수학강의록 제 2 권

THE INTERPLAY BETWEEN TOPOLOGICAL DYNAMICS AND THEORY OF C^*-ALGEBRAS

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Preface

The present note is based on the author's series of lectures given at Seoul National University under the project of the Global Analysis Research Center in Seoul.

Comparing with the long history about the interplay between measurable dynamics and theory of von Neumann algebras or factors, there seems to have been considerable lack of results in C^*-versions for the interplay between topological dynamics and theory of C^*-algebras though there are intensive studies about the structure of transformation group C^*-algebras by Effros-Hahn and others. Restricting to a topological dynamical system with single homeomorphism it is thus the purpose of this series of lectures to fill out this lack to some extent providing recent aspects of the subject for both people who are working on topological dynamics and on operator algebras. We remark that references attached are quite optional. They are mainly chosen for those readers who are interested in further investigations in the subject.

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Jun Tomiyama

§1. Crossed products of C^*-algebras

Let X be a compact Hausdorff space. One may then recognize that the aspect of the space X reflects to the structure of the algebra of all complex valued continuous functions on X, $C(X)$. Suppose further that X admits the action of a homeomorphism σ. To consider the system $\Sigma = (X, \sigma)$ is then equivalent to consider the algebra $C(X)$ with the automorphism α induced by σ as $\alpha(f)(x) = f(\sigma^{-1}x)$, the action of the integer group $\mathbb{Z}$ on $C(X)$. In this aspect, however, functions and automorphisms are in different levels but we can often transplant them into the same kinds of objects, operators on a Hilbert space. The simplest way is to find the space $L^2(X, \mu)$ for a suitable measure μ on X so that a function f in $C(X)$ can be faithfully represented as a multiplication operator $\pi(f)$ on $L^2(X, \mu)$ and moreover there exists a unitary operator u such that $u\pi(f)u^* = \pi(\alpha(f))$. We have thus the situation of a so-called (faithful) covariant representation of the (commutative) C^*-dynamical system $(C(X), \mathbb{Z}, \alpha)$ and in principle the structure of the system $(C(X), \alpha)$ could reflect exactly to that of the C^*-algebra $C^*(\pi(C(X)), u)$ generated by $\pi(C(X))$ and u. This C^*-algebra becomes necessarily non-commutative unless the automorphism α is trivial. Now, by the covariant condition, $u\pi(f)u^* = \pi(\alpha(f))$, the algebra $C^*(\pi(C(X)), u)$ is regarded as the norm closure of the self-adjoint linear subspace consisting of those elements $\sum_{k=-n}^{n} \pi(f_k)u^k$. Among those C^*-algebras associated to many covariant representations of $(C(X), \mathbb{Z}, \alpha)$, it is then natural to ask (at least) the following conditions for the good generation of $\pi(C(X))$ and u, namely; the set $\{u^n \mid n \in \mathbb{Z}\}$ should be independent over the algebra $\pi(C(X))$ and moreover at least the norm condition $\|\sum_{k=-n}^{n} \pi(f_k)u^k\| \geq \|\pi(f_0)\|$. In this way, we finally reach the primitive concept of the C^*-crossed products $A(\Sigma) = C(X) \underset{\alpha}{\times} \mathbb{Z}$ concerned with the topological dynamical system $\Sigma = (X, \sigma)$.

Thus we start from the definition and basic properties of C^*-crossed products. We employ here a rather general context in order to understand the general situation. By a *C^*-dynamical system*, we mean a triplet (A, G, α) where A is a unital C^*-algebra, G a discrete group and the action α of G on A means a homomorphism from G into $\mathrm{Aut}(A)$, the group of all $*$-automorphisms. A pair $\{\pi, u\}$ of a representation π of A and a unitary representation u of G on a Hilbert space H is called a *covariant representation* of the system (A, G, α) if $u_s\pi(a)u_s^* = \pi(\alpha_s(a))$ for every $a \in A$ and $s \in G$. The *full crossed product* $A \underset{\alpha}{\times} G$ for the system (A, G, α) is then defined as the universal C^*-algebra for the family of covariant representations. In order to realize $A \underset{\alpha}{\times} G$ in a concrete way, we first consider the space of all A-valued functions on G with the ℓ^1-norm, $\ell^1(G, A)$. Define the product (twisted convolution) and $*$-operation in $\ell^1(G, A)$ so that it becomes a Banach $*$-algebra; for two functions $x = \{x(s)\}$ and $y = \{y(s)\}$,

$$x^*(s) = \alpha_s(x(s^{-1})^*), \quad xy(s) = \sum_t x(t)\alpha_t(y(t^{-1}s)),$$

where the norm convergence of the value $xy(s)$ in A as well as the sum $\sum_s \|xy(s)\|$ are assured by the ℓ^1-norm property. One may look at these operations somewhat technical but those are quite natural ones once we have a covariant situation. At first, the algebra A can be identified with the algebra of functions $\tilde{a}$'s defined as $\tilde{a}(e) = a$ (e is the unit of G) and vanishes elsewhere. Moreover, if we consider the function δ_s for $s \in G$ as a function vanishing on all points except at s where $\delta_s(s) = 1$, it becomes a unitary element of $\ell^1(G, A)$ satisfying the covariant relation

$$\delta_s a \delta_s^* = \alpha_s(a)$$

for every $a \in A = \tilde{A}$ (identified) and $s \in G$. With this covariant relation, we can expand those functions x and y as $x = \sum_s x(s)\delta_s$ and $y = \sum_s y(s)\delta_s$. The definitions of the above product and $*$-operation simply mean that we can proceed operations in $\ell^1(G, A)$ in a quite natural way, that is,

$$\begin{aligned} x^* &= \sum_s \delta_s^* x(s)^* = \sum_s \alpha_s^{-1}(x(s)^*)\delta_{s^{-1}} = \sum_s \alpha_s(x(s^{-1})^*)\delta_s, \\ xy &= (\sum_t x(t)\delta_t)(\sum_s y(s)\delta_s) = \sum_{s,t} x(t)\delta_t y(s)\delta_s \\ &= \sum_{s,t} x(t)\alpha_t(y(s))\delta_{ts} = \sum_s(\sum_t x(t)\alpha_t(y(t^{-1}s)))\delta_s. \end{aligned}$$

Let E be the projection of norm one from $\ell^1(G, A)$ to the (embedded) algebra A defined as $E(x) = x(e)$. The map E has obviously the module property, $E(axb) = aE(x)b$ for $a, b \in A$ and positivity in the sense,

$$E(x^*x) = x^*x(e) = \sum_s x^*(s)\alpha_s(x(s^{-1}))$$
$$= \sum_s \alpha_s(x(s^{-1})^* x(s^{-1})) \geq 0.$$

Hence E is *faithful*, namely $E(x^*x) = 0$ implies $x = 0$. It follows that the algebra $\ell^1(G, A)$ has sufficiently many representations and we can consider the C^*-envelope of $\ell^1(G, A)$, $C^*(\ell^1(G, A))$, as the completion of $\ell^1(G, A)$ with the norm

$$\|x\|_\infty = \sup \|\tilde{\pi}(x)\| \leq \|x\|_1$$

where $\tilde{\pi}$ is ranging over all representations of $\ell^1(G, A)$. Now one may easily verify that any covariant representation $\{\pi, u\}$ of (A, G, α) gives rise to a representation $\tilde{\pi}$ of $\ell^1(G, A)$ (hence of $C^*(\ell^1(G, A))$ defined for a finitely ranging function x as

$$\tilde{\pi}(x) = \sum_s \pi(x(s))u_s.$$

Note that any representation of $C^*(\ell^1(G, A))$ has the above form. Thus we finally reach the following

DEFINITION 1.1. The *full C^*-crossed product* $A \underset{\alpha}{\times} G$ is $C^*(\ell^1(G, A))$.

Henceforth, we write the representation $\tilde{\pi}$ as $\tilde{\pi} = \pi \times u$. It is to be noticed that the linear space

$$\mathcal{D} = \{\sum a_s\delta_s \,|\, \text{all are finite sums, } a_s \in A\}$$

turnes out to be a dense $*$-subalgebra of $A \underset{\alpha}{\times} G$ and the projection E extends to $A \underset{\alpha}{\times} G$, which is however not faithful in general.

We next define the *reduced crossed product* $A \underset{\alpha r}{\times} G$. Suppose that A is acting on a Hilbert space H as a concrete C^*-algebra. Let $K = \ell^2(G) \otimes H$, which is also regarded as the H-valued ℓ^2-space on G, $\ell^2(G, H)$. Define a representation π_α (actually a $*$-isomorphism) of A and a unitary representation λ_s on K by

$$\begin{aligned} (\pi_\alpha(a)\xi)(s) &= \alpha_{s^{-1}}(a)\xi(s) \quad \xi \in K,\ a \in A \\ (\lambda_s\xi)(t) &= \xi(s^{-1}t). \end{aligned}$$

The pair $\{\pi_\alpha, \lambda_s\}$ becomes then a covariant representation.

DEFINITION 1.2. The *reduced crossed product* $A \underset{\alpha r}{\times} G$ with respect to the action α is the C^*-algebra on K generated by the family $\{\pi_\alpha(a), \lambda_s \mid a \in A,\ s \in G\}$.

It is then naturally proved that this definition does not depend on the acting space H.

Now in the simplest case where $A = \mathbb{C}$ with the trivial action of G, the algebra $\ell^1(G, A)$ coincides with $\ell^1(G)$ and $\{\lambda_s\}$ is the left regular representation of G so that $A \underset{\alpha}{\times} G$ and $A \underset{\alpha r}{\times} G$ are nothing but the group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C^*_r(G)$. Therefore one may call out the difference between $C^*(G)$ and $C^*_r(G)$ and this is also the case for $A \underset{\alpha}{\times} G$ and $A \underset{\alpha r}{\times} G$. Namely we have;

THEOREM 1.1. *The canonical homorphism $\Phi = \pi_\alpha \times \lambda$ from $A \underset{\alpha}{\times} G$ onto $A \underset{\alpha r}{\times} G$ becomes an isomorphism if and only if the group G is amenable.*

Since in the following we shall only rely on the fact of this deep theorem and are not concerned with details of the amenability property, we employ here rather a heuristic definition of the amenability for a discrete group;

. *"Every finite group and every abelian group are amenable. A semidirect product of two amenable groups is amenable as well as its subgroups."*

The free group on two generators is not amenable and there is a long standing conjecture that every non-amenable group may contain such a subgroup.

Now in order to proceed our discussions we need more detailed knowledge about the structure of $A \underset{\alpha r}{\times} G$. Let w_s be an operator of K on H such that

$$w_s\xi = \xi(s^{-1}), \quad \xi \in K.$$

Then, $w_s^* w_s$ is the projection on the subspace of K which consists of those functions having only nonzero values at s^{-1}. On the other hand, $w_s w_s^* = 1_H$ and $\{w_s^* w_s \mid s \in G\}$ is a family of orthogonal projections with sum 1_K. Futhermore, the following basic rules are easily verified.

LEMMA 1.2. *We have*

$$w_s\lambda_t = w_{st}, \quad w_s\pi_\alpha(a)w_s^* = \alpha_s(a),$$
$$\sum_s w_s^*\alpha_s(a)w_s = \pi_\alpha(a),$$

where the sum means in the sense of strong operator topology.

Put $\varepsilon(x) = w_e x w_e^*$ for $x \in B(K)$. The map ε is a normal positive map of $B(K)$ to $B(H)$. Write also as ε, the restriction of ε to $A \underset{\alpha r}{\times} G$.

THEOREM 1.3.

(1) *The map ε is a positive faithful norm one projection of $A \underset{\alpha r}{\times} G$ to A (identifying A with $\pi_\alpha(A)$) with the property,*

$$\pi_\alpha \circ E(x) = \varepsilon \circ \Phi(x) \quad \textit{for} \quad x \in A \underset{\alpha}{\times} G,$$

and

$$\varepsilon(\lambda_s a \lambda_s^*) = \alpha_s(\varepsilon(a)) \quad \textit{for} \quad a \in A \underset{\alpha r}{\times} G$$

(2) *Define*

$$a(s) = \varepsilon(a\lambda_s^*)$$

for an element $a \in A \underset{\alpha r}{\times} G$, *then the family* $\{a(s) \mid s \in G\}$ *in* A *determines the element* a *uniquely and the algebraic operations of* $A \underset{\alpha r}{\times} G$ *are expressed by this family as follows; denote the above correspondence by* $a \sim \{a(s)\}$, $b \sim \{b(s)\}$, *then*

$$a^* \sim \{\alpha_s(a(s^{-1})^*)\} \quad \text{and} \quad ab \sim \{\sum_t a(t)\alpha_t(b(t^{-1}s))\},$$

where the sum $\sum_t$ *is taken in the strong operator topology.*

Proof. For a finite sum $a = \sum_s \pi_\alpha(a_s)\lambda_s$, one may easily verify by Lemma 1.2 that $\varepsilon(a) = a_e$. It follows that $\pi_\alpha \circ E(x) = \varepsilon \circ \Phi(x)$ for any $x \in \mathcal{D}$, hence for any $x \in A \underset{\alpha}{\times} G$. Moreover, we also see that $\varepsilon(\lambda_s a \lambda_s^*) = \alpha_s(\varepsilon(a))$ for any $a \in A \underset{\alpha r}{\times} G$. Next suppose that $a(s) = 0$ for every $s \in G$, then for any s and t, we have

$$\begin{aligned} w_s a w_t^* &= w_e \lambda_s a \lambda_t^* w_e^* = \varepsilon(\lambda_s a \lambda_{s^{-1}t}^* \lambda_s^*) \\ &= \alpha_s \circ \varepsilon(a\lambda_{s^{-1}t}^*) = \alpha_s(a(s^{-1}t)) = 0. \end{aligned}$$

Hence $a = 0$. Furthermore, for a and b in $A \underset{\alpha r}{\times} G$,

$$\begin{aligned} ab(s) &= \varepsilon(ab\lambda_s^*) = w_e ab \lambda_s^* w_e^* \\ &= \sum_t w_e a w_t^* w_t b w_s^* \\ &= \sum_t a(t)\alpha_t(b(t^{-1}s)). \end{aligned}$$

Note that though the last member is concerned with an infinite sum with respect to the strong topology it is still in the C^*-algebra A as $\varepsilon(ab\lambda_s^*)$. Similarly, $a^*(s) = \alpha_s(a(s^{-1})^*)$. We assert at last that ε is faithful. This will be seen from the following identities,

$$\varepsilon(a^*a) = a^*a(e) = \sum_s a^*(s)\alpha_s(a(s^{-1})) = \sum_s \alpha_s(a(s^{-1})^* a(s^{-1})).$$

The above set of coefficients $\{a(s)\}$ is called the *Fourier coefficient* of an element a and we write the correspondence sometimes as $a =$

$\sum_s a(s)\delta_s$ as a formal sum. Here one has to be careful for the fact that the above sum is neither convergent in norm nor even in the strong topology in general. In fact, when $A = \mathbb{C}$ and $G = \mathbb{Z}$, the integer group, one can verify that $A \underset{\alpha r}{\times} Z = C^*(\mathbb{Z}) = C(\mathbb{T})$ for the torus $\mathbb{T}$ and the set $\{f(n) \,|\, n \in \mathbb{Z}\}$ for $f \in C(\mathbb{T})$ is nothing but the usual Fourier coefficients of the function f.

Finally, we put some remarks how the above situation changes when we plan to discuss the interplay between topological dynamics about flow, that is, dynamical systems with actions of the real line $\mathbb{R}$ and the theory of C^*-algebras. In this case we have a one parameter automorphism group $\{\alpha_t\}$ on $C(X)$ so that we must naturally replace sums in our discussion by integrals on the real line. Thus starting from a topological dynamical system $\Sigma = (X, \mathbb{R}, \sigma_t)$ we have to handle with the space $L^1(\mathbb{R}, C(X))$ instead of $\ell^1(\mathbb{Z}, C(X))$ with definitions of operations given by integrals such as

$$xy(s) = \int_{\mathbb{R}} x(t)\alpha_t(y(t^{-1}s))dt$$

and we reach the concept of the continuous crossed product $C(X) \underset{\alpha}{\times} \mathbb{R}$ as the C^*-envelope of the Banach $*$-algebra $L^1(\mathbb{R}, C(X))$. Amenability of the group $\mathbb{R}$ also works in this context. There appears however considerable difference between discrete and continuous crossed products. Indeed, in the latter case we can expect no more the embedding of the original algebra $C(X)$ into $C(X) \underset{\alpha}{\times} \mathbb{R}$ nor realization of those automorphisms α_t by means of unitary operators inside the crossed product. Moreover, the existence of the connecting projection map E from $C(X) \underset{\alpha}{\times} \mathbb{R}$ to $C(X)$ also goes out of context. Thus, even a starting point to discuss about state extensions of the evaluation functional μ_x takes different aspects from our present arguments. We shall have to work mainly with the multiplier algebra of $C(X) \underset{\alpha}{\times} \mathbb{R}$ and the author is afraid of admitting simply this crossed product as the transformation group C^*-algebra $A(\Sigma)$ canonically associated with the dynamical system $\Sigma = (X, \mathbb{R}, \sigma)$.

§2. Topological dynamical systems and their transformation group C^*-algebras

Henceforth we shall be always concerned with the topological dynamical system $\Sigma = (X, \sigma)$ for a single homeomorphism σ on a compact Hausdorff space X. The system gives rise to a C^*-dynamical system $(C(X), \mathbb{Z}, \alpha)$ for a single automorphism α defined by $\alpha(f)(x) = f(\sigma^{-1}x)$ as the action of the integer group $\mathbb{Z}$ on $C(X)$. Thus we may consider the full crossed product $C(X) \underset{\alpha}{\times} \mathbb{Z}$ and, since $\mathbb{Z}$ is amenable, it is isomorphic with the reduced crossed product $C(X) \underset{\alpha r}{\times} \mathbb{Z}$. Therefore identifying these two kinds of crossed products we denote this algebra by $A(\Sigma)$ and call it the *transformation group C^*-algebra* associated with the dynamical system $\Sigma = (X, \sigma)$. Write $\delta = \delta_1$, the unitary element corresponding to 1. It is to be noticed that in $A(\Sigma)$ the projection map E is faithful and moreover we can freely make use of both the universality of covariant representations and Fourier coefficients of elements of $A(\Sigma)$. The algebra $A(\Sigma)$ is considered as a transplantation of the dynamical system Σ into an algebraic frame of C^*-algebra. Here we do not impose the countability condition on X contrary to usual topological dynamical systems because once we are concerned with both the theory of C^*-algebras and that of topological dynamical systems we sometimes meet the system $\Sigma = (X, \sigma)$ for a quite big compact space such as $\Sigma = (\beta\mathbb{Z}, \sigma_0)$ where $\beta\mathbb{Z}$ is the Čech compactification of $\mathbb{Z}$ and σ_0 the extension of the shift map on $\mathbb{Z}$ to the whole space $\beta\mathbb{Z}$.

In the reduced crossed product $C(X) \underset{\alpha r}{\times} \mathbb{Z}$, the unitary operator $\lambda_1 (= \delta)$ is written as $s \otimes 1$ for the unitary operator s on $\ell^2(\mathbb{Z})$, and

$$C(X) \underset{\alpha r}{\times} \mathbb{Z} = C^*(\pi_\alpha(C(X)),\ s \otimes 1).$$

Here sometimes the automorphism α happens to be *spatial* on H, that is, there exists a unitary operator u on H such that $\alpha(f) = ufu^*$ for every $f \in C(X)$. Then the unitary operator w on $\ell^2(\mathbb{Z}, H)$ defined by

$$w(\xi)(n) = u^{-n}\xi(n) \quad \text{for} \quad \xi \in \ell^2(\mathbb{Z}, H)$$

intertwines the algebra $C(X) \underset{\alpha r}{\times} \mathbb{Z}$ and the C^*-algebra $C^*(1 \otimes f, s \otimes u \mid f \in C(X))$. In fact, one may easily verify that

$$w^{-1}\pi_\alpha(f)w = 1 \otimes f \quad (f \in C(X)), \quad w^{-1}(s \otimes 1)w = s \otimes u.$$

Therefore, in this case we can treat $A(\Sigma)$ also as the C^*-algebra generated by $1 \otimes C(X)$ and the unitary operator $s \otimes u$.

Now the first basic problem towards the interplay between Σ and $A(\Sigma)$ is the question what is the relation between two systems $\Sigma_1 = (X_1, \sigma_1)$ and $\Sigma_2 = (X_2, \sigma_2)$ when $A(\Sigma_1)$ is isomorphic to $A(\Sigma_2)$. Topological conjugacy of σ_1 and σ_2 implies of course an isomorphism between $A(\Sigma_1)$ and $A(\Sigma_2)$, but the converse is not valid in general. The former relation is too strong. The first aspect causing this difference seems to be breaking of symmetry. In the case of crossed products, universality of covariant representation assures that $C(X) \underset{\alpha}{\times} \mathbb{Z}$ is isomorphic to the crossed product $C(X) \underset{\alpha^{-1}}{\times} \mathbb{Z}$ with respect to the action α^{-1}, which corresponds to the dynamical system $\Sigma' = (X, \sigma^{-1})$. It is however not always true that σ and σ^{-1} are topologically conjugate. In fact, for instance let X be the three-points-compactification of $\mathbb{Z}$, $X = (\mathbb{Z}, w_1, w_2, w_3)$ where w_1 and w_2 are the limit points of even and odd positive integers respectively and w_3 is the limit point of negative integers. We consider in $\mathbb{Z}$ the shift operator, $\sigma(n) = n + 1$, and its extension on X such that $\sigma(w_1) = w_2$, $\sigma(w_2) = w_1$ and $\sigma(w_3) = w_3$. One may then easily verify that σ and σ^{-1} are not topologically conjugate. On the other hand, if X is the two-points-compactification of $\mathbb{Z}$ with ∞ and $-\infty$ (symmetric compactification) the extended shift homeomorphism is easily seen to be topologically conjugate to its inverse. Now it may be worth while to recall here what had happened in the case of the measurable dymanical system $\Phi = (\Omega, \sigma, \mu)$ where σ is an ergodic nonsingular bimeasurable transformation in the Lebesgue space (Ω, μ). Starting from the algebra $L^\infty(\Omega, \mu)$ acting on the Hilbert space $L^2(\Omega, \mu)$ as the algebra of multiplication operators with the induced automorphism α by σ, the von Neumann crossed product $\mathcal{R}(\Phi)$ on the space $\ell^2(\mathbb{Z}, L^2(\Omega, \mu))$ is defined as the σ-weak closure of the reduced crossed product $L^\infty(\Omega, \mu) \underset{\alpha r}{\times} \mathbb{Z}$. With further condition, the action being free, the von Neumann algebra $\mathcal{R}(\Phi)$ turns out to be a *factor*, that is, its center is trivial. After a long history about the interplay between the theory of von Neumann algebras and that of ergodic actions since Murray-von Neumann's works, it is then the celebrated result by Krieger [**10**] that the isomorphism class of $\mathcal{R}(\Phi)$ can be completely determined by the orbit equivalence class of the system Φ, where the discovery of the concept of orbit equivalence is due to Dye [**2**].

In connection with this result, recently the notion of topological orbit equivalence has been proposed by Skau [**18**]. For two topological dynamical systems $\Sigma_1 = (X_1, \sigma_1)$ and $\Sigma_2 = (X_2, \sigma_2)$, Σ_1 is said to be *topologically orbit equivalent* to Σ_2 if there exists a homeomorphism Φ such that $\Phi(O_1(x)) = O_2(\Phi(x))$ for all $x \in X_1$, where $O_1(x)$ and $O_2(\Phi(x))$ mean the orbit of x by σ_1 and the orbit of $\Phi(x)$ by σ_2 respectively. With this definition, the result is satisfactory at least for the minimal systems (see Definition 2.1 below) on the Cantor set. Indeed, Theorem 4 in [**18**] says;

. *"For two minimal dynamical systems $\Sigma_1 = (X_1, \sigma_1)$ and $\Sigma_2 = (X_2, \sigma_2)$ where X_1 and X_2 are Cantor sets, if $A(\Sigma_1)$ is isomorphic to $A(\Sigma_2)$ then Σ_1 is topologically orbit equivalent to Σ_2. The converse holds with other (technical) condition on K_0-groups of $A(\Sigma_1)$ and $A(\Sigma_2)$. (This extra condition is suspected to be superfluous)."*

The author feels however that for general dynamical systems including periodic points we should consider more about the counter part of the concept of "almost everywhere" in the case of measurable dynamical systems as in the case of topologically free actions, which will be discussed in §5. We shall be concerned later with the problem in case of rational and irrational rotation C^*-algebras. At any rate, as of now rather restricted results are known for this important problem. We mention [**4**] besides Skau's paper for the very recent development.

In what follows we shall be mainly concerned with the following types of topological dynamical systems. Write $\mathrm{Per}^n(\Sigma) = \{x \in X \mid \sigma^n x = x\}$ and $\mathrm{Per}_n(\Sigma)$ the set of all (exactly) n-periodic points in Σ. We also write $\mathrm{Per}^\infty(\Sigma)$ the set of all aperiodic points. We denote by $\mathrm{Per}(\Sigma)$ the set of all periodic points in Σ. As we have already used, the orbit of a point x is denoted by $O(x)$. By the *isotropy group* $\mathbb{Z}_x$ for $x \in X$ we mean the subgroup of $\mathbb{Z}$ defined by $\{n \in \mathbb{Z} \mid \sigma^n x = x\}$.

DEFINITION 2.1. (a) We call Σ to be *minimal* if every orbit is dense in X;

(b) Σ is said to be *topologically transitive* if for every pair of open sets $\{U, V\}$ there exists an integer n such that $\sigma^n U \cap V \neq \phi$;

(c) Σ is said to be *topologically free* if $\mathrm{Per}^\infty(\Sigma)$ is dense in X.

We say that the action is *effective* if for every integer n the homeomorphism σ^n does not coincide with the identity map except for $n = 0$. We also notice that our definition of topological transitivity is a little different from usual one, which takes as its definition the condition (1) of the next Proposition 2.1.

One of the most fruitful example of a minimal system in the interplay between Σ and $A(\Sigma)$ is the case of an irrational rotation $\sigma_\theta : x \mapsto x + \theta$ on the torus $\mathbb{T}$ for an irrational number θ (or $e^{2\pi i x} \mapsto e^{2\pi i(x+\theta)}$ on the circle $\mathbb{S}^1$). We call this algebra $A(\Sigma_\theta)$ an *irrational rotation C^*-algebra* and write as A_θ. We shall later discuss this algebra in detail. It may be also worth while to mention the result by Denjoy (cf. [**16**]) that all C^2-diffeomorphisms σ of the circle with the rotation number $\theta = \rho(\sigma)$ and with no periodic points are conjugate to the rigid rotation σ_θ. Thus, they are all minimal and all their transformation group C^*-algebras are isomorphic to A_θ.

As for a topologically transitive action, we mention the following Proposition.

PROPOSITION 2.1. *Consider the following three assertions for a system $\Sigma = (X, \sigma)$.*

(1) *There exists a point in X with dense orbit;*

(2) *Σ is topologically transitive;*

(3) *The set $\{x \in X \mid \overline{O(x)} \neq X\}$ is of first category.*

Then (1) $\Rightarrow$ (2) *and* (3) $\Rightarrow$ (1). *If X is second countable, the assertion* (2) $\Rightarrow$ (3) *holds, hence all three conditions are equivalent in this case.*

The result is standard and we omit the proof. The implication (2) $\Rightarrow$ (3) or (2) $\Rightarrow$ (1) does not hold in general. The well known symbolic dynamical system $\{X(k), \sigma_k\}$ of Bernoulli shift is an example of a topologically transitive system. Here the compact space $X(k)$ stands for the product space of infinite copies of the finite set of integers $\{0, 1, 2, \ldots, k-1\}$ and σ_k is the right (or left) shift homeomorphism. The space $X(k)$ turns out to be a metric space and if we regard the points of $X(k)$ as k-adic expansions of the points in the interval $[0, 1]$ the point x_0 whose expansion contains every finite portion in such k-adic expansions satisfies the condition $\overline{O(x_0)} = X(k)$.

The name, topologically free action, has not appeared in literature but actually this action has appeared in several papers ([**9**], [**T**] etc.) in the form below.

PROPOSITION 2.2. *The following two assertions are equivalent;*
(a) $\Sigma = (X, \sigma)$ *is topologically free;*
(b) *For every positive integer* n, *the set* $\mathrm{Per}_n(\Sigma)$ *does not contain an interior point.*

Proof. It suffices to show the implication (b) $\Rightarrow$ (a). Suppose that the set $\mathrm{Per}^\infty(\Sigma)$ is not dense in X. Since X is regular, we can find an open set U whose closure $\overline{U}$ is also disjoint from $\overline{\mathrm{Per}^\infty(\Sigma)}$. We have then

$$\overline{U} = \bigcup_{n=1}^{\infty} (\mathrm{Per}^n(\Sigma) \cap \overline{U}),$$

and, by the category theorem, there exists an integer n such that $\mathrm{Per}^n(\Sigma) \cap \overline{U}$ contains an interior point in the space $\overline{U}$. Let m be the smallest one among such integers, then we can say that the set $\mathrm{Per}_m(\Sigma) \cap \overline{U}$ contains an interior point. Thus we finally see that the set $\mathrm{Per}_m(\Sigma)$ contains an interior point in X, contradicting the assumption.

In case of topological dynamical systems on manifolds, the set Per(Σ) becomes quite often countable (though it often becomes also dense), hence those systems are all topologically free. We shall see later that this notion plays a central rôle in the interplay between Σ and $A(\Sigma)$. On the other hand, in C^*-theory the notion of the *free action* (the isotropy group $\mathbb{Z}_x$ is trivial for every $x \in X$) is often used, but the author feels that an appropriate counter notion to the free action in a measurable dynamical system could be the above topologically free action and not the free action itself.

Obviously, if a system Σ is minimal (hence free provided that X consists of infinite points) it is topologically transitive and, if the action is effective, topological transitivity implies topological freeness of the action. The converse is naturally not true. An important example to show this difference is the dynamical system $\Sigma = (\mathbb{T}^2, \sigma)$ where $\mathbb{T}^2$ means the two dimensional torus and σ the homeomorphism: $(s,t) \mapsto (s, t+s)$. The map induces an irrational rotation at each level of an irrational number s, whereas it induces a rational rotation at each level of a rational number s. This is also an example of those maps coming from the action of $SL(2, \mathbb{Z})$ to $\mathbb{T}^2$.

We note that, throughout this lecture, an *ideal* of a C^*-algebra A means always a closed ideal of A. Thus simplicity means that there is no proper closed ideal in A but this coincides with algebraic simplicity because as is well known an element a which is near to the unit such as $\|1 - a\| < 1$ becomes invertible in A. We mean by an *invariant set* both σ and σ^{-1} invariant set and not a mere σ-invariant set.

§3. Structure of the state space $S(A(\Sigma))$ of $A(\Sigma)$, pure state extensions of point evaluations on $C(X)$ and uniqueness of the trace on $A(\Sigma)$

A link to connect the system Σ with the structure of $A(\Sigma)$ is to consider the state space $S(A(\Sigma))$ together with GNS-representations of $A(\Sigma)$ for states as the extension of the state space $S(C(X))$. In this connection, notice that the space X is embedded into $S(C(X))$ as the compact subset of pure states $\{\mu_x \mid x \in X\}$ consisting of point evaluations μ_x for $x \in X$. In this section, we investigate this relation.

Recall that a scalar valued function Φ on $\mathbb{Z}$ is said to be *positive definite* if for arbitrary finite sets $\{s_1, s_2, \ldots, s_n\}$ in $\mathbb{Z}$ and complex numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ we have

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \lambda_j \Phi(s_j - s_i) \geq 0.$$

In the context of crossed products beyond group C^*-algebras, we need to generalize this notion to the following

DEFINITION 3.1. Let φ be a positive linear functional on $A(\Sigma)$. A function Φ on $\mathbb{Z}$ to $C(X)^*$, the dual of $C(X)$, is said to be *positive definite with respect to the action α* if it is defined as

$$\Phi(n)(f) = \varphi(f\delta^n) \quad \text{for} \quad f \in C(X),\ n \in \mathbb{Z}.$$

Write this relation by $\Phi = \Phi_\varphi$. Then as the generalization of the correspondence between the set of scalar valued positive definite functions on $\mathbb{Z}$ and that of positive functionals on $C^*(\mathbb{Z})$ we have the following

PROPOSITION 3.1. *A bounded function Φ on $\mathbb{Z}$ to $C(X)^*$ is positive definite, i.e., $\Phi = \Phi_\varphi$ for a positive functional φ on $A(\Sigma)$ if and only if*

$$\sum_{i,j} \Phi(s_j - s_i)(\alpha^{-s_i}(\overline{f}_i f_j)) \geq 0$$

for all finite sets $\{s_1, s_2, \ldots, s_n\}$ in $\mathbb{Z}$ and $\{f_1, f_2, \ldots, f_n\}$ in $C(X)$.

Proof. If $\Phi = \Phi_\varphi$ for a positive functional φ on $A(\Sigma)$, then for a finite sum $x = \sum_i f_i \delta^{s_i}$,

$$0 \leq \varphi(x^* x) = \varphi(\sum_{i,j} \alpha^{-s_i}(\overline{f}_i f_j)\delta^{s_j - s_i})$$
$$= \sum_{i,j} \Phi(s_j - s_i)(\alpha^{-s_i}(\overline{f}_i f_j)).$$

On the other hand, given a bounded function Φ, define the functional φ by

$$\varphi(x) = \sum_i \Phi(s_i)(f_i)$$

for a finite sum $x = \sum_i f_i \delta^{s_i}$. Since Φ is bounded, φ extends to a bounded linear functional on $\ell^1(\mathbb{Z}, C(X))$, which becomes naturally positive by the assumption. Hence it can be extended to a positive functional, say φ, and by definition $\Phi = \Phi_\varphi$. This completes the proof.

The proposition means that each state φ on $A(\Sigma)$ determines a distribution of a bounded family of signed measures on X, $\{\mu_n \mid n \in \mathbb{Z}\}$. We write this relation as $\varphi = \sum_n \oplus \mu_n$, then φ is an extension of the state μ_0 on $C(X)$ and if it is unique, it is the state $\mu \circ E$.

Let $\mu_n = \mu_n^1 + i\mu_n^2$ be a decomposition of μ_n into its real and imaginary parts. Define the positive measure $|\mu_n|$ as

$$|\mu_n| = |\mu_n^1| + |\mu_n^2|.$$

We say that μ_n is *absolutely continuous* with respect to a probability measure μ, $\mu_n < \mu$, if both μ_n^1 and μ_n^2 are absolutely continuous to μ. This is in fact equivalent to say that $|\mu_n^1| < \mu$ and $|\mu_n^2| < \mu$.

PROPOSITION 3.2. *With the above notations, we have for every integer n,*

(1) $\mu_n < \mu_0$ *and* $\mu_n < \mu_0 \circ \alpha^{-n}$;

(2) $\mu_{-n}(f) = \overline{\mu_n(\alpha^n(\overline{f}))}$ *for* $f \in C(X)$.

Proof. Let E be a Borel set of X and suppose that $\mu_0(E) = 0$. For a positive number ε, by the regularity of μ_0 and $|\mu_n|$ there exist then a compact set C and an open set U such that

$$C \subset E \subset U \quad \text{and}$$
$$\mu_0(U \backslash C) < \varepsilon, \quad |\mu_n|(U \backslash C) < \varepsilon.$$

Take a continuous function f on X such that $f(x) = 1$ on C, $f(x) = 0$ on U^c and $0 \leq f(x) \leq 1$. Then,

$$\begin{aligned}\varphi(f^2) = \mu_0(f^2) &= \int_C f^2 d\mu_0 + \int_{U\backslash C} f^2 d\mu_0 \\ &\leq \mu_0(C) + \mu_0(U\backslash C) < \varepsilon,\end{aligned}$$

and

$$\begin{aligned}|\mu_n(f)| = \left|\int_U f d\mu_n\right| &\geq \left|\int_C f d\mu_n\right| - \left|\int_{U\backslash C} f d\mu_n\right| \\ &\geq |\mu_n(C)| - \varepsilon.\end{aligned}$$

Hence,

$$\begin{aligned}|\mu_n(C)| &\leq |\mu_n(f)| + \varepsilon = |\varphi(f\delta^n)| + \varepsilon \\ &\leq \sqrt{\varphi(f^2)} + \varepsilon < \sqrt{\varepsilon} + \varepsilon\end{aligned}$$

and

$$|\mu_n(E)| \leq |\mu_n(C)| + |\mu_n|(U\backslash C) < 2\varepsilon + \sqrt{\varepsilon}.$$

It follows that $\mu_n(E) = 0$. Moreover if we make use of another identity $\mu_n(f) = \varphi(\delta^n\alpha^{-n}(f))$ almost the same argument leads us to the other conclusion, $\mu_n < \mu_0 \circ \alpha^{-n}$.

For the assertion (2), we note that by Proposition 3.1 the matrix

$$\begin{bmatrix} \mu_0(1) & \mu_{-n}(f) \\ \mu_n(\alpha^n(\overline{f})) & \mu_0(\alpha^n(|f|^2)) \end{bmatrix}$$

is positive for pairs $\{0, -n\}$ and $\{1, f\}$. Hence

$$\mu_{-n}(f) = \overline{\mu_n(\alpha^n(\overline{f}))}.$$

Now to construct a positive linear functional φ for a probability measure μ on X there is a simple way when μ is an invariant measure. In fact, we have the following

PROPOSITION 3.3. *Let Φ be a scalar valued positive definite function on $\mathbb{Z}$, μ an invariant probability measure on X, then the functional $\varphi = \sum_n \oplus\Phi(n)\mu$ on $A(\Sigma)$ is positive.*

Proof. Let $\{s_i\}$ and $\{f_i\}$ be finite sets in $\mathbb{Z}$ and $C(X)$, respectively. Then,

$$\sum_{i,j} \Phi(s_j - s_i)\mu(\alpha^{-s_i}(\overline{f}_i f_j)) = \sum_{i,j} \Phi(s_j - s_i)\mu(\overline{f}_i f_j)$$
$$= [\Phi(s_j - s_i)] \circ [\mu(\overline{f}_i f_j)] \geq 0,$$

where the above product means the expansion of Schur product of matrices. The Schur product of positive matrices as well as its expansion is again positive. Thus, by Proposition 3.1 the function: $n \mapsto \Phi(n)\mu$ is positive definite and φ is positive.

This method is not available for arbitrary positive measure. We can however look for another way and obtain the result about unicity of state extensions.

THEOREM 3.4. *Let μ be a probability measure on X. Then μ has a unique extension if and only if the measure $\mu \circ \alpha^{-n}$ is singular with respect to μ for every $n \in \mathbb{Z}$ except $n = 0$.*

We leave the detail to the paper [**6**].

For a point $x \in X$, let μ_x be the pure state of point evaluation on $C(X)$. Let $\varphi = \sum_n \oplus\mu_n$ be a state extension of μ_x (i.e., $\mu_0 = \mu_x$). Then, by Proposition 3.2

$$\operatorname{supp}|\mu_n| \subseteq \operatorname{supp}\mu_0 = \operatorname{supp}\mu_x = \{x\}.$$

Hence, $\mu_n = \Phi(n)\mu_x$ for a scalar $\Phi(n)$. Moreover, we also have the inclusion

$$\operatorname{supp}|\mu_n| \subseteq \operatorname{supp}\mu_x \circ \alpha^{-n}.$$

Hence, if $\sigma^n x \neq x$ $\mu_n = 0$. Namely, the function Φ is supported in the isotropy group $\mathbb{Z}_x$. With this result, combining Proposition 3.3 and the above theorem one can see a fact that the state extension is unique if and only if the isotropy group $\mathbb{Z}_x$ is trivial, that is, $\mathbb{Z}_x = \{0\}$. But we shall clarify this situation in a more concrete way.

THEOREM 3.5. *There exists a bijective correspondence between the set of state extensions of μ_x on $A(\Sigma)$ and that of scalar valued positive definite functions Φ's on $\mathbb{Z}_x$ with $\Phi(0) = 1$. The correspondence : $\Phi \leftrightarrow \varphi$ is given by $\varphi = \sum_{n\in\mathbb{Z}_x} \oplus\Phi(n)\mu_x$ with $\mu_n = 0$ for $n \notin \mathbb{Z}_x$.*

Proof. We first show that the induced function Φ from a state extension φ of μ_x is a positive definite function on $\mathbb{Z}_x$. Let $\{s_i\}$ and $\{f_i\}$ be finite subsets in $\mathbb{Z}_x$ and $C(X)$ respectively. Then

$$\begin{aligned} 0 &\leq \sum_{i,j} \mu(s_j - s_i)(\alpha^{-s_i}(\overline{f}_i f_j)) \\ &= \sum_{i,j} \Phi(s_j - s_i)\overline{f}_i(\sigma^{s_i}x)f_j(\sigma^{s_i}x) \\ &= \sum_{i,j} \Phi(s_j - s_i)\overline{f}_i(x)f_j(x) \\ &= [\Phi(s_j - s_i)] \circ [\overline{f}_i(x)f_j(x)]. \end{aligned}$$

Since the set $\{f_i\}$ is arbitrary, this means that the matrix $[\Phi(s_j - s_i)]$ is positive. Namely Φ is a positive definite function. Conversely take a scalar valued positive definite function Φ on $\mathbb{Z}_x$ with $\Phi(0) = 1$. Let $\hat{\Phi}$ be the function on $\mathbb{Z}$ such that $\hat{\Phi}\,|_{\mathbb{Z}_x} = \Phi$ and vanishes elsewhere. One can verify that $\hat{\Phi}$ is a positive definite function on $\mathbb{Z}$ (A general fact about the extension of a positive definite function on a subgroup to that on the whole group). Define the functional φ on the set of finite sums $\sum_i f_i\delta^{s_i}$ by

$$\varphi(\sum_i f_i\delta^{s_i}) = \sum_i \hat{\Phi}(s_i)f_i(x).$$

One then easily sees that the functional φ extends to $\ell^1(\mathbb{Z}, C(X))$ and bounded. Now again for a finite sum $a = \sum_i f_i\delta^{s_i}$, we have

$$\begin{aligned} \varphi(a^*a) &= \sum_{i,j} \hat{\Phi}(s_j - s_i)\overline{f}_i(\sigma^{s_i}x)f_j(\sigma^{s_i}x) \\ &= \sum{}_1 \Phi(s_j - s_i)\overline{f}_i(\sigma^{s_i}x)f_j(\sigma^{s_i}x) \end{aligned}$$

where the $\sum_1$ is ranging over $\mathbb{Z}_x$, that is, over only those integers such that $s_j - s_i \in \mathbb{Z}_x$. Thus, $\sigma^{s_i}x = \sigma^{s_j}x$ in the sum $\sum_1$ and finally the above sum becomes

$$\sum_{i,j} \hat{\Phi}(s_j - s_i)\overline{f}_i(\sigma^{s_i}x)f_j(\sigma^{s_j}x)$$
$$= [\hat{\Phi}(s_j - s_i)] \circ [\overline{f}_i(\sigma^{s_i}x)f_j(\sigma^{s_j}x)] \geq 0.$$

Therefore, φ extends to a state on $A(\Sigma)$, which is clearly a state extension of μ_x. This completes all proofs.

COROLLARY. *The pure state extensions of the pure state μ_x is unique if and only if $\mathbb{Z}_x$ is trivial.*

Notice that a pure state extension of μ_x is obtained as an extreme point of the set of all state extensions of μ_x. Hence, by Krein-Milman's theorem, μ_x has a unique pure state extension if and only if it has a unique state extension.

The so-called irrational rotation C^*-algebra $A_\theta = C(T) \underset{\alpha_\theta}{\times} \mathbb{Z}$ for the map : $x \mapsto x + \theta$ on $\mathbb{T}$ (θ, irrational) provides a typical example of the above corollary. On the other hand, when the isotropy group is not trivial, say $\mathbb{Z}_x = \{nk_0 \mid n = 0, \pm 1, \dots\}$, the ambiguity of state extensions of μ_x depends on the variety of scalar valued positive definite functions on $\mathbb{Z}_x$ and the latter corresponds to the state space of the group C^*-algebra $C^*(\mathbb{Z}_x) \cong C(\widehat{\mathbb{Z}_x}) = C(\mathbb{T})$. Therefore, referring the property of extreme points for pure states the set of pure state extensions of μ_x corresponds to the set of characters on the group $\mathbb{Z}_x$, which coincides with the torus $\mathbb{T}$.

Now in a quite different feature comparing with irrational rotations, the shift dynamical system $\Sigma = (\beta\mathbb{Z}, \sigma_0)$ mentioned in §2 also provides an example of the system in which there is no periodic point. In fact, suppose that there was a p-periodic point w in $\beta\mathbb{Z}\backslash\mathbb{Z}$. Let f be a bounded function on $\mathbb{Z}$ such that

$$f(n(p+1)+j) = j \quad 0 \leq j \leq p, \quad n \in \mathbb{Z}.$$

Let $\{n_\alpha\}$ be a net in $\mathbb{Z}$ converging to w. Then, regarding f as a continuous function on $\beta\mathbb{Z}$, we have

$$f(n_\alpha) \to f(w) \quad \text{and}$$
$$f(n_\alpha + p) = f(\sigma^p(n_\alpha)) \to f(\sigma^p(w)) = f(w),$$

whereas by definition of f

$$|f(n_\alpha) - f(n_\alpha + p)| \geq 1 \quad \text{for every} \quad n_\alpha.$$

This is a contradiction.

We next discuss existence of traces on A. Here we call a state τ a *trace* if $\tau(ab) = \tau(ba)$ for all a and b in $A(\Sigma)$. If one considers the forms $E(a^*a)$ and $E(aa^*)$, one may recognize that the composition $\mu \circ E$ with an invariant probability measure μ is a trace on $A(\Sigma)$. On the other hand, since the action α is inner in $A(\Sigma)$ the restriction of a trace to $C(X)$ is an invariant measure. Thus the only candidates for traces are state extensions of invariant probability measures.

PROPOSITION 3.6. *Let μ be an invariant probability measure in X and τ be a state extension of μ to $A(\Sigma)$, $\tau = \sum_n \oplus \mu_n$. Then τ is a trace if and only if every measure μ_n is invariant and its support is containd in* $\mathrm{Per}^{|n|}(\Sigma)$.

Proof. The assertion that τ is a trace is equivalent to the following series of identities;

$$\begin{aligned} & \tau(f\delta^n g\delta^m) = \tau(g\delta^m f\delta^n) \quad f, g \in C(X),\ m, n \in \mathbb{Z} \\ \iff \quad & \tau(f\alpha^n(g)\delta^{m+n}) = \tau(g\alpha^m(f)\delta^{m+n}) \\ \iff \quad & \mu_{m+n}(f\alpha^n(g)) = \mu_{m+n}(g\alpha^m(f)). \end{aligned}$$

Hence, putting $g = 1$

$$\mu_n(f) = \mu_{m+(n-m)}(f) = \mu_{m+(n-m)}(\alpha^m(f)) = \mu_n(\alpha^m(f))$$

Note also that $\mu_n(f\alpha^n(g)) = \mu_n(fg)$. Take a point $x \notin \mathrm{Per}^{|n|}(\Sigma)$, then there exists a neighborhood $U(x)$ of x such that $U(x) \cap \sigma^n(U(x)) = \phi$. Hence if we take a function f with $\mathrm{supp}\, f \subseteq U(x)$, we have

$$\mu_n(f^2) = \mu_n(f\alpha^n(f)) = 0.$$

It follows that $\mu_n(f) = 0$ for a positive function f, and $\mu_n(f) = 0$ whenever $\mathrm{supp}\, f \subseteq U(x)$. Therefore, the point x does not belong to $\mathrm{supp}\, \mu_n$ and $\mathrm{supp}\, \mu_n \subset \mathrm{Per}^{|n|}(\Sigma)$.

Conversely suppose two conditions hold, then

$$\begin{aligned}
\tau(f\delta^n g\delta^m) &= \mu_{m+n}(f\alpha^n(g)) \\
&= \mu_{m+n}(\alpha^m(f)\alpha^{m+n}(g)) \\
&= \int_{\mathrm{Per}^{|m+n|}(\Sigma)} \alpha^m(f)(x)\alpha^{m+n}(g)(x)d\mu_{m+n} \\
&= \int_{\mathrm{Per}^{|m+n|}(\Sigma)} f(\sigma^{-m}x)g(x)d\mu_{m+n} \\
&= \mu_{m+n}(\alpha^m(f)g) \\
&= \tau(g\delta^m f\delta^n).
\end{aligned}$$

This completes the proof.

With this result we can determine the case where the extension of μ is unique.

THEOREM 3.7. *Let μ be an invariant probability measure on X, then μ has a unique trace extension if and only if $\mu(\mathrm{Per}^n(\Sigma)) = 0$ for every natural integer n.*

An immediate consequence of this theorem is the following assertion,

. *"The C^*-algebra $A(\Sigma)$ has a unique trace if and only if Σ is uniquely ergodic and, for this unique ergodic measure μ, we have that $\mu(\mathrm{Per}^n(\Sigma)) = 0$ for every natural integer n."*

We notice that the existence of a unique invariant ergodic measure is equivalent to the existence of a unique invariant measure because an invariant ergodic measure is an extreme point of the weak-$*$ compact convex set of invariant probability measures. In the theory of C^*-algebras, the existence of a trace and its uniqueness plays an important rôle.

If all points of X are aperiodic such as those systems mentioned before, every invariant measure on X has a unique trace extension. In particular, in case of an irrational rotation σ_θ the Lebesgue measure is the only one invariant ergodic measure (uniquely ergodic). Hence the C^*-algebra A_θ has a unique trace. On the contrary, if all points are periodic such as the case of a rational rotation then for any invariant probability measure μ there exists an integer n such that $\mu(\mathrm{Per}^n(\Sigma)) \neq 0$. Therefore, in this case, trace extensions of μ are always not unique.

Proof of Theorem. Let $\tau = \sum_n \oplus \mu_n$ be a trace extension of μ. If $\mu(\mathrm{Per}^n(\Sigma)) = 0$ for every natural integer n, then $\mu_n = 0$ for every $n \neq 0$ by Proposition 3.6 and $\tau = \mu \circ E$. Suppose next that there exists a natural integer n with $\mu(\mathrm{Per}^n(\Sigma)) \neq 0$. We shall construct another trace extension of μ. Put $\mu_0 = \mu$ and

$$\mu_n = \mu_{-n} = \frac{1}{2}\mu\,|_{\mathrm{Per}^n(\Sigma)}.$$

Let τ be the bounded linear functional on $\ell^1(\mathbb{Z}, C(X))$ by the distribution of measures;

$$\tau = \mu_0 \oplus \mu_n \oplus \mu_{-n}.$$

Since $\mathrm{Per}^n(\Sigma)$ is an invariant closed subset we have

$$\mu_n(\alpha^k(f)) = \mu_{-n}(\alpha^k(f)) = \mu_n(f).$$

The functional τ satisfies the conditions in Proposition 3.6. Therefore it suffices to show that τ is positive, hence extends to a trace on $A(\Sigma)$. Take an element $a = \sum f_k \delta^k$ of finite sum, then

$$\begin{aligned}
\tau(a^*a) &= \tau(\sum_{k,l} \alpha^{-k}(\overline{f}_k f_{k+l})\delta^l) \\
&= \mu(\sum_k \alpha^{-k}(\overline{f}_k f_k)) + \mu_n(\sum_k \alpha^{-k}(\overline{f}_k f_{k+n})) \\
&\qquad + \mu_{-n}(\sum_k \alpha^{-k}(\overline{f}_k f_{k-n})) \\
&= \sum_k (\mu(|f_k|^2) + \mu_n(\overline{f}_k f_{k+n}) + \mu_{-n}(\overline{f}_{k+n} f_k)) \\
&\geq \frac{1}{2}\sum_k \int_{\mathrm{Per}^n(\Sigma)} (\overline{f}_k(x) f_k(x) + \overline{f}_k(x) f_{k+n}(x) + \overline{f}_{k+n}(x) f_k(x) \\
&\qquad + \overline{f}_{k+n}(x) f_{k+n}(x)) d\mu \\
&= \frac{1}{2}\sum_k \int_{\mathrm{Per}^n(\Sigma)} |f_k(x) + f_{k+n}(x)|^2 d\mu \geq 0.
\end{aligned}$$

This completes the proof.

§4. Induced representations of $A(\Sigma)$ arised from isotropy groups.

There is a general theory about covariant representation of a C^*-dynamical system (A, G, α) induced from covariant representation of the system $(A, K, \alpha|_K)$ for a subgroup K of G ([**21**], [**25**]) but we discuss here particular features of induced representations of $A(\Sigma)$ arised from isotropy subgroups. In order to clarify the context, however, we shall first define them for an extended topological dynamical system $\Sigma = (X, G, \sigma)$ where G is a discrete group and $s \mapsto \sigma_s$ is the action of G on X as a group of homeomorphisms. Let G_x be the isotropy group for a point x. Write the left coset space $G/G_x = \{s_\alpha G_x\}$ for the representatives $S = \{s_\alpha\}$ where $s_0 = e$ (unit of G). Let H_0 be the Hilbert space with $\dim H_0 = |G/G_x|$. For a unitary representation u of G_x on the space H_u, put $H = H_0 \otimes H_u$. Then every vector ξ in H is expanded as $\sum_\alpha e_\alpha \otimes \xi_\alpha$ with respect to a fixed completely orthonormal basis $\{e_\alpha\}$ in H_0 where the sum is ranging actually over countable union of indices α for which $\xi_\alpha \neq 0$. We define the unitary representation L_u^S of G on H induced by u in the following way;

$$L_u^S(s)(e_\alpha \otimes \xi) = e_\beta \otimes u_t \xi$$

if $ss_\alpha = s_\beta t$ for $t \in G_x$. One may easily verify that this is in fact a unitary representation of G. Next consider the orbit of x,

$$O(x) = \{s_\alpha x \mid \alpha \in I\},$$

where we use abbreviation $s_\alpha x$ instead of $\sigma_{s_\alpha} x$, and define the representation π_x^S of $C(X)$ on H by

$$\pi_x^S(f)(e_\alpha \otimes \xi) = f(s_\alpha x) e_\alpha \otimes \xi.$$

One then sees that the pair (π_x^S, L_u^S) turns out to be a covariant representation of $(C(X), G, \alpha)$, which gives rise to a representation of $A(\Sigma)$, $\tilde{\pi}_{x,u}^S = \pi_x^S \times L_u^S$.

We state basic results about this representation in the following

THEOREM 4.1. *With the above notations, we have*

(a) *The unitary equivalence class of the representation $\tilde{\pi}_{u,x}^S$ does not depend on the choice of the representatives $S = \{s_\alpha\}$.*

Hence we write the representation simply as $\tilde{\pi}_{x,u}$.

(b) *Two representations* $\tilde{\pi}_{x,u}$ *and* $\tilde{\pi}_{y,v}$ *are unitarily equivalent if and only if* $O(x) = O(y)$ *and, putting* $x = s_{\alpha_0} y$, *the representations of* $G_x : t \mapsto u_t$ *and* $t \mapsto v_{s_{\alpha_0}^{-1} t s_{\alpha_0}}$ *are unitarily equivalent.*

(c) $\tilde{\pi}_{x,u}$ *is irreducible if and only if the representation* u *of* G_x *is irreducible.*

We remark that the assertion (c) does not necessarily hold for a usual induced covariant representation.

Though essentially the same as usual one, the above approach to induced covariant representations may have a little advantage, for once representation spaces are same for different unitary representations of G_x the representation space for our induced representations remains the same, whereas in usual case it is changing according to each representation u. We leave details of proofs to [**T**] since we are treating here very special case $\Sigma = (X, \sigma)$. Note that for Σ the isotropy subgroup $\mathbb{Z}_x$ is just trivial or the group $\{nk_0 \mid n \in \mathbb{Z}\}$. Moreover, since an irreducible unitary representation of an abelian group becomes necessarily one dimensional, it is simply a character of this group. Therefore for the system $\Sigma = (X, \sigma)$ the assertion (b) in the above theorem says;

. *"Two irreducible representation* $\tilde{\pi}_{x,u}$ *and* $\tilde{\pi}_{y,v}$ *are equivalent if and only if* $O(x) = O(y)$ *and* $u = v$ *as characters of* $\mathbb{Z}_x = \mathbb{Z}_y$."

Here the representation space H appears as $H = H_0 \otimes \mathbb{C} = H_0$. Now take the unit vector e_0 in H corresponding to the coset $\mathbb{Z}_x$, then the state

$$\varphi_{x,u}(a) = (\tilde{\pi}_{x,u}(a)e_0, e_0)$$

is a pure state extension of μ_x. We shall show that the converse is also true. Namely, every pure state extension of μ_x has the above form. Thus, let φ be a state extension of μ_x and $\{H_\varphi, \pi_\varphi, \xi_\varphi\}$ be the GNS-representation of φ.

LEMMA 4.2. *For* $a \in A(\Sigma)$ *and* $f \in C(X)$ *we have*

$$\varphi(af) = \varphi(fa) = f(x)\varphi(a).$$

Proof. It suffices to show that

$$\pi_\varphi(f)\xi_\varphi = f(x)\xi_\varphi \quad \text{for every} \quad f \in C(X).$$

In fact, once we have this relation,

$$\varphi(af) = (\pi_\varphi(af)\xi_\varphi, \xi_\varphi) = (\pi_\varphi(a)f(x)\xi_\varphi, \xi_\varphi) = f(x)\varphi(a),$$

and similarly $\varphi(fa) = f(x)\varphi(a)$.

For the first assertion,

$$\begin{aligned}|\varphi(f)|^2 &= |f(x)|^2 \le \|\pi_\varphi(f)\xi_\varphi\|^2 \\ &= (\pi_\varphi(|f|^2)\xi_\varphi, \xi_\varphi) = |f(x)|^2.\end{aligned}$$

Hence,

$$|(\pi_\varphi(f)\xi_\varphi, \xi_\varphi)| = \|\pi_\varphi(f)\xi_\varphi\| \; \|\xi_\varphi\|.$$

Therefore,

$$\pi_\varphi(f)\xi_\varphi = \lambda\xi_\varphi \quad \text{for some scalar} \quad \lambda,$$

and we have that $\lambda = f(x)$.

PROPOSITION 4.3. *Keep the above notations. Define the subspace H_n as*

$$H_n = \{\xi \in H_\varphi \,|\, \pi_\varphi(f)\xi = f(\sigma^n x)\xi \quad \textit{for every} \quad f \in C(X)\}$$

and write $u = \pi_\varphi(\delta)$. We have then

(1) $\xi_\varphi \in H_0$, $H_n = u^n H_0$ and the restriction π_φ to the group $\{\delta^n \,|\, n \in \mathbb{Z}_x\}$ gives rise to a unitary representation of $\mathbb{Z}_x$ on H_0 (write also as u and henceforth we continue this notation).

(2) If x is aperiodic,

$$H_\varphi = \sum_{n\in\mathbb{Z}} \oplus H_n \quad \textit{(direct sum)}$$

and if x is a p-periodic point

$$H_\varphi = H_0 \oplus H_1 \oplus \cdots \oplus H_{p-1}.$$

Therefore, π_φ is unitarily equivalent to the induced representation $\tilde{\pi}_{x,u}$ arised from the unitary representation of $\mathbb{Z}_x$ in (1) and $\varphi = \varphi_{x,u}$;

(3) *The space H_0 becomes one dimensional if and only if φ is a pure state extension.*

Proof. Lemma 4.2 and covariant relation for $\pi_\varphi(f)$ and u imply the assertion (1). As for the orthogonality of subspaces (common eigenspaces) in (2), take two vectors $\xi \in H_n$ and $\eta \in H_m$ with $m \neq n$ (or $0 \leq m < n \leq p-1$ if x is a periodic point). We can find a continuous function f on X such that $f(\sigma^n x) = 1$ and $f(\sigma^m x) = 0$. Then

$$(\xi, \eta) = (\pi_\varphi(f)\xi, \eta) = (\xi, \pi_\varphi(\overline{f})\eta) = 0.$$

The assertion (3) follows from (c) in Theorem 4.1 and the remark mentioned before.

If x is a periodic point, the unitary operator u acts on H in a cyclic way but $u^p\xi$ for $\xi \in H_0$ may not coincide with ξ even if H_0 is one dimensional. In this case, however, $u^p\xi = \lambda\xi$ with $|\lambda| = 1$ and this λ generates the character associated to the one dimensional irreducible unitary representation u of $\mathbb{Z}_x$.

We shall show that conversely every finite dimensional irreducible representation of $A(\Sigma)$ arises from a periodic point. Before going into this discussion, we need some preparations.

Let $\tilde{\pi} = \pi \times u$ be a representation of $A(\Sigma)$ on a space H and let I_π be the kernel of π. The image $\pi(C(X))$ is expressed as the algebra of continuous functions, $C(X'_\pi)$, on a compact space X'_π. On the other hand, as a closed invariant ideal of $C(X)$, I_π is written as the kernel $k(X_\pi)$ for a closed invariant subset X_π of X, where $k(X_\pi)$ means the set of all functions vanishing on X_π. We have

$$\pi(C(X)) = C(X'_\pi) \cong C(X)/I_\pi = C(X_\pi).$$

It follows that we can identify X'_π with X_π together with the action $\sigma_\pi = \sigma|_{X_\pi}$ and that on X'_π induced from the action of $u = \tilde{\pi}(\delta)$ on $C(X'_\pi)$. With this identification, $\pi(f)$ on X'_π is nothing but the restriction of f on X_π and $\|\pi(f)\| = \|f|_{X_\pi}\|$. We denote this dynamical system by $\Sigma_\pi = (X_\pi, \sigma_\pi)$.

PROPOSITION 4.4. *With the above notations, if $\tilde{\pi}$ is irreducible, the system Σ_π is topologically transitive.*

Proof. Suppose there exist two non-empty open subsets U and V in X_π such that $\sigma_\pi^n U \cap V = \phi$ for every $n \in \mathbb{Z}$. This means that there exists a closed invariant set S in X_π such that $S \supset U$ and $S \cap V = \phi$. Put $I = k(S)$, then the closed linear span $[IH]$ turns out to be an invariant subspace for $\tilde{\pi}(A(\Sigma))$ and $[IH] = H$. If we take, however, a function $\pi(f)$ such that $\operatorname{supp} \pi(f) \subset U$, then $\pi(f)I = 0$, a contradiction. This completes the proof.

PROPOSITION 4.5. *Every finite dimensional irreducible representation of $A(\Sigma)$ is unitarily equivalent to an induced representation $\tilde{\pi}_{x,u}$ arised from a periodic point x.*

Proof. Let $\tilde{\pi}$ be a p-dimensional irreducible representation of $A(\Sigma)$ on a space H. Put $\pi = \tilde{\pi}\,|_{C(X)}$ and $u = \tilde{\pi}(\delta)$. Since $\pi(C(X))$ is finite dimensional, its spectrum X_π must consist of isolated points of finite number and moreover the system Σ_π is by Proposition 4.4 topologically transitive. Therefore, there exists a periodic point x in X_π with $O(x) = X_\pi$, say

$$X_\pi = \{x, \sigma x, \sigma^2 x, \ldots, \sigma^{n-1} x\}.$$

Let $p_i = \pi(f_i)$ be the characteristic function at the point $\sigma^i x$. One then easily verifies that $u^i p_0 (u^i)^* = p_i$ and if we denote

$$\begin{gathered} H_i = p_i H = \{\xi \in H \,|\, \pi(f)\xi = f(\sigma^i x)\xi \quad \forall f \in C(X)\}, \\ H_i = u^i H_0 \quad (0 \leq i \leq n-1) \quad \text{and} \quad H_0 = u^n H_0. \end{gathered}$$

We assert that p_0 is a one dimensional projection and $n = p$. In fact, let K be an invariant subspace of H_0 for the associated unitary representation v of the isotropy group $\mathbb{Z}_x$. The closed linear span $[\tilde{\pi}(A(\Sigma))K]$ is then an invariant subspace of H, hence

$$H = [\tilde{\pi}(A(\Sigma))K].$$

Considering the action of $\tilde{\pi}(A(\Sigma))$ on H, this means that $H_0 = K$ and $\mathbb{Z}_x$ acts irreducibly on H_0. Hence H_0 is one dimensional, and $n = p$. Thus, $\tilde{\pi}$ is unitarily equivalent to the induced representation $\tilde{\pi}_{x,v}$.

In case of infinite dimensional irreducible representations of $A(\Sigma)$ we can not say in general that they are all arised from representations of isotropy groups. It would be interesting to clarify the condition of a topological dynamical system for which every irreducible representation of $A(\Sigma)$ is induced from a unitary representation of an isotropy group. There are related results in literature (cf. [**21**]) but most of those results are referred about *smooth actions* meaning that the orbit space $X/\mathbb{Z}$ has a rich structure of Borel sets, whereas to settle down the above question completely we have to deal with actions such as irrational rotations for which the orbit spaces $\mathbb{T}/\mathbb{Z}$ become extremely wild.

So far finite dimensional representations are concerned, we have partial answers.

THEOREM 4.6. (1) *Every irreducible representations of $A(\Sigma)$ is finite dimensional if and only if the system $\Sigma = (X, \sigma)$ consists of periodic points, that is, $X = \mathrm{Per}(\Sigma)$.*

(2) *$A(\Sigma)$ has sufficiently many finite dimensional irreducible representations if and only if the set $\mathrm{Per}(\Sigma)$ is dense in X.*

Proof. Since we have already those results of Proposition 4.3 and Proposition 4.5, what we need to show for the assertion (1) is to prove that any irreducible representation $\tilde{\pi} = \pi \times u$ of $A(\Sigma)$ is finite dimensional when $X = \mathrm{Per}(\Sigma)$. From the assumption, $X_\pi = \mathrm{Per}(\Sigma_\pi)$ hence by category theorem there exists an integer n such that $\mathrm{Per}^n(\Sigma_\pi)$ contains an interior point. Let n be the smallest one among such integers, then the set $\mathrm{Per}_n(\Sigma_\pi) = \mathrm{Per}^n(\Sigma_\pi) \backslash \mathrm{Per}^{n-1}(\Sigma_\pi)$ contains an interior point, say x. We choose a compact neighborhood U of x such that $U \subset \mathrm{Per}_n(\Sigma_\pi)$ and the sets $\{\sigma^i U \mid 0 \leq i \leq n-1\}$ are mutually disjoint. It follows that $\cup_{i=0}^{n-1} \sigma^i U$ is an invariant closed subset of X_π, hence $X_\pi = \cup_{i=0}^{n-1} \sigma^i U$ by Proposition 4.4. Furthermore, if U contains another point y, choosing a smaller compact neighborhood V of x which does not contain y, we reach the same conclusion, $X_\pi = \cup_{i=0}^{n-1} \sigma^i V$, a contradiction. Therefore, $X_\pi = O(x)$, and the rest of the proof proceeds as in the proof of Proposition 4.5.

For the assertion (2), suppose that there are sufficiently many finite dimensional irreducible representations of $A(\Sigma)$ and let f be a continuous function on X vanishing on the set $\mathrm{Per}(\Sigma)$. Then by considering the structure of representations induced from periodic points in Proposition 4.3 we see that f is in the kernels of those representations, hence $f = 0$. This shows that $\mathrm{Per}(\Sigma)$ is dense in X. Next suppose that

Per(Σ) is dense in X and let J be the intersection of all kernels of finite dimensional irreducible representations. We assert that $J = \{0\}$. Take a periodic point x with period p and let J_x be the intersection of all kernels of associated irreducible representations. Let I_x be the ideal of $A(\Sigma)$ generated by $k(O(x))$. It is then not hard to see that the quotient image of $C(X)$ in the C^*-algebra $A(\Sigma)/I_x$ has its spectrum $O(x)$, the orbit of x, in the sense discussed before Proposition 4.4. On the other hand, as those functions in $k(O(x))$ are contained in the kernels of induced representations, we have that $I_x \subset J_x$. Let P be an arbitrary primitive ideal of $A(\Sigma)$ containing I_x and π be an associated irreducible representation of $A(\Sigma)$. The image $\pi(C(X))$ is then isomorphic to the quotient image of $C(O(x))$, hence it remains the same as $C(O(x))$. This means that π is a p-dimensional irreducible representation associated to the point x. Therefore, P contains J_x. Since any closed ideal in a C^*-algebra is the intersection of primitive ideals which contain that ideal we see that $J_x = I_x$. It follows that $E(J_x) = k(O(x))$ for the projection map E on $A(\Sigma)$ and since J is the intersection of all ideals J_x's for $x \in \mathrm{Per}(\Sigma)$, the set $E(J)$ is contained in the intersection of all ideals $k(O(x))$'s. Therefore, for any element a of J, $E(a)$ vanishes on the set Per(Σ) and $E(a) = 0$. This means that $J = \{0\}$ because J is linearly generated by positive elements and E is faithful. This completes all proofs.

This theorem with Proposition 4.5 shows that isomorphisms between transformation group C^*-algebras bring at least (almost) all informations about the set of periodic points, in particular if a system Σ consists of periodic points all other systems whose transformation group C^*-algebras are isomorphic to $A(\Sigma)$ also consist all of periodic points. The author does not know whether or not this situation leads us to topological conjugacy of those relevant homeomorphisms.

One might sometimes misunderstand that if a compact space X consists of periodic points their periods have to be bounded. This is however not the case, in fact let X be the union of circles in the unit disk having radii $\frac{1}{n}$ $(n = 1, 2, \dots)$ together with the origin and define the homeomorphism σ at each n-th circle by the rotation; $e^{2\pi i x} \mapsto e^{2\pi i (x + \frac{1}{n})}$ and $\sigma(0) = 0$. Then the system consists of periodic points and moreover contains periodic points for any period n.

Typical examples of topological dynamical systems satisfying the condition of Theorem 4.6 (1) are those for rational rotations of the unit circle $\mathbb{S}^1$. In this case, the situation becomes even more simple, namely, for a rational rotation of $\frac{q}{p}$ every point of $\mathbb{S}^1$ (or $\mathbb{T}$) becomes a periodic point of period p and irreducible representations of its transformation group algebra are all p-dimensional. On the other hand, an example of a dynamical system satisfying the condition (2) of the theorem is the Bernoulli shift as we mentioned before.

There was a conjecture by Effros and Hahn starting from separable transformation group C^*-algebras remained open until the work of Gootman and Rosenberg [**3**] in a quite general situation. In our present context, it is the problem whether or not every primitive ideal P in $A(\Sigma)$ for a metric space X becomes the kernel of an induced irreducible representation arised from the isotropy group. In this case we can see directly from our arguments that the conjecture is true. In fact, if the quotient algebra A/P is finite dimensional this is seen from Proposition 4.5. If an associate irreducible representation $\tilde{\pi}$ of $A(\Sigma)$ for P is infinite, Σ_π is topologically transitive and by (1) of Proposition 2.1 there exists a point x in X_π whose orbit is dense in X_π. Then if we consider the induced irreducible representation for x, its kernel coincides with P as will be seen from Corollary 5.1.B. For a general compact space, the problem is still remained open even in this restricted case.

There is a basic classification for C^*-algebras. A C^*-algebra A is said to be *liminal* (or CCR) if every image of its irreducible representation consists of compact operators. When A is unital, this means that every irreducible representation of A is finite dimensional. In this terminology, the assertion of the above Theorem 4.6(1) is nothing but a characterization of the system Σ for which $A(\Sigma)$ becomes a liminal C^*-algebra. The algebra A is called a *postliminal* (or GCR) algebra if every quotient algebra of A contains a nonzero liminal ideal. When A is separable it is well known that the following assertions are equivalent;

(1) *A is postliminal,*

(2) *A is of type I, that is, every representation of A generates a von Neumann algebra of type I,*

(3) *The image of every irreducible representation of A contains the algebra of compact operators (or equivalently the image contains a nonzero compact operator),*

(4) *Every two irreducible representations of A are unitarily equivalent if their kernels coincide each other.*

The C^*-algebra A contains always the unique largest ideal K which is postliminal and when K becomes trivial A is said to be *antiliminal.* Roughly speaking, this classification means that every reasonable behavior (such as those properties appeared above) of C^*-algebras is attached to the class of C^*-algebras of type I, whereas we have to face most bad phenomena in the case of C^*-algebras of non-type I.

We shall give characterizations of dynamical systems whose transformation group C^*-algebras become of type I and antiliminal. Recall that a point x is *recurrent* if for every neighborhood V of x there exists an (non-zero) integer n such that $\sigma^n x$ belongs to V. We denote by $C(\sigma)$ the set of all recurrent points. The closure of $C(\sigma)$ is usually called the *Birkhoff center.* Here by old Birkhoff's theorem the set $C(\sigma)$ is always non-empty. Note that $\mathrm{Per}(\Sigma)$ is a subset of $C(\sigma)$.

THEOREM 4.7. (Aoki-Tomiyama). *For a dynamical system $\Sigma = (X, \sigma)$ where X is a compact metric space, we have;*

(1) *$A(\Sigma)$ is of type I if and only if $C(\sigma) = \mathrm{Per}(\Sigma)$,*

(2) *$A(\Sigma)$ is antiliminal if and only if the set $C(\sigma)\backslash\mathrm{Per}(\Sigma)$ is dense in X.*

Thus the size of the postliminal part as well as that of the antiliminal part within $A(\Sigma)$ depends completely on the size of the set $C(\sigma)\backslash\mathrm{Per}(\Sigma)$. Details of proofs of the above theorem will be published elsewhere.

On the other hand, there is another important class of C^*-algebras called *nuclear* algebras. A C^*-algebra A is said to be nulear if it determines the unique C^*-crossnorm in the tensor product $A \otimes B$ for any C^*-algebra B (there are usually the maximal and minimal (spatial) ones among C^*-cross norms). There had been tremendous discussion about characterizations of a nuclear C^*-algebra such as the one which has the approximation property with respect to the family of completely positive maps of finite ranks and also as the one whose representations always generate injective von Neumann algebras. The algebra $C(X)$ is the simplest example of a nuclear C^*-algebra as well as the matrix algebra M_n. What we emphasize here is that so far we are working on the system $\Sigma = (X, \sigma)$ the algebra $A(\Sigma)$ always remains within the category of nuclear C^*-algebras because crossed products of a nuclear algebra by amenable groups fall in the same class.

§5. Topologically free action and qualitative properties of $A(\Sigma)$.

In our setting $\Sigma = (X, \sigma)$, the topologically free action is in fact equivalent to the action known as the properly outer action for a C^*-dynamical system (A, G, α). The latter plays an essential rôle in determining the simplicity of the general crossed product $A \underset{\alpha}{\times} G$, but as a terminology related to topological dynamical systems we prefer the present name, though it has not been used before in literature even in the author's book [**T**]. In this section we shall illustrate the results which reveal the importance of this concept in the interplay between Σ and $A(\Sigma)$. The following is a key result in our discussions.

THEOREM 5.1. *Let $\tilde{\pi} = \pi \times u$ be a representation of $A(\Sigma)$ on a Hilbert space H. If the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is topologically free, there exists a projection of norm one ε_π from the C^*-algebra $\tilde{\pi}(A(\Sigma))$ to $\pi(C(X))$ such that*

$$\varepsilon_\pi \circ \tilde{\pi}(a) = \pi \circ E(a) \quad \textit{for} \quad a \in A(\Sigma).$$

Proof. We note first that the space X_π consists of infinite points by the assumption. Put $u = \pi(\delta)$ (this may not cause confusion with the writing $\tilde{\pi} = \pi \times u$). From the covariant relation, the set

$$\{\sum_{-n}^{n} \pi(f_k) u^k \mid f_k \in C(X), \quad n \in \mathbb{Z}\}$$

is dense in $\tilde{\pi}(A(\Sigma))$, hence for the existence of a projection map ε_π it is enough to show the inequality,

$$\|\sum_{-n}^{n} \pi(f_k) u^k\| \geq \|\pi(f_0)\|.$$

Write $a = \Sigma_{-n}^{n} \pi(f_k) u^k$ and assume that $\pi(f_0) \neq 0$. We choose a positive number ε such that

$$\|\pi(f_0)\| - 2\varepsilon > 0.$$

Let x_0 be a point of X_π with $|f_0(x_0)| = \|\pi(f_0)\|$ and define a neighborhood Q of x_0 as

$$Q = \{x \in X_\pi \mid |f_0(x) - f_0(x_0)| < \varepsilon\}.$$

There exists then an aperiodic point y_0 in Q. Therefore we can find an open subset P_0 of Q (a neighborhood of y_0) such that if we put $P_j = \sigma_\pi^j(P_0)$ the sets $\{P_j \mid -2n \leq j \leq 2n\}$ become mutually disjoint. Let g be the continuous function on X whose restriction to X_π, that is $\pi(g)$, satisfies the conditions,

$$\begin{aligned} &\operatorname{supp} \pi(g) \subset P_0, \quad \|\pi(g)\| = 1 \quad \text{and} \\ &\|\pi(g)\pi(f_0)\| \geq \|\pi(f_0)\| - \varepsilon. \end{aligned}$$

Now consider the unit vector ξ in H such that

$$\|\pi(gf_0)\xi\| \geq \|\pi(gf_0)\| - \varepsilon \geq \|\pi(f_0)\| - 2\varepsilon > 0.$$

We assert that vectors $\{\pi(f_j)u^j\pi(g)\xi \mid -n \leq j \leq n\}$ are orthogonal. In fact, for any pair (k, j) with $k \neq j$ and $-n \leq k,\ j \leq n$ we have by the condition for $\{P_j\}$

$$\begin{aligned} &(\pi(f_j)u^j\pi(g)\xi \mid \pi(f_k)u^k\pi(g)\xi) \\ &= (\pi(\overline{g})u^{*k}\pi(\overline{f}_k f_j)u^j\pi(g)\xi \mid \xi) \\ &= (u^{*k}\pi(\alpha^k(\overline{g})\overline{f}_k f_j)u^j\pi(g)\xi \mid \xi) \\ &= (u^{*k}\pi(\overline{f}_k f_j)u^j\pi(\alpha^{k-j}(\overline{g})g)\xi \mid \xi) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \|a\pi(g)\xi\|^2 &= \sum_{-n}^{n} \|\pi(f_k)u^k\pi(g)\xi\|^2 \\ &\geq \|\pi(f_0 g)\xi\|^2 \geq (\|\pi(f_0)\| - 2\varepsilon)^2, \end{aligned}$$

and

$$\|a\pi(g)\xi\| \geq \|\pi(f_0)\| - 2\varepsilon.$$

Hence

$$\|a\| \geq \|\pi(f_0)\| - 2\varepsilon \quad \text{and} \quad \|a\| \geq \|\pi(f_0)\|.$$

This completes the proof.

An immediate consequence of this theorem is the fact that if the representation π is faithful then the whole representation $\tilde{\pi}$ becomes faithful, too. For if $\tilde{\pi}(a^*a) = 0$, then

$$\varepsilon_\pi \circ \tilde{\pi}(a^*a) = \pi \circ E(a^*a) = 0,$$

and $E(a^*a) = 0$. As E is faithful, this means that $a = 0$. Therefore we have the following corollaries.

COROLLARY 5.1.A. *Keep the above notations, then the image* $\tilde{\pi}(A(\Sigma))$ *is isomorphic to the transformation group* C^**-algebra* $A(\Sigma_\pi)$ $(= C(X_\pi) \underset{\alpha_\pi}{\times} \mathbb{Z})$.

Proof. It is enough to mention that because of the covariance relation there exists a natural homomorphism from $A(\Sigma_\pi)$ onto $\tilde{\pi}(A(\Sigma))$ whose restriction to $C(X_\pi)$ is faithful.

Combining this result with Proposition 4.4 we have

COROLLARY 5.1.B. *Let* $\tilde{\pi}$ *be an infinite dimensional irreducible representation of* $A(\Sigma)$, *then the image* $\tilde{\pi}(A(\Sigma))$ *is isomorphic to the transformation group* C^**-algebra* $A(\Sigma_\pi)$.

In case of a finite dimensional irreducible representation $\tilde{\pi}$ we know its structure by Proposition 4.5, and the image $\pi(C(X))$ turns out to be a diagonal matrix algebra for some basis in the representation space. Thus we also obtain a projection of norm one from $\tilde{\pi}(A(\Sigma))$ to $\pi(C(X))$ but this projection is not compatible with the map E in the sense of the above theorem.

Corollary 5.1.A allows us to describe the kernels of representations of $A(\Sigma)$ for which associated dynamical systems are topologically free.

PROPOSITION 5.2. *Let* P *be the kernel of a representation* $\tilde{\pi} = \pi \times u$ *for which the system* $\Sigma_\pi = (X_\pi, \sigma_\pi)$ *is topologically free, then*

(1) P *coincides with the closure of the set*

$$\{\sum_{-n}^{n} f_k \delta^k \mid f_k \in C(X),\ n \in \mathbb{Z} \quad \text{and} \quad f_k \mid_{X_\pi} = 0\}.$$

Namely it is the closed ideal generated by $k(X_\pi)$.

(2) *An element* a *of* $A(\Sigma)$ *belongs to* P *if and only if every Fourier coefficient of* a *vanishes on* X_π.

Proof. Let I be the closure of the set in (1). We easily see that $I = I(k(X_\pi))$, the closed ideal generated by $k(X_\pi)$, because X_π is an invariant subset of X. There is a natural embedding of $C(X_\pi)$ with covariant relation into the quotient algebra $A(\Sigma)/I$. On the other hand, since the quotient algebra $A(\Sigma)/P$ is the crossed product $C(X_\pi) \underset{\alpha_\pi}{\times} \mathbb{Z}$ there exists a canonical homomorphism from $A(\Sigma)/P$ to $A(\Sigma)/I$. This means that the quotient homomorphism from $A(\Sigma)/I$ to $A(\Sigma)/P$ is actually an isomorphism and $P = I$.

Next let ε_π be the projection map associated to $\tilde{\pi}(A(\Sigma))$. Then the Fourier coefficients of the element $\tilde{\pi}(a)$ is computed as,

$$\begin{aligned}\tilde{\pi}(a)(n) &= \varepsilon_\pi(\tilde{\pi}(a)\tilde{\pi}(\delta)^{-n}) = \varepsilon_\pi \circ \tilde{\pi}(a\delta^{-n}) \\ &= \pi \circ E(a\delta^{-n}) = \pi(a(n)) = a(n)\,|\,_{X_\pi}.\end{aligned}$$

Hence, a belongs to P if and only if $a(n)$ vanishes on X_π for every $n \in \mathbb{Z}$. This completes the proof.

THEOREM 5.3. *The algebra* $A(\Sigma)$ *is simple if and only if the system* Σ *is minimal, provided that* X *consists of infinite points.*

Proof. Suppose that Σ is minimal, then any representation $\tilde{\pi}$ of $A(\Sigma)$ becomes infinite dimensional. Furthermore, its restriction to $C(X)$ is faithful because $X_\pi = X$. Therefore by Theorem 5.1 (as well as by Corollary 5.1.B) $\tilde{\pi}$ is faithful and $A(\Sigma)$ is simple.

Next suppose that Σ is not minimal. There exists a proper invariant closed subset of X and this implies the existence of a proper invariant ideal I of $C(X)$. Let J be a closed ideal of $A(\Sigma)$ generated by I. Since I is proper we see that

$$\|1 - f\| \geq 1 \quad \text{for every} \quad f \in I.$$

As $E(J) = I$ this time, we also have the inequalities,

$$\|1 - a\| \geq \|1 - E(a)\| \geq 1$$

for every $a \in J$. Hence J is also proper in $A(\Sigma)$ and $A(\Sigma)$ is not simple. This completes the proof.

Besides simplicity, other basic qualitative properties of C^*-algebras are primeness and primitivity. A C^*-algebra is said to be *prime* if every pair of nonzero closed ideals $\{I, J\}$ has nonzero intersection $I \cap J$, whereas it is said to be *primitive* if it has a faithful irreducible representation. In order to discuss these properties, we need to know the structure of ideals in $A(\Sigma)$. In fact, a standard procedure to get ideals of $A(\Sigma)$ is to consider the ideals generated by those invariant ideals of $C(X)$, the latter of which come from invariant closed subsets of X. It is therefore always a big problem whether or not this procedure exhausts all ideals in $A(\Sigma)$. In this connection, we meet again the meaning of topologically free actions. The following result gives the answer to this question.

THEOREM 5.4. *The following three assertions are equivalent;*

(1) Σ *is topologically free;*

(2) *For any closed ideal I of $A(\Sigma)$, $I \cap C(X) \neq \{0\}$ if and only if $I \neq \{0\}$;*

(3) *$C(X)$ is a maximal abelian C^*-subalgebra of $A(\Sigma)$.*

Proof. Let $a \sim \sum_n f_n\delta^n$ be the Fourier expansion of an element a of $A(\Sigma)$. The equivalence of (1) and (3) is the consequence of the following series of equivalent assertions;

$$\begin{aligned}
& ag = ga \quad \text{for every } g \in C(X) \\
\Longleftrightarrow\ & \sum_n f_n\alpha^n(g)\delta^n = \sum_n g f_n\delta^n \quad \text{for every } g \in C(X) \\
\Longleftrightarrow\ & f_n\alpha^n(g) = g f_n \quad \text{for every } n \in \mathbb{Z},\ g \in C(X) \\
\Longleftrightarrow\ & f_n(x)g(\sigma^{-n}x) = f_n(x)g(x) \quad \text{for every } x \in X,\ n \in \mathbb{Z},\ g \in C(X), \\
\Longleftrightarrow\ & \operatorname{supp} f_n \subset \operatorname{Per}^{|n|}(\Sigma) \quad \text{for every nonzero } n.
\end{aligned}$$

Now suppose that Σ is not topologically free. By Proposition 2.2 we can find the positive integer n such that $\operatorname{Per}_n(\Sigma)$ contains an interior point. Let f be a nonzero continuous function on X such that its support is contained in $\operatorname{Per}_n(\Sigma)$ and let I be the closed ideal generated by $\{f - f\delta^n\}$. We shall shows that $I \cap C(X) = \{0\}$, which leads us to the assertion (2) $\Rightarrow$ (1). For each point x in $\mathrm{Per}_n(\Sigma)$ we choose the n-dimensional irreducible representation $\tilde{\pi}_x$ of $A(\Sigma)$ associated to x such that, with notations of Proposition 4.3, $u^n\xi = \xi$ for $\xi \in H_0$, i.e., $u^n = id_H$. For other point $x \notin \mathrm{Per}_n(\Sigma)$, take the GNS-representation $\tilde{\pi}_x$ of an arbitrary state extension of μ_x. With this choice if x belongs to $\mathrm{Per}_n(\Sigma)$ we have,

$$\tilde{\pi}_x(f - f\delta^n) = \tilde{\pi}_x(f) - \tilde{\pi}_x(f) = 0,$$

whereas if x does not belong to $\mathrm{Per}_n(\Sigma)$ we have,

$$O(x) \cap \mathrm{Per}_n(\Sigma) = \phi \quad \text{and} \quad \tilde{\pi}_x(f) = 0.$$

Therefore, the representation $\tilde{\pi}_x$ vanishes on I for every x. Since the family $\{\tilde{\pi}_x \mid x \in X\}$ is total on $C(X)$, we see that

$$I \cap C(X) = \{0\}.$$

Next suppose that the system Σ is topologically free and let I be a closed ideal of $A(\Sigma)$ such that

$$I \cap C(X) = \{0\}.$$

Let q be the quotient homomorphism: $A(\Sigma) \to A(\Sigma)/I$. The algebra $C(X)$ is naturally embedded into $A(\Sigma)/I$. For each point x, let μ'_x be the pure state on $q(C(X))$ defined by $\mu'_x(q(f)) = f(x)$. We consider a pure state extension φ_x of μ'_x on $A(\Sigma)/I$, then the pure state $\varphi_x \circ q$ on $A(\Sigma)$ is a pure state extension of μ_x. If $I \neq \{0\}$, take an element a of I for which $E(a) \neq 0$. From the assumption, we can find an aperiodic point x such that $E(a)(x) \neq 0$. Since x is aperiodic, the state μ_x has the unique pure state extension by Corollary of Theorem 3.5, and $\mu_x \circ E = \varphi_x \circ q$. But this is a contradiction because $\varphi_x \circ q(a) = 0$. This completes all proofs.

Since a minimal dynamical system is necessarily topologically free, we can also derive the hard part of Theorem 5.3 from the above theorem. For, in this case minimality assumption implies that there is no invariant closed ideal in $C(X)$ so that by the theorem there exists no proper ideal in $A(\Sigma)$.

THEOREM 5.5. *$A(\Sigma)$ is prime if and only if Σ is topologically transitive, provided that X consists of infinite points.*

Proof. Suppose that Σ is not topologically transitive. There exist then two disjoint invariant open sets O_1 and O_2 such that $\overline{O}_1 \cup \overline{O}_2 = X$. Let I_1 and I_2 be closed ideals of $A(\Sigma)$ generated by $k(\overline{O}_1)$ and $k(\overline{O}_2)$. Then, they are both nonzero ideals, whereas

$$\begin{aligned} E(I_1 \cap I_2) &\subset E(I_1) \cap E(I_2) \\ &= k(\overline{O}_1) \cap k(\overline{O}_2) \\ &= k(\overline{O}_1 \cup \overline{O}_2) \\ &= k(X) = \{0\}. \end{aligned}$$

Hence, $I_1 \cap I_2 = \{0\}$ and $A(\Sigma)$ is not prime.

Next suppose that Σ is topologically transitive. Then the action of $\mathbb{Z}$ on X is effective because if it were not effective X would consist of a single orbit $O(x)$ for a periodic point x as seen from the proof of (1) of Theorem 4.6. Thus, the action is topologically free. Let I and J be nonzero closed ideals of $A(\Sigma)$. Then by (2) of Theorem 5.4 both $I \cap C(X)$ and $J \cap C(X)$ are nonzero ideals of $C(X)$. Write $I \cap C(X) = k(E)$ and $J \cap C(X) = k(F)$ for invariant proper closed subsets E and F in X. The complements E^c and F^c are invariant (nonempty) open subsets, hence

$$E^c \cap F^c \neq \phi.$$

It follows that

$$E \cup F \neq X, \quad \text{and} \quad I \cap J \neq \{0\}.$$

This completes the proof.

It is well known that for a separable C^*-algebra A it is prime if and only if it is primitive, and it has been remained open for a long time whether or not this is true in general. In C^*-theory, the problem is deeply concerned with the old question of Naimark which asks whether or not the algebra of all compact operators on a non-separable Hilbert space could be characterized as a unique C^*-algebra having the only one equivalence class of irreducible representations. For a separable Hilbert space, the question was answered quite before by A. Rosenberg (1953). On the other hand, in our case for $A(\Sigma)$, separability means the condition X being a metric space. Therefore, with the above theorem coincidence of primeness and primitiveness for $A(\Sigma)$ where X is a metric space is nothing but the consequence of the equivalence (1) and (2) in Proposition 2.1 together with argument given before about the Effros-Hahn conjecture in §4. Moreover, it is known that the above equivalence in Proposition 2.1 does not hold in general. The author suspect that study of suitable examples of topological dynamical systems which illustrate the gap of (1) and (2) well would lead to the (negative) solution of the problem. Thus, as of now we have not had yet the characterization of a topological dynamical system $\Sigma = (X, \sigma)$ for which $A(\Sigma)$ is primitive except Theorem 5.5 for a metric space X.

§6. Irrational rotation C^*-algebras and representations of three dimensional Heisenberg group

Before going into discussions of various important aspects of irrational rotation C^*-algebras whose importance has been first notified by Reiffel [**17**] we sketch the case of rational rotations. Take a rational number $\theta = \frac{m}{n}$ (m and n are relatively prime) and denote by A_θ the algebra $A(\Sigma_\theta)$ for the rational rotation $\sigma_\theta : x \mapsto x + \theta$ on $\mathbb{T}$ (or $e^{2\pi ix} \mapsto e^{2\pi i(x+\theta)}$ on $\mathbb{S}^1$). As we have already shown, all points of $\mathbb{T}$ are n-periodic and all irreducible representations of A_θ are n-dimensional. Such a C^*-algebra is called an *n-homogeneous* C^*-algebra. In this case, the *dual* of A_θ, $\widehat{A_\theta}$, the space of all unitary equivalence classes of irreducible representations of A_θ with Mackey topology (or the space of all primitive ideals with hull-kernel topology) turns out to be two dimensional torus $\mathbb{T}^2$. This would be guessed from the facts that the orbit space $\mathbb{T}/\mathbb{Z}$ is obviously homeomorphic to $\mathbb{T}$ whereas the ambiguity of induced irreducible representations yields another torus (cf. Theorem 4.2.1 in [**T**]). The algebra A_θ is then isomorphic to the C^*-algebra of cross-sections in the fibre bundle (called the structure bundle of A_θ) over $\mathbb{T}^2$ with the fibre M_n, $n\times n$ matrix algebra, and the structure group U_n, the n-dimensional unitary group. Thus the isomorphism class of A_θ's is determined by the isomorphism class of bundle maps of these structure bundles. As in the case of irrational rotation C^*-algebras for which we shall discuss later, two rational rotation C^*-algebras A_θ and $A_{\theta'}$ (for $\theta = \frac{m}{n}$ and $\theta' = \frac{m'}{n'}$) are isomorphic if and only if $\theta = \theta'$ or $\theta = 1 - \theta'$, namely when θ and θ' are topologically conjugate (cf. [**1**], [**5**]).

Let now θ be an irrational number with $0 < \theta < 1$. Then A_θ is a simple C^*-algebra by Theorem 5.3. The following proposition shows an aspect of A_θ.

PROPOSITION 6.1. *A_θ is the C^*-algebra $C^*(u, v)$ generted by two unitary operators u and v with the commutation relation,*

$$uv = e^{2\pi i\theta} vu.$$

The algebra does not depend on the choice of generators.

Proof. It suffices to show that the above relation gives rise to a covariant representation of $(C(\mathbb{S}^1), \alpha_\theta)$ to $C^*(u, v)$. Write the assumption as

$$vuv^* = e^{-2\pi i\theta}u.$$

We have,

$$\mathrm{sp}(u) = \mathrm{sp}(vuv^*) = e^{-2\pi i\theta}\mathrm{sp}(u),$$

which shows the invariance of the spectrum of u by the rotation θ. Since Σ_θ is minimal, we see that $\mathrm{sp}(u) = \mathbb{S}^1$ and we may regard $C(\mathbb{S}^1)$ as the C^*-algebra generated by u. Moreover, in this situation the relation

$$vuv^* = e^{-2\pi i\theta}u$$

simply means that the action of v on $C(\mathbb{S}^1)$ (as $\mathrm{ad}\, v(f) = vfv^*$) coincides with the action α_θ on $C(\mathbb{S}^1)$ induced by σ_θ. Hence, $(C^*(u), \mathrm{ad}\, v)$ is a covariant representation of $(C(\mathbb{S}^1), \alpha_\theta)$.

The second aspect of A_θ is that it has a unique faithful trace τ by Theorem 3.7. Since

$$N_\tau = \{a \in A_\theta \,|\, \tau(a^*a) = 0\} = \{0\},$$

A_θ is embedded into the Hilbert space H_τ in the GNS-construction for τ. Let u and v be generating unitary operators for A_θ. Since τ has the form $\tau = dt \circ \varepsilon$ where dt is the Lebesgue measure on $\mathbb{T}$,

$$\tau(u^m v^n) = 0 \quad \text{except for the case} \quad m = n = 0.$$

It follows by the commutation relation that the set $\{u^m v^n \,|\, m, n \in \mathbb{Z}\}$ constitutes an complete orthonormal basis in H_τ. Therefore, each element a of A_θ has the expansion

$$a = \sum_{m,n} a_{mn} u^m v^n$$

in the Hilbert space H_τ and $\tau(a) = a_{00}$. Now define the action of $\mathbb{T}^2$ on A_θ by

$$\alpha_{(s,t)}(u^m v^n) = e^{2\pi i(ms+tn)}u^m v^n.$$

It turns out to be a continuous effective ergodic action. Conversely it is known that a C^*-algebra admitting a continuous effective ergodic action of $\mathbb{T}^2$ is either an irrational ratation C^*-algebra or else a rational rotation C^*-algebra. (cf. [**5**], [**13**]). The above action of $\mathbb{T}^2$ naturally induces two canonical derivations δ_1 and δ_2 (densely defined and unbounded) along the directions s and t respectively. They have forms,

$$\delta_1(u^m v^n) = 2\pi i m u^m v^n, \quad \delta_1(v^n) = 0$$
$$\delta_2(u^m v^n) = 2\pi i n u^m v^n, \quad \delta_2(u^m) = 0.$$

These two derivations play basic rôle in the set of all derivations in A_θ (cf. [**19**]). Thus we may recognize that a non-commutative differential structure is given by the pair (δ_1, δ_2) and may regard the following subalgebra

$$A_\theta^\infty = \{a = \sum a_{mn} u^m v^n \in A_\theta \mid \{a_{mn}\} \quad \text{are rapidly decreasing}\}$$

as the natural C^∞-object. This was a starting point of non-commutative differential geometry initiated by A. Connes.

The another important aspect of A_θ is that it has surprisingly rich structure for projections.

THEOREM 6.2. (Rieffel). *For any number α in the set $(\mathbb{Z}+\mathbb{Z}\theta)\cap[0,1]$ there exists a projection p such that $\tau(p) = \alpha$.*

Proof. By Proposition 6.1 we may regard A_θ as the C^*-algebra generated by multiplication operators of $C(\mathbb{T})$ and the shift operators s defined by

$$m_f g = fg \quad \text{and} \quad s(g)(t) = g(t-\theta).$$

Henceforth we identify m_f with f. With these notations the value of the trace τ is given by

$$\tau(\sum_{-n}^{n} f_k s^k) = \tau(f_0) = \int_0^1 f_0(t)dt.$$

We shall first construct a projection p with $\tau(p) = \theta$ in the form,

$$p = hs^{-1} + f + gs \quad h, f, g \in C(\mathbb{T}).$$

Suppose first that we have such a projection. Then the condition $p = p^*$ implies the conditions that f is real and $h = s^*\bar{g}s$. Combining with another condition $p = p^2$, we finally reach the relations,

(1) $g(t)g(t-\theta) = 0$,

(2) $g(t)(1 - f(t) - f(t-\theta)) = 0$,

(3) $f(t)(1 - f(t)) = |g(t)|^2 + |g(t+\theta)|^2 \quad t \in \mathbb{T}$.

Conversely if we can find two functions f and g satisfying the above three conditions with f being real, then putting $h = s^*\bar{g}s$ the element

$$p = hs^* + f + gs$$

becomes a projection. We shall construct this kind of pair (f, g) satisfying further condition that $\tau(f) = \theta$. Here we may assume that $0 < \theta < \frac{1}{2}$ because A_θ is isomorphic to $A_{1-\theta}$. Take a positive number ε such that

$$0 < \varepsilon < \theta \quad \text{and} \quad \theta + \varepsilon < \frac{1}{2}.$$

Let f be a real valued continuous function on $\mathbb{T}$ defined as follows. On the interval in $[0, \varepsilon]$, f may be any continuous function with values in $[0, 1]$, $f(0) = 0$ and $f(\varepsilon) = 1$. The function f should be constant value 1 on the interval $[\varepsilon, \theta]$ and 0 on $[\theta + \varepsilon, 1]$ respectively, and finally define f on $[\theta, \theta + \varepsilon]$ by

$$f(t) = 1 - f(t - \theta).$$

We next define the function g by

$$g(t) = (f(t)(1 - f(t)))^{\frac{1}{2}} \quad \text{on} \quad [\theta, \theta + \varepsilon]$$

and let g be zero elsewhere on $[0, 1]$. Then f and g satisfy relations (1), (2) and (3) above, so that they define a projection p. By construction,

$$\tau(p) = \int_0^1 f(t)dt = \theta.$$

For the general case, let $\{n\theta\}$ be the fractional part of $n\theta$ and suppose n is positive. We have,

$$u^n v = e^{2\pi i n\theta} v u^n = e^{2\pi i \{n\theta\}} v u^n.$$

Hence by Proposition 6.1 the algebra $C^*(u^n, v)$ is regarded as the algebra $A_{\{n\theta\}}$ and by the above arguments there exists a projection p in $C^*(u^n, v)$ such that

$$\tau'(p) = \{n\theta\}$$

for the trace τ' on $C^*(u^n, v)$. The trace τ' coincides however with τ by the unicity of the trace on these algebras.

In case $n < 0$, it is enough to find a projection p with $\tau(p) = \{-n\theta\}$ and to consider the projection $1 - p$. This completes all proofs.

Actually, more about the above result is known for the range of τ on the set of all projections of A_θ, $(A_\theta)_p$. In fact, the range $\tau((A_\theta)_p)$ coincides with the above set $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$. We need however another serious argument to prove this fact (cf. [**T**: Theorem 5.3.2]) and we do not enter further arguments. We have to mention here that once we know the range of $\tau((A_\theta)_p)$, then we can determine the isomorphism class of A_θ's. Namely,

. *"For two irrational numbers θ_1 and θ_2, A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$, namely when σ_{θ_1} and σ_{θ_2} are topologically conjugate."*

Let $\mathbb{R}$ be the real line. Recall that the *suspension* of a dynamical system $\Sigma = (X, \sigma)$ consists of the space $Y = (X \times \mathbb{R})/\sim$ factored through the equivalence relation;

$$(\sigma^n(x), s) \sim (x, s + n)$$

and the flow β_t on Y defined as

$$\beta_t[(x, s)] = [(x, s + t)],$$

where $[(x, s)]$ means the equivalence class of the point (x, s). If we consider the suspension of the system Σ_θ the space Y turns out to be $\mathbb{T}^2$ and the flow β_t becomes the so-called *Kronecker flow*. This flow forms a simple example of a foliation $\mathcal{F}$ on $\mathbb{T}^2$, and if we construct the *foliation C^*-algebra* $C^*(\mathbb{T}^2, \mathcal{F})$ from $\mathcal{F}$ it turns out to be isomorphic to the tensor product $A_\theta \otimes C(H)$.

We skip the other aspects of A_θ such as the case occuring in the projective representations of locally compact abelian groups but we can not overlook the next important result by Pimsner [**14**]. Some definitions both from C^*-theory and topological dynamics are in order before stating the result.

DEFINITION 6.1. A C^*-algebra is called an AF (approximately finite dimensional) algebra if it is spanned by an increasing sequence of finite dimensional C^*-algebras.

Since a finite dimensional C^*-algebra is nothing but a direct sum of full matrix algebras, the class of AF-algebras is the most tractable class of C^*-algebras. It is known by Pimsner-Voiculescu that the algebra A_θ can be embedded into an AF-algebra though A_θ itself is not an AF-algebra.

Recall that a point x is said to be *non-wandering* for σ if, for any neighborhood U of x and any natural number k, there exists an integer n with $|n| > k$ such that $\sigma^n(U) \cap U \neq \phi$. The set of non-wandering points, $\Omega(\sigma)$, is an invariant closed subset of X.

DEFINITION 6.2. (1) Let $\mathcal{V} = (V_i)_{i \in I}$ be an open cover of X. A sequence $w = (w(n))_{n \in \mathbb{Z}}$, $w(n) \in I$ is called a *$\mathcal{V}$-pseudo-orbit* of σ if

$$V_{w(n)} \cap \sigma^{-1}(V_{w(n+1)}) \neq \phi \quad \text{for every} \quad n \in \mathbb{Z}.$$

If the $\mathcal{V}$-pseudo-orbit is periodic we denote by $p(w)$ the smallest natural number p such that $w(n+p) = w(n)$ for every $n \in \mathbb{Z}$.

(2) A point x is said to be *pseudo-non-wandering* for σ if for every open cover $\mathcal{V} = (V_i)_{i \in I}$ and any $i \in I$ such that $x \in V_i$ there exists a periodic $\mathcal{V}$-pseudo-orbit $w = w(n)_{n \in Z}$ such that $w(0) = i$.

The set $X(\sigma)$ of all pseudo-non-wandering points also turns out to be an invariant closed subset of X. Pimsner's result clarifies in the following way the situation why the algebra A_θ is embedded into an AF-algebra.

THEOREM 6.3. *Let X be a compact metric space. Then the transformation group C^*-algebra $A(\Sigma)$ is embedded into an AF-algebra in a unit preserving way if and only if $X = X(\sigma)$.*

The proof is a little complicated and we leave full details to the Pimsner's paper [**14**].

Actually in the case of an irrational rotation σ_θ, we have that $\Omega(\sigma_\theta) = \mathbb{T}$. Every non-wandering point is pseudo-non-wandering but the converse is not true in general. In fact, consider the action of the shift σ_0 on the one point compactification of $\mathbb{Z}$, $X = (\mathbb{Z}, \infty)$, that is, $\sigma_0(n) = n + 1$ and $\sigma_0(\infty) = \infty$. In this case the only non-wandering point is ∞, whereas every point of X is pseudo-non-wandering ($\Omega(\sigma) = \{\infty\}$ and $X(\sigma) = X$).

There is another class of homeomorphisms of the circle $\mathbb{S}^1$ called Denjoy homeomorphisms for which the isomorphism class of their transformation group C^*-algebras is determined. Let σ be an orientation-preserving homeomorphism of $\mathbb{S}^1$. Then σ can be lifted to a strictly increasing function $\tilde{\sigma}$ on the real line $\mathbb{R}$ satisfying $\tilde{\sigma}(x+1) = \tilde{\sigma}(x) + 1$. Normalized as $0 \leq \tilde{\sigma}(0) < 1$, such a function is uniquely determined and the *rotation number* $\rho(\sigma)$ for the map σ is defined as the limit

$$\rho(\sigma) = \lim_{n\to\infty} \frac{\tilde{\sigma}^n(x)}{n},$$

which exists and is independent of $x \in \mathbb{R}$. The number $\rho(\sigma)$ is rational if and only if σ has a periodic point.

DEFINITION 6.3. A *Denjoy homeomorphism* is a homeomorphism σ of $\mathbb{S}^1$ with no periodic points.such that σ is not topologically conjugate to a regid rotation.

Thus, for this homeomorphism σ the rotation number $\rho(\sigma) = \theta$ is irrational and, by the famous Theorem of Poincare about the order of the set $\{\sigma^n(x)\}$, there exists an orientation-preserving continuous onto map h of $\mathbb{S}^1$ such that

$$h \circ \sigma = \sigma_\theta \circ h.$$

Namely, σ is *semiconjugate* to $\Sigma_\theta = (\mathbb{S}^1, \sigma_\theta)$. The measure $\mu = dh$ defined in particular $\mu([a,b]) = h(b) - h(a)$ is a unique σ-invariant probability measure on $\mathbb{S}^1$.

THEOREM 6.4. (Putnam-Schmidt-Skau). *Let σ_1 and σ_2 be two Denjoy homeomorphisms of the circle $\mathbb{S}^1$, and let $A(\Sigma_1)$ and $A(\Sigma_2)$ be their transformation group C^*-algebras respectively. Then $A(\Sigma_1)$ is isomorphic to $A(\Sigma_2)$ if and only if σ_1 is toplogically conjugate to σ_2 or σ_2^{-1}.*

We leave details to their joint paper [**16**] but illustrate here how to construct some typical Denjoy homeomorphisms. The observation is also quite important in their analysis.

Denote by S_μ the support of μ, then it turns out to be a Cantor set for which $h(S_\mu) = \mathbb{S}^1$ ($S_\mu \subsetneq \mathbb{S}^1$). Let $S_\mu = \mathbb{S}^1 \setminus \cup_{n=1}^{\infty} I_n$, where $\{I_n\}$ are countable disjoint open intervals with end points $\{a_n, b_n \,|\, n \in \mathbb{Z}\}$. The connecting map h naturally collapses each of the interval I_n into a single point, hence $h(a_n) = h(b_n)$. Set

$$\begin{aligned} Q(\sigma) &= \{h(x) \,|\, x \text{ is } a_n \text{ or } b_n \text{ for some } n\} \\ &= \{h(I_n) \,|\, n \in \mathbb{Z}\}. \end{aligned}$$

The set is uniquely determined by σ up to a rigid rotation. It is countable and invariant under σ_θ. Let $n(\sigma)$ be the number of disjoint orbits in $Q(\sigma)$ by σ_θ ($n(\sigma)$ could be infinite). Thus, $Q(\sigma) = \cup_{i=1}^{\infty} Q_i$, where each orbit Q_i is of the form $\gamma_i + n\theta$. Now consider the space $S'_{Q(\sigma)}$ which is the circle $\mathbb{S}^1$ with the points $Q(\sigma)$ being doubled in the following way. At each point of $Q(\sigma)$ we cut the circle with two end points adjoined and connect the pair of end points by an arc. The lengths of attached arcs should tend to zero as n increases to ∞. We then define the continuous map $\hat{\sigma}_\theta$ of $S'_{Q(\sigma)}$ as the rotation σ_θ on $\mathbb{S}^1$ but at the doubled point $x \in Q(\sigma)$ as a homeomorphism sending the arc at x onto the arc at the doubled point $x + \theta$, so that $\Sigma = (\mathbb{S}^1, \sigma)$ is conjugate to $\Sigma' = (S'_{Q(\sigma)}, \hat{\sigma}_\theta)$. Here the conjugation map netween Σ and Σ' maps the component $I_n = (a_n, b_n)$ onto one of the attached arcs with the pair a_n, b_n corresponding to the relevant doubled point of $\mathbb{S}^1$. Thus to obtain an example of a Denjoy homeomorphism for each n, say $n = 2$, we can start with the dynamical system $(S'_Q, \hat{\sigma}_\theta)$ for $Q = Q_1 \cup Q_2$, where $Q_1 = \{n\theta \,|\, n \in Z\}$ and $Q_2 = \{\gamma + n\theta \,|\, n \in Z\}$ with $\gamma \notin Q_1$. We then pull back this system to the situation $\mathbb{S}^1 \setminus \cup_{n=1}^{\infty} I_n$ assigning each double point to the appropriate pair a_n, b_n. The resulting map of $\mathbb{S}^1$ becomes a Denjoy homeomorphism with $\rho(\sigma) = \theta$.

Let H be the three dimensional discrete Heisenberg group expressed by matrices, that is,

$$H = \left\{ \begin{pmatrix} 1 & \ell & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \ell, m, n \in \mathbb{Z} \right\}.$$

There are two generators a and b in H, where

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The element

$$c = aba^{-1}b^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

belongs to the center of H. Let G be the abelian group generated by b and c, then

$$G = \left\{ \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, m, n \in \mathbb{Z} \right\}$$

and G is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. On the other hand, the group generated by a is isomorphic to $\mathbb{Z}$ and it acts on G as

$$axa^{-1} = \begin{pmatrix} 1 & 0 & m+n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Thus we get an action of $\mathbb{Z}$ on G and since

$$\begin{pmatrix} 1 & \ell & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the group H is considered as the semidirect product of G and $\mathbb{Z}$. Therefore, H is an amenable group. Let α be the action of $\mathbb{Z}$ on the group C^*-algebra $C^*(G)$ $(= C^*_r(G))$ lifted from the action : $x \mapsto axa^{-1}$. Here the algebra $C^*(G)$ coincides with the algebra $C(\mathbb{T}^2)$ of continuous functions on $\mathbb{T}^2$ $(= \widehat{G}$, the dual of $G)$. It is known then

$$C^*(H) \cong C^*(G) \underset{\alpha}{\times} \mathbb{Z} = C(\mathbb{T}^2) \underset{\alpha}{\times} \mathbb{Z}.$$

We shall prove later this fact for general case (Proposition 6.5). Before proving the above result, we determine the homeomorphism σ on

$\mathbb{T}^2$ induced by the action α and describe the structure of irreducible representations of $C^*(H)$ induced from those points of $\mathbb{T}^2$. We regard first an element of G as a continuous function on $\mathbb{T}^2$, which we denote by $f_{m,n}$ for $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. We have

$$\alpha(f_{m,n}) = f_{m+n,n}.$$

Therefore

$$\begin{aligned}\alpha(f_{m,n})(s,t) &= f_{m+n,n}(s,t)\\ &= e^{2\pi i((m+n)s+nt)}\\ &= e^{2\pi i(ms+n(s+t))}\\ &= f_{m,n}(s,t+s)\\ &= f_{m,n}(\sigma^{-1}(s,t)).\end{aligned}$$

Since the linear span of those functions $\{f_{m,n}\}$ is dense in $C(\mathbb{T}^2)$, the above relation determines the homeomorphism σ as

$$\sigma^{-1}(s,t) = (s, t+s) \quad \text{and} \quad \sigma(s,t) = (s, t-s).$$

Thus we have the homeomorphism which induces a rational or irratonal rotation along t-axis according to the condition, s being rational or irrational. Hence we obtain the following

THEOREM 6.5. *The group C^*-algebra $C^*(H)(= C^*_r(H))$ is regarded as the transformation group C^*-algebra arised from the topological dynamical system $\Sigma = (\mathbb{T}^2, \sigma)$ where the homeomorphism σ is the one described above. It therefore has every rational and irrational rotation C^*-algebra as its quotient image. Futhermore, $C^*(H)$ has sufficiently many finite dimensional irreducible representations with ranging all dimensions.*

For the proof we just mention that the restriction of the dynamical system $\Sigma = (\mathbb{T}^2, \sigma)$ to the torus

$$\mathbb{T}_s = \{(s,t) \mid t \in \mathbb{T}\}$$

at level s naturally induces a dynamical system $\Sigma_s = (\mathbb{T}_s, \sigma_s)$ for $\sigma_s = \sigma \mid \mathbb{T}_s$, and we get the canonical homomorphism from $A(\Sigma)$ to

$A(\Sigma_s)$, the latter of which is a rational or irrational rotation C^*-algebra according to the level s. The last conclusion follows from (2) of Theorem 4.6.

Let u, v and z be unitaries in $C^*(H)$ corresponding to a, b and c. For each point (s,t) for an irrational number s the pure state extension of $\mu_{(s,t)}$ is unique by Corollary of Theorem 3.5 and two induced irreducible representations $\pi_{(s,t_1)}$ and $\pi_{(s,t_2)}$ are unitarily equivalent if and only if $O((s,t_1)) = O((s,t_2))$. Here we can say moreover that there exists precisely one state on $C^*(H)$ satisfying

$$\varphi(z) = e^{2\pi is} \quad \text{and} \quad \varphi(v) = e^{2\pi it}.$$

For, an argument similar to the proof of Lemma 4.2 leads us to the assertions,

$$\begin{aligned} \varphi(az) &= \varphi(za) = e^{2\pi is}\varphi(a) \quad \text{and} \\ \varphi(av) &= \varphi(va) = e^{2\pi it}\varphi(a) \end{aligned}$$

for every $a \in C^*(H)$, which implies the fact that the restriction of φ on $C(\mathbb{T}^2)$ coincides with the point evaluation $\mu_{(s,t)}$. In the next section we shall shows that the family $\{A(\Sigma_s) \,|\, s \in \mathbb{T}\}$ is actually connected in a continuous way and $A(\Sigma)$ is expressed as the algebra of all (noncommutative) continuous fields having values in $A(\Sigma_s)$ at each point $s \in \mathbb{T}$.

Finally we prove the realization of $C^*(H)$ as the crossed product in a general context for reference since the detailed proof of this fact is hardly found in literature except in [**T**].

PROPOSITION 6.6. *Let $G = K \underset{\alpha}{\times} L$ be a discrete group which is a semidirect product of groups K and L by the action α of L on K. Then*

$$C^*(G) \cong C^*(K) \underset{\alpha}{\times} L.$$

Proof. Note first that by the definition of semidirect product the action α of L on K is expressed as $\alpha_a(x) = axa^{-1}$ for $x \in K$ and $a \in L$. Consider the algebra $\ell^1(G)$ as a usual convolution algebra with $*$-operation $a^*(g) = \overline{a(g^{-1})}$ and the algebra $\ell^1(L, \ell^1(K))$ as a twisted convolution Banach $*$-algebra for construction of the crossed product $C^*(K) \underset{\alpha}{\times} L$. We shall define an isomorphism Φ between $\ell^1(G)$ and $\ell^1(L, \ell^1(K))$. For an element $a = (a(g))$ in $\ell^1(G)$ put $\Phi(a)(l) = a_l$ for $l \in L$ where a_l is a function of $\ell^1(K)$ defined as $a_l(k) = a(kl)$. One then easily verify that Φ is a linear isometry. Moreover, we have for a and b in $\ell^1(G)$,

$$\begin{aligned}
(\Phi(a)\Phi(b))(l)(k) &= (\sum_{l_1} \Phi(a)(l_1)\alpha_{l_1}(\Phi(b)(l_1^{-1}l)))(k) \\
&= \sum_{k_1}\sum_{l_1} \Phi(a)(l_1)(k_1)\alpha_{l_1}(\Phi(b)(l_1^{-1}l))(k_1^{-1}k) \\
&= \sum_{l_1,k_1} a(k_1 l_1)b(l_1^{-1}k_1^{-1}kl) \\
&= ab(kl) = \Phi(ab)(l)(k).
\end{aligned}$$

Hence, $\Phi(ab) = \Phi(a)\Phi(b)$ and Φ is an isomorphism. Similarly we can show that $\Phi(a^*) = \Phi(a)^*$. We assert next that Φ is an isometry for the C^*-norm $\|a\|_\infty$. Let π be a representation of $\ell^1(G)$, then there exsits a unitary representation u of G such that $\pi(a) = \Sigma_g a(g)u(g)$. Let v and w be the restrictions of u to K and L respectively. Let ρ be the representation of $\ell^1(K)$ associated with v. We see that (ρ, w) is a covariant representation of $(\ell^1(K), L, \alpha)$ such that $(\rho \times w) \circ \Phi = \pi$. In fact,

$$\begin{aligned}
\rho \times w(\Phi(a)) &= \sum_{l} \rho(\Phi(a))(l)w(l) \\
&= \sum_{k,l} a(kl)v(k)w(l) \\
&= \sum_{k,l} a(kl)u(kl) \\
&= \pi(a).
\end{aligned}$$

Therefore, $\|\Phi(a)\|_\infty \geq \|a\|_\infty$. Conversely, let $\rho \times w$ be a representation of $C^*(K) \underset{\alpha}{\times} L$. Put $u(g) = v(k)w(l)$ for an element $g = kl \in G$, then u becomes a unitary representation of G because (ρ, w) is a covariant representation and G is the semidirect product of K and L. Thus we obtain a representation π of $\ell^1(G)$ through u and as mentioned above $(\rho \times w) \circ \Phi = \pi$. It follows that $\|\Phi(a)\|_\infty = \|a\|_\infty$. Therefore, with this mapping Φ we obtain a $*$-isomorphism of $C^*(G)$ to $C^*(K) \underset{\alpha}{\times} L$. This completes the proof.

§7. Decompositions of topological dynamical systems and their transformation group C^*-algebras

In studying topological dynamical systems we sometimes meet the situation that a given topological dynamical system is decomposed into a sum of closed invariant subsets, or even into a sum of minimal dynamical systems. Actually when a system $\Sigma = (X, \sigma)$ is decomposed into the sum of minimal systems, it is said to be *semisimple*.

In this section, we discuss how the algebra $A(\Sigma)$ is decomposed in terms of continuous fields of operators when the system Σ is decomposed into a sum of invariant closed subsets. Here we restrict the space X to be a metric space for simplicity although all results in this section are valid without any countability condition for X if we make use of the uniform structure attached to the compact space.

DEFINITION 7.1. (1) A topological dynamical system $\Sigma = (X, \sigma)$ for a metric space (X, d) is said to be *distal* if the condition

$$\inf\{d(\sigma^n x, \sigma^n y) \mid n \in \mathbb{Z}\} = 0$$

implies that $x = y$.

(2) Σ is said to be *equicontinuous* if for any positive number ε there exists a positive number δ such that

$$d(x, y) < \delta \Longrightarrow d(\sigma^n x, \sigma^n y) < \varepsilon \quad \text{for every} \quad n \in \mathbb{Z}.$$

It is obvious that an equicontinuous system is distal. To illustrate their difference we consider the so-called *Anzai skew product* of an irrational rotation σ_θ on $\mathbb{T}$, that is, the mapping σ on $\mathbb{T}^2 : (s, t) \mapsto (\sigma_\theta s, t + s)$. For those points $x = (s_1, t_1)$ and $y = (s_2, t_2)$ in $\mathbb{T}^2$ put the distance, $d(x, y) = |s_1 - s_2| + |t_1 - t_2|$.

We see then for every n,

$$\begin{aligned} d(\sigma^n x, \sigma^n y) &= |s_1 - s_2| + |t_1 - t_2 + n(s_1 - s_2)| \\ &\geq |s_1 - s_2|, \\ \text{and} \quad &\geq |t_1 - t_2| \quad \text{if} \quad s_1 = s_2. \end{aligned}$$

Hence the system $\Sigma = (\mathbb{T}^2, \sigma)$ is distal but if we consider the sequence $x_k = (\frac{1}{k}, s)$ $(k = 1, 2, \dots)$ converging to $(0, s)$ we have that

$d(\sigma^k x_k, \sigma^k(0,s)) \geq 1$. Therefore, Σ is not equicontinuous. A typical example of an equicontinuous system is the case for an isometric homeomorphism. Thus most of those examples discussed before are equicontinuous, hence distal systems.

We shall prove first that a distal system is necessarily semisimple. We need however some preparation before this assertion.

Let X^X be the space of all maps in X with the pointwise convergence topology. By Tychonoff's theorem the space is a compact Hausdorff space with the semigroup structure. For a dynamical system $\Sigma = (X, \sigma)$, we consider the closure $E(\Sigma)$ of the set $\{\sigma^n \mid n \in \mathbb{Z}\}$ in X^X. The set $E(\Sigma)$ becomes a commutative semigroup called the *Ellis semigroup* of Σ.

PROPOSITION 7.1. *The semigroup $E(\Sigma)$ becomes a group if and only if Σ is distal.*

Proof. Suppose that $E(\Sigma)$ is a group and consider points x, y and z in X such that both nets $\{\sigma^{n_v}(x)\}$ and $\{\sigma^{n_v}(y)\}$ converge to z. Since $E(\Sigma)$ is a compact space, by passing to a subnet we may assume that the net of mappings $\{\sigma^{n_v}\}$ converges to a map g in $E(\Sigma)$. Then

$$g(x) = \lim_v \sigma^{n_v}(x) = z = \lim_v \sigma^{n_v}(y) = g(y)$$

and as g is invertible we have that $x = y$. Namely, Σ is distal. Conversely suppose that Σ is distal. It follows that each map g in $E(\Sigma)$ is injective and $E(\Sigma)$ has a cancellation law; $gg_1 = gg_2$ implies $g_1 = g_2$. Therefore the only possible idempotent element in $E(\Sigma)$ is the identity. We assert that every element of $E(\Sigma)$ is invertible. Take a map h in $E(\Sigma)$ and put $E_1 = \{gh \mid g \in E(\Sigma))$. Consider the family

$$\Phi = \{S \mid \text{closed nonempty subset of } E_1, S^2 \subset S\}.$$

Obviously $E_1 \in \Phi$ and Φ is not empty. Assume the order in Φ by inclusion, then we can apply Zorn's lemma to Φ finding a minimal set S_0. Take an element g in S_0 and put

$$S_0 g = \{fg \mid f \in S_0\}.$$

Since $S_0^2 \subset S_0$, $S_0 g$ is a subset of S_0 and moreover $(S_0 g)^2 \subset S_0 g$. Hence $S_0 g = S_0$, which means that there is an element $f \in S_0$ such that $fg = g$. Therefore, the set $W = \{k \in S_0 \mid kg = g\}$ is a nonempty subset of S_0 satisfying the condition, $W^2 \subset W$. Hence, $W = S_0$ and $g \in W$. This means that $g^2 = g$ and as mentioned before E_1 contains the identity. Thus, h has the inverse in $E(\Sigma)$ and $E(\Sigma)$ is a group.

With this proposition we have the following

THEOREM 7.2. *A distal system is semisimple.*

Proof. Suppose that Σ is distal and take a point x. It is enough to shows that the closure $\overline{O(x)}$ becomes a minimal system. Let $y \in \overline{O(x)}$ and consider a net $\{\sigma^{n_v}(x)\}$ converging to y. By passing to a subnet, we may assume that the net $\{\sigma^{n_v}\}$ converges to a map g in $E(\Sigma)$ and $g(x) = y$. By the above proposition we have $x = g^{-1}(y)$ with $g^{-1} \in E(\Sigma)$. It follows that $x \in \overline{O(y)}$ and $\overline{O(x)} = \overline{O(y)}$ meaning that the system $\Sigma' = (\overline{O(x)}, \sigma|_{\overline{O(x)}})$ is minimal. This completes the proof.

Now suppose that the system Σ is decomposed into the disjoint union of subdynamical systems $\{\Sigma_\gamma = (X_\gamma, \sigma_\gamma) \mid \gamma \in \Gamma\}$ where each X_γ is an invariant closed subset of X and $\sigma_\gamma = \sigma|_{X_\gamma}$. Write the corresponding action of $\mathbb{Z}$ to $C(X_\gamma)$ by α_γ. We shall be concerned with the discussions how and in what circumstances those corresponding transformation group C^*-algebras $A(\Sigma_\gamma)$'s are connected to build up the original algebra $A(\Sigma)$. We have to consider first the quotient space of X by the relation R ($x \sim y \iff x, y \in X_\gamma$ for some $\gamma \in \Gamma$) assuming each subset X_γ being a point of the space, still denoted by Γ. Let E_γ be the canonical projection of norm one of $A(\Sigma_\gamma)$ to $C(X_\gamma)$. The restriction map $\rho_\gamma : C(X) \to C(X)|_{X_\gamma} = C(X_\gamma)$ is compatible with the action α and it induces the natural homomorphism $\tilde{\rho}_\gamma$ from $A(\Sigma)$ to $A(\Sigma_\gamma)$ such that $\rho_\gamma \circ E = E_\gamma \circ \tilde{\rho}_\gamma$. Let I_γ be the kernel of $\tilde{\rho}_\gamma$. Then although the system Σ_γ may not be topologically free we can apply the proof of Proposition 5.2 and get the results that I_γ is generated by $k(X_\gamma)$ and an element a belongs to I_γ if and only if every Fourier coefficient of a vanishes on X_γ. Let q be the quotient map of X to Γ. We recall here that a subset S in the quotient space Γ is open if and only if $q^{-1}(S)$ is open in X.

Now our object is the fibred space $\{\Gamma \mid A(\Sigma_\gamma)\}$. Write

$$a(\gamma) = \tilde{\rho}_\gamma(a)$$

for $a \in A(\Sigma)$. We are aiming, under suitable conditions, to represent $A(\Sigma)$ as the C^*-algebra of continuous operator fields with respect to this fibred space. Thus we have to explain what we mean by the above expression. Let $\{A(t) \mid t \in Y\}$ be a fibred space on a compact space Y with a C^*-algebra $A(t)$. Let a be a cross section or an operator field on Y meaning the function such that $a(t) \in A(t)$ for every t. In general, we can not consider the distance between $a(t)$ and $a(s)$ for different points t, s, hence can not talk about the continuity of a over Y. This situation is not improved even in the case where $A(t)$ is isomorphic to a fixed C^*-algebra A because the isomorphism θ_t between $A(t)$ and A may vary along t. Therefore, in order to define a continuity of operator fields in this fibred space we need the following family $\mathcal{F}$ of operator fields satisfying the conditions;

(1) $\|a(t)\|$ is continuous on Y for every $a \in \mathcal{F}$;
(2) At each point t, the image $\{a(t) \mid a \in \mathcal{F}\}$ is dense in $A(t)$;
(3) $\mathcal{F}$ forms a $*$-algebra under pointwise operations.

We say then an operator field a *continuous* at a point t_0 in Y with respect to $\mathcal{F}$ if for any positive number ε and for any operator field b in $\mathcal{F}$ with $\|a(t_0) - b(t_0)\| < \varepsilon$ there exists a neighborhood U of t_0 such that $\|a(t) - b(t)\| < \varepsilon$ for every $t \in U$. It then turns out that the set of all continuous operator fields with respect to $\mathcal{F}$ forms a C^*-algebra $C_{\mathcal{F}}(Y \mid A(t))$ with the norm $\|a\| = \sup \|a(t)\|$. We refer the details to [**22**]. Note that this continuity depends on the family $\mathcal{F}$ and moreover it may happen that there could exist no such a family for $\{Y \mid A(t)\}$. On the other hand, in particular case where any $A(t)$ is isomorphic to $\mathbb{C}$, the complex number field, the isomorphism between $A(t)$ and $\mathbb{C}$ does not depend on t so that we can talk about the distance between $a \in A(t)$ and $b \in A(s)$. Therefore, in this case the continuity is unique and the algebra of all continuous functions, $C(Y)$, is nothing but the algebra $C_{\mathcal{F}}(Y \mid A(t) = \mathbb{C})$ with respect to the family $\mathcal{F}$ of constant functions on Y. As we have mentioned above we may replace this family $\mathcal{F}$ by other family of continuous functions satisfying the starting conditions. Now in order to find the continuity of the function $\|a(\gamma)\|$ for $a \in A(\Sigma)$ we need the following lemmas.

LEMMA 7.3. *The following assertions are equivalent;*

(1) Γ *is Hausdorff;*

(2) *The map q is a closed map;*
(3) *The function* : $\gamma \to \|a(\gamma)\|$ *is upper semicontinuous for every element* $a \in A(\Sigma)$.

Proof. Since the map q is continuous it maps a closed hence compact subset of X to a compact subset of Γ, and if Γ is Hausdorff the latter set becomes closed in Γ. Thus, (1) implies (2). For the assertion, (2) $\Longrightarrow$ (3), consider the set $H = \{\gamma \in \Gamma \mid \|a(\gamma)\| < \varepsilon\}$ and take a point γ_0 in H. Because $\ker \tilde{\rho}_{\gamma_0}$ is generated by $k(X_{\gamma_0})$ as mentioned before we can find an element $b = \Sigma f_k \delta^k$ of finite sum such that $f_k|_{X_{\gamma_0}} = 0$ for every k and $\|a + b\| < \varepsilon_1$ for a positive number ε_1 with $\|a(\gamma_0)\| < \varepsilon_1 < \varepsilon$. On the other hand, by the compactness of X_{γ_0}, there exists a neighborhood U of X_{γ_0} such that

$$\sum_k |f_k(x)| < \varepsilon_1 - \|a + b\| \quad \text{for every} \quad x \in U.$$

Here by [**8**; Theorem 3.10(c)] the union of all members X_γ's contained in U forms an open set containing X_{γ_0}. Hence its quotient image V becomes a neighborhood of γ_0 such that for every $\gamma \in V$ we have

$$\begin{aligned} \|a(\gamma)\| &\leq \|a(\gamma) + b(\gamma)\| + \|\sum_k (f_k|_{X_\gamma})\tilde{\rho}_\gamma(\delta)^k\| \\ &\leq \|a + b\| + \sup_{x \in U} \sum_k |f_k(x)| \leq \varepsilon_1 < \varepsilon. \end{aligned}$$

This means that H is an open set.

Now suppose that the function $\|a(\gamma)\|$ is upper semicontinuous and take two different points γ_1 and γ_2 in Γ. Since X is normal, there exists a continuous function f such that $0 \leq f \leq 1$, $f|_{X_{\gamma_1}} = 0$ and $f|_{X_{\gamma_2}} = 1$. From the assumption the sets $V_1 = \{\gamma \in \Gamma \mid \|\rho_\gamma(f)\| < \frac{1}{2}\}$ and $V_2 = \{\gamma \in \Gamma \mid \|\rho_\gamma(1 - f)\| < \frac{1}{2}\}$ are disjoint open subsets of Γ which contain γ_1 and γ_2, respectively. Hence, Γ is a Hausdorff space.

The next lemma clarifies the situation of the another half of the continuity of those functions $\|a(\gamma)\|$'s.

LEMMA 7.4. *The following assertions are equivalent;*
(1) *q is an open map,*

(2) *The closure of any saturated subset of* X *with respect to the retation* R *is also saturated,*

(3) *The function* $\|a(\gamma)\|$ *is lower semicontinuous.*

Proof. The implication (1) $\Longrightarrow$ (2). Take a set $S = \cup_{\gamma\in\Lambda} X_\gamma$ and let x be a point of the closure $\overline{S}$. Let y be a point of X equivalent to x and take a neighborhood U of y. The image $q(U)$ is then a neighborhood of $q(y)$, and $q(U)\cap\Lambda \neq \phi$ because $q(y) = q(x) \in \overline{\Lambda}$. Hence,

$$U \cap q^{-1}(\Lambda) = U \cap S \neq \phi$$

which implies that $y \in \overline{S}$. Namely, $\overline{S}$ is also a saturated set. For the implication (2) $\Longrightarrow$ (3) we assert that for any $\varepsilon > 0$ the set $F = \{\gamma \in \Gamma \mid \|a(\gamma)\| \leq \varepsilon\}$ is closed. Here we may assume that a is positive. Suppose that there exists a point $\gamma_0 \in \overline{F}$ such that $\|a(\gamma_0)\| > \varepsilon$. We consider the continuous function h on the real line defined as

$$h(t) = \begin{cases} 0 & \text{if} \quad t \leq (\varepsilon + \|a(\gamma_0)\|)/2 \\ 1 & \text{if} \quad t \geq \|a(\gamma_0)\| \\ \text{linear} & \text{otherwise} \end{cases}$$

Then,

$$h(a)(\gamma) = \tilde{\rho}_\gamma(h(a)) = h(\tilde{\rho}_\gamma(a)) = 0 \quad \text{for every} \quad \gamma \in F$$

hence every Fourier coefficient of $h(a)$ vanishes on X_γ ($\gamma \in F$). On the other hand, from the assumption for (2) the set X_{γ_0} is contained in the closure of $q^{-1}(F)$. Therefore every Fourier coefficient of $h(a)$ vanishes on X_{γ_0}, and $h(a)(\gamma_0) = 0$. However, the property of the function $h(t)$ tells us that $h(a)(\gamma_0) \neq 0$, a contradiction. Hence, F is a closed set.

Next suppose the assumption (3) and consider an open set G in X. We assert that $q(G)$ is an open set in Γ. Thus, take a net $\{\gamma_\alpha\}$ in $q(G)^c$, the complement of $q(G)$, converging to γ_0. Then, $X_{\gamma_\alpha} \cap G = \phi$ and every X_{γ_a} is contained in the closed set G^c. Suppose that γ_0 belong to $q(G)$ and take a point x_0 in G with $q(x_0) = \gamma_0$. There exists then a continuous function f on X such that $0 \leq f \leq 1$, $f|_{G^c} = 0$ and $f(x_0) = 1$. Put $F = \{\gamma \in \Gamma \mid \|\rho_\gamma(f)\| \leq \frac{1}{2}\}$. By definition, every γ_α belongs to F whereas γ_0 does not belong to F because $\|\rho_{\gamma_0}(f)\| \geq 1$.

This is a contradiction and γ_0 belongs to $q(G)^c$. It follows that $q(G)^c$ is a closed set, and $q(G)$ is open. This completes all proofs.

Now suppose further that, in the decomposition $X = \cup_{\gamma \in \Gamma} X_\gamma$, both conditions in Lemma 7.3 and 7.4 hold. Then the function $\|a(\gamma)\|$ is continuous for every element $a \in A(\Sigma)$ and we can talk about continuous fields of operators with respect to the family

$$\mathcal{F} = \{a(\gamma) \,|\, a \in A(\Sigma)\}.$$

Let $C_{\mathcal{F}}(\Gamma \,|\, A(\Sigma_\gamma))$ be the C^*-algebra of all continuous operator fields with respect to $\mathcal{F}$. Obviously, we may regard the algebra $A(\Sigma)$ as a C^*-subalgebra of $C_{\mathcal{F}}(\Gamma \,|\, A(\Sigma_\gamma))$.

In order to show that $A(\Sigma)$ actually coincides with $C_{\mathcal{F}}(\Gamma \,|\, A(\Sigma_\gamma))$ we need the following non-commutative Stone-Weierstrass theorem (cf. [**22**]).

. *"Let A be a C^*-subalgebra of the C^*-algebra $C_{\mathcal{F}}(Y \,|\, A(t))$. Suppose that for any two points t_1 and t_2 and for any elements $a \in A(t_1)$ and $b \in A(t_2)$ there exists an element z of A such that $z(t_1) = a$ and $z(t_2) = b$, then the algebra A coincides with $C_{\mathcal{F}}(Y \,|\, A(t))$."*

With the help of this theorem we can prove our decomposition theorem.

THEOREM 7.5. *Suppose that in the decomposition $X = \cup_{\gamma \in \Gamma} X_\gamma$, both conditions in Lemma 7.3 and 7.4 hold, then the algebra $A(\Sigma)$ is isomorphic to the C^*-algebra $C_{\mathcal{F}}(\Gamma \,|\, A(\Sigma_\gamma))$ of continuous operator fields.*

Proof. Let Φ be the map from $A(\Sigma)$ into $C_{\mathcal{F}}(\Gamma \,|\, A(\Sigma_\gamma))$ defined by $\Phi(a) = \{a(\gamma)\}$. If $a(\gamma) = 0$ for all γ, all Fourier coefficients of a vanish on X_γ for all γ, hence on X and $a = 0$. Namely Φ is a $*$-isomorphism. Hence, it suffices to show that Φ is surjective. Thus, take two different points γ_1 and γ_2 in Γ and a continuous function f on X such that $f|_{X_{\gamma_1}} = 0$ and $f|_{X_{\gamma_2}} = 1$. Then, $f \in I_{\gamma_1}$ and $1 - f \in I_{\gamma_2}$, which means that $I_{\gamma_1} + I_{\gamma_2} = A(\Sigma)$. Take an arbitrary pair (c, d) in $(A(\Sigma_{\gamma_1}), A(\Sigma_{\gamma_2}))$ and consider those elements a_1 and a_2 in $A(\Sigma)$ such that $a_1(\gamma_1) = c$ and $a_2(\gamma_2) = d$.

We then easily see that the element $a = (1 - f)a_1 + fa_2$ satisfies the conditions that $a(\gamma_1) = c$ and $a(\gamma_2) = d$. By the above non-commutative Stone-Weierstrass theorem, we have that $\Phi(A(\Sigma)) = C_{\mathcal{F}}(\Gamma \mid A(\Sigma_\gamma))$.

Applying this theorem to the dynamical system associated to the three dimensional discrete Heisenberg group H discussed in §6 we have the following corollary.

COROLLARY 7.6. *The group C^*-algebra $C^*(H)$ is isomorphic to the C^*-algebra of all continuous operator fields over the fibred space $\{\mathbb{T} \mid A(\Sigma_s)\}$ where $\Sigma_s = \{(s,\mathbb{T}),\ \sigma_s = \sigma|_{(s,\mathbb{T})}\}$. At each level s an irrational rotation C^*-algebra or a rational rotation C^*-algebra appears as the algebra $A(\Sigma_s)$ according to the condition s being irrational or rational.*

The maps σ_s in the above corollary are actually isometries in $(s, \mathbb{T}^2)$ so that by Theorem 7.2 each dynamical system $\Sigma_s = ((s,\mathbb{T}), \sigma_s)$ for a rational number s is further decomposed into minimal systems. In this case the precise description of the corresponding representation of $A(\Sigma_s)$ is nothing but the representation of the homogeneous C^*-algebra $A(\Sigma_s)$ cited in §6 as the C^*-algebra of all continuous cross-sections in its structure bundle over the torus $\mathbb{T}$.

Now in general when a given dynamical system Σ is distal one may easily verify that the decomposition of Σ in Theorem 7.2 satisfies the condition (2) of Lemma 7.4. As for the condition of Lemma 7.3 we have

PROPOSITION 7.7. *Suppose that the system $\Sigma = (X, \sigma)$ is equicontinuous, then the decomposition of Σ satisfies both conditions in Lemmas 7.3 and 7.4.*

Hence, in this case, $A(\Sigma)$ is expressed as a continuous connection of those algebras $A(\Sigma_\gamma)$'s.

Proof. It suffices to show that the quotient map $q : X \to \Gamma$ is a closed map. Thus, let F be a closed subset of X. We must show that the saturation $R(F)$ of F is closed, too. Let $\{x_n\}$ be a sequence in $R(F)$ converging to a point x_0. We may assume here that the sequence $\{x_n\}$ is not eventually contained in any set X_γ in $R(F)$. For, otherwise, the point x_0 belongs obviously to some set X_γ in $R(F)$. Now choose an element $y_n \in F \cap X_{\gamma_n}$ for each $x_n \in X_{\gamma_n}$. We may assume then that $\{y_n\}$ also converges to a point y_0 in F. For an arbitrary positive number ε there exists a positive number δ such that

$$d(x,y) < \delta \Longrightarrow d(\sigma^n x, \sigma^n y) < \varepsilon \text{ for every } n \in \mathbb{Z}.$$

Choose a natural number k such that both $d(x_k, x_0)$ and $d(y_k, y_0)$ are less than δ and ε. Since x_k is equivalent to y_k, that is, $\overline{O(x_k)} = \overline{O(y_k)}$ we can find an integer n_k such that $d(x_k, \sigma^{n_k}(y_k)) < \varepsilon$. Then,

$$d(\sigma^{n_k}(y_0), x_0) \leq d(\sigma^{n_k}(y_0), \sigma^{n_k}(y_k)) + d(\sigma^{n_k}(y_k), x_k) + d(x_k, x_0) < 3\varepsilon.$$

Hence, $x_0 \in \overline{O(y_0)}$ and x_0 belongs to $R(F)$. This completes the proof.

In this kind of decomposition, most fibre algebras are simple provided that those corresponding component subsets X_γ's consist of infinite points.

Notes and Remarks

In this note, although we restrict our discussions to the case of a topological dynamical system with single homeomorphism for simplification all results except Theorem 4.7 are proved imposing no countability condition on the space X contrary to standard arguments in topological dynamics (condition for §7 is just conventional as mentioned there). This is one of our basic standpoint based on a fact that we sometimes have to handle with and to apply our results to topological dynamical systems defined on large compact spaces coming from operator algebras such us shift dynamical systems (cf. [**T**: Chapter 5]).

Most materials in §3 and §4 are taken from the author's book [**T**]. The uniqueness of traces on $A(\Sigma)$, Theorem 3.7 is extended in [**6**] to the case of a topological dynamical system for an arbitrary discrete group (in [**T**] the abelian case is proved).

§5 is a revised version for a corresponding part in [**T**] emphasizing the rôle of topologically free actions in Definition 2.1, whereas this concept has been used in literature as the condition (b) in Proposition 2.2. Since in topological dynamics problems about periodic points are always occupying important rôles, the author feels that there is a strict difference between free actions (no nontrivial isotropy group) and topologically free actions. As is seen in the discussions of this section the useful observation Proposition 2.2 is due to my student K. Mise. The key step in the form of Theorem 5.1 (existence of the compatible projection of norm one) was first treated in S. C. Power [**S9**] in case Σ_π being minimal, while in [**T**] proved when Σ_π was topologically transitive. In a form Proposition 2.2(b) the topologically free action is equivalent to the properly outer action in the context of C^*-algebras. For this type of actions G. Elliott has proved a similar result to Theorem 5.1 for an arbitrary discrete group but when the relevant C^*-algebra A is either separable or simple AF-algebra [**S1**] (thus leading him to the proof of the simplicity of $A \times_{\alpha r} G$ in these cases). We note that the algebra $C(X)$ hardly becomes an AF-algebra unless X is totally disconnected. The simplicity of $A(\Sigma)$, Theorem 5.3 is extended in [**7**] to the case of a topological dynamical system for an arbitrary amenable group. Here, contrary to abelian case, we need minimality and topological freeness (in their appropriate extended sense) altogether as the necessary and sufficient condition for simplicity. In the context of a C^*-dynamical system (A, G, α) for a separable C^*-algebra A with G being discrete, D. Olesen and G. K. Pedersen has given the sufficient condition for the simplicity of $A \underset{\alpha r}{\times} G$ where the action α is properly outer and A is G-simple [**S8**]. In our context however it is also important to know how the qualitative property of $A(\Sigma)$ determines the property of the dynamical system Σ. In this sense corresponding generalization of Theorem 5.5 parallel to the result in [**7**] has not been obtained yet. Proposition 5.2 is not mentioned in [**T**]. The equivalence (1) $\Longleftrightarrow$ (3) of Theorem 5.4 is an old result by G. Zeller-Meier [**25**] and though the proof given here makes no use of advanced results for C^*-algebras the noncommutative version (acting group is $\mathbb{Z}$) of the equivalence (1) $\Longleftrightarrow$ (2) (considering the transposed action of the automorphism on the dual of a C^*-algebra A) was stated in [**S6**]. As pointed out by D. Olesen and G. K. Pedersen [**S7**, Remark 4.8], however, the implication (2) $\Longrightarrow$ (1) contained the gap which has been overcome in the article at least for a C^*-algebra A of type I. The general case seems to be still open. On the other hand, the assertion (2) is also shown in their paper to be equivalent to the condition for the Connes spectrum, that is, $\Gamma(\alpha) = \hat{G}$ for an abelian discrete group G.

When $G = \mathbb{Z}$ and $A = C(X)$, this spectrum condition is naturally equivalent to our assertion (1). Even in the case $G = \mathbb{Z}$, however, it seems to be not known whether or not the condition $\Gamma(\alpha) = \mathbb{T}$ implies topological freeness of the action whereas the other implication is true.

In connection with §6, for those people who are interested in more advanced results about irrational rotation C^*-algebras a series of works by K. Kodaka (still continuing as of now) is indispensable. Most of his papers are published in Tokyo J. Math. such as [**S3, S4**, etc.]. There is however one notable paper appeared in [**S5**], which we shall introduce in the following. An automorphism α of an irrational rotation algebra A_θ is called a diffeomorphism if $\alpha(A_\theta^\infty) = A_\theta^\infty$. There are then three kinds of typical diffeomorphisms; one is the adjoint automorphism ad$w$ for a smooth unitary element w, that is, $w \in A_\theta^\infty$, the next one the automorphism α_g defined for an element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $SL(2, \mathbb{Z})$ as $\alpha_g(u) = u^a v^c$ and $\alpha_g(v) = u^b v^d$ and the last one is the canonical automorphism $\alpha_{(s,t)}$ for $(s,t) \in \mathbb{T}^2$. On the other hand, an irrational number θ is said to be generic if it is a Liouville number, that is, if there are $r > 1$ and $c > 0$ such that

$$|e^{2\pi i n\theta} - 1| \geq \frac{c}{|n|^r}$$

for any integer $n \neq 0$. G. Elliott proved before that any diffeomorphism of A_θ for a generic number θ is composed of the above three diffeomorphisms [**S2**]. In the above cited paper, Kodaka has shown that there exists a nongeneric irrational number θ and an automorphism α in A_θ which is different from those composed automorphims $\mathrm{ad}w \circ \alpha_g \circ \alpha_{(s,t)}$ for any smooth unitary element w, any element $g \in SL(2, \mathbb{Z})$ and any $(s,t) \in \mathbb{T}^2$. This means a remarkable fact that this rotation C^*-algebra A_θ inherits another exotic (noncommutative) differential structure different from that in irrational rotation C^*-algebras for generic numbers.

Results in §7 are based on the author's preprint [**23**] in which corresponding results are proved for topological dynamical systems with arbitrary amenable discrete groups of homeomorphisms.

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