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held at the Seoul National University

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PREFACE

The second GARC Symposium on Pure and Applied Mathematics was held at Seoul National University from February 4 to 20, 1993.

The symposium was organized by the Global Analysis Research Center which was founded in 1991 as one of 30 centers of excellence under the supports of the Korea Science and Engineering Foundation.

The symposium covered a broad range of topics in the fields of mathematical analysis and global analysis. It was carried out in 6 sessions ; nonlinear analysis, operator algebras, partial differential equations, topology and geometry of manifolds, differential geometry and complex algebraic varieties and several complex variable.

Among them the session of partial differential equations was held in the form of the first Korea-Japan joint conference. We expect the second joint conference will be held in Japan next year. We are pleased to express here our thanks to those participants from Japan whose collaboration made the conference a successful one.

The GARC symposium was actively attended by more than 200 participants including 16 foreign mathematicians. This proceedings of three issues contains research articles which were presented. The content will be of interest both to the members of the Global Analysis Research Center and to mathematicians working in the various fields of current mathematics.

We wish to express our gratitude to all contributors and especially to those mathematicians from abroad. We also express our thanks to the Korea Science and Engineering Foundation for having made this symposium possible, to Professors Sage Lee and Hyuk Kim for their endeavor in organizing this symposium and to Miss Jin Young Bae and Mr. Kyung Whan Park for their help in editing the proceedings.

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ACTIONS OF INFINITE DISCRETE GROUPS OF PROJECTIVE TRANSFORMATIONS

HYUK KIM

1. Introduction

In this paper, we will discuss some results about the actions of infinite discrete groups of projective transformations such as the limit singular projective transformations of the groups and the domains of discontinuity which arise naturally by these limit transformations.

After some results about these domains have bean obtained, it was found that the subject was already studied by Myrberg several decades ago [4], and more recently the domains of discontinuity for a discrete group action in a more general topological setting by Kulkarni [3]. Since the subject seems to be quite important to those who are interested in the geometric structures on manifolds, we reintroduce some of their works and make a comparison of their domains of discontinuity.

In general, the difficult part in the study of an infinite group action lies at the behavior of the action at infinity which is closely related to the structure of the end of the group. The detailed analysis of the limit singular projective transformations given as accumulation points of the group of projective transformations would then be expected to serve as a key for understanding of projective actions at infinity.

While the study of subgroups of projective transformations and their actions should be of fundamental importance in various branches of geometry, our motivation stems from the holonomy actions of projectively flat manifolds. (See [2, 5] for the notion of projectively flat manifolds.) The class of projectively flat manifolds is quite broad including the classical space forms (i.e., Riemannian or pseudo-Riemannian manifolds with constant curvature)

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and affinely flat manifolds. If we restrict the projective actions to subvarieties, we will have more list of interesting examples such as conformally flat manifolds. Also the essential part of the investigation of the structure of projectively flat bundles or foliations lies in the study of the associated holonomy actions. All the known special cases and detailed study about these examples will thus suggest a guideline for a future development of uniformizing theory which hopefully provide comparison and interrelations between the various substructures through the extrinsic viewpoint in a broader setting, namely in the projective space.

Let $PGL(n+1, \mathbb{F})$ be the group of projective transformations over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and Γ be an infinite subgroup of $PGL(n+1, \mathbb{F})$. If Γ acts properly discontinuously on a subset of the projective space $\mathbb{P}^n$, then Γ is clearly discrete. But a discrete subgroup Γ in general does not act on $\mathbb{P}^n$ properly discontinuously and we want to find a canonical domain on which Γ acts properly discontinuously. This problem is studied in a general topological setting in [3]. But if we concentrate on the projetive actions, we can formulate such domains in a canonical way using the limit singular transformations as observed in [4]. Let $M(n+1, \mathbb{F})$ be the space of $(n+1)$ by $(n+1)$ matrices identified with $\mathbb{F}^{(n+1)^2}$, and let $PM(n+1, \mathbb{F})$ be its projectivization which can be considered as a canonical compactification of $PGL(n+1, \mathbb{F})$. Let Γ' be the set of accumulation points of Γ in $PM(n+1, \mathbb{F})$. Then an element γ of Γ' will be a singular projective transformation and the kernel of γ will be denoted by $K(\gamma)$. Let $K(\Gamma)$ be the union of $K(\gamma)$ for $\gamma \in \Gamma'$. Then the complement of $K(\Gamma)$ is the desired domain on which Γ acts properly discontinuously. (See 2.6 below.) It can also be shown that this domain in general is contained in the domain of discontinuity in the sense of Kulkarni [3]. These two domains agree if the orbit space is compact and connected. See 2.9 and 2.10 for the precise statements. We also present some examples in section 3 illustrating these domains and other interesting invariant sets, including the examples where Γ occurs as the holomony group of various 2-dimensional affine tori.

2. Domains of discontinuity for discrete subgroups

If we projectivize a singular linear transformation $A \in M(n+1, \mathbb{F}) - GL(n+1, \mathbb{F})$, its projectivization α is not defined on the projectivization of the kernel of A which we denote by $K(\alpha)$, the *kernel* of α. Hence the *domain* of α is $\mathbb{P}^n - K(\alpha)$ and the *range* $R(\alpha)$ is the projectivization of the range of A. Such an $\alpha \in PM(n+1,\mathbb{F}) - PGL(n+1,\mathbb{F})$ will be called a *singular* projective transformation.

The natural action of $GL(n+1,\mathbb{F})$ on $\mathbb{F}^{n+1}$ can be extended to $M(n+1,\mathbb{F})$ canonically so that if $A_n \to A$ in $M(n+1,\mathbb{F})$ and $v_n \to v$ in $\mathbb{F}^{n+1}$ then $A_n v_n \to Av$.

This, of course, follows from

$$\begin{aligned} ||Av - A_n v_n|| &\leqq ||Av - A_n v|| + ||A_n v - A_n v_n|| \\ &\leqq ||A - A_u|| \; ||v|| + ||A_n|| \; ||v - v_n||, \end{aligned}$$

where $|| \; ||$ stands for the usual l_2-norm on Euclidean space. When we projectivize this statement, we need to be careful about the domain of singular transformation as follows.

2.1. *If $\alpha_n \to \alpha$ in $PM(n+1,\mathbb{F})$ and $x_n \to x$ in $\mathbb{P}^n$ for $x_n \notin K(\alpha_n)$, then $\alpha_n x_n \to \alpha x$ whenever $x \notin K(\alpha)$.*

Let Γ be a subgroup of $PGL(n+1,\mathbb{F})$. In order to investigate the cannonical extension of the action of Γ on $\mathbb{P}^n$ to the set Γ' of accumulation points of Γ in $PM(n+1,\mathbb{F})$, we need to consider the kernel and range of the elements of Γ' first. Let $K(\gamma)$ and $R(\gamma)$ be the kernel and range of $\gamma \in \Gamma'$ respectively. If $\gamma \in \Gamma'$ is non-singular, $K(\gamma) = \emptyset$ and $R(\gamma) = \mathbb{P}^n$, and hence we are interested in only singular γ. Let $K(\Gamma)$ and $R(\Gamma)$ be the union of $K(\gamma)$ and $R(\gamma)$ respectively for singular $\gamma \in \Gamma'$. Now we have the followings as observed in [4].

2.2. *$K(\Gamma)$ and $R(\Gamma)$ are Γ-invariant.*

Proof. In general, for $\alpha \in PGL(n+1,\mathbb{F})$ and $\beta \in PM(n+1,\mathbb{F})$, $\alpha(R(\beta)) = R(\alpha\beta\alpha^{-1})$ and $\alpha(K(\beta)) = K(\alpha\beta\alpha^{-1}) = K(\beta\alpha^{-1})$. Now the Γ-invariance of $K(\Gamma)$ and $R(\Gamma)$ follows from the Γ-invariance of Γ'. q.e.d.

2.3. *$K(\Gamma)$ is a closed subset of $\mathbb{P}^n$.*

Proof. Let x be an accumulation point of $K(\Gamma)$, say $x_n \to x$ with $x_n \in K(\alpha_n)$ and $\alpha_n \in \Gamma'$. We may assume that α_n are distinct and $\alpha_n \to \alpha \in PM(n+1,\mathbb{F})$ by taking a subsequence if necessary. Note that $\alpha \in \Gamma'$ since Γ' is closed. Choose $A_n \to A$ in $GL(n+1,\mathbb{F})$ and $v_n \to v$ in $\mathbb{F}^{n+1}$ whose projectivization corresponds to $\alpha_n \to \alpha$ and $x_n \to x$ respectively. Since $0 = A_n v_n \to Av$, we have $Av = 0$ and $x \in K(\alpha) \subset K(\Gamma)$. q.e.d.

Now suppose that $\alpha \in PM(n+1,\mathbb{F})$ is singular and let $\alpha_n \to \alpha$, $\alpha_n \in PGL(n+1,\mathbb{F})$. In general, we can not guarantee that α_n^{-1} converges unless α is non-singular. But we can always find a subsequence which converges in compact $PM(n+1,\mathbb{F})$. Hence we may assume $\alpha_n^{-1} \to \overline{\alpha} \in PM(n+1,\mathbb{F})$ by taking a subsequence. Now then $\alpha_n^{-1}\alpha_n \to \overline{\alpha}\alpha$ and this is absurd because $\alpha_n^{-1}\alpha_n = id$ and $\overline{\alpha}\alpha$ is singular. We look at the problem in $M(n+1,\mathbb{F})$. We can choose $A_n \to A$ whose projectivization is $\alpha_n \to \alpha$, but no A_n^{-1} can converge in $M(n+1,\mathbb{F})$ because of the above contradiction. However we can choose a subsequence again denoted by A_n and $\lambda_n \in \mathbb{F}$ such that $\lambda_n A_n^{-1} \to \overline{A}$. Now $\lambda_n I = \lambda_n A_n^{-1} A_n \to \overline{A}A$ and the projectivization forces that $\lambda_n \to 0$ and $\overline{A}A = 0$. Hence $R(\alpha) \subset K(\overline{\alpha})$. Now apply this argument to Γ.

2.4. *If $\gamma_n \to \gamma$ and $\gamma_n^{-1} \to \overline{\gamma}$ for $\gamma_n \in \Gamma$ and singular $\gamma, \overline{\gamma} \in \Gamma'$, then $R(\gamma) \subset K(\overline{\gamma})$ and $R(\overline{\gamma}) \subset K(\gamma)$. In particular, $R(\Gamma) \subset K(\Gamma)$.*

When Γ is a discrete subgroup of $PGL(n+1,\mathbb{F})$, we have the following simple characterization as observed in [4].

2.5. *Γ is discrete if and only if each element of Γ' is singular.*

Proof. Suppose that an element $\gamma \in \Gamma'$ is non-singular and let a sequence of distinct elements γ_n of Γ converges to γ. Then clearly γ_n^{-1} converges to γ^{-1} and $\gamma_n^{-1}\gamma_{n+1}$ converges to the identity. Then Γ can not be discrete. The converse is obvious since the identity is an isolated point of Γ. q.e.d.

Note that Γ is infinte if and only if $\Gamma' \neq \emptyset$.

Now we are ready to show that the complement of $K(\Gamma)$ is the domain of discontinuity on which a discrete Γ acts properly discontinuously as desired. (Compare [4].)

2.6. *Let Γ be a discrete subgroup of $PGL(n+1,\mathbb{F})$. Then Γ acts properly discontinuously on $K(\Gamma)^c = \mathbb{P}^n - K(\Gamma)$.*

Proof. Let C be a compact subset of $K(\Gamma)^c$. Suppose that the number of $\alpha \in \Gamma$ such that $\alpha C \cap C \neq \emptyset$ is infinte. Then we can choose a sequence x_n in C and a sequence of distinct α_n in Γ with $\alpha_n x_n \in C$. Since C is compact, we may assume $x_n \to x$ and $\alpha_n x_n \to y$ in C by taking subsequences. We can further assume $\alpha_n \to \alpha$ again by taking subsequence. By 2.1, $\alpha_n x_n \to \alpha x$ since $x \in C \subset K(\Gamma)^c$ and hence $y = \alpha x$. But then $y \in R(\alpha) \subset R(\Gamma)$ since α is singular by 2.5. This shows that $y \in K(\Gamma)$ by 2.4 and gives a contradiction since $y \in C \subset K(\Gamma)^c$. q.e.d.

It would be interesting to compare $K(\Gamma)^c$ for a discrete Γ with the domain of discontinuity in the sense of Kulkarni. Following [3], a point $x \in \mathbb{P}^n$ is called a *cluster point* of an indexed family $\{\gamma A | \gamma \in \Gamma\}$ for a given set $A \subset \mathbb{P}^n$ if every neighborhood of x intersects γA for infinitely many γ in Γ. Let $L_\circ(\Gamma)$ be the closure of the set of points in $\mathbb{P}^n$ with infinite isotropy group, and $L_1(\Gamma)$ be the closure of the set of cluster points of $\{\gamma z | \gamma \in \Gamma\}$ for all points z in the complement of $L_\circ(\Gamma)$. It is easy to show that these sets are Γ-invariant closed sets which lie in the complement of any open set on which Γ acts properly discontinuously by standard argument. Also we can show directly that $L_\circ(\Gamma) \cup L_1(\Gamma) \subset K(\Gamma)$ as follows.

2.7. *(i) If $x \in \mathbb{P}^n$ has an infinite isotropy group, then $x \in K(\Gamma)$. (ii) If $x \in \mathbb{P}^n$ is a cluster point of $\{\gamma z | \gamma \in \Gamma\}$ for a point $z \in \mathbb{P}^n$, then $x \in K(\Gamma)$.*

Proof. (i) Choose $\gamma_n \in \Gamma_x$ such that $\gamma_n \to \gamma$. Then $\gamma_n x \to \gamma x$ if $x \notin K(\gamma)$. Thus either $x \in K(\gamma) \subset K(\Gamma)$ or $x \in R(\gamma) \subset K(\Gamma)$. Note that γ is singular since Γ is discrete. (ii) Let $\gamma_n z \to x$ with distinct γ_n in Γ. If $z \in K(\Gamma)$, then $x \in K(\Gamma)$ since $K(\Gamma)$ is a closed Γ-invariant set. If $z \notin K(\Gamma)$, we may assume $\gamma_n \to \gamma$ by passing to a subsequence and $\gamma_n z \to \gamma z$ by 2.1. Thus $x = \gamma z \in R(\gamma) \subset K(\Gamma)$. q.e.d.

Again following Kulkarni, let $L_2(\Gamma)$ be the closure of the set of cluster points of $\{\gamma C | \gamma \in \Gamma\}$ for all compact C in the complement of $L_\circ(\Gamma) \cup L_1(\Gamma)$. Now the sets $\Lambda(\Gamma) := L_\circ(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)$ and $\Omega(\Gamma) := \mathbb{P}^n - \Lambda(\Gamma)$ are called the *limit set* of Γ and the *domain of discontinuity* of Γ respectively by Kulkarni.

2.8. *Let C be a compact set in the complement of the set of cluster points of $\{\gamma z | \gamma \in \Gamma\}$ for all $z \in K(\Gamma)^c$, and let x be a cluster point of $\{\gamma C | \gamma \in \Gamma\}$. Then $x \in K(\Gamma)$.*

Proof. Let $\gamma_n y_n \to x$ with y_n in C and distinct γ_n in Γ. By taking subsequences, we may assume $\gamma_n^{-1} \to \overline{\gamma}$ and $y_n \to y \in C$. If $x \in K(\Gamma)^c, y_n = \gamma_n^{-1}(\gamma_n y_n) \to \overline{\gamma} x$ by 2.1 and hence $y = \overline{\gamma} x$, which shows that $\gamma_n^{-1} x \to \overline{\gamma} x = y$, i.e., y is a cluster point of $\{\gamma x | \gamma \in \Gamma\}$, a contradiction. q.e.d.

2.8 shows immediately that $L_2(\Gamma) \subset K(\Gamma)$ and we can conclude as follows.

2.9. $\Lambda(\Gamma) \subset K(\Gamma)$, *or equivalently,* $K(\Gamma)^c \subset \Omega(\Gamma)$.

It is not clear at the moment whether the equality holds in 2.9 in general, but this is obviously true in the following situation.

2.10. *Suppose the quotient space $\Omega(\Gamma)/\Gamma$ is connected and $K(\Gamma)^c/\Gamma$ is compact. Then $K(\Gamma)^c = \Omega(\Gamma)$.*

Proof. The inclusion $i : K(\Gamma)^c \to \Omega(\Gamma)$ induces an open map $\bar{i} : K(\Gamma)^c/\Gamma \to \Omega(\Gamma)/\Gamma$ since the orbit projection map is always open. The quotient space $\Omega(\Gamma)/\Gamma$ is Hausdorff since the Γ-action is properly discontinuous, and hence $\bar{i}(K(\Gamma)^c/\Gamma)$ is an open and closed subset of a connected space $\Omega(\Gamma)/\Gamma$. This shows i is onto. q.e.d.

3. Examples

In this section, we will discuss some examples and describe Myrberg's and Kulkarni's domain of discontinuity for each of the examples. We use the homogeneous coordinates for a point in a projective space : The projectivization of a point $(a_1, a_2, \cdots, a_{n+1}) \in \mathbb{F}^{n+1}$ is denoted by $[a_1, a_2, \cdots, a_{n+1}] \in \mathbb{P}^n$ and similarly the projectivization of a linear transformation $A = (a_{ij}) \in M(n+1, \mathbb{F})$ is denoted by $\alpha = [a_{ij}] \in PM(n+1, \mathbb{F})$.

When we compute the limit transformation α of a given convergent sequence $\alpha_n \in PM(n+1, \mathbb{F})$, we can use any l_p-norm on $\mathbb{F}^{(n+1)^2}$ since these are equivalent. For most of the example we consider, it is more convenient to use l_2-norm or l_∞-norm, i.e., $||A||_\infty = max\,\{|a_{ij}|\}$ for $A = (a_{ij}) \in M(n+1, \mathbb{F})$.

3.1. *Let Γ be a subgroup of $GL(2, \mathbb{F})$ generated by $\alpha = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.*

If we projectivize Γ, still denoted by Γ, we obtain a discrete group of projective transformations acting on $\mathbb{P}^1$. Since $\alpha^n = \begin{bmatrix} 2^n & \\ & 2^{-n} \end{bmatrix}$ converges to $\gamma = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$ as $n \to \infty$ and to $\overline{\gamma} = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$ as $n \to -\infty$, we have two limit singular transformations γ and $\overline{\gamma}$. Clearly $K(\gamma) = R(\overline{\gamma}) = [0,1] \in \mathbb{P}^1$ and $K(\overline{\gamma}) = R(\gamma) = [1,0] \in \mathbb{P}^1$. Hence $K(\Gamma) = R(\Gamma) = \Lambda(\Gamma) = \{[1,0],[0,1]\}$ and $K(\Gamma)^c/\Gamma = \Omega(\Gamma)/\Gamma$ is two copies of incomplete affinely flat circles when $\mathbb{F} = \mathbb{R}$.

Now this time, consider the action of Γ as an affine part of 2-dimensional projective action by compactifying α as $\begin{bmatrix} 2 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \in PGL(3, \mathbb{F})$. Then the same computation as above gives two limit singular transformations $\gamma = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$

and $\overline{\gamma} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}$, and thus $K(\gamma)$ is the line passing through the points $[0,1,0]$ and $[0,0,1]$ and $R(\gamma)$ is the point $[1,0,0]$ in $\mathbb{P}^2$, and similarly for $K(\overline{\gamma})$ and $R(\overline{\gamma})$. Hence $K(\Gamma)$ is the union of two lines and $R(\Gamma)$ consists of two points. $K(\Gamma)^c = \Omega(\Gamma)$ consists of two convex domains when $\mathbb{F} = \mathbb{R}$. Note that the line at infinity is a Γ-invariant subset and the action of Γ restricted on this line is the 1-dimensional projective action described above. Also note that γ (*or* $\overline{\gamma}$) has the property that $R(\gamma) \cap K(\gamma) = \emptyset$. A singular transformation with this property may be called *hyperbolic.*

3.2. *Let Γ be an abelian subgroup of $PGL(3,\mathbb{R})$ generated by* $\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

For each $[p,q] \in \mathbb{P}^1$, there is a subsequence of $\alpha^n\beta^m = \begin{bmatrix} 1 & 0 & n \\ & 1 & m \\ & & 1 \end{bmatrix}$ which converges to $\gamma = \begin{bmatrix} 0 & 0 & p \\ 0 & 0 & q \\ 0 & 0 & 0 \end{bmatrix} \in \Gamma'$ and these are the only limit singular transformations. Thus Γ' is homeomorphic to $\mathbb{P}^1$ and each $\gamma \in \Gamma'$ has the range $[p,q,0] \in \mathbb{P}^2$ and the line at infinity, i.e., the line passing though $[1,0,0]$ and $[0,1,0]$ as the common kernel for all $\gamma \in \Gamma'$. And $K(\Gamma)^c = \Omega(\Gamma)$ is an affine subspace of $\mathbb{P}^2$ whose quotient with Γ is a complete affinely flat 2-torus. Note that for each $\gamma \in \Gamma', R(\gamma) \subset K(\gamma)$ and a singular transformation with this property many be called *parabolic.*

3.3. *Let Γ be an abelian subgroup of $PGL(3,\mathbb{R})$ generated by* $\alpha = \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$ *and* $\beta = \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$.

In this case, $\alpha^n\beta^m = \begin{bmatrix} 1 & m & n+\frac{1}{2}m(m-1) \\ & 1 & m \\ & & 1 \end{bmatrix}$ and it is clear that for each $[p,q] \in \mathbb{P}^1$, there is a sequence in Γ converging to $\gamma_{[p,q]} = \begin{bmatrix} 0 & q & p \\ & 0 & q \\ & & 0 \end{bmatrix} \in \Gamma'$ and these limits exhaust Γ'. This shows again that Γ' is homeomorphic to $\mathbb{P}^1$, and $K(\Gamma) = R(\Gamma) = \Lambda(\Gamma)$ is the line at infinity. But this example is essentially different from the example of 3.2 in that all $\gamma_{[p,q]} \in \Gamma'$ except $\gamma_{[1,0]}$ has 1-dimensional range in $\mathbb{P}^2$. This shows that the quotient affine torus obtained from 3.3 is *not projectively equivalent* to that of 3.2. It is well known that the affine torus of 3.2 and 3.3 are not affinely equivalent. Nevertheless they are algebraically equivalent, i.e., there is an equivariant polynomial diffeomorphism : $\mathbb{R}^2 \to \mathbb{R}^2$. (See [1].)

However, this diffeomorphism can not be extended to even a homeomorphism on $\mathbb{P}^2$ equivariantly since the corresponding $L_\circ(\Gamma)$ are certainly different.

3.4. *Let Γ be an abelian subgroup of $PGL(3, \mathbb{R})$ generated by* $\alpha = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ *and* $\beta = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}$.

In this case, $\alpha^n \beta^m = \begin{bmatrix} 2^n & & \\ & 2^m & \\ & & 1 \end{bmatrix}$ and it is easy to show that $\gamma \in \Gamma'$ is of the form $\begin{bmatrix} 2^p & & \\ & 2^q & \\ & & 0 \end{bmatrix}$, $p, q \in \mathbb{Z} \cup \{-\infty\}$, $\begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$. Hence $K(\Gamma)$ consists of three lines determined by the three coordinate planes and $R(\Gamma)$ is equal to $K(\Gamma)$. In this case, $K(\Gamma)^c = \Omega(\Gamma)$ consists of four triangular regions, each of which corresponds to a quadrant in an affine plane determined by $x_3 = 1$.

Notice that if we take logarithm to this action on the first quadrant, it becomes the example of 3.2. Hence the intrinsic actions on a domain of discontinuity are analytically equivalent.

3.5. *Let $\Gamma < PO(2,1) < PGL(3, \mathbb{R})$ be a Fuchsian representation of a closed surface group.*

Let D and ∂D be the projectivizations of the inside of the light cone and the light cone respectively so that D serves as projective model for hyperbolic plane. Let γ be a hyperbolic isometry of D, and let $\gamma^n \to \alpha$ and $\gamma^{-n} \to \overline{\alpha}$ as $n \to \infty$. Let l be the invariant hyperbolic axis for γ and let $l \cap \partial D = \{p, q\}$ where p is the source and q is the sink of γ. Since γ^n sends all the points in D to q, α sends $D - K(\alpha)$ to q and hence $R(\alpha) = \{q\}$. Similarly, $\{p\} = R(\overline{\alpha}) \subset K(\alpha)$. Therefore $K(\alpha)$ is a line passing through p which is clearly invariant under γ. It is clear that the only invariant lines passing through p are the axis l and the line k tangent to ∂D since ∂D and p are invariant under γ. Thus k and l are invariant under $\alpha = \lim \gamma^n$. If l were the $K(\alpha)$, α has to send $k - \{p\} \subset \mathbb{P}^2 - l$ to $q = R(\alpha)$, which contradicts the invariance of k under α. Thus we conclude that $K(\alpha) = k$. Similarly, $K(\overline{\alpha})$ is the tangent line passing through q. Now if Γ is a Fuchsian group of the first kind, the hyperbolic fixed points form a dense set in ∂D, the limit set of Γ and hence $K(\Gamma) = \mathbb{P}^2 - D$ since it is closed. Thus $K(\Gamma)^c = D$ and

$R(\Gamma) = \partial D$ since each point of ∂D is a cluster point of $\{\gamma C | \gamma \in \Gamma\}$ for any compact set C in D. Myrberg discussed this phenomenon in a general setting with arbitrary invariant quadratic forms in [4]. For this example, Kulkarni showed that $\Lambda(\Gamma) = \mathbb{P}^2 - D$ and hence $\Omega(\Gamma) = D$ [3]. Again Myrberg's and Kulkarni's domain of discontinuity agree.

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VANISHING THEOREMS FOR EULER CHARACTERISTIC

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In this note we discuss rank invariants (1) of finitely generated projective modules via a 'trace function' on the endomorphism group, (2) of finitely generated free modules of a group ring via a non-central localization, and (3) of $\ell_2(G)$-modules.

I. Hattori-Kaplansky-Stallings Ranks

Let A denote a ring; A-modules are understood to be right A-modules.

1. The group $T(A)$. It is the quotient $A/[A,A]$ of A by the additive subgroup $[A,A]$ generated by all commutators $[a,b] = ab - ba$, $a,b \in A$. We write T or T_A for the canonical projection $A \to T(A)$.

2. Coordinate systems of projective modules. An A-module P is projective if and only if there is a family (x_i) in P and (f_i) in $P^* = \mathrm{Hom}_A(P,A)$ such that, for all $x \in P$, $f_i(x) = 0$ for all but finitely many i, and $x = \sum_i x_i \cdot f_i(x)$. The system $(x_i),(f_i)$ will be called an *A-coordinate system* on P. Clearly, P is finitely generated if and only if there is a finite coordinate system $(x_i),(f_i)$. Let $\mathcal{P}(A)$ denote the category of finitely generated projective A-modules.

3 Traces T_P. Let $P \in \mathcal{P}(A)$. Define $t: P \times P^* \to T(A)$ by $t(x,f) = T(f(x))$. If $a \in A$ then $t(x \cdot a, f) = T(f(x \cdot a)) = T(f(x) \cdot a) = T((a \cdot f)(x)) = t(x, a \cdot f)$. Thus t induces an additive map $P \otimes_A P^* \to T(A)$. Since $P \in \mathcal{P}(A)$, $P \otimes_A P^* \to \mathrm{End}_A(P), x \otimes f(y) = x \cdot f(y)$, is an isomorphism. We have thus defined a homorphism

$$T_P : \mathrm{End}_A(P) \longrightarrow T(A),$$

called the *trace* on the A-module P. It is characterized by $T_P(x \otimes f) = T(f(x))$ for $x \in P$ and $f \in P^*$. Choose any finite coordinate system $(x_i),(f_i)$ of P and write $1_P = \sum x_i \otimes f_i$. If $u \in \mathrm{End}_A(P)$, then $u = u \circ 1_P =$

$\sum u(x_i) \otimes f_i$, so $T_P(u) = T(\sum f_i(u(x_i)))$. Note that if x_i is a free basis of P then f_i is the dual basis and the $u_{ji} = f_j(u(x_i))$ are the coefficients of the matrix representing u: $u(x_i) = \sum_j x_j u_{ji}$. Hence $T_P(u) = T(\sum u_{ii})$. If $P \in \mathcal{P}(A)$ with coordinate system $(x_i)_{i=1}^n, (f_i)_{i=1}^n$, then there is a canonical A-map $\pi : A^n \twoheadrightarrow P$. Note that $[f_j(x_i)]$ is idempotent in $g\ell_n(A)$ and 1_P extends to $[f_j(x_i)] : A^n \to A^n$.

4. Hattori-Stallings ranks r_P. If $P \in \mathcal{P}(A)$ its *Hattori-Stallings rank*, denoted r_P, is the element $r_P = T_P(1_P) \in T(A)$. If $(x_i), (f_i)$ is a finite coordinate system of P then $r_P = \sum_i T(f_i(x_i))$. If $P \cong A^n$, then $r_P = T(n)$.

5. Example. Let G be a finite group of order n and let k be a ring in which n is invertible. Then the multiplication by $1/n \sum_{g \in G} g$ is a splitting of the augmentation map $\epsilon : kG \to k$. Hence k is a projective kG-module with coordinate system $x_0 = 1, f_0 =$ the multiplication by $1/n \sum_{g \in G} g$ so that $1/n \sum_{g \in G} g$ is idempotent in kG. For any $u \in \mathrm{End}_{kG}(k)$, $T_k(u) = T(1/n \sum_{g \in G} u(1)g) = T(1/n \sum_{g \in G} g)$. Hence $r_k = T_k(1_k) = T(1/n \sum_{g \in G} g)$.

6. Euler characteristics $\chi(\mathcal{C})$ and Lefschetz numbers $\mathrm{L}(\phi)$. Let $\mathcal{C} = \{P_n \to \cdots \to P_0\}$ be a finite projective A-complex; that is, a finite dimensional chain complex of finitely generated projective A-modules. Then the *Euler characteristic* of $\mathcal{C}$ is defined to be $\chi(\mathcal{C}) = \sum (-1)^i r_{P_i}$. Let $\phi : \mathcal{C} \to \mathcal{C}$ be an endomorphism of a finite projective A-complex. The *Lefschetz number* of ϕ is defined to be $\mathrm{L}(\phi) = \sum (-1)^i T_{P_i}(\phi_i)$. In particular, $\chi(\mathcal{C}) = \mathrm{L}(1_{\mathcal{C}})$.

7. Theorem. *Let f nad g be chain-homotopic endomorphisms of $\mathcal{C}$. Then* $\mathrm{L}(f) = \mathrm{L}(g)$.

Proof. Let $\{d_i : P_i \to P_{i+1}\}$ be a chain homotopy between f and g. Then $f_i - g_i = \partial_{i+1} d_i + d_{i-1}\partial_i$. So, $\mathrm{L}(f) - \mathrm{L}(g) = \sum (-1)^i (T_{P_i}(f_i) - T_{P_i}(g_i)) = \sum (-1)^i (T_{P_i}(\partial_{i+1} d_i) + T_{P_i}(d_{i-1}\partial_i)) = \sum (-1)^i (T_{P_i}(\partial_{i+1} d_i) - T_{P_{i+1}}(d_i \partial_{i+1})) = 0$. □

8. Corollary. *If the finite projective complexes $\mathcal{C}$ and $\mathcal{D}$ have the same homotopy type, then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

Proof. If $f : \mathcal{C} \to \mathcal{D}$ is a homotopy equivalence and $\mathcal{M}$ is its mapping cone, then $1_{\mathcal{M}}$ and $0_{\mathcal{M}}$ are chain-homotopic. By Theorem 7, $\chi(\mathcal{D}) - \chi(\mathcal{C}) = \chi(\mathcal{M}) = \mathrm{L}(1_{\mathcal{M}}) - \mathrm{L}(0_{\mathcal{M}}) = 0$. □

9. Let G be a group, k be a commutative ring, and A the group algebra kG. Since G is a k-basis for A it follows that the k-module $[A, A]$ is generated by the elements $[s, t] = st - ts = sus^{-1} - u = [su, u^{-1}]$ where $s, t, u \in G$

and $u = ts$. Thus, for $s, t \in G$, one has $T(s) = T(t)$ if and only if s and t are conjugate in G. Thus we shall identify $T(s)$ with the G-conjugacy class of s, and identify $T(kG)$ with the free k-module $k^{T(G)}$ having the set $T(G)$ of conjugacy classes of G as a basis. Each $r \in T(kG)$ thus has a unique expression

$$r = \sum_{\tau \in T(G)} r(\tau) \cdot \tau \in T(kG)$$

where $\tau \to r(\tau)$ is a function $T(G) \to k$ with finite support.

10. Definitions. We say that G is of *type* (FP) over k or k is a kG-module of *type* (FP) if k (with trivial G-action) admits a finite projective kG-resolution. In this case we call $r_k = \sum(-1)^i r_{P_i} \in T(kG)$ the *complete Euler characteristic* of G over k, and $r_k(1) = \sum(-1)^i r_{P_i}(1) \in k$, which we denote $\chi(G)$, the *Euler characteristic* of G over k. We also define the *homological Euler characteristic* of G over k to be $\widetilde{\chi}(G) = \sum_{\tau \in T(G)} r_k(\tau) \in k$. The complete Euler characteristic of G is the Euler characteristic of the complex $\{P_n \to \cdots \to P_0\}$.

11. Remark. If k' is a commutative k-algebra then $k' \otimes_k -$ yields a resolution of k' over $k'G$, so G is of type (FP) over k' and $r_{k'} \in T(k'G)$ is the image of $r_k \in T(kG)$ under the natural isomorphism of k'-modules $k' \otimes_k T(kG) \to T(k'G)$ sending $a \otimes T(g)$ to $T(ag)$ for $a \in k'$ and $g \in G$. Similarly for $\chi(G)$ and $\widetilde{\chi}(G)$.

12. Examples. (i) If G is finite then G is of type (FP) over k if and only if its order $|G|$ is invertible in k. By Example 5, $r_k = T(1/|G| \sum_{g \in G} g) \in T(kG)$. In particular $r_k(g) = 1/|C_G(g)|$ for all $g \in G$. We have $\chi(G) = 1/|G|$ and $\widetilde{\chi}(G) = 1$.

(ii) Suppose that G is abelian and of type (FP) over k. Then as G is finitely generated $G = H \times K$ with H finite and F free abelian. Since H has finite cohomological dimension over k, $|H|$ must be invertible in k. Therefore $kH = k \oplus R$ for a certain ring R. Similarly $kG = (kH)[F] = k[F] \oplus R[F]$, and the kG-module k is annihilated by $0 \oplus R[F]$. Therefore $r_k \in k[F] \oplus 0$. A free resolution of k over kF can be obtained from the Koszul complex associated to the sequence $1 - s_1, \cdots, 1 - s_n$ where $s_1, \cdots, s_n$ is a free basis of F. It follows that $r_k = 0$ if $n > 0$, i.e., if $F \neq \{1\}$. In conculsion, $r_k = 0$ unless G is finite.

13. Theorem. ([**Ba**]) *If k is a field and a kG-module of type (FP), then*

$$\widetilde{\chi}(G) = \sum_{i\geq 0}(-1)^i \dim H_i(G,k) = \sum_{i\geq 0}(-1)^i \dim H^i(G,k).$$

This theorem motivates the terminology "homological Euler characteristic" for $\widetilde{\chi}(G)$.

14. Theorem. ([**Sta**]) *Suppose k is a kG-module of type (FP).*

(a) *If $c \in \mathcal{Z}(kG)$ has augmentation $c_0 \in k$, then $c \cdot r_k = c_0 \cdot r_k$. In particular $z \cdot r_k = r_k$ for all $z \in \mathcal{Z}(G)$.*
(b) *If $\chi(G) \neq 0$ then $\mathcal{Z}(G)$ is a finite subgroup whose order is invertible in k.*

When $k = \mathbb{Z}$ the conclusion of (b) implies that $\mathcal{Z}(G) = \{1\}$.

Proof. First we observe that the center $\mathcal{Z}(G)$ of kG acts on $T(kG)$ so that $T : kG \to T(kG)$ is $\mathcal{Z}(kG)$ linear. For if $T(s) = T(t)$, for some $s, t \in G$, so that $s = utu^{-1}$, and $c \in \mathcal{Z}(kG)$, then $cs = cutu^{-1} = uctu^{-1}$; so that $T(cs) = T(ct)$. Thus $c \cdot T(s) = T(cs)$.

Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a finite kG-projective resolution for k. Since $c - c_0 \in \mathcal{Z}(kG)$ annihilates k, $(c - c_0)\cdot 1_k = 0$. Hence $(c-c_0)\cdot r_k = \sum(-1)^i(c-c_0)\cdot T_{P_i}(1_{P_i}) = \sum(-1)^i T_{P_i}((c-c_0)\cdot 1_{P_i}) = T_k((c-c_0)\cdot 1_k) = T_k(0) = 0$.

(a) implies that $r_k(zs) = r_k(s)$ for all $s \in G$ and $z \in \mathcal{Z}(G)$; in particular $r_k(z) = r_k(1) = \chi(G)$ for all $z \in \mathcal{Z}(G)$. But only finitely many elements of $\mathcal{Z}(G)$ can belong to the support of r_k. Thus if $\chi(G) \neq 0$ then $\mathcal{Z}(G)$ is finite. Since the projective dimension of k as a kG-module and hence as a $k\mathcal{Z}(G)$-module is finite, the finite subgroup $\mathcal{Z}(G)$ must have order invertible in k. □

15. Theorem. ([**G**]) *If G is the fundamental group of a finite aspherical complex K, and $\chi(K) \neq 0$, then $\mathcal{Z}(G) = \{1\}$.*

Proof. The chain complex $C_*(\widetilde{K})$ of the universal cover $\widetilde{K}$ of K is a finite free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and $\chi(G) = \chi(K)$. By Theorem 14, since $\chi(K) \neq 0$, $\mathcal{Z}(G) = \{1\}$. □

16. Kaplansky ranks $\kappa(P)$. Let $P \in \mathcal{P}(kG)$. Then the *Kaplansky rank* of P is by deninition $\kappa(P) = r_P(1) \in k$. A theorem of Kaplansky (cf. [**M**]) states that if $P \in \mathcal{P}(\mathbb{C}G)$ with n generators, then $0 \leq \kappa(P) \leq n$; if $\kappa(P) = 0$ then $P = 0$, and if $\kappa(P) = n$ then $P \cong (\mathbb{C}G)^n$.

II. Localizations of Group Rings

1. Non-central localizations. Let B be a ring and let S be a multiplicative subset of B consisting of non-zero divisors in B. A *ring of fractions* of B with respect to S is a ring $S^{-1}B$ and an injective ring homomorphism $\phi : B \to S^{-1}B$ such that

(1) $\phi(s)$ is invertible for all $s \in S$ and
(2) every element of $S^{-1}B$ can be written $\phi(s)^{-1}\phi(b)$ for some $s \in S, b \in B$.

It is known (cf. [**Ste**]) that $S^{-1}B$ exists if and only if for every $s \in S, b \in B$ $Sb \cap Bs \neq \emptyset$. When it exists it is unique up to isomorphism and we supress the ϕ and identyfy B with its image in $S^{-1}B$. Since $S^{-1}B$ is the direct limit $\varinjlim \langle s^{-1} \rangle$ of the submodules $\langle s^{-1} \rangle$ of $S^{-1}B$ generated by s^{-1}, which is isomorphic to B, it follows that $S^{-1}B$ is a flat B-module.

2. Theorem. ([**Ros**]) *Let K be a finite aspherical complex and let $G = \pi_1(K)$. If G contains a non-trivial normal abelian subgroup then $\chi(G) = 0$.*

This theorem is a generalization of Gottlieb's theorem. Let A be a non-trivial normal abelian subgroup of G. Let $B = \mathbb{C}G$ and $S = \mathbb{C}A \setminus \{0\}$. Before proving Theorem 2, we shall prove the following: (1) $S^{-1}B$ exists, (2) $S^{-1}B \otimes_B \mathbb{C} = 0$ when $\mathbb{C}$ is the trivial B-module, and (3) free modules over $R = S^{-1}B$ have well-defined ranks.

3. Localizability. Let $B = \mathbb{C}G$ and $S = \mathbb{C}A \setminus \{0\}$. Clearly S is multiplicative. Let $\{r_i\}$ be a set of representatives for the cosets Ax in G. Then every element of B is uniquely expressible as a finite sum $\sum f_i r_i$ $(f_i \in \mathbb{C}A)$. Suppose $f \in S$ and $f \cdot (\sum f_i r_i) = 0$. Then $0 = f \cdot (\sum f_i r_i) = \sum (f f_i) r_i$ $(f f_i \in \mathbb{C}A)$, all $f f_i = 0$, and since $\mathbb{C}A$ is an integral domain, all $f_i = 0$. Thus $\sum f_i r_i = 0$. Similarly if $(\sum f_i r_i) \cdot f = 0$ with $f \in S$, then $\sum f_i r_i = 0$. Hence S consists of non-zero divisors in B.

Let $f \in S$ and $x \in B$. We shall show that $Sx \cap Bf \neq \emptyset$. Write $x = \sum_{j=1}^{k} x_j r_j$ $(x_j \in \mathbb{C}A)$, $f_j = r_j f r_j^{-1}$, $h = f_1 \cdots f_k$, and $h_j = f_1 \cdots \widehat{f_j} \cdots f_k$. Since A is normal in G all $f_j \in S$, and since S is multiplicative all h_j and $h \in S$. Hence $Sx \ni hx = \sum h_j f_j x_j r_j = \sum h_j x_j f_j r_j = (\sum h_j x_j r_j) f \in Bf$.

4. Triviality. We will show that if M, a finite dimensional vector space over $\mathbb{C}$, is a B-module, then $S^{-1}B \otimes_B M = 0$. By fixing a $\mathbb{C}$-basis of M, say $\{m_1, \cdots, m_r\}$, we define $\mathbb{C}A \to M^r$ by $f \mapsto (f \cdot m_1, \cdots, f \cdot m_r)$. Since A is an infinite group its kernel $\mathrm{ann}_{\mathbb{C}A}(M)$ is nontrivial. Let $0 \neq f \in \mathrm{ann}_{\mathbb{C}A}(M)$. Then f is invertible in $S^{-1}B$. For any $m \in M$, $1 \otimes m = f^{-1} \otimes fm = f^{-1} \otimes 0 = 0$. Hence $S^{-1}B \otimes_B M = 0$. In particular $S^{-1}B \otimes_B \mathbb{C} = 0$.

5. The completion of B. We endow B with an involution

$$(\sum_{x\in G} a_x \cdot x)^* = \sum_{x\in G} \overline{a_x} \cdot x^{-1}$$

and an inner product

$$(\sum a_x \cdot x, \sum b_x \cdot x) = \sum a_x \cdot \overline{b_x}.$$

Let H be the completion of B with respect to the norm defined by the inner product. The elements of G are an orthonormal basis of H. Thus $H \to \ell^2(G), h \mapsto (g \mapsto \widehat{h}(g) = (h, g))$, is a linear isometry by Riesz-Fisher Theorem.

Let W be the closure of $B = \{L_x\}_{x\in B}$ with respect to the operator norm in $B(H)$, where L_x is the left multiplication by x and $B(H)$ is the ring of bounded linear operators on H. According to Kaplansky (cf. [**M**]), the ring W has the following property: if $n \geq 1$ and $u, v \in g\ell_n(W)$ are such that $uv = 1$ then $vu = 1$. This property is called "finiteness" and we denote it by (F). By Roos ([**Roo**]), we can enlarge W to $\widetilde{W}$ in such a way that

(i) $\widetilde{W}$ is a ring of (densely defined) operators in H,
(ii) $\widetilde{W}$ has property (F), and
(iii) if I is a principal ideal of $\widetilde{W}$ then I is generated by an idempotent element in W.

6. Lemma. *Let $f \in S$. The operators L_f (R_f) defined by $L_f(h) = fh$ ($R_f(h) = hf$) are injective. That is, S consists of non-zero divisors in H.*

Proof. Suppose $L_f(h) = 0$ for some $h \in H$. Since $H \cong \ell^2(G)$ and $G = \cup Ar_j$, we can write $h = \sum_{g\in G} \lambda_g \cdot g$ ($\sum \lambda_x^2 < \infty = \sum_j \sum_{a\in A} \lambda_{ar_j} \cdot ar_j = \sum_j (\sum_{a\in A} \lambda_{ar_j} \cdot a) r_j = \sum_j f_j r_j$ ($f_j = \sum_{a\in A} \lambda_{ar_j} \cdot a \in \overline{\mathbb{C}A}$ and $\sum_j \|f_j\|^2 = \sum_{g\in G} \lambda_g^2 < \infty$). Then $0 = fh = \sum_j (ff_j) r_j$, $ff_j \in \overline{\mathbb{C}A}$, and $\sum_j \|ff_j\|^2 < \infty$. Hence all $ff_j = 0$. Let A_0 be the subgroup of A generated by the group elements appearing in $f \in S$. Then A_0 is finitely generated torsion free abelian. Let $\{s_i\}$ be a set of representatives for the cosets of A_0 in A. Thus any $g \in \overline{\mathbb{C}A}$ has a unique expression $g \in \sum \ell_i s_i$ with $\ell_i \in \overline{\mathbb{C}A_0}$ and $\sum \|\ell_i\|^2 < \infty$. Since $f \in \mathbb{C}A_0$, $0 = ff_j = \sum (f\ell_i) s_i$, and so all $f\ell_i = 0$. Since $\ell_i \in \overline{\mathbb{C}A_0} \cong \ell^2(A_0)$, ℓ_i has the Fourier expansion $\ell_i = \sum_{a\in A_0} (\ell_i, a) a$ with $\sum_{a\in A_0} (\ell_i, a)^2 < \infty$, and since $f \in \mathbb{C}A_0$, f has a finite sum $f = \sum_{a\in A_0} (f, a) a$. Hence $0 = f\ell_i =$ (a non-zero finite sum)$\cdot$(a Fourier expansion) implies that the Fourier expansion $\ell_i = 0$. Thus $h = 0$. $\square$

7. Finiteness property of $S^{-1}B$. We will show that $R = S^{-1}B$ has property (F). By Roos (ii), it suffices to show that R embeds in $\widetilde{W}$. Since we have $B = \{L_x\}_{x \in B} \subset W \subset \widetilde{W} \subset B(H)$, by the universal property of localizations it is enough to show that if $f \in S$ then f is invertible in $\widetilde{W}$. Let $f \in S$. By Roos (iii), $\widetilde{W} \cdot L_f = \widetilde{W} \cdot e$ for some idempotent $e \in W$. Suppose $e \neq 1_H$. Then there is $h \in H$ such that $(1-e)(h) \neq 0$. Since $L_f \in \widetilde{W} \cdot e$, $L_f = w \circ e$ for some $w \in \widetilde{W}$, and hence $L_f \circ (1-e) = w \circ e \circ (1-e) = w \circ (e - e^2) = 0$. However by Theorem 6, $L_f \circ (1-e)(h) \neq 0$, which is impossible. Thus $e = 1_H, \widetilde{W} = \widetilde{W} \cdot L_f$, and L_f has a left inverse in $\widetilde{W}$. We note that the property (F) for $R = S^{-1}B$ implies that free modules over R have well-defined ranks.

8. Proof of Theorem 2. Since K is a finite aspherical complex, the chain complex $C_*(\widetilde{K})$ over $\mathbb{C}$ gives rise to a finite free resolution of $\mathbb{C}$ over $B = \mathbb{C}G$ $0 \to F_n \to \cdots \to F_0 \to \mathbb{C} \to 0$ and $\chi(K) + \sum_j (-1)^j \mathrm{rank}_B F_j = \chi(G)$. Since $R = S^{-1}B$ is flat and $R \otimes_B \mathbb{C} = 0$, $R \otimes_B -$ yields an exact sequence $0 \to R \otimes_B F_n \to \cdots \to R \otimes_B F_0 \to 0$ of finitely generated free R-modules. By (F), $\chi(K) = \sum_j (-1)^j \mathrm{rank}_R (R \otimes_B F_j) = 0$.

III. Amenable Groups

Theorem II.2 states that if G admits a finite $K(G,1)$ and if G has a nontrivial normal abelian subgroup, then $\chi(G) = 0$. Cheeger and Gromov (**[CG]**) and Eckmann (**[E]**) considered infinite amenable groups G and free cocompact G-spaces X, i.e., connected complexes X on which G acts freely and simplicially with $G \backslash X$ being a finite complex. They showed that $\chi(G \backslash X)$ has some special properties due to the amenability of G. In particular, they showed that if the amenable group G admits a finite $K(G,1)$ then $\chi(G) = 0$.

1. Definitions. Let G be a locally compact group with a left Haar measure λ. For the discrete G we take λ to be the counting measure. Denote by $\mathfrak{M}(G)$ the family of λ-measurable subsets of G. Consider a positive, finitely additive measure $\mu : \mathfrak{M}(G) \to \mathbb{C}$ satisfying $\mu(G) = 1$ and $\mu(N) = 0$ for locally null N, i.e., for $N \in \mathfrak{M}(G)$ such that $\lambda(N \cap C) = 0$ if C is compact. We can regard μ as $m \in L_\infty(G, \lambda)'$ as follows: Since $\{\sum_{i=1}^n \alpha_i \chi_{E_i} \mid \alpha_i \in \mathbb{C}, E_i \in \mathfrak{M}(G)\}$ is norm dense in $L_\infty(G)'$, define $m(\sum \alpha_i \chi_{E_i}) = \sum \alpha_i \mu(E_i)$ and extend it to all of $L_\infty(G)$. Clearly $m(\chi_G) = \mu(G) = 1$. This functional m is called a *mean* on $L_\infty(G)$. We call μ or m *left invariant* for G if $m(\phi g) = m(\phi)$ for

$\phi \in L_\infty(G)$ and $g \in G$. The group G is called *amenable* if it admits a (left) invariant mean on $L_\infty(G)$.

2. Følner criterion. A *summing net* for G is a net of nonempty compact subsets $\{K_\delta\}_{\delta\in\Delta}$ with the properties that

(i) $K_\delta \subset K_\sigma$ if $\delta \leq \sigma$;
(ii) $G = \cup_{\delta\in\Delta} K_\delta^0$;
(iii) $\lambda(gK_\delta \,\Delta\, K_\delta)/\lambda(K_\delta) \to 0$ uniformly on compacta.

Theorem. ([**P**]) *A locally compact group G is amenable if and only if there exists a summing net for G. If G is σ-compact, then G is amenable if and only if there exists a summing sequence for G.*

3. Examples. Let $G = \mathbb{Z}$ and $K_n = \{-n, \cdots, 0, \cdots, n\}$. Then $\{K_n\}$ is a summing sequence for $\mathbb{Z}$. Put $f_n = 1/(2n+1)\cdot \chi_{K_n} \in \mathfrak{P}(\mathbb{Z}) := \{f \in \ell_1(\mathbb{Z}) \mid f \geq 0, \sum f(n) = 1\}$. To show the existence of an invariant mean for $\mathbb{Z}$, we use the following analysis facts:

(1) $L_1(G) \hookrightarrow L_1(G)'' \cong L_\infty(G)'$.
$f \longrightarrow \widehat{f} : \phi \in L_\infty(G) \mapsto \widehat{f}(\phi) = \int \phi f \, d\lambda$
(2) If $\mathfrak{P}(G) = \{f \in L_1(G) \mid f \geq 0, \int_G f\,d\lambda = 1\}$, then $\widehat{\mathfrak{P}(G)}$ is weak*-dense in $\mathfrak{M}(G)$, the set of all means on $L_\infty(G)$.
(3) $\mathfrak{M}(G)$ is weak*-compact in $L_\infty(G)'$.

For $\phi \in \ell_\infty(\mathbb{Z})$, $\widehat{f_n}(\phi) = \sum_{s\in\mathbb{Z}} \phi(s) f_n(s) = 1/(2n+1)\cdot \sum_{-n}^{n} \phi(r)$. If $s \geq 0$ in $\mathbb{Z}$, then

$$|\widehat{f_n}(\phi s) - \widehat{f_n}(\phi)| = \frac{1}{2n+1}|-\sum_{-n}^{-n+s-1} \phi(r) + \sum_{n+1}^{n+s} \phi(r)| \leq \frac{2s\|\phi\|}{2n+1} \to 0$$

as $n \to \infty$. A similar result holds for $s < 0$ in $\mathbb{Z}$. Since $\{f_n\} \subset \widehat{\mathfrak{P}(\mathbb{Z})} \subset \mathfrak{M}(\mathbb{Z}) \subset \ell_\infty(\mathbb{Z})'$, $\widehat{\mathfrak{P}(\mathbb{Z})}$ is weak*-dense in $\mathfrak{M}(\mathbb{Z})$, and $\mathfrak{M}(\mathbb{Z})$ is weak*-compact in $\ell_\infty(\mathbb{Z})'$, every weak*-culster point of $\{\widehat{f_n}\}$ is in $\mathfrak{M}(\mathbb{Z})$ and a left invariant mean.

For the monent, we will simply list some important examples of amenable groups:

(1) Every finite group is amenable.
(2) Every abelian group is amenable.
(3) The class of all discrete amenable groups is closed under subgroups, quotient groups, group extensions, and direct limits.

(4) Grigorchuck constructed examples of finitely generated amenable groups which cannot be obtained from finite and abelian groups by successive extensions and increasing unions.
(5) Every amenable group does not contain any non-abelian free subgroup. The converse was disproved by Ol'shanskii.

4. Følner sequences. Assume that G is an infinite amenable group and X is a free cocompact G-space. Then G must be finitely generated. By Theorem 2, we can fix a summing sequence $\{K_n\}$ for G.
Claim. X admits a Følner sequence $\{X_n\}$ associated to $\{K_n\}$. That is, there is an increasing sequence $\{X_n\}$ of nonempty finite subcomplexes of X such that

(1) X_n consists of k_n translates of D, where D is a closed fundamental domain for (G, X);
(2) $X = \cup X_n$;
(3) If $k'_n = \lambda(\{g \in G \mid gD \cap \partial X_n \neq \emptyset\})$, where ∂X_n is the topological boundary of X_n, then $\lim_{n\to\infty} k'_n/k_n = 0$.

Proof. Let $k_n = \lambda(K_n)$ and $X_n = \{gD \mid g \in K_n\}$. Then X_n is the union of k_n translates of D. Hence X_n is a finite subcomplexes of X and $X = \cup X_n$.

Since D is compact in X, we can choose a finite subset $\{x_1, \cdots, x_\ell\}$ of G so that $D \cap x_i D \neq \emptyset$, and $D \cap xD = \emptyset$ if $x \neq x_j$. If $gD \cap \partial X_n \neq \emptyset$, then there is $x \in K_n$ such that $gD \cap xD \neq \emptyset$, or $x^{-1}g = x_i$ for some $i = 1, \cdots, \ell$. Hence $g \in K_n x_j$. If $g \in K_n$, then $x^{-1}g$ fixes $D \cap x^{-1}gD$, which is nonempty. Since G acts freely X, $x^{-1}g = x_i = 1$. It follows that $k'_n \leq \sum_{i=1}^{\ell} \lambda(K_n x_i \vartriangle K_n)$ and hence $\lim_{n\to\infty} k'_n/k_n = 0$.

□

5. Theorem. *If G is an infinite amenable group which admits a finite $K(G,1)$ then $\chi(G) = 0$.*

This theorem is due to Morgan and Philips (unpublished).

Proof by Eckmann. ([E]) Let $X = \widetilde{K(G,1)}$ with a closed fundamental domain D and a Følner sequence $\{X_n\}$, and $m = \dim X$. Then $\chi(X_n) = k_n \cdot \chi(K(G,1)) + \Delta_n$ and $|\Delta_n| \leq k'_n \cdot \Delta$, where Δ_n comes from ∂X_n and Δ is the number of simplices of ∂D. Thus $\chi(G) = 1/k_n \cdot \chi(X_n) - \Delta_n/k_n$ and $|\Delta_n|/k_n \leq k'_n/k_n \cdot \Delta \to 0$ as $n \to \infty$. Hence

$$\chi(G) = \lim_{n\to\infty} \frac{\chi(X_n)}{k_n} = \lim_{n\to\infty} \sum_{i=1}^{m} (-1)^i \frac{\beta_i(X_n)}{k_n}.$$

Now the inclusion $(X \setminus X_n^0, \partial X_n) \to (X, X_n)$ yields the commutative diagram for homology with $\mathbb{Q}$-coefficients

$$\begin{array}{ccccc} H_{i+1}(X \setminus X_n^0, \partial X_n) & \longrightarrow & H_i(\partial X_n) & \xrightarrow{\ \phi\ } & H_i(X \setminus X_n^0) \\ \big\downarrow \text{excision isomorphism} & & \big\downarrow \psi & & \big\downarrow \\ H_{i+1}(X, X_n) & \longrightarrow & H_i(X_n) & \xrightarrow{\ \rho\ } & H_i(X). \end{array}$$

Thus

$$\begin{aligned} \beta_i(X_n) &= \dim_{\mathbb{Q}}(\ker\rho) + \dim_{\mathbb{Q}}(\mathrm{im}\rho) \\ &\le \dim_{\mathbb{Q}}(\ker\phi) + \dim_{\mathbb{Q}} H_i(X) \ (\because \psi : \ker\phi \twoheadrightarrow \ker\rho) \\ &\le \beta_i(\partial X_n) + \beta_i(X). \end{aligned}$$

Since $\beta_i(\partial X_n) \le$ the number of i-simplices of $\partial X_n \le k'_n \cdot d_i$, d_i is the number of the i-simplices of D, we have

$$\frac{1}{k_n}\beta_i(X_n) \le \frac{k'_n}{k_n} \cdot d_i + \frac{1}{k_n}\beta_i(X) \to 0 \text{ as } n \to \infty.$$

Here we must note that all $\beta_i(X)$ are finite. Hence $\chi(G) = 0$. □

IV. Simplicial ℓ_2-cohomology Spaces

1. Definitions. Let G be a countable group and let $\ell_2(G)$ denote the Hilbert space of real valued square summable functions on G. A Hilbert space P is called a *G-module* if:

(i) G acts on P by isometries, and
(ii) P is G-equivariantly isometric to a subspace of $\ell_2(G) \otimes H$ for some Hilbert space H on which G acts trivially.

For a G-module P with a G-equivariant embedding $P \hookrightarrow \ell_2(G) \otimes H$, let $p : \ell_2(G) \otimes H \to \ell_2(G) \otimes H$ be projection onto P and write p as a matrix (α_{ij}) with entries in $\ell_2(G)$. Then the *von Neumann dimension* of P is defined to be $\dim_G P = \sum \langle \alpha_{ii}, 1_G \rangle$, where $\langle , \rangle$ is the inner product on $\ell_2(G)$ and 1_G is the identity element of G. Let $\{h_i\}$ be a Hilbert basis of H and let $p_i : \ell_2(G) \otimes H \to \ell_2(G)$ be the projection $1 \otimes r_i h_i \mapsto r_i \cdot 1$. With $P_0 = P$, we define inductively P_{i+1} and I_{i+1} to be the kernel and the closure of the image, respectively, of $P_i \hookrightarrow \ell_2(G) \otimes H \overset{p_{i+1}}{\to} \ell_2(G)$. Then $P = \sum I_i$ and I_i is a G-module with a G-equivariant embedding $I_i \hookrightarrow \ell_2(G)$. For each i, we write $1 = e_i + (1 - e_i)$, where $e_i \in I_i$ and $(1 - e_i) \in I_i^{\perp}$, under the isometry $\ell_2(G) \cong I_i \oplus I_i^{\perp}$. Then $\dim_G I_i = \langle e_i, 1_G \rangle$ and $\dim_G P = \sum \dim_G I_i$. By Kaplansky, $0 \leq \dim_G I_i \leq 1$ and $\dim_G I_i = 0$ (resp. 1) if and only if $I = 0$ (resp. $\ell_2(G)$). Thus $\dim_G P \in [0, \infty]$.

2. Definitions. Let G be a countable group and X a free G-space. Denote by $X_{(i)}$ the set of all i-simplices of X. Define $C^i_{(2)}(X) = \{c \in C^i(X, \mathbb{R}) \mid \sum_{s \in X_{(i)}} c(s)^2 < \infty\}$ and call it the *space of ℓ_2-cochains*. Then $C^i_{(2)}(X) \cong \ell_2(G) \otimes H_i$ where H_i is a Hilbert space having a set S_i of representatives of $X_{(i)}$ mod G as a basis. Hence $C^i_{(2)}(X)$ is a free G-module and $\dim_G C^i_{(2)}(X) = \lambda(S_i)$. It is clear that the differentials $\delta^i C^i_{(2)}(X) \to C^{i+1}_{(2)}(X)$ commute with the G-action. We define the *(reduced) simplicial ℓ_2-cohomology spaces* by

$$\overline{H}^i_{(2)}(X; G) = \ker\delta^i / \overline{\mathrm{im}\delta^{i-1}}.$$

Since $C^i_{(2)}(X) \supset \ker\delta^i \cong \overline{\mathrm{im}\delta^{i-1}} \oplus \overline{H}^i_{(2)}(X; G)$, and $\ker\delta^i$ and $\overline{\mathrm{im}\delta^{i-1}}$ are G-modules, $\overline{H}^i_{(2)}(X; G)$ acquires the structure of a G-module and hence its von Neumann dimension is defined, denoted by $b^i_{(2)}(X; G)$, and called the *ith ℓ_2-Betti nember.* Observe that if X is a free cocompact G-space, then

$$\begin{aligned} \chi(G\backslash X) &= \sum(-1)^i \lambda(S_i) \ (\because G\backslash X \text{is a finite complex.}) \\ &= \sum(-1)^i \dim_G C^i_{(2)}(X) \\ &= \sum(-1)^i \dim_G \overline{H}^i_{(2)}(X; G) \\ &= \sum(-1)^i b^i_{(2)}(X; G). \end{aligned}$$

The third equality follows from the fact that the cochain complex $\{C^*_{(2)}(X)\}$ of G-modules is finite dimensional. When $X = \widetilde{K(G,1)}$ we simply put $\overline{H}^i_{(2)}(G) = \overline{H}^i_{(2)}(X; G)$ and $b^i_{(2)}(G) = b^i_{(2)}(X; G)$.

3. Example.

(1) If G is infinite and X is a connected free G-space, then $\overline{H}^0_{(2)}(X; G) = 0$. In praticular, $b^0_{(2)}(G) = 0$.

(2) If G is finite, then $b^0_{(2)}(G) = 1/|G|$.

Proof. If $c \in C^0(X;\mathbb{R})$ with $\delta(c) = 0$, then c is constant. If G is infinite, then $X_{(0)}$ is infinite, and hence $\sum_{s\in X_{(0)}} c(s)^2 < \infty$ implies $c = 0$. Suppose G is finite and let $X = \widetilde{K(G,1)}$ with one 0-simplex. Since $C^0_{(2)}(G) = C^0(X;\mathbb{R}) \cong \ell_2(G) = \mathbb{R}^{|G|}$, we have $\overline{H}^0_{(2)}(G) = H^0(X;\mathbb{R}) \cong \mathbb{R}$ and $b^0_{(2)}(G) = 1/|G|$. □

4. Theorem. ([**CG**]) *If G is amenable, then*

$$b^i_{(2)}(G) = \begin{cases} \frac{1}{|G|}, & i = 0 \\ 0, & i > 0. \end{cases}$$

In particular, if G is amenable which admits a finite $K(G,1)$ then $\chi(G) = \sum(-1)^i b^i_{(2)}(G) = 0$.

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CLASSIFICATION OF FREE ACTIONS ON THE 3-TORUS

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ABSTRACT. We classify free actions of finite abelian groups on the 3-torus, up to topological conjugacy. By the works of Bieberbach and Waldhausen, this classification problem is reduced to classifying all normal abelian subgroups of Bieberbach groups of finite index, up to affine conjugacy. All such actions are completely classified, see Theorems 2.1, 3.4, 4.1.

Introduction. The general question of classifying finite group actions on a closed 3-manifold is very hard. For example, it is not known if every finite action on S^3 is conjugate to a linear action. However, the actions on a 3-dimensional torus can be understood easily by the works of Bieberbach and Waldhausen. We shall study only free actions of finite abelian groups G on a 3-dimensional torus T.

The group of affine motions on the euclidean space $\mathbb{E}^n$ is $\mathrm{Aff}(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$. The group law is

$$(A, a) \cdot (B, b) = (AB, a + Ab),$$

and it acts on $\mathbb{E}^n$ by

$$(A, a) \cdot x = Ax + a$$

for $(A, a), (B, b) \in \mathrm{Aff}(n)$, and $x \in \mathbb{E}^n$, after a coordinat of $\mathbb{E}^n$ is specified. We shall denote the group of isometries of $\mathbb{R}^n$ by $E(n) = O(n) \ltimes \mathbb{R}^n \subset \mathrm{Aff}(n)$. A cocompact discrete subgroup Γ of $E(n)$ is called a *crystallographic group*. If Γ is torsion free, it is a *Bieberbach group*, and $\mathbb{R}^n/\Gamma$ is a flat Riemannian manifold. Conversely, let M be a flat Riemannian manifold of dimension n. Then $M = \mathbb{R}^n/\Gamma$ for some Bieberbach group $\Gamma \subset E(n)$.

A Bieberbach group Γ is torsion-free, contains a free abelian normal subgroup $\mathbb{Z}^n$ of finite index. In fact, $\mathbb{R}^n \cap \Gamma$ is the unique maximal free abelian normal subgroup of Γ. Conversely, any torsion-free group which contains a

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free abelian normal subgroup $\mathbb{Z}^n$ of finite index can be made into a Bieberbach group by embedding it into $E(n)$ as a cocompact discrete subgroup. We call a group which is isomorphic to a Bieberbach group an *abstract Bieberbach group.*

Let G be a finite group acting freely on a 3-torus T. Then clearly, $M = T/G$ is a topological manifold, and $\Gamma = \pi_1(T/G)$ is an abstract Bieberbach group. Let N be the subgroup of Γ corresponding to $\pi_1(T)$. Let Γ' be an embedding of Γ into $E(3)$ as a cocompact subgroup, and let N' be the image of N. Then the quotient group $G' = \Gamma'/N'$ acts freely on the flat torus $T' = \mathbb{R}^3/N'$. Moreover, $M' = T'/G'$ is a flat Riemannian manifold. Thus, a finite free topological action (G, T) gives rise to an isometric action (G', T') on a flat torus. Clearly, T/G and T'/G' are sufficiently large, see [Heil2, Proposition 2]. By works of Waldhausen and Heil [Heil1; Theorem A], M is homeomorphic to M'.

Definition. *Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is* **topologically conjugate** *to (G_2, M_2) if there exists an isomorphism $\theta\colon G_1 \to G_2$ and a homeomorphism $h\colon M_1 \to M_2$ such that $h(g \cdot x) = \theta(g) \cdot h(x)$ for all $x \in M_1$ and all $g \in G_1$.*

For T/G and T'/G' being homeomorphic implies that the two actions (G, T) and (G', T') are topologically conjugate. Consequently, a free finite action (G, T) gives rise to a topologically conjugate isometric action (G', T') on a flat torus T'. Such a pair (G', T') is not unique. However, by the following theorem of Bieberbach's, all the others are topologically conjugate.

Theorem 0.1. (Bieberbach) [Wolf, 3.2.2] *Any isomorphism between crystallographic groups on $\mathbb{R}^n$ is conjugation by an element of the affine group Aff(n).* $\square$

Consequenly, to classify all free actions by finite groups on a 3-torus, it is enough to classify only free isometric actions by finite groups on a flat torus.

When $\Gamma \subset \mathrm{Aff}(n)$ acts properly discontinuously and freely on $\mathbb{R}^n$, the manifold $\mathbb{R}^n/\Gamma$ is called an *affinely flat manifold.* Two affinely flat manifolds $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^n/\Gamma'$ are *affinely diffeomorphic* if there is an affine diffeomorphism between them. This is equivalent to saying that there is an element of $\mathrm{Aff}(n)$ which conjugates Γ onto Γ'. An abstract Bieberbach group Γ which is embedded in $\mathrm{Aff}(n)$ in such a way that $\mathbb{R}^n \cap \Gamma$ is a lattice of $\mathbb{R}^n$ is called an *affine Bieberbach group.* We can use affinely flat manifolds rather than flat Riemannian manifolds in the future discussions for simplicity. To justify this, we need the following generalization of the above theorem.

Theorem 0.2. *Let $\pi, \pi' \subset \mathrm{Aff}(n)$ be two affine Bieberbach groups. Then for any isomorphism $\theta : \pi \to \pi'$, there exists $\sigma \in \mathrm{Aff}(n)$ such that $\theta(\alpha) = \sigma \cdot \alpha \cdot \sigma^{-1}$ for all $\alpha \in \pi$.*

Proof. Clearly $\mathcal{Z} = \pi \cap \mathbb{R}^n$ and $\mathcal{Z}' = \pi' \cap \mathbb{R}^n$ are the unique maximal normal abelian subgroups. Therefore, θ maps $\mathcal{Z}$ onto $\mathcal{Z}'$. Let $\Psi = \pi/\mathcal{Z}$ and $\Psi' = \pi'/\mathcal{Z}'$ be the holonomy groups.

Since $\mathcal{Z}$ and $\mathcal{Z}'$ are lattices of $\mathbb{R}^n$, any homomorphism $\mathcal{Z} \to \mathcal{Z}'$ extends uniquely to an automorphism $D : \mathbb{R}^n \to \mathbb{R}^n$. Hence, $\theta(I, z) = (I, Dz)$ for all $(I, z) \in \mathcal{Z}$.

Let us denote the isomorphism on Ψ induced by θ by $\overline{\theta} : \Psi \to \Psi'$, and define a map $f : \Psi \to \mathbb{R}^n$ by

$$\theta(K, w) = (\overline{\theta}(K, w), Dw + f(K, w)) \tag{1}$$

For any $(I, z) \in \mathcal{Z}$ and $(K, w) \in \pi$, apply θ to both sides of $(K, w)(I, z)(K, w)^{-1} = (I, Kz)$ to get

$$\overline{\theta}(K, w)(Dz) = \theta(Kz) = DKz. \tag{2}$$

This is true for all $z \in \mathcal{Z}$, so it holds true for all $z \in \mathbb{R}^n$. It is also easy to see that $f(K, zw) = f(K, w)$ for all $z \in \mathcal{Z}$ so that $f : \pi \to \mathbb{R}^n$ does not depend on $\mathcal{Z}$. Thus, f factors through $\Psi = \pi/\mathcal{Z}$. We will use the notation $f : \Psi \to \mathbb{R}^n$ to denote this new map.

It is not hard to see that, with the Ψ-module structure on $\mathbb{R}^n$ via $\overline{\theta} : \Psi \to \Psi' \subset \mathrm{Aut}(\mathbb{R}^n)$, $f : \Psi \to \mathbb{R}^n$ is a crossed homomorphism, this is, $f(KK') = f(K) + \overline{\theta}(K)f(K')$ for all $K, K' \in \Psi$. So, $f \in Z^1(\Psi, \mathbb{R}^n)$.

However, $H^1(\Psi; \mathbb{R}^n) = 0$ since Ψ is a finite group. This means that any crossed homomorphism is principal. In other words, there exists $d \in G$ such that

$$f(K) = d - \overline{\theta}(K)(d) \tag{3}$$

Let $\sigma = (D, d) \in \mathrm{Aff}(n)$. Using (1), (2) and (3), one can show θ is conjugation by σ. That is, $\theta(K, w) = (D, d) \cdot (K, w) \cdot (D, d)$. This finishes the proof of theorem. □

The reason for using Aff(3) rather than $E(3)$ is obvious: We want to make the maximal normal abelian subgroup to be the standard lattice of $\mathbb{R}^3$. For such groups to act on $\mathbb{E}^3$, we need to introduce a coordinate system for $\mathbb{E}^3$, which may be non-standard.

We list all the 3-dimensional affine Bieberbach groups. (Some of these are not in the euclidean group $E(3) = O(3) \ltimes \mathbb{R}^3$, but they can be conjugated into $E(3)$ by Theorem 0.2). We list their holonomy groups Ψ and homology groups as well. There are 6 orientable ones and 4 non-orientable ones, see [Orlik] or [Wolf]. In the following LIST (A), $t_i = (I, e_i)$, $i = 1,2,3$, where $\{e_i\}$ is the standard basis in $\mathbb{R}^3$; namely,

$$t_1 = \left(I, \begin{bmatrix}1\\0\\0\end{bmatrix}\right), \qquad t_2 = \left(I, \begin{bmatrix}0\\1\\0\end{bmatrix}\right), \qquad t_3 = \left(I, \begin{bmatrix}0\\0\\1\end{bmatrix}\right),$$

where I is the 3×3 identity matrix.

LIST (A) *3-dimensional Bieberbach groups, their holonomy and first homology groups:*

$$\mathfrak{G}_1 := \langle t_1, t_2, t_3 \rangle, \quad \Psi = \{1\}, \qquad H_1(\mathfrak{G}_1; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}.$$

$$\mathfrak{G}_2 := \langle \alpha, t_1, t_2, t_3 \rangle, \qquad \alpha = \left(\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix}\right),$$

$$[\mathfrak{G}_2, \mathfrak{G}_2] = \langle t_2^{-2}, t_3^{-2} \rangle, \quad \Psi = \mathbb{Z}_2, \quad H_1(\mathfrak{G}_2; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$$\mathfrak{G}_3 := \langle \alpha, t_1, t_2, t_3 \rangle, \qquad \alpha = \left(\begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & -1\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\0\\0\end{bmatrix}\right),$$

$$[\mathfrak{G}_3, \mathfrak{G}_3] = \langle t_2 t_3^{-1}, t_3^3 \rangle, \quad \Psi = \mathbb{Z}_3, \quad H_1(\mathfrak{G}_3; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_3$$

$$\mathfrak{G}_4 := \langle \alpha, t_1, t_2, t_3 \rangle, \qquad \alpha = \left(\begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}\frac{1}{4}\\0\\0\end{bmatrix}\right),$$

$$[\mathfrak{G}_4, \mathfrak{G}_4] = \langle t_3 t_2^{-1}, t_2^{-1} t_3^{-1} \rangle, \quad \Psi = \mathbb{Z}_4, \quad H_1(\mathfrak{G}_4; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2$$

$\mathfrak{G}_5 := \langle \alpha, t_1, t_2, t_3 \rangle,$ $\alpha = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} \\ 0 \\ 0 \end{bmatrix} \right),$

$[\mathfrak{G}_5, \mathfrak{G}_5] = \langle t_2, t_3 \rangle, \quad \Psi = \mathbb{Z}_6, \quad H_1(\mathfrak{G}_5; \mathbb{Z}) = \mathbb{Z}.$

$\mathfrak{G}_6 := \langle \alpha, \beta, \gamma, t_1, t_2, t_3 \rangle,$ $\alpha = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right),$

$\beta = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right), \quad \gamma = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right),$

$[\mathfrak{G}_6, \mathfrak{G}_6] = \langle t_1 t_2^{-1} t_3^{-1}, t_1^{-2}, t_2^{-2}, t_3^{-2} \rangle, \quad \Psi = \mathbb{Z}_2 \times \mathbb{Z}_2,$

$H_1(\mathfrak{G}_6; \mathbb{Z}) = \mathbb{Z}_4 \times \mathbb{Z}_4.$

$\mathfrak{B}_1 := \langle \epsilon, t_1, t_2, t_3 \rangle,$ $\epsilon = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right),$

$[\mathfrak{B}_1, \mathfrak{B}_1] = \langle t_3^{-2} \rangle, \quad \Psi = \mathbb{Z}_2, \quad H_1(\mathfrak{B}_1; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2.$

$\mathfrak{B}_2 := \langle \epsilon, t_1, t_2, t_3 \rangle,$ $\epsilon = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right),$

$[\mathfrak{B}_2, \mathfrak{B}_2] = \langle t_1 t_2 t_3^{-2} \rangle, \quad \Psi = \mathbb{Z}_2, \quad H_1(\mathfrak{B}_2; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}.$

$\mathfrak{B}_3 := \langle \epsilon, \alpha, t_1, t_2, t_3 \rangle,$

$\alpha = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right), \quad \epsilon = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \right),$

$[\mathfrak{B}_3, \mathfrak{B}_3] = \langle t_2, t_3^{-2} \rangle, \quad \Psi = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad H_1(\mathfrak{B}_3; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathfrak{B}_4 := \langle \epsilon, \alpha, t_1, t_2, t_3 \rangle,$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right), \qquad \epsilon = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right),$$

$$[\mathfrak{B}_4, \mathfrak{B}_4] = \langle t_2 t_3, t_2^{-2}, t_3^{-2} \rangle,$$

$$\Psi = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad H_1(\mathfrak{B}_4; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_4.$$

The group Γ acts on $\mathbb{E}^3$ after a coordinate system of $\mathbb{E}^3$ has been specified. For all Γ's except for the following three, $\mathbb{E}^3$ has the standard coordinates $\{e_1, e_2, e_3\}$. The groups which uses non-standard coordinates on $\mathbb{E}^3$ are:

$$\mathfrak{G}_3 : \{e_1, \quad e_2, \quad -\frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3\}$$

$$\mathfrak{G}_5 : \{e_1, \quad e_2, \quad \frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3\}$$

$$\mathfrak{B}_2 : \{e_1, \quad e_2, \quad -\frac{1}{2}(e_1 + e_2) + e_3\}.$$

Here the second parts are bases for $\mathbb{E}^3$ on which the covering groups act.

In Section 1, all necessary basics are explained. The normalizer of each Bieberbach group is listed. Essentially, this section gives all the necessary idea for our classification problem. In subsequent sections, hard cases are worked out in detail; namely, actions (G, T) whose orbit spaces T/G are homeomorphic to $\mathbb{R}^3/\mathfrak{G}_2$, or $\mathbb{R}^3/\mathfrak{B}_1$ are worked out. These are the most difficult and interesting cases, which should give the reader enough of the idea for all the other cases. In the last section, we complete the remaining classification. This work contains the previous result of [Hempel], and supplies some missing ones there.

More details worked with K. B. Lee and S. Yokura will be published in Topology and Its Applications.

§1. Criteria for conjugacy.

In this section, we develop a technique for finding and classifying all possible group actions on a 3-torus T. The problem will be reduced to a purely group-theoretic one.

Definition. *Let $\Gamma \subset \mathrm{Aff}(3)$ be an affine Bieberbach group, and let N_1, N_2 be subgroups of Γ. We say that (N_1, Γ) is* **affinely conjugate** *to (N_2, Γ) if there exists an element $\sigma \in \mathrm{Aff}(3)$ such that $\sigma\Gamma\sigma^{-1} = \Gamma$ and $\sigma N_1 \sigma^{-1} = N_2$.*

Notation. *In a group Γ, $\mu(\sigma)$ denote the conjugation by σ. So for $\gamma \in \Gamma$, $\mu(\sigma)\gamma = \sigma\gamma\sigma^{-1}$.*

Let (G, T) be a free affine action of a finite abelian group G on a flat torus T. Then T/G is an affiley flat manifold. Let $\Gamma = \pi_1(T/G)$, and $N = \pi_1(T)$. Then Γ is an affine Bieberbach group. In fact, Since the covering projection $T \to T/G$ is regular, N is a normal (abelian) subgroup of Γ. Since the pure translations in Γ, $\mathcal{Z} = \Gamma \cap \mathbb{R}^3$, is the unique maximal normal abelian subgroup of Γ, the normal abelian subgroup N must be in $\mathcal{Z}$.

Our classification problem of free abelian group actions (G, T) with $\pi_1(T/G) \cong \Gamma$ can be solved by two steps:

(I) Find all normal free abelian subgroups N of Γ of finite index and classify (N, Γ) up to affine conjugacy.

(II) Realize the finite group Γ/N as an action on the torus $\mathbb{R}^3/N$.

For (I), we need the following. Let $\Gamma \subset \mathrm{Aff}(3)$ be an affine Bieberbach group; and let

$$t_1 = \left(I, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \qquad t_2 = \left(I, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \qquad t_3 = \left(I, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right),$$

where I is the 3×3 identity matrix. Then $\mathcal{Z} = \langle t_1, t_2, t_3 \rangle$ is the maximal normal free abelian subgroup of Γ. Let N be a subgroup of $\mathcal{Z}$ of rank 3 which is normal in Γ, and let $\mathcal{B} = \{t_1^{\ell_1} t_2^{m_1} t_3^{n_1},\ t_1^{\ell_2} t_2^{m_2} t_3^{n_2}, t_1^{\ell_3} t_2^{m_3} t_3^{n_3}\}$ be an ordered set of generators for N. More precisely,

$$t_1^{\ell_i} t_2^{m_i} t_3^{n_i} = \left(I, \begin{bmatrix} \ell_i \\ m_i \\ n_i \end{bmatrix}\right), \qquad i = 1, 2, 3.$$

We shall represent the particular ordered basis $\mathcal{B}$ of N as a 3×3 integral matrix

$$[\mathcal{B}] = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \iff \begin{array}{c} \langle t_1^{\ell_1} t_2^{m_1} t_3^{n_1},\ t_1^{\ell_2} t_2^{m_2} t_3^{n_2},\ t_1^{\ell_3} t_2^{m_3} t_3^{n_3} \rangle \\ \text{ordered basis of a subgroup} \\ N \text{ of } \mathcal{Z} \end{array}$$

The following Lemma is elementary but will be used repeatedly.

Lemma 1.1. *Any integral square matrix can be changed to an upper triangular matrix by integral column operations. Thus, any free abelian normal subgroup N of Γ has an ordered set of generators of the form*

$$\begin{bmatrix} \ell & * & * \\ 0 & m & * \\ 0 & 0 & n \end{bmatrix}. \quad \square$$

Let us denote the normalizer of Γ by $N_{\mathrm{Aff}(3)}(\Gamma)$. The maximal normal free abelian subgroup of Γ is characteristic (i.e., invariant under any automorphism of Γ). Under our representation of Γ into Aff(3), the subgroup Γ lies in $\mathbb{Z}^3 \subset \mathbb{R}^3$. Therefore, matrix parts of elements of $N_{\mathrm{Aff}(3)}(\Gamma)$ are integral.

To make the exposition easier, we introduce some more notations. Let N_1, N_2 be free abelian normal subgroups of Γ; $\mathcal{B}_1$, $\mathcal{B}_2$ be bases for N_1, N_2, respectively. If there is $Y \in \mathrm{GL}(3,\mathbb{Z})$ such that $[\mathcal{B}_2] = [\mathcal{B}_1]Y^{-1}$, then we say $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$. Similarly, if there exists $(X,x) \in N_{\mathrm{Aff}(3)}(\Gamma)$ so that $[\mathcal{B}_2] = X[\mathcal{B}_1]$, then we say $[\mathcal{B}_1] \overset{\Gamma}{\underset{R}{\sim}} [\mathcal{B}_2]$. Note that $\underset{C}{\sim}$ is the column operation so that it do not change Γ and its normal subgroup. It is an operation that picks a new set of generators. Therefore, if $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$, then $N_1 = N_2$. On the other hand, $\overset{\Gamma}{\underset{R}{\sim}}$ is the row operation on the matrix leaving Γ invariant. If (X,x) is in the normalizer of Γ, then it gives a new representation of Γ. Moreover, even if $[\mathcal{B}_1] \overset{\Gamma}{\underset{R}{\sim}} [\mathcal{B}_2]$, N_1 and N_2 will generally be different subgroups of Γ.

The following proposition is a working criterion for affine conjugacy. All calculations will be done by this method.

Proposition 1.2. *Let N_1, N_2 be free abelian normal subgroups of a Bieberbach group Γ. Then (N_1,Γ) is affine conjugate to (N_2,Γ) if and only if for any ordered set of generators $\mathcal{B}_1, \mathcal{B}_2$ for N_1, N_2, respectively, there exist $(X,x) \in N_{\mathrm{Aff}(3)}(\Gamma)$ and $Y \in GL(3,\mathbb{Z})$ such that*

$$[\mathcal{B}_2] = X[\mathcal{B}_1]Y^{-1}.$$

Proof. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be any ordered bases for N_1, N_2, respectively, say

$$\mathcal{B}_1 = \{(I,a_1),(I,a_2),(I,a_3)\}, \quad \mathcal{B}_2 = \{(I,b_1),(I,b_2),(I,b_3)\}.$$

Then $[\mathcal{B}_1] = [a_1,a_2,a_3]$ and $[\mathcal{B}_2] = [b_1,b_2,b_3]$. Suppose there exists $(X,x) \in$ Aff(3) giving rise to an affine conjugacy from (N_1,Γ) to (N_2,Γ). So the conjugation by (X,x) maps Γ to Γ itself and N_1 to N_2. Since $\mu(X,x)(I,a_i) = (I,Xa_i)$, for $i = 1,2,3$, clearly $\{Xa_1,Xa_2,Xa_3\}$ is a new ordered set of generators for N_2. This is related to the original ordered set of generators $\mathcal{B}_2$ by an integral matrix $Y \in \mathrm{GL}(3,\mathbb{Z})$. Thus

$$X[\mathcal{B}_1] = X[a_1,a_2,a_3] = [Xa_1,Xa_2,Xa_3] = [\mathcal{B}_2]Y.$$

The converse is easy. □

For convenience, in the rest of the paper we shall use the notation $N_1 \overset{\Gamma}{\sim} N_2$ if $[\mathcal{B}_2] = X[\mathcal{B}_1]Y^{-1}$ as in Proposition 1.2.

Now, the first step (I) is a purely group-theoretic problem and can be handled by Proposition 1.2. Firstly, we need to calculate the normalizer $N_{\mathrm{Aff}(3)}(\Gamma)$. Let $(X,x) \in \mathrm{Aff}(3)$. For (X,x) to normailize the maximal free abelian subgroup $\mathcal{Z}$ of Γ, it is necessary and sufficient that the matrix X to be in $\mathrm{GL}(3,\mathbb{Z})$. To take care of the rest, Pick a finite subset F of Γ whose image in the quotient (=holonomy) group $\Gamma/\mathbb{Z}^3$ is a set of generators. Find all $(X,x) \in \mathrm{GL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$ such that

$$(X,x)(A,a)(X,x)^{-1} \in \Gamma$$

for all $(A,a) \in F$. In dimension 3, F can be taken so that it has cardinality at most 2. For example, with $\mathfrak{G}_2$, one can take $F = \{\alpha = (A,a)\}$ a singleton, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and $a = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$. Now one needs to solve only the equations $XAX^{-1} = A$ and $(I - A)x + (X - I)a \in \mathbb{Z}^3$.

The following is a list of the linear parts of $N_{\mathrm{Aff}(3)}(\Gamma)$, denoted by $L(N_{\mathrm{Aff}(3)}(\Gamma))$, for all 3-dimensional Bieberbach groups.

LIST (B) *The linear parts of* $N_{\mathrm{Aff}(3)}(\Gamma)$:
(All matrices are in $\mathrm{GL}(3,\mathbb{Z})$, so their determinants are ± 1.)

$\mathfrak{G}_1$: $\mathrm{GL}(3,\mathbb{Z})$.

$\mathfrak{G}_2$: $\left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \mathrm{GL}(2,\mathbb{Z}) \end{bmatrix} \right\}$.

$\mathfrak{G}_3$: $\mathbb{Z}_6 \rtimes \mathbb{Z}_2$, the generators are $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$.

$\mathfrak{G}_4 : \mathbb{Z}_4 \rtimes \mathbb{Z}_2$, the generators are $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$.

$\mathfrak{G}_5 : \mathbb{Z}_6 \rtimes \mathbb{Z}_2$, the generators are $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$.

$\mathfrak{G}_6$: $(\mathbb{Z}_2)^3 \rtimes S_3$, where $(\mathbb{Z}_2)^3 = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$ and S_3 is the permutation group which acts on $(\mathbb{Z}_2)^3$ naturally.

$$\mathfrak{B}_1 : \left\{ \begin{bmatrix} \text{odd} & \mathbb{Z} & 0 \\ \text{even} & \text{odd} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{Z}) \right\}.$$

$$\mathfrak{B}_2 : \left\{ \begin{bmatrix} k & l & \frac{k+l\mp 1}{2} \\ m & n & \frac{m+n\mp 1}{2} \\ 0 & 0 & \pm 1 \end{bmatrix} \in \mathrm{GL}(3, \mathbb{Z}) \right\}.$$

$$\mathfrak{B}_3,\ \mathfrak{B}_4 : (\mathbb{Z}_2)^3 = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\}.$$

The second part (II) "Realization" can be done by the following procedure. Let Γ be an affine Bieberbach group, and N be a normal abelian subgroup of Γ with $G = \Gamma/N$ finite. To describe the natural affine action of G on the flat torus $\mathbb{R}^3/N$, we must make the torus the *standard torus*, and describe the action on the universal covering level. In other words, the action of G should be defined on $\mathbb{R}^3$ as affine maps (this is really explaining the liftings of a set of generators of G in Γ), and simply say that our action is the affine action modulo the standard $\mathbb{Z}^3$. It is quite easy to achieve this. Let $\{(I, a_1), (I, a_2), (I, a_3)\}$ be a generating set for N. Form a matrix B with the three column vectors a_1, a_2 and a_3. Then $B^{-1}a_i = e_i$, for i=1,2,3. Therefore, $\mu(B^{-1}, 0)$ maps Γ into another affine Bieberbach group in such a way that the generating set for N becomes the standard basis for $\mathbb{Z}^3$. Suppose

$\{\alpha_1, \cdots, \alpha_m\}, (m \leq 3)$ generates the quotient group G when project down via $\Gamma \to G$, then $\{(B^{-1},0)\alpha_1(B,0), (B^{-1},0)\alpha_2(B,0), (B^{-1},0)\alpha_3(B,0)\}$ describes the action of G on the standard torus. Hence,

Procedure 1.3. *Let Γ be an affine Bieberbach group, and N be a normal abelian subgroup of Γ with $G = \Gamma/N$ finite. The natural affine action of G on the flat torus $\mathbb{R}^3/N$ can be described by the following procedure:*

(1) *Find a generating set for N: $\{(I, a_1), (I, a_2), (I, a_3)\}$.*
(2) *Form a matrix B with the three column vectors a_1, a_2 and a_3 from (1).*
(3) *Find a set of elements $\{\alpha_1, \cdots, \alpha_m\}$ whose image in G is a generating set for G. (This set can be taken so that m is at most 3).*
(4) *Conjugate $\{\alpha_1, \cdots, \alpha_m\}$ by $(B^{-1}, 0) \in \mathrm{Aff}(3)$.*

One should interpret the resulting action of G on $\mathbb{R}^3$ modulo the standard $\mathbb{Z}^3$.

The above process involves many matrix calculations. These are done by a computer using *muMath*, and hand-checked.

Let N be a normal subgroup of Γ. Suppose that Γ/N is a finite abelian group. Then there is a surjective homomorphism of $H_1(\Gamma;\mathbb{Z})$ onto Γ/N. However LIST (A) shows that $H_1(\Gamma;\mathbb{Z})$ has p-rank at most 3. Therefore, Γ/N has p-rank at most 3. Thus, for any finite abelian group G acting freely on T, we only have to consider groups of the form $\mathbb{Z}_n \times \mathbb{Z}_m \times \mathbb{Z}_r$. Note $\mathbb{Z}_1 = \{1\}$.

The following observations eliminate some of the possible actions on a torus. Remarks (1) and (2) follow from the fact that $G = \Gamma/N$ is a quotient group of $H_1(\Gamma;\mathbb{Z})$. Remark (3) is true because the holonomy group Ψ is a quotient group of $G = \Gamma/N$. Note also that, by the composite map $\Gamma \to H_1(\Gamma;\mathbb{Z}) \to \Gamma/N$, the elements with non-trivial linear part map nontrivially into Γ/N.

REMARKS. (1) By virtue of $H_1(\mathfrak{G}_5;\mathbb{Z}) = \mathbb{Z}$, there is no free action of a non-cyclic group on T whose orbit manifold is $\mathbb{R}^3/\mathfrak{G}_5$.
(2) There is no free action of a group G of p-rank 3 on T whose $\pi_1(T/G)$ is $\mathfrak{G}_3$, $\mathfrak{G}_4$, $\mathfrak{G}_6$, $\mathfrak{B}_2$ or $\mathfrak{B}_4$. All these cases, $H_1(\Gamma,\mathbb{Z})$ has p-rank at most 2.
(3) There is no free action of a cyclic group G on T for which $\pi_1(T/G)$ is $\mathfrak{G}_6$, $\mathfrak{B}_3$ or $\mathfrak{B}_4$. This is due to the fact that their holonomy groups are not cyclic.

§2. Free actions of finite abelian groups G on T with $\pi_1(\mathrm{T}/\mathrm{G}) = \mathfrak{G}_2$.

Theorem 2.1. *The following table gives a complete list of all free actions (up to topological conjugacy) of finite abelian groups G on T which yield an orbit manifold homeomorphic to $\mathbb{R}^3/\mathfrak{G}_2$.*

Group G	*Conjugacy classes of normal free abelian subgroups*	
$\mathbb{Z}_{2n}$	all n	$K_1 = \langle \alpha^{2n}, t_2, t_3 \rangle$
	n even	$K_2 = \langle \alpha^{2n}, \alpha^n t_2, t_3 \rangle$
$\mathbb{Z}_{2n} \times \mathbb{Z}_2$	all n	$N_1 = \langle \alpha^{2n}, t_2^2, t_3 \rangle$
	n even	$N_2 = \langle \alpha^{2n}, t_2^2, \alpha^n t_3 \rangle$
$\mathbb{Z}_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$		$L = \langle \alpha^{2n}, t_2^2, t_3^2 \rangle$

The action of $\mathfrak{G}_2/K_i = \mathbb{Z}_{2n}$ on the torus $\mathbb{R}^3/K_i$ is given by $\langle h_i \rangle$, $(i = 1, 2)$:

$$h_1(x,y,z) = (x + \frac{1}{2n}, -y, -z), \qquad h_2(x,y,z) = (x + y + \frac{1}{2n}, -y, -z).$$

The action of $\mathfrak{G}_2/N_i = \mathbb{Z}_{2n} \times \mathbb{Z}_2$ on the torus $\mathbb{R}^3/N_i$ is given by $\langle f_i, g_i \rangle$, $(i = 1, 2)$:

$$f_1 = h_1, \qquad g_1(x,y,z) = (x, y + \frac{1}{2}, z);$$

$$f_2(x,y,z) = (x + z + \frac{1}{2n}, -y, -z), \qquad g_2 = g_1$$

The action of $\mathfrak{G}_2/L = \mathbb{Z}_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the torus $\mathbb{R}^3/L$ is given by $\langle \phi, \xi, \eta \rangle$:

$$\phi = h_1, \qquad \xi = g_1, \qquad \eta(x,y,z) = (x, y, z + \frac{1}{2}).$$

Proof. Let N be a normal free abelian subgroup of $\mathfrak{G}_2$ such that $\mathfrak{G}_2/N$ is abelian. Then $[\mathfrak{G}_2, \mathfrak{G}_2] = \langle t_2^2, t_3^2 \rangle \subset N \subset \langle t_1, t_2, t_3 \rangle$. Suppose N contains both $t_1^k t_2, t_1^{\ell_1} t_2^{\ell_2} t_3$. Then N can be represented by a matrix $\begin{bmatrix} n & k & \ell_1 \\ 0 & 1 & \ell_2 \\ 0 & 0 & 1 \end{bmatrix}$ by Lemma 1.1. By a column operation, N reduces to

$$\begin{bmatrix} n & k & \ell \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \text{that is,} \quad N = \langle \alpha^{2n},\ \alpha^{2k} t_2,\ \alpha^{2\ell} t_3 \rangle.$$

For convenience, we will abuse notation and use N instead of $[\mathcal{B}]$ for an ordered basis $\mathcal{B}$ of N, unless a confusion is likely. Since N contains t_2^2 and t_3^2, we have $\begin{pmatrix} 2k \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2\ell \\ 0 \\ 0 \end{pmatrix} \in N$. Thus $2k$ and 2ℓ must be multiples of n. If n is odd, then $k = \ell = 0$, and $N = \langle \alpha^{2n}, t_2, t_3 \rangle$. If n is even, $k = 0$, $\frac{n}{2}$, and $\ell = 0$, $\frac{n}{2}$. Therefore if n is even, the possible abelian normal subgroups are

$$K_1 = \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} n & \frac{n}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} n & 0 & \frac{n}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} n & \frac{n}{2} & \frac{n}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{bmatrix}$ are elements of $L(N_{\mathrm{Aff}(3)}(\mathfrak{G}_2))$ from LIST (B). Thus we have

$$K_3 = \begin{bmatrix} n & 0 & \frac{n}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{\mathfrak{G}_2}{\underset{R}{\sim}} \begin{bmatrix} n & 0 & \frac{n}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underset{C}{\sim} \begin{bmatrix} n & \frac{n}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = K_2,$$

$$K_4 = \begin{bmatrix} n & \frac{n}{2} & \frac{n}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{C}{\sim} \begin{bmatrix} n & \frac{n}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \overset{\mathfrak{G}_2}{\underset{R}{\sim}} \begin{bmatrix} n & \frac{n}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = K_2.$$

It is easy to see that $(K_1, \mathfrak{G}_2)$ is not affine conjugate to $(K_2, \mathfrak{G}_2)$, because there do not exist $X \in L(N_{\mathrm{Aff}(3)}(\mathfrak{G}_2))$ and $Y \in \mathrm{GL}(3, \mathbb{Z})$ for which $XK_2 = K_1Y$. Thus we get $K_1 = \langle \alpha^{2n}, t_2, t_3 \rangle$ and $K_2 = \langle \alpha^{2n}, \alpha^n t_2, t_3 \rangle$.

Suppose N does not contain $t_1^k t_2$, but $t_1^{\ell_1} t_2^{\ell_2} t_3$. Then, since $t_2^2 \in N$, N can be represented by a matrix $\begin{bmatrix} n & 0 & r \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that the $(2,3)$-entry was killed by a row operation using $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Since $t_3^2 \in N$, $r = 0$ or $n/2$. If $r = 0$, then $N = N_1$. If $r = n/2$ (with n even), then $N = N_2$. The case when $t_1^k t_2 \in N$, $t_1^{\ell_1} t_2^{\ell_2} t_3 \notin N$ is the same.

Lastly, suppose both $t_1^k t_2, t_1^{\ell_1} t_2^{\ell_2} t_3 \notin N$. Then N must be

$$L := \begin{bmatrix} n & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} ; \text{ that is, } \langle \alpha^{2n}, t_2^2, t_3^2 \rangle.$$

In this case, $\mathfrak{G}_2/L = G$ is of the form $\mathbb{Z}_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The realization of the action of $G \cong \mathfrak{G}_2/N$ on the torus $\mathbb{R}^3/N$, as an affine action on the standard torus, is easy provided that we follow the Procedure 1.3. For example, let $N = N_1$. Since G is generated by the images of α and t_2, it is enough to calculate conjugations of α and t_2 by $(N_1, 0)$.

For $\alpha = (A, a) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right)$,

$$(N_1, 0)^{-1}(A, a)(N_1, 0) = (N_1^{-1} A N_1, N_1^{-1} a) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2n} \\ 0 \\ 0 \end{bmatrix} \right).$$

For $t_2 = (I, e_2) = \left(I, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$,

$$(N_1, 0)^{-1}(I, e_2)(N_1, 0) = \left(N_1^{-1} I N_1, N_1^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \left(I, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \right).$$

Thus, for the pair $(N_1, \mathfrak{G}_2)$, we get an affine action of $G = \mathfrak{G}_2/N_1 \approx \mathbb{Z}_{2n} \times \mathbb{Z}_2$ on the standard torus:

$$f_1(x, y, z) = (x + \frac{1}{2n}, -y, -z) \text{ and } g_1(x, y, z) = (x, y + \frac{1}{2}, z).$$

The other cases are left to the reader. □

§3. Free actions of finite abelian groups G on T with $\pi_1(\mathrm{T}/\mathrm{G}) = \mathfrak{B}_1$.

We begin with a consideration of some fundamental facts for integral matrices. As usual $N_1 \overset{\mathfrak{B}_1}{\sim} N_2$ means that $(N_1, \mathfrak{B}_1)$ is affine conjugate to $(N_2, \mathfrak{B}_1)$. We need some lemmas.

Lemma 3.1. [Grosswald, p.234] *If* $(p,q)=1$, *then the set* $\{p+mq \mid m\in\mathbb{Z}\}$ *contains infinitely many primes.*

Corollary 3.2. *Let* $(p,q)=1$ *and* k *be any integer. Then there exist integers* m *and* n *such that* $mp+nq=1$ *and* $(k,n)=1$.

Proof. Assume that there exist integers a, b such that $ap+bq=1$. Since $(b,p)=1$, there is an integer ℓ such that $(b+\ell p,k)=1$ by Lemma 3.1. Thus, $m=a-\ell q$ and $n=b+\ell p$ satisfy the requirement. □

The next lemma shows how to eliminate the $(1,2)$-entries in upper triangular matrices using $L(N_{\mathrm{Aff}(3)}(\mathfrak{B}_1))$.

Lemma 3.3. *Let* $gcd(k,m,n)=c$.

$$\text{If } \frac{n}{c} \text{ is odd, } \begin{bmatrix} m & k & * \\ 0 & n & * \\ 0 & 0 & * \end{bmatrix} \overset{\mathfrak{B}_1}{\sim} \begin{bmatrix} \frac{mn}{c} & 0 & * \\ 0 & c & * \\ 0 & 0 & * \end{bmatrix}.$$

$$\text{If } \frac{n}{c} \text{ is even, } \begin{bmatrix} m & k & * \\ 0 & n & * \\ 0 & 0 & * \end{bmatrix} \overset{\mathfrak{B}_1}{\sim} \begin{bmatrix} c & 0 & * \\ 0 & \frac{mn}{c} & * \\ 0 & 0 & * \end{bmatrix}.$$

Proof. For convenience, we work only in the first 2×2 blocks of matrices involved. It is enough to check the case when $c=1$ so that $\begin{bmatrix} m & k \\ 0 & n \end{bmatrix}=\begin{bmatrix} dp & dq \\ 0 & \ell \end{bmatrix}$ with $(p,q)=1$ and $(d,\ell)=1$. By Corollary 3.2 (with $k=d$), there exist an integer a and an odd b such that $ap+bq=1$ and $(d,b)=1$. Note $\begin{bmatrix} a & -q \\ b & p \end{bmatrix}\in \mathrm{GL}(3,\mathbb{Z})$, thus we have

$$\begin{bmatrix} dp & dq \\ 0 & \ell \end{bmatrix} \underset{C}{\sim} \begin{bmatrix} d & 0 \\ \ell b & \ell p \end{bmatrix}.$$

Let ℓ be an odd number. Since $(d,\ell b)=1$ and ℓb is odd, there exist an integer r and an odd s such that $rd+s\ell b=1$. If r is even, then we have

$$\begin{bmatrix} \ell b & -d \\ r & s \end{bmatrix}\begin{bmatrix} d & 0 \\ \ell b & \ell p \end{bmatrix}=\begin{bmatrix} 0 & -d\ell p \\ 1 & s\ell p \end{bmatrix} \underset{C}{\sim} \begin{bmatrix} d\ell p & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $\begin{bmatrix} \ell b & -d \\ r & s \end{bmatrix}$ is an element of $L(N_{\mathrm{Aff}(3)}(\mathfrak{B}_1))$. If r is odd, then d is even. Set $r'=r-\ell b$ (so r' is even) and $s'=s+d$.

Then $r'd+s'\ell b = 1$. By proceeding as above with $\begin{bmatrix} \ell b & -d \\ r' & s' \end{bmatrix} \in L(N_{\mathrm{Aff}(3)}(\mathfrak{B}_1))$, we get $\begin{bmatrix} d & 0 \\ \ell b & \ell p \end{bmatrix} \overset{\mathfrak{B}_1}{\sim} \begin{bmatrix} d\ell p & 0 \\ 0 & 1 \end{bmatrix}$.
The case when ℓ is even is similar. □

Theorem 3.4. *The following table gives a complete list of all free actions (up to topological conjugacy) of finite abelian groups G on T yielding the orbit manifold $\mathbb{R}^3/\mathfrak{B}_1$.*

Group G	*Conjugacy classes of normal free abelian subgroups*	
$\mathbb{Z}_{2nm} \times \mathbb{Z}_n$	all m, n	$N_1 = \langle \epsilon^{2nm}, t_2^n, t_3 \rangle$
	nm even	$N_2 = \langle \epsilon^{2nm}, t_2^n, \epsilon^{nm} t_3 \rangle$
	n even	$N_3 = \langle \epsilon^{2nm}, t_2^n, t_2^{\frac{n}{2}} t_3 \rangle$
$\mathbb{Z}_{2n} \times \mathbb{Z}_{nm}$	all m, n	$N_4 = \langle \epsilon^{2n}, t_2^{nm}, t_3 \rangle$
	nm even	$N_5 = \langle \epsilon^{2n}, t_2^{nm}, t_2^{\frac{nm}{2}} t_3 \rangle$
	n even	$N_6 = \langle \epsilon^{2n}, t_2^{nm}, \epsilon^n t_3 \rangle$
$\mathbb{Z}_{2nm} \times \mathbb{Z}_n \times \mathbb{Z}_2$		$L_1 = \langle \epsilon^{2nm}, t_2^n, t_3^2 \rangle$
$\mathbb{Z}_{2n} \times \mathbb{Z}_{nm} \times \mathbb{Z}_2$		$L_2 = \langle \epsilon^{2n}, t_2^{nm}, t_3^2 \rangle$

When m is odd, N_1, N_2, N_3 and L_1 are affine conjugate to N_4, N_5, N_6 and L_2, respectively.
The action of $G \cong \mathfrak{B}_1/N_i$ on the torus $\mathbb{R}^3/N_i$ is given by $\langle f_i, g_i \rangle$, $(1 \leq i \leq 6)$:

$$f_1(x,y,z) = \left(x + \frac{1}{2nm}, y, -z\right), \qquad g_1(x,y,z) = \left(x, y + \frac{1}{n}, z\right)$$

$$f_2(x,y,z) = \left(x + z + \frac{1}{2nm}, y, -z\right), \qquad g_2 = g_1$$

$$f_3(x,y,z) = \left(x + \frac{1}{2nm}, y + z, -z\right), \qquad g_3 = g_1$$

$$f_4(x,y,z) = \left(x + \frac{1}{2n}, y, -z\right), \qquad g_4(x,y,z) = \left(x, y + \frac{1}{nm}, z\right)$$

$$f_5(x,y,z) = \left(x + \frac{1}{2n}, y + z, -z\right), \qquad g_5 = g_4$$

$$f_6(x,y,z) = \left(x + z + \frac{1}{2n}, y, -z\right), \qquad g_6 = g_4.$$

The action of $G \cong \mathfrak{B}_1/L_i$ *on the torus* $\mathbb{R}^3/L_i$ *is given by* $\langle \phi_i, \xi_i, \eta_i \rangle$, $(i = 1,2)$:

$$\phi_1 = f_1, \qquad \xi_1 = g_1, \qquad \eta_1(x,y,z) = \left(x,\, y,\, z + \frac{1}{2}\right)$$
$$\phi_2 = f_4, \qquad \xi_2 = g_4, \qquad \eta_2 = \eta_1.$$

Proof. Since $\langle t_3^2 \rangle = [\mathfrak{B}_1, \mathfrak{B}_1] \subseteq N$, by Lemm 1.1, N has bases of the form

$$\begin{bmatrix} a & b & k \\ 0 & c & \ell \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} p & q & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

By Lemma 3.3, there are 4 cases:

$$A_1 := \begin{bmatrix} nm & 0 & s \\ 0 & n & t \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 := \begin{bmatrix} n & 0 & s \\ 0 & nm & t \\ 0 & 0 & 1 \end{bmatrix},$$

$$L_1 := \begin{bmatrix} nm & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad L_2 := \begin{bmatrix} n & 0 & 0 \\ 0 & nm & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that when m is odd, A_1 and L_1 are affine conjugate to A_2 and L_2, respectively, by Lemma 3.3. If $s = 0$ and $t = 0$, then A_1 and A_2 yield N_1 and N_4, respectively. We shall use the fact $\langle t_3^2 \rangle = [\mathfrak{B}_1, \mathfrak{B}_1] \subseteq N$ repeatedly. If $s \neq 0$ and $t = 0$. Then nm even. So A_1 and A_2 yield N_2 and N_6, respectively. If $s = 0$ and $t \neq 0$ (so n is even), then A_1 and A_2 yield N_3 and N_5, respectively. When $s \neq 0$ and $t \neq 0$, a calculation yields $\langle \epsilon^{2nm},\, t_2^n,\, \epsilon^{nm} t_2^{\frac{n}{2}} t_3 \rangle \overset{\mathfrak{B}_1}{\sim} N_3$ and $\langle \epsilon^{2n},\, t_2^{nm},\, \epsilon^n t_2^{\frac{nm}{2}} t_3 \rangle \overset{\mathfrak{B}_1}{\sim} N_6$. Thus, A_1, A_2 do not yield new ones. Note that if n is odd, then $\mathfrak{B}_1/L_1 \cong \mathbb{Z}_{2nm} \times \mathbb{Z}_{2n}$.

Let N be one of the normal subgroups which we obtained. To get an affine action of $\mathfrak{B}_1/N$ on the standard torus, one proceeds as in Theorem 2.1. □

From this theorem, when $n = 1$ in $\mathbb{Z}_{2nm} \times \mathbb{Z}_n$, we get actions of $\mathbb{Z}_{2m}$. Compare [Hempel, Theorem 2].

Corollary 3.5. *For m even, there are exactly two free actions N_1, N_2 (up to topological conjugacy) of $\mathbb{Z}_{2m}$ on T whose orbit manifold is $\mathbb{R}^3/\mathfrak{B}_1$.* □

Example. (How to read the table?) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ act freely on T. Then there are 6 distinct affine conjugacy classes of free actions of G on T with $\pi_1(T/G) = \mathfrak{B}_1$. G can be viewed as $\mathbb{Z}_{2\cdot 2\cdot 1} \times \mathbb{Z}_2$ $(n = 2, m = 1)$; $\mathbb{Z}_{2\cdot 1} \times \mathbb{Z}_{1\cdot 4}$ $(n = 1, m = 4)$; and $\mathbb{Z}_{2\cdot 1\cdot 2} \times \mathbb{Z}_1 \times \mathbb{Z}_2$ $(n = 1, m = 2)$ to yield N_1, N_2, N_3; N_4, N_5; and L_1. They are

$$N_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

§4. Free actions of finite abelian groups G on T whose $\pi_1(T/G)$ is $\mathfrak{G}_1, \mathfrak{G}_3, \mathfrak{G}_4, \mathfrak{G}_5, \mathfrak{G}_6, \mathfrak{B}_2, \mathfrak{B}_3$ or $\mathfrak{B}_4$.

The method which was used in the previous sections gives enough idea for all the remaining cases. The following theorem is not hard and its proof is left to the reader.

Theorem 4.1. *The following table is a complete list of all free actions (up to topological conjugacy) of finite abelian groups G on T whose $\pi_1(T/G)$ is $\mathfrak{G}_1$, $\mathfrak{G}_3$, $\mathfrak{G}_4$, $\mathfrak{G}_5$, $\mathfrak{G}_6$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ or $\mathfrak{B}_4$.*

Γ	Group G		Conjugacy classes of normal free abelian subgroups
$\mathfrak{G}_1$:	$\mathbb{Z}_{nmr} \times \mathbb{Z}_{nm} \times \mathbb{Z}_n$		$L_1 = \langle t_1^{nmr}, t_2^{nm}, t_3^n \rangle$
$\mathfrak{G}_3$:	$\mathbb{Z}_{3n}$		$K_1 = \langle \alpha^{3n} t_2, t_3 \rangle$
		$n = 3k$	$K_2 = \langle \alpha^{3n}, \alpha^n t_2, \alpha^n t_3 \rangle$
	$\mathbb{Z}_{3n} \times \mathbb{Z}_3$	all n	$N_1 = \langle \alpha^{3n}, t_2^3, t_2^{-1} t_3 \rangle$
		$n = 3k$	$N_2 = \langle \alpha^{3n}, t_2^3, \alpha^n t_3 \rangle$

$\mathfrak{G}_4$:	$\mathbb{Z}_{4n}$	all n	$K_3 = \langle \alpha^{4n},\ t_2, t_3 \rangle$
		$n = 2k$	$K_4 = \langle \alpha^{4n}, \alpha^{2n} t_2, \alpha^{2n} t_3 \rangle$
	$\mathbb{Z}_{4n} \times \mathbb{Z}_2$	all n	$N_3 = \langle \alpha^{4n}, t_2^2, t_2^{-1} t_3 \rangle$
		$n = 2k$	$N_4 = \langle \alpha^{4n}, t_2^2, \alpha^{2n} t_3 \rangle$
$\mathfrak{G}_5$:	$\mathbb{Z}_{6n}$		$K_5 = \langle \alpha^{6n},\ t_2,\ t_3 \rangle$
$\mathfrak{G}_6$:	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$N_5 = \langle \alpha^2,\ \beta^2,\ \gamma^2 \rangle$
	$\mathbb{Z}_4 \times \mathbb{Z}_2$		$N_6 = \langle \alpha^4,\ \alpha^2 \beta^2,\ \gamma^2 \rangle$
	$\mathbb{Z}_4 \times \mathbb{Z}_4$		$N_7 = \langle \alpha^4,\ \beta^4,\ \alpha^2 \beta^2 \gamma^2 \rangle$
$\mathfrak{B}_2$:	$\mathbb{Z}_{2nm} \times \mathbb{Z}_n$	n odd	$N_8 = \langle \epsilon^{2nm},\ \epsilon^{2n} t_2^n,\ \epsilon^{n-1} t_2^{\frac{n-1}{2}} t_3 \rangle$
	$\mathbb{Z}_{2nm} \times \mathbb{Z}_{2n}$		$N_9 = \langle \epsilon^{2nm},\ \epsilon^{2n} t_2^n,\ \epsilon^2 t_2 t_3^{-2} \rangle$
			$N_{10} = \langle \epsilon^{2nm},\ t_2^n,\ \epsilon^2 t_2 t_3^{-2} \rangle$
$\mathfrak{B}_3$:	$\mathbb{Z}_{2n} \times \mathbb{Z}_2$	all n	$N_{11} = \langle \alpha^{2n},\ \epsilon^2,\ t_3 \rangle$
		$n = 2k$	$N_{12} = \langle \alpha^{2n}, \epsilon^2, \alpha^n t_3 \rangle$
	$\mathbb{Z}_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$		$L_2 = \langle \alpha^{2n},\ \epsilon^2,\ t_3^2 \rangle$
$\mathfrak{B}_4$:	$Z_{2n} \times \mathbb{Z}_2$	all n	$N_{13} = \langle \alpha^{2n},\ \epsilon^2,\ t_3 \rangle$
		$n = 2k$	$N_{14} = \langle \alpha^{2n}, \alpha^n \epsilon^2, \alpha^n t_3 \rangle$
	$\mathbb{Z}_{2n} \times \mathbb{Z}_4$		$N_{15} = \langle \alpha^{2n},\ \epsilon^4,\ \epsilon^2 t_3 \rangle$

Example. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ act freely on T. Then there are 18 distinct topological conjugacy classes of free actions. Recall that (G_1, T) and (G_2, T) are not affine conjugate unless T/G_1 and T/G_2 are homeomorphic. In fact, there exist 2 distinct free actions in each flat manifold whose $\pi_1(T/G)$ is $\mathfrak{G}_2$, $\mathfrak{B}_2$ or $\mathfrak{B}_3$, 3 in $\mathfrak{B}_4$, 6 in $\mathfrak{B}_1$ (see, Example in §3) and one in $\mathfrak{G}_1$, $\mathfrak{G}_4$, or $\mathfrak{G}_6$.

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THE LIMIT SET OF A DISCRETE SUBGROUP OF MÖBIUS GROUP

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In this survey article, we devote to a detailed discussion of the limit set of a discrete subgroup of Möbius group, $M(B^{m+1})$, and some interesting subsets.

We denote by $M(B^{m+1})$ the full group of Möbius transforms preserving B^{m+1}. A subgroup Γ of $M(B^{m+1})$ is discrete if the identity has a neighborhood whose intersection with Γ reduces to the identity.

A point $\xi \in S^m$ is a limit point for the discrete group Γ if for one, and hence every, point $x \in B^{m+1}$ the orbit $\Gamma(x)$ accumulates at ξ. The set of limit points is denoted by $\Lambda(\Gamma)$ or simply Λ.

Our analysis of the limit set will be based on the rate at which orbits approach the point in question. We will start by considering the most rapid rate possible and then succesively weaken the rate of approach.

Given a discrete group Γ acting in B^m and a point $\xi \in \Lambda$ then, for any $\gamma \in \Gamma$ and any $a \in B^{m+1}$ we have $1 - |\gamma(a)| \leq |\xi - \gamma(a)|$. In terms of orbital approach, the best we can hope for is that, on a sequence $\{\gamma_n\} \subset \Gamma$,

$$\frac{|\xi - \gamma_n(a)|}{1 - |\gamma_n(a)|} \to 1 \text{ as } n \to \infty.$$

DEFINITION 1: The point $\xi \in \Lambda$ is said to a line transitive point for Γ if for every pair $a, b \in B$ there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\lim_{n\to\infty} \frac{|\xi - \gamma_n(a)|}{1 - |\gamma_n(a)|} = 1 \text{ and } \lim_{n\to\infty} \frac{|\xi - \gamma_n(b)|}{1 - |\gamma_n(b)|} = 1.$$

To have a geometric interpretation of this definition, we need the following Lemma.

Lemma 2 [N]. *Suppose $a \in B$ and $\eta,\xi \in S$, $\eta \neq \xi$. Let s be the hyperbolic distance from a to the geodesic joining ξ and η then*

$$\cosh s = \frac{2|a-\xi||a-\eta|}{|\xi-\eta|(1-|a|^2)}.$$

Now suppose ξ is a line transitive point and σ is a geodesic ending at ξ (with η the other end point of σ) then from the Lemma 2, we have, on the sequence $\{\gamma_n\}$, $\rho(\gamma_n(a),\sigma) \to 0$ and, similarly $\rho(\gamma_n(b),\sigma) \to 0$ as $n \to \infty$. Thus, for any pair of points $a,b \in B$ there is a sequence of images of the geodesic σ coming arbitrarily close to both points.

We weaken the requirement that orbits approach a limit point almost radially and require that they approach within a cone.

DEFINITION 3: The point $\xi \in \Lambda$ is said to be a conical limit point (point of approximation) for Γ if for every $a \in B$ there exists a sequence $\{\gamma_n\} \subset \Gamma$ on which the sequence $\frac{|\xi-\gamma_n(a)|}{1-|\gamma_n(a)|}$ remains bounded.

The following result is s consequence of Theorem 1 of [**B-M**].

Thoerem 4. *The point $\xi \in S$ is a conical limit point for Γ if and only if there is a geodesic σ ending at ξ such that for any point $a \in B$ there are infinitely many Γ-images of σ within a bounded hyperbolic distance of a.*

To see the geometric significance of the conical limit set consider a line element l_1 determining a geodesic ending at a conical limit point. As the line element slides along the geodesic it keeps meeting images of some compact portion of B. On the quotient space this geodesic flow keeps returning to a compact part of the manifold.

Analogous to the notion of the orbits approaching the boundary in a conical region is that of the orbit approaching the boundary in a horoshere.

DEFINITION 5: Let Γ be a discrete group acting in B. A point $\xi \in S$ is a horospherical limit point for Γ if for every $a \in B$ there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\frac{|\xi-\gamma_n(a)|^2}{1-|\gamma_n(a)|} \to 0 \text{ as } n \to \infty.$$

We denote by H the horospherical limit set. Now we see the geometric interpretation of the horosherical limit point in dimension 2. Consider a line element l_1 determining a geodesic ending at a horosherical limit point ξ. Construct a horocycle at ξ passing through the carrier point of l_1. As the line element moves around the horocycle it comes arbitrarily close to group images of any other line element. On the quotient space B/Γ these ideas give rise to the horocyclic flow and the existence of a horospherical limit point means that there is a point with a dense trajectory under the flow. Sullivan has shown the remarkable equivalence regarding horospherical limit set and group actions.

Let $V = \mathbb{H}^n/\Gamma$ be a hyperbolic manifold and let $V(r)$ denote the volume of the points of V within a distance r of some base point p. Also $H(r)$ denote the volume of the ball of radius r in hyperbolic space.

Theorem 6 [S]. *The following are equivalent*

(1) *The fundamental domain has zero area at ∞.*
(2) *The ratio $V(r)/H(r) \to 0$ as $r \to \infty$.*
(3) *The action of Γ on the sphere at ∞ is conserative.*
(4) *The horospherrical limit set of Γ has full measure on the sphere.*

Now we introduce another subset of the limit set which can be characterized topologically. Here we will study two kinds of objects associated to a complete hyperbolic manifold $M = B^m/\Gamma$. The first are geodesic rays in M which satisfy a simple tangential recurrence condition. The other is a certain type of limit point of the universal covering group of the manifold. These limit points are defined purely in terms of the local topological dynamics of the action of the universal covering group on the sphere at infinity. We will give several characterizations which are directly analogous to characterizations of conical limit points. These show that every limit point of our type is a conical limit point. All geodesics and geodesic rays are assumed to have unit speed parametrizations, and to be oriented in the direction of increasing parameter. From now on, we denote by Γ a non-elementary torsionfree discrete group of hyperbolic isometries acting on the Poincaré ball $B^m = \{x \in \mathbb{R}^m : |x| < 1\}$, $m \geq 2$.

DEFINITION 7: An oriented geodesic or geodesic ray α in $M = B^m/\Gamma$ is said to be *recurrent* if for every (equivalently, for some) tangent vector v to α, there is a sequence of times $t_i \to \infty$ such that $\alpha'(t_i) \to v$. A geodesic or geodesic ray in B^m is called *recurrent* if its image in M is recurrent.

Equivalently, whenever α passes through a small open subset of the unit tangent space of M, it leaves and returns infinitely often. Our first result is Theorem 1.5, which characterizes the recurrent geodesic rays as those which are well approximated by closed geodesics in M (see §1 for precise definitions). This implies that the space of recurrent geodesic rays is topologically complete.

Now let $p \in \partial B^m$ be a limit point of Γ. By a neighborhood of p, we will always mean an open neighborhood of p in the "sphere at infinity" ∂B^m.

DEFINITION 8: One says that a neighborhood U of p *can be concentrated with control* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(V)$ and $\gamma(U) \subseteq V$. If such a neighborhood U exists, then p is called a *controlled concentration point* for Γ.

Obviously, the set of controlled concentration points of a Möbius group is invariant under the action of the group. Related properties of controlled concentration points are studied in [**A-H-M**] and [**H**].

The connection between recurrent geodesics and controlled concentration points is simple and is given as Theorem 13: a geodesic (or geodesic ray) in B^m is recurrent if and only if its endpoint is a controlled concentration point. The space of geodesic rays in a hyperbolic manifold M can be identified with the unit tangent bundle T_1M, and this endows it with a natural topology which coincides with the compact-open topology when the geodesic rays are regarded as maps $[0,\infty) \to T_1M$. In this section we will find a characterization of recurrent geodesic rays in terms of the closed geodesics in B^m/Γ. It implies that the recurrent geodesic rays form a G_δ subset of the space of all geodesic rays, and hence can be endowed with a complete metric.

From the definition, it is evident that a geodesic or geodesic ray in B^m is recurrent if and only if all of its translates are recurrent, and an (oriented) geodesic is recurrent if and only if each (or one) of its compatibly oriented geodesic subrays is recurrent. The following lemma gives two other descriptions of recurrent geodesics in B^m. Since they are direct consequences of the definitions, we omit the proofs.

Lemma 9. *The following are equivalent for an oriented geodesic α in B^m.*

(1) *α is recurrent.*
(2) *If $r \in \partial B^m$ is the starting point of α and $p \in \partial B^m$ is its ending point, then there is a sequence $\{\gamma_n\}$ of elements of Γ so that $\{\gamma_n(0)\}$ converges to r and $\{\gamma_n(p)\}$ converges to p.*
(3) *If α_0 is any subray of α, then there is a sequence $\{\gamma_n\}$ of elements of Γ such that the images of α_0 under the γ_n converge to α (with respect to the Hausdorff metric on compact subsets of $B^m \cup \partial B^m$) in an oriented sense.*

Fix a metric on the unit tangent bundle of hyperbolic space B^m which is invariant under the Möbius group, and let T_1M have the induced metric.

DEFINITION 10: Let β be a piecewise geodesic segment in M (possibly a closed loop) with initial point b and let γ be a closed geodesic. We say that γ *ϵ-approximates* β if there exists a point c on γ so that a pair of points starting at b and c and traveling at unit speed along β and γ respectively remain within hyperbolic distance ϵ until the point on γ reaches c again (necessarily, if β is not a closed loop, its length $\ell(\beta)$ must be at least $\ell(\gamma)$).

Lemma 11. *Fix $\epsilon > 0$ and let D be a geodesically convex disc in M. Given an oriented geodesic segment β_1 in M which starts and ends in D, denote by β_2 the geodesic segment in D running from the final point of β_1 to its initial point, and let β denote the piecewise geodesic loop $\beta_1 \cup \beta_2$. Then there are positive constants L and δ so that if $\ell(\beta_1) \geq L$ and the initial and final unit tangent vectors of β_1 are within distance δ, then*

(i) the element g of Γ represented by β is loxodromic, and

(ii) the unique closed geodesic γ in M which is freely homotopic to β satisfies $\ell(\beta) - \epsilon \leq \ell(\gamma) \leq \ell(\beta)$ and ϵ-approximates β.

PROOF: Assuming (i) we first prove (ii). Some lift $\tilde{\gamma}$ of γ is the axis of g in B^m. Lift β to a piecewise geodesic curve connecting the endpoints of $\tilde{\gamma}$, and fix a lift b of the initial point of β_1 which lies in the lift of β. Let α be the geodesic segment from b to $\tilde{\gamma}$ meeting $\tilde{\gamma}$ perpendicularly at a point a. Let γ_0 be the geodesic segment in γ from a to $g(a)$. Connecting b to $g(b)$ is a lift of β consisting of geodesic segments $\tilde{\beta}_1$ and $\tilde{\beta}_2$ which are lifts of β_1 and β_2. Let σ be the geodesic segment from b to $g(b)$. Note that the lengths satisfy $\ell(\gamma_0) \leq \ell(\sigma) \leq \ell(\tilde{\beta}_1) + \ell(\tilde{\beta}_2) = \ell(\beta)$. By making L large and δ small, we may assume that the initial tangent vectors of $\tilde{\beta}_1$ and σ are as close as we want. By making δ small, independent of L, we may force the final tangent vector of $\tilde{\beta}_1$ and the initial tangent vector of $g(\tilde{\beta}_1)$ to be close, hence also the final tangent vector of σ and the initial tangent vector of $g(\sigma)$. Therefore in the quadrilateral formed by σ, α, $g(\alpha)$, and γ_0, the (internal) angles at the initial and terminal points of σ must add up almost to π. This forces α to be short, so we may assume $\ell(\alpha) \leq \epsilon/6$. If δ is chosen to force $\tilde{\beta}_2$ to be shorter than $\epsilon/6$, then $\ell(\beta) \leq \ell(\sigma) + \epsilon/3 \leq \ell(\gamma) + 2\epsilon/3$. Hence points traveling along $\tilde{\gamma}$ and $\tilde{\beta}_1 \cup \tilde{\beta}_2$ starting from a and b respectively will stay within distance ϵ until the point from a reaches $g(a)$. This proves (ii) assuming (i).

To prove (i), suppose for contradiction that g is parabolic, fixing ∞ in the upper half-space model of hyperbolic space, and make a similar argument replacing α by the geodesic ray from b to ∞. Note that b and $g(b)$ lie on the same horosphere at ∞. Thus, when L is large, the initial and final tangent vectors of σ must point upward and downward at large angles, but then the final vector of $\tilde{\beta}_1$ and the image of its initial vector under g cannot be close.

We can now obtain the characterization of recurrent geodesics and geodesic rays.

DEFINITION 12: An oriented geodesic or geodesic ray β in a hyperbolic manifold M is said to be *approximable by closed geodesics* if for every $b \in \beta$ and every $\epsilon > 0$, there exists a closed oriented geodesic which ϵ-approximates the subray of β starting at b.

Theorem 13. *A geodesic or geodesic ray in M is recurrent if and only if it is approximable by closed geodesics.*

PROOF: Let b be an arbitrary point on the recurrent geodesic β and let $\epsilon > 0$. Choose L and δ as in the previous lemma (for a neighborhood of b). By recurrence there is a segment of β starting at b, of length greater than L, whose initial and final tangent vectors lie within distance δ in T_1M. The lemma yields a closed geodesic which ϵ-approximates β. The other direction is clear.

Given a geodesic ray $\beta\colon [0,\infty) \to M$, a closed geodesic $\gamma\colon \mathbb{R} \to M$, and $t_0 \in \mathbb{R}$, define the fellow traveler distance from (γ, t_0) to β to be

$$\max_{0 \leq t \leq \ell(\gamma)} d\left(\gamma(t_0 + t), \beta(t)\right) ,$$

where d denotes hyperbolic distance in M. (Remember that geodesics are parametrized by arc length.) This function is periodic in t_0, and we define $d_\gamma(\beta)$ to be the minimum fellow traveler distance over all $t_0 \in \mathbb{R}$. According to the previous theorem, β is recurrent if and only if there is a sequence γ_i of closed geodesics in M such that $\lim d_{\gamma_i}(\beta) = 0$.

For a closed geodesic γ define $B(\gamma, \epsilon)$ to be the open subset of the unit tangent space of M corresponding to the geodesic rays β such that $d_\gamma(\beta) < \epsilon$. If $\gamma_1, \gamma_2, \ldots$ is an enumeration of the closed geodesics in M, let $U_n = \cup_{k=1}^{\infty} B(\gamma_k, 1/n)$. Then by Theorem 13 the G_δ subset $\cap_{n=1}^{\infty} U_n$ is precisely the subspace corresponding to the recurrent geodesic rays. Since the unit tangent bundle admits a complete metric, the space of recurrent geodesic rays does also . We state this as the following.

Corollary 14. *The space of recurrent geodesic rays in M is topologically complete.*

Theorem 13 will give the basic connection between recurrent geodesics and controlled concentration points: the latter are precisely the endpoints of recurrent geodesics in B^m. As is well-known, if there is a sequence of distinct elements γ_n of Γ so that the images $\gamma_n(0)$ converge to a point r, then r must be in ∂B^m, since Γ is discrete. Moreover, for all x in B^m, the $\gamma_n(x)$ also converge to r. It is known [**M**, VI.B.4] that a limit point p is a conical limit point if and only if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n(p) \to q$ and $\gamma_n(0) \to r$ where $r \neq q$. We begin with an analogous characterization of controlled concentration points.

Theorem 15. *A limit point p is a controlled concentration point for Γ if and only if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n(p) \to p$ and $\gamma_n(0) \to r$ where $r \neq p$.*

PROOF: See Thoerem 2.1 in [**A-H-M**]

Combining Theorem 13 and the description of recurrent geodesic rays given in Lemma 9 (ii) yields immediately our unifying result:

Theorem 16. *A limit point p is a controlled concentration point for Γ if and only if there exists a recurrent geodesic (equivalently, a recurrent geodesic ray) in B^m which ends at p.*

obvious.

As one application of Theorem 13, we will give a second characterization of controlled concentration points. For $p \in \partial B^m$, define a subset $L(p)$ of the space of geodesics in B^m as follows. Let $\tilde{\alpha}$ be a geodesic ray in B^m ending at p and let α be its image in M. Define a subset $\ell(p)$ of the unit tangent bundle $T_1 M$ by

$$\ell(p) = \bigcap_{r>0} \overline{\alpha'([r,\infty))} \, .$$

($\ell(p)$ is the ω-limit set of $\alpha'(0)$ under the geodesic flow.) It is easy to check that $\ell(p)$ is independent of the choice of $\tilde{\alpha}$. Notice that if some tangent vector to a geodesic in M lies in $\ell(p)$, all of its tangent vectors do (i.e. $\ell(p)$ is invariant by the geodesic flow). Define $L(p)$ to be the collection of (oriented) geodesics in B^m such that the tangent vectors of their images in M lie in $\ell(p)$.

Lemma 17. *Let $G(p)$ be the set of geodesics in B^m which end at p. Then the set of recurrent geodesics ending at p equals $G(p) \cap L(p)$.*

PROOF: Suppose β is a recurrent geodesic ending at p. Since $L(p)$ is independent of the choice of ray ending at p, we may take the ray to lie in β, with initial vector v. A sequence of translates of the ray which limits onto the geodesic determines in M a sequence of tangent vectors to the image of the ray which limit onto the image of v, showing that β is in $L(p)$. Conversely, if $L(p)$ contains a geodesic ending at p, then taking the defining ray for $L(p)$ to be contained in this geodesic shows that the geodesic is recurrent.

Using this Lemma, Theorem 13 gives immediately

Corollary 18. *A point $p \in \partial B^m$ is a controlled concentration point if and only if $G(p) \cap L(p)$ is nonempty.*

There is an analogous characterization of conical limit points.

Proposition 19. *A point $p \in \partial B^m$ is a conical limit point if and only if $L(p)$ is nonempty.*

PROOF: p is a conical limit point if and only if for some geodesic ray $\tilde{\alpha}$ ending at p, there exists a sequence $t_i \to \infty$ such that the images $\alpha(t_i)$ in B^m/Γ lie in some compact region. Since the space of unit tangent vectors to the points in a compact region is compact, this is equivalent to the existence of a sequence $s_i \to \infty$ such that the $\alpha'(s_i)$ converge to a vector in the unit tangent bundle. Such limit vectors comprise $\ell(p)$.

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LOOP SPACES AND CATEGORIES

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ABSTRACT. The coherence theory in categories has a strong relation with the coherence problem in homotopy theory. This article is an elementary introduction to the long-standing problem: "What algebraic structure of a category corresponds to a n-fold loop space ?" This question has been resolved for $n = 1$ by Stasheff ([15]) and MacLane ([8]) and for $n = \infty$ by Segal ([14]).These answers to this question have been expolited in many areas like algebraic K-theory, homotopy theory, etc. This problem has recently been reilluminated by Fiedorowicz who found an answer for $n = 2$ ([8]) that is motivated by quantum groups. We recall braided tensor category (and quantum groups) and define iterated monoidal categories which seem to be the answer to the coherence problem.

1.INTRODUCTION

The recognition principle is to specify the appropriate internal structure such that a space X has such structure if and only if X is of the (weak) homotopy type of n-fold loop space. It has been known for years that there is a relation between coherence problems in homotopy theory and that in categories. It was shown by Stasheff ([15]) and MacLane ([8]) that monoidal categories give rise to 1-fold loop spaces. It was later proved that there is a similar correspondence between infinite loop spaces and symmetric monoidal categories ([14]). Precisely speaking, the group completion of the nerve of a symmetric monoidal category is an infinite loop space. This correpondence plays an important role in algebraic K-theory. This fact also provides new examples of infinite loop spaces and infinite loop maps in which many topologists are interested. Now we may naturally raise the following problem : "What algebraic structure on a category corresponds to an n-fold loop space ?" Fiedorowicz got the answer for $n = 2$ via braided tensor categories which play key roles in quantum groups. Although his proof looks simple, it is in some sense so mysterious that it is not immediate to see the generalization. Fiedorowicz, Schwänzl and Vogt later found the generalization of the notion of braided tensor category whose algebraic structure is more transparently analogous to the structure of a 2-fold loop space. They called this 2-fold monoidal category. The construction of 2-fold monoidal category endows us with an idea to consturct n-fold monoidal categories which seem to correspond to n-fold loop spaces. Although the main objective of this article is to deal with the coherence problem, quantum group theory is roughly introduced in section 2 because braided tensor categories have strong and important connection to quantum groups.

2.QUANTUM GROUPS

Quantum group is a catchall term used to describe mathematical developments arising from mathematical physics centered around the Yang-Baxter equation. Mathematcal physicists like to describe states of physical systems by using representations of Lie groups or Lie algebras. Thus the basic objects of study are k-modules V (k :fixed commutative ring, usually $\mathbb{C}$) with some extra structure, for example, action of Lie groups or Lie algebras. The basic operation on such objects is tensor product $V \otimes_k W$ that is associative and commutative up to isomorphism

$$(U \otimes V) \otimes W \overset{\cong}{\to} U \otimes (V \otimes W)$$
$$(u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w)$$

$$\tau : U \otimes V \overset{\cong}{\to} V \otimes U$$
$$u \otimes v \longmapsto v \otimes u$$

Around 1980, Yang and Baxter proposed that states of certain physical systems should be described by a tensor product which is commutative by an 'exotic' commutativity isomorphism $R : V \otimes V \overset{\cong}{\to} V \otimes V$ such that $R^2 \neq id$, but which still acts reasonably with respect to the associativity isomorphism. That means the following diagram commutes:

$$\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{id \otimes R} & V \otimes V \otimes V \\
{\scriptstyle R \otimes id}\downarrow & & \downarrow{\scriptstyle R \otimes id} \\
V \otimes V \otimes V & & V \otimes V \otimes V \\
{\scriptstyle id \otimes R}\downarrow & & \downarrow{\scriptstyle id \otimes R} \\
V \otimes V \otimes V & \xrightarrow[R \otimes id]{} & V \otimes V \otimes V
\end{array}$$

Hence if we let $R_{12} = R \otimes id$, $R_{23} = id \otimes R$, then the commutativity of the above diagram can be expressed by the following equation:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{1}$$

This equation is called Yang-Baxter equation. We can see it exists by the following example.

Example 2.1. Let V be a complex vector space with basis $\{v_1, \cdots, v_n\}$. Let $q \in \mathbb{C} - \{0\}$. Define $R : V \otimes V \to V \otimes V$ by

$$R(v_i \otimes v_j) = \begin{cases} qv_i \otimes v_i & \text{if } i = j \\ v_j \otimes v_i + (q - \frac{1}{q})v_i \otimes v_j & \text{if } j < i \\ v_j \otimes v_i & \text{if } j > i \end{cases}$$

Then R satisfies Yang-Baxter equation, but $R^2 \neq id$ unless $q = 1$.

Reamark 2.2. (Connection to braids) Given an R-matrix $R : V \otimes V \to V \otimes V$, and for a braid group B_n we get related group representation

$$\varphi : B_n \to \text{End}(V^{\otimes n})$$

as follows:
Let $\{\beta_i\}_{i=1}^{n-1}$ denote the standard braid generators of B_n, then define $\varphi_n(\beta_i) = R_i$, where $R_i \in \text{End}(V^{\otimes n})$ is defined by

$$R_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes R(x_i \otimes x_{i+1}) \otimes \cdots \otimes x_n$$

Then Yang-Baxter equation implies

$R_iR_j = R_jR_i \quad \text{if } |i-j| \geq 2$

$R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$

which are defining relations for β_i. Thus φ_n is well-defined. With some extra work, we can use these braid representations in defining knot and link polynomial of V.Jones.

2. Basic Definitions

Categories equipped with tensor product (call tensor categories or monoidal categories) have been studied for years. Tensor categories equipped with destinguished braidings, which are called *braided tensor categories*, have recently attracted many mathematician's attention since Drinfel'd's International Congress talk on quantums groups. He provided a class of new natural examples of braided tensor categories. In this section we define monoidal (or tensor) categories, symmetric monoidal categories and braided tensor categories.

Definition 2.1. A (strict) *monoidal* (or *tensor*) *category* $(\mathcal{C}, \Box, E)$ is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ (called the (tensor) product) and an object E (called the unit object) such that

(a) $\Box$ is strictly associative
(b) E is a strict 2-sided unit for $\Box$.

Definition 2.2. A (strict) *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \Box, E)$ such that

(c) There exists a natural commutativity isomorphism $C_{A,B} : A\Box B \to B\Box A$ satisfying

(i) $C_{A,E} = id_A = C_{E,A}$
(ii) associativity condition:

$$\begin{array}{ccccc} A\Box B\Box C & & \overset{C_{A\Box B,C}}{\longrightarrow} & & C\Box A\Box B \\ & id_A\Box C_{B,C} \searrow & & \nearrow C_{A,C}\Box id_B & \\ & & A\Box C\Box B & & \end{array}$$

(iii) $C_{A,B} = C_{B,A}^{-1}$

Definition 2.3. A *braided tensor category* is defined as follows:
Delete (iii) of Definition 2.2, and add another associativity condition

$$\begin{array}{ccccc} A\Box B\Box C & & \overset{C_{A,B\Box C}}{\longrightarrow} & & B\Box C\Box A \\ & C_{A,B}\Box id_C \searrow & & \nearrow id_B\Box_{A,C} & \\ & & B\Box A\Box C & & \end{array}$$

Example 2.4. Symmetric monoidal category.
Let F be a field. Let $\mathcal{C}$ be a category whose objects are F^n, $n \geq 0$ and morphisms are nonsingular square matrices. Then the product $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined by $F^m \oplus F^n = F^{m+n}$ and $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A \in GL_m(F)$, $B \in GL_n(F)$.
Then it is easy to see that $\mathcal{C} = \coprod_{n\geq 0} GL_n(F)$ is a symmetric monoidal category. This palys a key role in the definition of Quillen's algebraic K-theory. The nerve $B\mathcal{C} = \coprod_{n\geq 0} BGL_n(F)$ has a monoid structure induced by $\oplus$. It is known ([10]) that the group completion of $B\mathcal{C} = \coprod_{n\geq 0} BGL_n(F)$, which is

$\Omega B\left(\coprod_{n\geq 0} BGL_n(F), \oplus\right)$, is homotopy equivalent to $BGL(F)^+ \times Z$. Hence we have

$$\pi_i \Omega B\left(\coprod_{n\geq 0} BGL_n(F), \oplus\right) = K_i(F) \quad \text{for } i \geq 1$$

Moreover $\Omega B\left(\coprod_{n\geq 0} BGL_n(F), \oplus\right)$ is an infinite loop space. This comes from the general fact-Theorem 3.2.

Example 2.5. Braided tensor category.
Let $\mathcal{B}$ be a category whose objects are $[n]$, $n \geq 0$ and

$$hom_{\mathcal{B}}([m],[n]) = \begin{cases} \phi & \text{if } m \neq n \\ B_n & \text{if } m = n \end{cases}$$

$[m] \otimes [n] = [m+n]$ and the tensor product of braids is the disjoint union of braids. Let $E = [0]$. Define $\sigma_{[m],[n]}$ to be the braid connecting $1, \cdots, m$ to $n+1, \cdots, n+m$ and $m+1, \cdots, m+n$ to $1, \cdots, n$, respectively. Then $\mathcal{B}$ is the free braided tensor category on the object $[1]$.

Remark 2.6. V. Drinfel'd found natrual examples of braided tensor category. Let $\mathbb{C}_t = \mathbb{C}[t, t^{-1}]$. Let $q \in \mathbb{C} - \{0\}$. Let S_q be the funtor $S_q : \mathbb{C}_t$-$Mod \to \mathbb{C}$-$Mod, V \mapsto V \otimes_{\mathbb{C}_t} \mathbb{C}$, where $\mathbb{C}$ is given the $\mathbb{C}_t$-module structure by the ring homomorphism $\mathbb{C}_t \to \mathbb{C}$, $t \mapsto q$. Let $sl(n)$ denote a Lie algebra over $\mathbb{C}$. V. Drinfel'd proved that there is a braided tensor category $\mathcal{M}(sl(n,q)) \subseteq \mathbb{C}_t$-$Mod$ whose image under S_q is the category of finite dimensional representation of $sl(n)$. The case when q is a root of unity is of special interest to physicists.

3. Results on Coherence

In this section we mention a couple of classical results on the coherence problem and a recent result of Fiedorowicz. For all the results mentioned in this section, the converse statement is alse true.

Theorem 3.1. ([8], [15]) *The group completion of the nerve of a monoidal category is of the homotopy type of a loop space.*

Theorem 3.2. ([14]), [17]) *The group completion of the nerve of a symmetric monoidal category is of the homotopy type of an infinite loop space.*

These theorems are very important and have a plenty of applications. It has not been resolved for the coherence of n-fold loop spaces. Fiedorowicz proved a striking result:

Theorem 3.3. ([4]) *Let $\mathcal{C}$ be a braided tensor category. Then the group completion of its nerve $\Omega B(B\mathcal{C}, \Box)$ is a Ω^2-space up to homotopy.*

4. ITERATED MONOIDAL CATEGORY

Although Fiedorowicz proved Theorem 3.3 in a simple and short way, it was, in some sense, mysterious. In order to generalize the proof of Theorem 3.3, Fiedorowicz, Schwänzl and Vogt constructed a new category; 2-fold monoidal category. It is analogous to braided tensor category, but better than braided tensor category in the sense that it is more transparently analogous to the structure of a 2-fold loop space. We may regard a 2-fold loop space as a loop space in the category of loop spaces. Similarly we consider a monoidal category in the category of monoidal categories. This gives a notion of a 2-fold monoidal category. By mimicking the Segal-Thomason proof of Theorem 3., it is easy to prove that there is the same correspondence between 2-fold loop space and 2-fold monoidal category. Fiedorowicz, Schwänzl and Vogt generalized the definition of 2-fold monoidal category to construct n-fold monoidal category which seems to be the probable answer for the general coherence problem. The rest of this section consists of the definitions of 2-fold and n-fold monoidal categories.

Definition 4.1. A monoidal functor $(F, \eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(E) = E$ together with a natural transformation

$$\eta_{A,B} : F(A) \Box F(B) \to F(A \Box B),$$

which satisfies the following conditions

(1) Internal Associativity: The following diagram commutes

$$\begin{array}{ccc} F(A)\Box F(B)\Box F(C) & \xrightarrow{\eta_{A,B}\Box id_{F(C)}} & F(A\Box B)\Box F(C) \\ \Big\downarrow{\scriptstyle id_F(A)\Box\eta_{B,C}} & & \Big\downarrow{\scriptstyle \eta_{A\Box B,C}} \\ F(A)\Box F(B\Box C) & \xrightarrow{\eta_{A,B\Box C}} & F(A\Box B\Box C) \end{array}$$

(2) Internal Unit Conditions: $\eta_{A,E} = \eta_{E,A} = id_{F(A)}$.
Given two monoidal functors $(F,\eta) : \mathcal{C} \to \mathcal{D}$ and $(G,\zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF,\xi) : \mathcal{C} \to \mathcal{E}$,where ξ denotes the composite

$$GF(A)\Box GF(B) \stackrel{\xi_{F(A),F(B)}}{\longrightarrow} G(F(A)\Box F(B)) \stackrel{G(\eta_{A,B})}{\longrightarrow} GF(A\Box B).$$

We denote by **MonCat** the category of monoidal categories and monoidal functors.

Definition 4.2. A 2-*fold monoidal category* is a monoid in **MonCat**. This means that we are given a monoidal category $(\mathcal{C},\Box,E)$ and a monoidal functor $(\Box,\eta) : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which satisfies

(1) External Associativity: the following diagram commutes in **MonCat**

$$\begin{array}{ccc}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} & \xrightarrow{(\Box,\eta)\times id_c} & \mathcal{C}\times\mathcal{C} \\
\Big\downarrow{\scriptstyle id_c\times(\Box,\eta)} & & \Big\downarrow{\scriptstyle (\Box,\eta)} \\
\mathcal{C}\times\mathcal{C} & \xrightarrow{(\Box,\eta)} & \mathcal{C}
\end{array}$$

(2) External Unit Conditions: the following diagram commutes in **MonCat**

$$\begin{array}{ccccc}
\mathcal{C}\times E & \xrightarrow{\subseteq} & \mathcal{C}\times\mathcal{C} & \xleftarrow{\supseteq} & E\times\mathcal{C} \\
{\scriptstyle \cong}\Big\downarrow & & {\scriptstyle (\Box,\eta)}\Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\
\mathcal{C} & \xrightarrow{=} & \mathcal{C} & \xleftarrow{=} & \mathcal{C}
\end{array}$$

Explicitly this means that we are given a second associative binary operation $\boxdot : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, for which E is also a two-sided unit. Moreover we are given a natural transformation

$$\eta_{A,B,C,D} : (A\boxdot B)\Box(C\boxdot D) \to (A\Box C)\boxdot(B\Box D).$$

The internal unit conditions give $\eta_{A,B,E,E} = \eta_{E,E,A,B} = id_{A\boxdot B}$, while the external unit conditions give $\eta_{A,E,B,E} = \eta_{E,A,E,B} = id_{A\Box B}$, The internal associativity condition gives the commutative diagram

$$\begin{array}{ccc}
(U\boxdot V)\Box(W\boxdot X)\Box(Y\boxdot Z) & \xrightarrow{\eta_{U,V,W,X}\Box id_{Y\boxdot Z}} & ((U\Box W)\boxdot(V\Box X))\Box(Y\boxdot Z) \\
{\scriptstyle id_{U\boxdot V}\Box\eta_{W,X,Y,Z}}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta_{U\Box W,V\Box X,Y,Z}} \\
(U\boxdot V)\Box((W\Box Y)\boxdot(X\Box Z)) & \xrightarrow{\eta_{U,V,W\Box T,X\Box Z}} & (U\Box W\Box Y)\boxdot(V\Box X\Box Z)
\end{array}$$

The external associativity condition gives the commutative diagram

$$\begin{array}{ccc}
(U\Box V\Box W)\Box(X\Box Y\Box Z) & \xrightarrow{\eta_{U\Box V,W,X\Box Y,Z}} & ((U\Box V)\Box(X\Box Y))\Box(W\Box Z) \\
\eta_{U,V\Box W,X,Y\Box Z}\downarrow & & \downarrow \eta_{U,V,X,Y}\Box id_{W\Box Z} \\
(U\Box X)\Box((V\Box W)\Box(Y\Box Z)) & \xrightarrow{id_{U\Box X}\Box\eta_{V,W,Y,Z}} & (U\Box X)\Box(V\Box Y)\Box(W\Box Z)
\end{array}$$

Notice that we have natural transformations

$$\eta_{A,E,E,B} : A\Box B \to A\Box B \text{ and } \eta_{E,A,B,E} : A\Box B \to B\Box A$$

Definition 4.3. An *n-fold monoidal category* is a category $\mathcal{C}$ with the following structure.

(1) There are n distinct multiplications

$$\Box_1, \Box_2, \dots, \Box_n : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$$

which are strictly associative and $\mathcal{C}$ has an object E which is a strict unit for all the multiplications

(2) For each pair (i,j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{A,B,C,D} : (A\Box_j B)\Box_i(C\Box_j D) \to (A\Box_i C)\Box_j(B\Box_i).$$

These natural transformations η^{ij} satisfy the following conditions :

(a) Internal unit condition: $\eta^{ij}_{A,B,E,E} = \eta^{ij}_{E,E,A,B} = id_{A\Box_j B}$

(b) External unit condition: $\eta^{ij}_{A,E,B,E} = \eta^{ij}_{E,A,E,B} = id_{A\Box_i B}$

(c) Internal associativity condition: The following diagram commutes

$$\begin{array}{ccc}
(U\Box_j V)\Box_i(W\Box_j X)\Box_i(Y\Box_j Z) & \xrightarrow{\eta^{ij}_{U,V,W,X}\Box_i id_{Y\Box_j Z}} & ((U\Box_i W)\Box_j(V\Box_i X))\Box_i(Y\Box_j Z) \\
id_{U\Box_j V}\Box_i\eta^{ij}_{W,X,Y,Z}\downarrow & & \downarrow \eta^{ij}_{U\Box_i W,V\Box_i X,Y,Z} \\
(U\Box_j V)\Box_i((W\Box_i Y)\Box_j(X\Box_i Z)) & \xrightarrow{\eta^{ij}_{U,V,W\Box_i Y,X\Box_i Z}} & (U\Box_i W\Box_i Y)\Box_j(V\Box_i X\Box_i Z)
\end{array}$$

(d) External associativity condition: The following diagram commutes

$$\begin{array}{ccc}
(U\Box_j V\Box_j W)\Box_i(X\Box_j Y\Box_j Z) & \xrightarrow{\eta^{ij}_{U\Box_j V,W,X\Box_j Y,Z}} & ((U\Box_j V)\Box_i(X\Box_j Y))\Box_j(W\Box_i Z) \\
\eta^{ij}_{U,V\Box_j W,X,Y\Box_j Z}\downarrow & & \downarrow \eta^{ij}_{U,V,X,Y}\Box_j id_{W\Box_i Z} \\
(U\Box_i X)\Box_j((V\Box_j W)\Box_i(Y\Box_j Z)) & \xrightarrow{id_{U\Box_i X}\Box_j\eta^{ij}_{V,W,Y,Z}} & (U\Box_i X)\Box_j(V\Box_i Y)\Box_j(W\Box_i Z)
\end{array}$$

Finally it is required that for each triple (i,j,k) satisfying $1 \leq i < j < k \leq n$ the big hexagonal interchange diagram commute, which we omit here. (See [6]).

Definition 4.4. An *n-fold monoidal functor* $(F, \lambda^1, \ldots, \lambda^n) : \mathcal{C} \to \mathcal{D}$ between n-fold monoidal categories consists of a functor F such that $F(E) = E$ together with natural transformations

$$\lambda^i_{A,B} : F(A)\Box_i F(B) \to F(A\Box_i B) \quad i = 1, 2, \ldots, n$$

satisfying the same associativity and unit conditions as monoidal functors. In addition the following interchange diagram commutes:

$$\begin{array}{ccc}
(F(A)\Box_j F(B))\Box_i(F(C)\Box_j F(D)) & \xrightarrow{\eta^{ij}_{F(A),F(B),F(C),F(D)}} & (F(A)\Box_i F(C))\Box_j(F(B)\Box_i F(D)) \\
\Big\downarrow{\scriptstyle \lambda^j_{A,B}\Box_i\lambda^j_{C,D}} & & \Big\downarrow{\scriptstyle \lambda^i_{A,C}\Box_j\lambda^i_{B,D}} \\
F(A\Box_j B)\Box_i F(C\Box_j D) & & F(A\Box_i C)\Box_j F(B\Box_i D) \\
\Big\downarrow{\scriptstyle \lambda^i_{A\Box_j B,C\Box_j D}} & & \Big\downarrow{\scriptstyle \lambda^j_{A\Box_i C,B\Box_i D}} \\
F((A\Box_j B)\Box_i(C\Box_j D)) & \xrightarrow{F(\eta^{ij}_{A,B,C,D})} & F((A\Box_i C)\Box_j(B\Box_i D))
\end{array}$$

Composition of n-fold monoidal functors is defined in exactly the same way as for monoidal functors.

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A NOTE ON RATIONAL L-S CATEGORY

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1. Introduction

In this paper every topological space will be assumed to be path connected, pointed and to have the homotopy type of CW complex of finite type. The Lusternik-Schnirelmann category of a topological space $\mathcal{S}$, cat($\mathcal{S}$), is the least integer m so that $\mathcal{S}$ is covered by $m+1$ open subsets each of which is contractible in $\mathcal{S}$. For example, contractible spaces have category 0, spheres have category one, etc. The properties of L-S category have been studied extensively by [3].

An equivalent definition was given by G. Whitehead [6]; Let $T^{m+1}(\mathcal{S})$ denote the subspace of $\mathcal{S}^{m+1}$ consisting of all $(m+1)-$tuples $(x_1, \cdots, x_{m+1})$ with at least one x_i equal to specified base point in $\mathcal{S}$. It is usually called the fat wedge. Then cat($\mathcal{S}$) is defined as the least integer m so that we have the following homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S}^{m+1} \\ & {\scriptstyle \tilde{\Delta}} \searrow & \uparrow j \\ & & T^{m+1}(\mathcal{S}) \end{array} \qquad (*)$$

where Δ and j denote the diagonal map and an inclusion respectively.

2. Rational Homotopy

For details on the material on Sullivan minimal model and the homotopy of commutative differential graded algebra (CDGA) the reader is referred

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to [2] and [4]. Sullivan associated to every morphism $(A,d) \longrightarrow (B,d)$ of c-connected CDGA's $(i,eH^{\circ}(C)=0)$ a minimal model

$$(A,d) \xrightarrow{\ i\ } (A \otimes \Lambda X, d) \underset{\simeq}{\xrightarrow{\ \psi\ }} (B,d)$$

where i is a KS-extension defined by $i(a) = a \otimes 1$ for $a \in A$, and X has a well ordered, homogeneous vector space basis $\{x_\alpha | \alpha \in I\}$ such that $dx_\alpha \in A \otimes \Lambda X_{<\alpha}$ and $\alpha < \beta$ implies $deg\ x_\alpha \leq deg\ x_\beta$, where $X_{<\alpha}$ denotes span $\{x_\beta | \beta < \alpha\}$. The $' \simeq '$ indicates a cohomology isomorphism.

Sullivan minimal model of a space $\mathcal{S}$ is defined by a minimal model of $(\mathbb{Q},0) \to (A(\mathcal{S}),d)$ where $A(\mathcal{S})$ denotes the rational polynomial forms on $\mathcal{S}$. We usually denote a minimal model simply by $(\Lambda X, d)$. The fundamental theorem of rational homotopy theory is then

Theorem. *Each space $\mathcal{S}$ has a minimal model ΛX and for nilpotent spaces of finite type there is a natural isomorphism*

$$X^i \cong Hom(\pi_i \mathcal{S}, \mathbb{Q}), \quad i > 1$$

where $\pi_i \mathcal{S}$ is the i^{th} homotopy group.

Examples.

$$(1)\ M\left(\mathcal{S}^{2n+1}\right) = \Lambda\left(x_{2n+1}\right), dx = 0$$
$$(2)\ M\left(\mathcal{S}^{2n}\right) = \Lambda\left(x_{2n}, y_{4n-1}\right), dx = 0, dy = x^2$$

3. Rational Category

Let $\gamma : \Lambda X \to A(\mathcal{S})$ be a minimal model of a space $\mathcal{S}$. Then $\gamma^{(m+1)} : (\Lambda X)^{\otimes m+1} \to A\left(\mathcal{S}^{m+1}\right)$ is a minimal model for $\mathcal{S}^{m+1}$ and the multiplication map $\mu : (\Lambda X)^{\otimes m+1} \to \Lambda X$ represents the diagonal map. The previous diagram $(*)$ translates into a homotopy commutative drigram of minimal CDGA's

$$\begin{array}{ccc} \Lambda X & \xleftarrow{\ \mu\ } & (\Lambda X)^{\otimes m+1} \\ & \nwarrow_{\rho} & \downarrow \xi \\ & & \Lambda Y \end{array} \qquad (**)$$

where ΛY is a minimal model of $T^{m+1}(\mathcal{S})$ and ξ represents the inclusion j.

Definition. The rational category of $\mathcal{S}, cat_\circ(\mathcal{S})$, is the least integer m so that the above diagram (**) exists; that is, there exists ρ with $\rho\xi \simeq \mu$.

Remarks. (1) $cat_\circ(\mathcal{S}) \leq cat(\mathcal{S})$

(2) $cat_\circ(\mathcal{S}) = cat\,(\mathcal{S}_{\mathbb{Q}})$ if $\mathcal{S}$ is 1-connected where $\mathcal{S}_{\mathbb{Q}}$ is the $\mathbb{Q}$-localization of $\mathcal{S}$.

Definition. A morphism $f : A \to B$ of c-connected CDGA's makes A into a retract of B if there are morphisms

$$\Lambda X \xrightarrow{\alpha} \Lambda Y \xrightarrow{\beta} \Lambda X$$

(ΛX a model for A, ΛY a model for B) such that α represents f and $\beta\alpha \sim id.$ (If ΛX is minimal we can always modify β so that $\beta\alpha = id.$ [1])

Let $\Lambda^{>m}X$ denote the d-stable ideal of ΛX consisting of all products of length greater than m. We now describe a rational homotopy criterion for category giving a different proof of the following theorem given in [1]. The following lemma is well known in rational homotopy theory.

Lemma. *If $\theta : A \to B$ is a cohomology isomorphism, then $\theta_* : [\Lambda, A] \to [\Lambda, B]$ is bijective for any mininal Λ.*

Theorem. $cat_\circ(\mathcal{S}) \leq m$ *if ΛX is a retract of $\Lambda X/\Lambda^{>m}X$ where ΛX is a minimal model of $\mathcal{S}$.*

Proof. Consider the projection $p : \Lambda X \to \Lambda X/\Lambda^{>m}X$ and a minimal model $\theta : \Lambda Z \to \Lambda X/\Lambda^{>m}X$. By the lemma there exists a lift $\tilde{p} : \Lambda X \to \Lambda Z$ with $\theta\tilde{p} \simeq p$. Let $\gamma : \Lambda Z \to \Lambda X$ denote a retraction, with $\gamma\tilde{p} \simeq 1_{\Lambda X}$. We have the following homotopy commutative diagram

$$\tilde{p}\left(\begin{array}{ccc} \Lambda X & \xleftarrow{\mu} & (\Lambda X)^{\otimes m+1} \\ \downarrow p & & \downarrow \pi \\ \dfrac{\Lambda X}{\Lambda^{>m}X} & \xleftarrow{\bar{\mu}} & \dfrac{(\Lambda^+X)^{\otimes m+1}}{(\Lambda X)^{\otimes m+1}} \\ \simeq\uparrow\theta & & \simeq\uparrow\phi \\ \Lambda Z & \xleftarrow{\tilde{\mu}} & \Lambda Y \end{array}\right)\xi$$

where $\bar{\mu}$ is the map induced by μ and $\tilde{\mu}$ is a lift. In order to prove that $cat_o(\mathcal{S}) \leq m$ we must find $\rho : \Lambda Y \to \Lambda X$ with $\rho\xi \simeq \mu$. Let $\rho = \gamma\tilde{\mu}$ where γ is the retraction. Then we have

$$\theta\tilde{p}\mu \simeq p\mu = \bar{\mu}\pi \simeq \bar{\mu}\phi\xi \simeq \theta\mu\xi$$

Since θ is a cohomology isomorphism $\tilde{p}\mu \simeq \tilde{\mu}\xi$ also by the above lemma. Now

$$p\xi = \gamma\tilde{\mu}\xi \simeq \gamma\tilde{p}\mu \simeq 1_{\Lambda X}\mu = \mu.$$

Remark. The converse of the theorem is also true [1].

Before the work of Felix and Halperin on $cat_o(\mathcal{S})$ Toomer constructed an approximation, $e_o(\mathcal{S})$, to L-S category of a space $\mathcal{S}$ in terms of the Milnor-More spectral sequence. It can also be defined to be the least integer m so that $p : \Lambda X \to \Lambda X/\Lambda^{>m}X$ induces an injection in cohomology where ΛX is a minimal model of $\mathcal{S}$. The above remark clearly implies that $e_o(\mathcal{S}) \leq cat_o(\mathcal{S})$.

Theorem. *If the space $\mathcal{S}$ has a minimal model of the form $\Lambda = \big(\Lambda(x_1, \cdots, x_n), \alpha\big)$ with $deg\ x_i = odd$ for all i, then $e_o(\mathcal{S}) = cat_o(\mathcal{S}) = n$.*

Proof. The formal top dimension of the algebra Λ is $\sum deg\ x_i$ and the only element which can reach the dimension is $x_1 \cdots x_n$. By the hypothesis the fundamental class is maximally represented by a product of length n and hence $e_o(\mathcal{S}) = n$. On the other hand we have an inequality

$$cat_o\left(\Lambda\left(x_1, \cdots, x_{i+1}\right)\right) \leq cat_o\left(\Lambda\left(x_1, \cdots, x_i\right)\right) + 1$$

by [1]. Hence $cat_o\left(\Lambda\left(x_1, \cdots, x_n\right)\right) \leq n$ and the theorem follows immediately from the inequalities

$$n = e_o(\mathcal{S}) \leq cat_o(\mathcal{S}) \leq n.$$

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THE MORAVA K–THEORY AND SOME COMPUTATIONS

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INTRODUCTION

Among many spectra, the spectrum MU for complex bordism has played an important role in stable homotopy. Localized the spectrum MU at a prime p, it splits as wedges of suspensions of the similiar spectra BP which we call the Brown-Peterson Spectrum. The corresponding homology theory is called Brown-Peterson homology. BP was also proven to be very useful in stable homotopy, especially Adams Novikov spectral sequence. Quillen[7] found some strong connection between the bordism theory and the formal group law. But practically it never be easy to compute the BP theory. Early 1970's Morava developed a more manageable generalized homology theories known as the Morava K–theories, a sequence of homology theories which are represented by the spectrum $K(n)$ and satisfies the Kunneth isomorphisms for all spaces. In this paper we will give an elementary introduction to the Morava K–theories and compute the Atiyah–Hirzebruch spectral sequence for the Morava K–theories of $\Omega^2 Sp(n)$.

1. THE SPECTRUM $K(n)$.

In this section we will give some basic facts about the spectra MU, BP and $K(n)$. The good reference for these spectra is [8]. According to Brown's representation theorem every homology theory has its corresponding spectrum, a collections of spaces with structure maps. The spectrum for the complex bordism is the sequences of the Thom space $MU(n)$ of the classifying space $BU(n)$ for the unitary group $U(n)$ with the structure maps $\Sigma^2 MU(n-1) \to MU(n)$ induced by the map from $BU(n-1)$ with the universal bundle $\xi_{n-1} \bigoplus C$ into $BU(n)$ with ξ_n. Exploiting the map $CP^\infty \simeq MU(1) \to MU$ and the fact that $H_*(CP^\infty; Z)$ is free on $\beta_i \in H_{2i}(CP^\infty : Z)$, $i \geq 0$, we get

$$H_*(MU; Z) = Z[b_1, b_2, \dots].$$

$\pi_*(MU)$ was computed by Milnor using the Adams spectral sequence with

$$E_2 = Ext_{A_*}(Z/(p), H_*(MU; Z/(p)))$$

,where A_* is the dual of the Steenrod algebra A^*, converging to p-primary part of $\pi_*(MU)$. Due to the nice A_* comodule algebra structure of $H_*(MU)$ we can compute the E_2–term easily and the Adams spectral sequence collapes from E_2–term because of the even dimensionality of the surviving generators.

Theorem [5].

$$\pi_*(MU) = MU_* = Z[x_2, x_4, \ldots]$$

,where dim $x_{2i} = 2i$.

Localized the spectrum MU at a prime p, Quillen constructed a multiplicative idempotent map ϵ of ring spectra:

$$\epsilon : MU_{(p)} \to MU_{(p)}.$$

For any space X consider the map $\epsilon \wedge 1 : MU_{(p)} \wedge X \to MU_{(p)} \wedge X$. Then the image of ϵ_* become a natural direct summand of $MU_*(X)_p$ and it satisfies all the axioms for the generalized homology theory, so by the Browm's representation theorem it has its representing spectrum. We denote it by BP and homology theory by $BP_*(X)$ with

$$\pi_*(BP) = BP_* = Z_{(p)}[v_1, v_2, \cdots]$$

,where dim $v_i = 2(p^i - 1)$. Historically Brown and Peterson[2] first constructed BP spectrum starting with Eilenberg Maclane spectrum localized at p and building a postnikov tower to get a spectrum with torsion free in homology. We can also construct BP in a homotopy aspect like Priddy's construction[6]. From the BP spectrum we can follow the Sullivan–Bass way[1] to construct Morava K–theories spectrum $K(n)$. Each element x in $\pi_*(MU)$ can be represented by a manifold M.

A closed n-dimensional manifold W with singularity of type (x), $(n > k)$, is a space of the form $A \cup (B \times CM)$ where the boundary of the manifold A is $B \times M$ and CM is the cone space of M. Consider the following the cofibration

$$S_k \wedge MU \xrightarrow{x} MU \to C(MU, x).$$

If x is not a zero divisor, the exact sequence in homotopy yields a short exact. So we get $\pi_*(C(MU, x)) = \pi_*(MU)/(x)$. Note that in spectrum level a fibration is a cofibration and vice versa. We can interpreted above cofibration in geometric aspect, i.e., for each manifold N the manifold $N \times M$ is cobordant to empty set ϕ in $C(MU, x)$, where we regard empty set as an n-dimensional smooth manifold for all n.

We can iterated this process for elements $x_1, x_2, \ldots x_n$ in $\pi_*(MU)$. If each element x_k is not a zero divisor in $\pi_*(C(MU, x_1, \ldots, x_{k-1})$,

$$\pi_*(C(MU, x_1, \ldots, x_k) = \pi_*(MU)/(x_1, \ldots, x_k).$$

From the spectrum BP we can constuct a tower of spectra by killing certain bordism classes. This means that we can build a tower of bordism theories based on manifold with certain type of singularities. If we kill $(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots)$ in BP_* and we donote the corresponding spectrum by $k(n)$, $\pi_*(k(n)) = Z/(p)[v_n]$, where $|v_n| = 2(p^n - 1)$. If we kill v_n from $k(n)$, we get the Eilenberg–Maclane spectrum $H(Z/(p))$ for the mod p singular homology. In this constuction we may say that the mod p homology theory is also a bordism theory allowing infinitely many types of singularities. We have

$$S^{2(p^n-1)} \wedge k(n) \xrightarrow{v_n} k(n) \to H(Z)/(p).$$

We define the spectrum

$$K(n) = \varinjlim_{v_n} \sum^{-2i(p^n-1)} k(n).$$

This $K(n)$ is just the spectrum for Morava K–theories. Obviously these spectra are periodic , that is, $\sum^{2(p^n-1)} K(n) = K(n)$ and we have a sequence of homology theories for each n. We can also construct $K(n)$ using Landweber exact funtor method. Morava K–theories satisfies many nice properties. $K(n)_*$ is the graded field in the sense that all graded module over $K(n)_*$ are free. So $Tor_1^{K(n)_*}(K(n)_*(X), K(n)_*(Y)) = 0$ for all space X, Y. Hence from the Kunneth spectral sequence we get

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes K(n)_*(Y).$$

In fact besides the ordinary homology with field coefficients Morava K–theory is essentially only homology theories with the kunneth isomorphism. And

$$K(n)^*(X) \cong Hom_{K(n)_*}(K(n)_*(X), K(n)_*).$$

For the case $n = 0$, $K(0)_*(X) = H_*(X;Q)$ and $K(1)_*(X)$ is one of $p-1$ isomorphic summands of mod p complex K–theory for all p. Moreover suppose that X is a finite CW-complex. Then consider the Atiyah Hirzebruch spectral sequence converging to $K(n)_*(X)$ with

$$E_2 = H_*(X; Z/(p)) \otimes K(n)_*.$$

If the dimension of the X is less than $2p^n - 1$, there is no differential from the dimension reason. So

$$K(n)_*(X) \cong K(n)_* \otimes_{Z/(p)} H_*(X; Z/(p))$$

Therefore we may consider the spectrum $K(\infty)$ as the Eilenderg–Maclane spectrum with coefficient $Z/(p)$.

2. SOME COMPUTATION

Now we will turn to the computation of the Morava K–theory. For a space X and a generalized homology theory h_*, there exist a spectral sequence, we call the Atiyah–Hirzebruch spectral sequence, converging to $h_*(X)$ with

$$E_2 = H_*(X; h_*(*)).$$

Hence if h_* is the coefficient ring for the Morava K–theory, then

$$E_2 = H_*(X; Z/(p)) \otimes K(n)_*.$$

Like the classical K–theory the first non-trivial differential is determined by the Milnor operation $\mathcal{Q}_n$[4],where $\mathcal{Q}_n$ is defined inductively as the commutator for

$$\begin{aligned} \mathcal{Q}_0 &= Sq^1_*, \\ \mathcal{Q}_{k+1} &= [\mathcal{Q}_k, Sq^{2^k}_*] \quad ,\text{for} \quad p = 2, \\ \mathcal{Q}_0 &= \beta, \\ \mathcal{Q}_{k+1} &= [\mathcal{Q}_k, \mathcal{P}^{p^k}_*] \quad ,\text{for} \quad p > 2. \end{aligned}$$

Hence there is no differential until the E_{2p^n-1} stage because the dimension of v_n is $2p^n - 2$ and we have first non trivial differential: $d_{2p^n-1}(x \otimes v_n^k) = \mathcal{Q}_n x \otimes v_n^{n+1}$ in E_{2p^n-1}–term. Of course above argument also hold dually for a cohomology version of Morava K–theory $K(n)^*(X)$.

Now we will compute the Atiyah–Hirzebruch spectral sequence converging to $K(m)_*(\Omega^2 Sp(n))$ for $p = 2$ with

$$E_2 = H_*(\Omega^2 Sp(n); Z/(2)) \otimes K(m)_*.$$

From [3],

$$H_*(\Omega^2 Sp(n); Z/(2)) = P(Q_1^b x_{4i+1} : 0 \le i \le n-1, \quad b \ge 0)$$

$$\mathcal{Q}_m(x_{2^{k+1}i+2^k+2^{m+1}-1}) = a_{2i+1}^{2^k}, \text{i.e,}$$

$$\mathcal{Q}_m(a_{2^j s-1}) = \begin{cases} a_{s-2^{m+1-j}}^{2^j} & ,j \le m \\ a_t^{2^{m+k+1}} & ,j = m+1 \quad \text{and} \quad s-1 = 2^k t \\ a_{2^{j-m-1}s-1}^{2^{m+1}} & ,j > m+1 \end{cases}$$

,where $P(x)$ is the polynomial algebra on x. Let $T_m(x)$ be the truncated polynomial on x of the height 2^m and Q_i be the homology operation defined for $(n+1)$–fold loop space as

$$Q_i : H_q(\Omega^{n+1}X; Z/(2)) \longrightarrow H_{2q+i}(\Omega^{n+1}X; Z/(2)) \quad ,0 \le i \le n.$$

Proposition 2.1. *The $E_{2^{m+1}}$–term of the Atiyah–Hirzebruch spectral sequence for $K(m)_*(\Omega^2 Sp(n))$ is*

$$T_1(x_{2k_1+1} : 0 \le k_1 \le n-1-2^{m-1} \quad \text{and} \quad x_{4k+1} = 0 \quad \text{if} \quad k \ge 2^{m-1})$$
$$\otimes$$
$$T_2(x_{2k_2+1} : n-1-2^{m-1} < k_2 \le n-1-2^{m-2})$$
$$\otimes$$
$$\vdots$$
$$T_\ell(x_{2k_\ell+1} : n-1-2^{m-(\ell-1)} < k_\ell \le n-1-2^{m-\ell})$$
$$\otimes$$
$$\vdots$$
$$T_m(x_{2k_m+1} : n-3 < k_m \le n-2)$$
$$\otimes$$
$$T_{m+1}(x_{(2s+1)2^{M(s)+t+1}-1} : s = 0,1,2,\dots,n-1, \quad t = 0,1,2\dots m$$
$$x_{(2s+1)2^{M(s)+t+1}-1} \ne x_{(2k_j+1)2^j+2^{m+1}-1} \quad for \quad any\, j)$$
$$\otimes$$
$$T_m(x^2_{2^{M(s)+m+j+2}(2s+1)-1} : s = 0,1,2,\dots n-1, \quad j = 0,1,2,\dots)$$
$$\otimes$$
$$T_m(x^2_{(2k_i+1)2^i+2^{m+1}-1} : i = 2,\dots,m.)$$
$$\otimes$$
$$T_m(x^2_{(2k_1+1)2+2^{m+1}-1} : ((2k_1+1)2 + 2^{m+1}-1)^2 + 2^{m+1} - 1 > 4n-3)$$

where for each s, $M(s)$ is the largest number a such that $(2s+1)2^a - 1$ is of the form $2k_\ell + 1$ for some ℓ and if there is no such a, $M(s) = 0$.

Proof. Note that $|Q_1^a x_{4k+1}| \equiv 3 \bmod 4$ for all $a > 0$.

From the above Milnor operation we get

$$\mathcal{Q}_m(x_{4i+2^{m+1}+1}) = a_{2i+1}^2 \,.$$

For T_1 part since $(2i+1)2+2^{m+1}-1 = 4i+2^{m+1}+1 \equiv 1 \bmod 4$, $4i+2^{m+1}+1 \leq 4n-3$. Hence we get $i \leq n - 2^{m-1} - 1$. If $|x_{4k+1}| \geq 2^{m+1}+1$, that is, $k \geq 2^{m-1}$, x_{4k+1} is the source of the differential. In general for T_ℓ such that $\ell \leq m$, let define $\nu(|x|) = a$ if $|x| = 2^a t$ where t is odd. Then we have

$\nu(1+|Q_1^a(x)|) = a + \nu(1+|x|)$. Moreover $\nu((2k_\ell+1)2^\ell + 2^{m+1}) = \ell$ and $\nu(|Q_1^\ell x_{4i+1}|+1) = \ell+1$. Hence $(2k_\ell+1)2^\ell + 2^{m+1} - 1 \leq |Q_1^{\ell-1} x_{4n-3}| = (4n-3)2^{\ell-1} + 2^{\ell-1} - 1$. So $k_\ell \leq n - 1 - 2^{m-\ell}$. The remaining computation is very routine except the last part. Note that in these cases we have choices of generators. But here we abuse the notations and still use x_i for those choices. For last part if $(2k_1+1)2 + 2^{m+1} - 1)^2 + 2^{m+1} - 1 \leq 4n-3$, we have a differential from $x_{((2k_1+1)2+2^{m+1}-1)^2+2^{m+1}-1}$ to $v_m x^2_{(2k_1+1)2+2^{m+1}-1}$.

Corollary 2.2. *$K(m)_*(\Omega^2 Sp)$ is*

$$E(x_{4s+3} : s \geq 0) \otimes E(x_{4s+1} : 0 \leq s < 2^{m-1}) \,.$$

Proof. Since $\Omega^2 Sp = \lim_{n\to\infty} \Omega^2 Sp(n)$ and the homology theory preserves the direct limit, $E_{2^{m+1}}$–term for $K(m)_*(\Omega^2 Sp)$

$$T_1(x_{2k_1+1} : 0 \leq k_1 \leq n-1-2^{m-1} \quad \text{and} \quad x_{4k+1} = 0 \quad \text{if} \quad k \geq 2^{m-1})$$

This spectral sequence is also the spectral sequence of a coalgebra. Thus since there are no even dimensional primitive element the spectral sequence collapses, i.e, $E_{2^{m+1}} = E_\infty$. But Morava K–theories are not commutative when $p = 2$. So there would be some multiple extensions in the E_∞–term . In fact the deviation from the commutativity of $K(m)_*()$ is determined by $\widetilde{\mathcal{Q}}_{m-1}$ action[9], that is $[x,y] = v_m \widetilde{\mathcal{Q}}_{m-1}(x)\widetilde{\mathcal{Q}}_{m-1}(y)$ where $\widetilde{\mathcal{Q}}_{m-1}$ is the operation in $K(m)$–theory which is similar to the Milnor operation $\mathcal{Q}_{m-1}$ in mod p homology. Since $\widetilde{\mathcal{Q}}_{m-1}(x_{2i+1})$ is even dimensional primitive and the E_∞–term has only odd dimensional primitive element, there are no deviations from the commutativity. So every element in $K(m)_*(\Omega^2 Sp)$ is commutative and there is no multiple extension.

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YANG-MILLS THEORY ON BRANCHED COVERING SPACES

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§1. Introduction

Let X be a smooth oriented 4-manifold. Suppose that a cyclic group $\mathbb{Z}_n$ acts smoothly on X. Let $P : E \to X$ be an $SU(2)$-vector bundle over X with a smooth G-action such that the projection P is a G-map. Let $\pi : X \to X' = X/\mathbb{Z}_n$ be the projection and $E \to E' = E/\mathbb{Z}_n$ be the quotient bundle of E.

In this paper we would like to study the smooth structures on X', the relation between the moduli space of $\mathbb{Z}_n$-invariant anti-self-dual connections on E and the moduli space of anti-self-dual connections on E', and the relation between the polynomial invariants which is defined regarding the invariant moduli space $\mathcal{M}^{\mathbb{Z}_n}$ and the polynomial invariants which is defined by the moduli space $\mathcal{M}'$ on the quotient bundle E'.

In [F.S] and [C1] they showed that there exists a Baire set in the G-invariant metrics on X, when the manifold X has a finite group G-action, such that the moduli space $\mathcal{M}^G$ of G-invariant self-dual connections is smooth except the reducible singularities. In [C1] by using the G-transversality argument of T. Petrie, we identify cohomology obstructions to globally perturbe the full moduli space $\mathcal{M}$ of all self-dual connections into a G-manifold when $G = \mathbb{Z}_2$ and the fixed point set of the G-action on X is a non-empty collection of isolated points and Riemann surfaces. In [C2] we find generic metrics on X such that the moduli space $\mathcal{M}$ is smooth in a G-invariant neighborhood of the fixed point set $\mathcal{M}^G$ when $G = \mathbb{Z}_{2^n}$, for a Baire set of invariant metrics on X. In [H.L] they show that when G is a finite group, the G-equivariant moduli space $\mathcal{M}^*$ has a Whitney stratification with invariant subspaces $\mathcal{M}^*_{G'}$, $G' \subseteq G$ as its strata, by perturbing the self-dual equations and Bierstone's

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general position argument of equivariant maps in finite dimensional manifolds. In [W] when the group G is $\mathbb{Z}_2$ he gives the relations between the invariant moduli space $\mathcal{M}^G$ on a G-bundle $E \to X$ and the moduli space $\mathcal{M}'$ on the quotient bundle $E' \to X'$ and gives a relation between polynomial invariants defined by them. In [C4] he find generic G-invariant metric on X such that the moduli space $\mathcal{M}$ is a smooth G-manifold except the reducible singularities if the instanton number $c_2(E)$ is large enough.

§2. Lipschitz structure

A topological manifold X of dimension 4 is *Lipschitz* if there is a maximal atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ on X, where $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^4$ is a homeomorphism from an open set $U_\alpha \subset M$ onto an open set V_α of $\mathbb{R}^4$, and the changes of coordinates $\phi_\beta \circ \phi_\alpha^{-1}$ are Lipschitz functions, i.e., $|\phi_\beta\phi_\alpha^{-1}(x) - \phi_\beta\phi_\alpha^{-1}(y)| \leq K_{\alpha\beta}|x-y|$ for any $x, y \in \phi_\alpha(U_\alpha \cap U_\beta)$ with $K_{\alpha\beta}$ a constant.

In [S] Sullivan defined L_2-forms, exterior derivatives and differential forms on the Lipschitz manifolds. An L_2-form w of degree r on X is a system, $w = \{w_\alpha\}_{\alpha\in\Lambda}$, where each w_α is a L_2-differential form of degree r on the open subset $V_\alpha = \phi_\alpha(U_\alpha)$ of $\mathbb{R}^4$, and they satisfy the compatibility conditions:

$$(\phi_\beta\phi_\alpha^{-1})^* w_\beta = w_\alpha.$$

Proposition 2.1(Rademacher). *Let U be an open subset of $\mathbb{R}^4$, and let $\varphi : U \to \mathbb{R}^4$ be a Lipschitz map, then;*

(i) *φ is differentiable almost everywhere on U*

(ii) *$\nabla\varphi$ is a weak derivative :*

$$\int_U f\frac{\partial\varphi}{\partial x_i} = -\int \frac{\partial f}{\partial x_i}\varphi$$

for smooth compactly supported test functions f on U.

(ii) *φ preserves Lebesgue null sets.*

Theorem 2.2 [S]. *Any topological manifold of dimension $\neq 4$ has a Lipschitz atlas of coordinates, and for any two such Lipschitz structures $\mathcal{L}_i$, $i = 1, 2$, there exits a Lipschitz homeomorphim $h : \mathcal{L}_1 \to \mathcal{L}_2$ close to the identity.*

Theorem 2.3 [D.S]. (i) *There are topological* 4*-manifolds which do not admit any Lipschitz structure.*

(ii) *There are Lipschitz* 4*-manifolds which are homeomorphic but not Lipschitz equivalent.*

In [D.S] Donaldson and Sullivan studied the gauge theory on the quasiconformal 4-manifolds. As consequences they showed that the compact simply connected topological 4-manifolds with negative definite, even intersection forms do not admit quasiconformal structure. They showed that the complex Barlow surface is not quasiconformally equivalent to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ (they are homeomorphic). Similarly we may establish the gauge theory on the Lipschitz 4-manifolds. We may apply their results to the Lipschitz 4-manifolds. Then we will get Theorem [D.S].

§3. Smooth structure on quotient spaces

Let X be a smooth oriented, closed 4-manifold. Suppose that the cyclic group $\mathbb{Z}/n$ of order n acts semifreely on X with a 2-dimensional submanifold B as its fixed point set, and let $X' = X/\mathbb{Z}_n$ be its quotient space. Then we have an n-fold ramified covering space:

$$\pi : X \to X'$$

with branching locus $\pi(B) = B'$.

To study the smooth structures on X' we consider a small tubular neighborhood N of B in X which is isomorphic to the normal bundle of B in X. The projection π gives rise to a tubular neighborhood $\pi(N) = N'$ of $\pi(B) = B'$ in X' which is also isomorphic to the normal bundle of B' in X'. For a coordinate system $\{B_\alpha\}$ of B, let $N_\alpha \to B_\alpha \times \mathbb{C}$ be a local trivialization of the normal bundle $N \to B$, given by $(b, v) \mapsto (b, \varphi_\alpha(v))$.

Theorem 3.1. *If we give a local trivialization* $\pi(N_\alpha) = N'_\alpha \to \pi(B_j) \times \mathbb{C} = B'_\alpha \times \mathbb{C}$ *on the normal bundle* $N' \to B'$ *by* $(\pi(b), \pi(v)) \mapsto (\pi(b), \varphi_\alpha(v)^n)$, *then*

(i) *the quotient space* X' *is a smooth* 4*-manifold,*
(ii) *the projection map* $\pi : X \to X'$ *is smooth,*
(iii) *but* π *is not Lipschitz.*

Proof. (i) Suppose that the fixed point set B is orientable. Then the normal bundle $N \to B$ is a $U(1)$-bundle. The transition map $\varphi_\beta \varphi_\alpha^{-1} : B_\alpha \cap B_\beta \to U(1)$ gives an attaching linear map $(b, v) \mapsto (b, e^{i\theta} v)$ where $e^{i\theta} = \varphi_\beta \varphi_\alpha^{-1}(b)$ on the normal bundle N. By definition this transition map induces the transion map $\varphi'_\beta {\varphi'_\alpha}^{-1} : B'_\alpha \cap B'_\beta \to U(1)$ given by $(b', v') \mapsto (b', e^{in\theta} v')$, i.e., $\varphi'_\beta {\varphi'_\alpha}^{-1}(b') = e^{in\theta}$, and $|v|^n = |v'|$ if $\pi(b, v) = (b', v')$. Thus $N' \to B'$ is a smooth $U(1)$-bundle.

If B is not orientable, then we consider the orientation line bundle $L \to B$ which is given by transition functions, the Jacobian determinant of the matrix of partial derivatives of the transition functions of the tangent bundle TB. We can get the same result by tensoring the orientation bundle L to the normal bundle N.

(ii) On the free part the quotient space X' has the smooth structure induced by the smooth structure of X. Near the fixed point B the projection map π is just the bundle map between the normal bundles N and N'. Thus the projection map π is smooth.

(iii) Near the fixed point set, $\pi : N \to N'$ is the normal bundle map. We consider the local trivializations as above (i)

$$\begin{array}{ccccc} B_\alpha & \longleftarrow & N|_{B_\alpha} & \xrightarrow{\cong} & B_\alpha \times \mathbb{C} \\ {\scriptstyle \pi = id}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle id}\downarrow \\ B'_\alpha & \longleftarrow & N'|_{B'_\alpha} & \xrightarrow{\cong} & B'_\alpha \times \mathbb{C} \end{array}$$

since $\pi =$ identity on the fixed point set, $\pi(b, z) = (b, z^n)$ and $|(b, 0) - (b, z)| = |z|$, $|\pi(b, 0) - \pi(b, z)| = |z|^n$. Thus π is not Lipschitz.

Theorem 3.2. *If we give a local trivialization $N'_\alpha \equiv \pi(N_\alpha) \to \pi(B_\alpha) \times \mathbb{C}$ on the normal bundle $N' \to B'$ in X' by*

$$(\pi(b), \pi(v)) \mapsto (\pi(b), \frac{\varphi_\alpha(v)^n}{|\varphi_\alpha(v)|^{n-1}}),$$

then (i.e., in the polar coordinate $(\pi(b), \pi(v)) \mapsto (\pi(b), re^{in\theta})$ if $\varphi_\alpha(v) = re^{i\theta}$)

(i) *X' is smooth,*
(ii) *π is not smooth,*
(iii) *but π is bi-Lipschitz.*

Proof. (i) On the free part the Theorem is obvious because the structure on X' comes from the structure on X. It is enough to prove the theorem on a small tubular neighborhood of the fixed point set B. The proof is similar to the proof of above theorem. As a normal bundle, if $N \to B$ has the transition function $e^{i\theta} \in U(1)$, then the transition function of $N' \to B'$ is $e^{in\theta} \in U(1)$. Thus the X' is smooth.

(ii) In the local trivialization of the normal bundle. The projection $\pi(b, re^{i\theta}) = (\pi(b), re^{in\theta})$ is not smooth because $re^{in\theta} = \frac{\varphi_\alpha(v)^n}{|\varphi_\alpha(v)|^{n-1}}$ is not smooth when $\varphi_\alpha(v) = 0$, where $\varphi_\alpha(v) = re^{i\theta}$. Thus π is not smooth on the fixed point set B.

(iii) Let $\varphi(v_1) = r_1 e^{i\theta_1}$, $\varphi(v_2) = r_2 e^{i\theta_2}$. Then

$$|\varphi(v_1) - \varphi(v_2)| = (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1))^{\frac{1}{2}} \quad \text{and}$$
$$\left|\frac{\varphi(v_1)^n}{|\varphi(v_1)|^{n-1}} - \frac{\varphi(v_2)^n}{|\varphi(v_2)|^{n-1}}\right| = (r_1^2 + r_2^2 - 2r_1 r_2 \cos n(\theta_2 - \theta_1))^{\frac{1}{2}}.$$

There are constants c_1 and c_2 so that

$$c_1|\varphi(v_1) - \varphi(v_2)| \leq \left|\frac{\varphi(v_1)^n}{|\varphi(v_1)|^{n-1}} - \frac{\varphi(v_2)^n}{|\varphi(v_2)|^{n-1}}\right| \leq c_2|\varphi(v_1) - \varphi(v_2)|.$$

In this paper we will assume that X' has the smooth structure of Theorem 3.2. Since π is a bi-Lipschitz map, π sends sets of measure zero into sets of measure zero, moreover π sends measurable sets into measurable sets and so is π^{-1}.

From Theorem 3.2 we have following corollary.

Corollary 3.3. (i) *The $\mathbb{Z}/n$-invariant smooth metrics or differential forms on X pushdown to the metrics or forms on X' which are smooth away from B', and have bounded coefficients near B'.*

(ii) *The projection $\pi : X \to X'$ induces an 1-1 correspondence between $\mathbb{Z}/n$-invariant L^p-metrics (forms) on X and L^p-metrics (forms) on X'.*

Proof. (i) Since the smoothness is a local property and the restriction $\pi : X\backslash B \to X'\backslash B'$ is locally diffeomorphic, it is sufficient to prove the theorem near B'. At near B the metric is of the form $dx_1^2 + dx_2^2 + dr^2 + d\theta^2$, the pushdown metric is of the form $dx_1'^2 + dx_2'^2 + dr'^2 + d\theta'^2 = dx_1^2 + dx_2^2 + dr^2 + nd\theta^2$ since $\pi(x_1, x_2, r, \theta) = (x_1', x_2', r', \theta') = (x_1, x_2, r, n\theta)$. In differential forms since $\theta' = n\theta$, they have bounded coefficients near B'.

(ii) If we push down the $\mathbb{Z}_n$-invariant L^p-metrics (forms) on X then they are the L^p-metrics (forms) on X'. Conversely the pullback of the L^p-metric (form) on X' is a $\mathbb{Z}_n$-invariant L^p-metric (form) on X.

We consider some topological consequences on the quotient space X'. If X is simply connected, then $\pi_* : \pi_1(X) \to \pi_1(X')$ is zero. Indeed if we choose a base point in the branching locus B', any loop at the base point can meet only the point with B' by small perturbation. That loop lifts n-distinct loops in X with a base point in B. Thus X' is also simply connected. In [C3] to prove the following proposition we use the Hirzebruch G-signature Theorem.

Theorem 3.4 [C3].
(1) *The signature of* X' : $sign(X') = \frac{1}{n}(sign(X) + \frac{n^2-1}{2} B \circ B)$
(2) *The Eular characteristic of* X' : $\chi(X') = \frac{1}{n}(\chi(X) + (n-1)\chi(B))$
(3) *The rank of the maximal subspace of* $H^2(X)$ *which consists of the self dual harmonic 2-forms:*

$$b_2^+(X') = \frac{1}{n}[b_2^+(X) + \frac{n-1}{2}\{\chi(B) + \frac{n+1}{3} B \circ B - 2\}]$$

where $B \circ B$ *is the self-intersection number of* B *in* X.

§4. Comparison between Equivariance and Quotient

Let $E' \to X'$ be an $SU(2)$-vector bundle with the second chern number $c_2(E') = k'$ on the quotient X'. Let the pull-back bundle $E = \pi^* E'$ and $\mathbb{Z}/n = \langle\sigma\rangle$ be generated by σ and its lifting $\tilde{\sigma}$ on E. Then the second Chern number of E, $c_2(E) = nk' = k$. The lifting $\tilde{\sigma}$ is the identity on B, and is smooth away from B and Lipschitz around B as a bundle map $\tilde{\sigma} : E \to E$.

For $p > 4$, we would like to define a modified Sobolev space of 1-forms on X;

$$\tilde{L}^p(\Omega_X^1) = \{\alpha \in L^p(\Omega_X^1) \mid d\alpha \in L^{p/2}(\Omega_X^2)\},$$

$\tilde{\mathcal{A}}^p = \{A_0 + a \mid a \in \tilde{L}^p(\Omega_X^1(ad\,E))\}$, where A_0 is smooth, $\tilde{\mathcal{A}}^{p,\sigma}$ is the $\mathbb{Z}/n$-invariant subspace of $\tilde{\mathcal{A}}^p$.

Proposition 4.1. (i) *The projection map* $\pi : E \to E'$ *induces a bijection* $\tilde{\mathcal{A}}^{p,\sigma} \to \tilde{\mathcal{A}}'^p$ *where* $\tilde{\mathcal{A}}'$ *is the modified Sobolev space of connections on* E'.
(ii) *For an anti-self-dual,* σ*-invariant connection* $A \in \tilde{\mathcal{A}}^{p,\sigma}$ *the push down connection* A' *on* X' *is also anti-self-dual.*

(iii) *The diagram*

$$\begin{array}{ccccc} L^p_1(ad\,E)^\sigma & \xrightarrow{d_A} & \hat{L}^p(\Omega^1_X(ad\,E))^\sigma & \xrightarrow{d^+_A} & L^{p/2}_+(\Omega^2_X(ad\,E))^\sigma \\ \downarrow & & \downarrow & & \downarrow \\ L^p_1(ad\,E') & \xrightarrow{d_{A'}} & \tilde{L}^p(\Omega^1_{X'}(ad\,E')) & \xrightarrow{d^+_{A'}} & L^{p/2}_+(\Omega^2_{X'}(ad\,E')) \end{array}$$

is commutative for an anti-self-dual connection A in $\tilde{\mathcal{A}}^{p,\sigma}$. The above two complexes are isomorphic and elliptic.

In [D.S] to compute the index of the above elliptic complex we may use the parametrix $Q_{A'}$ for $d^+_{A'}$ instead of the adjoint operator $d^*_{A'} = - * d_{A'} *$ since the push down metric on X' of a G-invariant metric on X is singular. If we use the excision principle of the Atiyah-Singer index Theorem, we have the index.

Theorem 4.2. *The index of the above elliptic complex is*

$$\begin{aligned} i_{E'} &= 8c_2(E') - 3[1 - b_1(X') + b_2^+(X')] \\ &= \frac{1}{n}[i_E - \frac{3(n-1)}{2}\{\chi(B) + \frac{n+1}{3} B \circ B\}] \\ &= i^\sigma_E . \end{aligned}$$

In [D.S] Donaldson and Sullivan use a similar method to compute the index for the quasi-conformal setting.

§5. Equivariant generic metric

We would like to introduce equivariant generic metric on a 4-manifold X to define Donaldson polynomial invariant by regarding the equivariant moduli space $\mathcal{M}^{\mathbb{Z}/n} \subset \mathcal{B}^{\mathbb{Z}/n}$ as carrying a distinguished invariant homology class, independent of the choice of metric used to define $\mathcal{M}^{\mathbb{Z}/n}$.

Let $E \xrightarrow{p} X$ be an $SU(2)$ vector bundle with $c_2 = nk' = k$ over a simply connected closed smooth 4-manifold X. Suppose a cyclic group $\mathbb{Z}/n$ acts on X with a 2-dimensional submanifold B as the fixed point set. Let $\mathbb{Z}/n$ lift to the bundle E such that the projection p is a $\mathbb{Z}/n$-map. Choose $\mathbb{Z}/n$-invariant metrics on X and E. Recall that the notations $\mathcal{A}(\mathcal{A}^{\mathbb{Z}/n})$, $\mathcal{G}(\mathcal{G}^{\mathbb{Z}/n})$ the space of ($\mathbb{Z}/n$-invariant) connetions and the group of ($\mathbb{Z}/n$-invariant) gauge transformations on E respectively.

Consider the fundamental elliptic complexes

$$0 \to \Omega^0(ad\,E) \to \Omega^1(ad\,E) \to \Omega^2_+(ad\,E) \to 0$$

and

$$0 \to \Omega^0(ad\,E)^{\mathbb{Z}/n} \to \Omega^1(ad\,E)^{\mathbb{Z}/n} \to \Omega^2_+(ad\,E)^{\mathbb{Z}/n} \to 0$$

Then the space $\mathcal{A}$ and $\mathcal{A}^{\mathbb{Z}/n}$ are affine spaces modeled on the spaces $\Omega^1(ad\,E)$ and $\Omega^1(ad\,E)^{\mathbb{Z}/n}$, and the groups of gauge transformations $\mathcal{G}(E)$ and $\mathcal{G}(E)^{\mathbb{Z}/n}$ are modeled on the spaces $\Omega^0(ad\,E)$ and $\Omega^0(ad\,E)^{\mathbb{Z}/n}$ respectively. Since the gauge transformations act on the connection spaces, we have the orbit spaces

$$\mathcal{B} = \mathcal{A}/\mathcal{G}(E) \quad \text{and} \quad \mathcal{B}^{\mathbb{Z}/n} = \mathcal{A}^{\mathbb{Z}/n}/\mathcal{G}(E)^{\mathbb{Z}/n}$$

We have some immediate consequences.

Proposition 5.1. *Let $E' \xrightarrow{p'} X'$ be the quotient bundle of $E \xrightarrow{p} X$ under the $\mathbb{Z}/n$-action.*

(i) *X' has a smooth structure such that π is smooth.*
(ii) *$c_2(E) = nc_2(E')$.*
(iii) *The natural map $\mathcal{B}^{\mathbb{Z}/n} \to \mathcal{B}$ is injective if we restrict to irreducible connections and if the center of gauge group is trivial.*
(iv) *If we choose the pullback metric g on X from a smooth metric g' on X', then the metric g is $\mathbb{Z}/n$-invariant.*

Proof.

(i) Let $\pi : X \to X'$ be the projection map. If N is a tubular neighborhood of B, then $N \to B$ is an $U(1)$-bundle and $N' = \pi(N) \to B' = \pi(B)$ is also an $U(1)$-bundle. If we identify B and B', then N' is isomorphic to $N \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} N$ of n-copies of N
(ii) Since $E = \pi^*(E')$, $p_1(E) = p_1\pi^*E' = \pi^*p_1(E') = np_1(E')$ and $c_2(E) = nc_2(E')$.
(iii) Suppose $A_1, A_2 \in B^{*\mathbb{Z}/n}$ and $A_1 = gA_2g^{-1}$ for some $g \in \mathcal{G}(E)$. Then $A_1 = h^{-1}ghA_2h^{-1}g^{-1}h = gA_2g^{-1}$, for any $h \in \mathbb{Z}/n$ and $g^{-1}h^{-1}ghA_2hg^{-1}hg = A_2$. $g^{-1}h^{-1}gh$ is an element of the center of the gauge group. Since the center is trivial, $gh = hg$ for any $h \in \mathbb{Z}/n$ and $g \in \mathcal{G}(E)^{\mathbb{Z}/n}$.
(iv) Since $\pi^*g' = g$, for any $v, w \in T_pX$, $g_p(v,w) = (\pi^*g')_p(v,w) = g'_{\pi(p)}(\pi_*v, \pi_*w)$ and for any $h \in \mathbb{Z}/n$, $gh_{(p)}(h_*v, h_*w) = (\pi^*g')_{h(p)}(h_*v, h_*w) = g'_{\pi(h(p))}(\pi_*h_*v, \pi_*h_*w) = g'_{\pi(p)}(h_*v, h_*w)$.

Let $\mathcal{U} = \mathcal{U}(GL(TX))$ be the set of c^k-automorphisms of the tangent bundle for a sufficient large k. Let $\mathcal{U}^{\mathbb{Z}/n}$ be the subspace of $\mathbb{Z}/n$-invariant metrics in $\mathcal{U}$.

In fact if g is a fixed $\mathbb{Z}/n$-invariant metric on X, then every $\mathbb{Z}/n$-invariant metric on X is realized by a pull-back metric $\phi^*(g)$ of g for some $\phi \in \mathcal{U}^{\mathbb{Z}/n}$.

Let $p_+ : \Omega^{2,\mathbb{Z}/n} \to \Omega_+^{2,\mathbb{Z}/n}$ be the projection onto the self-dual $\mathbb{Z}/n$-invariant 2-forms with respect to the $\mathbb{Z}/n$-invariant metric g. Then $\phi^* p_+ \phi^{-1*}$ is the projection onto self-dual, $\mathbb{Z}/n$-invariant 2-forms with respect to the metric $\phi^*(g)$.

We define a map $\Phi : B^{*\mathbb{Z}/n} \times \mathcal{U}^{\mathbb{Z}/n} \to \Omega^{2+}(ad E)^{\mathbb{Z}/n}$ by $\Phi(A, \phi) = p_+ \phi^{-1*} F_A$. The map Φ is well-defined. Clearly a connection ∇ is anti-self-dual if and only if $h(\nabla)$ is anti-self-dual with respect to the metric $(h\phi)^*(g)$ [C3].

Theorem 5.2 [F.U],[C1]. *The map Φ is smooth and has zero as a regular value. The inverse image $\Phi^{-1}(0)$ is an infinite dimensional Banach manifold of anti-self-dual connections parametrized by the space $\mathcal{U}^{\mathbb{Z}/n}$ of all $\mathbb{Z}/n$-invariant metrics on X.*

Consider the projection map

$$\phi : \Phi^{-1}(0) = \cup_{\phi \in \mathcal{U}^{\mathbb{Z}/n}} \mathcal{M}^{*\mathbb{Z}/n}_{\phi^*(g)} \to \mathcal{U}^{\mathbb{Z}/n}$$

which is a Fredholm map with $\mathbb{Z}/n$-invariant index of the fundamental $\mathbb{Z}/n$-invariant elliptic complex as its index. By the Sard-Smale Theorem for a Fredholm map between paracompact Banach manifolds, we have the following theorem.

Theorem 5.3 [C1]. *There is a Baire set $\mathcal{U}'$ of $\mathcal{U}^{\mathbb{Z}/n}$ such that $\pi^{-1}(\phi) = \mathcal{M}^{\mathbb{Z}/n}_{\phi^*(g)}$ is a smooth manifold in the moduli space $\mathcal{M}_{\phi^*(g)}$ of the irreducible anti-self-dual connections for each metric $\phi \in \mathcal{U}'$.*

Let $g_0, g_1 \in \mathcal{U}'$, and $\gamma : [0,1] \to \mathcal{U}^{\mathbb{Z}/n}$ be a path between them. If $b_2^+(X') > 1$, by the similar proof as the above proposition we can perturb the path γ to a new path γ' which lies in $\mathcal{U}'$, that is, tansverse to π. We may assume $\gamma(0) = \gamma'(0) = g_0$ and $\gamma(1) = \gamma'(1) = g_1$. Then we have a cobordism W_γ between the moduli spaces $\mathcal{M}^*(g_0)^{\mathbb{Z}/n}$ and $\mathcal{M}^*(g_1)^{\mathbb{Z}/n}$. For any ℓ we can choose the parth γ' to have the transverse condition for all bundle E with $c_2(E) \le \ell$. Also if $b_2^+(X') > 1$ we may choose the path γ' to lie in $\mathcal{U}'$. Let us consider the reducible connection of E. If ∇ is a $\mathbb{Z}/n$-invariant reducible connection, then there is an equivariant bundle decomposition $E = L \oplus L^{-1}$

and an equivariant decomposition $\nabla = \nabla_0 \oplus \overline{\nabla}_0$ of the connection ∇. In addition if ∇ is anti-self-dual, then the curvature form $(i/2\pi)F$ represents the Euler class of the line bundle L and is a $\mathbb{Z}/n$-invariant ASD harmonic 2-form on X. The $\mathbb{Z}/n$-invariant fundamental elliptic complex

$$0 \to \Omega^0(ad\,E)^{\mathbb{Z}/n} \to \Omega^1(ad\,E)^{\mathbb{Z}/n} \to \Omega^2_+(ad\,E)^{\mathbb{Z}/n} \to 0$$

reduces to

$$(*) \qquad 0 \to \Omega^{0,\mathbb{Z}/n} \to \Omega^{1,\mathbb{Z}/n} \to \Omega^2_+(da\,E)^{\mathbb{Z}/n} \to 0$$

since the adjoint bundle is trivial. The index of $(*)$ is $(1/2)(\chi^{\mathbb{Z}/n}+\sigma^{\mathbb{Z}/n}) = 1+ b_2^{+.\mathbb{Z}/n}$, since $\dim H^{0,\mathbb{Z}/n} = 1$ and $\dim H^{1,\mathbb{Z}/n} - \dim H^{2,\mathbb{Z}/n} = -b_2^{+.\mathbb{Z}/n}$.

If $b_2^{+.\mathbb{Z}/n} > 0$ the Sard-Smale Theorem induces that generically there are no anti-self-dual solutions. Since the dimension of $\mathbb{Z}/n$-invariant, self-dual harmonic 2-forms on X equal to the dimension of self-dual harmonic 2-forms on the quotient space X' :

$$\begin{aligned} & b_2^{+\mathbb{Z}/n}(X) \\ = {} & b_2^+(X') \\ = {} & (1/n)[b_2^+(X) + ((n-1)/2)\{\chi(B) + ((n+1)/3)B \circ B - 2\}]. \end{aligned}$$

Thus we have the following proposition.

Proposition 5.4. (1) *If $b_2^+(X) > ((n-1)/2)\{2-\chi(B)-((n+1)/3)B\circ B\}$, then there is an open dense subset $\mathcal{U}'$ of the $\mathbb{Z}/n$-invariant metrics $\mathcal{U}^{\mathbb{Z}/n}$ on X such that $\mathcal{M}_g^{\mathbb{Z}/n}$ does not have a reducible $\mathbb{Z}/n$-invariant ASD-connection for each $g \in \mathcal{U}'$.*

(2) *If $b_2^+(X) > n+(n-1)/n\{2-\chi(B)-((n+1)/3B\circ B\}$, then any path in $\mathcal{U}^{\mathbb{Z}/n}$ of metrics in $\mathbb{Z}/n$-invariant metrics joining two metrics g_1 and g_2 in $\mathcal{U}'$ can be perturbed into a new path in $\mathcal{U}'$.*

(3) *The moduli spaces $\mathcal{M}^{\mathbb{Z}/n}$ of the equivariant classes of $\mathbb{Z}/n$-invariant ASD-connections are smooth manifolds under the condition (1), are cobordant under the condition (2) for the invariant generic metrics in $\mathcal{U}'$.*

§6. Polynomial invariants

We need the following conditions to compute Donaldson polynomial invariant for the quotient bundle $E' \to X'$ (see [D.K]).

(i) The dimension of the moduli space, $\dim \mathcal{M}_{E'} = i_{E'} = 2d'$ is even.

(ii) To avoid nontrivial reducible connections and to get generic metric

$$b_2^+(X) > n - \frac{n-1}{2}[\chi(B) + \frac{n+1}{3} B \circ B - 2].$$

(iii) The stable range

$$4k' \geq 2 + 3[1 + \frac{1}{n} b_2^+(X) + \frac{n-1}{2n}\{\chi(B) + \frac{n+1}{3} B \circ B - 2\}].$$

To define the Donaldson invariant for the invariant setting we would like to introduce the Donaldson μ-map.

For $\Sigma \in H_2(X;\mathbb{Z})^{\mathbb{Z}/n}$ with $\pi_*\Sigma = n[\Sigma/\mathbb{Z}_n] \in H_2(X';\mathbb{Z})$, if A is $\mathbb{Z}/n$-invariant connection in $\tilde{\mathcal{A}}^p$, we have a coupled Dirac operator

$$\partial\!\!\!/_A : \Gamma(S^+ \otimes E)^\sigma \to \Gamma(S^- \otimes E)^\sigma$$

where $S^\pm$ is the $(\pm\frac{1}{2})$-spinor bundle on Σ.

We have the determinant line bundle

$$\begin{array}{ccc} \gamma^*(\mathcal{L}_\Sigma) & & \mathcal{L}_\Sigma \\ \downarrow & & \downarrow \\ \mathcal{B}(X)^{\mathbb{Z}/n} & \xrightarrow{\ \gamma\ } & \mathcal{B}(\Sigma) \end{array}$$

with the fiber $(\det \operatorname{ind} \partial\!\!\!/_A)^{-1}$, where γ is the restriction map.

Note that we should use a small tubular neighborhood $N(\Sigma)$ of Σ instead of Σ. There is a generic section s_Σ of this bundle such that

$$s_\Sigma^{-1}(0) = V_\Sigma \text{ is a codimension 2-submanifold of } \mathcal{M}_k^\sigma \subset \mathcal{B}_k^\sigma.$$

V_Σ is the Poincaré dual of the first Chern class $c_1(\mathcal{L}_\Sigma)$ of the determinant line bundle $\mathcal{L}_\Sigma$. This is the image $\mu(\Sigma)$ of the Donaldson map $\mu : H_2(X;\mathbb{Z})^\sigma \to H^2(\mathcal{B}^\sigma;\mathbb{Z})$. Define the Donaldson invariant for the invariant setting

$$q^\sigma : \operatorname{sym}^{d'}(H_2(X;\mathbb{Z})) \to \mathbb{Z} \quad \text{by}$$

$$q^\sigma(\Sigma_1, \ldots, \Sigma_{d'}) = \#\mathcal{M}_k^\sigma \cap V_{\Sigma_1} \cap, \ldots, \cap V_{\Sigma_{d'}}.$$

Here the number is counted with sign, and $\mathcal{M}_k^\sigma \cap V_{\Sigma_1}, \ldots, V_{\Sigma_{d'}}$ is a zero-dimensional compact manifold, hence finite.

In [C1] and [F.S] they showed that for the generic G-invariant metrics on X, the moduli space $\mathcal{M}_k^\sigma$ of G-invariant anti-self-dual connections is a smooth manifold.

Proposition 6.1 [C3]. *The image of the first Chern class is*

$$\pi^*(c_1(\mathcal{L}_\Sigma)) = n.c_1(\mathcal{L}_{\Sigma/\mathbb{Z}_n}), \text{ and}$$

$$\begin{array}{ccc} H^2(X,\mathbb{Z})^\sigma & \xleftarrow[\pi^*]{} & H^2(X':\mathbb{Z}) \\ {\scriptstyle PD}\downarrow & & \downarrow{\scriptstyle n.PD} \\ H_2(X;\mathbb{Z})^\sigma & \xrightarrow{\pi_*} & H_2(X':\mathbb{Z}) \end{array}$$

is commutative where PD stands for Poincaré dual.

Theorem 6.2. *If $\pi_* : H_2(X;\mathbb{Z})^{\mathbb{Z}/n} \to H_2(X';\mathbb{Z})$ and*

$$\pi_*(\xi_i) = n.\eta_i, \quad i = 1,\dots,d', \quad \textit{then}$$
$$q^\sigma(\xi_i,\dots,\xi_{d'}) = q'(\eta_1,\dots,\eta_{d'}).$$

Sketch of the proof. Take a smooth generic metric g' on X'. The pull-back metric $g = \pi^* g'$ is a bounded measurable σ-invariant metric on X. Let $\{g_n\}$ be a sequence of σ-invariant generic smooth metric on X such that $g_n \to g$ in C^r-sense. Then the push down metric g'_n of g_n on X' is bounded measurable and g'_n converges to g'. So far we showed that

(i) $q_n^\sigma(\xi_1,\dots,\xi_{d'}) = \tilde{q}'_n(\eta_1,\dots,\eta_{d'})$ for the metric g_n and g'_n respectively, where $\tilde{q}'_n$ is the polynomial invariant with respect to g'_n.

(ii) For large n

$$\tilde{q}'_n(\eta_1,\dots,\eta_{d'}) = q'(\eta_1,\dots,\eta_{d'}) \text{ (see [D.S] for detail).}$$

Combining (i) and (ii), we have the required result,

$$q^\sigma(\xi_1,\dots,\xi_{d'}) = q'(\eta_1,\dots,\eta_{d'}).$$

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ANALYTIC TORSION FOR ELLIPTIC OPERATORS

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ABSTRACT. Analytic torsion was first introduced by Ray and Singer [RS71, RS73, RS, Ray, Sin, Che, Mül78, Mül91] in the study of Reidemeister-Franz torsion [deR, DFN, Fra, Mil61, Mil62, Mil66, Rei]. Recently, it has become more and more interesting [Bra, Fal, Fay, FG, Fre, Joh, Qui, Ros, Wit]. In this elementary article we study finite dimensional linear algebra of analytic torsions and its analogy for elliptic operators.

1. LINEAR ALGEBRA

1.1. Regularized determinant. In this section we fix a real or complex field as a scalar field $\mathbb{F}$. For an endomorphism P of a vector space V of finite dimension n, the *regularized* or *renormalized determinant*[1] of P, denoted by $\operatorname{\text{đet}} P$, is the product of all nonzero (complex) eigenvalues of P counted with multiplicity. For a nilpotent endomorphism, the regularized determinant is equal to 1.

One can see easily that

$$\text{đet}(P^i) = (\text{đet}\, P)^i \qquad \forall i = 0, 1, 2, \ldots .$$

The regularized determinant is nonzero and equal, up to sign, to the coefficient of the lowest degree monomial in the characteristic polynomial.

[1] The notation $\det^{\times} P$ is also suggestive.

1.1.1. Proposition. *The map*

$$\text{đet} : \operatorname{End} V \to \mathbb{F}^{\times}$$

is upper semi-continuous.

Proof. Let $\det(t-P) = t^n - a_{n-1}t^{n-1} + \cdots + (-1)^{n-k}a_k t^k$ be the characteristic polynomial of P, where $a_k \neq 0$. Then it should be obvious that $\text{đet}\, P = a_k$ and $\det(t+P) = t^n + a_{n-1}t^{n-1} + \cdots + a_k t^k$. Thus

$$\text{đet}\, P = \lim_{t\to 0} \frac{\det(t+P)}{t^k} \tag{1.1.2}$$

where $k = k(P)$ depends on P. Observe that $k(P) = \lim_{i\to\infty} \dim\ker(P^i) = \dim\ker(P^n)$. Thus the "stable nullity" $k(P)$ is an upper semi continuous function of P. Now the result follows from (1.1.2).

Note that đet is continuous when restricted to each subvariety of $\operatorname{End} V$ with fixed "stable nullity".

1.1.3. Proposition. *For any nonzero scalar c,*

$$\text{đet}(cP) = c^l\, \text{đet}\, P,$$

where $l = \lim_{i\to\infty} \dim P^i(V) = \dim P^n(V)$ *denotes the "stable rank" of* P.

Proof. Note that the stable nullities k of P and cP are equal and hence their stable ranks $l = n - k$ are also equal. Now

$$\begin{aligned}\text{đet}(cP) &= \lim_{t\to 0} \frac{\det(t+cP)}{t^k} = \lim_{t\to 0} \frac{\det(ct+cP)}{(ct)^k} = \frac{c^n}{c^k} \lim_{t\to 0} \frac{\det(t+P)}{t^k} \\ &= c^l\, \text{đet}\, P.\end{aligned}$$

This completes the proof.

We will see later (2.2.4) that the stable rank of some elliptic operator Q is equal to the value of the zeta function $\zeta_Q(s)$ at $s = 0$.

1.1.4. Proposition. *If* P_1 *is an endomorphism of* V_1 *and* P_2 *is an endomorphism of* V_2*, then*

$$\text{đet}(P_1 \oplus P_2) = \text{đet}\, P_1 \cdot \text{đet}\, P_2.$$

1.1.5. Proposition. *If two endomorphisms $P_1 : V_1 \to V_1$ and $P_2 : V_2 \to V_2$ are equivalent in the sense that there exists an isomorphism $h : V_1 \to V_2$ such that $hP_1 = P_2 h$, then the regularized determinant of P_1 and P_2 are the same, i.e.,*

$$\det(h \circ P_1 \circ h^{-1}) = \det P_1.$$

1.2. Zeta function and the regularized determinant. For an endomorphism P of a finite dimensional vector space V over $\mathbb{C}$, define the *zeta function* ζ_P of P by

$$\zeta_P(s) := \sum_{\lambda} \lambda^{-s}, \qquad \forall s \in \mathbb{C},$$

where λ runs through all nonzero eigenvalues of P counted with multiplicities and $\lambda^{-s} = |\lambda|^{-s} e^{-is \arg \lambda}$ (we fix an argument $\arg \lambda$ for each λ). Then $\zeta_P(0)$ is equal to the stable rank of P and

$$\det P = \exp(-\zeta_P'(0)).$$

1.3. The analytic torsion of a linear map. Suppose that we are given a linear map

$$P : V_0 \to V_1$$

between *inner product spaces* V_0 and V_1 of finite dimension. Then we have the adjoint

$$P^* : V_1 \to V_0$$

of P. Since the nonzero eigenvalues of P^*P and PP^* are the same with the same multiplicity, they have the same determinant, which is a *positive* real number.

We now define the *analytic torsion* of $P : V_0 \to V_1$ between inner product spaces to be

$$\tau(P) := \sqrt{\det(P^*P)} = \sqrt{\det(PP^*)} \in \mathbb{R}_+.$$

It is obvious that

$$\tau(P) = \tau(P^*).$$

For another interpretation of the analytic torsion see [Kim93], [BZ].

The analytic torsion for a single map $P : V_0 \to V_1$ between inner product spaces is clearly an upper semi-continuous function of P. If the analytic torsion is restricted only to isomorphisms, then it is continuous and C^∞.

1.3.1. Remark. Suppose h_i is an automorphism of V_i for $i = 0, 1$. Then it defines a new inner product $\langle \cdot, \cdot \rangle^{\sim} := \langle h_i(\cdot), h_i(\cdot) \rangle$ on V_i and hence we have a new adjoint $P^{\tilde{*}}$ of P. Then

$$P^{\tilde{*}} = (h_0^* h_0)^{-1} P^* (h_1^* h_1).$$

In particular, if h_i is a nonzero scalar μ_i, then the new torsion $\tilde{\tau}(P)$ is equal to $\left|\frac{\mu_1}{\mu_0}\right|^l \tau(P)$, where l is the stable rank of P^*P, which is equal to the rank of P.

We list some properties of the analytic torsion.

1.3.2. Proposition. *For two linear maps $P : V_0 \to V_1$ and $Q : W_0 \to W_1$,*

$$\tau(P \oplus Q) = \tau(P) \cdot \tau(Q).$$

If there exist isometries $h_0 : V_0 \to W_0$ and $h_1 : V_1 \to W_1$ such that $Q = h_1 \circ P \circ h_0{}^{-1}$, then

$$\tau(h_1 \circ P \circ h_0{}^{-1}) = \tau(P).$$

1.3.3. Proposition. *If P is an endomorphism of an inner product space V such that $P^*P = PP^*$, then*

$$\tau(P) = |\det P|.$$

In this case we have a geometric interpretation of $\tau(P)$. Note that if P is normal, i.e., $PP^* = P^*P$, then

$$P : (\ker P)^{\perp} \to (\ker P)^{\perp}$$

is an isomorphism and $\tau(P)$ is equal to the volume change ratio of this isomorphism.

1.4. Analytic torsion of a chain complex. Now suppose that we are given a chain complex

$$P : 0 \to V_0 \xrightarrow{P_0} V_1 \xrightarrow{P_1} \dots \xrightarrow{P_{l-1}} V_l \to 0 \tag{1.4.1}$$

of finite dimensional inner product spaces. Then the *analytic torsion* of this chain complex is defined as the alternating product

$$\tau P := \prod_{q \geq 0} (\tau P_q)^{(-1)^q}.$$

Let P_q^* be the adjoint of P_q and let

$$\Box_q := P_q^* P_q + P_{q-1} P_{q-1}^*.$$

Then by the Hodge theory, $V_q = \operatorname{im}(P_{q-1}) \oplus \operatorname{im}(P_q^*) \oplus \ker(\Box_q)$ is a direct sum of $\Box_q$-invariant subspaces. Since $\Box_q | \operatorname{im}(P_{q-1}) = P_{q-1} P_{q-1}^*$ and $\Box_q | \operatorname{im}(P_q^*) = P_q^* P_q$, we have, by the proposition (1.1.4)

1.4.2. Proposition. $\operatorname{\text{đ}et} \square_q = (\tau(P_{q-1}) \cdot \tau(P_q))^2$.

Thus we have

$$\tau P = \prod_{q\geq 0} (\operatorname{\text{đ}et} \square_q)^{(-1)^{q+1} q/2}. \tag{1.4.3}$$

This identity will serve as a definition for the infinite dimensional case.

1.4.4. If V_+ and V_- denote the direct sum of all V_q for even q and for odd q, respectively, then we have maps

$$P_+ : V_+ \to V_-, \quad P_- : V_- \to V_+$$

where $P_\pm = (P + P^*)|V_\pm$. Now $P_- = (P_+)^*$ and

$$(P_+)^* P_+ = \square_0 \oplus \square_2 \oplus \ldots, \qquad (P_-)^* P_- = \square_1 \oplus \square_3 \oplus \ldots,$$

and hence

$$\tau(P_+) = \prod_{q\geq 0} \tau(P_q) = \tau(P_-).$$

1.4.5. More generally, one may be interested in

$$\tau_t := \prod_{q\geq 0} \tau(P_q)^{t^q}$$

for $t \in \mathbb{R}$.

1.4.6. From the above proposition, one can see easily that

$$\prod_{q=0}^{l} (\operatorname{\text{đ}et} \square_q)^{(-1)^q} = 1$$

in the complex (1.4.1). This phenomenon is also true for infinite dimensional cases (2.2.5).

1.4.7. If we only consider the torsions of exact sequences $P : 0 \to V_0 \xrightarrow{P_0} V_1 \xrightarrow{P_1} \cdots \to V_l \to 0$, then $P_+ : V_+ \to V_-$ is an isomorphism and hence $\tau(P_+) = \prod_{q\geq 0} \tau(P_q)$ is a continuous function of P. Since $\prod_{q:\text{even}} \tau(P_q)$ and $\prod_{q:\text{odd}} \tau(P_q)$ are upper semi-continuous, $\tau(P) = \tau(P_+)/\left(\prod_{q:\text{odd}} \tau(P_q)\right)^2 = \left(\prod_{q:\text{even}} \tau(P_q)\right)^2/\tau(P_+)$ is a lower and upper semi-continuous function of P and hence is a continuous function of P.

1.4.8. If we use the zeta function (1.2), then the torsion of the chain complex (1.4.1) is given by

$$\tau P = \prod_{q\geq 0}(\det \Box_q)^{(-1)^{q+1}q/2} = \exp\left(\frac{1}{2}\sum_{q\geq 0}(-1)^q q\zeta'_{\Box_q}(0)\right).$$

We now show that

$$\sum_{q\geq 0}(-1)^q q\zeta_{\Box_q}(0)$$

is equal to the alternating sum of the ranks of P_q's.

Let $z_q = \dim\ker P_q$, $b_q = \dim\operatorname{im} P_{q-1}$ and $v_q = \dim V_q$. Then $V_q = \ker P_q \oplus \operatorname{im} P_q^*$ and hence $\dim\operatorname{im} P_q^* = v_q - z_q = b_{q+1}$. Now $\zeta_{\Box_q}(0)$ is the dimension of $\operatorname{im}\Box_q = \operatorname{im} P_{q-1} \oplus \operatorname{im} P_q^*$ and hence

$$\sum_{q\geq 0}(-1)^q q\zeta_{\Box_q}(0) = \sum(-1)^q q(b_q + b_{q+1}) = \sum(-1)^q b_q$$

as claimed.

1.5. Torsion vector and analytic torsion. For a finite dimensional vector space V, let $\det V$ be the highest exterior power of V and let V^{-1} be the dual space of V. If $V_\bullet = \{V_0, \dots, V_l\}$ is a sequence of finite dimensional vector spaces, then

$$\det V_\bullet := \det V_0 \otimes (\det V_1)^{-1} \otimes \dots \otimes (\det V_l)^{(-1)^l}.$$

Suppose we are given a chain complex

$$P : 0 \to V_0 \xrightarrow{P_0} V_1 \xrightarrow{P_1} \dots \xrightarrow{P_{l-1}} V_l \to 0$$

of finite dimensional vector spaces. For the moment we do not assume inner products. Let $H^\bullet(P) = \{H^0(P), \dots, H^l(P)\}$ be the cohomology spaces of the complex. Then

1.5.1. Proposition. *There is a canonical isomorphism* [BGS, BZ, Kim93, Mül91]

$$\det V_\bullet \simeq \det H^\bullet(P).$$

We may regard this isomorphism as an element $\vec{\tau}(P)$ of $(\det V_\bullet)^{-1} \otimes \det H^\bullet(P)$, which will be called the *torsion vector* of the chain complex P.

If we assume inner products on each vector spaces V_k, then $\det V_\bullet$ inherits a canonical inner product, and by the Hodge theory, $\det H^\bullet(P)$ also inherits an inner product. Thus one can measure the length of an element in $(\det V_\bullet)^{-1} \otimes \det H^\bullet(P)$. One can see easily that [Mül91]

1.5.2. Proposition. $|\vec{\tau}(P)| = \tau(P)$.

2. Analytic torsion for elliptic complexes

Suppose we have an elliptic complex[2]

$$(2.1)\qquad P : 0 \to \mathcal{C}^\infty(E_0) \xrightarrow{P_0} \mathcal{C}^\infty(E_1) \xrightarrow{P_1} \dots \xrightarrow{P_{l-1}} \mathcal{C}^\infty(E_l) \to 0$$

for hermitian vector bundles $E_0, E_1, \dots, E_l$ over a compact Riemannian manifold M, where $\mathcal{C}^\infty$ denotes the space of smooth sections. Then we have the formal adjoints P_q^* and the 'Laplacians' $\Box_q$. Motivated by (1.4.3), the *analytic torsion* of the elliptic complex P is defined by

$$\tau(P) := \prod_{q\geq 0} (\det \Box_q)^{(-1)^{q+1}q/2},$$

where the meaning of $\det \Box$ is explained in the following.

2.2. Zeta function and the regularized determinant of a self-adjoint, positive semi-definite elliptic operator. Let

$$Q : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$$

be a self-adjoint, positive semi-definite elliptic (pseudo differential linear) operator of order $m > 0$ on a hermitian vector bundle E over a compact Riemannian manifold (M, g) of dimension n. Let λ_k denote the k-th *non zero* eigenvalue of Q counted with multiplicity. Then it is well known [Shu, p. 124] that the limit

$$\lim_{k\to\infty} \frac{{\lambda_k}^{n/m}}{k}$$

exists as a finite positive real number. Thus the series

$$\zeta_Q(s) := \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}$$

converges and holomorphic for complex numbers s with large real part.

We now show that $\zeta_Q(s)$ has a meromorphic extension for all $s \in \mathbb{C}$ and regular at $s = 0$. For this, consider the *heat operator* of Q;

$$e^{-tQ} : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E).$$

[2]We assume the order of the linear differential operators P_j's are all the same and positive.

It is an integral operator with kernel

$$K_t(x,y) : E_y \to E_x, \qquad x,y \in M,\ t>0$$

so that for any section s of E,

$$(e^{-tQ}s)(x) = \int_M K_t(x,y)s(y)\,\delta g(y)$$

satisfies the heat equation $(\frac{d}{dt}+Q)s_t = 0$, $\lim_{t\searrow 0} s_t = s$, where δg denotes the canonical density on (M,g). Then for each $x \in M$, there exists an asymptotic expansion (cf. Appendix, [Shu, p. 114], [Gil], [Roe]) as $t \searrow 0$

$$K_t(x,x) \sim \sum_{i=0}^{\infty} \theta_i(x) t^{(i-n)/m}, \qquad \theta_i \in C^\infty(\operatorname{End} E)$$

uniformly in $x \in M$.

2.2.1. Corollary. *Let*

$$h_Q(t) := \sum_{k=1}^{\infty} e^{-\lambda_k t} = \operatorname{Tr}(e^{-tQ}) - \dim\ker Q.$$

Then as $t \searrow 0$ there exists an asymptotic expansion

$$h_Q(t) \sim \sum_{i=0}^{\infty} a_i t^{(i-n)/m}, \quad a_i \in \mathbb{R}$$

as $t \searrow 0$, where $a_i = \int_M \operatorname{tr}(\theta_i)\,\delta g$ if $i \neq n$ and $a_n = \int_M \operatorname{tr}(\theta_n)\,\delta g - \dim\ker Q$.

2.2.2. Now by the Mellin transform, we have

$$\zeta_Q(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} h_Q(t)dt, \quad \operatorname{Re} s >> 0.$$

The zeta function is regular at $s=0$ and $\zeta_Q(0) = a_n = \int_M \operatorname{tr}(\theta_n)\,\delta g - \dim\ker Q$ and $\zeta_Q'(0)$ is a *real number*.

Now the *regularized determinant*[3] of Q is defined by

$$\operatorname{det} Q = \exp(-\zeta_Q'(0)).$$

Thus the torsion of the elliptic complex (2.1) is

$$\tau P = \exp\Big(\frac{1}{2}\sum_{q=0}^{l}(-1)^q q \zeta_{\square_q}'(0)\Big). \tag{2.2.3}$$

[3] Also called a *functional determinant*.

2.2.4. Proposition. *Let E be a hermitian vector bundle over a compact Riemannian manifold M. Let $Q : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ be a positive semi-definite self-adjoint elliptic operator. Then for any positive real number c,*

$$\zeta_{cQ}(s) = c^{-s}\zeta_Q(s), \quad s \in \mathbb{C}$$

and

$$\text{đet}(cQ) = c^{\zeta_Q(0)}\,\text{đet}(Q).$$

Proof. Let λ denote the eigenvalues of Q so that $c\lambda$'s are the eigenvalues of cQ. Then for $\operatorname{Re} s >> 0$,

$$\zeta_{cQ}(s) = \sum_{\lambda>0} \frac{1}{(c\lambda)^s} = c^{-s}\sum \frac{1}{\lambda^s} = c^{-s}\zeta_Q(s)$$

and hence it is true for all $s \in \mathbb{C}$. Now

$$\zeta'_{cQ}(s) = -c^{-s}(\log c)\zeta_Q(s) + c^{-s}\zeta'_Q(s)$$

at regular point s and hence $\zeta'_{cQ}(0) = -(\log c)\zeta_Q(0) + \zeta'_Q(0)$. Finally,

$$\text{đet}(cQ) = \exp(-\zeta'_{cQ}(0)) = \exp((\log c)\zeta_Q(0)) \cdot \exp(-\zeta'_Q(0)) = c^{\zeta_Q(0)}\,\text{đet}(Q).$$

This completes the proof.

This proposition says that $\zeta_Q(0)$ is the 'stable rank' of Q (cf. Proposition (1.1.3)). Note that

$$\int_M \operatorname{tr}(\theta_n)\,\delta g = \dim\ker Q + \zeta_Q(0).$$

2.2.5. Proposition. *In the elliptic complex (2.1),*

$$\sum_{q\geq 0}(-1)^q\zeta_{\Box_q}(s) \equiv 0, \quad \forall s \in \mathbb{C}$$

and

$$\prod_{q=0}^{l}(\text{đet}\,\Box_q)^{(-1)^q} = 1.$$

Proof. For $\lambda > 0$, let $\Gamma_\lambda(E_q)$ denote the λ-eigenspace of $\Box_q$. Then we have an exact sequence

$$0 \to \Gamma_\lambda(E_0) \xrightarrow{P_0} \Gamma_\lambda(E_1) \xrightarrow{P_1} \cdots \to \Gamma_\lambda(E_l) \to 0.$$

The spaces $\Gamma_\lambda(E_q)$ are finite dimensional and if $c_{\lambda,q}$ denote the dimension of these spaces, then

$$\sum_{q\geq 0}(-1)^q c_{\lambda,q} = 0.$$

Now for $s \in \mathbb{C}$ with large real part,

$$\sum_{q\geq 0}(-1)^q \zeta_{\Box_q}(s) = \sum_q (-1)^q \sum_{\lambda>0} \frac{c_{\lambda,q}}{\lambda^s} = \sum_\lambda \sum_q (-1)^q \frac{c_{\lambda,q}}{\lambda^s} = 0.$$

This shows the first assertion.

Now we have

$$\sum_{q\geq 0}(-1)^q \zeta'_{\Box_q}(0) = 0$$

and the second assertion is clear. This completes the proof.

2.2.6. Proposition. *Let $Q : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ and $Q' : \mathcal{C}^\infty(E') \to \mathcal{C}^\infty(E')$ be positive semi-definite self-adjoint elliptic operators on M. Suppose there exists an isometric bundle homomorphism $h : E \to E'$ such that $Q'h = hQ$. Then*

$$\det Q = \det Q'.$$

2.2.7. Corollary. *Suppose that an elliptic complex (2.1) is equivalent to another elliptic complex*

$$P' : 0 \to \mathcal{C}^\infty(E'_0) \xrightarrow{P'_0} \mathcal{C}^\infty(E'_1) \xrightarrow{P'_1} \ldots \xrightarrow{P'_{l-1}} \mathcal{C}^\infty(E'_l) \to 0$$

in the sense that there exist isometric bundle homomorphisms $h_q : E_q \to E'_q$ such that $h_{q+1}P_q = P'_q h_q$, then

$$\tau(P) = \tau(P').$$

Proof. Note that $P'_q = h_{q+1} P_q {h_q}^{-1}$ and $(P'_q)^* = h_q P^*_q {h_{q+1}}^{-1}$. Thus

$$\Box'_q = h_q \Box_q {h_q}^{-1}$$

and the result follows from the above proposition.

2.3. Duality. Let

$$P : 0 \to \mathcal{C}^\infty(E_0) \xrightarrow{P_0} \mathcal{C}^\infty(E_1) \xrightarrow{P_1} \dots \xrightarrow{P_{l-1}} \mathcal{C}^\infty(E_l) \to 0$$

be an elliptic complex for hermitian vector bundles $E_0, E_1, \dots, E_l$ over a compact Riemannian manifold M. Then we have the adjoint

$$P^* : 0 \to \mathcal{C}^\infty(E_l) \xrightarrow{P^*_{l-1}} \dots \xrightarrow{P^*_0} \mathcal{C}^\infty(E_0) \to 0.$$

2.3.1. Proposition. $\tau(P) = \tau(P^*)^{(-1)^{l+1}}$

Proof. Note that

$$\tau(P^*) = \prod_{q\geq 0} (\operatorname{\text{đ}et} \square_{l-q})^{(-1)^{q+1}q/2} = \prod_{q} (\operatorname{\text{đ}et} \square_q)^{(-1)^{l-q+1}(l-q)/2}$$

and hence

$$\tau(P^*)^{(-1)^{l+1}} = \Big(\prod_{q\geq 0} (\operatorname{\text{đ}et} \square_q)^{(-1)^q}\Big)^{l/2} \Big(\prod_{q\geq 0} (\operatorname{\text{đ}et} \square_q)^{(-1)^{q+1}q/2}\Big),$$

which is equal to $\tau(P)$ by the Proposition (2.2.5). This completes the proof.

2.4. Change of the metric. Let $P : \mathcal{C}^\infty(E_0) \to \mathcal{C}^\infty(E_1)$ be a differential operator between hermitian vector bundles $(E_q, \langle\ ,\ \rangle_q)$, $q = 0, 1$, over a compact Riemannian manifold (M, g). Let $\langle\ ,\ \rangle\tilde{}_q$ be a new hermitian structure on E_q. Then $\langle \cdot, \cdot \rangle\tilde{}_q = \langle H_q \cdot, \cdot \rangle_q$ for some self-adjoint positive definite endomorphism H_q of $(E_q, \langle\ ,\ \rangle_q)$. Let $\tilde{g}$ be a new Riemannian metric on M so that the new Riemannian density satisfies $\delta\tilde{g} = \lambda\delta g$ for some positive function λ on M. Then the new adjoint of P is given by (cf. (1.3.1))

$$P^{\tilde{*}} = \frac{1}{\lambda} {H_0}^{-1} P^* H_1 \lambda.$$

In particular, if λ is constant and $H_q = \rho_q \operatorname{id}$ for some constant $\rho_q > 0$, then

$$P^{\tilde{*}} = \frac{\rho_1}{\rho_0} P^*.$$

Now in the elliptic complex (2.1), if we use the new Riemannian density $\delta\tilde{g} = \lambda\delta g$ ($\lambda \equiv \text{const} > 0$) and new hermitian structures $\langle\ ,\ \rangle\tilde{}_q = \rho_q \langle\ ,\ \rangle_q$, then

$$\tilde{\square}_q = \frac{\rho_{q+1}}{\rho_q} P_q^* P_q + \frac{\rho_q}{\rho_{q-1}} P_{q-1} P_{q-1}^*.$$

2.4.1. Corollary. *Suppose we have a homothetic change of the Riemannian density $\delta\tilde{g} = \lambda\delta g$ on M and a homothetic change of hermitian structures $\langle\ ,\ \rangle_q^{\sim} = \rho^{q-1}\rho_0\langle\ ,\ \rangle_q$ on E_q for some constants $\rho, \rho_0 > 0$, then*

$$\tilde{\Box}_q = \rho\Box_q, \quad \zeta_{\tilde{\Box}_q}(s) = \rho^{-s}\zeta_{\Box_q}(s)$$
$$\zeta_{\tilde{\Box}_q}(0) = \zeta_{\Box_q}(0), \quad \zeta'_{\tilde{\Box}_q}(0) = -(\log\rho)\zeta_{\Box_q}(0) + \zeta'_{\Box_q}(0)$$
$$\operatorname{\bar{d}et}\tilde{\Box}_q = \rho^{\zeta_{\Box_q}(0)}\operatorname{\bar{d}et}\Box_q, \quad \tilde{\tau}(P) = \rho^{-\frac{1}{2}\sum(-1)^q q\zeta_{\Box_q}(0)}\tau(P)$$

In particular, $\tilde{\tau}(P)$ is independent of λ.

3. Dependence on Riemannian metric

Note that the eigenvalues of an elliptic operator depends on the Riemannian metric of the base manifold and the hermitian structure of the bundle.

In this section we study the dependence of the analytic torsion of the elliptic complex (2.1) on the Riemannian metric on M.

In this case, we fix hermitian structures on the bundles E_q. Note that in the cases of flat bundles and holomorphic vector bundles, if we change the Riemannian metric on M, then the hermitian structures on the corresponding bundles also change.

3.1. Lemma. *Let E and F be hermitian vector bundles over a compact smooth manifold M. $P : \Gamma(E) \to \Gamma(F)$ be a differential operator. Then with respect to a density δ on M, $\Gamma(E)$ and $\Gamma(F)$ are prehilbert spaces and the adjoint P^* is defined. If $\tilde{\delta} := \alpha\delta$ for some positive function α on M, then the adjoint $P^{\tilde{*}}$ of P with respect to the new density $\tilde{\delta}$ is given by*

$$P^{\tilde{*}} = \alpha^{-1} \circ P^* \circ \alpha.$$

In particular, if α is constant, then $P^{\tilde{*}} = P^*$.

Now we investigate the variance of the analytic torsion of the elliptic complex (2.1) under the change of the Riemannian metric on M. We fix a Riemannian density δ and consider the 1-parameter family $\delta_u = \alpha_u\delta$ for positive functions α_u on M. Thus we will compute

$$\begin{aligned}\frac{d}{du}\log\tau(P_u) &= \frac{d}{du}\log\Big(\prod_{q\geq 0}(\operatorname{\bar{d}et}\Box_{q,u})^{(-1)^{q+1}q/2}\Big)\\ &= \frac{d}{du}\frac{1}{2}\sum_{q\geq 0}(-1)^q q\zeta'_{\Box_{q,u}}(0)\\ &= \frac{1}{2}\frac{d}{ds}\Big|_0\sum_{q\geq 0}(-1)^q q\frac{d}{du}\zeta_{\Box_{q,u}}(s).\end{aligned}$$

3.2. Proposition. *Let denote the derivative with respect to u-variable. Then for* $\beta = \beta_u := \dot{\alpha}/\alpha$,

(1) $\dot{P^*} = P^*\beta - \beta P^*$.

(2) $\dot{\Box}_{q,u} = -\beta P_q^* P_q + P_q^*\beta P_q - P_{q-1}\beta P_{q-1}^* + P_{q-1}P_{q-1}^*\beta$.

(3)

$$\sum_{q\geq 0}(-1)^q q \operatorname{Tr}\left(\dot{\Box}_{q,u}e^{-t\Box_{q,u}}\right) = \sum_{q\geq 0}(-1)^q \operatorname{Tr}\left(\beta\Box_{q,u}e^{-t\Box_{q,u}}\right)$$

$$= -\sum_{q\geq 0}(-1)^q \frac{d}{dt}\operatorname{Tr}\left(\beta e^{-t\Box_{q,u}}\right) = -\sum_{q\geq 0}(-1)^q \frac{d}{dt}\operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right),$$

where $H_{q,u}$ *denotes the orthogonal projection onto* $\ker\Box_{q,u} \subset L^2(M, E_q)$.

Now for $\operatorname{Re} s >> 0$ and $h_q := \dim\ker\Box_{q,u}$,

$$\begin{aligned}\frac{d}{du}\zeta_{\Box_{q,u}}(s) &= \frac{d}{du}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\operatorname{Tr}(e^{-t\Box_{q,u}}) - h_q\right)dt\\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}\left(-t\dot{\Box}_{q,u}e^{-t\Box_{q,u}}\right)dt\\ &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s}\operatorname{Tr}\left(\dot{\Box}_{q,u}e^{-t\Box_{q,u}}\right)dt\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{q\geq 0}(-1)^q q\frac{d}{du}\zeta_{\Box_{q,u}}(s)\\ &= -\frac{1}{\Gamma(s)}\int_0^\infty t^s \sum_{q\geq 0}(-1)^q q\operatorname{Tr}\left(\dot{\Box}_{q,u}e^{-t\Box_{q,u}}\right)dt\\ &=\frac{1}{\Gamma(s)}\sum_q(-1)^q\int_0^\infty t^s\frac{d}{dt}\operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right)dt\\ &=\frac{1}{\Gamma(s)}\sum_q(-1)^q\Big(t^s\operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right)\Big|_0^\infty\\ &\qquad - s\int_0^\infty t^{s-1}\operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right)dt\Big)\\ &=\frac{-s}{\Gamma(s)}\sum_q(-1)^q\int_0^\infty t^{s-1}\operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right)dt.\end{aligned}$$

Now if we can show that the meromorphic extension of

$$B : s \mapsto \int_0^\infty t^{s-1} \operatorname{Tr}\left(\beta(e^{-t\Box_{q,u}} - H_{q,u})\right) dt, \quad \operatorname{Re} s >> 0$$

is analytic at $s = 0$, then the analytic torsion is independent of the Riemannian metric on M, since $\frac{s}{\Gamma(s)}$ has a double zero at $s = 0$.

4. FLAT BUNDLES

Let $E \to M$ be a hermitian vector bundle with a flat connection D. Then we have the associated elliptic complex

$$d_D : 0 \to A^0(E) \xrightarrow{d_D} A^1(E) \xrightarrow{d_D} \cdots \to A^n(E) \to 0.$$

The analytic torsion of this complex will be denoted by $\tau(D)$.

4.1. Proposition. *Let D be a flat hermitian connection over an even dimensional oriented M. Then $\tau(D) = 1$.*

Proof. The Hodge star gives a commutative diagram

$$\begin{array}{ccccccc} 0 \to A^0(E) & \xrightarrow{d_D} & A^1(E) & \xrightarrow{d_D} & \cdots \to & A^n(E) \to 0 \\ {\scriptstyle *}\downarrow & & {\scriptstyle -*}\downarrow & & & \downarrow{\scriptstyle *} \\ 0 \to A^n(E) & \xrightarrow{d_D^*} & A^{n-1}(E) & \xrightarrow{d_D^*} & \cdots \to & A^0(E) \to 0. \end{array}$$

Thus from Corollary (2.2.7) and Proposition (2.3.1),

$$\tau(D) = \tau(D^*) = \tau(D)^{-1}.$$

Thus $\tau = 1$. This completes the proof.

4.2. Remark. If D_1 is another flat hermitian connection on E equivalent to D, then there exists an isometry $h : E \to E$ such that

$$d_{D_1} = h^{-1} \circ d_D \circ h.$$

Thus $\tau(D_1) = \tau(D)$ by Corollary (2.2.7). Thus τ is a function on the *moduli space* of flat hermitian connections.

4.3. Homothetic change of the metric. Suppose D is a flat connection on a hermitan vector bundle E over a Riemannian manifold (M,g) . If $\tilde{g}=\mu^2 g$ for some constant $\mu>0$, then

$$\langle\ ,\ \rangle_q^{\sim}=\mu^{-2q}\langle\ ,\ \rangle_q \quad \text{on } A^q(E)$$

and hence $\Delta_q^{\sim}=\mu^{-2}\Delta_q$. Thus (cf. 2.4.1)

$$\tilde{\tau}(D)=\mu^{\sum(-1)^q q\zeta_{\Delta_q}(0)}\tau(D).$$

4.4. Example. For example, consider the Laplacian $\Delta=-\left(\frac{\partial}{\partial\theta}\right)^2$ acting on the space of smooth functions on the circle $\mathbf{S}^1(r)=\{re^{i\theta}:\theta\in\mathbb{R}\}=\mathbb{R}/2\pi r\mathbb{Z}$ of radius $r>0$. Then the spectrum of Δ is $\{\frac{k^2}{r^2}:k\in\mathbb{Z}\}$ and hence

$$\zeta_\Delta(s)=2\sum_{k=1}^{\infty}\frac{r^{2s}}{k^{2s}}=2r^{2s}\zeta_R(2s), \qquad \operatorname{Re}s>\frac{1}{2},$$

where ζ_R denotes the Riemann zeta function. Since [Tit]

$$\zeta_R(0)=-\frac{1}{2}, \quad \zeta_R'(0)=-\log\sqrt{2\pi},$$

we have

$$\zeta_\Delta(0)=-1, \quad \zeta_\Delta'(0)=-2\log 2\pi r, \quad \det\Delta=(2\pi r)^2.$$

Thus the analytic torsion of the de Rham complex on the circle is

$$\tau(\mathbf{S}^1(r),d)=\sqrt{\det\Delta}=2\pi r.$$

If we consider a complex line bundle L over $\mathbf{S}^1(r)$, then L must be trivial and hence we may assume that $L=\mathbf{S}^1(r)\times\mathbb{C}$. We also assume the standard Hermitian structure on L. Then a Hermitian connection D on L is given by

$$D=d+i\omega$$

for some (real valued) 1-form ω on $\mathbf{S}^1(r)$. One can see easily[4] that any Hermitian connection on L is equivalent to

$$D_u:=d+iud\theta$$

[4] If $\frac{1}{2\pi}\int_{\mathbf{S}^1(r)}\omega=k+u$ for $k\in\mathbb{Z}$ and $0\le u<1$, then

$$\frac{1}{2\pi}\int_{\mathbf{S}^1}\omega+ig_k{}^{-1}dg_k-ud\theta=0,$$

and hence we have an analytic torsion $\tau(D)$ for each Einstein-Hermitian connection. The group of the bundle isometries of E acts on the space of Einstein-Hermitian connections and the analytic torsion is invariant under the action. Thus the analytic torsion is a function defined on the moduli space of Einstein-Hermitian connections. This moduli space is closely related to the moduli space of stable bundles.

APPENDIX: ASYMPTOTIC EXPANSION

Let $\mathcal{F}$ be the set of all (real or complex valued) functions defined on an interval $(0, \epsilon)$ for some $\epsilon > 0$. For $f, g \in \mathcal{F}$, let $f \equiv g$ if there exists $\epsilon > 0$ such that $f(t) = g(t)$ for all $t \in (0, \epsilon)$. Then $\equiv$ is an equivalence relation and the quotient $\mathcal{F}/\equiv$ becomes an algebra.

For $g \in \mathcal{F}$, let

$$O(g) := \{f \in \mathcal{F} : \exists C > 0,\ \exists \epsilon > 0,\ |f(t)| \le C|g(t)|\ \forall t \in (0, \epsilon)\}$$

and

$$o(g) := \{f \in \mathcal{F} : \forall C > 0,\ \exists \epsilon > 0,\ |f(t)| \le C|g(t)|\ \forall t \in (0, \epsilon)\}.$$

Now let $\{g_i\}_{i=0}^{\infty}$ be a sequence in $\mathcal{F}$ such that $g_{i+1} \in o(g_i)$. We write, for $f \in \mathcal{F}$ and for scalars a_i,

$$f \sim \sum_{i=0}^{\infty} a_i g_i$$

if for any $N \ge 0$

$$f - \sum_{i=0}^{N} a_i g_i \in o(g_N)$$

or *equivalently*

$$f - \sum_{i=0}^{N} a_i g_i \in O(g_{N+1}).$$

We list some properties. Suppose $f, h \in \mathcal{F}$ and

$$f \sim \sum_{i=0}^{\infty} a_i g_i, \quad h \sim \sum_{i=0}^{\infty} b_i g_i.$$

Then

(1) for any scalar c, $cf \sim \sum_{i=0}^{\infty} c a_i g_i$.
(2) $hg_{i+1} \in o(hg_i)$ and $hf \sim \sum_{i=0}^{\infty} a_i (hg_i)$.
(3) $f + h \sim \sum_{i=0}^{\infty} (a_i + b_i) g_i$.
(4) If $g_i \not\equiv 0$ and $f \equiv h$, then $a_i = b_i$.

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COMPLEX ANALYTIC MANIFOLDS AND VARIETIES

CAUCHY TRANSFORMS ON NON-LIPSCHITZ CURVES, T1-THEOREM AND THE HAAR SYSTEM

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ABSTRACT. The content of this note is twofold. We first summarize the result by Kang and Seo on the L^2-boundedness of the Cauchy transform on smooth non-Lipshitz curves. We then give a discrete proof of $T1$-theorem in 1-dimension using the Haar system based on known ideas.

1. INTRODUCTION

Let Γ be a curve defined by $y = A(x)$ in $\mathbb{R}^2$. The Cauchy transform $\mathcal{C}_A$ on the curve Γ is a singular integral operator defined by the singular integral kernel

$$k(x,y) = \frac{1 + iA'(y)}{(x-y) + i(A(x) - A(y))}. \tag{1.1}$$

If A is a Lipschitz function, i.e., $\|A'\|_\infty < \infty$, then $\mathcal{C}_A$ makes the most significant example of non-convolution type singular integral operators (SIO) which has enough generality in itself. The problem of L^2-boundedness of the Cauchy transform was raised and solved when $\|A'\|_\infty$ is small by A. P. Calderón in relation to the Dirichlet problem on Lipschitz domains [**Cal1, Cal2**]. Since then, it has been a central problem in the theory of singular integral operators and several significant techniques has been developed to deal with this problem [**DJ, CMM, CJS**]. We refer to [**Chr, Mur**] for a history of development in the last decades on the theory of the Cauchy transform. One and the most significant one of such development is the $T(1)$-Theorem of David and Journé.

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T(1)-Theorem (David and Journé). *Let T be the integral operator defined by*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x,y) f(y) dy$$

for any bounded function with compact supports where the kernel $k(x,y)$ satisfy the so called standard estimates

(S1) $|k(x,y)| \leq C|x-y|^{-n}$,
(S2) $|\nabla_{x,y} k(x,y)| \leq C|x-y|^{-n-1}$, $\forall x \neq y \in \mathbb{R}^n$,

Then, T can be extended as an operator bounded on $L^2(\mathbb{R}^n)$ if and only if T satisfies

(1) *Weak Boundedness Property: there exist constants N and C such that for any pair of functions φ and ψ in $C_0^\infty(\mathbb{R})$ satisfying $\varphi(x) = \psi(x) = 0$ if $|x| > 1$ and $\|\varphi\|_{C^N} \leq 1$ and $\|\psi\|_{C^N} \leq 1$, for any $x \in \mathbb{R}$ and $t > 0$,*

$$| < T\varphi^{x,t}, \psi^{x,t} > | \leq Ct$$

where $\varphi^{x,t}(y) = \varphi(\frac{x+y}{t})$,

(2) $T1 \in BMO$,
(3) $T^*1 \in BMO$.

$T1$-Theorem deals with a very hard question of L^2-boundedness of singular integral operators and gives a necessary and sufficient condition even if checking $T1, T^*1 \in BMO$ is not, in general, an easy matter.

If $\|A'\|_\infty = \infty$, then the Cauchy kernel does not satisfy the standard estimates. So, the theory of the singular integral operators may not be applied directly. In [**KS**], we deal with the Cauchy transform defined on non-Lipschitz curve and obtain the following theorem.

Theorem A. *Let $A(x)$ be a polynomial of the form*

$$(1.2) \qquad \begin{cases} A(x) \text{ is any polynomial if } d \text{ is an odd integer,} \\ A(x) = \sum_{i=1}^{n} a_i x^{2i} \text{ if } d = 2n \text{ is an even integer.} \end{cases}$$

Then, the Cauchy transform $\mathcal{C}_A$ is bounded on L^2.

In this note, we briefly summarize the idea and steps to prove Theorem A.

We then turn to $T1$-Theorem and gives a discrete proof of it using the Haar wavelet basis. Since David and Journé found $T1$-Theorem, the proof has been improved by Coifman and Meyer. This improved proof can be found in [**DK**] or [**Chr**], and it is based on the the following identity of A. P. Calderón; if T is an SIO, then

$$T = \int_0^\infty T_t \frac{dt}{t} \tag{1.3}$$

where

$$T_t f = (Tf) * \psi_t * \psi_t, \qquad \psi_t(x) = t^{-n}\psi(t^{-1}x)$$

for some nice function ψ. If one discretize the parameter t in (1.3), (1.3) is a wavelet decomposition (or Paley-Littlewood decomposition) of T. So, we can discretize the proof. The proof in this note is not new. Since the concepts of wavelets and multiresolution approximations were formulated by Y. Meyer [**Mey**] and S. Mallat [**Mal**] in late 1980s, the idea of discretization prevails among harmonic analysts like Y. Meyer [**Mey**] and R. Coifman [**BCR**]. But, no complete proof seems available. So, we make a complete proof.

I would like to take this opportunity to thank Jin Keun Seo for all the exciting discussions and collaborations.

2. Cauchy Transform on Non-Lipschitz Curves

For clarity, let us first consider the Cauchy transform defined on the curve $y = A(x) = x^2$. On this curve, the Cauchy kernel is

$$k(x,y) = \frac{1+2iy}{(x-y)+i(x^2-y^2)}.$$

This kernel has two kinds of singularities: one at $x = y$ as one can expect, and a weaker one at $x = -y$. Because of this weaker singularity, $k(x,y)$ does not satisfy the standard estimates, for example, it does not satisfy $|k(x,-x)| \le C|x|^{-1}$. However, we can decompose $k(x,y)$ into two standard kernels;

$$k(x,y) = \frac{1+2iy}{(x-y)+i(x^2-y^2)} = \frac{1}{x-y} + \frac{-i}{1+i(x+y)}. \tag{2.1}$$

The first kernel of the right hand side is the Hilbert kernel while the second one is a kernel of Poisson type. So, if $A(x) = x^2$, then $\mathcal{C}_A$ is bounded on L^2. This is the beginning of the whole story. We perform the same kind of decomposition for general Cauchy kernels. The key idea in this process is how to separate the singularity at $x = -y$ from the one at $x = y$. We proceed as follows; let $\tilde{\phi}$ be a C^∞ smooth function such that

$$\text{(2.2)} \qquad \begin{cases} \tilde{\phi}(x) = 1 & \text{if } |x| < \dfrac{1}{2} \\ \tilde{\phi}(x) = 0 & \text{if } |x| \geq \dfrac{4}{5} \\ \|\tilde{\phi}'\|_{L^\infty} \leq 10. \end{cases}$$

and we let

$$\text{(2.3)} \qquad \phi(x, y) = \tilde{\phi}(\frac{x-y}{1+|x|}).$$

We then docompose the Cauchy kernel $k(x, y)$ as

$$k = k_1 + k_2 := k\phi + k(1 - \phi).$$

Let us see why this decomposition does the job. If $x = -y$ and $|x|$ is large, then $\frac{|x-y|}{1+|x|} \geq 1$ and hence $\phi = 0$. So, k_1 does not have a singularity at $x = -y$. On the other hand, if $x = y$, then $\phi = 1$ and hence k_2 does not have a singularity at $x = y$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the integral operators defined by k_1 and k_2 respectively. We show that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are bounded on L^2 by using $T(1)$-theorem. In fact, we prove the following theorem.

Theorem 2.1. *$\mathcal{C}_1, \mathcal{C}_2, k_1$, and k_2 satisfy all the conditions of $T1$-Theorem, namely, the standard estimates, the weak boundedness property, $\mathcal{C}_1 1, \mathcal{C}_2 1 \in BMO$, and $\mathcal{C}_1^* 1, \mathcal{C}_2^* 1 \in BMO$.*

In order to prove Theorem 2.1, we use the following estimates for polynomials.

Lemma 2.2. *Let $A(x)$ be d-th degree polynomial of the form:*

$$\text{(2.4)} \qquad \begin{cases} A(x) \text{ is any polynomial if } d \text{ is an odd integer} \\ A(x) = \displaystyle\sum_{i=1}^{n} a_i x^{2i} \quad \text{if } d = 2n \text{ is an even integer.} \end{cases}$$

Then,

(1) *If d is odd, then*

$$|A(x) - A(y)| \approx |x - y|(|x|^{d-1} + |y|^{d-1}).$$

Moreover, there exists a positive number M such that

(2) *If $|x| \geq M$, then $|A(x)| \approx |x|^d$, $|A'(x)| \approx |x|^{d-1}$, and $|A''(x)| \approx |x|^{d-2}$.*

(3) *If d is even and if either $|x| \geq M$ or $|y| \geq M$, then*

$$|A(x) - A(y)| \approx |x-y||x+y|(|x|^{d-2} + |y|^{d-2}).$$

(4) *If $|x| < M$ and $|y| > 2M$, then*

$$|A(x) - A(y)| \approx |A(y)| \approx |y|^d.$$

(3) is the most important one. Among the polynomials which are not covered in [**KS**] is $A(x) = x^4 - x^3$. This polynomial does not satisfy Lemma 2.2. In fact, it does not satisfy the estimate $|A(x) - A(y)| \approx |x-y||x+y|(|x|^2 + |y|^2)$ when $|x| + |y|$ is large.

Regarding the proof of the fact that $\mathcal{C}_1 1, \mathcal{C}_2 1 \in BMO$, let us make a comment. Showing that a function is in BMO is a fairly hard task. One of the reasons is that being a BMO function is not just a size condtion. For example, even if $|f| \in$ BMO, f may not be a BMO function. It can also be shown easily that even if $0 \leq f \leq g$ and $g \in$ BMO, f may not be a BMO function. In particular, that $f(x) = O(\log|x|)$ as $x \to \infty$ does not imply $f \in$ BMO. However, since A is a polynomial, we could use the smoothness of the A to check that $\mathcal{C}_1 1, \mathcal{C}_2 1 \in BMO$, and $\mathcal{C}_1^* 1, \mathcal{C}_2^* 1 \in BMO$. We show that if $f'(x) = O(|x|^{-1})$ as $x \to \infty$, then $f \in$ BMO. In fact, we proved the following lemma which is interesting in itself.

Lemma 2.3. *Suppose that there exists a positive number m such that f is bounded on $[-m, m]$ and f is continuously differentiable if $|x| \geq m$. If $|f'(x)| = O(|x|^{-1})$ as $x \to \infty$, then $f \in BMO$.*

3. The Haar System

Let

$$\chi = \chi_{[0,1)} \text{ and } h = \chi_{[0,1/2)} - \chi_{[1/2,1)}. \tag{3.1}$$

where χ_E is the characteristic function on E. When $I = [2^{-j}k, 2^{-j}(k+1))$ where j and k are integers, we define

$$\varphi_I(x) = \varphi_{j,k}(x) = 2^{j/2}\chi(2^j x - k), \tag{3.2}$$

$$\psi_I(x) = \psi_{j,k}(x) = 2^{j/2}h(2^j x - k). \tag{3.3}$$

We note that φ_I and ψ_I are defined by translating and dilating functions χ and h. We then define for each integer j

$$(3.4)\quad V_j = \{ \sum_{|I|=2^{-j}} \alpha_I \varphi_I : \sum_{|I|=2^{-j}} |\alpha_I|^2 < \infty \}$$
$$= \{ f \in L^2(\mathbb{R}) : f|_{[2^{-j}k, 2^{-j}(k+1))} = \text{constant for each } k \in \mathbb{Z} \},$$

and

$$(3.5)\qquad W_j = \{ \sum_{|I|=2^{-j}} \alpha_I \psi_I : \sum_{|I|=2^{-j}} |\alpha_I|^2 < \infty \}.$$

Then, one can easily observe the followings (proofs can be found in [**Dau**]).

(1) Each V_j is a closed subspace of L^2,
(2) $\cup_{j\in\mathbb{Z}} V_j$ is dense in L^2,
(3) for each j, $V_j \perp W_j$,
(4) for each j, $V_j \bigoplus W_j = V_{j+1}$,
(5) $L^2 = \bigoplus_{j\in\mathbb{Z}} W_j$,
(6) moreover, φ_I and ψ_I are defined by translating and dilating functions χ and h.

The sequences $\{V_j\}$ and $\{W_j\}$ of closed subspaces of L^2 with the properties (1)-(6) are called a multiresolution approximation and a wavelet decompostion of L^2, respectively.

4. A Discrete Proof of $T1$-Theorem

In this section, we give a discrete proof of $T1$-theorem using the Haar system. We confine ourselves to skew-symmetric ones on $\mathbb{R}^1$ for convenience. An SIO T is skew-symmetric if $T^* = -T$. If an SIO T is skew symmetric, then it is automatically weakly bounded. Therefore, $T1$-theorem for a skew-symmetric SIO is as follows:

T1-Theorem. *Let T be a skew-symmetric singular integral operator defined by the kernel $k(x,y)$ satisfying* (S1) *and* (S2) *with $n = 1$. Then, T can be extended as an operator bounded on L^2 if and only if $T1 \in BMO$.*

Let us begin the proof. Since the only if part follows from the general theory of SIOs, significance of the T1-Theorem lies in the if part. Let $\{V_j\}$ and $\{W_j\}$ be a multiresolution approximation and a wavelet decompostion of L^2 as defined in (3.4) and (3.5), and let P_j and Q_j be the orthogonal projections onto V_j and W_j, respectively. We need to prove that

$$|\langle Tf, g\rangle| \leq C\|f\|\|g\| \qquad \forall f, g \in C_0^\infty(\mathbb{R}^1).$$

Here, $\|f\|$ denotes the L^2 norm of f.

Lemma 4.1. *For any $f, g \in C_0^\infty(\mathbb{R}^1)$, we have*

$$\lim_{n\to\infty}\langle P_nTP_nf, g\rangle = \langle Tf, g\rangle,$$
$$\lim_{n\to-\infty}\langle P_nTP_nf, g\rangle = 0.$$

We omit the proof of Lemma 4.1. Note that $P_{j+1} - P_j = Q_j$ since $V_j \bigoplus W_j = V_{j+1}$ for each j. Since $\lim_{n\to\infty} P_nTP_n = T$ and $\lim_{n\to-\infty} P_nTP_n = 0$ in the sense of Lemma 1, we have

$$\begin{aligned}
T &= \lim_{n\to\infty}(P_nTP_n - P_{-n}TP_{-n}) \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n-1}(P_{j+1}TP_{j+1} - P_jTP_j) \\
&= \sum_{-\infty}^{\infty}[(P_{j+1} - P_j)T(P_{j+1} - P_j) + P_jT(P_{j+1} - P_j) + (P_{j+1} - P_j)TP_j] \\
&= \sum_{-\infty}^{\infty}[Q_jTQ_j + P_jTQ_j + Q_jTP_j]
\end{aligned}$$

Since P_j and Q_j are orthogonal projections and T is skew-symmetric, we have, by the Cauchy- Schwartz inequality,

$$
\begin{aligned}
|\langle Tf, g\rangle| &= \left|\sum_{-\infty}^{\infty}\langle Q_jTQ_jf, g\rangle + \langle P_jTQ_jf, g\rangle + \langle Q_jTP_jf, g\rangle\right| \\
&\le \sum_{-\infty}^{\infty}|\langle Q_jTQ_jf, Q_jg\rangle| + \sum_{-\infty}^{\infty}|\langle Q_jf, Q_jTP_jg\rangle| \\
&\quad + \sum_{-\infty}^{\infty}|\langle Q_jTP_jf, Q_jg\rangle| \\
&\le \sum_{-\infty}^{\infty}\|Q_jTQ_jf\|\|Q_jg\| + \sum_{-\infty}^{\infty}\|Q_jf\|\|Q_jTP_jg\| \\
&\quad + \sum_{-\infty}^{\infty}\|Q_jTP_jf\|\|Q_jg\| \\
&\le \Big(\sum_j \|Q_jTQ_jf\|^2\Big)^{1/2}\Big(\sum_j \|Q_jg\|^2\Big)^{1/2} \\
&\quad + \Big(\sum_j \|Q_jf\|^2\Big)^{1/2}\Big(\sum_j \|Q_jTP_jg\|^2\Big)^{1/2} \\
&\quad + \Big(\sum_j \|Q_jTP_jf\|^2\Big)^{1/2}\Big(\sum_j \|Q_jg\|^2\Big)^{1/2}.
\end{aligned}
$$

Since $\sum_j \|Q_jf\|^2 = \|f\|^2$, it suffices to prove that

$$
\begin{aligned}
&\sum_j \|Q_jTQ_jf\|^2 \le C\|f\|^2, \\
&\sum_j \|Q_jTP_jf\|^2 \le C\|f\|^2.
\end{aligned}
$$

We prove the following theorem.

Theorem 4.2. *There exists a universal constant C such that*

$$\|Q_jTQ_jf\|^2 \le C\|Q_jf\|^2 \qquad \forall j \in \mathbb{Z} \tag{4.1}$$

and

$$\sum_j \|Q_jTP_jf\|^2 \le C\|f\|^2. \tag{4.2}$$

To prove Theorem 4.2, we need a lemma. Throughout this paper, $|I|$ denotes the length of the interval I.

Lemma 4.3. *If* $|I| = |I'|$, *then*

$$|\langle T\varphi_{I'}, \psi_I\rangle| \le \frac{C|I|^2}{|I|^2 + dist(I,I')^2}, \tag{4.3}$$

and

$$|\langle T\psi_{I'}, \psi_I\rangle| \le \frac{C|I|^2}{|I|^2 + dist(I,I')^2}. \tag{4.4}$$

Proof. We first prove (4.4). We first suppose that $\mathrm{dist}(I, I') > 0$. Then, since $\int \psi_I(x)dx = 0$, we have

$$\begin{aligned}\langle T\psi_{I'}, \psi_I\rangle &= \int\int k(x,y)\psi_{I'}(y)\psi_I(x)dxdy \\ &= \int\int [k(x,y) - k(x_I, y)]\psi_{I'}(y)\psi_I(x)dxdy\end{aligned}$$

where x_I is the center of I. It then follows from the mean value theorem and the standard estimate (S1) that

$$\begin{aligned}|\langle T\psi_{I'}, \psi_I\rangle| &\le |I|^{-1}\int_{I'}\int_I |k(x,y) - k(x_I,y)|dxdy \\ &\le |I|^{-1}\int_{I'}\int_I |\nabla_x k(\xi, y)||x - x_I|dxdy \\ &\qquad \text{for some } \xi \text{ in between } x \text{ and } x_I \\ &\le |I|^{-1}\int_{I'}\int_I \frac{|x - x_I|}{|\xi - y|^2}dxdy \\ &\le \frac{C|I|^2}{|I|^2 + \mathrm{dist}(I,I')^2}.\end{aligned}$$

We now suppose that $\mathrm{dist}(I, I') = 0$ but $I \ne I'$. Note that

$$\frac{1}{d}\int_d^{2d}\int_0^d \frac{1}{y - x}dx \le C. \tag{4.5}$$

By (4.5), we have

$$\begin{aligned}|\langle T\psi_{I'}, \psi_I\rangle| &\le |I|^{-1}\int_{I'}\int_I |k(x,y)|dxdy \\ &\le |I|^{-1}\int_{I'}\int_I \frac{1}{|x - y|}dxdy \le C.\end{aligned}$$

Finally, suppose that $I = I'$. Then, since T is skew symmetric, we have

$$\langle T\psi_{I'}, \psi_I\rangle = \frac{1}{2}\int\int k(x,y)[\psi_{I'}(y)\psi_I(x) - \psi_{I'}(x)\psi_I(y)]dxdy = 0.$$

This completes the proof of (4.4)

To prove (4.3), it suffices to consider only the case when $I = I'$ since the rest can be proved in the same way as above. Suppose that $I = I'$. Then,

$$\langle T\varphi_I, \psi_I\rangle = \frac{1}{2}\int\int k(x,y)[\varphi_I(y)\psi_I(x) - \varphi_I(x)\psi_I(y)]dxdy.$$

By translating I if necessary, we may assume that $I = [0, 2d)$. Note that $\varphi_I(y)\psi_I(x) - \varphi_I(x)\psi_I(y) = 0$ on $[0,d) \times [0,d)$ and $[d, 2d) \times [d, 2d)$. Therefore, we have

$$|\langle T\varphi_I, \psi_I\rangle| \leq C|I|^{-1}\left(\int_d^{2d}\int_0^d + \int_0^d\int_d^{2d} \frac{1}{|y-x|}dxdy\right) \leq C.$$

This completes the proof. □

We now start to prove (4.1) and (4.2). We will clearly see for which the condition $T1 \in \mathrm{BMO}$ is required and for which it is not.

Proof of (4.1). (4.1) is an easier part. Note that

$$\begin{aligned} Q_jTQ_jf &= Q_jT\Big(\sum_{|I|=2^{-j}} \langle f, \psi_I\rangle\psi_I\Big) \\ &= \sum_{|I'|=2^{-j}}\sum_{|I|=2^{-j}} \langle f, \psi_I\rangle\langle T\psi_I, \psi_{I'}\rangle\psi_{I'} \end{aligned}$$

We regard this as an matrix multiplication. Then, by Shur's lemma, it suffices to show

$$\sup_{I'}\Big(\sum_{|I|=|I'|} |\langle T\psi_{I'}, \psi_I\rangle| + \sum_{|I|=|I'|} |\langle T\psi_I, \psi_{I'}\rangle|\Big) \leq C. \tag{4.6}$$

But (4.6) follows immediately from (4.4). This proves (4.1). □

Proof of (4.2). Note that

$$\begin{aligned} Q_jTP_jf &= Q_jT\Big(\sum_{|I|=2^{-j}} \langle f,\varphi_I\rangle\varphi_I\Big) \\ &= \sum_{|I'|=2^{-j}}\sum_{|I|=2^{-j}} \langle f,\varphi_{I},\rangle\langle T\varphi_I,\psi_{I'}\rangle\psi_{I'} \end{aligned}$$

and hence

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \|Q_jTP_jf\|^2 &= \sum_I \Big|\sum_{\substack{I' \\ |I'|=|I|}} \langle f,\varphi_{I'}\rangle\langle T\varphi_{I'},\psi_I\rangle\Big|^2 \\ &\le 2\sum_I \Big|\sum_{\substack{I' \\ |I'|=|I|}} \langle f,\varphi_{I'}-\varphi_I\rangle\langle T\varphi_{I'},\psi_I\rangle\Big|^2 \\ &\quad + 2\sum_I \Big|\sum_{\substack{I' \\ |I'|=|I|}} \langle f,\varphi_I\rangle\langle T\varphi_{I'},\psi_I\rangle\Big|^2. \end{aligned}$$

Sublemma 4.4. *If T is a skew symmetric standard SIO, then*

$$\sum_I \Big|\sum_{\substack{I' \\ |I'|=|I|}} \langle f,\varphi_{I'}-\varphi_I\rangle\langle T\varphi_{I'},\psi_I\rangle\Big|^2 \le C\|f\|^2 \tag{4.7}$$

Proof. We first note that we do not need the condition $T1 \in$ BMO for this lemma. We first fix a notation for convenience. When I is a dyadic interval, we denote by I_n the interval J such that $dist(I,J) = |n-1||I|$ and if $n>0$ then I_n is on the right hand side of I, if $n<0$ then I_n is on the left hand side of I. Suppose that f is real without loss of generality. By Jensen's inequality, we have

$$\begin{aligned}
&\sum_I \Big| \sum_{\substack{I' \\ |I'|=|I|}} \langle f, \varphi_{I'} - \varphi_I \rangle \langle T\varphi_{I'}, \psi_I \rangle \Big|^2 \\
&\le C \sum_I \Big(\sum_{\substack{I' \\ |I'|=|I|}} |\langle f, \varphi_{I'} - \varphi_I \rangle| \frac{|I|^2}{|I|^2 + \mathrm{dist}(I, I')^2} \Big)^2 \\
&= C \sum_I \Big(\sum_{n \neq 0} |\langle f, \varphi_{I_n} - \varphi_I \rangle| \frac{|I|^2}{|I|^2 + (n-1)^2 |I|^2} \Big)^2 \\
&\le C \sum_I \Big(\sum_{n \neq 0} |\langle f, \varphi_{I_n} - \varphi_I \rangle| \frac{1}{1+n^2} \Big)^2 \\
&\le C \sum_{n \neq 0} \frac{1}{1+n^2} \sum_I \langle f, \varphi_{I_n} - \varphi_I \rangle^2 .
\end{aligned}$$

Therefore, it is enough to prove the following:

$$\sum_{n \neq 0} \frac{1}{1+n^2} \sum_I \langle f, \varphi_{I_n} - \varphi_I \rangle^2 \le C \|f\|^2 . \tag{4.8}$$

Based on the observation that $\{ \varphi_{I_n} - \varphi_I \}_I$ is an almost orthogonal system, we use the Cotlar-Stein lemma (for the Cotlar-Stein lemma, refer to [**Chr**] or [**Tor**]). In fact, if $|I| = |J|$, then

$$\langle \varphi_{J_n} - \varphi_J, \varphi_{I_n} - \varphi_I \rangle = \begin{cases} 2 & \text{if } I = J \\ -1 & \text{if either } I = J_n \text{ or } I_n = J \\ 0 & \text{if } I \neq J. \end{cases}$$

Define, for each integer j,

$$k_j(x,y) = \sum_{|I|=2^{-j}} (\varphi_{I_n} - \varphi_I)(x)(\varphi_{I_n} - \varphi_I)(y),$$

and define T_j by

$$T_j f(x) = \int_{-\infty}^{\infty} k_j(x,y) f(y) dy.$$

Then,

$$\|T_j f\|^2 = 2 \sum_{|I|=2^{-j}} \langle f, \varphi_{I_n} - \varphi_I \rangle^2 - 2 \sum_{|I|=2^{-j}} \langle f, \varphi_{I_n} - \varphi_I \rangle \langle f, \varphi_I - \varphi_{I_{-n}} \rangle,$$

and hence

$$\sum_{|I|=2^{-j}} \langle f, \varphi_{I_n} - \varphi_I \rangle^2 \leq \|T_j f\|^2,$$

Therefore,

$$\sum_I \langle f, \varphi_{I_n} - \varphi_I \rangle^2 = \sum_{j\in\mathbb{Z}} \sum_{|I|=2^{-j}} \langle f, \varphi_{I_n} - \varphi_I \rangle^2 \leq \sum_{j\in\mathbb{Z}} \|T_j f\|^2.$$

Note that

$$\| \sum_{j=-N}^{N} T_j f\|^2 = \sum_{j=-N}^{N} \|T_j f\|^2 + 2 \sum_{-N\leq j<i\leq N} \langle T_j f, T_i f \rangle.$$

Therefore, since $T_j^* = T_j$ for any j, we have

$$\sum_{j=-N}^{N} \|T_j f\|^2 \leq \| \sum_{j=-N}^{N} T_j f\|^2 + 2 \sum_{-N\leq j<i\leq N} \|T_i T_j f\| \|f\|. \tag{4.9}$$

Let

$$\begin{aligned} k_{j,i}(x,y) &= \int k_j(x,z) k_i(z,y) dy \\ &= \sum_{|I|=2^{-i}} \sum_{|J|=2^{-j}} (\varphi_{J_n} - \varphi_J)(x)(\varphi_{I_n} - \varphi_I)(y) \\ &\qquad \times \int (\varphi_{J_n} - \varphi_J)(z)(\varphi_{I_n} - \varphi_I)(z) dz. \end{aligned}$$

Then,

$$T_j T_i f(x) = \int k_{j,i}(x,y) f(y) dy.$$

We may suppose that $j \geq i$. We claim that

$$\|T_j T_i\| \leq C\sqrt{n} 2^{\frac{i-j}{2}} \qquad \forall i, j. \tag{4.10}$$

In order to prove the claim, we first deal with the case when $n|J| \geq |I|$, i.e., $n2^{-j} \geq 2^{-i}$.

We note that

$$\int (\varphi_{J_n} - \varphi_J)(z)(\varphi_{I_n} - \varphi_I)(z)dz \neq 0$$

only if one of the followings occurs.

$$J \subset I, \qquad J_n \subset I, \qquad J \subset I_n, \qquad J_n \subset I_n.$$

Therefore,

$$\begin{aligned}
&|k_{j,i}(x,y)| \\
&\leq \sqrt{\frac{|J|}{|I|}} \left(\sum_{J \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I_n} \right) (\varphi_{J_n} + \varphi_J)(x)(\varphi_{I_n} + \varphi_I)(y).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int |T_j T_i f(x)|^2 dx \\
&\leq \left(\sum_{J \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I_n} \right) \int_{J \cup J_n} \frac{|J|}{|I|} \frac{1}{|I||J|} \left| \int_{I \cup I_n} f(y)dy \right|^2 dx \\
&\leq \left(\sum_{J \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I} + \sum_{J_n \subset I_n} \right) \frac{|J|}{|I|} \int_{I \cup I_n} |f(y)|^2 dy \\
&\leq C \|f\|^2.
\end{aligned}$$

Now suppose that $n|J| < |I|$, i.e., $n2^{-j} < 2^{-i}$. Then,

$$\int (\varphi_{J_n} - \varphi_J)(z)(\varphi_{I_n} - \varphi_I)(z)dz \neq 0$$

only if one of the followings occurs:

(1) $J \subset I$ and $J_n \cap I = \emptyset$,
(2) $J \subset I_n$ and $J_n \cap I_n = \emptyset$.

Therefore,

$$\int |T_jT_if(x)|^2dx \le \Big(\sum_{\substack{J\subset I\\ J_n\not\subset I}} + \sum_{\substack{J_n\subset I\\ J_n\not\subset I_n}} \Big) \int_{J\cup J_n} \frac{|J|}{|I|}\frac{1}{|I||J|}\left|\int_{I\cup I_n} f(y)dy\right|^2 dx$$

$$\le C\Big(\sum_{\substack{J\subset I\\ J_n\not\subset I}} + \sum_{\substack{J_n\subset I\\ J_n\not\subset I_n}} \Big)\frac{|J|}{|I|}\int_{I\cup I_n} |f(y)|^2dy$$

$$\le C\sum_{|I|=2^{-i}} \frac{1}{|I|}\int_{I\cup I_n} |f(y)|^2dy\Big(\sum_{\substack{J\subset I\\ J_n\not\subset I}} + \sum_{\substack{J_n\subset I\\ J_n\not\subset I_n}} \Big)|J|$$

$$\le Cn2^{i-j}\|f\|^2.$$

This proves (4.10).

It then follows from the Cotlar-Stein lemma that

$$\sum_{j=-N}^{N} \|T_jf\|^2 \le \|\sum_{j=-N}^{N} T_jf\|^2 + 2\sum_{-N\le j<i\le N} \|T_iT_jf\|\|f\| \le C\sqrt{n}\|f\|^2$$

for all N. So, we have

$$\sum_I \langle f, \varphi_{I_n} - \varphi_I\rangle^2 \le C\sqrt{n}\|f\|^2$$

and hence, we have (4.8) and the proof of sublemma 4.4 is completed.

Sublemma 4.5. *If T is a skew symmetric SIO and if $T1 \in BMO$, then*

$$\sum_I \Big| \sum_{\substack{I'\\ |I'|=|I|}} \langle f, \varphi_I\rangle\langle T\varphi_{I'}, \psi_I\rangle\Big|^2 \le C\|f\|^2. \tag{4.11}$$

Let us find a necessary condition for (4.11). Let J be a dyadic interval and let $f = \chi_J$, the characteristic function on J. Then, (4.11) implies that

$$\sum_{I\cap J\neq\emptyset} \Big| \sum_{\substack{I'\\ |I'|=|I|}} \langle \chi_J, \varphi_I\rangle\langle T\varphi_{I'}, \psi_I\rangle\Big|^2$$

$$= \sum_{I\subset J} |I|\Big| \sum_{\substack{I'\\ |I'|=|I|}} \langle T\varphi_{I'}, \psi_I\rangle\Big|^2 + \sum_{J\subsetneq I} |I|^{-1}|J|^2\Big| \sum_{\substack{I'\\ |I'|=|I|}} \langle T\varphi_{I'}, \psi_I\rangle\Big|^2$$

$$\le C|J|.$$

That the second term in the second line is less that $C|J|$ follows from (4.3). So, a necessary condition for (4.11) is

$$\sum_{I\subset J} |I| \, \Big| \sum_{\substack{I' \\ |I'|=|I|}} \langle T\varphi_{I'}, \psi_I\rangle \Big|^2 \le C|J|. \tag{4.12}$$

We will show that (4.12) is, in fact, sufficient. We first show the following.

Lemma 4.6. *The condition* (4.12) *is equivalent to the condition to* $T1 \in BMO$.

Proof. A proof can be found in [**BCR**]. But we reproduce it here. Note that

$$\sum_{\substack{I' \\ |I'|=|I|}} T\varphi_{I'} = |I|^{-1/2} T1.$$

Hence, (4.12) is equivalent to

$$|J|^{-1} \sum_{I\subset J} \big|\langle T1, \psi_I\rangle\big|^2 \le C. \tag{4.13}$$

Put

$$m_J = |J|^{-1} \int_J (T1).$$

Then, since $\int \psi_I = 0$, (4.13) is equivalent to

$$|J|^{-1} \sum_{I\subset J} \big|\langle T1 - m_J, \psi_I\rangle\big|^2 \le C$$

which is

$$|J|^{-1} \int_J |T1(x) - m_J|^2 dx \le C$$

since $\{\psi_I\}_{I\subset J}$ is an orthonormal basis for $L^2(J)$. This completes the proof. □

Proof of Lemma 4.5. Define an indicator function $\theta(I,t)$ by

$$\theta(I,t) = \begin{cases} 1 & \text{if } 0 < t < |I|^{-1/2} |\langle f, \varphi_I\rangle|, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\sum_I |\langle f,\varphi_I\rangle|^2| \sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2$$
$$= 2\int_0^\infty t\sum_I \theta(I,t)|I|\ |\sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2 dt.$$

Let $E_t = \cup\{ I : |I|^{-1/2}|\langle f,\varphi_I\rangle| > t\}$ and let $\{I_k\}$ be the maximal dyadic intervals in E_t. Then, by (4.11),

$$\sum_I \theta(I,t)|I|\ |\sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2 \le \sum_{I\subset E_t} |I||\sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2$$
$$\le \sum_k \sum_{I\subset I_k} |I||\sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2$$
$$\le C\sum_k |I_k| \le C|E_t|.$$

Note that $E_t \subset \{ x : Mf(x) > t \}$ where

$$Mf(x) = \sup_{x\in I}\frac{1}{|I|}\int_I |f(y)|dy,$$

the dyadic Hardy-Littlewood maximal function. Therefore, we have

$$\sum_I |\langle f,\varphi_I\rangle|^2| \sum_{\substack{I'\\|I'|=|I|}} \langle T\varphi_{I'},\psi_I\rangle|^2$$
$$\le \int_0^\infty t|E_t|dt \le C\int |Mf(x)|^2 dt \le C\|f\|^2$$

This completes the proof.

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A REPRESENTATION FOR THE SECOND DERIVATIVE OF ENERGY DENSITY FUNCTION IN NONLINEAR HOMOGENIZATION

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ABSTRACT. Using the representation formula, the Hölder continuity of the second derivative of the homogenized energy density function in no nlinear homogenization is proved.

1. INTRODUCTION

In this paper we consider the homogenization problems and compactness. For given ε, define

$$I^{\varepsilon}(u) = \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u : \Omega \to \mathbb{R}^n$ is a vector valued function. We assume that $W : \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}$ is measurable and periodic on $Y = [0,1]^N$ with respect to its first variable and strongly convex on its second variable. We know that as $\varepsilon \to 0$, I^{ε} converges to the homogenized functional

$$I[u] = \int_{\Omega} \overline{W}(\nabla u) dx \tag{1.2}$$

in the sense of $\Gamma - convergence$, where $\overline{W}$ is the homogenized energy density function.

Here we want to show that the homogenized energy density $\overline{W}$ is twice differentiable and its second derivatives are represented by an integral formula. In the linear case, that is,

$$W(y, Q) = A^{ij}_{\alpha,\beta} Q^i_{\alpha} Q^j_{\beta}$$

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for some matrix A, we know that $\overline{W}$ is characterized by the corrector matrix(see Tartar[Tar]). Here we find a similar corrector term for nonlinear cases which will compensate the discrepancy between strong convergence and weak convergence.

The characterization of the homogenized energy density function has been solved by Marcellini([Mar]) when W is convex and of polynomial growth with respect to the deformation gradient. Müller([Mul]) has considered the case that W is not necessarily convex and satisfies a polynomial growth condition. He found a representation formula for $\overline{W}$ similar to Marcellini. We also note that Weinan([Wei]) considered the case that W depends on u, that is, $I^\varepsilon[u]$ is of the form

$$I^\varepsilon[u] = W\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, \nabla u\right).$$

The characterization of the second derivative of the homogenized energy density function $\overline{W}$ is closely related to bounding the effective moduli of composite materials. For instance the effective conductivity tensor A is modeled by the variational principle

$$\langle A\xi, \xi\rangle = \inf_{\phi} \frac{1}{|Y|}\int_Y a|\xi + \nabla\phi|^2 dy \tag{1.3}$$

for any $\xi \in \mathbb{R}^n$, where ϕ ranged over $Y-$ periodic functions and

$$a(y) = \alpha\chi_\alpha + \beta\chi_\beta.$$

We note that

$$A = \frac{\partial^2 \overline{W}}{\partial Q \partial Q}.$$

It is rather an important and interesting question to bound the effective conductivity tensor for nonlinear cases. We shall consider this question for nonlinear cases in the forthcoming papers.

NOTATIONS AND PRELIMINARIES

After De Giorgi introduced the $\Gamma -$ *convergence*, it has been applied to many questions such as convex relaxation, homogenization, optimizations and so forth. We review briefly about the $\Gamma -$ *convergence* and $\Gamma -$ *limits*.

Let $\{I^\varepsilon\}_{\varepsilon>0}$ be a family of functionals on the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$. I^ε converges to a functional I on $W^{1,p}(\Omega)$ in the sense of $\Gamma -$ *convergence* if

i) (lower semicontinuity) for every sequence $\{u^\varepsilon\}_{\varepsilon>0}$ that converges to u in the weak topology of $W^{1,p}(\Omega)$ we have

$$\liminf_{\varepsilon\to 0} I^\varepsilon[u^\varepsilon] \geq I[u], \tag{2.1}$$

ii) (realization) for every $u \in W^{1,p}(\Omega)$, there is a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ such that $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon\to 0} I^\varepsilon[u^\varepsilon] = I[u]. \tag{2.2}$$

The topology of $W^{1,p}(\Omega)$ for $\Gamma -$ *convergence* is the weak topology of $W^{1,p}(\Omega)$ and $\rightharpoonup$ means the weak convergence. The following lemma is useful for the calculus of variation problems.

Lemma 2.1. *Suppose that $\{I^\varepsilon\}_{\varepsilon>0}$ is $\Gamma -$ convergent to I and g is a continuous linear function on $W^{1,p}(\Omega)$. Assume $\{u^\varepsilon\} \subset W^{1,p}(\Omega)$ satisfy*

$$I^\varepsilon[u^\varepsilon] + g(u^\varepsilon) < \inf\left\{I^\varepsilon[u] + g(u) : u \in W^{1,p}(\Omega)\right\} + \varepsilon. \tag{2.3}$$

Moreover if $\{u^{\varepsilon_k}\}$ is a weakly convergent subsequence of $\{u^\varepsilon\}$ and $u^{\varepsilon_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$ as $\varepsilon_k \to 0$, then

$$I[u] + g[u] \leq I[v] + g[v] \text{ for all } v \in W^{1,p}(\Omega) \tag{2.4}$$

$$\min\left\{I[u] + g[u] : u \in W^{1,p}(\Omega)\right\} = \liminf_{\varepsilon\to 0}\left\{I^\varepsilon[u] + g[u] : u \in W^{1,p}(\Omega)\right\}. \tag{2.5}$$

For a proof for lemma 2.1 see Attouch([Att]).

For the composite materials the procedure from (1.1) to (1.2) is named as homogenization and the $\Gamma -$ *convergence* idea follows passing to the limits. Thus it is important to characterize the $\Gamma -$ *limit* I. Several authors are successful in finding the representation formula for I. The following result is due to Marcellini([Mar]).

Lemma 2.2. *(Marcellini) Suppose that $W(x,Q)$ is convex and has polynomial growth condition with respect to Q. Then $I^\varepsilon[u]$ is Γ – convergent to*

$$I[u] = \int_\Omega \overline{W}(\nabla u)dx \tag{2.6}$$

with

$$\overline{W}(Q) = \inf_{\psi \in W^{1,p}_{per}(Y)} \int_Y W(y, Q + \nabla\psi)dy \tag{2.7}$$

where $W^{1,p}_{per}(Y)$ is the Sobolev space $W^{1,p}(Y)$ with periodic boundary data.

When the energy density function W is not convex, Müller ([Mul]) and Weinan ([Wei]) found an integral representation formula similar to (2.7). The integral representation formula is closely related to the relaxation problem in the calculus of variations as is shown by Dacorogna([Da]). In particular Müller proved the following lemma in ([Mul]).

Lemma 2.3. *Suppose that $W : \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}$ is a locally Lipschitz function and satisfies:*

$$a|Q|^p \le W(y,Q) \le b(1 + |Q|^p), \ a > 0, \ 1 < p < \infty$$

$$|W(y,P) - W(y,Q)| \le C\left(1 + |P|^{p-1} + |Q|^{p-1}\right)|P - Q|.$$

Assume Ω is a bounded $C^{0,1}$ domain. Then $\{I^\varepsilon\}_{\varepsilon>0}$ given by (1.1) is Γ – convergent to

$$I[u] = \int_\Omega \overline{W}(\nabla u)dx, \tag{2.8}$$

where

$$\overline{W}(Q) = \inf_{k\in\mathbb{N}} \inf_{\psi\in W^{1,p}_0(kY)} \frac{1}{k^n}\int_{kY} W(y, Q + \nabla\psi)dy. \tag{2.9}$$

In the next section using the characterization lemma 2.2 by Marcellini([Mar]) we prove a representation theorem for the second derivative of homogenized energy density function $\overline{W}$.

As usual the double indices mean summation and if there is no confusion, we omit x_0 in various symbols. In particular the following symbols are used.

$$\mathbb{Z} : \text{ the set of integers}$$

$$Y = [0,1]^n$$

$$B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$$

$$D(x,r) = B(x,r) \cap \Omega$$

$$W_{Q^i_\alpha}(y,Q) = \frac{\partial W}{\partial Q^i_\alpha}(y,Q)$$

$$W_Q(y,Q) = \left[W_{Q^i_\alpha}\right]$$

$$W_Q \cdot \xi = W_{Q^i_\alpha} \xi^i_\alpha$$

$$W_{QQ}\xi\xi = W_{Q^i_\alpha Q^j_\beta} \xi^i_\alpha \xi^j_\beta$$

$$\partial\Omega : \text{ boundary of } \Omega$$

$$|E| : \text{ the Lebesgue measure of } E$$

$$(f)_{B(x,r)} = \frac{1}{|B(x,r)|} \int f(y)dy$$

$$x_0 : \text{ generic point}$$

3. Representation for the second derivative of $\overline{W}$

In this section we characterize the second derivative of $\overline{W}$ using the integral representation formula derived by Marcellini (see lemma 2.2). We assume that $W(y,Q)$ is measurable with respect to y and continuous with respect to Q. Furthermore we assume that W satisfies the following :

i) (Periodicity)

$$W(y+z,Q) = W(y,Q), \text{ for all } Q \in \mathbb{R}^{nN}, y \in \mathbb{R}^n \text{ and } z \in \mathbb{Z}^n. \tag{3.1}$$

ii) (Strong convexity)

$$\lambda|\xi|^2 \le W_{QQ}(y,Q)\xi\xi \le \lambda^{-1}(|\xi|^2+1) \tag{3.2}$$

for some $\lambda > 0$, for all $y \in \mathbb{R}^n$ and for all $Q, \xi \in \mathbb{R}^{nN}$.

iii) (Continuity)

$$|W_{QQ}(y,Q_1) - W_{QQ}(y,Q_2)| \le c|Q_1 - Q_2|^\mu \tag{3.3}$$

for some $\mu > 0$, for all $y \in \mathbb{R}^n$ and for all $Q_1, Q_2 \in \mathbb{R}^{nN}$.

Hence from lemma 2.2,

$$I^{\varepsilon}[u] = \int_{\Omega} W\left(\frac{x}{\varepsilon}, \nabla u\right) dx \tag{3.4}$$

is Γ − *convergent* to

$$I[u] = \int_{\Omega} \overline{W}(\nabla u) dx, \tag{3.5}$$

where

$$\overline{W}(Q) = \inf_{\psi \in W^{1,2}_{per}(Y)} \int_{Y} W(y, Q + \nabla\psi) dy. \tag{3.6}$$

Since W is strongly convex, we see immediately that there is a unique minimizer $\psi \in W^{1,2}_{per}(Y)$ for the functional

$$I^{Q}[\psi] = \int_{Y} W(y, Q + \nabla\psi) dy$$

for each Q. We denote ψ^Q by the unique minimizer ψ for $I^Q[\psi]$. Employing the integral representation of $\overline{W}$ and using the Euler-Lagrange equation for the minimizer ψ^Q, we prove that $\overline{W} : \mathbb{R}^{nN} \to \mathbb{R}$ is differentiable.

Theorem 3.1. *The homogenized energy density function $\overline{W}$ is differentiable everywhere and*

$$\overline{W}_Q(Q) = \int_{Y} W_Q(y, Q + \nabla\psi^Q) dy. \tag{3.7}$$

proof. We set

$$A(Q) = \int_{Y} W_Q(y, Q + \nabla\psi^Q) dy.$$

and naturally we expect that A is the derivative of $\overline{W}$. To prove that $A(Q) = \overline{W}_Q(Q)$, we need only to show that for each $H \in \mathbb{R}^{nN}$ with $|H| = 1$,

$$\frac{1}{\varepsilon}|\overline{W}(Q + \varepsilon H) - \overline{W}(Q) - \varepsilon A \cdot H| = o(\varepsilon). \tag{3.8}$$

Since ψ^Q is a minimizer of

$$\int_Y W(y, Q + \nabla\psi)dy,$$

ψ^Q satisfies the Euler- Lagrange equation

$$(W_{Q^i_\alpha}(y, Q + \nabla\psi^Q))_{y_\alpha} = 0, \ i = 1, ..., N \tag{3.9}$$

with respect to the periodic test function class $W^{1,2}_{per}(Y)$. Similarly we have

$$(W_{Q^i_\alpha}(y, Q + \varepsilon H + \nabla\psi^{Q+\varepsilon H}))_{y_\alpha} = 0, \ i = 1, ..., N$$

with respect to the periodic test function class $W^{1,2}_{per}(Y)$. Since $\psi^{Q+\varepsilon H} - \psi^Q \in W^{1,2}_{per}(Y)$, we obtain

(3.10)

$$\int_Y [W_{Q^i_\alpha}(y, Q+\varepsilon H+\nabla\psi^{Q+\varepsilon H}) - W_{Q^i_\alpha}(y, Q+\nabla\psi^Q)] \left(\psi^{Q+\varepsilon H,i} - \psi^{Q,i}\right)_{y_\alpha} dy = 0.$$

By the mean value theorem we have

$$W_{Q^i_\alpha}(y, Q + \varepsilon H + \nabla\psi^{Q+\varepsilon H}) - W_{Q^i_\alpha}(y, Q + \nabla\psi^Q)$$

$$= \int_0^1 W_{Q^i_\alpha Q^j_\beta}\left(y, Q+\nabla\psi^Q+t\left(\varepsilon H+\nabla\psi^{Q+\varepsilon H}-\nabla\psi^Q\right)\right) dt \cdot \left(\varepsilon H^j_\beta + \psi^{Q+\varepsilon H,j}_{y_\beta} - \psi^{Q,j}_{y_\beta}\right)$$
$$= B^{ij}_{\alpha\beta}(y, Q, \varepsilon H)\left(\varepsilon H^j_\beta + \psi^{Q+\varepsilon H,j}_{y_\beta} - \psi^{Q,j}_{y_\beta}\right),$$

where B is defined by

$$B^{ij}_{\alpha\beta}(y, Q, \varepsilon H) = \int_0^1 W_{Q^i_\alpha Q^j_\beta}(y, Q + \nabla\psi^Q + t(\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q))dt.$$

So using the strong convexity condition (3.2) we have

$$\lambda \int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy$$

$$\leq \int_Y B^{ij}_{\alpha\beta}\left(\psi^{Q+\varepsilon H,i}_{y_\alpha} - \psi^{Q,i}_{y_\alpha}\right)\left(\psi^{Q+\varepsilon H,j}_{y_\beta} - \psi^{Q,j}_{y_\beta}\right) dy$$

$$= -\varepsilon \int_Y B^{ij}_{\alpha\beta} H^j_\beta \left(\psi^{Q+\varepsilon H,i}_{y_\alpha} - \psi^{Q,i}_{y_\alpha}\right) dy$$
$$\leq C\varepsilon \int_Y |H| \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right| dy.$$

Hence Young's inequality gives

$$\int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy \leq C\varepsilon^2 \tag{3.11}$$

for some $C > 0$.

On the other hand we have

$$|\overline{W}(Q+\varepsilon H) - \overline{W}(Q) - \varepsilon A \cdot H| \tag{3.12}$$

$$= \left|\int_Y W(y, Q+\varepsilon H+\nabla\psi^{Q+\varepsilon H}) - W(y, Q+\nabla\psi^Q)dy - \int_Y W_Q(y, Q+\nabla\psi^Q)\cdot \varepsilon H dy\right|$$

$$= \Bigg|\int_Y \int_0^1 W_Q\left(y, Q+\nabla\psi^Q + t\left(\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right)\right) dt$$
$$\cdot \left(\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right) dy - \int_Y W_Q(y, Q+\nabla\psi^Q)\cdot \varepsilon H dy\Bigg|$$

$$\leq \int_0^1 dt \int_Y \left|W_Q(y, Q+\nabla\psi^Q + t\left(\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right)) - W_Q(y, Q+\nabla\psi^Q)\right|$$
$$\left|\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right| dy + \left|\int_Y W_Q(y, Q+\nabla\psi^Q)\cdot(\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q)dy\right|.$$

Since ψ^Q satisfies the Euler-Lagrange equation and $\psi^{Q+\varepsilon H} - \psi^Q \in W^{1,2}_{per}(\Omega)$, we see that

$$\int_Y W_Q(y, Q+\nabla\psi^Q)\cdot(\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q)dy = 0 \tag{3.13}$$

Therefore from (3.11) and (3.12) we conclude that

$$\begin{aligned} &|\overline{W}(Q+\varepsilon H) - \overline{W}(Q) - \varepsilon A \cdot H| \\ &\leq C\left(|\varepsilon H|^2 + \int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy\right) \\ &\leq C\varepsilon^2 \end{aligned} \tag{3.14}$$

and this completes the proof.

Now we proceed to prove that $\overline{W}$ is twice differentiable. This is crucial to get regularity results for minimizers of functional. The following lemma is useful for the characterization of the second derivative of $\overline{W}$.

Lemma 3.1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and $\{u^\varepsilon\}_{\varepsilon>0}$ is a sequence in $L^2(\Omega)$ such that $u^\varepsilon \to u$ strongly in L^2. Then we have*

$$\lim_{\varepsilon\to 0}\int_\Omega f(u^\varepsilon)|u^\varepsilon|^2 dx = \int_\Omega f(u)|u|^2 dx. \tag{3.15}$$

proof. Let $\eta_0 > 0$. Since $|u|^2$ is integrable, there is $\theta_0 > 0$ such that if $|E| \le \theta_0$, then $\int_E |u|^2 dx \le \eta_0$. Since u^ε converge u strongly in L^2, there exist ε_0 and μ_0 such that

$$|\{x\in\Omega : |u^\varepsilon(x)| \ge \mu_0\}| \le \theta_0$$

for all $\varepsilon < \varepsilon_0$ and

$$|\{x\in\Omega : |u(x)| \ge \mu_0\}| \le \theta_0.$$

Hence we get

$$\int_{\{|u^\varepsilon|\ge\mu_0\}\cup\{|u|\ge\mu_0\}} |f(u^\varepsilon)-f(u)|\,|u|^2\,dx \le c\eta_0$$

for some c depending on $\sup|f|$.

Since f is uniformly continuous on $[-\mu_0,\mu_0]$, there exists δ_0 such that if $|u^\varepsilon - u| \le \delta_0$, then

$$|f(u^\varepsilon) - f(u)| \le \eta_0.$$

We also note that since u^ε converge to u strongly in L^2, there exists ε_0 such that

$$|\{x\in\Omega : |u^\varepsilon(x) - u(x)| \ge \delta_0\}| \le \theta_0$$

for each $\varepsilon < \varepsilon_0$. Hence we obtain that for all $\varepsilon \le \varepsilon_0$

$$\int_\Omega |f(u^\varepsilon) - f(u)|\,|u^2|\,dx$$

$$\le \int_{\{|u^\varepsilon|\ge\mu_0\}\cup\{|u|\ge\mu_0\}} |f(u^\varepsilon) - f(u)|\,|u^2|\,dx$$

$$+ \int_{\{|u^\varepsilon-u|\le\delta_0,|u^\varepsilon|\le\mu_0,|u|\le\mu_0\}} |f(u^\varepsilon) - f(u)|\,|u|^2\;dx$$

$$+ \int_{\{|u^\varepsilon-u|\ge\delta_0\}} |f(u^\varepsilon) - f(u)|\,|u|^2\;dx$$

$$\leq c\eta_0$$

for some c. Therefore if ε_0 is small enough, we see that for each $\varepsilon \leq \varepsilon_0$

$$\left| \int_\Omega f(u^\varepsilon)|u^\varepsilon|^2 - f(u)|u|^2 dx \right|$$

$$\leq \int_\Omega |f(u^\varepsilon)| \left| |u^\varepsilon|^2 - |u|^2 \right| dx + \int_\Omega |f(u^\varepsilon) - f(u)| \, |u|^2 dx$$

$$\leq c\eta_0$$

for some c and this completes the proof.

Let $Q \in \mathbb{R}^{nN}$. Let us define $w^{(k,\gamma)} \in W^{1,2}_{per}(Y \to \mathbb{R}^N)$ to be the solution to

$$\left(W_{Q^i_\alpha Q^j_\beta}(y, Q + \nabla\psi^Q) w^{(k,\gamma),j}_{y_\beta} \right)_{y_\alpha} = -\left(W_{Q^i_\alpha Q^k_\gamma}(y, Q + \nabla\psi^Q) \right)_{y_\beta} \tag{3.16}$$

in Y with periodic boundary data. In the linear cases $w^{(k,\gamma)}$ is the corrector matrix as is discussed by Tartar(see [Tar]). Hence we may consider $w^{(k,\gamma)}$ as a corrector term for nonlinear cases. Next we define

$$A^{ij}_{\alpha\beta}(Q) = \int_Y W_{Q^i_\alpha Q^k_\gamma}(y, Q + \nabla\psi^Q) \left[\delta^{jk}\delta_{\beta\gamma} + w^{(k,\gamma),j}_{y_\beta} \right] dy$$

for each $Q \in \mathbb{R}^{nN}$, where δ is the Kronecker delta function. The following theorem is our main result in this section.

Theorem 3.2. *$\overline{W}$ is twice differentiable everywhere in $\mathbb{R}^{nN}$ and*

$$\overline{W}_{QQ}(Q) = A(Q) \tag{3.17}$$

for each $Q \in \mathbb{R}^{nN}$.

proof. As in the proof of theorem 3.1 we need only to show that for each $Q \in \mathbb{R}^{nN}$ and $H \in \mathbb{R}^{nN}$ with $|H| = 1$

$$\frac{1}{\varepsilon} \left| \overline{W}_Q(Q + \varepsilon H) - \overline{W}_Q(Q) - \varepsilon A(Q) \cdot H \right| = o(\varepsilon).$$

From theorem 3.1 we know that

$$\overline{W}_Q(Q + \varepsilon H) = \int_Y W_Q(y, Q + \varepsilon H + \nabla\psi^{Q+\varepsilon H}) dy$$

and

$$\overline{W}_Q(Q) = \int_Y W_Q(y, Q + \nabla\psi^Q)dy.$$

Now define

(3.18)

$$B^{ij}_{\alpha\beta}(y, Q, \varepsilon H) = \int_0^1 W_{Q^i_\alpha Q^j_\beta}\left(y, Q+\nabla\psi^Q+t\left(\varepsilon H+\nabla\psi^{Q+\varepsilon H}-\nabla\psi^Q\right)\right) dt.$$

Hence from the mean value theorem we get

$$\overline{W}_Q(Q+\varepsilon H) - \overline{W}_Q(Q)$$

$$= \int_Y W_Q(y, Q+\varepsilon H + \nabla\psi^{Q+\varepsilon H}) - W_Q(y, Q+\nabla\psi^Q)dy$$

$$= \int_Y B(y, Q, \varepsilon H)\cdot(\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q)dy.$$

From the definition of $A(Q)$, we have

$$\left|\overline{W}_Q(Q+\varepsilon H) - \overline{W}_Q(Q) - \varepsilon A(Q)\cdot H\right| \tag{3.19}$$

$$\le \left|\int_Y [B(y,Q,\varepsilon H) - B(y,Q,0)]\cdot\left[\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right] dy\right|$$

$$+ \left|\int_Y B(y,Q,0)\cdot\left[\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \nabla w\cdot\varepsilon H\right] dy\right|$$

$$= I + II.$$

Since $W_{QQ}(y, Q)$ is in C^μ with respect to Q, we have

$$|B(y,Q,\varepsilon H) - B(y,Q,0)| \le c\left|\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^\mu$$

for some c. From (3.11) we already know that

$$\int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy \le c\varepsilon^2$$

for some c. Hence from Young's inequality we have

$$|I| \le C|\varepsilon|^{1+\mu}.$$

Now we proceed to estimate II. Since $|W_{QQ}| \le M$, we see that

$$|II| \le c \int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \varepsilon\nabla w\cdot H\right| dy. \tag{3.20}$$

To estimate the right handside of (3.20), we recall that for each $i = 1, 2, ..., N$

$$\left[W_{Q^i_\alpha}(y, Q+\varepsilon H + \nabla\psi^{Q+\varepsilon H}) - W_{Q^i_\alpha}(y, Q+\nabla\psi^Q)\right]_{y_\alpha}$$

$$= \left[B^{ij}_{\alpha\beta}(y, Q, \varepsilon H)\cdot\left(\varepsilon H^j_\beta + \psi^{Q+\varepsilon H, j}_{y_\beta} - \psi^{Q,j}_{y_\beta}\right)\right]_{y_\alpha} = 0$$

and

$$\left[B^{ij}_{\alpha\beta}(y, Q, 0) w^{(k,\gamma),j}_{y_\beta}\varepsilon H^k_\gamma\right]_{y_\alpha} = -\left(B^{ij}_{\alpha\beta}(y, Q, 0)\varepsilon H^j_\beta\right)_{y_\alpha}.$$

Hence we have

$$\left[B^{ij}_{\alpha\beta}(y, Q, 0)\left(\psi^{Q+\varepsilon H, j}_{y_\beta} - \psi^{Q,j}_{y_\beta} - \varepsilon w^{(k,\gamma),j}_{y_\beta} H^k_\gamma\right)\right]_{y_\alpha}$$

$$= \left\{\left[B^{ij}_{\alpha\beta}(y, Q, 0) - B^{ij}_{\alpha\beta}(y, Q, \varepsilon H)\right]\left(\psi^{Q+\varepsilon H, j}_{y_\beta} - \psi^{Q,j}_{y_\beta}\right)\right\}_{y_\alpha}$$

$$+\varepsilon\left\{\left[B^{ij}_{\alpha\beta}(y, Q, 0) - B^{ij}_{\alpha\beta}(y, Q, \varepsilon H)\right] H^j_\beta\right\}_{y_\alpha}$$

for each i. Note that $\psi^{Q+\varepsilon H} - \psi^Q - \varepsilon w\cdot H \in W^{1,2}_{per}(Y)$. Hence taking $\psi^{Q+\varepsilon H} - \psi^Q - \varepsilon w\cdot H$ as a test function to (3.21), we have

$$\int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \varepsilon\nabla w\cdot H\right|^2 dy \tag{3.22}$$

$$\le \int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|\left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|\left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \varepsilon\nabla w\cdot H\right| dy$$

$$+\varepsilon\int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|\left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right| dy = III + IV.$$

Since $W_{QQ} \in C^\mu$, we have

$$|B(y,Q,\varepsilon H) - B(y,Q,0)| \le C\left|\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^\mu.$$

Since

$$\int_Y |\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q|^2 dy \le c\varepsilon^2,$$

we obtain that

$$
\begin{aligned}
IV &\leq \varepsilon \int_Y |\varepsilon H + \nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q|^\mu \ |\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q| \ dy \\
&\leq c\varepsilon^{2+\mu} + c\varepsilon \int_Y |\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q|^{1+\mu} dy \\
&\leq c\varepsilon^{2+\mu}
\end{aligned}
$$

for some c.

Finally we estimate III. By Young's inequality we have

$$
\begin{aligned}
III &\leq \frac{c}{\delta} \int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|^2 \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy \\
&\quad + \delta \int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \varepsilon\nabla w \cdot H\right|^2 dy
\end{aligned}
\tag{3.23}
$$

for any $\delta > 0$. Hence combining (3.22) and (3.23), we conclude that

$$
\begin{aligned}
&\int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q - \varepsilon\nabla w \cdot H\right|^2 dy \\
&\leq C \int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|^2 \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q\right|^2 dy + c\varepsilon^{2+\mu} \\
&= V + c\varepsilon^{2+\mu}.
\end{aligned}
\tag{3.24}
$$

To estimate V, we set v^ε to be

$$
v^\varepsilon = \frac{\psi^{Q+\varepsilon H} - \psi^Q}{\varepsilon}.
$$

We note that $\|v^\varepsilon\|_{W^{1,2}_{per}(Y)} < C$ independent of ε and v^ε satisfies

$$
\left(B^{ij}_{\alpha\beta}(y,Q,\varepsilon H) v^{\varepsilon,j}_{x_\beta}\right)_{x_\alpha} = -\left(B^{ij}_{\alpha\beta}(y,Q,\varepsilon H) H^j_\beta\right)_{x_\alpha}
\tag{3.25}
$$

in Y. Since W_{QQ} is continuous and bounded as a function of Q, we see easily that

$$
B^{ij}_{\alpha\beta}(y,Q,\varepsilon H) \to B^{ij}_{\alpha\beta}(y,Q,0)
$$

strongly in $L^2(Y)$. Let $v^\varepsilon \rightharpoonup v$ weakly in $W^{1,2}_{per}(Y)$. Then the weak limit v satisfies

$$(3.26) \qquad \left(B^{ij}_{\alpha\beta}(y,Q,0)v^j_{x_\beta}\right)_{x_\alpha} = -\left(B^{ij}_{\alpha\beta}(y,Q,0)H^j_\beta\right)_{x_\alpha}.$$

Now we claim that $\nabla v^\varepsilon \to \nabla v$ strongly in L^2. From (3.25) and (3.26) we have

$$(3.27) \qquad \left[B^{ij}_{\alpha\beta}(y,Q,\varepsilon H)\left(v^{\varepsilon,j}_{x_\beta} - v^j_{x_\beta}\right)\right]_{x_\alpha}$$

$$= \left[\left(B^{ij}_{\alpha\beta}(y,Q,0) - B^{ij}_{\alpha\beta}(y,Q,\varepsilon H)\right)\left(v^j_{x_\beta} + H^j_\beta\right)\right]_{x_\alpha}$$

and taking $v^\varepsilon - v$ as a test function in (3.27), we get

$$\int_Y |\nabla v^\varepsilon - \nabla v|^2 dy \le C\int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|^2\left(|\nabla v|^2 + 1\right) dy.$$

Note that B is bounded and

$$B(y,Q,\varepsilon H) - B(y,Q,0) \to 0 \text{ a.e. as } \varepsilon \to 0.$$

So from the dominated convergence theorem we have

$$\int_Y |B(y,Q,\varepsilon H) - B(y,Q,0)|^2\left(|\nabla v|^2 + 1\right) dy \to 0$$

as ε goes to zero and we conclude

$$\nabla v^\varepsilon \to \nabla v \text{ strongly in } L^2(Y).$$

Finally, let us set g to be the modulus of continuity of B at Q. Then from lemma 3.1,

$$(3.28) \qquad \begin{aligned} \frac{1}{\varepsilon^2}V &\le \int_Y g(\varepsilon|H| + \varepsilon|\nabla v^\varepsilon|)\left|\frac{\nabla\psi^{Q+\varepsilon H} - \nabla\psi^Q}{\varepsilon}\right|^2 dy \\ &= \int_Y g(\varepsilon|H| + \varepsilon|\nabla v^\varepsilon|)|\nabla v^\varepsilon|^2 dy = o(\varepsilon). \end{aligned}$$

Combining (3.24) and (3.28) gives

$$
\begin{aligned}
(3.29)\qquad & \int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^{Q} - \varepsilon\nabla w\cdot H\right| dy \\
&\le \left[\int_Y \left|\nabla\psi^{Q+\varepsilon H} - \nabla\psi^{Q} - \varepsilon\nabla w\cdot H\right|^2 dy\right]^{\frac{1}{2}} \\
&\le \left(\varepsilon^2 o(\varepsilon) + c\varepsilon^{2+\mu}\right)^{\frac{1}{2}} \le \varepsilon o(\varepsilon).
\end{aligned}
$$

Therefore from (3.19), (3.20) and (3.29) we conclude that

$$
\frac{1}{\varepsilon}\left|\overline{W}_Q(Q+\varepsilon H) - \overline{W}_Q(Q) - \varepsilon A(Q)\cdot H\right| = o(\varepsilon)
$$

and this completes the proof.

Now it is rather simple to prove that $\overline{W}$ is strongly convex. From the mean value theorem we have that

$$
\left|\overline{W}(Q+H) - \overline{W}(Q) - \overline{W}(Q)\cdot H\right|
$$

$$
\begin{aligned}
&= \left|\int_Y W(y, Q+H+\nabla\psi^{Q+H}) - W(y, Q+\nabla\psi^{Q}) - W_Q(y, Q+\nabla\psi^{Q})\cdot H dy\right| \\
&= \Big| \int_Y \int_0^1 W_Q\left(y, Q+\nabla\psi^{Q} + t(H+\nabla\psi^{Q+H} - \nabla\psi^{Q})\right) dt\cdot(H+\nabla\psi^{Q+H} - \nabla\psi^{Q}) \\
&\qquad - W_Q(y, Q+\nabla\psi^{Q})\cdot(H+\nabla\psi^{Q+H} + \nabla\psi^{Q}) dy\Big| \\
&\ge \Big|\int_Y\int_0^1 t\int_0^1 W_{Q^i_\alpha Q^j_\beta}\left(y, Q+\nabla\psi^{Q} + ts(H+\nabla\psi^{Q+H} - \nabla\psi^{Q})\right) ds\, dt \\
&\qquad (H^i_\alpha + \psi^{Q+H,i}_{y_\alpha} - \psi^{Q,i}_{y_\alpha})(H^j_\beta + \psi^{Q+H,j}_{y_\beta} - \psi^{Q,j}_{y_\beta})\, dy\Big| \\
&\qquad \ge \frac{1}{2}\lambda\int_Y \left|H+\nabla\psi^{Q+H} - \nabla\psi^{Q}\right| \\
&\qquad \ge \frac{\lambda}{2}|H|^2
\end{aligned}
$$

for all $Q, H \in \mathbb{R}^{nN}$, where we used the fact that ψ^{Q+H} and ψ^{Q} are both periodic. Hence we have the following lemma.

Lemma 3.2. *$\overline{W}$ is strongly convex and*

$$\overline{W}_{Q^i_\alpha Q^j_\beta}(Q)\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2$$

for all $Q, \xi \in \mathbb{R}^{nN}$.

Using the representation theorem for the second derivative of the homogenized energy density function, we prove the homogenized density function is $C^{2,\mu}$. Define $[W_{QQ}]_{C^\mu}$ as Hölder norm of W_{QQ} with respect to Q.

Theorem 3.3. *We have $\overline{W}_{QQ} \in C^\mu$ and*

$$[\overline{W}_{QQ}]_{C^\mu} \leq c[W_{QQ}]_{C^\mu}$$

for some c.

proof. We define $f_\alpha^{Q,(k,\gamma),i}$ by

$$f_\alpha^{Q,(k,\gamma),i} = W_{Q^i_\alpha, Q^j_\beta}(y, Q + \nabla\psi^Q)w_{y_\beta}^{Q,(k,\gamma),j} \in W^{1,2}_{per}(Y).$$

Considering (3.16), it immediately follows that

$$(f_\alpha^{Q,(k,\gamma),i})_{y_\alpha} = -(W_{Q^i_\alpha, Q^{k_\gamma}}(y, Q + \nabla\psi^Q))_{y_\alpha}.$$

Next we define $\phi^{Q,(k,\gamma)} \in W^{1,2}_{per}(Y)$ by

$$\Delta\phi = \text{ div } (f).$$

Again from (3.16) we have

$$(f_\alpha^{Q+\varepsilon H,(k,\gamma),i} - f_\alpha^{Q,(k,\gamma),i})_{y_\alpha} \tag{3.30}$$

$$= -\left(W_{Q^i_\alpha Q^k_\gamma}(y, Q + \varepsilon H + \nabla\psi^{Q+\varepsilon H}) - W_{Q^i_\alpha Q^k_\gamma}(y, Q + \nabla\psi^Q)\right)_{y_\alpha}.$$

Taking $\phi^{Q+\varepsilon H,(k,\gamma)} - \phi^{Q,(k,\gamma)}$ as a test function to (3.30) we have

$$\int_Y \left|f^{Q+\varepsilon H,(k,\gamma)} - f^{Q,(k,\gamma)}\right|^2 dy$$

$$\leq c\int_Y \left|W_{QQ}(y, Q + \varepsilon H + \nabla\psi^{Q+\varepsilon H}) - W_{QQ}(y, Q + \nabla\psi^Q)\right|^2 dy$$

$$\leq c \int_Y [W_{QQ}]^2_{C^\mu} \left(\varepsilon^{2\mu} + \left| \nabla \psi^{Q+\varepsilon H} - \nabla \psi^Q \right|^{2\mu} \right) dy$$

for some c. Using (3.11), we have

$$\int_Y \left| f^{Q+\varepsilon H} - f^Q \right|^2 dy \leq c\varepsilon^{2\mu}.$$

From the representation theroem 3.2 we obtain

$$\left| \overline{W}_{QQ}(Q + \varepsilon H) - \overline{W}_{QQ}(Q) \right|$$

$$\leq \int_Y \left| W_{QQ}(y, Q + \varepsilon H + \nabla \psi^{Q+\varepsilon H}) - W_{QQ}(y, Q + \nabla \psi^Q) \right| dy + \int_Y \left| f^{Q+\varepsilon H} - f^Q \right| dy$$

$$\leq c \int_Y [W_{QQ}]_{C^\mu} \left(\varepsilon^\mu + \left| \nabla \psi^{Q+\varepsilon H} - \nabla \psi^Q \right|^\mu \right) dy + c \left(\int_Y \left| f^{q+\varepsilon H} - f^Q \right|^2 dy \right)^{\frac{1}{2}}$$

$$\leq c \, [W_{QQ}]_{C^\mu} \, \varepsilon^\mu$$

for some c and this concludes the proof.

Now we consider a natuarl application. From the convexity and smoothness of homogenized energy density function we have a partial regularity result for the weak limit of minimizers(see [Gia]).

Corollary 3.4. *Suppose u^ε is a minimizer of I^ε. Since $\overline{W}$ is in $C^{2,\mu}$, the weak limit u of u^ε is $C^{1,\alpha}(\Omega \setminus E)$ for some α, where the exceptional set E is closed and its Lebesgue measure is zero.*

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WEAKLY CONTINUOUS HOLOMORPHIC FUNCTIONS ON BANACH SPACES

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ABSTRACT. Let E be an infinite dimensional complex Banach space. Let $H_w(E)$ be the space of entire complex-valued functions on E which are weakly continuous when restricted to any bounded subset of E. Each $f \in H_w(E)$ is characterized in terms of its differential $\hat{d}f$.

Let E be a complex Banach space. For $x \in E$ and $r > 0$ $B_r(x)$ denotes the open ball with center x and radius r. Let $H_w(E)$ ($H_{wu}(E)$) be the space of entire complex-valued functions on E which are weakly (uniformly) continuous when restricted to any bounded subset of E. Let $H_{lwu}(E)$ be the space of entire complex-valued functions on E such that for each $x \in E$, there is a neighborhood of x such that f is weakly uniformly continuous on V. It was known that $H_{wu}(E) \subset H_w(E) \subset H_{lwu}(E)$, but it remains still an open problem whether $H_{wu}(E)$ is a proper subspace of $H_w(E)$ or not, which was suggested in [A-H-V]. In connection with this problem, we would like to characterize each function f in terms of its differential $\hat{d}f$. Following such characterization of f in $H_{wu}(E)$ or $H_{lwu}(E)$ was studied by Aron [A1, A2]. For general background on holomorphic functions we refer to [D] and [M].

Proposition A. *$f \in H_{wu}(E)$ if and only if $\hat{d}f(B)$ is relatively compact in E' for every bounded subset B of E.*

Proposition B. *$f \in H_{lwu}(E)$ if and only if for each $x \in E$ there exists $r > 0$ such that $\hat{d}f(B_r(x))$ is relatively compact in E'.*

Recall that for a complex Banach space F, $H_K(E,F)$ is the space of all entire mappings $f : E \to F$ satisfying that for each $x \in E$ there is a neighborhood U such that $F(U)$ is relatively compact in F. For details see [A-S]. Hence Proposition B means that $f \in H_{lwu}(E)$ if and only if $\hat{d}f \in H_K(E, E')$.

For each $n \in \mathbf{N}$ let $P(^nE;F)$ denote the Banach space of all continuous n-homogeneous polynomials from E into F. Let $P_{wu}(E;F)$ be the space of all continuous n-homogeneous polynomials P such that for any bounded subset B of E, P is weakly uniformly continuous on B, and $P_K(^nE;F)$ be the space of all continuous n-homogeneous polynomials P such that $P(B_1(O))$ is relatively compact in F. The following results are also due to Aron [A1, A2].

Proposition 1. *Let $P \in P(^nE)$. Then the following are equivalent.*
(a) $P \in P_{wu}(E)$ (b) $\hat{d}p \in P_{wu}(^{n-1}E;E')$ (c) $\hat{d}p \in P_K(^{n-1}E;E')$

Proposition 2. *$f \in H_{lwu}(E)$ if and only if $\hat{d}^n f(O) \in P_{wu}(^nE)$ for each $n \in \mathbf{N}$ (equivalently, $\hat{d}^n f(x) \in P_{wu}(^nE)$ for each $x \in E$ and $n \in \mathbf{N}$).*

Compared with Proposition A and Proposition B we have the following result about $f \in H_w(E)$.

Theorem 3. *$f \in H_w(E)$ if and only if given $r > 0$ and $x \in B_r(O)$, there exists a weak neighborhood W of O such that $\hat{d}f(B_r(O) \cap (x+W))$ is relatively compact in E'.*

Proof. ($\Rightarrow$) Let $r > 1$ and $x \in B_r(O)$ be given. Since $f \in H_w(E)$, there exist a convex balanced weak neighborhood V of O and $M > 0$ such that $|f(y)| \leq M$ for all $y \in (x+V) \cap B_{6r}(O)$. Let W be a convex balanced weak neighborhood of O with $W + W + W \subset V$. From the Taylor series of f at x, we have that for $y \in E$,

$$\begin{aligned}\hat{d}f(y) &= \sum_{n=1}^{\infty} \hat{d}\left(\frac{\hat{d}^n f(x)}{n!}\right)(y-x) \\ &= \sum_{n=1}^{\infty} nA_n(y-x, y-x, \cdots, y-x, \cdot),\end{aligned}$$

where $\hat{A}_n = \frac{\hat{d}^n f(x)}{n!}$. Let $0 < \alpha < 1$ such that $\alpha B_1(O) \subseteq W$.

For $y \in (x+W) \cap B_r(O)$,

$$\begin{aligned}
&\|\hat{d}(\frac{\hat{d}^n f(x)}{n!})(y-x)\| \\
&= \sup_{\|t\|\leq 1} |nA_n(y-x,\cdot,y-x,t)| \\
&= \sup_{\|t\|\leq 1} |\frac{n}{\alpha}A_n(y-x,\cdots,y-x,\alpha t)| \\
&= \sup_{\|t\|\leq 1} |\frac{1}{\alpha}(\frac{1}{2\pi i})^n \int_{\substack{|\lambda_1|=2\\|\lambda_2|=1}} \frac{f(x+\lambda_1(y-x)+\lambda_2(\alpha t))}{\lambda_1^n\lambda_2^2} d\lambda_1 d\lambda_2| \\
&\leq \frac{1}{\alpha}\cdot(\frac{1}{2\pi})^n \cdot \frac{(4\pi)(2\pi)M}{2^n},
\end{aligned}$$

because $\|x+\lambda_1(y-x)+\lambda_2(\alpha t)\| \leq 6r$ and $\lambda_1(y-x)+\lambda_2(\alpha t) \in V$. Hence $\hat{d}f(y) = \sum_{n=1}^{\infty} d(\frac{\hat{d}^n f(x)}{n!})(y-x)$ converges uniformly on $(x+W)\cap B_r(O)$. $(\star)$ Since $\frac{\hat{d}^n f(x)}{n!} \in P_{wu}({}^nE)$, $\hat{d}(\frac{\hat{d}^n f(x)}{n!}) \in P_K({}^{n-1}E, E')$ by Proposition 1 and hence $\hat{d}(\frac{\hat{d}^n f(x)}{n!})((x+W)\cap B_r(O))$ is relatively compact in E' for each n. $(\star\star)$ ¿From $(\star)$, $(\star\star)$, we can show that $df((x+W)\cap B_r(O))$ is totally bounded in E'.

$(\Leftarrow)$ Let $r>0$ and $x \in B_r(O)$. By hypothesis, there exists a convex balanced weak neighborhood V of O such that $\hat{d}f((x+V)\cap B_{5r}(O))$ is relatively compact in E'. Then f is bounded on $(x+V)\cap B_{5r}(O)$, because for $y \in B_{5r}(O)\cap(x+V)$,

$$|f(y)-f(x)| \leq \sup_{c\in[x,y]} \|\hat{d}f(c)\|\|x-y\|.$$

Let W be a convex balanced weak neighborhood of O with $W+W \subset V$. For $y \in (x+W)\cap B_r(O)$,

$$\begin{aligned}
|\frac{\hat{d}^n f(x)}{n!}(y-x)| &= |\frac{1}{2\pi i}\int_{|\lambda|=2} \frac{f(x+\lambda(y-x))}{\lambda^{n+1}} d\lambda| \\
&\leq \frac{4\pi}{2\pi}\cdot\frac{1}{2^{n+1}}\cdot\max_{|\lambda|=2}|f(x+\lambda(y-x))| \\
&\leq \frac{M}{2^n},
\end{aligned}$$

where $M = \sup\{|f(y)| : y \in (x+V) \cap B_{5r}(O)\} < +\infty$. Given $\epsilon > 0$, Choose N such that $\sum_{N+1}^{\infty} \frac{M}{2^n} < \frac{\epsilon}{3}$. ¿From hypothesis, we have that $df \in H_K(E, E')$ and hence $\hat{d}(\frac{\hat{d}^n f(x)}{n!}) \in P_K({}^{n-1}E, E')$ for each n by Proposition 1 and Proposition 2. Let $K = \cup_{j=1}^{N}\{A_n(y-x, \cdots, y-x, \cdot) \,|\, y \in (x+W) \cap B_r\} \subset E'$, where $\hat{A}_n = \frac{\hat{d}^n f(x)}{n!}$. Then K is relatively compact in E' and so there exist $\phi_1, \cdots, \phi_k$ in E' such that $\cup_{i=1}^{k} B_{\epsilon/6Nr}(\phi_i) \supset K$. If $y \in (x+W) \cap B_r(O)$ and $|\phi_i(y-x)| < \epsilon/3N$ $(i = 1, \cdots, k)$, then

$$\begin{aligned}
|f(y) - f(x)| &\leq |\sum_{n=1}^{N} \frac{\hat{d}^n f(x)}{n!}(y-x)| + |\sum_{N+1}^{\infty} \frac{\hat{d}^n f(x)}{n!}(y-x)| \\
&\leq \sum_{n=1}^{N}(|(A_n(y-x, \cdot, y-x, \cdot) - \phi_i)(y-x)| + |\phi_i(y-x)|) + \frac{\epsilon}{3} \\
&\leq \sum_{n=1}^{N}(\|A_n(y-x, \cdots, y-x, \cdot) - \phi_i\| \|y-x\|) + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq N \cdot \frac{\epsilon}{6Nr} \cdot 2r + \frac{2}{3}\epsilon = \epsilon.
\end{aligned}$$

Hence $f \in H_w(E)$. □

Corollary 4. *$f \in H_w(E)$ if and only if (a) given $r > 0$ and $x \in B_r(O)$, there exists a weak neighborhood W of O such that $\hat{d}((x+W) \cap B_r(O))$ is bounded in E' and (b) $\hat{d}(\frac{\hat{d}^n f(x)}{n!}) \in P_{wu}({}^nE)$ for all n.*

Proof. It is straightforward from the proof of Theorem 3. □

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BOUNDARY BEHAVIOR OF THE BERGMAN KERNEL FUNCTION IN $\mathbb{C}^n$.

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ABSTRACT. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r and let $z_0 \in b\Omega$ be a point of finite type. We also assume that the Levi-form $\partial\bar{\partial}r(z)$ has $(n-2)$–positive eigenvalues at z_0. Then we get a quantity which bounds from above and below the Bergman Kernel function in a small constant and large constant sense.

1. Introduction.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. A natural operator on Ω is the orthogonal projection

$$P : L^2(\Omega) \longrightarrow H(\Omega) \cap L^2(\Omega) = A^2(\Omega)$$

where $H(\Omega)$ denotes the holomorphic functions on Ω. There is a corresponding Kernel function $K(z, \bar{z})$, the Bergman Kernel function, given by;

$$K_\Omega(z, \bar{z}) = sup\{|f(z)|^2; f \in A^2(\Omega), \|f\|_{L^2(\Omega)} \leq 1\}.$$

Since the important paper of Fefferman [10], the singularity of the Bergman Kernel function on strongly pseudoconvex domain at the boundary is quite well known. For weakly pseudoconvex domains, however, much less is known. In [3], Catlin got a result which completely characterized the boundary behavior of $K_\Omega(z, \bar{z})$ for weakly pseudoconvex domains in $\mathbb{C}^2$. Estimates have also been obtained for some weakly pseudoconvex domains in $\mathbb{C}^n$, but in each case the lower bounds are different from the upper bounds [1,7,8,9].

Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r and let $z_0 \in b\Omega$ be a point of finite type m in the sense

I would like to thank D.W. Catlin and Alan Noell for several conversations we had about the material in this paper.

of D'Angelo [6]. We assume that the Levi-form $\partial\overline{\partial}r(z)$ has $(n-2)$-positive eigenvalues at z_0 and that $\frac{\partial r}{\partial z_1} \neq 0$ for all z in a neighborhood U of z_0. After a linear change of coordinates, we can find a coordinate functions $z_1, ..., z_n$ defined on U such that

$$\begin{aligned}(1.1)\qquad L_1 &= \frac{\partial}{\partial z_1},\\ L_j &= \frac{\partial}{\partial z_j} + b_j\frac{\partial}{\partial z_1}, \quad L_j r \equiv 0, \quad b_j(z_0) = 0, \quad j = 2, \ldots, n,\end{aligned}$$

which form a basis of $\mathbb{C}T(U)$ and satisfies

$$(1.2)\qquad \partial\overline{\partial}r(z_0)(L_i, \overline{L}_j) = \delta_{ij}, \quad 2 \leq i, j \leq n-1,$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. For any integers $j, k > 0$, set

$$\mathcal{L}_{j,k}\partial\overline{\partial}r(z) = \underbrace{L_n \ldots L_n}_{(j-1)times}\underbrace{\overline{L}_n \ldots \overline{L}_n}_{(k-1)times}\partial\overline{\partial}r(z)(L_n, \overline{L}_n),$$

and define

$$(1.3)\qquad C_l(z) = max\{|\mathcal{L}_{j,k}\partial\overline{\partial}r(z)|; j + k = l\}.$$

We can state the main result as follows;

Theorem 1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let z_0 be a point of finite type m on $b\Omega$. Also assume that the Levi-form $\partial\overline{\partial}r(z)$ has $(n-2)$–positive eigenvalues at z_0. Then there exist a neighborhood U of z_0 and constant C such that*

$$(1.4)\qquad \frac{1}{C}\sum_{l=2}^{m}|C_l(z)|^{\frac{2}{l}}|r(z)|^{-n-\frac{2}{l}} \leq K_\Omega(z,\overline{z}) \leq C\sum_{l=2}^{m}|C_l(z)|^{\frac{2}{l}}|r(z)|^{-n-\frac{2}{l}}$$

for all $z \in U$, where $C_l(z)$ is defined as in (1.3).

Remark 1.1. Since $z_0 \in b\Omega$ is a point of finite type m, we have $C_m(z_0) > 0$. Therefore (1.4) says, in particular, that

$$K_\Omega(z,\overline{z}) \geq c'|r(z)|^{-n-\frac{2}{m}}$$

for all $z \in U$, for some $c' > 0$.

Remark 1.2. In [9], Diederich, Herbort and Ohsawa proved that the Bergman Kernel function satisfies

$$K_\Omega(z,\overline{z}) \geq c|r(z)|^{-n-\frac{1}{N}} \left[log\left(\frac{1}{r(z)}\right)\right]^{-1} \tag{1.5}$$

if $z_0 \in b\Omega$ has (n-2)–positive eigenvalues and Ω is uniformly extendable in a pseudoconvex way of order N near a point $z_0 \in b\Omega$ (Of course, they proved more than this). If $z_0 \in b\Omega$ is a point of 1-type, then it holds that $N \geq m$. In [7], Diederich and Fornaess proved that the pseudoconvex domain with real analytic boundary can uniformly extendable. Recently, the author showed the same result in case $b\Omega$ is pseudoconvex and finite type [5]. The main theorem completely characterizes the boundary behavior of $K_\Omega(z,\overline{z})$ in $\mathbb{C}^n$, in case $z_0 \in b\Omega$ is of finite type with (n-2)–positive eigenvalues, while (1.5) says a lower bound of $K_\Omega(z,\overline{z})$.

A key idea to prove Theorem 1 is that the terms mixed with strongly pseudoconvex direction and weakly pseudoconvex direction can be negligible. This result will be proved in several propositions in section 2. Then the proof of the Theorem 1 is based on the construction of special polydiscs and weighted L_2-estimates of Hörmander which Catlin has employed to get a result for $K_\Omega(z,\overline{z})$ in $\mathbb{C}^2$.

I would like to thank D.W. Catlin and Alan Noell for several conversations we had about the material in this paper.

2. Special Coordinates and Polydiscs.

In this section we want to show that about each point z' in U, there is a polydisc (more precisely, the biholomorphic image of a polydisc) of maximal size on which the function changes by no more than some prescribed small number $\delta > 0$. First we show how to construct the coordinates about z' which will be used to define a polydisc.

Let us take the coordinate functions $z_1, ..., z_n$ about z_0 so that (1.2) holds. Therefore $L_1 r(z) \geq c > 0$ for all $z \in U$, and $\partial\overline{\partial}r(z)(L_i, \overline{L}_j)_{2\leq i,j\leq n-1}$ has $(n-2)$-positive eigenvalues in U where

$$L_1 = \frac{\partial}{\partial z_1} \quad \text{and}$$

$$L_j = \frac{\partial}{\partial z_j} - \left(\frac{\partial r}{\partial z_1}\right)^{-1} \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_1}, \quad j = 2, \ldots, n.$$

Set

$$w_1 = z_1 + \sum_{j=2}^{n} \left[\left(\frac{\partial r}{\partial z_1} \right)^{-1} \frac{\partial r}{\partial z_j}(z') \right] z_j, \text{ and} \tag{2.1}$$
$$w_j = z_j \text{ for } j = 2, \dots, n.$$

Then L_j can be written as

$$L_j = \frac{\partial}{\partial w_j} + b'_j \frac{\partial}{\partial w_1}, \; 2 \leq j \leq n,$$

where $b'_j(z') = 0$. In $w_1, \ldots, w_n$ coordinates, $A = (\frac{\partial^2 r(z')}{\partial w_i \partial \overline{w}_j})_{2 \leq i,j \leq n-1}$ is Hermitian matrix and there is an unitary matrix $P = (P_{ij})_{2 \leq i,j \leq n-1}$ such that $P^* A P = D$, where D is a diagonal matrix whose entries are positive eigenvalues of A. Set

$$z_1 = w_1, \quad z_n = w_n \text{ and}$$
$$z_j = \sum_{k=2}^{n-1} \overline{P}_{kj} w_k, \text{ for } j = 2, \dots, n-1.$$

Then $\frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}(z') = \lambda_i \delta_{ij}$, $2 \leq i, j \leq n-1$, where $\lambda_i > 0$ is an $i-th$ entry of D(we may assume that $\lambda_i \geq c > 0$ in U for all i). Finally set $w_j = \lambda_j^{-\frac{1}{2}} z_j$, $j = 2, \dots, n-1$, $w_1 = z_1$, $w_n = z_n$. Then

$$\frac{\partial^2 r}{\partial w_i \partial \overline{w}_j}(z') = \delta_{ij}, \; 2 \leq i, j \leq n-1. \tag{2.2}$$

Remark 2.1. If we take the above coordinate changes to get (2.2) with z' replaced by z_0, then this coordinate function satisfies (1.2).

Proposition 2.1. *For each positive number $\epsilon > 0$, there is a neighborhood U_ϵ of z_0 such that*

$$|\partial \overline{\partial} r(z)(L_i, \overline{L}_j)| \leq \epsilon \tag{2.3}$$

for all $z \in U_\epsilon$ and $2 \leq i, j \leq n-1$, $i \neq j$.

Proof. From the Remark 2.1, and from the coordinate changes up to (2.2), one has $L_j = \sum_{k=2}^{n-1} b_{jk} \frac{\partial}{\partial w_k} + b'_j \frac{\partial}{\partial w_1}$, where $b'_j(z_0) = 0$ and $\partial \overline{\partial} r(L_i, \overline{L}_j)(z_0) = \delta_{ij}$. So (2.3) holds provided one takes U_ϵ sufficiently small. □

Proposition 2.2. *For each $z' \in U$ and positive even integer m, there is a biholomorphism $\Phi_{z'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_{z'}^{-1}(z') = 0$, $\Phi_{z'}^{-1}(z) = (\zeta_1, \dots, \zeta_n)$ such that*

$$
\begin{aligned}
r(\Phi_{z'}(\zeta)) = r(z') + Re\zeta_1 &+ \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leq \frac{m}{2} \\ j,k>0}} Re\left(b_{j,k}^{\alpha}(z')\zeta_n^j\overline{\zeta}_n^k\zeta_\alpha\right) \\
&+ \sum_{\substack{j+k\leq m \\ j,k>0}} a_{j,k}(z')\zeta_n^j\overline{\zeta}_n^k + \sum_{\alpha=2}^{n-1}|\zeta_\alpha|^2 \qquad (2.4)\\
&+ \mathcal{O}(|\zeta_1||\zeta| + |\zeta''|^2|\zeta| + |\zeta''||\zeta_n|^{\frac{m}{2}+1} + |\zeta_n|^{m+1}).
\end{aligned}
$$

Proof. We may assume that $z' = 0 \in b\Omega$. Let us take the coordinate functions $w_1, \dots, w_n$ about 0 so that (2.2) holds. After a linear change, $r(w)$ can be written as

$$
\begin{aligned}
r(w) = Rew_1 &+ \sum_{\alpha=2}^{n-1} \sum_{1\leq j\leq \frac{m}{2}} Re\left[\left(a_j^{\alpha}w_n^j + b_j^{\alpha}\overline{w}_n^j\right)w_\alpha\right] \\
&+ \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leq \frac{m}{2} \\ j,k>0}} Re(a_{j,k}^{\alpha}w_n^j\overline{w}_n^k w_\alpha) \qquad (2.5)\\
&+ \sum_{2\leq j+k\leq m} b_{j,k}w_n^j\overline{w}_n^k + \sum_{\alpha=2}^{n-1}|w_\alpha|^2 \\
&+ \mathcal{O}(|w_1||w| + |w''|^2|w| + |w''||w_n|^{\frac{m}{2}+1} + |w_n|^{m+1}),
\end{aligned}
$$

where $w'' = (0, w_2, \dots, w_{n-1}, 0)$. It is standard to perform the change of coordinates

$$
\begin{aligned}
z_1 &= w_1 + \sum_{2\leq k\leq m} \frac{2}{k!}\frac{\partial^k r(0)}{\partial w_n^k}w_n^k + \sum_{\alpha=2}^{n-1}\sum_{1\leq k\leq m} \frac{2}{k!}\frac{\partial^{k+1} r(0)}{\partial w_\alpha \partial w_n^k}w_\alpha w_n^k \\
z_j &= w_j, \quad j = 2, \dots, n,
\end{aligned}
$$

which serves to remove the pure terms from (2.5), i.e, it removes w_n^k, $\overline{w}_n^k$ terms as well as $w_n^k w_\alpha$, $\overline{w}_n^k\overline{w}_\alpha$ terms from the summation in (2.5). We may also perform a change of coordinates,

$$\zeta_1 = z_1, \quad \zeta_n = z_n \quad \zeta_\alpha = z_\alpha + \sum_{1 \le k \le \frac{m}{2}} \frac{1}{k!} \frac{\partial^{k+1} r(0)}{\partial \overline{w}_\alpha \partial w_n^k} z_n^k$$

to remove terms of the form $\overline{w}_n^j w_\alpha$ from the summation in (2.5), and hence $r(\zeta)$ has the desired expression as in (2.4) in ζ-coordinates. □

Remark 2.2. The coordinate changes in the proof of the proposition 2.2 are unique and hence the map $\Phi_{z'}$ is defined uniquely.

Set $\rho(\zeta) = r \circ \Phi_{z'}(\zeta)$, and set

(2.6)
$$\begin{aligned} A_l(z') &= max\{|a_{j,k}(z')|; j+k=l\}, \ 2 \le l \le m, \\ B_{l'}(z') &= max\{|b^\alpha_{j,k}(z')|; j+k=l', \ 2 \le \alpha \le n-1\}, \ 2 \le l' \le \frac{m}{2}. \end{aligned}$$

For each $\delta > 0$, we define $\tau(z', \delta)$ as follows;

(2.7) $$\tau(z',\delta) = min\{(\delta/A_l(z'))^{\frac{1}{l}}, \ (\delta^{\frac{1}{2}}/B_{l'}(z'))^{\frac{1}{l'}}; 2 \le l \le m, \ 2 \le l' \le \frac{m}{2}\}.$$

Since $A_m(z_0) \ge c > 0$, it follows that $A_m(z') \ge c' > 0$ for all $z' \in U$ if U is sufficiently small. This gives the inequality,

(2.8) $$\delta^{\frac{1}{2}} \lesssim \tau(z',\delta) \lesssim \delta^{\frac{1}{m}}, \quad z' \in U.$$

The definition of $\tau(z', \delta)$ easily implies that if $\delta' < \delta''$, then

(2.9) $$(\delta'/\delta'')^{\frac{1}{2}} \tau(z',\delta'') \le \tau(z',\delta') \le (\delta'/\delta'')^{\frac{1}{m}} \tau(z',\delta'').$$

Now set $\tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{\frac{1}{2}}, \ \tau_n = \tau(z',\delta) = \tau$ and define

(2.10) $$\begin{aligned} R_\delta(z') &= \{\zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k, \ k = 1,2,\ldots,n\}, \text{ and} \\ Q_\delta(z') &= \{\Phi_{z'}(\zeta); \zeta \in R_\delta(z')\}. \end{aligned}$$

In the sequal we denote D_k^l any partial derivative operator of the form $\frac{\partial^{\mu+\nu}}{\partial \zeta_k^\mu \partial \overline{\zeta}_k^\nu}$, where $\mu + \nu = l$, $k = 1, 2, \ldots, n$.

Proposition 2.3. *Let $z' \in U$. Then the function $\rho = r \circ \Phi_{z'}(\zeta)$ satisfies;*

$$|\rho(\zeta) - \rho(0)| \lesssim \delta, \quad \zeta \in R_\delta(z'), \text{ and} \tag{2.11}$$
$$|D_k^i D_n^l \rho(\zeta)| \lesssim \delta \tau_n^{-l} \tau_k^{-i}, \ \zeta \in R_\delta(z'),$$

for $l + \frac{im}{2} \leq m$, $i = 0, 1$, $k = 2, \dots, n-1$.

Proof. The definitions in (2.6) and (2.7) imply that $|D_n^j \rho(0)| \lesssim \delta\tau^{-j}$ and $|D_k D_n^j \rho(0)| \lesssim \delta^{\frac{1}{2}}\tau^{-j} = \delta\tau^{-j}\tau_k^{-1}$. Since $|D_k D_n^{\frac{m}{2}+1} \rho(0, \dots, \zeta_k, 0, \dots, \zeta_n)| \lesssim 1$, for $k = 2, \dots, n-1$, and $|D_n^{m+1}\rho(0, \zeta_n)| \lesssim 1$, we may use (2.4) and Taylor's expansion theorem to prove (2.11). □

In order to study how $\tau(z, \delta)$ depends on z for $z \in Q_\delta(z')$, it is convenient to introduce an analogous quantity $\eta(z, \delta)$ that is defined more intrinsically. Recall that L_n is given by

$$L_n = \frac{\partial}{\partial z_n} - \left(\frac{\partial r}{\partial z_1}\right)^{-1} \frac{\partial r}{\partial z_n} \frac{\partial}{\partial z_1}.$$

For any j, k with $j, k > 0$, define

$$\mathcal{L}_{j,k}\partial\overline{\partial}r(z) = \underbrace{L_n \dots L_n}_{(j-1)times} \underbrace{\overline{L}_n \dots \overline{L}_n}_{(k-1)times} \partial\overline{\partial}r(L_n, \overline{L}_n)(z),$$

and define

$$C_l(z) = max\{|\mathcal{L}_{j,k}\partial\overline{\partial}r(z)|; j + k = l\}, \ l = 2, \dots, m, \tag{2.12}$$

$$\eta(z, \delta) = min\{(\delta/C_l(z))^{\frac{1}{l}}; l = 2, \dots, m\}. \tag{2.13}$$

Set $L'_n = (d\Phi_{z'})^{-1}L_n$, and define

$$\mathcal{L}'_{j,k}\partial\overline{\partial}\rho(\zeta) = \underbrace{L'_n \dots L'_n}_{(j-1)-times} \underbrace{\overline{L}'_n \dots \overline{L}'_n}_{(k-1)-times} \partial\overline{\partial}\rho(\zeta)(L'_n, \overline{L}'_n).$$

Then

$$\mathcal{L}_{j,k}\partial\overline{\partial}r(\Phi_{z'}(\zeta)) = \mathcal{L}'_{j,k}\partial\overline{\partial}\rho(\zeta) \tag{2.14}$$

by functoriality. Notice that

$$(\Phi_{z'}^{-1})_* L_n = L'_n = \frac{\partial}{\partial \zeta_n} + b(\zeta)\frac{\partial}{\partial \zeta_1} \quad \textit{and}$$

(2.15)

$$(\Phi_{z'}^{-1})_* L_k = L'_k = \sum_{j=2}^{n-1} \overline{P}_{kj}\lambda_j^{-\frac{1}{2}}\frac{\partial}{\partial \zeta_j} - \left(\frac{\partial r}{\partial \zeta_1}\right)^{-1}\sum_{j=2}^{n-1} \overline{P}_{kj}\lambda_j^{-\frac{1}{2}}\frac{\partial r}{\partial \zeta_j}\frac{\partial}{\partial \zeta_1}$$

$$= \sum_{j=1}^{n-1} b_{kj}\frac{\partial}{\partial \zeta_j}, \quad k = 2, \dots, n-1,$$

where $b(\zeta) = -\left(\frac{\partial \rho}{\partial \rho_1}\right)^{-1}\left(\frac{\partial \rho}{\partial \zeta_n}\right)$ and $P = (P_{kj})$ is an unitary matrix. Since $\frac{\partial \rho}{\partial \zeta_1}(\zeta) \neq 0$ in $\Phi_{z'}^{-1}(U)$, we obtain from Leibniz's identity and (2.11) that

$$|D_k^i D_n^l b(0)| \lesssim \delta\tau^{-l-1}\tau_k^{-i}, \tag{2.16}$$

for $i = 0, 1$, $l + \frac{im}{2} \le m - 1$, $k = 2, \dots, n-1$. Since

$$\partial\overline{\partial}\rho(L'_n, \overline{L}'_n) = \frac{\partial^2 \rho}{\partial \zeta_n \partial \overline{\zeta}_n} + \mathcal{O}(b),$$

one gets by induction and (2.16) that

$$\mathcal{L}'_{j,k}\partial\overline{\partial}\rho = \frac{\partial^{j+k}\rho}{\partial \zeta_n^j \partial \overline{\zeta}_n^k} + E_{j+k-1}, \tag{2.17}$$

where

$$|D_k^i D_n^l E_s(0)| \lesssim \delta\tau^{-l-s}\tau_k^{-i}, \ l + \frac{im}{2} \le m - s, \tag{2.18}$$

for $i = 0, 1$, $1 \le s \le m - 1$, $k = 2, \dots, n-1$. With (2.17), (2.18) and by induction one will get

$$|D_k^i D_n^l \mathcal{L}'_{j,k}\partial\overline{\partial}\rho(0)| \lesssim \delta\tau^{-(l+j+k)}\delta^{-\frac{i}{2}},$$

and hence one obtains that

$$|\mathcal{L}_{j,k}\partial\overline{\partial}r(z)| \lesssim \delta\tau^{-(j+k)}, \ z \in Q_\delta(z'),$$

by Taylor series method and (2.14). Since this means that $C_l(z) \lesssim \delta\tau^{-l}$, $z \in Q_\delta(z')$, $l = 2, ..., m$, one concludes that

$$\eta(z,\delta) \gtrsim \tau(z',\delta), \tag{2.19}$$

when $z \in Q_\delta(z')$. We want to prove the opposite inequality in (2.19). To show this, we first show that the quantities $B_{l'}(z')$ in (2.6) is less important term than $A_l(z')$ for the definition of $\tau(z',\delta)$ in (2.7). This is a key point in the rest of this section. Recall that $L'_n = \frac{\partial}{\partial\zeta_n} + b(\zeta)\frac{\partial}{\partial\zeta_1}$, where $b(\zeta)$ satisfies (2.16). This implies that

$$|b(\zeta)| \lesssim \delta\tau^{-\frac{5}{4}} \tag{2.20}$$

for $\zeta \in P_\tau = \{\zeta; |\zeta_1| \le \delta, |\zeta_k| \le \delta^{\frac{1}{2}}\tau^{-\frac{1}{4}}, |\zeta_n| \le \tau\}$. Define a map $\Lambda_\delta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ by

$$\Lambda_\delta(\zeta) = (\delta^{-1}\zeta_1, \delta^{-\frac{1}{2}}\zeta_2, \dots, \delta^{-\frac{1}{2}}\zeta_{n-1}, \tau^{-1}\zeta_n) = (\tilde{\zeta}_1, \dots, \tilde{\zeta}_n).$$

Then

$$\tilde{L}_n = \tau(\Lambda_\delta)_* L'_n = \frac{\partial}{\partial\zeta_n} + b\left(\Lambda_\delta^{-1}(\zeta)\right)\delta^{-1}\tau\frac{\partial}{\partial\zeta_1},$$

where we have dropped ($\tilde{}$'s) in ζ-variables. With (2.20), one has

$$|b\left(\Lambda_\delta^{-1}(\zeta)\right)\delta^{-1}\tau| \lesssim \tau^{-\frac{1}{4}} \tag{2.21}$$

for $\zeta \in Q_\tau = \Lambda_\delta(P_\tau) = \{\zeta; |\zeta_1| \le 1, |\zeta_n| \le 1, |\zeta_k| \le \tau^{-\frac{1}{4}}, k = 2, \dots, n-1\}$. If we set $\rho^\delta_{z'}(\zeta) = \delta^{-1}((\Lambda_\delta^{-1})^*\rho_{z'}(\zeta))$, then

$$\begin{aligned}
\rho^\delta_{z'}(\zeta) = {} & Re\zeta_1 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\le\frac{m}{2}\\ j,k>0}} Re\left[b^\alpha_{j,k}(z')\,\delta^{-\frac{1}{2}}\tau^{j+k}\zeta_n^j\bar{\zeta}_n^k\zeta_\alpha\right] \\
& + \sum_{\substack{j+k\le m\\ j,k>0}} a_{j,k}(z')\delta^{-1}\tau^{j+k}\zeta_n^j\bar{\zeta}_n^k + \sum_{\alpha=2}^{n-1}|\zeta_\alpha|^2 \\
& + \tau\mathcal{O}(|\zeta_1||\zeta| + \delta|\zeta_1|^2|\zeta| + |\zeta''|^2|\zeta|) \\
& + \tau\mathcal{O}(|\zeta''||\zeta_n|^{\frac{m}{2}+1} + |\zeta_n|^{m+1}),
\end{aligned} \tag{2.22}$$

for all $\zeta \in Q_\tau$. From the expression in (2.22), we set

$$A^\delta(\zeta_n,\overline{\zeta}_n) = \sum_{\substack{j+k\le m\\ j,k>0}} a_{j,k}(z')\delta^{-1}\tau^{j+k}\zeta_n^j\overline{\zeta}_n^k, \text{ and}$$

$$B_\alpha^\delta(\zeta_n,\overline{\zeta}_n) = \sum_{\substack{j+k\le \frac{m}{2}\\ j,k>0}} b_{j,k}^\alpha(z')\delta^{-\frac{1}{2}}\tau^{j+k}\zeta_n^j\overline{\zeta}_n^k, \ \alpha = 2,\dots,n-1.$$

Since the level sets of $\rho_{z'}^\delta(\zeta)$ are pseudoconvex and since $\tilde{L}_n = \tau(\Lambda_\delta)_* L_n'$ is a tangential vector field on the level sets of $\rho_{z'}^\delta$, we have $\partial\overline{\partial}\rho_{z'}^\delta(\zeta)(\tilde{L},\overline{\tilde{L}}_n) \ge 0$. By combining (2.21) and (2.22), one can get

(2.23)
$$\begin{aligned}\partial\overline{\partial}\rho_{z'}^\delta(\zeta)(\tilde{L}_n,\overline{\tilde{L}}_n) &= \frac{\partial^2\rho_{z'}^\delta}{\partial\zeta_n\partial\overline{\zeta}_n} + \mathcal{O}\left(\tau\tilde{b}\frac{\partial^2\rho_{z'}^\delta}{\partial\zeta_n\partial\overline{\zeta}_1}\right) + \mathcal{O}\left(\delta\tilde{b}^2\frac{\partial^2\rho_{z'}^\delta}{\partial\zeta_1\partial\overline{\zeta}_1}\right)\\ &= \frac{\partial^2 A^\delta}{\partial\zeta_n\partial\overline{\zeta}_n} + Re\left(\sum_{\alpha=2}^{n-1}\frac{\partial^2 B_\alpha^\delta(\zeta_n,\overline{\zeta}_n)}{\partial\zeta_n\partial\overline{\zeta}_n}\zeta_\alpha\right) + \mathcal{O}\left(\tau^{\frac{1}{2}}\right),\end{aligned}$$

for all $\zeta \in Q_\tau$ where $\tilde{b} = \delta^{-1}\tau b\left(\Lambda_\delta^{-1}(\zeta)\right)$.

Lemma 2.4. $|B_\delta^\alpha(\zeta_n,\overline{\zeta}_n)| \le \tau^{\frac{1}{10}}$ *for all* $\alpha = 2,\dots,n-1$, $\zeta \in Q_\tau$, *provided* τ *is sufficiently small.*

Proof. From (2.6) we know that the coefficients of A^δ and B_α^δ are bounded by one. At first, let's show that $|\frac{\partial^2 B_\alpha^\delta(\zeta_n,\overline{\zeta}_n)}{\partial\zeta_n\partial\overline{\zeta}_n}| \le \tau^{\frac{1}{10}}$ for $\tau \in Q_\tau$. Suppose, on the contrary, that $|\frac{\partial^2 B_\alpha^\delta}{\partial\zeta_n\overline{\zeta}_n}(\zeta_n,\overline{\zeta}_n)| > \tau^{\frac{1}{10}}$ for some ζ_n and α. Then $\frac{\partial^2 B_\alpha^\delta(\zeta_n,\overline{\zeta}_n)}{\partial\zeta_n\partial\overline{\zeta}_n}\zeta_\alpha < -|\mathcal{O}(\tau^{-\frac{1}{10}})|$, provided one takes $|\zeta_\alpha|$ sufficiently large (say $\tau^{-\frac{1}{5}} < |\zeta_\alpha| < \tau^{-\frac{1}{4}}$), with appropriate argument. If one combines this fact and (2.23), then $\partial\overline{\partial}\rho_{z'}^\alpha(\tilde{L}_n,\overline{\tilde{L}}_n) < 0$ at that point provided τ is sufficiently small. This contradicts to the fact that the level sets of $\rho_{z'}^\delta$ are pseudoconvex and hence $|\frac{\partial^2 B_\alpha^\delta(\zeta_n,\overline{\zeta}_n)}{\partial\zeta_n\partial\overline{\zeta}_n}| \le \tau^{\frac{1}{10}}$. This implies that $|B_\alpha^\delta(\zeta_n,\overline{\zeta}_n)| \le \tau^{\frac{1}{10}}$ because $|\zeta_n| \le 1$. □

Using this lemma, one can show that the coefficients of B_δ^α can be made arbitrary small provided δ is sufficiently small. First we prove the following lemma.

Lemma 2.5. *Let $P_k(z,\overline{z}) = \sum_{i+j=k} a_{i,j} z^i \overline{z}^j$ be a homogeneous polynomial of order k in z and $\overline{z}$, and suppose that $|P_k(z,\overline{z})| \le \epsilon$ for all z on the unit circle on $\mathbb{C}^1$. Then $|a_{i,j}| \le \epsilon$.*

Proof. $P_k(z,\overline{z}) = \sum_{l+j=k} a_{l,j} e^{i(l-j)\theta}$ on the unit circle in $\mathbb{C}^1$. So

$$\begin{aligned} |a_{l,j}| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_k(z,\overline{z}) e^{i(l-j)\theta} d\theta\right| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |P|_\infty d\theta \le \epsilon. \end{aligned}$$

□

Proposition 2.6. *Let $P(z,\overline{z}) = \sum_{i+j\le n} a_{ij} z^i \overline{z}^j$ be a polynomial of order n with $|a_{ij}| \le 1$. Suppose $|P(z,\overline{z})| \le \epsilon^2$ for all $|z| \le 1$ for some small number $\epsilon > 0$. Then $|a_{ij}| \le C_n \epsilon^\alpha$, where $\alpha = \frac{1}{n!}$.*

Proof. Let $P = \sum_{k=0}^n P_k$, where P_k is a homogeneous polynomial of order k. It is clear that $|P_0| \le \epsilon^2$. Since $|\sum_{l=2}^n P_l| \lesssim \epsilon^2$ on $|z| = \epsilon$, we have $|P_1(z,\overline{z})| \le |P| + |P_0| + |\sum_{l=2}^n P_l| \lesssim \epsilon^2$ on $|z| \le \epsilon$. This implies that $|P_1(z,\overline{z})| \le \epsilon$ for all $|z| \le 1$, and therefore $|a_{ij}| \le \epsilon$, $i+j \le 1$, by Lemma 2.5. Similarly one can prove that $|P_2(z,\overline{z})| \le \epsilon^{\frac{1}{2}}$ for all $|z| \le 1$ and hence $|a_{ij}| \le \epsilon^{\frac{1}{2}}$, $i+j \le 2$. Let $k \ge 2$ and suppose by induction that $|a_{ij}| \lesssim \epsilon^{\frac{1}{k!}}$ for all $i+j \le k$. Then $|\sum_{l=k+2}^n P_l| \lesssim \epsilon^{\frac{k+2}{(k+1)!}}$ on $|z| \le \epsilon^{\frac{1}{(k+1)!}}$, and so

$$|\sum_{l=0}^{k} P_l| \lesssim |P_0| + |P_1| + |\sum_{l=2}^{k} P_l| \lesssim \epsilon^{\frac{1}{k!} + \frac{2}{(k+1)!}}$$

on $|z| \le \epsilon^{\frac{1}{(k+1)!}}$. Therefore

$$|P_{k+1}| \lesssim |P| + |\sum_{l=k+2}^{n} P_l| + |\sum_{l=0}^{k} P_l| \lesssim \epsilon^{\frac{k+2}{(k+1)!}}$$

on $|z| \le \epsilon^{\frac{1}{(k+1)!}}$. This implies that $|P_{k+1}| \lesssim \epsilon^{\frac{k+2}{(k+1)!} - \frac{k+1}{(k+1)!}} = \epsilon^{\frac{1}{(k+1)!}}$, for all $|z| \le 1$, and hence $|a_{ij}| \le \epsilon^{\frac{1}{(k+1)!}}$ for all $i+j \le k+1$ by Lemma 2.5. So we get the Proposition 2.6 by induction. □

If one combines Lemma 2.4, Lemma 2.5 and Proposition 2.6, then

$$|b^\alpha_{j,k}(z')\delta^{-\frac{1}{2}}\tau^{j+k}| \lesssim \tau^{\frac{1}{20 \times m!}} \tag{2.24}$$

for all $2 \le \alpha \le n-1$, $2 \le j+k \le \frac{m}{2}$. So $\left(\delta^{\frac{1}{2}}/B_{l'}(z')\right) >> \tau$, $l' = 2,\ldots,\frac{m}{2}$, if δ(and hence τ) is sufficiently small and therefore $\tau(z',\delta) = min\{\left(\frac{\delta}{A_l(z')}\right)^{\frac{1}{l}}; 2 \le l \le m\}$. Now define

$$T(z',\delta) = min\{l; (\delta/A_l(z'))^{\frac{1}{l}} = \tau(z',\delta)\}. \tag{2.25}$$

Then there exists j,k with $j+k = T(z',\delta)$ so that

$$|a_{j,k}(z')| = |\frac{\partial^{j+k}\rho}{\partial\zeta_n^j\partial\bar{\zeta}_n^k}(0)(j!k!)^{-1}| = \delta\tau^{-j-k}. \tag{2.26}$$

From (2.17), (2.18) and (2.26), one has

$$|\mathcal{L}'_{j,k}\partial\bar{\partial}\rho(0)| \ge \frac{1}{2}(j!k!)\delta\tau^{-j-k}.$$

Again by Taylor's theorem and by the fact that $|\zeta_n| < \tau(z',b\delta) \le b^{\frac{1}{m}}\tau(z',\delta)$ for $\zeta \in R_{b\delta}(z')$, one has

$$|\mathcal{L}'_{j,k}\partial\bar{\partial}\rho(\zeta) - \mathcal{L}'_{j,k}\partial\bar{\partial}\rho(0)| \lesssim b^{\frac{1}{m}}\delta\tau^{-j-k},$$

and hence $|\mathcal{L}'_{j,k}\partial\bar{\partial}\rho(\zeta)| \approx \delta\tau^{-j-k}$ for $\zeta \in R_{b\delta}(z')$, if b is sufficiently small. This implies that $\eta(z,\delta) \lesssim \tau(z',\delta)$, for $\zeta \in Q_{b\delta}(z')$, and hence

$$\eta(z,\delta) \lesssim b^{-\frac{1}{2}}\tau(z',\delta), \quad z \in Q_\delta(z'), \tag{2.27}$$

by (2.9). With (2.19) and (2.27), we have proved the following proposition.

Proposition 2.7. *Let z' and z be any two points with $z \in Q_\delta(z')$. Then*

$$\tau(z',\delta) \lesssim \eta(z,\delta) \lesssim \tau(z',\delta).$$

Corollary 2.8. *Suppose that $z \in Q_\delta(z')$. Then*

$$\tau(z',\delta) \approx \tau(z,\delta).$$

Proof. By Proposition 2.7, $\tau(z',\delta) \approx \eta(z,\delta) \approx \tau(z,\delta)$. □

Using the definitions of $\eta(z',\delta)$, $\tau(z',\delta)$, $T(z',\delta)$ and with Proposition 2.7, Corollary 2.8, we can show the following semi-continuous result for the integer $T(z,\delta)$ as a method similar to Proposition 1.5 in [3].

Proposition 2.9. *There exists a small constant $b > 0$ so that if $z \in Q_{b\delta}(z')$, then*

$$T(z, \epsilon) \leq T(z', \delta) \tag{2.28}$$

for all $\epsilon \leq b\delta$.

3. Estimates of the Bergman Kernel Function.

In this section we prove the main theorem of this article. The following proposition is the local version of the problem constructing a function with large Hessian near the boundary. For z near the boundary of Ω, we denote the closest point in $b\Omega$ to z by $\pi(z)$. Let us take the vector fields $L_1, \ldots, L_n$ as in (1.1).

Proposition 3.1. *Suppose $z' \in U \cap b\Omega$. Then there exist a small constant $a > 0$ and a smooth function $g_{z',\delta}$ on $\overline{\Omega}$ that satisfies*

(i) $|g_{z',\delta}(z)| \leq 1$ *and* $g_{z',\delta} \in C_0^\infty(Q_\delta(z'))$.

(ii) *If $-a\delta \leq r(z) \leq a\delta$ and if $g_{z',\delta}$ is not plurisubharmonic at z, then*

$$T(\pi(z), a\delta) < T(z', \delta). \tag{3.1}$$

(iii) *If $z \in Q_{a\delta}(z')$, $-a\delta \leq r(z) \leq a\delta$, and if the inequality*

$$\partial\overline{\partial} g_{z',\delta}(L, \overline{L})(z) \gtrsim (\tau(z', \delta))^{-2}|b_n|^2 + \delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \delta^{-2}|b_1|^2 \tag{3.2}$$

fails to hold at z for $L = \sum_{j=1}^n b_j L_j$, then

$$T(\pi(z), a\delta) < T(z', \delta).$$

(iv) *For all $z \in Q_\delta(z')$ and all $L = \sum_{j=1}^n b_j L_j$ at z,*

$$|\partial\overline{\partial} g_{z',\delta}(L, \overline{L})| \lesssim (\tau(z', \delta))^{-2}|b_n|^2 + \delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \delta^{-2}|b_1|^2. \tag{3.3}$$

(v) *If Φ' denotes the map associated with z', then*

$$|D^\alpha g_{z',\delta} \circ \Phi'(\zeta)| \leq C_\alpha \tau^{-\alpha_n}\delta^{-\alpha_1}\delta^{-\frac{1}{2}(\alpha_2+\ldots+\alpha_{n-1})} \tag{3.4}$$

where $\alpha_i = \beta_i + \gamma_i$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ and $D_i^{\alpha_i} = D_i^{\beta_i}\overline{D}_i^{\gamma_i}$.

Proof. The proof will be similar to that of dimension two case of Proposition 2.1 in [3]. We will sketch the proof briefly here. Set $\tau(z',\delta) = \tau$ for the convenience. From (2.15), Proposition 2.3 and Lemma 2.4, one has

$$\partial\bar{\partial}r(L_k,\bar{L}_n) = \partial\bar{\partial}\rho(L'_k,\bar{L}'_n) = \mathcal{O}(\delta^{\frac{1}{2}}\tau^{-1}), \text{ and}$$
$$\partial\bar{\partial}r(L_k,\bar{L}_k) = \partial\bar{\partial}\rho(L'_k,\bar{L}'_k) = 1+\mathcal{O}(\delta^{\frac{1}{2}}),$$

for $k = 2,\dots,n-1$. Therefore from Proposition 2.1 and the fact that $Lr = b_1L_1r$, we obtain that if δ is small,

(3.5)
$$\begin{aligned}
&\lambda\delta^{-1}\partial\bar{\partial}r(L,\bar{L}) + (\lambda\delta^{-1})^2|Lr|^2\\
&= \lambda\delta^{-1}\partial\bar{\partial}r(L_n,\bar{L}_n)|b_n|^2 + 2\lambda\delta^{-1}Re\sum_{j=1}^{n}\partial\bar{\partial}r(L_1,\bar{L}_j)b_1\bar{b}_j\\
&+\mathcal{O}(\epsilon)\lambda\delta^{-1}\sum_{2\le j<k\le n} b_j\bar{b}_k + \lambda\delta^{-1}\sum_{k=2}^{n-1}\partial\bar{\partial}r(L_k,\bar{L}_k)|b_k|^2 + \lambda^2\delta^{-2}|b_1L_1r|^2\\
&\approx \lambda\delta^{-1}\partial\bar{\partial}r(L_n,\bar{L}_n)|b_n|^2 + \lambda\delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \lambda^2\delta^{-2}|b_1|^2 + \mathcal{O}(\epsilon)|b_n|^2.
\end{aligned}$$

Let $\psi(\zeta)$ be defined by

$$\psi(\zeta) = \chi(\delta^{-2}|\zeta_1|^2 + \delta^{-1}\sum_{k=2}^{n-1}|\zeta_k|^2 + \tau^{-2}|\zeta_n|^2),$$

where $\chi(t) = 1$ for $t < \frac{b}{2}$ and $\chi(t) = 0$ for $t \ge b$. Here $b > 0$ is the small constant as in Proposition 2.9. Now set $\Psi(z) = \psi((\Phi_{z'})^{-1}(z))$. Then by Proposition 2.3 and (2.15), one has

(3.6)
$$|\partial\bar{\partial}\Psi(L,\bar{L})| = |\partial\bar{\partial}\psi(L',\bar{L}')| \lesssim |b_1|^2\delta^{-2} + \delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \tau^{-2}|b_n|^2,$$

$$|L\Psi| = |L'\psi| \lesssim |b_1|\delta^{-1} + \delta^{-\frac{1}{2}}\sum_{k=2}^{n-1}|b_k| + \tau^{-1}|b_n|.$$

Suppose at first that $T(z',\delta)=2$. Then we conclude from (2.25) that

$$\partial\bar{\partial}r(L_n,\bar{L}_n)(z)\approx\delta\tau^{-2},\quad z\in Q_{b\delta}(z'). \tag{3.7}$$

For $\lambda\geq 1$ we have

(3.8)

$$\partial\bar{\partial}(\Psi e^{\lambda\delta^{-1}r})(L,\bar{L})=e^{\lambda\delta^{-1}r}[\partial\bar{\partial}\Psi(L,\bar{L})+\lambda\delta^{-1}\sum_{i,j=1}^{n}2Re((L_i\Psi)(\bar{L}_jr))b_i\bar{b}_j$$
$$+\lambda\delta^{-1}\Psi\partial\bar{\partial}r(L,\bar{L})+(\lambda\delta^{-1})^2\Psi|Lr|^2].$$

Combining (3.5)–(3.8), one will get

$$\partial\bar{\partial}(\Psi e^{\lambda\delta^{-1}r})(L,\bar{L})\approx\delta^{-2}|b_1|^2+\delta^{-1}\sum_{k=2}^{n-1}|b_k|^2+\tau^{-2}|b_n|^2 \tag{3.9}$$

provided λ is sufficiently large and $\Psi(z)\geq\frac{1}{4}$.

Let h denote a convex increasing smooth function such that $h(t)=0$ for $t\leq\frac{1}{2}$ and $h(t)>0$ for $t>\frac{1}{2}$, and set $g_{z',\delta}(z)=h\left(\Psi(z)e^{\lambda\delta^{-1}r(z)}\right)$. Then $\frac{1}{2}<e^{\lambda\delta^{-1}r(z)}<\frac{3}{2}$ and hence $\Psi(z)\geq\frac{1}{4}$ for those z with $-a\delta\leq r(z)\leq a\delta$ and $z\in supp\ g_{z',\delta}$, provided $a>0$ is sufficiently small. Therefore $g_{z',\delta}(z)$ is smooth plurisubharmonic with support in $Q_\delta(z')$. It also satisfies property (v) in Proposition 3.1 and hence this proves for $T(z',\delta)=2$.

When $T(z',\delta)=l>2$, one has $|\mathcal{L}_{j,k}\partial\bar{\partial}r(z')|\approx\delta\tau^{-l}$ for some positive integers j,k with $j+k=l$. This implies that at least one of the inequalities

$$|L_n(Re\mathcal{L}_{j-1,k}\partial\bar{\partial}r)(z')|\approx\delta\tau^{-l}\quad\text{and} \tag{3.10}$$

$$|L_n(Im\mathcal{L}_{j-1,k}\partial\bar{\partial}r)(z')|\approx\delta\tau^{-l} \tag{3.11}$$

is valid. (When $j=1$, we replace $\mathcal{L}_{j-1,k}$ by $\mathcal{L}_{1,k-1}$). We may assume that (3.10) is valid. Now set $G(z)=Re\mathcal{L}_{j-1,k}\partial\bar{\partial}r(z)$ and suppose that $T(z,e\delta)=l$, for e still to be chosen. Then by (2.6), (2.7), (2.12), (2.13) and Proposition 2.7, Corollary 2.8, one has

$$|\mathcal{L}_{j-1,k}\partial\bar{\partial}r(z)|\leq C_{l-1}(z)\lesssim e^{\frac{1}{l}}\delta\tau^{-l+1},$$

and this implies that

$$\partial\bar{\partial}G^2(L_n,\bar{L}_n)(z)\geq c'\delta^2\tau^{-2l},$$

provided $e > 0$ is sufficiently small. Also from (2.15) and Proposition 2.3, one has $|L_k G(z)| \lesssim \delta^{\frac{1}{2}}\tau^{-l+1}$, for $l = 2, \dots, m$, $k = 2, \dots, n-1$. Therefore we get

$$\partial\bar{\partial}G^2(L,\bar{L})(z) \geq c'\delta^2\tau^{-2l}|b_n|^2 - C'\sum_{k=2}^{n-1}\delta\tau^{-2l+2}|b_k|^2 - C'|b_1|^2. \tag{3.12}$$

Set

$$G_{z',\delta}(z) = \Psi(z)e^{\lambda\delta^{-1}r(z)} + \phi(\delta^{-2}\tau^{2l-2}(G(z))^2)$$

and set $g_{z',\delta}(z) = h(G_{z',\delta}(z))$, where $\phi(t)$ is a smooth function that satisfies $\phi(t) = t$, $t \leq \frac{1}{16}$, $\phi(t) = 0$ for $t \geq 1$ and $\phi(t) \leq \frac{1}{8}$ for all t. If one combines (3.5), (3.6), (3.8), (3.12) and the fact that $\delta^{-2}\tau^{2l-2}G(z)^2 < \frac{1}{16}$, provided e is sufficiently small, one will get (3.2) and (3.3) and hence $g_{z',\delta}$ is plurisubharmonic for those $z \in Q_{b\delta}(z')$ with $T(z, e\delta) = l$. Since $e \leq b$, Proposition 2.9 implies that

$$T(\pi(z), e^2\delta) \leq T(z, e\delta) \leq T(z', \delta) = l$$

for $z \in Q_{a\delta}(z')$, with $a \leq e^2$ and $|r(z)| \leq a\delta$. Therefore if $T(z, e\delta) < l$ then $T(\pi(z), a\delta) < l$ and hence this proves (ii)–(iv). For (i), we divide $g_{z',\delta}$ by some constant. Since $g_{z',\delta}(z)$ is a composition of the functions which satisfy (3.4), it also satisfies (3.4) and this proves (v). □

Using Proposition 3.1, we can prove the following proposition which says that there is a bounded plurisubharmonic weight function such that the Hessian satisfies certain essentially maximal bounds in a thin strip near the boundary of Ω. For $\epsilon > 0$, we let $\Omega_\epsilon = \{z; r(z) < \epsilon\}$ and set

$$S(\epsilon) = \{z : -\epsilon < r(z) < \epsilon\}.$$

Theorem 3.2. *For all small $\delta > 0$, there is a plurisubharmonic function $\lambda_\delta \in C^\infty(\Omega_\delta)$ with the following properties,*

(i) $|\lambda_\delta(z)| \leq 1$, $z \in U \cap \Omega_\delta$.

(ii) *For all $L = \sum_{j=1}^{n} b_j L_j$ at $z \in U \cap S(\delta)$,*

$$\partial\bar{\partial}\lambda_\delta(z)(L,\bar{L}) \approx \delta^{-2}|b_1|^2 + \delta^{-1}\sum_{k=2}^{n-1}|b_k|^2 + \tau^{-2}|b_n|^2,$$

(iii) *If $\Phi_{z'}$ is the map associted with a given $z' \in U \cap S(\delta)$, then for all $\zeta \in R_\delta(z')$ with $|\rho(\zeta)| < \delta$,*

$$|D^\alpha(\lambda_\delta \circ \Phi_{z'})(\zeta)| \lesssim C_\alpha \delta^{-\alpha_n} \delta^{-\frac{1}{2}(\alpha_2+\dots+\alpha_{n-1})} \tau^{-\alpha_n}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$.

The proof is very cose to that of Theorem 3.1 in [3] and we omit the proof here. The following theorem was essentially done by Hörmander and Catlin has modified it in [3].

Theorem 3.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that $z' = (z'_1, \dots, z'_n)$ is a given point in Ω, that $\tau_1, \dots, \tau_n$ are given positive numbers, and that there is a function $\phi \in C^3(\overline{\Omega})$ that satisfies the following properties;*

(i) $|\phi(z)| \lesssim 1$, $z \in \Omega$.
(ii) *ϕ is plurisubharmonic in Ω.*
(iii) *Ω contains the polydisc $B = \{z; |z_i - z'_i| < \tau_i, i = 1, \dots, n\}$.*
(iv) *In Ω, ϕ satisfies*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j}(z) t_i \bar{t}_j \gtrsim \sum_{i=1}^{n} \tau_i^{-2} |t_i|^2, \quad z \in B.$$

(v) *If $D_i^{\alpha_i}$ denotes any mixed partial derivatives in z_i and $\overline{z}_i$ of total order α_i, then $D^\alpha \phi = D_1^{\alpha_1} \dots D_n^{\alpha_n} \phi$ satisfies*

$$|D^\alpha \phi(z)| \lesssim C_\alpha \prod_{i=1}^{n} \tau_i^{-\alpha_i}, \quad z \in B, \quad |\alpha| \le 3.$$

Then $K_\Omega(z', \overline{z}')$, the Bergman kernel function of Ω at z', satisfies

$$K_\Omega(z', \overline{z}') \approx \prod_{i=1}^{n} \tau_i^{-2} \tag{3.13}$$

We now ready to prove Theorem 1; Let $z \in U$ with $r(z) = -\frac{b\delta}{2}$ and $\pi(z) = z' \in b\Omega$ where U is a small neighborhood of $z_0 \in b\Omega$ and $b > 0$ is the number as in Proposition 2.9. Set $\phi'_\delta(\zeta) = \lambda_\delta \circ \Phi_{z'}(\zeta)$. Then ϕ'_δ will satisfy Theorem 3.3 in ζ-coordinates. So we will work on $\Omega_{z'} = (\Phi_{z'}^{-1})(\Omega)$. Set $\zeta = (-\frac{b\delta}{2}, 0, \ldots, 0)$. Then $\zeta = \Phi_{z'}^{-1}(z)$ and by (2.11) there is a constant $0 < c < 1$ such that the polydisc $B = \{\zeta : |\zeta_1 + b\delta/2| < c\delta, \quad |\zeta_k| < c\delta^{\frac{1}{2}}, \quad |\zeta_n| < c\tau(z', \delta), \; k = 2, \ldots, n-1\}$ lies in $\Omega_{z'}$. Hence $K_{\Omega_{z'}}(\zeta, \overline{\zeta}) \approx \delta^{-2}\delta^{-(n-2)}\tau(z', \delta)^{-2} = \delta^{-n}\tau(z', \delta)^{-2}$ by (3.13). Since the Jacobian of $\Phi_{z'}$ at ζ satisfies

$$|J_\zeta(\Phi_{z'})| = \left| det \left[\frac{\partial \Phi_{z'}^i}{\partial \zeta_j}(\zeta) \right] \right| \approx 1,$$

the transformation identity of the Bergman kernel function implies that

$$K_\Omega(z, \overline{z}) = |J_\zeta(\Phi_{z'})|^{-2} K_{\Omega_{z'}}(\zeta, \overline{\zeta}) \approx \delta^{-n}\tau(z', \delta)^{-2}.$$

Since $|r(z)| \approx \delta$ and since $z \in Q_\delta(z')$,

$$K_\Omega(z, \overline{z}) \approx |r(z)|^{-n}(\eta(z, \delta))^{-2} \approx \sum_{l=2}^{m} |C_l(z)|^{\frac{2}{l}} |r(z)|^{-n-\frac{2}{l}}$$

by (2.29), and hence we have proved Theorem 1. □

Remark 3.1. Theorem 3.2 says that the optimal subelliptic estimates (of order m) holds near z_0 according to the Catlin's theorem in [2].

Remark 3.2. The optimal estimates for the Caratheodory, Kobayashi and Bergman metrics will be obtained in the forthcoming article using the theorems in this article and [4].

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EXISTENCE OF RATIONAL FUNCTIONS OF RELATIVELY HIGH ORDER ON COMPLEX ALGEBRAIC CURVES*

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By the basic theorem of Griffiths and Harris in Brill-Noether theory, a general curve of genus g has a base-point-free pencil of degree $d \geq \left[\frac{g+3}{2}\right]$, but some special curve (in the sense of moduli) may not have such a pencil. On the other hand, by the Riemann-Roch theorem for curves it is quite easy to see that every curve of genus g possesses a base-point free pencil of degree $d \geq g$. Thus it is quite natural to ask what kind of special curves do not have a base-point-free pencil of certain degree d where d is a number close to g.

In this article, we present a new proof of the fact that unless the given curve C is hyperelliptic of odd genus, C admits a rational function of order $g-1$, which is equivalent to the existence of base point free pencils of degree $g-1$.

It should be noted that the answer to the above question or the existence (or non-existence) of base-point-free pencil of degree $g-1$ for every non-hyperelliptic curves has been known for many years (see e.g. [ACGH]), but we provide a simpler proof for statement in this article.

Here the issue is not the statement itself. The importance we want to stress is the new method in the proof, namely the enumerative method which we hope will have wider applications in similar circumstances. Indeed, the case for $d = g-2$ can be proved in the same kind of spirit (see [BK]) and this

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* This paper is in final form and no version of it will be submitted for publication elsewhere.

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improves and simplifies the results proved by the author jointly with Coppens and Martens [CKM].

Because there are many good references on the topic we are going to discuss (e.g. again [ACGH]), we will restrict ourselves to the proof of the more difficult part, i.e. we will only concentrate in providing new proofs for the following two statements. And it is left to the reader to check the other details.

Theorem 1. *Let C be a trigonal complex algebraic curve of genus $g \geq 4$. Then C has a base point free pencil of degree $g-1$.*

Theorem 2. *Let C be an elliptic-hyperlliptic curve of genus $g \geq 6$. Then C has a base point free pencil of degree $g-1$.*

The idea of our new proof is relatively simple. In fact, one needs to prove that there exists a component in $W^1_{g-1}(C)$ other than the component whose general element has a base locus. The proof as in [ACGH] is based on calculating the intersection numbers on the subvariety of $J(C)$, the Jacobian variety of C, and this causes some technical difficulties when one goes to the lower degree cases, especially when one deals with curves which are multiple covers of non-rational curves.

On the other hand, as far as the component we are considering are of pure and expected dimension one can do the intersection theory on the appropriate symmetric product of C_d of C, thanks to the fundamental class formula for the class of $C^r_d := \{D \in Div^d(C) : r(D) \geq r\}$ in case C^r_d has the expected dimension $\rho(g,r,d)+r$, where $\rho(g,r,d) := g-(r+1)(g-d+r)$ is the Brill-Noether number. And it turns out in general that this is relatively easy to handle.

Proof of Theorem 1 and Theorem 2.

We first recall some of the notations used in [ACGH]. Let $u : C_d \to J(C)$ be the abelian sum map and let θ be the class of the theta divisor in $J(C)$. Let $u^* : H^*(J(C), \mathbf{Q}) \to H^*(C_{g-1}, \mathbf{Q})$ be the homomorphism induced by u. By abusing notation, we use the same letter θ for the class $u^*\theta$. By fixing a point P on C, one has the map $\iota : C_{d-1} \to C_d$ defined by $\iota(D) = D + P$. We denote the class of $\iota(C_{d-1})$ by x.

Let $\pi : C \to E$ be the 2-sheeted covering, genus $(E) = 1$. By the various H. Martens and Mumford type dimension theorems on the subvariety of $J(C)$ or just by the well-known classical fact that the singular locus of the theta divisor in the $J(C)$ has dimension $g-4$ for non-hyperelliptic curve $C, W^1_{g-1}(C)$ is of pure dimension $g-4 = \rho(g,1,g-1)$, hence the variety $C^1_{g-1} (\subseteq C_{g-1})$ is of pure dimension $g-3$. Also it is easy to show that the only component of $W^1_{g-1}(C)$ whose general element has a base point is $\pi^*\left(W^1_2(E)\right) + W_{g-5}(C)$ and hence the only component of C^1_{g-1} consisting of divisors whose complete linear series have base points is $\pi^*\left(E^1_2\right) + C_{g-5}$, whose class in C^1_{g-1} we denote by γ. Because C^1_{g-1} is of pure (and expected) dimension $\rho(g,1,g-1)+1$, the fundamental class c^1_{g-1} of C^1_{g-1} is known (cf. [ACGH], Theorem p326) : $c^1_{g-1} = \frac{1}{2}\theta^2 - x\theta$. Let's also recall that given a cycle Z in C_d, the assignments

$$Z \mapsto A_k(Z) := \{E \in C_{d+k} : E - D \geq 0 \text{ for some } D \in Z\},$$
$$Z \mapsto B_k(Z) := \{E \in C_{d-k} : D - E \geq 0 \text{ for some } D \in Z\}$$

induce maps

$$A_k : H^{2m}(C_d, \mathbf{Q}) \to H^{2m+2k}(C_{d+k}, \mathbf{Q}),$$
$$B_k : H^{2m}(C_d, \mathbf{Q}) \to H^{2m-2k}(C_{d-k}, \mathbf{Q})$$

and the so called push-pull formulas for symmetric products hold (cf. [ACGH] p367-369). Thus by the push-pull formulas,

$$B_{g-5}\left(x^{g-3}\right) = \frac{(g-3)!}{2(g-5)!}x^2.$$

Denoting $\tilde{\gamma}$ by the class of $\pi^*\left(E^1_2\right)$ in C_4, we have

$$\begin{aligned}\left(\gamma \cdot x^{g-3}\right)_{C_{g-1}} &= \left(A_{g-5}(\tilde{\gamma}) \cdot x^{g-3}\right)_{C_{g-1}} = \left(\tilde{\gamma} \cdot B_{g-5}\left(x^{g-3}\right)\right)_{C_4} \\ &= \left(\tilde{\gamma} \cdot \frac{(g-3)!}{2(g-5)!}x^2\right)_{C_4} = \frac{(g-3)(g-4)}{2}\end{aligned}$$

by noting the simple fact that $\left(\tilde{\gamma} \cdot x^2\right)_{C_4} = 1$.

On the other hand

$$\left(c^1_{g-1}\cdot x^{g-3}\right)_{C_{g-1}}=\left(\frac{1}{2}\theta^2-x\theta\cdot x^{g-3}\right)_{C_{g-1}}=\frac{1}{2}\theta^2x^{g-3}-\theta x^{g-2}$$
$$=\frac{g!}{2(g-2)!}-\frac{g!}{(g-1)!}=\frac{1}{2}\left(g^2-3g\right),$$

by the Poincaré's formula. Comparing the above intersection numbers we have

$$\left(\gamma\cdot x^{g-3}\right)_{C_{g-1}}<\left(c^1_{g-1}\cdot x^{g-3}\right)_{C_{g-1}}$$

and this shows that there exists a component other than $\pi^*\left(E^1_2\right)+C_{g-5}$ in C^1_{g-1} which in turn shows that there exist divisors of degree $g-1$ which move in a pencil and whose complete linear system does not have a base point.

For the trigonal curve case, one should remark that there is an elementary proof without using the enumerative methods, which is also left to the reader. On the other hand, since the purpose of this note is to introduce the enumerative calculation on the subvariety of the symmetric product on the given curve C, we also present a enumerative proof for the trigonal case.

By the same reason as in the elliptic-hyperelliptic case, we note that the only component of $W^1_{g-1}(C)$ whose general element has a base locus is of the form $W^1_3(C)+W_{g-4}(C)$. Hence the only component of C^1_{g-1} consisting of divisors whose complete linear series have base points is $C^1_3+C_{g-4}$ whose class in C^1_{g-1} we denote by η. By the push-pull formulas, one has $B_{g-4}\left(x^{g-3}\right)=(g-3)x$. On the other hand, denoting $\tilde{\eta}$ by the class of C^1_3 in C_3, we have

$$\left(\eta\cdot x^{g-3}\right)_{C_{g-1}}=\left(A_{g-4}(\tilde{\eta})\cdot x^{g-3}\right)_{C_{g-1}}=\left(\tilde{\eta}\cdot B_{g-4}\left(x^{g-3}\right)\right)_{C_3}$$
$$=\left(\tilde{\eta}\cdot(g-3)x\right)_{C_3}=g-3$$

again by noting the simple fact that $(\tilde{\eta}\cdot x)_{C_3}=1$.

Comparing the intersection number $\left(\eta\cdot x^{g-3}\right)_{C_{g-1}}$ with $\left(c^1_{g-1}\cdot x^{g-3}\right)_{C_{g-1}}$ which we already have, we see that

$$\left(\eta\cdot x^{g-3}\right)_{C_{g-1}}<\left(c^1_{g-1}\cdot x^{g-3}\right)_{C_{g-1}}$$

and this shows that there exists a component other than $C^1_3+C_{g-4}$ in C^1_{g-1} which in turn show that there exist divisors of degree $g-1$ which moves in a pencil and whose complete linear system does not have a base point.

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STABLE REDUCTION OF SOME FAMILIES OF PLANE QUARTICS IN ORDER TO GET A COMPLETE CURVE OF $\mathcal{M}_3$

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Let $\mathcal{M}_g$ be the moduli space of isomorphism classes of genus g smooth curves. It is a quasi-projective variety of dimension 3g-3, when $g > 2$. It is known that a complete subvariety of $\mathcal{M}_g$ has dimension $< g - 1[D]$. In general it is not known whether this bound is rigid. For example, it is not known whether $\mathcal{M}_4$ has a complete surface in it. But one knows that there is a complete curve through any given finite points [H]. Recently, an explicit example of a complete curve in moduli space is given in [G-H]. In [G-H] they constructed a complete curve of $\mathcal{M}_3$ as an intersection of five hypersurfaces of the Satake compactification of $\mathcal{M}_3$.

One way to get a complete curve of $\mathcal{M}_3$ is to find a complete one dimensional family $p : X \to B$ of plane quartics which gives a nontrivial morphism from the base space B to the moduli space $\mathcal{M}_3$. This is because every non-hyperelliptic smooth curve of genus three can be realized as a nonsingular plane quartic and vice versa. Since nonsingular quartics form an affine space some fibers of p must be singular ones. In this paper, due to semistable reduction theorem [M], we search singular irreducible plane quartics which can occur as a singular fiber of the family above.

Let P^{14} be the projective space parametrizing all plane quartics, C a singular quartic and E an equisingular stratum containing C. Let Δ be an open unit disk of C. We embed Δ locally in P^{14} in such a way that $\Delta - \{0\}$ is contained in the locus of smooth curves and 0 maps to C. Pulling back the universal family over P^{14} to a family over Δ we get a family $p : X \to \Delta$ of smooth plane quartics degenerating to C. We call C the central fiber of p and P the singular point of C we examine. Note that the total surface X is either

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nonsingular or singular at P according how we embed Δ in P^{14}. We also take a family whose generic fibers are not projectively equivalent to each other. For a chosen family as above we make X nonsingular and do semistable reduction (i.e., base change, desingularization, blowups and blowdowns, and repetition of these), which gives us a semistable curve. A semistable curve is a reduced nodal curve without (-1) rational component. It is allowed to have (-2) rational components. Now by contracting (-2) rational components we get a stable curve of genus three. We call it *a stable model* of C. From it we can determine the map Δ into $\overline{\mathcal{M}_3}$, the Deligne-Mumford compactification of $\mathcal{M}_3$, i.e., the moduli space of all genus three stable curves.

The purpose of this talk is to find singular *irreducible* plane quartics which have a nonsingular curve as one of its stable model.

In the following we list all possible distinct irreducible plane quartics in terms of singularities.

Proposition 1. *There are 55 equisingular strata in P^{14}, 20 among which are loci of irreducible curves, besides nonsingular ones. They are irreducible quartics*

0) which are nonsingular
*1) with one node (1N)**
2) with two nodes (2N)
3) with three nodes (3N)
4) with one cusp (2C)
5) with one cusp and one node (3C)
6) with one cusp and two nodes (4C)
7) with a tacnode (3T)
8) with a tacnode and a node (4T)
9) with a triple point (4Tr)
10) with two cusps (4CC)
11) with two cusps and a node (5CC)
12) with a double cusp (4dC)
13) with a double cusp and a node (5dC)
14) with a tacnode and a cusp (5CT)
15) with an osnode (5oN)
16) with a cusp with a smooth branch (5Cs)
17) with three cusps (6CCC)
18) with a cusp and a double cusp (6CdC)
19) with a triple cusp (6tC)
20) with an ordinary cusp of multiplicity three ($6C_3$)

* *The number is the codimension in P^{14} and letters represent the singularities of the corresponding curves.*

Proof. For irreducible quartcs, it is classical, or see Namba [N].

Remark. (1) If one wants to get reducible ones, combine all possible plane curves of degree < 4 with Bezout theorem.

(2) Each equisingular stratum of the space of all plane quartics is irreducible.

(3) Some reducible ones can have a smooth stable model. A well known example is a double conic; the canonical model of genus three smooth curves approaching a hyperelliptic one will tend to a double conic. For other candidates we leave them in [K].

(4) A stable model is not unique. It depends on the given curve C and the embedding of Δ.

(5) A stable model of C will always contain as its components the normalization of all nonrational components of C and components produced by each singular point other than a node of C in the process of seminstable reduction.

(6) If C does not have a unique tangent line at a singulaity and the pont at infinity then the normalization must meet other components at more than two points, so it cannot have a nonsingular stable model. More precise statement will appear in a paper [K] on a rational map $f : P^{14} \to \overline{\mathcal{M}}_3$.

Therefore, for irreducible plane quartics to have a nonsingular stable model, it should be rational with only one singular point with unitangent line (cuspidal singular point). From Proposition 1, there remain only two curves; a plane quartic with a triple cusp, and one with an ordinary cusp of multipliticity three.

A k-tuple cusp (of a plane curve) is a double point with a unique tangent line which becomes (k-1)-tuple cusp after a blowup. An ordinary cusp of multiplicity three is a triple point with a unique tangent line which becomes smooth point by taking a blowup once, i.e., a point whose local equation is $y^3 = x^4$.

Theorem. *The above two candidates have a smooth stable model.*

Proof. It is enough to give a family $p : X \to \Delta_t$ of smooth quartics degenerating to each candidate, the stable reduction of which replaces C with a smooth curve of genus three.

For a plane quartic with a triple cusp we take C $y^2z^2+2x^2yz+x^4+xy^3=0$. For a detail, see [K]. It has a triple cusp at (0:0:1) with a tangent line $y=0$. Let us take X as $y^2z^2+2x^2yz+x^4+xy^3+txz^3=0$ in $P^2 \mathrm{x} \Delta_t$, or $y^2z^2+2x^2yz+x^4+xy^3+tz^4=0$ ($y^2+2x^2y+x^4+xy^3+tx=0$, or $y^2+2x^2y+x^4+xy^3+t=0$ in $A^3_{(x,y,t)}$ resp.). We note that these are not the only possibilities for X for the chosen C. For other cases, see [K] too. For several reasons we take the former X.

Since X has a unique singular point, it is a normal. So, blow up A^3 at origin and compute the proper transform X' of X. Here

$$X' = \{y^2+2x^2y+x^4+xy^3+tx=0,\ xy'-x'y=xt'-x't=yt'-y't=0\}$$

in $A^3 \times P^2_{(x',y',t')}$. When x' is not equal to zero, the local equation of X' becomes

$$y'^2+2xy'+x^2+x^2y'^3+t'=0$$

with local coordinates x, y' and t'. Over $t=t'x=0$, we have two components- the proper transform C_1 of $C\left\{t'=0, y'^2+2xy'+x^2+x^2y'^3=0\right\}$ and the exceptional divisor $L_1\left\{x=0, y'^2+t'=0\right\}$, which meet at (0,0,0) [figure 1]. Note that C_1 has a double cusp. For an affine neighborhood that y' or t' is not zero, we do the same computation and do not get any more information. With X' smooth we apply semistable reduction theorem. We explain this process by chasing the central fiber with the following figures. We first blow up the intersection point of two components of the central fiber. In figure 2, the vertical line is a triple exceptional line, L_2 the proper transform of L_1, and C_2 the proper transform of C_1, which has an ordinary cusp. Blow up P_2. The new central fiber is in figure 3. Note the number next each component represents the multiplicity of the corresponding one. We also denote M_{i+1} (or L_{i+1}) the proper transform of M_i (or L_i). Blow up P_3 (figure 4) and then P_4 (figure 5). In figure 5, there is no other singular point other than nodes.

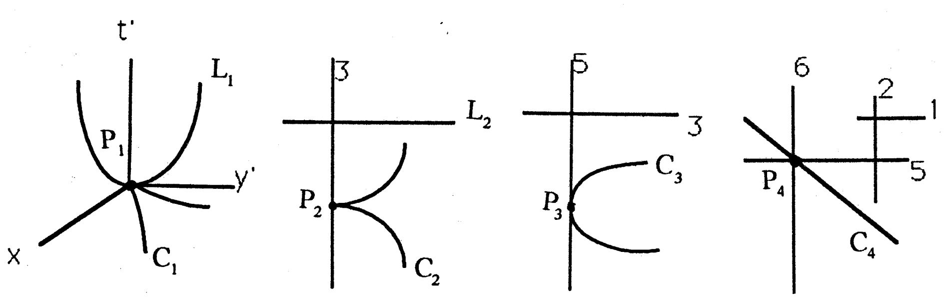

figure 1 figure 2 figure 3 figure 4

We now make base changes to remove multiple components. The base change of order two gives us figure 6, where the vertical curve is two fold cover of the previous vertical curve. By Riemann-Hurwitz, it is rational. The base change of order three replaces figure 6 with figure 7. Here, the vertical double line is over the previous vertical one totally branched at two points Q_1 and Q_2. By Riemann-Hurwitz again it is rational too; 2g - 2 = 3(-2) + 2(2). The base change of order two to get figure 8: the genus of the vertical curve has genus three; 2g - 2 = 2(-2) + 8.

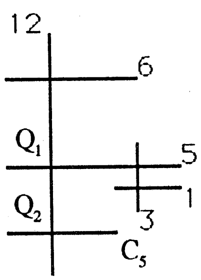

figure 5

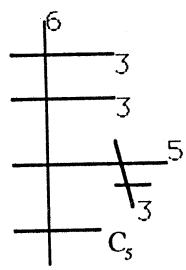

figure 6

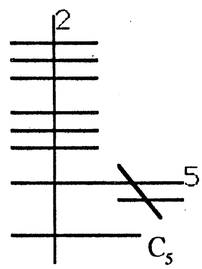

figure 7

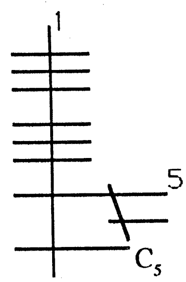

figure 8

In figure 8, seven horizontal (-1) rational curve can be blow-downed. After base change of order 5, you may make remaining rational components reduced, so they can be blown down too. Therefore, we have a smooth curve of genus three. In particular we get a hyperelliptic one.

For a plane quartic C with an ordinary cusp of multiplicity three, there are two up to projective equivalence [N]: $y^3z + y^4 + x^4 = 0$ and $y^3z + x^4 = 0$. Here we take C $y^3z + y^4 + x^4 = 0$ and X $y^3z + y^4 + x^4 + tz^4 = 0$, or for similar reason, $y^3z + y^4 + x^4 + txz^3 = 0$. Doing the exactly same process as before, we get a smooth curve of genus three as a stable model. In this case it is a trigonal.

Remark. The above theorem does not claim that all possible stable models of two curves in theorem are smooth. In fact, this happens rather rare. If we take X $y^2 + 2x^2y + x^4 + xy^3 + ty = 0$ when C is a quartic with a triple cusp, we get, as a stable model, a reducible curve consisting of a genus two curve and an elliptic curve meeting at one point.

For a quartic with an ordinary cusp of multiplicity three, the similar example $y^3 + y^4 + x^4 + ty = 0$ gives a smooth one too. But if we take X as $y^3 + x^4 + t^2x + tx^2 = 0$, we get a reducible curve consisting of a genus two curve and an elliptic curve meeting at one point.

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ON THE LUROTH SEMIGROUP OF ALGEBRAIC CURVES

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Abstract. In this note, we calculate the Luroth semigroup of a trigonal curve, an elliptic-hyperelliptic, a 2-hyperelliptic crve. And we study which curves are characterized by the difference in Luroth semigroup from a general k-gonal curves.

In the study of smooth irreducible complete algebraic curve C it is interesting and classical problem to know for which integers n there are linear series of degree n without base points (equivalently: rational functions of degree n).

The Luroth semigroup S_C of C is the additive sub-semigroup of $\mathbf{N}$ consisting of all the degrees of the linear series on C without base points.

The knowledge of this semigroup S_C or, equivalently, the set of gaps of C, that is $N \setminus S_C$, gives a good deal of information on the geometry of C.

In few cases, S_C is completely determined.

Example. (i) $S_C = N$ if and only if $C \simeq \mathbf{P}^1$.

(ii) If $g = 1$, then $n = 1$ is the only gap.

(iii) If $g \geq 2$ and C is hyperelliptic, then S_C is generated by 2, $g+1$ and $g+2$.

(iv) If C is non-hyperelliptic, then $g \in S_C$ and for every $n \geq g+1$, $n \in S_C$.

(V) If C is a general curve, then $S_C = \{ n \mid n \geq \frac{g+1}{2} \}$.

Remark. In case C is a smooth plane curve and C is a smooth curve lying on a smooth quadric surface, a lot of informations about S_C are known (see [GR], [PRR]).

Lemma 1. *(1) Let C, B_1, B_2 be curves of respective genera g, g_1, g_2. Assume that $\pi_i : C \to B_i$ $(i = 1, 2)$ is a d_i-sheeted covering such that $\pi = \pi_1 \times \pi_2 : C \to B_1 \times B_2$ is birational onto its image. Then $g \leq (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2$.*

(2) Let C be a n-sheeted covering of a curve C' of genus g' of genus $g > (n-1)^2 + 2ng'$. If f is a meromorphic function on C with order $o(f) < \frac{g-ng'}{n-1} + 1$, then f is a lift of a meromorphic function on C'.

proof. See [M].

Theorem 2. *If C is a trigonal curve with $g \geq 4$, then*

$$S_C = \{ n \mid n \geq \left[\tfrac{g+1}{2}\right] + 1 \text{ or } n = 3k \text{ for some } k \}.$$

proof. If there exists a meromorphic function of order $n < \left[\frac{g+1}{2}\right]$ and $n \neq 3k$, then $g \leq 2(n-1) \leq g-1$ by the lemma 1 (1). Thus n with $n < \left[\frac{g+1}{2}\right]$, $n \neq 3k$ is not contained in S_C. On the other hand by the result of [H] it holds for a trigonal curve of $g \geq 4$ that $W_r^1 = (W_3^1 + W_{r-3}) \cup (k - W_3^1 - W_{2g-r-5}^{g-r-1})$ $(3 \leq r \leq g-1)$. Then $W_{g-1}^1 = (W_3^1 + W_{g-4}) \cup (k - W_3^1 - W_{g-4})$ and $W_{g-2}^1 = (W_3^1 + W_{g-5}) \cup (k - W_3^1 - W_{g-3}^1)$. By the way, $\dim\,(k - W_3^1 - W_{g-4}) = g-4 > \dim\,\{(k - W_3^1 - W_{g-3}^1) + W_1\} = g-5$. Thus there exists a base-point-free g_{g-1}^1, i.e. $g-1 \in S_C$. By the similar way we have $g-2, \cdots, \left[\frac{g+1}{2}\right] + 2 \in S_C$.

Now consider $\left[\frac{g+1}{2}\right] + 1$. Then we have

$$W^1_{\left[\frac{g+1}{2}\right]+1} = (W_3^1 + W_{\left[\frac{g+1}{2}\right]-2}) \cup (k - W_3^1 - W^{\left[\frac{g}{2}\right]-2}_{\left[\frac{3}{2}g\right]-6}).$$

And $W^1_{\left[\frac{g+1}{2}\right]} = W_3^1 + W_{\left[\frac{g+1}{2}\right]-3}$ because there exists no meromorphic function of order $\left[\frac{g+1}{2}\right]$, Moreover $W_3^1 + W_{\left[\frac{g+1}{2}\right]-2} \neq k - W_3^1 - W^{\left[\frac{g}{2}\right]-2}_{\left[\frac{3}{2}g\right]-6}$ because otherwise $W^{\left[\frac{g}{2}\right]-2}_{\left[\frac{3}{2}g\right]-6} = k - 2W_3^1 - W_{\left[\frac{g+1}{2}\right]-2}$ and hence $k - 2W_3^1 \subseteq W^{\left[\frac{g}{2}\right]-2}_{\left[\frac{3}{2}g\right]-6} \ominus (-W_{\left[\frac{g+1}{2}\right]-2}) = W^{g-4}_{2g-8}$, which is a contradiction beacuse C is non-hyperelliptic (see [H]). Therefore there exists a base-point-free $g^1_{\left[\frac{g+1}{2}\right]+1}$, i.e. $\left[\frac{g+1}{2}\right] + 1 \in S_C$. □

Remark. In the above proof, we know that there exists a base-point-free $g^1_{\left[\frac{g+1}{2}\right]+1}$. And $g^1_3 \times g^1_{\left[\frac{g+1}{2}\right]+1}$ is birational onto its image in $\mathbf{P}^1 \times \mathbf{P}^1 = Q$. If g is even, C is birationl to smooth $(3, \frac{g}{2}+1)$ type curve on a smooth quadric surface Q. Thus we can get the same result about S_C of a trigonal curve with even genus by using the results in [PRR].

Theorem 3. *Let C be an elliptic-hyperelliptic curve of $g \geq 6$. Then*

$$S_C = \{\, n \mid n \textit{ is even with } 4 \leq n \leq g-2 \textit{ and } n \geq g-1 \,\}.$$

proof. By the lemma 1 (2), every meromorphic function on C with order $n \leq g-2$ is a lift of a meromorphic fuction on the elliptic base curve if n is even, and no meromorphic function of order $n \leq g-2$ exists if n is odd. Moreover an elliptic-hyperelliptic curve of $g \geq 6$ admits a base-point-free g^1_{g-1} (see [H]) and hence we get the theorem. □

Theorem 4. *Let C be a 2-hyperelliptic curve with $g \geq 13$ then*

$$S_C = \{\, n \mid n \textit{ is even with } 4 \leq n \leq g-4 \textit{ and } n \geq g-3 \,\}.$$

proof. By the lemma 1 (2), every meromorphic function on C with order $n \leq g-4$ is a lift of a meromorphic fuction on the base curve of genus 2 if n is even, and no meromorphic function of order $n \leq g-4$ exists if n is odd. Moreover a 2-hyperelliptic curve of $g \geq 13$ admits a base-point-free g^1_{g-1}, g^1_{g-2}, g^1_{g-3} (see [CKM1]). □

Let C be a general k-gonal curve. By the Brill-Neothon theory we know that if $\rho(g,1,n) \geq 0$ (i.e. $n \geq \frac{g}{2}+1$), then C has a complete and base-point-free pencil g^1_n and if $\rho(g,1,n) < 0$ (i.e. $n < \frac{g}{2}+1$) and $n \neq mk$ for some m, then n is a gap (see [CKM2]). By the theorem 2, we see that the Luroth semigroup of a trigonal curve is the same as that of a general k-gonal curve. But in the case that C is either elliptic-hyperelliptic or 2-hyperelliptic, they are different (see Theorem 2,3). So we want to know which curves are characterized by the difference in Luroth semigroup from the general k-gonal curve.

Firstly we need the following lemma

Lemma 5. *Let C be an irreducible curve of type (a, b) with $a+b>9$ lying on a smooth quadric surface Q in $\mathbf{P}^3$. If g^s_m is a linear series on C with $m \le a+b$, then g^s_m is induced by the hyperplane system.*

proof. Let D be a generic divisor in g^s_m and let x_0 be the least integer for which the following property holds: there exists an irreducible surface G of degree x_0 in $\mathbf{P}^3$ such that

1. $G \not\supset C$
2. G cuts out on C a divisor containing D.

We denote by $D+D'$ the divisor cut out by G on C. Consider the linear system Σ formed by all surfaces of degree x_0 which cut out on C a divisor containing D'. Let F' and F'' be generic elements in Σ and C' the complete intersection curve of F' and F''. Then $h^0(I_{F'\cap Q}(x_0)) = h^0(\mathcal{O}_{\mathbf{P}^3}(x_0)) - h^0(\mathcal{O}_{F'\cap Q}(x_0)) = \binom{x_0+3}{3} - \{2x_0^2 - (x_0^2 - 2x_0 + 1) + 1\} = \frac{x_0+2}{6}(x_0^2 - 2x_0 + 3)$ because $\mathcal{O}_{F'\cap Q}(n)$ is nonspecial for $n \ge x_0 - 1$. Let n be the number of conditions imposed by D' on hypersurfaces of degree x_0 in $\mathbf{P}^3$. Then n is the same as the number of conditions which D' imposes on the complete intersection curves of the hypersurfaces of degree x_0 in $\mathbf{P}^3$ and Q because Q contains D'. Consequently, we have

$$h^0(I_{D'}(x_0)) - h^0(I_{F'\cap Q}(x_0)) \ge \binom{x_0+3}{3} - n - \frac{x_0+2}{6}(x_0^2 - 2x_0 + 3)$$
$$= x_0^2 + 2x_0 - n > 0$$

since $dim|x_0E - D'| = \binom{x_0+3}{3} - \binom{x_0+1}{3} - 1 - n = x_0(x_0+2) - n > 0$ where E is of type $(1,1)$ i.e., the divisor cut out on Q by the hyperplanes in $\mathbf{P}^3$. Therefore we can choose F'' in Σ such that F'' does not contain $F'\cap Q$, and hence the cycle of intersection $Q\cap C'$ is well defined and $Q \cap C' \supset D'$. Then this yields $deg(Q\cap C') \ge degD'$ and hence $2x_0^2 \ge (a+b)x_0 - m \ge (a+b)x_0 - (a+b)$ since deg $D' = (a+b)x_0 - m$. Thus we have

$$2x_0^2 - (a+b)x_0 + (a+b) \ge 0. \qquad (*)$$

and (*) can only happen if either $x_0 = 1$ or $x_0 > \frac{a+b}{2} - 2$ because $\frac{a+b}{2} > 4$. But the linear series $|([\frac{a+b}{2}] - 2)E - D|$ is non-empty, for

$$dim\left|([\frac{a+b}{2}] - 2)E\right| \ge \frac{1}{2}\left(([\frac{a+b}{2}] - 2)E\right).\left(([\frac{a+b}{2}] - 2)E - K\right) + 1$$
$$= \frac{1}{2}\left([\frac{a+b}{2}] - 2, [\frac{a+b}{2}] - 2\right)\left([\frac{a+b}{2}], [\frac{a+b}{2}]\right) + 1$$
$$\ge \frac{(a+b)^2}{4} - \frac{3}{2}(a+b) + \frac{5}{4}$$

and $\frac{a+b)^2}{4} - \frac{3}{2}(a+b) - \frac{5}{4} - (a+b) > 0$ since $a+b > 9$. Therefore $x_0 \leq \frac{a+b}{2} - 2$, and this yields $x_0 = 1$ and hence the conclusion follows. □

Theorem 6. *Let C be a k-gonal curve with $k \geq 4$. If $l > k+1$ and $l \neq 2k$ is contained in S_C and $k+1, \cdots, k+l-3$ except l are gaps of C, then C is birational to a smooth curve of type (k, l) on a smooth quadric surface Q in $\mathbf{P}^3$.*

proof. Since $l \in S_C$, there exists a base-point-free g_l^r for some r. If $r \geq 3$, then we have at least two linear series g_{l-1}^{r-1}, g_{l-r}^1 in case g_l^r is birational and g_{l-m}^{r-1}, g_{l-mr}^1 in case g_l^r is not birational because any reduced irreducible and nondegenerate curve of $d \geq r+2$ in $\mathbf{P}^r$, $r \geq 3$ has an r-secant $(r-2)$-plane. This is a contradiction because there is only one element k in S_C less than l. Thus $r \leq 2$. If $r = 2$, then g_l^2 cannot be birational because $l-1$ is a gap. Then g_l^2 gives a m-sheeted map and we get $l-m = k$ and l is a multiple of m. Thus it follows that $l = 2k$, which contradicts to our hyphothesis. Therefore $r = 1$.

Now consider $g_k^1 \times g_l^1$. Then it is birational onto its image on a smooth quadric Q. Because otherwise it defines a morphism of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^3$ of degree $\frac{k+l}{m}$. Then we can have a $g^2_{\frac{k+l}{m}-1}$ on C' by substracting a generic point from $g^3_{\frac{k+l}{m}}$ and hence g^2_{k+l-m} on C. Also we have g^1_{k+l-2m} by substracting two generic points. Since C has only g_k^1, g_l^1 with degree less than $k+l$, we have $k+l-m = l$ and $k+l-2m = k$ and hence $l = 2k$, which is a contradiction. Moreover the image curve cannot be singular because otherwise we can have g_n^1 with $n \leq l+k-3$ by sustracting a singular point and a smooth point from g^3_{k+l}, which is a contradiction. □

Remark. If C is an irreducible smooth curve of type (k, l) with $k \geq 4$ on a smooth quadric surface Q in $\mathbf{P}^3$, then $k+1, \cdots, k+l-3$ except l are gaps. Because otherwise they are induced by the hyperplane system by the lemma 5 and hence C has a n-secant line L with $3 \leq n \leq k-1$, $k+1 \leq n \leq l-1$. Then L is contained in Q because $n \geq 3$. So n is either k or l, which is a contradiction.

Theorem 7. *Let C be a k-gonal curve with $g \geq 22$ and $k \geq 4$. If $k+1$, $2k-2$ are contained in S_C and $k+2, \cdots, 2k-3$ are gaps of C, then C is either a smooth plane curve of degree $k+1$ or a singular curve of type $(k, k+1)$ with a multiplicity 2 singular point lying on a smooth quadric surface Q in $\mathbf{P}^3$.*

proof. Since $k+1 \in S_C$, there exists a base-point-free g_{k+1}^r for some r. If $r \geq 3$, then we have a either g^1_{k+1-r} or g^1_{k+1-mr} for some m because any reduced irreducible and nondegenerate curve of degree $\geq r+2$ in $\mathbf{P}^r$ $r \geq 3$, has an r-secant $(r-2)$-plane, which is a contradiction. Thus $r \leq 2$. If $r=2$, then g^2_{k+1} has to be very ample and C is a smooth plane curve of degree $k+1$.

In case $r=1$, consider g^s_{2k-2} for some s. If $s \geq 2$, we have a base-point-free g^{s-1}_{2k-2-l} for some l $(l \geq 1)$ by substracting a generic point. Hence $s=2$ and $l = k-2$ or $k-3$ because $k+2, \cdots, 2k-3$ are gaps of C. If $l=k-2$, then C is a $(k-2)$-sheeted covering of a plane $\frac{2k-2}{k-2}$-tic curve C' and hence $k = 3$, 4. If $k=4$, then C is a 2-sheeted covering of a plane cubic curve. Since $g \geq 22$, g^1_5 is a lifting of a meromorphic function on C' by Lemma 1 (2), which cannot be happened. If $l = k-3$, then C is a $(k-3)$-sheeted covering of a plane $\frac{2k-2}{k-3}$-tic curve C' and hence $k = 4$, 5, 7. If $k=4$, then g^2_6 is birational and hence $g \leq 10$, which contradicts our genus bound. If $k=5$, C is a 2-sheeted covering of a plane quartic curve C' and by lemma 1 (2) g^1_5 is a lifting of a meromorphic function on C' because $g \geq 22$, which cannot be happened. If $k=7$, C is a 4-sheeted covering of a plane cubic curve and by lemma 1 (2) g^1_7 is a lifting of a meromorphic function on C' because $g \geq 22$, which is a contradiction. Therefore $s=1$ and hence C has g^1_k, g^1_{k+1} and g^1_{2k-2}.

Now consider $g^1_k \times g^1_{k+1}$. It is birational onto its image on a smooth quadric surface in $\mathbf{P}^3$. Because otherwise it defines a morphism of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^3$ of degree $\frac{2k+1}{m}$. Then we can have a $g^2_{\frac{2k+1}{m}-1}$ on C' by substracting a generic point from $g^3_{\frac{2k+1}{m}}$ and hence g^2_{2k+1-m} on C. Also we have $g^1_{2k+1-2m}$ by substracting two generic points. Then by the hypothesis on S_C we have $m=2$. But in this case $g^1_{2k+1-2m} = g^1_{2k-3}$, which is a contradiction. Thus C is birational to a curve of type $(k, k+1)$ on a smooth quadric surface Q. By the lemma 5, g^1_{2k-2} is induced by the hyperplane system and hence C has a trisecant line L. If C is smooth, then three points lying on L are distinct and hence L is contained in Q. Thus L is either k or $(k+1)$-secant line and so g^1_{2k-2} has a base point, which is a contradiction. Therefore C is singular. And from the condition "$2k-3 \notin S_C$" it follows that C has only one multiplicity 2 singular point. □

Remark. If C is a smooth plane curve of degree $k+1$, then C has g^1_k, g^2_{k+1} and g^1_{2k-2}. Moreover we see that $k+2 \cdots, 2k-3$ are gaps because there is no base-point-free g^1_n with $5 \leq d \leq n \leq 2d-5$ except $n=d$ and no g^2_n with $n \leq 2d-4$, $d \geq 4$ except $n=d$ and no g^r_d with $d \geq 2$, $r \geq 3$, and $1 \leq n \leq 2d-3$ (see [N]).

If C is a singular curve of type $(k, k+1)$ with one multiplicity 2 singular point lying on a smooth quadric surface in $\mathbf{P}^3$, then we can see $k+2, \cdots, 2k-3$ are gaps of C because the linear series with degree $\leq 2k+1$ are induced by the hyperplane system by the lemma 5.

Theorem 8. *Let C be a k-gonal curve. If $k+1$ is contained in S_C and $k+2, \cdots, 2k-3$ and $2k+3$ are gaps of C, then C is a smooth plane curve of degree $k+1$.*

proof. Since $k+1 \in S_C$, there exists a base-point-free g^r_{k+1} for some r. Then $r \leq 2$ because otherwise we have a either g^1_{k+1-r} or g^1_{k+1-mr} for some m with $m \geq 2$ because any reduced irreducible and nondegenerate curve of degree $\geq r+2$ in $\mathbf{P}^r$ $r \geq 3$, has an r-secant $(r-2)$-plane, which is a contradiction.

If $r = 2$, then g^2_{k+1} must be very ample and C is a smooth plane curve of degree $k+1$.

In case $r = 1$, consider $g^1_k \times g^1_{k+1}$. It is birational onto its image on a smooth quadric surface in $\mathbf{P}^3$. Because otherwise it defines a morphism of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^3$ of degree $\frac{2k+1}{m}$. Then we can have a $g^2_{\frac{2k+1}{m}-1}$ on C' by substracting a generic point from $g^3_{\frac{2k+1}{m}}$ and hence g^2_{2k+1-m} on C. Also we have $g^1_{2k+1-2m}$ by substracting two generic points. Then by the hypothesis on S_C we have $m = 2$. But in this case $g^1_{2k+1-2m} = g^1_{2k-3}$, which is a contradiction. Thus C is birational to a curve of type $(k, k+1)$ on a smooth quadric surface in $\mathbf{P}^3$. Now consider $3g^1_{k+1} = g^r_{3k+3}$ $(r \geq 3)$. Since C has a k-secant line, we can obtain g^{r-2}_{2k+3} by projecting from a k-secant line. Hence $2k+3 \in S_C$, which is a contradiction.

Remark. If C is a smooth plane curve of degree $k+1$, then $k+2, \cdots, 2k-3$ and $2k+3$ are gaps (see remark below theorem 8 and [GR]).

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NORMAL QUINTIC ENRIQUES SURFACES WITH MODULI NUMBER 6

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ABSTRACT. We present a families of normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces. These Enriques surfaces are characterized by a special type of divisors D. We then show that the space of Enriques surfaces obtained from the above family of normal quintic surfaces is of dimension 6.

1. Introduction

An Enriques surface S is a non-singular surface S over a complex number field $\mathbb{C}$ satisfying one of the following equivalent conditions:

(1) $2K_S \sim \mathcal{O}_S$, but $K_S \nsim \mathcal{O}_S$, and $q(S) = 0$.
(2) $K_S \equiv 0$ and $b_2(S) = 10$.
(3) S is minimal with $\varkappa(S) = 0$ and $b_2(S) = 10$.
(4) S is minimal with $\varkappa(S) = 0$ and $p_g = 0\,,\ q = 0$.

Normal quintic Enriques surfaces are then normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces.

Enriques surfaces were dicovered by Federigo Enriques. They are one of the first examples of non-rational surfaces with vanishing geometric genus. The following theorem says that the dimension of the moduli space of Enriques surfaces is 10.

Local Moduli Theorem. *The Kuranishi family for an Enriques surface S is universal at all points in a small neighborhood U around the point corresponding to S. The base space is smooth and has dimension 10. The period map is a local isomorphism at each point of U.*

Let (V, p) be a normal surface singularity and $\pi : M \to V$ a minimal resolution of (V, p).

Definition. The *geometric genus* $h(p)$ of V at p is the dimension of the complex vector space $H^1(M, \mathcal{O}_M)$. This number is finite and independent of the choice of resolution of singularity $\pi : M \to V$.

If $h(p) = 0$, then p is called a rational singularity. A rational singularity embeds in codimension 1 if and only if it is a double point. And among all surface singularities rational double points are the simplest ones. They are classified into the following five types with well-known dual graphs :

$$\begin{aligned}
A_n(n \geq 1) &: \quad z^2 + x^2 + y^{n+1} = 0 \\
D_n(n \geq 4) &: \quad z^2 + y(x^2 + y^{n-2}) = 0 \\
E_6 &: \quad z^2 + x^3 + y^4 = 0 \\
E_7 &: \quad z^2 + x(x^2 + y^3) = 0 \\
E_8 &: \quad z^2 + x^3 + y^5 = 0
\end{aligned}$$

Definition. A cycle $D > 0$ on X is rational if $\chi(\mathcal{O}_X(D)) = 1$, and elliptic if $\chi(\mathcal{O}_X(D)) = 0$, minimally elliptic if $\chi(\mathcal{O}_X(D)) = 0$ and $\chi(\mathcal{O}_X(C)) > 0$ for all cycles C such that $0 < C < D$. Let Z be the fundamental cycle of an isolated singular point $p \in X$, then p is called *rational (weakly elliptic, minimally elliptic)* point if Z is rational (elliptic, minimally elliptic).

In **[7]**, H. Laufer shows that a singular point p of a surface F if $\mathbf{P}^3$ is minimally elliptic if and only if $h(p) = 1$. Let Z be a fundamental cycle of a minimally elliptic point p. He then shows that if $Z \cdot Z = -1$ or -2, then p is a double point.

According to the classical definition, a point p of a surface X is called *tacnode* if it is a double point and X has an infinitesimal double line $\mathbf{L}$ in the first neighborhood of p (page 426, **[8]**). Following this definition, all minimally elliptic double points are tacnodes. In this paper, we will consider only those tacnodes which are minimally elliptic double points. From now on, we only deal isolated singularities of a surface in $\mathbf{P}^3$.

For our purpose, we define tacnodes and triple points as follows.

Definition. *Tacnodes* are minimally elliptic double points with $Z^2 = -2$. And *triple points* are minimally elliptic triple points.

In general, the equation of tacnodes is given by the equation,

$$z^2 + f(x, y) = 0,$$

where $f(x, y)$ are polynomials of degree four or five. Then *tacnodal planes* are defined to be the planes given by the equation "$z = 0$" in the above equation.

2. Normal quintic Enriques surfaces

Theorem 1. *(E. Stagnaro [9]) Let F be a normal quintic surface in $\mathbf{P}^3$ with the following condition $\mathcal{P}$:*

> *F has four tacnodal points at the vertices A_1, A_2, A_3, A_4 of a tetrahedron T such that tacnodal planes to F at A_1, A_2 and A_3, A_4 are identical.*

If S is a minimal non-singular model of F, then S is an Enriques surface.

We prove this theorem by showing that the surface invariants $p_g(S) = 0, q(S) = 0$ and $\varkappa(S) = 0$. The condition $\mathcal{P}$ at the above theorem is essential. As soon as we drop the condition $\mathcal{P}$, the minimal non-singular model of F becomes a rational surface.

Corollary. *Let X be a minimal non-singular model of a normal quintic surface F which has four tacnodes in general position and does not satisfy the property $\mathcal{P}$. Then X is a rational surface.*

We now fix four points of the tetrahedron T, say $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 0, 1, 0), A_3 = (0, 1, 0, 0), A_4 = (0, 0, 0, 1)$, and two tacnodal planes to F,

$$\alpha_1 : x_1 + x_3 = 0 \text{ and } \alpha_2 : x_2 + x_4 = 0.$$

Proposition 2. *F contains three lines L_1, L_1' and L_2 ; the lines $L_1 = \overline{A_1A_2}$ and $L_1' = \overline{A_3A_4}$ are lines joining two vertices of the tetrahedron T and L_2 is the intersection of two tacnodal planes α_1 and α_2. Furthermore, if tacnodes are of type I_0, i.e. simple elliptic singularities, then the normal quintic surface F has the following equation:*

$$\begin{aligned} F :\ & (x_2^3 + x_4^3)(x_1 + x_3)^2 \\ & +(x_1^3 + x_3^3)(x_2 + x_4)^2 \\ & +\big(a_1x_1x_2x_3 + a_2x_1x_2x_4 + a_3x_1x_3x_4 + a_4x_2x_3x_4\big)(x_1 + x_3)(x_2 + x_4) \\ & +a_5x_2^2x_4^2(x_1 + x_3) + a_6x_1^2x_3^2(x_2 + x_4) = 0\,;\ a_5 \neq 0\,,\ a_6 \neq 0. \end{aligned}$$

Let $\sigma : \tilde{S} \longrightarrow F$ be the minimal desingularization of F, which is a composition of blow-ups at points and along double lines. Then the proper transforms of lines L_1, L_1' and L_2 become exceptional curves of the first kind. Let $\rho : \tilde{S} \longrightarrow S$ the blow-down of $\tilde{S}$ to S, contracting exceptional curves of the first kind. Then by Theorem 1, S is an Enriques surface.

Let H be a hyperplane section of the normal quintic surface F, $\tilde{H}$ the proper tranform of H by the map $\sigma : \tilde{S} \longrightarrow F$, and D the divisor which is the image of $\tilde{H}$ bt the map $\rho : \tilde{S} \longrightarrow S$. Then the divisor D has the configuration as follows.

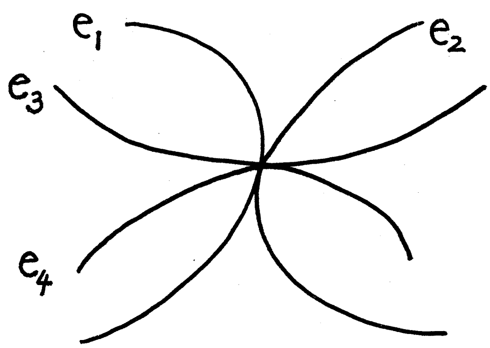

Figure 1

Proposition 3. *If S is the Enriques surface obtained from the normal quintic surface F satisfying the condition $\mathcal{P}$ of Theorem 1, then S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in the Figure 1, where $e_1, \ldots, e_4$ are ioslated elliptic curves.*

Now we show that conversely a generic Enriques surface S with a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in Figure 1 is birationally isomorphic to a normal quintic surface F in $\mathbf{P}^3$ satisfying the condition $\mathcal{P}$ of Theorem 1.

Theorem 4. *Let S be an Enriques surface with a divisor $D = e_1+e_2+e_3+e_4$ with the configuration in Figure 1, that is, e_1, e_2, e_3, e_4 are isolated elliptic curves and $e_1 \cdot e_3 = e_1 \cdot e_4 = e_2 \cdot e_3 = e_2 \cdot e_4 = 1$, and $e_1 \cdot e_2 = e_3 \cdot e_4 = 2$, where e_1, e_2 and e_3, e_4 meet tangentially at a point p.*

Then the following statements are true:

(1) *If the adjoints ${e_1}', {e_2}', {e_3}'$ and ${e_4}'$ do not have a common point, then S is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ satisfying the property $\mathcal{P}$ of Theorem 1, where four tacnodes are of type I_n $(0 \leq n \leq 9)$. F may have finitely many isolated rational double points.*

(2) *If the adjoints e_1', e_2', e_3' and e_4' have a common point, then S is birationally two to one onto a quadric surface Q in $\mathbf{P}^3$.*

Notice that e_1', e_2', e_3' and e_4' can not have more than one common point, if any, because $e_1' \cdot e_3' = e_1' \cdot e_4' = e_2' \cdot e_3' = e_2' \cdot e_4' = 1$.

It is well known that every Enriques surface has a divisor $D = e_1 + e_2 + e_3 + e_4$ satisfying all conditions of Theorem 4 except the requirement of tangential contact at a point.

Since isolated elliptic curves e_i are generically non-singular, tacnodes of the normal quintic surface F in Theorem 4 are generically of type I_0, that is, simple elliptic singularities.

3. The linear independence of four tacnodes

Proposition 5. *In the space of all quintic surfaces of $\mathbf{P}^3$, tacnodal singularities of type I_0, i.e. simple elliptic singularities at four points P_1, P_2, P_3, P_4 of $\mathbf{P}^3$, which are in general position, give 40 linearly independent conditions.*

There is a similar but more general result on rational double points of hypersurfaces in $\mathbf{P}^3$ by D. Burns and J. Wahl **[2]**. It is likely that four tacnodes of type I_0 are a maximum number on normal quintic surfaces in $\mathbf{P}^3$ which give linearly independent conditions, and we do not expect the same result for other types of tacnodes since tacnodes of type I_0 are generic.

By applying Proposition 5, we get the following theorem.

Theorem 6. *Let $\mathcal{F}$ be the space of normal quintic surfaces in $\mathbf{P}^3$ which satisfy the condition $\mathcal{P}$ of Theorem 1. Then the dimension of the space $\mathcal{F}$ is 6.*

Corollary. *Let $\mathcal{E}$ be the moduli space of Enriques surfaces which are the minimal non-singular models of normal quintic surfaces in $\mathcal{F}$. Then the dimension of $\mathcal{E}$ is 6.*

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ON THE INDEX OF THE WEIERSTRASS SEMIGROUP OF A PAIR OF POINTS ON A CURVE

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ABSTRACT. We obtain the exact formulas for the cardinality of the complement of the Weierstrass semigroup of a pair (p,q) of points on a curve C. Using these formulas we obtain lower bounds and upper bounds on the cardinalities of such sets. Moreover, considering examples, we show that these bounds are sharp.

1. Introduction and Preliminaries.

Let C be a nonsingular complex projective curve (or a compact Riemann surface) of genus g. For a divisor D on C, $\dim D$ means the dimension of the complete linear series $|D|$ containing D, which is the same as the projective dimension of the vector space of meromorphic functions f on C with divisor of poles $(f)_\infty \leq D$.

Let $\mathcal{M}(C)$ denote the field of meromorphic functions on C. For points $p, q \in C$, we define the Weierstrass semigroup of a point and the Weierstrass semigroup of a pair of points by

$$H(p) = \{\alpha \in N \mid \text{ there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha p\},$$

$$H(p,q) = \{(\alpha,\beta) \in N \times N \mid \text{ there exists } f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha p + \beta q\},$$

where N denotes the set of non-negative integers. Indeed, these sets form sub-semigroups of N and $N \times N$, respectively. We know that the cardinality of $G(p) = N \setminus H(p)$ is equal to the genus g of the given curve C. But the cardinality of $G(p,q) = N \times N \setminus H(p,q)$ is not determined; that is, it depends on the points p and q. In [1] we can find only the lower bound $\operatorname{card} G(p,q) \geq \binom{g+2}{2} - 1$. In this paper, we obtain formulas for $\operatorname{card} G(p,q)$ depending on points p and q in Theorem 2.6 and Theorem 3.1. Using these formulas, we find lower and upper bounds on the cardinalities of such sets.

As usual, the weight of p is defined by $w(p) = \sum_{\alpha \in G(p)} \alpha \; - \; \frac{g(g+1)}{2}$.

We will often use the following lemmas.

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Lemma 1.1. *[1] The weight of p, $w(p)$ can be expressed as*

$$w(p) = \sum_{\alpha=1}^{\infty}(\dim(\alpha p) - \max\{0, \alpha - g\}).$$

Lemma 1.2. *[2] For any divisor D and any point p on C, we have either*

$$\dim(D+p) = \dim D + 1 \quad \text{and} \quad i(D+p) = i(D)$$

or

$$\dim(D+p) = \dim D \quad \text{and} \quad i(D+p) = i(D) - 1,$$

where $i(D) = \dim D + g - \deg D$, which is called the index of speciality of a divisor D.

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2. Lower Bounds on the Indices.

In this section, we prove a formula for the index of the Weierstrass semigroup $H(p,q)$, and give some examples to illustrate this formula. As an application, we obtained lower bounds on the indices of Weierstrass semigroups.

Since the set $H(p,p)$ is just $\{(\alpha,\beta) \mid \alpha+\beta \in H(p)\}$, it is completely determined by the set $H(p)$ which we are already familiar with. Therefore, we consider only the case $p \neq q$ in the following lemmas.

Lemma 2.1. *For $(\alpha,\beta) \in N \times N \setminus \{(0,0)\}$, the following are equivalent:*

(1) $(\alpha,\beta) \in H(p,q)$.

(2) The complete linear series $|\alpha p + \beta q|$ is base point free.

(3) $\dim(\alpha p + \beta q) = \dim((\alpha-1)p + \beta q) + 1 = \dim(\alpha p + (\beta - 1)q) + 1$.

proof. Recall that, for any divisor D on C, the following are equivalent;

(1) There exists $f \in \mathcal{M}(C)$ such that $(f)_\infty = D$.

(2) The complete linear series $|D|$ is base point free.

(3) $\dim D = \dim(D - x) + 1$ for any point $x \leq D$.

Then the proof is obvious. □

Lemma 2.2. *If* $(\alpha,\beta),(\alpha',\beta') \in H(p,q)$ *with* $\alpha \geq \alpha'$, $\beta \leq \beta'$, *then* $(\alpha,\beta') \in H(p,q)$.

proof. We may assume $\alpha > \alpha'$, $\beta < \beta'$. Let f and g be meromorphic functions satisfying $(f)_\infty = \alpha p + \beta q$ and $(g)_\infty = \alpha' p + \beta' q$. Then $(f+g)_\infty = \alpha p + \beta' q$, and hence $(\alpha,\beta') \in H(p,q)$. □

Lemma 2.3. *Let* $\beta \geq 1$. *Then* $\dim(\alpha p + \beta q) = \dim(\alpha p + (\beta - 1)q) + 1$ *if and only if* $(\gamma,\beta) \in H(p,q)$ *for some* γ, $0 \leq \gamma \leq \alpha$. *In this case, if there is an element* $(\alpha,\beta') \in H(p,q)$ *for* $\beta' \leq \beta$, *then* $(\alpha,\beta) \in H(p,q)$.

proof. Suppose that $\dim(\alpha p+\beta q) = \dim(\alpha p+(\beta-1)q)+1$ and that $\dim(\alpha p+\beta q) = r$. Then $\dim(\alpha p+(\beta-1)q) = r-1$ and so $\dim(\alpha' p+\delta q) \leq r-1$ for all $\alpha' \leq \alpha$, $\delta \leq \beta-1$. Let γ be the smallest number such that $\dim(\gamma p+\beta q) = r$. Then (γ,β) is an element of $H(p,q)$ by Lemma 2.1.

Conversely, suppose that $(\gamma,\beta) \in H(p,q)$ for some γ, $0 \leq \gamma \leq \alpha$. Then q is not a base point of $|\alpha p + \beta q|$, since $|\alpha p + \beta q| = \big||\gamma p + \beta q| + (\alpha - \gamma)p\big|$ and $|\gamma p + \beta q|$ is base point free. Hence $\dim(\alpha p + \beta q) = \dim(\alpha p + (\beta - 1)q) + 1$.

If there is an element $(\alpha,\beta') \in H(p,q)$ for $\beta' \leq \beta$, then, by Lemma 2.2, $(\alpha,\beta) \in H(p,q)$. □

Lemma 2.4. *If* (α,β) *and* (α',β) *belong to* $H(p,q)$ *with* $\alpha > \alpha'$ *and* $\beta \geq 1$, *then there exists an element* (α,δ) *in* $H(p,q)$ *with* $\delta < \beta$.

proof. Let f and g be meromorphic functions satisfying that $(f)_\infty = \alpha p+\beta q$ and $(g)_\infty = \alpha' p + \beta q$. Then we can choose suitable complex numbers a and b such that $(af + bg)_\infty = \alpha p + \delta q$ with $\delta < \beta$. Hence (α,δ) is an element in $H(p,q)$ with $\delta < \beta$. □

Lemma 2.5. *For* $\alpha \in G(p)$, *let* $\beta_\alpha = \min\{\beta \mid (\alpha,\beta) \in H(p,q)\}$. *Then* $(\gamma,\beta_\alpha) \notin H(p,q)$ *for all* $\gamma < \alpha$. *That is,* $\alpha = \min\{\gamma \mid (\gamma,\beta_\alpha) \in H(p,q)\}$.

proof. Notice that $\beta_\alpha \geq 1$ since $\alpha \notin H(p)$. Suppose that $(\alpha',\beta_\alpha) \in H(p,q)$ for some $\alpha' < \alpha$. Then, by Lemma 2.4, there exists an integer $\delta < \beta_\alpha$ such that $(\alpha,\delta) \in H(p,q)$, which contradicts the minimality of β_α. □

Lemma 2.6. *With the same notation* β_α *defined in Lemma 2.5,* $\{\beta_\alpha \mid \alpha \in G(p)\} = G(q)$.

proof. Lemma 2.5 implies that $\beta_\alpha \notin H(q)$, and that $\beta_\alpha \neq \beta_\gamma$ for $\alpha \neq \gamma$. Hence the set $\{\beta_\alpha \mid \alpha \in G(p)\}$ is contained in $G(q)$ and its cardinality is just g, therefore it must be $G(q)$. □

Now we prove a formula for the index of a Weierstrass semigroup. Recall that $i(D) = \dim D + g - \deg D$.

Theorem 2.7. *Let p and q be distinct points on a smooth curve C of genus g. Then*

$$\operatorname{card} G(p,q) = \binom{g+2}{2} - 1 + w(p) + w(q) + \sum_{\alpha \in G(p)} i(\alpha p + \beta_\alpha q),$$

where $\beta_\alpha = \min\{\beta \mid (\alpha,\beta) \in H(p,q)\}$, as in Lemma 2.5.

proof. For each $\alpha \geq 1$, consider two sets

$$\begin{aligned} A_\alpha &= \{\beta \mid \beta \geq 1, (\alpha,\beta) \notin H(p,q)\}, \\ B_\alpha &= \{\beta \mid \beta \geq 1, \dim(\alpha p + \beta q) = \dim(\alpha p + (\beta - 1)q)\}. \end{aligned}$$

By Lemma 2.1, it is obvious that $B_\alpha \subseteq A_\alpha$. For all sufficiently large integers β, $\dim(\alpha p + \beta q) = \alpha + \beta - g$. So we have $\operatorname{card} B_\alpha = \dim(\alpha p) - (\alpha - g)$.

Hence

$$\begin{aligned} \operatorname{card} G(p,q) &= \operatorname{card} G(p) + \operatorname{card} G(q) + \sum_{\alpha=1}^{\infty} \operatorname{card} A_\alpha \\ &\geq 2g + \sum_{\alpha=1}^{\infty} \operatorname{card} B_\alpha \\ &= \sum_{\alpha=1}^{\infty} (\dim(\alpha p) - (\alpha - g)) + 2g \\ &= \sum_{\alpha=1}^{\infty} \big(\dim(\alpha p) - \max\{0, \alpha - g\} + \max\{0, \alpha - g\} - (\alpha - g)\big) + 2g \\ &= w(p) + \binom{g+2}{2} - 1. \end{aligned}$$

In the last equality, we used Lemma 1.1.

Now it remains to examine the difference of the two sets A_α and B_α. By Lemma 2.3, if $\alpha \in H(p)$, then $A_\alpha = B_\alpha$. So, we consider only elements in $G(p)$.

Let $G(p) = \{n_1, n_2, \cdots, n_g\}$, where $n_1 < n_2 < \cdots < n_g$. By Lemma 2.3, if $\beta \geq \beta_{n_k}$, then $\beta \in A_{n_k}$ if and only if $\beta \in B_{n_k}$. Hence the set $A_{n_k} \setminus B_{n_k}$ is the set of all numbers β satisfying $1 \leq \beta < \beta_{n_k}$ and $\beta \notin B_{n_k}$. Note that $\beta \notin B_{n_k}$ means $\dim(\alpha p + \beta q) = \dim(\alpha p + (\beta - 1)q) + 1$, by Lemma 1.2. Thus the cardinality of the set $A_{n_k} \setminus B_{n_k}$ is just $\dim(n_k p + \beta_{n_k} q) - \dim(n_k p) - 1$.

Finally, we prove that

$$\sum_{k=1}^{g} \operatorname{card}(A_{n_k} - B_{n_k}) = w(q) + \sum_{k=1}^{g} i(n_k p + \beta_{n_k} q).$$

To prove it, we use the following facts;

(1) $i(n_k p + \beta_{n_k} q) = g - (n_k + \beta_{n_k}) + \dim(n_k p + \beta_{n_k} q)$ by definition of the index of speciality.

(2) $\sum_{k=1}^{g} \beta_{n_k} = \sum_{l=1}^{g} m_l$, by Lemma 2.6, where $\{m_1, m_2, \cdots, m_g\} = G(q)$.

(3) $w(q) = \sum_{l=1}^{g}(m_l - l)$, by definition of the weight at q.

(4) $\dim(n_k p) = n_k - k$ by definition of its notation.

Now, we obtain the equality.

$$\begin{aligned}
w(q) + \sum_{k=1}^{g} i(n_k p + \beta_{n_k} q) &= w(q) + \sum_{k=1}^{g} [g - (n_k + \beta_{n_k}) + \dim(n_k p + \beta_{n_k} q)] \\
&= w(q) + g^2 - \sum_{k=1}^{g} n_k - \sum_{k=1}^{g} \beta_{n_k} + \sum_{k=1}^{g} \dim(n_k p + \underline{e}ta_{n_k} q) \\
&= w(q) + g^2 - \sum_{k=1}^{g} n_k - \sum_{l=1}^{g} m_l + \sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= w(q) + g^2 - \sum_{k=1}^{g} n_k - \sum_{l=1}^{g} (m_l - l) - \sum_{l=1}^{g} l + \sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= \sum_{k=1}^{g} (g - k) - \sum_{k=1}^{g} n_k + \sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= \sum_{k=1}^{g} (k - 1) - \sum_{k=1}^{g} n_k + \sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= \sum_{k=1}^{g} (-(n_k - k) - 1) + \sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= \sum_{k=1}^{g} (-\dim(n_k p) - 1) + \ sum_{k=1}^{g} \dim(n_k p + \beta_{n_k} q) \\
&= \sum_{k=1}^{g} \operatorname{card}(A_{n_k} - B_{n_k}).
\end{aligned}$$

Thus the proof is complete. □

Theorem 2.8. *Let p be a point on a smooth curve C of genus g. Then*

$$\operatorname{card} G(p,p) = \binom{g+2}{2} - 1 + w(p).$$

proof. Notice that $(\alpha, \beta) \in H(p,p)$ if and only if $\alpha + \beta \in H(p)$. Thus, for each $k \in G(p)$, there exist $k+1$ elements $(\alpha, \beta) \in G(p,p)$ where $\alpha + \beta = k$. Hence

$$\operatorname{card} G(p,p) = \sum_{k \in G(p)} (k+1) = \binom{g+2}{2} - 1 + w(p).$$

□

Corollary 2.9. *Let p and q be, not necessarily distinct, points on a smooth curve C of genus g. Then*

$$\operatorname{card} G(p,q) \geq \binom{g+2}{2} - 1 + \max\{w(p), w(q)\}.$$

proof. If $p = q$, then it is obvious by Theorem 2.8. If $p \neq q$, then the inequality follows from Theorem 2.7 and the fact $i(D) \geq 0$ for all divisor D. □

Corollary 2.10. *Let p and q be distinct points on a smooth curve C of genus g. Then* $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1 + w(p) + w(q)$ *if and only if, for each $(\alpha, \beta) \in H(p,q)$, the divisor $\alpha p + \beta q$ is non-special or $(\alpha, \beta) \in H(p) \times H(q)$.*

proof. Suppose that $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1 + w(p) + w(q)$, or equivalently, by Theorem 2.7, $i(\alpha p + \beta_\alpha q) = 0$ for all $\alpha \in G(p)$. This is also equivalent to that all the divisors $\alpha p + \beta_\alpha q$, $\alpha \in G(p)$ are non-special.

If $(\alpha, \beta) \in H(p,q)$ and $\alpha \notin H(p)$, then $\beta \geq \beta_\alpha$ by Lemma 2.5. Since $\alpha p + \beta_\alpha q$ is non-special, $\alpha p + \beta q$ is also non-special.

If $(\alpha, \beta) \in H(p,q)$ and $\beta \notin H(q)$, then $\beta = \beta_\gamma$ for some $\gamma \in G(p)$ by Lemma 2.6. By Lemma 2.5, we have $\alpha \geq \gamma$. Since $\gamma p + \beta_\gamma q = \gamma p + \beta q$ is non-special, $\alpha p + \beta q$ is also non-special.

Conversely, for each $\alpha \in G(p)$, since $(\alpha, \beta_\alpha) \in H(p,q)$ and $(\alpha, \beta_\alpha) \notin H(p) \times H(q)$, $\alpha p + \beta_\alpha q$ is non-special. Hence $i(\alpha p + \beta_\alpha q) = 0$ for all $\alpha \in G(p)$. Thus we have the equality by Theorem 2.7. □

Corollary 2.11. *Let p and q be, not necessarily distinct, points on a smooth curve C of genus g. Then* $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1$ *if and only if* $\dim(\alpha p + \beta q) = 0$ *for all non-negative integers α, β with $\alpha + \beta \leq g$.*

proof. If $p = q$, Theorem 2.8 implies that $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1$ if and only if $w(p) = 0$, i.e., $\dim(gp) = 0$.

Assume that $p \neq q$ and that $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1$, or equivalently, by Theorem 2.7, $w(p) = 0$, $w(q) = 0$, and $i(\alpha p + \beta_\alpha q) = 0$ for all $\alpha \in G(p)$. Note that $w(p) = 0$ and $w(q) = 0$ implies $G(p) = G(q) = \{1, 2, \cdots, g\}$.

Suppose that $\dim(\alpha p + \beta q) > 0$ for some non-negative integers α, β with $\alpha + \beta \leq g$. Choose an element (γ_0, δ_0) in the set $I = \{(\gamma, \delta) \mid \gamma \leq \alpha,\ \delta \leq \beta,$ and $\dim(\gamma p + \delta q) = \dim(\alpha p + \beta q)\}$ such that $\gamma_0 + \delta_0 = \min\{\gamma + \delta \mid (\gamma, \delta) \in I\}$. Then $(\gamma_0, \delta_0) \in H(p,q)$, by Lemma 2.1. Since $\dim(gq) = 0$, $\gamma_0 \neq 0$. Hence $\gamma_0 \in G(p)$. Now we have $\beta_{\gamma_0} \leq \delta_0$ and hence

$$i(\gamma_0 p + \beta_{\gamma_0} q) \geq i(\gamma_0 p + \delta_0 q) \geq i(\alpha p + \beta q) > 0,$$

which contradicts our assumption.

Conversely, suppose that $\dim(\alpha p + \beta q) = 0$ for all (α, β) with $\alpha + \beta \leq g$. Then $G(p,q) = \{(\alpha, \beta) \in N \times N \mid \alpha + \beta \leq g\} \setminus \{(0,0\}$. Thus $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1$. □

Now we give some examples of Weierstrass semigroups which illustrate the formula in Theorem 2.7.

Example 2.1. Let C be a non-singular plane quartic curve with a bitangent line which is tangent to C at p and q. Then the genus of C is three, and its canonical series is $K = |2p + 2q|$ which is cut out by lines. Using the Riemann-Roch theorem and Lemma 2.1, we get

$$\begin{aligned} G(p) &= G(q) = \{1, 2, 3\}, \\ G(p,q) &= \{(0,1), (0,2), (0,3), (1,0), (1,1), (1,3), \\ &\qquad (2,0), (2,3), (3,0), (3,1), (3,2)\}. \end{aligned}$$

And

$$\begin{aligned} &\beta_1 = 2, \quad \beta_2 = 1, \quad \beta_3 = 3, \\ &i(p + 2q) = 1, \quad i(2p + q) = 1, \quad i(3p + 3q) = 0. \end{aligned}$$

Example 2.2. Let C be a non-singular plane quartic curve with a flex p of order 1. Suppose that the other intersection point q of C and the tangent line at p is not a flex. Then the genus of C is three and its canonical series is $K = |3p+q|$ which is cut out by lines. Using the Riemann-Roch theorem and Lemma 2.1, we get

$$\begin{aligned} G(p) = \{1,2,4\}, \qquad G(q) = \{1,2,3\}, \\ G(p,q) = \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2), \\ (2,0),(2,2),(3,2),(4,0),(4,1)\}. \end{aligned}$$

And

$$\begin{aligned} &\beta_1 = 3, \quad \beta_2 = 1, \quad \beta_4 = 2, \\ &i(p+3q) = 0, \quad i(2p+q) = 1, \quad i(4p+2q) = 0. \end{aligned}$$

Remark. Those curves in Example 2.1 and Example 2.2 can be constructed easily.

Example 2.3. Let C be a hyperelliptic curve of genus g. We use the fact that any special linear series is compounded of the g_2^1, the linear series of dimension 1 and degree 2.

(1) If p and q are distinct points and $\dim(p+q) = 1$, then

$$\begin{aligned} G(p) = G(q) = \{1,2,\cdots,g\}, \\ G(p,q) = \{(\alpha,\beta) \mid 0 \le \alpha \le g,\ 0 \le \beta \le g,\ \alpha \ne \beta\}. \end{aligned}$$

For each k, $k = 1,2,\cdots,g$, we have $\beta_k = k$ and $i(kp+\beta_k q) = g-k$. Thus $\operatorname{card} G(p,q) = g(g+1) = \binom{g+2}{2} - 1 + \sum_{\alpha\in G(p)} i(\alpha p + \beta_\alpha q)$.

(2) Let p and q be distinct points satisfying $\dim(p+q) = 0$, $\dim(2p) = 0$, and $\dim(2q) = 0$. Then

$$\begin{aligned} G(p) = G(q) = \{1,2,\cdots,g\}, \\ G(p,q) = \{(\alpha,\beta) \mid \alpha+\beta \le g\} - \{(0,0)\}. \end{aligned}$$

For each k, $k = 1,2,\cdots,g$, we have $\beta_k = g-k+1$, and $i(kp+\beta_k q) = 0$. Thus $\operatorname{card} G(p,q) = \binom{g+2}{2} - 1$.

(3) Let p and q be distinct points satisfying $\dim(2p) = 1$ and $\dim(2q) = 1$. Then

$$\begin{aligned} G(p) = G(q) = \{2k-1 \mid k = 1,2,\cdots,g\}, \\ G(p,q) = \{(\alpha,\beta) \mid \alpha+\beta \le 2g-1,\ \alpha \text{ or } \beta \text{ is odd. }\}. \end{aligned}$$

For each k, $k = 1, 2, \cdots, g$, we have $\beta_{2k-1} = 2(g-k)$ and $i((2k-1)p + \beta_{2k-1}q) = 0$. Thus $\operatorname{card} G(p,q) = g(3g+1)/2 = \binom{g+2}{2} - 1 + w(p) + w(q)$.

(4) Let p and q be distinct points satisfying $\dim(2p) = 1$ and $\dim(2q) = 0$. Then

$$G(p) = \{2k-1 \mid k = 1, 2, \cdots, g\},$$
$$G(q) = \{1, 2, \cdots, g\},$$
$$G(p,q) = [\bigcup_{k=0}^{g-1} \{(2k, \beta) \mid 1 \le \beta \le g-k\}]$$
$$\bigcup [\bigcup_{k=1}^{g} \{(2k-1, \beta) \mid 0 \le \beta \le g-k\}].$$

For each k, $k = 1, 2, \cdots, g$, we have $\beta_{2k-1} = g-k+1$ and $i((2k-1)p + \beta_{2k-1}q) = 0$. Thus $\operatorname{card} G(p,q) = g(g+1) = \binom{g+2}{2} - 1 + w(p)$.

3. Upper Bounds on the Indices.

In this section, we change the formula in Theorem 2.7 and find another expression of it to get upper bounds on the indices of Weierstrass semigroups.

Theorem 3.1. *Under the same assumption as in Theorem 2.7,*

$$\operatorname{card} G(p,q) = \binom{g+2}{2} - 1 - g + \sum_{\alpha \in G(p)} \dim(\alpha p + \beta_\alpha q).$$

proof. By Lemma 2.6, we have that $w(q) = \sum_{\alpha \in G(p)} \beta_\alpha - g(g+1)/2$. Hence

$$\begin{aligned} w(p) + w(q) + \sum_{\alpha \in G(p)} i(\alpha p + \beta_\alpha q) &= \Big(\sum_{\alpha \in G(p)} \alpha + \beta_\alpha + i(\alpha p + \beta_\alpha q)\Big) - g(g+1) \\ &= \Big(\sum_{\alpha \in G(p)} \dim(\alpha p + \beta_\alpha q)\Big) + g^2 - g(g+1) \\ &= \sum_{\alpha \in G(p)} \dim(\alpha p + \beta_\alpha q) - g. \end{aligned}$$

Then the proof follows from Theorem 2.7. □

Now we get upper bounds on the indices of Weierstrass semigroups.

Theorem 3.2. *Let C be a smooth curve of genus g and p, q be, not necessarily distinct, points on C. Then*

$$\operatorname{card} G(p,q) \leq \binom{g+2}{2} - 1 - g + g^2.$$

Moreover, the equality holds if and only if the curve C is a hyperelliptic curve and $2p$, $2q \in g_2^1$, and $p \neq q$.

proof. Suppose firstly that $p \neq q$. We claim that $\dim(\alpha p + \beta_\alpha q) \leq g$ for each $\alpha \in G(p)$. If $\alpha p + \beta_\alpha q$ is special, then $\dim(\alpha p + \beta_\alpha q) \leq g - 1$. Now consider the case that $\alpha p + \beta_\alpha q$ is non-special. By way of contradiction, suppose that $\dim(\alpha p + \beta_\alpha q) \geq g + 1$. Then $\alpha + \beta_\alpha \geq 2g + 1$, and hence the degrees of the two divisors $\alpha p + (\beta_\alpha - 2)q$ and $(\alpha - 1)p + (\beta_\alpha - 1)q$ are not less than $2g - 1$, so both of them are non-special divisors. Then $\dim(\alpha p + (\beta_\alpha - 1)q) = \dim(\alpha p + (\beta_\alpha - 2)q) + 1 = \dim((\alpha - 1)p + (\beta_\alpha - 1)q) + 1$, by the Riemann-Roch theorem. Hence, by Lemma 2.1, $(\alpha, \beta_\alpha - 1) \in H(p,q)$, which contradicts the minimality of β_α. Thus the inequality follows from Theorem 3.1.

In the case $p = q$, the inequality follows from Theorem 2.8 and the fact $w(p) \leq \frac{g(g-1)}{2}$.

In Example 2.3.(3), we saw that if C is a hyperelliptic curve and $2p, 2q \in g_2^1$, and $p \neq q$, then we have equality in the above formula.

Conversely, suppose that the equality holds. Then, from the above proof, we have the equalities $\dim(\alpha p + \beta_\alpha q) = g$, for all $\alpha \in G(p)$. That is, for each $\alpha \in G(p)$, there is no special divisor $\alpha p + \delta q$ such that $(\alpha, \delta) \in H(p,q)$. By the choice of β_α, we have $\dim(\alpha p + (\beta_\alpha - 1)q) = g - 1$ and $(\alpha, \beta_\alpha - 1) \in G(p,q)$. By Lemma 2.1, we have $\dim(\alpha p + (\beta_\alpha - 2)q) = g - 1$ or $\dim((\alpha - 1)p + (\beta_\alpha - 1)q) = g - 1$.

If $\dim(\alpha p + (\beta_\alpha - 2)q) = g - 1$, then the divisor $\alpha p + (\beta_\alpha - 2)q$ is a canonical divisor. Since the canonical series has no base points, by Lemma 2.1, $(\alpha, \beta_\alpha - 2) \in H(p,q)$, which contradicts the minimality of β_α. Thus we conclude that $\dim(\alpha p + (\beta_\alpha - 2)q) = g - 2$ and $\dim((\alpha - 1)p + (\beta_\alpha - 1)q) = g - 1$.

Now $\dim((\alpha - 1)p + (\beta_\alpha - 1)q) = g - 1$ implies that $(\alpha - 1)p + (\beta_\alpha - 1)q$ is a canonical divisor and hence $(\alpha - 1, \beta_\alpha - 1) \in H(p,q)$. Then, since $\dim((\alpha - 1)p + (\beta_\alpha - 1)q) = g - 1 \neq g$, $\alpha - 1$ is not an element of $G(p)$. Thus we conclude that no two consecutive integers are in $G(p)$. Thus the fact that 1 is an element of $G(p)$ implies that 2 is an element of $H(p)$. Hence C is a hyperelliptic curve. Furthermore, letting $\alpha = 1$ in the above proof, we have $(2g - 2)q$ is a canonical divisor and hence $2q \in g_2^1$. And the fact that $p \neq q$ follows from knowing that $\dim(\alpha p + (\beta_\alpha - 2)q) = g - 2$ and $\dim((\alpha - 1)p + (\beta_\alpha - 1)q) = g - 1$. Thus the proof is complete. □

Corollary 3.3. *Let C be a smooth curve of genus g, and let p and q be, not necessarily distinct, points. Then*

$$\binom{g+2}{2} - 1 \leq \operatorname{card} G(p,q) \leq \binom{g+2}{2} - 1 - g + g^2.$$

proof. This follows from Corollary 2.9 and Theorem 3.2.

Theorem 3.4. *Let C be a smooth curve of genus g. Suppose that neither of two distinct points p and q is a Weierstrass point on C. Then*

$$\operatorname{card} G(p,q) \leq \binom{g+2}{2} - 1 - g + g(g+1)/2.$$

Moreover, the equality holds if and only if the curve C is a hyperelliptic curve and $p + q \in g_2^1$.

proof. Since $G(p) = G(q) = \{1, 2, \cdots, g\}$, it is obvious that $2 \leq \deg(\alpha p + \beta_\alpha q) \leq 2g$ and $1 \leq \dim(\alpha p + \beta_\alpha q) \leq g$. Moreover, for each $\alpha \in G(p)$, we have $\dim(\alpha p + \beta_\alpha q) \leq (\alpha + \beta_\alpha)/2$ by Clifford's Theorem [1]. Hence we obtain

$$\sum_{\alpha \in G(p)} \dim(\alpha p + \beta_\alpha q) \leq \sum_{\alpha \in G(p)} \frac{\alpha + \beta_\alpha}{2} = \frac{g(g+1)}{2}.$$

By Theorem 3.1, the inequality follows.

In Example 2.3.(1), we saw that if the curve C is a hyperelliptic curve and $p + q \in g_2^1$, then the equality holds. Conversely, if the equality holds, then $\dim(\alpha p + \beta_\alpha q) = (\alpha + \beta_\alpha)/2$ for all $\alpha \in G(p)$, by the above proof. In particular, for $\alpha = 1$, $\dim(p + \beta_1 q) = (1 + \beta_1)/2$. But this value must be just 1 because there is no element $(\gamma, \delta) \in H(p,q) - \{(0,0)\}$ with $\gamma \leq 1$, $\delta \leq \beta_1$. Thus $\beta_1 = 1$ and hence $p + q \in g_2^1$ and C is a hyperelliptic curve. □

Corollary 3.5. *Let C be a smooth curve of genus g, and let p and q be, not necessarily distinct, non-Weierstrass points on C. Then*

$$\binom{g+2}{2} - 1 \leq \operatorname{card} G(p,q) \leq \binom{g+2}{2} - 1 - g + g(g+1)/2.$$

proof. If $p \neq q$, this follows from Corollary 2.9 and Theorem 3.4. If $p = q$, this follows from Theorem 2.8.

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MARTENS INDEX OF AN ALGEBRAIC CURVE

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ABSTRACT. Let C be a curve of Clifford index c and Martens index $c+\alpha$ with Martens dimension $r > 2$. Then we obtain the following result as in the case of the Clifford index and the Clifford dimension. Suppose that the linear series $|D| = g^r_d$ on C gives the Matens index and the Martens dimension $r \geq 3$. Then $d \geq 4r - 5 + \beta$, $\beta = (g-1) - d$ in case $h^1(C, \mathcal{O}(2D)) \geq 2$ and $d \geq 6r - 8$ in case $h^1(C, \mathcal{O}(2D)) \leq 3$.

1. Introduction.

Let C be a complete smooth curve curve of genus $g \geq 4$. For the linear series $|D|$ on C, the Clifford index of $|D|$ is defined by Cliff(D) =deg $D -$ 2r(D). The minimum of all Cliff(D) with deg$D \leq g-1$ is called the Clifford index of C, denoted by Clff(C). If C has the pencil of degree $c+2$, then the Clifford index of C is given by that pencil. But if C has no such a pencil, then Cliff(C) is given by the linear series g^r_{c+2r} with $r \geq 2$. The smallest dimension of the linear series computing the Clifford index of C is called the Clifford dimension $r(C)$ of C. Then the linear series $|D|$ computing the Clifford index and the Clifford dimension of C is very ample and the image curve of C via the morphism $\varphi_{|D|}$ is not contained in any quadric hypersurface of rank no more than 4.(See [ELMS].)

We also define the $r-$Clifford index Cliff$_r(C), r \geq 1$, by the minium of all Cliff(D) with deg$D \leq g-1$ and $r(D) \geq r$.(See [B].) We set Cliff$_2(C) = c+\alpha$. Then $0 \leq \alpha \leq c$. In particular, if C has at most finitely many base point free pencils of of Clifford index of no more than $c+\alpha+2$, then we say that the curve C is of Martens index $c+\alpha$. In this case, any linear series $|D|$ of Clifford index $c+\alpha$ with $r(D) \geq 2$ is birationally very ample.(See Theorem 3.1.) The minimum r for which there is a $|D| = g^r_d$ computing the Martens index of C is called the Martens dimension of C. We can see that if $|D|$ computes the Martens index of C with $r(D)$ the Martens dimension of C, then the image curve $\varphi_{|D|}(C)$ is contained in at most finitely many quadric hypersurfaces of rank no more than 4.(See the proof of Theorem 3.2) In the case of $\alpha = 0$, a lot is known by the various papers.(See [Ma], [ELMS], [CM] and [KKM].) On the other hand, for $\alpha = 1$ Ballico has recently proved the similar result as in $\alpha = 0$.

We now show that for any α, the linear series $|D|$ which computes the Martens index of C with the Martens dimension has the similar properties as in $\alpha = 0, 1$.

2.Result for $\alpha = 0, 1$

At first, we demonstrate only the theorems for $\alpha = 0, 1$ which can be similarly generalized for any α in this paper. In this section, the curve C is of Clifford index c and of genus g.

Theorem 2.1 [KKM]. *Let the linear series $|D|$ on C of Clifford index c. If $r(D) \geq 3$, then $|D|$ is birationally very ample, unless C is hyperelliptic or bielliptic.*

Theorem 2.2 [ELMS]. *Let C be the curve of Clifford index c and Clifford dimension $r \geq 3$. Then for a $|D| = g^r_d$ of Clifford index c, the image curve of C via the morphism $\varphi_{|D|}$ is not contained any quadric hypersurface of rank less than 4.*

Theorem 2.3 [CM]. *Suppose the linear series $|D|$ computes the Clifford index of C with $r(D) \geq 2$. Then $degD \leq 4r - 2$.*

In fact, in the above theorems C has the Martens index c.(; That is the case $\alpha = 0$. In case $\alpha = 1$, Ballico has recently proved the following theorem.)

Theorem 2.4 [B]. *Let C be of the Martens index $c+1$ and the linear series $|D|$ compute the Martens index(i.e., Cliff$(D) = c + \alpha$ and $degD \leq g - 1$.)*

(1)Assume $c \leq 2r - 5$; then $c = 2r - 5, d = g - 1$ and D is a theta-characteristic; then $d \geq 4r - 4 + \beta$.

(2)Assume $c \geq 2r - 4$ and $h^1(2D) \leq 1$; then $d \geq 4r - 4 + \beta$, $\beta(g-1) - d$.

(2)Assume $c \geq 2r - 4$ and $h^1(2D) > 1$; then $d \geq 6r - 9$ with equqlity only if $2D \cong K \otimes G^{-1}$ with G pencil computing the Clifford index of C.

3. A curve of Martens index $c + \alpha$

In this section, the curve C is of Clifford index c and Martens index $c + \alpha$.

Theorem 3.1. *Let the linear series $|D|$ compute the martens index $c + \alpha$. If $r(D) \geq 2$, then $|D|$ is birationally very ample.*

proof. Let φ be the morphism from C to $\mathbb{P}^r$ which is associated with $|D|$. Suppose φ is not birational. Set deg$\varphi = k$ and $\varphi(C) = C'$. Then the degree d' of C' is equal to d/k. Then the linear series which consists of the divisors cut out by the hyperplanes in $\mathbb{P}^r$ is a complete $|D'|$. By the uniform position theorem, for the generic $(r-1)-$tuple of the points $\sum P_i'$ on C', $|D'|\setminus\sum P_i'$ is a base point free pencil of degree $d'-r+1$. Then the linear series $|D|\setminus\varphi^{-1}(\sum P_i')$ has degree no more than $c+\alpha+2$ and so it is a base point free pencil. But because C has only finitely many base point free pencils of degree no more than $c+\alpha+2$, for the generic $(r-1)-$tuples of the points $\sum P_i'$, the linear series $|D'\setminus\sum P_i'|$ are the same linear series which we denote by $|F'|$. And then the generic $(r-1)-$tuple of the points $\sum P_i'$ is contained in $|D'\setminus F'|$ Thus the dimension of $|P_i'|$ is at least $(r-1)$ and so C' is a rational curve, for deg $\sum P_i' = r-1$. Thus the curve C' has the 1-dimensional family of g_1^1's , Then for the pull-back G on C of a divisor in g_1^1, $|G|$ has the degree less than $c+\alpha+2$ since $d' \geq r'$ and $kd' = d = c+\alpha+2r$. Thus by the definition of the Martens index $|G|$ is a pencil, and so the pullbacks of g_1^1's are the distinct pencils of degree no more than $c+\alpha+2$. Thus C has the $1-$dimensional fmily of such pencils, which cannot happen. Therefore, the morphism φ is birational. □

Theorem 3.2. *Let C be a curve of Martens index $c+\alpha$ and Martens dimension $r \geq 3$. Suppose $|D| = g_d^r$ computes the Martens index of C. Then :*

(1) If $h^1(C,\mathcal{O}(2D)) \leq 2$, then $d \geq 4r-5+\beta$, $\beta = (g-1)-d$.
(2) If $h^1(C,\mathcal{O}(2D)) \geq 3$, then $d \geq 6r-8$.

proof. Since $|D|$ computes the Martens index with the Martes dimension $r \geq 3$, the morphism $\varphi_{|D|}$ is biregular. Suppose there is a quadric hypersuface of rank more than 4 containing the image curve $\varphi_{|D|}(C)$. Let $|F|$ be the base point free pencil on C, which is a subseries of the linear series generated by the rullings of Q. Then $h^0(D\setminus F) \geq 2$.

Suppose Cliff$(D\setminus F) \geq c+\alpha+1$. Then $r(D\setminus F) \leq r-f/2-1$. Then by the base point free pencil trick, $r(D+F) \geq r+f/2+1/2$ and so Cliff$(D+F) \leq c+\alpha-1$ with $h^1(C,\mathcal{O}(D+F)) \geq 2$, which cannot happen. Thus Cliff$(D\setminus F) \leq c+\alpha$ and $|D\setminus F|$ is a pencil since $|D|$ has the Martens dimension. We set B the base locus of $|D|\setminus F$ and $|D\setminus F|\setminus B = |G|$. Then, as in the above, $|D\setminus G| = |F|+B$ is also of Clifford index no more than $c+\alpha$ and a pencil. And so both pencils $|F|$ and $|G|$ consist of the divisors cut out by the two family of rullings of the quadric hypersurface Q. If there is another quadric hypersurface Q' of rank no more than 4 containing C', then C has also two pencils which consist of the divisors cut out by the rullings of Q'. Thus two pairs of pencils are distinct if $Q \neq Q'$. Hence, there are at most finitely many quadric hypersurfaces of rank no more than 4 containing $\varphi_{|D|}$, since C has only finitely many pencils of Clifford index no more than $c+\alpha$. Thus $r(2D) \geq 4r-4$, by the sheaf exact sequence : $0 \to \mathcal{I}_C(2) \to \mathcal{O}_{P^r}(2) \to 0$.

Therefore, If $h^1(C, \mathcal{O}(2D)) \geq 3$, then $\text{Cliff}(D) \leq c + \alpha$ and so $d \geq 6r - 8$. If $h^1(C, \mathcal{O}(2D)) \leq 2$ then $d \geq 4r - 4 + \beta$, $\beta = (g-1) - d$, by Riemann-Roch theorem. □

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CLASSIFICATION OF 3-DIMENSIONAL COMPACT NONSINGULAR TORIC VARIETIES WITH PICARD NUMBER 6

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ABSTRACT. The point of the theory of toric varieties lies in its ability of translating meaningful algebro-geometric and analytic phenomena into very simple statements about the combinatorics of cones in affine spaces over the reals. We classify 3-dimensional compact nonsingular toric varieties by classifying corresponding complete nonsingular fans in 3-dimensional affine space over the reals.

§1 Introduction

A toric variety was first introduced by Demazure and then by Mumford et al., Satake and Miyake-Oda. It is a normal algebraic variety containing algebraic torus T_N as an open dense subset with an algebraic action of T_N which is an extension of the group law of T_N. A toric variety can be described in terms of a certain collection, which is called a fan, of cones. From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators. That is, we can translate the difficult algebro-geometric properties of toric varieties into very simple properties about the combinatorics of cones in affine spaces over the reals. Also, the classification of 3-dimensional compact nonsingular toric varieties is reduced to that of complete nonsingular fans.

Now we introduce some basic definitions which are used throughout this paper. Let N be a free $\mathbf{Z}$-module of rank r over the ring $\mathbf{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ its dual $\mathbf{Z}$-module with the canonical bilinear pairing

$$\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}.$$

We denote the scalar extensions of N and M to the field $\mathbf{R}$ of real numbers by $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, respectively.

A subset σ of $N_{\mathbf{R}}$ is called a *rational convex polyhedral cone* (or a *cone*, for short), if there exist a finite number of elements $n_1, n_2, ..., n_s$ in N such that

$$\begin{aligned}\sigma &= \mathbf{R}_{\geq 0}n_1 + \mathbf{R}_{\geq 0}n_2 + ... + \mathbf{R}_{\geq 0}n_s \\ &:= \{a_1n_1 + ... + a_sn_s \mid a_i \in \mathbf{R}, a_i \geq 0 \text{ for all } i\},\end{aligned}$$

where we denote by $\mathbf{R}_{\geq 0}$ the set of nonnegative real numbers. σ is said to be *strongly convex* if it contains no nontrivial subspace of $\mathbf{R}$, that is, $\sigma \cap (-\sigma) = \{0\}$.

A subset τ of σ is called a *face* and denoted by $\tau \prec \sigma$, if

$$\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma \mid \langle m_0, y\rangle = 0\}$$

for an $m_0 \in \sigma^{\vee}$, where

$$\sigma^{\vee} := \{x \in M_{\mathbf{R}} \mid \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}$$

is the *dual* cone of σ.

Definition. A finite collection Δ of strongly convex cones in $N_{\mathbf{R}}$ is called a *fan* if it satisfies the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.
(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The *support* of a fan Δ is defined to be $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$.

A cone σ is said to be *nonsingular* if there exist a $\mathbf{Z}$-basis $\{n_1, n_2, ..., n_r\}$ of N and $s \leq r$ such that

$$\sigma = \mathbf{R}_{\geq 0}n_1 + \cdots + \mathbf{R}_{\geq 0}n_s.$$

We say that a fan Δ is *nonsingular* if every cone $\sigma \in \Delta$ is nonsingular. A fan Δ is said to be *complete* if $|\Delta| = N_{\mathbf{R}}$.

If a fan Δ is given, then there exists a toric variety $X := T_N\text{emb}(\Delta)$ determined by Δ over the field $\mathbf{C}$ of complex numbers. For the precise definition of toric varieties, see [2], [6] and [7].

It is known that the toric variety corresponding to a nonsingular fan is nonsingular. Also, the toric variety is compact if and only if the corresponding fan is complete (cf. [7]).

§2 Some properties

In this section, we state some properties which are necessary for classification of toric varieties. For the proof, see [6] and [7].

Proposition 2.1. *The set of isomorphic classes of 2-dimensional compact nonsingular toric varieties is in one-to-one correspondence with the set of weighted circular graph of the following form:*

(*i*) *the circular graph having three vertices with* $1, 1, 1$ *as weights;*
(*ii*) *the circular graph having four vertices with weights* $a, 0, -a, 0$ *in this order, where* a *is a nonnegative integer;*
(*iii*) *the weighted circular graph with* $s \geq 5$ *vertices which we obtained from those with* $s-1$ *vertices by adding a vertex of weight* -1 *and substracting* 1 *from the weight of each of the two adjacent vertices.*

From now on, we fix $N_{\mathbf{R}} \cong \mathbf{R}^3$.

Let $X = T_N\mathrm{emb}(\Delta)$ be a 3-dimensional compact nonsingular toric variety. The corresponding fan is nonsingular and complete. If we intersect Δ with a sphere $S \subset N_{\mathbf{R}}$ centered at 0, we get a triangulation of S

$$S = \bigcup_{\sigma \in \Delta} (\sigma \cap S).$$

We have a canonical N-weighting for this triangulation. Indeed, each spherical vertex is of the form $\mathbf{R}_{\geq 0} n \cap S$ for a primitive $n \in N$, which we attach to the vertex as an *N-weight*.

On the other hand, an N-weighting for a triangulation of S gives rise to a double $\mathbf{Z}$-weighting. Indeed, each spherical edge is of the form $\tau \cap S$ for a 2-dimensional cone $\tau \in \Delta$. There exist exactly two 3-dimensional cones $\sigma, \sigma' \in \Delta$ satisfying $\sigma \cap \sigma' = \tau$. Let $\{n, n_1, n_2\}$ and $\{n', n_1, n_2\}$ be the primitive elements in N which generate σ and σ', respectively. Since σ and σ' are nonsingular, there exist $a, b \in \mathbf{Z}$ such that

$$n + n' + an_1 + bn_2 = 0.$$

We attach a pair (a, b) to the edge $\tau \cap S$ as a *double* $\mathbf{Z}$*-weight*, with a on the side of $\mathbf{R}_{\geq 0} n_1 \cap S$ and b on the side of $\mathbf{R}_{\geq 0} n_2 \cap S$. Consider a vertex v with N-weight n in the triangulation. Let $v_1, v_2, ..., v_s$ be vertices which are adjacent to v in this order, with N-weights $n_1, n_2, ..., n_s$, respectively. We have

$$(*) \qquad n_{i-1} + n_{i+1} + a_i n_i + b_i n = 0 \qquad 1 \leq i \leq s,$$

where $a_i, b_i \in \mathbf{Z}$ and $n_0 := n_s, n_{s+1} := n_1$. Note that the weights $a_1, a_2, ..., a_s$ are those appeared in Proposition 2.1.

Definition. A doubly **Z**-weighted triangulation of S is called *admissible* if, around each vertex, the equation $(*)$ in unknowns $n, n_1, \ldots, n_s$ are compatible and if the weighted link of each vertex is a weighted circular graph obtained as in Proposition 2.1.

Proposition 2.2. *We have canonical bijection between the set of isomorphic classes of 3-dimensional compact nonsingular toric varieties and the set of combinatorial isomorphic classes of admissible doubly* **Z***-weighted triangulations of* S.

The following is handy in deciding when a double **Z**-weighting is admissible:

Proposition 2.3. *A double* **Z***-weighting for a combinatorial triangulation of* S *is admissible if and only if, around each vertex, the weights satisfy the following under the same notation as above:*

(1) $\sum_{j=1}^{s} a_j = 12 - 3s$

(2) $$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_s & 0 \\ 0 & -b_s & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_2 & 0 \\ 0 & -b_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_1 & 0 \\ 0 & -b_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

§ 3 Classification

Now we classify 3-dimensional compact nonsingular toric varieties by using the classification of triangulations of S. According to Grünbaum [3], the numbers of the combinatorial equivalence classes of the triangulations of S are the following, where d is the number of the vertices in the triangulation:

d	4	5	6	7	8	9	10	11	12
	1	1	2	5	14	50	233	1249	7595

It is known that for a 3-dimensional compact nonsingular toric variety, the Picard number is equal to $\#\Delta(1) - 3$, where $\#\Delta(1)$ is the cardinality of one-dimensional cones in the corresponding fan.

Oda [6] [7] have classified 3-dimensional compact nonsingular toric varieties with Picard number five or less which are minimal in the sense of equivarient blowing-ups. We classify those with Picard number 6 using the classification of the combinatorically different triangulations of S with $d = 9$, which is due to Y. Kado-oka and M. Ohshima. Now we state the result of the classification without proof:

Theorem 3.1. *There are 79 different kinds of 3-dimensional compact non-singular toric varieties with Picard number 6 which are minimal in the sense of equivarient blowing-ups. We can describe the corresponding admissible doubly* **Z***-weighted triangulations with the label* $\Pi_{s\geq 3}s^{p(s)}$ *which means that the triangulation has* $p(s)$ *elements of* s*-valent vertices of it (i.e., vertices incident with exactly* s *edges) for each integer* $s \geq 3$.

Remark Kleinschmidt and Sturmfels [4] have proved that r-dimensional compact toric variety X with Picard number ≤ 3 must be projective, while Ewald [1] constructed an r-dimensional nonsingular, non-projective toric variety with Picard number $= 4$ by using Gale diagrams (cf. [5]). We know the sufficient condition for non-projectivity (cf. [6]), which is very convenient in concrete applications. We found 22 different kind of non-projective toric varieties in the list in Theorem 3.1 by applying above sufficient condition. There may be, however, more non-projective toric varieties in the list. It is desirable to get a necessary condition for non-projectivity for toric variety, which can be written in the concrete combinatorial language.

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DIFFERENTIAL GEOMETRY AND DYNAMICAL SYSTEM

THREE SHARP ISOPERIMETRIC INEQUALITIES FOR STATIONARY VARIFOLDS AND AREA MINIMIZING FLAT CHAINS MOD k

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Preliminary Report*

It is well known [C, H, Ch1] that a smooth minimal surface Σ spanned by a rectifiable Jordan curve C satisfies the isoperimetric inequality

$$4\pi Area(\Sigma) \leq Length(C)^2, \tag{1}$$

where equality holds if and only if C is a circle and Σ is a disk. Some *smooth* minimal surfaces in $\mathbf{R}^3$ can be physically realized as soap films. However, the soap film formed by dipping a connected wire frame consisting of *two* or more closed curves in a soap solution represents a minimal surface which contains interior *singular* curves. Here arises the main question of this paper: Does (1) still hold optimally for this soap-film-like surface with singularities? In 1986 Almgren [A1] answered this question affirmatively if C bounds a soap- film-like surface which is an area minimizing flat chain mod k. In this paper we extend his result and show that (1) holds also for two-dimensional *stationary* varifolds with boundary multiplicity ≥ 1 (Theorem 2). Moreover, if the spanning curve C consists of k curves having the same end points, we obtain a new type of sharp isoperimetric inequality for area minimizing flat chains mod k spanned by C. Here, unlike (1), equality holds only for the union of k flat half disks with a common diameter (Theorem 3). Finally it is shown that m-dimensional stationary varifolds with boundary lying on a sphere centered at a point in their support satisfy a third sharp isoperimetric inequality (Theorem 4).

*Full text with complete proofs will appear elsewhere.

1 Arcs and sectors

In this section we derive sharp isoperimetric inequalities for domains in the plane where only a specific part of the boundary counts toward the length of the boundary.

Lemma 1 *(a) Let $\overline{PQ}$ be a line segment in $\mathbf{R}^2$. The only curve in $\mathbf{R}^2$ from P to Q that maximizes the area of the domain bounded by the curve and $\overline{PQ}$ among all curves of the same length is the arc.*
(b) For any curve C from P to Q and the domain D bounded by C and $\overline{PQ}$,

$$2\pi Area(D) \leq Length(C)^2.$$

Equality holds if and only if C is the semicircle from P to Q.

Lemma 2 *Let l_1 and l_2 be the rays emanating from a point O with an angle of $\theta \leq \pi$. Let C be a curve from a point of l_1 to a point of l_2 without self-intersection.*
(a) Suppose that C lies in the smaller sector of the two formed by the rays (C may lie in either sector if $\theta = \pi$). Define D as the domain bounded by l_1, l_2, and C. Then

$$2\theta Area(D) \leq Length(C)^2,$$

and equality holds if and only if C is the arc perpendicular to the rays.
C lies in the larger sector, then

$$2\pi Area(D) \leq Length(C)^2,$$

where equality holds if and only if C is a semicircle perpendicular to only one ray of the two.

2 Cones with vertex on the boundary

Some two-dimensional cones satisfy the classical isoperimetric inequality (see [Ch1, Theorem 1]). This is because two-dimensional cones, being flat, can be flattened (i.e., developed) to become a planar domain provided its density at the vertex is not smaller than 1. However, if the vertex lies on the boundary of the cone, the density hypothesis can be dropped [Ch1, Corollary 1].

Definition 1 A one-dimensional rectifiable connected set C in $\mathbf{R}^n$ is said to be a *compound Jordan curve* if for any point p of C there exists a Jordan curve (= a homeomorphic image of a circle) C' such that $p \in C' \subset C$. For $q \in \mathbf{R}^n$, $q \divideontimes C$ is the *cone* from q over C, the set of all line segments from q to the points of C.

Lemma 3 *If C is a compound Jordan curve in $\mathbf{R}^n$ and p a point of C, then*

$$4\pi Area(p \divideontimes C) \leq Length(C)^2.$$

Equality holds if and only if $p \divideontimes C$ can be developed, by cutting and inserting, one-to-one onto a disk.

3 Stationary varifolds

Our purpose in this section is to prove the isoperimetric inequality for a stationary 2-dimensional varifold in $\mathbf{R}^n$. Every stationary varifold V is rectifiable [A2, All], and if its density is bounded away from zero, an open dense subset of the support of V is a continuously differentiable submanifold of $\mathbf{R}^n$ [All]. Following [All] and [S], we briefly introduce varifolds in $\mathbf{R}^n$, define stationary varifolds and their generalized boundary, and derive an area estimate of a stationary varifold from the first variation formula.

m-dimensional varifolds in $\mathbf{R}^n$ are simply Radon measures on $G_m(\mathbf{R}^n) = \mathbf{R}^n \times G(n,m)$, where $G(n,m)$ is the space of m-dimensional subspaces of $\mathbf{R}^n$. Given such an m-varifold V on $\mathbf{R}^n$, there corresponds a Radon measure μ_V on $\mathbf{R}^n$ defined by

$$\mu_V(A) = V(\pi^{-1}(A)), \quad A \subset \mathbf{R}^n,$$

where π is the projection $(x, S) \mapsto x$ of $G_m(\mathbf{R}^n)$ onto $\mathbf{R}^n$. The *mass* $\mathbf{M}(V)$ of V is defined by

$$\mathbf{M}(V) = \mu_V(\mathbf{R}^n) = V(G_m(\mathbf{R}^n)).$$

If $M = \mathrm{spt}\mu_V$ is rectifiable, then $\mu_V = \mathcal{H}^m \llcorner \theta$, where $\mathcal{H}^m$ is the m-dimensional Hausdorff measure, θ vanishes on $\mathbf{R}^n \sim M$ and is a positive locally $\mathcal{H}^m$-integrable function on M. θ is called the *multiplicity function* of μ_V. The support M and multiplicity θ of μ_V completely determine V when V is rectifiable. So the varifold V is also denoted by $\underline{\underline{v}}(M, \theta)$. When θ is integer valued almost everywhere, V is called an *integral* varifold.

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable. Then we define the image varifold $f_\sharp V$ of V by

$$f_\sharp V(A) = \int_{F^{-1}(A)} J_S f(x) dV(x,S), \quad A \subset G_m(\mathbf{R}^n),$$

where $F : G_m(\mathbf{R}^n) \to G_m(\mathbf{R}^n)$ is defined by $F(x,S) = (f(x), df_x(S))$ and where

$$J_S f(x) = \sqrt{\det((df_x|S)^* \circ (df_x|S))}, \quad (x,S) \in G_m(\mathbf{R}^n),$$

$(df_x|S)^*$ being the adjoint of $df_x|S$.
The *first variation* δV of V is a linear functional on the set $\mathcal{X}(\mathbf{R}^n)$ of continuous vector fields with compact support on $\mathbf{R}^n$, defined by

$$\delta V(Y) = \frac{d}{dt}\mathbf{M}(\phi_{t\sharp}V)|_{t=0},$$

where $\{\phi_t\}_{-1<t<1}$ is any 1-parameter family of diffeomorphisms in $\mathbf{R}^n$ with Y as the initial velocity vector field. Differentiation under the integral gives

$$\delta V(Y) = \int_{G_m(\mathbf{R}^n)} \mathrm{div} dV(x,S),$$

where

$$\mathrm{div}_S Y = \sum_{i=1}^{m} < \tau_i, D_{\tau_i} Y >,$$

$\tau_1, ..., \tau_m$ being an orthonormal basis for S, and D the Euclidean connection.
V is said to be *stationary* in U if $\delta V(Y) = 0$ for any $Y \in \mathcal{X}(\mathbf{R}^n)$ with spt$Y \subset U$.
Now we want to define $||\delta V||$, the *total variation measure* of δV. Assume that V has *locally bounded first variation* in $\mathbf{R}^n$, that is, for each $W \subset\subset \mathbf{R}^n$ there is a constant $c < \infty$ such that $|\delta V(Y)| \le c \sup |Y|$ for any $Y \in \mathcal{X}(\mathbf{R}^n)$ with spt$|Y| \subset W$. Then the Riesz representation theorem says that there exist a Radon measure $||\delta V||$ on $\mathbf{R}^n$ and a $||\delta V||$-measurable vector field ν on $\mathbf{R}^n$ such that $|\nu| = 1$ $||\delta V||$-a.e. and

$$\delta V(Y) = \int_{\mathbf{R}^n} \nu \cdot Y d||\delta V||,$$

where $||\delta V||$ is characterized by

$$||\delta V||(W) = \sup\{\delta V(Y) : Y \in \mathcal{X}(\mathbf{R}^n), \ |Y| \le 1, \text{ and spt}|Y| \subset W\}$$

for any open $W \subset\subset \mathbf{R}^n$. Differentiating $||\delta V||$ with respect to μ_V, we see that

$$\frac{d||\delta V||}{d\mu_V}(x) = \lim_{\rho \to 0} \frac{||\delta V||(B_\rho(x))}{\mu_V(B_\rho(x))}$$

exists μ_V-a.e. and that

$$\int_{\mathbf{R}^n} \nu \cdot Y d\|\delta V\| = -\int_{\mathbf{R}^n} \vec{H} \cdot Y d\mu_V + \int_{\mathbf{R}^n} \nu \cdot Y d\sigma,$$

where

$$\vec{H}(x) = -\frac{d\|\delta V\|}{d\mu_V}(x)\nu(x), \ \sigma = \|\delta V\| \llcorner Z,$$

$$Z = \{x \in \mathbf{R}^n : \frac{d\|\delta V\|}{d\mu_V}(x) = \infty\}, \text{ and } \mu_V(Z) = 0.$$

Thus for $Y \in \mathcal{X}(\mathbf{R}^n)$ we can write

$$\delta V(Y) = -\int_{\mathbf{R}^n} \vec{H} \cdot Y d\mu_V + \int_Z \nu \cdot Y d\sigma. \tag{2}$$

By analogy with the classical first variation formula for a smooth submanifold of $\mathbf{R}^n$, we call $\vec{H}$ the *generalized mean curvature* of V, Z the *generalized boundary* of V, σ the *generalized boundary measure* of V, and $\nu|Z$ the *generalized unit conormal* of V. We can easily see that V is stationary in U if and only if $\vec{H}|U = 0$ and $Z \cap U = \emptyset$.

Definition 2 (a) Let V be an m-dimensional varifold of locally bounded first variation in $\mathbf{R}^n$ and Z the generalized boundary of V with the generalized boundary measure σ. Assume Z is $(m-1)$-rectifiable. Let

$$\psi(x) = \lim_{\rho \to 0} \frac{\sigma(B_\rho(x))}{\mathcal{H}^{m-1}(Z \cap B_\rho(x))}, \ x \in Z.$$

Then define ∂V to be the varifold $\underline{\underline{v}}(Z, \psi)$. In other words, ∂V is the $(m-1)$-dimensional rectifiable varifold with support Z and multiplicity ψ. Clearly $\mu_{\partial V} = \sigma$. ∂V is called the *varifold boundary* of V.
(b) For an m-varifold $V = \underline{\underline{v}}(M, \theta)$, the *varifold cone* $p \times\!\!\!\times V$ from p over V is the $(m+1)$-varifold $\underline{\underline{v}}(p \times\!\!\!\times M, \bar{\theta})$, where $\bar{\theta}(y) = \theta(x)$ whenever y lies on the line segment from p to $x \in M$.

Example Given a cube I^3 of volume 1 in $\mathbf{R}^3$, let F be the union of the faces of I^3, E the union of the edges of I^3. Define V to be the 2-dimensional varifold with support F and multiplicity 1 everywhere, i.e., $V = \underline{\underline{v}}(F, 1)$. Then one can see that i) the generalized mean curvature $\vec{H}$ of V vanishes on $F \sim E$, ii) E is the generalized boundary of V, iii) $\sigma = (\mathcal{H}^1 \llcorner E) \llcorner \sqrt{2}$ is the generalized boundary measure of V, and iv) the generalized unit conormal

ν of V makes an angle of 45 degrees with the outward unit normals to F along E. It follows that V is stationary in $\mathbf{R}^3 \sim E$, the multiplicity of ∂V is $\sqrt{2}$, i.e., $\partial V = \underline{\underline{v}}(E, \sqrt{2})$, and $p⋇\partial V = \underline{\underline{v}}(p⋇E, \sqrt{2})$, $p \in \mathbf{R}^3$. Moreover $\mathbf{M}(V) = 6$, $\mathbf{M}(\partial V) = 12\sqrt{2}$, and if p_0, p_1 are the center of gravity and a vertex of I^3 respectively, $\mathbf{M}(p_0⋇\partial V) = 6$, $\mathbf{M}(p_1⋇\partial V) = 3 + 3\sqrt{2}$.

In [Ch1, Proposition 1] we proved a volume estimate for minimal submanifolds in $\mathbf{R}^n$. We extend this estimate to stationary varifolds in $\mathbf{R}^n$ as follows.

Theorem 1 *Let V be an m-varifold of locally bounded first variation in $\mathbf{R}^n$. If the generalized boundary Z of V is rectifiable and V is stationary in $\mathbf{R}^n \sim Z$, then for any $p \in \mathbf{R}^n$*

$$\mathbf{M}(V) \leq \mathbf{M}(p⋇\partial V).$$

Lemma 4 *Let $W = \underline{\underline{v}}(Z, \psi)$ be a rectifiable 1-varifold in $\mathbf{R}^n$ with $\psi \geq 1$ and let p be a point in Z. If Z is a compound Jordan curve, then*

$$4\pi\mathbf{M}(p⋇W) \leq \mathbf{M}(W)^2.$$

Theorem 2 *Suppose that V is a 2-varifold of locally bounded first variation in $\mathbf{R}^n$, the generalized boundary Z of V is rectifiable, and V is stationary in $\mathbf{R}^n \sim Z$. If the multiplicity of ∂V is ≥ 1 and Z is a compound Jordan curve, then*

$$4\pi\mathbf{M}(V) \leq \mathbf{M}(\partial V)^2.$$

We conjecture that the theorem above can be extended in two ways: i) The theorem should hold without the hypothesis on the multiplicity of ∂V if the multiplicity of V is assumed to be 1 a.e.; ii) The optimal case (equality) should occur only when spt$V$ is a disk. In case the multiplicity of ∂V is less than 1, one can modify the theorem as follows.

Corollary 1 *Let V be a 2-varifold of locally bounded first variation in $\mathbf{R}^n$ such that V is stationary outside the rectifiable generalized boundary Z. Write $\partial V = \underline{\underline{v}}(Z, \theta)$ and define $\bar{\partial} V = \underline{\underline{v}}(Z, \bar{\theta})$, $\bar{\theta} = \max\{\theta, 1\}$. If Z is a compound Jordan curve, then*

$$4\pi\mathbf{M}(V) \leq \mathbf{M}(\bar{\partial} V)^2.$$

4 Area minimizing flat chains mod k

In this section we derive a different type of sharp isoperimetric inequalities for certain area minimizing flat chains mod k. Roughly speaking, flat chains, or currents, are obtained by assigning an orientation to the tangent space of varifolds. First let us briefly define currents and related terminology.

Let $\mathcal{D}^m$ be the space of smooth differential m-forms with compact support in $\mathbf{R}^n$. An m-dimensional *current* in $\mathbf{R}^n$ is a continuous linear functional on $\mathcal{D}^m$. The set of such m-currents will be denoted $\mathcal{D}_m$. Any oriented m-dimensional rectifiable set M may be viewed as a current T_M in the following way. Let $\vec{S}(x)$ denote the unit m-vector associated with the oriented tangent space to M at x. Then for any differential m-form ω, define

$$T_M(\omega) = \int_M < \vec{S}(x), \omega > d\mathcal{H}^m.$$

Furthermore, we will allow T_M to carry a positive integer multiplicity $\theta(x)$, and define

$$T_{M,\theta}(\omega) = \int_M < \vec{S}(x), \omega > \theta(x) d\mathcal{H}^m. \tag{3}$$

Motivated by the classical Stokes' theorem, we are led to define the *boundary* $\partial T \in \mathcal{D}_{m-1}$ of an m-current T by

$$\partial T(\omega) = T(d\omega), \quad \omega \in \mathcal{D}^{m-1}.$$

Again motivated by the example above, T_M, we define the *mass* of T, $\mathbf{M}(T)$, for $T \in \mathcal{D}_m$ by

$$\mathbf{M}(T) = \sup\{T(\omega) : |\omega| \leq 1, \ \omega \in \mathcal{D}^m\},$$

where $|\omega| = \sup_{x\in\mathbf{R}^n} < \omega(x), \omega(x) >^{1/2}$. The *support* of a current T, spt$T$, is the complement in $\mathbf{R}^n$ of the largest open set on which $T = 0$. T is called a *rectifiable* current if spt$T$ is rectifiable. The mass of a rectifiable current is just the Hausdorff measure of the associated rectifiable support (counting multiplicities). The integer multiplicity rectifiable currents $T_{M,\theta}$ as defined in (10) are characterized by the property that they agree, to within a set of arbitrarily small $\mathcal{H}^m$ measure, with m-dimensional C^1 singular chainnteger coefficients. Notice that one can associate $T_{M,\theta}$ with the integer multiplicity varifold $V = \underline{\underline{v}}(M, \theta)$ in $\mathbf{R}^n$.

$\mathcal{R}_m$ denotes the set of integer multiplicity rectifiable m-currents in $\mathbf{R}^n$. And

$\mathcal{I}_m^k$ denotes the space of m-dimensional rectifiable *flat chains modulo k* whose boundaries are also rectifiable flat chains modulo k, that is,

$$\mathcal{I}_m^k = \{T : T \in \mathcal{R}_m/k\mathcal{R}_m,\ \partial T \in \mathcal{R}_{m-1}/k\mathcal{R}_{m-1}\}$$

(see [F, 4.2.26]). We write the same notations spt, ∂, $\mathbf{M}$ for flat chains mod k as we do for currents. One says $T \in \mathcal{I}_m^k$ is *area minimizing* if

$$\mathbf{M}(T) \le \mathbf{M}(S) \text{ for every } S \in \mathcal{I}_m^k \text{ with } \partial S = \partial T.$$

Definition 3 Let $Y^k \subset \mathbf{R}^3$ be the union of k great semicircles on a sphere meeting at the north and south poles at equal angles of $2\pi/k$. Define $\mathcal{Y}_2^k \subset \mathcal{I}_2^k$ to be the set of 2-dimensional flat chains T mod k in $\mathbf{R}^n$with multiplicity 1 almost everywhere such that spt∂T is homeomorphic to Y^k and the associated varifold $V = \underline{\underline{v}}(\text{spt}T, \theta)$ is locally of bounded first variation in $\mathbf{R}^n$.

Theorem 3 *Suppose that T is a 2-dimensional area minimizing flat cahin mod k in $\mathcal{Y}_2^k$. If $C_1, C_2, ..., C_k$ are the curves that constitute* spt∂T *and have common end points p, p', then*

$$2\pi\mathbf{M}(T) \le \sum_{i=1}^{k} Length(C_i)^2.$$

And equality holds if and only if spt$T$ *is the union of k flat half disks.*

Lemma 5 *Let $T \in \mathcal{Y}_2^k$ be a 2-dimensional area minimizing flat chain mod k and V the varifold associated with T. Then* spt$\partial V \subset$ spt∂T *and the multiplicity ψ of ∂V is less than or equal to 1 almost everywhere on* spt∂T.

Let Y be a union of three half disks meeting each other along their common diameter at equal angles of 120 degrees. Let T (T^1, respectively) be the intersection with the unit ball $B_1(O)$ ($\partial B_1(O)$, respectively) of an infinite cone from O through the 1-skeleton of a regular tetrahedron with its center of mass at O. In [T2] J. Taylor proved that the disk, Y, and T are the only three cones that are area minimizing under Lipschitz maps leaving the boundary fixed. In view of this fact we raise the following problem as an analogue of Theorem 3.

Open Problem: Suppose that V is a 2-varifold with multiplicity 1 almost everywhere and is locally of bounded first variation in $\mathbf{R}^n$ such that V is stationary outside the rectifiable boundary spt∂V. Suppose also that spt∂V is homeomorphic to T^1. Let $C_1, C_2, ..., C_6 \subset$ spt∂V be the curves that constitute spt∂V and lie between 4 junctions of spt∂V. Show that

$$[2\cos^{-1}(-\frac{1}{3})]\mathbf{M}(V) \leq \sum_{i=1}^{6} Length(C_i)^2,$$

where equality holds if and only if spt$V$ is a homothetic expansion (or contraction) of T.

5 Monotonicity of stationary varifolds

We derived the mass estimate of Theorem 1 from the first variation formula (2) with $Y = x - p$. If one uses $Y = \phi(|x-p|)(x-p)$ with ϕ equal to the characteristic function of the interval $(-\infty, \rho)$, then one can obtain the well known monotonicity of the mass of a stationary m-varifold $V : \rho^{-m}\mu_V(B_\rho(p))$ is a nondecreasing function of ρ (see [S, p.236]). In this section we combine the mass estimate and the monotonicity of mass to prove the third sharp isoperimetric inequality for an m-varifold in a ball.

Theorem 4 *Suppose V is an m-varifold of locally bounded first variation in $\mathbf{R}^n$ and p is a point in* spt$V$. *If V is stationary in an open ball B centered at p and* spt∂V *lies on the sphere ∂B, then*

$$m^m \omega_m \Theta^m(\mu_V, p)\mathbf{M}(V)^{m-1} \leq \mathbf{M}(\partial V)^m. \tag{4}$$

Here ω_m is the volume of the unit ball in $\mathbf{R}^m$, and equality holds if and only if V coincides with the varifold cone $p \divideontimes \partial V$.

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ON CONFORMAL GEOMETRY

JOSÉ F. ESCOBAR

In this lecture I will report on some progress I have obtained in some problems in conformal geometry. Let $\Omega \subset R^2$ be an open, non-empty connected domain which is not the whole plane. The Riemann mapping theorem asserts that there exists a conformal diffeomorphism $F : \Omega \to B$ where B is a ball. It is well known that the Riemann mapping theorem does not hold in higher dimensions. Indeed if $\Omega \subset R^n$ is an open, simply connected, bounded domain with C^2 boundary and there exists a conformal diffeomorphism $F : \Omega \to B$ then Ω is a ball. In order to see that we let $\delta_{ij}(B)$ and $\delta_{ij}(\Omega)$ represent the Euclidean metric on the ball and Ω respectively. Since F is conformal then

$$F^*(\delta_{ij}(B)) = |DF|^2 \delta_{ij}(\Omega). \tag{1}$$

The boundary of the ball is umbilic (the second fundamental form is proportional to the metric). Umbilicity is a conformal invariant property. Thus, the boundary of Ω is umbilic. It is elementary to see that $\partial\Omega$ is locally a piece of sphere or a hyperplane. Since Ω is simply connected, bounded and $\partial\Omega$ is C^2 then Ω is a ball.

We consider the right-hand side of (1), that is, metrics of the form $g = u^{\frac{4}{n-2}} \delta_{ij}(\Omega)$ and ask: can we find a function u defined on Ω such that the metric g satisfies that the scalar curvature vanishes and the mean curvature of the boundary is constant? These two conditions are clearly satisfied by the Euclidean metric on the ball. We answer the above question with the following Theorem:

Theorem 1. *Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $n > 5$ or $n = 3$. There exists a smooth metric $\bar{g}$ conformally related to the Euclidean metric such that the scalar curvature of $\bar{g}$ is zero and the mean curvature of the boundary with respect to the metric $\bar{g}$ is constant.*

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The existence of the metric $\bar{g}$ is obtained as a minima for a variational problem. The functional we study is the "Total scalar curvature + total mean curvature".

Let M be a smooth n–dimensional compact manifold with boundary. Let $\mathcal{M}$ denote the space of all smooth Riemannian metrics on M. For $g \in \mathcal{M}$ we denote by $R(g)$ the scalar curvature of g and by $h(g)$ the mean curvature of ∂M with respect to the metric g. We define

$$F(g) = c_1(n)\int_M R(g)dv(g) + c_2(n)\int_{\partial M} h(g)d\sigma(g) \tag{2}$$

where dv and $d\sigma$ represent the Riemannian measure on M and on ∂M induced by the metric g, $c_1(n) = \frac{n-2}{4(n-1)}$ and $c_2(n) = \frac{n-2}{2}$.

We denote by $C_{a,b} = \{g \in \mathcal{M} |\ a\mathrm{Vol}(M,g) + b\mathrm{Vol}(\partial M, g) = 1\}$.

Let $g \in \mathcal{M}$ be a fixed metric and consider $\bar{g} \in C_{a,b}$ such that $\bar{g}$ is within the conformal class of g. In this case the metric $\bar{g}$ can be written as $\bar{g} = u^{\frac{4}{n-2}}g$, where u is a smooth positive function defined on M.

The functional F takes the following form $F(\bar{g}) = F(u^{\frac{4}{n-2}}g) = E(u)$ where

$$E(u) = \int_M |\nabla u|^2 dv + c_1(n)\int_M Ru^2 dv + c_2(n)\int_{\partial M} hu^2 d\sigma$$

The constraint set $C_{a,b}$ is

$$\{u \in C^\infty(\bar{M}), u > 0, a\int_M u^p dv + b\int_{\partial M} u^q d\sigma = 1\}$$

where $p = \frac{2n}{n-2}$ and $q = \frac{2(n-1)}{n-2}$. The natural space to study the functional E is $H_1(M)$, the Hilbert space of functions in $L^2(M)$ and with first derivatives in $L^2(M)$. For that reason we redefine the constraint set as

$$C_{a,b} = \{\varphi \in H_1(M) |\ a\int_M |\varphi|^p + b\int_{\partial M} |\varphi|^q d\sigma = 1\}$$

The Euler-Lagrange equation associated to the functional E defined on the constraint set $C_{a,b}$ is

$$\begin{cases} \Delta u - c_1(n)Ru + \lambda n a u^{\frac{n+2}{n-2}} = 0 & \text{on } M. \\ \frac{\partial u}{\partial \eta} + c_2(n)hu = \lambda(n-1)bu^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{3}$$

where λ is a Lagrange multiplier.

Our first proposition shows that the functional E is bounded from below.

Proposition 1. *There exists a constant C_0 such that*

$$E(\varphi) \geq C_0, \qquad \varphi \in C_{a,b}$$

Proof. In order to prove the claim is enough to observe that for any $\varphi \in C_{a,b}$ and any $\varepsilon > 0$, there exists a constant $C_1 = C_1(\varepsilon, M, p, q, a, b)$ such that

$$\int_M \varphi^2 dv \leq \varepsilon \int_M |\nabla\varphi|^2 + C_1 \tag{4}$$

$$\int_{\partial M} \varphi^2 d\sigma \leq \varepsilon \int_M |\nabla\varphi|^2 + C_1. \tag{5}$$

Hölder's inequality implies that

$$\begin{aligned}\left(\int_M \varphi^2 dv\right)^{\frac{1}{2}} &\leq \left(\int_M |\varphi|^p dv\right)^{1/p} \mathrm{Vol}(M)^{\frac{p-2}{2p}}. \\ &\leq \frac{\mathrm{Vol}(M)^{\frac{p-2}{2p}}}{a^{1/p}} \left[1 - b\int_{\partial M} |\varphi|^q d\sigma\right]^{1/p} \\ &\leq C_0 + C_0|b|^{1/p}\left(\int_{\partial M} |\varphi|^q d\sigma\right)^{1/p}\end{aligned}$$

Using the Sobolev inequality we have

$$\left(\int_{\partial M} |\varphi|^q d\sigma\right) \leq C_1\left[\left(\int_M |\nabla\varphi|^2\right)^{q/2} + \left(\int_M \varphi^2\right)^{q/2}\right.$$

Thus

$$\left(\int_{\partial M} |\varphi|^q d\sigma\right)^{1/p} \leq C_2\left[\left(\int_M |\nabla\varphi|^2\right)^{q/2p} + \left(\int_M \varphi^2\right)^{q/2p}\right]$$

and therefore we conclude that

$$\left(\int_M \varphi^2 dv\right)^{1/2} \leq C_0 + C_3\left[\left(\int_M |\nabla\varphi|^2\right)^{q/2p} + \left(\int_M \varphi^2\right)^{q/2p}\right]$$

Proposition 3 implies Theorem 1. In order to show inequality (7), it is enough to exhibit a function $\varphi \in H_1(M), \varphi \in C_{0,1}(M)$ for which

$$E(\varphi) < G_{0,1}(B).$$

To construct the function φ, we first observe that $G_{0,1}(B)$ is minimized by the standard metric, that is by a constant function (see [B] and [E1]). The conformal change of variables, given by the inversion map $x \to \frac{x}{|x|^2}$ coupled with the conformal invariance of the Sobolev quotient show that if $R^n_+ = \{(x,t)|x \in R^{n-1}, t > 0\}$ then $v^{\frac{4}{n-2}}\delta_{ij}$, are the minimizers for $G_{0,1}(R^n_+)$ where

$$v(x,t) = c\left(\frac{\varepsilon}{(\varepsilon+t)^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}} \tag{8}$$

and c is a suitable constant.

On a compact Riemannian manifold with boundary, using Fermi coordinates one can transplant the approximate extremal functions for R^n_+ given by (8) to M and deduce

$$G_{0,1}(M) \leq G_{0,1}(B).$$

From now on we assume that $G_{0,1}(M)$ is positive because $G_{0,1}(B) > 0$.

In [E3] we showed that strict inequality in (7) holds provided that $n > 6$ and there exists a non-umbilic point on the boundary. This is the generic case. To proof this we exhibited a function ψ supported in a small neighborhood of a boundary point. Let $(x^1, x^2, \ldots, t)$ be Fermi coordinates around a point $0 \in \partial M$. We define $\psi = v_\varepsilon \varphi$ where φ is a standard cut-off function and v_ε is given in (8) for $x_0 = 0$. To proof our inequality we use the norm of the trace free part of the second fundamental form as a correction term. We can use this norm because at a non-umbilic point this tensor does not vanishes. Recently we improved our previous estimate. We consider a small perturbation of the function φ and is defined by

$$\varphi = \left(\frac{\varepsilon}{(\varepsilon+t)^2 + |x|^2 - \delta x_1^2}\right)^{\frac{n-2}{2}} \psi \tag{9}$$

where ψ is as before.

The parameter δ plays an important role in distinguishing the eigenvalues of the second fundamental form in the $\frac{\partial}{\partial x^1}$ direction. Using the test function φ defined in (9) our proof succeded for $n > 5$ (see [E4]).

When ∂M is umbilic and M is locally conformally flat, we use a global test function. If we assume that (M, g) is locally conformally flat, then for a suitable metric within the conformal class of g, the Green's function G, for the conformal Laplacian has the following asymptotic expansion for $|x|$ small

$$G(x) = |x|^{2-n} + A + O(|x|).$$

In the appendix of [E2] we show that the Positive Mass Theorem holds for locally conformally flat manifolds with umbilic boundary. In fact we show that when $G_{0,1}(M) > 0$, $A \geq 0$ and equality holds only if M is conformally equivalent to B. The constant A is the crucial ingredient. However, when M is a bounded domain in R^n, we do not need to use the Positive Mass Theorem. We show that $E(c) < G_{0,1}(B)$ where $c = (b\mathrm{Vol}(M))^{\frac{-n-2}{2(n-1)}}$. In the general case we construct a function φ that coincides with v_ε in a small neighborhood of a point $0 \in \partial M$, so that we can get arbitrarily close to $G_{0,1}(B)$. Outside of the small neighborhood of the point $0 \in \partial M$, the function φ coincides with a small multiple of the Green's function. We use as a correction term the constant A in this case. Since v_ε *is not* rotationally symmetric around the point 0 and G *is* (asymptotically) gives rise a great number of error terms.

The estimates can be carried out for a general 4- or 5-dimensional Riemannian manifold with umbilic boundary.

Our proof when $n = 3$ is quite different from the case $n > 5$. When $n = 3$ we first show that, if there exist an umbilic point on the boundary and M^3 is not conformally equivalent to B^3, then $G_{0,1}(M) \leq G_{0,1}(B)$. To show this we use again the Positive Mass Theorem that holds for 3 dimensional manifolds having an umbilic point on the boundary. (This is not true when $n \geq 4$. One needs to assume that the boundary is umbilic on a neighborhood.) The other case is when the boundary is non-umbilic. This case follows by approximating the manifold M by a sequence of manifolds having one umbilic point and not conformally equivalent to B^3. To this sequence we apply the previous result. One proves then that strict inequality in (7) is preserved upon passage to the limit.

Recently using a perturbation argument we were able to show the following theorem **Theorem 3**. *Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $n \geq 3$. There exists a smooth metric $\bar{g}$ conformally related to the Euclidean metric such that the scalar curvature of $\bar{g}$ is constant and the mean curvature of the boundary with respect to the metric $\bar{g}$ is constant.*

The proof of the above theorem is obtained by a perturbation argument. In the proof of Theorem 3 one shows that for b small (positive or negative) and for (M, g) not conformally equivalent to the Ball we have

$$G_{a,b}(M) < G_{a,b}(B)$$

In order to prove that we relay in the estimates obtained in our previous paper [E2].

As a consequence of the previous theorem is that if a bounded domain in R^n admits a conformal metric to the euclidean with positive scalar curvature and minimal boundary then it admits a metric with positive constant scalar curvature and positive constant mean curvature on the boundary. The first condition is satisfied provided that the first eigenvalue for the linear operator associtaed with Euler-Lagrange equation is positive.

For compact manifolds without boundary we consider the functionals we defined before but without the boundary integrals. A consequence of the work of Aubin, [A] and Schoen [S] on the solution of the Yamabe problem for closed compact manifolds is that

$$G(M) < G(S^n)$$

when (M, g) is not conformally equivalent to (S^n, g_0), where g_0 is the standard round metric.

It is clear that by perturbing the standard metric of S^n we can construct manifolds with Sobolev quotient arbitray close to the Sobolev quotient of the sphere. However, it is unclear to decide whether or not the $G(M)$ of an arbitrary compact Einstein manifold can be arbitrary close to $G(S^n)$. We show by using a compactness method that

Theorem 4. *Let E_n denote the space of n-dimensional compact Einstein manifolds, which are not conformally diffeomorphic to S^n, $n \geq 3$. Then there exists $\varepsilon(n) > 0$ such that*

$$\sup_{(M,g)\in E_n} G(M) \leq G(S^n) - \varepsilon$$

Theorem 4 holds for compact manifolds with boundary provided that the boundary is totally geodesic.

Critical points for the total scalar curvature functional (2) on closed compact manifolds with a fixed volume are Einstein metrics.

Theorem 4 implies that there is a gap between the value of the total scalar curvature functional (when restricted to a set of metrics with a fixed volume) on the sphere with the standard metric and the value of any other critical point which is obtained as a critical point when we minimize in the set of conformal metrics and then maximize in the ortogonal complemet to the confomal metrics (modulus diffeormorphisms).

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HEAT KERNERL AND CONVERGENCE OF RIEMANNIAN MANIFOLDS

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Since M. Gromov introduced a notion of Hausdorff distance on the set of Riemannian manifolds, or more generally metric spaces, this decade has seen intensive activities around the convergence theory of Riemannian manifolds. These include some works from the viewpoint of spectral geometry or probability theory.

In this report, we would like to discuss the convergence theory in connection with the Laplace operators of Riemannian manifolds. After quick rewiew on Gromov-Hausdorff distance and measured Hausdorff topology in Section 1, we shall introduce a new distance on the set of Riemannian manifolds, making use of the heat kernels , and mention some basic properties of the distance in Sections 2 and 3. In Section 4, we shall focus on the study of Laplace operators of manifolds with bounded curvature, and its applications will be given in the last two sections.

1. Gromov-Hausdorff distance and measured Hausdorff topology

To begin with, we recall the definition of the Hausdorff distance on the set $\mathcal{MET}$ (of isometry classes) of compact metric spaces introduced by Gromov [14]. Given two metric spaces X and Y, a distance δ on the disjoint union $X \sqcup Y$ is said to be admissible if its restriction to X and Y are equal to the original distances d_X and d_Y in X and Y respectively. The Gromov-Hausdorff distance $HD(X,Y)$ is by definition the lower bound inf $H^\delta(X,Y)$, where δ runs over all admissible distances on $X \sqcup Y$ and $H^\delta(X,Y)$ stands for the Hausdorff distance in $(X \sqcup Y, \delta)$, namely, the lower bound of the numbers $\varepsilon > 0$ such that $\delta(x,Y) < \varepsilon$ and $\delta(y,X) < \varepsilon$ for all $x \in X$ and $y \in Y$. We observe that if $HD(X,Y) < \varepsilon$, then there exist mappings $f : X \to Y$ and $h : Y \to X$ such that the 2ε-neighborhood of $f(X)$ covers Y and for all x, $y \in X$,

$$|d_X(x,y) - d_Y(f(x),f(y))| < 2\varepsilon;$$

and also the 2ε-neighborhood of $h(Y)$ covers X and for all a, $b \in Y$,

$$|d_X(h(a),h(b)) - d_Y(a,b)| < 2\varepsilon.$$

In fact, we take an admissible distance δ on $X \sqcup Y$ such that $H^{\delta}(X,Y) < \varepsilon$ and then choose mappings $f : X \to Y$ and $h : Y \to X$ in such a way that $\delta(x, f(x)) < \varepsilon$ and $\delta(a, h(a)) < \varepsilon$ for all $x \in X$ and $a \in Y$. We call (not necessarily continuous) maps $f : X \to Y$ and $h : Y \to X$ possessing the above properties *2ε-Hausdorff approximations* between X and Y. Let us denote by $HD'(X,Y)$ the lower bound of the numbers $\varepsilon > 0$ for which there exist ε-Hausdorff approximations between X and Y. Then we have

$$\frac{1}{2}HD'(X,Y) \le HD(X,Y) \le 2HD'(X,Y).$$

For this reason, HD and HD' induce the same uniform topology on $\mathcal{MET}$.

Given a positive integer n and a positive number D, we shall denote by $\mathcal{S}(n,D)$ the set of isometry classes of compact Riemannian manifolds M of dimension n such that the diameter $diam(M) \le D$ and the Ricci curvature $Ric_M \ge -(n-1)$. Then Gromov [14] showed the following

Theorem 1. *The set $\mathcal{S}(n,D)$ is precompact in $\mathcal{MET}$ with respect to the Gromov-Hausdorff distance HD.*

Now we shall consider pairs (X,μ) of a compact metric space X and a Borel measure μ on X with unit total mass $\mu(X) = 1$. We say that a sequence of such pairs $\{(X_i,\mu_i)\}$ converges to a pair (X,μ) with respect to the *meaured Hausdorff topology* if there exist Borel measurable mappings $f_i : X_i \to X$ and positive numbers ε_i with $\lim_{i\to\infty} \varepsilon_i = 0$ such that f_i gives an ε-Hausdorff approximation of X_i to X, and the push-forward $f_{i*}\mu_i$ of the measure μ_i converges to μ with respect to the weak* topology, namely, for each continuous function ϕ on X, we have

$$\lim_{i\to\infty} \int_{X_i} \phi \circ f_i d\mu_i = \int_X \phi d\mu.$$

This topology was introduced by Fukaya in [10], where he discussed the convergence of the spectra of compact Riemannian manifolds. The main result of [10] is stated in the following

Theorem 2. *Let M_i be a sequence of compact Riemannian manifolds in $\mathcal{S}(n,D)$ and μ_{M_i} the normarized Riemannnian measure of M_i. Suppose (M_i,μ_{M_i}) converges to a pair (X,μ) of a compact metric space X and a Borel measrure μ on X with respect to the measured Hausdorff topology as i goes to infinity. In addition, suppose the sectional curvature of M_i is bounded in its absolute values uniformly in i. Then the following assertions hold:*

(1) For each $k \in \mathbf{N}$, the k-th eigenvalue $\lambda_k(M_i)$ of M_i converges as i goes to infinity.

(2) There exists an (unbounded) selfadjoint operator $P_{(X,\mu)}$ on $L^2(X,\mu)$ such that the limit $\lim_{i\to\infty}\lambda_k(M_i)$ is equal to the k-th eigenvalue $\lambda_k(X,\mu)$ of the operator $P_{(X,\mu)}$.

(3) Let $f_i : M_i \to X$ be a Borel measurable ε_i-Hausdorff approximation such that $\lim_{i\to\infty}\varepsilon_i = 0$ and $f_{i*}\mu_{M_i}$ converges weakly to the measure μ. Let ϕ_{k,M_i} be a normarized eigenfunction of Δ_{M_i}. Put $\Lambda_{k,i} = \{\phi\circ f_i | \phi \in L^2(X,\mu), P_{(X,\mu)}\phi = \lambda_k(X,\mu)\phi\}$. Then the distance between $\Lambda_{k,i}$ and $\phi_{k,i}$ in $L^2(M_i,\mu_{M_i})$ goes to zero as i tends to infinity.

It is asked in [10] if the uniform boundedness of the sectional curvatures could be dropped in this theorem (cf. Sections 3 and 4).

Here and after, given a compact Riemannian manifold M, we write μ_M for the normarized Riemannian measure on M with unit total mass, $\mu_M = dvol_M/Vol(M)$.

2. A Spectral distance

Now we would like to introduce a new distance on the set $\mathcal{M}$ of (isometry classes of) compact Riemannian manifolds, making use of the heat kernels. The contents of the present and the next sections are taken from [21].

Let M be a compact Riemannian manifold of dimension n and $p_M(t,x,y)$ the heat kernel of M in $L^2(M,\mu_M)$. By the Strum-Liouville decomposition, we have the eigenfunction expansion of the kernel:

$$p_M(t,x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k(M)t}\phi_k(x)\phi_k(y).$$

Here $0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \ldots \nearrow \infty$ are the eigenvalues of M and $\{\phi_k\}$ is a complete orthonormal system of $L^2(M,\mu_M)$ consisting of eigenfunctions with ϕ_k having eigenvalue $\lambda_k(M)$.

Given two compact Riemannian manifolds M and N, a mapping $f : M \to N$ is called *an ε-spectral approximation* if

$$e^{-1/t}\left|p_M(t,x,y) - p_N(t,f(x),f(y))\right| < \varepsilon$$

for all $t > 0$, and for all points x, y of M. The *spectral distance* between M and N, denoted by $SD(M,N)$, is by definition the infimum of the positive numbers ε such that there exist ε-spectral aproximations $f : M \to N$ and $h : N \to M$. Then SD becomes a distance on $\mathcal{M}$. In fact, if $SD(M,N) = 0$, we have ε_i-spectral approximations $f_i : M \to N$ with $\lim_{i\to\infty}\varepsilon_i = 0$. Take a countable and dense subset M_0 of M. Then we may assume that the mapping f_i converges to a mapping $f_0 : M_0 \to N$ pointwise on M_0. Clearly f_0 preserves the heat kernels on M_0, namely,

$$p_M(t, x, y) = p_N(t, f_0(x), f_0(y))$$

for all $t > 0$, and for all x, $y \in M_0$. This implies that M and N have the same dimension, because of the asymptotic behaviour of the heat kernels as t goes to zero. Moreover recall a fundamental result on the heat kernels of complete Riemannian manifolds, which asserts that

$$\lim_{t \to \infty} 4t \, \log p_M(t, x, y) = -dis_M(x, y)^2$$

(cf. Varadhan[26], Cheng, Li and Yau [7]). Applying this result to our manifolds M and N, we see that the mapping f_0 preserves the distances on M_0, namely,

$$dis_M(x, y) = dis_N(f_0(x), f_0(y))$$

for all x, $y \in M_0$. Since M_0 is dense, f_0 extends uniquely to an isometry between M and N.

The above argument suggests some connections between the spectral distance SD and the Gromov-Hausdorff distance HD. In fact, if we restrict our attention to the set $\mathcal{S}(n, D)$ for given n and D, we have the following

Theorem 3([21]). *(1) The identity mapping of the metric space* $(\mathcal{S}(n, D), SD)$ *to the metric space* $(\mathcal{S}(n, D), HD)$ *is uniformly continuous.*
(2) The metric space $(\mathcal{S}(n, D), SD)$ *is precompact.*

We can also show that the topology on $\mathcal{S}(n, D)$ induced by the spectral distance SD is finer than that of measured Hausdorff convergence. Moreover these two topologies actually coincide on the subspace $\mathcal{K}(n, D)$ of $\mathcal{S}(n, D)$ which consists of manifolds with sectional curvature bounded by 1 in its absolute values. This can be verified by the results mentioned below together with Theorem 2. However it is not clear at the present stage whether these two topologies coincide or not on $\mathcal{S}(n, D)$.

As for the completion $\mathcal{C}(\mathcal{S}(n, D), SD)$ of the metric space $(\mathcal{S}(n, D), SD)$, we have the following

Theorem 4 ([21]). *A (boundary) element of* $\mathcal{C}(\mathcal{S}(n, D), SD)$ *can be regarded as a triad* (X, μ, p) *consisting of a compact metric space* X *(of length), a Borel measure* μ *of unit total mass on* X*, and a positive Lipschitz function* p *on* $(0, \infty) \times X \times X$ *which is the heat kernel of a* C_0*-semigroup on* $L^2(X, \mu)$.

In fact, let (X, μ, p) be the limit element to which a sequence $\{M_i\}$ in $(\mathcal{S}(n, D), SD)$ converges. Then there exist Borel measurable ε_i-Hausdorff approximations $f_i : M_i \to X$ and $h_i : X \to M_i$ with $\lim_{i\to\infty} \varepsilon_i = 0$ such that (M_i, μ_{M_i}) converges to (X, μ) with respect to the measured Hausdorff topology via the approximation f_i, and further they give ε_i-spectral approximations:

$$e^{-t}|p_{M_i}(t, x, y) - p(t, f_i(x), f_i(y))| < \varepsilon_i$$

$$e^{-t}|p_{M_i}(t, h_i(a), h_i(b)) - p(t, a, b)| < \varepsilon_i$$

for all $t > 0$, and for all x, $y \in M_i$ and a, $b \in X$. In addition, the limit measure μ satisfies

$$\frac{\mu(B(a,r))}{\mu(B(a,R))} \geq \frac{V_n(r)}{V_n(R)}$$

for every $a \in X$ and for all r, R, $0 < r \leq R$, where $B(a, r)$ stands for the metric ball in X around a of radius r, and $V_n(r) = \int_0^r (\sinh t)^{n-1} dt$. This estimate is well known as the Bishop-Gromov's inequality for the case: $(X, \mu) = (M, \mu_M)$ (cf.[14]). As for the kernel p, it has the following properties:

$$p(t, a, b) \leq C(n, \varepsilon) \frac{\exp\left(-(1-\varepsilon) dis_X(a, b)^2/4t + \varepsilon t\right)}{\mu(B(a, \sqrt{t}))^{1/2} \mu(B(b, \sqrt{t}))^{1/2}};$$

$$p(t, a, b) \geq C'(n, \varepsilon) \frac{\exp\left(-(1+\varepsilon) dis_X(a, b)^2/4t - (\varepsilon + C'')t\right)}{\mu(B(a, \sqrt{t}))^{1/2} \mu(B(b, \sqrt{t}))^{1/2}}$$

$(C'' = (n-1)^2/4)$, for any $\varepsilon > 0$, and for all $t > 0$ and a, $b \in X$, where $C(n, \varepsilon)$ and $C'(n, \varepsilon)$ are some positive constants depending only on n and ε. Moreover we have

$$\begin{aligned} |p(s, a, b) - \quad & p(t, a', b')| \leq C^{(3)}(n, D) d \left(d^2 + (\tfrac{d}{\sqrt{s}})^{n+1} + (\tfrac{d}{\sqrt{t}})^{n+1}\right) \times \\ & \{dis_X(a, a') + dis_X(b, b')\} + C^{(4)}(n, D) d^n \left|(\tfrac{1}{\sqrt{s}})^n - (\tfrac{1}{\sqrt{t}})^n\right| \end{aligned}$$

for all s, $t > 0$ and for all a, a', b, $b' \in X$. Here we set $d = diam(X)$ and $C^{(3)}$ and $C^{(4)}$ are some positive constants depending only on n and D. For the case $(X, \mu, p) = (M, \mu_M, p_M)$, these estimates for the heat kernel have been shown by several authors, especially by Li and Yau [22] (cf. also [8] and the references therein). In fact, these geometric estimates for manifolds are crucial to the proofs of Theorems 3 and 4.

3. Convergence of eigenvalues and eigenfunctions

In this section, we shall first introduce a distance on the set of pairs of compact Riemannian manifolds M and complete orthonormal systems $\Phi =$

$\{\phi_k\}$ in $L^2(M, \mu_M)$ consisting of the eigenfunctions. For this, we fix some notations.

Let ℓ^2 be a Hilbert space defined by

$$\ell^2 = \{\{a_k\}_{k=1,2,\dots} : \sum_{k=1}^{\infty} {a_k}^2 < +\infty\}.$$

Let us consider the space $C_\infty([0,\infty), \ell^2)$ of continuous curves $\gamma : [0,\infty) \to \ell^2$ such that the ℓ^2 norm $|\gamma(t)|_{\ell^2}$ of $\gamma(t)$ decays to zero as t goes to infinity. This space is endowed with the distance d_∞:

$$d_\infty(\gamma, \sigma) = \sup\{|\gamma(t) - \sigma(t)|_{\ell^2} : t \geq 0\}.$$

For a subset A of $C_\infty([0,\infty), \ell^2)$ and a positive number r, $\mathcal{N}_r(A)$ stands for the r-neighborhood of A, $\mathcal{N}_r(A) = \{\gamma \in C_\infty([0,\infty), \ell^2) : d_\infty(A, \gamma) < r\}$. The Hausdorff distance δ_H on the set of bounded closed subsets of the metric space $C_\infty([0,\infty), \ell^2)$ is defined by

$$\delta_H(A, B) = \inf\{r > 0 : A \subset \mathcal{N}_r(B), B \subset \mathcal{N}_r(A)\}.$$

Given a compact Riemannian manifold M, we embed M into the metric space $C_\infty([0,\infty), \ell^2)$ as follows. Let $0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$ be the eigenvalues of M and $\Phi = \{\phi_k\}$ a complete orthonormal system of eigenfunctions of M in $L^2(M, \mu_M)$ with ϕ_k having eigenvalue $\lambda_k(M)$. For a point x of M, we define an element $F_\Phi[x]$ of $C_\infty([0,\infty), \ell^2)$ by

$$F_\Phi[x](t) = \{\zeta_0(t) e^{-\lambda_k(M)t/2} \phi_k(x)\}_{k=1,2,\dots},$$

where we put $\zeta_0(t) = e^{-1/2t}$. Then the map F_Φ of M into $C_\infty([0,\infty), \ell^2)$ defined by $x \to F_\Phi$ gives rise to a continuous imbedding of M, since the eigenfunctions separate the points of M. Given two such pairs $(M, \Phi = \{\phi_k\})$ and $(N, \Psi = \{\psi_k\})$, we put

$$SD^*((M, \Phi), (N, \Psi)) = \delta_H(F_\Phi[M], F_\Psi[N]).$$

We observe from the definition of SD^* that $SD^*((M, \Phi), (N, \Psi)) = 0$ if and only if there exist mappimgs $f : M \to N$ and $h : N \to M$ which preserve the heat kernels and further the given orthonormal systems respectively, that is, $f^*\psi_k = \phi_k$ and $h^*\phi_k = \psi_k$ for all k. Thus identifying such pairs, we obtain a metric space consisting of (the equivalence classes of) the pairs of compact Riemannian manifolds M and complete orthonormal systems Φ of the eigenfunctions in $L^2(M, \mu_M)$.

Let us now restrict ourselves to manifolds in $\mathcal{S}(n,D)$, and denote by $\mathcal{FS}(n,D)$ the set of equivalence classes of pairs (M,Φ), where $M \in \mathcal{S}(n,D)$.

Theorem 5 ([21]). *(1) The metric space $\mathcal{FS}(n,D)$ equipped with the distance SD^* is precompact.*

(2) The projection Π of $(\mathcal{FS}(n,D), SD^*)$ onto $(\mathcal{S}(n,D), SD)$ which sends (M,Φ) to M is uniformly continuous, so that Π extends uniquely to a continuous mapping $\overline{\Pi}$ of the completion $\mathcal{C}(\mathcal{FS}(n,D), SD^*)$ onto the completion $\mathcal{C}(\mathcal{S}(n,D), SD)$.

(3) An element of $\mathcal{C}(\mathcal{FS}(n,D), SD^*)$ can be regarded as a pair of an element (X,μ,p) in $\mathcal{C}(\mathcal{S}(n,D), SD)$ and a complete orthonomarl system $\Phi = \{\phi_k\}$ of eigenfunctions of $\mathcal{L}_p$ in $L^2(X,\mu)$, where $\mathcal{L}_p$ stands for the infinitessimal generator of the C_0 semigroup in $L^2(X,\mu)$ with kernel p.

The first assertion follows from the fact that for any $(M,\Phi) \in \mathcal{FS}(n,D)$, $F_\Phi[M]$ is contained in some compact subset of $C_\infty([0,\infty),\ell^2)$ depending only on n and D.

Now we shall mention the continuity of eigenvalues and eigenfunctions with respect to the spectral distance. Given two elements σ and τ of $\mathcal{C}(\mathcal{S}(n,D), SD)$, we set

$$\Gamma(\sigma,\tau) = \max\{SD^*(\alpha, \overline{\Pi}^{-1}(\tau)) : \alpha \in \overline{\Pi}^{-1}(\sigma)\},$$

$$\Lambda(\sigma,\tau) = \min\{SD^*(\alpha,\beta) : \alpha \in \overline{\Pi}^{-1}(\sigma), \beta \in \overline{\Pi}^{-1}(\tau)\}.$$

Then we can show that given a sequence $\{\sigma_i\}$ and an element τ in $\mathcal{C}(\mathcal{FS}(n,D), SD^*)$, the following three conditions are mutually equivalent: (i) $\lim_{i\to\infty} SD(\sigma_i, \tau) = 0$, (ii) $\lim_{i\to\infty} \Gamma(\sigma_i,\tau) = 0$, (iii) $\lim_{i\to\infty} \Lambda(\sigma_i,\tau) = 0$. This yields the following

Theorem 6 ([21]). *(1) For each positive integer k, the k-th eigenvalue λ_k, which is regarded as a function on $(\mathcal{S}(n,D), SD)$, extends continuously to the completion $\mathcal{C}(\mathcal{S}(n,D), SD)$.*

(2) Let $\{M_i\}$ be a sequence in $(\mathcal{S}(n,D), SD)$ which converges to an element (X,μ,p). Then for any complete orthonormal system $\Phi_i = \{\phi^{(i)}{}_k\}$ in $L^2(M_i,\mu_{M_i})$ which consists of eigenfunctions $\phi^{(i)}{}_k$ of the k-th eigenvalue $\lambda_k(M_i)$, there exists such a system $\Psi_i = \{\psi^{(i)}{}_k\}$ in $L^2(X,\mu)$ with $\psi^{(i)}{}_k$ having the k-th eigenvalue λ_k of $\mathcal{L}_p$, and ε_i-spectral approximations $f_i : M_i \to X$ and $h_i : X \to M_i$ satisfying

$$e^{-1/t} \sum_{k=1}^{\infty} \left| e^{-\lambda_k(M_i)t/2} \phi^{(i)}{}_k(h_i(a)) - e^{-\lambda_k t/2} \psi^{(i)}{}_k(a) \right|^2 \le \varepsilon_i$$

for all $t > 0$ *and* $a \in X$,

$$e^{-1/t}\sum_{k=1}^{\infty}\left|e^{-\lambda_k(M_i)t/2}\phi^{(i)}{}_k(x) - e^{-\lambda_k t/2}\psi^{(i)}{}_k(f_i(x))\right|^2 \le \varepsilon_i$$

for all $t > 0$ *and* $x \in M_i$. *Here* ε_i *does not depend on the choice of* Φ_i *and goes to zero as* i *tends to infinity.*

Remarks. (1) Conditions for the convergence of a sequence in $\mathcal{C}(\mathcal{S}(n, D), SD)$ are given in terms of the resolvents or the heat semigroups (as a Trotter-Kato type convergence theorem). (2) We refer the reader to [4] and [23] for different definitions of spectral distance, and also [16] and [23] for related topics on diffusion processes. (3) To simplify the exposition, we insist on the canonical measures of Riemannian manifolds. However in view of the results mentioned in these two sections, it would be natural to introduce a spectral distance in a similar manner on the set of Riemannian manifolds equipped with measures of some regularity conditions.

4. Bounded curvature and Laplace operators

The results mentioned in the last two sections are concerning manifolds with Ricci curvature bounded below uniformly. However they are intermediate in understanding the Laplace operators from the view points of the convergence theory of Riemannian manifolds. In the rest of the report, we would like to stay in the set of manifolds with curvature bounded uniformly and investigate more closely the Laplace operators. In this section, we recall first a fundamental fact on manifolds with bounded curvature, and secondly we review some of geometric structure theorems due to Cheeger, Fukaya and Gromov. Then we discuss the Laplace operators of manifolds collapsing to a lower dimensional space while their curvature keeping bounded.

Let $M = (M, g)$ be a complete Riemannian manifold of dimension n, the sectional curvature of which is bounded in its absolute values by a constant, say 1. Let x be a point of M and $\exp_x : T_xM \to M$ the exponential mapping of M at x. We shall identify the tangent space T_xM with Euclidean space $\mathbf{R}^n$ by a linear isometry. Then for a positive number r less than π, the exponential mapping induces a local diffeomorphism of the Euclidean ball $B^n(r)$ onto the geodesic ball $B(x, r)$ in M around x with radius r. Let g^* be the pull-back metric on $B^n(r)$. If we take r less than a positive constant depending only on n, then we have a coordinate system $H = (h_1, \dots, h_n)$ on $B^n(r)$ which possesses the following properties (cf. e.g. [17]):

(i) Each component h_ν is harmonic.

(ii) If we write the metric g^* in this coordinate system as

$$g^* = \sum_{j,k}^{n} g^*_{j,k}(x) dh_1 dh_n,$$

then the coefficients $g^*_{j,k}$ satisfy

$$\frac{1}{C_1}|\xi|^2 \le \sum_{j,k}^{n} g^*_{j,k}\xi^j\xi^k \le C_1|\xi|^2$$

$$|g^*_{j,k}(x) - \delta_{j,k}| \le C_2 r^2$$

for all $x \in H(B^n(r))$ and $\xi \in \mathbf{R}^n$, where C_1 and C_2 are some positive constants depending only on n.

(iii) Given $\alpha \in (0, \infty)$, there exists a constant C_3 depending only on n and α such that the $C^{1,\alpha}$ norms of $g^*_{j,k}$ satisfy

$$|g^*_{j,k}|_{C^{1,\alpha}(H(B^n(r)))} \le C_3$$

and given $p \in (1, \infty)$, there is a positive constant C_4 depending only on n and p which bounds the Sobolev norms of $g^*_{j,k}$:

$$\|g^*_{j,k}\|_{W^{2,p}(H(B^n(r)))} \le C_4.$$

(iv) If the m-th covariant derivatives of the Ricci tensor of M is bounded by a constant Λ_m, then for some constant $C_{3,m}$ depending only on n, m, α and Λ_m,

$$|g^*_{j,k}|_{C^{1+m,\alpha}(H(B^n(r)))} \le C_{3,m}.$$

We remark that if the injectivity radius of M is greater than or equal to a positive constant ι, and if we take r which is less than some positive constant depending only on n and ι, then the above coordinate system may be assumed to be defined on $B(x, r)$. Hence such a manifold M is covered by coordinate neighborhoods with the uniform estimates described as above, and if in addition, the diameter of M is bounded above by a given positive constant D, then the number of such coverings may be assumed to be not greater than some constant depending only on n, ι and D. This leads to the following

Theorem 7 ([14], [13], [25], [18]). *Let $\{M_i = (M_i, g_{M_i})\}$ be a sequence of compact Riemannian manifolds of dimension n which converges to a compact metric space M_∞ with respect to the Gromov-Hausdorff distance. Suppose that the sectional curvature of M_i is bounded in its absolute values by a constant* of measured Hausdorff convergence and the topology given by the spectral distance coincide on $\mathcal{K}(n,D)$. Moreover we want to show that $\mathcal{L}_\infty$ is the limit of the Laplace operator Δ_{M_i} of M_i in a sense. To make it precise, we shall replace the given fibration f_i with another appropriate one for analysis. To begin with, we cover M_∞ with a finite number of geodesic balls B_ν of sufficiently small radius. On each ball B_ν is defined a coordinate system H_ν whose components are $\mathcal{L}_\infty$ *– harmonic*, $\mathcal{L}_\infty H_\nu = 0$. Set $B_{i,\nu} = f_i{}^{-1}(B_\nu)$ and let $F_{i,\nu}$ be the solution of the Dirichlet problem:

$$\begin{aligned} \Delta_{M_i} F_{i,\nu} &= 0 \qquad \text{in } B_{i,\nu} \\ F_{i,\nu} &= H_\nu \circ f_i \qquad \text{on } \partial B_{i,\nu} \end{aligned}$$

Then we have (local) approximations $\Phi_{i,\nu}$ for f_i in $B_{i,\nu}$. Taking a partition of unity $\{\xi_{i,\nu}\}$ subordinate to the covering $\{B_\nu\}$ with uniformly bounded C^2 norms, we obtain uniquely a smooth mapping Φ_i of M_i onto M_∞, called a center-of-mass of $\{\Phi_{i,\nu}\}$ with weights $\{\xi_{i,\nu}\}$, which is given by

$$\sum_\nu \xi_{i,\nu}(x) \exp_{\Phi_i(x)}{}^{-1} \Phi_{i,\nu}(z) = 0$$

for $x \in M_i$. Now employing the standard elliptic regularity estimates together with the fact mentioned at the begining of this section, we can prove

Theorem 10 ([19-a]). *Let $\{M_i\}$ be a sequence in $\mathcal{K}(n,D)$. Suppose that M_i endowed with its normalized Riemannian measure $\mu_i (= \mu_{M_i})$ converges to a metric space M_∞ with a Borel measure μ_∞ with respect to the measure Hausdorff topology, and further that M_∞ is a smooth manifold with metric of class $C^{1,\alpha}$ (any $\alpha \in (0,1)$). Then there exists a fibration $\Phi_i : M_i \to M_\infty$ (for large i) having the following property: given $\beta \in [0,1)$, there exist a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and a positive constant C such that for all smooth function h on M_∞,*

(1) $\quad (1-\varepsilon_i)\Phi_i^*(|dh|) \le |d\Phi_i^*(h)| \le (1+\varepsilon_i)\Phi_i^*(|dh|)$ *on* M_i;

(2) $\quad |\Phi_i^*(h)|_{C^{k,\beta}(M_i)} \le |h|_{C^{k,\beta}(M_\infty)}$ *(k=0,1,2);*

(3) $\quad |\Delta_i \Phi_i^*(h) - \Phi_i^*(calL_\infty h)| \le \varepsilon_i \Phi_i^*(|Ddh| + |dh|)$ *on* M_i,

where Δ_i stands for the Laplace operator of M_i and $\mathcal{L}_\infty$ is the operator associated with a quadratic form

$$\int_{M_\infty} |dh|^2 d\mu_\infty.$$

Let M_i, M_∞ and Φ_i be as in this theorem. For a smooth function ψ on M_i, we define a smooth function $\Theta_i(\psi)$ on M_∞ by

$$\Theta_i(\psi)(z) = \frac{1}{Vol(\Phi_i{}^{-1}(z))} \int_{\Phi_i{}^{-1}(z)} \psi \quad (z \in M_\infty).$$

Then we have the following

Theorem 11 ([19-a]). *Let M_i, M_∞, $\mathcal{L}_\infty$ and $\Theta_i : C^\infty(M_i) \to C^\infty(M_\infty)$ be as above. Given $\beta \in (0,1)$ and $p \in (1,\infty)$, there exists a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty} \varepsilon_i = 0$ such that for large i and for all $\psi \in C^\infty(M_i)$,*

(1) $|\Theta_i(\psi)|_{C^1(M_\infty)} \leq (1+\varepsilon_i)|\psi|_{C^1(M_i)}$;

(2) $\|\Theta_i(\psi)\|_{W^{2,p}(M_\infty,\mu_\infty)} \leq (1+\varepsilon_i)\|\psi\|_{W^{2,p}(M_i,\mu_i)}$;

(3) $\|L_\infty\Theta_i(\psi) - \Theta_i(\Delta_i\psi)\|_{L^p(M_\infty,\mu_\infty)} \leq \varepsilon_i\|\psi\|_{W^{2,p}(M_i,\mu_i)}$;

(4) $\|\psi - \Phi_i^* \circ \Theta_i(\psi)\|_{W^{2,p}(M_i,\mu_i)} \leq \varepsilon_i \left(|\psi|_{C^0(M_i)} + |\Delta_{M_i}\psi|_{C^{0,\beta}(M_i)}\right)$.

We should here explain the notations used in this theorem. For a smooth function h on a Riemannian manifold M with a Borel measure μ, we set

$$\|h\|_{L^p(M,\mu)} = \left(\int_M |h|^p d\mu\right)^{1/p};$$

$$\|h\|_{W^{2,p}(M,\mu)} = \|h\|_{L^p(M,\mu)} + \|dh\|_{L^p(M,\mu)} + \|Ddh\|_{L^p(M,\mu)}.$$

It is not hard to deduce Theorem 2 from Theorems 10 and 11. For details, see [19-b]. Moreover in the following sections, we shall give two applications of Theorems 10 and 11.

5. Energy spectra of harmonic mappings

In Section 3, we have discussed the continuity of spectra of Riemannian manifolds in $\mathcal{S}(n,D)$ with respect to the spectral distance. We may ask a similar question on the energy spectrum of harmonic mappings into nonpositively curved manifolds. Let $M = (M,g)$ and $N = (N,h)$ be two compact Riemannian manifolds. Given a smooth mapping ϕ of M into N, the energy density $e(\phi)$ is a function defined by the trace of the induced tensor ϕ^*h with respect to the metric g and the energy of the mapping ϕ is given by

$$E(\phi) = \int_M e(\phi) dvol_g.$$

A smooth mapping $\phi : M \to N$ is said to be harmonic if the energy functional E is stationary at ϕ, or equivalently if the tension field $\tau(\phi)$ vanishes.

From now on we assume the target manifold N has nonpositive sectional curvature. Then the fundamental theorem due to Eells and Sampson [9] asserts that any smooth mapping $\phi : M \to N$ is homotopic to a harmonic mapping which has minimum energy in its homotopy class. In addition, Hartman [15] showed a uniqueness theorem saying that if ϕ_0 and ϕ_1 are homotopic harmonic mappings, then they are smoothly homotopic through harmonic mappings; and the energy is constant on any arcwise connected set of harmonic mappings. We shall denote by $\mathcal{H}(M,N)$ the set of harmonic mappings of M into N, and consider the energy spectrum $\{E(\phi) : \phi \in \mathcal{H}(M,N)\}$. In view of the above results, we may set $E(\mathcal{C}) = E(\phi)$ for a component $\mathcal{C}$ of $\mathcal{H}(M,N)$, where ϕ belongs to $\mathcal{C}$. A theorem of Adachi and Sunada [1] states that there are explicit positive constants C_1 and C_2 depending only on the diameters $diam(M)$, $diam(N)$, the volumes $Vol(M)$, $Vol(N)$ and the lower bounds on the Ricci curvatures of M, N such that

$$\sharp\{\mathcal{C} \subset \mathcal{H}(M,N) : E(\mathcal{C}) \le \lambda^2\} \le C_1 \exp C_2\lambda$$

for any λ. By virtue of this result, we may put the connected components $\{\mathcal{C}_k\}$ $(k = 0,1,2,\ldots,\nu-1)$ of $\mathcal{H}(M,N)$ in order as follows:

$$E(\mathcal{C}_k) \le E(\mathcal{C}_{k'}) \qquad \text{if} \quad k \le k'$$

(hence $\mathcal{C}_0$ consists of constant mappings). Here ν, $1 \le \nu \le +\infty$, stands for the number of the connected components of $\mathcal{H}(M,N)$. Set

$$\sigma_k(M,N) = E_{\mu_M}(\mathcal{C}_k) = \int_M e(\phi)d\mu_M \qquad (\phi \in \mathcal{C}_k).$$

Here for convenience, we understand $\sigma_k(M,N) = +\infty$ if $k \ge \nu$.

Now for the target manifold N of nonpositive curvature being fixed, we may ask if $\sigma_k(M,N)$ is continuous as a function of the domain manifold M in $\mathcal{S}(n,D)$ with respect to the spectral distance SD. At the present stage, using Theorems 10 and 11, we are able to show a partial answer for this.

Theorem 12([19-a]). *Let $\{M_i\}$ be a sequence in $\mathcal{K}(n,D)$ which converges to a triad (X,μ,p) with respect to the spectral distance. Then given a compact Riemannian manifold N of nonpositive sectional curvature, the limit of $\sigma_k(M_i,N)$ exists for $k < \nu$ and diverges for $k \ge \nu$, where ν denotes the number of the arcwise connected components of continuous mappings of X into N.*

We can also discuss the convergence of harmonic mappings themselves in a certain sense. For details, see [19-a]. We remark also that in case $N = \mathbf{R}/\mathbf{Z}$, the harmonic mappings of M into $\mathbf{R}/\mathbf{Z}$ correspond to the harmonic one forms on M with integral periods. In the next section, we would like to ask a question concerning harmonic one forms and the spectral distance.

6. Convergence of Albanese tori

Given a compact Riemannian manifold M, we denote by $H^1(M, \mathbf{R})$ the space of harmonic one forms on M equipped with an inner product $< \, , \, >$ defined by

$$< \omega, \eta > = \int_M (\omega, \eta) d\mu_M.$$

Let $H^1(M, \mathbf{Z})$ be a lattice of $H^1(M, \mathbf{R})$ which consists of harmonic one forms with integral periods. Dividing the dual space $H^1(M, \mathbf{R})^*$ by the dual lattice $H^1(M, \mathbf{Z})^*$, we obtain the Albanese torus

$$\mathcal{A}(M) = H^1(M, \mathbf{R})^* / H^1(M, \mathbf{Z})^*.$$

Let $\widetilde{M}$ be the universal covering space of M and π the projection of $\widetilde{M}$ onto M. We fix a point p of M, and take a point $\tilde{p}$ of $\widetilde{M}$ with $\pi(p) = \tilde{p}$. Then we have a mapping $\widetilde{J}_M$ of $\widetilde{M}$ into the dual space $H^1(M, \mathbf{R})^*$ defined by

$$\widetilde{J}_M(\tilde{x})(\omega) = \int_{\tilde{p}}^{\tilde{x}} \pi^* \omega.$$

This map is obviously harmonic and it induces a harmonic mapping J_M of M into the Albanese torus $\mathcal{A}(M)$, called the Albanese map, in such a way that $J_M(p) = 0$ and $\pi' \circ \widetilde{J}_M = J_M \circ \pi$, where $\pi' : H^1(M, \mathbf{R})^* \to \mathcal{A}(M)$ is the natural projection.

We shall consider $\mathcal{A}(M)$ as a mapping of the set of compact Riemannian manifolds onto the set of flat tori. Then we would like to ask if this map is continuous in $\mathcal{S}(n, D)$ with respect to the spectral distance. We note that the Gromov-Hausdorff distance, the measured Hausdorff topology and the spectral distance induce the same topology on the set of flat tori. We remark also that the first Betti number $b_1(M)$, namely $\dim \mathcal{A}(M)$, for a manifold M in $\mathcal{S}(n, D)$ is bounded above uniformly (cf. [12]). Moreover we claim that *the diameter of* $\mathcal{A}(M)$ *is also bounded above uniformly in* $\mathcal{S}(n, D)$. In fact, for any nonconstant harmonic mapping ϕ of M into $\mathbf{R}/\mathbf{Z}$,

$$e(\phi) \leq C E_{\mu_M}(\phi)$$

for some positive constant C depending only on n and D (cf. e.g., [1]). Since the energy density $e(\phi)$ must be greater than or equal to $1/4diam(M)^2$ somewhere in M, it follows that

$$E_{\mu_M}(\phi) \geq \frac{1}{2CD}.$$

Inother words, we have

$$\lambda_1(\mathcal{A}(M)) \geq \frac{2\pi^2}{CD}.$$

On the other hand, we know that

$$\lambda_1(\mathcal{A}(M)) \leq \frac{C'}{diam(\mathcal{A}(M))^2}$$

for some constant C' depending only on $\dim \mathcal{A}(M)$. Thus the claim is clear.

At the present stage, we are able to answer the above question partially.

Theorem 13 ([19-c]). *Let $\{M_i\}$ be a sequence in $\mathcal{K}(n, D)$ which converges with respect to the spectral distance. Then the Albanese torus $\mathcal{A}(M_i)$ converges to a point or a flat torus T^m of positive dimension m, where $0 < m \leq \liminf_{i\to\infty} b_1(M_i)$.*

We refer the reader to [19-c] for the more precise statement of the theorem, and also [27,28] for some related results.

In this report, we restrict ourselves to compact manifolds. However it is possible to discuss noncompact manifolds from the same point of view. For example, in [20], we investigated harmonic functions of polynomial growth on a noncompact complete Riemannian manifold and as an application of Theorems 10 and 11, we gave some geometric conditions for the space of such functions to be of finite dimension if the order is fixed.

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HARMONIC MAP HEAT EQUATION OF COMPLETE MANIFOLDS INTO COMPACT MANIFOLDS OF NONPOSITIVE SECTIONAL CURVATURE

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One of the interesting questions in the geometry of noncompact complete manifold is that of infinity behavior of geometric objects. Since the concept of infinity is rather hazy, one may envision it as approaching ideal points at infinity. As a noncompact complete manifold has no boundary at infinity, one somehow artificially attach an ideal boundary at infinity to form a compactification of the original complete manifold. One typical example is the so-called Eberlein-O'Neill boundary of complete simply connected manifold of nonpositive sectional curvature [E-O]. There are many other construction of ideal boundaries, and the geometric study of such ideal boundary is an interesting subject in itself.

However, certain analytic property of geometric objects depends only on some very general structure of the ideal boundary, regardless of its specific geometric construction. One such example is the following very crude property of the ideal boundary points.

Definition. *Let M be a complete manifold without boundary, and let $\overline{M}$ be a compactification of M with the ideal boundary $M(\infty)$. We say $\overline{M}$ satisfies the ball convergence criterion, if for any sequence $\{x_n\}$ in M with $x_n \to \overline{x} \in M(\infty)$ and for all $r > 0$, the geodesic ball $B(x_n, r)$ centered at x_n with radius r converges to $\overline{x}$.*

It was Donnelly and Li who first gave the above definition [D-L]. They proved that the over-determined problem of initial boundary value problem of the heat equation can be solved as long as each ideal boundary point

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able to overcome this difficulty by carefully choosing the function ψ which replaces the role of the naively defined distance function. By doing so, the terms involving $\tilde{u}$ miraculously disappears in our calculation and we are left with terms expressible solely in terms of h. In carrying out our program, we must be able to control the growth of the Christoffel symbol of $\widetilde{N}$. It turns out that the curvature bound and the bound on the covariant derivative of the curvature tensor are enough to prove that the Christoffel symbol grows exponentially, which is enough for us.

The following is one of our main results.

Theorem 1.1. *Suppose that M is a complete Riemannian manifold with Ricci curvature bounded below by a constant, and N is a compact Riemannian manifold with nonpositive sectional curvature. Let $f \in C(\overline{M}, N) \cap C^1(M, N)$ such that $e(f) < \infty$. Let u be the unique solution of (1.2) with the initial map f. If $\overline{M}$ satisfies the ball convergence criterion, then $u(\bar{x}, t) = f(\bar{x})$ for all $(\bar{x}, t) \in M(\infty) \times [0, \infty)$, namely, u is a solution of* (1.3).

Remark. In the course of the proof, we have to estimate the growth property of the Christoffel symbol of the universal cover of N. To do that, we need the bounds of the covariant derivative of the curvature tensor as well as the sectional curvature of N. Thus the assumption on N can be replaced with N being a complete manifold with $-c \le K_N \le 0$ for some nonnegative constant c, where K_N is the sectional curvature of N, and the covariant derivative of the curvature tensor bounded by a constant.

When the target has nonpositive sectional curvature, the existence of the solution of (1.2) was proved by Li and Tam. However, if the target manifold has positive curvature, it is no longer true that the (1.2) has a solution for all time. The solution may blow up in finite time, and it may develop singularities. However, when it is confined to a convex geodesic ball $B_\tau(p)$ of radius τ and center at p in N, the following long time existence holds.

Theorem 1.2. *Suppose that M is a complete Riemannian manifold with Ricci curvature bounded below by a constant and N is a complete Riemannian manifold with sectional curvature bounded above by a constant $\mu > 0$. Assume that $\tau < \min\{\frac{\pi}{2\sqrt{\mu}}$, injectivity radius of N at $p\}$. Let $f \in C^1(M, N)$ be such that $f(M) \subset B_\tau(p)$ and $\sup_M e(f) < \infty$. Then (1.2) has a unique solution u defined on $M \times [0, \infty)$ with the initial map f such that u has the following properties:*

(a) $u(M \times [0,T]) \in B_\tau(p)$ *for all* $T > 0$

(b) $\sup_{M\times[0,T]} e(u) < \infty$ *for all* $T > 0$

With this long time existence theorem, the maximum principle technique which is similar to, but harder than, those used in the proof of Theorem 1.1 gives the following result:

Theorem 1.3. *Suppose* N *is a complete Riemannian manifold whose sectional curvature is bounded above by a constant* $\mu > 0$. *Let* $p \in M$. *Assume that* $\tau < \min\{\frac{\pi}{2\sqrt{\mu}}$, *injectivity radius of* N *at* $p\}$. *Let* $f \in C^1(M, B_\tau(p)) \cap C(\overline{M}, B_\tau(p))$. *Let* u *be the unique solution of (1.2) given by Theorem 1.2 with the initial map* f. *If* $\overline{M}$ *satisfies the ball convergence criterion, then* $u(\bar{x},t) = f(\bar{x})$ *for all* $(\bar{x},t) \in M(\infty) \times [0,\infty)$, *namely,* u *is a solution of* (1.3).

Our results, Theorem 1.1 and 1.3, are related to the earlier result of the second author with Aviles and Micallef. But, one crucial drawback of the argument in [A-C-M] is the assumption that the map must lie in a convex ball of the target manifold, which exclude most interesting geometric situation. Thus, one of our main contributions, among others, is that we were able to remove in Theorem 1.1 the smallness assumption of the image in the context of harmonic map heat equation.

The following maximum principle due to Liao and Tam [L-T] will be extensively used in this paper:

Theorem 1.4. *Let* M *be a complete noncompact Riemannian manifold such that there exists a point* $p \in M$ *and a constant* $k > 0$ *satisfying that* $Vol(B_r(p)) \leq \exp(k(1+r^2))$ *for all* $r > 0$. *Let* f *be a function on* $M \times [0,T), T > 0$. f *is smooth on* $M \times (0,T)$ *and continuous on* $M \times [0,T)$. *Suppose* f *satisfies the following conditions;*

(a) $(\Delta - \frac{\partial}{\partial t})f \geq 0$ *on* $M \times (0,T)$;

(b) $f(x,0) \leq 0$ *for all* $x \in M$; *and*

(c) $\int_0^T(\int_M \exp(-\alpha r^2(p,y))|\nabla f|^2(y)dV_M(y))dt < \infty$, *for some* $\alpha > 0$.

Then $f \leq 0$ *on* $M \times [0,T)$.

Rough sketch of proof of Theorem 1.1:

We now give a very rough sketch of the proof of Theorem 1.1: First, the existence of the solution u of heat equation for harmonic map on $M \times [0,\infty)$ which satisfies that $u(x,0) = f(x)$ is proved by Li and Tam [Li-T]. So we have only to control the boundary behavior. Let $\widetilde{M}$ and $\widetilde{N}$ be the universal covers of M and N respectively. Consider the lifting $\tilde{f}$ and $\tilde{u}$ of f and u respectively. An easy application of the maximum principle shows that $\tilde{u}$ is the solution of (1.1) with the initial data $\tilde{f}$. Choose global normal coordinates on $\tilde{N}$. Then with respect to these coordinates the map $\tilde{u} : \widetilde{M} \to \tilde{N}$ is represented as $\tilde{u} = (u^1, \ldots, u^n)$, and the initial map $\tilde{f} : \widetilde{M} \to \tilde{N}$ is also represented as $\tilde{f} = (f^1, \ldots, f^n)$. Let $h^i : \widetilde{M} \times [0, \infty) \to \mathbb{R}$ be the solution of the linear heat equation with the initial value f^i, for $i = 1, \ldots, n$. Let $\tilde{x}$ be the lifting in $\widetilde{M(\infty)}$ of a point $\bar{x} \in M(\infty)$, and let $x_n \in M$ be a sequence of points converging to $\tilde{x}$. Let U be a neighborhood of $\tilde{x}$. Then for sufficiently large n, $x_n \in U$. Then probabilistic arguments can be cooked up to show that most Brownian particles originating from x_n does not get out of U. In fact, one can estimate the escaping probability rather accurately to prove that $\lim_{n\to\infty} h(x_n, t) = \tilde{f}(\tilde{x})$.

Define now $\psi : \tilde{N} \times \tilde{N} \longrightarrow \mathbb{R}$ by $\psi(y_1, y_2) = \phi(\rho(y_1, y_2))$ for $(y_1, y_2) \in \tilde{N} \times \tilde{N}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that for some constant $C' > 0$ $\phi(0) = \phi'(0) = \phi''(0) = 0$, $0 \le \phi'(t) \le C'$, and $\phi''(t) \ge 0$ for all $t \in [0, \infty)$. Note that the condition $\phi'(0) = 0$ guarantees that ψ is smooth along the diagonal. It is not hard to prove that

$$
\begin{aligned}
&(\Delta - \frac{\partial}{\partial t})\psi(\tilde{u}(x,t), h(x,t)) \\
\ge& \frac{\partial \psi}{\partial y^j}\Gamma^j_{kl} g^{\alpha\beta} \frac{\partial h^k}{\partial x^\alpha}\frac{\partial h^l}{\partial x^\beta} \\
\ge& - C_1 e^{C_2|h(x,t)|}|\nabla h|^2,
\end{aligned}
$$

Here, we use the following Lemma 1.5 to control the growth of the Christoffel symbol Γ^i_{jk} of $\tilde{N}$. We then apply the maximum principle technique to prove that

$$
\psi(\tilde{u}(x,t), h(x,t)) \le C_4 \left\{ v(x,t) - e^{C_3|h|^2(x,t)} \right\},
$$

where v be the solution of the linear heat equation with the initial data $v(x, 0) = e^{C_3|\tilde{f}|^2}$. Now as we mentioned above, out probabilistic argument shows that $v(x, t)$ converges to $e^{C_3|\tilde{f}(\tilde{x})|^2}$ as x converges to $\tilde{x}$. Therefore, $\psi(\tilde{u}(x, t), h(x, t))$ converges to 0 as x converges to $\tilde{x}$. Since we have proved above $h(x, t)$ converges to $\tilde{f}(\tilde{x})$ as x converges to $\tilde{x}$, the theorem follows.

As mentioned above, the following Lemma shows how to control the growth properties of the Christoffel symbol. Its proof is based on the Jacobi equation argument.

Lemma 1.5. *Suppose (N, h) is a simply connected complete Riemannian manifold such that its sectional curvature K_N satisfies $-\kappa \leq Sect_N \leq 0$ and the covariant derivative of the curvature tensor is bounded. Let $(x^1, \ldots, x^n)$ be a normal coordinate centered at a fixed point $p \in N$. Let $ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta$ be the Riemannian metric. Let $\Gamma^\alpha_{\beta\gamma}$ be the Christoffel symbol. Then $\Gamma^\alpha_{\beta\gamma} = O(e^{C\gamma(x)})$ on N for some constant depending only on the dimension, the bounds of the sectional curvature and the covariant derivative of the curvature tensor of N, where γ is the distance function from p in N.*

Remark. This note is based on the lecture at the GARC conference given by the second author, and the details of this work will be published elsewhere.

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ON THE ROLE OF SINGULAR SPACES IN RIEMANNIAN GEOMETRY

KARSTEN GROVE

There are many examples of problems in mathematics, where solutions are found only after enlarging the collection of objects the problem is concerned with.

Our aim here is to discuss several types of such phenomena in riemannian geometry, where of course the basic objects are riemannian manifolds.

In general, the right class of spaces to consider will depend on the particular problem, but is often found by "completing" the class of interest relative to an "appropriate topology", or relative to "desirable operations".

When trying to construct a suitable topology on a class of riemannian manifolds, one must keep in mind, that its utility is balanced between "having easy convergence", and "getting sufficient structure" on the limit objects. Another issue, of course, is generality. The so-called Gromov-Hausdorff topology, introduced in [G1], (cf. also [Pe]), is defined on the class of all compact (separable) metric spaces, and is coarse enough to satisfy the first criteria (cf. Section 1). However, in general it does not satisfy the second criteria: For example, any compact inner metric space (i.e., a space in which distances are measured by the infimum of lengths of curves) is the Gromov-Hausdorff limit of closed surfaces [C]. So far, the largest classes of closed riemannian manifolds for which Gromov-Hausdorff limits inherit sufficient structure, are the classes of manifolds whose sectional curvatures are bounded below by an arbitrary, but fixed real number (cf. Sections 1,2). Specifically, if X is the Gromov-Hausdorff limit of a sequence of (closed) riemannian n-manifolds $\{M_i\}$ with $\sec M_i \geq k$, then X is an inner metric space with curvature, $\operatorname{curv} X \geq k$ in distance comparison sense, and the Hausdorff dimension of X is at most n [GP1].

We refer to finite Hausdorff dimensional inner metric spaces with a lower curvature bound simply as *Alexandrov spaces.* As we shall see, their importance to riemannian geometry is not only due to the fact mentioned above,

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(cf. Section 3), but also because there are other natural geometric operations, which are closed in Alexandrov geometry but not in riemannian geometry. These operations include *taking quotients* by isometric group actions and *forming joins* of positively curved spaces (cf. Section 4).

By itself, the collection of Alexandrov spaces form a large and interesting class of spaces. They do not only allow metric singularities, but topological singularities as well. In fact, locally such a space X, is homeomorphic to a cone on a positively curved Alexandrov space, and globally X is stratisfied into topological manifolds [P1,2] (cf. Section 2).

The problems discussed here all deal with relations between geometry and topology of riemannian manifolds/Alexandrov spaces. Specifically, they are concerned with questions of the type: How do restrictions on metric invariants of a riemannian M manifold restrict the underlying (differential) topology of M? Answers are sought in the form of finiteness theorems, structure theorems or recognition theorems. Rather than trying to be complete, we have just chosen a few illustrative examples.

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1. THE GROMOV-HAUSDORFF TOPOLOGY

Without reference to any particular problem, it is natural to ask whether there are any reasonable topologies on say the collection of all isometry classes of closed riemannian manifolds.

To get an idea of what seems reasonable, let us consider the following examples.

Example 1.1. In each of the examples below we will construct sequences of metrically singular spaces from which smooth examples are easily obtained.

(a) Set $M_1 = S_1^2$ the unit two-sphere. For each i, let M_i be obtained from S_1^2 by replacing two antipodal spherical caps of radius, $(1 - 1/i)\pi/2$ by the corresponding euclidean discs of radius $\sin(1 - 1/i)\pi/2$. The double of the flat unit disk in $\mathbb{R}^2$ is a limit candidate.

(b) For each i, let M_i be the orbit space $S_1^2/\mathbb{Z}_i$; where the generator of $\mathbb{Z}_i$ acts as a rotation by angle $2\pi/i$. The interval $[0, \pi]$ is a limit candidate.

(c) Let X be the compact set in $\mathbb{R}^2$ obtained from the unit disc centered at $(0,0)$ by deleting the open $1/2^{n+2}$ - discs with centers at $(1-1/2^n, 0), n = 0, 1, \cdots$. For each i, let M_i be the set of points in $\mathbb{R}^3$ at distance $1/i$ from $X \subset \mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}$. The double of X is a limit candidate.

(d) Set $M_i = S_1^3$, the unit 3-sphere and consider the Hopf action $S^1 \times S_1^3 \to S_1^3$. For each i, let M_i be the riemannian manifold obtained from S_1^3 by scaling all Hopf orbits by the factor i, but keeping the riemannian metric unchanged perpendicular to the orbits. The 3-sphere with Carnot-metric induced by the distribution perpendicular to the Hopf fibers is a limit candidate.

(e) For each i, let $X_i \subset [0,1]^3$ be the grid consisting of points, two of whose coordinates are of the form p/i, $p = 0, \cdots, i$. If M_i is the set of points in $\mathbb{R}^3$ at distance say $(10i)^{-1}$ to X_i, then the limit candidate of these surfaces is $[0,1]^3$ with distances induced by the ℓ^1-norm.

To accommodate the desire that each of these sequences (a)-(e) above be convergent with the given candidates as limit spaces, the topology will necessarily have to be fairly coarse. Such a topology was proposed by Gromov in connection with his celebrated work on groups with polynomial growth [G1]. In fact, he extended Hausdorff's classical notion of distance between compact subsets of a compact metric space, to the situation where no ambient space is given: Indeed, if X and Y are compact metric spaces, the *Gromov-Hausdorff distance*, $d_{GH}(X,Y)$ between X and Y is the infimum of all Hausdorff distances between X and Y, when isometrically embedded in compact metric spaces Z. For Z, it actually suffices to take $Z = X \amalg Y$. Thus

$$d_{GH}(X,Y) = \inf d_H(X,Y), \tag{1.2}$$

where the infimum is taken over all metrics on $X \amalg Y$ extending those of X and Y.

Using (1.2) it is easy to see that d_{GH} is indeed a metric on the collection of all isometry classes of separable compact metric spaces. It is also not difficult to verify that the limit candidates of Example 1.1 are indeed limits relative to the Gromov-Hausdorff topology.

Note that if $N \subset X$ is an ϵ-net in X, i.e.,

$$D(N,\epsilon) = \{x \in N \mid \operatorname{dist}(x,N) \le \epsilon\} = X,$$

then $d_{GH}(X, N) \le \epsilon$. In particular, any compact X can be approximated arbitrarily well by finite subsets of X. If we let

$$\mathrm{Cov}(X, \epsilon) = \min\{n \mid X \text{ is covered by } n \text{ closed } \epsilon - \text{balls}\}$$

$$\mathrm{Cap}(X, \epsilon) = \max\{n \mid X \text{ is contains } n \text{ disjoint open } \epsilon/2 - \text{balls}\}$$

the following precompactness criteria due to Gromov are easy to derive (cf. e.g., [Pe]).

Theorem 1.3. *Let $\mathcal{C}$ be a class of compact metric spaces. Then $\mathcal{C}$ is precompact relative to d_{GH} if and only if the following equivalent conditions hold*

(i) *There is a function $N : (0, \alpha) \to (0, \infty)$ such that $\mathrm{Cap}(X, \epsilon) \le N(\epsilon)$ for all $\epsilon \in (0, \alpha)$ and all $X \in \mathcal{C}$.*

(ii) *There is a function $N: (0, \beta) \to (0, \infty)$ such that $\mathrm{Cov}(X, \epsilon) \le N(\epsilon)$ for all $\epsilon \in (0, \beta)$ and all $X \in \mathcal{C}$.*

This criterion applies to large interesting classes of riemannian manifolds. For example:

Corollary 1.4. *For any $k \in \mathbb{R}$, $D > 0$ and integer $n \ge 2$, the class $\mathcal{R}_k^D(n) = \{M^n \mid \mathrm{Ric}\, M \ge (n-1)k,\ \mathrm{diam}\, M \le D\}$ is precompact.*

Corollary 1.5. *For any $i > 0$, $V > 0$ and integer $n \ge 2$ the class $\mathcal{M}_i^V(n) = \{M^n \mid \mathrm{inj}\, M \ge i,\ \mathrm{vol}\, M \le V\}$ is precompact.*

The first of these corollaries follows from the relative volume comparison theorem (cf. [BC], [GLP] or [Gr1]), and the second from a volume estimate due to Croke [Cr] (cf. [GPW]).

As mentioned earlier, the utility of this topology depends to a large extent on how much structure is carried over to the limit objects. Looking back at the examples in (1.1) one might get the impression that little or nothing is carried over. In fact, the dimension can go down as in (b), go up as in (d) and (e). Even when the dimension is preserved, the limit space can have bad local properties such as not being locally contractible, (c).

Nevertheless some metric properties are indeed preserved. First of all, any Gromov-Hausdorff limit of inner metric spaces is an inner metric space. Moreover, if for example the α-dimensional Hausdorff measure is bounded, say $\mathcal{H}^\alpha(M_i) \le c$, then $\mathcal{H}^\alpha(X) \le c$ when $X = \lim M_i$. Finally, if the sectional curvatures are bounded from below, say $\sec M_i \ge k$, then the curvature of X satisfies $\mathrm{curv}\, X \ge k$ in (global) distance comparison sense ([GP1]).

Our main concern here is the following class of closed riemannian manifold

$$\mathcal{C}_k^D(n) = \{M^n \mid \sec M \geq k,\ \operatorname{diam} M \leq D\}.$$

From 1.4 we know that the Gromov-Hausdorff closure $\overline{\mathcal{C}_k^D(n)}$ is compact, and from the discussion above we know that any $X \in \overline{\mathcal{C}_k^D(n)}$ is a locally compact inner metric space with Hausdorff dimension, $\dim X \leq n$ and $\operatorname{curv} X \geq k$. As we shall see in the next section, it turns out that these few general properties alone are restrictive enough to yield a rich geometric structure.

2. Alexandrov Spaces

In this section we give a brief overview of the geometric and topological structure of Alexandrov spaces, i.e., finite Hausdorff dimensional complete inner metric spaces with a lower curvature bound in distance comparison sense: Following Berestowski [B] (cf. also [W]), we say that a metric space X has curvature, $\operatorname{curv} X \geq k$ if every point $p \in X$ has a neighborhood U such that for any four (distinct) points a, b, c, d in U, the *comparison angles* of a satisfies:

$$\tilde{\sphericalangle}\,\mathrm{bac} + \tilde{\sphericalangle}\,\mathrm{bad} + \tilde{\sphericalangle}\,\mathrm{cad} \leq 2\pi. \tag{2.1}$$

Here the comparison angle, $\tilde{\sphericalangle}\,\mathrm{bac}$ is the angle at $\tilde{a}$ in the triangle $(\tilde{a}, \tilde{b}, \tilde{c})$ in the k-plane S_k^2 with sidelengths equal to the corresponding sidelengths in (a, b, c). Independent and different approaches to the basic theory can be found in [BGP] and [Pl]. The local and global structure results are due to Perelman (cf. [P1,2]).

It is a nontrivial fact, that any Alexandrov space as defined above is locally compact ([BGP], [Pl]). By Ascoli's theorem this implies in particular, that any two points in such a space can be joined by a *segment*, i.e., a shortest curve or *minimal geodesic*. Given the existence of segments, there are several useful equivalent ways in which distance comparison can be formulated. One of them, "the hinge version" (2.2), gives rise to the notion of angle (2.3):

Two segments $c_i\colon [0, \ell_i] \to X, i = 1, 2$ with common initial point $p = c_1(0) = c_2(0)$ in an Alexandrov space X is called a *hinge* at p. The assumption $\operatorname{curv} X \geq k$ can now be formulated as follows

$$(t_1, t_2) \to \tilde{\sphericalangle}\,(c_1(t_1), p, c_2(t_2)), \quad t_i \in [0, \ell_i] \tag{2.2}$$

is nonincreasing for any hinge (c_1, c_2) in X. The angle between c_1 and c_2 at p can then be defined as

$$\sphericalangle(c_1, c_2) = \lim_{t_1, t_2 \to 0} \tilde{\sphericalangle}(c_1(t_1), p, c_2(t_2)). \tag{2.3}$$

We are now ready to describe the *infinitesimal structure* of X. Fix $p \in X$ and let G_p denote the set of geodesic germs at p. This is well defined because geodesics cannot bifurcate in an Alexandrov space. It is easy to see that (2.3) defines a metric on G_p, which in general is incomplete. The completion $\overline{G_p} = S_p$ is called the *space of directions* of X at p (the unit sphere if X is a riemannian manifold). It turns out, that S_p is also an Alexandrov space and $\operatorname{curv} S_p \geq 1$. Moreover, $\dim S_p = \dim X - 1$, where dim is the topological as well as the Hausdorff dimension (cf. [BGP] or [Pl]). The euclidean cone $C_0 S_p$ on S_p is called the *tangent space* to X at p, and $T_p X = C_0 S_p$ can also be viewed as the pointed Gromov-Hausdorff limit of the scaled spaces $(\lambda X, p)$, $\lambda \to \infty$ [BGP]. Clearly $T_p X$ is an Alexandrov space with $\operatorname{curv} T_p X \geq 0$ and $\dim T_p X = \dim X$.

To come to grips with the local structure, one has to extend critical point theory for distance functions as known in riemannian geometry (cf. [Gr2], [Ch]) to Alexandrov spaces. This is done in [P1,2]. In fact, a general local fibration theorem for m-tuples of distance type functions $1 \leq m \leq \dim X$, near a "regular point" is established by inverse induction in m, i.e., starting with $m = n$ and ending with $n = 1$.

The main results derived from this are:

Local Theorem 2.4. *Let X be an Alexandrov space. Then any $p \in X$ has an open neighborhood which is homeomorphic to the tangent space $T_p X = C_0 S_p X$ to X at p.*

Stability Theorem 2.5. *Let X be an n-dimensional Alexandrov space with $\operatorname{curv} X \geq k$. There is an $\epsilon = \epsilon(X) > 0$ such that any other compact n-dimensional Alexandrov space Y with $\operatorname{curv} Y \geq k$ and $d_{GH}(X, Y) < \epsilon$ is homeomorphic to X.*

It should be noted that globally any Alexandrov space admits a natural *topological stratification* into topological manifolds: the m-dimensional strata consist of points whose canonical neighborhoods splits off a euclidean factor (topologically) of dimension at most m. This was used in [GP2] to generalize the classical sphere theorem in riemannian geometry to Alexandrov spaces.

To make further progress in understanding the structure of Alexandrov spaces it seems necessary to find a stratification, which is more geometric in nature. This is currently being pursued by Perelman and Petrinin (cf. [PP]). One also hopes to be able to extend Theorem 2.5 to some kind of "stratified fibration" theorem when $\dim Y > \dim X$ (cf. Theorem 3.1, [Y1,2], [Sh] and [Wi]).

It should also be mentioned that the isometry group of an Alexandrov space is a Lie group [FY].

As we will see in the next sections, the knowledge we have already about Alexandrov spaces, has strong implication in riemannian geometry. This is not only due to the fact that the class $\mathcal{A}_k(n)$ of Alexandrov spaces X with $\dim X \leq n$ and $\operatorname{curv} X \geq k$ is closed in the Gromov-Hausdorff topology and $\mathcal{M}_k(n) \subset \mathcal{A}_k(n)$ (cf. Section 3), but also because there are other natural geometric operations which are closed in Alexandrov geometry but not in riemannian geometry. One of these is taking quotients by a group of isometries (immediate from (2.1)). Another is the operation of forming (spherical) joins of positively curved spaces. The quickest way to see this is as follows: Let X and Y be Alexandrov spaces with $\operatorname{curv} X \geq 1$, $\operatorname{curv} Y \geq 1$. Then C_0X and C_0Y are Alexandrov spaces with $\operatorname{curv} \geq 0$, and so is $C_0X \times C_0Y = Z$. Now $X * Y$ can be identified with the space of directions $S_{(p,q)}$ in Z where p and q are the cone points in C_0X and C_0Y respectively. Applications of these constructions to riemannian geometry will be given in Section 4.

3. Convergence Techniques

So far, the applications of taking Gromov-Hausdorff limits from the class of riemannian manifolds with a lower curvature and an upper diameter bound, hinge on two results: The Stability Theorem 2.5 (or its predecessor in [GPW]) and the following Fibration Theorem due to Yamaguchi [Y1].

Fibration Theorem 3.1. *Let N be a closed riemannian manifold with $\sec N \geq k$. There is an $\epsilon = \epsilon(N) > 0$ such that for any other closed riemannian manifold M with $\sec M \geq k$ and $d_{GH}(M,N) < \epsilon$ there is $\tau(\epsilon)$-almost riemannian submersion $f : M \to N$, where $\tau(\epsilon) \to 0$ as $\epsilon \to 0$. In particular, if $\dim M = \dim N$, $f : M \to N$ is a diffeomorphism.*

Recall that for the class $\mathcal{C}_k^D(n)$ of closed riemannian n-manifolds M with $\sec M \geq k$ and $\operatorname{diam} M \leq D$ there is a $C = C(n,k,D)$ such that

$\dim H_*(M,F) \le C$ for any $M \in \mathcal{C}_k^D(n)$ and field F [G2] (cf. also [Ls]). To get more information, consider for example the volume function

$$\mathrm{vol} : \mathcal{C}_k^D(n) \to \mathbb{R}.$$

For the superlevel sets $\mathcal{C}_{k,v}^D(n) = \{M \in \mathcal{C}_k^D(n) \mid \mathrm{vol}\, M \ge v > 0\}$ we have (cf. [GPW], [P1] and also [F]).

Finiteness Theorem 3.2. *For every $k \in \mathbb{R}, D, v > 0$ and integer $n \ge 2$, the class $\mathcal{C}_{k,v}^D(n)$ contains at most finitely many homeomorphism types.*

Indeed by 1.4, $\overline{\mathcal{C}_{k,v}^D(n)}$ is compact. Moreover, since each $X \in \overline{\mathcal{C}_{k,v}^D(n)}$ is an n-dimensional Alexandrov space, the claim follows from 2.5.

Note that it is also a consequence of 2.5 that each $X \in \overline{\mathcal{C}_{k,v}^D(n)}$ is a topological manifold.

To investigate the extremal case, where $M \in \mathcal{C}_k^D(n)$ and $\mathrm{vol}\, M \ge \sup\{\mathrm{vol}\, N \mid N \in \mathcal{C}_k^D(n)\} - \epsilon$, the strategy is to first find this supremum and then attempt to classify the corresponding Alexandrov spaces. So far, however, it is only known that $\sup\{\mathrm{vol}\, N \mid N \in \mathcal{C}_k^D(n)\} < \mathrm{vol}\, D_k^n(D)$, where $D_k^n(D) \subset S_k^n$ is the closed ball in S_k^n of radius D (cf. [GP1]). This problem is a natural extension of Alexandrov's classical and unsolved area problem for convex surfaces in $\mathbb{R}^3$ [A].

If we replace the assumption $\mathrm{diam}\, M \le D$ with

$$\mathrm{rad}\, M = \min_p \max_q \mathrm{dist}(p,q) \le R,$$

a complete solution to the extremal volume problem was found in [GP3].

Volume Recognition Theorem 3.3. *Fix $n \ge 2$. Then*

$$\text{(i)} \qquad \overline{\mathrm{vol}\,(\mathcal{C}_k^D(n))} = \begin{cases} a)\ [0, \frac{R}{\pi\sqrt{k}} \mathrm{vol}\, S_k^n] & \text{if } k > 0 \text{ and } R > \frac{\pi}{2\sqrt{k}} \\ b)\ [0, \mathrm{vol}\, D_k^n(R)] & \text{otherwise.} \end{cases}$$

Moreover, if $\mathrm{vol}\, M$ is almost extremal, then

$$\text{(ii)} \qquad M \text{ is } \begin{cases} \text{diffeomorphic to } S^n \text{ in case a)} \\ \text{homeomorphic to } S^n \text{ or to } \mathbb{R}P^n \text{ in case b).} \end{cases}$$

The proof follows the strategy described above. Using a volume comparison theorem for non-star-like sets proved in [GP1] (cf. also [D]), one shows that in case (a) the only possible limit space is $S_k^{n-2} * S^1(R/\pi)$, whereas in the second case the space must be of the form $D^n_k(R)/x \sim I(x)$, $x \in \partial D^n_k(R)$ where $I : \partial D \to \partial D$ is an isometric involution. From 2.5, however, only the antipodal map and reflexions are possible, corresponding to singular metrics on $\mathbb{R}P^n$ and S^n respectively. The diffeomorphism claim does not follow from this argument, but uses the description above together with the packing radius theorem of [GW] (see 4.6).

In order to fully understand the structure of manifolds $M \in \mathcal{C}^D_k(n)$ with $\operatorname{vol} M < \epsilon$, $\epsilon > 0$ small, it seems necessary to extend 3.1 and 2.5 to the case where $X = N$ is an Alexandrov space. Note, however, that in general $\overline{\mathcal{C}^D_k(n)} \subsetneqq \mathcal{A}^d_k(n)$ (cf. [PWZ]).

However, using the Fibration Theorem 3.1 and the Splitting Theorem for non-negatively curved Alexandrov spaces (cf. [GP4] and [ShW]) as the key tools, Fukaya and Yamaguchi [Y1], [FY2] were able to probe the following results for manifolds with almost non-negative curvature (cf. also [Wu]).

Structure Theorem 3.4. *There is an $\epsilon = \epsilon(n) > 0$ such that any closed riemannian n-manifold M with $\sec M \cdot (\operatorname{diam} M)^2 > -\epsilon$ satisfies*

(i) *A finite cover of M covers a $b_1(N)$-torus, where $b_1(M)$ is the first Betti number of M.*
(ii) *If $b_1(M) = n$, then M is diffeomorphic to the n-torus.*
(iii) *the fundamental group $\pi_1(M)$ is almost nilpotent.*

The volume function is just one natural function which can be used to investigate the classes $\mathcal{C}^D_k(n)$ or $\mathcal{C}^R_k(n)$. In principle, any metric invariant which assigns to a metric space a number, can be used in a similar fashion. When applied to the classes above, or subclasses defined e.g., as sub/super-level sets of other invariants, one is lead to investigate extremal problems which either should give rise to structure results (as in 3.4) or to recognition results (as in 3.3 and 4.3). The general strategy towards the classification of riemannian manifolds via metric invariants is described in [GM] as the *recognition program.* To make any progress in this direction it is of course necessary to introduce and investigate numerous new and old invariants. As specific examples of such invariants we mention Urysohn's intermediate diameters or widths (cf. [Pe] and [P3]), Berger's systoles (cf. [G3]), Gromov's Filling radius [G4], excess invariants [GP4], extents [GM], packing radii [GM,GW], etc.

Of course, these basic invariants play important roles in a variety of problems, and their use in the recognition program has only just begun.

4. OTHER TECHNIQUES

In the remaining part of this paper we will discuss applications of other natural operations which are closed in Alexandrov geometry but not in riemannian geometry.

We begin with the operation of taking quotients by an isometric group action. Using (2.1) it is trivial to see that if G is a compact Lie group (cf. [FY1]) of isometries acting on an Alexandrov space X with $\operatorname{curv} X \geq k$ then X/G is an Alexandrov space with $\operatorname{curv} X/G \geq k$ as well.

Here we will consider the case where $X = M$ is a riemannian n- manifold with positive curvature, i.e., $\sec M > 0$. The classification of such manifolds is far from complete and the list of known examples is rather slim. All of the known examples, however, are constructed by means of groups and it only seems natural to seek a classification of smaller classes of positively curved manifolds with large groups of isometries. The fact that M/G is an Alexandrov space with $\operatorname{curv} M/G > 0$, turns out to shed light on problems of this type.

If for example largeness is measured in terms of the dimension of a maximal torus in the isometry group, $\operatorname{Iso} M$, of M (called the *symmetry rank* of M, $\operatorname{symrank} M = \operatorname{rank} \operatorname{Iso} M$) we have [GS1]:

Theorem 4.1. *Let M be a closed riemannian n-manifold with $\sec M > 0$. Then $\operatorname{symrank} M \leq \left[\frac{n+1}{2}\right]$, and equality holds if and only if M is diffeomorphic to one of S^n, $S^n/\mathbb{Z}_k$ or $\mathbb{C}P^{[\frac{n+1}{2}]}$.*

The proof is divided into the cases $n = 2m$ and $n = 2m + 1$. The first claim is proved by induction on m and is based on the simple fact that the fixed point set of an isometric group action is a disjoint union of totally geodesic submanifolds. The key point in the second claim is that if $\operatorname{symrank} M = \left[\frac{n+1}{2}\right]$ then M admits an isometric S^1-action with fixed point set of codimension 2. This is also proved by induction on m. Using the geometry of M/S^1 one finds that $S^1 \times M \to M$ has precisely one fixed point component N of codimension 2, that there is exactly one orbit $S^1 \cdot p_0$ at maximal distance from N, and that S^1 acts freely on $M - (N \cup S^1 p_0)$. Furthermore, since $\partial(M/G) = N$, M/G is the cone on the space of directions $S_{\bar{p}_0}$ at $\bar{p}_0 \in M/G$ corresponding to the orbit $S^1 \cdot p_0$ (cf. the Soul Theorem for Alexandrov spaces [P1]). $S_{\bar{p}_0}$ however, is easily seen to be $S^{\perp}_{p_0}/S^1_{p_0}$, where $S^{\perp}_{p_0}$ is the normal sphere to the orbit $S^1 \cdot p_0$ at p_0 and $S^1_{p_0}$ is the isotropy group of S^1 at p_0. Combining this with a closer look at the quotient map $M \to M/S^1$ completes the proof.

There is another case related to the Hopf conjecture, where a similar situation arises (cf. [HK], [K]).

Theorem 4.2. *Let M be a closed simply connected 4- manifold with* $\sec M \geq 0$. *If* $\operatorname{symrank} M \geq 1$, *i.e.,* $\operatorname{Iso}(M)$ *is not finite, then M is homeomorphic to either S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$. Here only the first two can occur if* $\sec M > 0$.

By assumption there is an isometric S^1-action on M. From the topological classification of 4-manifolds [Fr] it suffices to show that the Euler character $\chi(M) = \chi(\operatorname{Fix} S^1)$ of M is bounded above by 4. This on the other hand can be proved using the geometry of the Alexandrov space $X = M/S^1$:

When $\operatorname{Fix} S^1$ is finite, the argument boils down to estimating extent invariants of S^3/S^1 (cf. [GM] and [GS2] for other applications). If e.g., $\sec M > 0$ one needs to show that $\operatorname{Fix} S^1$ can have at most 3 isolated fixed points. Assume on the contrary that $p_0, \cdots, p_3$ are isolated fixed points. Join $\bar{p}_0, \cdots, \bar{p}_3$ pairwise by a minimal geodesic in X. There are 4 triangles in this configuration, so for the total sum of angles we get, $\sum \sphericalangle > 4\pi$, since $\operatorname{curv} X > 0$. However, $\sum \sphericalangle \leq 4\pi$ since the sum of angles at each $\bar{p}_i$ is $\leq \pi$ (the space of directions $S_{\bar{p}_i}$ has 3- extent, $xt_3\, S_{\bar{p}_i} \leq \frac{\pi}{3}$ since $S_{\bar{p}_i} = S^3/S^1$ and S^1 acts almost freely on S^3).

There are other natural ways in which to express largeness of $\operatorname{Iso}(M)$ (cf. , e.g., [S], [GS2]). It is only natural that the Alexandrov geometry of the orbit space is likely to play a significant role as in the above examples. An exception of course is the case of homogeneous manifolds, but here a classification is already known [AW], [Be], [BB], and [Wa].

We conclude with a brief discussion of the significance of the join operation between positively curved spaces.

Recall that the q'th packing radius of a compact metric space X is defined by

$$\operatorname{pack}_q X = \frac{1}{2} \max_{(p_1, \cdots, p_q)} \min_{i<j} \operatorname{dist}(p_i, p_j).$$

For an inner metric space, X, $\operatorname{pack}_q X$ is clearly the maximal radius of q disjoint open balls in X. Of course

$$\frac{1}{2} \operatorname{diam} X = \operatorname{pack}_2 X \geq \cdots \geq \operatorname{pack}_q X \geq \cdots \to 0. \tag{4.3}$$

If X is an n-dimensional Alexandrov space with $\operatorname{curv} X \geq 1$, a simple distance comparison argument yield the estimate

$$\operatorname{pack}_q X \leq \operatorname{pack}_q S_1^n. \tag{4.4}$$

To determine $\mathrm{pack}_q S_1^n$ in general is an exceedingly hard problem. For $q \leq n+2$, however, it is not difficult to see that $\mathrm{pack}_q S_1^n$ is realized by the vertices of a $(q-1)$-simplex in $\mathbb{R}^{q-1}$ regularly inscribed in S_1^{q-2}.

Using the rigidity comparison distance Theorem from [GM] and critical point theory for distance functions in Alexandrov spaces (cf. [P1,2]) one can prove [GW].

Theorem 4.5. *Let X be an n-dimensional Alexandrov space with* $\mathrm{curv}\, X \geq 1$. *For each $q \leq n+2$ we have*

(i) $\mathrm{pack}_q X = \mathrm{pack}_q S_1^n$, *if and only if X is isometric to $S_1^{q-2} * E$ for some $(n-q+1)$-dimensional Alexandrov space E with* $\mathrm{curv}\, X \geq 1$.

(ii) *If $\mathrm{pack}_q X > \frac{\pi}{4}$, then X is homeomorphic to $S_1^{q-2} * E$ for some E as in (i).*

Using convergence techniques as discussed in Section 3 one can use 4.5(i) to prove a diffeomorphism theorem for riemannian n-manifolds M with $\sec M \geq 1$ and $\mathrm{pack}_{n+2} M$ or $\mathrm{pack}_{n+1} M$ almost maximal. It turns out, however that one can do much better (cf. [GW]):

Packing Radius Sphere Theorem 4.6. *Any riemannian n-manifold M with $\sec M \geq 1$ and $\mathrm{pack}_{n-1} M > \frac{\pi}{4}$ is diffeomorphic to S^n.*

Note that this result is optimal in the sense that $\mathrm{pack}_{n-1} M > \frac{\pi}{4}$ cannot be replaced by $\mathrm{pack}_{n-1} M = \frac{\pi}{4}$. Indeed, even $\mathrm{pack}_2 \mathbb{R}P_1^n = \mathrm{pack}_{n+1} \mathbb{R}P_1^n = \frac{\pi}{4}$. Morever, even $\mathrm{pack}_n M$ about maximal allows collapse to $D_1^{n-1}(\frac{\pi}{2})$.

The proof exploits in an essential way the geometry of the $(n+1)$-dimensional Alexandrov space $X = \sum_1 M = S^0 * M$ the spherical suspension on M. Using this geometry together with smoothing theory for distance functions it is shown that M can be smoothly embedded in $\mathbb{R}^{n+1}$.

The equality discussion $\mathrm{pack}_q X = \frac{\pi}{4}$ is difficult even for $q < n+2$. In the maximal case, in the sense of (4.3), it is easy to see however, that $\mathrm{diam}\, X \leq \frac{\pi}{2}$ if $\mathrm{curv}\, X \geq 1$. For $\mathrm{pack}_{n+1} X = \frac{1}{2}\mathrm{diam}\, X = \frac{\pi}{4}$ there is a complete classification (cf. [GM]).

Theorem 4.7. *An n-dimensional Alexandrov space X with* $\mathrm{curv}\, X \geq 1$ *has $\mathrm{pack}_{n+1} X = \frac{1}{2}\mathrm{diam}\, X = \frac{\pi}{4}$ if and only if X is isometric S_1^n/H for some finite abelian group $H \subset O(n+1)$ of involutions acting without fixed points on S_1^n.*

It is worth mentioning, that the class of spaces in Theorem 4.7 is closed under the spherical join operation. Also, it turns out that the only topological manifolds among these spaces are spaces homeomorphic to S^n or isometric to $\mathbb{R}P_1^n$ (see [GM]).

¿From the convergence techniques, i.e., 2.5 and 3.1, combined with the diameter sphere theorem [GS] one therefore gets

Corollary 4.8. *There is an* $\epsilon = \epsilon(n) > 0$, *such that any riemannian n-manifold* M *with* $\sec M \geq 1$ *and* $\text{pack}_{n+1} M \geq \frac{\pi}{4} - \epsilon$ *is homeomorphic to* S^n *or diffeomorphic to* $\mathbb{R}P^n$.

In view of 4.6, one might hope to be able to replace homeomorphism by diffeomorphism in this corollary.

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