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EXACT C*-ALGEBRAS AND RELATED TOPICS

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Preface

It has seemed to me for some time that it would be valuable to have an exposition of the theory of nuclear and exact C*-algebras, incorporating both background material on tensor products and completely bounded maps on the one hand, and a treatment of such related topics as the lifting results of Arveson, Choi, Effros and Haagerup on the other. With the appearance of Kirchberg's remarkable work on exact C*-algebras, this whole circle of ideas has reached a form which can be considered to some extent complete. It has therefore been a pleasure to be able to collect together some of this material here. Although I hope to give a comprehensive treatment of these topics in the reasonably near future, my aims here are of necessity more limited.

These notes are the amplified text of a series of lectures given at Seoul National University between 20th and 30th December 1993 under the auspices of the Global Analysis Research Centre. There were essentially two strands to the lectures: a description of certain families of exact and inexact C*-algebras, and a proof of Kirchberg's characterisation of separable exact C*-algebras. The first two chapters deal mostly in outline with fundamental concepts, such as nuclearity and exactness. Chapter 3 is concerned with some of the main examples of inexact C*-algebras that arise from countable groups. The existence of such examples is, of course, the basis for the study of exact C*-algebras. Chapter 5 is devoted to the work of Archbold and Batty on property C, and chapter 6 to the afore-mentioned results on completely positive liftings. These apparently disparate topics are a vital element in establishing the characterisation of exact C*-algebras, and the corollary that exactness passes to quotients, in chapter 9. Another essential ingredient is the result, proved in chapter 7, that exactness implies nuclear embeddability. In chapter 8 an isometric lifting result, originally proved independently by Brown and Kirchberg, is established by elementary methods.

The appendix contains a short, self-contained and fairly elementary K-theoretic derivation, based on Cuntz's approach, of the K-groups of the regular C*-algebras of the free groups on finitely many generators. My purpose in including this is two-fold. Firstly, the existing proofs in the literature are either quite involved or require relatively sophisticated apparatus, such as KK-theory. Secondly, the corollary that these regular C*-algebras are mutually non-isomorphic for different numbers of free generators is important in the context of chapter 3.

I was aided in my preparation of the final version of these notes by feedback from the audience. Dr Jang Sun-Young, Professor Kye Seung-Hyeok and Professor Lee Sa-Ge, in particular, made a number of valuable comments. I should also like to record my appreciation of the very warm hospitality they extended to me during my visit to Korea. I should like to thank Professor Kye especially for his careful reading of earlier drafts of the notes, which resulted in many errors and imprecisions being corrected, and for inviting me to make a most enjoyable visit to Korea.

Contents.

1. Tensor products of C*-algebras, completely positive and completely bounded maps.

In this chapter basic results on tensor products of C*-algebras, and completely positive and completely bounded mappings will be reviewed. Since the two areas are so intimately connected, it seems appropriate to give a combined treatment. Much of this ground is well trodden, so proofs will in general only be included when they differ significantly from published proofs of the same results. Good sources for much of this material are the notes of Kye in this series [**Kye2**, §§4.1, 4.2] and the monograph of Paulsen [**Pau**].

1.1. Tensor products. The tensor product of vector spaces E and F will be denoted by $E \odot F$. If A and B are algebras over $\mathbb{C}$, then $A \odot B$ is an algebra with product

$$(\sum_i a_i \otimes b_i)(\sum_j a'_j \otimes b'_j) = \sum_{i,j} a_i a'_j \otimes b_i b'_j,$$

for $\sum_i a_i \otimes b_i, \sum_j a'_j \otimes b'_j \in A \odot B$. If A and B are, in addition, $*$-algebras then $A \odot B$ is a *-algebra with involution

$$(\sum_i a_i \otimes b_i)^* = \sum_i a_i^* \otimes b_i^*.$$

If E and F are normed spaces a norm $\| \ \|_\alpha$ on $E \odot F$ is a *cross-norm* if

$$\|e \otimes f\|_\alpha = \|e\|\|f\| \quad (e \in E, f \in F).$$

The completion of $E \odot F$ with respect to $\| \ \|_\alpha$ will be denoted by $E \otimes_\alpha F$. If E and F are Banach spaces, two cross-norms on $E \odot F$, the *projective* norm $\| \ \|_\pi$ and the *injective* norm $\| \ \|_\varepsilon$, are of particular interest. For $x = \sum_i e_i \otimes f_i \in E \odot F$, these norms are given by

$$\|x\|_\varepsilon = \sup\{\|\sum_i f(e_i)g(f_i)\| : f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\},$$

and

$$\|x\|_\pi = \inf\{\sum_i \|e'_i\|\|f'_i\| : x = \sum_i e'_i \otimes f'_i\}.$$

It is not difficult to see that for any cross-norm $\| \ \|_\alpha$ on $E \odot F$, $\|x\|_\varepsilon \leq \|x\|_\alpha \leq \|x\|_\pi$ for $x \in E \odot F$.

1.2. Tensor products of C*-algebras. If A and B are C*-algebras, a norm $\| \ \|_\alpha$ on $A \odot B$ is a *C*-norm* if (i) $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$ and (ii) $\|x^*x\|_\alpha = \|x\|_\alpha^2$ for $x, y \in A \odot B$. The completion $A \otimes_\alpha B$ is then a C*-algebra. If A and B are unital, then $A \otimes 1 \subseteq A \odot B \subseteq A \otimes_\alpha B$, and for $a \in A$, $\|a \otimes 1\|_\alpha = \|a\|$, since the restriction of $\| \ \|_\alpha$ to $A \otimes 1$ coincides with the norm on A by the uniqueness of the C*-norm on a C*-algebra. Similarly $\|1 \otimes b\|_\alpha = \|b\|$ for $b \in B$. Thus

$$\|a \otimes b\|_\alpha = \|(a \otimes 1)(1 \otimes b)\|_\alpha \leq \|a\|\|b\|,$$

i.e. the norm $\| \ \|_\alpha$ is sub-cross. Vowden [**Vow**] has shown that this inequality holds also in the non-unital case.

1.3. The spatial norm. Let A and B be C*-algebras, and let $\pi : A \to B(H)$ and $\sigma : B \to B(K)$ be faithful representations of A and B on Hilbert spaces H and K, respectively. The map $\pi \odot \sigma : A \odot B \to B(H \odot K)$ given by

$$((\pi \odot \sigma)(a \otimes b))\,(\xi \otimes \eta) = \pi(a)\xi \otimes \sigma(b)\eta$$

$(a \in A, b \in B, \xi \in H, \eta \in K)$ is a $*$-isomorphism, and so a C*-norm $\| \ \|_\alpha$ on $A \odot B$ is given by

$$\|x\|_\alpha = \|(\pi \odot \sigma)(x)\|_{B(H \otimes K)}.$$

It might seem from this definition that the norm thus defined depends on the particular choice of Hilbert spaces H, K and representations π, σ, but as the following classical result of Takesaki shows, $\| \ \|_\alpha$ is in fact independent of such choices, is minimal among C*-norms on $A \odot B$ and is a cross-norm.

Theorem 1.1. [Tak1] *For $x \in A \odot B$*

$$\|x\|_\alpha^2 = \sup\{(f \otimes g)(y^*x^*xy)\},$$

where the supremum is over all $f \in S(A), g \in S(B)$ and $y \in A \odot B$ such that $(f \otimes g)(y^*y) \leq 1$. For any C-norm $\| \ \|_\beta$ on $A \otimes B$, $\|x\|_\beta \geq \|x\|_\alpha \geq \|x\|_\epsilon$ for $x \in A \odot B$. In particular, $\| \ \|_\beta$ is a cross-norm.*

Definition 1.2. For C*-algebras A and B, the norm $\| \ \|_\alpha$ is denoted by $\| \ \|_{min}$ and is known as the *minimal* (or *spatial*) C*-norm on $A \odot B$. The completion $A \otimes_{min} B$ will mostly be written $A \otimes B$ in what follows.

From the spatial definition of the minimal norm it is easy to see that it is hereditary, in the sense that if A and B are C*-algebras with C*-subalgebras C and D, respectively, then the restriction of $\| \ \|_{min}$ on $A \odot B$ to $C \odot D$ is just the minimal norm on $C \odot D$, i.e. the embedding $C \odot D \to A \odot B$ extends to an isometry $C \otimes D \to A \otimes B$. Moreover if $\mathcal{S}$ and $\mathcal{T}$ are faithful families of states on A and B, respectively, then the set $\{f \otimes g : f \in \mathcal{S}, g \in \mathcal{T}\}$ is faithful on $A \otimes B$.

1.4. Completely positive and completely bounded maps. Let $M_n = M_n(\mathbb{C})$ be the C*-algebra of $n \times n$ complex matrices, the norm being the operator norm resulting from the isomorphism $M_n \cong B(\mathbb{C}^n)$. If A is a C*-algebra, then $M_n(A)$ denotes the *-algebra of A-valued $n \times n$ matrices, and there is a natural *-isomorphism $A \odot M_n \to M_n(A)$ given by

$$\sum a_{i,j} \otimes e_{i,j} \longrightarrow [a_{i,j}],$$

where $\{e_{i,j} : 1 \le i, j \le n\}$ is the standard basis of matrix units in M_n. Since M_n is finite dimensional, $A \odot M_n$ is complete in any cross-norm, in particular $\| \ \|_{min}$. By the uniqueness of the C*-norm, $M_n(A)$ has $\| \ \|_{min}$ as its unique C*-norm.

Let $\phi : A \to B$ be a linear map, where A and B are C*-algebras, and for $n = 1, 2, \ldots$ let $\phi_n : M_n(A) \to M_n(B)$ be the linear map $[a_{i,j}] \to [\phi(a_{i,j})]$. Identifying $M_n(A)$ and $M_n(B)$ with $A \otimes M_n$ and $B \otimes M_n$, respectively, ϕ_n is just the map $\phi \otimes id_n : A \otimes M_n \to B \otimes M_n$.

Definition 1.3. 1. The map ϕ is *completely bounded* (c.b.) if ϕ_n is bounded for $n \ge 1$ and $\sup_{n \ge 1} \|\phi_n\| < \infty$. If finite this supremum, denoted by $\|\phi\|_{cb}$, is the *completely bounded* norm of ϕ.

2. The map ϕ is *completely positive* (c.p.) if $\phi_n \ge 0$ for $n \ge 1$.

3. The map ϕ is *completely isometric* (c.i.) if ϕ_n is isometric for $n \ge 1$.

4. The map ϕ is *completely contractive* (c.c.) if $\|\phi_n\| \le 1$ for $n \ge 1$.

A map $\phi : A \to B$ between unital C*-algebras A and B is *unital* if $\phi(1) = 1$. In later chapters we shall be concerned particularly with unital completely positive (u.c.p.), unital completely contractive (c.c.p.) and unital completely isometric (u.c.i.) maps. We now review some of the important properties of completely positive maps that we shall require.

1.5. Properties of completely positive maps. **1.5.1.** If $\phi : A \to B$ is completely positive, then ϕ is completely bounded and $\|\phi\|_{cb} = \|\phi\| = \|\phi_n\|$ for $n = 1, 2, \ldots$. If A has a unit, $\|\phi\| = \|\phi(1)\|$.

1.5.2. If A is unital, a complete contraction $\phi : A \to B$ is completely positive if and only if $\|\phi(1)\| = \|\phi\|$.

1.5.3. If A, B and C are C*-algebras and $\phi : A \to B$ is completely bounded, then the linear map $\phi \odot id : A \odot C \to B \odot C$ extends to a bounded linear map $\phi \otimes id : A \otimes C \to B \otimes C$. The map $\phi \otimes id$ is completely bounded and $\|\phi \otimes id\| = \|\phi\|_{cb}$. If ϕ is completely positive then $\phi \otimes id$ is completely positive and $\|\phi \otimes id\| = \|\phi\|$.

1.5.4. Examples of completely bounded and completely positive maps.

(a) A bounded linear functional f on a C*-algebra A is completely bounded with $\|f\|_{cb} = \|f\|$. If f is positive, it is completely positive.

(b) If A and B are C*-algebras and $f \in A^*$, the map $f \otimes id : A \otimes B \to \mathbb{C} \otimes B$ is completely bounded. Identifying $\mathbb{C} \otimes B$ with B, this map, denoted by R_f, is the *right slice map* corresponding to f. If f is positive, then R_f is completely positive. For $g \in B^*$ the left slice map $L_g : A \otimes B \to A$ is the map $id \otimes g$. Each of the sets $\{R_f : f \in A^*\}$ and $\{L_g : g \in B^*\}$ separates $A \otimes B$.

(c) $*$-homomorphisms from one C*-algebra to another are completely positive.

(d) Let B be a C*-subalgebra of a C*-algebra A and let $\pi : A \to B$ be a projection from A onto B such that $\|\pi\| = 1$. Then π is completely positive.

1.5.5. For many purposes one can restrict attention to completely bounded and completely positive maps between unital C*-algebras, as the following observation shows. Let $\phi : A \to B$ be a completely bounded map, where A is a non-unital C*-algebra and B is a von Neumann algebra. For $f \in B_*$ let $\overline{f}$ denote the corresponding element of B^*. If B is identified with its canonical image in B^{**}, then the dual of the map $\iota : f \to \overline{f}; B_* \to B^*$ is a projection $\iota^* : B^{**} \to B$ of norm 1 (see §5.1). It is easy to see that for any C*-algebra C, $M_n(C)^{**}$ is naturally isomorphic to $M_n(C^{**})$ for $n = 1, 2, \ldots$. Then $(\phi^{**})_n : M_n(A^{**}) \to M_n(B^{**})$ is bounded with $\|(\phi^{**})_n\| = \|\phi_n\|$ for

$n \in \mathbb{N}$, i.e. ϕ^{**} is completely bounded with $\|\phi^{**}\|_{cb} = \|\phi\|_{cb}$. Identifying the unitization $\tilde{A}$ of A with the C*-subalgebra of A^{**} generated by A and 1, the map $\tilde{\phi} : \tilde{A} \to B$ given by $\tilde{\phi} = \iota^*\phi^{**}|_{\tilde{A}}$ is a completely bounded extension of ϕ to $\tilde{A}$ with $\|\tilde{\phi}\|_{cb} = \|\phi\|_{cb}$. If ϕ is completely positive, then $\tilde{\phi}$ is completely positive.

1.5.6. The Cauchy-Schwartz inequality. If A and B are C*-algebras and $\phi : A \to B$ is a completely positive map, then

$$\phi(a^*)\phi(a) \leq \|\phi\|\phi(a^*a) \quad (a \in A).$$

If A and B are unital, $\phi(1) = 1$, and for a particular $a \in A$ we have equality, then

$$\phi(xa) = \phi(x)\phi(a) \quad (x \in A).$$

The set of such $a \in A$ forms a subalgebra of A, the *mutiplicative domain* of ϕ. Choi [**Ch1**] proves these results in the more general setting of 2-positive maps, but for completely maps they follow simply from Stinespring's theorem (Corollary 1.7).

Lemma 1.4. *Let $\theta : M_n \to A$ be a linear map. Then θ is completely positive if and only if the element $\sum_{i,j=1}^n \theta(e_{ij}) \otimes e_{ij}$ of $A \otimes M_n$ is positive.*

Proof: 1. Let $p = \sum_{i,j=1}^n e_{ij} \otimes e_{ij} \in M_n \otimes M_n$. Then $p = p^*$ and $p^2 = np$, so $p \geq 0$. If θ is completely positive, then

$$(\theta \otimes id_n)(p) = \sum_{i,j=1}^{n} \theta(e_{ij}) \otimes e_{ij} \geq 0.$$

2. Conversely, suppose that $(\theta \otimes id)(p) \geq 0$ and let B be any C*-algebra. Then for $b_1, \ldots, b_n \in B$ the element

$$(\sum_{i,j} \theta(e_{ij}) \otimes e_{ij} \otimes 1_n \otimes 1_B)(\sum_{k,l} 1_A \otimes 1_n \otimes e_{kl} \otimes b_k^* b_l)$$

of $A \otimes M_n \otimes M_n \otimes B$ is positive, since it is the product of two commuting positive elements. Let $V_{rs} = \sum_{i=1}^n e_{ri} \otimes e_{si}$ in $M_n \otimes M_n$. Then

$$\begin{aligned}
0 &\leq \sum_{r,s}(1_A \otimes V_{rs} \otimes 1_B)(\sum_{i,j,k,l} \theta(e_{ij}) \otimes e_{ij} \otimes e_{kl} \otimes b_k^* b_l)(1_A \otimes V_{rs}^* \otimes 1_B) \\
&= \sum_{r,s,i,j} \theta(e_{ij}) \otimes e_{rr} \otimes e_{ss} \otimes b_i^* b_j \\
&= \sum_{i,j} \theta(e_{ij}) \otimes 1_n \otimes 1_n \otimes b_i^* b_j,
\end{aligned}$$

i.e. $\sum_{i,j} \theta(e_{ij}) \otimes b_i^* b_j \geq 0$. Thus if $x = \sum_{i,j} e_{ij} \otimes b_{ij} \in M_n \otimes B$,

$$(\theta \otimes id)(x^*x) = \sum_k (\sum_{i,j} \theta(e_{ij}) \otimes b_{ki}^* b_{kj}) \\ \geq 0,$$

i.e. $\theta \otimes id \geq 0$. □

1.6. Stinespring's Theorem. We shall prove a version of Stinespring's theorem which is more general than the usual one and has an important application to the maximal tensor product of C*-algebras (see §1.9). The proof is essentially the same as that of the usual version. The following simple fact will be required in its proof.

Lemma 1.5. *Let A, B and C be C*-algebras, and let $\phi : A \to C$ and $\psi : B \to C$ be completely positive maps such that $\phi(a)\psi(b) = \psi(b)\phi(a)$ $(a \in A, b \in B)$. If $a = [a_{ij}] \in M_n(A)$ and $b = [b_{ij}] \in M_n(B)$ are positive, then*

$$\sum_{i,j=1}^{n} \phi(a_{ij})\psi(b_{ij}) \geq 0.$$

Proof: Let $c_1, \ldots, c_n \in A$ and $d_1, \ldots, d_n \in B$. Then the element

$$(\sum_{i,j} \phi(c_i^* c_j) \otimes e_{ij} \otimes 1_n)(\sum_{k,l} \psi(d_k^* d_l) \otimes 1_n \otimes e_{kl}) = \sum_{i,j,k,l} \phi(c_i^* c_j)\psi(d_k^* d_l) \otimes e_{ij} \otimes e_{kl}$$

of $C \otimes M_n \otimes M_n$ is positive, since it is the product of commuting positive elements. With $V_{rs} = \sum_{i=1}^{n} e_{ri} \otimes e_{si}$,

$$0 \leq \sum_{r,s} (1_C \otimes V_{rs})(\sum_{i,j,k,l} \phi(c_i^* c_j)\psi(d_k^* d_l) \otimes e_{ij} \otimes e_{kl})(1_C \otimes V_{rs}^*) \\ = \sum_{i,j} \phi(c_i^* c_j)\psi(d_i^* d_j) \otimes 1_n \otimes 1_n,$$

i.e. $\sum_{i,j} \phi(c_i^* c_j)\psi(d_i^* d_j) \geq 0$. Let $a = c^*c, b = d^*d$, where

$$c = \sum_{i,j} c_{ij} \otimes e_{ij}, \quad d = \sum_{i,j} d_{ij} \otimes e_{i,j}.$$

Then

$$a = \sum_r \sum_{i,j} c_{ri}^* c_{rj} \otimes e_{ij},$$
$$b = \sum_s \sum_{k,l} s_{sk}^* d_{sl} \otimes e_{kl}$$

and

$$\sum_{i,j} \phi(a_{ij})\psi(b_{ij}) = \sum_{r,s} \sum_{i,j} \phi(c_{ri}^* c_{rj})\psi(d_{si}^* d_{sj})$$
$$\geq 0.$$

□

Theorem 1.6. *Let A_1 and A_2 be C^*-algebras and let $\phi_i : A_i \to B(H)$ be completely positive maps such that $[\phi_1(a_1), \phi_2(a_2)] = 0$ $(a_1 \in A_1, a_2 \in A_2)$, for some Hilbert space H. There are a Hilbert space K, representations $\pi_i : A_i \to B(K)$ such that $[\pi_1(a_1), \pi_2(a_2)] = 0$ $(a_1 \in A_1, a_2 \in A_2)$ and a bounded linear operator $V : H \to K$ such that*

$$\phi_1(a_1)\phi_2(a_2) = V^*\pi_1(a_1)\pi_2(a_2)V \quad (a_i \in A_i)$$

for $i = 1, 2$, and $\|V\|^2 = \|\phi_1\|\|\phi_2\|$. If ϕ_1 and ϕ_2 are unital, then V is an isometry.

Proof: By 1.5.5 we can assume that A_1 and A_2 are unital. A semidefinite sesquilinear form is defined on $A_1 \odot A_2 \odot H$ by

$$(a_1 \otimes a_2 \otimes \xi_1 | b_1 \otimes b_2 \otimes \xi_2) = (\phi_1(b_1^* a_1)\phi_2(b_2^* a_2)\xi_1 | \xi_2).$$

By Lemma 1.5 this form is positive. Let

$$K_0 = \{\eta \in A_1 \odot A_2 \odot H : (\eta|\eta) = 0\},$$

and let K be the completion of $(A_1 \odot A_2 \odot H)/K_0$ with respect to the resulting norm. Let representations $\pi_i : A_i \to B(K)$ be defined by

$$\pi_1(a)(a_1 \otimes a_2 \otimes \xi + K_0) = (aa_1 \otimes a_2 \otimes \xi + K_0)$$
$$\pi_2(a)(a_1 \otimes a_2 \otimes \xi + K_0) = (a_1 \otimes aa_2 \otimes \xi + K_0)$$

and $V : H \to K$ by

$$V\xi = 1 \otimes 1 \otimes \xi + K_0.$$

It is easy to verify that π_1, π_2 and V have the required properties. □

Corollary 1.7. (Stinespring's theorem) *Let A be a C^*-algebra and $\phi : A \to B(H)$ a completely positive map, for some Hilbert space H. Then there exist a Hilbert space K, a representation $\pi : A \to B(K)$ and a bounded linear operator $V : H \to K$ such that*

$$\phi(x) = V^*\pi(x)V \quad (x \in A).$$

If A is unital and $\phi(1) = 1$, then V is an isometry and we can assume, identifying H with its image VH in K, that $H \subseteq K$ and $V = E_H$, the orthogonal projection onto H.

Proof: It suffices to take $A_1 = A$, $A_2 = \mathbb{C}$ and $\phi_2(\lambda) = \lambda 1_H$. □

1.7. Operator systems. An *operator space* is a linear subspace X of a C*-algebra A. If A is unital, $X = X^*$ and $1 \in X$, then X is an *operator system.*

Let X be an operator system. For $x \in X$, $x_1 = (1/2)(x + x^*)$ and $x_2 = (1/2i)(x - x^*)$ are self-adjoint elements of X and $x = x_1 + ix_2$. If $x \in X$ is self adjoint, then the elements $x_+ = \frac{1}{2}(\|x\|.1 + x)$ and $x_- = \frac{1}{2}(\|x\|.1 - x)$ are positive, and $x = x_+ - x_-$. It follows that X is the linear span of its cone of positive elements. A linear map ϕ from X to another operator system Y is *positive* if $\phi(x) \geq 0$ for each positive $x \in X$. For $n \in \mathbb{N}$, $X \odot M_n$ is an operator system and, as in the case of C*algebras, ϕ_n is defined to be the map $\phi \odot id_n : X \odot M_n \to Y \odot M_n$. As before ϕ is *completely positive* (respectively *completely bounded*, *completely isometric*) if ϕ_n is positive (repectively bounded, isometric) for $n \in \mathbb{N}$.

Theorem 1.8. (Arveson's extension theorem) *Let A be a unital C^*-algebra, X an operator system in A, and $\phi : X \to B(H)$ a completely positive map. There exists a completely positive map $\overline{\phi} : A \to B(H)$ which extends ϕ.*

Definition 1.9. An operator system X is *injective* if, whenever Y and Z are operator systems with $Y \subseteq Z$ and $\phi : Y \to X$ is completely positive, then there is a completely positive extension $\overline{\phi} : Z \to X$ of ϕ.

Remarks. 1. Theorem 1.4 says that $B(H)$ is an injective C*-algebra for any Hilbert space H.

2. If X is an injective operator system in $B(H)$, the identity map $id : X \to X$ extends to a completely positive surjective map $\pi : B(H) \to X$. The map π is a projection of norm 1. Conversely, if X is an operator system in $B(H)$ which is the image of of such a π, it follows from Arveson's extension theorem that X is injective. In fact if $Y \subseteq Z$ and $\phi : Y \to X$ is completely positive, there is a completely positive extension $\overline{\phi} : Z \to B(H)$ of ϕ, regarded as a map into $B(H)$. Then $\pi\overline{\phi} : Z \to X$ is a completely positive extension of ϕ.

3. An injective operator system has a product relative to which it is a C*-algebra. If $X \subseteq B(H)$ and $\pi : B(H) \to X$ is a norm 1 projection onto X, a product on X is given by $x \circ y = \pi(xy)$ for $x, y \in X$. It is straightforward to verify that with this product X is a C*-algebra.

1.8. Tensor products of operator systems. Let X and Y be operator systems, with $X \subseteq B(H)$ and $Y \subseteq B(K)$ for suitable Hilbert spaces H and K. If X and Y are also operator systems in $B(H')$ and $B(K')$, respectively, then the identity maps on X and Y have completely positive extensions $\phi : B(H') \to B(H)$ and $\psi : B(K') \to B(K)$, respectively, by Arveson's extension theorem. The map $\phi \otimes \psi$ is a contraction. Thus if $x \in X \odot Y$, $\|x\|_{B(H' \otimes K')} \leq \|x\|_{B(H \otimes K)}$. Interchanging H and H', K and K', we get the opposite inequality. It follows that if we define $\|x\|_{min}$ by $\|x\|_{min} = \|x\|_{B(H \otimes K)}$ for $x \in X \odot Y$, then $\| \ \|_{min}$ is independent of the particular representations of X and Y as operator systems. The completion of $X \odot Y$ relative to $\| \ \|_{min}$ is denoted by $X \otimes Y$. Clearly $X \otimes Y$ is also an operator system.

If X' and Y' are operator systems and $\phi : X \to X'$ and $\psi : Y \to Y'$ are completely positive maps, an application of Theorem 1.8 similar to that above shows that $\phi \odot \psi$ has a completely positive extension $\phi \otimes \psi : X \otimes Y \to X' \otimes Y'$.

We shall be interested later in the case where $\phi = id_X$, Y is a C*-algebra B, $Y' = B/J$ for some ideal J of B, and $\psi : B \to B/J$ is the quotient map. If X is an operator system in a unital C*-algebra A then $A \otimes J$ is an ideal of $A \otimes B$ and is contained in the kernel of the homomorphism $id \otimes \psi : A \otimes B \to A \otimes (B/J)$. This homomorphism thus has a factorisation

$$A \otimes B \xrightarrow{\Psi} (A \otimes B)/(A \otimes J) \xrightarrow{T_A} A \otimes (B/J),$$

where Ψ and T_A are *-homomorphisms. Now $(A \odot B) \cap \ker(id \otimes \psi) = A \odot J$, which implies that $\Psi(A \odot B) \cong A \odot (B/J)$. It follows that

$$(A \otimes B)/(A \otimes J) \cong A \otimes_\nu (B/J)$$

for some C*-norm $\| \; \|_\nu$ on $A \odot (B/J)$. With the obvious identifications, $T_A : A \otimes_\nu (B/J) \to A \otimes (B/J)$ extends the identity map on $A \odot (B/J)$.

We now consider the restriction of $id \otimes \psi$ to $X \otimes B$. For this we require the following technical lemma.

Lemma 1.10. $\text{dist}(x, A \otimes J) = \text{dist}(x, X \odot J)$ for $x \in X \odot B$.

Proof: Let J have an approximate identity $\{e_\lambda\}$ with $0 \le e_\lambda \le 1$. Given $x \in X \odot B$ and $\varepsilon > 0$, there is an $a \in A \otimes J$ such that $\|x - a\| < \delta + \varepsilon$, where $\delta = \text{dist}(x, A \otimes J)$. Choose e_λ so that $\|a - a(1 \otimes e_\lambda)\| < \varepsilon$. Then

$$\begin{aligned} \|x - x(1 \otimes e_\lambda)\| &\le \|(x - a)(1 - 1 \otimes e_\lambda)\| + \|a(1 - 1 \otimes e_\lambda)\| \\ &< \|x - a\| + \varepsilon \\ &< \delta + 2\varepsilon. \end{aligned}$$

Since $x(1 \otimes e_\lambda) \in X \odot J$ and ε is arbitrary, this shows that $\text{dist}(x, X \odot J) \le \text{dist}(x, A \otimes J)$. The opposite inequality is obvious, so the result follows. □

It follows from this lemma that $\Psi(X \otimes B)$ is isometric to $(X \otimes B)/(X \otimes J)$, and the restriction of T_A to $\Psi(X \otimes B)$ is a unital completely positive map $T_X : (X \otimes B)/(X \otimes J) \to X \otimes (B/J)$. Since $\Psi(X \odot B)$ is naturally isomorphic to $X \odot (B/J)$, identifying these two operator systems, T_X is an extension of the identity map on $X \odot (B/J)$. As we shall see in chapter 3, T_X is not, in general, isometric.

1.9. The maximal C*-norm. Let A and B be C*-algebras, and let π be a faithful representation of $A \otimes B$ on a Hilbert space H. Let $\sigma : A \odot B \to B(K)$ be a representation (i.e. a $*$-homomorphism) of $A \odot B$ on a Hilbert space K. If $\overline{\pi} = \pi|_{A \odot B}$, then $\overline{\pi} \oplus \sigma$ is a faithful representation of $A \odot B$ on $H \oplus K$. Thus $\overline{\pi} \oplus \sigma$ defines a C*-norm on $A \odot B$, and, by Theorem 1.1, $\|\overline{\pi}(x) \oplus \sigma(x)\| \le \|x\|_\pi$, so that $\|\sigma(x)\| \le \|x\|_\pi$ for $x \in A \odot B$. It follows that for $x \in A \odot B$ the supremum

$$\|x\|_\nu = \sup\{\|\pi(x)\|_{B(H)} : \pi \text{ a representation of } A \odot B\}$$

is finite, and defines a C*-norm on $A \odot B$. It is immediate from this definition that for any C*-norm $\| \ \|_\beta$ on $A \odot B$, $\|x\|_\beta \leq \|x\|_\nu$ $(x \in A \odot B)$. Accordingly $\| \ \|_\nu$ is called the *maximal* C*-norm on $A \odot B$ and is denoted by $\| \ \|_{max}$.

It follows by the method of [**Vow**] (see also [**La2**]) that the norm $\| \ \|_{max}$ on $A \odot B$ is the restriction of $\| \ \|_{max}$ on $\tilde{A} \odot \tilde{B}$, so that $A \otimes_{max} B \subseteq \tilde{A} \otimes_{max} \tilde{B}$ canonically, and $A \otimes_{max} B$ is an ideal of $\tilde{A} \otimes_{max} \tilde{B}$. If π is a non-degenerate representation of $A \odot B$ on $B(H)$, π extends to a representation of $A \otimes_{max} B$, and hence to a representation $\tilde{\pi}$ of $\tilde{A} \otimes_{max} \tilde{B}$ on H. Let $\pi_A : A \to B(H)$ and $\pi_B : B \to B(H)$ be the representations given by

$$\pi_A(a) = \tilde{\pi}(a \otimes 1), \ \pi_B(b) = \tilde{\pi}(1 \otimes b) \quad (a \in A, b \in B).$$

Then $\{\pi_A, \pi_B\}$ is a *commuting pair* of representations of the pair $\{A, B\}$ on H, i.e.

$$\pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a) \quad (a \in A, b \in B),$$

and

$$\pi(\sum_i a_i \otimes b_i) = \sum_i \pi_A(a_i)\pi_B(b_i).$$

Thus for $\sum_i a_i \otimes b_i \in A \odot B$,

$$\|\sum_i a_i \otimes b_i\|_{max} = \sup_{\pi_1, \pi_2} \|\sum_i \pi_1(a_i)\pi_2(b_i)\|,$$

where the supremum on the right is taken over all commuting pairs $\{\pi_1, \pi_2\}$ of representations of $\{A, B\}$.

Proposition 1.11. *Let A, B, C and D be C*-algebras, and let $\phi : A \to C$ and $\psi : B \to D$ be completely positive maps. The map $\phi \odot \psi : A \odot B \to C \odot D$ extends to a completely positive map $\phi \otimes_{max} \psi : A \otimes_{max} B \to C \otimes_{max} D$ with $\|\phi \otimes_{max} \psi\| = \|\phi\|\|\psi\|$.*

Proof: Let $\sigma : C \otimes_{max} D \to B(H)$ be a faithful representation, and let σ_1 and σ_2 be the restrictions of σ to C and D, respectively. Then if $\Phi = \sigma_1\phi$ and $\Psi = \sigma_2\psi$, Φ and Ψ are commuting completely positive maps of A and B, respectively into $B(H)$. By Theorem 1.6 there are a Hilbert space K, commuting representations $\pi_1 : A \to B(K)$ and $\pi_2 : B \to B(K)$ and a bounded linear operator $V : H \to K$ with $\|V\|^2 = \|\Phi\|\|\Psi\| \leq \|\phi\|\|\psi\|$ such that

$$\Phi(a)\Psi(b) = V^*\pi_1(a)\pi_2(b)V|_H \quad (a \in A, b \in B).$$

Let π be the representation of $A \otimes_{max} B$ in $B(K)$ such that

$$\pi(a \otimes b) = \pi_1(a)\pi_2(b) \quad (a \in A, b \in B).$$

Then for $x \in A \odot B$,

$$\begin{aligned}
\|(\phi \odot \psi)(x)\|_{max} &= \|(\sigma(\phi \odot \psi))(x)\|_{B(H)} \\
&= \|(\sigma_1\phi \odot \sigma_2\psi)(x)\| \\
&= \|(\Phi \odot \Psi)(x)\| \\
&= \|V^*\pi(x)V\| \\
&\leq \|\pi(x)\| \\
&\leq \|x\|_{max},
\end{aligned}$$

and so $\phi \otimes_{max} \psi$ exists. By continuity

$$(\sigma(\phi \otimes_{max} \psi))(x) = V^*\pi(x)V|_H,$$

for $x \in A \otimes_{max} B$, from which it follows that $\phi \otimes_{max} \psi$ is completely positive, with the stated norm. □

The map $\phi \otimes_{max} id$ is of particular interest in the following special cases.

1. If $A \subseteq B$ and ϕ is the inclusion map, then $\phi \otimes_{max} id$ is a homomorphism of $A \otimes_{max} C$ into $B \otimes_{max} C$. In general this homomorphism is not injective (see chapter 2).

2. If moreover $A = J$, an ideal of B, then $\phi \otimes_{max} id$ is injective, and we have $J \otimes_{max} C \subseteq B \otimes_{max} C$, identifying $J \otimes_{max} C$ with its image under $\phi \otimes_{max} id$.

3. If $\pi : B \to B/J$ is the quotient map, then $\pi \otimes_{max} id$ is a homomorphism of $B \otimes_{max} C$ onto $(B/J) \otimes_{max} C$. Since $(B \odot C) \cap (J \otimes_{max} C) = J \odot C$,

$$(B \otimes_{max} C)/(J \otimes_{max} C) \cong (B/J) \otimes_\nu C$$

for some C*-norm $\| \ \|_\nu$ on $(B/J) \odot C$. It is also easy to see that $J \otimes_{max} C \subseteq \ker(\pi \otimes_{max} id)$. These two facts together imply that $\| \ \|_\nu = \| \ \|_{max}$ and $J \otimes_{max} C = \ker(\pi \otimes_{max} id)$. The coincidence of these two ideals is equivalent to saying that the sequence

$$0 \longrightarrow J \otimes_{max} C \longrightarrow B \otimes_{max} C \longrightarrow (B/J) \otimes_{max} C \longrightarrow 0$$

is exact.

If $\otimes_{max}$ is replaced by $\otimes_{min}$ in this sequence, exactness can fail (see chapter 3).

1.10. Other C*-norms. Let A and B be C*-algebras and suppose that A is a von Neumann algebra. A representation $\pi : A \odot B \to B(H)$ is *(left-) normal* if the map $a \to \pi(a \otimes b)$ is normal for each $b \in B$. The *(left-)normal* C*-norm $\| \ \|_{nor}$ on $A \odot B$ is defined by

$$\|x\|_{nor} = \sup\{\|\pi(x)\| : \pi \text{ a normal representation of } A \odot B\}.$$

If B is a von Neumann algebra also, a representation π of $A \odot B$ is *binormal* if the map $a \to \pi(a \otimes b)$ is normal for each $b \in B$, and the map $b \to \pi(a \otimes b)$ is normal for each $a \in A$. The *binormal* C*-norm $\| \ \|_{binor}$ on $A \odot B$ is then given by taking the above supremum over all binormal representations π. It follows from these definitions that for $x \in A \odot B$,

$$\|x\|_{min} \leq \|x\|_{binor} \leq \|x\|_{nor} \leq \|x\|_{max}.$$

If $A = B(H)$, or, more generally, A is an injective von Neumann algebra, then $\| \ \|_{nor} = \| \ \|_{min}$ on $A \odot B$ for any C*-algebra B, and if B is a von Neumann algebra, $\| \ \|_{binor} = \| \ \|_{min}$ on $A \odot B$.

If A and B are arbitrary C*-algebras and $\{\pi_1, \pi_2\}$ is a commuting pair of representations of $\{A, B\}$ on a Hilbert space H, then π_1 has a natural extension to a normal representation $\overline{\pi}_1$ of A^{**} on H (see §5.1). Then $\{\overline{\pi}_1, \pi_2\}$ is a commuting pair of representations of the pair $\{A^{**}, B\}$. If, conversely, $\{\overline{\pi}_1, \pi_2\}$ is a commuting pair of representations of the pair $\{A^{**}, B\}$ with $\overline{\pi}_1$ normal, and $\pi_1 = \overline{\pi}_1|_A$, then $\{\pi_1, \pi_2\}$ is a commuting pair of representations of $\{A, B\}$. It follows that the restriction of the norm $\| \ \|_{nor}$ on $A^{**} \odot B$ to $A \odot B$ is the norm $\| \ \|_{max}$. Thus the inclusion $A \odot B \subseteq A^{**} \odot B$ extends to an isometric inclusion $A \otimes_{max} B \subseteq A^{**} \otimes_{nor} B$. Analogous considerations show that $A^{**} \otimes_{nor} B \subseteq A^{**} \otimes_{binor} B^{**}$.

1.11. Completely bounded maps. We give here a brief summary of the theory of completely bounded maps and, in particular, results that will be required in later chapters. Full details can be found in [**Pau**, Chapter 7]. The following results can be deduced from analogous properties of completely positive maps using a dilation technique due to Paulsen, the basis of which is

Proposition 1.12. [**Pau**, Lemma 7.1]* *Let A and B be unital C*-algebras, M an operator space in A and $\phi : M \to B$ a completely bounded map. Then* *the subspace S of $M_2(A)$ defined by*

$$S = \left\{ \begin{bmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{bmatrix} : \lambda, \mu \in \mathbb{C}, a, b \in M \right\}$$

is an operator system, and if $\|\phi\|_{cb} \leq 1$ (i.e. ϕ is a complete contraction), then the map $\Phi : S \to M_2(B)$ given by

$$\Phi\left(\begin{bmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{bmatrix} \right) = \begin{bmatrix} \lambda 1 & \phi(a) \\ \phi(b)^* & \mu 1 \end{bmatrix}$$

is completely positive.

Theorem 1.13. (Wittstock extension theorem) *If M is an operator space in a C^*-algebra A, and $\phi : M \to B(H)$ is completely bounded, then ϕ has a completely bounded extension $\psi : A \to B(H)$ such that $\|\psi\|_{cb} = \|\phi\|_{cb}$.*

Proof: This follows by applying the extension theorem for completely positive maps (Theorem 1.8) to be the completely positive map $\Phi : S \to M_2(B(H))$ of Proposition 1.12, where ϕ is assumed completely contractive (without loss of generality).

Theorem 1.14. (Decomposition theorem.) *Let A be a unital C^*-algebra and $\phi : A \to B(H)$ a completely bounded map. There exist completely positive maps $\phi_i : A \to B(H)$ $(i = 1, 2)$ such that $\|\phi_i\|_{cb} = \|\phi\|_{cb}$ or, alternatively when $\|\phi\|_{cb} \leq 1$, such that $\phi_1(1) = \phi_2(1) = 1$, and such that the map $\Phi : M_2(A) \to M_2(B(H))$ given by*

$$\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \phi_1(a) & \phi(b) \\ \phi(c^*)^* & \phi_2(d) \end{bmatrix}$$

is completely positive.

Proof: This is proved by applying Proposition 1.12 with $M = A$, followed by Arveson's extension theorem.

Theorem 1.15. (Stinespring's theorem for completely bounded maps.) *With A and ϕ as in Theorem 1.14, there are a Hilbert space K, a representation $\pi : A \to B(K)$ and bounded linear maps $V_i : H \to K$ $(i = 1, 2)$ such that $\|V_1\|\|V_2\| = \|\phi\|_{cb}$ and*

$$\phi(a) = V_1^* \pi(a) V_2 \quad (a \in A).$$

If $\|\phi\|_{cb} \leq 1$, then V_1 and V_2 may be chosen to be isometric.

Proof: Assuming, without loss of generality, that $\|\phi\|_{cb} \leq 1$, the map $\Phi : M_2(A) \to M_2(B(H))$ given by Theorem 1.14 is u.c.p. The result now follows by applying Stinespring's theorem to Φ.

Theorem 1.16. (Wittstock decomposition theorem.) *Let A be a unital C^*-algebra and let $\phi : A \to B(H)$ be a completely bounded self-adjoint map. Then there are completely bounded maps $\phi_i : A \to B(H)$ $(i = 1, 2)$ such that $\phi = \phi_1 - \phi_2$ and $\|\phi_1 + \phi_2\|_{cb} \leq \|\phi\|_{cb}$.*

Proof: Let V_1, V_2 and π be as in Theorem 1.15, with $\|V_1\| = \|V_2\| = \|\phi\|_{cb}^{\frac{1}{2}}$. Let $\psi(a) = \frac{1}{2}(V_1^*\pi(a)V_1 + V_2^*\pi(a)V_2)$ $(a \in A)$. The map ψ is completely positive and $\|\psi\|_{cb} = \|\psi(1)\| \leq \|\phi\|_{cb}$. Since ϕ is self-adjoint, $\phi(a^*) = V_1^*\pi(a^*)V_2 = (\phi(a))^* = V_2^*\pi(a^*)V_1$ for $a \in A$. Let

$$\phi_1(a) = \frac{1}{2}(\psi(a) + \phi(a)) = \frac{1}{4}(V_1 + V_2)^*\pi(a)(V_1 + V_2)$$

and

$$\phi_2(a) = \frac{1}{2}(\psi(a) - \phi(a)) = \frac{1}{4}(V_1 - V_2)^*\pi(a)(V_1 - V_2).$$

Then ϕ_1 and ϕ_2 are completely positive, $\phi = \phi_1 - \phi_2$ and $\|\phi_1 + \phi_2\|_{cb} = \|\psi\|_{cb} \leq \|\phi\|_{cb}$. □

We conclude this chapter with two perturbation results for self-adjoint unital completely bounded maps. The first will be required in chapter 6, the second in chapter 7. The following preliminary lemma is needed.

Lemma 1.17. [**E-H**, Lemma 2.4] *Let E be an n-dimensional operator system, A a unital C^*-algebra and $\phi : E \to A$ a self-adjoint completely bounded map. Then there is an element $w^* \in E^*$ such that $\|w^*\| \leq n\|\phi\|_{cb}$ and such that the maps $w^*1 \pm \phi$ are completely positive.*

Proof: Let $E \subseteq B(H)$, let $\{e_1, \ldots, e_n\}$ be a basis of E such that $e_i = e_i^*$ and $\|e_i\| = 1$, and let $\{e_1', \ldots, e_n'\}$ be the dual basis of E^*. Then e_i' is self-adjoint, and so has a self-adjoint extension to $B(H)$ with the same norm. Taking the Hahn decomposition of the extension and restricting to E, it follows that there are positive $f_i', g_i' \in E^*$ such that $e_i' = f_i' - g_i'$ and $\|f_i'\| + \|g_i'\| = \|e_i'\| = 1$. Let

$$w^* = \|\phi\|_{cb} \sum_{i=1}^{n} (f_i' + g_i').$$

A simple calculation shows that the maps $w^*1 \pm \phi$ are completely positive. □

Lemma 1.18. (cf. **[E-H]**) *Let E be an n-dimensional operator system, B a unital C^*-algebra and $\phi : E \to B$ a self-adjoint unital linear map such that $\|\phi\|_{cb} \leq 1 + \delta$, where $0 < \delta < 1/n$. Then there is a u.c.p. map $\psi : E \to B$ such that $\|\psi - \phi\|_{cb} \leq 4n\delta$.*

Proof: Letting $B \subseteq B(H)$, there are c.p. maps $\phi_1, \phi_2 : E \to B(H)$ such that $\phi = \phi_1 - \phi_2$ and $\|\phi_1 + \phi_2\| \leq \|\phi\|_{cb}$, by Theorem 1.16. Then $\phi_1(1) - \phi_2(1) = 1$ and

$$\|\phi_1(1)\| \leq \|\phi_1(1) + \phi_2(1)\| \leq \|\phi\|_{cb} \leq 1 + \delta.$$

Since $\phi_1(1) = 1 + \phi_2(1)$,

$$1 + \|\phi_2(1)\| = \|\phi_1(1)\| \leq 1 + \delta,$$

and so $\|\phi_2\| = \|\phi_2(1)\| \leq \delta$. By Lemma 1.17 there is a $w^* \in E^*$ with $\|w^*\| \leq n\delta$, such that the maps $w^*.1 \pm \phi_2$ are completely positive. Let $\psi_0 = \phi + w^*.1$. Then $\psi_0(E) \subseteq B$, $\psi_0 = \phi_1 + (w^*.1 - \phi_2)$ is completely positive, and

$$\|\psi_0 - \phi\|_{cb} = \|w^*\|_{cb} = \|w^*\| \leq n\delta.$$

Let $b = \psi_0(1)$. Then $b \geq 0$ and

$$\|b - 1\| = \|\psi_0(1) - \phi(1)\| \leq n\delta < 1,$$

so that b^{-1} exists. Also

$$\|b^{\frac{1}{2}} - 1\| \leq \|b - 1\| \leq n\delta,$$

which implies that $\|b^{\frac{1}{2}}\| \leq 1 + n\delta < 2$. For $y \in B \otimes M_m$

$$\begin{aligned}\|(b^{\frac{1}{2}} \otimes 1_m)y(b^{\frac{1}{2}} \otimes 1_m) - y\| &= \|(b^{\frac{1}{2}} \otimes 1_m - 1)y(b^{\frac{1}{2}} \otimes 1_m) + y(b^{\frac{1}{2}} \otimes 1_m - 1)\| \\ &\leq (\|b^{\frac{1}{2}} - 1\|\|b^{\frac{1}{2}}\| + \|b^{\frac{1}{2}} - 1\|)\|y\| \\ &\leq 3n\delta.\end{aligned}$$

Let $\psi = b^{-\frac{1}{2}}\psi_0 b^{-\frac{1}{2}}$. Then ψ is u.c.p., $\psi(E) \subseteq B$ and

$$\|\psi - \psi_0\|_{cb} = \|\psi - b^{\frac{1}{2}}\psi b^{\frac{1}{2}}\|_{cb} \leq 3n\delta\|\psi\| = 3n\delta.$$

Thus $\|\psi - \phi\|_{cb} \leq 4n\delta$. □

Proposition 1.19. (cf. **[Kir4]**) *Let A be a unital C^*-algebra and let $W : A \to B(H)$ be a completely bounded self-adjoint unital map. Then there is a u.c.p. map $U : A \to B(H)$ such that*

$$\|U - W\|_{cb} \leq \|W\|_{cb} - 1.$$

Proof: By Theorem 1.16, there are completely positive maps $S, T : A \to B(H)$ such that $W = S - T$ and $\|S + T\|_{cb} \leq \|W\|_{cb}$. Then $S(1) = W(1) + T(1) = 1 + t$, where t is the positive operator $T(1)$. Let

$$U(a) = (1+t)^{-\frac{1}{2}}S(a)(1+t)^{-\frac{1}{2}} \ (a \in A).$$

The map U is u.c.p.n and

$$\begin{aligned} \|W - U\|_{cb} &\leq \|T\|_{cb} + \|s - U\|_{cb} \\ &= \|t\| + \|LU\|_{cb} \\ &\leq \|t\| + \|L\|_{cb}, \end{aligned}$$

where

$$L(b) = (1+t)^{-\frac{1}{2}}b(1+t)^{-\frac{1}{2}} - b \quad (b \in B(H)).$$

Then

$$\begin{aligned} \|L\|_{cb} &\leq \|(1+t)^{-\frac{1}{2}}\|\|(1+t)^{-\frac{1}{2}} - 1\| + \|(1+t)^{-\frac{1}{2}} - 1\| \\ &= (1 + \|t\|)^{\frac{1}{2}}\left((1+\|t\|)^{\frac{1}{2}} - 1\right) + (1+\|t\|)^{\frac{1}{2}} - 1 \\ &= \|t\| \end{aligned}$$

since $t \geq 0$. Thus $\|W - U\|_{cb} \leq 2\|t\|$. Also

$$\|W\|_{cb} \geq \|S + T\|_{cb} = \|S(1) + T(1)\| = \|1 + 2t\| = 1 + 2\|t\|,$$

from which it follows that $\|W - U\|_{cb} \leq \|W\|_{cb} - 1$. □

2. Nuclear and exact C*-algebras.

2.1. Nuclear C*-algebras. Since M_n is a finite dimensional C*-algebra, for any C*-algebra B, $M_n(B) \cong B \otimes M_n$ is complete in any C*-norm, so that all norms are equivalent, hence identical. A C*-algebra A is said to be *nuclear* if $\| \; \|_{max} = \| \; \|_{min}$ on $A \odot B$ for any B. The following basic facts about nuclear C*-algebras are well-known and mostly straightforward to prove.

(a) M_n is nuclear $(n = 1, 2, \ldots)$. Any finite-dimensional C*-algebra is nuclear.

(b) Inductive limits of nuclear C*-algebras are nuclear. In particular all AF C*-algebras are nuclear. If H is an infinite–dimensional Hilbert space, the algebra $K(H)$ of compact linear operators on H is an inductive limit of matrix algebras, hence nuclear.

(c) Abelian C*-algebras are nuclear [**Tak1**].

(d) Type I C*-algebras are nuclear [**Tak1**].

(e) If A is a C*-algebra with an ideal J such that J and A/J are nuclear, then A is nuclear, i.e. an extension of a nuclear C*-algebra by a nuclear C*-algebra is nuclear.

(f) Recall that the Cuntz algebra $\mathcal{O}_n$ is the C*-algebra generated by n elements $S_1, ..., S_n$ satisfying the relations $1 = S_1S_1^* + ... + S_nS_n^*$, $S_r^*S_r = 1$ $(r = 1, \ldots, n)$. $\mathcal{O}_n$ is simple and nuclear [**Cu1**].

2.2. Approximation Properties. A unital C*-algebra A is said to have the *completely positive approximation property* (CPAP) if there are positive integers $\{n_\lambda\}$ and nets of u.c.p. maps $\psi_\lambda : A \to M_{n_\lambda}, \varphi_\lambda : M_{n_\lambda} \to A$ such that

$$\lim_\lambda \|(\varphi_\lambda \psi_\lambda)(x) - x\| \to 0 \quad (x \in A).$$

When A is non-unital, the maps φ_λ and ψ_λ are required to be completely positive contractions. When A is separable, the nets $\{\varphi_\lambda\}, \{\psi_\lambda\}$ can be taken to be sequences $\varphi_r : M_{n_r} \to A, \psi_r : A \to M_{n_r}$. When A is separable and unital we can find finite dimensional operator systems $X_1 \subset X_2 \subset \ldots$ in A such that $A = \overline{\bigcup_1^\infty X_i}$. Passing to subsequences of the sequences $\{\varphi_r\}, \{\psi_r\}$ if necessary, we can assume that

$$\|\varphi_r \psi_r|_{X_r} - id_{X_r}\| < \frac{1}{r} \quad (r = 1, 2, \ldots).$$

A formally weaker approximation property, the *completely contractive approximation property* (CCAP), is obtained from the CPAP by requiring only that the maps φ_λ and ψ_λ be complete contractions.

Proposition 2.1. *If A has the CPAP, then A is nuclear.*

Proof: If $\psi_\lambda : A \to M_{n_\lambda}, \varphi_\lambda : M_{n_\lambda} \to A$ are completely positive contractions as in the above definition, and B is any C*-algebra, then the maps $\psi_\lambda \odot id : A \odot B \to M_{n_\lambda} \odot B$ and $\varphi_\lambda \odot id : M_{n_\lambda} \odot B \to A \odot B$ have completely positive extensions $\psi_\lambda \otimes id : A \otimes B \to M_{n_\lambda} \otimes B$ and $\varphi_\lambda \otimes_{max} id : M_{n_\lambda} \otimes_{max} B \to A \otimes_{max} B$, by 1.5.3 and Proposition 1.11. The maps $\psi_\lambda \otimes id$ and $\varphi_\lambda \otimes_{max} id$ are complete contractions. Since $\| \ \|_{min} = \| \ \|_{max}$ on $M_{n_\lambda} \odot B$,

$$(\varphi_\lambda \otimes_{max} id)(\psi_\lambda \otimes id) : A \otimes B \to A \otimes_{max} B$$

is defined and contractive. For $x = \sum_i a_i \otimes b_i \in A \odot B$,

$$\begin{aligned} \| [(\varphi_\lambda \otimes_{max} id)(\psi_\lambda \otimes id)] (x) - x \|_{max} &= \| \sum_i [(\varphi_\lambda \psi_\lambda)(a_i) - a_i)] \otimes b_i \|_{max} \\ &\leq \sum_i \|(\varphi_\lambda \psi_\lambda)(a_i) - a_i\| \|b_i\| \\ &\to 0. \end{aligned}$$

Since $\| [(\varphi_\lambda \otimes_{max} id)(\psi_\lambda \otimes id)] (x) \|_{max} \leq \|x\|_{min}$, it follows that $\|x\|_{max} \leq \|x\|_{min}$, and so $\|x\|_{max} = \|x\|_{min}$, and $\| \ \|_{max} = \| \ \|_{min}$ on $A \odot B$. □

Remark. Smith [**Sm**] has shown that the stronger implication A has the CCAP $\Rightarrow$ A nuclear holds. A proof which is somewhat simpler than the proof in [**Sm**] may be given by combining the first part of the Smith's proof with the technique used in the proof of Proposition 2.1.

The following converse of Proposition 2.1 was proved independently by Choi and Effros [**C-E1**], and Kirchberg [**Kir1**].

Proposition 2.2. *A nuclear C*-algebra has the CPAP.*

Definition 2.3. If A and B are unital C*-algebras, a unital completely positive map $\phi : A \to B$ is *nuclear* if there are integers n_λ and nets $\psi_\lambda : A \to M_{n_\lambda}$, $\varphi_\lambda : M_{n_\lambda} \to B$ of u.c.p. maps such that $\lim_\lambda \|(\varphi_\lambda \psi_\lambda)(x) - \phi(x)\| = 0$ for $x \in A$. If A is non-unital and ϕ is a completely positive contraction, ϕ

is nuclear if completely positive contractions φ_λ and ψ_λ exist such that ϕ is the point-norm limit of $\varphi_\lambda\psi_\lambda$.

Remark. By Propositions 2.1 and 2.2, a C*-algebra A is nuclear if and only if the identity map $id_A : A \to A$ is a nuclear map. In chapter 7 we will consider nuclear embeddings $A \to B$.

2.3. Injectivity of the second dual. One of the deepest results about nuclear C*-algebras is the following characterisation involving the second dual. That nuclearity of a C*-algebra A implies injectivity of A^{**} was proved by Effros and Lance [**E-L**]; a short proof using an idea of Lance is given in [**Wa1**]. The converse implication makes use of Connes' celebrated result that injective factors on separable Hilbert spaces are hyperfinite [**Co**]. Choi and Effros [**C-E2, C-E3**] gave a proof which built directly on Connes' theorem. An alternative approach [**Wa2**] makes use of a generalisation of part of Connes' results to arbitrary Hilbert spaces.

Proposition 2.4. *A C*-algebra A is nuclear if and only if A^{**} is injective.*

Corollary 2.5. *If A is nuclear, A/J is nuclear for any ideal J of A.*

Proof: $(A/J)^{**} \cong eA^{**}e$ for some central projections e of A^{**}. It follows easily from the definition of injectivity that if A^{**} is injective then $eA^{**}e$ injective. □

2.4. Examples. (a) Let G be a countable group. The left- and right-regular representations λ and ρ of G on $l^2(G)$ are given by

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h), \quad (\rho(g)\xi)(h) = \xi(gh) \quad (g, h \in G, \xi \in \ell^2(G)).$$

If $C_r^*(G)$ is the regular C*-algebra of G, i.e. the algebra $C^*(\lambda(G)) \subseteq B(\ell^2(G))$, Lance [**La1**] has shown that $C_r^*(G)$ is nuclear if and only if G is an amenable group. The free group $\mathbb{F}_2$ on two generators is non-amenable, and so $C_r^*(\mathbb{F}_2)$ is not nuclear. This was first shown by Takesaki [**Tak1**], and can be demonstrated explicitly as follows. For an arbitrary group G, $\{\lambda, \rho\}$ is a commuting pair of representations of G and so defines a commuting pair of representations of $\{C^*(\lambda(G)), C^*(\rho(G))\}$, also denoted by $\{\lambda, \rho\}$. If U is the self-adjoint unitary on ℓ^2 given by

$$(U\xi)(g) = \xi(g^{-1}),$$

then $\rho(g) = U\lambda(g)U$ ($g \in G$) and $C^*(\rho(G)) = UC^*(\lambda(G))U \cong C_r^*(G)$, so that a representation π of $C_r^*(G) \odot C_r^*(G)$ on $\ell^2(G)$ is defined by

$$\pi(\lambda(g) \otimes \lambda(h)) = \lambda(g)\rho(h) \quad (g, h \in G).$$

The C*-algebra $C_r^*(\mathbb{F}_2)$ is simple [**Pow**], so that, with $G = \mathbb{F}_2$, π is injective, and a C*-norm $\| \ \|_\nu$ on $C_r^*(\mathbb{F}_2) \odot C_r^*(\mathbb{F}_2)$ is defined by $\|x\|_\nu = \|\pi(x)\|_{B(\ell^2(\mathbb{F}_2))}$. If u and v are the generators of $\mathbb{F}_2$, it can be shown [**Wa3**] that the self-adjoint contraction

$$c = \frac{1}{4}\left(\lambda(u)\rho(u) + \lambda(v)\rho(v) + \lambda(u^{-1})\rho(u^{-1}) + \lambda(v^{-1})\rho(v^{-1})\right)$$

has the property that $c\xi_e = \xi_e$ and $\mathrm{sp}(c|_{(\mathbb{C}\xi_e)^\perp}) \subseteq [-1, 1 - 42^{-4}]$, where ξ_e is the element of $\ell^2(\mathbb{F}_2)$ such that $\xi_e(h) = \delta_{e,h}$ ($h \in \mathbb{F}_2$). It then follows easily by spectral theory that the rank 1 projection P_e of $\ell^2(\mathbb{F}_2)$ onto $\mathbb{C}\xi_e$ is in $C^*(\lambda(\mathbb{F}_2), \rho(\mathbb{F}_2))$, and hence so is the ideal $K(\ell^2(\mathbb{F}_2))$ of compact operators on $\ell^2(\mathbb{F}_2)$, since $C^*(\lambda(\mathbb{F}_2), \rho(\mathbb{F}_2))$ acts irreducibly on $\ell^2(\mathbb{F}_2)$. Thus $C_r^*(\mathbb{F}_2) \otimes_\nu C_r^*(\mathbb{F}_2)$ contains a non-trivial ideal. However $C_r^*(\mathbb{F}_2) \otimes C_r^*(\mathbb{F}_2)$ is simple, by [**Tak1**], since $C_r^*(\mathbb{F}_2)$ is. Thus $\| \ \|_\nu \neq \| \ \|_{min}$.

(b) Choi [**Ch2**] has constructed an embedding of $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ as a C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$. Since $\mathbb{Z}_2 * \mathbb{Z}_3$ is not amenable, $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is not nuclear. Moreover the free group $\mathbb{F}_2$ can be embedded in $\mathbb{Z}_2 * \mathbb{Z}_3$. Indeed if a and b are the generators of $\mathbb{Z}_2 * \mathbb{Z}_3$, with $a^2 = b^3 = 1$, the elements bab and $ababa$ generate a subgroup isomorphic to $\mathbb{F}_2$. Thus $C_r^*(\mathbb{F}_2) \subseteq C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3) \subseteq \mathcal{O}_2$. This shows that the property of nuclearity does not in general pass to C*-subalgebras.

2.5. Exact C*-algebras. If A and B are C*-algebras and J is an ideal of B, we have seen in chapter 1 that $(A \otimes B)/(A \otimes J) \subseteq A \otimes_\nu (B/J)$ for some C*-norm $\| \ \|_\nu$ on $A \odot (B/J)$. If A is nuclear then $\| \ \|_\nu = \| \ \|_{min}$. This last condition is equivalent to saying that if $\sigma : B \to B/J$ is the quotient-map, then the kernel of the homomorphism $id \otimes \sigma : A \otimes B \to A \otimes (B/J)$ is just $A \otimes J$. This is, in turn, equivalent to saying that the sequence

$$0 \longrightarrow A \otimes J \longrightarrow A \otimes B \longrightarrow A \otimes (B/J) \longrightarrow 0 \qquad (*)$$

is exact. A C*-algebra A such that $(*)$ is exact for arbitrary B and $J \triangleleft B$ is said to be *exact*. Clearly nuclear C*-algebras are exact.

A useful criterion for the exactness of $(*)$ can be given in terms of slice maps. For $\varphi \in A^*$, $R_\varphi(id \otimes \sigma) = \sigma R_\varphi$. Since the family $\{R_\varphi : \varphi \in A^*\}$ is faithful, it follows that $x \in \ker(id \otimes \sigma)$ if and only if $R_\varphi(x) \in \ker J$ for $\varphi \in A^*$, i.e.

$$\ker(id \otimes \sigma) = \{x \in A \otimes B : R_\varphi(x) \in J \ (\varphi \in A^*)\}.$$

Proposition 2.6. *Let A, B and J be such that $(*)$ is exact. Then if D is a C*-subalgebra of A, the sequence*

$$0 \longrightarrow D \otimes J \longrightarrow D \otimes B \longrightarrow D \otimes (B/J) \longrightarrow 0$$

is exact.

Proof: Let $x \in \ker(id_D \otimes \sigma)$. For $\varphi \in A^*$, let $\overline{\varphi} = \varphi|_D$. Then $R_\varphi(x) = R_{\overline{\varphi}}(x) \in J$, and since φ is arbitrary, $x \in \ker(id \otimes \sigma) = A \otimes J$, since $(*)$ is exact. Thus $\ker(id_D \otimes J) \subseteq (A \otimes J) \cap (D \otimes B)$. Let $\{e_\lambda\}$ be an approximate unit for J with $0 \leq e_\lambda \leq 1$, and let $x \in (A \otimes J) \cap (D \otimes B)$. Then $\{1 \otimes e_\lambda\}$ is an approximate identity for $A \otimes J$, so that $\lim_\lambda \|x(1 \otimes e_\sigma) - x\| = 0$. However $x(1 \otimes e_\lambda) \in (D \otimes B)(1 \otimes e_\sigma) \subseteq D \otimes J$ for each λ. Thus $x \in \overline{D \otimes J} = D \otimes J$, and so $\ker(id_D \otimes \sigma) \subseteq D \otimes J$. Since the opposite inclusion is clear, $\ker(id_D \otimes \sigma) = D \otimes J$, and the given sequence is exact. □

Properties of exact C*-algebras.

2.5.1. Nuclear C*-algebras are exact.

2.5.2. If A is an exact C*-algebra, and D is a C*-subalgebra of A, then for any C*-algebra B and ideal J of B, $(*)$ is exact. By Proposition 2.6 D is exact.

More generally, If X is an operator system in A, by the discussion of §1.8, $(X \otimes B)/(X \otimes J)$ is canonically isometric to the image of $X \otimes B$ in $(A \otimes B)/(A \otimes J)$, which is canonically isomorphic to $A \otimes (B/J)$, since A is exact. It follows that the corresponding map $T_X : (X \otimes B)/(X \otimes J) \to X \otimes (B/J)$ is isometric.

2.5.3. There exist C*-algebras which are exact but not nuclear. In fact, since $C_r^*(\mathbb{F}_2) \subseteq \mathcal{O}_2$ (see §2.4), $C_r^*(\mathbb{F}_2)$ is such an algebra, by 2.5.2. It is conjectured that $C_r^*(G)$ is exact for any countable group G.

2.5.4. There exist C*-algebras which are not exact, an example being the full C*-algebra $C^*(\mathbb{F}_2)$ of $\mathbb{F}_2$ (see chapter 3). Since $C^*(\mathbb{F}_2)$ can be embedded as a C*-subalgebra in $B(H)$ when H is an infinite dimensional Hilbert space, it follows that for such H, $B(H)$ is not exact and hence not nuclear. By Proposition 2.2, $B(H)$ does not have the CPAP. This result has been strengthened in two ways. Szankowski [**Sz**] has shown that $B(H)$ does not have the approximation property of Grothendieck, which is weaker than the CPAP. Very recently Junge and Pisier [**J-P**] have shown that the norms $\| \ \|_{max}$ and $\| \ \|_{min}$ on $B(H) \odot B(H)$ are distinct (see chapter 10).

2.5.5. It is straight-forward to prove that the class of exact C*-algebras is closed under the operations of forming inductive limits, restricted direct sums and minimal tensor products.

2.5.6. If A is a C*-algebra with an ideal J such that J and A/J are exact, A need not be exact, i.e. extensions of exact C*-algebras by exact C*-algebras are not necessarily exact (see chapter 4). However if A/J and J are exact and the quotient map $\pi : A \rightarrow A/J$ has a contractive completely positive lifting (i.e. a c.c.p. right inverse), then it can be shown that A is exact.

2.5.7. Quotients of exact C*-algebras are exact. This deep result of Kirchberg will be proved in chapter 9.

2.5.8. Let M be the finite von Neumann algebra

$$\oplus_{i=1}^{\infty} M_i = \{(x_i) : x_i \in M_i, \sup_i \|x_i\| < \infty\}.$$

If I_0 is the ideal $\{(x_i) \in M : \lim_{i\to\infty} \|x_i\| = 0\}$, then a C*-algebra A is exact if and only if the sequence

$$0 \longrightarrow A \otimes I_0 \longrightarrow A \otimes M \longrightarrow A \otimes (M/I_0) \longrightarrow 0$$

is exact. The exactness of this sequence is in turn equivalent to the exactness of the sequence

$$0 \longrightarrow A \otimes K(H) \longrightarrow A \otimes B(H) \longrightarrow A \otimes (B(H)/K(H)) \longrightarrow 0,$$

where $H = \ell^2_{\mathbb{N}}$ [**Kir2**].

3. C*-algebras arising from discrete groups.

3.1. The factorisation property. Let G be a discrete group. A *representation* of G will always be taken to be a unitary representation. Recall that the left- and right-regular representations $\lambda,\ \rho : G \to \ell^2(G)$ are given by

$$(\lambda(g)\xi)(h) = \xi(g^{-1}h), \quad (\rho(g)\xi(h) = \xi(hg) \quad (g, h \in G, \xi \in \ell^2(G)),$$

and if U is the self-adjoint unitary on $\ell^2(G)$ such that $(U\xi)(h) = \xi(h^{-1})$, then $\rho(g) = U\lambda(g)U$ $(g \in G)$, i.e. λ and ρ are unitarily equivalent.

Let π_G be the universal representation of G on the Hilbert space H_G. The *full* (or *universal*) C*-algebra of G is the algebra $C^*(\pi_G(G))$. If G is identified with the unitary subgroup $\pi_G(G)$ of $C^*(G)$, then any representation $\pi : G \to B(H)$ has a canonical extension to a representation $\overline{\pi} : C^*(G) \to B(H)$, and $\overline{\pi}(C^*(G)) = C^*(\pi(G))$ (this universal property defines $C^*(G)$ up to isomorphism). For ease of notation we shall write π instead of $\overline{\pi}$ to denote this extension.

Now the commuting pair $\{\lambda, \rho\}$ of representations of $C^*(G)$ defines a representation π_r of $C^*(G) \otimes_{max} C^*(G)$ on $B(\ell^2(G))$ such that

$$\pi_r(x \otimes y) = \lambda(x)\rho(y) \quad (x, y \in C^*(G)). \tag{3.1}$$

Definition 3.1. The group G has the *factorisation property* if the representation π_r has the factorisation

$$\begin{array}{ccc} C^*(G) \otimes_{max} C^*(G) & \xrightarrow{\quad\pi_r\quad} & B(\ell^2(G)) \\ & \searrow^{\psi} \qquad \nearrow_{\tilde{\pi}_r} & \\ & C^*(G) \otimes C^*(G) & \end{array}$$

where ψ is the canonical morphism and $\tilde{\pi}_r$ is a homomorphism satisfying (3.1) with π_r replaced by $\tilde{\pi}_r$.

Proposition 3.2. *If G is amenable, then G has the factorisation property.*

Proof : This is an immediate consequence of the fact that if G is amenable, then the canonical morphism $\lambda : C^*(G) \to C_r^*(G)$ is injective and $C_r^*(G)$ is nuclear [**Kye2**, Theorems 4.5.7, 4.5.8]. □

Recall that a discrete group G is *residually finite* if, given $g_1, \ldots, g_n \in G$, there is a homomorphism θ of G into a finite group G' such that the elements $\theta(g_1), \ldots, \theta(g_n)$ of G' are distinct. For a finitely generated discrete group residual finiteness of G is equivalent to the formally weaker condition that G be *maximally almost periodic*, i.e. that G have a separating family of finite-dimensional representations, by a theorem of Malcev (see [**Alp**]).

Proposition 3.3. [**Wal**] *If G is residually finite, then G has the factorisation property.*

Proof: Since G is residually finite, there are finite groups $\{G_i\}_{i\in I}$ and homomorphisms $\theta_i : G \to G_i$ such that $\bigcap_{i\in I} \ker \theta_i = e$. Let λ_i and ρ_i be the left- and right-regular representations of G_i on $H_i = \ell^2(G_i)$ $(i \in I)$. For each i let π_i be the representation π_r of $C_r^*(G_i) \otimes C_r^*(G_i)$ (since G_i is finite, $C_r^*(G_i) = C^*(G_i)$ is finite dimensional), so that

$$\pi_i(x \otimes y) = \lambda_i(x)\rho_i(y) \quad (x, y \in C^*(G_i)).$$

Then $\lambda_i\theta_i$, $\rho_i\theta_i$ extend to a commuting pair $\{\lambda_i', \rho_i'\}$ of representations of $\{C^*(G), C^*(G)\}$ on $\ell^2(G_i)$. Let σ_i be the corresponding representation of $C^*(G) \otimes_{max} C^*(G)$. Then

$$\sigma_i = \pi_i(\theta_i \otimes \theta_i)\psi,$$

where $\theta_i \otimes \theta_i : C^*(G) \otimes C^*(G) \to C_r^*(G_i) \otimes C_r^*(G_i)$. Let $\pi = \oplus_{i\in I}\sigma_i$. Then

$$\|\pi(x)\| \leq \sup_{i\in I} \|(\theta_i \otimes \theta_i)(\psi(x))\|_{min} \leq \|x\|_{min}$$

for $x \in C^*(G) \odot C^*(G)$, and the proof will be complete if we can show that

$$\|\pi_r(x)\| \leq \|\pi(x)\| \tag{3.2}$$

for $x \in C^*(G) \odot C^*(G)$.

Since any element $a \in C^*(G)$ is a limit of elements of form $\sum_{i=1}^n \alpha_i g_i$, where $\alpha_i \in \mathbb{C}$ and $g_i \in G$, for any $x \in C^*(G) \odot C^*(G)$ and $\varepsilon > 0$ there is an element $z = \sum_{i=1}^m \alpha_i(g_i \otimes g_i')$ such that $\|x - z\|_{max} < \varepsilon$. If $\|\pi_r(z)\| \leq \|\pi(z)\|$, then

$$\|\pi_r(x)\| \leq \|\pi_r(z)\| + \varepsilon \leq \|\pi(z)\| + \varepsilon \leq \|\pi(x)\| + 2\varepsilon.$$

Thus if (3.2) holds for all z of this form, (3.2) will hold in general. So to prove (3.2), we can assume that $x = \sum_{r=1}^{k} \alpha_r(g_r \otimes g_r')$, and show that if $\xi \in \ell^2(G)$, then

$$\|\pi_r(x)\xi\| \le \|\pi(x)\|\|\xi\|. \tag{3.3}$$

Let ξ_g be the unit vector in $\ell^2(G)$ such that $\xi_g(h) = \delta_{gh}$. Then the elements of form $\sum_{s=1}^{l} \beta_s \lambda(h_s)\xi_e$ are dense in $\ell^2(G)$, and to prove (3.3) it suffices to take ξ of this form. Now

$$\begin{aligned}\pi_r(x)\xi &= \sum_{r,s} \alpha_r\beta_s\lambda(g_r)\rho(g_r')\lambda(h_s)\xi_e \\ &= \sum_{r,s} \alpha_r\beta_s\xi_{g_r h_s g_r'^{-1}}.\end{aligned}$$

Choose $i \in I$ so that the restriction of θ_i to the set

$$\{g_r h_s g_r'^{-1} : 1 \le r \le k, 1 \le s \le l\} \cup \{h_s : 1 \le s \le l\}$$

is injective, and let

$$\begin{aligned}\overline{\xi} &= \sum_{s=1}^{l} \beta_s\lambda_i(\theta_i(h_s))\overline{\xi}_e \\ &= \sum_{s=1}^{l} \beta_s\overline{\xi}_{\theta_i(h_s)}\end{aligned}$$

where $\overline{\xi}_g$ is the vector in $\ell^2(G_i)$ such that $\overline{\xi}_g(h) = \delta_{gh}$. Then

$$\begin{aligned}\sigma_i(x)\overline{\xi} &= \sum_{r,s} \alpha_r\beta_s\lambda_i(\theta_i(g_r))\rho_i(\theta_i(g_r'))\lambda_i(\theta_i(h_s))\overline{\xi}_e \\ &= \sum_{r,s} \alpha_r\beta_s\overline{\xi}_{\theta_i(g_r h_s g_r'^{-1})}.\end{aligned}$$

and

$$\|\pi_r(x)\xi\|^2 = \sum_{S} \alpha_r\beta_s\overline{\alpha}_t\overline{\beta}_u = \|\sigma_i(x)\overline{\xi}\|^2,$$

where

$$\begin{aligned}S &= \{(r,s,t,u) : g_r h_s {g_r}'^{-1} = g_t h_u {g_t}'^{-1}\} \\ &= \{(r,s,t,u) : \theta_i(g_r h_s {g_r}'^{-1}) = \theta_i(g_t h_u {g_t}'^{-1})\}.\end{aligned}$$

Also
$$\|\xi\|^2 = \sum |\beta_s|^2 = \|\overline{\xi}\|^2,$$
since $\theta_i(h_r) \neq \theta_i(h_s)$ for $r \neq s$. Thus
$$\begin{aligned}\|\pi_r(x)\xi\|^2 &= \|\sigma_i(x)\overline{\xi}\|^2 \\ &\leq \|\sigma_i(x)\|^2\|\overline{\xi}\|^2 \\ &\leq \|\pi(x)\|^2\|\xi\|^2,\end{aligned}$$
and (3.3) follows. □

Proposition 3.4. *Let G be a discrete group which has the factorisation property. If J is the kernel of the canonical morphism $\lambda : C^*(G) \to C^*_r(G)$, then the sequence*
$$0 \longrightarrow J \otimes C^*(G) \longrightarrow C^*(G) \otimes C^*(G) \xrightarrow{\lambda \otimes id} C^*_r(G) \otimes C^*(G) \longrightarrow 0 \quad (3.4)$$
is exact if and only if G is amenable.

Proof: 1. If G is amenable, $J = 0$, and the exactness of the sequence (3.4) is trivial.
2. If the sequence $(*)$ is exact, let $\pi_r : C^*(G) \otimes C^*(G) \to B(\ell^2(G))$ be the canonical representation. Now
$$(C^*(G) \otimes C^*(G))/(J \otimes C^*(G)) \cong (C^*(G)/J) \otimes_\nu C^*(G) \cong C^*_r(G) \otimes_\nu C^*(G)$$
for some C*-norm ν on $C^*_r(G) \odot C^*(G)$. Since $J \otimes C^*(G) \subseteq \ker \pi_r$, π_r has a factorisation $\pi_r = \phi\sigma$, where $\sigma : C^*(G) \otimes C^*(G) \to C^*_r(G) \otimes_\nu C^*(G)$ is the quotient map and ϕ is a representation of $C^*_r(G) \otimes_\nu C^*(G)$ on $\ell^2(G)$ which satisfies
$$\phi(a \otimes b) = a\rho(b) \quad (a \in C^*_r(G), b \in C^*(G)).$$
If (3.4) is exact, then $\| \ \|_\nu = \| \ \|_{min}$ on $C^*_r(G) \odot C^*(G)$. Since
$$C^*_r(G) \otimes C^*(G) \subseteq B(\ell^2(G)) \otimes C^*(G)$$
canonically, there are a Hilbert space K such that $\ell^2(G) \subseteq K$ and a representation $\overline{\phi}$ of $B(\ell^2(G)) \otimes C^*(G)$ on K such that
$$\phi(x) = E\overline{\phi}(x)|_{\ell^2(G)}$$

for $x \in C_r^*(G) \otimes C^*(G)$, where E is the orthogonal projection onto $\ell^2(G)$. For $x \in B(\ell^2(G))$ let $\varepsilon(x) = E\overline{\phi}(x \otimes 1)|_{\ell^2(G)}$. It is readily verified that the map ε of $B(\ell^2(G))$ into itself is a weak expectation for $C^*(\lambda(G))$, i.e. ε is a unital completely positive map into $C^*(\lambda(G))''$ such that $\varepsilon(axb) = a\varepsilon(x)b$ for $a, b \in C^*(\lambda(G))$ and $x \in B(\ell^2(G))$ (cf. [**La1**]).

For $f \in \ell^\infty(G)$ let T_f be the bounded linear operator on $\ell^2(G)$ given by

$$(T_f\xi)(g) = f(g)\xi(g) \quad (g \in G).$$

If τ is the canonical trace state on $C^*(\lambda(G))''$ obtained by restricting the vector state defined by ξ_e, then the map $m : f \to \tau(\varepsilon(T_f))$ is a left-invariant mean on $\ell^\infty(G)$. To see this, note that for $f \in \ell^\infty(G)$ and $g \in G$, $\lambda(g)T_f\lambda(g^{-1}) = T_{f_g}$, where $f_g(h) = f(g^{-1}h)$, and

$$\begin{aligned} m(f_g) &= \tau(\varepsilon(\lambda(g)T_f\lambda(g^{-1}))) \\ &= \tau(\lambda(g)\varepsilon(T_f)\lambda(g^{-1})) \\ &= \tau(\varepsilon(T_f)) \\ &= m(f). \end{aligned}$$

It follows that G is amenable. □

Corollary 3.5. *For a residually finite group G, $C^*(G)$ is exact if and only if G is amenable.*

3.2. Free groups.

Lemma 3.6. *Let F be a free group. Then F is residually finite.*

Proof: Let F have generators $\{g_\lambda\}_{\lambda \in \Lambda}$, and let $h_1, \ldots, h_n \in F$. We can find a finite subset $g_1, \ldots, g_m$ of the generators such that all the h_j are in the subgroup F' generated by $g_1, \ldots, g_m$. If $h = g_{i_1}^{n_1} \ldots g_{i_l}^{n_l} \in F'$, where $i_r \neq i_{r+1}$ for $r = 1, \ldots, l-1$, the *length* $|h|$ of h is $|n_1| + \ldots + |n_l|$. Let $k = \max_{1 \le i \le n} |h_i|$ and let $S = \{h \in F' : |h| \le k\}$. For each generator g_i of F', let $S_i = \{h \in S : g_i h \in S\}$. Then $S_i \subseteq S$, $g_i S_i \subseteq S$ and the map $h \to g_i h$ takes S_i onto $g_i S_i$ injectively. Thus $|S_i| = |g_i S_i|$ and $|S \setminus S_i| = |S \setminus g_i S_i|$. Let Q_i be a bijective map from $S \setminus S_i$ to $S \setminus g_i S_i$, and define a permutation P_i of S by

$$P_i h = \begin{cases} g_i h & (h \in S_i) \\ Q_i h & (h \in S \setminus S_i). \end{cases}$$

A homomorphism ϕ of F into the finite group $\Pi(S)$ of permutations of S is defined by

$$\begin{cases} \phi(g_i) = P_i & (1 \leq i \leq m) \\ \phi(g_\lambda) = 1 & (g_\lambda \notin \{g_1, \ldots, g_m\}). \end{cases}$$

For each h_i, $\phi(h_i)e = h_i$, and so $\phi(h_i) \neq \phi(h_j)$ for $i \neq j$. □

Corollary 3.7. *1. A free group has the factorisation property.*

2. If F is a free group, then the sequence

$$0 \longrightarrow J \otimes C^*(F) \longrightarrow C^*(F) \otimes C^*(F) \xrightarrow{\lambda \otimes id} C^*_r(F) \otimes C^*(F) \longrightarrow 0$$

is not exact. In particular, $C^*(F)$ is not exact.

Remarks. 1. If $F = \mathbb{F}_2$, the free group on two generators u and v, and $\| \ \|_\nu$ is the C*-norm on $C^*(\mathbb{F}_2) \odot C^*(\mathbb{F}_2)$ such that

$$(C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2))/(J \otimes C^*(\mathbb{F}_2)) \cong C^*_r(\mathbb{F}_2) \otimes_\nu C^*(\mathbb{F}_2)$$

canonically, it can be shown by an explicit computation [**Wa3**] that if c is the self-adjoint element

$$\frac{1}{4}(u \otimes u + u^{-1} \otimes u^{-1} + v \otimes v + v^{-1} \otimes v^{-1})$$

of $C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$, then the image of c in $C^*_r(\mathbb{F}_2) \otimes_\nu C^*(\mathbb{F}_2)$ is a contraction containing 1 in its spectrum, whereas the image of c in $C^*_r(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$ is a contraction with spectrum contained in the interval $[-1, 1 - 42^{-4}]$. If $f : [-1, 1] \to [0, 1]$ is a continuous function such that $f(t) = 0$ for $-1 \leq t \leq 1 - 42^{-4}$ and $f(1) = 1$, then $f(c) \in \ker(\lambda \otimes id) \setminus (J \otimes C^*(\mathbb{F}_2))$.

2. For any free groups F, $C^*_r(F)$ is a simple C*-algebra [**A-L**]. If $\mathbb{F}_n$ denotes the free group on n generators, then $K_1(C^*_r(\mathbb{F}_n)) = \mathbb{Z}^n$ [**P-V**]. This implies that $C^*_r(\mathbb{F}_m) \not\cong C^*_r(\mathbb{F}_n)$ for $m \neq n$. (A self-contained derivation of the K-groups of $C^*_r(\mathbb{F}_n)$ is given in appendix A).

3.3. Groups with Kazhdan's property T. In this section all the groups we consider will be discrete and countable. The principal reference is [**Wa4**].

Definition 3.8. A countable group G has *property* T if there are a finite subset $g_1, \ldots, g_n$ of G and and $\varepsilon > 0$ such that, whenever π is a (unitary) representation of G on a Hilbert space H containing a unit vector ξ for which

$$\|\pi(g_i)\xi - \xi\| < \varepsilon \quad (i = 1, \ldots, n), \tag{3.5}$$

then there is a non-zero vector in H invariant under $\pi(G)$.

Remarks 3.9. 1. This definition says that if a representation π of G approximately contains the trivial representation τ (in the sense of (3.5)), then π actually contains τ.

2. It follows from this definition that if G has property T, then it is finitely generated [**dH-V**, Théorème 1.10], and we can (and will) take the elements $g_1, \ldots, g_n$ to be a generating set for G with $g_1 = e$.

3. Examples of countable groups with property T are the matrix groups $SL_n(\mathbb{Z})$ $(n = 3, 4, \ldots)$ (see [**dH-V**]).

4. Property T and amenability are mutually exclusive properties for infinite groups. If a group G has both properties, amenability implies the existence of a unit vector $\xi \in \ell^2(G)$ such that (3.5) holds with $\pi = \lambda$ [**Kye2**, Lemma 4.5.9], and then property T implies the existence of a non-zero invariant vector $\zeta = \sum_{h \in G} \alpha_h \xi_h$ with $\sum |\alpha_h|^2 < \infty$ in $\ell^2(G)$. Since

$$\lambda(g)\zeta = \sum \alpha_h \xi_{gh} = \zeta,$$

it follows that $\alpha_g = \alpha_e$ for all $g \in G$, which implies that $|G| < \infty$.

Let π be a representation of a group G on a Hilbert space H with orthonormal basis $\{\xi_i\}_{i \in I}$, and let J be the conjugate linear isometry on H given by

$$J(\sum \alpha_i \xi_i) = \sum \overline{\alpha_i} \xi_i.$$

A representation π' of G is defined by

$$\pi'(g) = J\pi(g)J \quad (g \in G).$$

The unitary equivalence class of π' is independent of the particular choice of orthonormal basis of H, and depends only on the unitary equivalence class of π. The representation π' is the representation *conjugate* to π.

Lemma 3.10. *If G has property T and the generating set $1 = g_1, \ldots, g_n$ and $\varepsilon > 0$ are as in Definition 3.8, then there is a positive constant $C < n$ such that if π_1 and π_2 are unitary representations of G on Hilbert spaces H_1 and H_2, respectively, satisfying*

$$\left\| \sum_{i=1}^{n} \pi_1(g_i) \otimes \pi_2'(g_i) \right\|_{min} \geq C,$$

then there is a finite-dimensional unitary representation π of G which is contained in both π_1 and π_2. If, conversely, π is a finite dimensional representation of G, then

$$\left\| \sum_{i=1}^{n} \pi(g_i) \otimes \pi'(g_i) \right\|_{min} = n.$$

Proof: If

$$\left\| \sum_{i=1}^{n} \pi_1(g_i) \otimes \pi_2'(g_i) \right\|_{min} \geq C,$$

there is a unit vector ξ in H such that

$$\left\| \sum_{i=1}^{n} (\pi_1(g_i) \otimes \pi_2'(g_i))\, \xi \right\| \geq C.$$

Since $g_1 = e$, $(\pi_1(g_1) \otimes \pi_2'(g_1))\xi = \xi$. A simple calculation then shows that

$$\|(\pi_1(g_i) \otimes \pi_2'(g_i))\xi - \xi\| \leq 2\sqrt{n - C} \quad (i = 2, \ldots, n).$$

Since G has property T, if C is chosen so that $2\sqrt{n-C} < \varepsilon$, there is a non-zero η in $H_1 \otimes H_2$ fixed by $(\pi_1 \otimes \pi_2')(G)$. If $\{e_1, e_2, \ldots\}$ and $\{f_1, f_2, \ldots\}$ are orthonormal bases of H_1 and H_2, respectively, with the f_i chosen so that

$$(\pi_2'(g)f_i|f_j) = \overline{(\pi_2(g)f_i|f_j)}$$

for $g \in G$ and $i, j \in \mathbb{N}$, then $\eta = \sum_{i,j} \lambda_{ij} e_i \otimes f_j$, where $\sum_{i,j} |\lambda_{ij}|^2 < \infty$. The linear operator $T : H_2 \to H_1$ given by

$$T\zeta = \sum_{i,j} \lambda_{ij} (\zeta|f_j) e_i$$

is compact (in fact Hilbert-Schmidt), and it is easily verified that $\pi_1(g)T = T\pi_2(g)$ $(g \in G)$. Since $T \neq 0$, T^*T is non-zero, commutes with π_2 and is compact. If E is the projection onto the eigenspace of T^*T corresponding to the eigenvalue $\|T^*T\|$, then E is non-zero, of finite rank and commutes with π_2. It follows that if $U = \|T\|^{-1}TE$, then $U\pi_1U^*|_{UH_2}$ and $E\pi_1E|_{EH_2}$ are equivalent finite-dimensional subrepresentations of π_1 and π_2, respectively.

For the converse, if π is a representation if G on the finite dimensional Hilbert space H with orthonormal basis $\{e_1, \dots, e_m\}$, let π' satisfy

$$(\pi'(g)e_i|e_j) = \overline{(\pi(g)e_i|e_j)},$$

and let $\xi = \sum_{j=1}^m e_j \otimes e_j$. Then $(\pi(g) \otimes \pi'(g))\xi = \xi$ for $g \in G$, so that

$$\left\| \sum_{i=1}^n \pi(g_i) \otimes \pi'(g_i) \right\|_{min} = n.$$

□

Corollary 3.11. *With the notation of Lemma 3.10, if π_1 and π_2 are irreducible representations of G such that*

$$\left\| \sum_{i=1}^n \pi_1(g_i) \otimes \pi_2'(g_i) \right\|_{min} \geq C,$$

then π_1 and π_2 are unitarily equivalent and $\dim\pi_1 = \dim\pi_2 < \infty$.

Corollary 3.12. *There is an $\varepsilon' > 0$ such that if π_1 and π_2 are irreducible representations of G on a finite dimensional Hilbert space H, then*

$$\|\pi_1(g_i) - \pi_2(g_i)\| \leq \varepsilon' \quad (i = 1, \dots, n)$$

implies that π_1 and π_2 are unitarily equivalent.

Proof: If $e_1, \dots, e_m$ is an orthonormal basis of H, and ξ is the unit vector $(1/\sqrt{m})\sum_{j=1}^m e_j \otimes e_j$, then

$$\begin{aligned} \left\| \sum_i (\pi_1(g_i) \otimes \pi_2'(g_i))\xi \right\| &\geq \left\| \sum_i (\pi_2(g_i) \otimes \pi_2'(g_i))\xi \right\| \\ &\quad - \left\| \sum_i \left((\pi_1(g_i) - \pi_2(g_i)) \otimes \pi_2'(g_i)\right)\xi \right\| \\ &\geq (n - n\varepsilon'). \end{aligned}$$

It is thus sufficient to take $\varepsilon' = 1 - C/n$. □

Corollary 3.13 *For $m \in \mathbb{N}$ there are at most finitely many equivalence classes of irreducible representations of G of dimension m.*

Proof: This is an immediate consequence of Corollary 3.12 and the compactness of the unit ball of $B(\ell^2_m)$. □

From now on let G be a countably infinite, residually finite group with property T, for example $SL_3(\mathbb{Z})$. Then G has infinitely many equivalence classes of finite-dimensional irreducible representations, and by Corollary 3.13 the number of these classes is countably infinite. Let $\pi_1, \pi_2, \ldots$ be a sequence of mutually inequivalent finite-dimensional irreducible representations of G which includes a representative from each such equivalence class, let the generating set $e = g_1, \ldots, g_n$ and $\varepsilon > 0$ be as in Definition 3.8, and let C be as in Lemma 3.10. If π_i acts on the Hilbert space H_i of dimension n_i ($i = 1, 2, \ldots$), let $H = \oplus_{i=1}^{\infty} H_i$. The representation $\rho_k = \oplus_{i=k}^{\infty} \pi_i$ acts on the Hilbert subspace $\oplus_{i=k}^{\infty} H_i$ of H, and so we can regard each ρ_k as a degenerate representation of G by isometries on H. Let $M = \oplus_{i=1}^{\infty} B(H_i)$ and let I_0 be the ideal $M \cap K(H)$ of M, so that $I_0 = \{(x_i) \in M : \lim_{i\to\infty} \|x_i\| = 0\}$. Let $\phi : M \to M/I_0$ be the quotient map. Then the homomorphisms $\phi\rho_k : G \to M/I_0$ coincide for $k = 1, 2, \ldots$.

Lemma 3.14. *For any irreducible representation σ of G,*

$$\left\| \sum_{i=1}^{n} \sigma(g_i) \otimes (\phi\rho_1)(g_i) \right\|_{min} \leq C.$$

Proof: Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \left\| \sum_i \sigma(g_i) \otimes (\phi\rho_1)(g_i) \right\|_{min} &= \left\| \sum_i \sigma(g_i) \otimes (\phi\rho_k)(g_i) \right\|_{min} \\ &\leq \left\| \sum_i \sigma(g_i) \otimes \rho_k(g_i) \right\|_{min} \\ &= \sup_{l \geq k} \left\| \sum_i \sigma(g_i) \otimes \pi_l(g_i) \right\|_{min} \\ &\leq C \end{aligned}$$

for all k if σ is infinite dimensional, and for large enough k if σ is finite-dimensional, by Corollary 3.11. □

Let $A = \rho_1(C^*(G)) \subseteq M$.

Lemma 3.15. $I_0 \subseteq A$.

Proof: Let $k \in \mathbb{N}$ and consider the element

$$x = \frac{1}{n}\sum_{i=1}^{n} \rho_1(g_i) \otimes \pi'_k(g_i)$$

of $A \otimes B(H_k)$. Since $M \otimes B(H_k) \cong \oplus_{i=1}^{\infty}(B(H_i) \otimes B(H_k))$, x is identified with the sequence (x_j), where

$$x_j = \frac{1}{n}\sum_{i=1}^{n} \pi_j(g_i) \otimes \pi'_k(g_i) \in B(H_j) \otimes B(H_k).$$

By Corollary 3.11, $\|x_j\| < C/n$ if $j \neq k$ and $\|x_k\| = 1$, and so $|||x_j||| < C/n$ for $j \neq k$ and $|||x_k||| = 1$. Now let $f : [0,1] \to [0,1]$ be continuous and such that $f(1) = 1$ and $f(t) = 0$ for $0 \leq t \leq C/n$. If $z = f(|x|)$, then $z = (z_j)$, where $z_j = 0$ $(j \neq k)$ and $\|z_k\| = 1$. Since $z \neq 0$, there is a $\varphi \in B(H_k)^*$ such that $L_\varphi(z)$ is a non-zero element of A. Let $y = L_\varphi(z) = (y_j)$, where $y_j = 0$ $(j \neq k)$ and $y_k \neq 0$. Now $y_k \in C^*(\pi_k(G)) = B(H_k)$, $y_k C^*(\pi_k(G)) = B(H_k)$, since the representation π_k is irreducible, and

$$yA = \{(x_j) \in M : x_j = 0\ (j \neq k), x_k \in B(H_k)\}.$$

It follows that A contains all elements $x = (x_j)$ in M with only finitely many entries non-zero. Every element of I_0 is the limit of a sequence of such x, and so $I_0 \subseteq A$. □

Let D be a C*-algebra through which the morphism $\rho_1 : C^*(G) \to A$ factors, so that $\rho_1 = \gamma\psi$, where ψ and γ are morphisms of $C^*(G)$ onto D and of D onto A, respectively. Let $B = A/I_0 \cong \phi(A)$.

Theorem 3.16. *The sequence*

$$0 \longrightarrow I_0 \otimes D \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow 0$$

is not exact.

Proof: The quotient $(A \otimes D)/(I_0 \otimes D)$ is canonically isomorphic to $B \otimes_\nu D$ for some C*-norm $\| \ \|_\nu$ on $B \odot D$. To show that the given sequence is not exact, it is sufficient to show that $\| \ \|_\nu \neq \| \ \|_{min}$ on $B \odot D$, i.e. to show that $\|c\|_\nu > \|c\|_{min}$ for some $c \in B \odot D$. Let

$$c = \sum_{i=1}^{n} (\phi\rho_1)(g_i) \otimes \psi(g_i).$$

If σ is an irreducible representation of D, then $\sigma\psi$ is an irreducible representation of $C^*(G)$ and

$$\begin{aligned}\|(id \otimes \sigma)(c)\|_{min} &= \left\| \sum_{i=1}^{n} (\phi\rho_1)(g_i) \otimes (\sigma\psi)(g_i) \right\|_{min} \\ &\leq C\end{aligned}$$

by Lemma 3.14. Since the set of morphisms $id \otimes \sigma$, where σ varies over all irreducible representations of D, is faithful on $B \otimes D$, it follows that

$$\|c\|_{min} \leq C < n.$$

Now let e_k be the projection onto the finite dimensional subspace $\oplus_{i=1}^{k} H_i$ of H, for $k = 1, 2, \ldots$. Then $\{e_k\}_{k \in \mathbb{N}}$ is a (sequential) approximate unit for I_0 and $\{e_k \otimes 1\}$ is an approximate unit for $I_0 \otimes D$. Also

$$\begin{aligned}\left\| \left(\sum_i \rho_1(g_i) \otimes \psi(g_i) \right) ((1 - e_k) \otimes 1) \right\|_{min} &= \left\| \sum_i \rho_k(g_i) \otimes \psi(g_i) \right\|_{min} \\ &\geq \left\| \sum_i \pi_k(g_i) \otimes \pi'_k(g_i) \right\|_{min} \\ &= n,\end{aligned}$$

by corollary 3.11, since ρ_1, and hence π'_k, factor through D. Now if $x \in I_0 \otimes D$ and $\varepsilon > 0$, there is a k such that $\|x((1 - e_k) \otimes 1)\| < \varepsilon$. Then

$$\begin{aligned}\left\| \left(\sum_i \rho_1(g_i) \otimes \psi(g_i) \right) + x \right\|_{min} &\geq \left\| \left\{ \left(\sum_i \rho_1(g_i) \otimes \psi(g_i) \right) + x \right\} ((1 - e_k) \otimes 1) \right\| \\ &\geq n - \varepsilon.\end{aligned}$$

Since ε is arbitrary,

$$\left\| \left(\sum_i \rho_1(g_i) \otimes \psi(g_i) \right) + x \right\|_{min} \geq n.$$

It follows that

$$\begin{aligned} \|c\|_\nu &= \left\| \sum_i (\phi\rho_1)(g_i) \otimes \psi(g_i) \right\|_\nu \\ &= \left\| \left(\sum_i \rho_1(g_i) \otimes \psi(g_i) \right) + I_0 \otimes D \right\|_{min} \\ &= n. \end{aligned}$$

Thus $\| \ \|_\nu \neq \| \ \|_{min}$ on $B \odot D$. □

Remarks 3.17. 1. If $A_1 = A + K(H)$, then $A_1/K(H) \cong A/(A \cap K(H)) = A/I_0 = B$. The sequence $\{e_k\}$ is an approximate unit for $K(H)$ contained in A, so that if $x \in (A \otimes D) \cap (K(H) \otimes D)$, then $x = \lim_{k\to\infty} x(e_k \otimes 1) \in I_0 \otimes D$, and $(A \otimes D) \cap (K(H) \otimes D) = I_0 \otimes D$. Thus

$$(A_1 \otimes D)/(K(H) \otimes D) \cong (A \otimes D)/\left((A \otimes D) \cap (K(H) \otimes D)\right) \cong B \otimes_\nu D,$$

and $\|c\|_\nu = \|c + K(H) \otimes D\|_{min}$. It follows that the sequence

$$0 \longrightarrow K(H) \otimes D \longrightarrow A_1 \otimes D \longrightarrow B \otimes D \longrightarrow 0$$

is not exact, and that the natural embedding ι of B in the Calkin algebra $B(H)/K(H)$ is non-liftable, i.e. the quotient map $A_1 \to B$ does not have a unital completely positive right inverse (see chapter 6). This implies that ι is a non-invertible extension of B in the sense of the Brown-Douglas-Fillmore theory [**BDF**], so that Ext(B) is not a group.

2. Kirchberg [**Kir7**] has recently shown that if G is a discrete group with property T , then G has the factorisation property if and only if G is maximally almost periodic, which is equivalent, in view of the remark following Proposition 3.2, to G being residually finite. Now Gromov has shown that there exist countable groups with property T which are not residually finite (see [**dH-V**, chap.3, §19–21]). It follows that these groups do not have the factorisation property. The following short proof of Kirchberg's result is

based on a proof kindly communicated to me by A. Valette which uses an idea of M.E.B. Bekka.

Theorem 3.18. *Let G be a discrete group with property T which has the factorisation property. Then G is maximally almost periodic.*

Proof: Let $T = \{\sigma \otimes \pi : \sigma, \pi$ irreducible representations of $C^*(G)\}$, and let $F = \{\sigma \otimes \sigma' : \sigma$ a finite-dimensional irreducible representation of $C^*(G)\}$. Let ω_0 and ω_1 be the representations of $C^*(G) \otimes C^*(G)$ given by

$$\omega_0 = \oplus_{\pi \in F} \pi, \; \omega_1 = \oplus_{\pi \in T \setminus F} \pi.$$

The representation $\omega_0 \oplus \omega_1$ of $C^*(G) \otimes C^*(G)$ is faithful. Let

$$A = \omega_0(C^*(G) \otimes C^*(G)), \quad B = \omega_1(C^*(G) \otimes C^*(G)).$$

Then $C^*(G) \otimes C^*(G) \cong (\omega_0 \oplus \omega_1)(C^*(G) \otimes C^*(G)) \subseteq A \oplus B$. A state f on $C^*(G) \oplus_{max} C^*(G)$ is defined by

$$f(\sum_i \alpha_i h_i \otimes h_i') = \sum_i \alpha_i (\lambda(h_i)\rho(h_i')\xi_e | \xi_e) \quad (h_i, h_i' \in G).$$

Since G has the factorisation property, f factors through $C^*(G) \otimes C^*(G)$, i.e. there is a state f' on $C^*(G) \otimes C^*(G)$ such that

$$f'(\sum_i \alpha_i h_i \otimes h_i') = \sum_i \alpha_i (\lambda(h_i)\rho(h_i')\xi_e | \xi_e) \quad (h_i, h_i' \in G).$$

Identifying $C^*(G) \otimes C^*(G)$ with its image in $A \oplus B$ under $\omega_0 \oplus \omega_1$, f' extends, by the Hahn-Banach theorem, to a state f'' on $A \oplus B$. Then $f'' = f_0 + f_1$, where f_0 and f_1 are positive linear functionals on $A \oplus B$ such that $f_0((0,1)) = f_1((1,0)) = 0$.

Let G_Δ denote the diagonal subgroup $\{(g,g) : g \in G\}$ of $G \times G$. Then $G \cong G_\Delta$, and $C^*(G) \cong C^*(G_\Delta) \subseteq C^*(G) \otimes_{max} C^*(G)$ ($\cong C^*(G \times G)$). Let φ, φ_0 and φ_1 be the restrictions of $f, f_0\omega_0$ and $f_1\omega_1$ to $C^*(G_\Delta)$, respectively. Then

$$\varphi((g,g)) = (\lambda(g)\rho(g)\xi_e | \xi_e) = 1$$

for $g \in G$, i.e. φ is the trivial character of $C^*(G_\Delta)$, and so a pure state. Since $\varphi = \varphi_0 + \varphi_1$, it follows that $\varphi_0 = \|\varphi_0\|\varphi$ and $\varphi_1 = \|\varphi_1\|\varphi$ (this also follows easily from the fact that φ is a homomorphism with range $\mathbb{C}$). Suppose that $f_1 \neq 0$. Also

$$\begin{aligned}\|f_1\|\|\omega_1(\sum_1^n g_i \otimes g_i)\| &\geq f_1(\omega_1(\sum_1^n g_i \otimes g_i)) \\ &= \varphi_1(\sum_1^n g_i \otimes g_i) \\ &= \|\varphi_1\|\varphi(\sum_1^n g_i \otimes g_i) \\ &= n\|\varphi_1\| \\ &= n\|f_1\|.\end{aligned}$$

This implies that $\|\omega_1(\sum_1^n g_i \otimes g_i)\| = n$ if $f_1 \neq 0$. Now if σ and π are irreducible representations of $C^*(G)$ such that $\sigma \otimes \pi \in T \setminus F$, by Corollary 3.11 $\|(\pi \otimes \sigma)(\sum_1^n g_i \otimes g_i)\| < C$ so that $\|\omega_1(\sum_1^n g_i \otimes g_i)\| \leq C < n$. Thus $f_1 = 0$ and $f = f_0$. Now

$$f_0(\omega_0(g \otimes 1)) = f(g \otimes 1) = (\lambda(g)\rho(1)\xi_e|\xi_e) = (\xi_g|\xi_e) = 0$$

if $g \neq e$. Thus $\omega_0(g \otimes 1) \neq 1$, and so for some finite-dimensional representation σ of $C^*(G)$, $\sigma(g) \neq 1$. It follows that the finite-dimensional irreducible representations separate the elements of G, i.e. G is maximally almost periodic. □

4. Quasidiagonal C*-algebras.

4.1. Block-diagonal and quasidiagonal C*-algebras. I shall consider only separable C*-algebras and representations on separable Hilbert spaces. For such a Hilbert space H a set $\Omega \subseteq B(H)$ is *block-diagonal* if there is a sequence $P_1 \leq P_2 \leq \ldots$ of finite-rank projections converging strongly to I such that $[P_n, T] = 0$ $(T \in \Omega,\ n \in \mathbb{N})$, and *quasidiagonal* if $\lim_{n\to\infty} \|[P_n, T]\| = 0$ for each $T \in \Omega$. Equivalently, letting $E_n = P_{n+1} - P_n$ $(n = 1, 2, \ldots)$, Ω is block-diagonal if there is a sequence $E_1, E_2, \ldots$ of mutually orthogonal finite-rank projections such that $\sum_{i=1}^{\infty} E_i$ is strongly convergent to I and $[E_i, T] = 0$ for each $i \in \mathbb{N}$ and $T \in \Omega$; and Ω is quasidiagonal if there is a sequence $\{E_i\}$ such that $T - \sum_{i=1}^{\infty} E_i T E_i$ is a compact operator for $T \in \Omega$.

A C*-algebra A is *quasidiagonal* if there is a faithful representation π of A on a separable Hilbert space H such that $\pi(A)$ is quasidiagonal as a subset of $B(H)$. It is a simple consequence of Voiculescu's theorem (see [**Arv**]) that for A separable, the quasidiagonality of $\pi(A)$ for one faithful π implies that of the image of A under any faithful representation of infinite multiplicity on a separable Hilbert space. A quasidiagonal C*-algebra preserves, in some sense, a vestige of finite dimensional behaviour.

One question which remained unanswered for some time was whether the image of a quasidiagonal C*algebra $A \in B(H)$ in the Calkin algebra $B(H)/K(H)$ is necessarily quasidiagonal. This question has also been investigated in relation to specific C*-algebras. In §4.2 we shall look at a specific instance, involving the algebras constructed from residually finite Kazhdan groups in chapter 3, where quasidiagonality is not preserved on passing to the quotient by $K(H)$.

Using a technique similar to that used to prove the residual finiteness of free groups in chapter 3 (Lemma 3.6), Choi [**Ch3**] proved

Theorem 4.1. *If F is a free group on at most a countably infinite number of generators, then $C^*(F)$ has a faithful block-diagonal representation on a separable Hilbert space.*

Proof: Let u_1, u_2 be the generators of $\mathbb{F}_2$. Then the elements $\{u_1 u_2^n u_1 : n \in \mathbb{N}\}$ are free generators of a subgroup of $\mathbb{F}_2$ isomorphic to $\mathbb{F}_\infty$. Thus $\mathbb{F}_n \subseteq \mathbb{F}_2$ for $n = 3, 4, \ldots, \infty$, and it is sufficient to consider the case $F = \mathbb{F}_2$.

We show that for $x \in C^*(F)$ there is a finite-dimensional representation

σ of $C^*(F)$ such that

$$\|\sigma(x)\| \geq \|x\| - \varepsilon. \tag{4.1}$$

Given $x \in C^*(F)$ and $\varepsilon > 0$, there is a z of form $z = \sum_{i=1}^n \alpha_i h_i$, where the h_i are elements of F, such that $\|x - z\| < \varepsilon$. If (4.1) holds for z then

$$\|\sigma(x)\| \geq \|\sigma(z)\| - \varepsilon \geq \|z\| - 2\varepsilon \geq \|x\| - 3\varepsilon.$$

It thus suffices to prove (4.1) for $x = \sum_{i=1}^n \alpha_i h_i$. Let π be a faithful representation of $C^*(F)$ on a Hilbert space H, and let $\xi_0 \in H$ be a unit vector such that $\|\pi(x)\xi_0\| \geq \|x\| - \varepsilon$. With $k = \max_{1\leq i\leq n} |h_i|$, let K be the finite-dimensional subspace of H spanned by the set $\{\pi(h)\xi_0 : |h| \leq k\}$. If $K_i = \{\xi \in K : \pi(u_i)\xi \in K\}$ $(i = 1, 2)$ then K_i is a linear subspace of K, $\pi(u_i)K_i \subseteq K$ and $\pi(u_i)|_{K_i}$ is an isometry. Since $\dim K_i = \dim \pi(u_i)K_i$, $\dim(K \ominus K_i) = \dim(K \ominus \pi(u_i)K_i)$, and so there is an isometry u_i' of $K \ominus K_i$ onto $K \ominus \pi(u_i)K_i$. Define a unitary U_i on K by

$$U_i\xi = \begin{cases} \pi(u_i)\xi & (\xi \in K_i) \\ u_i'\xi & (\xi \in K_i^\perp). \end{cases}$$

A representation σ of $C^*(F)$ on K is defined by $\sigma(u_i) = U_i$ $\quad (i = 1, 2)$. It is easy to see that for $h \in F$ with $|h| \leq k$,

$$\sigma(h)\xi_0 = \pi(h)\xi_0,$$

so that

$$\|\sigma(x)\xi_0\| = \|\pi(x)\xi_0\| \geq \|x\| - \varepsilon,$$

and $\|\sigma(x)\| \geq \|x\| - \varepsilon$.

Now let $\{a_1, a_2, \ldots\}$ be a dense subset of $C^*(F)$. For $n = 1, 2, \ldots$ there are finite-dimensional representations σ_n of $C^*(F)$ on Hilbert spaces H_n such that

$$\|\sigma(a_i)\| \geq \|a_i\| - \frac{1}{n} \quad (1 \leq i \leq n).$$

Let $M = \oplus_{n=1}^\infty B(H_n)$. Then the $*$-homomorphism $\sigma : x \to (\sigma_n(x)); C^*(F) \to M$ is isometric on each a_i, hence on $C^*(F)$. Since M is block-diagonal, so is σ. □

Proposition 4.2. *The algebra $C_r^*(\mathbb{F}_2)$ is not quasidiagonal.*

Proof: To prove this result, we actually prove the more general result that if G is a countable group and the C*-algebra $C_r^*(G)$ is quasidiagonal, then G is amenable. The stated result will then follow, since $\mathbb{F}_2$ is not amenable. If $C_r^*(G)$ is quasidiagonal, there are finite rank projections $P_1 \leq P_2 \leq \ldots$ in $B(\ell^2(G))$ with strong limit I such that for $x \in C_r^*(G)$, $\lim_{n\to\infty} \|[P_n, x]\| = 0$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let m be the linear functional on $\ell^\infty(G)$ given by

$$m(f) = \lim_{\mathcal{U}} \tau_n(P_n T_f P_n) \quad (f \in \ell^\infty(G),$$

where τ_n is the trace state on $B(P_n\ell^2(G))$. Then

$$\lim_n \|\lambda(g)P_n - P_n\lambda(g)\| = 0 \quad (g \in G)$$

so that

$$\lim_n \|\lambda(g)P_n\lambda(g^{-1}) - P_n\| = 0,$$

and so

$$\lim_n |\tau_n(P_n\lambda(g)T_f\lambda(g^{-1})P_n) - \tau_n(P_nT_fP_n)| = 0 \quad (f \in \ell^\infty(G)).$$

Thus

$$m(f_g) = \lim_{\mathcal{U}} \tau_n(P_n\lambda(g)T_f\lambda(g^{-1})P_n) = m(f),$$

and m is a left-invariant mean on G. Thus G is amenable. □

4.2. Quasidiagonal C*-algebras associated with Kazhdan groups.

Let G be a countably infinite residually finite Kazhdan group with generating set $1 = g_1, \ldots, g_n$ and Kazhdan constant $\varepsilon < 1$ as in Definition 3.8. Recall that if, with the notation of §3.3, $A = C^*(\rho_1(G))$, $B = A/I_0$ and $A_1 = A + K(H)$, then $B \cong A_1/B(H)$.

Theorem 4.3. *The algebra B has no quasidiagonal representation, and is, in particular, not quasidiagonal.*

Proof: Suppose that B has a quasidiagonal representation π on a separable Hilbert space K. Clearly π can be assumed non-degenerate. Then K has a decomposition $K = \oplus_{j=1}^\infty K_j$, with $\dim K_j < \infty$ for each j, such that

$$\pi(g) \in \oplus_{j=1}^\infty B(K_j) + K(K),$$

for $g \in G$. If $\omega = \pi\phi\rho_1$, and for each generator g_i of G we let $v_i = \omega(g_i)$, then

$$v_i = (\oplus_j v_j^i) + k_i,$$

where $(v_j^i)_{j\in\mathbb{N}} \in \oplus_{j=1}^{\infty} B(K_j)$ and $k_i \in K(K)$. Let F_p be the projection onto $\oplus_{j=1}^{p} K_j$. Then for each i, $\|k_i - F_p k_i F_p\| \to 0$ as $k \to \infty$. It follows that we can find a p such that

$$\|v_i - (\oplus_j v_j^i) - F_p k_i F_p\| = \|k_i - F_p k_i F_p\| < 1,$$

for $i = 1, \dots, n$, and so if $f = F_p$,

$$\|1 - v_i^*(\oplus_j v_j^i + f k_i f)\| < 1.$$

This implies that $v_i^*(\oplus_j v_j^i + f k_i f)$, and thus $(\oplus_j v_j^i + f k_i f)$, are invertible. Then for fixed $i = 1, \dots, n$, v_j^i is invertible for $j > p$, and if v_j^i has polar decomposition $v_j^i = u_j^i |v_j^i|$, u_j^i is unitary for $j > p$, $i = 1, \dots, n$. For $j \leq k$ let u_j^i be any unitary in $B(K_j)$. Since v_i is unitary, $(1 - \oplus_j v_j^{i*} v_j^i) \in K(H)$, which implies that $\|1_j - v_j^{i*} v_j^i\| \to 0$ as $j \to \infty$. This in turn implies that $\lim_{j\to\infty} \|1_j - |v_j^i|\| = 0$ and $\lim_{j\to\infty} \|u_j^i - v_j^i\| = 0$. Thus

$$\omega(g_i) = (\oplus_j u_j^i) + k_i'$$

for some $k_i' \in K(H)$ $(i = 1, \dots, n)$. Let E_j be the projection onto K_j. Let j be chosen sufficiently large that $\|k_i' E_j\| < \varepsilon/3$ for $i = 1, \dots, n$. If $e_1, \dots, e_{d(j)}$ is an orthonormal basis of K_j and

$$\xi = \sum_{k=1}^{d(j)} e_k \otimes e_k,$$

then

$$\begin{aligned}
\|\left(\omega(g_i) \otimes \omega'(g_i)\right)\xi - \xi\| &= \|(u_j^i + k_i' E_j) \otimes J_j(u_j^i + k_i' E_j) J_j \xi - \xi\| \\
&\leq \|(u_j^i \otimes J_j u_j^i J_j)\xi - \xi\| \\
&\qquad + 2\|k_i' E_j\|\|\xi\| + \|k_i' E_j\|^2 \|\xi\| \\
&< \left(\frac{2\varepsilon}{3} + \frac{\varepsilon^2}{9}\right)\|\xi\| \\
&< \varepsilon\|\xi\|,
\end{aligned}$$

where we can take ω' to be the representation $J\pi\phi\rho_1 J$ with J the conjugate linear isometry (J_i) corresponding to the given bases of the K_j. Since G has property T, it follows that there is a non-zero $\xi_0 \in K \otimes K$ invariant for $(\omega\otimes\omega')(G)$. Then $\|\sum_i \omega(g_i)\otimes\omega'(g_i)\| = n$ and, by Lemma 3.10, π contains a finite-dimensional subrepresentation π_0 of B. The representation $\sigma = \pi_0\phi\rho_1$ is a finite-dimensional representation of $C^*(G)$ and

$$\begin{aligned}\|\sum_i \sigma'(g_i)\otimes\sigma(g_i)\| &= \|\sum_i \sigma'(g_i)\otimes(\pi_0\phi\rho_1)(g_i)\| \\ &\leq \|\sum_i \sigma'(g_i)\otimes(\phi\rho_1)(g_i)\| \\ &< C \\ &< n,\end{aligned}$$

by Lemma 3.14. This contradicts the fact that $\|\sum_i \sigma'(g_i)\otimes\sigma(g_i)\| = n$, by Lemma 3.10. It follows that B has no quasidiagonal representation, and is, in particular, not quasidiagonal. □

Remarks 4.4. 1. For groups G of the type just considered, it is not difficult to show that the canonical morphism $\lambda : C^*(G) \to C^*_r(G)$ factors through B. It follows that $C^*_r(G)$ is not quasidiagonal. This fact also follows by Proposions 3.3, 3.4 and Remark 3.9 (4).

2. It would be interesting to know if the algebra B is exact for any G of the type considered here, in particular if $G = SL_n(\mathbb{Z})$ for $n \geq 3$.

3. Is there a quasidiagonal C*-algebra A in $B(H)$ containing $K(H)$ such that $C^*_r(\mathbb{F}_2) \cong A/K(H)$? (See [**Wa5**] for an example of a quasidiagonal A such that $A/K(H)$ has a C*-subalgebra isomorphic to $C^*_r(\mathbb{F}_2)$.)

4. For any C*-algebra B, let Cone(B) be the C*-algebra $C((0,1],B)\check{}$. Voiculescu [**Voi1**] has shown that for any separable C*-algebra B, Cone(B) is quasidiagonal. It is also quite simple to show that if B is exact, then so is Cone(B). Indeed if B is exact, so are $\tilde{B}$ and $C([0,1],\tilde{B}) \cong C([0,1]) \otimes \tilde{B}$. Since Cone$(B)$ is a C*-subalgebra of $C([0,1],\tilde{B})$, it follows that Cone(B) is exact. Now suppose that A, B and C are separable C*-algebras and J is an ideal of A such that $B \cong A/J$. Kirchberg [**Kir6**, Proposition 5.1] has shown that if the sequence

$$0 \longrightarrow J\otimes C \longrightarrow A\otimes C \longrightarrow B\otimes C \longrightarrow 0 \qquad (4.2)$$

is not exact, then there are a finite type I von Neumann algebra $M = \oplus_{i=1}^{\infty} M_{n_i}$ and a C*-subalgebra D of M such that $M_0 \subseteq D$, with M_0 the ideal of zero-sequences in M, such that $\mathrm{Cone}(B) \cong D/M_0$ and such that the sequence

$$0 \longrightarrow M_0 \otimes C \longrightarrow D \otimes C \longrightarrow \mathrm{Cone(B)} \otimes C \longrightarrow 0$$

is not exact. It will follow by theorem 9.1, Corollary 5.6 and Proposition 5.2, that D cannot be exact. There are examples of sequences (4.2) in chapter 3 with exact B (e.g. with $A = C = C^*(G)$, $B = C_r^*(\mathbb{F}_2)$) which are not exact. For such a B, Cone(B) is exact, but the algebra D is an inexact extension of Cone(B) by the nuclear ideal M_0. This shows that an extension of an exact C*-algebra by an exact C*-algebra need not be exact.

5. Property C.

In this chapter we consider property C, a rather subtle property of C*-algebras introduced by Archbold and Batty [**A-B**]. Property C is defined in terms of mappings between the second duals of C*-algebras, the properties of which are recalled briefly in §5.1.

5.1. The second dual of a C*-algebra. Let A be a C*-algebra, and for a state f on A, let $\{\pi_f, H_f, \xi_f\}$ be the cyclic representation of A associated with f by the GNS construction, so that for $a \in A$, $f(a) = (\pi_f(a)\xi_f|\xi_f)$. Let $H_A = \oplus_{f\in S(A)} H_f$, where $S(A)$ denotes the set of states on A, and let π_A be the representation $\oplus_{f\in S(A)}\pi_f$ of A on H_A. The representation π_A is the *universal representation* of A, and is faithful. Let $\overline{\pi_A(A)}$ be the weak closure of $\pi_A(A)$ in $B(H_A)$. Then the von Neumann algebra $\overline{\pi_A(A)}$ is canonically linearly isometric to A^{**}. To see this, let $f \in \overline{\pi_A(A)}_*$. Since $\pi_A(A)$ is weakly dense in $\overline{\pi_A(A)}$, the unit ball of $\pi_A(A)$ is weakly dense in that of $\overline{\pi_A(A)}$ by Kaplansky's density theorem. Thus the functional $\overline{f} = f\pi_A$ has the same norm as f, and the map $\tau : f \to f\pi_A; \overline{\pi_A(A)} \to A^*$ is linear and isometric (and positive). For any state g on A, $g = \omega_{\xi_g}\pi_A$, where ω_{ξ_g} is the vector state defined by ξ_g. Since any element of A^* is a linear combination of states, it follows that $\tau(\overline{\pi_A(A)}_*) = A^*$. The linear map $\tau^* : A^{**} \to \overline{\pi_A(A)}$ is isometric, and if A^{**} is given the (unique) C*-algebra structure making τ^* a *-isomorphism, it is easy to see that, restricted to A under its canonical embedding in A^{**}, this C*-structure is just the C*-structure of A. It is usual to identify A^{**} with $\overline{\pi_A(A)}$, so that A^{**} is a von Neumann algebra and τ^* is the (unique) normal extension of π_A to A^{**}.

If N is a von Neumann algebra with predual N_*, any $f \in N_*$ is an element of N^*. If $\varepsilon : N_* \to N^*$ denotes the embedding map, then $\varepsilon^* : N^{**} \to N$ is a normal contraction. If N is identified with its canonical image in N^{**} and $x \in N$, then for $f \in N_*$, $f(\varepsilon^*(x)) = (\varepsilon(f))(x) = f(x)$, i.e. $\varepsilon^*(x) = x$, and ε^* is a projection of norm 1. Using the fact that the map ε^* is normal and that it satisfies the module property of Tomiyama, since it is a projection of norm 1 (see [**Kye2**]), it is straightforward to show that ε^* is a *-homomorphism with kernel a weakly-closed ideal J of N^{**}. Then $J = eN^{**}$, where e is a central projection in N^{**}, and $N = \varepsilon^*(N^{**}) = (1-e)N^{**}$.

Now let σ be a *-homomorphism of a C*-algebra A into N. Then σ^{**} is a normal *-homomorphism of A^{**} into N^{**}, and $\overline{\sigma} = \varepsilon^*\sigma^{**} : A^{**} \to N$ is a normal *-homomorphism extending σ. If $\sigma(A)$ is weakly dense in N, then $\overline{\sigma}(A^{**}) = N$. The map $\overline{\sigma}$, which is uniquely determined by σ, is the *canonical extension* of σ to A^{**}. If σ is a representation of A on a Hilbert space H, taking N to be the weak closure of $\sigma(A)$ in $B(H)$, the canonical extension $\overline{\sigma}$ of σ is a normal representation of A^{**} on H with image N.

If A is a C*-algebra and J is an ideal of A, then the weak closure $\overline{J}$ of J in A^{**} is a weakly closed ideal, and $\overline{J} = eA^{**}$ for some central projection e of A^{**}. By the Hahn-Banach theorem, $A \cap \overline{J} = J$, and if A is identified canonically with its image in A^{**}, then $J = \{a \in A : ea = a\}$.

5.2. The norm $\| \|_C$. Let A and B be C*-algebras and let π_1 and π_2 be the restrictions of the universal representation $\pi_{A\otimes B} : A \otimes B \to B(H_{A\otimes B})$ to A and B, respectively. Then $\{\pi_1, \pi_2\}$ is a commuting pair of representations of of the pair $\{A, B\}$. The representations π_1 and π_2 extend to normal representations ι_A and ι_B of A^{**} and B^{**}, respectively, and the pair $\{\iota_A, \iota_B\}$ is a commuting pair of normal representations of the pair $\{A^{**}, B^{**}\}$. There is thus a $*$-homomorphism $\pi : A^{**} \otimes_{binor} B^{**} \to (A \otimes B)^{**}$ such that

$$\pi(a \otimes b) = \iota_A(a)\iota_B(b) \quad (a \in A^{**}, b \in B^{**}).$$

If $x \in A^{**} \odot B^{**}$, there are normal states f on A^{**} and g on B^{**}, i.e. $f \in A^*$, $g \in B^*$, such that $(f \otimes g)(x) \neq 0$. The state $f \otimes g$ has a unique normal extension φ to $(A \otimes B)^{**}$ and for $a \in A$ and $b \in B$,

$$\varphi(\iota_A(a)\iota_B(b)) = f(a)g(b).$$

Since the bilinear maps on $A^{**} \times B^{**}$ defined by the two sides are separately normal in each variable, this equality holds for $a \in A^{**}$ and $b \in B^{**}$ also, and so $\varphi(\pi(x)) = (f \otimes g)(x) \neq 0$, which implies that $x \neq 0$. Thus $\pi|_{A^{**}\odot B^{**}}$ is injective. It follows that a C*-norm $\| \|_C$ on $A^{**} \odot B^{**}$ is defined by

$$\|x\|_C = \|\pi(x)\|_{(A\otimes B)^{**}} \quad (x \in A^{**} \odot B^{**}),$$

and $\pi|_{A^{**}\odot B^{**}}$ extends to a $*$-monomorphism $\iota_{A,B} : A^{**} \otimes_C B^{**} \to (A \otimes B)^{**}$. The homomorphism π has a factorisation $\pi = \iota_{A,B}\overline{\pi}$, where $\overline{\pi} : A^{**} \otimes_{binor} B^{**} \to A^{**} \otimes_C B^{**}$ is an extension of the identity map on $A^{**} \odot B^{**}$.

Proposition 5.1. *Let A and B be C^*-algebras such that $\| \ \|_C = \| \ \|_{min}$ on $A^{**} \odot B^{**}$. Then*

(i) for any C^*-subalgebra C of A, $\| \ \|_C = \| \ \|_{min}$ on $C^{**} \odot B^{**}$;

(ii) for any ideal J of A, $\| \ \|_C = \| \ \|_{min}$ on $(A/J)^{**} \odot B^{**}$.

Proof: (i) Since $C \otimes B \subseteq A \otimes B$, it follows that $(C \otimes B)^{**} \subseteq (A \otimes B)^{**}$ canonically, and we have the following diagram of inclusions

$$\begin{array}{ccc} C^{**} \odot B^{**} & \longrightarrow & A^{**} \odot B^{**} \\ {\scriptstyle \iota_{C,B}} \downarrow & & \downarrow {\scriptstyle \iota_{A,B}} \\ (C \otimes B)^{**} & \longrightarrow & (A \otimes B)^{**}. \end{array}$$

To see that this diagram is commutative, let $c \in C^{**}$ and let $\{c_\lambda\}$ be a net in C converging to c weakly. Let $\varphi \in (A \otimes B)^*$ and let $\overline{\varphi} = \varphi|_{C \otimes B}$. Then $\lim_\lambda \overline{\varphi}(c_\lambda \otimes 1) = \overline{\varphi}(\iota_{C,B}(c \otimes 1))$, i.e. $\iota_{C,B}(c_\lambda \otimes 1) \to \iota_{C,B}(c \otimes 1)$ in $(C \otimes B)^{**}$, and, regarding c_λ and c as elements of A^{**}, $\lim_\lambda \varphi(c_\lambda \otimes 1) = \varphi(\iota_{A,B}(c \otimes 1))$, i.e. $\iota_{A,B}(c_\lambda \otimes 1) \to \iota_{A,B}(c \otimes 1)$ in $(A \otimes B)^{**}$. Now since $c_\lambda \in C$, the image of $\iota_{C,B}(c_\lambda \otimes 1)$ in $(A \otimes B)^{**}$ is $\iota_{A,B}(c_\lambda \otimes 1)$. Thus

$$\begin{aligned} \varphi(\iota_{C,B}(c \otimes 1)) &= \overline{\varphi}(\iota_{C,B}(c \otimes 1)) \\ &= \lim_\lambda \overline{\varphi}(\iota_{C,B}(c_\lambda \otimes 1)) \\ &= \lim_\lambda \varphi(\iota_{A,B}(c_\lambda \otimes 1)) \\ &= \varphi(\iota_{A,B}(c \otimes 1)), \end{aligned}$$

i.e. the image of $\iota_{C,B}(c \otimes 1)$ in $(A \otimes B)^{**}$ is $\iota_{A,B}(c \otimes 1)$. Similarly the image of $\iota_{C,B}(1 \otimes b)$ in $(A \otimes B)^{**}$ is $\iota_{A,B}(1 \otimes b)$ for $b \in B^{**}$, and so $\iota_{C,B}(c \otimes b) = \iota_{C,B}(c \otimes 1)\iota_{C,B}(1 \otimes b)$ has image $\iota_{A,B}(c \otimes 1)\iota_{A,B}(1 \otimes b) = \iota_{A,B}(c \otimes b)$ in $(A \otimes B)^{**}$. The commutativity of the diagram now follows by the linearity of the maps involved. When $A^{**} \odot B^{**}$ and $C^{**} \odot B^{**}$ have their respective norms $\| \ \|_C$, the maps $\iota_{A,B}$ and $\iota_{C,B}$ are isometric, as is the lower inclusion. Thus the norm $\| \ \|_C$ on $C^{**} \odot B^{**}$ is just the restriction of $\| \ \|_C$ on $A^{**} \odot B^{**}$. Since, by assumption, the latter norm coincides with $\| \ \|_{min}$, $\| \ \|_C = \| \ \|_{min}$ on $C^{**} \odot B^{**}$.

(ii) The $\sigma(A^{**}, A^*)$-closure $\overline{J}$ of J is a weakly-closed ideal of A^{**}, and so $\overline{J} = (1 - e)A^{**}$ for some central projection $e \in A^{**}$ ($\overline{J} = \ker \sigma^{**}$). Since

$J = A \cap \overline{J}$, letting $\sigma : A \to A/J$ be the quotient map, we have

$$\|\sigma(a)\| = \|a + J\| = \|a + \overline{J}\| = \|ea\|,$$

and the map $\sigma(a) \to ea$ is a well-defined $*$-monomorphism of A/J onto the C*-subalgebra eA of A^{**}. This morphism has a normal extension $\phi : (A/J)^{**} \to eA^{**}$. For $a \in A^{**}$, $(\phi\sigma^{**})(a) = ea$.

Similarly the $*$-homomorphism $\sigma \otimes id : A \otimes B \to (A/J) \otimes B$ has the normal extension $(\sigma \otimes id)^{**} : (A \otimes B)^{**} \to ((A/J) \otimes B)^{**}$. Then $\ker(\sigma \otimes id)^{**} = (1-f)(A \otimes B)^{**}$ for some central projection f in $(A \otimes B)^{**}$, and there is a $*$-isomorphism $\psi : f(A \otimes B)^{**} \to ((A/J) \otimes B)^{**}$ such that $(\sigma \otimes id)^{**}(x) = \psi(fx)$ for $x \in (A \otimes B)^{**}$. Since it is a $*$-isomorphism between von Neumann algebras, the morphism ψ is necessarily normal.

We now show that $f = \iota_A(e) = \iota_{A,B}(e \otimes 1)$. Let $\{a_\alpha\}$ be a net in J with $\sigma(A^{**}, A^*)$-limit $1 - e$, and let $\{b_\beta\}$ be a net in B with $\sigma(B^{**}, B^*)$-limit 1. Then

$$\begin{aligned}(\sigma \otimes id)^{**}(\iota_{A,B}(a_\alpha \otimes b_\beta)) &= (\sigma \otimes id)^{**}(a_\alpha \otimes b_\beta) \\ &= \sigma(a_\alpha) \otimes b_\beta \\ &= 0.\end{aligned}$$

Also

$$\begin{aligned}\lim_\alpha \iota_{A,B}(a_\alpha \otimes b_\beta) &= \lim_\alpha \iota_A(a_\alpha)\iota_B(b_\beta) \\ &= \iota_A(1-e)\iota_B(b_\beta).\end{aligned}$$

Thus

$$\begin{aligned}(\sigma \otimes id)^{**}(\iota_A(1-e)\iota_B(b_\beta)) &= \psi(f\iota_A(1-e)\iota_B(b_\beta)) \\ &= 0,\end{aligned}$$

which implies that $f\iota_A(1-e)\iota_B(b_\beta) = 0$ since ψ is injective. Taking $\lim_\beta$, we have $f\iota_A(1-e) = 0$, and so $f \leq \iota_A(e)$.

Conversely, let $\{x_\lambda\}$ be a net in $\ker(\sigma \otimes id)$ tending weakly to $1 - f$ in $(A \otimes B)^{**}$. Then

$$\|(e \otimes 1)x_\lambda\|_{min} = \|(\sigma \otimes id)x_\lambda\|_{min} = 0,$$

so that $(e \otimes 1)x_\lambda = 0$, and thus $\iota_{A,B}((e \otimes 1)x_\lambda) = \iota_A(e)x_\lambda = 0$. Then

$$\begin{aligned} 0 &= \lim_\lambda \iota_A(e)x_\lambda \\ &= \iota_A(e)(1-f), \end{aligned}$$

i.e. $\iota_A(e) \leq f$.

By hypothesis the map $\iota_{A,B}$ is an embedding of $A^{**} \otimes B^{**}$ in $(A \otimes B)^{**}$. Consider the map θ of $(A/J)^{**} \odot B^{**}$ into $((A/J) \otimes B)^{**}$ given by

$$\theta(x) = \psi\left(f \iota_{A,B}((\phi \otimes id)(x))\right).$$

Since ϕ and ψ are normal, and $\iota_{A,B}$ is binormal, it follows that θ is binormal. Let $c \in A/J$, $b \in B$ and let $a \in A$ be such that $c = \sigma(a) = \sigma^{**}(ea)$. Then

$$\begin{aligned} \theta(c \otimes b) &= \psi(f \iota_{A,B}(\phi(c) \otimes b)) \\ &= \psi(f \iota_{A,B}((\phi\sigma^{**})(ea) \otimes b)) \\ &= \psi(f \iota_{A,B}(ea \otimes b)) \\ &= \psi(f \iota_A(e)(a \otimes b)) \\ &= (\sigma \otimes id)^{**}(a \otimes b) \\ &= \sigma(a) \otimes b \\ &= c \otimes b. \end{aligned}$$

By linearity $\theta(x) = x$ for $x \in (A/J) \odot B$. Since θ is binormal, i.e. the map $(c, b) \to \theta(c \otimes b)$ is separately normal for $c \in (A/J)^{**}$, $b \in B^{**}$, it follows that θ is the restriction of $\iota_{A/J,B}$ to $(A/J)^{**} \odot B^{**}$. For $x \in (A/J)^{**} \odot B^{**}$,

$$\begin{aligned} \|x\|_C &= \|\iota_{A/J,B}(x)\| \\ &= \|\theta(x)\| \\ &\leq \|(\phi \otimes id)(x)\|_{A^{**} \otimes B^{**}} \\ &= \|(\phi \otimes id)(x)\|_{min} \\ &\leq \|x\|_{min}. \end{aligned}$$

Thus $\| \|_C = \| \|_{min}$ on $(A/J)^{**} \odot B^{**}$. □

Proposition 5.2. *Let A and B be C^*-algebras such that $\| \|_C = \| \|_{min}$ on $A^{**} \odot B^{**}$. Then if J is an ideal of B, the sequence*

$$0 \longrightarrow A \otimes J \longrightarrow A \otimes B \longrightarrow A \otimes (B/J) \longrightarrow 0$$

is exact.

Proof: Let e be the central projection in B^{**} such that $\overline{J} = (1-e)B^{**}$. Then if $\sigma : B \to B/J$ is the quotient morphism, $\|\sigma(b)\| = \|eb\|$ $(b \in B)$ and $\|(id \otimes \sigma)(x)\| = \|(1 \otimes e)x\|$ for $x \in A \otimes B \subseteq A \otimes B^{**}$. Thus for $x \in A \otimes B$, $x \in \ker(id \otimes \sigma) \Leftrightarrow (1 \otimes e)x = 0 \Leftrightarrow x = (1 \otimes (1-e))x \Leftrightarrow x \in A \otimes \overline{J}$. Thus $\ker(id \otimes \sigma) = (A \otimes B) \cap (A \otimes \overline{J}) \subseteq A^{**} \otimes B^{**}$.

By assumption the map $\iota_{A,B} : A^{**} \otimes B^{**}$ is a binormal monomorphism. Thus $\iota_{A,B}(A \otimes \overline{J})$ is a subset of the weak closure $\overline{A \otimes J}$ of $A \otimes J$ in $(A \otimes B)^{**}$. Now $(A \otimes B) \cap \overline{(A \otimes J)} = A \otimes J$ (see §5.1), from which it follows that $\ker(id \otimes \sigma) \subseteq A \otimes J$, so that the given sequence is exact. □

5.3. Property C.

Definition 5.3. A C*-algebra A has *property C* if for any C*-algebra B, $\| \ \|_C = \| \ \|_{min}$ on $A^{**} \odot B^{**}$.

Corollary 5.4. *If a C*-algebra A has property C, then it is exact.*

Proof: This follows immediately from Proposition 5.2. □

Proposition 5.5. *Every nuclear C*-algebra has property C.*

Proof: Let B be any C*-algebra. We defined a canonical binormal *-homomorphism $\pi : A^{**} \otimes_{binor} B^{**} \to (A \otimes B)^{**}$ in §5.2 which has a factorisation $\pi = \iota_{A,B}\overline{\pi}$. Now if A is nuclear, then A^{**} is injective, and so $\| \ \|_{binor} = \| \ \|_{min}$ on $A^{**} \odot B^{**}$ (cf. chapter 1). This implies that $\overline{\pi}$ and $\iota_{A,B}$ are isometric. Thus $\| \ \|_C = \| \ \|_{min}$ on $A^{**} \odot B^{**}$. □

Corollary 5.6. *Every C*-algebra A which is the quotient of a C*-subalgebra of a nuclear C*-algebra is exact.*

Proof: Let B be a nuclear C*-algebra, and let G be a C*-subalgebra of B with ideal J such that $A \cong G/J$. Then B has property C, by Proposition 5.5, so that G and G/J have property C by Proposition 5.1. By Corollary 5.4, $A \cong G/J$ is exact. □

We shall see in chapter 9 that every separable exact C*-algebra is a quotient of a C*-subalgebra of a nuclear C*-algebra, and that the converse of Corollaries 5.4 and 5.6 hold in the separable case.

6. Completely positive liftings.

In this chapter we prove the lifting theorems of Choi and Effros [**C-E4**] and Effros and Haagerup [**E-H**]. For the former we use the elegant approach of Arveson [**Arv**], while for the latter we follow the original treatment of Effros and Haagerup, which also uses Arveson's technique. One of the main technical ideas underlying Arveson's methos is that of a central approximate unit.

6.1. Central approximate units.

Definition 6.1. Let J be an ideal in a C*-algebra B. A net $\{e_\lambda\}_{\lambda\in\Lambda}$ is a *central approximate unit* in J if $e_\lambda \in J$ with $0 \le e_\lambda \le 1$ $(\lambda \in \Lambda)$, $\{e_\lambda\}$ is an approximate unit for J, i.e. $\lim \|e_\lambda x - x\| = 0$ $(x \in J)$, and $\lim_\lambda \|e_\lambda b - b e_\lambda\| = 0$ $(b \in B)$.

Theorem 6.2. [**Arv**, Theorem 1] *For any B and ideal J of B, there is a central approximate unit in J. If B is separable the approximate unit can be taken to be a sequence $e_1 \le e_2 \le \ldots$.*

Lemma 6.3. *With B, J and $\{e_\lambda\}_{\lambda\in\Lambda}$ as above,*

$$\|b + J\| = \lim_\lambda \|b(1 - e_\lambda)\| = \lim_\lambda \|(1 - e_\lambda)^{\frac{1}{2}} b (1 - e_\lambda)^{\frac{1}{2}}\|.$$

The first equality is proved by the method of the proof of Lemma 1.10. For the second, see [**Arv**, proof of Lemma 3.1].

6.2. Lifting problems. Given a unital C*-algebra B, an operator system E, an ideal K of B and a u.c.p. map $\phi : E \to B/K$, the lifting problem for ϕ is to find a u.c.p. map $\psi : E \to B$ such that $\phi = \pi\psi$, where $\pi : B \to B/K$ is the quotient map. The map ψ is a *lifting* of ϕ. For a given ϕ such a ψ need not exist (cf. Remark 3.15(1)), but if it does ϕ is said to be *liftable*.

Suppose that E is separable, and let $\{a_1, a_2, \ldots\}$ be a sequence dense in the unit ball of E. For u.c.p. maps $\phi, \psi : E \to B$ let

$$d_B(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \|\phi(a_n) - \psi(a_n)\|.$$

Then d_B is a metric on the set of u.c.p. maps from E to B, and $d_B(\phi_n, \phi) \to 0$ as $n \to \infty$ if and only if $\|\phi_n(a) - \phi(a)\| \to 0$ as $n \to \infty$ for $a \in E$. It is immediate from the definition that

$$d_{B/K}(\pi\phi, \pi\psi) \leq d_B(\phi, \psi)$$

for u.c.p. maps $\phi, \psi : E \to B$, and

$$d_{B/K}(\pi\phi, \pi\psi) = \inf\{d_B(\phi, \psi') : \psi' : A \to B \text{ u.c.p.}, \pi\psi' = \pi\psi\}$$

(see [**Arv**, Lemma 3.1]). The following result shows that the set of liftable u.c.p. map from E to B/K is closed in the $d_{B/K}$- topology.

Proposition 6.4. [**Arv**, Theorem 6] *If $\phi_n : E \to B/K$ are liftable u.c.p. maps for $n = 1, 2, \ldots$, and $\phi : E \to B/K$ is a u.c.p. map such that $d_{B/K}(\phi_n, \phi) \to 0$ as $n \to \infty$, then ϕ is liftable.*

Proof: We can assume that $d_{B/K}(\phi_n, \phi) < 2^{-(n+1)}$ $(n \geq 1)$, and construct inductively u.c.p. maps $\psi_n : E \to B$ such that $\phi_n = \pi\psi_n$ and $d(\psi_n, \psi_{n+1}) < 2^{-n}$. Let ψ_1 be any u.c.p. lifting of ϕ_1. If $\psi_1, ..., \psi_n$ have been defined, let θ be a u.c.p. lifting of ϕ_{n+1}. Since

$$\inf\{d_B(\psi_n, \theta') : \pi\theta' = \phi_{n+1}\} = d_{B/K}(\pi\psi_n, \pi\theta) = d_{B/K}(\phi_n, \phi_{n+1}) < 2^{-n},$$

there is a $\phi' : E \to B$ such that $\pi\phi' = \phi_{n+1}$ and $d(\psi_n, \phi') < 2^{-n}$. Let $\psi_{n+1} = \phi'$.

Now for $a \in E$, $\{\psi_n(a)\}$ is Cauchy sequence, which has a limit $\psi(a)$. The map $\psi : A \to B$ is u.c.p. and $\pi\psi = \phi$. □

Lemma 6.5. *Let $\phi : M_n \to B/K$ be completely positive and unital (respectively contractive). Then there is a unital (respectively contractive) completely positive lifting $\psi : M_n \to B$ of ϕ.*

Proof: Let $\{e_{ij}\}$ be the standard set of matrix units in M_n, and let

$$p = (1/n)\sum_{i,j} e_{ij} \otimes e_{ij} \in M_n \otimes M_n.$$

Then $p = p^* = p^2$, so that p is positive, and

$$(\phi \otimes id)(p) = (1/n)\sum_{i,j} \phi(e_{ij}) \otimes e_{ij} \geq 0$$

in $(B/K)\otimes M_n$. Since $(B/K)\otimes M_n \cong (B\otimes M_n)/(K\otimes M_n)$, there is a positive element $\sum b_{ij}\otimes e_{ij}$ in $B\otimes M_n$ such that

$$\sum \phi(e_{ij})\otimes e_{ij} = \sum \pi(b_{ij})\otimes e_{ij}.$$

Let $\theta : M_n \to B$ be the map $[\lambda_{ij}] \to \sum \lambda_{ij} b_{ij}$. Then $\phi = \pi\theta$. Since

$$\sum \theta(e_{ij})\otimes e_{ij} = \sum b_{ij}\otimes e_{ij} \geq 0,$$

θ is completely positive, by Lemma 1.4. Since $\phi(1) \geq 0$ and $\|\phi(1)\| \leq 1$, there is a $b \in B$ such that $b \geq 0, \|b\| \leq 1$ and $\pi(b) = \phi(1)$ (see Remark 8.6 (1)). Then $\theta(1) - b \in K$, so that $\theta(1) = b + k$ where $k^* = k \in K$. There are positive elements k_1, k_2 in K such that $k = k_1 - k_2$. If f is a state on M_n, let

$$\psi(x) = (1+k_1)^{-\frac{1}{2}}(\theta(x) + f(x)k_2)(1+k_1)^{-\frac{1}{2}}.$$

Then $\psi : M_n \to B$ is completely positive, $\pi\psi = \phi$ and

$$\psi(1) = (1+k_1)^{-\frac{1}{2}}(b+k_1)(1+k_1)^{-\frac{1}{2}},$$

so that $\|\psi(1)\| \leq 1$, and ψ is contractive. If $\phi(1) = 1$, we can take $b = 1$, so that then $\psi(1) = 1$. □

6.3. The Choi-Effros Lifting theorem.

Theorem 6.6 [**C-E4**, Theorem 3.10]. *Let B be a unital C*-algebra, K an ideal of B and E a separable operator system. If $\phi : E \to B/K$ is a nuclear u.c.p. map, then ϕ has a u.c.p. lifting $\psi : E \to B$.*

Proof: (cf. [**Arv**, Theorem 7]). Since ϕ is nuclear, there are sequences of natural numbers $\{n_r\}$, and u.c.p. maps $\psi_r : E \to M_{n_r}$, $\phi_r : M_{n_r} \to B/K$ such that $\lim_{r\to\infty} d_{B/K}(\phi_r\psi_r, \phi) = 0$. Now by Lemma 6.5, ϕ_r has a u.c.p. lifting $\theta_r : M_{n_r} \to B$. Then $\theta_r\psi_r$ is a u.c.p. lifting of $\phi_r\psi_r$, so that $\phi_r\psi_r$ is liftable. Since ϕ is the $d_{B/K}$-limit of liftable maps, it is itself liftable, by Proposition 6.4. □

Remark 6.7. A unital C*-algebra A has the *lifting property* (LP) if for any unital C*-algebra B and ideal K of B, every u.c.p. map $\phi : A \to B/K$ has a u.c.p. lifting $\psi : A \to B$. By Theorem 6.6, every separable unital nuclear C*-algebra has the LP. If F is a free group on finitely or countably many generators then the algebra $C^*(F)$ has the LP [**C-E5**, Lemma 4.4], [**Kir6**, Lemma 2.1].

6.4. The Effros-Haagerup lifting theorem.

Proposition 6.8. *Let A and B be unital C^*-algebras such that $A = B/J$ for some ideal J in A. Then the following conditions are equivalent.*

(i) For every finite dimensional operator system E in A, $id_E : E \to A$ is liftable.

(ii) For every C^*-algebra C, the sequence

$$0 \longrightarrow J \otimes C \longrightarrow B \otimes C \longrightarrow A \otimes C \longrightarrow 0 \tag{6.1}$$

is exact.

Proof: (i) $\Rightarrow$ (ii): For some C*-norm $\| \|_\nu$ on $A \odot C$, $(B \otimes C)/(J \otimes C) \cong A \otimes_\nu C$. Let $x = \sum a_i \otimes b_i \in A \odot C$. Then there is a finite-dimensional operator system E in A containing all the a_i, and by assumption there is a u.c.p. lifting $\phi : E \to B$ of $id_E : E \to A$. Then

$$\|\sum \phi(a_i) \otimes b_i\|_{\min} \leq \|\sum a_i \otimes b_i\|_{\min}.$$

Also

$$\|\sum a_i \otimes b_i\|_\nu = \|\sum \phi(a_i) \otimes b_i + J \otimes C\| \leq \|\sum \phi(a_i) \otimes b_i\|_{\min}$$

i.e. $\|x\|_\nu \leq \|x\|$. Thus $\| \|_\nu = \| \|_{\min}$ on $A \odot C$, i.e. (6.1) is exact.

(ii) $\Rightarrow$ (i): Let E be a finite-dimensional operator system in A. Since E is finite dimensional, there is a linear lifting $\psi : E \to B$ of id_E. Replacing ψ by $\frac{1}{2}(\psi + \psi^*)$, we can assume that $\psi(e^*) = \psi(e)^*$ for $e \in E$. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a central approximate unit in J. If $\pi : B \to B/J$ is the quotient map, then $\pi(1 - e_\lambda) = 1$, so that $\pi((1 - e_\lambda)^{\frac{1}{2}}) = 1$. Thus the map $\psi_\lambda : E \to B$ defined by

$$\psi_\lambda(x) = (1 - e_\lambda)^{\frac{1}{2}} \psi(x) (1 - e_\lambda)^{\frac{1}{2}}$$

is a lifting of id_E for each $\lambda \in \Lambda$. Since ψ_λ is a lifting, $\|\psi_\lambda\|_{cb} \geq 1$. It is a consequence of the hypothesis that, in fact,

$$\lim_\lambda \|\psi_\lambda\|_{cb} = 1. \tag{6.2}$$

To see this, suppose, to the contrary, that $\limsup_\lambda \|\psi_\lambda\| > 1$. Then replacing Λ by a subnet, we can assume that for some $\varepsilon > 0$

$$\|\psi_\lambda\|_{cb} \geq 1 + \varepsilon$$

for $\lambda \in \Lambda$. So for each λ we may choose $n_\lambda \in \mathbb{N}$ and $x_\lambda \in E \otimes M_{n_\lambda}$ such that $\|x_\lambda\| \leq 1$ and

$$\|(\psi_\lambda \otimes id_{n_\lambda})(x_\lambda)\| \geq 1 + \frac{\varepsilon}{2}. \tag{6.3}$$

Now let $C = \oplus_\lambda M_{n_\lambda}$. Then $(x_\lambda) \in \oplus_\lambda(E \otimes M_{n_\lambda}) \cong E \otimes (\oplus_\lambda M_{n_\lambda}) = E \otimes C$ and for $\nu \in \Lambda, \|(\psi_\nu \otimes id_C)(x)\| \geq 1 + (\varepsilon/2)$. Also

$$\begin{aligned}(\psi_\lambda \otimes id_C)(x) &= ((\psi_\nu \otimes id_{n_\lambda}(x_\lambda))_{\lambda\in\Lambda} \\ &= ((1-e_\nu)^{\frac{1}{2}}\psi(1-e_\nu)^{\frac{1}{2}} \otimes id_{n_\lambda})(x_\lambda)_{\lambda\in\Lambda} \\ &= ((1-e_\nu)^{\frac{1}{2}} \otimes 1)((\psi \otimes id_C)(x))((1-e_\nu)^{\frac{1}{2}} \otimes 1).\end{aligned}$$

By Lemma 6.3, it follows that

$$\begin{aligned}\lim_\nu \|(\psi_\nu \otimes id_C)(x)\| &= \lim_\nu \|(\psi \otimes id_C)(x)((1-e_\nu)\otimes 1)\| \\ &= \|(\psi \otimes id_C)(x) + J \otimes C\| \\ &= \|(\pi \otimes id_C)(\psi \otimes id_C)(x)\| \\ &\qquad \text{(by the exactness assumption)} \\ &= \|x\|,\end{aligned}$$

where we have used the fact that $(e_\nu \otimes 1)_{\nu\in\Lambda}$ is a central approximate unit in the ideal $J \otimes C$ of $B \otimes C$. However this contradicts (6.3), since by (6.3),

$$\|(\psi_\nu \otimes id_C)(x)\| \geq \|(\psi_\nu \otimes id_{n_\nu})(x_\nu)\| \geq \frac{\varepsilon}{2}.$$

So (6.2) holds.

Let $\varepsilon > 0$ and let ω be a state on E. Replacing ψ by some ψ_λ, we can assume that $\|\psi\|_{cb} \leq 1 + \varepsilon$. Let

$$\psi'_\lambda(x) = (1-e_\lambda)^{\frac{1}{2}}\psi(x)(1-e_\lambda)^{\frac{1}{2}} + \omega(x)e_\lambda \quad (x \in E).$$

Then ψ'_λ is a lifting of id_E and

$$\|\psi'_\lambda\|_{cb} \leq \|\psi\|_{cb}$$

for $\lambda \in \Lambda$. Moreover

$$\begin{aligned}1 - \psi'_\lambda(1) &= (1-e_\lambda)^{\frac{1}{2}}(1-\psi(1))(1-e_\lambda)^{\frac{1}{2}} \\ &\to 0,\end{aligned}$$

since $1-\psi(1)\in J$. It follows that for some $\lambda\in\Lambda$, $\|1-\psi'_\lambda(1)\|<\varepsilon$, and

$$\psi''(x)=\psi'_\lambda(x)+(1-\psi'_\lambda(1))\omega(x)$$

defines a lifting ψ'' of id_E such that $\psi''(1)=1$ and $\|\psi''\|_{cb}\leq 1+2\varepsilon$. By Lemma 1.18 there is a u.c.p. map $\theta: E\to B$ such that $\|\theta-\psi''\|_{cb}\leq 8\varepsilon\dim E$. The u.c.p. map $\pi\theta: E\to B/J$ is, trivially, liftable and $\|\pi\theta-id_E\|_{cb}\leq 8\varepsilon\dim E$. Since ε is arbitrary, id_E is the limit of liftable maps in the sense of Proposition 6.4. By that proposition, id_E is liftable. □

Remark 6.9. Let A, B and $J\lhd B$ be as in statement (i) of Proposition 6.8. Then the sequence (6.1) is exact for a given C if and only if for any element $\sum_1^n b_i\otimes c_i\in B\odot C$,

$$\|\sum_1^n\pi(b_i)\otimes c_i\|=\|\sum_1^n(b_i\otimes c_i)+J\otimes C\|, \tag{6.4}$$

where $\pi: B\to A$ is the quotient map. If X is a finite dimensional operator system in C containing $c_1,\ldots,c_n$, then

$$\|\sum_1^n(b_i\otimes c_i)+J\otimes C\|=\|\sum_1^n(b_i\otimes c_i)+J\otimes X\|,$$

by Lemma 1.10, so that (6.4) holds for all $\sum b_i\otimes c_i\in B\odot X$ if and only if the map $T_X:(B\otimes X)/(J\otimes X)\to A\otimes X$ is isometric. Thus (6.1) is exact if and only if T_X is isometric for every finite-dimensional operator system X in C. Let $M=\oplus_{n=1}^\infty M_n$. If $H=\ell^2_{\mathbb{N}}$, with orthonormal basis $\xi_1,\xi_2,\ldots$, and P_k is the projection onto the finite-dimensional subspace spanned by the first k of the basis members, then the map $\iota: B(H)\to M; T\to(P_iTP_i)$ is a u.c.p. complete isometry. The operator system X is contained in some C*-algebra, and the C*-subalgebra C that it generates is separable. Thus we have unital u.c.p. completely isometric embeddings $X\subseteq C\subseteq B(H)\subseteq M$, where the final embedding is that defined by ι. It follows that conditions (i) and (ii) of Proposition 6.8 are equivalent to the further conditions

(iii) For every separable C-algebra C, the sequence (6.1) is exact.*

(iv) The sequence (6.1) is exact when $C=M$.

(v) The sequence (6.1) is exact when $C=B(H)$.

Lemma 6.10. *Let B be a C^*-algebra with a nuclear ideal J such that the sequence*

$$0 \longrightarrow J \otimes C \longrightarrow B \otimes C \longrightarrow (B/J) \otimes C \longrightarrow 0$$

is exact for any C^*-algebra C. Then if $E_1 \subseteq E_2$ are finite-dimensional operator systems in B/J and $\psi_1 : E_1 \to B$ is a u.c.p. lifting of id_{E_1}, given $\varepsilon > 0$, there is a u.c.p. lifting ψ_2 of id_{E_2} such that $\|\psi_2|_{E_1} - \psi_1\| < \varepsilon$.

Proof: Let ψ be a u.c.p. lifting of id_{E_2} (ψ exists by Proposition 6.8). For $x \in E_1$, $\psi_1(x) - \psi(x) \in J$, and so

$$\lim_\lambda \|(1 - e_\lambda)^{\frac{1}{2}}(\psi_1(x) - \psi(x))(1 - e_\lambda)^{\frac{1}{2}}\| = 0$$

and

$$\lim_\lambda \|\psi_1(x) - (1 - e_\lambda)^{\frac{1}{2}}\psi_1(x)(1 - e_\lambda)^{\frac{1}{2}} - e_\lambda^{\frac{1}{2}}\psi_1(x)e_\lambda^{\frac{1}{2}}\| = 0.$$

Thus

$$\lim_\lambda \|\psi_1(x) - (1 - e_\lambda)^{\frac{1}{2}}\psi(x)(1 - e_\lambda)^{\frac{1}{2}} - e_\lambda^{\frac{1}{2}}\psi_1(x)e_\lambda^{\frac{1}{2}}\| = 0.$$

Let $\delta < \min\{\varepsilon/5, 1\}$. Then with $e = e_\lambda$ for a suitable λ,

$$\|\psi_1(x) - (1 - e)^{\frac{1}{2}}\psi(x)(1 - e)^{\frac{1}{2}} - e^{\frac{1}{2}}\psi_1(x)e^{\frac{1}{2}}\| < \delta\|x\| \quad (x \in E_1),$$

where we have used the fact that E_1 is finite-dimensional, so that the point-norm and operator norm topologies on the space of contractive linear maps from E_1 to B coincide.

Now the map $x \to e^{\frac{1}{2}}\psi_1(x)e^{\frac{1}{2}}; E_1 \to J$ is completely positive. Since J is nuclear, there are a matrix algebra M_n, a u.c.p. map $\rho : E_1 \to M_n$ and a contractive c.p. map $\phi : M_n \to J$ such that $\|\phi\rho - e^{\frac{1}{2}}\psi_1 e^{\frac{1}{2}}\| \leq \delta$. Since M_n is injective, ρ has a u.c.p. extension $\overline{\rho} : E_2 \to M_n$. If $\theta = \phi\overline{\rho} : E_2 \to J$, then

$$\|\theta(x) - e^{\frac{1}{2}}\psi_1(x)e^{\frac{1}{2}}\| \leq \delta\|x\| \quad (x \in E_1)$$

and $\|\theta(1) - e\| \leq \delta$. Let

$$\psi'(x) = (1 - e)^{\frac{1}{2}}\psi(x)(1 - e)^{\frac{1}{2}} + \theta(x) \quad (x \in E_2).$$

Then ψ' is a c.p. lifting of id_{E_2},

$$\begin{aligned} \|\psi_1(x) - \psi'(x)\| &\leq \|\psi_1(x) - (1 - e)^{\frac{1}{2}}\psi(x)(1 - e)^{\frac{1}{2}} - e^{\frac{1}{2}}\psi_1(x)e^{\frac{1}{2}}\| \\ &\quad + \|e^{\frac{1}{2}}\psi_1(x)e^{\frac{1}{2}} - \theta(x)\| \\ &\leq 2\delta\|x\| \end{aligned}$$

for $x \in E_1$, and

$$\|\psi'(1) - 1\| = \|(1 - e) + \theta(1) - 1\| \leq \delta < 1.$$

Thus $b = \psi'(1)$ is invertible, and if $\psi_2 = b^{-\frac{1}{2}}\psi' b^{-\frac{1}{2}}$, then $\|\psi_2 - \psi'\| \leq 3\delta$. Since $\pi(b) = 1$, ψ_2 is a lifting of id_{E_2}, and

$$\begin{aligned}\|\psi_2(x) - \psi_1(x)\| &\leq 3\delta\|x\| + \|\psi'(x) - \psi_1(x)\| \\ &\leq 5\delta\|x\| \\ &\leq \varepsilon\|x\|\end{aligned}$$

for $x \in E_1$, so that $\|\psi_2|_{E_1} - \psi_1\| \leq \varepsilon$. □

Theorem 6.10. *Let B be a unital C^*-algebra and let J be a nuclear ideal in B such that for any C^*-algebra C the sequence*

$$0 \longrightarrow J \otimes C \longrightarrow B \otimes C \longrightarrow (B/J) \otimes C \longrightarrow 0$$

is exact. Then for any separable operator system $E \subseteq B/J$, id_E has a u.c.p. lifting $\psi : E \to B$.

Proof: Let $\{a_1, a_2, \ldots\}$ be a dense subset of E. There are finite dimensional operator systems $E_1 \subseteq E_2 \subseteq \ldots$ in E such that $\{a_1, \ldots, a_n\} \subseteq E_n$ for $n = 1, 2, \ldots$. By induction using the previous lemma, there are u.c.p. maps $\psi_n : E_n \to B$ such that (i) ψ_n is a lifting of id_{E_n} and (ii) $\|\psi_{n+1}|_{E_n} - \psi_n\| \leq 2^{-n}$. For $x \in E_n$, $\{\psi_n(x), \psi_{n+1}(x), \ldots\}$ is a Cauchy sequence which converges to some element $\psi(x) \in B$. Then $\psi : \bigcup_{n=1}^{\infty} E_n \to B$ is a u.c.p. lifting of $id_E|_{\bigcup E_n}$. By linearity and continuity, ψ extends to a u.c.p. lifting of id_E. □

7. Nuclear embeddability and exactness.

7.1. Let A be a nuclear C*-algebra. We saw in chapter 2 that any C*-subalgebra D of A is exact. Moreover, since A is nuclear, the embedding map $\iota : D \to A$ is nuclear as a map into A. Let $\pi : D \to B(H)$ be a faithful representation of D on a Hilbert space S. Then by Arveson's extension theorem (Theorem 1.8) there is a completely positive contractive extension $\overline{\pi} : A \to B(H)$ of π. Since $\pi = \overline{\pi}\iota$, and ι is a nuclear map, so is π. This motivates

Definition 7.1. A C*-algebra D is *nuclearly embeddable* if for some C*-algebra A there is an embedding ι of D as a C*-subalgebra of A with ι a nuclear map.

Using the technique above, the following properties of nuclear embeddability are easily established:

1. D is nuclearly embeddable if and only if D has a nuclear embedding as a C*-subalgebra of $B(H)$ for some Hilbert space.

2. D is nuclearly embeddable if and only if there is a C*-algebra A and a completely positive complete isometry $\theta : D \to A$ with θ nuclear.

An obvious question that arises is the relationship between nuclear embeddability and exactness. Our main purpose in this chapter is to show that these properties are equivalent for any C*-algebra. The easier implication to prove is

Proposition 7.2. *If a C*-algebra D is nuclearly embeddable, then D is exact.*

Proof: Let $\iota : D \to A$ be a nuclear embedding of D in a unital C*-algebra A. Let B be any C*-algebra and let J be an ideal of B. If x is any element of the kernel of the morphism $id \otimes \pi : D \otimes B \to D \otimes (B/J)$, where $\pi : B \to B/J$ is the quotient map, $R_\varphi(x) \in J$ for $\varphi \in D^*$, by the observations preceding Proposition 2.6. Since ι is nuclear, given $\varepsilon > 0$ there are $n \in \mathbb{N}$, and completely positive contractions $\psi : D \to M_n$, $\phi : M_n \to A$ such that $\|(\phi\psi \otimes id)(x) - (\iota \otimes id)(x)\| < \varepsilon$. For $f \in M_n^*$, $f\psi \in D^*$ and $R_{f\psi} = R_f(\psi \otimes id)$. Thus $R_f((\psi \otimes id)(x)) \in J$ for $f \in M_n^*$. Since M_n is finite dimensional, $M_n \otimes B = M_n \odot B$, and it is easy to see that

$(\psi \otimes id)(x) \in M_n \otimes J$. It follows that $(\phi\psi \otimes id)(x) \in A \otimes J$. Since ε is arbitrary, it follows that $(\iota \otimes id)(x) \in A \otimes J$. Let $\{e_\lambda\}$ be an approximate unit for J. Then $\lim_\lambda(\iota \otimes id)(x)(1 \otimes e_\lambda) = (\iota \otimes id)(x)$ and, since $(\iota \otimes id)(x)(1 \otimes e_\lambda) \in \iota(D) \otimes J$ for each λ, $(\iota \otimes id)(x) \in \iota(D) \otimes J$. Thus $x \in D \otimes J$, $\ker(id \otimes \pi) = D \otimes J$, and D is exact. □

7.2. The main result of this chapter is the following converse of Proposition 7.2.

Theorem 7.3. [**Kir4**, Theorem 4.1] *If a C*-algebra A is exact, then A is nuclearly embeddable.*

To prove this theorem following Kirchberg's argument, we need some preliminary results. If A is exact, so is $\tilde{A}$, and so we can assume that A is a unital C*-subalgebra of $B(H)$ for some Hilbert space H. Nuclear embeddability means that for any finite subset $\mathcal{F} = \{x_1, \ldots, x_n\}$ of A and $\varepsilon > 0$, there are $m \in \mathbb{N}$, and u.c.p. maps $\psi : A \to M_n, \varphi : M_n \to B(H)$ such that $\|\phi(\psi(x_i)) - x_i\| < \varepsilon$ for $i = 1, ..., m$. Adding elements to $\mathcal{F}$ if necessary, we can assume that $X = \mathrm{span}(\mathcal{F})$ is a finite-dimensional operator system in A. Since all norms on a finite dimensional vector space are equivalent, it follows that $\|(\phi\psi)|_X - id_X\| < C\varepsilon$, where C is a positive constant depending on X but not on ε. If, conversely, $\psi : X \to M_m$ and $\phi : M_m \to B(H)$ are u.c.p. maps such that $\|\phi\psi - id_X\| < \varepsilon$, ϕ has a u.c.p. extension $\overline{\psi}$ to A, by Arveson's extension theorem, and $\|\phi(\overline{\psi}(x_i)) - x_i\| < \varepsilon\|x_i\|$ for $i = 1, 2, ..., n$. To show that A is nuclearly embeddable, it is thus sufficient to show that for any finite-dimensional operator system X in A and $\varepsilon > 0$, there are $n \in \mathbb{N}$ and u.c.p. maps $\psi : X \to M_n, \phi : M_n \to B(H)$ such that

$$\|\phi\psi - id_X\| < \varepsilon. \tag{7.1}$$

Fix such an X and let

$$\mathrm{fin}(X) = \inf \|\phi\psi - id_X\|,$$

where the infimum is over all $n \in \mathbb{N}$ and all u.c.p. maps $\phi : X \to M_n$, $\psi : M_n \to B(H)$. We need to show that $\mathrm{fin}(X) = 0$.

Let B be an arbitrary C*-algebra, and let J be an ideal of B. Exactness of A is equivalent to saying that the canonical morphism

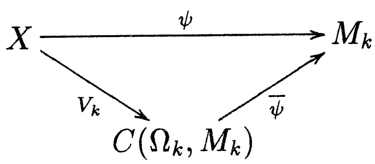
$$(A \otimes B)/(A \otimes J) \to A \otimes (B/J)$$

is an isometry (see 2.5.2). By Lemma 1.10, $\mathrm{dist}(x, A \otimes J) = \mathrm{dist}(x, X \otimes J)$ for $x \in X \odot B$, so that the image of $X \odot B$ in $(A \otimes B)/(A \otimes J)$ is naturally isometric to $(X \otimes B)/(X \otimes J)$. Thus the canonical map

$$T_X : (X \otimes B)/(X \otimes J) \to X \otimes (B/J)$$

is an isometry for any B and $J \triangleleft B$. We shall deduce the existence of u.c.p. maps ϕ and ψ for which (7.1) holds from this fact.

7.3. Let X be a finite-dimensional operator system in $B(H)$ and for $k = 1, 2, \ldots$, let $\Omega_k = \Omega(X, k)$ be the set of u.c.p. maps from X into M_k with the point-weak topology. Ω_k is compact and a u.c.p. map $V_k : X \to C(\Omega_k, M_k)$ is defined by $V_k(x) = f_x$ where, for $x \in X$, $f_x(\phi) = \phi(x)$ $(\phi \in \Omega_k)$. If $\psi : X \to M_k$ is any u.c.p. map, let $\overline{\psi}$ be the evaluation map $f \to f(\psi)$ $(f \in C(\Omega_k, M_k))$. Then $\psi = \overline{\psi} V_k$, i.e. ψ has the factorisation



If $k \geq 2$, V_k is isometric, as the following argument shows. Given $x \in X$ and $\varepsilon > 0$, there are unit vectors $\xi, \eta \in H$ such that $|(x\xi|\eta)| \geq \|x\| - \varepsilon$. If e is the projection onto the subspace of H spanned by ξ and η, the map

$$\phi_e : X \to B(eH); x \to exe|_{eH}$$

is u.c.p and $\|\varphi_e(x)\| \geq \|x\| - \varepsilon$. Thus $\|f_x\| \geq \|x\| - \varepsilon$ for arbitrary $\varepsilon > 0$, so that $\|f_x\| = \|x\|$. A similar argument shows that V_k is in fact $[k/2]$-isometric. V_1 is injective, but in general not isometric.

Lemma 7.4. *For any C^*-algebra B and $x \in X \otimes B$,*

$$\|x\| = \sup_k \|(V_k \otimes id_B)(x)\|.$$

Proof: Since V_k is completely contractive,

$$\|(V_k \otimes id_B)(x)\| \leq \|x\| \quad (k \in \mathbb{N}).$$

Given $\varepsilon > 0$, there is a finite-rank projection $e \in B(H)$ such that

$$\|(e \otimes 1)x(e \otimes 1)\| \geq \|x\| - \varepsilon.$$

If $\phi : X \to B(eH)$ is the map $x \to exe|_{eH}$, then $\phi = \overline{\phi}V_k$, where $k = \dim(eH)$ and $\overline{\phi}$ is the evaluation map corresponding to ϕ. Thus

$$\begin{aligned} \|(V_k \otimes id_B)(x)\| &\geq \|(\overline{\phi} \otimes id_B)(V_k \otimes id_B)(x)\| \\ &= \|(e \otimes 1)x(e \otimes 1)\| \\ &\geq \|x\| - \varepsilon. \end{aligned}$$

Since ε is arbitrary, the result follows. □

Now let $X_k = V_k(X) \subseteq C(\Omega_k, M_k)$ for $k = 1, 2, \ldots$. Since the map V_k is injective, the inverse map $W_k = V_k^{-1} : X_k \to X$ exists. The map W_k is self-adjoint and completely bounded, for $k = 1, 2, \ldots$. (An elementary argument shows that in fact $\|W_k\|_{cb} \leq \dim(X) < \infty$ for $k = 1, 2, \ldots$.)

Lemma 7.5. *For $k = 1, 2, \ldots, \|W_{k+1}\|_{cb} \leq \|W_k\|_{cb}$.*

Proof: Fix k, let e be a fixed rank-k projection in M_{k+1}, and let $\overline{e}$ be the function in $C(\Omega_{k+1}, M_{k+1})$ taking constant value e. For $\phi \in \Omega_{k+1}$ let ϕ_e be the map $x \to e\phi(x)e; X \to eM_{k+1}e \cong M_k$. Identifying $eM_{k+1}e$ and M_k, $\phi_e \in \Omega_k$.

For $f \in C(\Omega_k, M_k)$ let $\overline{f} : \Omega_{k+1} \to M_{k+1}$ be given by

$$\overline{f}(\phi) = f(\phi_e).$$

Then $\overline{f} \in C(\Omega_{k+1}, M_{k+1})$ since the map $\phi \to \phi_e; \Omega_{k+1} \to \Omega_k$ is continuous, and the map $\theta : f \to \overline{f}; C(\Omega_k, M_k) \to \overline{e}C(\Omega_{k+1}, M_{k+1})\overline{e}$ is a $*$-homomorphism. Since any $\psi \in \Omega_k$ is of the form ϕ_e, with ϕ given by

$$\phi(x) = \begin{bmatrix} \psi(x) & 0 \\ 0 & \omega(x) \end{bmatrix},$$

where ω is a state on X, θ is injective. Moreover if $x \in X$, then $(\theta(f_x))(\phi) = f_x(\phi_e) = \phi_e(x) = e\phi(x)e = ef_x(\phi)e$, so that $\theta(V_k(x)) = \overline{e}V_{k+1}(x)$.

Thus $\theta(X_k) = \overline{e}X_{k+1}$, and a u.c.p map $V_{k,k+1} : X_{k+1} \to X_k$ is defined by $V_{k,k+1}(x) = \theta^{-1}(\overline{e}x\overline{e})$. Clearly $V_{k,k+1}V_{k+1} = V_k$, so that $W_kV_{k,k+1} = W_{k+1}$, from which the required inequality follows. □

Proposition 7.6. *If X is a finite-dimensional operator system in an exact C^*-algebra A, then* $\lim_{k\to\infty} \|W_k\|_{cb} = 1$.

Proof: For $k \in \mathbb{N}$ there are $n_k \in \mathbb{N}$ and $y_k \in X_k \otimes M_{n_k}$ such that $\|y_k\| = 1$ and

$$\|W_k\|_{cb} \geq \|(W_k \otimes id_{n_k})(y_k)\| \geq \|W_k\|_{cb} - \frac{1}{k}.$$

Let $h_k = (W_k \otimes id_{n_k})(y_k) \in X \otimes M_{n_k}$, $B = \oplus_{k=1}^{\infty} M_{n_k}$ and $J = \{(x_1, x_2, ...) \in B : \lim_{i\to\infty} \|x_i\| = 0\}$. Then $h = (h_1, h_2, ...) \in \oplus_{k=1}^{\infty}(X \otimes M_{n_k}) \cong X \otimes B$, J is an ideal of B and for the element $h + X \otimes J$ of $(X \otimes B)/(X \otimes J)$

$$\|h + X \otimes J\| = \limsup_k \|h_k\| = \lim_{k\to\infty} \|W_k\|_{cb}. \tag{7.2}$$

For the element $(V_k \otimes id)(h) + X_k \otimes J$ of $(X_k \otimes B)/(X_k \otimes J)$

$$\begin{aligned} \|(V_k \otimes id)(h) + X_k \otimes J\| &= \limsup_l \|(V_k \otimes id_l)(h_l)\| \\ &\leq \lim_{l\to\infty} \|y_l\| = 1, \end{aligned} \tag{7.3}$$

since for $k < l$,

$$\|(V_k \otimes id_l)(h_l)\| \leq \|(V_{k,k+1} \dots V_{l-1,l} V_l \otimes id_l)(h_l)\| \leq \|(V_l \otimes id_l)(h_l)\| = \|y_l\|.$$

If $T_X : (X \otimes B)/(X \otimes J) \to X \otimes (B/J)$ is the natural morphism, then T_X is isometric, by 2.5.2. Similarly the natural maps $T_{X_k} : (X_k \otimes B)/(X_k \otimes J) \to X_k \otimes (BJ)$ are isometric for $k = 1, 2, \dots$, since X_k is an operator system in $C(\Omega_k, M_k)$, which is nuclear hence exact. For each k,

$$(V_k \otimes id_B)(X \otimes J) = X_k \otimes J,$$

and so a u.c.p. map $\Phi_k : (X \otimes B)/(X \otimes J) \to (X_k \otimes B)/(X_k \otimes J)$ is defined by

$$\Phi_k(x + X \otimes J) = (V_k \otimes id_B)(x) + X_k \otimes J.$$

It is straightforward to verify that the diagram

$$\begin{array}{ccccc} X \otimes B & \longrightarrow & (X \otimes B)/(X \otimes J) & \xrightarrow{T_X} & X \otimes (B/J) \\ \downarrow{\scriptstyle V_k \otimes id_B} & & \downarrow{\scriptstyle \Phi_k} & & \downarrow{\scriptstyle V_k \otimes id_{B/J}} \\ X_k \otimes B & \longrightarrow & (X_k \otimes B)/(X_k \otimes J) & \xrightarrow{T_{X_k}} & X_k \otimes (B/J) \end{array}$$

commutes, i.e. that $(V_k \otimes id_{B/J})T_X = T_{X_k}\Phi_k$. Thus

$$\begin{aligned}\|T_X(h + X \otimes J)\| &= \sup_k \|(V_k \otimes id_{B/J})(T_X(h + X \otimes J))\| \\ &\quad \text{(by Lemma 7.4)} \\ &= \sup_k \|T_{X_k}(\Phi_k(h))\| \\ &= \sup_k \|(V_k \otimes id_B)(h) + X_k \otimes J\| \\ &\quad \text{(since } T_{X_k} \text{ is isometric)} \\ &= 1\end{aligned}$$

by (7.3). Since T_X is isometric, $\lim_{k\to\infty} \|W_k\|_{cb} = 1$, by (7.2). □

Proof of Theorem 7.3: Let $A \subseteq B(H)$ be an exact C*-algebra. Adjoining an identity if necessary, we can assume that A is unital. If X is a finite-dimensional operator system in A, with the notation of Proposition 7.6, $\lim_{k\to\infty} \|W_k\|_{cb} = 1$. Let $\varepsilon > 0$, and choose k so that $\|W_k\|_{cb} < 1 + \varepsilon$. The map $W_k : X_k \to X \subseteq B(H)$ is self-adjoint and completely bounded. By the Wittstock extension theorem (Theorem 1.13) W_k has a completely bounded self-adjoint extension $W : C(\Omega_k, M_k) \to B(H)$ such that $\|W\|_{cb} = \|W_k\|_{cb}$. By Proposition 1.19 there is a u.c.p. map $U : C(\Omega_k, M_k) \to B(H)$ such that $\|W - U\|_{cb} \leq \|W\|_{cb} - 1 < \varepsilon$. Then

$$\|id_X - UV_k\| = \|(W - U)V_k\| < \varepsilon.$$

Since $C(\Omega_k, M_k)$ is nuclear there are an $n_k \in \mathbb{N}$ and u.c.p. maps $\psi : C(\Omega_k, M_k) \to M_{n_k}$ and $\phi : M_{n_k} \to C(\Omega_k, M_k)$ such that

$$\|\phi\psi|_{X_k} - id_{X_k}\| < \varepsilon.$$

Then $\psi V_k : X \to M_{n_k}$ and $U\phi : M_{n_k} \to B(H)$ are u.c.p. and

$$\|id_X - (U\phi)(\psi V_k)\| < 2\varepsilon.$$

Since $X \subseteq A$ and ε are arbitrary, this completes the proof. □

8. Lifting elements in quotients isometrically.

In this chapter we establish a lifting result which will be crucial in proving the main theorem of chapter 9. Let L and R be closed left and right ideals, respectively, in a C*-algebra A. The following lemma is well-known (see, for example, [**Kir3**, Lemma 4.9(iv)]), and although we shall, in the course of proving Proposition 8.3, obtain a proof of the special case which we require, it seems appropriate to include also the simple self-contained proof of the general case that follows.

Lemma 8.1. *The subspace $L + R$ of A is norm-closed.*

Proof: For $a \in L$, $\text{dist}(a, R) \leq \text{dist}(a, L \cap R)$. If $\{e_\lambda\}_{\lambda \in \Lambda}$ is a right approximate unit for L, with $0 \leq e_\lambda \leq 1$ for each λ, and $\varepsilon > 0$, there is an e_λ such that $\|a - ae_\lambda\| \leq \varepsilon$. For $r \in R$,

$$\begin{aligned} \|a - r\| &\geq \|ae_\lambda - re_\lambda\| \\ &\geq \|a - re_\lambda\| - \|a - ae_\lambda\| \\ &\geq \text{dist}(a, L \cap R) - \varepsilon. \end{aligned}$$

Thus $\text{dist}(a, L \cap R) = \text{dist}(a, R)$. If $\phi : A \longrightarrow A/R$ is the quotient map, this implies that $\phi|_L$ has a factorisation

$$L \longrightarrow L/(L \cap R) \longrightarrow \overline{(L + R)}/R,$$

where the first map is the quotient map and the second is an isometry. Thus $\phi(L)$ is closed, and so also is $L + R = \phi^{-1}(\phi(L))$. □

Since $L + R$ is closed, the quotient $A/(L + R)$ is a Banach space, and if $x \in A/(L + R)$, for any $\varepsilon > 0$ there is an $\overline{x} \in A$ which is mapped onto x by the quotient map and such that $\|\overline{x}\| \leq (1 + \varepsilon)\|x\|$. An important step in the proof of Theorem 9.1 will be to show that when L is the left ideal generated by an increasing sequence of projections and $R = L^*$, $\bar{x}$ can be chosen to have the same norm as x. Kirchberg [**Kir3**] has proved, using quite elaborate methods, that such an isometric lifting always exists for general L and R, and Brown [**Br**] has given an alternative, shorter proof of this general result. In the special case that we consider here, the proof, which is based on some of Brown's observations in [**Br**], is particularly elementary.

Lemma 8.2. *Let A be a unital C*-algebra, and p a non-zero projection in A. If $\varepsilon > 0$ and $x \in A$ with $\|x\| \leq 1+\varepsilon$ and $\|px\| \leq 1$, then there is a $y \in A$ such that $py = 0$, $\|y\| \leq \sqrt{2\varepsilon+\varepsilon^2}$, and $\|x-y\| \leq 1$.*

Proof: Let $a = px, b = (1-p)x$ and

$$b' = b(1-a^*a)^{\frac{1}{2}}((1+\varepsilon)^2 \cdot 1 - a^*a)^{-\frac{1}{2}}.$$

Letting $y = b - b'$, it follows that $a + b' = x - y$, and it is readily verified that y has the required properties. □

Lemma 8.3. *Let A be a unital C*-algebra, and p and q non-zero projections in A. If $0 < \varepsilon \leq 1$, and $x \in A$ with $\|x\| \leq 1+\varepsilon$ and $\|pxq\| \leq 1$, then there are y and z in A such that $yq = pz = 0$, $\|y\| \leq 4\varepsilon^{\frac{1}{4}}, \|z\| \leq 4\varepsilon^{\frac{1}{4}}$ and $\|x-(y+z)\| \leq 1$.*

Proof: Let $a = pxq$ and $b = px(1-q)$. Then

$$\|a^* + b^*\| = \|x^*p\| \leq 1+\varepsilon,$$

$a^* = qx^*p$ and $\|a^*\| \leq 1$. By Lemma 8.2, applied to x^*p and the projection q, there is a $y \in A$ such that $qy^* = 0, \|y\| \leq \sqrt{2\varepsilon+\varepsilon^2} \leq 4\varepsilon^{\frac{1}{4}}$ and $\|x^*p - y^*\| \leq 1$, i.e. $yq = 0$ and $\|px - y\| \leq 1$. Replacing y by py, we can assume that $py = y$. Letting $c = px - y, d = (1-p)x$ and $x' = c + d = x - y$,

$$\|px'\| = \|pc\| \leq 1$$

and

$$\|x'\| \leq 1 + \varepsilon + \sqrt{2\varepsilon+\varepsilon^2} = 1 + \varepsilon'$$

where

$$\varepsilon' = \varepsilon + \sqrt{2\varepsilon+\varepsilon^2} \leq 3\sqrt{\varepsilon}.$$

By Lemma 8.2, there is a $z \in A$ such that $pz = 0, \|x' - z\| \leq 1$, and

$$\begin{aligned}\|z\| &\leq \sqrt{2\varepsilon' + \varepsilon'^2} \\ &\leq \sqrt{6\sqrt{\varepsilon} + 9\varepsilon} \\ &\leq 4\varepsilon^{\frac{1}{4}}.\end{aligned}$$

Then $\|x-(y+z)\| = \|x'-z\| \leq 1$. □

Proposition 8.4 *Let $p_1, p_2, \ldots$ be non-zero projections in a unital C^*-algebra A such that $p_1 \leq p_2 \leq \ldots$, and let $L = \overline{\bigcup Ap_i}$. For each $x \in A$ there is an $\overline{x} \in L + L^*$ such that*

$$\|x - \overline{x}\| = \operatorname{dist}(x, L + L^*) = \inf_{a \in L+L^*} \|x - a\|.$$

Proof: Assume, without loss of generality, that $\operatorname{dist}(x, L+L^*) \leq 1$. Choose $y_0 \in L$ and $z_0 \in L^*$ such that $\|x-(y_0+z_0)\| \leq 2$ and let $x' = x - (y_0 + z_0)$. Since $\operatorname{dist}(x', L+L^*) = \lim_{i\to\infty} \|(1-p_i)x'(1-p_i)\|$, we can assume, passing to a subsequence of the p_i if necessary, that

$$\|(1-p_i)x'(1-p_i)\| \leq 1 + 2^{-4i} \quad (i = 1, 2, \ldots).$$

Sequences $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ such that $y_i(1-p_i) = (1-p_i)z_i = 0$, $\|y_i\| \leq 9 \cdot 2^{-i}$; $\|z_i\| \leq 9 \cdot 2^{-i}$, and

$$\|x' - (y_1 + \ldots + y_k + z_1 + \ldots + z_k)\| \leq 1 + 2^{-4k}$$

are defined inductively as follows. If $y_0, \ldots, y_k, z_0, \ldots, z_k$ have been chosen, where $k \geq 0$, let $x_k = x' - (y_1 + \ldots + y_k + z_1 + \ldots + z_k) = x - (y_0 + \ldots + y_k + z_0 + \ldots + z_k)$. Then

$$\begin{aligned}\|(1-p_{k+1})x_k(1-p_{k+1})\| &= \|(1-p_{k+1})x'(1-p_{k+1})\| \\ &\leq 1 + 2^{-4(k+1)}.\end{aligned}$$

Applying Lemma 8.3 with x the element $x'_k = (1/1 + 2^{-4(k+1)})x_k$ and $p = q = 1 - p_{k+1}$, since

$$\|x'_k\| \leq \frac{1 + 2^{-4k}}{1 + 2^{-4(k+1)}} \leq 1 + 2^{-4k},$$

there are y'_{k+1} and z'_{k+1} such that $\|y'_{k+1}\|, \|z'_{k+1}\| \leq 6.2^{-k}$,

$$(1-p_{k+1})y'_{k+1} = z'_{k+1}(1-p_{k+1}) = 0,$$

and $\|x'_k - (y'_{k+1} + z'_{k+1})\| \leq 1$. Letting

$$y_{k+1} = (1 + 2^{-4(k+1)})y'_{k+1}, \quad z_{k+1} = (1 + 2^{-4(k+1)})z'_{k+1},$$

it follows that y_{k+1} and z_{k+1} have the required properties. This completes the inductive step.

Now let $y = \sum_0^\infty y_i$ and $z = \sum_0^\infty z_i$. Then $y \in L, z \in L^*$, and if $\bar{x} = y + z$,

$$\|x - \bar{x}\| = \lim_{i \to \infty} \|x - (y_0 + \ldots + y_i + z_0 + \ldots + z_i)\| \leq 1.$$

□

Corollary 8.5. *With A and L as above, $L + L^*$ is closed in norm. If $\rho : A \longrightarrow A/(L + L^*)$ is the quotient map, then for each $x \in A/(L + L^*)$ there is an $\bar{x} \in A$ such that $x = \rho(\bar{x})$ and $\|\bar{x}\| = \|x\|$.*

Remarks 8.6. 1. When L is a closed two-sided ideal corollary 6 is well-known, and easy to prove. A simple proof goes as follows. Let $x \in A/L$ with $\|x\| = 1$, and let $y \in A$ with $x = \rho(y)$, where $\rho : A \longrightarrow A/L$ is the quotient morphism. If $f : [0, \infty) \longrightarrow [0, 2]$ is the function given by $f(t) = \min\{2, t\}$, then the element

$$z = y(1 + y^*y)^{-1} f(1 + y^*y)$$

satisfies $\|z\| = 1$ and $x = \rho(z)$.

2. Brown [**Br**,Lemma 3.1(b)] states Lemma 8.3 with the bounds on $\|y\|$ and $\|z\|$ replaced by $\|y + z\| \leq 2\sqrt{2\varepsilon + \varepsilon^2}$. While a bound of this order follows from Kirchberg's penetrating analysis of the geometry of the quotient map $A \longrightarrow A/(L + L^*)$ in [**Kir3**], the proof in [**Br**] is incomplete. However using the weaker bound of $8\varepsilon^{\frac{1}{4}}$ given by Lemma 8.3, all the results in [**Br**] which require [**Br**, Lemma 3.1(b)] follow with obvious slight modifications to the proofs.

9. A characterisation of separable exact C*-algebras.

The main result of this chapter is the following remarkable result of Kirchberg [**Kir4**, Corollaries 1.4, 1.5].

Theorem 9.1. *For a separable unital C*-algebra A the following conditions are equivalent:*

(i) A is exact;

(ii) A is nuclearly embeddable;

(iii) there are a unital C-subalgebra G of the CAR algebra B and a 2-sided closed AF ideal J of G such that A is $*$-isomorphic to G/J;*

(iv) there is a unital completely isometric linear map $\theta : A \longrightarrow B$.

Moreover A is nuclear if and only if there is a unital completely isometric linear map $\theta : A \longrightarrow B$ such that $\theta(A)$ is the image in B of a completely positive projection.

The proof we shall give is taken from [**Wa5**] and uses techniques from [**Kir3**], but is somewhat simpler than Kirchberg's original proof. The following preliminary proposition, implicit in [**Kir3**, Prop. 2.3, Cor. 2.4, Prop. 3.4], will be needed.

Proposition 9.2. *Let A be a separable unital nuclearly embeddable C*-algebra and B a UHF C*-algebra. Then there are a closed left ideal L of B, an isometry ι of the quotient Banach space $B/(L + L^*)$ into a unital C*-algebra D, and a unital complete isometry $\sigma : A \longrightarrow D$ such that if $\rho : B \longrightarrow B/(L + L^*)$ is the quotient map,*

(i) $\iota \circ \rho : B \longrightarrow D$ is unital and completely positive,

(ii) $\sigma(A) \subseteq \iota(\rho(B))$, and if A is nuclear, then σ, ι and L can be chosen so that $\sigma(A) = \iota(\rho(B))$.

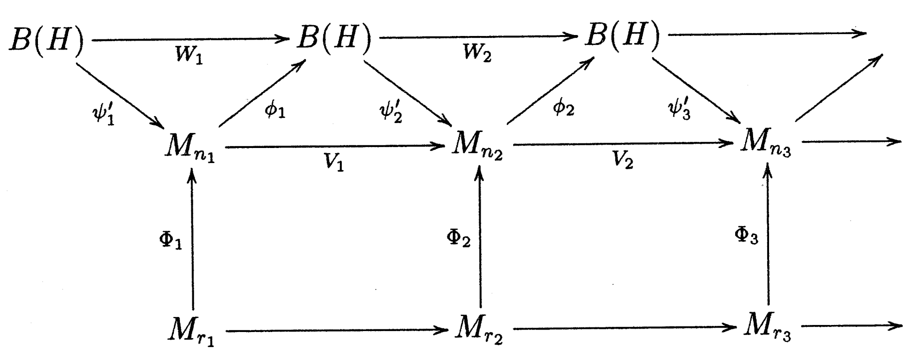
Proof: 1. We can assume that for some Hilbert space H, $A \subseteq B(H)$ unitally. If μ is a nuclear embedding of A in a C*-algebra D, the embedding of A in $B(H)$ extends to a completely positive contractive mapping ϕ of D into $B(H)$, by Arveson's extension theorem (Theorem 1.8). Then $\phi\mu$ is a nuclear embedding of A in $B(H)$. We can thus assume in what follows that $D = B(H)$ and that μ is the embedding map of A in $B(H)$.

Since A is separable and μ is nuclear, μ is the limit in the point-norm topology of a sequence of unital completely positive maps on A which factorise through finite matrix algebras. This implies that $\mu \otimes id_n : A \otimes M_n \longrightarrow B(H) \otimes M_n$ is the limit in the point norm topology of these maps tensored with id_n, for $n = 1, 2, \ldots$. It follows that there are a sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite-dimensional self-adjoint unital subspaces of A with union dense in A and, for $i = 1, 2, \ldots$, an $n_i \in \mathbb{N}$, and unital completely positive maps $\psi_i : A \longrightarrow M_{n_i}$ and $\phi_i : M_{n_i} \longrightarrow B(H)$ such that

$$\|((\mu - \phi_i\psi_i) \otimes id_j)|_{(A_i \otimes M_j)}\|_j \leq 2^{-i}$$

for $1 \leq j \leq i$, where $\| \ \|_j$ is the C*-norm on $B(H) \otimes M_j$.

Since B is UHF, there are a sequence $1 < s_1 < s_2 < \ldots$ of integers such that $s_i | s_{i+1}$ for each i and a sequence $B_1 \subseteq B_2 \subseteq \ldots$ of unital subalgebras of B such that for each k, $B_k \cong M_{s_k}$ and $B = \overline{\bigcup_k B_k}$. By Arveson's extension theorem, the map $\psi_i : A \longrightarrow M_{n_i}$ extends to a completely positive unital map $\psi_i' : B(H) \longrightarrow M_{n_i}$ for $i = 1, 2, \ldots$. Let $V_i = \psi_{i+1}'\phi_i : M_{n_i} \longrightarrow M_{n_{i+1}}$ and $W_i = \phi_i\psi_i' : B(H) \longrightarrow B(H)$. We now show inductively that there is a subsequence $0 < r_1 < r_2 < \ldots$ of the sequence $s_1, s_2, \ldots$ with $n_i \leq r_i$, so that we can regard M_{n_i} as a $*$-subalgebra of M_{r_i}, and projections $p_i \in M_{r_i}$ such that $(1 - p_i)M_{r_i}(1 - p_i) = M_{n_i}$ and such that, if $\Phi_i : M_{r_i} \longrightarrow M_{n_i}$ denotes the compression $x \longrightarrow (1 - p_i)x(1 - p_i)$, the following diagram is commutative:



The map in the bottom row from M_{r_i} to $M_{r_{i+1}} \cong M_{r_i} \otimes M_{r_{i+1}/r_i}$ is the embedding $x \longrightarrow x \otimes 1$. Identifying each M_{r_i} with its image in $M_{r_{i+1}}$, the sequence $\{p_i\}$ will satisfy $p_1 \leq p_2 \leq \ldots$.

To start the construction, let r_1 be the smallest s_i such that $n_1 \leq s_i$. Let q_1 be a projection in M_{r_1} of rank n_1, and let $p_1 = 1 - q_1$. Identifying M_{n_1} with $q_1 M_{r_1} q_1$, the image of the completely positive unital map $\Phi_1 : x \longrightarrow q_1 x q_1$ is M_{n_1}. Now suppose that M_{r_i} and p_i have been constructed for $i \leq k$. Then $V_k \Phi_k : M_{r_k} \longrightarrow M_{n_{k+1}}$ is unital and completely positive. By Stinespring's theorem there are a Hilbert space K, a unital representation π of M_{r_k} in $B(K)$, and a projection $q_{k+1} \in B(K)$ of rank n_{k+1} such that for $x \in M_{r_k}$,

$$V_k(\Phi_k(x)) = q_{k+1}\pi(x)q_{k+1}|_{q_{k+1}K},$$

where $M_{n_{k+1}}$ is identified with $B(q_{k+1}K) \subseteq B(K)$. It is apparent from the proof of Stinespring's theorem (Theorem 1.6) that K can be taken to be finite-dimensional, since $V_k\Phi_k$ is a mapping between finite-dimensional C*-algebras. Since matrix algebras are simple, it follows that π is an isomorphism. Thus $B(K) \cong M_{r_k} \otimes M_r$ for a suitable $r \in \mathbb{N}$, and with this identification π is the map $x \longrightarrow x \otimes 1_r$. Let r_{k+1} be the smallest s_i such that $r_k r < s_i$. Then $r_k | r_{k+1}$, since r_k and r_{k+1} are elements of the sequence $s_1, s_2, \ldots$ and $r_k \leq r_{k+1}$; thus $B(K) \cong M_{r_k} \otimes M_r \subseteq M_{r_k} \otimes M_{r_{k+1}/r_k} \cong M_{r_{k+1}}$. With the obvious identifications $M_{n_{k+1}} \subseteq M_{r_{k+1}}$, and the above equation becomes

$$\begin{aligned} V_k(\Phi_k(x)) &= (1 - p_{k+1})(x \otimes 1)(1 - p_{k+1}) \\ &= \Phi_{k+1}(x \otimes 1), \end{aligned}$$

where, for each i, $p_i = 1 - q_i$. This completes the inductive step and proves commutativity of the above diagram at the kth square. Identifying p_k with $p_k \otimes 1$,

$$\begin{aligned} (1 - p_{k+1})p_k(1 - p_{k+1}) &= V_k(\Phi_k(p_k)) \\ &= 0. \end{aligned}$$

Thus $p_k \leq p_{k+1}$.

2. In the above diagram we can assume that the M_{r_i} and M_{n_i} are $*$-subalgebras of B with $1 \in M_{r_1} \subset M_{r_2} \subset \ldots$ and $\bigcup M_{r_i}$ dense in B. Then $p_i \in B$, so that $L = \overline{\bigcup_i Bp_i}$ is a closed left ideal of B. Let B_∞ be the C*-algebra of bounded sequences of elements of B with norm $\|(x_i)\| = \sup_i \|x_i\|$, and let I be the ideal of zero-sequences in B_∞, i.e. the ideal of sequences

(x_i) such that $\lim_{i\to\infty}\|x_i\| = 0$. Let τ be the quotient map $B_\infty \longrightarrow B_\infty/I$, and for $x \in B$, let $\Psi : B \longrightarrow B_\infty/I$ be the map given by

$$\Psi(x) = \tau((q_i x q_i)).$$

Since τ and the map

$$x \longrightarrow (q_1 x q_1, q_2 x q_2, \ldots)$$

are both completely positive and contractive, the same is true of Ψ, and $\Psi(1)$ is the projection $e = \tau((q_i))$ in B_∞/I. Let $D = e(B_\infty/I)e$. Then $\Psi(B) \subseteq D$, and Ψ, as a map to D, is unital.

If $x \in B$, since $p_1 \leq p_2 \leq \ldots$, the sequence $\{\|q_i x q_i\|\}$ is decreasing, and so tends to a limit as $i \to \infty$. It follows that

$$\|\Psi(x)\| = \lim_{i\to\infty} \|q_i x q_i\|,$$

so that $\operatorname{dist}(x, L+L^*) \leq \|\Psi(x)\|$. The opposite inequality holds since $\Psi(L) = \Psi(L^*) = \{0\}$, and so $\|\Psi(x)\| = \|\rho(x)\|$ for $x \in B$, where $\rho : B \longrightarrow B/(L+L^*)$ is the quotient map. It is now immediate that there is a well-defined linear isometry $\iota : B/(L+L^*) \longrightarrow D$ such that $\Psi = \iota\rho$.

3. It remains to construct the unital complete isometry $\sigma : A \longrightarrow D$. With the above identifications, $M_{n_i} \subseteq B$ and for $y \in M_{n_i}$, $V_i(y) = q_{i+1} y q_{i+1}$, for each i. If $x \in A_k$ has unit norm, then $\|x - W_i(x)\| \leq 2^{-i}$ for $i \geq k$, and

$$\begin{aligned}\|\psi_{i+1}(x) - V_i(\psi_i(x))\| &= \|\psi'_{i+1}(x - \phi_i(\psi_i(x)))\| \\ &\leq \|x - W_i(x)\| \\ &\leq 2^{-i}.\end{aligned}$$

Since $q_j V_i(\psi_i(x)) q_j = q_j(\psi_i(x)) q_j$ for $j > i$, it follows that $\Psi(\psi_i(x)) = \Psi(V_i(\psi_i(x)))$, so that

$$\begin{aligned}\|\Psi(\psi_{i+1}(x)) - \Psi(\psi_i(x))\| &= \|\Psi(\psi_{i+1}(x)) - \Psi(V_i(\psi_i(x)))\| \\ &\leq \|\psi_{i+1}(x) - (V_i(\psi_i(x))\| \\ &\leq 2^{-i}.\end{aligned}$$

This shows that for $x \in \bigcup A_k$, the sequence $\{\Psi(\psi_i(x))\}$ is Cauchy in D. Since $\bigcup A_k$ is dense in A and the maps $\Psi\psi_i$ are all contractive, it follows that for any $x \in A, \sigma(x) = \lim_{i\to\infty} \Psi(\psi_i(x))$ exists. The map $\sigma : A \longrightarrow D$, being a point-norm limit of unital completely positive maps, is itself unital, completely positive, and therefore completely contractive. By construction, $\sigma(A) \subseteq \iota(\rho(B))$.

To show that σ is a complete isometry, it suffices to show that if $n \in \mathbb{N}$ and $x \in A \otimes M_n$ with $\|x\| = 1$, then $\|\sigma^{(n)}(x)\| \geq 1$, where $\sigma^{(n)} = \sigma \otimes id_n$; and since σ is completely contractive, it is enough to show this for $x \in (\bigcup A_k) \otimes M_n$. Thus it is enough to show that for $k \in \mathbb{N}$ and $x \in A_k \otimes M_n$ with $\|x\| = 1$, for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for $i \geq N$,

$$\begin{aligned}\|\Psi^{(n)}(\psi_i^{(n)}(x))\| &= \lim_{j\to\infty} \|q_j^{(n)}\psi_i^{(n)}(x)q_j^{(n)}\| \\ &\geq 1 - \varepsilon,\end{aligned}$$

where $q_j^{(n)} = q_j \otimes 1_n, \Psi^{(n)} = \Psi \otimes id_n$, etc. Now for $x \in A_k \otimes M_n$ of unit norm and $\varepsilon > 0$, let $N \in \mathbb{N}$ satisfy $2^{-N} < \varepsilon/3$ and $N \geq \max(k, n)$. Then $A_k \otimes M_n \subseteq A_i \otimes M_n \subseteq A_i \otimes M_i$ for $i \geq N$, so that $x \in A_i \otimes M_i$ for such i. Fix $i \geq N$. Then

$$\begin{aligned}\|\psi_{l+1}^{(n)}(x) - q_{l+1}^{(n)}\psi_l^{(n)}(x)q_{l+1}^{(n)}\| &= \|(\psi_{l+1}')^{(n)}(x - \phi_l^{(n)}(\psi_l^{(n)}(x)))\| \\ &\leq 2^{-l}\end{aligned}$$

for $l \geq i$. For $j > i + 1$, applying this for $l = i, \ldots, j - 2$, and using the fact that for such $l, q_j \leq q_{l+1}$, it follows that

$$\|q_j^{(n)}\psi_{l+1}^{(n)}(x)q_j^{(n)} - q_j^{(n)}\psi_l^{(n)}(x)q_j^{(n)}\| \leq 2^{-l}$$

and

$$\|\psi_j^{(n)}(x) - q_j^{(n)}\psi_{j-1}^{(n)}(x)q_j^{(n)}\| \leq 2^{-j+1},$$

so that, summing,

$$\|\psi_j^{(n)}(x) - q_j^{(n)}\psi_i^{(n)}(x)q_j^{(n)}\| \leq 2^{-i+1}.$$

Thus

$$\|\phi_j^{(n)}(\psi_j^{(n)}(x) - q_j^{(n)}\psi_i^{(n)}(x)q_j^{(n)})\| \leq 2^{-i+1}$$

and since also

$$\|x - \phi_j^{(n)}(\psi_j^{(n)}(x))\| \leq 2^{-j} \leq 2^{-i},$$

it follows, finally, that

$$\begin{aligned}\|q_j^{(n)}\psi_i^{(n)}(x)q_j^{(n)}\| &\geq \|\phi_j^{(n)}(q_j^{(n)}\psi_i^{(n)}(x)q_j^{(n)})\| \\ &\geq \|x\| - 2^{-i+1} - 2^{-i} \\ &\geq 1-\varepsilon.\end{aligned}$$

This implies that $\|\Psi^{(n)}(\psi_i^{(n)}(x))\| \geq 1-\varepsilon$ for $i \geq N$, so that $\|\sigma^{(n)}(x)\| \geq 1-\varepsilon$. Since ε is arbitrary, $\|\sigma^{(n)}(x)\| \geq 1$, from which it follows that σ is completely isometric.

4. When A is nuclear, it satisfies the CPAP and so the maps ϕ_i and ψ_i can be chosen to satisfy all the previous properties and also $\phi_i(M_{n_i}) \subseteq A$ for all i. The finite-dimensional operator systems A_k will be assumed chosen so that for each i, $\phi_i(M_{n_i}) \subseteq A_{i+1}$ (it is easy to see that this is possible since the A_k are defined inductively). To show that $\sigma(A) = \Psi(B)$, it suffices to show that for $x \in M_{r_i}$ with $\|x\| = 1$, there is an $a \in A$ such that $\sigma(a) = \Psi(x)$. Let $x \in M_{r_i}$, with $\|x\| = 1$ and consider the sequence $\{\phi_j(q_jxq_j)\}_{j\in\mathbb{N}}$. For $j > i$,

$$\begin{aligned}\|\phi_j(q_jxq_j) - \phi_{j-1}(q_{j-1}xq_{j-1})\| &= \|(\phi_j\psi_j)(\phi_{j-1}(q_{j-1}xq_{j-1})) \\ &\qquad -\phi_{j-1}(q_{j-1}xq_{j-1})\| \\ &\leq 2^{-j}\|\phi_{j-1}(q_{j-1}xq_{j-1})\| \\ &\leq 2^{-j}.\end{aligned}$$

Thus $\{\phi_j(q_jxq_j)\}$ is a Cauchy sequence in A, and has a limit a with $\|a\| \leq 1$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $2^{-k} \leq \varepsilon$ and

$$\|a - \phi_j(q_jxq_j)\| \leq \varepsilon$$

for $j \geq k$. This last inequality implies that for $j > k$,

$$\begin{aligned}\|\psi_j(a) - q_{j+1}xq_{j+1}\| &= \|\psi_j(a - \phi_j(q_jxq_j))\| \\ &\leq \varepsilon.\end{aligned}$$

Let $a' = \phi_k(q_kxq_k) \in A_{k+1}$, so that $\|a - a'\| \leq \varepsilon$ and $\|a'\| \leq 1$. For $i > k$ and $l \geq i$,

$$\|\psi_{l+1}(a') - q_{l+1}\psi_l(a')q_{l+1}\| \leq 2^{-l}$$

so that, for $j > i + 1$,

$$\|\psi_j(a') - q_j\psi_i(a')q_j\| \leq 2^{-i+1}.$$

Combining these inequalities gives

$$\|q_j\psi_i(a')q_j - q_jxq_j\| \leq \varepsilon + 2^{-i+1} \leq 3\varepsilon$$

and

$$\|q_j\psi_i(a)q_j - q_jxq_j\| \leq 4\varepsilon$$

for $i > k$ and $j > i + 1$, from which it is immediate that

$$\|\Psi(\psi_i(a)) - \Psi(x)\| \leq 4\varepsilon$$

for $i \geq k$. Thus

$$\begin{aligned}\|\sigma(a) - \Psi(x)\| &= \lim_{i\to\infty}\|\Psi(\psi_i(a)) - \Psi(x)\| \\ &\leq 4\varepsilon\end{aligned}$$

and, since ε is arbitrary, it follows that $\sigma(a) = \Psi(x)$, as desired. □

Proof of Theorem 9.1:

(i) $\Leftrightarrow$ (ii): This is immediate from Proposition 7.2 and Theorem 7.3.

(ii) $\Rightarrow$ (iii): Let A be a separable unital nuclearly embeddable C*-algebra and let B be the CAR algebra. Thus $B = \overline{\bigcup_k B_k}$, where B_k is a unital $*$-subalgebra of B $*$-isomorphic to M_{2^k} and $B_1 \subset B_2 \subset \ldots$. Let A be faithfully unitally represented on a Hilbert space H, so that $1 \in A \subset B(H)$. By Proposition 9.2, there are a unital C*-algebra D, a unital complete isometry $\sigma : A \longrightarrow D$ and a unital completely positive map $\iota\rho : B \longrightarrow D$ such that $\sigma(A) \subseteq \iota(\rho(B))$, with equality if A is nuclear. Now $\sigma(A)$ is a closed operator subsystem of D and $\sigma^{-1} : \sigma(A) \longrightarrow A \subseteq B(H)$ is a unital complete isometry, hence completely positive. By Arveson's extension theorem σ^{-1} extends to a completely positive map $\tau : D \longrightarrow B(H)$. Let $\pi = \tau\iota\rho$ and let $U(A)$ be the group of unitaries in A. For $u \in U(A)$, $\sigma(u) \in \iota(\rho(B))$, so that $\iota^{-1}(\sigma(u)) \in \rho(B) = B/(L + L^*)$ and

$$\|\iota^{-1}(\sigma(u))\| = \|u\| = 1.$$

By Corollary 8.5, there is an element $x \in B$ such that $\|x\| = 1$ and $\rho(x) = \iota^{-1}(\sigma(u))$, i.e. $u = \pi(x)$. Clearly π is unital and completely positive, and by Choi's generalised Cauchy-Schwartz inequality (cf. 1.5.6),

$$1 = u^*u = \pi(x^*)\pi(x) \leq \pi(x^*x) \leq \|x\|^2\pi(1) = 1,$$

so that $\pi(x^*x) = \pi(x^*)\pi(x)$. Similarly $\pi(xx^*) = \pi(x)\pi(x^*)$. This implies that x and x^* are in the multiplicative domain of π, i.e. for $a \in B$, $\pi(xa) = \pi(x)\pi(a)$ and $\pi(ax) = \pi(a)\pi(x)$ (see 1.5.6.).

Letting $X = \{x \in B : \|x\| = 1, \pi(x) \in U(A)\}$, X is self-adjoint and closed under multiplication, so that its linear span $\mathrm{sp}(X)$ is a *-subalgebra of B. Since X lies in the multiplicative domain of π, $\pi|_{\mathrm{sp}(X)}$ is a $*$-homomorphism, so that, if $F = C^*(X) = \overline{\mathrm{sp}(X)}$, $\pi|_F$ is a *-homomorphism, by continuity. Since $\pi(X) = U(A)$, $\pi(F) = A$. Let $K = F \bigcap \ker\pi$, so that K is a two-sided closed ideal of F and $A \cong F/K$. Let $J = \mathrm{C}^*(KBK)$, the hereditary C*-subalgebra of B generated by K. By **[El]**, J is approximately finite; also $FJ \subseteq J$, and $\pi(J) = \{0\}$. Letting $G = F + J$, it follows by standard arguments that G is a C*-subalgebra of B, J is a closed ideal of G, and $G/J \cong A$. Indeed $\overline{F+J}$ is a C*-subalgebra of B with J as a closed ideal. If $\pi' : \overline{F+J} \longrightarrow \overline{(F+J)}/J$ is the quotient morphism, then $K = F \cap J$ and

$$\pi'(F) \cong F/(F \cap J) = F/K \cong A.$$

It follows that $F + J = \pi'^{-1}(A)$, so that $F + J$ is closed in B.

(iii) $\Rightarrow$ (iv): If $A \cong G/J$, where $G \subseteq B$ and J is AF, J is nuclear and G has property C, since it is a C*-subalgebra of B, which is nuclear, and so has property C. It follows by Proposition 5.2 and Theorem 6.8 that the quotient morphism $\pi : G \longrightarrow G/J$ has a unital completely positive right inverse $\theta : A \longrightarrow G$. Since $\pi\theta$ is the identity map on A, θ is completely isometric.

(iv) $\Rightarrow$ (ii): Let $\pi_k : B \longrightarrow B_k \cong M_{2^k}$ be a contractive projection, for $k = 1, 2, \ldots$. Then for $x \in B$, $\|\pi_k(x) - x\| \to 0$ as $k \to \infty$. If $A \subseteq B(H)$, $\theta : A \longrightarrow B$ is unital and completely isometric, and so $\theta^{-1} : \theta(A) \longrightarrow A \subseteq B(H)$ is unital, completely isometric, and therefore completely positive. By Arveson's extension theorem, θ^{-1} extends to a unital completely positive $\tau : B \longrightarrow B(H)$. For $x \in A$,

$$\|(\tau\pi_k\theta)(x) - x\| \leq \|\pi_k(\theta(x)) - \theta(x)\| \to 0$$

as $k \to \infty$. Since $\pi_k\theta : A \longrightarrow M_{2^k}$ and $\tau : M_{2^k} \longrightarrow B(H)$ are completely contractive, it follows that A is nuclearly embeddable.

When A is nuclear, $\sigma(A) = \iota(\rho(B))$ in the previous notation. The existence of the map $\theta : A \longrightarrow B$ follows as above, but is also a consequence of the Choi-Effros lifting theorem if A is nuclear. Then $\theta\sigma^{-1}\iota\rho$ is a completely positive projection of B onto $\theta(A)$.

Conversely, let $\theta : A \longrightarrow B$ be a unital complete isometry and $\phi : B \longrightarrow \theta(A)$ a completely positive projection. Then

$$\begin{aligned}\|(\theta^{-1}\phi\pi_k\theta)(x) - x\| &= \|\phi(\pi_k(\theta(x))) - \theta(x)\| \\ &\to \|\phi(\theta(x)) - \theta(x)\| \\ &= 0\end{aligned}$$

for $x \in A$, as $k \to \infty$. Since $\pi_k\theta : A \longrightarrow M_{2^k}$ and $\theta^{-1}\phi : M_{2^k} \longrightarrow A$ are unital and completely positive, A has the CPAP and is thus nuclear. □

Corollary 9.3. *Any quotient of an exact C*-algebra is exact.*

Proof: 1. Let A be separable and exact and let J be an ideal of A. Then $\tilde{A}$ is exact and J is an ideal of $\tilde{A}$. It thus suffices to consider the case where A is unital. By theorem 9.1 there is a C*-subalgebra G of the CAR algebra B such that $G/I \cong A$ for some ideal I of G. Then $A/J \cong G/K$ for some ideal K of G with $I \subseteq K$. By theorem 9.1 G/K, and hence A/J, are exact.

2. If A is an arbitrary exact C*-algebra with J an ideal of A, and C is a separable C*-subalgebra of A/J, there is a separable C*-subalgebra D of A such that $C = \pi(D)$, where $\pi : A \to A/J$ is the quotient map. Being a C*-subalgebra of an exact C*-algebra, D is exact, and so C is exact, by part 1 of the proof. Thus every separable C*-subalgebra of A/J is exact, and this implies that A/J is exact, since an inductive limit of exact C*-algebras is exact, and A/J is the inductive limit of its separable C*-subalgebras. □

Corollary 9.4. *1. The CAR algebra B has a C*-subalgebra which is not nuclear*

2. If A is a non-type I C-algebra and D is any separable exact C*algebra, there is a completely positive complete isometry $\omega : D \to A$.*

Proof: 1. let D be a separable unital C*-algebra which is exact but not nuclear, e.g. let $D = C_r^*(\mathbb{F}_2)$. By Theorem 9.1 there is a C*-subalgebra G of B such that $D = G/J$ for some ideal J of G. Since D is not nuclear, neither is G.

2. By the Glimm-Sakai theorem [**Sak**], A has a C*-subalgebra H such that $B \cong H/K$ for some ideal K of H. Since B is nuclear, the quotient map $H \to H/K$ has a completely positive right inverse τ, by the Choi-Effros lifting theorem. The map $\tau : B \to H$ is completely isometric. If D is any separable exact C*-algebra, then by Theorem 9.1 there is a completely positive, completely isometric linear map $\theta : D \to B$. The composition $\tau\theta : D \to A$ is the required map ω. □

Remarks 9.5. 1. An alternative route from (iii) to (i) of Theorem 9.1 is as follows. If a separable C*-algebra A is $*$-isomorphic to a quotient of a C*-subalgebra of the CAR algebra, A is exact, by Corollary 5.6. By Theorem 7.3, A is nuclearly embeddable.

2. By Propositions 5.1 and 5.5, any C*-algebra which is a subquotient of the CAR algebra has property C. Thus any separable exact C*-algebra has property C. It can be shown that an inseparable C*-algebra has property C if all its separable C*-subalgebras have property C. It follows that if A is an arbitrary exact C*-algebra, each separable C*-subalgebra of A is exact and has property C, so that A has property C. Thus a C*-algebra is exact if and only if it has property C.

3. Blackadar [**Bla**] showed that any C*-algebra A which is not type I has a C*-subalgebra with a quotient isomorphic to the Cuntz algebra $\mathcal{O}_2$. Since $\mathcal{O}_2$ has a non-nuclear C*-subalgebra, it follows that any non-type I C*-algebra, in particular an AF algebra, has a non-nuclear subalgebra.

By Corollary 9.4 it follows that given a non-type I C*-algebra A and a separable non-nuclear exact C*-algebra C, there is a C*-subalgebra D of A with a quotient isomorphic to C. Clearly D cannot be nuclear.

10. Further results and open problems.

1. Kirchberg gives two rather different proofs in **[Kir4]** that an exact C*-algebra is nuclearly embeddable, the second of which is given chapter 7. The other proof makes use of the remarkable result, proved in **[Kir5]**, that there is a unique C*-norm on $B(H) \odot C^*(\mathbb{F}_\infty)$, where H is any Hilbert space. The proof of this result in **[Kir5]** is quite complicated. It would be valuable to have a simpler and more concise proof.

2. Junge and Pisier **[J-P]** have recently shown that, with $H = \ell^2_{\mathbb{N}}$, the norms $\| \ \|_{max}$ and $\| \ \|_{min}$ on $B(H) \odot B(H)$ are distinct. Their method of proof, which we now outline, is interesting.

Fix $n \geq 2$, let X and Y be n-dimensional operator spaces, and define $d_{cb}(X,Y)$ to be the infimum of $\|\phi\|_{cb}\|\phi^{-1}\|_{cb}$ over all linear bijections $\phi : X \to Y$. Then $d_{cb}(X,Y) = 1$ if and only if X and Y are completely isometrically isomorphic. Let OS_n denote the set of equivalence classes of n-dimensional operator spaces, the equivalence relation being completely isometric isomorphism. A metric δ_{cb} on OS_n is defined by

$$\delta_{cb}(\overline{X},\overline{Y}) = \log d_{cb}(X,Y).$$

With this metric OS_n was known to be complete and non-compact, and Junge and Pisier have now shown that for $n > 2$, OS_n is not separable. Their proof, which comes down to a surprising application of the Baire category theorem, is rather involved and relies on much previous work.

It is quite simple to derive the non-uniqueness of the C*-norms on $B(H) \odot B(H)$ from the inseparability of OS_n. For a given a separable C*-algebra A, the set of equivalence classes of n-dimensional operator spaces contained in A is easily shown to be a separable subspace of OS_n. The non-separability of OS_n then implies that there is an n-dimensional operator space which has no completely isometric embedding in A. Let Λ be the unitary group of $B(H)$, regarded as a set, and for $\lambda \in \Lambda$ let u_λ be the corresponding unitary in $B(H)$. Since $B(H)$ is the linear span of its unitary group, there is a $*$-epimorphism $\Phi : C^*(\mathbb{F}_\Lambda) \to B(H)$ such that $\Phi(g_\lambda) = u_\lambda$ ($\lambda \in \Lambda$), where $\mathbb{F}_\Lambda$ is the free group on generators $\{g_\lambda\}_{\lambda\in\Lambda}$. If the norms $\| \ \|_{min}$ and $\| \ \|_{max}$ coincided on $B(H) \odot B(H)$, the sequence

$$0 \longrightarrow \ker\Phi \otimes B(H) \longrightarrow C^*(\mathbb{F}_\Lambda) \otimes B(H) \xrightarrow{\Phi\otimes id} B(H) \otimes B(H) \longrightarrow 0$$

would be exact and, by remark 6.9, each finite-dimensional operator system in $B(H)$ would have a completely isometric embedding in $C^*(\mathbb{F}_\Lambda)$. Since an n-dimensional operator space X is contained in a finite dimensional operator system, this would imply that each such X had a completely isometric embedding ϕ in $C^*(\mathbb{F}_\Lambda)$. For a suitable countable subset I of Λ, $\phi(X)$ would be contained in $C^*(\{g_\lambda : \lambda \in I\}) \cong C^*(\mathbb{F}_\infty)$, which is separable. Since every n-dimensional operator space X has a completely isometric embedding in $B(H)$, this would imply that every such X has a completely isometric embedding in $C^*(\mathbb{F}_\infty)$, which contradicts the existence indicated above of an X without this property. It follows that $\| \ \|_{max} \neq \| \ \|_{min}$ on $B(H) \odot B(H)$.

Since OS_n is not separable, there exist an $\varepsilon > 0$, an uncountable set K, and a family $\{X_k\}_{k \in K}$ of n-dimensional operator spaces such that $\delta_{cb}(X_k, X_{k'}) \geq \varepsilon$ if $k \neq k'$. The inseparability proof in [**J-P**] is an existence proof, and no example of such a family $\{X_k\}$ is given. Presumably the set K can be taken to be the interval $(0, 1)$. It would be satisfying to have an explicit example of a such a family with this K.

4. Can every separable exact C*-algebra be embedded as a C*-subalgebra of a nuclear C*-algebra? In particular, is every exact C*-algebra $*$-isomorphic to a C*-subalgebra of the Cuntz algebra $\mathcal{O}_2$? By Proposition 9.2, if A is separable and exact, then (with the notation of Proposition 9.2)

$$\sigma(A) \subseteq (\iota\rho)(B) \subseteq D.$$

Now $(\iota\rho)(B) \cong \rho(B) = B/(L + L^*)$, and if the $\sigma(B^{**}, B^*)$-closure of L in B^{**} is $\overline{L}$, then $\overline{L} = B^{**}e$ for some projection $e \in B^{**}$. Thus $((\iota\rho)(B))^{**}$ is unitally completely isometric to $(1-e)B^{**}(1-e)$, which is an injective von Neumann algebra. Thus $(1-e)B^{**}(1-e)$ is semidiscrete and has the normal CPAP [**Wa2**]. From this it follows that $(\iota\rho)(B)$ is a nuclear operator system, i.e. the identity map on $(\iota\rho)(B)$ is nuclear (see [**Kir4**]). Thus A has a u.c.i. embedding in a nuclear operator system, and one might hope that any nuclear operator system could be embedded as an operator subsystem of a unital nuclear C*-algebra. However Kirchberg has shown that there is a nuclear operator system which does not have a u.c.i. embedding in any nuclear C*-algebra.

5. Is an arbitrary (inseparable) exact C*-algebra a C*-subalgebra or a subquotient of a nuclear C*-algebra?

6. If G is a countable discrete group, is $C_r^*(G)$ exact? Let α be the action of G on $\ell^\infty(G)$ given by

$$(\alpha_g(\xi))(h) = \xi(g^{-1}h) \quad (\xi \in \ell^\infty(G).$$

Then $C_r^*(G)$ is a C*-subalgebra of the crossed product $\ell^\infty(G) \rtimes_{\alpha,r} G$. There has been speculation that this crossed product algebra is always nuclear, which would imply the exactness of $C_r^*(G)$, but a proof has remained elusive.

Appendix: The K-groups of $C_r^*(\mathbb{F}_n)$ for $n = 1, 2, \ldots$

In this appendix we give a self-contained derivation of the K-groups of $C_r^*(\mathbb{F}_n)$ for $n \in \mathbb{N}$. Our method, though based on the KK-theoretic treatment of Cuntz [**Cu3**], uses only K-theory, and in particular the notion of difference maps. The original derivation of these K-groups by Pimsner and Voiculescu [**P-V**] also used K-theory; and Lance [**La3**] obtained the K-groups of a more general class of groups using a method based on that of [**P-V**] involving difference maps. However the present approach is somewhat simpler and more concise. A good reference for K-theory is the recent book of Wegge-Olsen [**We**].

For a C*-algebra A, $K_*(A)$ will denote the graded group $K_0(A) \oplus K_1(A)$. If B is another C*-algebra and $\phi : A \to B$ a $*$-homomorphism, then ϕ_* will denote the induced homomorphism $K_*(A) \to K_*(B)$.

A.1 The K-groups of $C^*(\mathbb{F}_n)$. Recall that $K_0(C^*(\mathbb{F}_n)) = \mathbb{Z}$ and $K_1(C^*(\mathbb{F}_n)) = \mathbb{Z}^n$ [**Cu2**]. For completeness a sketch of the proof is included. To keep the notation simple, only the case $n = 2$ will be considered; the general case is analogous.

Let $C^*(\mathbb{F}_2) = C^*(u, v)$, where u and v are the generators of $\mathbb{F}_2$. Then $C^*(u) \cong C^*(v) \cong C(\mathbb{T})$. Let

$$D = \{(f, g) : f \in C^*(u), g \in C^*(v), f(1) = g(1)\} \subseteq C^*(u) \oplus C^*(v),$$

and let $j : D \to M_2(C^*(\mathbb{F}_2))$ and $k : C^*(\mathbb{F}_n) \to D$ be the homomorphisms given by

$$j((f,g)) = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix},$$

and

$$k(u) = (u, 1), \quad k(v) = (1, v).$$

The homomorphism $k_2 j : D \to M_2(D)$ is homotopic to the homomorphism

$$(f, g) \longrightarrow \begin{bmatrix} (f, g) & 0 \\ 0 & (g(1)1, f(1)1) \end{bmatrix}$$

via the continuous path of homomorphisms

$$(f, g) \longrightarrow (1, V_\theta^*) \begin{bmatrix} (f, f(1)1) & 0 \\ 0 & (g(1)1, g) \end{bmatrix} (1, V_\theta) \quad (\theta \in [0, \pi/2]), \qquad (1)$$

where V_θ is the unitary $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (it is not hard to verify that the right-hand side of (1) is in $M_2(D)$ for $(f,g) \in D$ and $\theta \in [0,1]$). The homomorphism $jk : C^*(\mathbb{F}_2) \to M_2(C^*(\mathbb{F}_2))$, which sends u and v to $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix}$, respectively, is homotopic to the homomorphism

$$x \longrightarrow \begin{bmatrix} x & 0 \\ 0 & \psi(x) \end{bmatrix},$$

via the continuous path of homomorphisms ϕ_θ given by

$$\phi_\theta(u) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi_\theta(v) = V_\theta^* \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} V_\theta \quad (\theta \in [0, \pi/2]),$$

where ψ is the canonical character of $C^*(\mathbb{F}_2)$ extending the trivial representation of $\mathbb{F}_2$. If $\hat{\psi} : D \to C^*(\mathbb{F}_2)$ is given by $\hat{\psi}((f,g)) = f(1)$, then $\psi = \hat{\psi}k$, and so

$$j_*k_* = id_{K_*(C^*(\mathbb{F}_2))} + \psi_*,$$

so that

$$id_{K_*(C^*(\mathbb{F}_2))} = j_*k_* - \psi_* = (j_* - \hat{\psi}_*)k_*.$$

Also

$$k_*j_* = id_{K_*(D)} + k_*\hat{\psi}_*,$$

so that

$$id_{K_*(D)} = k_*(j_* - \hat{\psi}_*).$$

It follows that the homomorphism $k_* : K_*(C^*(\mathbb{F}_2)) \to K_*(D)$ is a bijection.

Let $I = \{(f,g) \in D : f(1) = g(1) = 0\}$. Then $I \cong C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ and $I \lhd D$. The short exact sequence

$$0 \longrightarrow I \longrightarrow D \xrightarrow{\hat{\psi}} \mathbb{C} \longrightarrow 0$$

is split, so that

$$K_i(D) \cong K_i(\mathbb{C}) \oplus K_i(C_0(\mathbb{R}) \oplus K_i(C_0(\mathbb{R}).$$

Since $K_0(\mathbb{C}) = \mathbb{Z}$, $K_1(\mathbb{C}) = \{0\}$, $K_0(C_0(\mathbb{R})) = \{0\}$, and $K_1(C_0(\mathbb{R})) = \mathbb{Z}$, it follows that $K_0(C^*(\mathbb{F}_2)) = \mathbb{Z}$ and $K_1(C^*(\mathbb{F}_2)) = \mathbb{Z}^2$.

A.2 Difference Maps. If C is a unital C*-algebra and J an ideal of C which is stably isomorphic to itself, i.e. $J \cong B \otimes K(\ell^2_{\mathbb{N}})$ for some C*-algebra B, let

$$D(C,J) = \{(c,c') : c,c' \in C, c - c' \in J\},$$

and let $*$-homomorphisms $j : J \to D(C,J)$, $\pi : D(C,J) \to C$, and $\theta : C \to D(C,J)$ be defined by

$$j(x) = (x,0), \quad \pi((c,c')) = c', \quad \theta(c) = (c,c).$$

Then the sequence

$$0 \longrightarrow J \xrightarrow{j} D(C,J) \xrightarrow{\pi} C \longrightarrow 0$$

is exact and split, since $\pi\theta = id_C$. Thus

$$K_*(D(C,J)) \cong K_*(J) \oplus K_*(C)$$

and a natural group homomorphism $\sigma : K_*(D(C,J)) \to K_*(J)$ is given by $\sigma = j_*^{-1}(id_{K_*(D(C,J))} - \theta_*\pi_*)$. We now obtain explicit formulae for the action of σ.

Since C, and hence $D(C,J)$, are unital, the elements of $K_1(D(C,J))$ will be equivalence classes of unitaries in $M_n(D(C,J))$, for $n \in \mathbb{N}$. A typical unitary in $M_n(D(C,J)) \cong D(M_n(C), M_n(J))$ is of the form (u,u'), where u and u' are unitaries in $M_n(C)$ such that $1 - uu'^{-1} \in M_n(J)$. Then $uu'^{-1} \in M_n(\tilde{J})$ and $[(u,u')] = [(uu'^{-1},1)] + [(u',u')]$, so that

$$\sigma([(u,u')]) = [uu'^{-1}]$$

The group $K_0(D(C,J))$ is generated by equivalence classes of projections in $M_n(D(C,J))$, for $n \in \mathbb{N}$. A typical projection in $M_n(D(C,J))$ is of the form (e,f), where e and f are projections in $M_n(C)$ such that $e - f \in M_n(J)$. Then

$$[(e,f)] = [(e,f)] - [(f,f)] + [(f,f)]$$

and

$$[(e,f)]-[(f,f)] = ([(e,f)]+[(1-f,1-f)]) - ([(f,f)]+[(1-f,1-f)])$$
$$= [\begin{bmatrix} (e,f) & 0 \\ 0 & (1-f,1-f) \end{bmatrix}] - [\begin{bmatrix} (f,f) & 0 \\ 0 & (1-f,1-f) \end{bmatrix}].$$

Let

$$U = \begin{bmatrix} f & 1-f \\ 1-f & f \end{bmatrix},$$

and $V=(U,U)\in M_{2n}(D(C,J))$. Then V is a self-adjoint unitary and

$$V\begin{bmatrix} (e,f) & 0 \\ 0 & (1-f,1-f) \end{bmatrix}V = (P,\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}),$$

where

$$P = \begin{bmatrix} 1+f(e-f)f & fe(1-f) \\ (1-f)ef & (1-f)e(1-f) \end{bmatrix}.$$

Thus

$$[\begin{bmatrix} (e,f) & 0 \\ 0 & (1-f,1-f) \end{bmatrix}] - \begin{bmatrix} (f,f) & 0 \\ 0 & (1-f,1-f) \end{bmatrix}]$$
$$= [(P,\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})] - [(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})],$$

and so

$$\sigma([(e,f)]) = [P] - [\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}],$$

since $\tilde{J}\cong j(J)+\mathbb{C}(1,1)$.

Now let A be another C*-algebra, which, for simplicity, will be assumed unital. If $\phi,\phi' : A\to C$ are $*$-homomorphisms such that $\phi(a)-\phi'(a)\in J$ for $a\in A$, then the map $(\phi,\phi') : a\to(\phi(a),\phi'(a)); A\to D(C,J)$ is a $*$-homomorphism, and so induces a group homomorphism $(\phi,\phi')_* : K_*(A)\to K_*(D(C,J))$. Let $[\phi,\phi'] : K_*(A)\to K_*(J)$ be the homomorphism $\sigma(\phi,\phi')_*$. If $\phi(1)=p$ and $\phi'(1)=q$, then for u a unitary in $M_n(A)$, $\phi_n(u)+(1-p)\otimes 1_n$ and $\phi'_n(u)+(1-q)\otimes 1_n$ are unitaries in $M_n(C)$ and

$$[\phi,\phi']_1([u]) = [(\phi_n(u)+(1-p)\otimes 1_n)(\phi'_n(u)+(1-q)\otimes 1_n)^{-1}], \qquad (2)$$

and for e a projection in $M_n(A)$,

$$[\phi, \phi']_0([e]) = [P_e] - [\begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}], \tag{3}$$

where $P_e \in M_{2n}(\tilde{J})$ is the projection

$$P_e = \begin{bmatrix} 1_n + \phi'(e)(\phi(e) - \phi'(e))\phi'(e) & \phi'(e)\phi(e)(1_n - \phi'(e)) \\ (1_n - \phi'(e))\phi(e)\phi'(e) & (1_n - \phi'(e))\phi(e)(1_n - \phi'(e)) \end{bmatrix}.$$

It is simple to verify directly that P_e is a projection and a perturbation of $\begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}$ by an element of $M_{2n}(J)$. Indeed equations (2) and (3) can be taken as the definition of the difference map $[\phi, \phi']$. The following properties of difference maps then follow easily from the basic definitions of K-theory.

1. If $\phi_t : A \to C$ ($t \in [0,1]$) is a continuous path of $*$-homomorphisms such that $\phi_0 = \phi_t \ mod\ J$ for $t \in [0,1]$, and $\phi' = \phi_0 \ mod\ J$, then

$$[\phi_0, \phi'] = [\phi_1, \phi'].$$

2. If $\phi : A \to J$ is a $*$-homomorphism, then $[\phi, 0] = \phi_*$ and $[0, \phi] = -\phi_*$.
3. For any $*$-homomorphism $\phi : A \to C$, $[\phi, \phi] = 0$.
4. If $\phi_i, \phi_i' : A \to C$ are $*$-homomorphism such that $\phi_i = \phi_i' \ mod\ J$ ($i = 1, 2$), and $\phi_1 \oplus \phi_2 : A \to M_2(C)$ is given by

$$(\phi_1 \oplus \phi_2)(a) = \begin{bmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(a) \end{bmatrix},$$

then $\phi_1 \oplus \phi_2 = \phi_1' \oplus \phi_2' \ mod\ M_2(J)$ and

$$[\phi_1 \oplus \phi_2, \phi_1' \oplus \phi_2'] = [\phi_1, \phi_1'] + [\phi_2, \phi_2'].$$

5. If $\psi : A_1 \to A$ is a $*$-homomorphism, and $\phi, \phi' : A \to C$ are equal $mod\ J$, then $\phi\psi = \phi'\psi \ mod\ J$ and

$$[\phi\psi, \phi'\psi] = [\phi, \phi']\psi_*.$$

6. If $\phi, \phi' : A \to C$ are equal $mod\ J$, and $\psi : C \to D$ is a $*$-homomorphism, let $K = \psi(J)$. Then $\psi\phi = \psi\phi' \ mod\ K$ and

$$[\psi\phi, \psi\phi'] = \psi_*[\phi, \phi'],$$

where $\psi_* : K_*(J) \to K_*(K)$ is the homomorphism induced by $\psi|_J$.

A.3 The K-groups of $C_r^*(\mathbb{F}_n)$. Let $n \in \mathbb{N}$ be fixed, and let $\ell_0^2(\mathbb{F}_n) = \ell^2(\mathbb{F}_n) \ominus (\mathbb{C}\xi_e)$. A unitary representation $\lambda_0 : \mathbb{F}_n \to B(\ell_0^2(\mathbb{F}_n))$ is given by

$$\lambda_0(u_i)\xi_g = \begin{cases} \xi_{u_i} & (g = u_i^{-1}) \\ \xi_{u_i g} & (g \neq u_i^{-1}) \end{cases} \quad (1 \le i \le n),$$

where $u_i, \ldots, u_n$ are the generators of $\mathbb{F}_n$. Let S_i be the set of those elements of $\mathbb{F}_n$ which, when written as reduced words in the u_j, end in a non-zero power of u_i, for $i = 1, \ldots, n$. Then $S_1, \ldots, S_n$ is a partition of $\mathbb{F}_n \setminus \{e\}$. Let H_i be the closed subspace of $\ell_0^2(\mathbb{F}_n)$ generated by $\{\xi_g : g \in S_i\}$. Then H_i is a reducing subspace for λ_0 and $\lambda_0|_{H_i} \simeq \lambda$. In fact if $V_i : \ell^2(\mathbb{F}_n) \to H_i$ is given by

$$V_i\xi_g = \begin{cases} \xi_g & (g \text{ ends in a positive power of } u_i) \\ \xi_{gu_i^{-1}} & (\text{otherwise}), \end{cases}$$

then V_i is unitary and for $g \in \mathbb{F}_n$, $\lambda(g) = V_i^*\lambda_0(g)V_i$. Since $H_i \perp H_j$ for $i \neq j$, it follows that $\lambda_0 \simeq \lambda \oplus \ldots \oplus \lambda$ (n copies).

If $\pi : \mathbb{F}_n \to B(H)$ is a any representation of $\mathbb{F}_n$, let $\overline{\pi}$ denote the canonical extension of π to a representation of $C^*(\mathbb{F}_n)$ on H. Now let $\omega : C^*(\mathbb{F}_n) \to B(K)$ be a faithful representation of $C^*(\mathbb{F}_n)$, and let $\pi = \omega|_{\mathbb{F}_n}$. Recall that $\pi \otimes \lambda$ is equivalent to a multiple of λ. In fact, if U is the unitary on $\ell^2(\mathbb{F}_n) \otimes K \cong \ell^2(\mathbb{F}_n, K)$ given by

$$(U\xi)(g) = \pi(g)\xi(g) \quad (\xi \in \ell^2(\mathbb{F}_n),$$

then

$$U^*(\pi(g) \otimes \lambda(g))U = 1_K \otimes \lambda(g) \quad (g \in \mathbb{F}_n).$$

Thus $\pi \otimes \lambda$ and $\pi \otimes \lambda_0$ are equivalent to multiples of λ, and their extensions $\overline{\pi \otimes \lambda}$ and $\overline{\pi \otimes \lambda_0}$ to $C^*(\mathbb{F}_n)$ factor through $C_r^*(\mathbb{F}_n)$. We can regard λ_0 as a degenerate representation of $\mathbb{F}_n$ on $\ell^2(\mathbb{F}_n)$, and $\pi \otimes \lambda_0$ as a degenerate representation of $C^*(\mathbb{F}_n)$ on $K \otimes \ell^2(\mathbb{F}_n)$. Thus there are $*$-homomorphisms $\theta, \theta_0 : C_r^*(\mathbb{F}_n) \to C^*(\mathbb{F}_n) \otimes B(\ell^2(\mathbb{F}_n))$ such that $\overline{\pi \otimes \lambda} = \theta\overline{\lambda}$ and $\overline{\pi \otimes \lambda_0} = \theta_0\overline{\lambda}$. Now $\lambda(u_i) - \lambda_0(u_i)$ is of finite rank, and so $(\pi \otimes \lambda)(u_i) - (\pi \otimes \lambda_0)(u_i) \in C^*(\mathbb{F}_n) \otimes K(\ell^2(\mathbb{F}_n))$, for $1 \le i \le n$. Thus $\overline{\pi \otimes \lambda} = \overline{\pi \otimes \lambda_0}$ (respectively, $\theta = \theta_0$) *mod* $C^*(\mathbb{F}_n) \otimes K(\ell^2(\mathbb{F}_n))$.

Let $t : \mathbb{F}_n \to \mathbb{C}$ be the trivial representation. If the representation $\lambda_0 \oplus t$ is identified with the representation on $\ell^2(\mathbb{F}_n)$ whose restrictions to

$\mathbb{C}\xi_e$ and $(\mathbb{C}\xi_e)^\perp$ are t and λ_0, respectively, then $\lambda(u_i)^*(\lambda_0(u_i) \oplus t(u_i))$ is in $\mathbb{C}1 + K(\ell^2(\mathbb{F}_n)$ for each i. By spectral theory $\lambda(u_i)^*(\lambda_0(u_i) \oplus t(u_i))$ can be connected to 1 by a norm-continuous path of unitaries in $\mathbb{C}1 + K(\ell^2(\mathbb{F}_n))$. We thus obtain a path of representations $\{\lambda^s\}_{0\le s\le 1}$ of $\mathbb{F}_n$ such that $\lambda^0 = \lambda$, $\lambda^1 = \lambda_0 \oplus t$, $\lambda^s(g) - \lambda(g) \in K(\ell^2(\mathbb{F}_n))$ for $g \in \mathbb{F}_n$ $(0 \le s \le 1)$, with $s \to \lambda^s(g)$ norm-continuous for each g. We can regard t and λ as degenerate representations of $\mathbb{F}_n$ on $\ell^2(\mathbb{F}_n)$ and $\ell_0^2(\mathbb{F}_n) \oplus \ell^2(\mathbb{F}_n)$, respectively. Then

$$\overline{\pi \otimes (\lambda_0 \oplus t)} \simeq (\overline{\pi \otimes \lambda_0}) \oplus (\overline{\pi \otimes t}),$$

and

$$\begin{aligned}
[\theta, \theta_0]\overline{\lambda}_* &= [\theta\overline{\lambda}, \theta_0\overline{\lambda}] \\
&= [\overline{\pi \otimes \lambda}, \overline{\pi \otimes \lambda_0}] \\
&= [\overline{\pi \otimes (\lambda_0 \oplus t)}, \overline{\pi \otimes \lambda_0}] \\
&= [\overline{\pi \otimes t}, 0] + [\overline{\pi \otimes \lambda_0}, \overline{\pi \otimes \lambda_0}] \\
&= (\pi \otimes t)_* \\
&= \overline{\pi}_* \\
&= \omega_*,
\end{aligned}$$

i.e. $\omega_* = [\theta, \theta_0]\overline{\lambda}_*$.

Now $(\overline{\lambda}\otimes 1)\theta|_{\mathbb{F}_n} = \lambda\otimes\lambda$, $(\overline{\lambda}\otimes 1)\theta_0|_{\mathbb{F}_n} = \lambda\otimes\lambda_0$ and $\lambda\otimes\lambda \simeq (\lambda\otimes\lambda_0)\oplus(\lambda\otimes t) \simeq (\lambda \otimes \lambda_0) \oplus \lambda$. Thus $(\overline{\lambda} \otimes 1)\theta$ and $(\overline{\lambda} \otimes 1)\theta_0$ are multiples of the identity representation π_r of $C_r^*(\mathbb{F}_n)$, and $(\overline{\lambda} \otimes 1)\theta \simeq (\overline{\lambda} \otimes 1)\theta_0 \oplus \pi_r$. Hence

$$\begin{aligned}
[(\overline{\lambda} \otimes 1)\theta, (\overline{\lambda} \otimes 1)\theta_0] &= [\pi_r, 0] + [(\overline{\lambda} \otimes 1)\theta_0, (\overline{\lambda} \otimes 1)\theta_0] \\
&= (\pi_r)_*,
\end{aligned}$$

i.e. $\overline{\lambda}_*[\theta, \theta_0] = (\pi_r)_*$. Since ω_* and $(\pi_r)_*$ are the identity maps on $K_*(C^*(\mathbb{F}_n))$ and $K_*(C_r^*(\mathbb{F}_n))$, respectively, it follows that $\overline{\lambda}_* : K_*(C^*(\mathbb{F}_n)) \to K_*(C_r^*(\mathbb{F}_n))$ is an isomorphism. Thus by the results of §A.1,

$$\boxed{K_0(C_r^*(\mathbb{F}_n)) = \mathbb{Z},\ K_1(C_r^*(\mathbb{F}_n)) = \mathbb{Z}^n.}$$

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