), we have

$$\begin{aligned}
&\sup\left\{\left(\sum_{\alpha=1}^{\infty}|\varphi_\alpha(z)|^2\right)^{1/2} : z\epsilon K\right\}\\
&= \sup\left\{\left|\sum_{\alpha=1}^{\infty} a_\alpha\varphi_\alpha(z)\right| : z\epsilon K,\quad \sum_{\alpha=1}^{\infty}|a_\alpha|^2 = 1\right\}\\
&= \sup\{|f(z)| : z\epsilon K,\ \|f\|_2 = 1\}\\
&= \sup\{K(z,z)^{1/2} : z\epsilon K\} \leqq C(K)
\end{aligned}$$

for some constant $C(K) > 0$ which depends only on K. The convergence of the series in (1.1h) is uniform on $K \times K$, since

$$\sum_{\alpha=1}^{\infty}|\varphi_\alpha(z)\overline{\varphi_\alpha(w)}| \leqq \left(\sum_{\alpha=1}^{\infty}|\varphi_\alpha(z)|^2\right)^{1/2}\left(\sum_{\alpha=1}^{\infty}|\varphi_\alpha(w)|^2\right)^{1/2}.$$

For each $f\epsilon H$, the series

$$f = \sum_{\alpha=1}^{\infty}(f,\varphi_\alpha)\varphi_\alpha$$

converges in the Hilbert space and thus it converges uniformly on compact subsets of Ω by (1.1f). In particular, if $z\epsilon\Omega$

$$f(z) = \sum_{\alpha=1}^{\infty}(f,\varphi_\alpha)\varphi_\alpha(z) = \left(f, \sum_{\alpha=1}^{\infty}\overline{\varphi_\alpha(z)}\varphi_\alpha\right).$$

Since $\sum_{\alpha=1}^{\infty}\varphi_\alpha(z)\varphi_\alpha\epsilon H$, the uniqueness of the Riesz representation shows that

$$K(w,z) = K_z(w) = \sum_{\alpha=1}^{\infty}\overline{\varphi_\alpha(z)}\varphi_\alpha(w)$$

from which (1.1h) follows by (1.1c).

Proposition 1.2. *If every function in H is continuous, then the following are equivalent :*

(i) $(z,w) \to K(z,w)$ is continuous;
(ii) $z \to K(z,z)$ is continuous;
(iii) $z \to K_z$ is a continuous mapping Ω into H.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. To show (ii) $\Rightarrow$ (iii) fix z. If $w \to z$ then $K(w,w) \to K(z,z)$ by (ii) and $K(w,z) = K_z(w) \to K_z(z) = K(z,z)$ because $K_z \epsilon H$. Hence, from (1.1b)

$$(1.2) \qquad \begin{aligned} ||K_w - K_z||^2 &= (K_w, K_w) + (K_z, K_z) - 2\mathrm{Re}\,(K_z, K_w) \\ &= K(w,w) + K(z,z) - 2\mathrm{Re}\,K(w,z) \to 0, \end{aligned}$$

as $z \to w$. The implication (iii) $\Rightarrow$ (i) is immediate again by (1.1b). □

Let μ be a measure on Ω and let $L^2(\mu) = L^2(\Omega, \mu)$ denote the space of square integrable measurable functions on Ω with respect to the measure μ. Suppose that H is a closed subspace of $L^2(\mu)$ such that the point evaluations are continuous on H. Note that the functions in H thus are well-defined everywhere, although functions in $L^2(\mu)$ are defined only almost everwhere on Ω.

Let P denote the orthogonal projection $L^2(\mu) \to H$. Then (1.1c) implies that for any $f \epsilon L^2(\mu)$ and $z \epsilon \Omega$,

$$(1.3) \qquad \begin{aligned} (Pf)(z) &= (Pf, K_z) = (f, PK_z) = (f, K_z) \\ &= \int_\Omega f(w)\overline{K_z(w)}d\mu(w) = \int_\Omega K(z,w)f(w)d\mu(w). \end{aligned}$$

2. Change of Gauge and Change of Variables.

Definition 2.1. Let φ be a non-zero measurable function on Ω. Then the map

$$(2.1) \qquad f \to \varphi f, \quad \mu \to |\varphi|^{-2}\mu$$

maps $L^2(\mu)$ isometrically onto $\varphi L^2(\mu) = L^2(|\varphi|^{-2}\mu)$, and H onto the subspace $\varphi H = \{f : \varphi^{-1} f \in H\}$ of $L^2(|\varphi|^{-2}\mu)$. The map given by (2.1) is called a <u>change of gauge</u>.

Since $\{\varphi\varphi_\alpha\}$, where $\{\varphi_\alpha\}$ is an orthonormal basis of φH, the reproducing kernel for the space φH is given by

$$\varphi(z)\overline{\varphi(w)}K(z,w). \tag{2.2}$$

Consequently, the measure

$$d\lambda(z) = K(z,z)d\mu(z)$$

is invariant under all changes of gauge. It follows from the fact that a change of gauge transforms by $d\mu(z) \to |\varphi(z)|^{-2}d\mu(z)$.

Let ψ be a bijection of Ω onto Ω'. Then under the change of variables ψ, a measure μ on Ω maps onto the measure $\mu \circ \psi^{-1}$ on Ω' and the space H is mapped isometrically onto the space

$$H \circ \psi^{-1} \subset L^2(\mu) \circ \psi^{-1} = L^2(\Omega', \mu \circ \psi^{-1}).$$

The reproducing kernel for $H \circ \psi^{-1}$ is clearly

$$K(\psi^{-1}(z), \psi^{-1}(w)), \quad \text{for } z, w \epsilon \Omega'. \tag{2.3}$$

3. Basic Hilbert Space.

In the remainder of this lecture note, we assume the following :

(C. 0) : Ω is a domain in $\mathbb{C}^n$ and μ is an absolutely continuous measure on Ω which has continuous strictly positive Radon-Nikodym derivative $d\mu/d\nu$ with respect to the Lebesgue measure ν.

Our basic Hilbert space will be the space of square integrable holomorphic functions on Ω. Namely,

$$A^2(\mu) = L^2(\mu) \cap H(\Omega) = \left\{ f \epsilon H(\Omega) : \int_\Omega |f(z)|^2 d\mu(z) < \infty \right\}. \tag{3.1a}$$

with the inner product

$$(f,g) = \int_\Omega f(z)\overline{g(z)}d\mu(z), \tag{3.1b}$$

and the norm $||f|| = \sqrt{(f,f)}$. It is an easy consequence of the mean value property of holomorphic functions that $A^2(\mu)$ is a closed subspace of the Hilbert space $L^2(\mu)$ and that point evaluations are continuous. In fact, the embedding $A^2(\mu) \to L^2(\mu)$ is continuous.

Definition 3.1. The reproducing kernel for the space $A^2(\mu)$ with the Lebesgue measure $\mu = \nu$ is known as the Bergman kernel and the space $A^2(\nu)$ is the Bergman space on Ω.

From now on we will only consider analytic changes of gauge and analytic changes of variables. Note that if φ is holomorphic and nonzero in Ω, then $\varphi A^2(\mu) = A^2(|\varphi|^{-2}\mu)$. The Bergman kernel satisfies the following :

Proposition 3.2. *$K(z,w)$ is continuous on $\Omega \times \Omega$, holomorphic in z and anti-holomorphic in w. Consequently, the map $z \to K_z$ is a continuous map of Ω into $A^2(\mu)$.*

Proof : $K(z,w) = K_w(z)$ is holomorphic in z because $K_w \epsilon A^2(\mu)$. By (1.1c), $K(z,w)$ is anti-holomorphic in w. Hence $K(z,\bar{w})$ is holomorphic in each variable on $\Omega \times \Omega$. By Hartog's theorem, $K(z,\bar{w})$ is holomorphic and, in particular, continuous in $\Omega \times \Omega$.

Proposition 3.3. *If $J(z,w)$ is holomorphic in z and anit-holomorphic in w on $\Omega \times \Omega$ and that $J(z,z) = K(z,z)$ for $z \epsilon \Omega$, then $J(z,w) = K(z,w)$ for all $z, w \epsilon \Omega$.*

Proof. We may assume that $0 \epsilon \Omega$. The function $f(z,w) = J(z,\bar{w}) - K(z,\bar{w})$ is holomorphic, and $f(z,\bar{z}) = 0$ in a neighborhood of 0. Hence, $f \equiv 0$ by identity theorem.

Let

$$(3.2) \qquad G(\mu) = \{g \epsilon Aut(\Omega) : \mu \circ g^{-1} = |\varphi|^2 \mu, \quad \text{for some } \varphi \epsilon H(\Omega)\}.$$

Note that $G(\mu)$ is in general strictly smaller than $Aut(\Omega)$, while $G(\mu) = Aut(\Omega)$ for the Bergman space with $\mu =$ Lebesgue measure ν on Ω and φ being the Jacobian of g^{-1}.

Proposition 3.4. *Let $g \epsilon G(\mu)$ and let $\varphi \epsilon H(\Omega)$ be such that $\mu \circ g^{-1} = |\varphi|^2 \mu$. Then the following transformation formula for the kernel function K holds :*

$$(3.3) \qquad K(g^{-1}(z),\ g^{-1}(w)) = \varphi(z)^{-1}\overline{\varphi(w)}^{-1} K(z,w), \qquad z, w \epsilon \Omega$$

Proof. It follows from the fact that the change of gauge induced by φ^{-1} and the change of variables induced by g map the space $A^2(\mu)$ onto the same space, and hence they transform K into the same kernel.

Corollary 3.5. (a) The measure $d\lambda(z) = K(z,z)d\mu(z)$ is invariant for all $g \epsilon G(\mu)$. (b) $|K(z,w)|^2 / K(z,z)K(w,w)$ is a $G(\mu)$ - invariant function of $(z,w) \epsilon \Omega \times \Omega$.

Definition 3.2. $G(\mu)$ is called transitive if for every $z, w \epsilon \Omega$, there exists $g \in G(\mu)$ with $g(z) = w$.

Lemma 3.6. *If $G(\mu)$ is transitive and $A^2(\mu) \neq \{0\}$, then $K(z,z) \neq 0$ for all $z \epsilon \Omega$.*

Proof. If not, then by Proposition 3.4, $K(z,z) = 0$ for all $z \epsilon \Omega$, which contradicts (1.1f).

4. Weighted Bergman Spaces on the Ball.

In this section we consider the case where our domain Ω is given by the open unit ball B in $\mathbb{C}^n$ and the measure μ is the weighted Lebesgue measure μ_q :

(4.1a) $$d\mu_q = c_q(1 - |z|^2)^q d\nu(z),$$

where $q > -1$ is fixed, and c_q is a normalization constant such that $\mu_q(B) = 1$. It is given by

(4.1b) $$c_q = \frac{\Gamma(n+q+1)}{\Gamma(n+1)\Gamma(q+1)} = \binom{n+q}{n}.$$

Then

(4.2) $$A^2(B, \mu_q) = A^2_q(B) = \left\{ f \epsilon H(B) : \int_B |f|^2 d\mu_q < \infty \right\}$$

is the (weighted) Bergman space in the unit ball B.

We remark that if $q = 0$, then $d\mu_\circ = d\nu$ and $A^2(B) = A^2_\circ(B)$ is the usual Bergman space on B. If $q = -1$, we define formally $d\mu_{-1}$ by the surface measure $d\sigma$ on S, i.e., $d\mu_{-1}$ is defined as a measure on $\bar{B}$ by

$$d\mu_{-1} = \lim_{q \to -1} d\mu_q = d\sigma$$

in the weak* sense. This can be verified easily by a calculation based on polar coordinates. In particular, if f is a continuous function on $\bar{B}$, then

(4.3) $$\int_B f d\mu_{-1} = \lim_{q \to -1} \int_b f d\mu_q = \int_S f d\sigma.$$

For $f, g \epsilon A^2_q(B)$, we define the inner product by

$$< f, g >_q = \int_B f \bar{g} d\mu_q$$

and the norm by $||f||_{A^2_q} = \sqrt{< f, f >}$.

Theorem 4.1. *For all* $f \epsilon L^1_q(B) \cap H(B)$ *and* $q \geqq -1$,

$$f(z) = \int_B K_q(z,w) f(w) d\mu_q(w), \tag{4.4a}$$

where

$$K_q(z,w) = (1 - < z, w >)^{-(q+n+1)}. \tag{4.4b}$$

Moreover,

$$f(z) = \int_B B_q(z,w) f(w) d\mu_q(w), \tag{4.4c}$$

where

$$B_q(z,w) = \frac{|K_q(z,w)|^2}{K_q(z,z)}. \tag{4.4d}$$

Proof. If $f \epsilon H(B)$, by the mean value theorem

$$f(0) = \int_B f(r\zeta) d\sigma(\zeta), \quad 0 < r < 1. \tag{4.5a}$$

If $f \epsilon \left(L^1_q \cap H\right)(B)$, then by integrating both sides of (4.5a) with respect to the measure $2n(1-r^2)^q r^{2n-1} dr$ over [0, 1], we have

$$2n \int_0^1 \int_S f(r\zeta)(1-r^2)^q r^{2n-1} d\sigma(\zeta) dr = f(0) c_q^{-1}. \tag{4.5b}$$

Namely,

$$f(0) = \int_B f(z) d\mu_q(z). \tag{4.5c}$$

Replacing f by $f \circ \varphi_{z'}$ we find

$$f(z) = \int_B f[\varphi_z(w)] d\mu_q(w). \tag{4.5d}$$

By the change of variables formula,

$$\begin{aligned} (4.5\text{e}) \qquad f(z) &= \int_B f(w)d\mu_q[\varphi_z(w)] \\ &= c_q \int_B f(w)(1-|\varphi_z(w)|^2)^q d\nu(\varphi_z(w)). \end{aligned}$$

From (4.5e) together with (3.2b) and (3.3b) of Ch.1, and the fact that

$$(4.5\text{f}) \qquad d\nu(\varphi_z(w)) = |J\varphi_z(w)|^2 d\nu(w),$$

we find that

$$(4.5\text{g}) \qquad f(z) = (1-|z|^2)^{q+n+1} \int_B \frac{f(w)}{|1-<w,z>|^{2(q+n+1)}} d\mu_q(w).$$

Replacing $f(w)$ by $f(w)[1-<w,z>]^{q+n+1}$ in (4.5g), we obtain (4.4a). To prove (4.4c), fix $z\epsilon B$ and put

$$(4.5\text{h}) \qquad g(w) = \frac{K_q(w,z)}{K_q(z,z)} f(w), \quad w\epsilon B.$$

Then $g\epsilon \left(L_q^1 \cap H\right)(B)$ and $g(z) = f(z)$. Hence,

$$f(z) = g(z) = \int_B K_q(z,w)g(w)d\mu_q(w) = \int_B B_q(z,w)f(w)d\mu_q(w). \ \square$$

From Theorem 4.1, we have the following corollary which in turn implies the completeness of the Hilbert space $A_q^2(B)$.

Corollary 4.2. *Let $f\epsilon A_q^2(B)$. For each compact subset K of B, there exists a constant $C(K) > 0$ such that*

$$(4.6) \qquad |f(z)| \leqq C(K)||f||_{A_q^2}, \quad \textit{for all } z\epsilon K.$$

Proof. From Theorem 4.1,

$$(4.7\text{a}) \qquad |f(z)|^2 \leqq \int_B |K_q(z,w)|^2 |f(w)|^2 d\mu_q(w).$$

But, $|K_q(z,w)|^2 \leqq (1-|z||w|)^{-2(q+n+1)}$. *Since* K *is compact in* B, *there exists an* $r\epsilon(0,1)$ *such that* $|z|<r$ *for all* $z\epsilon K$. *Therefore,*

$$|K_q(z,w)|^2 \leqq (1-r)^{-2(q+n+1)} \quad \text{for } w\epsilon B. \tag{4.7b}$$

The corollary now follows from (4.7a) and (4.7b). □

5. Weighted Bergman Kernel.

As a consequence of Corollary 4.2, convergence in norm implies uniform convergence on compact subsets of B, from which the completeness of Hilbert space $A_q^2(B)$ follows. Furthermore, point evaluation functional $e_z(f)=f(z)$ of $f\epsilon A_q^2(B)$ is continuous at each fixed point $z\epsilon B$. Thus by the Riesz representation theorem there exists the (weighted). Bergman kernel : $K_{q,z}\epsilon A_q^2(B)$ such that

$$f(z) = < f, K_{q,z} > = \int_B f(w)\overline{K_{q,z}(w)}d\mu_q(w), \quad z\epsilon B. \tag{5.1}$$

We often write the Bergman kernel by $K_q(w,z)=K_{q,z}(w)$. The Bergman kernel $K_q(z,w)$ clearly satisfies all the properties (1.1a) through (1.1h). Thus, $K_q(z,w)=\overline{K_q(w,z)}$. Consequently, for each fixed $w\epsilon B$, $z\to K_q(z,w)$ is in $A^2(\mu_q)$. Furthermore, (5.1) implies

$$K_q(z,z) = \int_B |K_q(z,w)|^2 d\mu_q(w), \tag{5.2}$$

and by the Cauchy-Schwarz inequality

$$|K_q(z,w)|^2 \leqq K_q(z,z)K_q(w,w). \tag{5.3}$$

From (1.1d) it follows that $K_q(z,z)\geqq 0$. Since B is bounded, $A_q^2(B)$ satisfies the hypothesis of Lemma 3.6 and hence $K_q(z,z)>0$ for all $z\epsilon B$.

Since $A_q^2(B)$ is separable Hilbert space, it contains a countable complete orthonormal system $\{\varphi_j\}$ so that every $f\epsilon A_q^2(B)$ has a Fourier series expansion :

$$f=\sum_{j=1}^{\infty} a_j\varphi_j, \quad a_j = < f, \varphi_j >,$$

where the convergence is in the norm of $A^2_q(B)$. But as a consequence of Proposition 1.1 and Corollary 4.2, we also have

$$f(z) = \sum_{j=1}^{\infty} a_j \varphi_j(z), \tag{5.4}$$

where the series converges absolutely in B, and uniformly on compact subsets of B. In particular, for the Bergman kernel $K_q(z,w)$, we have

Proposition 5.1. *Let $\{\varphi_j\}$ be a countable complete orthonormal system for $A^2_q(B)$. Then*

$$K_q(z,w) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}, \tag{5.5}$$

where the series converges absolutely and uniformly on compact subsets of $B \times B$.

We now compute the (weighted) Bergman kernel of the unit ball B. It is well-known [30] that a complete Reinhardt circular domain $R \subset \mathbb{C}^n$ with center $0 \epsilon R$ admits a complete orthonormal system given by monomials of the form $\varphi_\alpha(z) = \gamma_\alpha z^\alpha$. Hence, the reproducing kernel of any such domain is given by

$$K_q(z,w) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{z^\alpha \bar{w}^\alpha}{||z^\alpha||_{A^2(\mu_q)}}. \tag{5.6a}$$

In particular, if $R = B$, then

$$\begin{aligned} ||z^\alpha||^2_{A^2(\mu_q)} &= \int_B |z^\alpha|^2 (1-|z|^2)^q dv(z) \\ &= \frac{\alpha!\Gamma(q+n+1)}{\Gamma(|\alpha|+q+n+1)}. \end{aligned} \tag{5.6b}$$

Therefore,

(5.6c)

$$\begin{aligned} K_q(z,w) &= \sum_{k=0}^{\infty} < z, w >^k \Gamma(k+q+n+1)/\Gamma(q+n+1) \\ &= (1- < z, w >)^{-q-n-1}. \end{aligned}$$

In particular, if $q = 0$, we obtain the usual Bergman kernel of B:

$$K(z,w) = K_{\circ}(z,w) = (1- < z,w >)^{-n-1}, \tag{5.6d}$$

and if $q = -1$,

$$S(z,w) = K_{-1}(z,w) = (1- < z,w >)^{-n} \tag{5.6e}$$

is the Cauchy-Szego kernel of B.

For $\Omega = B$, every $\varphi \epsilon \mathrm{Aut}(\Omega) = \mathrm{Aut}(B)$ acts on μ_q as a holomorphic gauge transformation, i.e., $G(\mu_q) = \mathrm{Aut}(B) = \mathrm{PSU}(n,1)$. The group Aut(B) is described in §1.3. The corresponding invariant measure is given by

$$\begin{aligned} d\lambda_q(z) &= K_q(z,z)d\mu_q(z) \\ &= c_q(1-|z|^2)^{-q-n-1}(1-|z|^2)^q d\nu(z) \\ &= c_q(1_|z|^2)^{-n-1}d\nu(z) \\ &= c_q d\lambda(z). \end{aligned} \tag{5.7}$$

6. Weighted Bergman p-spaces.

For $0 < p \leqq \infty$ we denote by L^p_q the L^p-space with respect to the probability measure $d\mu_q$ for $q \geqq -1$, and the corresponding norm by

$$||f||_{p,q} = \left\{ \int_B |f|^p d\mu_q \right\}^{1/p}. \tag{6.1}$$

The term "norm " is used loosely here, since $|| \cdot ||_{p,q}$ does not satisfy the triangle inequality for $0 < p < 1$, but in this case $\rho(f,g) = ||f-g||^p_{p,q}$ defines a metric in L^p_q which turns it into a Frechet space [8].

The weighted Bergman p-space is defined by $A^p_q(B) = H(B) \cap L^p_q(B)$. Namely, $A^p_q(B)$ is the closed subspace of $L^p_q(B)$ consisting of holomorphic functions on B. In particular, when $q = -1$, we obtain the Hardy p-class $H^p(B) = A^p_{-1}(B)$, which we identify as subspace of $L^p_{-1}(B) = L^p(\partial B)$ [8], and when $q = 0$, we obtain the usual Bergman p-space $A^p(B)$.

Proposition 6.1. *For $q \geqq -1$, the following transformation formulas hold under $\varphi_a \epsilon Aut(B)$.*

(6.2a) $$K_q(z,w) = K_q(\varphi_a(z),\varphi_a(w)) \left[J\varphi_a(z)\overline{J\varphi_a(w)}\right]^{\frac{q+n+1}{n+1}}$$

(6.2b) $$K_q(\varphi_a(z),\varphi_a(w)) = \frac{K_q(z,w)K_q(a,a)}{K_q(z,a)K_q(a,w)}.$$

(6.2c) $$B_q(\varphi_a(z),\varphi_a(w)) = \frac{B_q(z,w)}{B_q(a,w)}.$$

In particular,

(6.2d) $$B_q(\varphi_a(0),\varphi_a(w)) = \frac{1}{B_q(a,w)},$$

(6.2e) $$B_q(\varphi_a(z),\varphi_a(w)) = B_q(z,w)B_q(\varphi_a(0),\varphi_a(w)),$$

(6.2f) $$B_q(\varphi_a(z),w) = B_q(z,\varphi_a(w))B_q(\varphi_a(0),w).$$

(6.2g) $$\begin{aligned} d\mu_q(\varphi_a(z)) &= |J\varphi_a(z)|^{\frac{2(q+n+1)}{n+1}} d\mu_q(z) \\ &= \frac{|K_q(a,z)|^2}{K_q(a,a)} d\mu_q(z). \end{aligned}$$

Proof. Replacing g in (3.3) by $\varphi_{a'}$ we obtain

(6.3a) $$K_q(\varphi_a(z),\varphi_a(w)) = \varphi(\varphi_a(z))K_q(z,w)\overline{\varphi(\varphi_a(w))}.$$

After some manupulations, using (I.3.2b) and (I.3.2a), we find

(6.3b)
$$\begin{aligned} K_q(\varphi_a(z),\varphi_a(w)) &= K_q(z,w)\left[\frac{(1-<z,a>)(1-<a,w>)}{1-|a|^2}\right]^{q+n+1} \\ &= \frac{K_q(z,w)K_q(a,a)}{K_q(z,z)K_q(a,w)}, \end{aligned}$$

which proves (6.2b). (6.2c), (6.2d) and (6.2e) are immediate consequences of (6.2b) and (4.4d), the definition of B_q. Replacing w by $\varphi_a(w)$ in (6.2e) we get (6.2f). Comparing (6.3a) and (6.3b), we obtain

$$\varphi(\varphi_a(z)) = \left[\frac{1-<z,a>}{\sqrt{1-|a|^2}}\right]^{q+n+1} = \frac{\sqrt{K_q(a,a)}}{K_q(z,a)} \tag{6.3c}$$

and hence, from (I.3.3b).

$$|\varphi(\varphi_a(z))|^2 = |J\varphi_a(z)|^{-\frac{2(q+n+1)}{n+1}} = \frac{K_q(a,a)}{|K_q(z,a)|^2} \tag{6.3d}$$

(6.2g) follows from (3.2), together with (6.3c) and (6.3d). Namely,

$$\begin{aligned} d\mu_q(\varphi_a(z)) &= |\varphi(\varphi_a(z))|^{-2} d\mu_q(z) \\ &= \left[\frac{1-|a|^2}{|1-<z,a>|^2}\right]^{q+n+1} d\mu_1(z) \\ &= \frac{|K_q(a,z)|^2}{K_q(a,a)} d\mu_q(z) \\ &= |J\varphi_a(z)|^{\frac{2(q+n+1)}{n+1}} d\mu_q(z). \end{aligned} \tag{6.3e}$$

(6.2a) follows from (6.3a) and (6.3d). □

Theorem 6.2 [8]. *For $0 < p \leqq \infty$, $q \geqq -1$, and $\varphi \epsilon Aut(B)$, the mapping :*

$$T_\varphi : L^p_q(B) \to L^p_q(B),$$

given by $T_\varphi f = (f \circ \varphi)[J\varphi(z)]^{\frac{2(q+n+1)}{p(n+1)}}$, is a linear isometry of L^p_q onto L^p_q (and also A^p_q onto A^p_q) and involutive.

Proof. The theorem is trivial for $p = \infty$. For $0 < p < \infty$, we use (6.2g) with $w = \varphi(z)$, $z \epsilon B$:

$$\begin{aligned} ||T_\varphi f||^p_{p,q} &= \int_B |f(\varphi(z))|^p |J\varphi(z)|^{\frac{2(q+n+1)}{n+1}} d\mu_q(z) \\ &= \int_B |f(w)|^p d\mu_q(w) = ||f||^p_{p,q}. \end{aligned}$$

The operator T_φ is surjective, since if $g \epsilon A^p_q$ and $f = T_\varphi g$, then $f \epsilon L^p_q$. That T_φ is an involution follows from $\varphi \circ \varphi = I$. □

Definition 6.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. a C^2 function $f : \Omega \to \mathbb{R}$ is harmonic if it satisfies the Laplace equation $\Delta f = 0$, where

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) = 4 \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \tag{6.4}$$

is the usual Laplace operator in $\mathbb{R}^{2n} = \mathbb{C}^n$.

It is well-known that a function $f \epsilon C(\Omega)$ is harmonic if and only if it satisfies the mean value property, that is, for every $z \epsilon \Omega$ and $r > 0$ with $B(z, r) \subset\subset \Omega$, it holds that

$$f(z) = \int_S f(z + r\zeta) d\sigma(\zeta). \tag{6.5a}$$

Definition 6.2. A function $f : \Omega \to [-\infty, \infty), f \not\equiv -\infty$ is said to be subharmonic if it is upper semicontinuous in Ω and satisfies the sub-mean value propoperty: for every $z \epsilon \Omega$ and $r > 0$ with $B(z, r) \subset\subset \Omega$, it holds that

$$f(z) \leqq \int_S f(z + r\zeta) d\sigma(\zeta). \tag{6.5b}$$

It is clear that f is harmonic in Ω if and only if both $\pm f$ are subharmonic in Ω.

Definition 6.3. An upper semicontinuous function

$$f : \Omega \to [-\infty, \infty), \quad f \not\equiv -\infty$$

is plurisubharmonic if for every $z \epsilon \Omega$ and $w \epsilon \mathbb{C}^n$, the function $\lambda \to f(z + \lambda w)$ is subharmonic in a neighborhood of 0 in $\mathbb{C}$. A continuous function $f : \Omega \to \mathbb{R}$ is pluriharmonic if the above function is harmonic in a neighborhood of 0 in $\mathbb{C}$ for every $z \epsilon \Omega$ and $w \epsilon \mathbb{C}^n$.

A C^2 function f is plurisubharmonic in Ω if and only if it satisfies :

$$\sum_{i=1}^{n} \frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \geqq 0 \tag{6.6}$$

for all $z \epsilon \Omega$ and $\xi \epsilon \mathbb{C}^n$, and pluriharmonic in Ω if and only if the inequality in (6.6) is replaced by the equality.

By $h(\Omega), sh(\Omega), psh(\Omega) \subset sh(\Omega)$ and $ph(\Omega)$, we denote the spaces of harmonic functions, subharmonic functions, plurisubharmonic functions and pluriharmonic functions on Ω, respectively. Evidently,

$$h(\Omega) \subset sh(\Omega), \quad psh(\Omega) \subset sh(\Omega) \text{ and } ph(\Omega) = psh(\Omega) \cap h(\Omega). \tag{6.7}$$

For $\Omega = B$ and $q > -1$, we define $(sh)^1_q = sh \cap L^1_q$, while for $q = -1$ we let $(sh)^1_{-1}$ denote the space of all $u \epsilon sh$ such that

$$\sup_{0<r<1} \int_S |u(r\zeta)| d\sigma(\zeta) < \infty. \tag{6.8}$$

In this case, $u \epsilon (sh)^1_{-1}$ if and only if there exists a finite Borel measure $\hat{u}$ on S such that its Poisson integral $P[\hat{u}]$ is the least harmonic majorant of u on B. Similarly we let

$$h^1_q = h \cap (sh)^1_q, \quad (psh)^1_q = psh \cap (sh)^1_q, \quad \text{and } (ph)^1_q = ph \cap (sh)^1_q. \tag{6.9}$$

Theorem 6.3 [8]. *Let u be a plurisubharmonic function in $L^1_q(B)$ for $q \geqq -1$. Then for any $z \epsilon B$,*

$$u(z) \leqq \int_B u(w) B_q(z, w) d\mu_q(w). \tag{6.10}$$

Equality holds at some point $z \epsilon B$ if and only if u is pluriharmonic in B. Here, for $q = -1$, $u d\mu_q$ in the above integral has to be replaced by a finite Borel measure $d\hat{u}$ on $S = \partial B$.

Proof. Let ψ be a subharmonic function in B. Then by the sub-mean value property we have

$$\psi(0) \leqq \int_S \psi(r\zeta) d\sigma(\zeta), \quad 0 \leqq r < 1. \tag{6.11a}$$

In particular, if $\psi \epsilon L^1_q(B)$, $q > -1$, then by integrating both sides over the interval [0, 1] with respect to the measure $2nr^{2n-1}(1-r^2)^q dr$ we have

$$\psi(0) \leqq \int_B \psi(z) d\mu_q(z). \tag{6.11b}$$

This inequality is also true for $q = -1$, provided that $\psi d\mu_q$ is replaced by $d\hat{\psi} = \lim_{r\to 1} \psi(r\zeta)d\mu_q(r\zeta)$. It is also clear that the equality in (6.11a) holds if and only if $\psi\epsilon(sh)^1_q(B)$ is harmonic in B. Since $u\circ\varphi\epsilon L^1_q(B)$ is subharmonic,

$$u(z) = u \circ \varphi_z(0) \leqq \int_b u \circ \varphi_z(w) d\mu_q(w). \tag{6.11c}$$

By the change of variables formula and (6.2g),

$$\begin{aligned}\int_B u(\varphi_z(w))d\mu_q(w) &= \int_B u(w)d\mu_q(\varphi_z(w)) \\ &= \int_B u(w)\frac{K_q(z,w)|^2}{K_q(z,z)}d\mu_q(w).\end{aligned} \tag{6.12}$$

By passing to the limit as $q \to -1^+$, (6.12) remains valid for $q = -1$, provided that $ud\mu_q$ is replaced by $d\hat{u}$. Combining (6.11c) and (6.12) yields (6.10). If $u\epsilon L^1_q(B)$ is pluriharmonic, then (6.10) must also hold with u replaced by $-u$ so that we have equality in (6.11c). Conversely, if equality holds for some $z\epsilon B$, then it follows from (6.11c) that for some $\varphi_z\epsilon Aut(B)$

$$u(\varphi_z(0)) = \int_B u(\varphi_z(w))d\mu_q(w) \tag{6.13}$$

and hence $u \circ \varphi_z\epsilon L^1_q(B)$ is harmonic. Since u is plurisubharmonic, $u \circ \varphi_z$ is also plurisubharmonic. So, $u \circ \varphi_z$ and hence $u = (u \circ \varphi_z) \circ \varphi_z^{-1}\epsilon L^1_q(B)$ is pluriharmonic in B. □

Corollary 6.4. Let $0 < p < \infty$, $q \geqq -1$ and let $f\epsilon A^p_q(B)$. Then

(i) For any $z\epsilon B$,

$$|f(z)|^p \leqq \int_B |f(w)|^p B_q(z,w)d\mu_q(w) \tag{6.14}$$

with equality at some point $z\epsilon B$ if and only if f is constant on B. Here for $q = -1$, the above integral is regarded as the Poisson integral of the boundary value function f^* of $f\epsilon A^p_0 = H^p$.

(ii) For any $z\epsilon B$,

$$|f(z)| \leqq \{K_q(z,z)\}^{1/p}||f||_{p,q} \tag{6.15}$$

with equality at some $z \epsilon B$ if and only if $f(w) = \lambda(1- < w, z >)^{-2(q+n+1)/p}$ for some constant $\lambda \epsilon \mathbb{C}$ and every $z \epsilon B$.

Proof. Since $|f|^p \epsilon L^1_q$ is plurisubharmonic in B, inequality in (i) follows directly from Theorem 6.3. Moreover, by the same theorem, equality in (i) holds for some $z \epsilon B$ if and only if $|f|^p$ is also pluriharmonic on B. Since $f \epsilon H(B)$, this is equivalent to f being a constant on B, and (i) follows. To prove (ii) we fix $z \epsilon B$ and define a function g on B by $g(w) = f(w)(1- < w, z >)^{-2(q+n+1)/p}$. Statement (ii) now follows from (i) with g in place of f. □

Corollary 6.5 [56]. Let a ϵB. The following extremal problem

$$\sup \left\{ f(a) : f \epsilon A^p_q, \quad ||f||_{p,q} = 1, \quad f(a) > 0 \right\} \tag{6.16a}$$

has a unique solution for $p \epsilon (0, \infty)$ and $q \geqq -1$, given by

$$F(z) = (T_\varphi 1)(z) = \left[\frac{K_q(z, a)}{K_q(a, a)} \right]^{1/p} . \tag{6.16b}$$

Proof. The equality in (6.14) holds at $0 \epsilon B$ if and only if f is constant and positive. Hence $f \equiv 1$ is the unique extremal function at $a = 0$. For arbitrary a ϵB, Theorem 6.2 implies :

$$\left\{ f \epsilon A^p_q : ||f||_{p,q} = 1, \quad f(a) > 0 \right\} = \left\{ f \epsilon A^p_q : ||T_{\varphi_a} f||_{p,q} = 1, \quad T_{\varphi_a} f(0) > 0 \right\} .$$

Therefore, $f \epsilon A^p_q$ is extremal for point evaluation at a ϵB if and only if $T_{\varphi_a} f$ is extremal for point evaluation at 0. By the above argument, there exists a unique extremal function F that satisfies $(T_{\varphi_a} F)(z) \equiv 1$, and thus by the involutive property of T_{φ_a} we have

$$F(z) = T_{\varphi_a}(1)(z) = [J\varphi_a(z)]^{\frac{2(q+n+1)}{p(n+1)}} = \left[\frac{K_q(z, a)}{K_q(a, a)} \right]^{1/p} . \ \square$$

Chapter III The Bergman Geometry

In this chapter we discuss the basic properties of the Bergman metric on a domain Ω in $\mathbb{C}^n$ and related complex geometry in the analysis view point. In constructing the Bergman metric, we take a non-standard approach by embedding Ω into an infinite dimensional projective space by a holomorphic mapping. We also give an inequality between the Bergman metric and the Caratheodory metric. Finally, we consider the Bergman Geometry on the ball and introduce various geometric quantities which are invariant under the automorphisms of the ball.

1. The Bergman Metric

We assume that the domain Ω satisfies :

(C.1) For each $z \epsilon \Omega$, there exists an $f \epsilon A^2(\mu)$ such that $f(z) \neq 0$, i.e., $K(z,z) > 0$ for every $z \epsilon \Omega$ by (1.1g) of Ch.2.

The Bergman (pseudo) metric $b_\Omega : \Omega \times \mathbb{C}^n \to \mathbb{R}$ is defined by the differential form :

$$b_\Omega(z,\xi) = \sum_{i,j=1}^{n} b_{ij}(z)\xi_i\bar{\xi}_j, \tag{1.1}$$

where $b_{ij} = \frac{\partial^2 \log K}{\partial z_i \partial \bar{z}_j}$ $(i,j = 1,\cdots,n)$ and $K = K(z,z)$ denotes the Bergman kernel of Ω.

Let $\{\varphi_\nu\}$ be a complete orthonormal system of the Hilbert space $A^2(\mu)$. Then every $f \epsilon A^2(\mu)$ may be represented uniquely by the series

$$f(z) = \sum_\nu c_\nu \varphi_\nu(z), \quad c_\nu = (f,\varphi_\nu). \tag{1.2a}$$

which converges absolutely and uniformly in every compact subset of Ω. Moreover,

$$||f||_{A^2(\mu)} = \sum_\nu |c_\nu|^2 = ||c||_2^2, \tag{1.2b}$$

where $||c||_2$ denotes the $\ell^2(\mathbb{C})$-norm of the infinite series $c = \{c_\nu\}$.

It is clear from (1.2b), that the map

$$\sigma : f \to c, \quad c = \{c_\nu\}, \quad c_\nu = (f, \varphi_\nu), \tag{1.3}$$

is a linear isometry between $A^2(\mu)$ and the space $\ell^2(\mathbb{C})$ with the usual inner product :

$$< a, b >= \sum_{\nu=0}^{\infty} a_\nu \bar{b}_\nu, \quad a, b \epsilon \ell^2(\mathbb{C}).$$

Under σ, each φ_ν corresponds to the standard orthonormal basis

$$e_\nu = \sigma(\varphi_\nu) = \{\delta_{\nu\mu}\}_{\mu=0}^{\infty}$$

in $\ell^2(\mathbb{C})$. Therefore, the kernel function K_z is mapped by σ to

$$\sigma(K_z) = \sum_{\nu=0}^{\infty} \varphi_\nu(z) e_\nu = \varphi(z), \tag{1.4a}$$

and

$$||\sigma(K_z)||_2^2 =< \varphi(z), \varphi(z) >= ||\varphi(z)||_2^2 = K(z, z), \quad z \epsilon \Omega. \tag{1.4b}$$

Clearly, $K(z, z) > 0$ under the condition (C.1).

Therefore, the mapping

$$\varphi : \Omega \to \ell^2(\mathbb{C})$$

omits $0 \epsilon \ell^2(\mathbb{C})$ for all $z \epsilon \Omega$. The mapping φ is also continuous on Ω. It follows from the fact that

$$||\varphi(z) - \varphi(w)||_2^2 = K(z, z) - K(z, w) - K(w, z) + K(w, w) \tag{1.5}$$

and that $K(z, w)$ is holomorphic in $(z, \bar{w}) \epsilon \Omega \times \bar{\Omega}$. Since each component φ_ν of φ is holomorphic in Ω, φ is a holomorphic mapping of Ω into $\ell^2(\mathbb{C}) - \{0\}$. Under this mapping $\varphi : \Omega \to \ell^2(\mathbb{C})$, the Hilbert metric in $\ell^2(\mathbb{C})$ induces the following metric in Ω :

(1.6)

$$\begin{aligned} dh^2(z) =< d\varphi, d\varphi > &= \sum_{\mu,\nu=1}^{n} < \partial\varphi/\partial z_\mu, \quad \partial\varphi/\partial z_\nu > dz_\mu d\bar{z}_\nu \\ &= \sum_{\mu,\nu=1}^{n} (\partial^2 K(z, z)/\partial z_\mu \partial \bar{z}_\nu) dz_\mu d\bar{z}_\nu, \end{aligned}$$

which is positive definite Kaehler metric in Ω. In general, dh^2 is not invariant under $Aut(\Omega)$.

In order to equip Ω with an invariant metric, we construct the infinite dimensional projective space $P(\ell^2)$ from $\ell^2 = \ell^2(\mathbb{C})$. Let ζ and ζ' be two points in $\ell^2 - \{0\}$. We say they are equivalent if $\zeta' = \lambda\zeta$ for some complex number λ. The quotient space of $\ell^2 - \{0\}$ by this equivalence relation is the projective space $P(\ell^2)$. We furnish $P(\ell^2)$ with the standard hermitian metric, called the Fubini-Study metric :

(1.7)

$$d\chi^2(w) = \frac{< w, w >< dw, dw > - < w, dw >< dw, w >}{< w, w >^2}, \quad w\epsilon\ell^2(\mathbb{C}) - \{0\}$$

Then the induced holomorphic mapping :

$$\tilde{\varphi} : \Omega \to P(\ell^2), \quad \tilde{\varphi} = p \circ \varphi,$$

where $p : \ell^2(\mathbb{C}) - \{0\} \to P(\ell^2)$ is the usual projection map, pulls the Fubini-Study metric back to Ω. Indeed,

(1.8)

$$\begin{aligned} db^2_\Omega(z) = d\chi^2(\varphi(z)) &= \frac{< \varphi, \varphi >< d\varphi, d\varphi > -| < \varphi, d\varphi > |^2}{< \varphi, \varphi >^2} \\ &= \sum_{\mu,\nu=1}^{n} \left(\partial^2 \log ||\varphi(z)||_2^2 / \partial z_\mu \partial \bar{z}_\nu\right) dz_\mu d\bar{z}_\nu \\ &= \sum_{\mu,\nu=1}^{n} \left(\partial^2 \log K(z,z) / \partial z_\mu \partial \bar{z}_\nu\right) dz_\mu d\bar{z}_\nu \end{aligned}$$

is a well-defined positive semi-definite hermitian form which is Kaehler and invariant under $Aut(\Omega)$, as Theorem 2.3 below shows. The metric db^2_Ω is positive definite if Ω satisfies the following condition (see Corollary 2.2 below):

(C.2) For every holomorphic tangent vector ξ at $z\epsilon\Omega$, there exists an $f\epsilon A^2(\mu)$ such that $\nabla f(z) \cdot \xi \neq 0$.

Therefore, any domain Ω in $\mathbb{C}^n$ with properties (C.1) and (C.2) can be furnished with an invariant Kaehler metric, called the Bergman metric [35].

2. Comparison Between the Metrics of Bergman and Carathéodory.

Let $Hol(\Omega, U)$ be the set of holomorphic mappings $f : \Omega \to U$, where U denotes the open unit disc in $\mathbb{C}$. The Caratheodory (pseudo) differential metric $c_\Omega : \Omega \times \mathbb{C}^n \to \mathbb{R}$ is defined by

$$c_\Omega(z,\xi) = \sup\{|\nabla f(z)\xi| : f \epsilon Hol(\Omega, U)\}. \tag{2.1}$$

Then the Bergman metric dominates the Caratheodory metric as shown in the following :

Theorem 2.1 [19]. *Let Ω be a domain in $\mathbb{C}^n$ with properties (C.1) and (C.2). For each $z \epsilon \Omega$ and $\xi \epsilon \mathbb{C}^n$,*

$$c_\Omega(z,\xi) \leqq b_\Omega(z,\xi). \tag{2.2}$$

Proof. For any $f \epsilon Hol(\Omega, U)$, let

$$\alpha(t) = f(t) K_z(t) \tag{2.3a}$$

$$\beta(t) = \sum_{\nu=1}^{n} \bar{\xi}_\nu \frac{\partial}{\partial \bar{z}_\nu} \left[\frac{K_z(t)}{K_z(z)}\right]. \tag{2.3b}$$

By the Schwarz inequality,

$$|(\alpha,\beta)|^2 \leqq (\alpha,\alpha)(\beta,\beta). \tag{2.4}$$

Using the reproducing property of the kernel K_z, we have

$$(\alpha,\alpha) = (fK_z, fK_z) \leqq K(z,z), \quad \text{since } |f(t)| \leqq 1 \quad \text{in } \Omega. \tag{2.5a}$$

(2.5b)

$$\begin{aligned}(\beta,\beta) &= K^{-3} \sum_{\mu,\nu=1}^{n} \xi_\mu \bar{\xi}_\nu \left[K \frac{\partial^2 K}{\partial z_\mu \partial \bar{z}_\nu} - \frac{\partial K}{\partial z_\mu}\frac{\partial K}{\partial \bar{z}_\nu}\right], \quad K = K(z,z), \\ &= b^2(z,\xi)/K(z,z)\end{aligned}$$

$$(\alpha, \beta) = \nabla f(z)\xi. \tag{2.5c}$$

From (2.4), together with (2.5a) - (2.5c), we obtain

$$|\nabla f(z)\xi|^2 \leqq b^2(z, \xi), \tag{2.6}$$

which proves the theorem. □

As a consequence of inequality (2.6), we obtain

Corollary 2.2. The Bergman metric b_Ω is positive definite if (C.2) holds.

Let $\gamma : [0, 1] \to \Omega$ be a piecewise $\mathbb{C}^1$ curve. We define the Bergman length of γ by

$$|\gamma|_b = \int_0^1 b_\Omega(\gamma(t),\ \gamma'(t))dt. \tag{2.7}$$

If $z, w \epsilon \Omega$, then the Bergman distance between z and w is given by

$$\beta_\Omega(z, w) = \inf\{|\gamma|_b : \gamma(0) = z, \quad \gamma(1) = w\}, \tag{2.8}$$

where the infimum is taken over all piecewise C^1 curves between z and w in Ω. Due to the invariant nature of the Bergman metric b_Ω, we have the following.

Theorem 2.3. *Let Ω and D be domains in $\mathbb{C}^n$ and let φ be a biholomorphic mapping of Ω onto D. Then for all $z, w \epsilon \Omega$,*

$$\beta_\Omega(z, w) = \beta_D(\varphi(z), \varphi(z)). \tag{2.9}$$

Proof. From (II, Proposition 3.4), the kernel function admits the following transformation formula :

$$K_\Omega(z, w) = K_D(\varphi(z), \varphi(w))J\varphi(z)\overline{J\varphi(w)}, \tag{2.10a}$$

In particular,

$$K_\Omega(z, z) = K_D(\varphi(z), \varphi(z))|J\varphi(z)|^2. \tag{2.10b}$$

Therefore,

$$(2.11a)\qquad \begin{aligned} b_{ij}^{\Omega}(z) &= \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_{\Omega}(z,z) \\ &= \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_D(\varphi(z),\varphi(z))|J\varphi(z)|^2 \\ &= \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_D(\varphi(z),\varphi(z)), \end{aligned}$$

since $\log|J\varphi(z)|^2 = \log J\varphi(z) + \log \overline{J\varphi(z)}$ + constant locally and hence is annihilated by the mixed second derivative. Using the chain rule and Cauchy-Riemann conditions for holomorphic functions, the last term of (2.11a) reduces to

$$(2.11b)\qquad \sum_{\alpha=1}^{n} b_{\alpha\beta}^{D}(\varphi(z)) \frac{\partial \varphi_\alpha(z)}{\partial z_i} \frac{\partial \overline{\varphi_\beta(z)}}{\partial \bar{z}_j}$$

which implies :

$$(2.11c)\qquad db_D^2(\varphi(z)) = db_\Omega^2(z), \quad z\epsilon\Omega,$$

for all biholomorphic mappings $\varphi : \Omega \to D$. Theroem 2.3 now follows by standard integration. □

3. Invariant Laplacian.

Let Ω be any bounded domain in $\mathbb{C}^n$ furnished with the Bergman metric b_Ω. Then we may regard (Ω, b_Ω) as a Riemannian manifold. In particular, it is a Kaehler manifold. Theorefore, the corresponding gradient is the vector field, given by

$$(3.1a)\qquad \tilde{\nabla}_\Omega f(z) = 2\sum_{i,j=1}^{n} b^{ij}\left(\frac{\partial f}{\partial \bar{z}_i}\frac{\partial}{\partial z_j} + \frac{\partial f}{\partial z_j}\frac{\partial}{\partial \bar{z}_i}\right), \quad \text{for } f\epsilon C^1(\Omega),$$

where $(b^{ij}) = (b_{ij})^{-1}$ is the inverse matrix of the Bergman metric tensor (b_{ij}). See [49], [51]. For $f\epsilon H(\Omega)$, $\frac{\partial f}{\partial \bar{z}_i} = 0$ so that (3.1a) is reduced to the following form :

$$(3.1b)\qquad \tilde{\nabla}_\Omega f(z) = 2\sum_{i,j=1}^{n} b^{ij} \frac{\partial f}{\partial z_j}\frac{\partial}{\partial \bar{z}_i}.$$

If $f, g \epsilon C^1(\Omega)$, a routine computation yields :

$$(3.1c)\quad 2(\tilde{\nabla} f)\bar{g} =< \tilde{\nabla} f, \tilde{\nabla} g >= 4\sum_{i,j=1}^{n} b^{ij}\left(\frac{\partial f}{\partial \bar{z}_i}\frac{\partial \bar{g}}{\partial z_j} + \frac{\partial f}{\partial z_j}\frac{\partial \bar{g}}{\partial \bar{z}_i}\right), \tilde{\nabla} = \tilde{\nabla}_\Omega.$$

If $f = g$, then

$$(3.1d)\quad |\tilde{\nabla} f|^2 =< \tilde{\nabla} f, \tilde{\nabla} f >= 2(\tilde{\nabla} f)\bar{f} = 4\sum_{i,j=1}^{n} b^{ij}\left(\frac{\partial f}{\partial \bar{z}_i}\frac{\partial f}{\partial z_j} + \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_i}\right).$$

In particular, if $f \epsilon H(\Omega)$,

$$(3.1e)\qquad |\tilde{\nabla} f|^2 = 4\sum_{i,j=1}^{n} b^{ij}\left(\frac{\partial f}{\partial \bar{z}_i}\right)\frac{\partial f}{\partial z_j}.$$

The corresponding Laplace-Beltrami operator is given by

$$(3.2)\qquad \tilde{\Delta}_\Omega f(z) = 4\sum_{i,j=1}^{n} b^{ij}\frac{\partial^2 f}{\partial \bar{z}_i \partial z_j}, \quad \text{for } f \epsilon C^2(\Omega),$$

Due to the invariant property of the Bergman metric, these operators are invariant under the group $G(\mu) = Aut(\Omega)$. That is,

$$(3.3a)\qquad (\tilde{\nabla}_\Omega f)\circ\varphi = \tilde{\nabla}_\Omega(f\circ\varphi), \quad \text{for } f \epsilon C^1(\Omega),$$

$$(3.3b)\qquad (\tilde{\Delta}_\Omega f)\circ\varphi = \tilde{\Delta}_\Omega(f\circ\varphi), \quad \text{for } f \epsilon C^2(\Omega).$$

for all $\varphi \epsilon Aut(\Omega)$. For this reason the operators $\tilde{\nabla}_\Omega$ and $\tilde{\Delta}_\Omega$ are often referred to as the <u>invariant gradient</u> and <u>invariant Laplacian</u> on Ω, respectively.

Further computations show that if f and g are real valued in Ω,

$$(3.4a)\qquad \tilde{\Delta}(fg) = g\tilde{\Delta} f + 2(\tilde{\nabla} f)g + f\tilde{\Delta} g,$$

$$(3.4b)\qquad \tilde{\Delta}(f^2) = 2f\tilde{\Delta} f + 2|\tilde{\nabla} f|^2.$$

In particular, if $f \epsilon H(\Omega)$, then

$$(3.5c)\qquad \tilde{\Delta}|f|^2 = 2(\tilde{\nabla} f)\bar{f} = 2|\tilde{\nabla} f|^2.$$

4. Bergman Geometry on the Unit Ball B.

For the case where Ω is the open unit ball B in $\mathbb{C}^n$, the Bergman metric tensor can be written explicitly by

(4.1a)
$$\begin{aligned} b_{ij}(z) &= \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_q(z,z), \\ &= \frac{q+n+1}{(1-|z|^2)^2}[(1-|z|^2)\delta_{ij} + z_i \bar{z}_j], \quad (i,j = 1,\cdots,n), \end{aligned}$$

and thus, the Bergman metric (1.1) becomes :

$$b_B^2(z,\xi) = \frac{q+n+1}{(1-|z|^2)^2}[(1-|z|^2)|\xi|^2 + |<z,\xi>|^2] \tag{4.1b}$$

When $q = 0$, (4.1b) is the usual Bergman metric. These metrics differ only by constant factors. In view of the insignificance of these constant factors in dealing with the Bergman geometry, we normalize the Bergman metric by choosing the constant factor to be 1. Namely,

$$b^2(z,\xi) = [(1-|z|^2)|\xi|^2 + |(z,\xi)|^2]/(1-|z|^2)^2. \tag{4.1c}$$

To compute the Bergman distance $\beta = \beta_B$ on the unit ball B (see (2.8)), we observe that the Bergman distance joining 0 and z in B is attained by the line segment $\sigma(t) = tz,\ 0 \leqq t \leqq 1$. Hence,

$$\beta(0,z) = \int_0^1 b(\sigma(t),\sigma'(t))dt = \frac{1}{2}\log\frac{1+|z|}{1-|z|} = \tanh^{-1}|z|. \tag{4.1d}$$

The distance between z and w in B is now obtained from the invariant property of the Bergman distance β :

$$\beta(z,w) = \beta(\varphi_z(z),\varphi_z(w)) = \beta(0,\varphi_z(w)) = \tanh^{-1}|\varphi_z(w)|. \tag{4.1e}$$

It is easily seen that the volume element of the Bergman metric (4.1c) on B is given by

$$d\lambda(z) = |\det(b_{ij})|d\nu(z) = \frac{d\nu(z)}{(1-|z|^2)^{n+1}} \tag{4.1f}$$

which coincides with (4.4a), Ch.1, given earlier.

From (4.1c) it follows that for $(z,\xi)\epsilon B\times\mathbb{C}^n$

$$\frac{|\xi|}{\sqrt{1-|z|^2}}\leqq b(z,\xi)\leqq\frac{|\xi|}{1-|z|^2}. \tag{4.2}$$

The inverse matrix $(b^{ij}=(b_{ij})^{-1}$ of the Bergman metric (4.1c) is

$$b^{ij}(z)=(1-|z|^2)[\delta_{ij}-\bar z_i z_j]. \tag{4.3a}$$

Therefore, from (3.2) we obtain

$$\tilde\Delta_B=4(1-|z|^2)\sum_{i,j=1}^{n}[\delta_{ij}-\bar z_i z_j]\frac{\partial^2}{\partial z_j\partial\bar z_i}. \tag{4.3b}$$

From (3.3b) and (3.2), the invariant Laplacian $\tilde\Delta$ of B satisfies :

$$\tilde\Delta f(z)=\Delta(f\cdot\varphi_z)(0),\quad f\in C^2(B), \tag{4.3c}$$

where $\Delta=4\sum_{i=1}^n\frac{\partial^2}{\partial z_i\partial\bar z_i}$ is the usual euclidean Laplacian in $\mathbb{C}^n$.

When $n=1$, we have

$$\tilde\Delta f(z)=4(1-|z|^2)^2\frac{\partial^2 f(z)}{\partial z\partial\bar z}. \tag{4.3d}$$

Similary, from (3.3a) and (3.1a), the invariant gradient $\tilde\nabla$ of B satisfies :

$$\tilde\nabla f(z)=\nabla(f\circ\varphi_z)(0),\quad f\epsilon C^1(B). \tag{4.4a}$$

If $f\epsilon H(B)$, then

$$|\tilde\nabla f(z)|^2=4(1-|z|^2)\left[\sum_{i=1}^n\left|\frac{\partial f}{\partial z_i}\right|^2-\left|\sum_{i=1}^n z_i\frac{\partial f}{\partial z_i}\right|^2\right]. \tag{4.4b}$$

When $n=1$, we have

$$|\tilde\nabla f(z)|=2(1-|z|^2)|f'(z)| \tag{4.4c}$$

Definition 4.1. The radial derivative of a $C^1(\Omega)$ function u at z is defined by

(4.5a) $$(\Re u)(z) = \partial u(z) \cdot z + \bar{\partial} u \cdot \bar{z} = \sum_{i=1}^{n} \left(z_i \frac{\partial u}{\partial z_i} + \bar{z}_i \frac{\partial u}{\partial \bar{z}_i} \right).$$

The radial derivative $\Re^m$ of order m is defined inductively by $\Re(\Re^{m-1})$ for $m = 1, 2, \cdots$.

For a holomorphic function $f \epsilon H(\Omega)$, we have

(4.5b) $$(\Re f)(z) = \partial f(z) \cdot z = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z).$$

In addition to the radial derivative $\Re f$, gradient ∇f and invariant gradient $\tilde{\nabla} f$, we introduce the tangential derivative $\mathcal{D}_\tau f$ for $f \epsilon H(B)$ which measures the size of the gradient ∇f in the complex tangential directions.

Definition 4.2. Let $T_{ij} = (\bar{z}_i \partial_j - \bar{z}_j \partial_i)/\sqrt{2}$ for $i, j = 1, \cdots, n,\ i \neq j$. The tangential derivative of a holomorphic function $f \epsilon H(B)$ is defined by

(4.6a) $$\mathcal{D}_\tau f(z)^2 = \sum_{i,j=1}^{n} |T_{ij} f(z)|^2 \quad ([5], [41]).$$

Then an elementary calculation yields :

(4.6b) $$\begin{aligned} \mathcal{D}_\tau f(z)^2 &= \sum_{i,j=1}^{n} (|z|^2 \delta_{ij} - z_i \bar{z}_j) \frac{\partial f}{\partial z_i} \cdot \overline{\frac{\partial f}{\partial z_j}} \\ &= |z|^2 \sum_{i=1}^{n} \left| \frac{\partial f}{\partial z_i} \right|^2 - \left| \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} \right|^2 \\ &= |z|^2 |\nabla f(z)|^2 - |\Re f(z)|^2. \end{aligned}$$

Moreover, we find

(4.6c)

$$\begin{aligned} |\tilde{\nabla} f(z)|^2 &= 2(1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial f}{\partial z_i} \overline{\frac{\partial f}{\partial z_j}} \\ &= 2(1 - |z|^2)^2 |\nabla f(z)|^2 + 2(1 - |z|^2)(\mathcal{D}_\tau f(z))^2. \end{aligned}$$

In particular, we obtain the following inequalities :

$$\text{(4.6d)} \qquad \begin{gathered} |\Re f(z)| \leqq |z||\nabla f(z)| \leqq |\nabla f(z)| \\ (1-|z|^2)|\nabla f(z)| \leqq \tilde{\nabla} f(z) \leqq (1-|z|^2)^{1/2}|\nabla f(z)|. \end{gathered}$$

Chapter IV Bloch and Besov Spaces of M-Harmonic Functions

In this chapter we consider M-harmonic (invariant harmonic) and M-subharmonic (invariant sub-harmonic) functions on the unit ball and discuss their basic properties. It should be noted that the space of M-harmonic functions is stable under the complex conjugation. Therefore, it must contain both holomorphic functions and pluriharmonic functions. We also introduce a formal definition of Besov p-spaces for M-harmonic functions.

1. M-harmonic Functions.

Definition 1.1. A complex-valued C^2 functions u defined on an open set $\Omega \subset \mathbb{C}^n$ is called (invariant) harmonic or M-harmonic on Ω (adapting Rudin's terminology on the ball [44]) if $\tilde{\Delta}_\Omega u(z) = 0$ on Ω.

Clearly, the notion of harmonicity is invariant under the actions of $Aut(\Omega)$. We denote by $\tilde{h}(\Omega)$ the space of all invariant harmonic functions on Ω.

For the case where $\Omega = B, u \epsilon C^2(B)$ is M-harmonic if and only if it satisfies :

$$\tilde{\Delta}_B u = 4(1-|z|^2)\sum_{i,j=1}^{n}[\delta_{ij} - \bar{z}_i z_j]\frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_i} = 0, \quad z \epsilon B. \tag{1.1}$$

Definition 1.2. We denote by $\tilde{h}^p(S) = \tilde{h}^p(\partial B)$ the space of all M-harmonic functions on B that satisfies the growth condition :

$$\sup_{0<r<1}\int_S |u(r\zeta)|^p d\sigma(\zeta) < \infty, \quad p > 0. \tag{1.2}$$

we call $\tilde{h}^p(S)$ the Hardy p-space of M-harmonic functions on B.

Further, we let

$$\tilde{h}^p_q(B) = L^p_q(B) \cap \tilde{h}(B) \quad \text{for } q \geqq -1. \tag{1.3a}$$

In particular, we let

$$\tilde{h}^p(B) = \tilde{h}^p_0(B) \quad \text{for } q = 0, \tag{1.3b}$$

$$\tilde{h}^p(S) = \tilde{h}^p_{-1}(B) \quad \text{for } q = -1. \tag{1.3c}$$

Definition 1.3. If μ is a Borel measure on $S = \partial B$, the Poisson integral $P[\mu]$ of μ is defined by

$$P[\mu](z) = \int_S P(z,\zeta)d\mu(\zeta), \tag{1.4}$$

where $P(z,\zeta) = \frac{(1-|z|^2)^n}{|1-<z,\zeta>|^{2n}}$ is called the Poisson-Szego kernel of B. If $d\mu(\zeta) = f(\zeta)d\sigma(\zeta)$, where $f \epsilon L^1(\partial B, \sigma)$, we denote $P[fd\sigma]$ by $P[f]$.

Definition 1.4. A function $f \epsilon C(B)$ is said to have the invariant mean value property if

$$f(\psi(0)) = \int_S f(\psi(r\zeta))d\sigma(\zeta) \tag{1.5a}$$

for every $\psi \epsilon Aut(B)$ and $0 < r < 1$. Let $\psi(0) = a$. Then $\psi = \varphi_a U$ for some $U \epsilon \mathcal{U}$. By the $\mathcal{U}$-invariance of $d\sigma$,

$$f(a) = \int_S f(\varphi_a(r\zeta))d\sigma(\zeta). \tag{1.5b}$$

Integrating both sides of (1.5b) with respect to the measure $2n(1-r^2)^q r^{2n-1}dr$ over $[0,1]$

$$f(a) = \int_B f(\varphi_a(w))d\mu_q(w). \tag{1.5c}$$

Lemma 1.1 [44]. *Let $f \epsilon C^2(B)$. Then f is M-harmonic if it has the invariant mean value property.*

Proof. For each fixed $z \epsilon B$, let $h = f \circ \psi$. By the Taylor series expansion of h about 0,

$$\begin{aligned} h(z) = h(0) &+ \sum_{i=1}^n \left[z_i \frac{\partial h}{\partial z_i}(0) + \bar{z}_i \frac{\partial h}{\partial \bar{z}_i}(0) \right] \\ &+ \frac{1}{2}\sum_{i=1}^n \left[z_i^2 \frac{\partial^2 h}{\partial z_i^2}(0) + \bar{z}_i^2 \frac{\partial^2 h}{\partial \bar{z}_i^2}(0) \right] \\ &+ \sum_{i,j=1}^n z_i \bar{z}_j \frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}(0) + 0(|z|^3). \end{aligned}$$

Let $z = r\zeta$, $\zeta \epsilon S$, $0 < r < 1$. Then by (I.4.3b) and (I.4.3a)

$$\int_S h(r\zeta)d\sigma(\zeta) = h(0) + \frac{r^2}{4n}(\Delta h)(0) + 0(r^3).$$

Thus,

$$\lim_{r\to 0} \frac{4n}{r^2} \int_S [f \circ \psi(r\zeta) - f(\psi(0))]d\sigma(\zeta) = \tilde{\Delta} f(z). \tag{1.6}$$

from which Lemma 1.1 follows. □

The Poisson integral $P[\mu]$ gives arise to an M-harmonic function on B. See [44, p.49].

In the following we state a few basic properties of M-harmonic functions on B :

Theorem 1.2 [44, Theorem 4.3.3]. *Let $u \epsilon \tilde{h}(B)$ be such that (1.2) holds for some $p, 1 \leqq p \leqq \infty$. If $p > 1$, then there is an $f \epsilon L^p(S)$ such that $u = P[f]$. If $p = 1$, then there exists a measure μ such that $u = P[\mu]$.*

The following is a local version of the invariant mean value property which follows from a minor modification of Lemma 1.1

Corollary 1.3 [55, Theorem 1.5]. Suppose that $\Omega \subset B$ is open and u is a locally bounded measurable function on Ω. Then u is M-harmonic on Ω if and only if the following mean value theorem holds :

$$u(z) = \int_S u(\varphi_z(r\zeta))d\sigma(\zeta) \tag{1.7}$$

for every $z \epsilon \Omega$ and $r > 0$ such that $\varphi_z(r\bar{B}) \subset \Omega$.

Corollary 1.4. Let $\{f_i\}$ be a sequence of M-harmonic functions on an open set $\Omega \subset B$ which converges to f uniformly on compact subsets of Ω, then f is M-harmonic on Ω.

Integrating both sides of (1.7) with respect to the measure $2nr^{2n-1}(1 - r^2)^{-n-1}dr$ over [0, 1], we obtain

Corollary 1.5. If u is M-harmonic in an open subset Ω of B, and if $\varphi_z(r\bar{B}) \subset \Omega$, then

$$u(z) = \frac{1}{\lambda[E(z,r)]} \int_{E(z,r)} u(w)d\lambda(w), \tag{1.8}$$

where $d\lambda(z) = (1 - |z|^2)^{-n-1}d\nu(z)$ is an invariant measure introduced in (3.6a), Ch.1, and $E(z,r) = \varphi_z(rB)$.

Proposition 1.6. *If $u \in L^1_q(B)$ is an M-harmonic function on B, then for all $z \epsilon B$ and $q \geqq -1$*

$$u(z) = \int_B u(w) B_q(z,w) d\mu_q(w) \tag{1.9}$$

Proof. If u is M-harmonic on B, then by Corollary 1.3 with $z = 0$, we obtain

$$u(0) = \int_S u(r\zeta) d\sigma(\zeta). \tag{1.9a}$$

Integrating both sides with respect to the measure $2nr^{2n-1}(1-r^2)^q dr$, we have

$$u(0) = \int_B u(w) d\mu_q(w). \tag{1.9b}$$

Replacing u by $u \circ \varphi_z, z\epsilon B$, in (1.9b), we get

$$\begin{aligned} u(z) = (u \circ \varphi_z)(0) &= \int_B (u \circ \varphi_z)(w) d\mu_q(w) \\ &= \int_B u(\varphi_z(w)) d\mu_q(w) \\ &= \int_B u(w) \frac{|K_q(z,w)|^2}{K_q(z,z)} d\mu_q(w), \quad \text{by}(II.6.2g). \end{aligned} \tag{1.9c}$$ □

Proposition 1.6 was proved in [23] for the case $q = 0$.

Definition 1.5. <u>The Berezin transform</u> B_q <u>with weight q</u> of $u \epsilon L^1_q(B)$ is defined by

$$\mathcal{B}_q[u](z) = \int_B u(w) B_q(z,w) d\mu_q(w), \quad z\epsilon B, \quad q \geqq -1 \tag{1.10}$$

In particular, when $q = 0$, (1.10) coincides with the usual definition of the Berezin transform of u. When $q = -1$, it gives the Poisson integral $P[u*]$, where $u*$ is the boundary function of u on S defined by the radial limits almost everywhere on S.

A consequence of Proposition 1.6 is that the Berezin transform of an M-harmonic function $u \epsilon L^1_q(B)$ is itself. Namely, we have

Corollary 1.7. If $u\epsilon\tilde{h}^1_q(B)$, then $\mathcal{B}_q[u] = u$.

Remark 1.8. (a) Every M-harmonic functions on B has the invariant mean value property [44, Theorem 4.2.4].

(b) Every $f\epsilon C(\bar{B})$ that has the invariant mean value property is M-harmonic on B (see Corollary 2 of [44, Theorem 4.2.4]).

(c) Every $f\epsilon C(\bar{B})$ that satisfies : $\mathcal{B}_0[f] = f(q=0)$ is M-harmonic on B (see [23, Corollary 3.5]).

(d) For $p \geqq 1, \tilde{h}^p_q(B) = L^p_q(B) \cap \tilde{h}(B)$ is a Banach space, since the space $\tilde{h}(B)$ is closed under the topology of uniform convergence on compact sets (see Corollary 1.4).

(e) Evidently, the space of M-harmonic functions is stable under complex çonjugation. Therefore, it must contain both holomorphic functions and pluriharmonic functions.

(f) In genral, extending known results of holomorphic functions to M-harmonic functions are nontrivial, due to the limitation of tools available for M-harmonic functions. An example of such a limitation occurs from the fact that an M-harmonicity is not stable under differentiation.

2. M-subharmonic Functions.

Definition 2.1. Let $\Omega \subset B$ be an open set. A function $u : \Omega \to [-\infty, \infty)$ is said to be M-subharmonic if it is upper semicontinuous and for each $z\epsilon\Omega$, there exists $r(z) > 0$ such that for all $0 < r \leqq r(z)$,

$$u(z) \leqq \int_S u(\varphi_z(r\zeta))d\sigma(\zeta), \tag{2.1}$$

and none of the integrals in (2.1) is $-\infty$ [55, Definition 1.15]. A function u is M-superharmonic on Ω if $-u$ is M-subharmonic on Ω.

Let $(\tilde{s}h)(B)$ denote the space of M-subharmonic functions on B and $(\tilde{s}h)^p_q(B) = L^p_q \cap (\tilde{s}h)(B)$.

It follows from Corollary 1.3 that equality in (2.1) holds on Ω if and only if u is M-harmonic there. In fact, if $u\epsilon C^2(\Omega)$, then u is subharmonic on Ω if and only if $\tilde{\Delta}u \geqq 0$ there, as it can be shown easily using (1.6).

Proposition 2.1. *Let $\Omega \subset B$ be an open set and let $u \epsilon C^2(\Omega)$. Then u is M-subharmonic on Ω if and only if it satisfies : $\tilde{\Delta} u(z) \geqq 0$ for all $z \epsilon \Omega$.*

Proposition 2.2. *If u M-subharmonic on an open set $\Omega \subset B$, then so is $u \circ \psi$ for all $\psi \epsilon Aut(B)$.*

Proof. Let a $\epsilon \Omega$ and let $b = \psi(a)$. Then $(\varphi_b \circ \psi \circ \varphi_a)(0) = 0$. By Cartan's theorem (Theorem 3.2, Ch.1), $\varphi_b \circ \psi \circ \varphi_a = U \epsilon \mathcal{U}$ or $\psi(\varphi_a(z)) = \varphi_b(Uz)$. Thus,

$$\begin{aligned}\int_S (u \circ \psi)(\varphi_a(r\zeta))d\sigma(\zeta) &= \int_S u(\varphi_b(Ur\zeta))d\sigma(\zeta) \\ &= \int_S u(\varphi_b(r\zeta))d\sigma(\zeta) \geqq u(b) = (u \circ \psi)(a). \quad \square\end{aligned}$$

Proposition 2.3. *u is M-subharmonic on an open set $\Omega \subset B$ if and only if it satisfies the following sub-mean value property : for every $z \epsilon \Omega$*

$$u(z) \leqq \frac{1}{\lambda[E(z,r)]} \int_{E(z,r)} u(w) d\lambda(w) \tag{2.2}$$

for $r > 0$ sufficiently small.

Proof. It follows from (2.1) by using the same method as in the proof of Corollary 1.5.

Definition 2.2. <u>The Green's function</u> G for the invariant Laplace operator $\tilde{\Delta}$ is given by $G(z,w) = (g \circ \varphi_z)(w)$, where

$$g(z) = \int_{|z|}^1 t^{-2n+1}(1-t^2)^{n-1} dt. \tag{2.3}$$

The Green potential $G\mu$ of a Borel measure μ on B is defined by

$$G\mu(z) = \int_B G(z,w) d\mu(w), \quad z \epsilon B. \tag{2.4}$$

The following analogue of the Riesz decomposition theorem for M-subharmonic functions was proved by D. Ullrich in [55].

Theorem 2.4 [55, Theorem 2.16]. *If μ is in a subharmonic Hardy 1-space then*

$$u(z) = P[u*](z) - G\mu(z), \tag{2.5a}$$

where $P[u*]$ is the least M-harmonic majorant of u and μ is the Riesz measure of u, i.e., $d\mu = \tilde{\Delta} u d\lambda$. Moreover, the Riesz measure μ satisfies :

$$\int_B (1 - |w|^2)^n d\mu(z) < \infty. \tag{2.5b}$$

Conversely, for any positive measure μ on B that satisfies (2.5b), the Green potential $G\mu$ is M-superharmonic on B.

3. Bloch and Besov Space of M-harmonic Functions.

Definition 3.1. Let $f \epsilon C^1(\Omega)$ and $\xi \epsilon \mathbb{C}^n$. The maximal derivative of f with respect to the Bergman metric b_Ω is defined by

$$\hat{Q}f(z) = \sup_{|\xi|=1} \frac{|df(z)\xi|}{b_\Omega(z,\xi)}, \quad z\epsilon\Omega, \tag{3.1a}$$

where

$$df(z) \cdot \xi = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial z_i}(z)\xi_i + \frac{\partial f}{\partial \bar{z}_i}(z)\bar{\xi}_i \right] = \partial f(z) \cdot \xi + \bar{\partial} f(z) \cdot \bar{\xi}. \tag{3.1b}$$

If $f \epsilon H(\Omega)$, then the quantity $\hat{Q}f$ is reduced to

$$Qf(z) = \sup_{|\xi|=1} \frac{|\nabla f(z)\xi|}{b_\Omega(z,\xi)}, \quad z\epsilon\Omega. \tag{3.1c}$$

The quantity $\hat{Q}f$ is invariant under $Aut(\Omega)$, due to the invariant nature of b_Ω. More precisely,

Proposition 3.1 [22], [23]. *For a $C^1(\Omega)$ function f and $\varphi \epsilon Aut(\Omega)$, we have*

$$\hat{Q}(f \circ \varphi) = (\hat{Q}f) \circ \varphi, \quad \text{and } \hat{Q}\bar{f} = \hat{Q}f. \tag{3.2}$$

In particular, if $f \epsilon H(B)$ and $\varphi \epsilon Aut(B)$, then

$$Qf(z) = \frac{1}{2}\sqrt{\tilde{\Delta}}|f|^2(z) = |\tilde{\nabla} f(z)|, \quad z \epsilon B. \tag{3.3a}$$

Proof. Let $f \epsilon C^1(\Omega)$ and $\varphi \epsilon Aut(\Omega)$. For $z \epsilon \Omega$ and $\xi \epsilon \mathbb{C}^n$ with $|\xi| = 1$, set $\eta = \varphi'(z)\xi$. Then

$$b_\Omega(z, \xi) = b_\Omega(\varphi(z), \varphi'(z)\xi) = b_\Omega(\varphi(z), \eta)$$

so that

$$\frac{|d(f \circ \varphi)(z)\xi|}{b_\Omega(z,\xi)} = \frac{|(df) \circ \varphi(z)\eta|}{b_\Omega(\varphi(z), \eta)} = \frac{|d(f \circ \varphi)(z)\eta/|\eta||}{b_\Omega(\varphi(z), \eta/|\eta|)} \leqq (\hat{Q}f) \circ \varphi,$$

showing that $\hat{Q}(f \circ \varphi) \leqq \hat{Q}(f) \circ \varphi$. Applying this inequality to the function $f \circ \varphi^{-1}$, we obtain the reversed inequality. The second equality of (3.2) is clear. If $f \epsilon H(B)$, then the following identity holds :

$$\Delta(|f|^2)(0) = 4 \sum_{i=1}^{n} |\partial f / \partial z_i(0)|^2,$$

which implies

$$Qf(0) = \sup_{|\xi|=1} |\nabla f(0)\xi| = |\nabla f(0)| = \left[\sum_{i=1}^{n} |\partial f / \partial z_i(0)|^2\right]^{1/2} = \frac{1}{2}\sqrt{\Delta}|f|^2(0). \tag{3.3b}$$

Replacing f by $f \circ \varphi_z, z \epsilon B$, in (3.3b) we get (3.3a). □

Let $\delta_\Omega(z)$ be the euclidean distance from z to the bounday $b\Omega$.

Definition 3.2. Let $0 < p < \infty, s \epsilon \mathbb{R}$. The Besov p-space $\mathcal{B}_p^s(\Omega)$ with weight s is defined by the space of all locally integrable functions f on Ω such that

$$\|f\|_{p,s} = \left\{ \int_\Omega (\hat{Q}f)^p(z)\delta(z)^s d\lambda(z) \right\}^{1/p} < \infty. \tag{3.4a}$$

Here $\hat{Q}f$ is defined in the sense of distributions and $d\lambda$ is defined by

$$d\lambda(z) = B(z)d\nu(z), \tag{3.4b}$$

where $B(z) = \det(b_{ij})$ is an invariant volume measure with respect to the Bergman metric b_Ω.

We shall denote by $M\mathcal{B}_p^s(\Omega)$ a Besov p-space of M-harmonic functions, and by $H\mathcal{B}_p^s(\Omega)$ a Besov p-space of holomorphic functions on Ω.

In particular, if $s = 0$, then the spaces $M\mathcal{B}_p^0(\Omega)$ are invariant under the actions of $Aut(\Omega)$ and constitute most interesting spaces. We denote these spaces simply by $M\mathcal{B}_p$. If $p = \infty$, then the corresponding space is the Bloch space of M-harmonic functions.

Definition 3.3. The Bloch space $M\mathcal{B}(\Omega)$ of M-harmonic functions consists of all M-harmonic functions on Ω such that

$$\sup_{z \epsilon \Omega} (\hat{Q}f)(z) < \infty. \tag{3.4c}$$

The little Bloch space $M\mathcal{B}_0(\Omega)$ is as usual defined by those M-harmonic functions f in $M\mathcal{B}(\Omega)$ such that

$$\lim_{z \to \partial\Omega} (\hat{Q}f)(z) = 0. \tag{3.4d}$$

The Bloch space $H\mathcal{B}$ of holomorphic functions were first studied in [1] and extended to the general homogeneous domains in [20]. But it was R. Timoney [52], [53] who gave a complete description of the Bloch space of holomorphic functions on the bounded symmetric domains. Recently, Krantz and Ma [39] gave a definition of holomorphic Bloch functions on strongly pseudoconvex domains and proved several interesting characterizations of Bloch functions.

It is well-known [20], [52], [53] that the Bloch space $H\mathcal{B}$ is a Banach space with respect to the Bloch norm :

$$\|f\|_\mathcal{B} = |f(0)| + \sup_{z \epsilon B} (Qf)(z).$$

The little Bloch space $M\mathcal{B}_0$ is precisely the closure of the polynomials in the Bloch space $M\mathcal{B}$.

Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with at least C^1- boundary $\partial\Omega$. Let r be a defining function of Ω, i.e.,

$$\Omega = \{z \epsilon \mathbb{C}^n : r(z) < 0\},$$

where $r \epsilon C^1(\bar{\Omega})$ and grad $r(z) \neq 0$ for all $z \epsilon \partial\Omega$. Let $\delta(z)$ be the euclidean distance from z to the boundary $\partial\Omega$. Then $\delta(z)$ is equivalent to $|r(z)| = -r(z)$, i.e., there exists a constant $c > 0$ such that for all $z \epsilon \Omega$

$$c^{-1}\delta(z) \leqq |r(z)| \leqq c\delta(z) \quad (see[39]). \tag{3.5}$$

We denote this equivalence by $\delta(z) \sim |r(z)|$. The invariant measure $d\lambda$ used to define the Besov spaces in (3.4b) is equivalent to $d\nu/\delta(z)^{n+1}$ [39]. The Besov norm (3.4a) can be written in the following more explicit form :

$$||f||_{p,s} = \left\{\int_\Omega |Qf|^p(z)|r(z)|^s d\lambda(z)\right\}^{1/p} \tag{3.6a}$$

on a strongly pseudoconvex domain Ω, where $d\lambda(z) \sim \delta(z)^{-n-1}dV(z)$ is a measure on Ω invariant under $Aut(\Omega)$, and on the unit ball B

$$||f||_{p,s} = \left\{\int_S |Qf|^p(z)(1-|z|^2)^s d\lambda(z)\right\}^{1/p}, \tag{3.6b}$$

where $d\lambda(z) = K(z,z)d\nu(z) = (1-|z|^2)^{-n-1}d\nu(z)$ is the invariant measure under $Aut(B)$.

Chapter V Properties of Besov Spaces.

In this chapter we give various characterizations of the Besov p-space $M\mathcal{B}_p^s(B)$ on the unit ball B of weight $s\epsilon\mathbb{R}$, and prove that it is a Banach space for $1 \leqq p \leqq \infty$. We also consider the "modified " Besov p-spaces $\tilde{\mathcal{B}}_p^s$ and "diagonal " Besov p-spaces B_p^s. In general, $M\mathcal{B}_p^s \subset M\tilde{\mathcal{B}}_p^s$ and $M\mathcal{B}_p^s = M\tilde{\mathcal{B}}_p^{n-sp}$. If $s > n$ and $p > 2n$, then

$$M\mathcal{B}_p^s = M\tilde{\mathcal{B}}_p^s = MB_p^\beta = \tilde{h}_q^p \quad \left(\beta = \frac{n-s}{p},\ q = s-n-1\right).$$

Let $1 < p \leqq q < \infty$. It is shown that (i) for a fixed $s \leqq 0, M\mathcal{B}_p^s \subset M\mathcal{B}_q^s \subset M\mathcal{B}_\infty = M\mathcal{B}$, (ii) for a fixed $s > n, M\mathcal{B} = M\mathcal{B}_\infty \subset M\mathcal{B}_p^s$, and (iii) for $s = n, H^p \subset H\tilde{\mathcal{B}}_p^n$ when $p \geqq 2, H\mathcal{B}_p \subset H\tilde{\mathcal{B}}_p^n \subset H^p$ when $0 < p \leqq 2$, and $H^2 = H\tilde{\mathcal{B}}_2^n$.

1. Characterizations of Besov p-Spaces with Weights.

Lemma 1.1. *Let $1 < p < \infty$. Then there exists a positive constant C_p such that for all $f\epsilon C^1(B)$*

$$\int_B |f(z) - f(0)|^p d\nu(z) \leqq C_p \int_B \frac{(\hat{Q}f)^p(z)}{|z|^{2n-1/2}} d\nu(z).$$

Proof. For $z\epsilon B$ and a C^1-function f, we have

$$\begin{aligned} |f(z) - f(0)| &= \left|\int_0^1 \frac{df(tz)}{dt} dt\right| = |\int_0^1 df(tz)\cdot z dt|, \quad \text{see (IV.3.1b)}, \\ &= \left|\int_0^1 \frac{df(tz)\cdot z}{b(tz,z)} b(tz,z) dt\right| \\ &\leqq \int_0^1 (\hat{Q}f)(tz) b(tz,z) dt \\ &\leqq \int_0^1 (\hat{Q}f)(tz) \frac{|z|}{1-|tz|} dt, \quad \text{by } (III.4.2). \end{aligned}$$

Let q be the conjugate exponent of p. Then Holder's inequality implies

$$(1.1)\quad |f(z) - f(0)| \leqq \left[\int_0^1 \frac{|z|^q}{(1-|tz|)^{(1+q)/2}} dt\right]^{1/q} \left[\int_0^1 \frac{(\hat{Q}f)^p(tz)}{(1-|tz|)^{1/2}} dt\right]^{1/p}.$$

After some manupulation on the first integral of (1.1), we find

$$|f(z) - f(0)| \leqq \left[\frac{C|z|}{\sqrt{1-|z|}} \int_0^1 \frac{(\hat{Q}f)^p(tz)}{\sqrt{1-|tz|}} dt \right]^{1/p}$$

for some positive constant C independent of f. Therefore,

$$\begin{aligned}
\int_B |f(z) - f(0)|^p d\nu(z) &\leqq C \int_B \int_0^1 \frac{(\hat{Q}f)^p(tz)|z|}{\sqrt{(1-|z|)(1-|tz|)}} dt d\nu(z) \\
&\leqq C \int_0^1 \int_{tB} \frac{(\hat{Q}f)^p(z)|z|}{\sqrt{(1-|z|/t)(1-|z|)}} d\nu(z) dt / t^{2n+1} \\
&\leqq C \int_B \frac{(\hat{Q}f)^p(z)|z|}{\sqrt{1-|z|}} \int_{|z|}^1 \frac{1}{\sqrt{t-|z|} t^{2n+1/2}} dt d\nu(z) \\
&\leqq C \int_B \frac{(\hat{Q}f)^p(z)}{\sqrt{1-|z|}|z|^{2n-1/2}} d\nu(z) \int_{|z|}^1 \frac{dt}{\sqrt{t-|z|}} \\
&\leqq 2C \int_B \frac{(\hat{Q}f)^p(z)}{|z|^{2n-1/2}} d\nu(z). \ \square
\end{aligned}$$

Lemma 1.2. *For $-1 < s < n+1$, there exists a positive constant C such that for each $z \epsilon B$*

$$\int_B \frac{|K(z,w)|^2}{|\varphi_w(z)|^{2n-1/2}} (1-|w|^2)^s d\nu(w) \leqq C(1-|z|^2)^s K(z,z).$$

Proof. Since $|\varphi_w(z) = |\varphi_z(w)|$, the change of variables formula yields :

(1.2)

$$\begin{aligned}
&\int_B \frac{|K(z,w)|^2}{|\varphi_w(z)|^{2n-1/2}} (1-|w|^2)^s d\nu(w) \\
&= \int_B \frac{|K(z,\varphi_z(w))|^2}{|w|^{2n-1/2}} (1-|\varphi_z(w)|^2)^s d\nu(\varphi_z(w)) \\
&= K(z,z)(1-|z|^2)^s \int_B \frac{(1-|w|^2)^s}{|(1-<z,w>)|^{2s}|w|^{2n-1/2}} d\nu(w).
\end{aligned}$$

It is clear that the integrand of the last integral has no singularity at $w = 0$. On the other hand, the following integral :

$$I_s(z) = \int_B \frac{(1-|w|^2)^s}{|1-<z,w>|^{2s}} d\nu(w)$$

is bounded whenever $-1 < s < n+1$, according to [44, Proposition 1.4.10]. Therefore, the last integral of (1.2) is bounded by a constant and proves the lemma. □

Lemma 1.3. *Let $1 < p < \infty$ and $-1 < s < n+1$. Then there exists a positive constant A_p such that for every $f \epsilon C^1(B)$*

$$\int_B \int_B |f \circ \varphi_z(w) - f(z)|^p (1-|z|^2)^s d\nu(w) d\lambda(z) \leqq A_p \int_B (\hat{Q}f)^p(z)(1-|z|^2)^s d\lambda(z).$$

Proof. Let $f \epsilon C^1(B)$. Lemma 1.1 with f replaced by $f \circ \varphi_z$ implies

$$\begin{aligned}
&\int_B \int_B |f \circ \varphi_z(w) - f(z)|^p (1-|z|^2)^s d\nu(w) d\lambda(z) \\
&\leqq C_p \int_B \int_B \frac{\hat{Q}(f \circ \varphi_z)^p(w)}{|w|^{2n-1/2}} (1-|z|^2)^s d\nu(w) d\lambda(z) \\
&\leqq C_p \int_B \int_B \frac{(\hat{Q}f)^p(\zeta)}{|\varphi_z(\zeta)|^{2n-1/2}} (1-|z|^2)^s d\nu(\varphi_z(\zeta)) d\lambda(z), \quad \zeta = \varphi_z(w), \\
&\leqq C_p \int_B \int_B (\hat{Q}f)^p(\zeta) \frac{|K(\zeta,z)|^2}{|\varphi_z(\zeta)|^{2n-1/2}} (1-|z|^2)^s d\nu(\zeta) d\nu(z), \quad \text{by } (II.2.2g), \\
&\leqq C_p \int_B (\hat{Q}f)^p(\zeta) \int_B \frac{|K(\zeta,z)|^2}{|\varphi_z(\zeta)|^{2n-1/2}} (1-|z|^2)^s d\nu(z) d\nu(\zeta), \\
&\leqq CC_p \int_B (\hat{Q}f)^p(\zeta)(1-|\zeta|^2)^s K(\zeta,\zeta) d\nu(\zeta), \quad \text{by Lemma 1.2}, \\
&\leqq A_p \int_B (\hat{Q}f)^p(\zeta)(1-|\zeta|^w)^s d\lambda(\zeta). \ \square
\end{aligned}$$

Definition 1.1. Two functional quanttities $Q_1 f$ and $Q_2 f$ defined on B are said to be <u>equivalent</u> on B, write $Q_1 f \approx Q_2 f$, if there are positive constants C_1 and C_2 independent of f such that

$$Q_2 f \leqq C_1 Q_1 f \leqq C_2 Q_2 f.$$

Theorem 1.4. *Let $s\epsilon\mathbb{R}$ be such that $-1 < s < n+1$. For $1 < p < \infty$ and $f\epsilon\tilde{h}(B)$, the following semi-norms are equivalent :*

$$||f||_1 = ||f||_{p,s} = \left(\int_B (\hat{Q}f)^p(w)(1-|w|^2)^s d\lambda(w)\right)^{1/p}$$

$$||f||_2 = \left(\int_B \int_B |f(z)-f(w)|^p(1-|z|^2)^s d\mu(z,w)\right)^{1/p},$$

where $d\mu(z,w) = \frac{|K(z,w)|^2}{K(z,z)K(w,w)} d\lambda(z)d\lambda(w)$ is an invariant probability measure over $B \times B$.

$$||f||_3 = \left(\int_B \int_B |f\circ\varphi_z(w) - f(z)|^p(1-|z|^2)^s d\nu(w)d\lambda(z)\right)^{1/p}$$

$$||f||_4 = \left(\int_B (\tilde{\Delta}|f|^2)^{p/2}(z)(1-|z|^2)^s d\lambda(z)\right)^{1/p}$$

$$||f||_5 = \left(\int_B (\mathcal{B}[|f|^2](z) - |f|^2(z))^{p/2}(1-|z|^2)^s d\lambda(z)\right)^{1/p} \quad (p \geqq 2).$$

Proof. By the change of variables $\zeta = \varphi_z(w)$ and (6.2g), Ch.3, with $q=0$, (1.3)

$$||f||_3 = \int_B \int_B |f(\zeta) - f(z)|^p (1-|z|^2)^s \frac{|K(z,\zeta)|^2}{K(z,z)} d\nu(\zeta) d\lambda(z)^{1/p}$$
$$= \left(\int_B \int_B |f(\zeta)-f(z)|^p(1-|z|^2)^s d\mu(z,\zeta)\right)^{1/p} = ||f||_2.$$

To prove the rest, let $f\epsilon\tilde{h}(B)$. An elementary calculation shows :

(1.4a)
$$(\Delta|f|^2)(0) = 4\sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(f\bar{f})$$
$$= 4[|\nabla f|^2(0) + |\nabla \bar{f}|^2(0)].$$

If $f\epsilon h(B)$ is real valued, then

$$(\hat{Q}f)(0) = \sup_{|\zeta|=1} |df(0)\zeta|$$
$$= \sup_{|\zeta|=1} |\nabla f(0)\zeta + \overline{\nabla \bar{f}(0)\zeta}|$$
$$= 2 \sup_{|\zeta|=1} |Re(\nabla f(0)\zeta)| = 2|\nabla f(0)|$$

so that by (1.4a)

$$(\Delta|f|^2)(0) = 8|\nabla f|^2(0) = 2(\hat{Q}f)^2(0).$$

Replacing f by $f \circ \varphi_z, z\epsilon B$, we have for all $z\epsilon B$

$$(1.4b) \qquad (\tilde{\Delta}|f|^2)(z) = 2(\hat{Q}f)^2(z) = 8|\tilde{\nabla} f|^2(z), \quad (\hat{Q}f)(z) = 2|\tilde{\nabla} f|(z).$$

For an arbitrary M-harmonic function f in B,

$$\hat{Q}(\text{Ref}) = \hat{Q}\left(\frac{f+\bar{f}}{2}\right) \leqq \hat{Q}f, \quad \hat{Q}(Imf) = \hat{Q}\left(\frac{f+\bar{f}}{2i}\right) \leqq \hat{Q}f,$$
$$\hat{Q}f = \hat{Q}(\text{Re f} + i\text{Im f}) \leqq \hat{Q}(\text{Re f}) + \hat{Q}(\text{Im f}).$$

Therefore, by (1.4b)

$$\begin{aligned}\tilde{\Delta}|f|^2 &= \tilde{\Delta}(\text{Re f})^2 + \tilde{\Delta}(\text{Im f})^2 \\ &= 2\{(\hat{Q}(\text{Re f}))^2 + (\hat{Q}(\text{Im f}))^2\} \leqq 4(\hat{Q}f)^2.\end{aligned}$$

On the other hand,

$$(\hat{Q}f)^2 \leqq 2\{(\hat{Q}(\text{Re f}))^2 + (\hat{Q}(\text{Im f}))^2\} = \tilde{\Delta}(|f|^2).$$

Putting all together, we have

$$(1.5) \qquad \frac{1}{2}\sqrt{\tilde{\Delta}|f|^2} \leqq \hat{Q}f = 2|\tilde{\nabla} f| \leqq \sqrt{\tilde{\Delta}|f|^2}.$$

which proves : $||f||_1 \approx ||f||_4$.

Let $f\epsilon\tilde{h}^p(B)$ for $1 \leqq p < \infty$. (IV, Proposition 1.10), with $q = 0$ yields :

$$\frac{\partial f}{\partial z_i}(0) = \int_B f(w)\left[\frac{\partial}{\partial z_i}B(z,w)\right]_{z=0} d\nu(w), \quad i = 1, \cdots, n.$$

Setting $M_i = \sup_{w\epsilon B}\left|\left[\frac{\partial}{\partial z_i}B(z,w)\right]_{z=0}\right|$, $i = 1, \cdots, n$, we have

$$\left|\frac{\partial f}{\partial z_i}(0)\right| \leqq M_i\left|\int_B f(w)d\nu(w)\right| \leqq M_i||f||_{L^p(\nu)}.$$

Therefore,

$$(1.6a) \qquad |\nabla f(0)|^2 \leqq M||f||^2_{L^p(\nu)}, \quad \text{where } M = \left(\sum_{i=1}^n M_i^2\right)^{1/2}.$$

Similar arguments show that (1.6a) is also true with $\bar{f}$ in place of f so that by (1.3)

$$\sqrt{\tilde{\Delta}|f|^2(0)} = \sqrt{|\nabla f(0)|^2 + |\nabla f(0)|^2} \leqq \sqrt{2}M||f||_{L^p(\nu)}.$$

Replacing f by $f \circ \varphi_z - f(z)$ yields :

$$\sqrt{(\tilde{\Delta}|f|^2)(z)} \leqq \sqrt{2}M||f \circ \varphi_z - f(z)||_{L^p(\nu)}. \tag{1.6b}$$

Putting together (1.5), (1.6b), Lemma 1.3, and the fact that $||f||_2 = ||f||_3$, we find the equivalence of $||f||_{p,s}$ and $||f||_i$, $i = 2, 3, 4$.

If $p \geqq 2$, then it holds that

(1.7)

$$\begin{aligned} ||f \circ \varphi_z - f(z)||_{L^2(\nu)} &= \left\{\int_B |f \circ \varphi_z(w) - f(z)|^2 d\nu(w)\right\}^{1/2} \\ &= \left\{\int_B |f(\zeta) - f(z)|^2 d\nu(\varphi_z(\zeta))\right\}^{1/2} \quad \text{(with } w = \varphi_z(\zeta)) \\ &= \left\{\int_B |f(\zeta) - f(z)|^2 B(\zeta, z) d\nu(\zeta)\right\}^{1/2}, \quad \text{by (6.2g), Ch.2,} \\ &= \{B[|f|^2](z) - |f|^2(z)\}^{1/2} \\ &\leqq ||f \circ \varphi_z - f(z)||_{L^p(\nu)} \quad \text{for } p \geqq 2, \end{aligned}$$

which proves : $||f||_3 \approx ||f||_5$ for $p \geqq 2$. □

Theorem 1.5. *Let $s \in \mathbb{R}$ be such that $-1 < s < n+1$, and let $1 < p < \infty$, $0 < q \leqq p$. For $f \in \tilde{h}(B)$, the following semi-norms are equivalent to $||f||_{p,s}$.*

$$||f||_6 = \left\{\int_B \left(\int_{E(z,r)} |f(w) - f(z)|^q \frac{d\nu(w)}{|E(z,r)|}\right)^{p/q} (1 - |z|^2)^s d\lambda(z)\right\}^{1/p}$$

$$||f||_7 = \left\{\int_B \left(\int_{E(z,r)} |f(w) - \hat{f}(z,r)|^q \frac{d\nu(w)}{|E(z,r)|}\right)^{p/q} (1 - |z|^2)^s d\lambda(z)\right\}^{1/p}$$

where $|E(z,r)| = \nu(E(z,r))$, $E(z,r) = \varphi_z(rB)$, and

$$\hat{f}(z,r) = \int_{E(z,r)} f(w) d\nu_{z,r}(w), \quad d\nu_{z,r}(w) = \frac{d\nu(w)}{|E(z,r)|}.$$

Proof. The reproducing kernel of rB is given by

$$(1.8) \qquad K_{rB}(z,w) = r^{-2n} K_B\left(\frac{z}{r}, \frac{w}{r}\right), \quad z, \quad w \epsilon rB.$$

The mean value property of $f \epsilon \tilde{h}(B)$ (see [51]) and (1.8) imply that

$$f(z) = r^{-2n} \int_{rB} f(w) K_B\left(\frac{z}{r}, \frac{w}{r}\right) d\nu(w), \quad z \epsilon rB.$$

Using similar arguments as in the proof of (1.6a), we have that

$$(1.9) \qquad |\nabla f(0)| \leqq \tilde{M} ||f||_{L^p(rB,\nu)} \leqq \tilde{M} ||f||_{L^p(B,\nu)},$$

where $\tilde{M}$ is a positive constant independent of f. Replacing f by $f \circ \varphi_z - f(z)$ yields :

$$(1.10) \quad (\hat{Q}f)(z) = \tilde{M} ||f \circ \varphi_z - f(z)||_{L^p(rB,\nu)} \leqq \tilde{M} ||f \circ \varphi_z - f(z)||_{L^p(B,\nu)}.$$

Since $B(z,w)|E(z,r)|$ and $|E(z,r)|/B(z,w)$ are both bounded by a constant independent of $z \epsilon B$ and $w \epsilon rB$, (1.9) and the change of variables formula imply that for some constant $\tilde{M}_0 > 0$

$$(1.11) \qquad (\hat{Q}f)(z) \leqq \tilde{M} \left(\int_{E(z,r)} |f(w) - f(z)|^p \frac{d\nu(w)}{|E(z,r)|} \right)^{1/p}$$

$$\leqq \tilde{M}_o \left(\int_B |f \circ \varphi_z(w) - f(z)|^p d\nu(w) \right)^{1/p}.$$

Replacing f by $f \circ \varphi_z - \hat{f}(z,r)$ in the first inequality of (1.9) yields :

$$(\hat{Q}f)(z) \leqq \tilde{M} \left(\int_{E(z,r)} |f(w) - \hat{f}(z,r)|^p \frac{d\nu(w)}{|E(z,r)|} \right)^{1/p}$$

$$\leqq \tilde{M} \left\{ \left(\int_{E(z,r)} |f(w) - f(z)|^p \frac{d\nu(w)}{|E(z,r)|} \right)^{1/p} + \left(\int_{E(z,r)} |f(z) - \hat{f}(z,r)|^p \frac{d\nu(w)}{|E(z,r)|} \right)^{1/p} \right\}$$

$$\leqq 2\tilde{M} \left(\int_{E(z,r)} |f(w) - f(z)|^p \frac{d\nu(w)}{|E(z,r)|} \right)^{1/p}.$$

Theorem 1.5 now follows from (1.11), Lemma 1.3 and Theorem 1.4. □

Lemma 1.6. *For each compact set $K \subset B$, there exists a constant $C > 0$ such that*

$$\sup_{z \epsilon K} |f(z)| \leqq C\|f\|_{p,s}, \quad 1 \leqq p < \infty, \quad s \epsilon \mathbb{R}, \tag{1.12}$$

for all M-harmonic functions $f \epsilon \tilde{h}(B)$.

Proof. The mean value property of M-harmonic function f implies that for $|\zeta| = 1$, $0 < t < 1$,

$$f(t\zeta) = r^{-2n} \int_{rB} f \circ \varphi_{t\zeta}(w) d\nu(w).$$

Therefore,

$$df(0)\zeta = \frac{d}{dt} f(t\zeta)\big|_{t=0} = r^{-2n} \int_{rB} df(-w) \cdot \frac{d}{dt} \varphi_{t\zeta}(w)\big|_{t=0} d\nu(w). \tag{1.13a}$$

A routine calculation shows that

$$a_\zeta(w) = \frac{d}{dt} \varphi_{t\zeta}(w)\big|_{t=0} = \zeta - < w, \zeta > w, \tag{1.13b}$$

and hence, is a bounded holomorphic mapping from B into $\mathbb{C}^n$ that satisfies $a_\zeta(0) = \zeta$. From (1.13a) and (1.13b),

$$df(0)\zeta = r^{-2n} \int_{rB} df(-w) \cdot a_\zeta(w) \frac{b(-w, a_\zeta(w))}{b(-w, a_\zeta(w))} d\nu(w),$$

which implies

$$|df(0)\zeta| \leqq C \int_{rB} (\hat{Q}f)(-w) d\lambda(w) = C \int_{rB} (\hat{Q}f)(w) d\lambda(w)$$

where C is a positive constant independent of f. Therefore,

$$(\hat{Q}f)(0) \leqq C \int_{rB} (\hat{Q}f)(w) d\lambda(w).$$

Replacing f by $f \circ \varphi_z$ and using the change of variables formula, we have

$$(\hat{Q}f)(z) \leqq \tilde{C} \int_{rB} (\hat{Q}f) \circ \varphi_z(w) d\lambda(w) = \tilde{C} \int_{E(z,r)} (\hat{Q}f)(w) d\lambda(w)$$

where $\tilde{C}$ is a constant depending only on r and n. Holder's inequality implies that

$$(\hat{Q}f)(z) \leqq \tilde{C}\left\{\int_{E(z,r)}(1-|w|^2)^{\frac{-s}{p-1}}d\lambda(w)\right\}^{\frac{p-1}{p}}\left\{\int_{E(z,r)}(\hat{Q}f)^p(1-|w|^2)^s d\lambda(w)\right\}^{1/p}.$$

Therefore,

$$(\hat{Q}f)(z) \leqq (1-|z|^2)^{\frac{-s}{p}}||f||_{p,s}, \tag{1.14}$$

where M is a constant depending only on r and n. Hence,

$$\begin{aligned}|f(z)-f(0)| &\leqq \int_0^1 |df(tz)z|dt \\ &\leqq \int_0^1 \frac{|df(tz)z|}{b(tz,z)}b(tz,z)dt \\ &\leqq \int_0^1 (\hat{Q}f)(tz)\frac{|z|}{1-|tz|^2}dt \\ &\leqq M||f||_{p,s}\int_0^1 (1-|tz|^2)^{-\frac{s}{p}}\frac{|z|}{1-|tz|^2}dt \quad (\text{by}(1.14)).\end{aligned}$$

This fact proves Lemma 1.6. □

Theorem 1.7. *Let $s\epsilon\mathbb{R}$ and $1 \leqq p < \infty$.*

Then the space $M\mathcal{B}_p^s(B)/\mathbb{C}$ is a Banach space under the quotient norm associated with the norm $||\cdot||_{p,s}$.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $M\mathcal{B}_p^s(B)$. By Lemma 1.6, we see that $\{f_n\}$ converges uniformly on each compact subset of B to an M-harmonic function f in B. Now the convergence $f_n \to f$ with respcet to the semi-norm $||\cdot||_{p,s}$ follows by standard arguments. □

Theorems 1.4, 1.5, and 1.7 have been proved in [23] for the case where $s=0$. The results similar to Theorem 1.4, 1.5, and 1.7 are proved in [25] for the bounded symmetric domains.

In the case where $s=0$ and $p=\infty$, The most of the results obtained in Theorems 1.4, 1.5, and 1.7 remain valid. More precisely, we have

Corollary 1.8. *Let $1 \leqq p < \infty$ and $0 < r < 1$. Let $f \epsilon \tilde{h}(B)$. Then $f \epsilon M\mathcal{B}$ if and only if any of the following hold :*

(a) $||f||_{\mathcal{B}} = \sup_{z \epsilon B}(\hat{Q}f)(z) < \infty$
(b) $\sup_{z \epsilon B} ||f - f(z)||_{L^p(\mu(z,\cdot))} < \infty$
(c) $\sup_{z \epsilon B} ||f \circ \varphi_z - f(z)||_{L^p(\nu)} < \infty$
(d) $\sup_{z \epsilon B} \sqrt{\tilde{\Delta}}(|f|^2)(z) < \infty$
(e) $\sup_{z \epsilon B} \sup_{w \epsilon E(z,r)}(\hat{Q}f)(w) < \infty$ for any $r \epsilon (0,1)$
(f) $\sup_{z \epsilon B} \left\{\int_{E(z,r)}(\hat{Q}f)^p(w)d\lambda(w)\right\} < \infty$
(g) $\sup_{z \epsilon B}(1 - |z|^2)|\nabla f(z)| < \infty$
(f) $\sup_{z \epsilon B}(1 - |z|^2)|\Re(z)| < \infty$.

2. More Properties of Besov p-Spaces.

By modifying Hardy's inequality [27, Theorem 330], we obtain

Lemma 2.1. *Let $q > -1$, $1 \leqq p < \infty$, and let $G(r) = \int_0^r g(t)dt, 0 < r < 1$. Then*

$$\int_0^1 |G(r)|^p(1-r)^q dr \leqq C \int_)^1 |(1-r)g(r)|^p(1-r)^q dr. \tag{2.1a}$$

If in addition $-1 < s < p - 1$, or $p = 1$ and $s = 0$, then

$$\int_0^1 |G(r)|^p(1-r)^q r^s dr \leqq C \int_0^1 |(1-r)g(r)|^p(1-r)^q r^s dr. \tag{2.1b}$$

Proof. Inequality (2.1a) follows by making change of variable : $x = 1 - r$ in the original Hardy's inequality, while (2.1b) follows from (2.1a) by the change of variable $r = t^\alpha$ for a suitable α.

Theorem 2.2 [5]. *Let $1 \leqq p < \infty$ and $q > -1$. Then there exists a constant $C > 0$ such that for every $f \epsilon C^1(B)$*

(2.2a)
$$\int_B |f(z) - f(0)|^p d\mu_q(z) \leqq C \int_B [(1 - |z|^2)|\Re f(z)|/|z|]^p |z|^{1-2n} d\mu_q(z)$$
$$\leqq C \int_B (\hat{Q}f)^p(z)|z|^{1-2n} d\mu_q(z).$$

If $p > 2n$, then

(2.2b)
$$\int_B |f(z) - f(0)|^p d\mu_q(z) \leqq C \int_B [(1-|z|^2)|\Re f(z)|/|z|]^p d\mu_q(z)$$
$$\leqq C \int_B (\hat{Q}f)^p(z) d\mu_q(z).$$

If $f \epsilon H(B)$, both (2.2a) and (2.2b) hold for every $p > 0$ even without the negative powers of $|z|$. Namely,

(2.2c)
$$\int_B |f(z) - f(0)|^p d\mu_q(z) \leqq C \int_B [(1-|z|^2)|\Re f(z)|]^p d\mu_q(z)$$
$$\leqq C \int_B (\hat{Q}f)^p(z) d\mu_q(z).$$

Proof. Using (2.1a) on each radius to obtain

$$\begin{aligned}
\int_B |f(z) - f(0)|^p d\mu_q(z) &\leqq C \int_S \int_0^1 |f(r\zeta) - f(0)|^p (1-r)^q dr d\sigma(\zeta) \\
&\leqq C \int_S \int_0^1 \left[(1-r)\left|\frac{d}{dr} f(r\zeta)\right|\right]^p (1-r)^q dr d\sigma(\zeta) \\
&= C \int_S \int_0^1 [(1-r)|\Re f(r\zeta)|/r]^p (1-r)^q dr d\sigma(\zeta) \\
&= C \int_B [(1-|z|)|\Re f(z)|/|z|]^p (1-|z|)^q |z|^{1-2n} d\nu(z) \\
&\leqq C \int_B (\hat{Q}f)^p(z) |z|^{1-2n} d\mu_q(z).
\end{aligned}$$

The latter inequality follows from (III.4.6d) and (IV.3.3a).

To prove inequalities (2.2b), we apply (2.1b) with $s = 2n - 1 < p - 1$, or

$p > 2n$, to obtain

$$\begin{aligned}\int_B |f(z)-f(0)|^p d\mu_q(z) &\leqq 2nC\int_S\int_0^1 |f(r\zeta)-f(0)|^p(1-r)^q r^{2n-1}dr d\sigma(\zeta)\\ &\leqq 2nC\int_S\int_0^1 \left[(1-r)\left|\frac{d}{dr}f(r\zeta)\right|\right]^p(1-r)^q r^{2n-1}dr d\sigma(\zeta)\\ &\leqq 2nC\int_S\int_0^1 [(1-r)|\Re f(r\zeta)|/r]^p(1-r)^q r^{2n-1}dr d\sigma(\zeta)\\ &\leqq C\int_B [(1-|z|)|\Re f(z)|/|z|]^p(1-|z|)^q d\nu(z)\\ &\leqq C\int_B (\hat{Q}f)^p(z)d\mu_q(z),\end{aligned}$$

by (III.4.6d) and (IV.3.3a). □

Corollary 2.3. *Let $2n < p < \infty$ and $s > n$. Then there exists a constant $C > 0$ such that for every $f\epsilon\tilde{h}(B)$*

$$\int_B |f(z)|^p(1-|z|^2)^s d\lambda(z) \leqq C\int_B (\hat{Q}f)^p(z)(1-|z|^2)^s d\lambda(z) + |f(0)|,$$

i.e.,

$$||f||_{A_q^p} \leqq c||f||_{B_p^s}, \quad \text{with } s = q+n+1.$$

Proof. Choose $q = s-n-1 > -1$ in (2.2b) and use the triangle inequality of the norm $||f||_{A_q^p}$. □

Theorem 2.4. *Let $0 < p < \infty$ and $s\epsilon\mathbb{R}$. Then there exists a constant $C > 0$ such that for every $f\epsilon\tilde{h}(B)$*

$$\int_B (\hat{Q}f)^p(z)(1-|z|^2)^s d\lambda(z) \leqq C\int_B\int_B |f(w)-f(z)|^p(1-|z|^2)^s d\mu(z,w),$$

where $d\mu(z,w) = \frac{|K(z,w)|^2}{K(z,z)K(w,w)}d\lambda(z)d\lambda(w)$ is given as in Theorem 1.4.

Proof. From (1.5) and (1.6b),

$$\begin{aligned}(\hat{Q}f)^p(z) &\leqq C\int_B |f\circ\varphi_z(w)-f(z)|^p d\nu(w)\\ &\leqq C\int_B |f(\zeta)-f(z)|^p\frac{|K(z,\zeta)|^2}{K(z,z)}d\nu(\zeta), \quad \zeta = \varphi_z(w),\end{aligned}$$

Integrating the both sides with respect to the measure $(1-|z|^2)^s d\lambda(z)$, we obtain

$$\int_B (\hat{Q}f)^p(z)(1-|z|^2)^s d\lambda(z) \leqq \int_B \int_B |f(\zeta)-f(z)|^p(1-|z|^2)^s |K(z,\zeta)|^2 d\nu(\zeta)d\nu(z)$$

which proves the theorem. □

The following corollary is a consequence of Lemma 2.1.

Corollary 2.5. *Let $1 \leqq p < \infty$ and $s > n$. There exist constants $C_1, C_2 > 0$ such that for every $f \epsilon C^1(B)$*

$$\int_B |f(z)|^p(1-|z|^2)^s d\lambda(z) \leqq C_1 \int_B [(1-|z|^2)|df(z)|]^p(1-|z|^2)^s d\lambda(z) + |f(0)|^p$$
$$\leqq C_2 \int_B |f(z)|^p(1-|z|^2)^s d\lambda(z).$$

If $f \epsilon H(B)$, then the above inequalities hold for all $p > 0$ with df replaced by $\partial f = \nabla f$ in the second integral.

The above result was proved in [8, Theorem 5.12] for holomorphic functions f and remarked also in [5, Remark 5.2]. For general case, see [41, Theorem I.4.6].

An immediate consequence of Corollary 2.5 is the following

Corollary 2.6. *Let $1 \leqq p \leqq \infty$, $s > n$. If $f \epsilon \tilde{h}(B)$, then for every $k, m > n/p$*

$$\int_B [(1-|z|^2)^k |d^k f(z)|]^p (1-|z|^2)^s d\lambda(z) < \infty$$

if and only if

$$\int_B [(1-|z|^2)^m |d^m f(z)|]^p (1-|z|^2)^s d\lambda(z) < \infty.$$

If $f \epsilon H(B)$, then the above equivalence holds for $p, 0 < p \leqq \infty$, with $d^k f$ and $d^m f$ replaced by $\partial^k f$ and $\partial^m f$, respectively.

Lemma 2.7 [8]. *Let $0 < p < \infty$ and $s > n$. Then for every $f \epsilon H(B)$*

$$\int_B |\Re f(z)|^p (1-|z|^2)^s d\lambda(z) \approx \int_B |\nabla f(z)|^p (1-|z|^2)^s d\lambda(z)$$

Proof. The fact that the second integral dominates the first comes from (III.4.6d). The reverse inequality is obtained in [8]. See also [41, Theorem I.4.2]. □

Lemma 2.8 [8]. *Let $p > 2n$ and $s > n$. Then for every $f \epsilon H(B)$*

$$\int_B (Qf)^p(z)(1-|z|^2)^s d\lambda(z) \approx \int_B [(1-|z|^2)|\nabla f(z)|]^p (1-|z|^2)^s d\lambda(z).$$

Proof. It is clear form (III.4.6d) that the integral on the left dominates that on the right. To prove the reversed inequality, apply (III.4.6c) and (III.4.6d), Lemma 2.7, and [5, Lemma 5.1]. □

Lemma 2.9 [5]. *Let V be a linear subspace of $H(B)$ which satisfies the following two conditions :*

(i) *If $f \epsilon V$ and $\varphi \epsilon Aut(B)$, then $f \circ \varphi \in V$.*
(ii) *If $f \epsilon V$, then $g(z) = \int_0^2 e^{-i\theta} f(e^{i\theta}) d\sigma / 2\pi \epsilon V$.*

Then either V contains only constant functions or V contains the linear function z_1.

Proof. Assume that $f \epsilon V$ is not constant. Then $\nabla f(z) \neq 0$ for some $z \epsilon B$, and if $h = f \circ \varphi_z$, then $h \epsilon V$ and $|\nabla h(0)| = (Qf)(z) \neq 0$. If g is given as in (ii), then $g(z) = \sum_{i=1}^n c_i z_i$, where $c_i = \frac{\partial f}{\partial z_i}(0)$ are not all 0. Finally, a suitable rotation φ gives $g \circ \varphi(z) = c z_1$, with $c \neq 0$, and thus $z_1 = c^{-1} g \circ \varphi \epsilon V$. □

Lemma 2.10 [5]. *Let $n \geqq 2$. Then $\int_B (Qz_1)^p(z) d\lambda(z) < \infty$ if and only if $p > 2n$.*

Proof. By (III.4.6c), $(Qz_1)^2(z) = 2(1-|z|^2)(1-|z_1|^2)$. Hence,

$$\int_B (Qz_1)^p(z) d\lambda(z) = c \int_B (1-|z_1|^2)^{p/2}(1-|z|^2)^{p/2} d\lambda(z)$$

which is less than $\int_B (1-|z_1|^2)^{p/2} d\lambda(z)$ and larger than

$$\int_{|z_1|<|z_2|} (1/2)^{p/2}(1-|z|^2)^{p/2} d\lambda(z) = c \int_B (1-|z|^2)^{p/2} d\lambda(z).$$

These integrals are finite if and only if $p/2 - n - 1 > -1$. □

Lemma 2.11 [5]. *Let $n \geqq 2$ and $0 < p \leqq 2n$. If $f \epsilon H(B)$ and if*

$$\int_B (Qf)^p(z)d\lambda(z) < \infty,$$

then f is constant.

Proof. Let $V_p = \{f\epsilon H(B) : \int_B (Qf)^p(z)d\lambda(z) < \infty\}$. Since $d\lambda$ is invariant, it follows from (IV.3.2) that (i) of Lemma 2.9 holds for V_p. The result will follow from Lemma 2.9 and Lemma 2.10 as soon as (ii) is verified. To this end, we assume first that $1 \leqq p \leqq 2n$. Then $V_p/\mathbb{C}$ is a Banach space and if $f\epsilon V_p$ and $f_\theta(z) = f(e^{i\theta}z)$, then $\theta \to f_\theta$ is a continuous mapping of $[0, 2\pi]$ into V_p. Hence, $e^{-i\theta}f_\theta$ is Bochner integrable, and (ii) follows. To prove for the case $0 < p < 1$, we observe that if $f\epsilon V_p$, the function $g_z, z\epsilon B$, defined by

$$g_z(\zeta) = \frac{f(\zeta z) - f(0)}{\zeta}, \quad \zeta \neq 0,$$

$$g_z(0) = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} f(0), \quad \zeta = 0,$$

is holomorphic and that the maximum principle (pointwise or in L^p) shows that $g = g_z \epsilon V_p$, as a function of z. Thus, (ii) holds also for $p < 1$, which proves the lemma. □

Corollary 2.12. *Let $n \geqq 2$ and $0 < p \leqq 2n$. $f\epsilon H\mathcal{B}_p^0(B)$ if and only if f is constant.*

Proof. If $f\epsilon H\mathcal{B}_p^0(B)$, then f is constant by Lemma 2.11. The converse is trivial.

Lemma 2.13. *For $p > 2n$, every $f\epsilon H(B)$ having bounded ∇f on B belongs to the space $H\mathcal{B}_p^0$.*

Proof. By (III.4.6d), and (IV.3.3a),

$$(Qf)(z) \leqq (1 - |z|^2)^{1/2}|\nabla f(z)| \leqq C(1 - |z|^2)^{1/2},$$

which implies $f\epsilon H\mathcal{B}_p^0$ for $p > 2n$. □

3. Modified Besov p-Spaces $\tilde{\mathcal{B}}_p^s$ with Weights.

In addition to the Besov spaces introduced in Chapter 4, we define a modified Besov p-spaces $\tilde{\mathcal{B}}_p^s$ with weight $s\epsilon\mathbb{R}$ as follows :

Definition 3.1. For $s\epsilon\mathbb{R}$ and $p\epsilon(0,\infty)$, let

$$\tilde{\mathcal{B}}_p^s = \{f\epsilon C^\infty(B) : |||f|||_{p,s} < \infty\},$$

where

$$|||f|||_{p,s} = \left\{\int_B [(1-|z|^2)^m |d^m f(z)|]^p (1-|z|^2)^s d\lambda(z)\right\}^{1/p},$$

where m is any integer larger than $\beta = (n-s)/p$.

It turns out that different values of $m > \beta$ define the same semi-norm (see Corollary 2.6).

The diagonal Besov p-spaces are defined by

$$B_p^s = \{f\epsilon C^\infty(B) : (1-|z|^2)^{m-s}|\Re^m f(z)|\epsilon L^p(d\nu/(1-|z|^2))\}$$

for any integer $m > s$. Notice that $B_p^s = \tilde{\mathcal{B}}_p^{n-sp}$. The "diagonal " Besov spaces were originally defined for holomorphic functions and studied by many authors, see [5], [41] for example.

It is clear from (III.4.6d), and (IV.3.3a) that $\mathcal{B}_p^s \subset \tilde{\mathcal{B}}_p^s$ and from the definition that for a fixed $p\epsilon(0,\infty)$, both $\mathcal{B}_p^s$ and $\tilde{\mathcal{B}}_p^s$ are increasing families of $s\epsilon\mathbb{R}$, that is, if $-\infty < s \leqq t < \infty$, then

$$\mathcal{B}_p^s \subset \mathcal{B}_p^t \quad \text{and} \quad \tilde{\mathcal{B}}_p^s \subset \tilde{\mathcal{B}}_p^t.$$

In general, similar results do not hold for $\mathcal{B}_p^s$ and $\tilde{\mathcal{B}}_p^s$ when these spaces are regarded as functions of $p\epsilon(0,\infty)$ with fixed $s\epsilon\mathbb{R}$. In fact, we establish the following :

Lemma 3.1. *Let* $1 < p < \infty$, $s\epsilon\mathbb{R}$, *and* $f\epsilon\tilde{h}(B)$.

(a) If $s < -np$, then $(\hat{Q}f)(z) \leqq C(1-|z|^2)^n ||f||_{\mathcal{B}_p^s}$.

(b) If $-np < s < n$, then $(\hat{Q}f)(z) \leqq C(1-|z|^w)^{-s/p} ||f||_{\mathcal{B}_p^s}$.

(c) If $s = -np$, then $(\hat{Q}f)(z) \leqq C(1-|z|^2)^n [\log(1-|z|^2)^{-1}]^{\frac{p-1}{p}} ||f||_{\mathcal{B}_p^s}$.

Proof. Let $f\epsilon\tilde{h}(B)$ and $\zeta\epsilon S$. The mean value property implies

$$f(t\zeta) = \int_B f\circ\varphi_{t\zeta}(w)d\nu(w), \quad t\epsilon(0,1). \tag{3.1a}$$

Therefore, using the definition of $\hat{Q}f$ and the estimate (III.4.2) for the Bergman metric, we see

$$|df(z)\cdot\zeta| = \left|\frac{d}{dt}f(t\zeta)\right|_{t=0} \leqq \int_b \frac{(\hat{Q}f)(w)}{1-|w|^2}d\nu(w),$$

and hence

$$(\hat{Q}f)(0) \leqq \int_b (\hat{Q}f)(w)(1-|w|^2)^n d\lambda(w). \tag{3.1b}$$

Replacing f by $f\circ\varphi_z$ in the above equation, we obtain

(3.1c)

$$\begin{aligned}(\hat{Q}f)(z) &\leqq (1-|z|^2)^n \int_B (\hat{Q}f)(w)\frac{(1-|w|^2)^n}{|1-<z,w>|^{2n}}d\lambda(w)\\ &\leqq (1-|z|^2)^n \left[\int_b (\hat{Q}f)^p(w)(1-|w|^2)^s d\lambda(w)\right]^{1/p}\\ &\times\left[\int_B \frac{(1-|w|^2)^{s+q(n-s)}}{|1-<w,z>|^{2nq}}d\lambda(w)\right]^{1/q}.\end{aligned}$$

It follows from [44, Proposition I.4.10] that the second term $I_{s,p}(z)$ of the latter expression is bounded for $s<-np$ and behaves like

$$I_{s,p}(z)\approx(1-|z|^2)^{-n-s/p}, \quad \text{as } |z|\to 1,$$

for $-np<s<n$, and for $s=-np$

$$I_{s,p}(z)\approx[\log(1-|z|^2)^{-1}]^{\frac{p-1}{p}}, \quad \text{as } |z|\to 1.$$

Lemma 3.1 now follows from (3.1c) when $I_{s,p}(z)$ is substituted. □

Theorem 3.2. *Let $1<p\leqq q<\infty$, $s\leqq 0$, and $f\epsilon\tilde{h}(B)$. Then*

$$M\mathcal{B}_p^s \subset M\mathcal{B}_q^s \subset M\mathcal{B}_\infty = M\mathcal{B}.$$

Proof. It follows from Lemma 3.1 that if $s\leqq 0$, then there exists a constant $C>0$, such that for all $z\epsilon B$

$$(\hat{Q}f)(z)\leqq ||f||_{\mathcal{B}_\infty}\leqq C||f||_{\mathcal{B}_p^s}. \tag{3.2a}$$

If $p < q$, then

$$(3.2b) \qquad \begin{aligned} \|f\|^q_{\mathcal{B}^s_q} &= \int_B (\hat{Q}f)^q(z)(1-|z|^2)^s d\lambda(z) \\ &= \int_B (\hat{Q}f)^p(z)(1-|z|^2)^s (\hat{Q}f)^{q-p}(z) d\lambda(z) \\ &\leqq C^{q-p}\|f\|^{q-p}_{\mathcal{B}^s_p}\|f\|^p_{\mathcal{B}^s_p} = C^{q-p}\|f\|^q_{\mathcal{B}^s_p}. \end{aligned}$$

The theorem now follows from (3.2a) and (3.2b). □

Lemma 3.3. *Let* $1 < p < \infty$, $s \epsilon \mathbb{R}$, *and* $f \epsilon \tilde{h}(B)$.

(a) *If* $s < -p(n+2)$, *then* $(\hat{Q}f)(z) \leqq C(1-|z|^2)^{n+1}\|f\|_{\tilde{\mathcal{B}}^s_p}$.

(b) *If* $-p(n+2) < s < n$, *then* $(\hat{Q}f)(z) \leqq C(1-|z|^2)^{-1-s/p}\|f\|_{\tilde{\mathcal{B}}^s_p}$.

(c) *If* $s = -p(n+1)$, *then* $(\hat{Q}f)(z) \leqq C(1-|z|^2)^{n+1}[\log(1-|z|^2)^{-1}]^{\frac{p-1}{p}}\|f\|_{\tilde{\mathcal{B}}^s_p}$.

Proof. From (1.13a) and (1.13b),

$$\begin{aligned} |df(0)\zeta| &\leqq \int_B |df(w)||\zeta - <w,\zeta> w| d\nu(w) \\ &\leqq C\int_B |df(w)|(1-|w|^2)^{n+1} d\lambda(w) \\ \hat{Q}f(0) &\leqq C\int_b |df(w)|(1-|w|^2)^{n+1} d\lambda(w). \end{aligned}$$

Replacing f by $f \circ \varphi_z$ in the above inequality, it follows from the change of variable formula that

$$\begin{aligned} \hat{Q}f(z) &\leqq C\int_B |df(w)|(1-|\varphi_z(w)|^2)^{n+1} d\lambda(w) \\ &\leqq C\int_B |df(w)| \frac{(1-|w|^2)^{n+1}(1-|z|^2)^{n+1}}{|1-<w,z>|^{2(n+1)}} d\lambda(w) \\ &\leqq C(1-|z|^2)^{n+1}\int_B [(1-|w|^2)|df(w)|](1-|w|^2)^s \frac{(1-|w|^2)^{n-s}}{|1-<w,z>|^{2(n+1)}} d\lambda(w) \\ &\leqq C(1-|z|^2)^{n+1}\left[\int_B [(1-|w|^2)|df(w)|]^p (1-|w|^2)^s d\lambda(w)\right]^{1/p} \\ &\times \left[\int_B \frac{(1-|w|^2)^{s+q(n-s)-n-1}}{|1-<w,z>|^{2q(n+1)}} d\nu(w)\right]^{1/q}. \end{aligned}$$

By proceeding as in the proof of Lemma 3.1, we find the desired results. □

Theorem 3.4. *Let $1 < p \leqq q < \infty$, $s \leqq -p$, and $f \epsilon \tilde{h}(B)$. Then*

$$M\tilde{\mathcal{B}}_p^s \subset M\tilde{\mathcal{B}}_q^s \subset M\mathcal{B}_\infty \subset M\mathcal{B}.$$

Proof. It is clear that if $s \leqq -p$, then $(\hat{Q}f)(z) \leqq C||f||_{\tilde{\mathcal{B}}_p^s}$ for some constant $C > 0$ which implies : $M\tilde{\mathcal{B}}_p^s \subset M\mathcal{B}_\infty$. The rest follows as in Theorem 3.2. □

Definition 3.2. The uniform space of a linear space X of functions on B, denoted by $\mathbf{U}(X)$, endowed with a semi-norm $||\cdot||_X$ is defined by

$$\mathbf{U}(X) = \{f \epsilon X : \sup ||f \circ \varphi||_X < \infty \quad \text{for all } \varphi \epsilon Aut(B)\}.$$

In general, $\mathbf{U}(X)$ is a proper subspace of X. There are spaces whose uniform spaces coincde with themselves. The Bloch space $\mathcal{B}$ and the Besov p-spaces $\mathcal{B}_p$ are such examples

Theorem 3.5. *Let $s > n$ be fixed. Then for all $p \epsilon (0, \infty)$*

$$\mathcal{B} \subset \mathbf{U}(\mathcal{B}_p^s) \subset \mathcal{B}_p^s.$$

Moreover, if $2n < p \leqq q < \infty$, then $H\mathcal{B}_q^s \subset H\mathcal{B}_p^s$.

Proof.

(3.3)

$$\begin{aligned} ||f \circ \varphi_z||_{\mathcal{B}_p^s}^p &= \int_B (\hat{Q}f)^p(w)(1 - |\varphi_z(w)|^2)^s d\lambda(w) \\ &= (1 - |z|^2)^s \int_B (\hat{Q}f)^p(w) \frac{(1 - |w|^2)^s}{|1 - <w, z>|^{2s}} d\lambda(w) \\ &\leqq (1 - |z|^2)^s \sup_{w \epsilon B} (\hat{Q}f)^p(w) \int_B \frac{(1 - |w|^2)^s}{|1 - <w, z>|^{2s}} d\lambda \\ &\leqq C||f||_{\mathcal{B}}^p \end{aligned}$$

for all $s > n$ and $p \epsilon (0, \infty)$, since the last integral behaves like $(1 - |z|^2)^{-s}$ as $|z| \to 1$. This implies that $\mathcal{B} \subset \mathbf{U}\left(\mathcal{B}_p^s\right)$ for all $s > n$. That $\mathbf{U}\left(\mathcal{B}_p^s\right) \subset \mathcal{B}_p^s$ is clear. To prove the second statement, we observe that if $s > n$ and $p > 2n$, then $H\mathcal{B}_p^s = H\tilde{\mathcal{B}}_p^s$ by Lemma 2.8. Also, by Corollary 2.5, $H\tilde{\mathcal{B}}_p^s = A_q^p$ for $q = s - n - 1$. It is well-known that A_q^p is a non-decreasing family of function spaces of $p\epsilon(0,\infty)$ with a fixed $q > -1$, and hence, $H\mathcal{B}_p^s$ is also non-decreasing in $p\epsilon(0,\infty)$ for a fixed $s = q+n+1$. □

Remark 3.6. (a) It is easy to see that if $p \leqq n$, then the space $H\tilde{\mathcal{B}}_p^0$ consists only of the constant functions and non-trivial for $p > n$. This contrasts to the fact that the invariant space $H\mathcal{B}_p^0$ consists of constant functions for $p \leqq 2n$ (Corollary 2.12) and non-trivial for $p > 2n$ (Lemma 2.10). When $n = 1$, $H\mathcal{B}_p^0 = H\tilde{\mathcal{B}}_p^0$ for all $0 < p < \infty$, and there are non-constant functions if and only if $p > 1$ [3].

(b) If $f\epsilon H(B)$, it follows from Corollary 2.6 that for every integer m satisfying $mp > n$ and for $p\epsilon(0,\infty]$, any of the following

$$||f||_{\tilde{\mathcal{B}}_p} = \left[\int_B [(1-|z|^2)^m |\partial^m f(z)|]^p (1-|z|^2)^s d\lambda(z)\right]^{1/p} + |f(0)|$$

defines an equivalent norm for the space $H\tilde{\mathcal{B}}_p^s$, provided that m is an integer satisfying $m > \beta$. Therefore, for the sake of simplicity we usually take $m = 1$ in the definition of norm.

(c) In view of the previous results, the characters of the Besov spaces $M\mathcal{B}_p^s$ are strongly influenced by the values of $s\epsilon\mathbb{R}$.

Indeed, we have the following phenomena :

(i) For a fixed $s \leqq 0$, the family of Besov spaces $M\mathcal{B}_p^s$ is an increasing function of $p\epsilon(1,\infty)$ and satisfies : $\bigcup_{p>1} M\mathcal{B}_p^s \subset M\mathcal{B}_\infty$, by Theorem 3.2.

(ii) For a fixed $s > n$, $H\mathcal{B}_p^s$ is a non-decreasing function of $p\epsilon(2n,\infty)$, and satisfies : $M\mathcal{B}_\infty \subset \mathrm{U}(M\mathcal{B}_p^s) \subset M\mathcal{B}_p^s$, by Theorem 3.5.

(iii) For $s = n$, $H^p \subset H\tilde{\mathcal{B}}_p^n$ when $p \geqq 2$, $H\mathcal{B}_p^n \subset H\tilde{\mathcal{B}}_p^n \subset H^p$ when $0 < p \leqq 2$, and $H^2 = H\tilde{\mathcal{B}}_2^n$.

Similar results are not available for each fixed value of s between 0 and n.

We shall call the weights $s = 0$ and n the critical weights for $M\mathcal{B}_p^s$ and describe the spaces corresponding to these weights more in detail in the next chapter.

Chapter VI The Besov p-Spaces with Critical Weights.

In this chapter we consider the Besov p-spaces $\tilde{\mathcal{B}}_p^s = B_p^\beta(s = n - \beta p)$ of weights $s = n$, $n+1$, and 0. As it is noted in Remark 3.6 in the previous chapter, these weights make up in some sense threshold values for the behavior of the spaces $M\tilde{\mathcal{B}}_p^s$. Furthermore, the Besov spaces corresponding to these weights constitute the most interesting spaces.

1. The $BMO(\partial B)$ and $VMO(\partial B)$ Spaces

We consider the Besov spaces $\tilde{\mathcal{B}}_p^n$ with weight n. These spaces correspond to diagonal Besov spaces B_p^0 which are closely related to the Hardy spaces H^p, $BMOA(\partial B)$, and $VMOA(\partial B)$ spaces.

The Hardy space $H^p(\partial B)$, $0 < p < \infty$, is the space of holomorphic functions f on B which satisfy

$$||f||_p^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

For $\zeta \epsilon S$ and $0 < \delta \leqq 2$, let

$$S_\delta(\zeta) = \{\eta \epsilon S : |1 - <\eta, \zeta>| < \delta\}, \tag{1.1a}$$

be the Koranyi ball centered at $\zeta \epsilon S$ with radius $\sqrt{\delta}$, and let

$$B_\delta(\zeta) = \{z \epsilon B : |1 - <z, \zeta>| < \delta\} \quad [44, 5.1.1]. \tag{1.1b}$$

Definition 1.1. A positive Borel measure μ on B is called a Carleson measure if there exists a positive constant C_μ such that

$$\mu(B_\delta(\zeta)) \leqq C_\mu \delta^n, \quad \text{for all } \zeta \epsilon S, \quad 0 < \delta \leqq 2. \tag{1.2}$$

The following result concerning the Carleson measure is due to Hörmander. See [29] for example.

Theorem 1.1. *A finite measure μ is a Carleson measure on ∂B if and only if there exists a constant $C > 0$ depending on μ such that for all $f \epsilon H^2(\sigma)$*

$$\int_B |f|^2 d\mu \leqq C||f||_{H^2}^2. \tag{1.3}$$

Definition 1.2. The space $BMO(\partial B)$ of functions of bounded mean oscillation on B consists of $L^1(\sigma)$ functions f which satisfy

$$||f||_{BMO} = \sup\{M_\delta(\zeta) : \zeta \epsilon S, \quad \delta > 0\} < \infty,$$

where $M_\delta(\zeta) = \frac{1}{\sigma[S_\delta(\zeta)]} \int_{S_\delta(\zeta)} |f(\eta) - \hat{f}_\delta| d\sigma(\eta)$, and $\hat{f}_\delta$ is the average of f over $S_\delta(\zeta)$. The space $VMO(\partial B)$ of functions of vanishing mean oscillation on B is a subspace of $BMO(\partial B)$ for which $\lim_{\delta\to 0} M_\delta(f)(\zeta) = 0$ uniformly for all $\zeta \epsilon S$. We denote the corresponding spaces of holomorphic functions by $BMOA(\partial B) = BMO(\partial B) \cap H(B)$ and $VMOA(\partial B) = VMO(\partial B) \cap H(B)$.

One of the fundamental rsults in the theory of Hardy spaces is the Fefferman-Stein duality theorem between H^1 and BMO [16]. The corresponding result for holomorphic functions was proved by Coifman-Rochberg-Weiss [15].

Theorem 1.2 [15]. *$(H^1)^* = BMOA(\partial B)$ under the usual pairing*

$$< f, g >= \lim_{r\to 1} \int_S f(\zeta)\overline{g(r\zeta)} d\sigma(\zeta) \tag{1.4}$$

for $f \epsilon H^1$ and $g \epsilon BMOA(\partial B)$.

There are many different characterizations of $BMOA(\partial B)$ space in the literature. We shall gather them in the following :

Theorem 1.3. *Let $f \epsilon H^2(B)$. Then $f \epsilon BMOA(\partial B)$ if and only if any of the following conditions hold :*

$||f||_1 = ||f||_{BMO} < \infty$

$||f||_2 = ||f||_G = \sup_{z\epsilon B} \left\{\int_S P(z,\zeta)|f(\zeta) - f(z)|^2 d\sigma(\zeta)\right\}^{1/2} < \infty$

$||f||_3 = ||f||_{\mathcal{L}} = \sup_{z\epsilon B} ||f \circ \varphi_z||_{B_2^n} < \infty$

$||f||_4 = \sup_{z\epsilon B} \left\{\int_S P(z,\zeta)|f(\zeta)|^2 d\sigma(\zeta) - |f(z)|^2\right\}^{1/2} < \infty$

$||f||_5 = \sup_{z\epsilon B} ||f \circ \varphi_z - f \circ \varphi_z(0)||_{H^2} < \infty$

$||f||_6 = \sup_{z\epsilon B} \left\{\int_B G(z,w)(Qf)^2(w) d\lambda(w)\right\}^{1/2} < \infty$

$||f||_7 = \sup_{z\epsilon B} \left\{\int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}} d\mu_f(w)\right\} < \infty,$

where $d\mu_f(w) = (Qf)^2(w)(1-|w|^2)^n d\lambda(w)$

$||f||_8 = \sup\{(\mu_f[B_\delta(\zeta)]/\delta^n) : \delta > 0, \quad \zeta\epsilon S\} < \infty$

$||f||_9 = \sup_{z\epsilon B} \left\{\int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}} (|\nabla f|^2 - |\Re f|^2)(w) d\nu(w)\right\} < \infty$

$||f||_{10} = \sup_{z\epsilon B} \left\{\int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}} (1-|w|^2)|\nabla f(w)|^2 d\nu(w)\right\} < \infty$

$||f||_{11} = \sup\left\{\left|\int_S f\bar{g} d\sigma\right| : g\epsilon H^2(B), \quad ||g||_{H^2} = 1\right\} < \infty$

Proof. The proof of the equivalences between $||f||_{BMO}$, $||f||_1$, $||f||_2$, and $||f||_{10}$ can be found in [33] when one uses Lemma 4.1 of [13], while the equivalences of $||f||_2$, $||f||_3$, $||f||_5$, $||f||_7$, and $||f||_9$ are proved in [13]. The equivalence between $||f||_2$ and $||f||_4$ follows from a simple computation. The equivalences of $||f||_{BMO}$, $||f||_8$, and $||f||_{11}$ are proved in [15].

Therefore, it remains to prove the equivalence of $||f||_6$ and $||f||_7$. It is easy to see that there are constants C_1 and C_2 such that Green's function $G(z,w)$ satisfies :

$$C_1(1-|\varphi_z(w)|^2)^n \leqq G(z,w) \quad (z,\, w\epsilon B), \tag{1.5a}$$

and

$$C_2(1-|\varphi_z(w)|^2)^n \geqq G(z,w) \quad (\text{if } |\varphi_z(w)| \geqq c > 0), \tag{1.5b}$$

see [21] for example. The equivalence of $||f||_6$ and $||f||_7$ now follows from (1.5a) and (1.5b) together with (I.3.2b). □

Corollary 1.4. *Let $f \epsilon H^2(B)$. Then $f \epsilon VMOA(B)$ if and only if any of the following conditions holds :*

(a) $\lim_{\delta\to 0} M_\delta(f)(\zeta) = 0$ *uniformly for all* $\zeta \epsilon S$.

(b) $\lim_{|z|\to 1} \int_S P(z,\zeta)|f(\zeta)-f(z)|^2 d\sigma(\zeta) = 0$

(c) $\lim_{|z|\to 1} ||f\circ\varphi_z||_{B_2^n} = 0$

(d) $\lim_{|z|\to 1} \{\int_S P(z,\zeta)|f(\zeta)|^2 d\sigma(\zeta) - |f(z)|^2\} = 0$

(e) $\lim_{|z|\to 1} ||f\circ\varphi_z - f\circ\varphi_z(0)||_{H^2} = 0$

(f) $\lim_{|z|\to 1} \int_B G(z,w)(Qf)^2(w)d\lambda(w) = 0$

(g) $\lim_{|z|\to 1} \int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}}(Qf)^2(w)(1-|w|^2)^n d\lambda(w) = 0$

(h) $\lim_{\delta\to 0} \mu_f[B_\delta(\zeta)]/\delta^n = 0$ uniformly for all $\zeta\epsilon S$.

(i) $\lim_{|z|\to 1} \int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}}(|\nabla f|^2 - |\Re f|^2)(w)d\nu(w) = 0$

(j) $\lim_{|z|\to 1} \int_B \frac{(1-|z|^2)^n}{|1-<z,w>|^{2n}}(1-|w|^2)|\nabla f(w)|^2 d\nu(w) = 0$

Theorem 1.5. *$VMO(\partial B) \subset M\mathcal{B}_0(B)$ and $BMO(\partial B) \subset M\mathcal{B}_\infty(B)$.*

Proof. From (V.1.5) and (V.1.6b), there exists a constant $C > 0$ such that for all $p \epsilon [1, \infty)$ and $f \epsilon \tilde{h}(B)$

$$(\hat{Q}f)(z) \leqq C \| f \circ \varphi_z - f(z) \|_{L^p(\nu)}.$$

From the fact that $\|f\|_{L^p(\nu)} \leqq \sqrt{2n} \|f\|_{L^p(\sigma)}$, we obtain

$$(\hat{Q}f)(z) \leqq C\sqrt{2n} \| f \circ \varphi_z - f(z) \|_{L^p(\sigma)}$$

from which the theorem follows. □

Theorem 1.6. *For all $p \geqq 1$, $M\mathcal{B}_p \subset VMO(\partial B) \subset M\mathcal{B}_0$.*

Proof. Fix $p > 1$ and let $1/p + 1/q = 1$. First, we shall prove : $M\mathcal{B}_{2p} \subset VMO(\partial B)$. Let $f \epsilon M\mathcal{B}_{2p}$ and let $d\mu(z) = (\hat{Q}f)^2(z)(1 - |z|^2)^n d\lambda(z)$. Then

$$\begin{aligned} \mu[B_\delta(\zeta)] &= \int_{B_\delta(\zeta)} (\hat{Q}f)^2(z)(1-|z|^2)^n d\lambda(z) \\ &= \int_{B_\delta(\zeta)} \frac{(\hat{Q}f)^2(z)}{(1-|z|^2)^{(n+1)(1/p+1/q)-n}} d\nu(z) \\ &\leqq \left(\int_{B_\delta(\zeta)} \frac{(\hat{Q}f)^{2p}(z)}{(1-|z|^2)^{n+1}} d\nu \right)^{1/p} \left(\int_{B_\delta(\zeta)} \frac{d\nu(z)}{(1-|z|^2)^{n+1-qn}} \right)^{1/q}. \end{aligned}$$

The first integral approaches 0 as $\delta \to 0$, since $f \epsilon M\mathcal{B}_{2p}$ and the second integral is finite for $q > 1$. Thus, $M\mathcal{B}_{2p} \subset VMO$. But since $M\mathcal{B}_p \subset M\mathcal{B}_{2p}$ by (V, Theorem 3.2), we have $M\mathcal{B}_p \subset VMO \subset M\mathcal{B}_0$ for all $p \geqq 1$. □

Theorem 1.6 has been proved in [60] for the case $n = 1$.

2. The $BMO(B)$ and $VMO(B)$ Spaces.

The holomorphic Besov p-spaces : $H\tilde{\mathcal{B}}_p^{n+1} = H\mathcal{B}_p^{n+1} = HB_p^{-1/p}$ with weight $s = n + 1$ are precsely the Bergman spaces $A^p = A_0^p$, due to (V, Corollary 2.5). As we have proved in (V, Theorem 3.5), the spaces $M\mathcal{B}_p^{n+1}$ contain the Bloch space $M\mathcal{B}$ as a subspace and the inclusions are continuous for $p \epsilon (0, \infty)$.

Definition 2.1. For a fixed $r\epsilon(0,1)$, $1 \leqq p < \infty$, and $f\epsilon L^1(\nu)$, we define

$$\hat{f}_r(z) = \frac{1}{|E(z,r)|}\int_{E(z,r)} f(w)d\nu(w),$$

$$MO_pf(z,r) = \left\{\int_{E(z,r)} |f(w) - \hat{f}_r(z)|^p \frac{d\nu(w)}{|E(z,r)|}\right\}^{1/p}, \quad r > 0,$$

$$MO_pf(z) = \left\{\int_B\int_B |f(u) - f(v)|^p \frac{|K(u,z)|^2|K(v,z)|^2}{K(z,z)^2} d\nu(u)d\nu(v)\right\}^{1/p}.$$

The space $BMO(B)$ of functions of bounded mean oscillation on B consists of $L^1(\nu)$ functions which satisfy :

$$||f||_{BMO} = \sup\{MO_1f(z,r) : z\epsilon B\} < \infty,$$

here the quantities on the right hand side of the equality are independent of $r > 0$.

The space $VMO(B)$ of functions of vanishing mean oscillation is the subspace of $BMO(B)$ for which

$$\lim_{|z|\to 1} MO_1f(z,r) = 0 \quad \text{for some } r > 0.$$

We denote the corresponding spaces of holomorphic functions on B by $BMOA(B)$ and $VMOA(B)$.

Theorem 2.1 [34]. *Let $1 \leqq p < \infty$ and $f\epsilon L^1(\nu)\cap\tilde{h}(B)$. Then the following statements are equivalent to $f\epsilon BMO(B)\cap\tilde{h}(B)$.*

(a) $\sup_{z\epsilon B} MO_pf(z,r) < \infty$, for some $r\epsilon(0,1)$
(b) $\sup_{z\epsilon B} MO_pf(z,r) < \infty$, for all $r\epsilon(0,1)$
(c) $\sup_{z\epsilon B} MO_pf(z) < \infty$
(d) $\sup_{z\epsilon B} MO_2f(z) < \infty$
(e) $\sup_{z\epsilon B} \left\{\int_B |f\circ\varphi_z(w) - f(z)|^p d\nu(w)\right\}^{1/p} < \infty$
(f) $f\epsilon M\mathcal{B}$
(g) $f\epsilon \mathrm{Lip}\,\beta = \{f\epsilon C(B) : ||f||_\beta < \infty\}$,

where $||f||_\beta = \inf\{\alpha : |f(z) - f(w)| \leqq \alpha\beta(z,w)\}$.

Theorem 2.2. *Let* $1 \leqq p < \infty$ *and let* $f \epsilon L^1(\nu) \cap \tilde{h}(B)$. *The following statements are equivalent to* $f \epsilon VMO(B) \cap \tilde{h}(B)$.

(a) $\lim_{|z|\to 1} MO_p f(z,r) = 0$ for some $r\epsilon(0,1)$

(b) $\lim_{|z|\to 1} MO_p f(z,r) = 0$ for all $r\epsilon(0,1)$

(c) $\lim_{|z|\to 1} MO_p f(z) = 0$

(d) $\lim_{|z|\to 1} MO_2 f(z) = 0$

(e) $\lim_{|z|\to 1} \int_B |f \circ \varphi_z(w) - f(z)|^p d\nu(w) = 0$

(f) $f \epsilon M\mathcal{B}_0$

From Theorem 2.1 and Corollary 2.2, we obtain

Corollary 2.3 [15]. $BMOA(B) = H\mathcal{B}$ *and* $VMOA(B) = H\mathcal{B}_0$.

The following duality theorem has been proved by several authors. See [61, Theorem 3], [57] for example.

Theorem 2.4 [15]. $A^1(B)^* = H\mathcal{B}$ *and* $(H\mathcal{B}_0)^* = A^1(B)$ *under the pairing* :

$$< f, g >_{A^2} = \int_B f(z)\overline{g(z)} d\nu(z).$$

3. The Invariant Besov p-Spaces

Consider the modified Besov p-spaces $\tilde{\mathcal{B}}_p^0$ of weight 0. These spaces correspond to the diagonal Besov spaces $B_p^{n/p}$. If $p > 2n$, then

$$HB_p^{n/p} = H\tilde{\mathcal{B}}_p^0 = H\mathcal{B}_p^0 = H\mathcal{B}_p.$$

and the spaces $H\mathcal{B}_p$ are the only holomorphic Besov p-spaces $H\mathcal{B}_p^s$ with weight $s\epsilon\mathbb{R}$ which are invariant in the sense that $||f \circ \varphi_z||_{p,s} = ||f||_{p,s}$ for all $\varphi\epsilon Aut(B)$ and all $f\epsilon H\mathcal{B}_p^s$. Therefore, we often call $H\mathcal{B}_p$ the invariant Besov p-space. As we have already observed in the previous section, there are others. The Bloch space $H\mathcal{B}$ is such a space. In fact, the Bloch space is maximal in the sense that it contains all such invariant spaces. The space $H\mathcal{B}$ contains the little Bloch space $H\mathcal{B}_0$ which is the closure of polynomials in $H\mathcal{B}$ (see § 4.3). The Bloch space $H\mathcal{B}$ also contains $BMOA(\partial B)$ and $VMOA(\partial B)$.

We now introduce the notion of Mobius invariant spaces. It was Arazy-Fisher-Peetre [3] who first introduced the Mobius invariant Banach spaces of holomorphic functions on the unit disc U and showed that $H\mathcal{B}_p$, $\mathcal{B}$, $\mathcal{B}_0$, $BMOA(\partial D)$, and $VMOA(\partial D)$ are among the Mobius invariant spaces.

Definition 3.1. Let X be a linear space of functions on B with a semi-norm ρ. X is said to be Mobius invariant if the following conditions hold.

(a) X is complete in the the topology generated by ρ and embedded continuously in the Bloch space $\mathcal{B}$, i.e., there exists a constant $A > 0$ such that

$$||f||_{\mathcal{B}} \leqq A\rho(f) \quad \text{for all } f \epsilon X.$$

(b) For all $f\epsilon X$ and $\varphi \epsilon Aut(B)$, $f \circ \varphi \epsilon X$.

(c) There exists a constant $C > 0$ such that for all $f \epsilon X$ and $\varphi \epsilon Aut(B)$,

$$\rho(f \circ \varphi) \leqq C\rho(f) \tag{3.1a}$$

(d) For each $f \epsilon X$, the composition map $C_f : Aut(B) \to X$, defined by $C_f(\varphi) = f \circ \varphi$, is consinuous.

Note that if (c) holds then we can define a new, and equivalent, semi-norm ρ' on X by

$$\rho'(f) = \sup\{\rho(f \circ \varphi) : \varphi \epsilon Aut(B)\}. \tag{3.1b}$$

With this new semi-norm ρ' in place of ρ in (c), $C = 1$. Namely,

$$\rho'(f \circ \varphi) = \rho'(f) \tag{3.1c}$$

for all $f \epsilon X$ and $\varphi \epsilon Aut(B)$. When ρ in (c) satisfies (3.1c), X is called a strictly Mobius invariant space. Evidently, every Mobius invariant space can be made into a strictly Mobius invariant space.

In the following we exhibit some well-known spaces which are Mobius invariant spaces and also some which are not. See [42].

Examples 3.1. (a) The Besov p-spaces $H\mathcal{B}_p(B)$ of holomorphic functions on the unit ball $B \subset \mathbb{C}^n$ is a Mobius invariant space for $1 \leqq p < \infty$. In particular, when $n = 1$ and $p = 2$, the Besov 2-space is known as the Dirichlet space and is denoted by $\mathcal{D}$. It is given by

$$\mathcal{D} = \left\{ f\epsilon H(U) : \int_U |f'(z)|^2 \mathrm{dxdy} < \infty \right\}, \quad z = x + iy.$$

The fact that $H\mathcal{B}_p$ is Mobius invariant is proved in [3] for the case of disc in $\mathbb{C}$ and in [22] for the case of the unit ball $B \subset \mathbb{C}^n$. The fact that $H\tilde{\mathcal{B}}_p(B)$ is Mobius invariant is proved in [42, Corollary 4.4]. Notice that $H\mathcal{B}_p = H\tilde{\mathcal{B}}_p$ for the case $n = 1$.

(b) Let $A(B) = H(B) \cap C(\bar{B})$ be the ball algebra. Then $A(B)$ is a strictly Mobius invariant space with the sup-norm, as is the case for the space H^∞ of bounded holomophic functions on B.

(c) The Bloch space $H\mathcal{B}$ is (strictly) Mobius invariant, but it does not satisfy the condition (d). Indeed, let $f(z) = \log(1 - z)^{-1}$, $z\epsilon U \subset \mathbb{C}$. Then $f\epsilon H\mathcal{B}(U)$, but for $\theta, \theta' \epsilon \mathbb{R}$

$$||f \circ e^{i\theta} - f \circ e^{i\theta'}||_{\mathcal{B}} \geqq 2.$$

Hence the group action in condition (d) is not continuous. However, the group action is continuous in the weak*-topology.

(d) The little Bloch space $H\mathcal{B}_0$ is (strictly) Mobius invariant. The group action in this case is continuous, since if $f\epsilon H\mathcal{B}_0$ is non-constant, then there exists a point $z_0 \epsilon B$ that realizes the semi-norm, i.e., $\sup_{z\epsilon B}(Qf)(z) = (Qf)(z_0)$.

(e) The space $BMOA(\partial B)$ is (strictly) Mobius invariant as is its subspace $VMOA(\partial B)$.

Note that neither H^∞ nor $BMOA$ satisfy condition (d) although in both cases the group action is continuous when the space is given the weak*-topology. The group action is also continuous when the function space is given the topology of uniform convergence on compact subsets.

(f) The classical Hardy spaces H^p do not satisfy conditon (c) for $p\epsilon[1, \infty)$, since for $f\epsilon H^p$

$$||f \circ \varphi||_{H^p} \leqq C(1 - |z|^2)^{-n/p} ||f||_{H^p}.$$

As is mentioned in the beginning of this section, H^p is in general not contained in $H\mathcal{B}$. Neither is $H\mathcal{B}$ contained in H^p.

(g) The Besov p-spaces $H\mathcal{B}_p^s$ with weight $s > n$ satisfy condition (c) by (V.3.3), but these spaces are not contained in $H\mathcal{B}$. In fact, the opposite is true by (V. Theorem 3.5).

In the following we construct the smallest Mobius invariant space, in the sense that every Mobius invariant space contains it.

As it is remarked in Chapter 1, each $a \epsilon B$, the map $a \to \varphi_a \epsilon Aut(B)$ is continuous from B into $Aut(B)$. It is also true that $\varphi_a \to a$ uniformly on compact subsets of B as $|a| \to 1$. Thus we identify φ_a with a on ∂B.

Definition 3.1. Let $e_i(z) = z_i$, $i = 1, \cdots, n$, be the coordinate functions and define the subspace $\mathcal{M}$ of $H(B)$ by

$$\mathcal{M} = \left\{ f \epsilon H(B) : f = \sum_{i=1}^{\infty} c_i \varphi_i, \quad \varphi \epsilon Aut(B), \quad c_i \epsilon \mathbb{C}, \quad \sum_{i=1}^{\infty} |c_i| < \infty \right\},$$

and the norm by

$$||f||_{\mathcal{M}} = \inf \left\{ \sum_{i=1}^{\infty} |c_i| : f = \sum_{i=1}^{\infty} c_i \varphi_i \right\}.$$

Then we have the following theorem of M. Peloso [42, Theorem 2.5] who extended an earlier one variable result of Arazy-Fisher-Peetre [3] to the case of the unit ball in $\mathbb{C}^n$.

Theorem 3.1. *The space $\mathcal{M}$ is a Mobius invariant space. Moreover, it is minimal in the sense that if X is any Mobius invariant space that contains a nonconstant function, then there exists a constant $C > 0$ such that for all $f \epsilon \mathcal{M}$*

$$||f||_X \leqq C ||f||_{\mathcal{M}}.$$

Corollary 3.2. *Let X be a Mobius invariant space with invariant semi-norm $|| \cdot ||_X$. Suppose that X contains a nonconstant function. Then there exists two positive constants C_i, $i = 1, 2$, such that for every f, holomorphic in the neighborhood of $\bar{B}$,*

$$||f||_{\mathcal{B}} \leqq C_1 ||f||_X \leqq C_2 ||f||_{\mathcal{M}}.$$

It is shown in [42, Theorem 4.1] that the minimal invariant space $\mathcal{M}$ can be identitified with the modified 1-Besov space $H\tilde{\mathcal{B}}_1$ which can be in the following form :

$$H\tilde{\mathcal{B}}_1 = \left\{ f \epsilon H(B) : \int_B |\partial^m f(z)| d\nu(z) < \infty \right\},$$

by [V, Remark 3.6 (b)], with $m = n + 1$. Consequently, we obtain

Theorem 3.3 [42, Theorem 4.2]. *The modified Besov 1-space $H\tilde{\mathcal{B}}_1$ is a Mobius invariant space.*

Using complex interpolation and the same argument used in [62, Theorem 5], we obtain

Theorem 3.4 [42, Corollary 4.4]. *For $1 \leqq p < \infty$, the modified Besov p-spaces $H\tilde{\mathcal{B}}_p$ are Mobius invariant.*

Theorem 3.4 has been proved earlier in [5] and [59] for the case $p > 2n$.

Introducing a modified radial differential operators $\tilde{\Re}^k$, Peloso [42] proved that $\tilde{\mathcal{B}}_2$ is actually the unique Mobius invariant Hilbert space which realizes the following duality relations :

Theorem 3.5 [42, Theorem 5.13 & Proposition 5.14]. *Let $p > 1$ and $1/p + 1/q = 1$. Then*

$$\mathcal{M}^* = H\mathcal{B}/\mathbb{C}, \quad (H\mathcal{B}_0/\mathbb{C})^* = \mathcal{M}, \quad \text{and} \quad (H\tilde{\mathcal{B}}_p)^* = H\tilde{\mathcal{B}}_q,$$

using the invariant Hilbert space inner product pairing :

$$< f, g >_2 = \int_B \frac{(1-|z|^2)^n}{|z|^n} \tilde{\Re}^n f(z) \frac{(1-|z|^2)^n}{|z|^n} \overline{\tilde{\Re} g(z)} d\lambda(z), \quad \text{for } f, g \epsilon H\tilde{\mathcal{B}}_2,$$

where $\tilde{\Re}^k$, $k = 0, 1, \cdots$, are defined by setting $\tilde{\Re}^0 = 1$, $\tilde{\Re}^1 = \Re$, and for $k > 1$,

$$\tilde{\Re}^k = [\Re/(k-1) + I]\tilde{\Re}^{k-1},$$

inductively.

Chapter VII Boundary Behavior of Besov Functions

This chapter deals with the boundary behavior of M-harmonic functions in the Besov p-spaces of various weights. It is shown that (i) if $s = n$, then any M-harmonic function in $\mathcal{B}_p^n$ has admissible limits a.e. on the boundary $S = \partial B$ of the unit ball B in the sense of Koranyi for all $p > 2n$ [14], (ii) if $0 < s < n$, $1 < p < \infty$, and $\eta > 1$, then any M-harmonic function in $M\mathcal{B}_p^s$ has tangential limits almost every where on S along a tangential approach region $\Omega_{\alpha,\eta}(\zeta)$, $\zeta \epsilon S$, having the degree $\eta < n/s$ of tangency with S, (iii) if $s = 0$, then $f \epsilon M\mathcal{B}_p$ has tangential limits along every direction in B, i.e., along an approach region having infinite degree of tangency with S, (iv) if $-p < s < 0$, $1 < p < \infty$, then $f \epsilon M\mathcal{B}_p^s$ satisfies the Lipschitz condition of order $-s/p$ and has a continuous extension to the closure $\bar{B}$, and (v) if $s > n$, then the spaces $M\mathcal{B}_p^s$ include the Bloch space $M\mathcal{B}$ as a subspace. Therefore, there are functions in $M\mathcal{B}_p^s$ which do not have radial limits on a set of positive measure on S.

1. Admissible Boundary Behavor of M-subharmonic Functions.

In the following we define various approach regions in B which will be used in this and the rest of the chapter. For any $z \epsilon B$, we let $z = r\zeta$ for some $\zeta \epsilon S$ and $0 \leqq r < 1$.

Definition 1.1. For $\zeta \epsilon S$, $\alpha > 1$, and $0 < \delta < 1$, let

(1.1a) $$S_\delta(\zeta) = \{\eta \epsilon S : |1 - <\eta, \zeta>| < \delta\}$$

be the Koranyi ball centered at ζ with radius $\sqrt{\delta}$, and for $\eta > 1$, let

(1.1b) $$D_\alpha(\zeta) = \{z \epsilon B : |1 - <z, \zeta>| < \alpha(1 - |z|)\},$$

(1.1c) $$\Omega_{\alpha,\eta}(\zeta) = \{z \epsilon B : |1 - <z, \zeta>|^\eta < \alpha(1 - |z|)\}.$$

The regions $D_\alpha(\zeta)$ are equivalent to the Koranyi admissible regions which are non-tangential to S in the complex radial direction and tangential in the other directions. On the other hand, the regions $\Omega_{\alpha,\eta}(\zeta)$ are tangential in all directions with the boundary S.

The regions $\Omega_{\alpha,\eta}(\zeta)$ are considered by Nagel et al. [40] for $n = 1$ and by Shaw [47] and Hahn-Youssfi [24] for general $n \geqq 1$.

First we state Ullrich's analogue of Littlewood's theorem on the existence of radial limits of subharmonic functions on the unit disc U in $\mathbb{C}$.

Proposition 1.1 [55]. *Suppose u is M-subharmonic on B and satisfies the growth condition (IV. 1.2) with $p = 1$. Then $\lim_{r\to 1} u(r\zeta)$ exists for almost every $\zeta \epsilon S$.*

The main result of this section is the following theorem of Cima-Stanton.

Theorem 1.2 [14]. *Suppose f is M-subharmonic and satisfies the condition (IV. 1.2) with $p = 1$. If $\tilde{\Delta} f$ is absolutely continuous with respect to the invariant measure $d\lambda$ and*

$$\int_B (\tilde{\Delta} f)^p(z)(1-|z|^2)^n d\lambda(z) < \infty \tag{1.2}$$

for some $p > n$, then f has admissible limits a.e. on S, that is,

$$\lim_{D_\alpha(\zeta)\ni z\to\zeta} u(z) \quad \text{exists a.e. on } S.$$

Proof. By (IV, Theorem 2.4), the function f can be written as the sum of an M-harmonic function h and a Green potential

$$G\mu(z) = \int_B G(z,w)d\mu(w) \tag{1.3a}$$

where $d\mu(w) = \tilde{\Delta} f d\lambda$ is a positive Borel measure defined in the sense of distributions and satisfies :

$$\int_B (1-|w|^2)^n d\mu(w) < \infty. \tag{1.3b}$$

The function h satisfies the condition (IV. 1.2) and thus, by Koranyi's result [36], has admissible limits a.e. on S. Therefore, it remains to prove that the Green potential has admissible limit 0 almost everywhere on S. We need the following three technical lemmas :

Lemma 1.3. *Let $\alpha > 1$, $0 < r < 1$. For any $a \epsilon D_\alpha(\zeta)$ there exists a constant $c = c_{\alpha,r} > 0$ such that for all $\beta > c$*

$$\varphi_a(rB) \subset D_\beta(\zeta)$$

Proof. Let $z \epsilon rB$ and $a \epsilon D_\alpha(\zeta)$. From the identity (I. 3.2a), we have

$$\begin{aligned} |1-<\varphi_a(z),\zeta>| &= \frac{|1-<a,\zeta>||1-<z,\varphi_a(\zeta)>|}{|1-<z,a>|} \\ &\leqq \alpha\left(\frac{1+r}{1-r}\right)\frac{(1-|a|^2)(1-|z|^2)}{|1-<z,a>|^2} \\ &\leqq \beta(1-|\varphi_a(z)|), \end{aligned}$$

which is the required inequality. □

Lemma 1.4. *Let $z\epsilon B$, $\alpha > 1$, and $\tilde{D}_\alpha(z) = \{\zeta\epsilon S : z\epsilon D_\alpha(\zeta)\}$. There exists a constant $C = C_{\alpha,n} > 0$ such that*

$$\sigma(\tilde{D}_\alpha(z)) \leqq C(1-|z|^2)^n. \tag{1.4}$$

Proof. Let $\zeta\epsilon\tilde{D}_\alpha(z)$. Then $z\epsilon D_\alpha(\zeta)$. Set $\eta = z/|z|$. Then by [44, Lemma 5.4.3],

$$|1-<\zeta,\eta>| < 4\alpha|1-<z,\eta>| \leqq 4\alpha(1-|z|^2) \tag{1.5}$$

so that $\tilde{D}_\alpha(z) \subset S(\eta, \sqrt{4\alpha(1-|z|^2)})$. Using [44, Proposition 5.1.4], there exists a constant $A_0 > 0$ such that

$$\sigma(S(\eta, \sqrt{4\alpha(1-|z|^2)})) \leqq A_0[4\alpha(1-|z|^2)]^n$$

which proves the lemma. □

Lemma 1.5. *Let $p \geqq 1$ and $D_{\beta,\rho}(\zeta) = D_\beta(\zeta) \cap \{z\epsilon B : |z| > \rho\}$. If*

$$\int_B f(z)^p(1-|z|^2)^n d\lambda(z) < \infty, \tag{1.6a}$$

then

$$\int_{D_\beta(\zeta)} f(z)^p d\lambda(z) < \infty \tag{1.6b}$$

and hence

$$\lim_{\rho\to 1}\int_{D_{\beta,\rho}(\zeta)} f(z)^p d\lambda(z) = 0. \tag{1.6c}$$

Proof. By Lemma 1.4, $\sigma(\tilde{D}_\beta(z)) \leqq C(1-|z|^2)^n$. Hence, by Fubini' theorem,

$$\begin{aligned}\int_S\left(\int_{D_\beta(\zeta)} f(z)^p d\lambda(z)\right) d\sigma(\zeta) &= \int_B\int_S \chi_{\tilde{D}_\beta(z)}(\zeta)d\sigma(\zeta)f(z)^p d\lambda(z)\\ &\leqq C\int_B (1-|z|^2)^n f(z)^p d\lambda(z).\end{aligned}$$

Since the last integral is finite by our assumption, the conclusion of lemma holds. □

Proof of Theorem 1.2. Let $G(z,w) = (g \circ \varphi_z)(w)$, where g is defined by (IV. 2.3), be the Green's function of B. Define

$$g_0(t) = g(t)\chi_{(0,1/2)}(t), \quad g_1(t) = g(t) - g_0(t), \tag{1.7}$$
$$G_0(z,w) = g_0(|\varphi_z(w)|), \quad G_1(z,w) = g_1(|\varphi_z(w)|),$$

so that the Green potential can be written as

$$\int_B G(z,w)\tilde{\Delta} f(w) d\lambda(w) = \psi_0(z) + \psi_1(z) \tag{1.8a}$$

with

$$\psi_0(z) = \int_B G_0(z,w)\tilde{\Delta} f(w) d\lambda(w) \tag{1.8b}$$

$$\psi_1(z) = \int_B G_1(z,w)\tilde{\Delta} f(w) d\lambda(w). \tag{1.8c}$$

We will show both ψ_0 and ψ_1 have admissible limits almost everywhere. By (VI. 1. 5b) and (I. 3.2b),

$$\begin{aligned}\psi_1(z) &\leqq C \int_B (1 - |\varphi_z(w)|^2)^n \tilde{\Delta} f(w) d\lambda(w) \\ &= C \int_B \left(\frac{1-|z|^2}{|1- < z, w > |^2} \right)^n (1 - |w|^2)^n \tilde{\Delta} f(w) d\lambda(w).\end{aligned}$$

We define a measure τ on S by

$$\tau(A) = \int_{\tilde{A}} (1 - |w|^2)^n \tilde{\Delta} f(w) d\lambda(w), \tag{1.9}$$

where $\tilde{A} = \{w \epsilon B : w/|w| \epsilon A\}$. Since the measure $d\mu(w) = \tilde{\Delta} f(w) d\lambda(w)$ satisfies (I.3b), τ is a finite measure. From the inequality $|1- < z, w/|w| > | < 2|1- < z, w > |$, we find

$$\left(\frac{1-|z|^2}{|1- < z, w > |^2} \right)^n \leqq 2^n P(z, w/|w|),$$

and hence

$$\psi_1(z) \leqq C' \int_S P(z,\zeta)d\tau(\zeta).$$

We define the maximal function of ψ_1 by

$$[M_\alpha \psi_1](\zeta) = \sup\{\psi_1(z) : z\epsilon D_\alpha(\zeta)\}.$$

From [44, Theorem 5.4.5], we have

$$\sigma\{M_\alpha \psi_1 > t\} < C\frac{||\tau||}{t}.$$

It is now standard that ψ_1 has admissible limits a.e. on S. To see that ψ_0 has admissible limits almost everywhere on S, we observe that $G_0(z,w)$ is supported on $\varphi_z(B/2)$. Therefore, if $1/p + 1/q + 1$, then

$$\psi_0(z) \leqq \left(\int_B [G_0(z,w)]^q d\lambda(w)\right)^{1/q} \left(\int_B [\tilde{\Delta} f(w)]^p d\lambda(w)\right)^{1/p}.$$

Due to the invariance of $d\lambda$, we obtain

$$\begin{aligned}\int_B [G_0(z,w)]^q d\lambda(w) &= \int_B [g_0(|\varphi_z(w)|)]^q d\lambda(w) \\ &= \int_B [g_0(|w|)]^q d\lambda(w).\end{aligned}$$

From the following estimates of Green's function :

$$g_0(t) \approx \begin{cases} \log(1/t) & \text{for } 0 < t < 1/2,\ n = 1 \\ t^{2-2n} & \text{for } 0 < t < 1/2,\ n \geqq 2 \\ 0 & \text{for } 1/2 \leqq t < 1 \end{cases} \tag{1.10}$$

we find that $G_0(z,\cdot)\epsilon L^q(B,\lambda)$, provided that $q < n/(n-1)$. Hence, if $p > n$,

$$\psi_0(z) \leqq C\left(\int_{\varphi_z(B/2)} [\tilde{\Delta} f(w)]^p d\lambda(w)\right)^{1/p}.$$

Let $\alpha > 1$ be fixed and set $\beta = 3\alpha$. If $w\epsilon B/2$, then $|\varphi_z(w)|^2 \geqq 2|z|^2 - 1$. From Lemma 1.3, if $\rho \leqq 2|z|^2 - 1$, we have the inclusion

$$\varphi_z(B/2) \subset D_{\beta,\rho}(\zeta)$$

and hence

$$\psi_o(z) \leqq C\left(\int_{D_{\beta,\rho}(\zeta)} [\tilde{\Delta}f(w)]^p d\lambda(w)\right)^{1/p}.$$

Lemma 1.5 with $\tilde{\Delta}f$ in place of f and $z \epsilon D_\alpha(\zeta)$ implies :

$$\lim_{z\to\zeta} \psi_0(z) = 0 \quad \text{a.e. on } S.$$

Since $f = h + \psi_0 + \psi_1$ and each function of the sum has admissible limits almost everywhere on S, so does f. □

As a consequence of Theorem 1.2, we have

Theorem 1.6. *If $p > 2n$, every $f \epsilon M\mathcal{B}_p^n(B)$ has admissible limits almost everywhere on S. If $s > n$, then there are functions in $M\mathcal{B}_p^s(B)$ which have no radial limits on a set of positive measure on S.*

Proof. If $f = u + iv \epsilon \tilde{h}(B)$, then u and v are both real valued M-harmonic functions on B and $|f|^2 = u^2 + v^2$. Consider first the real part u. It satisfies:

$$\frac{1}{2}\sqrt{\tilde{\Delta}u^2}(z) \leq (\hat{Q}u)(z) \leq \sqrt{\tilde{\Delta}u^2}, \tag{1.11}$$

by (V.1.5) and hence

$$\int_B (\hat{Q}u)^p(z)(1-|z|^2)^n d\lambda(z) \approx \int_B [\tilde{\Delta}u^2(z)]^{p/2}(1-|z|^2)^n d\lambda(z). \tag{1.12}$$

Therefore, by Theorem 1.2, if $p/2 > n$, then u^2 has admissible limits almost everwhere on S and so does u. Since we can draw the same conclusion for v, we prove the first part of the theorem. The second half follows from [V, Theorem 3.5] and the well-known fact that the holomorphic Bloch space $H\mathcal{B}(U)$ on the unit disc U contains a function which has no radial limits on a set of positive measure on $T = \partial U$. In fact, the Riemann mapping function f that maps the unit disc U onto the domain $D = \mathbb{C} - \{(m,n) : m, n \in \mathbb{Z}\}$ provides such a function. □

2. Tangential boundary behavior of M-harmonic functions.

Definition 2.1. For a function $f : B \to \mathbb{C}$ and $\zeta \epsilon S$, let

$$(M_\alpha f)(z) = \sup\{|f(z) - f(0)| : z \epsilon D_\alpha(\zeta)\} \tag{2.1a}$$

denote the admissible maximal function (modulo constants) of f and let

$$(M_{\alpha,\tau} f)(\zeta) = \sup\{|f(z) - f(0)| : z \epsilon \Omega_{\alpha,\tau}(\zeta)\} \tag{2.1b}$$

be the tangential maximal function (modulo constants) of f. That is, $M_{\alpha,\tau} f$ is the maximal function of f with respect to the tangential approach region $\Omega_{\alpha,\tau}(\zeta)$. Note that $D_\alpha(\zeta) = \Omega_{\alpha,1}(\zeta)$ is an admissible region.

We prove the following L^p estimate of the tangential maximal function.

Theorem 2.1 [24]. *Let $p \epsilon (1, \infty)$, $0 < m \leqq n$, and $s \epsilon \mathbb{R}$. Suppose that μ is a positive measure on S such that for all $\zeta \epsilon S$ and $\delta > 0$*

$$\mu(S_\delta(\zeta)) \leqq C\delta^m, \tag{2.2}$$

where C is a positive constant.

Let $s < m$, then there exists a positive constant C_α such that

$$||M_\alpha f||_{L^p(S,\mu)} \leqq C_\alpha ||f||_{B_p^s}. \tag{2.3a}$$

for all M-harmonic functions f on B. If, in addition, $s < m/\tau$, then there exists a positive constant $C_{\alpha,\tau}$ such that

$$||M_{\alpha,\tau} f||_{L^p(S,\nu)} \leqq C ||f||_{B_p^s} \tag{2.3b}$$

for all M-harmonic functions f on B.

To prove the theorem we need a series of preliminary lemmas.

Lemma 2.2. *Let $z = |z|\eta \epsilon B$ with $\eta \epsilon S$, and let $\{\eta = \eta_1, \eta_2, \cdots \eta_n\}$ be an orthonormal basis for $\mathbb{C}^n$. If $r \epsilon (0, 1)$ and $(v_1, \cdots, v_n) \epsilon \mathbb{C}^n$ are such that $v = \sum_{j=1}^n v_j \eta_j \epsilon rB$, then*

$$\varphi_z(v) = \frac{|z| - v_1}{1 - |z|v_1}\eta_1 - \frac{\sqrt{1 - |z|^2}}{1 - |z|v_1} \sum_{j=1}^n v_j \eta_j. \tag{2.4a}$$

$$\frac{1 - r}{1 + r}(1 - |z|^2) \leqq 1 - |\varphi_z(v)|^2 \leqq \frac{1 + r}{1 - r}(1 - |z|^2). \tag{2.4b}$$

Proof. (2.4a) follows from the definition of φ_z and (2.4b) is an immediate consequence of (I. 3.2b). □

Lemma 2.3. *Let $1 < \alpha < \beta$, $\zeta \epsilon S$, and let $z = |z|\eta \epsilon D_\alpha(\zeta) \cap \{|z| > 3/4\}$. Then there exists $r\epsilon(0,1)$ such that $\varphi_z(rB) \subset D_\beta(\zeta)$.*

Proof. Let z and $\{\eta = \eta_1, \eta_2, \cdots, \eta_n\}$ be as in Lemma 2.2, and let $0 < r \leqq 2/3$. Then each $w = |w|\xi\epsilon\varphi_z(rB)$ can be written as

$$w = \varphi_z(v) \quad \text{with} \quad v = \sum_{j=1}^{n} v_j \eta_j \epsilon r B.$$

If $|z| \geq 3/4$, then by (2.4a) we have

$$|w||1 - |z|v_1| \geqq ||z| - v_1| \geqq 1/12. \tag{2.5}$$

By (2.4b),

$$\frac{1-r}{2(1+r)}(1-|z|) \leqq 1 - |w| \leqq \frac{2(1+r)}{1-r}(1-|z|), \tag{2.6}$$

from which it follows that

$$1 - |z||w| \leqq 1 - |z| + 1 - |w| \leqq \frac{3+r}{1-r}(1-|w|).$$

But since $|z| \geqq 3/4$,

$$\begin{aligned} ||w| - |z|| &= \frac{|1 - |w|^2 - (1 - |z|^2)|}{|z| + |w|} \\ &\leqq \frac{4}{3}|1 - |w|^2 - (1 - |z|^2)| \\ &\leqq \frac{16r}{3(1-r)}(1 - |w|), \quad \text{by (I.3.2b)}. \end{aligned} \tag{2.7}$$

Therefore, by (2.4a) we obtain

$$\begin{aligned} |1 - <\xi, \eta>| &= |1 - <\varphi_z(v), \eta> /|w|| \\ &= \frac{||w|(1 - |z|v_1) - |z| + v_1|}{|w||1 - |z|v_1|} \\ &\leqq 12||w|(1 - |z|v_1) - |z| + v_1|, \quad \text{by (2.5)}, \\ &\leqq 4r\frac{25 + 3r}{1 - r}(1 - |w|). \end{aligned} \tag{2.8}$$

From (2.7), we also obtain

$$1-|z| \leqq 1-|w|+||w|-|z|| \leqq \frac{3+13r}{3(1-r)}(1-|w|). \tag{2.9}$$

By the triangle inequality,

(2.10)

$$\begin{aligned}|1-<\xi,\zeta>| &\leqq [|1-<\xi,\eta>|^{1/2}+|1-<\eta,\zeta>|^{1/2}]^2 \\ &\leqq [|1-<\xi,\eta>|^{1/2}+\sqrt{\alpha}(1-|z|)]^2, \quad \text{since } z\epsilon D_a(\zeta), \\ &\leqq \left[\left(\frac{4r(25+3r)}{1-r}\right)^{1/2}+\left(\frac{(3+13r)\alpha}{3(1-r)}\right)^{1/2}\right]^2(1-|w|).\end{aligned}$$

It is possible to choose $r>0$ so small that the expression within the bracket in (2.10) can be made smaller than $\sqrt{\beta}$. For such an $r>0$, we have

$$|1-<\xi,\zeta>|<\beta(1-|w|) \tag{2.11}$$

so that $\varphi_z(rB)\subset D_\beta(\zeta)$. □

Lemma 2.4. *Let $\beta>1$, $\eta>1$, and $0<r<2/3$. Then there exists $\epsilon>1$ such that for all $\zeta\epsilon S$, $\alpha>1$, and $z\epsilon\Omega_{\alpha,r}(\zeta)-D_\beta(\zeta)$ with $|z|\geqq 3/4$*

$$\varphi_z(rB)\subset\Omega_{\epsilon\alpha,r}(\zeta).$$

Proof. Here we use the same notations as in the proof of Lemma 2.3. Assume that z is not in $D_\beta(\zeta)$, i.e.,

$$1-|z|\leqq|1-<\eta,\zeta>|/\beta. \tag{2.12}$$

Then by the triangle inequality together with (2.8), (2.6), and (2.11)

(2.13a)

$$\begin{aligned}|1-<\xi,\eta>| &\leqq [|1-<\xi,\eta>|^{1/2}+|1-<\eta,\zeta>|^{1/2}]^2 \\ &\leqq \left[\left(\frac{8r(1+r)(25+3r)}{(1-r)^2\beta}\right)^{1/2}+1\right]^2|1-<\eta,\zeta>|.\end{aligned}$$

Denoting the quantity inside the bracket of (2.13a) by $\sqrt{\epsilon_0}$, we obtain

$$(2.13b) \qquad |1-<\xi,\eta>| \leqq \epsilon_0|1-<\eta,\zeta>|, \quad \text{with } \epsilon_0 > 1.$$

Since $z\epsilon\Omega_{\alpha,\tau}(\zeta)$, it follows that

$$\begin{aligned} |1-<\xi,\zeta>|^\tau \leq \epsilon_0^\tau|1-<\tau,\zeta>|^\tau &< \alpha\epsilon_0^\tau(1-|z|) \\ &< \alpha\epsilon_0^\tau\left(1+\frac{16r}{3(1-r)}\right)(1-|w|), \quad \text{by (2.9).} \end{aligned}$$

Therefore, $\varphi_z(rB) \subset \Omega_{\epsilon\alpha,\tau}(\zeta)$, with $\epsilon = \epsilon_0^\tau\left(\frac{3+13r}{3(1-r)}\right) > 1$. □

Lemma 2.5. *Suppose that* $z\epsilon B$ *and* $0 < t_1 < \cdots < t_{N+1} = 1$ *satisfy*

$$(2.13) \qquad 1-t_k|z| = 2^{N-k+1}(1-|z|), \quad k = 1,\cdots,N+1.$$

Then

(a) for each $r\epsilon(0,1)$ *and* $t\epsilon[t_j,t_{j+1}], j = 1,\cdots,N$, *we have*

$$\varphi_{tz}(rB) \subset \varphi_{t_jz}(\delta B),$$

where $\delta = \delta(r) = \sqrt{(7r+1)/4(1+r)}$.

(b) If, in addition, $r < 7/47$ *and* $\alpha > 1$, *then there exists* $\epsilon = \epsilon(r,\alpha)$ *such that for all* $\tau > 1$, $\zeta\epsilon S$, *and* $z\epsilon\Omega_{\alpha,\tau}(\zeta)$ *if* $t_jz\epsilon D_\alpha(\zeta)$ *and* $|z| \geqq 3/4$, *then* $t_jz\epsilon\Omega_{\epsilon\alpha2^{-N+j-1},\tau(\zeta)}$ *for all* $j = 1,\cdots,N+1$.

Proof. Let $z\epsilon B$ and $\{\eta = \eta_1,\eta_2,\cdots,\eta_n\}$ be as in Lemma 2.2, and let $w = \varphi_{tz}(v) \in \varphi_{tz}(rB)$, where $v = \sum_{j=1}^n v_j\eta_j\epsilon rB$. Then

$$(2.14) \qquad w\epsilon\varphi_{t_jz}(rB) \quad \text{if and only if} \quad |\varphi_{t_jz}\circ\varphi_{tz}|(v) \leqq \delta(r).$$

But $\varphi_{t_jz}\circ\varphi_{tz} = -\varphi_{-b}$, where

$$b = (\varphi_{t_jz}\circ\varphi_{tz})(0) = \varphi_{t_jz}(tz) = \frac{z(t_j-t)}{1-tt_j|z|^2}.$$

This combined with (2.13) shows that

$$|b| \leqq \frac{(t_{j+1}-t_j)|z|}{1-t_j|z|} = \frac{|t_{j+1}|z|-1+1-t_j|z||}{2^{N-j+1}(1-|z|)} = 1/2$$

so that

$$1-|\varphi_{-b}(v)|^2 = \frac{(1-|b|^2)(1-|v|^2)}{|1-<b,v>|^2} > \frac{(1-1/4)(1-r^2)}{(1+r)^2} = \frac{3(1-r)}{4(1+r)}.$$

Thus, $|\varphi_{-b}(v)| < \delta(r)$ so that (2.14) follows which proves (a). To prove (b), observe that if $z\epsilon\Omega_{\alpha,r}(\zeta)$, then $t_j z\epsilon\Omega_{\alpha 2^{-N+j-1},r(\zeta)}$ for all $j = 1,\cdots,N+1$, by (2.13), and that if $r < 7/47$, then $\delta(r) < 2/3$. Part (b) now follows from Lemma 2.5 by taking α in place of β and $2^{-N+j-1}\alpha$ in place of α. □

Proof of Theorem 2.1. Let $z\epsilon\Omega_{\alpha,r}(\zeta)\cap\{|z|>3/4\}$ and let $t_0\epsilon[0,1]$ be the unique number defined by $t_0 z\epsilon\partial D_\alpha(\zeta)$, the boundary of $D_\alpha(\zeta)$, if $z\epsilon D_\alpha(\zeta)$, and $t_0 = 1$ if $z\epsilon D_\alpha(\zeta)$. Let $0 < r \leqq 1$ and $1 < p < \infty$. If f is an M-harmonic function on B, by the mean value property, there exists a constant $C > 0$ such that

$$(\hat{Q}f)(z) \leqq C\int_{E(z,r)} (\hat{Q}f)(w)d\lambda(w). \tag{2.15a}$$

Furthermore,

$$|f(z)-f(0)| \leqq \int_0^1 (\hat{Q}f)(tz)\frac{|z|}{1-|tz|^2}dt. \tag{2.15b}$$

For the derivatons of (2.15a) and (2.15b), see (V, Lemma 1.6).

Substituting (2.15a) into (2.15b), we have

$$|f(z)-f(0)| \leqq C\int_0^1\int_{E(tz,r)} \frac{(\hat{Q}f)(w)}{(1-|w|^2)^{n+2}}d\nu(w)dt.$$

Let $1/p+1/q=1$. By Holder's inequality, we have

(2.16a)

$$|f(z)-f(0)| \leqq C\lambda(rB)^{1/q}\int_0^1\left\{\int_{E(tz,r)}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{p+n+1}}d\nu(w)\right\}^{1/p}dt$$
$$= \tilde{C}\{A(z)+B(z)\} \quad \text{for some constant } \tilde{C},$$

where

$$A(z) = \int_0^{t_0}\left\{\int_{E(tz,r)}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{p+n+1}}d\nu(w)\right\}^{1/p}dt \tag{2.16b}$$

$$B(z) = \int_{t_0}^{t} \left\{ \int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{p+n+1}} d\nu(w) \right\}^{1/p} dt \tag{2.16c}$$

Fix $\beta > \alpha$. Since $tz \epsilon D_\alpha(\zeta)$ for $0 < t < t_0$, Lemma 2.3 implies that there exists $0 < r < 7/47$ such that $\varphi_{tz}(rB) \subset D_\beta(\zeta)$ for all $0 < t < t_0$. It follows from this fact and (2.6) that there exists a constant $C = C(r,n) > 0$ such that

$$\begin{aligned} A(z) &= \int_0^{t_0} \left\{ \int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{p+n+1}} d\nu(w) \right\}^{1/p} dt \\ &\leqq C \int_0^{t_0} (1-|tz|^2)^{-1-(s-m)/p} \left\{ \int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m+n+1}} d\nu(w) \right\}^{1/p} dt. \end{aligned}$$

Since $(s-m)/p < 0$ and $E(tz,r) \subset D_\beta(\zeta)$ for all $t \epsilon (0, t_0)$,

$$A(z) \leqq C \left\{ \int_{D_\beta(\zeta)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m+n+1}} d\nu(w) \right\}^{1/p},$$

where C is a constant depending only on r, s, p, and n. By (2.2), we have

$$\mu(\tilde{D}_\beta(w)) \leqq C\beta^m (1-|w|^2)^m \quad \text{for all} \quad w\mathcal{B}. \tag{2.17}$$

Therefore,

(2.18)

$$\begin{aligned} &\int_S \sup\{(A(z))^p : z \epsilon \Omega_{\alpha,r}(\zeta)\} d\mu(\zeta) \\ &\leqq C \int_S \int_{D_\beta(\zeta)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{m-s}} d\lambda(w) d\mu(\zeta) \\ &= C \int_B \mu(\tilde{D}_\beta(w)) \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{m-s}} d\lambda(w), \quad \text{by Fubini's theorem} \\ &\leqq C \int_B (\hat{Q}f)^p(w)(1-|w|^2)^s d\lambda(w), \quad \text{by (2.17).} \end{aligned}$$

If $z \epsilon D_\beta(\zeta)$, then $t_0 = 1$ and $B(z) = 0$, so that by (2.16a) and (2.18) we find

$$\int_S (M_\alpha f)^p(\zeta) d\mu(\zeta) \leqq \int_B (\hat{Q}f)^p(w)(1-|w|^2)^s d\lambda(w),$$

so that (2.3a) follows. Next we establish an estimate for $B(z)$. By (2.6), there exists a constant $C = C(n,r) > 0$ such that

$$\begin{aligned}
B(z) &= \int_{t_0}^{1}\left\{\int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{p+n+1}}d\nu(w)\right\}^{1/p} dt \\
&\leqq \int_{t_0}^{1}(1-|tz|^2)^{-1+\frac{-s+1+m/\tau}{p}}\left\{\int_{E(tz,r)} \frac{(\hat{Q}f)^p}{(1-|w|^2)^{-s+m/\tau+n+2}}d\nu(w)\right\}^{1/p} dt \\
&\leqq C\left\{\int_{t_0}^{1}(1-|tz|^2)^{(-1+\frac{-s+1+m/\tau}{p})/q}dt\right\}^{1/q} \\
&\times\left\{\int_{t_0}^{1}\int_{E(tz,r)} \frac{(\hat{Q}f)^p}{(1-|w|^2)^{-s+m/\tau+n+2}}d\nu(w)dt\right\}^{1/p}, \quad 1/p+1/q=1.
\end{aligned}$$

Since $s < m/\tau$, it follows that

$$B(z) \leqq C\left\{\int_{t_0}^{1}\int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+2}}d\nu(w)dt\right\}^{1/p}, \tag{2.19}$$

where C is a constant depending only on r, s, p, τ and n. There exists a nonnegative integer N such that

$$1 - t_0|z| > 2^N(1-|z|) \geqq (1-t_0|z|)/2.$$

We now break up $[t_0, 1]$ into a partition $0 < t_1 < \cdots < t_{N+1} + 1$ that satisfies (2.13). By Lemma 2.5, we obtain that $E(tz,r) \subset E(t_jz, \delta(r))$ for all $t\epsilon[t_j, t_{j+1}], j = 1, \cdots, N$, and then by (2.19) we obtain

$$\begin{aligned}
(B(z))^p &\leqq \sum_{j=0}^{N+1}\int_{t_j}^{j+1}\int_{E(tz,r)} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+2}}d\nu(w)dt \\
&\leqq \sum_{j=0}^{N+1}(t_{j+1}-t_j)\int_{E(t_jz,\delta(r))} \frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+2}}d\nu(w).
\end{aligned}$$

Since $t_{j+1} - t_j \leqq 1 - t_j|z| \leqq C(1-|w|^2)$, for $w\epsilon E(t_jz, \delta(r))$, where C is a constant depending only on r, it follows that

$$\begin{aligned}(B(z))^p &\leqq \sum_{j=0}^{N+1}\int_{E(t_j z,\delta(r))}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+1}}d\nu(w)\\ &\leqq \sum_{j=0}^{N+1}\int_{\Omega_{\epsilon\alpha 2^{-N+j-1},\tau}}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+1}}d\nu(w)\\ &\leqq \sum_{j=0}^{\infty}\int_{\Omega_{\epsilon\alpha 2^{-k},\tau}}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+1}}d\nu(w).\end{aligned}$$

Therefore,

(2.20)

$$\begin{aligned}&\int_S \sup\{(B(z))^p : z\epsilon\Omega_{\alpha,\tau}(\zeta)\}d\mu(z)\\ &\leqq C\int_S\left\{\sum_{k=0}^{\infty}\int_{\Omega_{\epsilon\alpha 2^{-k},\tau(\zeta)}}\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+m/\tau+n+1}}d\nu(w)\right\}d\mu(\zeta)\\ &\leqq C\sum_{k=0}^{\infty}\int_B \mu\left(\left\{\tilde{\Omega}_{\epsilon\alpha 2^{-k},\tau(w)}\right\}\right)\frac{(\hat{Q}f)(w)}{(1-|w|^2)^{-s+m/\tau+n+1}}d\nu(w)\\ &\leqq C\sum_{k=0}^{\infty}(\epsilon\alpha 2^{-k})^m\int_B\frac{(\hat{Q}f)^p(w)}{(1-|w|^2)^{-s+n+1}}d\nu(w), \text{by } (3.17)\\ &\leqq C\int_B(\hat{Q}f)^p(w)(1-|w|^2)^s d\lambda(w),\end{aligned}$$

where $\tilde{\Omega}_{\alpha,\tau}(z) = \{\zeta\epsilon S : z\epsilon\Omega_{\alpha,\tau}(\zeta)\}, z\epsilon B$. Equation (2.3b) now follows from (2.16a), (2.18), and (2.20). □

The following theorem now follows from Theorem 2.1 by a standard argument, see [49].

Theorem 2.6 [24]. *Let $p\epsilon(1,\infty), 0 < m \leqq n, s\epsilon\mathbb{R}$, and $\tau > 1$. Suppose that μ is a positive measure on S that satisfies (2.2). Let $f\epsilon M\mathcal{B}_p^s$. If $s < m$, then admissible limit : $\lim f(z)$ as $z \to \zeta$, $z\epsilon D_\alpha(\zeta)$, exists a.e. $[\mu]$ on S. If, in addition, $s < m/\tau$, then the tangential limit : $\lim f(z)$ as $z \to \zeta$, $z\epsilon\Omega_{\alpha,\tau}(\zeta)$, exists a.e. $[\mu]$ on S.*

Since the normalized surface measure σ clearly satisfies condition (2.2) with $m = n$, we have the following corollary.

Corollary 2.7. *Let* $p\epsilon(1,\infty), s\epsilon\mathbb{R}$ *and* $\tau > 1$. *If* $s < n/\tau$, *then each* $f\epsilon M\mathcal{B}_p^s$ *has the tangential limit almost everywhere* $[\sigma]$ *on* S *along an approach region* $\Omega_{\alpha,\tau}(\zeta)$.

Theorem 2.8 [24]. *Let* $p\epsilon(1,\infty), 0 < m \leqq n$, *and* $-p < s < 0$. *Then there exists a positive constant* C *such that*

$$|f(z) - f(w)| \leqq C|z-w|^{-s/p}||f||_{\mathcal{B}_p^s}, \quad \text{for all } z, w\epsilon B. \tag{2.21}$$

for all M-harmonic functions f *on* B. *In particular,* $f\epsilon\mathcal{B}_p^s$ *satisfies the Lipschitz condition of order* $-s/p$.

Proof. Let $z\epsilon B$ be fixed. By (2.4b) and (2.15), there exists a positive constant C such that

$$\begin{aligned}(\hat{Q}f)(z) &\leqq C(1-|z|^2)^{-s/p}\int_{E(z,r)}(\hat{Q}f)(w)(1-|w|^2)^{s/p}d\lambda(w)\\ &\leqq C\lambda(rB)^{1/q}(1-|z|^2)^{-s/p}||f||_{\mathcal{B}_p^s}, \quad 1/p+1/q=1.\end{aligned}$$

Since $(1-|z|^2)|df(z)| \leqq (Qf)(z)$ for all $z\epsilon B$, we have

$$|df(z)| \leqq C(1-|z|^2)^{-1-s/p}||f||_{\mathcal{B}_p^s} \tag{2.22}$$

for all $z\epsilon B$. If $-p < s < 0$, then (2.22) and [44, Lemma 6.4.8] imply (2.21). □

Chapter VIII Qustions and Comments

In this final chapter we pose a number of questions and comments related to the subject described earlier in the text.

1. Notations and Definitions

For a function $f \epsilon C^\infty(B)$ and $m = 1, 2, \cdots$, we define

$$\begin{aligned} \partial^m f(z) &= (\partial^\alpha f(z))_{|\alpha|=m}, \bar{\partial}^m f(z) = (\bar{\partial}^\alpha f(z))_{|\alpha|=m}, \\ d^m f(z) &= (\partial^\alpha \bar{\partial}^\beta f(z))_{|\alpha|+|\beta|=m}, \end{aligned} \tag{1.1}$$

where $\partial^\alpha f(z) = \frac{\partial^{|\alpha|} f(z)}{\partial z_\alpha}, \tilde{\partial}^\alpha f(z) = \frac{\partial^{|\alpha|} f(z)}{\partial \bar{z}_\alpha}$, and α and β are multiindicies.
Further we define

(1.2)

$$\begin{aligned} |\partial^m f(z)| &= \sum_{|\alpha|=m} \left|\frac{\partial^{|\alpha|} f(z)}{\partial z_\alpha}\right|, \quad |\tilde{\partial}^m f(z)| = \sum_{|\alpha|=m} \left|\frac{\partial^{|\alpha|} f(z)}{\partial \bar{z}_\alpha}\right|, \\ |d^m f(z)| &= \sum_{|\alpha|+|\beta|=m} \left|\frac{\partial^m f(z)}{\partial z_\alpha \partial \bar{z}_\beta}\right|. \end{aligned}$$

The radial derivative $\Re^m$ of order m is defined inductively by $\Re(\Re^{m-1})$ for $m = 1, 2, \cdots$, where

$$\Re f(z) = \sum_{j=1}^{n} \left(z_j \frac{\partial f(z)}{\partial z_j} + \bar{z}_j \frac{\partial f(z)}{\partial \bar{z}_j} \right) = z \cdot \partial f(z) + \bar{z} \cdot \bar{\partial} f(z). \tag{1.3a}$$

Notice that if $f \epsilon H(B)$, then

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f(z)}{\partial z_j} = z \cdot \partial f(z). \tag{1.3b}$$

If $f \epsilon H(B)$ has a homogeneous expansion : $f(z) = \sum_{k=0}^{\infty} f_k$, then

$$\Re^m f(z) = \sum_{k=0}^{\infty} k^m f_k. \tag{1.3c}$$

Let $0 < p < \infty$ and $s\epsilon\mathbb{R}$. Recall that the Besov p-space of M-harmonic functions $f\epsilon\tilde{h}(B)$ with weight s is defined by

$$\text{(1.4a)} \qquad \mathcal{B}_p^s(B) = \{f\epsilon\tilde{h}(B) : \hat{Q}f\epsilon L^p((1-|z|^2)^s d\lambda(z))\},$$

the "modified " Besov p-space by

$$\text{(1.4b)} \qquad \tilde{\mathcal{B}}_p^s(B) = \{f\epsilon\tilde{h}(B) : (1-|z|^2)^m |d^m f(z)| \epsilon L^p((1-|z|^2)^s d\lambda(z))\},$$

for $m > (n-s)/p$, and the "diagonal " Besov p-space by

$$\text{(1.4c)} \qquad B_p^s(B) = \{f\epsilon\tilde{h}(B) : (1-|z|^2)^m |\Re^m f(z)| \epsilon L^p((1-|z|^2)^{n-ps} d\lambda(z))\},$$

for $m > s$, where $d\lambda(z) = (1-|z|^2)^{-n-1} d\nu(z)$ is the invariant volume measure on B.

Recall that the modified Besov spaces $\tilde{\mathcal{B}}_p^s$ are defined for any integer $m > (n-s)/p$, independently of m, see (V, Remark 3.6, (b)), and clearly $B_p^s = \tilde{\mathcal{B}}_p^{n-sp}$. From the following inequalities :

$$\text{(1.5a)} \qquad (1-|z|^2)|df(z)| \leqq \hat{Q}f(z) \leqq (1-|z|^2)^{1/2}|df(z)|,$$

we have

$$\text{(1.5b)} \qquad \tilde{\mathcal{B}}_p^{s-\frac{p}{2}} \subset \mathcal{B}_p^s \subset \tilde{\mathcal{B}}_p^s.$$

2. The Besov p-spaces and Their Weights

It is clear that for each fixed $p\epsilon(0,\infty)$, both $\mathcal{B}_p^s$ and $\tilde{\mathcal{B}}_p^s$ are increasing families of $s\epsilon\mathbb{R}$, as s increases. In general, similar results do not hold when these spaces are looked at as functions of $p\epsilon(0,\infty)$ with a fixed $s\epsilon\mathbb{R}$. In fact, we have following phenomena :

(i) For a fixed $s \leqq 0$, $M\mathcal{B}_p^s$ is an increasing function of $p\epsilon(1,\infty)$ and satisfies : $\bigcup_{p>1} M\mathcal{B}_p^s \subset M\mathcal{B}_\infty$, see (V, Theorem 3.2).

See (V, Theorem 3.4) for a similar result in the case of modified Besov $M\tilde{\mathcal{B}}_p^s$.

(ii) For a fixed $s > n$, $H\mathcal{B}_p^s$ is a decreasing function of $p\epsilon(2n,\infty)$ and satisfies : $M\mathcal{B}_\infty \subset M\mathcal{B}_p^s$ for all $p\epsilon(0,\infty)$. See (V, Theorem 3.5).

(iii) For $s = n$, $H^p \subset H\tilde{\mathcal{B}}_p^n$ when $p \geqq 2$, $HB_p^n \subset H\tilde{\mathcal{B}}_p^n \subset H^p$ when $0 < p \leqq 2$, and $H^2 = H\tilde{\mathcal{B}}_2^n$ when $p = 2$. See [8].

Question 2.1. *The natural question to ask is whether or not the first statement of (ii) holds for the M-harmonic Besov space $M\mathcal{B}_p^s$. That is, is $M\mathcal{B}_p^s$ a decreasing function of $p\epsilon(2n, \infty)$ for each fixed $s > n$? A similar question may be asked for the modified Besov spaces $M\tilde{\mathcal{B}}_p^s$.*

In proving (ii), we used the fact that the space $H\mathcal{B}_p^s$ and the Bergman space A_q^p coincide if $q = s - n - 1$. Therefore, the question is wether or not this same result holds for M-harmonic case.

Question 2.2. *For a fixed $s\epsilon(0, n)$, nothing much is known about the behavior of the spaces $M\mathcal{B}_p^s$, the behavior similar to those described in (i) and (ii), as a function of $p\epsilon(1, \infty)$, not even for the holomorphic case $H\mathcal{B}_p^s$. We do know, however, that if $1 < p < \infty$ and $s\epsilon(0, n)$, then $f\epsilon M\mathcal{B}_p^s$ has the tangential limits of degree $\tau < n/s$ of tangency almost everywhere on S. See [24, Corollary 1.3] for details. It is interesting to know more about the functions in the class $M\mathcal{B}_p^s$ for $s\epsilon(0, n)$.*

In the case where the weight $s = 0$, the following relationship exist between $H\mathcal{B}_p^s$ and $H\tilde{\mathcal{B}}_p^s$.

(i) For $0 < p \leqq n, \mathcal{B}_p^0 = \tilde{\mathcal{B}}_p^0 = \mathbb{C}$, i.e., the spaces consist only of constant functions.

(ii) For $p > 2n, \mathcal{B}_p^0 = \tilde{\mathcal{B}}_p^0 =$ non-trivial, and contains all polynomials.

(iii) For $n < p \leqq 2n, \mathcal{B}_p^0$ is contained in $\tilde{\mathcal{B}}_p^0$ properly as a subspace.

Therefore, we may ask the following question.

Question 2.3. *Find a relationship between $\mathcal{B}_p^s$ and $\tilde{\mathcal{B}}_p^s$, analogous to* (i) - (iii) *above, for the case where $s \neq 0$.*

3. Zhu's Conjecture

Zhu's conjecture [22] states that for $n \geqq 2$, the Besov p-spaces $H\mathcal{B}_p^0(B)$ (with weight $s = 0$) of holomorphic functions on B is non-trivial if and only if $p > 2n$. This conjecture has been answered positively by Hahn-Youssfi [22] and by Arazy et al. [5] independently. The M-harmonic analogue of this conjecture is likely to be true. This fact has already been conjectured in [23] and is still open as far as we know.

It is also proved in [5] that for $n \geqq 2$ the modified Besov spaces $H\tilde{\mathcal{B}}_p^0$ (with $s = 0$) is non-trivial if and only if $p > n$.

Question 3.1. *We conjecture that the M-harmonic analogue of the above result for $H\tilde{\mathcal{B}}_p^0$ holds. Namely, $M\tilde{\mathcal{B}}_p^0$ is non-trivial if and only if $p > n$.*

4. Uniform Spaces

The Bloch space $M\mathcal{B}$, the little Bloch space $M\mathcal{B}_0$ and the invariant Besov p-spaces $M\mathcal{B}_p$ (with $s = 0$) of M-harmonic functions are clearly uniform spaces, see (V, Definition 3.2) for definition.

Question 4.1. *It is of some interests to find the uniform spaces of $M\mathcal{B}_p^s$ for general weight $s \epsilon \mathbb{R}$.*

An attempt may be made by looking at three different cases :
(i) The case where $s \leqq 0$, (ii) the case $s > n$, and (iii) the case $0 < s \leqq n$.

(i) Since $U(\mathcal{B}_p^s) \subset \mathcal{B}_p^s \subset \mathcal{B}_p^0$ for $s \leqq 0$ and $H\mathcal{B}_p^0 = \mathbb{C}$ for $p \leqq 2n, U(H\mathcal{B}_p^s) = \mathbb{C}$ for $s \leqq 0$ and $p \leqq 2n$. On the other hand, for $s \leqq 0$ and for $p > 2n, H\mathcal{B}_p^s$ is non-trivial and contains all polynomials. Thus, $U(H\mathcal{B}_p^s)$ must contain all the polynomials. It would be more feasible to find $U(H\mathcal{B}_p^s)$ first and then $U(M\mathcal{B}_p^s)$.

(ii) Since $M\mathcal{B} \subset U(M\mathcal{B}_p^s) \subset M\mathcal{B}_p^s$ for $s > n$ (V, Theorem 3.5), $U(M\mathcal{B}_p^s)$ contain the Bloch space $M\mathcal{B}$. On the other hand, for $s > n$, by (V, Corollary 2.5),

$$||f||_{p,s}^p \approx \int_B |f(z)|^p (1 - |z|^2)^s d\lambda(z).$$

Hence, $||f \circ \varphi_z||_{p,s} \leqq ||f||_{L^p(B,\lambda)}$ which implies that $U(M\mathcal{B}_p^s)$ contain M-harmonic $L^p(B, \lambda)$ spaces for all $s > n$. Therefore, $U(M\mathcal{B}_p^s)$ contain both $M\mathcal{B}$ and M-harmonic $L^p(B, \lambda)$ spaces. It turns out that nothing is new, because M-harmonic $L^p(B, \lambda)$ spaces may be identified with $M\mathcal{B}_p^0$ under the Berezin transform, see (IV, Definition 1.5), in which case we already know that $M\mathcal{B}_p^0 \subset M\mathcal{B}$ by (V, Theorem 3.2).

(iii) As of now there is no definite clue found in this case, due to the unpredictability of the behavior of $M\mathcal{B}_p^s$ for $s \epsilon (0, n)$. For the case where $s = n$, this question is even unpredictable, see (iii) of Question 2.2.

5. The Space $\mathcal{B}_0 \bigcap H^\infty$

Question 5.1. *We know that for $p \leqq 2n, H\mathcal{B}_p^0 = \mathbb{C} \subset H^\infty$ and for $p > 2n, H\mathcal{B}_p^0$ contain all the polynomials, but so does the space H^∞. Therefore, it is reasonable to expect that there may be a value $p_0 > 2n$ such that $H\mathcal{B}_p^0 \subset H^\infty$ for all $p \leqq p_0$. If such p_0 exists, then $H\mathcal{B}_p^0 \subset H\mathcal{B}_0 \bigcap H^\infty$ for all $p \leqq p_0$. Recall that the space $(\mathcal{B}_0 \bigcap H^\infty)(U)$ is a subalgebra of $H^\infty(U)$ on the unit disc $U \subset \mathbb{C}$. The space $(\mathcal{B}_0 \bigcap H^\infty)(U)$ is often called COP (constant on parts), because it consists of the functions in $H^\infty(U)$ which are constants on each Gleason part (except U) of the maximal ideal space of $H^\infty(U)$, see[B]. Another question which is not unreasonable is to find $q_0 > 2n$ such that $\mathcal{B}_0 \bigcap H^\infty \subset \mathcal{B}_q$ for all $q \geqq q_0$. Any aforemative answer to these questions may provide a powerful tool in the study of the Blaschke products of holomorphic function theory on the unit disc $U \subset \mathbb{C}$.*

6. The Boundary Behavior of Functions in $M\mathcal{B}_p^s$

A description of the boundary behavior of M-harmonic Besov functions in $M\mathcal{B}_p^s$ is given in Chapter 7. It is rather surprising to see the boundary behavior of M-harmonic functions in the class $M\mathcal{B}_p^s$ being characterized by their weight $s \epsilon \mathbb{R}$.

In the following we recapture this description :

(i) If $s > n$, then there is no result that describes a "good" boundary behavior of functions in $M\mathcal{B}_p^s$. In fact, there is a Bloch function f in $H\mathcal{B}(U) \subset M\mathcal{B}_p^s(U)$ on the unit disc $U \subset \mathbb{C}$ which has radial limits almost nowhere on the boundary $T = \partial U$, see (VII, Theorem 1.6).

(ii) If $s = n$, then any M-harmonic function in $M\mathcal{B}_p^s$ has admissible limits almost everywhere on the boundary $S = \partial B$ of the unit ball in the sense of Koranyi for all $p > 2n$, see (VII, Theorem 1.6) and [14].

(iii) If $0 < s < n, 1 < p < \infty$, and $\tau > 1$, then any M-harmonic function in $M\mathcal{B}_p^s$ has tangential limits almost everywhere on S along a tangential approach region $\Omega_{\alpha,\tau}(\zeta), \zeta \epsilon S$, having the degree $\tau < n/s$ of tangency with S for all $\alpha > 1$, see (VII, Corollary 2.7) or [24].

(iv) If $s = 0$, then $f \epsilon M\mathcal{B}_p^0 = M\mathcal{B}_p$ has tangential limits along every direction in B, i.e., along an approach region having "infinite" degree of tangency with S, see (VII, Corollary 2.7) or [24].

(v) If $-p < s < 0$ and $1 < p < \infty$, then $f \epsilon M\mathcal{B}_p^s$ satisfies the Lipschitz condition of order $-s/p$ and has a continuous extension to the closure $\bar{B}$, see (VII, Theorem 2.8) or [24].

The tangential boundary behavior of holomorphic functions similar to the case (ii) has been considered by Nagel et al [40] for the case $n = 1$ on the unit disc $U \subset \mathbb{C}$ and by K. Shaw [47] for general $n \geqq 1$ on unit ball $B \subset \mathbb{C}^n$.

Question 6.1. *Let* $1 \leqq p < \infty, pq = p+q, F \epsilon L^p(S, \sigma)$, $S = \partial B, B \subset \mathbb{C}^n, 0 < \alpha < 1$, *and*

$$h_\alpha(z) = \int_S \frac{F(\zeta)}{(1 - \langle z, \zeta \rangle)^{n-\alpha}} d\sigma(\zeta).$$

It is shown for the case where $n = 1$, *[40], that*

(i) if $\alpha p < 1$, *then* h_α *has the tangential limits of degree* $\gamma = \frac{1}{1-\alpha p}$ *almost everywhere on* $T = \partial U$ *in the sense of [40],*

(ii) if $\alpha p = 1$, *then* h_α *has the tangential limits of "exponential contact with* T" *almost everwhere on* T, *and*

(iii) if $\alpha p > 1$, *then* h_α *is continuous on* $\bar{U}$.

It is interesting to see if the above result of [40] can be extended to the case of several complex variables. For a given $\alpha, 0 < \alpha < n$, it is equally interesting to find the values of s and p for which $h_\alpha \epsilon H\mathcal{B}_p^s$.

Question 6.2. *It is curious to see the statement made in (iii) in the begining of this section still holds for the case* $\tau = n/s$. *The result of Shaw [47] shows that it is likely to be true. However, it is not immediately clear from the proof given in [24].*

Supplementary Reference

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