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NOTES ON ABSTRACT HARMONIC ANALYSIS

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김종식 선생님을 기억하며

NOTES ON ABSTRACT HARMONIC ANALYSIS

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PREFACE

This is the collection of notes which have been distributed during the lectures on abstract harmonic analysis in the fall semester of the academic year 1993 at Seoul National University. The main topic of the lecture was to introduce measure theoretic or functional analysis approach to the group representation theory. It has been assumed that the audience have good backgrounds on abstract measure theory and elementary functional analysis with Hahn-Banach and Banach-Steinhaus Theorems. Some advanced functional analysis techniques such as Banach-Alaoglu, Krein-Milman, Stone-Weierstrass Theorems and the spectral decomposition theorem have been discussed briefly during the course.

One of the breakthrough in the group representation theory was the H. Weyl's observation that the multiplication of the group ring is nothing but the convolution in Fourier analysis. This observation leads him to study the representations of compact groups, generalizing those of finite groups. The existence of left invariant measure for an arbitrary locally compact group by Haar enables us to define the convolution and involution on the Banach space $L^1(G)$, to get a Banach $*$-algebra. We begin this note with the proof of the existence and uniqueness of the Haar measure, and examine elementary properties of the convolution and involution. Every unitary representation of a group G naturally induces a $*$-representation of the Banach $*$-algebra $L^1(G)$, where positive linear functionals play crucial roles. We conclude Chapter I with elementary properties of positive linear functionals on $L^1(G)$, or equivalently positive definite functions on G.

In Chapter II, we exclusively consider locally compact abelian (LCA) groups, whose representation theory amounts to the Fourier transform which converts L^1-functions (respectively complex regular Borel measures) on G to continuous functions on the dual group $\widehat{G}$ consisting of characters which vanish at infinity (respectively which is bounded). The Fourier transform on $L^1(G)$ may be generalized to the Gelfand transform on arbitrary commutative Banach algebras. Classical inversion formula for periodic functions and Plancherel transform on the real line will be proved for arbitrary LCA groups, using Bochner Theorem on positive definite functions. The central theme of Chapter II is the Pontryagin duality, which says that the double dual of an LCA group is topologically isomorphic to the original group. With this duality in hand, every result in Fourier transform has the dual interpretation. The critical point of the Pontryagin duality is that there are sufficiently many characters, and this will be proved using the inversion formula. The Fourier transform may be extended to the distributions if the involving group has the differential structures. We restrict ourselves to the circle group, and study the Fourier-Schwartz transform of periodic distributions. The range of this transform covers *every* slowly increasing sequence, and so every trigonometric series with slowly increasing coefficients defines a distribution in a suitable sense. This completes the idea of Jean Baptiste Joseph Fourier that every periodic *function* is represented by a trigonometric series.

In the case of non-abelian groups, the characters should be replaced by irreducible unitary representations. We begin Chapter III with the establishment of the correspondences between continuous unitary representations of G, non-degenerated $*$-representations of $L^1(G)$ and continuous positive definite functions on G. Irreducible representations correspond to continuous positive definite functions which are extreme in a sense. Employing functional analysis techniques such as Banach-Alaoglu and Krein-Milman Theorems, we show that there are sufficiently many irreducible representations for an arbitrary locally compact Hausdorff group. We will pay attention to compact groups, for which every irreducible representation is of *finite-dimensional*. We decompose the regular representation of a compact group into irreducible representations, which amounts to the Fourier series expansion for periodic functions. Compact groups also enjoy dualities, and we discuss here the classical Tannaka-Krein duality. We close this note by finding out all irreducible representations for the simplest non-abelian compact groups such as special unitary and orthogonal groups with low dimensions.

The lack of time prevents us to continue our study on general locally compact groups. The author hopes to continue this part in an another chance. The author would like to express his deep gratitude to all participants of the lecture. Another special thanks are due to Professors Jaihan Yoon, Dohan Kim, Hong-Jong Kim and Insok Lee. Discussions with them together with their comments were indispensable to prepare this notes.

CHAPTER I

GROUP ALGEBRAS

We prove in §1 that there exists a unique left invariant positive Borel measure on a locally compact Hausdorff group up to constant multiples. It should be noted that the existence will be proved via compactness argument using Tychonoff Theorem or equivalently Axiom of Choice. Therefore, it is an another job to construct the invariant measure for individual groups. In §2, we define convolution and involution on the space $L^1(G)$ of integrable functions and $M(G)$ of complex regular Borel measures, to get involutive Banach algebras. It turns out that $L^1(G)$ is an ideal of $M(G)$, which has always the identity for the convolution, the point mass δ_e on the identity of the group. Although $L^1(G)$ does not have the identity in general, we will see that there are approximate identities, which are nothing but various kernels in Fourier analysis. Every involutive algebra has the natural order structure as in the case of matrix algebras. A positive definite function on a group G is an L^∞-function which represents a positive linear functional on $L^1(G)$ with this order. An intrinsic characterization of positive definite functions will be discussed in §3 using the notion of semi-definite positivity of matrices.

1. Haar Integrals

A *topological group* is an abstract group together with a topological structure such that the group operations $(s,t) \mapsto st$ and $s \mapsto s^{-1}$ are continuous. An abstract group is a topological group with the discrete topology, called a *discrete group*. Throughout this note, we denote by $\mathbb{R}$ the additive group of all real numbers with the usual topology. The exponential map $t \mapsto e^{2\pi it}$ from $\mathbb{R}$ into the multiplicative group of all nonzero complex numbers is a continuous homomorphism. The range and the kernel of this homomorphism will be denoted by $\mathbb{T}$ and $\mathbb{Z}$, respectively.

From the definition, we see that the left and right translations $s \mapsto as$, $s \mapsto sa$ are homeomorphisms for $a \in G$, and so the local neighborhood systems are translated each other. By a *neighborhood*, we always mean a neighborhood of the identity e, unless stated otherwise. It is easy to see that for any neighborhood U there is a neighborhood $V \subseteq U$ such that $V = V^{-1}$. Such a neighborhood is called *symmetric*.

Exercise 1.1. Show that every nontrivial closed subgroup of $\mathbb{R}$ is topologically isomorphic to $\mathbb{Z}$.

Exercise 1.2. If a topological group G satisfies the T_0-separation axiom then G is Hausdorff.

From now on, a "group" means always a *locally compact Hausdorff* topological group. We also impose implicitly the countability condition such as σ-compactness whenever it is necessary. An abelian group is said to be an *LCA* group. The following proposition says that a continuous function on a group with a compact support is *uniformly* continuous in a sense.

Proposition 1.1. *Let f be a complex continuous function on a group G with a compact support K. Then for any $\epsilon > 0$ there is a neighborhood V such that*

$$st^{-1} \in V \Longrightarrow |f(s) - f(t)| < \epsilon. \tag{1.1}$$

For a function f on a group G, we denote by

$$f_t(s) = f(t^{-1}s), \qquad f^t(s) = f(st), \qquad s, t \in G.$$

Note that (1.1) is equivalent to say

$$s \in G,\ t \in V \Longrightarrow |f(ts) - f(s)| < \epsilon, \tag{1.2}$$

or equivalently, $\|f_{t^{-1}} - f\|_\infty < \epsilon$ for $t \in V$.

Proof. Choose a symmetric compact neighborhood U, and put

$$W = \{t \in G : |f(ts) - f(s)| < \epsilon \ \text{ for each } \ s \in UK\}.$$

Then it is easy to see that $V = U \cap W$ satisfies (1.2). Therefore, it suffices to show that W is a neighborhood. For each $s \in UK$, the function $(s,t) \mapsto f(ts) - f(s)$ is continuous at the point (s, e), and so we have a neighborhood V_s of s and a neighborhood W_s of e such that $|f(wv) - f(v)| < \epsilon$ for each $v \in V_s$ and $w \in W_s$. Because UK is compact, finitely many V_{s_i}'s cover UK. It is easy to see that W contains the intersection of the corresponding W_{s_i}'s □

In the remainder of this section, we show that every locally compact Hausdorff group admits a left-translation invariant (left-invariant, in short) measure. By the Riesz representation theorem, it suffices to show that there is a left-invariant positive linear functional on $C_c(G)$, the space of all continuous functions on G with compact supports. Such a *linear* functional will be exhibited as the limit of *sublinear* functionals as in the construction of the Riemann integral.

Lemma 1.2. *For given nonzero f, g in $C_c^+(G)$, there are $t_1, \dots, t_n \in G$ and $c_1, \dots, c_n \in \mathbb{R}^+$ such that*

$$f(s) \le \sum_{i=1}^{n} c_i g_{t_i}(s), \qquad s \in G. \tag{1.3}$$

Proof. Take $a \in G$ with $g(a) > 0$, and a neighborhood U such that $g(s) \ge \alpha > 0$ for $s \in aU$. If $t_1U, \dots, t_nU$ cover the support of f then we have

$$f(s) \le \sum_{i=1}^{n} \frac{\|f\|_\infty}{\alpha} g_{t_i a^{-1}}(s),$$

for each $s \in G$. Indeed, if $s \in t_iU$ then $at_i^{-1}s \in aU$, and so it follows that $g_{t_i a^{-1}}(s) \ge \alpha$. □

For nonzero $f, g \in C_c^+(G)$, we define the number $(f; g)$ by the infimum of the numbers $\{\sum_i c_i\}$ through $\{c_i\}$'s satisfying the relation (1.3). Then we have the following:

$$(f_t; g) = (f; g), \tag{1.4.i}$$

$$(f_1 + f_2; g) \leq (f_1; g) + (f_2; g), \tag{1.4.ii}$$

$$(cf; g) = c(f; g), \quad c > 0, \tag{1.4.iii}$$

$$f_1 \leq f_2 \Longrightarrow (f_1; g) \leq (f_2; g), \tag{1.4.iv}$$

$$(f; h) \leq (f; g)(g; h), \tag{1.4.v}$$

$$(f, g) \geq \|f\|_\infty / \|g\|_\infty, \tag{1.4.vi}$$

The first four relations are trivial. For (1.4.v), assume that $f \leq \sum c_i g_{t_i}$ and $g \leq \sum d_j h_{u_j}$. Then we have

$$f(s) \leq \sum c_i g(t_i^{-1}s) \leq \sum_{i,j} c_i d_j h_{u_j}(t_i^{-1}s) = \sum_{i,j} c_i d_j h_{t_i u_j}(s).$$

The last property is immediate, if we choose $s \in G$ with $f(s) = \|f\|_\infty$.

From now on, we fix a nonzero $f_0 \in C_c^+(G)$, and define

$$\Lambda_g(f) = \frac{(f; g)}{(f_0; g)},$$

for a nonzero $f, g \in C_c^+(G)$. Then Λ_g is a subadditive homogeneous functional on $C_c^+(G)$ which is left-invariant, that is, $\Lambda_g(f_s) = \Lambda_g(f)$ for $s \in G$. By the relation (1.4.v), we also have

$$\frac{1}{(f_0; f)} \leq \Lambda_g(f) \leq (f; f_0). \tag{1.5}$$

The following lemma shows that the functional Λ_g becomes approximately linear as the support of g becomes smaller. The notations $f \prec V$ (respectively $K \prec f$) means that $f \in C_c(G)$, $0 \leq f \leq 1$ and

$$\operatorname{supp} f \subseteq V \qquad (\text{respectively } \ f(x) = 1, \ x \in K),$$

where V (respectively K) is open (respectively compact) in G.

Lemma 1.3. *For given $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$, there is a neighborhood V such that*

$$g \prec V \implies \Lambda_g(f_1) + \Lambda_g(f_2) \le \Lambda_g(f_1 + f_2) + \epsilon.$$

Proof. Take $f' \succ \operatorname{supp}(f_1 + f_2)$ and $\delta, \epsilon' > 0$ arbitrary. Put $f = f_1 + f_2 + \delta f'$ and $h_i = f_i / f$ for $i = 1, 2$. If $f(s) = 0$ then we put $h_i(s) = 0$. By Proposition 1.1, there is a neighborhood V such that

$$t^{-1}s \in V \implies |h_i(s) - h_i(t)| < \epsilon', \quad i = 1, 2.$$

If $g \prec V$ and $f \le \sum_j c_j g_{t_j}$ then $|h_i(s) - h_i(t_j)| < \epsilon'$ for $i = 1, 2$ whenever $g_{t_j}(s) \ne 0$. Hence, for each $s \in G$, we have

$$f_i(s) = h_i(s)f(s) \le \sum_j c_j g_{t_j}(s) h_i(s) \le \sum_j c_j g_{t_j}(s)(h_i(t_j) + \epsilon').$$

Therefore, we have

$$(f_i; g) \le \sum_j c_j(h_i(t_j) + \epsilon'), \quad i = 1, 2,$$

and so,

$$(f_1; g) + (f_2; g) \le \sum_j c_j(1 + 2\epsilon'),$$

because $h_1 + h_2 \le 1$. This completes the proof by the estimate

$$\Lambda_g(f_1) + \Lambda_g(f_2) \le (1 + 2\epsilon')\Lambda_g(f) \le (1 + 2\epsilon')[\Lambda_g(f_1 + f_2) + \delta\Lambda_g(f')]. \quad \square$$

Theorem 1.4. *For any locally compact Hausdorff group G, there is a positive left-invariant linear functional on $C_c(G)$.*

Proof. For $f \in C_c^+(G)$, we denote by I_f the interval $[1/(f_0; f), (f; f_0)]$. Then we have

$$\Lambda_g \in \prod_{f \in C_c^+(G)} I_f, \qquad g \in C_c^+(G).$$

by (1.5). By the Tychonoff theorem, $\prod_f I_f$ is compact with respect to the product topology. For a neighborhood V, we also denote by $\mathcal{C}_V$ the compact closure of $\{\Lambda_g : g \prec V\}$ in $\prod_f I_f$. We see that the family $\{\mathcal{C}_V : V$ is a neighborhood$\}$ has the finite intersection property from the easy relation $\mathcal{C}_{V_1} \cap \dots \mathcal{C}_{V_n} = \mathcal{C}_{V_1 \cap \dots V_n}$. Therefore, there exists Λ which lies in $\mathcal{C}_V$ for any neighborhood V. From the definition of the product topology, this means that for any given neighborhood V, $\epsilon > 0$ and $f_1, f_2 \in C_c^+(G)$, there is $g \prec V$ such that $|\Lambda(f_i) - \Lambda_g(f_i)| < \epsilon$ for $i = 1, 2$, and $|\Lambda(f_1 + f_2) - \Lambda_g(f_1 + f_2)| < \epsilon$. By Lemma 1.3, we have

$$\Lambda(f_1) + \Lambda(f_2) \le \Lambda_g(f_1) + \Lambda_g(f_2) + 2\epsilon \le \Lambda_g(f_1 + f_2) + 3\epsilon \le \Lambda(f_1 + f_2) + 4\epsilon.$$

Therefore, Λ is linear and it extends to the linear functional of $C_c(G)$. $\square$

By the Riesz representation theorem, we see that there is a left-invariant measure μ on G in the sense

$$\int_G f(s)d\mu(s) = \int_G f_t(s)d\mu(s) = \int_G f(t^{-1}s)d\mu(s) = \int_G f(s)d\mu(ts),$$

for each $t \in G$. By the same argument, there is also a right-invariant measure ν in the sense $\displaystyle\int f d\nu = \int f^t d\nu$.

From now on, we discuss the uniqueness of the left-invariant measure. To do this, we fix a nonzero $g \in C_c^+(G)$ and a right-invariant measure ν such that

$$\int_G g(t^{-1})d\nu(t) = 1. \tag{1.6}$$

We define the function Γ on G by

$$\Gamma(s) = \int_G g(t^{-1}s)d\nu(t). \tag{1.7}$$

Then we see that Γ is continuous by Proposition 1.1, and $\Gamma(s) > 0$ for each $s \in G$, because the function $t \mapsto g(t^{-1}s)$ is positive on an open set. Note also that $\Gamma(e) = 1$. Denote by $\Delta = 1/\Gamma$.

Lemma 1.5. *Let μ be a left-invariant measure satisfying*

$$\int_G g(s)d\mu(s) = 1. \tag{1.8}$$

Then we have $d\mu = \Delta d\nu$.

Proof. Take $f \in C_c(G)$. Then we have

$$\begin{aligned}\int f(s)d\mu(s) &= \int f(s)\Delta(s)\Gamma(s)d\mu(s)\\ &= \int\left[\int f(s)\Delta(s)g(t^{-1}s)d\nu(t)\right]d\mu(s)\\ &= \iint f(s)\Delta(s)g(t^{-1}s)d\mu(s)d\nu(t)\\ &= \iint f(ts)\Delta(ts)g(s)d\mu(s)d\nu(t)\\ &= \int\left[\int f(ts)\Delta(ts)d\nu(t)\right]g(s)d\nu(s)\\ &= \int\left[\int f(t)\Delta(t)d\nu(t)\right]g(s)d\mu(s)\\ &= \int f(t)\Delta(t)d\nu(t).\end{aligned}$$

Note that we have used the Fubini theorem in the third equality. □

Assume that μ_1 and μ_2 are two left-invariant measure on a group G. If we normalize μ_i so that the equality (1.8) holds then we have $\mu_1 = \mu_2$ by Lemma 1.5. Therefore, we have the following:

Theorem 1.6 (Haar). *Every locally compact Hausdorff group admits a unique left-invariant measure up to constant multiples.*

We always denote by $ds, dt, \dots$ this unique left-invariant measure. If G is discrete then we always assume that every point has the unit mass. If G is compact then the constant function 1_G lies in $C_c(G)$, and so the whole measure of G is finite. In this case, we always assume that $\int_G ds = 1$. There should be no confusion when G is a finite group.

Exercise 1.3. If g' and ν' satisfy the condition (1.6) then show that

$$\Gamma(s) = \int_G g'(t^{-1}s)d\nu'(t).$$

Show also that

$$\int_G f(t^{-1}s)d\nu(t) = \Gamma(s)\int_G f(t^{-1})d\nu(t), \qquad f \in C_c(G),\ s \in G,$$

whenever ν is a right invariant measure.

Therefore, the function Γ in (1.7) is independent of the choice of g and ν under the assumption (1.6). The function $\Delta = 1/\Gamma$ is said to be the *modular* function of G. A group G is said to be *unimodular* if $\Delta(s) = 1$ for each $s \in G$.

Exercise 1.4. Show that the modular function of G is a continuous homomorphism from G into the multiplicative group of all positive real numbers. Show that every compact or discrete group is unimodular.

Exercise 1.5. Show the following relations:

$$\int_G f(st)ds = \Delta(t^{-1}) \int_G f(s)ds,$$
$$\int_G f(s^{-1})\Delta(s^{-1})ds = \int_G f(s)ds,$$

for $f \in L^1(G)$. [Hint: The measure $f \mapsto \displaystyle\int f(s^{-1})\Delta(s^{-1})ds$ is left-invariant.]

2. Convolutions

For complex-valued functions f and g on a group G, we define the *convolution* $f * g$ of f and g by

$$(2.1) \qquad (f * g)(t) =: \int_G f(s)g(s^{-1}t)ds = \int_G f(ts)g(s^{-1})ds, \quad t \in G.$$

If s is an element of a discrete group G then we denote by χ_s the characteristic function on the singleton $\{s\}$. If $f = \sum_s a_s\chi_s$ and $g = \sum_t b_t\chi_t$ with finite summations then we have

$$(f * g)(r) = \sum_s f(s)g(s^{-1}r) = \sum_s a_s b_{s^{-1}r} = \sum_{st=r} a_s b_t, \quad r \in G,$$

and so it follows that

$$(2.2) \qquad (\sum_s a_s\chi_s) * (\sum_t b_t\chi_t) = \sum_r(\sum_s a_s b_{s^{-1}r})\chi_r = \sum_{s,t} a_s b_t \chi_{st},$$

which is nothing but the multiplication of usual *group rings*. Note that χ_e is the identity for this multiplication in this case.

If $f, g \in L^1(G)$ then we may assume that f, g are Borel measurable, and so the function $(s,t) \mapsto f(s)g(s^{-1}t)$ is also Borel measurable. Applying the Fubini theorem, we get

$$\int \left[\int |f(s)g(s^{-1}t)|ds\right] dt = \iint |f(s)g(s^{-1}t)|dtds$$
$$= \iint |f(s)||g(t)|dtds = \|f\|_1\|g\|_1.$$

Therefore, we see that the value $(f * g)(t)$ is finite for almost all $t \in G$ and

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1, \quad f, g \in L^1(G). \tag{2.3}$$

In particular, the space $L^1(G)$ is closed under the convolution.

Exercise 2.1. Show that the convolution is associative.

This says that $L^1(G)$ is a *Banach algebra* under the convolution: It is a Banach space with an associative multiplication satisfying the relation $\|xy\| \leq \|x\|\|y\|$. Note that this relation says that the multiplication is jointly continuous. It is customary to assume that $\|1_A\| = 1$ if a Banach algebra A is unital.

We denote by $M(G)$ the Banach space of all (finite) complex regular Borel measures on G. Note that $L^1(G)$ is a subspace of $M(G)$ consisting of all absolutely continuous measure with respect to the left-invariant Haar measure, by the Radon-Nikodym theorem. Recall that every bounded linear functional on $C_0(G)$, the Banach space of all continuous functions on G vanishing at infinity, is represented by an element μ of $M(G)$:

$$h \mapsto \int_G h d\mu, \qquad h \in C_0(G).$$

For $\mu, \nu \in M(G)$, we define the linear functional $\mu * \nu$ on $C_0(G)$ by

$$\mu * \nu : h \mapsto \iint_{G \times G} h(st)d(\mu \times \nu)(s,t), \qquad h \in C_0(G). \tag{2.4}$$

Because

$$\left|\int hd(\mu * \nu)\right| \leq \|h\|_\infty |(\mu \times \nu)(G \times G)| \leq \|h\|_\infty \|\mu\|\,\|\nu\|,$$

we see that $\mu * \nu$ defines a bounded linear functional on $C_0(G)$ with

$$(2.5) \qquad \|\mu * \nu\| \le \|\mu\| \, \|\nu\|, \qquad \mu, \nu \in M(G).$$

If μ and ν are absolutely continuous measure (with respect to the left-invariant Haar measure) represented by $d\mu(s) = f(s)ds$ and $d\nu(t) = g(t)dt$, then we have

$$\int h d(\mu * \nu) = \iint h(st) f(s) ds g(t) dt, \quad h \in C_c(G).$$

On the other hand,

$$\int h(t)(f * g)(t)dt = \int h(t) \int f(s) g(s^{-1}t) ds dt = \int h(t) g(s^{-1}t) dt f(s) ds.$$

By the left-invariance, the above two quantities coincide, and so our definitions in (2.1) and (2.4) are consistent. We denote by δ_e the point mass at the identity. It is easy to see that δ_e is the identity for the convolution (2.4).

We define one more operation. For $\mu \in M(G)$, we define the *involution* μ^* of μ by

$$(2.6) \qquad \int_G h(s) d\mu^*(s) = \overline{\int_G \overline{h}(s^{-1}) d\mu(s)}, \quad h \in C_0(G).$$

Recall that a map $x \mapsto x^*$ of an associative algebra A over the complex field is said to be an *involution* if

$$(x+y)^* = x^* + y^*, \quad (\alpha x)^* = \overline{\alpha} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x,$$

for $x, y \in A$ and $\alpha \in \mathbb{C}$.

Exercise 2.2. Show that $(\mu * \nu) * \lambda = \mu * (\nu * \lambda)$ for $\mu, \nu, \lambda \in M(G)$ and that $\mu \mapsto \mu^*$ is an isometric involution.

This says that $M(G)$ is an *involutive Banach algebra* together with (2.5). If $d\mu(s) = f(s)ds$ for an L^1-function f, then we have

$$\int h(s) d\mu^*(s) = \int h(s^{-1}) \overline{f(s)} ds = \int \Delta(s^{-1}) h(s) \overline{f(s^{-1})} ds,$$

for $h \in C_0(G)$ by Exercise 1.5. Therefore, we have

$$(2.7) \qquad f^*(s) = \Delta(s^{-1}) \overline{f(s^{-1})}, \qquad f \in L^1(G).$$

Theorem 2.1. *Let G be a group. Then we have the following:*

(i) *$M(G)$ is an involutive unital Banach algebra.*

(ii) *$L^1(G)$ is a $*$-preserving left-ideal of $M(G)$.*

Proof. It remains to show that $L^1(G)$ is a left-ideal of $M(G)$. To do this, take $\mu \in M(G)$ and $f \in L^1(G)$. By the Fubini theorem, we have

$$\iint |f(t^{-1}s)|d\mu(t)ds = \iint |f(t^{-1}s)|dsd\mu(t) = \iint |f(s)|dsd\mu(t) = \|f\|_1\|\mu\|.$$

Therefore, it follows that the function $s \mapsto \int f(t^{-1}s)d\mu(t)$ defines an L^1-function. It is easy to see that

$$(\mu * f)(s) = \int f(t^{-1}s)d\mu(t). \quad \square \tag{2.8}$$

Exercise 2.3. Show that $L^1(G)$ is a two-sided ideal of $M(G)$ if G is a unimodular group.

Exercise 2.4. Show that G is abelian if and only if $M(G)$ is commutative. Show also that G is discrete if and only if $L^1(G)$ is unital.

In the remainder of this section, we show that $L^1(G)$ has an approximate identity. To do this, we first show that the function $s \mapsto f_s$ is uniformly continuous from G into $L^1(G)$ whenever $f \in L^1(G)$.

Lemma 2.2. *Let $f \in L^1(G)$. Then given $\epsilon > 0$, there is a neighborhood V such that $\|f_s - f\|_1 < \epsilon$ for each $s \in V$.*

Proof. We first consider the case $f \in C_c(G)$ with the compact support K. We fix a compact neighborhood W, and choose a compact neighborhood $V \subseteq W$ such that

$$s \in V \Longrightarrow \|f - f_s\|_\infty < \epsilon/\mu(KW),$$

by Proposition 1.1, where μ is the left-invariant measure. If $s \in V$ then the support of $f - f_s$ is contained in $KV \subseteq KW$. Therefore, we have $\|f - f_s\|_1 < \epsilon$. If $f \in L^1(G)$ then we approximate f by functions in $C_c(G)$ in the L^1-norm. $\square$

Exercise 2.5. If $f \in L^p(G)$ with $1 \le p < \infty$ then show that the map $s \mapsto f_s$ is uniformly continuous with respect to the L^p-norm.

Proposition 2.3. *Let $f \in L^1(G)$. Then given $\epsilon > 0$ there exists a neighborhood V with the property:*

$$u \in L^1(G),\ u \geq 0,\ \operatorname{supp} u \subseteq V,\ \int_G u = 1 \implies \|f - u * f\|_1 < \epsilon. \tag{2.9}$$

Proof. Take a neighborhood V as in Lemma 2.2. Note that

$$f(t) - (u * f)(t) = \int u(s)[f(t) - f_s(t)]ds, \qquad t \in G.$$

Therefore, we have

$$\begin{aligned} \|f - u * f\|_1 &\leq \iint u(s)|f(t) - f_s(t)|dsdt \\ &= \int_V u(s) \int_G |f(t) - f_s(t)|dtds \\ &\leq \int_V u(s)\|f - f_s\|_1 ds \leq \epsilon. \quad \square \end{aligned}$$

Exercise 2.6. For $f \in L^1(G)$ and $\epsilon > 0$, show that there is a neighborhood V such that

$$u \in L^1(G),\ u \geq 0,\ \operatorname{supp} u \subseteq V,\ \int_G u = 1 \implies \|f - f * u\|_1 < \epsilon.$$

We denote by $\mathcal{V}$ the directed set of the neighborhood system of the identity. For each $V \in \mathcal{V}$, take a nonnegative continuous function u_V satisfying the assumption of (2.9). The above proposition and exercise say that

$$\lim_V \|f - u_V * f\|_1 = \lim_V \|f - f * u_V\|_1 = 0,$$

for each $f \in L^1(G)$. A net $\{u_V : V \in \mathcal{V}\}$ satisfying the above relations is called an *approximate identity*.

Exercise 2.7. Let X be a locally compact Hausdorff space. Show that the Banach space $C_0(X)$ is an involutive Banach algebra with respect to the operations

$$(fg)(x) = f(x)g(x), \qquad \overline{f}(x) = \overline{f(x)}, \qquad f, g \in C_0(X), x \in X.$$

Show that $C_0(X)$ is unital if and only if X is compact. Construct an approximate identity of $C_0(X)$.

We close this section to consider the convolution of L^p-functions.

Proposition 2.4. *Assume that G is unimodular and $f \in L^p(G)$ and $g \in L^q(G)$, where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have $f * g \in C_0(G)$.*

Proof. Since G is unimodular, we have

$$|(f * g)(t)| \leq \int |f(ts)g(s^{-1})|ds \leq \|f\|_p \|g\|_q, \qquad t \in G,$$

by the Hölder inequality, and so the function $f * g$ is defined everywhere, with $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. We also have

$$|(f * g)(s) - (f * g)(t)| \leq \|f_{s^{-1}} - f_{t^{-1}}\|_p \|g\|_q.$$

Therefore, the function $f*g$ is continuous by Exercise 2.5. If we take sequences $\{f_n\}$ and $\{g_n\}$ of $C_c(G)$ converging to f and g with respect to the L^p and L^q norms, respectively, then it is easy to see that $f_n * g_n$ converges to $f * g$ uniformly, by an another application of Hölder inequality. If f_n and g_n vanish outside H and K, respectively then $f_n * g_n$ vanishes outside HK. Therefore, $f * g$ is the uniform limit of functions in $C_c(G)$. □

Exercise 2.8. Let $f \in L^1(G)$ and $g \in L^\infty(G)$. Show that $f * g$ is a continuous bounded function. If G is unimodular then $g * f$ is also continuous bounded.

Exercise 2.9. Show that $(\chi_V * \chi_{V^{-1}U})(t) = \mu(V)$ for $t \in U$, where μ is the Haar measure.

Exercise 2.10. Let a and b functions on $\mathbb{Z}$ so that $a(i) = b(i) = 0$ for $|i| > n$. Show that

$$\left(\sum_{i=-n}^{n} a(i)10^i\right)\left(\sum_{i=-n}^{n} b(i)10^i\right) = \sum_{i=-2n}^{2n} (a * b)(i)10^i.$$

3. Positive Definite Functions

One of the merits of the involution is that there is a natural order. *Positive* elements of an involutive algebra are those of the forms x^*x. For the case of $C_0(X)$, they are nothing but nonnegative functions. On the other hand, the positive cone of the matrix algebra consists of positive semi-definite matrices. In this sense, we may consider the notion of positive linear functionals on the involutive algebra $L^1(G)$. Recall that a bounded linear functional of $L^1(G)$ is represented by an L^∞-function. An L^∞-function ϕ of G is said to be *positive definite* if

$$\langle \phi, f^* * f \rangle \geq 0, \quad \text{for each } f \in L^1(G). \tag{3.1}$$

In other word, a positive definite function is an L^∞-function which induces a positive linear functional on $L^1(G)$. Note that

$$(f^* * f)(t) = \int \Delta(s^{-1})\overline{f(s^{-1})}f(s^{-1}t)ds = \int \overline{f(s)}f(st)ds,$$

and so, we have

$$\langle \phi, f^* * f \rangle = \int \phi(t)(f^* * f)(t)dt = \iint \phi(s^{-1}t)\overline{f(s)}f(t)dsdt. \tag{3.2}$$

Theorem 3.1. *Let ϕ be a continuous bounded function on a group G. Then the following are equivalent:*

(i) *The function ϕ is positive definite.*
(ii) *For each $\mu \in M(G)$, we have $\langle \phi, \mu^* * \mu \rangle \geq 0$.*
(iii) *For each $f \in C_c(G)$, we have $\langle \phi, f^* * f \rangle \geq 0$.*
(iv) *For any choice of elements $s_1, \ldots, s_n$ of G and complex numbers $\alpha_1, \ldots, \alpha_n$, we have $\sum_{i,j=1}^n \overline{\alpha_i}\alpha_j \phi(s_i^{-1}s_j) \geq 0$.*

Proof. The directions (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) are clear. For the implication (iii) $\Longrightarrow$ (iv), take $\mu = \sum_i \alpha_i \delta_{s_i}$, where δ_s denotes the point mass at $s \in G$. Then we have

$$(\mu * f)(s) = \sum_i \alpha_i f(s_i^{-1}s), \qquad s \in G$$

for any $f \in C_c(G)$ by (2.8). Since $\mu * f \in C_c(G)$, we also have

$$0 \leq \langle \phi, (\mu * f)^*(\mu * f) \rangle = \sum_{i,j} \overline{\alpha_i}\alpha_j \iint \phi(s^{-1}t)\overline{f(s_i^{-1}s)}f(s_j^{-1}t)dsdt,$$

by (3.2). If we take $f = u_V$ satisfying the assumption of (2.9) then the integral parts converge to the value $\phi(s_i^{-1}s_j)$, and this proves (iv).

For the converse (iv) $\Longrightarrow$ (ii), we need the following fact (see Remark below): Every complex regular Borel measure μ is the limit of $\{\mu_\iota\}$ with finite supports in the sense

$$\int f d\mu = \lim_\iota \int f d\mu_\iota \qquad \text{for each } \ f \in C_0(G). \tag{3.3}$$

Note that $\langle \phi, \mu^* * \mu \rangle = \iint \phi(s^{-1}t)\overline{d\mu(s)}d\mu(t)$. If $\mu = \sum \alpha_i \delta_{s_i}$ with finite support then we have

$$\langle \phi, \mu^* * \mu \rangle = \sum_{i,j} \overline{\alpha_i}\alpha_j \phi(s_i^{-1}s_j) \geq 0.$$

First, we assume that μ is supported by a compact set, and choose $\psi \in C_c(G)$ with $\psi \succ K$, where K is the support of $\mu^* * \mu$. Then we have

$$\begin{aligned}\langle \phi, \mu^* * \mu \rangle = \langle \phi\psi, \mu^* * \mu \rangle &= \iint (\phi\psi)(s^{-1}t)\overline{d\mu(s)}d\mu(t) \\ &= \lim_\iota \iint (\phi\psi)(s^{-1}t)\overline{d\mu_\iota(s)}d\mu_\iota(t) = \lim_\iota \langle \phi\psi, \mu_\iota^* * \mu_\iota \rangle \geq 0.\end{aligned}$$

The proof is complete, because a complex regular Borel measure is the norm limit of measures with compact supports. $\square$

Remark. The relation (3.3) is a direct consequence of the two fundamental principles of functional analysis: The Banach-Alaoglu theorem and the Krein-Milman theorem. We digress for a while to explain these two theorems. Let X be a normed space. Recall that the *dual* space X^* is defined by the vector space consisting of all *bounded* linear functionals of X. With the norm topology, X^* becomes a Banach space. We introduce an another topologies on X and X^*. A net $\{x_\iota\}$ of X is said to converge to an element $x \in X$ in the *weak* topology if

$$\lim_\iota \phi(x_\iota) = \phi(x) \qquad \text{for each } \ \phi \in X^*.$$

On the other hand, a net ϕ_ι of X^* is said to converge to an element $\phi \in X^*$ in the *weak**-topology if

$$\lim_\iota \phi_\iota(x) = \phi(x) \qquad \text{for each } \ x \in X.$$

Exercise 3.1. Recall that $(c_0)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$. Discuss the relations between the norm, weak and weak*-topologies of the space ℓ^1.

Banach-Alaoglu Theorem. *Let X be a normed space. Then the closed unit ball B of X^* is weak*-compact.*

Note that every element $\phi \in B$ is a function from the closed unit ball U of X into the closed unit disc D of the complex plane, and so, we see that

$$B \subseteq \prod_{x \in U} D_x,$$

where D_x is a copy of D for each $x \in U$. It is easy to see that B is a closed subset of $\prod D_x$ with respect to the product topology, which coincides with the weak*-topology on the subset B. These imply that B is weak*-compact by the Tychonoff theorem.

Krein-Milman Theorem. *Let X be a normed space. Then every compact convex subset K of X is the closure of the convex hull of its extreme points.*

The proof of this theorem also depends on the Maximal Principle. Recall that a point x of a convex set K in a vector space is said to be *extreme* if

$$y, z \in K, \ 0 < t < 1, \ ty + (1-t)z = x \implies y = z = x.$$

In other words, an extreme point is the point which is not the convex combination of other points. One of the important consequences of the Krein-Milman theorem is that every *compact* convex set has an extreme point. Note that the notion of extremity itself is irrelevant to the topology.

Exercise 3.2. Show that the closed unit ball of $(c_0)^*$ has no extreme point. Conclude that there is no normed space X such that $X^* = c_0$.

In the context of $C_0(G)^* = M(G)$, the relation (3.3) says that μ_ι converges to μ in the weak*-topology. In order to find a net $\{\mu_\iota\}$ with finite supports satisfying the relation (3.3), the following would suffice:

Exercise 3.3. Show that a measure $\mu \in M(G)$ is an extreme point of the closed unit ball of $M(G)^+$, the cone of all finite positive measures on G, if and only if μ is a point mass.

The above Theorem 3.1 says that a continuous bounded function ϕ is positive definite if and only if the $n \times n$ matrix $[\phi(s_i^{-1}s_j)]$ is a positive semi-definite matrix for arbitrary choice of a natural number n and element $s_1, \ldots, s_n$ of G. We denote by $P(G)$ the set of all continuous positive definite functions on a group G. It is clear that $P(G)$ is a positive cone:

$$\phi, \psi \in P(G), \ \alpha > 0 \implies \phi + \psi, \alpha\phi \in P(G).$$

If we choose two points $\{e, s\}$ of G then the matrix $\begin{pmatrix} \phi(e) & \phi(s) \\ \phi(s^{-1}) & \phi(e) \end{pmatrix}$ is positive semi-definite. Therefore, we have

$$\phi(s^{-1}) = \overline{\phi(s)}, \qquad |\phi(s)| \leq \phi(e), \qquad s \in G, \ \phi \in P(G).$$

In particular, we have

$$\|\phi\|_\infty = \phi(e), \qquad \phi \in P(G). \tag{3.4}$$

Exercise 3.4. Show that the entry-wise product of two positive semi-definite matrices is again positive semi-definite. Therefore, the pointwise product of continuous positive definite functions is again positive definite.

The entry-wise product of two matrices with the same sizes is said to be the *Hadamard product* of matrices.

Exercise 3.5. Let μ be a finite positive Borel measure on $\mathbb{R}$. Show that the function

$$\phi(s) = \int_{\mathbb{R}} e^{ist} d\mu(t), \qquad s \in \mathbb{R},$$

is continuous positive definite.

Later, we will show that every continuous positive definite function on $\mathbb{R}$ is in this form. This is also the case for any LCA group in terms of characters. In particular, the function $s \mapsto e^{ist}$ is positive definite for each $t \in \mathbb{R}$.

Exercise 3.6. Let $\phi = \chi_0 + \alpha\chi_1 + \overline{\alpha}\chi_{-1}$ be a function on $\mathbb{Z}$. Find the condition on α for which ϕ is positive definite.

Positive definite functions will be discussed again together with the representation theory. We close this section with an another example of positive definite functions. For a function f on a group G, we define

$$\tilde{f}(t) = \overline{f(t^{-1})}, \qquad t \in G.$$

If G is unimodular then $\tilde{f}$ is nothing but f^*. We note that $\tilde{f}$ need not to be integrable even if f is integrable, unless G is unimodular.

Proposition 3.2. *Let G be a unimodular group and $f \in L^2(G)$. Then the function $f * \tilde{f}$ is continuous positive definite.*

Proof. The function $f * \tilde{f}$ is continuous by Proposition 2.4. Note that

$$(f * \tilde{f})(t) = \int f(s)\overline{f(t^{-1}s)}ds, \qquad t \in G.$$

Therefore, we have

$$\begin{aligned}\sum_{i,j} \overline{\alpha_i}\alpha_j (f * \tilde{f})(t_i^{-1}t_j) &= \sum_{i,j} \overline{\alpha_i}\alpha_j \int f(s)\overline{f(t_j^{-1}t_i s)}ds \\ &= \sum_{i,j} \int \overline{\alpha_i f(t_i^{-1}s)}\alpha_j \overline{f(t_j^{-1}s)}ds \\ &= \int |\Sigma_i \overline{\alpha_i} f(t_i^{-1}s)|^2 ds \geq 0. \quad \square\end{aligned}$$

Exercise 3.7. Consider the convolution of the characteristic functions on intervals, to find examples of positive definite functions on $\mathbb{R}$ which are piecewise linear with compact supports.

Assume that G is a discrete group. For a function ϕ of G, we denote by A_ϕ the matrix with the size of the order of G whose (s,t)-entry is given by $\phi(s^{-1}t)$:

$$(A_\phi)_{s,t} = \phi(s^{-1}t). \tag{3.5}$$

Then a function ϕ is positive definite if and only if the matrix A_ϕ is positive semi-definite. It is clear that $A_{\tilde{\phi}} = (A_\phi)^*$. Note that

$$(\phi * \psi)(s_i^{-1}s_j) = \sum_{s\in G} \phi(s)\psi(s^{-1}s_i^{-1}s_j) = \sum_{t\in G} \phi(s_i^{-1}t)\psi(t^{-1}s_j).$$

This shows the relation

$$A_{\phi*\psi} = A_\phi A_\psi, \tag{3.6}$$

with the usual (formal) matrix multiplication. Note that $\phi(s)$ is nothing but the (e, s)-entry of the matrix A_ϕ, and so the correspondence $\phi \mapsto A_\phi$ is one-to-one. With this machinery in hands, it is easy to show the partial converse of Proposition 3.2 for discrete groups.

Proposition 3.3. *Assume that G is a discrete group. If ϕ is positive definite function with the finite support then there is a function f on G with the finite support such that $\phi = f * \widetilde{f} = f * f$.*

Proof. Note that A_ϕ may be considered as a matrix with finite size which is positive semi-definite. Define $f(s)$ by the (e, s)-entry of the square root of the matrix A_ϕ. □

We note that the above proposition also holds for any locally compact groups: Every function $\phi \in C_c(G) \cap P(G)$ is expressed as $\phi = f * \widetilde{f} = f * f$ for an L^2-function f.

We conclude this section with the Schwarz inequality: If ϕ is a positive linear functional on an involutive algebra A, that is, $\phi(x^*x) \geq 0$ for $x \in A$, then it is easy to see that ϕ induces a sesquilinear form on A by

$$\langle x, y \rangle = \phi(y^*x), \qquad x, y \in A. \tag{3.7}$$

Therefore, we have

$$\phi(y^*x) = \overline{\phi(x^*y)}, \qquad x, y \in A, \tag{3.8}$$

$$|\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y), \qquad x, y \in A. \tag{3.9}$$

If A is unital then we take $y = e$ in (3.8), to get

$$|\phi(x)|^2 \leq \phi(e)\phi(x^*x), \qquad x \in A.$$

If G is a locally compact group and $\phi \in P(G)$ then $\phi(\delta_e) = \phi(e) = \|\phi\|_\infty$, and so we have

$$|\langle \phi, f \rangle|^2 \leq \|\phi\|_\infty \langle \phi, f^* * f \rangle, \qquad \phi \in P(G), f \in L^1(G). \tag{3.10}$$

NOTE

We have followed [LOOMIS, §29] and [ROYDEN, §14.6] for the existence and uniqueness of the Haar measure, respectively. See also [HEWITT AND ROSS, §15], for another definition of the modular function and history. For examples of non-unimodular groups, we refer to [LOOMIS, §30D]. The proof of Theorem 3.1 was taken from [DIXMIER, §13.4]. It should be noted that a positive linear functional on an involutive Banach algebra with an approximate identity is automatically bounded, and so the boundedness condition is actually redundant in our definition of positive definite functions. A history in [DIEUDONNÉ81, Chapter VII] would be useful throughout this note.

CHAPTER II

ABELIAN GROUPS

Let G be a finite group with the group algebra $\mathbb{C}[G]$. Then every homomorphism $\pi : s \mapsto \pi_s$ from G into the group $U(n)$ of all $n \times n$ unitary matrices induces an algebra homomorphism from $\mathbb{C}[G]$ into the full matrix algebra by

$$\sum_{s \in G} a_s \chi_s \mapsto \sum_{s \in G} a_s \pi_s .$$

If G is the circle group $\mathbb{T}$ and π is given by $e^{it} \mapsto e^{-int} \in U(1)$ then the induced algebra homomorphism is nothing but the n-th Fourier coefficient assignment $f \mapsto \widehat{f}(n)$ for $f \in L^1(G)$.

When G is an abelian group, we define $\widehat{f}(\gamma)$ for each continuous homomorphism γ, called a character, from G into $\mathbb{T} = U(1)$. We endow the group $\widehat{G}$ of all characters with the smallest topology with which the function $\widehat{f}$ is continuous on $\widehat{G}$ for every $f \in L^1(G)$. It turns out that this topology is equivalent to the compact-open topology. In this way, $\widehat{G}$ becomes again an LCA group, and the Fourier transform $f \mapsto \widehat{f}$ defines a norm-decreasing $*$-homomorphism from $L^1(G)$ onto a dense $*$-subalgebra of $C_0(\widehat{G})$. This transform extends to $M(G)$, to get the Fourier-Stieltjes transform. Because every nonzero complex homomorphism of $L^1(G)$ is of the form $f \mapsto \widehat{f}(\gamma)$ for a character γ, we also topologize the set of all complex homomorphisms for an arbitrary commutative Banach algebra, to get the Gelfand transform, which generalize the Fourier transform. We study Gelfand transforms in §5 together with the spectral radius formula, with which we relates the L^1-norm of $f \in L^1(G)$ and L^∞-norm of $\widehat{f} \in C_0(\widehat{G})$.

In §6, we consider the dual map of the Fourier transform and characterize positive definite functions on an LCA group in terms of characters. Using this machinery, we prove the inversion formula for positive definite L^1-functions and prove the Plancherel theorem. As an another consequence of the inversion formula, we show that there are sufficiently many characters to distinguish points of an LCA group in §7, and establish the Pontryagin duality, which asserts that the double dual of an LCA group is topologically isomorphic to the original group. We discuss dual interpretations of the former results, and obtain the uniqueness theorem which says that the Fourier transform $f \mapsto \widehat{f}$ is injective.

In §8, we restrict ourselves to the circle group, and exploit the differential structures. We endow with a complete metric on the space $\mathcal{D}(\mathbb{T})$ of all periodic C^∞-functions which is invariant under the addition. The convolution, involution and the Fourier coefficients may be defined on the dual space $\mathcal{D}'(\mathbb{T})$ which is bigger than $M(\mathbb{T})$. In this way, we get the Fourier-Schwartz transform whose restriction to $M(\mathbb{T})$ is just the Fourier-Stieltjes transform. Because the range of the Fourier-Stieltjes transform does not cover even sequences converging to 0, there was no way to interpret trigonometric series with these coefficients. The Fourier-Schwartz transform gives us an isomorphism from the algebra of distributions *onto* the algebra of all slowly increasing sequences.

4. Dual Groups and the Fourier Transforms

Let G be a locally compact Hausdorff *abelian* (LCA) group throughout this section. A *character* γ of G is a continuous homomorphism from G into the group $\mathbb{T} = \{e^{it} : -\pi < t \leq \pi\}$. The set of all characters of G is denoted by $\widehat{G}$. With the pointwise multiplication, $\widehat{G}$ becomes an abelian group, which is called the *dual* group of G. It is not so difficult to see that $\widehat{G}$ is an LCA group with respect to the compact-open topology. We will prove this during the discussion of Fourier transforms. For $f \in L^1(G)$, we define the *Fourier transform* $\widehat{f}$ by

$$(4.1) \qquad \widehat{f}(\gamma) =: (f * \gamma)(0) = \int_G f(t)\gamma(-t)dt = \int_G f(t)\overline{\gamma(t)}dt, \qquad \gamma \in \widehat{G}.$$

Therefore, $\widehat{f}$ is a function on $\widehat{G}$. It is easy to see that the Fourier transform converts the multiplication by a character into the translation and vice versa:

$$(4.2) \qquad \widehat{f\gamma} = \widehat{f}_\gamma, \qquad \widehat{f_s}(\gamma) = \widehat{f}(\gamma)\gamma(-s) = \widehat{f}(\gamma)\overline{\gamma(s)}, \qquad \gamma \in \widehat{G},\ s \in G.$$

Exercise 4.1. Show that $\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma)$.

Proposition 4.1. *Let $\{\gamma_i\}$ be a net of characters of an LCA group G. Then the following are equivalent:*

(i) *The net $\{\gamma_i\}$ converges to $\gamma \in \widehat{G}$ pointwise.*
(ii) *For each $f \in L^1(G)$, we have $\lim_i \widehat{f}(\gamma_i) = \widehat{f}(\gamma)$.*
(iii) *The net $\{\gamma_i\}$ converges to $\gamma \in \widehat{G}$ uniformly on every compact sets in G.*

Proof. The direction (i) $\Longrightarrow$ (ii) is a direct consequence of the Lebesgue dominated convergence theorem, and (iii) $\Longrightarrow$ (i) is clear. For the converse (ii) $\Longrightarrow$ (iii), we fix $s \in G$ and $f \in L^1(G)$ such that $\widehat{f}(\gamma) \neq 0$. By Proposition 2.2, there is a neighborhood V of s such that $\|f_t - f_s\|_1 < \epsilon$ for each $t \in V$. By the assumption, there is an i_0 such that

$$|\widehat{f_s}(\gamma_i) - \widehat{f_s}(\gamma)| < \epsilon, \qquad i \geq i_0.$$

Now, for each $t \in V$ and $i \geq i_0$, we have

$$\begin{aligned}|\widehat{f}(\gamma_i)\overline{\gamma_i(t)} - \widehat{f}(\gamma)\overline{\gamma(t)}| &= |\widehat{f_t}(\gamma_i) - \widehat{f_t}(\gamma)| \\ &\leq |\widehat{f_t}(\gamma_i) - \widehat{f_s}(\gamma_i)| + |\widehat{f_s}(\gamma_i) - \widehat{f_s}(\gamma)| + |\widehat{f_s}(\gamma) - \widehat{f_t}(\gamma)| \\ &\leq \|f_t - f_s\|_1 + \epsilon + \|f_s - f_t\|_1 < 3\epsilon.\end{aligned}$$

Since $\widehat{f}(\gamma) \neq 0$, we see that $\{\gamma_i\}$ converges to γ uniformly on V. We have shown that for a given $s \in G$ there is a neighborhood V of s such that $\{\gamma_i\}$ converges to γ uniformly on V. The usual compactness argument completes the proof. □

This proposition says that the compact-open topology on $\widehat{G}$ is the smallest topology with which $\widehat{f}$ is continuous for each $f \in L^1(G)$.

Exercise 4.2. Show that $\widehat{G}$ is a Hausdorff topological group with respect to the compact-open topology.

Before going further, we examine several examples. It is clear that a character γ of $\mathbb{Z}$ is determined by the value of $\gamma(1) \in \mathbb{T}$. Therefore, every character of $\mathbb{Z}$ is of the form

$$\gamma_t : n \mapsto e^{int}, \qquad n \in \mathbb{Z},$$

for some $e^{it} \in \mathbb{T}$, and the correspondence $e^{it} \mapsto \gamma_t : \mathbb{T} \to \widehat{\mathbb{Z}}$ is a group isomorphism. It is clear that this is a homeomorphism by Proposition 4.1.(i).

In order to determine the dual groups of $\mathbb{T}$ and $\mathbb{R}$, we use Exercise 1.1. From this, we see that every nontrivial closed subgroup of $\mathbb{T}$ is a cyclic group consisting of the roots of unity. If γ is a character of $\mathbb{T}$ then the kernel of γ is a cyclic group of order n. It is easy to see that $\gamma(e^{it}) = e^{nit}$ or e^{-nit}. Therefore, every character of $\mathbb{T}$ is of the form

$$\gamma_n : e^{it} \mapsto e^{nit}, \qquad e^{it} \in \mathbb{T},$$

for an integer n.

Exercise 4.3. Show that the correspondence $n \mapsto \gamma_n$ defines a topological group isomorphism from $\mathbb{Z}$ onto $\widehat{\mathbb{T}}$. Show that every character of $\mathbb{R}$ is of the form

$$\gamma_t : s \mapsto e^{ist}, \qquad s \in \mathbb{R},$$

for a real number $t \in \mathbb{R}$. Show also that the map $t \mapsto \gamma_t$ is a topological isomorphism from $\mathbb{R}$ onto $\widehat{\mathbb{R}}$.

In short, we have $\widehat{\mathbb{T}} = \mathbb{Z}$, $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mathbb{R}} = \mathbb{R}$. These relations recover the ordinary Fourier transforms:

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, \qquad f \in L^1(\mathbb{T}),\ n \in \mathbb{Z}, \tag{4.3}$$

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(s)e^{-ist}ds, \qquad f \in L^1(\mathbb{R}),\ t \in \mathbb{R}, \tag{4.4}$$

where the coefficient $\dfrac{1}{\sqrt{2\pi}}$ will be explained later in the context of the inversion formula. Note that we confuse the variables $t \in [-\pi, \pi]$ and $e^{it} \in \mathbb{T}$.

Note that every $\gamma \in \widehat{G}$ induces a *complex homomorphism*

$$f \mapsto \widehat{f}(\gamma), \qquad f \in L^1(G) \tag{4.5}$$

of $L^1(G)$ by Exercise 4.1: It is a linear functional which preserves the multiplication. We will show that every nonzero complex homomorphism of $L^1(G)$ is in this form. To do this, we need the following simple fact about Banach algebras:

Lemma 4.2. *Every complex homomorphism ϕ of a Banach algebra is a bounded linear functional whose norm is at most* 1.

Proof. Assume that $|\phi(x_0)| > \|x_0\|$ for some $x_0 \in A$ and put $\lambda = \phi(x_0)$ and $x = x_0/\lambda$. Because $\|x\| < 1$, the sequence

$$s_n = -x - x^2 - \cdots - x^n, \qquad n = 1, 2, \ldots$$

converges to an element $y \in A$. Since $x + s_n = xs_{n-1}$ we have $x + y = xy$, and so it follows that $\phi(x) + \phi(y) = \phi(x)\phi(y)$. This is a contradiction, because $\phi(x) = 1$. □

For a Banach algebra A, we denote by Δ_A the set of all *nonzero* complex homomorphisms in A^*. Then Δ_A is a subset of the closed unit ball of A^* by the above lemma. It is easy to see that $\Delta_A \cup \{0\}$ is weak*-closed in this unit ball, and so $\Delta_A \cup \{0\}$ is weak*-compact. In the case of $A = L^1(G)$ we have the inclusions

$$\widehat{G} \subseteq \Delta_{L^1(G)} \subseteq L^1(G)^*, \tag{4.6}$$

with the relation (4.5). We also note that the compact-open topology on $\widehat{G}$ is nothing but the weak*-topology on $\Delta_{L^1(G)}$ induced by $L^1(G)^*$ by Proposition 4.1. Thus, we have shown that $\widehat{G}$ is a subset of a compact space.

Theorem 4.3. *Let Φ be a nonzero complex homomorphism of $L^1(G)$. Then there exists a unique $\gamma \in \widehat{G}$ such that $\Phi(f) = \widehat{f}(\gamma)$ for $f \in L^1(G)$.*

Therefore, we see that $\widehat{G} = \Delta_{L^1(G)}$, and so it follows that $\widehat{G}$ is locally compact space whose the one-point compactification is homeomorphic to $\Delta_{L^1(G)} \cup \{0\}$.

Proof. Take $\phi \in L^\infty(G)$ which represents Φ, that is,

$$\Phi(f) = \int f(t)\phi(t)dt, \qquad f \in L^1(G).$$

Then it suffices to show that ϕ is a character. Note that $\|\phi\|_\infty \leq 1$ by Lemma

4.2. For each $g \in L^1(G)$, we have

$$\begin{aligned}\int \Phi(f_t)g(t)dt &= \iint f_t(s)\phi(s)ds g(t)dt \\ &= \int (f * g)(s)\phi(s)ds = \Phi(f * g) = \Phi(f)\Phi(g) \\ &= \int \Phi(f)\phi(t)g(t)dt.\end{aligned}$$

This shows that $\Phi(f_t) = \Phi(f)\phi(t)$ for almost all $t \in G$. Because $t \mapsto \Phi(f_t)$ is continuous, we may assume that ϕ is continuous if we choose $f \in L^1(G)$ so that $\Phi(f) \neq 0$. Therefore, we have

$$\Phi(f)\phi(t) = \Phi(f_t), \qquad f \in L^1(G),\ t \in G.$$

Now, we have

$$\Phi(f)\phi(s)\phi(t) = \Phi(f_s)\phi(t) = \Phi((f_s)_t) = \Phi(f_{s+t}) = \Phi(f)\phi(s+t),$$

and so ϕ is a homomorphism from G into $\mathbb{C}$. From the condition $\|\phi\| \leq 1$, it is easy to see that the range of ϕ is the unit circle. Finally, it is also easy to see that if $\widehat{f}(\gamma) = \widehat{f}(\gamma')$ for each $f \in L^1(G)$ then $\gamma = \gamma'$, since they are continuous. □

Exercise 4.4. Show that $\widehat{f^*} = \overline{\widehat{f}}$ for $f \in L^1(G)$.

It is clear that $\|\widehat{f}\|_\infty \leq \|f\|_1$ for $f \in L^1(G)$. Therefore, we see that the Fourier transform $f \mapsto \widehat{f}$ is a norm-decreasing $*$-homomorphism from $L^1(G)$ into $C_0(\widehat{G})$.

Exercise 4.5. Define $D_n(e^{it}) = \sum_{k=-n}^{+n} e^{ikt}$ for $e^{it} \in \mathbb{T}$. Show that $\|D_n\|_1 \to \infty$ as $n \to \infty$, whereas $\|\widehat{D_n}\|_\infty = 1$ for each $n = 1, 2, \ldots$. [Hint: $D_n(e^{it}) = \sin(n + \frac{1}{2})t / \sin \frac{t}{2}$ for $t \neq 0$.]

By the Stone-Weierstrass theorem, we conclude the following:

Theorem 4.4. *The Fourier transform $f \mapsto \widehat{f}$ is a norm-decreasing $*$-homomorphism from $L^1(G)$ onto a dense $*$-subalgebra of $C_0(\widehat{G})$.*

We digress again to state and prove the Stone-Weierstrass theorem using the Banach-Alaoglu and Krein-Milman theorems.

Stone-Weierstrass Theorem. *Let X be a compact Hausdorff space. Assume that a norm-closed unital $*$-subalgebra A of $C(X)$ separates points: For any two different points x and y of X there is $f \in A$ such that $f(x) \neq f(y)$. Then we have $A = C(X)$.*

Define the *annihilator* $A^{\perp}$ of A by

$$A^{\perp} = \{\mu \in M(X) = C(X)^* : \int f d\mu = 0 \text{ for each } f \in A\}.$$

Applying the Hahn-Banach theorem, it suffices to show that $A^{\perp} = \{0\}$. Assuming contrary, the unit ball K of $A^{\perp}$ has a non-zero extreme point μ by the Banach-Alaoglu and Krein-Milman theorems. We show that μ is a point mass. We denote by E the support of μ and take $g \in A$ with $0 < g(x) < 1$ for $x \in E$. Because A is an algebra, we see that two measure $d\sigma = gd\mu$ and $d\tau = (1-g)d\mu$ are elements of $A^{\perp}$. It is easy to see that $\|\sigma\| + \|\tau\| = 1$, and so μ is the convex combination of $\sigma/\|\sigma\|$ and $\tau/\|\tau\|$. By the extremity of μ in K, we have $\mu = \sigma/\|\sigma\|$ or $gd\mu = \|\sigma\|d\mu$. Therefore, g is a constant function on E. We have thus shown that every real valued function in A is constant on E. (Note that A has constant functions.) Because A is self-adjoint, we see that real-valued functions in A separates points. This implies that E is a one-point set $\{x\}$, and $f(x) = 0$ for each $f \in A$. This contradiction completes the proof of the Stone-Weierstrass theorem.

Exercise 4.6. Let X be a locally compact Hausdorff space and A a $*$-subalgebra of $C_0(X)$ which separates points of X, and assume that for each $x \in X$ there is $f \in A$ such that $f(x) \neq 0$. Show that A is norm-dense in $C_0(X)$. [Hint: Consider the one-point compactification of X.]

Exercise 4.7. Show that the subalgebra $\{\widehat{f} : f \in L^1(G)\}$ of $C_0(\widehat{G})$ satisfies the assumptions of Exercise 4.6.

Later, we will see that the Fourier transform is injective. The range of the Fourier transform is called the *Fourier algebra* of $\widehat{G}$ and denoted by $A(\widehat{G})$, which is a *proper* $*$-subalgebra of $C_0(G)$, in general. The Fourier transform naturally extends to the whole $M(G)$, called the *Fourier-Stieltjes transform*, by the formula:

$$\widehat{\mu}(\gamma) =: (\mu * \gamma)(0) = \int \gamma(-t) d\mu(t), \qquad \gamma \in \widehat{G}. \tag{4.7}$$

Exercise 4.8. Show that $\widehat{\mu}$ is a bounded continuous function on $\widehat{G}$. Show also that $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ and $\widehat{\mu^*} = \overline{\widehat{\mu}}$.

Therefore, we see that the Fourier-Stieltjes transform also gives rise to a $*$-homomorphism from $M(G)$ into the involutive algebra $C_b(\widehat{G})$ of all bounded continuous functions on $\widehat{G}$ with the same operations as in Exercise 2.7. Note that the function $\widehat{\mu}$ need not to vanish at infinity, considering the point mass. The range of the Fourier-Stieltjes transform is said to be the *Fourier-Stieltjes algebra* of $\widehat{G}$, and denoted by $B(\widehat{G})$.

Exercise 4.9. Show that the dual group of a discrete group is compact. Assume that G is compact and so the constant function $1_G \in L^1(G)$. Show that $\widehat{1_G}(0) = 1$ and $\widehat{1_G}(\gamma) = 0$ for $\gamma \neq 0$. Conclude that the dual group of a compact group is discrete.

5. The Gelfand Transforms

The Fourier transform enables us to study the convolutions in terms of the pointwise multiplications, which are much easier to deal with. We generalize the Fourier transform for general commutative Banach algebras. For an element x of a *unital* Banach algebra A, we define the *spectrum* of x by

$$\mathrm{sp}_A(x) = \{\lambda \in \mathbb{C} : x - \lambda 1_A \text{ is not invertible in } A\}.$$

We also define the *spectral radius* $\|x\|_{\mathrm{sp}}$ of x by

$$\|x\|_{\mathrm{sp}} = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(x)\}.$$

Note that the spectrum of a continuous function $f \in C(X)$ is just the range of f. If $\|x\| < 1$ then the sequence $y_n = \sum_0^n x^k$ converges to an element $y \in A$ by the completeness, and the relation

$$y(1-x) = (1-x)y = 1 \tag{5.1}$$

holds. Especially, if $\|x\| < |\lambda|$ then $x - \lambda$ is invertible, and so $\lambda \notin \mathrm{sp}_A(x)$. This means that $\mathrm{sp}_A(x)$ is bounded and

$$\|x\|_{\mathrm{sp}} \leq \|x\|. \tag{5.2}$$

The relation (5.1) also means that the open ball $B(1_A, 1)$ centered at 1_A with radius 1 is contained in the group $\mathcal{G}(A)$ of all invertible elements of A. For $x \in \mathcal{G}(A)$, consider the left multiplication $L_x : A \to A$, which is a homeomorphism with the inverse $L_{x^{-1}}$. Because the image of $B(1_A, 1)$ under L_x is an open neighborhood of x contained in $\mathcal{G}(A)$, we see that $\mathcal{G}(A)$ is open. From this, it is easy to see that the complement of $\mathrm{sp}_A(x)$ is open. From the relation (5.1), we also get

$$\|(1-x)^{-1} - 1^{-1}\| \leq \sum_{1}^{\infty} \|x\|^k = \|x\|(1 - \|x\|)^{-1},$$

and we see that the map $x \mapsto x^{-1}$ is continuous at $1 \in G(A)$. By the similar argument using L_x, the map $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$.

We consider the function

$$f : \lambda \mapsto \rho((x - \lambda)^{-1}), \qquad \lambda \notin \mathrm{sp}_A(x), \tag{5.3}$$

for $\rho \in A^*$. From the relation

$$(x - \lambda')^{-1} - (x - \lambda)^{-1} = (\lambda' - \lambda)(x - \lambda')^{-1}(x - \lambda)^{-1},$$

it follows that

$$\lim_{\lambda' \to \lambda} \frac{f(\lambda') - f(\lambda)}{\lambda' - \lambda} = \rho((x - \lambda)^{-2}),$$

by the continuity of $x \mapsto x^{-1}$. Therefore, the function f in (5.3) in holomorphic on the complement of $\mathrm{sp}_A(x)$. It vanishes at infinity:

$$\lim_{|\lambda| \to \infty} f(\lambda) = \frac{\rho((\lambda^{-1}x - 1_A)^{-1})}{\lambda} = 0.$$

If $\mathrm{sp}_A(x)$ is empty, this function vanishes everywhere by the Liouville theorem, and especially we have $\rho(x^{-1}) = 0$ for each $\rho \in A^*$. It follows that $x^{-1} = 0$ by the Hahn-Banach theorem, a contradiction. Summing up, we have the following:

Proposition 5.1. *The set* $\mathrm{sp}_A(x)$ *is a non-empty compact subset of the complex plane.*

Theorem 5.2 (Gelfand-Mazur). *If every nonzero element of a unital Banach algebra A is invertible then A is isomorphic to the complex field.*

Proof. The assumption implies that $\mathrm{sp}_A(x)$ is a one point set, say $\{\lambda(x)\}$, for each $x \in A$. The singularity of $\lambda(x)1_A - x$ also implies that $x = \lambda(x)1_A$. The map $x \mapsto \lambda(x)$ is an isomorphism from A onto $\mathbb{C}$. □

The following formula relates the spectral radius and the ordinary norm in Banach algebras.

Theorem 5.3 (Spectral Radius Formula). *For an element x of a unital Banach algebra A, we have*

$$\|x\|_{\mathrm{sp}} = \lim_n \|x^n\|^{\frac{1}{n}}. \tag{5.4}$$

Proof. It is easy to see that if $\lambda \in \mathrm{sp}(x)$ then $\lambda^n \in \mathrm{sp}(x^n)$, and so it follows that $|\lambda^n| \leq \|x^n\|$. Hence, we have $|\lambda| \leq \|x^n\|^{\frac{1}{n}}$ for each $\lambda \in \mathrm{sp}(x)$, and so

$$\|x\|_{\mathrm{sp}} \leq \liminf \|x^n\|^{\frac{1}{n}}. \tag{5.5}$$

For each $|\lambda| < \dfrac{1}{\|x\|_{\mathrm{sp}}}$, we have $\dfrac{1}{\lambda} \notin \mathrm{sp}_A(x)$, and so the function

$$g : \lambda \mapsto \rho(1_A - \lambda x)^{-1}, \qquad |\lambda| < \frac{1}{\|x\|_{\mathrm{sp}}}$$

is holomorphic for $\rho \in A^*$ as in the case of (5.3). By the relation (5.1), we have

$$g(\lambda) = \rho\Big(\sum_{n=0}^{\infty}(\lambda x)^n\Big) = \sum_{n=0}^{\infty} \rho(x^n)\lambda^n, \qquad |\lambda| < \frac{1}{\|x\|}. \tag{5.6}$$

This Taylor series expansion also holds for $|\lambda| < \dfrac{1}{\|x\|_{\mathrm{sp}}}$. We fix such a $\lambda \in \mathbb{C}$ with $|\lambda| < \dfrac{1}{\|x\|_{\mathrm{sp}}}$. The relation (5.6) shows that $\{|\rho(\lambda_n x^n)| : n = 1, 2, \ldots\}$ is bounded for each $\rho \in A^*$. We apply the Banach-Steinhaus theorem to the family of bounded linear functionals $\{\rho \mapsto \rho(\lambda^n x^n) : n = 1, 2, \dots\}$ on A^*, to see that there is a number M such that

$$\|\lambda^n x^n\| \le M, \qquad n = 1, 2, \dots$$

From this, we have $\limsup_n \|x^n\|^{\frac{1}{n}} \le \frac{1}{|\lambda|}$. Because λ was arbitrary with $|\lambda| < \frac{1}{\|x\|_{\mathrm{sp}}}$, we see that $\limsup_n \|x^n\|^{\frac{1}{n}} \le \|x\|_{\mathrm{sp}}$. The proof is complete together with (5.5). $\square$

From now on throughout this section, we restrict our attention to *commutative* Banach algebras. For a unital commutative Banach algebra A, we denote by Δ_A the set of all *nonzero* complex homomorphisms of A. For each $x \in A$, we define the *Gelfand transform* $\widehat{x}$ by

$$\widehat{x}(h) = h(x), \quad h \in \Delta_A. \tag{5.7}$$

In the last section, we have seen that $\Delta_A \cup \{0\}$ is a compact space with the weak*-topology induced by A^*, the smallest topology that makes every function $h \mapsto h(x)$ on Δ_A is continuous. This says that every $\widehat{x}$ is continuous function on Δ_A. If A is unital then we have $h(1_A) = 1$ for each $h \in \Delta_A$, and so the zero homomorphism is an isolated point of $\Delta_A \cup \{0\}$. Therefore, Δ_A itself is compact, and the Gelfand transform

$$x \mapsto \widehat{x}, \qquad x \in A$$

is a homomorphism from A into $C(\Delta_A)$. In the case $A = L^1(G)$ with an abelian group G, the Gelfand transform is nothing but the Fourier transform by Theorem 4.3. The space Δ_A is said to be the *maximal ideal space* of A, because every complex homomorphism corresponds to a maximal ideal of A.

Proposition 5.4. *Let A be a unital commutative Banach algebra. Then the kernel of $h \in \Delta_A$ is a maximal ideal of A. Conversely, every maximal ideal is the kernel of some $h \in \Delta_A$.*

Proof. It is clear that $\operatorname{Ker} h$ is a maximal ideal of A for each complex homomorphism h of A, because it is of codimension one. Assume that I is a maximal ideal. Because the set $\mathcal{G}(A)$ of all invertible elements is open, we see that $\overline{I} \cap \mathcal{G}(A) = \emptyset$, especially $\overline{I}$ is a proper ideal. From the maximality of I, we see that I is a closed ideal, and A/I is also a Banach algebra. For the converse direction, it suffices to show that every nonzero element of A/I is invertible by the Gelfand-Mazur theorem. For $x \in A \setminus I$, put

$$J = \{ax + y : a \in A, y \in I\}.$$

Because J is an ideal of A which contains I strictly, we have $J = A$. This says that for each $x \in A$, there are $a \in A$ and $y \in I$ such that $ax + y = 1_A$, and so $\pi(a)\pi(x) = \pi(1_A)$, where $\pi : A \to A/I$ is the quotient map. □

Corollary 5.5. *Let x be an element of a unital commutative Banach algebra A. Then the spectrum $\mathrm{sp}_A(x)$ coincides with the range of the function $\widehat{x}$.*

Proof. We show that $\lambda \in \mathrm{sp}_A(x)$ if and only if $\lambda = h(x)$ for a $h \in \Delta_A$, and it suffices to show that x is singular if and only if $h(x) = 0$ for some $h \in \Delta_A$. If x is invertible then $h(x)$ is also invertible, and so is nonzero for each $h \in \Delta_A$. If x is singular then the set $\{ax : a \in A\}$ is a proper ideal. Using the Maximal Principle, we see that this ideal is contained in a maximal ideal, which is the kernel of an $h \in \Delta_A$. □

The above corollary shows that

$$\|x\|_{\mathrm{sp}} = \|\widehat{x}\|_\infty, \qquad x \in A. \tag{5.8}$$

Combining with the spectral radius formula, we also have

$$\|\widehat{x}\|_\infty = \lim_n \|x^n\|^{\frac{1}{n}}, \qquad x \in A. \tag{5.9}$$

Note that the relation (5.8) together with (5.2) shows that the Gelfand transform is a norm-decreasing homomorphism from A into $C(\Delta_A)$. A commutative Banach algebra is said to be *semi-simple* if the Gelfand transformation is injective. We will see later that $L^1(G)$ is semi-simple for each abelian group G. If G is not discrete then $L^1(G)$ is not unital. In this case, we consider the subspace $L^1(G) + \mathbb{C}\delta_e$ of $M(G)$, which is a unital commutative Banach algebra, In this way, we see that the formulae (5.4) and (5.9) hold for every $f \in L^1(G)$ even if $L^1(G)$ is not unital.

More generally, if an involutive Banach algebra A has no identity, then we may embed A into a unital algebra A_I. This is nothing but the direct sum of A and the complex field $\mathbb{C}$ endowed with

$$(5.10) \qquad \begin{aligned} (x,a)(y,b) &= (xy + bx + ay, ab), \\ (x,a)^* &= (x^*, \overline{a}), \\ \|(x,a)\| &= \|x\| + |a|. \end{aligned}$$

Exercise 5.1. Show that A_I is an involutive Banach algebra with the identity $(0,1)$. Show also that A is an ideal of A_I under the identification of $x \mapsto (x,0)$.

Exercise 5.2. Show that there is a $*$-isomorphism from the C^*-algebra $C_0(X)_I$ onto $C(X \cup \{\infty\})$, where $X \cup \{\infty\}$ denotes the one-point compactification of X.

We note the typical relation $\|\overline{f}f\|_\infty = \|f\|_\infty^2$ for $f \in C_0(X)$, which relates the involution and norm structures, where X is a locally compact Hausdorff space. An involutive Banach algebra A is said to be a *C^*-algebra* if the relation

$$(5.11) \qquad \|x^*x\| = \|x\|^2, \qquad x \in A$$

is satisfied.

Exercise 5.3. Show that the L^1-norm of a discrete group satisfies (5.11) only if G is the trivial group. [Hint: Consider the function $\chi_e + a\chi_s + b\chi_{s^{-1}}$.]

If x is a *normal* element of a C^*-algebra, that is, $x^*x = xx^*$, then the relation (5.2) becomes

$$(5.12) \qquad \|x\|_{sp} = \|x\|, \qquad \text{for normal } x \in A.$$

Indeed, for a normal element $x \in A$, we have

$$\|x^{2n}\|^2 = \|(x^{2n})^*(x^{2n})\| = \|(x^*x)^n(x^*x)^n\| = \|(x^*x)^n\|^2.$$

Therefore, we have $\|x^{2n}\| = \|(x^n)^*x^n\| = \|x^n\|^2$. By induction, we see that $\|x^m\| = \|x\|^m$ for each $m = 2^n, n = 1, 2, \dots$. Therefore, we have

$$\|x\|_{\text{sp}} = \lim_n \|x^n\|^{\frac{1}{n}} = \|x\|.$$

Note that every element is normal if A is commutative. Therefore, the relations (5.8) and (5.12) say that the Gelfand transform is an isometry if A is a unital commutative C^*-algebra. Actually, the Gelfand transform completely determines the structures of a commutative C^*-algebra.

Theorem 5.6 (Gelfand-Naimark). *Let A be a unital commutative C^*-algebra. Then the Gelfand transform is a $*$-preserving isometric isomorphism from A onto $C(\Delta_A)$.*

Proof. We first show that the Gelfand transform preserves the involution: $\widehat{x^*} = \overline{\widehat{x}}$ for $x \in A$, or equivalently

$$h(x^*) = \overline{h(x)}, \qquad x \in A, \ h \in \Delta_A.$$

Assume that u is *self-adjoint*, that is $u^* = u$. If $h(u) = a + ib$ with $a, b \in \mathbb{R}$, then we have

$$h(u + is1_A) = a + i(b + s), \qquad s \in \mathbb{R}.$$

Therefore, it follows that

$$\begin{aligned} a^2 + (b+s)^2 &= |h(u + is1_A)|^2 \le \|u + is1_A\|^2 \\ &= \|(u + is1_A)^*(u + is1_A)\| = \|u^2 + s^2 1_A\| \le \|u\|^2 + s^2, \end{aligned}$$

for each $s \in \mathbb{R}$. From this, we see that $b = 0$, and $h(u)$ is real. Note that every element x is written by $x = u + iv$ with self-adjoint elements u and v:

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2i}. \tag{5.13}$$

Now, we have $h(x^*) = h(u - iv) = h(u) - ih(v) = \overline{h(x)}$. Therefore, the range $\widehat{A}$ of the Gelfand transform is self-adjoint subalgebra of $C(\Delta_A)$, and so it is dense by the Stone-Weierstrass theorem. We have already seen that the Gelfand transform is an isometry, and from this we conclude that the range $\widehat{A}$ is complete, and so it is closed in $C(\Delta_A)$. Therefore, we have $\widehat{A} = C(\Delta_A)$. $\square$

Exercise 5.4. If the Gelfand transform of a commutative involutive Banach algebra A is an isometry then show that A is a C^*-algebra. If the Gelfand transform is bijective then there exists $\epsilon > 0$ such that $\|\widehat{x}\|_\infty \geq \epsilon\|x\|$ for each $x \in A$.

Exercise 5.5. Show that the algebra $C_b(X)$ becomes a unital C^*-algebra with the same operations as in Exercise 2.7. The maximal ideal space of $C_b(X)$ will be denoted by βX.

Exercise 5.6. Let h_x be a complex homomorphism on $C_b(X)$ defined by the formula $h_x(f) = f(x)$, for $f \in C_b(X)$. Show that the mapping $x \mapsto h_x$ is a homeomorphism from X onto a dense subspace of βX.

For a C^*-algebra A, the norm in (5.10) does not satisfy the C^*-norm condition $\|x^*x\|^2 = \|x\|^2$ in general. But, we can redefine a norm on A_I for a non-unital C^*-algebra A so that A_I becomes a C^*-algebra. Recall that A is an ideal of A_I. Therefore, every left translation $L_x : y \mapsto xy$ defines a linear map from A into A, for each $x \in A$. It is easy to see that $\|L_x\| = \|x\|$ for $x \in A$. If we define

$$\|x\| = \|L_x\| \qquad x \in A_I, \tag{5.14}$$

then one can show that this defines a C^*-norm on A_I.

Exercise 5.7. Let A be a non-unital commutative C^*-algebra. Show that there exists a unique $h_0 \in \Delta_{A_I}$ such that $h_0(x) = 0$ for each $x \in A$. Conclude that A can be identified with the ideal $\{\widehat{x} \in C(\Delta_{A_I}) : \widehat{x}(h_0) = 0\}$ of $C(\Delta_{A_I})$.

6. Inversion Formula and the Plancherel Transforms

If $f \in L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$ then the Riesz-Fisher theorem says that

$$f(t) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n)e^{int}, \tag{6.1}$$

in the sense of L^2-convergence. If $\widehat{f} \in \ell^1(\mathbb{Z})$ then it is also easy to see that the formula (6.1) also holds in the sense L^1-convergence. In this section, we prove the generalization of (6.1) for arbitrary abelian groups. To do this, we first consider the dual of the Fourier transform. For $\mu \in M(\widehat{G}) = C_0(\widehat{G})^*$, we define the function ϕ_μ on G by

$$\phi_\mu(t) = \int_{\widehat{G}} \gamma(t)d\mu(\gamma), \qquad t \in G. \tag{6.2}$$

Then we have $\phi_\mu \in L^\infty(G)$ with $\|\phi_\mu\|_\infty \leq \|\mu\|$, and

$$\langle \overline{\phi_\mu}, f\rangle = \int_G f(t) \int_{\widehat{G}} \overline{\gamma(t)d\mu(\gamma)}dt = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{d\mu(\gamma)} = \langle \overline{\mu}, \widehat{f}\rangle, \tag{6.3}$$

for each $f \in L^1(G)$. Therefore, the map $\mu \mapsto \phi_\mu : C_0(\widehat{G})^* \to L^1(G)^*$ is the dual of the Fourier transform if we adjust the $\langle L^1(G), L^\infty(G)\rangle$ and $\langle C_0(G), M(G)\rangle$ dualities by the sesqui-linear forms. Note that

$$\langle \phi_\mu, \overline{f}\rangle = \langle \mu, \overline{\widehat{f}}\,\rangle = \langle \mu, \widehat{f^*}\rangle, \tag{6.4}$$

If μ is a positive measure then

$$\langle \phi_\mu, \overline{f}^* * \overline{f}\rangle = \langle \phi_\mu, \overline{f^* * f}\rangle = \langle \mu, |\widehat{f}|^2\rangle \geq 0, \qquad f \in L^1(G),$$

and so ϕ_μ is positive definite. It is also easy to see that ϕ_μ is continuous. Furthermore, the correspondence $\mu \mapsto \phi_\mu$ is injective: If $\phi_\mu = 0$ then $\langle \overline{\mu}, \widehat{f}\rangle = 0$ for each $f \in L^1(G)$ by (6.3). Since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, we see that $\mu = 0$. In other word, we have

$$\mu \in M(\widehat{G}), \int_{\widehat{G}} \gamma(t)d\mu(\gamma) = 0 \ \text{ for each } t \in G \Longrightarrow \mu = 0. \tag{6.5}$$

Theorem 6.1 (Bochner). *The map $\mu \mapsto \phi_\mu : M(\widehat{G})^+ \to P(G)$ is a norm-preserving bijection.*

Proof. Let $\phi \in P(G)$. We may assume that $\|\phi\|_\infty = 1$. Put $h = f^* * f$. Then we have

$$|\langle \phi, f\rangle|^2 \leq \langle \phi, h\rangle \leq \langle \phi, h * h\rangle^{\frac{1}{2}} \leq \cdots \leq \langle \phi, h^{2^n}\rangle^{2^{-n}} \leq \|h^{2^n}\|_1^{2^{-n}},$$

for each $n = 1, 2, \ldots$ by (3.10), and so it follows that

$$|\langle \phi, f\rangle|^2 \leq \|\widehat{h}\|_\infty = \|\widehat{f}\|_\infty^2,$$

by the spectral radius formula (5.9). This says that $\widehat{f} = \widehat{g}$ implies $\langle \phi, f \rangle = \langle \phi, g \rangle$, and so $\overline{\phi}$ may be considered as a bounded linear functional on $A(\widehat{G})$, which extends to the whole $C_0(\widehat{G})$. Therefore, there is $\overline{\mu} \in M(\widehat{G})$ with $\|\phi\| = \|\mu\|$ such that $\langle \overline{\phi}, f \rangle = \langle \overline{\mu}, \widehat{f} \rangle$ for $f \in L^1(G)$, and so we see that $\phi = \phi_\mu$ by (6.3). This measure μ is positive, because

$$\|\phi\| = \phi(0) = \int_{\widehat{G}} d\mu(\gamma) = \mu(\widehat{G}) \leq \|\mu\| = \|\phi\|,$$

This also shows that $\|\phi_\mu\| = \|\mu\|$. □

Theorem 6.2 (Inversion Formula). *Let G be an LCA group. Then we have the following:*

(i) *For $f \in L^1(G) \cap P(G)$, we have $\widehat{f} \in L^1(\widehat{G})$.*

(ii) *There is a normalization of the Haar measure $d\gamma$ on $\widehat{G}$ such that the formula*

$$f(t) = \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(t)d\gamma$$

holds for each $f \in L^1(G) \cap P(G)$.

Proof. Note that every $f \in L^1(G) \cap P(G)$ corresponds to a unique positive measure $\mu_f \in M(\widehat{G})$ with the formula

$$f(t) = \int_{\widehat{G}} \gamma(t)d\mu_f(\gamma),$$

by Theorem 6.1. We fix $k \in C_c(\widehat{G})$ with the compact support K. For each $\gamma \in K$, we may choose $u \in C_c(G)$ with $\widehat{u}(\gamma) \neq 0$, and so there are finitely many $u_1, \dots, u_n \in C_c(G)$ such that the Fourier transform $\widehat{f}$ of the function $f = u_1 * \widetilde{u_1} + \cdots + u_n * \widetilde{u_n}$ is positive on K. By Proposition 3.2, we have $f \in C_c(G) \cap P(G)$. We have shown that for each $k \in C_c(\widehat{G})$ there is $f \in C_c(G) \cap P(G)$ such that $\widehat{f}(\gamma) > 0$ for $\gamma \in \operatorname{supp} k$. Define

$$T(k) = \int_{\widehat{G}} \frac{k}{\widehat{f}} d\mu_f, \qquad k \in C_c(\widehat{G}).$$

In order to show that T is well-defined, we choose $h \in L^1(G)$. Then we have

$$\int_{\widehat{G}} \widehat{h} d\mu_f = \int_{\widehat{G}} \int_G h(t)\overline{\gamma(t)} dt d\mu_f(\gamma) = \int_G h(t) f(-t) dt = (h * f)(0).$$

If $g \in C_c(G) \cap P(G)$ with $\widehat{g} > 0$ on the support of k, then we have

$$\int \widehat{h}\widehat{f} d\mu_g = ((h * f) * g)(0) = ((h * g) * f)(0) = \int \widehat{h}\widehat{g} d\mu_f, \qquad h \in L^1(G),$$

and so it follows that

$$\int \frac{k}{\widehat{f}} d\mu_f = \int \frac{k}{\widehat{f}\widehat{g}} \widehat{g} d\mu_f = \int \frac{k}{\widehat{f}\widehat{g}} \widehat{f} d\mu_g = \int \frac{k}{\widehat{g}} d\mu_g.$$

Therefore, T is well-defined. It is easy to see that T is a nonzero positive functional on $C_c(\widehat{G})$, which is invariant under the translation. Therefore, we have

$$T(k) = \int_{\widehat{G}} k d\gamma, \qquad k \in C_c(\widehat{G}),$$

for a translation invariant measure $d\gamma$ on $\widehat{G}$. If $f \in L^1(G) \cap P(G)$ then we have

$$\int k d\mu_f = \int \frac{k\widehat{f}}{\widehat{g}} d\mu_g = T(k\widehat{f}) = \int k\widehat{f} d\gamma, \qquad k \in C_c(G).$$

Hence, we have $d\mu_f = \widehat{f} d\gamma$, and it follows that $\widehat{f} \in L^1(\widehat{G})$ since μ_f is a finite measure. Furthermore, we have

$$f(t) = \int_{\widehat{G}} \gamma(t) d\mu_f(\gamma) = \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(t) d\gamma. \quad \square$$

Exercise 6.1. If G is discrete or compact then our conventions in §1 are the right normalizations for the inversion formulae. See Exercise 4.9.

Exercise 6.2. Normalize the Haar measure on $\widehat{\mathbb{R}} = \mathbb{R}$ so that the inversion formula holds. [Hint: Consider the function $e^{-|t|}$ or $e^{-t^2/2}$.]

Exercise 6.3. We denote by $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ the cyclic group of order n. Show that every character of $\mathbb{Z}_n$ is given by

$$\gamma_p : j \mapsto \zeta^{jp}, \qquad j \in \mathbb{Z}_n,$$

where $\zeta = e^{\frac{2\pi i}{n}}$ is the n-th root of unity. Show also that $p \mapsto \gamma_p$ is a group isomorphism from $\mathbb{Z}_n$ onto $\widehat{\mathbb{Z}_n}$. We normalize the Haar measure on $\mathbb{Z}_n$ and $\widehat{\mathbb{Z}_n}$ as compact and discrete groups, respectively. Prove the inversion formula by a direct computation.

Exercise 6.4. Show that the dual group of $G \oplus F$ is isomorphic to $\widehat{G} \oplus \widehat{H}$.

By the above two exercises, we see that the Fourier transform of a finite abelian group G of order n gives rise to a $*$-isomorphism from $\ell^1(G)$ onto the algebra $\mathbb{C}^n$ with the point multiplication and involution. Therefore, the group algebras of finite abelian groups are determined by their orders. It should be noted that this is not an isometry by Exercises 5.3 and 5.4.

Exercise 6.5. Consider the group $\mathbb{Z}_2$. If $f = a\chi_0 + b\chi_1$ then show that

$$\|\widehat{f}\|_\infty = \max\{|a+b|, |a-b|\}.$$

Show also that this is identical with the operator norm of the 2×2 matrix A_f in (3.5).

From now on throughout this chapter, we always assume that the Haar measures are normalized so that the inversion formula holds. Now, we generalize the Riesz-Fisher theorem for arbitrary abelian groups. Since $L^2(G)$ is not contained in $L^1(G)$ in general, the Fourier transform does not make sense for L^2-functions. The point is that the Fourier transform defines an L^2-isometry on $L^1(G) \cap L^2(G)$.

Theorem 6.3. *For each $f \in L^1(G) \cap L^2(G)$, we have $\widehat{f} \in L^2(\widehat{G})$ with $\|f\|_2 = \|\widehat{f}\|_2$. The range $\mathcal{R} = \{\widehat{f} : f \in L^1(G) \cap L^2(G)\}$ is dense in $L^2(\widehat{G})$.*

Proof. Note that $f * \widetilde{f} \in L^1(G) \cap P(G)$, and so we have

$$\|f\|^2 = \int_G f(t)\overline{f(t)}dt = (f * \widetilde{f})(0) = \int_{\widehat{G}} \widehat{f * \widetilde{f}}(\gamma)d\gamma = \int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\gamma = \|\widehat{f}\|^2,$$

by the inversion formula. Let $\psi \in L^2(\widehat{G})$ such that

$$(6.6) \qquad \langle \phi, \psi \rangle = \int \phi(\gamma)\overline{\psi(\gamma)}d\gamma = 0, \qquad \phi \in \mathcal{R}.$$

In order to show that $\mathcal{R}$ is dense in $L^2(\widehat{G})$, it suffice to show that $\psi = 0$ in $L^2(\widehat{G})$. If $\phi \in \mathcal{R}$ with $\phi = \widehat{f}$ then the function $\gamma \mapsto \phi(\gamma)\gamma(t) = \widehat{f_{-t}}(\gamma)$ also lies in $\mathcal{R}$, and so we have

$$\int \gamma(t)\phi(\gamma)\overline{\psi(\gamma)}d\gamma = 0, \qquad \phi \in \mathcal{R},\ t \in G,$$

by (6.6). This implies that

$$\phi\overline{\psi} = 0, \qquad \phi \in \mathcal{R}, \tag{6.7}$$

by (6.5), since $\phi(\gamma)\overline{\psi(\gamma)}d\gamma$ is a finite measure. If $\phi \in \mathcal{R}$ with $\phi = \widehat{f}$ then the translation $\phi_\gamma = \widehat{f\gamma} \in \mathcal{R}$ for each $\gamma \in \widehat{G}$. Therefore, every $\gamma \in \widehat{G}$ corresponds to a $\phi \in \mathcal{R}$ such that ϕ is nonzero on a neighborhood of γ. Now, the relation (6.7) says that $\psi = 0$ in $L^2(\widehat{G})$. □

Because $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, we see that there is a unique isometry from $L^2(G)$ onto $L^2(\widehat{G})$ which coincides with the Fourier transform on the dense subspace $L^1(G) \cap L^2(G)$. This isometry is called the *Plancherel transform*, denoted by the same symbol $f \mapsto \widehat{f}$ as the Fourier transform.

Exercise 6.6. Show that every isometric isomorphism between inner product spaces preserves the inner products. Deduce the Parseval identity:

$$\int_G f(t)\overline{g(t)}dt = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}d\gamma, \qquad f, g \in L^2(G). \tag{6.8}$$

Put $h(t) = \overline{g(t)}\,\gamma_0(t)$ then $\overline{\widehat{h}(\gamma)} = \widehat{g}(\gamma_0 - \gamma)$, and so we have

$$\widehat{fg}(\gamma_0) = \int_G f(t)\overline{h(t)}dt = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{h}(\gamma)}d\gamma = \int_{\widehat{G}} \widehat{f}(\gamma)\widehat{g}(\gamma_0 - \gamma)d\gamma = (\widehat{f} * \widehat{g})(\gamma_0).$$

In short, we have

$$\widehat{fg} = \widehat{f} * \widehat{g}, \qquad f, g \in L^2(G). \tag{6.9}$$

Compare with Exercise 4.1. This show that if $\widehat{f}, \widehat{g} \in L^2(\widehat{G})$ then $\widehat{f} * \widehat{g} \in A(\widehat{G})$, since $fg \in L^1(G)$ for $f, g \in L^2(G)$. Conversely, every L^1-function is a pointwise product of two L^2-function. Therefore, we have the following characterization of the Fourier algebra:

$$A(\widehat{G}) = \{\phi * \psi : \phi, \psi \in L^2(\widehat{G})\}. \tag{6.10}$$

Exercise 6.7. For any compact subset K of $\widehat{G}$ there is $f \in L^1(G)$ such that $\widehat{f} \equiv 1$ on K. [Hint: Exercise 2.9.]

7. Pontryagin-van Kampen Duality

Let G be an LCA group with the dual group $\widehat{G}$. Then every element $t \in G$ defines the function

$$e_t : \gamma \mapsto \gamma(t), \qquad \gamma \in \widehat{G}$$

from $\widehat{G}$ into the circle group $\mathbb{T}$. This is nothing but the Fourier-Stieltjes transform of the point mass δ_{-t}. By Exercise 4.8, e_t is a continuous homomorphism from $\widehat{G}$ into $\mathbb{T}$, and so gives us a character of $\widehat{G}$. It is easy to see that the map

$$e : t \mapsto e_t : G \to \widehat{\widehat{G}} \qquad t \in G, \tag{7.1}$$

called the *evaluation map*, is a group homomorphism. The Pontryagin-van Kampen duality theorem says that this is a homeomorphic group isomorphism from G onto the double dual $\widehat{\widehat{G}}$. Thus, every LCA group is the dual of its dual group. This enables us to replace the pair $(G, \widehat{G})$ by $(\widehat{G}, G)$ in our previous discussions.

Lemma 7.1. *For each neighborhood V of 0 in G, there exists a compact subset K of $\widehat{G}$ with the property:*

$$|\gamma(t) - 1| < \frac{1}{3} \text{ for each } \gamma \in K \implies t \in V. \tag{7.2}$$

Proof. Choose a compact neighborhood W such that $W - W \subseteq V$. Put $f = \dfrac{1}{\sqrt{\mu(W)}}\chi_W$ and $g = f * \widetilde{f}$, where μ is the Haar measure. Then $g \in P(G)$ by Proposition 3.2 with $\operatorname{supp} g \subseteq V$. By the inversion formula, we have $\displaystyle\int \widehat{g}(\gamma)d\gamma = g(0) = 1$ and $\widehat{g} = |\widehat{f}|^2 \geq 0$. Therefore, there is a compact subset K of $\widehat{G}$ such that

$$\int_K \widehat{g}(\gamma)d\gamma > \frac{2}{3} \qquad \text{and} \qquad \int_{\widehat{G}\setminus K} \widehat{g}(\gamma)d\gamma \leq \frac{1}{3}.$$

If $t \in G$ satisfies the assumption of (7.2) then $\operatorname{Re}\gamma(t) > \dfrac{2}{3}$ for each $\gamma \in K$, and so it follows that

$$\operatorname{Re} \int_K \widehat{g}(\gamma)\gamma(t)d\gamma > \frac{2}{3}\int_K \widehat{g}(\gamma)d\gamma > \frac{4}{9}.$$

Because $\left|\int_{\widehat{G}\setminus K} \widehat{g}(\gamma)\gamma(t)d\gamma\right| \le \int_{\widehat{G}\setminus K} |\widehat{g}(\gamma)|d\gamma \le \frac{1}{3}$, we have

$$g(t) = \int_{\widehat{G}} \widehat{g}(\gamma)\gamma(t)d\gamma = \int_K \widehat{g}(\gamma)\gamma(t)d\gamma + \int_{\widehat{G}\setminus K} \widehat{g}(\gamma)\gamma(t)d\gamma > \frac{1}{9}.$$

Therefore, we have $t \in \operatorname{supp} g \subseteq V$. □

As an immediate consequence, we have the following:

Theorem 7.2. *Every LCA group G admits sufficiently many characters to separate points of G: If $s \ne t$ in G then there is a character $\gamma \in \widehat{G}$ such that $\gamma(s) \ne \gamma(t)$.*

Proof. Assume that $\gamma(t) = 1$ for each $\gamma \in \widehat{G}$. Lemma 7.1 shows that t lies in every neighborhood of 0, and so $t = 0$. □

Lemma 7.3. *For each open subset U in $\widehat{G}$, there is a nonzero $\widehat{f} \in A(\widehat{G})$ such that $\widehat{f} = 0$ on $\widehat{G} \setminus U$.*

Proof. Take a compact set K with positive measure and a compact neighborhood V such that $K + V \subseteq U$. Then the function $\widehat{\chi_K} * \widehat{\chi_V}$ lies in $A(\widehat{G})$ by (6.10). This function satisfies the required conditions. □

Theorem 7.4. *The evaluation map $t \mapsto e_t$ is a homeomorphic group isomorphism from G onto $\widehat{\widehat{G}}$.*

Proof. Note that Theorem 7.2 says that the evaluation map is an isomorphism. If $t_i \to 0$ in G then $\gamma(t_i) \to 1$ for each $\gamma \in \widehat{G}$, and so we have $e_{t_i} \to 1$ in $\widehat{\widehat{G}}$ by Proposition 4.1.(i). Conversely, assume that $e_{t_i} \to 1$ in $\widehat{\widehat{G}}$ and V is a given neighborhood in G. Take a compact subset K of $\widehat{G}$ with the property (7.2). Then Proposition 4.1.(iii) says that t_i satisfies the assumption of (7.2) for sufficiently large i. Therefore, $t_i \in V$ for these sufficiently large i's, and so we have $t_i \to 0$. Thus, the evaluation map is a homeomorphism from G onto the range $e(G)$ by the translations. Especially, $e(G)$ is locally compact.

We proceed to show that $e(G)$ is dense in $\widehat{\widehat{G}}$. Assuming the contrary, there is a nonzero $F \in A(\widehat{\widehat{G}})$ such that $F \equiv 0$ on $e(G)$ by Lemma 7.3. Note that F is the Fourier transform of some $\phi \in L^1(\widehat{G})$. Now, we have

$$\int_{\widehat{G}} \overline{\gamma(t)}\phi(\gamma)d\gamma = \int_{\widehat{G}} \phi(\gamma)\overline{e_t(\gamma)}d\gamma = \widehat{\phi}(e_t) = F(e_t) = 0,$$

for each $t \in G$. By (6.5), we have $\phi = 0$, and so $F = 0$. This shows that $e(G)$ is a dense subgroup of $\widehat{\widehat{G}}$ with respect to the relative topology. From this, it is easy to conclude that $e(G) = \widehat{\widehat{G}}$. Indeed, if $e(G)$ is a proper dense subgroup of $\widehat{\widehat{G}}$ then we see that for any neighborhood U of $\widehat{\widehat{G}}$, the set $U \cap e(G)$ is not compact with respect to the relative topology, and so it follows that $e(G)$ is not locally compact. □

We consider several dual interpretations of the previous results. First of all, the relation (6.5) shows that

$$\mu \in M(G), \int_G \gamma(t)d\mu(t) = 0 \text{ for each } \gamma \in \widehat{G} \implies \mu = 0.$$

In other word, we have

$$\mu \in M(G),\ \widehat{\mu} = 0 \implies \mu = 0. \tag{7.3}$$

In particular, we also have

$$f \in L^1(G),\ \widehat{f} = 0 \implies f = 0. \tag{7.4}$$

This says that the Fourier transform is injective, and so $L^1(G)$ is semi-simple. Note that (7.3) shows that $M(G)$ is also semi-simple, because every character induces a homomorphism of M(G) by Exercise 4.8, as well as of $L^1(G)$.

For a measure $\mu \in M(G)$, the map ϕ_μ defined in (6.2) is a function on $\widehat{G}$:

$$\phi_\mu(\gamma) = \int_G \gamma(t)d\mu(t), \qquad \gamma \in \widehat{G}. \tag{7.5}$$

If μ is a positive measure of G then we have

$$\phi_\mu(\gamma) = \overline{\int \overline{\gamma(t)}d\mu(t)} = \overline{\widehat{\mu}(\gamma)} = \widehat{\mu^*}(\gamma),$$

and so, it follows that

$$\phi_\mu = \widehat{\mu^*}, \qquad \mu \in M(G)^+. \tag{7.6}$$

If $\phi \in P(\widehat{G})$ then the Bochner theorem says that $\phi = \widehat{\mu^*}$ for a $\mu \in M(G)^+$. Therefore, $\phi \in B(\widehat{G})$. Conversely, assume that $\phi \in B(\widehat{G})$ with $\phi = \widehat{\mu}$. If we decompose $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ with $\mu_j \in M(G)^+$ for $j = 1, 2, 3, 4$, then it follows that

$$\phi = \widehat{\mu} = (\widehat{\mu_1} - \widehat{\mu_2}) + i(\widehat{\mu_3} - \widehat{\mu_4}) = (\phi_{\mu_1^*} - \phi_{\mu_2^*}) + i(\phi_{\mu_3^*} - \phi_{\mu_4^*}).$$

If $\mu \in M(G)^+$ then $\mu^* \in M(G)^+$, and so ϕ_{μ^*} is a positive definite. We have thus shown that

$$B(\widehat{G}) = \{(\phi_1 - \phi_2) + i(\phi_3 - \phi_4) : \phi_j \in P(\widehat{G}),\ j = 1, 2, 3, 4\}. \tag{7.7}$$

Of course, the relation (7.7) is valid for the group G itself. Therefore, we see that the inversion formula holds for $f \in L^1(G) \cap B(G)$.

For each $\mu \in M(G)$ with $\widehat{\mu} \in L^1(\widehat{G})$, put $f(t) = \widehat{\widehat{\mu}}(-t)$, $t \in G$. Because $\widehat{\mu} \in L^1(\widehat{G}) \cap B(\widehat{G})$, we apply the inversion theorem to see that $f \in L^1(G)$ and

$$\widehat{f}(\gamma) = \int_G \widehat{\widehat{\mu}}(-t)\overline{\gamma(t)}dt = \int_G \widehat{\widehat{\mu}}(t)\gamma(t)dt = \widehat{\mu}(\gamma).$$

Therefore, we see that $\mu = f \in L^1(G)$ and

$$f(t) = \widehat{\widehat{\mu}}(-t) = \int_{\widehat{G}} \widehat{\mu}(t)\overline{\gamma(-t)}d\gamma = \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(t)d\gamma.$$

Especially, we have shown that the inversion formula holds whenever $\widehat{f}$ is an L^1-function:

$$f \in L^1(G), \widehat{f} \in L^1(\widehat{G}) \implies f(t) = \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(t)d\gamma. \tag{7.8}$$

If $G = \mathbb{T}$ then we have

$$f \in L^1(\mathbb{T}), \widehat{f} \in \ell^1(\mathbb{Z}) \implies f(t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{int}, \tag{7.9}$$

where the equality and the infinite sum should be, of course, taken in the L^1-space.

For a function f on G, we define the *convolution operator* λ_f by

$$\lambda(f) : \xi \mapsto f * \xi, \qquad \xi \in L^2(G). \tag{7.10}$$

The formula $\widehat{f * \xi} = \widehat{f}\,\widehat{\xi}$ says that the operator $\lambda(f)$ is converted to the (pointwise) *multiplication operator* $m_{\widehat{f}}$ through the Plancherel transform. Therefore, the Fourier transform may be understood as a diagonalization process, since the multiplication operators induce diagonal matrices. The following easy relation

$$f * \gamma = \widehat{f}(\gamma)\gamma, \qquad \gamma \in \widehat{G},\ f \in L^1(G) \tag{7.11}$$

says that every $\gamma \in \widehat{G}$ is a common eigenvector of the family $\{\lambda(f) : f \in L^1(G)\}$ with the eigenvalue $\widehat{f}(\gamma)$ in an informal sense. The situation becomes clear if G is *compact*.

Assume that G is compact. For each characteristic function $\chi_\gamma \in \ell^2(\widehat{G})$, we can take $\xi_\gamma \in L^2(G) \subseteq L^1(G)$ such that $\widehat{\xi_\gamma} = \chi_\gamma$ by the Plancherel theorem. Then we have

$$\xi_\gamma(t) = \sum_{\gamma' \in \widehat{G}} \chi_\gamma(\gamma')\gamma'(t) = \gamma(t)$$

by the inversion formula, and so $\widehat{\gamma} = \chi_\gamma$ for each $f \in \widehat{G}$. Therefore, the correspondence $\gamma \leftrightarrow \chi_\gamma$ gives us complete orthonormal bases for $L^2(G)$ and $\ell^2(\widehat{G})$, respectively. The relation (7.11) says that $\lambda(f)$ is a *bounded* linear operator with $\|\lambda(f)\| = \|\widehat{f}\|_\infty$. By the completeness of $\widehat{G}$ in $L^2(G)$, we have

$$f = \sum_{\gamma \in \widehat{G}} \langle f, \gamma \rangle \gamma = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\gamma, \qquad f \in L^2(G) \tag{7.12}$$

It should be noted that the right sides of (7.12) need not to define a function unless $\widehat{f} \in \ell^1(\widehat{G})$. This is a distinction with the L^1-inversion formula (7.8). The formula (7.12) just means that the finite sums of the right sides converge to the function f in the L^2-norm. Therefore, it is absurd to conclude that the formula (7.12) holds pointwise even in the sense of almost everywhere. Note that the equalities in (7.8) and (7.9) hold for almost all $t \in G$. A finite linear combination of characters is called a *trigonometric polynomial*. The relation (7.12) says that trigonometric polynomials are dense in $L^2(G)$ if G is compact.

Exercise 7.1. Assume that G is compact. Use the Stone-Weierstrass theorem to show that trigonometric polynomials are uniformly dense in $C(G)$. Deduce that they are also dense in $L^p(G)$ for $1 \le p < \infty$.

The statements (6.5), (7.3) and (7.4) are called the *Uniqueness theorems.* Now, we infer that the Fourier algebra $A(\mathbb{Z})$ is a proper subset of $c_0(\mathbb{Z})$ by Exercises 4.5 and 5.4. We can find an explicit example of c_0-sequence which is not an element of $A(\mathbb{Z})$. Let f be a 2π-periodic function on $\mathbb{R}$ given by $f(t) = t$ for $t \in [0, 2\pi]$ then we calculate

$$\sum_{k=-n}^{n} \widehat{f}(k)e^{ikt} = \pi - 2\sum_{k=1}^{n} \frac{1}{k} \sin kt := \pi - 2\phi_n(t).$$

Recall that a sequence converges to f in L^p-space, $1 \le p < \infty$, then a subsequence converges to f pointwise almost everywhere. Therefore, there is a subsequence of $\{\phi_n\}$ which converges almost everywhere on $[0, 2\pi]$ to the function $\dfrac{\pi - t}{2}$. Although this is sufficient to find an example of $c_0(\mathbb{Z}) \setminus A(\mathbb{Z})$, we proceed to conclude that $\{\phi_n\}$ is uniformly bounded on $\mathbb{R}$.

Exercise 7.2. Put $D'_n(t) = \sum_{k=1}^{n} \sin kt$ for $t \in \mathbb{R}$. Show that $D'_n(t) \le \dfrac{\pi}{t}$ for $n = 1, 2, \dots$ and $t \in [0, \pi]$. [Hint: $\sum_{k=1}^{n} e^{ikt} = e^{\frac{i(n+1)t}{2}} \sin \frac{nt}{2} / \sin \frac{t}{2}$.]

Lemma 7.5. *The sequence $\{\phi_n\}$ is uniformly bounded.*

Proof. For each $t > 0$, take a natural number N with $\dfrac{\pi}{N+1} < t \le \dfrac{\pi}{N}$. For $k \le N$, we have $\sin kt \le kt$, and so $|\sum_{k=1}^{N} \frac{1}{k} \sin kt| \le tN \le \pi$. On the other hand, we infer that

$$\left|\sum_{N+1}^{\infty} \frac{1}{k} \sin kt\right| = \left|\sum_{N}^{\infty} (\frac{1}{k} - \frac{1}{k+1}) D'_k(t) - \frac{1}{N} D'_N(t)\right| \le \frac{2}{N+1} \frac{\pi}{t} \le 2,$$

using the summation by parts and Exercise 7.2. □

The above lemma would be trivial if we have *pointwise* convergence theorems, by which we know that $\{\phi_n(t)\}$ converges pointwise to the function $\dfrac{\pi - t}{2}$ on $(0, 2\pi)$.

Proposition 7.6. *Let $f \in L^1(\mathbb{T})$ be an odd function: $f(e^{it}) = -f(e^{-it})$. Then the partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n} \widehat{f}(n)$ are bounded.*

Proof. Since f is an odd function, we have $\widehat{f}(n) = \frac{i}{\pi}\int_0^\pi f(t)\sin nt dt$. By Lemma 7.5, we have

$$\left|\sum_{n=1}^{N} \frac{1}{n}\widehat{f}(n)\right| \leq \frac{1}{\pi}\int_0^\pi |f(t)| \left|\sum_{n=1}^{N}\frac{1}{n}\sin nt\right| dt \leq M,$$

for a constant M, which is independent of N. □

Exercise 7.3. Show that $f \in L^1(G)$ is an odd function; $f(-t) = -f(t)$ if and only if $\widehat{f}$ is an odd function, that is, $\widehat{f}(\gamma^{-1}) = -\widehat{f}(\gamma)$ for each $\gamma \in G$. Find an example of a sequence $\{a_n\}$ in $c_0(\mathbb{Z})$ which is not in the Fourier algebra $A(\mathbb{Z})$.

Exercise 7.4. Modify the above argument to show $c_0(\mathbb{Z}) \not\subset B(\mathbb{Z})$.

Exercise 7.5. Assume that $f \in L^1(\mathbb{R})$ is an odd function. Show that there is a constant M such that

$$\left|\int_\epsilon^R \frac{1}{t}\widehat{f}(t)dt\right| \leq M, \qquad \text{whenever } \epsilon, R > 0.$$

Find examples of $C_0(\widehat{\mathbb{R}}) \setminus A(\widehat{\mathbb{R}})$ and $C_0(\widehat{\mathbb{R}}) \setminus B(\widehat{\mathbb{R}})$.

8. Smoothness

Up to now, we have exploited the *topological* structures of groups. The classical groups $\mathbb{T}$ and $\mathbb{R}$ have additional important structures; the smoothness. In this section, we use the differential structure on $\mathbb{T}$ to construct the *distribution space* $\mathcal{D}'(\mathbb{T})$ on $\mathbb{T}$, which is bigger than $M(\mathbb{T})$. The Fourier-Stieltjes transform will be extended on the whole $\mathcal{D}'(\mathbb{T})$, so that the range of this transform includes a very large class of sequences on $\mathbb{Z}$.

We will denote by $\mathcal{D}(\mathbb{T})$ the vector space of all C^∞-functions on $\mathbb{T}$, or equivalently, all periodic C^∞-functions on $\mathbb{R}$ with the period 2π. Put

$$d(f,g) = \sum_{p=0}^{\infty} \frac{2^{-p}\|D^p f - D^p g\|_\infty}{1 + \|D^p f - D^p g\|_\infty}, \qquad f, g \in \mathcal{D}(\mathbb{T}), \tag{8.1}$$

where D is the usual differentiation.

Proposition 8.1. *Let $\{f_n\}$ be a sequence in $\mathcal{D}(\mathbb{T})$ and $f \in \mathcal{D}(\mathbb{T})$. Then the following are equivalent:*

(i) $d(f_n, f) \to 0$ as $n \to \infty$.

(ii) *For each $p = 0, 1, \dots,$ the sequence $\{D^p f_n\}$ converges uniformly to the function $D^p f$.*

(iii) *Given $\epsilon > 0$, there is a natural number N such that*

$$n \geq N,\ \ p < \frac{1}{\epsilon} \implies \|D^p f_n - D^p f\|_\infty < \epsilon. \tag{8.2}$$

Proof. The directions (i) $\implies$ (ii) $\implies$ (iii) are easy. Assume that the condition (8.2) holds. If we take a natural number P with $P < \frac{1}{\epsilon} \leq P + 1$, then we have

$$\begin{aligned} d(f_n, f) &= \sum_{p=0}^{P} \frac{2^{-p}\|D^p f - D^p g\|_\infty}{1 + \|D^p f - D^p g\|_\infty} + \sum_{p=P+1}^{\infty} \frac{2^{-p}\|D^p f - D^p g\|_\infty}{1 + \|D^p f - D^p g\|_\infty} \\ &\leq \sum_{p=0}^{P} 2^{-p}\epsilon + \sum_{p=P+1}^{\infty} 2^{-p} \leq 2\epsilon + 2^{-P} \leq 2\epsilon + 2^{-\frac{1}{\epsilon}+1}, \end{aligned}$$

whenever $n \geq N$. □

Exercise 8.1. Show that (8.1) defines a metric on $\mathcal{D}(\mathbb{T})$ which is invariant under translations; $d(f + h, g + h) = d(f, g)$. Show also that the addition and scalar multiplication are continuous with respect to this metric. Finally, show that $\mathcal{D}(\mathbb{T})$ is a complete metric space, in which the unit ball is convex.

The above exercise says that $\mathcal{D}(\mathbb{T})$ is a *Fréchet space*. We say that $\{f_n\}$ *converges to f in* $\mathcal{D}(\mathbb{T})$ if the conditions in Proposition 8.1 are satisfied. A *distribution* ϕ on $\mathbb{T}$ is a *continuous* linear functional on $\mathcal{D}(\mathbb{T})$. The space of all distributions is denoted by $\mathcal{D}'(\mathbb{T})$.

Exercise 8.2. Let ϕ be a linear functional on $\mathcal{D}(\mathbb{T})$. Show that $\phi \in \mathcal{D}'(\mathbb{T})$ if and only if there is a constant C and a nonnegative integer P such that

$$|\langle \phi, f \rangle| \leq C\|D^p f\|_\infty, \qquad p = 0, 1, 2, \dots, P.$$

Show that if a sequence $\{\phi_n\}$ in $\mathcal{D}'(\mathbb{T})$ converges pointwise to a linear functional ϕ on $\mathcal{D}(\mathbb{T})$ then $\phi \in \mathcal{D}'(\mathbb{T})$.

It is clear that every measure is a distribution. Our first task is to define the convolution on $\mathcal{D}'(\mathbb{T})$. The definition (2.4) suggests the following: For $\phi, \psi \in \mathcal{D}'(\mathbb{T})$ and $f \in \mathcal{D}(\mathbb{T})$, we define

$$(8.3) \qquad (\phi \boxtimes f)(s) = \langle \phi, f_{-s} \rangle, \qquad \phi \in \mathcal{D}'(\mathbb{T}), f \in \mathcal{D}(\mathbb{T}), s \in T,$$

$$(8.4) \qquad \langle \phi * \psi, f \rangle = \langle \phi, \psi \boxtimes f \rangle, \qquad \phi, \psi \in \mathcal{D}'(\mathbb{T}), f \in \mathcal{D}(\mathbb{T}),$$

Compare with the relation (2.8). If ϕ and ψ are continuous linear functionals on $C(\mathbb{T})$ and $f \in C(T)$, then this is nothing but the definition of the convolution of measures. In order to show that the definition (8.4) is legitimate, we should check that $\phi \boxtimes f$ is a C^∞-function. If Df is continuous then it is easy to see that $\dfrac{f_{-h} - f}{h}$ converges uniformly to Df as $h \to 0$, by the mean-value theorem, and so we see that they converges to Df in $\mathcal{D}(\mathbb{T})$, whenever $f \in \mathcal{D}(\mathbb{T})$. Therefore, it follows that

$$\frac{(\phi \boxtimes f)(s+h) - (\phi \boxtimes f)(s)}{h} = \langle \phi, \frac{(f_{-s})_{-h} - f_{-s}}{h} \rangle \to \langle \phi, D(f_{-s}) \rangle,$$

as $h \to 0$. The last quantity is equal to $\langle \phi, (Df)_{-s} \rangle = \phi \boxtimes (Df)$, and so we have

$$(8.5) \qquad D(\phi \boxtimes f) = \phi \boxtimes (Df), \qquad \phi \in \mathcal{D}'(\mathbb{T}), f \in \mathcal{D}(\mathbb{T}).$$

The repeated use of (8.5) shows that $\phi \boxtimes f$ is a C^∞-function.

Exercise 8.3. Show that $\phi * f \in \mathcal{D}(\mathbb{T})$ for $\phi \in \mathcal{D}'(\mathbb{T})$ and $f \in \mathcal{D}(\mathbb{T})$.

For each $n = 1, 2, \ldots$, take a nonnegative C^∞-function h_n supported on the interval $[-\frac{1}{n}, \frac{1}{n}]$, whose integral is 1. If $f \in \mathcal{D}(\mathbb{T})$, then by the same argument as in the proof of Proposition 2.3, we see that $\{h_n \boxtimes f\}$ converges uniformly to f. By the relation (8.5), we conclude that it converges to f in $\mathcal{D}(\mathbb{T})$. Therefore, we have

$$\langle \phi * h_n, f \rangle = \langle \phi, h_n \boxtimes f \rangle \to \langle \phi, f \rangle, \qquad \phi \in \mathcal{D}'(\mathbb{T}), f \in \mathcal{D}(\mathbb{T}).$$

By Exercise 8.3, we see that every distribution is approximated by smooth functions with respect to the weak*-topology in $\mathcal{D}'(\mathbb{T})$.

Considering the definition (4.7) of the Fourier-Stieltjes transform, it is natural to define

$$\widehat{\phi}(n) = \langle \phi, \gamma_{-n} \rangle, \qquad n \in \mathbb{Z}, \tag{8.6}$$

where γ_n is the character of $\mathbb{T}$ given by $\gamma_n(t) = e^{int}$. This is called the *Fourier-Schwartz transform* on $\mathbb{T}$.

Exercise 8.4. Show that $\widehat{\phi * \psi}(n) = \widehat{\phi}(n)\widehat{\psi}(n)$ for $\phi, \psi \in \mathcal{D}'(\mathbb{T})$, $n \in \mathbb{Z}$.

Before the study of $\widehat{\phi}(n)$ for $\phi \in \mathcal{D}'(\mathbb{T})$, we first investigate the properties of the Fourier coefficients for functions in $\mathcal{D}(\mathbb{T})$. If f is a differentiable function, then it is easy to see that $\widehat{Df}(n) = in\widehat{f}(n)$ for each $n \in \mathbb{Z}$ using integration by parts. Therefore, we have

$$\widehat{D^p f}(n) = (in)^p \widehat{f}(n), \qquad f \in \mathcal{D}(\mathbb{T}). \tag{8.7}$$

Because $D^p f$ is, of course, an L^1-function, we see that

$$f \in \mathcal{D}(\mathbb{T}) \implies |n|^p \widehat{f}(n) \to 0 \text{ for each } p = 0, 1, 2, \ldots. \tag{8.8}$$

A sequence $\{c_n\}$ on $\mathbb{Z}$ is said to *decrease rapidly* if the sequence $\{|n|^p c_n\}$ is bounded, for each $p = 0, 1, 2, \ldots$.

Proposition 8.2. *The map $f \mapsto \widehat{f}$ is a one-to-one correspondence from $\mathcal{D}(\mathbb{T})$ onto the set of all rapidly decreasing sequences. For each $f \in \mathcal{D}(\mathbb{T})$, we have*

$$f = \lim_{|N| \to \infty} \sum_{n=-N}^{N} \widehat{f}(n)\gamma_n, \tag{8.9}$$

with respect to the topology of $\mathcal{D}(\mathbb{T})$.

Proof. For a rapidly decreasing sequence $\{c_n\}$, it is easy to see that the function $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ is a C^∞-function, and $\widehat{f}(n) = c_n$ by the orthogonality of $\{\gamma_n\}$. For the formula (8.9), we fix $p = 0, 1, 2, \ldots$ and note that the sequence

$$D^p \left(\sum_{n=-N}^{N} \widehat{f}(n)\gamma_n \right) = \sum_{n=-N}^{N} \widehat{f}(n)(in)^p \gamma_n = \sum_{n=-N}^{N} \widehat{D^p f}(n)\gamma_n$$

converges uniformly to $D^p f$ as $N \to \infty$, because $\sum_{n=-\infty}^{\infty} \widehat{D^p f}(n)$ is an absolutely summable series. □

We say that a sequence $\{c_n\}$ on $\mathbb{Z}$ *increases slowly* if $\{|n|^{-p} c_n\}$ is bounded for some $p = 0, 1, 2, \ldots$. It is clear that the set of all slowly increasing sequences is closed under the pointwise product. Now, we are ready to state and prove the following main theorem of this section.

Theorem 8.3. *The Fourier-Schwartz transform $\phi \mapsto \widehat{\phi}$ is an isomorphism from $\mathcal{D}'(\mathbb{T})$ onto the algebra of all slowly increasing sequence.*

Proof. By Exercise 8.2, we see that there are C and p such that

$$|n|^{-p}|\widehat{\phi}(n)| = |n|^{-p}|\langle \phi, \gamma_{-n}\rangle| \le |n|^{-p} C \|D^p \gamma_{-n}\|_\infty = C.$$

Now, we assume that $\{c_n\}$ is a slowly increasing sequence with $|n|^{-p}|c_n| \le C$ for each $n \in \mathbb{Z}$. Considering γ_n as an element of $\mathcal{D}'(\mathbb{T})$, we have

$$\langle \gamma_n, f\rangle = \lim_{|N| \to \infty} \sum_{k=-N}^{N} \langle \gamma_n, \widehat{f}(k)\gamma_k\rangle = \widehat{f}(-n),$$

for each $f \in \mathcal{D}(\mathbb{T})$ and $n \in \mathbb{Z}$, by Proposition 8.2. Now, we fix $f \in \mathcal{D}(\mathbb{T})$, then there is a constant C' such that

$$\sum_{n=-M}^{N} |\langle c_n \gamma_n, f\rangle| = \sum_{n=-M}^{N} |c_n|\,|\widehat{f}(-n)| \le \sum_{n=-M}^{N} |n|^p C |\widehat{f}(-n)| \le C'$$

for all M and N, because $\{\widehat{f}(-n)\}$ decreases rapidly. This shows that the series $\{\sum_{n=-M}^{N} c_n \gamma_n\}$ converges to a distribution $\phi \in \mathcal{D}'(\mathbb{T})$ as $M, N \to \infty$, by Exercise 8.2. It is clear that $\widehat{\phi}(n) = c_n$ by the orthogonality of $\{\gamma_n\}$.

It remains to show that $\phi \mapsto \widehat{\phi}$ is one-to-one. To do this, assume that $\widehat{\phi}(n) = 0$ and so $\langle \phi, \gamma_n\rangle = 0$ for each $n \in \mathbb{Z}$. By Proposition 8.2, we see that $\langle \phi, f\rangle = 0$ for each $f \in \mathcal{D}(\mathbb{T})$ from the continuity of ϕ. □

Exercise 8.5. Define the involution $\phi \mapsto \phi^*$ for $\phi \in \mathcal{D}'(\mathbb{T})$ so that $\widehat{\phi^*} = \overline{\widehat{\phi}}$.

NOTE

For the direct proof that the dual group of an LCA group is LCA with the compact-open topology, see [PONTRYAGIN, §34] or [MORRIS, §3]. The proof of Theorem 4.3 was taken from [RUDIN62, §1.2]. We have followed [LOOMIS, §24] and [RUDIN73, Chapter 11], for the spectral radius formula and the Gelfand transform. For the proof that (5.14) is a C^*-norm, see [DIXMIER, §1.3] or [TAKESAKI, §I.1]. The materials in Sections 6 and 7 were taken from [RUDIN62, Chapter 1].

For another proofs of Pontraygin duality and related topics such as structure theorems for LCA groups, we refer to the books [HEWITT AND ROSS], [MORRIS] and [PONTRYAGIN]. For a brief history on Pontryagin duality, see [HEWITT AND ROSS, §24 Notes]. Much larger class of abelian groups than LCA groups enjoys the Pontryagin duality. See the Remark in [MORRIS, §5] and references there. The abelian groups satisfying the Pontryagin duality have been characterized in [Venkataraman, Math. Z. 149(1976), 109-119]. The Pontryagin duality has the obvious similarity with the reflexivity of topological vector spaces, which are abelian groups under the addition. See [Kye, Chinese J. Math. 12(1984), 129-136; J. Math. Anal. Appl. 139(1989), 477-482].

For *Classical* Fourier Analysis on the groups $\mathbb{T}$, $\mathbb{Z}$ and $\mathbb{R}$, we refer to the books [CHANDRASEKHARAN], [EDWARDS], [HELSON], [KATZNELSON] and [ZYGMUND]. Lemma 7.5 was taken from [ZYGMUND, §V.1]. As for Proposition 7.6, the situation is quite different in the case of even functions. See [KATZNELSON, §I.4] for more details and an another proof of Proposition 7.6. For an example of $\phi \in C(\mathbb{T}) \setminus A(\mathbb{T})$, see [KATZNELSON, Exercise I.6.6].

We have followed [KHAVIN AND NIKOLSKIJ, Chapter I.1] for §8. For the general theory of distributions on $\mathbb{R}^n$ and the Fourier transform there, we refer to the book such as [RUDIN73] or [SCHWARTZ]. For the history of Fourier analysis, we also refer to [BOTTAZZINNI, Chapter 5], [DIEUDONNÉ81, Chapter VII] or [KHAVIN AND NIKOLSKIJ, Chapter I.4]. It should be noted that the analysis of Fourier series was one of the main motivations for the developments of the concepts of *function* and *set*. See the Introduction part of [CANTOR].

CHAPTER III

NON–ABELIAN GROUPS

The central theme of the study of abelian groups in the last chapter is the Pontryagin-van Kampen duality, which enables us to recover the original groups from its dual objects. The essential part of the duality theorem for LCA groups is Theorem 7.2, which says that there are sufficiently many characters to distinguish elements of an LCA group. If we consider a non-commutative simple group then the only homomorphism into $\mathbb{T}$ is the trivial one. This leads us naturally to consider the group $U(n, \mathbb{C})$ of all $n \times n$ unitary matrices. For non-compact groups, we need also *infinite-dimensional* unitary matrices, or equivalently unitary operators on an infinite-dimensional Hilbert space.

In §9, we establish the correspondences between continuous unitary representations of a locally group G and non-degenerated $*$-representations of the group algebra $L^1(G)$. For the case of abelian groups, this amounts to Theorem 4.3, where we have seen that every complex homomorphism of $L^1(G)$ is induced by a character. It also turns out that every representation of $L^1(G)$ is induced by a positive linear functional of $L^1(G)$, equivalently, by a continuous positive definite function on G. Therefore, a unitary representation of G corresponds to a positive definite function on G. In this situation, we show in §10 that an irreducible representation corresponds to a positive definite function which is extreme, called a pure positive definite function. With this machinery, it is easy to see that there are sufficiently many irreducible representations on a locally compact group. Considering Bochner Theorem, we see that characters of an abelian group play the roles of both irreducible representations and pure positive definite functions.

We restrict ourselves to compact groups in §11. Employing the spectral decomposition theorem, we show that every irreducible representation of a compact group is of finite dimensional. This enables us to decompose the regular representation of a compact group into the direct sum of finite dimensional irreducible representations, to get the noncommutative version of Riesz-Fisher Theorem (7.12) for L^2-functions. In this case, the notion of a character is generalized as the trace of an irreducible representation. In order to get the duality of a compact group, we take the set of all finite dimensional representations as the dual object. This set has the operations such as direct sum, tensor product and conjugate. With a suitable notions of representations of this dual objects, we get the Tannaka duality in §12. We close this note by exhibiting all irreducible representations for simplest non-abelian infinite compact groups such as special unitary and special orthogonal groups with low dimensions in §13.

9. Unitary Representations

From now on throughout this note, $\mathcal{H}$ and $\mathcal{B}(\mathcal{H})$ will always denote a Hilbert space and the Banach space of all *bounded* linear operators on $\mathcal{H}$. For each $x \in \mathcal{B}(\mathcal{H})$ and $\eta \in \mathcal{H}$, the map $\xi \mapsto \langle x\xi, \eta\rangle$ is a bounded linear functional whose norm is $\|x\|$. Therefore, there is a unique element, denoted by $x^*\eta$ in $\mathcal{H}$, such that

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle, \qquad x \in \mathcal{B}(\mathcal{H}), \quad \xi, \eta \in \mathcal{H}.$$

It is easy to see that $\eta \mapsto x^*\eta$ is a bounded linear operator on $\mathcal{H}$, and $x \mapsto x^*$ is an isometric involution. Because

$$\|x\xi\|^2 = \langle x\xi, x\xi\rangle = \langle x^*x\xi, \xi\rangle \le \|x^*x\|\|\xi\|^2,$$

we have $\|x\|^2 \le \|x^*x\|$. From the relation $\|x^*x\| \le \|x^*\|\|x\| = \|x\|^2$, it follows that

$$\|x^*x\| = \|x\|^2, \qquad x \in \mathcal{B}(\mathcal{H}). \tag{9.1}$$

This says that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. An element $u \in \mathcal{B}(\mathcal{H})$ is said to be a *unitary* if $u^*u = uu^* = 1_{\mathcal{H}}$, and the group of all unitaries on $\mathcal{H}$ will be denoted by $\mathcal{U}(\mathcal{H})$.

A *unitary representation* of a locally compact group G on a Hilbert space $\mathcal{H}$ is a group homomorphism $s \mapsto \pi_s$ from G into $\mathcal{U}(\mathcal{H})$ such that the map $s \mapsto \pi_s\xi$ is continuous from G into $\mathcal{H}$ for each fixed $\xi \in \mathcal{H}$. For $s \in G$ and $\xi \in L^2(G)$, we define

$$(\lambda_s\xi)(t) = \xi_s(t) = \xi(s^{-1}t), \qquad t \in G. \tag{9.2}$$

Then we have $\langle \lambda_s\xi, \eta\rangle = \langle \xi, \lambda_{s^{-1}}\eta\rangle$ for each $\xi, \eta \in L^2(G)$. Therefore, λ_s is a unitary operator on $L^2(G)$ with $(\lambda_s)^* = \lambda_{s^{-1}}$ for each $s \in G$. By Exercise 2.5, we see that $s \mapsto \lambda_s$ is a unitary representation of G on the Hilbert space $L^2(G)$. This is called the *left regular representation*, which will be always denoted by λ. The *right regular representation* $s \mapsto \rho_s$ is also defined by

$$(\rho_s\xi)(t) = \xi^s(t) = \xi(ts), \qquad s, t \in G,\ \xi \in L^2(G). \tag{9.3}$$

Exercise 9.1. Let $\{u_i\}$ be a net in $\mathcal{U}(\mathcal{H})$, and $u \in \mathcal{U}(\mathcal{H})$. Show that $u_i\xi \to u\xi$ for each $\xi \in \mathcal{H}$ if and only if $\langle u_i\xi, \eta\rangle \to \langle u\xi, \eta\rangle$ for each $\xi, \eta \in \mathcal{H}$.

A unitary representation π is said to be *irreducible* if there is no nontrivial closed subspace of $\mathcal{H}$ which is invariant under $\{\pi_s : s \in G\}$. In other words, if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ such that $\pi_s(\mathcal{K}) \subseteq \mathcal{K}$ for each $s \in G$, then $\mathcal{K} = \mathcal{H}$. It is clear that every one-dimensional representation is irreducible: Every character of an LCA group is an irreducible representation.

A *representation* of an involutive (normed) algebra A on a Hilbert space $\mathcal{H}$ is a continuous $*$-homomorphism from A into $\mathcal{B}(\mathcal{H})$. The notion of irreducibility is defined similarly as above. For a representation π of an involutive algebra A on $\mathcal{H}$, we denote by

$$\mathcal{K} = \text{the closure of the linear span of } \{\pi(x)\xi : x \in A, \xi \in \mathcal{H}\}.$$

Then we see that $\mathcal{K}$ is an invariant subspace of $\mathcal{K}$, and $\eta \in \mathcal{K}^\perp$ if and only if $\pi(x)\eta = 0$ for each $x \in A$. In other words, π acts trivially on the orthogonal complement of $\mathcal{K}$. We say that a representation is *non-degenerate* if $\mathcal{K} = \mathcal{H}$ in the above discussion, which says that every representation is the *sum* of a non-degenerate representation and the trivial one. If $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ are representations of A, then the sum $\pi_1 \oplus \pi_2$ is the representation of A on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by $x \mapsto (\pi_1(x), \pi_2(x))$.

For a given arbitrary family $\{\mathcal{H}_i : i \in I\}$ of Hilbert spaces, we define the *direct sum* $\sum_i^{\oplus} \mathcal{H}_i = \{\xi \in \prod_{i\in I} \mathcal{H}_i : \sum_i \|\xi_i\|^2 < \infty\}$, with the inner product

$$\langle \xi, \eta \rangle = \sum_{i\in I} \langle \xi_i, \eta_i \rangle, \qquad \xi, \eta \in \sum_{i\in I}{}^{\oplus} \mathcal{H}_i.$$

It is plain that $\sum_i^{\oplus} \mathcal{H}_i$ is a Hilbert space. If $x_i \in \mathcal{B}(\mathcal{H}_i)$ with $\|\xi_i\| \leq M$ for each $i \in I$ then it is also legitimate to define the sum $\sum_i^{\oplus} x_i$ by

$$\langle (\sum_i{}^{\oplus} x_i)\xi, \eta \rangle = \sum_{i\in I} \langle x_i \xi_i, \eta_i \rangle, \qquad \xi, \eta \in \sum_i{}^{\oplus} \mathcal{H}_i.$$

Then $\sum_i^{\oplus} x_i \in \mathcal{B}(\sum_i^{\oplus} \mathcal{H}_i)$ with $\|\sum_i^{\oplus} x_i\| \leq M$. The *direct sum* of representations $\{\pi_i, \mathcal{H}_i\}$ of A on the Hilbert space $\sum_i^{\oplus} \mathcal{H}_i$ is define by

$$\sum_i{}^{\oplus} \pi_i : x \mapsto \sum_i{}^{\oplus} \pi_i(x), \qquad x \in A.$$

Exercise 9.2. Show that a representation π of an involutive algebra A on $\mathcal{H}$ is non-degenerate, if and only if, for each nonzero $\xi \in \mathcal{H}$ there is $a \in A$ such that $\pi(x)\xi \neq 0$.

The following theorem relates unitary representations of a group G and representations of the group algebra $L^1(G)$. Compare with Theorem 4.3.

Theorem 9.1. *Let $s \mapsto \pi_s$ be a unitary representation of a locally compact group G on $\mathcal{H}$. Then the formula*

$$(9.4) \qquad \langle \pi(\mu)\xi, \eta \rangle = \int_G \langle \pi_s \xi, \eta \rangle d\mu(s), \qquad \mu \in M(G), \xi, \eta \in \mathcal{H}$$

defines a representation of the involutive algebra $M(G)$, whose restriction to $L^1(G)$ is non-degenerate. Conversely, if π is a non-degenerate representation of $L^1(G)$ on $\mathcal{H}$ then there exists a unique unitary representation $s \mapsto \pi_s$ of G on $\mathcal{H}$ satisfying the relation (9.4).

Proof. It is clear that $\pi(\mu) \in \mathcal{B}(\mathcal{H})$ with $\|\pi(\mu)\| \leq \|\mu\|$. By the straightforward calculation, we also have $\pi(\mu * \nu) = \pi(\mu)\pi(\nu)$ and $\pi(\mu)^* = \pi(\mu^*)$. Note that

$$\langle \pi(\delta_t * f)\xi, \eta \rangle = \iint \langle \pi_{sr}\xi, \eta \rangle d\delta_t(s) f(r) dr = \int \langle \pi_{tr}\xi, \eta \rangle f(r) dr,$$

for each $t \in G$ and $f \in L^1(G)$. If we take an approximate identity $\{u_i\}$ satisfying the assumptions of (2.9), then it follows that

$$\lim\langle \pi(\delta_t * u_i)\xi, \eta\rangle = \langle \pi_t\xi, \eta\rangle, \qquad t \in G, \quad \xi, \eta \in \mathcal{H}, \tag{9.5}$$

by the continuity of $r \mapsto \pi_{tr}\xi$. We denote by $\mathcal{K}$ the subspace of $\mathcal{H}$ generated by $\{\pi(f)\xi : f \in L^1(G), \xi \in \mathcal{H}\}$. If η is in the orthogonal complement of $\mathcal{K}$ then we see that $\eta = 0$ by (9.5) with $t = e$. This shows that the representation π restricted on $L^1(G)$ is non-degenerate.

Now, we proceed to prove the converse. The uniqueness comes out from (9.5). Assume that π is a non-degenerate representation of $L^1(G)$. Then $\mathcal{K}$ is a dense subspace of $\mathcal{H}$. We note that $(u_i)_s * f = (u_i * f)_s \to f_s$ in $L^1(G)$, and so it follows that

$$\pi(f_s) = \lim \pi((u_i)_s * f) = \lim \pi((u_i)_s)\, \pi(f), \qquad f \in L^1(G), \ s \in G$$

by the continuity of π. This shows that there is an operator π_s on $\mathcal{K}$ such that

$$\pi_s\pi(f) = \pi(f_s) = \pi(\delta_s * f), \qquad f \in L^1(G), \ s \in G.$$

Because $\|\pi((u_i)_s)\| \leq \|\pi\|$, we see that π_s extends to a bounded linear operator on $\mathcal{H}$ with $\|\pi_s\| \leq \|\pi\|$. Since $s \mapsto \pi(f_s)\xi$ is continuous, we see that $s \mapsto \pi_s\xi$ is also continuous for $\xi \in \mathcal{K}$, and hence for $\xi \in \mathcal{H}$. Now, we have

$$\pi_{st}\pi(f) = \pi(f_{st}) = \pi((f_t)_s) = \pi_s\pi(f_t) = \pi_s\pi_t\pi(f), \qquad f \in L^1(G),$$

and so $s \mapsto \pi_s$ is a homomorphism. From the relation $(\delta_s)^* * \delta_s = \delta_e$, it is easy to see $\langle \pi_s\pi(f)\xi, \pi_s\pi(g)\eta\rangle = \langle \pi(f)\xi, \pi(g)\eta\rangle$, and each π_s is a unitary operator. It remains to show that the relation (9.4) is satisfied for $\mu(s) = f(s)ds$ with $f \in L^1(G)$. We fix $\xi, \eta \in \mathcal{H}$. Note that $g \mapsto \langle \pi(g)\xi, \eta\rangle$ is a bounded linear functional in $L^1(G)$, and so there exist $h \in L^\infty(G)$ such that

$$\langle \pi(g)\xi, \eta\rangle = \int g(r)h(r)dr, \qquad g \in L^1(G).$$

Now, we have

$$\begin{aligned}\int \langle \pi_s\pi(g)\xi, \eta\rangle\, f(s)ds &= \iint f(s)g_s(r)h(r)drds \\ &= \int (f * g)(r)h(r)dr = \langle \pi(f * g)\xi, \eta\rangle = \langle \pi(f)\pi(g)\xi, \eta\rangle.\end{aligned}$$

By the density of $\mathcal{K}$, we may replace $\pi(g)\xi$ by ξ, and this completes the proof. □

Corollary 9.2. *Every representation of $L^1(G)$ is norm-decreasing.*

The representation π (we use the same notation) on $L^1(G)$ given by (9.4) is called the *induced* representation by the unitary representation $s \mapsto \pi_s$. If G is an LCA group with a character γ, then the Fourier transform $f \mapsto \widehat{f}(\gamma)$ is nothing but the induced representation of the unitary representation given by $s \mapsto \overline{\gamma(s)}$. Compare the formulae (9.4) and (4.7).

Exercise 9.3. Show that a closed subspace of $\mathcal{H}$ is invariant under $\{\pi_s : s \in G\}$ if and only if it is invariant under $\{\pi(f) : f \in L^1(G)\}$. Conclude that a unitary representation is irreducible if and only if its induced representation is irreducible.

It is easy to see that the induced representation of the left regular representation (9.3) of G is given by

$$\langle \lambda(f)\xi, \eta\rangle = \langle f * \xi, \eta\rangle, \qquad f \in L^1(G),\ \ \xi, \eta \in L^2(G).$$

The norm condition $\|\lambda(f)\| \le \|f\|_1$ implies that

$$\|f * \xi\|_2 \le \|f\|_1 \|\xi\|_2, \qquad f \in\in L^1(G),\ \ \xi \in L^2(G).$$

Exercise 9.4. Assume that $f \in L^1(G)$ and $g \in L^p(G)$ with $1 < p < \infty$. Show that $f * g \in L^p(G)$ with $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

Now, we relate representations of $L^1(G)$ with positive-definite functions on G, which are nothing but bounded positive linear functionals on the involutive algebra $L^1(G)$. If $\{\pi, \mathcal{H}\}$ is a representation of $L^1(G)$ and $\xi \in \mathcal{H}$ then we see that the function

$$\phi(s) = \langle \pi_s \xi, \xi\rangle, \qquad s \in G \tag{9.6}$$

defines a positive definite function on G. Indeed, we have

$$\langle \phi, f^* * f\rangle = \int_G \langle \pi_s \xi, \xi\rangle (f^* * f)(s)ds = \langle \pi(f^* * f)\xi, \xi\rangle = \|\pi(f)\xi\|^2 \ge 0, \tag{9.7}$$

by Theorem 9.1. In order to construct a representation of $L^1(G)$ from a given positive definite function ϕ, it is more convenient to consider general involutive algebras.

Let ϕ be a bounded positive linear functional on a unital involutive algebra A with the following property:

$$\phi(y^*x^*xy) \leq \|x\|^2\phi(y^*y), \qquad x, y \in A. \tag{9.8}$$

Note that the formula (3.7) induces a definite inner product on the quotient space A/L_ϕ, where

$$L_\phi = \{x \in A : \phi(x^*x) = 0\}.$$

We denote by $\mathcal{H}_\phi$ the Hilbert space obtained by the completion of A/L_ϕ. For every $x \in A$, denote by $\pi_\phi(x)$ the linear map on the pre-Hilbert space A/L_ϕ induced by the multiplication;

$$\pi_\phi(x)(y + L_\phi) = xy + L_\phi, \qquad y \in A,$$

which is well-defined by (3.9). Indeed, if $y \in L_\phi$ then we have

$$|\phi((xy)^*(xy))|^2 = |\phi((x^*xy)^*y)|^2 \leq \phi(y^*y)\phi((x^*xy)^*(x^*xy)) = 0,$$

and so $xy \in L_\phi$. The condition (9.8) implies

$$\|\pi_\phi(x)(y + L_\phi)\|^2_{\mathcal{H}_\phi} = \phi((xy)^*(xy)) \leq \|x\|^2\phi(y^*y) = \|x\|^2\|y + L_\phi\|^2_{\mathcal{H}_\phi}.$$

Therefore, $\pi_\phi(x)$ extends to a bounded linear map on $\mathcal{H}_\phi$ with $\|\pi_\phi(x)\| \leq \|x\|$. Also, if we denote by ξ_ϕ the vector in $\mathcal{H}_\phi$ represented by the identity 1 of A, then we have

$$\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle, \quad x \in A. \tag{9.9}$$

It is easily seen that $\{\pi_\phi, \mathcal{H}_\phi\}$ is a representation of A. Note that the set $\{\pi_\phi(x)\xi_\phi : x \in A\}$ is dense in $\mathcal{H}_\phi$, by the construction of $\mathcal{H}_\phi$. Such a vector is called a *cyclic* vector. Especially, a representation with a cyclic vector is non-degenerate. We have thus shown the following:

Proposition 9.3. *Let ϕ be a bounded positive linear functional on a unital involutive algebra A with the property* (9.8). *Then there is a representation $\{\pi_\phi, \mathcal{H}_\phi\}$ and a cyclic vector $\xi_\phi \in \mathcal{H}_\phi$ satisfying the relation* (9.9).

Exercise 9.5. Let ϕ be the linear functional of the C^*-algebra $C[0,1]$ given by the Lebesgue measure on the unit interval. Show that the condition (9.8) is satisfied and describe the induced representation. Do the same question for the normalized trace of the matrix algebra.

Exercise 9.6. Let G be an LCA groups. Describe the representation of $L^1(G)$ induced by the positive definite function ϕ_μ defined by (6.2) for each $\mu \in M(\widehat{G})^+$. What is the corresponding unitary representation of G?

Exercise 9.7. Show that a representation $\{\pi, \mathcal{H}\}$ of A is irreducible if and only if every nonzero cyclic vector of $\mathcal{H}$ is cyclic.

Exercise 9.8. Show that every non-degenerate representation of A is the direct sum of representations with cyclic vectors. [Hint: Zorn's lemma.]

For any positive definite function ϕ on a group G, we have

$$(9.10)\qquad \langle \phi, \mu^* * \mu \rangle = \iint \phi(s^{-1}t)\overline{d\mu(s)}d\mu(t) \le \phi(e)\|\mu\|^2, \qquad \mu \in M(G).$$

For $\nu \in M(G)$, we define the L^∞-function $\psi : t \mapsto \displaystyle\int \phi(str)d\nu^*(s)d\nu(r)$ for $t \in G$. Then we have

$$\langle \psi, \mu \rangle = \iiint \phi(str)d\nu^*(s)d\nu(r)d\mu(t) = \int \phi(t)d(\nu^* * \mu * \nu)(t) = \langle \phi, \nu^* * \mu * \nu \rangle,$$

and so $\langle \psi, \mu^* * \mu \rangle = \langle \phi, (\mu * \nu)^* * (\mu * \nu) \rangle \ge 0$. Therefore, ψ is also a positive definite function on G and we have $\langle \psi, \mu^* * \mu \rangle \le \psi(e)\|\mu\|^2$ by (9.10). In other words, we have

$$\langle \phi, \nu^* * \mu^* * \mu * \nu \rangle \le \psi(e)\|\mu\|^2 = \|\mu\|^2 \langle \psi, \delta_e \rangle = \|\mu\|^2 \langle \phi, \nu^* * \nu \rangle.$$

This shows that the unital involutive Banach algebra $M(G)$ satisfies the condition (9.8), and so we may apply Proposition 9.3 for $A = L^1(G) + \mathbb{C}\delta_e$.

Theorem 9.4. *Let ϕ be a continuous positive definite function on a locally compact group G. Then there exists a representation $\{\pi_\phi, \mathcal{H}_\phi\}$ of $L^1(G)$ and a cyclic vector $\xi_\phi \in \mathcal{H}_\phi$ such that*

$$\langle \phi, f \rangle = \langle \pi_\phi(f)\xi_\phi, \xi_\phi \rangle, \qquad f \in L^1(G).$$

Proof. It remains to show that ξ_ϕ is cyclic vector for $L^1(G)$. If $\{u_i\}$ is an approximate identity satisfying the assumption of (2.9), then it is plain that

$\langle \phi, u_i \rangle \to \phi(e)$ and $\langle \phi, u_i^* \rangle \to \phi(e)$. Now, we use the relations (3.10) and (9.10) to infer that

$$|\langle \phi, u_i \rangle|^2 \le \phi(e)\langle \phi, u_i^* * u_i \rangle \le \phi(e)^2 \|u_i\|_1^2 = \phi(e)^2,$$

and so $\langle \phi, u_i^* * u_i \rangle \to \phi(e)$. Therefore, we have

$$\langle \phi, (u_i - \delta_e)^* * (u_i - \delta_e) \rangle = \langle \phi, u_i^* * u_i \rangle - \langle \phi, u_i \rangle - \langle \phi, u_i^* \rangle + \phi(e) \to 0$$

as $i \to \infty$. This means that the vector ξ_ϕ is approximated in $\mathcal{H}_\phi$ by vectors induced by L^1-function. The proof is complete, because we already know that ξ_ϕ is cyclic vector for $L^1(G) + \mathbb{C}\delta_e$ and $\pi_\phi(\delta_e)(\xi_\phi) = \xi_\phi$. □

We say that two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ of A are *unitarily equivalent* each other if there is a Hilbert space isomorphism U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $\pi_1(x) = U^* \pi_2(x) U$ for each $x \in A$.

Exercise 9.9. Assume that two representations $\{\pi_1, \mathcal{H}_1\}$ and $\{\pi_2, \mathcal{H}_2\}$ have cyclic vectors ξ_1 and ξ_2, respectively, with the relation

$$\langle \pi_1(x)\xi_1, \xi_1 \rangle = \langle \pi_2(x)\xi_2, \xi_2 \rangle, \qquad x \in A.$$

Show that they are unitarily equivalent.

This exercise shows that the representation in Theorem 9.4 is determined uniquely up to unitary equivalence. The notions of cyclic vector and unitary equivalence are defined for unitary representations of groups by the similar way.

Exercise 9.10. Show that ξ is a cyclic vector for a unitary representation of G if and only if ξ is also a cyclic vector for the induced representation of $L^1(G)$.

Exercise 9.11. Show that the left and right regular representations are unitarily equivalent each other.

Theorem 9.5. *Every unitary representation $s \mapsto \pi_s$ of G defines a continuous positive definite function ϕ by the formula* (9.6). *Conversely, for each*

$\phi \in P(G)$ there exists a unique unitary representation $s \mapsto \pi_s$, up to unitary equivalence, and a cyclic vector ξ satisfying the relation (9.6).

Proof. The first statement has been already proved. By Theorem 9.4, there is a representation $\{\pi, \mathcal{H}\}$ of $L^1(G)$ and a cyclic vector $\xi \in \mathcal{H}$ such that $\langle \phi, f \rangle = \langle \pi(f)\xi, \xi \rangle$ for each $f \in L^1(G)$. By Theorem 9.1, this representation is induced by a unitary representation $s \mapsto \pi_s$, with the relation

$$\int_G \phi(s)f(s)ds = \langle \phi, f \rangle = \langle \pi(f)\xi, \xi \rangle = \int_G f(s)\langle \pi_s\xi, \xi \rangle ds, \qquad f \in L^1(G).$$

From the continuity of two functions ϕ and $s \mapsto \langle \pi_s\xi, \xi \rangle$, we have the required relation (9.6). □

10. Irreducible Representations

In the last section, we have established the correspondence between unitary representations and continuous positive definite functions for a locally compact group G. In this section, we study the subclass of $P(G)$ corresponding to irreducible representations, and generalize Theorem 7.2 to noncommutative groups: Every locally compact group admits sufficiently many irreducible representations to distinguish elements of G. For an element $s \neq e$ of G, take a compact neighborhood U of e such that $U \cap sU = \emptyset$, and denote by $\xi \in L^2(G)$ the characteristic function on U. Then we have

$$\langle \lambda_s\xi, \xi \rangle = \int \xi(s^{-1}t)\overline{\xi(s)}dt = 0 \neq \langle \lambda_e\xi, \xi \rangle. \tag{10.1}$$

If we denote by ϕ_ξ the continuous positive definite function (9.6) defined by the left regular representation λ and ξ, then the above formula (10.1) says that $\phi_\xi(s) \neq \phi_\xi(e)$.

We denote by $P(G)_1$ the set of all $\phi \in P(G)$ such that $\phi(e) = 1$. It is easy to see that $P(G)_1$ is a weak*-closed convex subset of $L^1(G)^*$. Therefore, we see that every element of $P(G)_1$ is approximated by convex combinations of extreme points of $P(G)_1$ in the weak*-topology by the Banach-Alaoglu and Krein-Milman theorems. An extreme point of $P(G)_1$ is said to be a *pure positive definite* function on G. If G is a discrete group and a net $\{\phi_i\}$ of $P(G)$ converges to ϕ in the weak*-topology then $\{\phi_i\}$ converges to ϕ uniformly on finite subsets, considering the point masses. This is the case for general locally compact groups. Compare with Proposition 4.1. We need the following general fact.

Exercise 10.1. Let X be a normed space and $\{\phi_i\}$ a bounded net in X^*. If $\{\phi_i\}$ converges to ϕ in the weak*-topology then show that $\{\phi_i\}$ converges to ϕ uniformly on every compacta of X.

Proposition 10.1. *Let $\{\phi_i\}$ be a net in $P(G)_1$ and $\phi \in P(G)_1$. Then the following are equivalent:*

(i) *The net $\{\phi_i\}$ converges to ϕ pointwise.*
(ii) *For each $f \in L^1(G)$, we have $\lim_i \langle \phi_i, f\rangle = \langle \phi, f\rangle$.*
(iii) *The net $\{\phi_i\}$ converges to ϕ uniformly on every compacta of G.*

Proof. The direction (i) $\Longrightarrow$ (ii) follows from the Lebesgue dominated convergence theorem, as was in Proposition 4.1, and so it suffices to show the direction (ii) $\Longrightarrow$ (iii). To do this, take a compact neighborhood V such that $|\phi(s) - 1| < \epsilon$ for $s \in V$, and put $h = a^{-1}\chi_V$ where a is the mass of V. We first show that

$$\psi \in P(G)_1,\ |\langle \psi - \phi, h\rangle| < \epsilon \implies |\psi(s) - (h * \psi)(s)| < 2\sqrt{\epsilon}\Delta(s^{-1}), \tag{10.2}$$

for each $s \in G$. Assume that ψ satisfy the assumption of (10.2). Then we have

$$\left|\int_V (\psi(t) - \phi(t))dt\right| \le |\langle \psi - \phi, ah\rangle| < a\epsilon,$$

and so it follows that

$$\left|\int_V (1 - \psi(t))dt\right| \le \left|\int_V (1 - \phi(t))dt\right| + \left|\int_V (\phi(t) - \psi(t))dt\right| < 2a\epsilon.$$

If we write $\psi(s) = \langle \pi_s\xi, \xi\rangle$ for a unitary representation π of G then $\|\xi\|^2 = \psi(e) = 1$. Therefore, we have

$$\begin{aligned}|\psi(t) - \psi(s)|^2 &= |\langle \pi_s\xi - \pi_t\xi, \xi\rangle|^2 \le \|\pi_s\xi - \pi_t\xi\|^2 \\ &= \|\pi_s\xi\|^2 + \|\pi_t\xi\|^2 - 2\mathrm{Re}\,\langle \pi_s\xi, \pi_t\xi\rangle = 2 - 2\mathrm{Re}\,\psi(t^{-1}s).\end{aligned}$$

Combining the above two inequalities, it follows that

$$\begin{aligned}
|(h*\psi)(s)-\psi(s)| &= \left|\int_G h(t)\psi(t^{-1}s)dt - \frac{1}{a}\int_V \psi(s)dt\right| \\
&\le \frac{1}{a}\int_V |\psi(t^{-1}s)-\psi(s)|dt \\
&\le \frac{\sqrt{2}}{a}\int_V (1-\operatorname{Re}\psi(s^{-1}ts))^{\frac{1}{2}}dt \\
&= \frac{\sqrt{2}}{a}\Delta(s^{-1})\left(\int_V (1-\operatorname{Re}\psi(t))dt\right)^{\frac{1}{2}}\left(\int_V dt\right)^{\frac{1}{2}} \\
&\le 2\sqrt{\epsilon}\Delta(s^{-1}).
\end{aligned}$$

If we denote $\check{\phi}(t)=\phi(t^{-1})$ then it is easy to see that $\{\check{\phi}_i\}$ also satisfies the condition (ii). We fix a compact set K of G. Then the set $L=\{h_{s^{-1}} : s\in K\}$ is compact in $L^1(G)$ by Lemma 2.2. Applying Exercise 10.1, we see that $\{h*\phi_i\}$ converges to $\{h*\phi\}$ uniformly on K by the relation

$$(h*\phi)(s)=\int h(st)\phi(t^{-1})dt = \langle\check{\phi}, h_{s^{-1}}\rangle, \qquad s\in G.$$

Note that ϕ_i satisfies the assumption of (10.2) for sufficiently large i, because $h\in L^1(G)$. The proof is thus complete by (10.2) together with the usual 3ϵ-technique. □

Exercise 10.2. Let $f\in C_c(G)$, and assume that $u_V\in C_c(G)$ satisfies the assumptions of (2.9). Show that $\{f*u_V\}$ converges to f uniformly as V becomes smaller. Use the *polarization identity*;

$$4f*g=\sum_{k=0}^{3} i^k(f+ig)*\widetilde{(f-ig)},$$

to conclude that every continuous function on G is the limit of linear combinations of pure positive definite functions, in the uniform limit on every compacta.

Proposition 10.2. *Let $s\neq e$ in a locally compact group G. Then there is a continuous pure positive definite function ϕ on G such that $\phi(s)\neq\phi(e)$.*

Proof. If $\phi(s)=\phi(e)$ for each pure positive definite function ϕ then the same relation holds for each $\phi\in P(G)$ by Proposition 10.1 and the discussion before the proposition. This is absurd by (10.1). □

Now, we investigate the properties of the unitary representation induced by a pure positive definite function. We write $\phi \ll \psi$ if $\psi - \phi$ is positive definite.

Lemma 10.3. *A positive definite function $\phi \in P(G)_1$ is pure if and only if every $\psi \in P(G)$ with $\psi \ll \phi$ is a scalar multiple of ϕ.*

Proof. If $0 \ll \psi \ll \phi$ then $0 \leq \psi(e) \leq \phi(e) = 1$. If $\psi(e) = 0$ then $\psi = 0$. If $\psi(e) = 1$ then $(\phi - \psi)(e) = 0$, and so $\phi = \psi$. If $0 < \psi(e) < 1$ then we infer that $\psi = \psi(e)\phi$ from the relation

$$\phi = (1 - \psi(e)) \left[\frac{1}{1 - \psi(e)} (\phi - \psi) \right] + \psi(e) \left[\frac{1}{\psi(e)} \psi \right],$$

and the extremity of ϕ. The converse is easier. □

Now, we assume that ϕ is a pure positive definite function with the associate unitary representation $\{\pi, \mathcal{H}\}$ and the cyclic vector ξ. Assume that E is a closed invariant subspace of $\mathcal{H}$ under $\{\pi_s : s \in G\}$, and denote by P the projection onto E. By a *projection*, we always mean a self-adjoint idempotent. By Exercise 9.3, E is also invariant under $\{\pi(f) : f \in L^1(G)\}$, and so $P\pi(f)P\eta = \pi(f)P\eta$ for each $f \in L^1(G)$ and $\eta \in \mathcal{H}$. Therefore, we have

$$P\pi(f) = (\pi(f^*)P)^* = (P\pi(f^*)P)^* = P\pi(f)P = \pi(f)P, \tag{10.3}$$

for each $f \in L^1(G)$. Now, we define the positive definite function ψ by $\psi(s) = \langle \pi_s P\xi, P\xi \rangle$ for $s \in G$. Then it follows that

$$\langle \psi, f^* * f \rangle = \|\pi(f)P\xi\|^2 = \|P\pi(f)\|^2 \leq \|\pi(f)\xi\|^2 = \langle \phi, f^* * f \rangle, \qquad f \in L^1(G),$$

by (9.7) and (10.3). This says that $\psi \ll \phi$, and so there is scalar λ such that $\phi = \lambda\psi$ by Lemma 10.3. Therefore, we have

$$\langle P\pi(f)\xi, \xi \rangle = \langle P\pi(f)\xi, P\xi \rangle = \langle \pi(f)P\xi, P\xi \rangle = \langle \lambda\pi(f)\xi, \xi \rangle, \qquad f \in L^1(G).$$

Because ξ is a cyclic vector, we have $P = \lambda 1_{\mathcal{H}}$. Since P is a projection, it follows that $P = 1_{\mathcal{H}}$ or $P = 0$. Therefore, we have shown the following:

Proposition 10.4. *The unitary representation associated by a pure positive definite function is irreducible.*

Theorem 10.5 (Gelfand-Raikov). *Every locally compact group G admits sufficiently many irreducible representations to separate elements of G.*

Proof. If $s \neq e$ then we can take a pure positive definite function ϕ such that $\phi(s) \neq \phi(e)$ by Proposition 10.2. If $\{\pi, \mathcal{H}\}$ is the irreducible representation associated by ϕ, then we have

$$\langle \pi_s \xi, \xi \rangle = \phi(s) \neq \phi(e) = \langle \pi_e \xi, \xi \rangle,$$

and so it follows that $\pi_s \neq \pi_e$. □

Corollary 10.6. *If every irreducible representation of a group G is one-dimensional then G is abelian.*

Proof. Note that one-dimensional representation is nothing but a multiplication by a scalar, and so they commutes each other. For any $s, t \in G$, we thus have

$$\pi_{st} = \pi_s \pi_t = \pi_t \pi_s = \pi_{ts}$$

for every irreducible representation π. By Theorem 10.5, we have $st = ts$. □

The converse is also true: Every irreducible representation of an abelian group is one-dimensional. Because the proof involves the spectral decomposition of unitary operators, we will prove here the partial converse: If $\{\pi, \mathcal{H}\}$ is a finite-dimensional irreducible representation of an LCA group then it is one-dimensional. For each $s \in G$, we take an eigenvalue $\lambda(s)$ of the unitary matrix π_s, with the corresponding eigenspace E_s. From the commutativity, we see that E_s is invariant under $\{\pi_t : t \in G\}$. By the irreducibility, we infer that $E_s = \mathcal{H}$. In other words, every π_s is a scalar operator, and so every subspace of $\mathcal{H}$ is invariant. This implies that $\mathcal{H}$ is one-dimensional space.

The converse of Proposition 10.4 also holds, and the proof involves the spectral decomposition again.

Exercise 10.3. Let ϕ be a continuous positive definite function whose associate unitary representation is irreducible and of finite dimension. Show that ϕ is pure.

Exercise 10.4. Assuming the converse of either Proposition 10.4 or Corollary 10.6, characterize all pure positive definite functions and irreducible representations of an LCA group.

Exercise 10.5. Show that one-dimensional representations π and ρ are unitarily equivalent if and only if $\pi = \rho$.

We will denote by $\widehat{G}$ the set of all equivalent class of irreducible unitary representations of a locally compact group G up to unitary equivalence. Note that $\widehat{G}$ is just a set (prove this!) without any additional structures. We confuse an irreducible representation σ and the equivalence class of σ in $\widehat{G}$ unless stated otherwise.

Parts of the following exercise have been already used in the proof of Proposition 10.4.

Exercise 10.6. Let E be a closed subspace of $\mathcal{H}$ with the projection p onto E. Show that E is invariant under $x \in \mathcal{B}(\mathcal{H})$ if and only if $xp = pxp$, and that E is invariant under $x \in \mathcal{B}(\mathcal{H})$ if and only if $E^{\perp}$ is invariant under x^*. Show also that both E and $E^{\perp}$ are invariant under x if and only if $xp = px$. Conclude that E is invariant under a $*$-subalgebra A of $\mathcal{B}(\mathcal{H})$ if and only if $xp = px$ for each $x \in A$.

Exercise 10.7. Under the above situation, show that E is invariant under a unitary representation $\{\pi, \mathcal{H}\}$ of a group G if and only if $p\pi_s = \pi_s p$ for each $s \in G$ if and only if $E^{\perp}$ is invariant under π. Finally, prove that every finite dimensional representation is the direct sum of irreducible representations.

Now, we exhibit a concrete example of an irreducible representation of the symmetric group S_n. We denote by sgn the homomorphism from S_n into $\mathbb{T}$ which assigns the signs of permutations. Because the alternating group A_n is simple, we see that the trivial homomorphism 1 and sgn are only two one-dimensional representations of S_n. It is clear that they are not unitarily equivalent. There is a natural representation of S_n on n-dimensional space which permutes the basis. We take the usual orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $\mathbb{C}^n$. Define the unitary representation of S_n on $\mathbb{C}^n$ by

$$\nu_\sigma : e_i \mapsto e_{\sigma(i)}, \qquad \sigma \in S_n,\ i = 1, 2, \dots, n. \tag{10.4}$$

There is an obvious invariant vector $\xi = e_1 + \cdots + e_n$. Assume that W is an invariant subspace in the orthogonal complement of the one dimensional subspace $[\xi]$ generated by ξ. If $w = \sum_{i=1}^n a_i e_i$ in W then $\sum_{i=1}^n a_i = 0$, and so there are i and j with $i \neq j$, $a_i \neq 0$, $a_j \neq 0$ and $a_i \neq a_j$. Considering the transposition $\sigma = (i, j)$, we see that $w - \nu_\sigma(w) = (a_i - a_j)(e_i - e_j) \in W$, and so it follows that $e_i - e_j \in W$. If we consider the permutation which sends i and j to k and ℓ, respectively, then $e_k - e_\ell \in W$ for any k, ℓ with $k \neq \ell$. It is clear that they span $[\xi]^\perp$. We have thus shown that the restriction of ν to $[\xi]^\perp$ is irreducible unitary representation on the $n-1$ dimensional space.

Now, we restrict our attention to the simplest noncommutative group S_3 which generated by $a = (1,2)$ and $b = (1,2,3)$ with the relations $a^2 = b^3 = e$ and $ab = b^2 a$. We list up elements of S_3 by

$$e, \quad b = (1,2,3), \quad b^2 = (1,3,2), \quad a = (1,2), \quad ab = (1,3), \quad ab^2 = (2,3).$$

With respect to the usual basis, the representation ν is expressed by

$$\nu_a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \nu_b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In order to express the above 2-dimensional irreducible representation with matrices, we choose orthonormal basis

$$\xi = \sqrt{\frac{1}{3}}\,(1,1,1), \quad \eta = \sqrt{\frac{2}{3}}\,(c,c,1), \quad \zeta = \sqrt{\frac{2}{3}}\,(-s,s,0),$$

where $c = \cos\frac{2}{3}\pi = -\frac{1}{2}$ and $s = \sin\frac{2}{3}\pi = \frac{\sqrt{3}}{2}$. We denote by P the orthogonal matrix whose columns are ξ, η and ζ. Then we have

$$P^{-1}\nu_a P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad P^{-1}\nu_b P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}.$$

Therefore, we get the following 2-dimensional irreducible representation of S_3:

$$a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} \cos\frac{2}{3}\pi & \sin\frac{2}{3}\pi \\ -\sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi \end{pmatrix}. \tag{10.5}$$

Exercise 10.8. Construct a 2-dimensional irreducible unitary representation of the dihedral group D_n which is generated by order 2 element a and order n element b with the relation $ab = b^{-1}a$.

We closed this section with a few comments on irreducible representations of finite groups. It is plain that every irreducible representation of a finite group is finite dimensional. We begin with the following simple lemma:

Lemma 10.7 (Schur). *Let $\{\pi, \mathcal{H}_\pi\}$ and $\{\rho, \mathcal{H}_\rho\}$ be finite dimensional irreducible unitary representations of a group G. If U is a linear map from $\mathcal{H}_\pi$ into $\mathcal{H}_\rho$ such that $\rho_s U = U\pi_s$ for each $s \in G$, then either U is the zero map or a bijection.*

Proof. It is easy to see that $\operatorname{Ker} U$ and $\operatorname{Im} U$ are invariant under π and ρ, respectively. This proves the lemma by the irreducibilities. □

Corollary 10.8. *Let $\{\pi, \mathcal{H}\}$ be a finite dimensional irreducible unitary representation of a group G. Assume that $x \in \mathcal{B}(\mathcal{H})$ commutes with every π_s, that is, $x\pi_s = \pi_s x$ for each $s \in G$. Then x is a scalar operator.*

Proof. Let λ be an eigenvalue of x. Then we have $(\lambda 1_\mathcal{H} - x)\pi_s = \pi_s(\lambda 1_\mathcal{H} - x)$ for each $s \in G$. By Lemma 10.7, we have $x = \lambda 1_\mathcal{H}$. □

Note that we have already proved the following, during the proof of Proposition 10.4: If the scalars are only operators which commute with every π_s then π is irreducible. See also Exercise 10.4. Corollary 10.8 provides a partial converse. A continuous linear map $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ with $U\pi_s = \sigma_s U$ for each $s \in G$ is called an *intertwining operator* for $\{\pi, \mathcal{H}_\pi\}$ and $\{\sigma, \mathcal{H}_\sigma\}$. Noting that $\mathcal{H}_\pi$ and $\mathcal{H}_\sigma$ are G-modules, an intertwining operator is nothing but a G-module map.

Exercise 10.9. Assume that $\{\pi, \mathcal{H}_\pi\}$ and $\{\sigma, \mathcal{H}_\sigma\}$ are two finite dimensional irreducible representations which are not unitarily equivalent. Show that the zero map is the only intertwining operator for π and σ.

For a subset S of $\mathcal{B}(\mathcal{H})$, we define the *commutant* S' of S by

$$S' = \{y \in \mathcal{B}(\mathcal{H}) : xy = yx \quad \text{for each} \quad x \in S\}.$$

Lemma 10.9. *Let $\mathcal{H}$ be an n-dimensional Hilbert space and A a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then we have $A'' = A$.*

Proof. It is clear that $A \subseteq A''$. We denote by $\mathcal{H}^{\oplus n}$ the direct sum of n-copies of $\mathcal{H}$. For $x \in \mathcal{B}(\mathcal{H})$, we also denote $\widetilde{x} \in \mathcal{B}(\mathcal{H}^{\oplus n})$, the direct sum of n copies of x, that is,

$$\widetilde{x}(\xi) = \sum_{i=1}^{n} {}^{\oplus} x\xi_i, \qquad \xi = \sum_{i=1}^{n} {}^{\oplus} \xi_i \in \mathcal{H}^{\oplus n}.$$

Then $\widetilde{A} = \{\widetilde{x} : x \in A\}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}^{\oplus n})$, and the subspace $\widetilde{A}\xi = \{\widetilde{x}\xi : x \in A\}$ is invariant under $\widetilde{A}$, and so the projection p onto $\widetilde{A}\xi$ lies in $(\widetilde{A})'$, by Exercise 10.6. If $x \in A''$ then it is easy to see that $\widetilde{x} \in (\widetilde{A})''$, and $\widetilde{x}p = p\widetilde{x}$. By Exercise 10.6 again, we see that $\widetilde{A}\xi$ is invariant under $\widetilde{x}$. Note that $\xi = \widetilde{I}\xi \in \widetilde{A}\xi$, and so $\widetilde{x}\xi \in \widetilde{A}\xi$. If we take $\xi = \sum_i^{\oplus} \xi_i$ with linearly independent $\{\xi_i\}$ in $\mathcal{H}$, then we see that $x \in A$. □

Exercise 10.10. Show that every $*$-subalgebra of a matrix algebra is $*$-isomorphic to a direct sum of matrix algebras.

Now, we consider the induced representation σ of $\ell^1(G)$ of an irreducible unitary representation $\{\sigma, \mathcal{H}_\sigma\}$ of a finite group G. If $f = \sum_{s\in G} a_s\chi_s \in \ell^1(G)$ then $\sigma(f) = \sum_{s\in G} a_s\sigma_s$, and so the range of σ is generated by $\{\sigma_s : s \in G\}$. Combining Corollary 10.8 and Lemma 10.9, we see that σ is a surjective $*$-homomorphism from $\ell^1(G)$ onto $\mathcal{B}(\mathcal{H}_\sigma)$. Applying Theorem 10.5, we have the following:

Proposition 10.10. *Let G be a finite group. Then the sum of all induced representations of irreducible representations gives rise to the $*$-isomorphism:*

$$\sum_{\sigma\in\widehat{G}}{}^{\oplus}\sigma : \ell^1(G) \xrightarrow{\simeq} \sum_{\sigma\in\widehat{G}}{}^{\oplus}\mathcal{B}(\mathcal{H}_\sigma). \tag{10.6}$$

As an immediate consequence, we have

$$\text{The order of } G = \sum_{\sigma\in\widehat{G}}(\dim\mathcal{H}_\sigma)^2. \tag{10.7}$$

Especially, we see that $\widehat{G}$ is a finite set whenever G is a finite group. The formula (10.7) is very useful to determine the group algebra $\ell^1(G)$ and $\widehat{G}$. For example, we have $\ell^1(S_3) \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$, and so this shows that the three irreducible representations 1, sgn and (10.5) list up $\widehat{S_3}$.

11. Compact Groups

In this section, we restrict our attention to representations of compact groups. It turns out that every representation of a compact group is finite dimensional. Let $\{\pi, \mathcal{H}\}$ be a unitary representation of a group G. During the proof of Proposition 10.4, we have seen that if $\pi_s x = x\pi_s$ for each $x \in \mathcal{B}(\mathcal{H})$ then π is irreducible. See also Exercise 10.4. The converse is also true, and this is a generalization of Corollary 10.8. For the proof, the spectral decomposition theorem for single operators are indispensable.

Spectral Theorem. *Let $x \in \mathcal{B}(\mathcal{H})$ be a normal operator; $x^*x = xx^*$. Then every Borel subset A of $\mathrm{sp}(x)$ corresponds to a projection P_A with the following properties:*

(i) $P_\emptyset = 0$ and $P_{\mathrm{sp}(x)} = 1_{\mathcal{H}}$.

(ii) $P_{A\cap B} = P_A P_B$. *If $A \cap B = 0$ then $P_{A\cup B} = P_A + P_B$.*

(iii) *An operator $y \in \mathcal{B}(\mathcal{H})$ commutes with x if and only if y commutes with every P_A.*

(iv) *For each $\xi, \eta \in \mathcal{H}$, the set function $A \mapsto \langle P_A\xi, \eta\rangle$ is a complex measure, denoted by $\mu_{\xi,\eta}$.*

(v) *For each $\xi, \eta \in \mathcal{H}$, we have*

$$\langle x\xi, \eta\rangle = \int_{\mathrm{sp}(x)} \lambda d\mu_{\xi,\eta}(\lambda). \tag{11.1}$$

The projections $\{P_A : A \text{ is a Borel subset of } \mathrm{sp}(x)\}$ are called the *spectral projections*. If $\mathcal{H}$ is finite dimensional then $\mathrm{sp}(x)$ is a finite set and the spectral projection associated to a point is nothing but the projection onto the corresponding eigenspace.

Proposition 11.1. *Let $\{\pi, \mathcal{H}\}$ be a unitary representation of a locally compact group G. Then the following are equivalent:*

(i) *π is irreducible.*

(ii) *If $x \in \mathcal{B}(\mathcal{H})$ commutes with π_s for each $s \in G$ then x is a scalar operator.*

Proof. It suffices to show the direction (i) $\Longrightarrow$ (ii). Assume that π is an irreducible representation of G and $x \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator which commutes with every π_s. Then every spectral projection P_A for x commutes with π_s for each $s \in G$. By Exercise 10.7, we see that the range of P_A is an invariant space, and so $P_A = 0$ or $P_A = 1_{\mathcal{H}}$ for each Borel set A. By the second property of the spectral projections, we see that there is a single point $\{\lambda\}$

in $\mathrm{sp}(x)$ whose spectral projection is $1_{\mathcal{H}}$, and this shows that $x = \lambda 1_{\mathcal{H}}$. The general case follows from the fact that every element is the sum of self-adjoint elements by (5.13). $\square$

Exercise 11.1. Prove the converses of Proposition 10.4 and Corollary 10.6.

Theorem 11.2. *Every irreducible unitary representation of a compact group G is finite dimensional.*

Proof. Assume that $\{\pi, \mathcal{H}\}$ is an irreducible unitary representation of G. We fix $\xi, \eta \in \mathcal{H}$. Then the map $\zeta \mapsto \int_G \langle \pi_s \eta, \zeta \rangle \overline{\langle \pi_s \eta, \xi \rangle}\, ds$ is a bounded conjugate-linear functional on $\mathcal{H}$ by the compactness of G, and so there is a vector, denoted by $B_\eta \xi$, such that

$$\langle B_\eta \xi, \zeta \rangle = \int_G \langle \pi_s \eta, \zeta \rangle \overline{\langle \pi_s \eta, \xi \rangle}\, ds \qquad \xi, \eta, \eta \in \mathcal{H}. \tag{11.2}$$

It is plain that $\xi \mapsto B_\eta \xi$ is a bounded linear operator on $\mathcal{H}$ for each $\eta \in \mathcal{H}$. Now, we have

$$\langle B_\eta \pi_t \xi, \zeta \rangle = \int_G \langle \pi_s \eta, \zeta \rangle \overline{\langle \pi_{t^{-1}s} \eta, \xi \rangle}\, ds = \int_G \langle \pi_{ts} \eta, \zeta \rangle \overline{\langle \pi_s \eta, \xi \rangle}\, ds = \langle \pi_t B_\eta \xi, \zeta \rangle,$$

and so $B_\eta \pi_t = \pi_t B_\eta$ for each $t \in G$ and $\eta \in \mathcal{H}$. By Proposition 11.1, B_η is a scalar operator $\lambda(\eta) 1_{\mathcal{H}}$. Taking $\zeta = \xi$ in (11.2), we obtain

$$\lambda(\eta) \|\xi\|^2 = \int_G |\langle \pi_s \eta, \xi \rangle|^2 ds. \tag{11.3}$$

Since G is unimodular, the roles of ξ and η may be replaced in the right side of (11.3), and so we have

$$\lambda(\eta) \|\xi\|^2 = \lambda(\xi) \|\eta\|^2 =: c, \qquad \xi, \eta \in \mathcal{H}.$$

If we take $\xi = \eta$ in (11.3), we see that c is a positive number.

For any orthonormal set $\{\xi_1, \ldots, \xi_n\}$ in $\mathcal{H}$, the set $\{\pi_s(\xi_i) : i = 1, 2, \ldots, n\}$ is also orthonormal, and so

$$nc = \sum_{i=1}^{n} \int_G |\langle \pi_s(\xi_i), \xi_1 \rangle|^2 ds = \int_G \sum_{i=1}^{n} |\langle \pi_s(\xi_i), \xi_1 \rangle|^2 ds \le \int_G ds = 1.$$

It follows that $\mathcal{H}$ is finite dimensional with $\dim \mathcal{H} \le \dfrac{1}{c}$. $\square$

As a special case of the Gelfand-Raikov theorem, we have the following:

Theorem 11.3 (Peter-Weyl). *Every compact group admits sufficiently many finite dimensional irreducible unitary representations.*

We continue to study properties of finite dimensional representations in the relation of regular representation of a compact group G. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then, the map $(x, y) \mapsto \mathrm{Tr}(y^*x)$ is a sesqui-linear form on $\mathcal{B}(\mathcal{H})$ which is positive definite. We denote by $\mathcal{T}(\mathcal{H})$ the Hilbert space $\mathcal{B}(\mathcal{H})$ with the inner product

$$(11.4) \qquad \langle x, y\rangle_{\mathrm{Tr}} = \mathrm{Tr}(y^*x), \qquad x, y \in \mathcal{B}(\mathcal{H}).$$

Note that $\mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ as sets since $\mathcal{H}$ is finite dimensional. For a finite dimensional representation $\{\sigma, \mathcal{H}_\sigma\}$, we define

$$(11.5) \qquad \ddot{\sigma}_s(x) = \sigma_s \circ x, \qquad s \in G,\ x \in \mathcal{T}(\mathcal{H}_\sigma).$$

Then it is easy to see that $\ddot{\sigma}$ is a unitary representation of G on the Hilbert space $\mathcal{T}(\mathcal{H}_\sigma)$. Now, for $x \in \mathcal{T}(\mathcal{H}_\sigma)$, we also define

$$\xi_x(s) = \langle \sigma_s, x^*\rangle_{\mathrm{Tr}} = \mathrm{Tr}(x \circ \sigma_s), \qquad s \in G.$$

Then ξ_x is a continuous function on G, and so $\xi_x \in L^2(G)$. Now, we have

$$[\rho_s(\xi_x)](t) = \xi_x(ts) = \mathrm{Tr}(x\sigma_t\sigma_s) = \mathrm{Tr}(\sigma_s x \sigma_t) = \xi_{\sigma_s x}(t),$$

for each $s, t \in G$, where $s \mapsto \rho_s$ is the right regular representation in (9.3), and so it follows that

$$(11.6) \qquad \rho(\xi_x) = \xi_{\sigma_s x}, \qquad s \in G,\ x \in \mathcal{T}(\mathcal{H}_\sigma).$$

This means that $V_\sigma : x \mapsto \xi_x : \mathcal{T}(\mathcal{H}_\sigma) \to L^2(G)$ is an intertwining operator for $\{\ddot{\sigma}, \mathcal{T}(\mathcal{H}_\sigma)\}$ and $\{\rho, L^2(G)\}$. We denote by E_σ the range of V_σ in $L^2(G)$. Especially, E_σ is an invariant subspace of $L^2(G)$ under the right regular representation.

Exercise 11.2. Let E be a finite dimensional subspace of $L^2(G)$ which is invariant under the right regular representation ρ of a group G. For each $\xi \in E$, show that there is $x \in \mathcal{B}(E)$ such that

$$\xi(s) = \mathrm{Tr}[x \circ (\rho|_E)_s], \qquad s \in G.$$

Exercise 11.3. Let σ be a unitary representation on the n-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_1, \ldots, e_n\}$. Show that the map $U : \mathcal{T}(\mathcal{H}) \to \mathcal{H}^{\oplus n}$ given by

$$U : x \mapsto (xe_1, \ldots, xe_n), \qquad x \in \mathcal{T}(\mathcal{H})$$

is a Hilbert space isomorphism. Conclude that $\{\ddot{\sigma}, \mathcal{T}(\mathcal{H})\}$ is unitarily equivalent to the direct sum $\{\sigma^{\oplus n}, \mathcal{H}^{\oplus n}\}$ of the n copies of $\{\sigma, \mathcal{H}\}$.

Exercise 11.4. Let $\{\sigma, \mathcal{H}_\sigma\}$ be an irreducible representation of a compact group G. Show that every $x \in \mathcal{B}(\mathcal{H}_\sigma\}$ is a finite linear combination of $\{\sigma_s : s \in G\}$. (See the paragraph preceding Proposition 10.10.) Conclude that if σ is irreducible then V_σ is a linear isomorphism from $\mathcal{T}(\mathcal{H}_\sigma)$ into $L^2(G)$.

We actually show that V_σ is a Hilbert space isomorphism (up to constant multiples) whenever $\sigma \in \widehat{G}$, and we also clarify the relations E_σ and E_τ for two irreducible representations σ and τ. These are, of course, reduced to calculate the inner product

$$(11.7) \qquad \langle V_\sigma(x), V_\tau(y)\rangle_{L^2(G)} = \int_G \operatorname{Tr}(x\sigma_s)\overline{\operatorname{Tr}(y\tau_s)}ds.$$

for $x \in \mathcal{T}(\mathcal{H}_\sigma)$ and $y \in \mathcal{T}(\mathcal{H}_\tau)$. It is plain that the results would follow if we consider rank one operators. For two vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, we denote by $\xi \otimes \eta$ the rank one operator from $\mathcal{H}$ into $\mathcal{K}$ by

$$(11.8) \qquad (\xi \otimes \eta)(\zeta) = \langle \zeta, \xi\rangle \eta, \qquad \zeta \in \mathcal{H}.$$

Exercise 11.5. Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Show that $\operatorname{Tr}[(u \otimes v)x] = \langle xv, u\rangle$ for $u, v \in \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$.

Note that the formula $\operatorname{Tr}(u \otimes v) = \langle v, u\rangle$ may be considered as the definition of the trace free from orthonormal basis. Now, for $u, v \in \mathcal{H}_\sigma$ and $c, d \in \mathcal{H}_\tau$, we calculate

$$(11.9) \qquad \begin{aligned} \operatorname{Tr}[(u \otimes v)\sigma_s]\, \overline{\operatorname{Tr}[(c \otimes d)\tau_s]} &= \langle \sigma_s v, u\rangle \, \langle c, \tau_s d\rangle \\ &= \langle \sigma_s(\langle \tau_{s^{-1}} c, d\rangle v), u\rangle = \langle \sigma_s(d \otimes v)\tau_{s^{-1}} c, u\rangle. \end{aligned}$$

Let $\phi : \mathcal{H}_\tau \to \mathcal{H}_\sigma$ be a linear map. The above calculation suggests to define the linear map $\Phi : \mathcal{H}_\tau \to \mathcal{H}_\sigma$ by

$$(11.10) \qquad \langle \Phi c, u\rangle = \int_G \langle \sigma_s \phi \tau_{s^{-1}} c, u\rangle \, ds, \qquad c \in \mathcal{H}_\tau, \ u \in \mathcal{H}_s.$$

Because G is compact, no integrability problem arises.

Lemma 11.4. *Under the above situation, the linear map Φ is an intertwining operator for τ and σ.*

Proof. The proof is the straightforward calculation using the left invariance of the Haar measure:

$$\begin{aligned}\langle \sigma_t \Phi c, u\rangle &= \langle \Phi c, \sigma_{t^{-1}} u\rangle \\ &= \int \langle \sigma_s \phi \tau_{s^{-1}} c, \sigma_{t^{-1}} u\rangle \, ds \\ &= \int \langle \sigma_{ts} \phi \tau_{s^{-1}} c, u\rangle \, ds \\ &= \int \langle \sigma_s \phi \tau_{s^{-1}t} c, u\rangle \, ds = \langle \Phi \tau_t c, u\rangle . \quad \square\end{aligned}$$

If σ and τ are irreducible representations which are not unitarily equivalent each other, then Φ is thus the zero map by Exercise 10.9. Therefore, we see that E_σ and E_τ are orthogonal subspaces of $L^2(G)$ by (11.7) and (11.9). If σ and τ are unitarily equivalent irreducible representations then it is clear that $E_\sigma = E_\tau$, and we may assume that $\sigma = \tau$ is an irreducible representation on $\mathcal{H}_\sigma$. By Corollary 10.8, we have $\Phi = \lambda 1_{\mathcal{H}_\sigma}$ for a scalar λ. In order to determine λ, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathcal{H}_\sigma$. Then it follows that

$$n\lambda = \sum_{i=1}^n \langle \Phi e_i, e_i\rangle = \int \sum_{i=1}^n \langle \phi \sigma_{s^{-1}} e_i, \sigma_{s^{-1}} e_i\rangle \, ds = \int \mathrm{Tr}(\phi) ds = \mathrm{Tr}(\phi),$$

and so, $\Phi = \dfrac{1}{n}\mathrm{Tr}(\phi) 1_{\mathcal{H}_\sigma}$, or we have

$$\int_G \langle \sigma_s \phi \sigma_{s^{-1}} c, u\rangle \, ds = \frac{\mathrm{Tr}(\phi)}{\dim \mathcal{H}_\sigma} \langle c, u\rangle, \qquad \phi \in \mathcal{B}(\mathcal{H}_\sigma),\ c, u \in \mathcal{H}_\sigma, \tag{11.11}$$

whenever $\{\sigma, \mathcal{H}_\sigma\}$ is an irreducible unitary representation of a compact group G. Therefore, we have

$$\langle V_\sigma(u \otimes v), V_\sigma(c \otimes d)\rangle_{L^2(G)} = \frac{1}{\dim \mathcal{H}_\sigma} \mathrm{Tr}(d \otimes v) \langle c, u\rangle = \frac{1}{\dim \mathcal{H}_\sigma} \langle v, d\rangle \, \langle c, u\rangle .$$

On the other hand, we also have

$$\langle u \otimes v, c \otimes d\rangle_{\mathrm{Tr}} = \sum_{i=1}^n \langle (u \otimes v) e_i, (c \otimes d) e_i\rangle = \langle c, u\rangle \, \langle v, d\rangle ,$$

and so, we get the following identity:

$$\langle V_\sigma(x), V_\sigma(y)\rangle_{L^2(G)} = \frac{1}{\dim \mathcal{H}_\sigma}\langle x, y\rangle_{\mathrm{Tr}}, \qquad x, y \in \mathcal{T}(\mathcal{H}_\sigma). \tag{11.12}$$

Theorem 11.5 (Peter-Weyl). *For a compact group G, we have*

$$L^2(G) = \sum_{\sigma\in\widehat{G}}{}^{\oplus} E_\sigma. \tag{11.13}$$

Furthermore, the right regular representation ρ is unitarily equivalent to the direct sum $\sum^{\oplus}_{\sigma\in\widehat{G}} \ddot{\sigma}$.

Proof. Note that every L^2-function on G is the limit of linear combinations of pure positive definite functions by Exercise 10.2. Now, we note by Proposition 10.4 that pure positive definite functions are obtained by elements of E_σ, because

$$\langle \sigma_s u, u\rangle = \mathrm{Tr}[(u\otimes u)\sigma_s] = \xi_{u\otimes u}(s) = [V_\sigma(u\otimes u)](s), \qquad u \in \mathcal{H}_\sigma,\ s \in G,$$

which exhaust all pure positive definite functions by Exercise 11.1. This proves (11.13). The last assertion follows from (11.13) and (11.12), because we already know that each V_σ is an intertwining operator by (11.6). □

By Exercises 11.3 and 9.11, we see that the regular representation of a compact group is decomposed by

$$\lambda \simeq \rho \simeq \sum_{\sigma\in\widehat{G}}{}^{\oplus}(\dim\sigma)\cdot\sigma, \tag{11.14}$$

where $n\cdot\sigma$ and "$\simeq$" denotes the direct sum of n copies of σ and the unitary equivalence, respectively. If G is a finite group then we recover (10.7) from (11.13) or (11.14). If G is a compact group which is not finite, then $\widehat{G}$ should be an infinite set: G has infinitely many non-equivalent irreducible representations. It does not mean that we need infinitely many irreducible representations in order to distinguish elements of G. Consider the *unitary group* $U(n)$ consisting of all $n\times n$ unitary matrices. The identity map σ on $U(n)$ is an irreducible representation which is *faithful*: $s\neq e$ implies $\sigma_s\neq 1$. Note that our proof of Theorem 11.5 is independent of Theorems 11.2 and

11.3, although we use the spectral theorem in order to show that the direct sum of (11.13) exhaust whole of $L^2(G)$. Theorem 11.5 or the relation (11.14) actually implies that there are sufficiently many finite dimensional irreducible representations.

Next thing to do is, of course, to find the explicit decomposition formula of L^2-functions with respect to (11.13). To do this, we introduce the notion of *character*, denoted by χ_π, of a finite dimensional unitary representation π: It is the function on G defined by

$$\chi_\pi(s) := V_\pi(1_{\mathcal{H}_\pi}) = \mathrm{Tr}(\pi_s), \qquad s \in G. \tag{11.15}$$

Now, assume that $\sigma, \tau \in \widehat{G}$. Then, for each $x \in \mathcal{T}(\mathcal{H}_\sigma)$, we have

$$\begin{aligned}
[V_\sigma(x) * \chi_\tau](t) &= \int_G \mathrm{Tr}(x\sigma_s)\,\mathrm{Tr}(\tau_{s^{-1}t})ds = \int_G \mathrm{Tr}(x\sigma_s)\,\overline{\mathrm{Tr}(\tau_{t^{-1}}\tau_s)}ds \\
&= \langle V_\sigma(x), V_\tau(\tau_{t^{-1}})\rangle_{L^2(G)} \\
&= \begin{cases} 0, & \text{if } \sigma \neq \tau \text{ in } \widehat{G}, \\ \dfrac{1}{\dim \mathcal{H}_\sigma}\langle x, \sigma_{t^{-1}}\rangle_{\mathrm{Tr}}, & \text{if } \sigma = \tau \text{ in } \widehat{G}, \end{cases}
\end{aligned}$$

by (11.7), (11.12) and Lemma 11.4. Note also that

$$\langle x, \sigma_{t^{-1}}\rangle_{\mathrm{Tr}} = \mathrm{Tr}(\sigma_t x) = [V_\sigma(x)](t), \qquad t \in G.$$

If we define the operator P_σ on $L^2(G)$ by

$$P_\sigma(\xi) = [\dim \mathcal{H}_\sigma]\,\xi * \chi_\sigma, \qquad \sigma \in \widehat{G},\ \xi \in L^2(G),$$

then the above calculations show that

$$P_\sigma(E_\tau) = 0 \ \text{ if } \ \sigma \neq \tau \ \text{ in } \ \widehat{G}, \qquad P_\sigma(\xi) = \xi \ \text{ if } \ \xi \in E_\sigma. \tag{11.16}$$

In other word, the set $\{P_\sigma : \sigma \in \widehat{G}\}$ is an orthogonal family of projections whose sum is equal to $1_{L^2(G)}$ by Theorem 11.5. Therefore, we have

$$\xi = \sum_{\sigma \in G} P_\sigma(\xi) = \sum_{\sigma \in G} [\dim \sigma]\xi * \chi_\sigma, \qquad \xi \in L^2(G). \tag{11.17}$$

Exercise 11.6. For an irreducible unitary representation σ, prove the following formula

$$[\dim\sigma]\int_G \chi_\sigma(rsr^{-1}t)dr = \chi_\sigma(s)\chi_\sigma(t), \tag{11.18}$$

for each $s, t \in G$. [Hint: Use (11.11).] Show also that $\langle\chi_\sigma, \chi_\tau\rangle = 0$ whenever $\sigma \neq \tau$ in $\widehat{G}$.

A function f on a group G is said to be *central* if it is invariant under inner automorphism. In other word, a central function is a function which is constant on every conjugacy class of G. For this reason, a central function is also called a *class function.* It is plain that every character is central. If $\xi \in L^2(G)$ is central then the formula (11.17) may be expressed more conveniently: We apply (11.18) to get

$$\begin{aligned}
[\dim\sigma](\xi * \chi_\sigma)(t) &= [\dim\sigma]\int \xi(s)\chi_\sigma(s^{-1}t)ds \\
&= [\dim\sigma]\iint \xi(r^{-1}sr)\chi_\sigma(s^{-1}t)dsdr \\
&= [\dim\sigma]\iint \xi(s)\chi_\sigma(rs^{-1}r^{-1}t)drds \\
&= \int \xi(s)\chi_\sigma(s^{-1})\chi_\sigma(t)ds = \langle\xi, \chi_\sigma\rangle\chi_\sigma(t).
\end{aligned}$$

Therefore, we have

$$\xi = \sum_{\sigma\in\widehat{G}}\langle\xi, \chi_\sigma\rangle\chi_\sigma, \qquad \text{whenever } \xi \in L^2(G) \text{ is central.} \tag{11.19}$$

This the noncommutative analogue of (7.12).

Exercise 11.7. Prove that $\chi_{\sigma_1\oplus\cdots\oplus\sigma_n} = \chi_{\sigma_1} + \cdots + \chi_{\sigma_n}$. Show that $\chi_\pi = \chi_\tau$ if and only if π and τ are unitarily equivalent. Show also that a finite-dimensional representation π is irreducible if and only if $\langle\chi_\pi, \chi_\pi\rangle = 1$. (Hint: Exercise 10.7.)

Exercise 11.8. If G is a finite group then show that the number of $\widehat{G}$ coincides with the number of conjugacy classes of G.

Exercise 11.9. Let G be a compact group. Show that $\widehat{G}$ is countable if and only if $L^2(G)$ is separable if and only if G is metrizable.

For an irreducible representation $\{\sigma, \mathcal{H}_\sigma\}$ of a compact group G, let $\{u_1, \dots, u_n\}$ be a fixed orthonormal basis of $\mathcal{H}_\sigma$. Then the set $\{u_i \otimes u_j : i, j = 1, \dots, n\}$ is an orthonormal basis for the Hilbert space $\mathcal{T}(\mathcal{H}_\sigma)$. The continuous function $C^\sigma_{i,j}$ defined by

$$C^\sigma_{i,j} : s \mapsto \langle \sigma_s u_j, u_i \rangle = [V_\sigma(u_i \otimes u_j)](s), \qquad s \in G \tag{11.20}$$

is called the *coefficient function* determined by σ and i, j. By Theorem 11.5, we see that $\{C^\sigma_{i,j} : \sigma \in \widehat{G}, i, j = 1, 2, \dots, \dim \sigma\}$ is an orthogonal basis for $L^2(G)$ consisting of continuous functions. If $G = \mathbb{T}$ then the coefficient function is nothing but a character. In this sense, a finite linear combination of coefficient functions is called a *trigonometric polynomial*, and we denote by $T(G)$ the dense space of all trigonometric polynomials on G. We will see in the next section that $T(G)$ is a dense $*$-subalgebra of $C(G)$.

12. Tannaka-Krein Duality

There are several versions for the dualities of compact groups. We study here one of the classical one, the Tannaka-Krein duality. Throughout this section, G is always a compact group unless mentioned otherwise. We take, as the dual object, the set V_G of all finite dimensional representations of G. We distinguish equivalent but different representations in V_G. We have already defined the direct sum in V_G. There is an another operation in V_G; the tensor product.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. We denote by $\mathcal{H}_1 \odot \mathcal{H}_2$ the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ as vector spaces. The *Hilbert space tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of of $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the unique inner product satisfying

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle, \qquad \xi_1, \xi_2 \in \mathcal{H}_1, \ \eta_1, \eta_2 \in \mathcal{H}_2. \tag{12.1}$$

Let $\{\eta_i : i \in I\}$ be an orthonormal basis of $\mathcal{K}$. Then $\mathcal{H} \otimes \mathcal{K}$ is nothing but the direct sum $\sum^{\oplus}_{i \in I} \mathcal{H}_i$, where $\mathcal{H}_i$ is the copy of $\mathcal{H}$, by the Hilbert space isomorphism

$$U : \sum_{i\in I}{}^{\oplus}\xi_i \mapsto \sum_{i\in I}\xi_i \otimes \eta_i : \sum_{i\in I}{}^{\oplus}\mathcal{H}_i \to \mathcal{H}\otimes\mathcal{K}. \tag{12.2}$$

The Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is characterized by the existence of a bilinear map $p : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ with the following property: For every *bounded* bilinear map $\phi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{K}$ into a Hilbert space $\mathcal{K}$, there exists a unique bounded linear map $\widetilde{\phi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$ such that $\phi = \widetilde{\phi} \circ p$. From this universal property, we may define the *tensor product* $x_1 \otimes x_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of $x_1 \in \mathcal{B}(\mathcal{H}_1)$ and $x_2 \in \mathcal{B}(\mathcal{H}_2)$ satisfying

$$(x_1 \otimes x_2)(\xi_1 \otimes \xi_2) = x_1\xi_1 \otimes x_2\xi_2, \qquad \xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2. \tag{12.3}$$

Exercise 12.1. Show that the tensor product of unitary operators is a unitary. More generally, show that $\|x_1 \otimes x_2\| = \|x_1\|\|x_2\|$ for $x_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$.

For unitary representations $\{\pi, \mathcal{H}_\pi\}$ and $\{\sigma, \mathcal{H}_\sigma\}$ of a locally compact group G, the *tensor product* $\pi \otimes \sigma$ is a unitary representation on the Hilbert space $\mathcal{H}_\pi \otimes \mathcal{H}_\sigma$ defined by

$$(\pi \otimes \sigma)_s = \pi_s \otimes \sigma_s, \qquad s \in G. \tag{12.4}$$

Exercise 12.2. Show that $\chi_{\pi\otimes\tau} = \chi_\pi\chi_\tau$ for finite dimensional unitary representations π and τ.

A *representation* of V_G is an operator Ω which assigns $\Omega(\pi) \in \mathcal{U}(\mathcal{H}_\pi)$ for each $\{\pi, \mathcal{H}_\pi\} \in \mathsf{V}_G$ with the following properties:

(A1) Whenever $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is an intertwining operator for π and σ in V_G, we have $U\Omega(\pi) = \Omega(\sigma)U$.

(A2) $\Omega(\pi \otimes \sigma) = \Omega(\pi) \otimes \Omega(\sigma)$ for any $\pi, \sigma \in \mathsf{V}_G$.

We denote by $\boldsymbol{\Omega}$ the set of all representations of V_G which do not vanish identically. We denote by $1 \in \mathsf{V}_G$ the trivial representation of G on $\mathbb{C}$.

Exercise 12.3. Show that $\Omega(1) = 1_{\mathbb{C}}$ for any $\Omega \in \boldsymbol{\Omega}$. Show also that $\Omega(\pi \oplus \sigma) = \Omega(\pi) \oplus \Omega(\sigma)$ for any $\pi, \sigma \in \mathsf{V}_G$.

Now, we define the multiplication in $\boldsymbol{\Omega}$ by

$$(\Omega_1\Omega_2)(\pi) = \Omega_1(\pi)\Omega_2(\pi), \qquad \Omega_1, \Omega_2 \in \boldsymbol{\Omega}, \ \pi \in \mathsf{V}_G.$$

We also topologize $\boldsymbol{\Omega}$ by defining that $\Omega_i \to \Omega$ if and only if

$$\Omega_i(\pi)\xi \to \Omega(\pi)\xi, \qquad \pi \in \mathsf{V}_G, \ \xi \in \mathcal{H}_\pi.$$

Exercise 12.4. Show that $\boldsymbol{\Omega}$ is a Hausdorff topological group under the above multiplication and topology.

Now, we fix a representative σ for each equivalent class of $\widehat{G}$ and an orthonormal basis $\{u_i^\sigma : i = 1, \dots, \dim\sigma\}$ of $\mathcal{H}_\sigma$. For each $\Omega \in \boldsymbol{\Omega}$, we define the linear functional ϕ_Ω on $T(G)$ by

$$\phi_\Omega : C_{i,j}^\sigma \mapsto \langle \Omega(\sigma)u_j^\sigma, u_i^\sigma\rangle, \qquad \sigma \in \widehat{G}, \ \ i, j = 1, 2, \dots, \dim\sigma,$$

where $C_{i,j}^\sigma$ is the coefficient function.

Lemma 12.1. *Under the above notations, we have the following:*

(i) *The map ϕ_Ω extends to a complex homomorphism on $C(G)$.*

(ii) *For $\pi \in \mathsf{V}_G$ and $\xi, \eta \in \mathcal{H}_\pi$, we define the continuous function $v_{\xi,\eta}^\pi$ on G by $v_{\xi,\eta}^\pi(s) = \langle \pi_s\eta, \xi\rangle$ for $s \in G$. Then we have*

$$\phi_\Omega(v_{\xi,\eta}^\pi) = \langle \Omega(\pi)\eta, \xi\rangle, \qquad \pi \in \mathsf{V}_G, \ \xi, \eta \in \mathcal{H}_\pi. \tag{12.5}$$

Before the proof of this lemma, we now state and prove the main theorem. Note that every $s \in G$ naturally defines a representation $\Omega_s \in \boldsymbol{\Omega}$ by

$$\Omega_s(\pi) = \pi_s, \qquad s \in G, \ \pi \in \mathsf{V}_G. \tag{12.6}$$

Theorem 12.2. *For a compact group G, the map $s \mapsto \Omega_s$ is a topological isomorphism from G onto $\boldsymbol{\Omega}$.*

Proof. It is plain that $s \mapsto \Omega_s$ is a continuous group homomorphism, which is injective by the Peter-Weyl theorem. In order to show that it is surjective, let $\Omega \in \mathbf{\Omega}$. Because ϕ_Ω is a complex homomorphism, we see that there is $s \in G$ such that

$$\phi_\Omega(f) = f(s), \qquad f \in C(G),$$

by Exercise 5.6. (Note that $\beta G = G$ since G is compact.) Now, we have

$$\langle \Omega(\pi)\eta, \xi\rangle = \phi_\Omega(v^\pi_{\xi,\eta}) = v^\pi_{\xi,\eta}(s) = \langle \pi_s\eta, \xi\rangle, \qquad \pi \in \mathsf{V}_G, \quad \xi, \eta \in \mathcal{H}_\pi,$$

and this completes the proof with $\Omega_s = \Omega$ $\square$

Let $\pi \in \mathsf{V}_G$ be irreducible. Then there is a representative $\sigma \in \widehat{G}$ and a Hilbert space isomorphism $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ which intertwine π and σ. For $\xi, \eta \in \mathcal{H}_\pi$, we denote by $\{a_i\}$ and $\{b_i\}$ the coordinates of $U\xi$ and $U\eta$ with respect to the basis $\{u^\sigma_i\}$. Then we have

$$\begin{aligned} v^\pi_{\xi,\eta}(s) &= \langle \pi_s\eta, \xi\rangle = \langle \sigma_s U\eta, U\xi\rangle \\ &= \sum_{i,j} \overline{a_i} b_j \langle \sigma_s u^\sigma_j, u^\sigma_i\rangle = \sum_{i,j} \overline{a_i} b_j C^\sigma_{i,j}(s), \end{aligned} \tag{12.7}$$

for each $s \in G$. By (A1), we have

$$\phi_\Omega(v^\pi_{\xi,\eta}) = \sum_{i,j} \overline{a_i} b_j \langle \Omega(\sigma) u^\sigma_j, u^\sigma_i\rangle = \langle \Omega(\sigma)U\eta, U\xi\rangle = \langle \Omega(\pi)\eta, \xi\rangle.$$

This proves the second assertion of Lemma 12.1 when $\pi \in \mathsf{V}_G$ is an irreducible representation of G. If $\pi = \pi_1 \oplus \pi_2$ is the direct sum of irreducible representations $\pi_1, \pi_2 \in \mathsf{V}_G$ then we see that $v^\pi_{\xi,\eta} = v^{\pi_1}_{\xi_1,\eta_1} + v^{\pi_2}_{\xi_2,\eta_2}$ for $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ in $\mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2}$. We use Exercise 12.3 to infer that

$$\phi_\Omega(v^\pi_{\xi,\eta}) = \langle \Omega(\pi_1)\eta_1, \xi_1\rangle + \langle \Omega(\pi_2)\eta_2, \xi_2\rangle = \langle [\Omega(\pi_1) \oplus \Omega(\pi_2)]\eta, \xi\rangle = \langle \Omega(\pi)\eta, \xi\rangle.$$

Note that every finite dimensional unitary representation is the direct sum of finitely many irreducible representations by Exercise 10.7. This completes the proof of the second assertion of Lemma 12.1.

For any $\sigma, \tau \in \widehat{G}$, we note that

$$C^\sigma_{i,j}(s)\, C^\tau_{k,\ell}(s) = \langle \sigma_s u^\sigma_j, u^\sigma_i\rangle \langle \tau_s u^\tau_\ell, u^\tau_k\rangle = \langle (\sigma \otimes \tau)_s (u^\sigma_j \otimes u^\tau_\ell), (u^\sigma_i \otimes u^\tau_k)\rangle, \tag{12.8}$$

for each $s \in G$. By the second part of Lemma 12.1 and (A2), we have

$$\begin{aligned}\phi_\Omega(C^\sigma_{i,j}\, C^\tau_{k,\ell}) &= \langle \Omega(\sigma \otimes \tau)(u^\sigma_j \otimes u^\tau_\ell), (u^\sigma_i \otimes u^\tau_k)\rangle \\ &= \langle [\Omega(\sigma) \otimes \Omega(\tau)](u^\sigma_j \otimes u^\tau_\ell), (u^\sigma_i \otimes u^\tau_k)\rangle \\ &= \langle \Omega(\sigma)u^\sigma_j, u^\sigma_i\rangle\, \langle \Omega(\tau)u^\tau_\ell, u^\tau_k\rangle = \phi_\Omega(C^\sigma_{i,j})\, \phi_\Omega(C^\tau_{k,\ell}).\end{aligned} \tag{12.9}$$

This shows that ϕ_Ω is multiplicative on $T(G)$. The relations (12.8) and (12.7) show that $T(G)$ is closed under multiplication because $\sigma \otimes \tau$ is the direct sum of irreducible representations as mentioned before, and the calculation (12.9) is legitimate. We proceed to show that the conjugate of a coefficient function lies in $T(G)$. Actually, we construct the irreducible representation $\overline{\sigma}$ whose coefficient functions are the conjugate of the coefficient functions of σ.

For $\xi \in \mathcal{H}$, we denote by $\overline{\xi}$ the element of $\mathcal{H}^*$ given by

$$\overline{\xi}(\eta) = \langle \eta, \xi\rangle, \qquad \eta \in \mathcal{H}. \tag{12.10}$$

Then $\xi \mapsto \overline{\xi} : \mathcal{H} \to \mathcal{H}^*$ is conjugate-linear; $\overline{a\xi} = \overline{a}\,\overline{\xi}$ for each $a \in \mathbb{C}$ and $\xi \in \mathcal{H}$. Therefore, we see that

$$\langle \overline{\xi}, \overline{\eta}\rangle_{\mathcal{H}^*} = \langle \eta, \xi\rangle_{\mathcal{H}}, \qquad \xi, \eta \in \mathcal{H} \tag{12.11}$$

defines an inner product on $\mathcal{H}^*$. With this inner product, the map $\xi \mapsto \overline{\xi}$ is a conjugate-linear isometry. For a representation $\{\pi, \mathcal{H}\}$ of a locally compact group G, we define the *conjugate representation*, denoted by $\overline{\pi}$, on the Hilbert space $\mathcal{H}^*$, by

$$(\overline{\pi}_s\, \overline{\xi})(\eta) = \overline{\xi}(\pi_{s^{-1}}\eta), \qquad \xi \in \mathcal{H}^*,\ \eta \in \mathcal{H}. \tag{12.12}$$

From the relation $(\overline{\pi}_s\, \overline{\xi})(\eta) = \langle \pi_{s^{-1}}\eta, \xi\rangle = \langle \eta, \pi_s\xi\rangle = \overline{\pi_s\xi}(\eta)$, we have

$$\overline{\pi}_s\, \overline{\xi} = \overline{\pi_s\xi}, \qquad s \in G,\ \ \xi \in \mathcal{H}. \tag{12.13}$$

From this, it is immediate that $\{\overline{\pi}, \mathcal{H}^*\}$ is a unitary representation of G.

Exercise 12.5. Show that E is an invariant subspace of $\mathcal{H}$ under π if and only if $\overline{E} = \{\overline{\xi} \in \mathcal{H}^* : \xi \in E\}$ is an invariant subspace under $\overline{\pi}$. Conclude that π is irreducible if and only if $\overline{\pi}$ is irreducible.

Now, for each $\sigma \in \widehat{G}$, we have the relation

$$\begin{aligned}\overline{C^\sigma_{i,j}(s)} = \overline{\langle \sigma_s u_j, u_i \rangle} &= \langle u_i, \sigma_s u_j \rangle \\ &= \langle \overline{\sigma_s u_j}, \overline{u_i} \rangle_{\mathcal{H}^*} = \langle \overline{\sigma}_s \overline{u_j}, \overline{u_i} \rangle_{\mathcal{H}^*} = C^{\overline{\sigma}}_{i,j}(s),\end{aligned}$$

for each $s \in G$, and so we see that the conjugate of a coefficient function is also a coefficient function. By the Stone-Weierstrass theorem, we have thus proved the following:

Proposition 12.3. *Let G be a compact group. Then the space $T(G)$ of all trigonometric polynomials becomes a unital dense $*$-subalgebra of $C(G)$.*

Exercise 12.6. Show that the direct sum $\sum^{\oplus}_{\sigma \in \widehat{G}} E_\sigma$ in Theorem 11.5 exhausts the whole $L^2(G)$, without Exercise 10.2.

Now, the proof of Lemma 12.1 would be complete by the following inequality:

$$|\phi_\Omega(f)| \le \|f\|_\infty, \qquad f \in T(G) \tag{12.14}$$

Recall that an $n \times n$ matrix $[u_{ij}]$ is unitary if and only if $\sum_k u_{ik}\overline{u_{jk}} = \sum_k \overline{u_{ki}} u_{kj} = \delta_{ij}$ for each $i, j = 1, 2, \dots, n$. Furthermore, a linear map x on a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{u_1, \dots, u_n\}$ is a unitary operator if and only if the $n \times n$ matrix $[\langle x u_i, u_j \rangle]$ is a unitary matrix. Because every σ_s is a unitary operator on $\mathcal{H}_\sigma$, we see that

$$\sum_k C^\sigma_{i,k}(s) \overline{C^\sigma_{j,k}(s)} = \delta_{ij}, \qquad s \in G, \;\; i, j = 1, \dots, n.$$

Note that 1_G is a coefficient function with $1_G = C^1_{1,1}$, and so $\phi_\Omega(1_G) = \langle \Omega(1)1, 1 \rangle = 1$ for each $\Omega \in \mathbf{\Omega}$, by Exercise 12.2. Therefore, we have

$$\sum_k \phi_\Omega(C^s_{i,k}) \phi_\Omega(\overline{C^\sigma_{j,k}}) = \phi_\Omega(\sum_k C^\sigma_{ik} \overline{C^\sigma_{j,k}}) = \delta_{ij}, \qquad i, j = 1, \dots, n. \tag{12.15}$$

Since every $\Omega(\sigma)$ is unitary, we see that the $\dim \sigma \times \dim \sigma$ matrix $[\phi_\Omega(C^\sigma_{i,j})]$ is a unitary matrix, whose inverse is $[\phi_\Omega(\overline{C^\sigma_{j,i}})]$ by (12.15), and so we have $\overline{\phi_\Omega(C^\sigma_{j,i})} = \phi_\Omega(\overline{C^\sigma_{j,i}})$ for each coefficient function $C^\sigma_{j,i}$. Now, we conclude that

$$\phi_\Omega(|f|^2) = \phi_\Omega(f\overline{f}) = \phi_\Omega(f)\phi_\Omega(\overline{f}) = \phi_\Omega(f)\overline{\phi_\Omega(f)} \ge 0,$$

for each trigonometric polynomial $f \in T(G)$. In other word, we have shown that ϕ_Ω is a unital positive linear functional on $T(G)$. If $f \in T(G)$ is real-valued then the relation $-\|f\|_\infty \le f \le \|f\|_\infty$ implies thus that $-\|f\|_\infty \le \phi_\Omega(f) \le \|f\|_\infty$. The desired relation (12.14) follows from

$$|\phi_\Omega(f)|^2 = \phi_\Omega(f)\phi_\Omega(\overline{f}) = \phi_\Omega(|f|^2) \le \|f\|_\infty^2, \qquad f \in T(G),$$

and this completes the proof of Lemma 12.1.

13. The Special Unitary and Orthogonal Groups

In this section, we study the *special unitary group* $SU(2)$ of order 2, which consists of all 2×2 unitary matrices s with $\det s = 1$.

Exercise 13.1. Show that every element of $SU(2)$ is of the form

$$s = \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix} \tag{13.1}$$

for some $(x,y,z,w) \in S^3 = \{(x,y,z,w) \in \mathbb{R}^4 : |x|^2+|y|^2+|z|^2+|w|^2 = 1\}$. Conclude that S^3 and $SU(2)$ are homeomorphic each other. Show also that this gives an isomorphism from the group of quaternions $x+yi+zj+wk$ with unit norms onto $SU(2)$.

In this way, $SU(2)$ and S^3 are topologically isomorphic each other. We define the vector space E by

$$E = \left\{ \widetilde{\mathbf{x}} = \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix} : \mathbf{x} = (x,y,z,w) \in \mathbb{R}^4 \right\}.$$

The the left translation $t \mapsto s^{-1}t$ on $SU(2)$ extends to a linear map L_s on E, which is also a linear map of $\mathbb{R}^4$ with the identification $\mathbf{x} \leftrightarrow \widetilde{\mathbf{x}}$. Note that $\|\mathbf{x}\|^2 = \det \widetilde{\mathbf{x}}$. Therefore, we have

$$\|L_s\mathbf{x}\|^2 = \det(L_s\widetilde{\mathbf{x}}) = \det s^{-1} \det \widetilde{\mathbf{x}} = \det \widetilde{\mathbf{x}} = \|\mathbf{x}\|^2, \qquad \mathbf{x} \in \mathbb{R}^4,$$

and so L_s is an orthogonal transformation of $\mathbb{R}^4$. Since the orthogonal group $O(4)$ has two component according to $\det = \pm 1$, we see that $L_s \in SO(4)$ for each $s \in SU(2)$. The same is true for the right translations. (Note that not every rotation arises in this way since $\dim SO(4) = 6$.) In other word, the translations on $SU(2)$ amount to the rotations on S^3 under the identification in Exercise 13.1. In order to construct the Haar measure on $SU(2)$, it thus suffices to find the rotation invariant measure on S^3.

Exercise 13.2. Write down the 4×4 orthogonal matrix L_s for each element $s \in SU(2)$ given by (13.1).

Exercise 13.3. Consider the map $r \mapsto s^{-1}rt$ of $SU(2)$ for two elements $s, t \in SU(2)$. Show that they induce rotations on S^3. Show also that every rotation of S^3 is obtained in this way.

If we put $x = \cos\theta$ with $0 \le \theta \le \pi$ in (13.1) then (y, z, w) is on the 2-sphere with the radius $\sin\theta$. Hence, we get the following parametrization:

(13.2)
$$x = \cos\theta, \quad y = \sin\theta\cos\phi, \quad z = \sin\theta\sin\phi\cos\psi, \quad w = \sin\theta\sin\phi\sin\psi,$$
$$\text{where } 0 \le \theta \le \pi, \quad 0 \le \phi \le \pi, \quad 0 \le \psi \le 2\pi.$$

It is a well-known fact in differential geometry that

(13.3)
$$\frac{1}{2\pi^2}\sin^2\theta\sin\phi d\theta d\phi d\psi$$

is the rotation invariant measure on S^3 whose total mass is 1. Therefore, we have

(13.4)
$$\int_{SU(2)} f(s)ds = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} f(\theta, \phi, \psi)\sin^2\theta\sin\phi d\theta d\phi d\psi.$$

Now, we determine the conjugacy classes of $SU(2)$. The characteristic function of $s \in SU(2)$ is given by $\lambda^2 - \lambda(\mathrm{Tr}s) + 1$, or equivalently $\lambda^2 - 2\lambda\cos\theta + 1$ with respect to (13.2). Therefore, there are $\theta \in [0, \pi]$ and a unitary matrix u such that

(13.5)
$$s = u^* h_\theta u \qquad \text{or} \qquad s = u^* h_{-\theta} u,$$

where $h_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Of course, the matrix u may be taken in $SU(2)$ by multiplying a constant in $\mathbb{T}$, and so every element of $SU(2)$ is conjugate to h_θ

for some $\theta \in [-\pi, \pi]$. Therefore, every central function f on $SU(2)$ may be considered on the function on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ by

$$f(s) = f(h_\theta) = f(\theta). \tag{13.6}$$

In other word, a central function is independent on the variables ϕ, ψ as a function on S^3. Therefore, we have

$$\int_{SU(2)} f(s)ds = \frac{2}{\pi}\int_0^\pi f(\theta)\sin^2\theta d\theta, \qquad \text{whenever } f \text{ is central.} \tag{13.7}$$

Exercise 13.4. Show that every central function f on $SU(2)$ is an even function of θ.

Now, we construct an $(n+1)$-dimensional irreducible representation for each natural number $n = 0, 1, 2, \ldots$. We denote by E_n the Hilbert space of all degree n homogeneous polynomials in two complex variables x and y, with the inner product

$$\left\langle \sum_{k=0}^n a_k x^k y^{n-k}, \sum_{k=0}^n b_k x^k y^{n-k} \right\rangle = \sum_{k=0}^n k!(n-k)! a_k \overline{b_k}. \tag{13.8}$$

Exercise 13.5. Let $\omega = e^{\frac{2\pi}{n}i}$ be the n-th root of unity. Show that the functions

$$(x+y)^n, \ (x+\omega y)^n, \ \ldots, \ (x+\omega^{n-1}y)^n, \ y^n$$

are linearly independent in E_n.

Now, we define the representation σ^n of $SU(2)$ on the space E_n by

$$(\sigma^n_s \xi)(z) = \xi(s^{-1}z), \qquad s \in SU(2), \ \xi \in E_n, \ z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2. \tag{13.9}$$

It is immediate that $s \mapsto \sigma^n_s$ is a homomorphism. For $u = (a, b) \in \mathbb{C}^2$, put

$$\xi_u(z) =: (uz)^n = (ax+by)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} x^k y^{n-k}.$$

For each $u = (a, b)$ and $v = (c, d)$ in $\mathbb{C}^2$, we have

$$\begin{aligned}\langle \xi_u, \xi_v \rangle_{E_n} &= \sum_{k=0}^{n} k!(n-k)! \binom{n}{k} a^k b^{n-k} \binom{n}{k} \overline{c}^k \overline{d}^{n-k} \\ &= n! \sum_{k=0}^{n} \binom{n}{k} (a\overline{c})^k (b\overline{d})^{n-k} \\ &= n!(a\overline{c} + b\overline{d})^n = n!\langle u, v\rangle_{\mathbb{C}^2}^n.\end{aligned}$$

Because $(\sigma_s^n \xi_u)(z) = \xi_u(s^{-1}z) = (us^{-1}z)^n = \xi_{us^{-1}}(z)$, it follows that

$$\langle \sigma_s^n \xi_u, \sigma_s^n \xi_v \rangle_{E_n} = n!\langle us^{-1}, vs^{-1} \rangle^n = n!\langle u, v\rangle^n = \langle \xi_u, \xi_v \rangle_{E_n},$$

since every s^{-1} is a unitary operator on $\mathbb{C}^2$. Note that every multiplications such as us, uz or sz in the above calculations are usual matrix multiplications. By Exercise 13.5, we have shown that every σ_s^n preserves the inner product on a basis of E_n, and so σ_s^n is a unitary for each $s \in SU(2)$.

Now, we proceed to find the character $\chi_n =: \chi_{\sigma^n}$ of σ^n. If we define $\xi_k(x, y) = x^k y^{n-k}$ for $k = 0, 1, \ldots, n$ then $\{\xi_0, \xi_1, \ldots, \xi_n\}$ is an orthogonal basis of E_n with $\|\xi_k\|^2 = k!(n-k)!$. Because every character is central, it suffices to find the value at the point h_θ for $\theta \in [-\pi, \pi]$. From the relation $\sigma_{h_\theta}^n \xi_k(x, y) = e^{i(n-2k)\theta} x^k y^{n-k}$, we have

$$\chi_n(h_\theta) = \sum_{k=0}^{n} \langle \sigma_{h_\theta}^n \xi_k, \xi_k \rangle \frac{1}{\|\xi_k\|^2} = \sum_{k=0}^{n} e^{i(n-2k)\theta}.$$

Therefore, it follows that

$$\chi_n(h_0) = n + 1 = \dim \sigma^n, \qquad \chi_n(h_\pi) = (-1)^n(n+1). \tag{13.10}$$

If $e^{i\theta} \neq \pm 1$ then

$$\chi_n(h_\theta) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{13.11}$$

Because

$$\langle \chi_n, \chi_n \rangle = \frac{2}{\pi} \int_0^{\pi} \sin^2(n+1)\theta d\theta = 1,$$

we conclude that σ^n is irreducible for each $n = 0, 1, 2, \dots$, by Exercise 11.7.

Assume that $\xi \in L^2(SU(2))$ is a central function with the property

$$(13.12) \qquad \langle \xi, \chi_n \rangle = 0, \qquad n = 0, 1, 2, \dots .$$

Note that

$$\langle \xi, \chi_n \rangle = \frac{2}{\pi} \int_0^\pi \xi(\theta) \chi_n(\theta) \sin^2 \theta d\theta = \frac{2}{\pi} \int_0^\pi \xi(\theta) \sin \theta \sin(n+1)\theta d\theta.$$

Since every central function is an even function of $\theta \in [-\pi, \pi]$, we see that $\eta(\theta) = \xi(\theta) \sin \theta$ is an odd function on $\mathbb{T}$, whose Fourier coefficients are given by

$$\widehat{\eta}(n) = \frac{i}{\pi} \int_o^\pi \eta(\theta) \sin n\theta d\theta = \frac{i}{2} \langle \xi, \chi_{n-1} \rangle = 0, \qquad n = 1, 2, \dots,$$

from the assumption (13.12), and so it follows that $\eta = 0$ and $\xi = 0$. By (11.19), we conclude that

$$(13.13) \qquad \widehat{SU(2)} = \{\sigma^n : n = 0, 1, 2, \dots\}.$$

Exercise 13.6. Express $\sigma^n \otimes \sigma^m$ in terms of the direct sum of irreducible representations up to unitary equivalence. [Hint: Use Exercises 12.2 and 11.7.]

Now, we consider an action of $SU(2)$ on the 3-dimensional vector space. Define

$$V = \left\{ \widetilde{\mathbf{x}} = \begin{pmatrix} x & y + iz \\ y - iz & -x \end{pmatrix} : \mathbf{x} = (x, y, z) \in \mathbb{R}^3 \right\}.$$

We also define a homomorphism $\tau : SU(2) \to \mathcal{H}(V)$ by

$$(13.14) \qquad \tau(s)(\widetilde{\mathbf{x}}) = s\widetilde{\mathbf{x}}s^*, \qquad s \in SU(2),\ \widetilde{\mathbf{x}} \in V.$$

Because $\det \widetilde{\mathbf{x}} = -\|\mathbf{x}\|_2^2$, we see that each $\tau(s)$ may be considered as a norm preserving linear map on $\mathbb{R}^3$. Because $O(3)$ has two components, we see that $\tau(s) \in SO(3)$ for each $s \in SU(2)$.

Exercise 13.7. Show that every element of $SO(3)$ arises in this way.

Because $\ker \tau = \{h_0, h_\pi\}$, we see that $SU(2)$ is a double covering of $SO(3)$. If n is an even natural number then $\sigma^n_{h_0} = \sigma^n_{h_\pi} = 1_{E_n}$, and so we see that there is a homomorphism $\widetilde{\sigma}^n : SO(3) \to \mathcal{U}(E_n)$ such that $\sigma^n = \widetilde{\sigma}^n \circ \tau$. Therefore, we have a sequence

$$\{\widetilde{\sigma}^n : n = 0, 2, 4, \dots\} \tag{13.15}$$

of irreducible representations of $SO(3)$.

Exercise 13.8. Show that the set (13.15) exhausts $\widehat{SO(3)}$.

NOTE

We have usually followed [DIXMIER, Chapter 13] in §9. The representation constructed in Proposition 9.3 is said to be the *Gelfand-Naimark-Segal construction*. It should be noted that every $*$-representation of a Banach $*$-algebra is automatically norm-decreasing, and so the continuity assumption is redundant in our definition of representation. The crucial condition (9.8) is also valid for any Banach $*$-algebra with bounded approximate identity. See [DIXMIER, Chapter 2] or [TAKESAKI, Chapter I]. The proof of Proposition 10.1 was also taken from [DIXMIER, §13.5]. The complete representation theory for symmetric groups may be found in [JAMES AND KERBER].

We have followed [HEWITT AND ROSS, Theorem 22.13] for the proof of Theorem 11.2, for which various proofs are available. For example, see [ROBERT, §4 and §8], or [VARADARAJAN, §2.1]. The spectral theorem is essential for these proofs. The compactness assumption in the Peter-Weyl theorem is crucial. For example, every finite dimensional unitary representation of the special linear group $SL(2,\mathbb{R})$ is trivial. See [ROBERT, §11]. The proof of the basic formula (11.12) was also taken from [ROBERT, §5]. For the further theory of characters on finite groups, we also refer to [FEIT], [LEDERMANN] or [SERRE]. The proof of Tannaka duality was extracted from [HEWITT AND ROSS, §30]. See also [ROBERT, §9].

We have followed [SUGIURA, Chapter II] for the representation theory of $SU(2)$. See also [ROBERT, §10] and [DYM AND MCKEAN, Chapter 4]. For the full descriptions of the duals of the unitary groups $U(n)$ and the orthogonal group $O(n)$ for arbitrary $n = 2, 3, \dots$, we refer to [HEWITT AND ROSS, §29], [TAYLOR], [VARADARAJAN] or [WEYL].

The lack of time prevents us to continue our study on general locally compact groups. One of the central theme is the notion of amenability. A locally compact group G is said to be *amenable* if there is a finitely additive left invariant measure m with $m(G) = 1$. Every abelian or compact group is amenable. Representations of amenable groups are relatively well-understood. We refer to the books such as [DIXMIER], [GREENLEAF], [PATERSON], [PIER] for further study of amenability.

The basic motivation of representation theory is to regard an element of an abstract group or a group algebra as a concrete operator on a Hilbert space. A crucial defect of the group algebra $L^1(G)$ is that its norm does not share the characteristic property of the operator norms: See (5.11) and (9.1). We endow $L^1(G)$ with various operator norms to get group operator algebras by taking completions. Group operator algebras of non-amenable groups have interesting properties in view of the theory of operator algebras. The free group is a typical example of non-amenable group, together with various matrix groups. We refer to [FIGÁ-TALAMANCA AND PICARDELLO] for further study on the free groups. The notes [KYE, §4.6] and [WASSERMANN] explain some recent results on group operator algebras of the free groups.

REFERENCES

1. C. BERG, J. P. R. CHRISTENSEN AND P. RESSEL, *Harmonic Analysis on Semigroups*, Graduate Texts Math., Vol 100, Springer-Verlag, 1984.
2. U. BOTTAZZINNI, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*, Springer-Verlag, 1986.
3. G. CANTOR, *Contributions to the Founding of the Theory of Transfinite Numbers*, Dover, 1955.
4. K. CHANDRASEKHARAN, *Classical Fourier Transforms*, Universitext, Springer-Verlag, 1989.
5. J. DIEUDONNÉ78, *Treatise on Analysis Vol. VI*, Pure Applied Math. Vol 10-VI, Academic Press, 1978.
6. J. DIEUDONNÉ81, *History of Functional Analysis*, North-Holland Math. Studies Vol. 49, North-Holland, 1981.
7. J. DIXMIER, *C^*-algebras*, North-Holland Math. Library, Vol. 15, A translation of "*Les C^*-algèbres et Leurs Représentations*, Gauthier-Villars, 1969", North-Holland, 1977.
8. H. DYM AND H. P. MCKEAN, *Fourier Series and Integrals*, Probab. and Math. Statist., Vol. 14, Academic Press, 1972.
9. R. E. EDWARDS, *Fourier Series, A Modern Introduction, I; II*, Holt, Rinehart and Winston, 1967.
10. W. FEIT, *Characters of Finite Groups*, Benjamin, 1967.
11. A. FIGÁ-TALAMANCA AND M. A. PICARDELLO, *Harmonic Analysis on Free Groups*, Lecture Notes Pure Appl. Math., Vol. 87, Dekker, 1983.
12. W. FULTON AND J. HARRIS, *Representation Theory*, Graduate Texts Math., Vol. 129, Springer-Verlag, 1991.
13. F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand, 1969.
14. H. HELSON, *Harmonic Analysis*, Addison-Wesley, 1983.
15. E. HEWITT AND K. A. ROSS, *Abstract Harmonic Analysis I, 2/e; II*, Grund. Math. Wiss., no. 115; 152, Springer-Verlag, 1979; 1970.
16. G. JAMES AND A. KERBER, *The Representation Theory of the Symmetric Group*, Encyclopedia Math. Appl., Vol. 16, Addison-Wesley, 1981.
17. Y. KATZNELSON, *An Introduction to Harmonic Analysis, 2/e*, Dover, 1976.
18. V. P. KHAVIN AND N. K. NIKOLSKIJ, *Commutative Harmonic Analysis, I*, Encyclopaedia Math. Sci., Vol. 15, Springer-Verlag, 1991.
19. S.-H. KYE, *Notes on Operator Algebras*, RIM-GARC Lecture Note Series, No. 7, Seoul National University, 1993.
20. W. LEDERMANN, *Introduction to Group Characters*, Cambridge Univ. Press, 1977.
21. L. H. LOOMIS, *Abstract Harmonic Analysis*, Univ. Series Higher Math., Van Nostrand, 1953.

22. D. A. MORRIS, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*, London Math. Soc. Lecture Note Series, no. 29, Cambridge Univ. Press, 1977.
23. A. T. Paterson, *Amenability*, Math. Surveys Monographs No. 29, Amer. Math. Soc., 1988.
24. J.-P. Pier, *Amenable Locally Compact Groups*, John Wiley & Sons, 1984.
25. L. S. PONTRYAGIN, *Topological Groups, 2/e*, Gordon and Breach, 1966.
26. C. E. RICKART, *General Theory of Banach Algebras*, Univ. Series Higher Math., Van Nostrand, 1960.
27. A. ROBERT, *Introduction to the Representation Theory of Compact and Locally Compact Groups*, London Math. Soc. Lecture Note Series, no. 80, Cambridge Univ. Press, 1983.
28. H. L. ROYDEN, *Real Analysis 3/e*, Maxwell McMillan, 1989.
29. W. RUDIN62, *Fourier Analysis on Groups*, Interscience Tracks Pure Appl. Math., Vol. 12, Interscience Publisher, 1962.
30. W. RUDIN73, *Functional Analysis*, McGraw-Hill, 1973.
31. W. RUDIN87, *Real and Complex Analysis, 3/e*, McGraw-Hill, 1987.
32. L. SCHWARTZ, *Mathematics for the Physical Sciences*, Addison-Wesley, 1966.
33. J.-P. SERRE, *Linear Representations of Finite Groups*, Graduate Texts Math., Vol. 42, A translation of "*Représentations Linéaires des Groupes Finis*, Hermann, 1971", Springer-Verlag, 1977.
34. M. SUGIURA, *Unitary Representations and Harmonic Analysis*, Kodansha, 1975.
35. M. TAKESAKI, *Theory of Operator Algebras I*, Springer-Verlag, 1979.
36. M. E. TAYLOR, *Noncommutative Harmonic Analysis*, Math. Surveys Monographs, No. 22, Amer. Math. Soc., 1986.
37. V. S. VARADARAJAN, *An Introduction to Harmonic Analysis on Semi-simple Lie Groups*, Cambridge Studies Adv. Math., Vol. 16, Cambridge Univ. Press, 1989.
38. S. WASSERMANN, *Exact C^*-algebras and Related Topics*, RIM-GARC Lecture Note Series, No. 19, Seoul National University, 1994.
39. H. WEYL, *The Classical Groups, 2/e*, Princeton Math. Series, Vol. 1, Princeton Univ. Press, 1953.
40. A. ZYGMUND, *Trigonometric Series, Vol. I and II*, Cambridge Univ. Press, 1968.

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