수 학 강 의 록 제 17 권

# Proceedings of the Second GARC SYMPOSIUM on Pure and Applied Mathematics

## PART I

**The first Korea-Japan Conference of Partial Differential Equations**

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서 울 대 학 교
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# PREFACE

The second GARC Symposium on Pure and Applied Mathematics was held at Seoul National University from February 4 to 20, 1993.

The symposium was organized by the Global Analysis Research Center which was founded in 1991 as one of 30 centers of excellence under the supports of the Korea Science and Engineering Foundation.

The symposium covered a broad range of topics in the fields of mathematical analysis and global analysis. It was carried out in 6 sessions ; nonlinear analysis, operator algebras, partial differential equations, topology and geometry of manifolds, differential geometry and complex algebraic varieties and several complex variable.

Among them the session of partial differential equations was held in the form of the first Korea-Japan joint conference. We expect the second joint conference will be held in Japan next year. We are pleased to express here our thanks to those participants from Japan whose collaboration made the conference a successful one.

The GARC symposium was actively attended by more than 200 participants including 16 foreign mathematicians. This proceedings of three issues contains research articles which were presented. The content will be of interest both to the members of the Global Analysis Research Center and to mathematicians working in the various fields of current mathematics.

We wish to express our gratitude to all contributors and especially to those mathematicians from abroad. We also express our thanks to the Korea Science and Engineering Foundation for having made this symposium possible, to Professors Sage Lee and Hyuk Kim for their endeavor in organizing this symposium and to Miss Jin Young Bae and Mr. Kyung Whan Park for their help in editing the proceedings.

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# PARTIAL DIFFERENTIAL EQUATIONS

# ON THE INTEGRAL ESTIMATES FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

DONGHO CHAE

ABSTRACT. In this note we review the author's previous results concerning the improvements of Foias-Guillopé-Temam's a priori estimates for the weak solutions of the Navier-Stokes equations in the periodic domain in $\mathbb{R}^3$. More specifically, we present theorems and the corollaries on the estimates for the upperbounds for the temporal averages of the Gevrey class norm for the weak solutions of the equations.

## 1. INTRODUCTION

We consider the Navier-Stokes equations in a periodic domain $Q = (0, L)^3$:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

$$u(\cdot, 0) = u_0 \tag{3}$$

The unknown functions are $u = u(x,t)$, $u = (u_1, u_2, u_3)$ and $p = p(x,t)$. The force f and the kinematic viscosity $\nu > 0$ are given. We assume that $u, p$ and $f$ are periodic functions with period $Q$. For the problem (1)-(3) the

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fundamental result due to Leray and Hopf states that for $u_0$ in $V_0$ we have global in time existence of weak solutions satisfying:

$$u \in L^\infty(0,T;V_0) \cap L^2(0,T;V_1) \tag{4}$$

$$\int_0^T |u(t)|_1^2 dt \leq c_1(1+T) \tag{5}$$

for all $T > 0$, where the constant $c_1$ depends on the data $\nu, f, Q$ and $u_0$.(See [C,F], or [T1] for proofs.) Above and in the sequel, $V_m, m \geq 0$, denotes a class of periodic, $H^m(Q)$- Sobolev functions. Further descriptions of the space will be given in the following section. Thus, for almost every $t \in [0,\infty)$ the weak solutions satisfy $u(t) \in V_m$ . On the other hand Foias-Guillopé -Temam[F,G,T] have proved the regularizing effect of the Navier- Stokes equations by showing $u(t) \in V_m$ for almost every $t \in [0,\infty]$, and for any large $m$'s. This follows from their estimate of type:

$$u \in L^{\alpha_m}(0,T;V_m) \quad , \int_0^T |u(t)|_m^{\alpha_m} dt \leq c_m(1+T) \tag{6}$$

for all $T > 0$ and $m \geq 1$, where $\alpha_m$ is given by

$$\alpha_m = \frac{2}{2m-1} \tag{7}$$

As one of crucial steps leading to (6) they established and used the fact that the strong solutions of (1)-(3) (i.e. solutions $u \in C([0,T_*];V_1)$ associated with initial data $u_0 \in V_0$ , where $T_*$ depends on the data ) with regular force $f \in L^\infty(0,T;V_m)$ have corresponding regularization properties:

$$u \in C((0,T_*] : V_m) \quad \text{for } u_0 \in V_0 \tag{8}$$

In their recent paper [F,T] C. Foias and R. Temam pushed such local in time regularization for the strong solutions even further up to Gevrey class regularity. This class of functions will be defined in the next section.

One main result in this paper is an integral estimate of type (6) with $V_m$ replaced by suitable Gevrey classes firstly for $m = 1$ and then, by the induction procedure similar to one used in [F,G,T], for arbitrary $m \geq 1$ by exploiting this result of local in time Gevrey class regularity and its strengthened version. The other of the main results is to use local in time analyticity result of strong solutions to obtain inegral estimats for the temporal derivatives of weak solutions.

## 2. Preliminaries

It is well-known that (1)-(3) are equivalent to an abstract evolution equation for $u$(see e.g. [C,F])

$$\frac{du}{dt} + \nu A u + B(u) = f \tag{9}$$

$$u(0) = u_0 \tag{10}$$

in a Hilbert space $H$ which consists of solenoidal vector fields in $L^2(Q)^3$ with scalar product and norm denoted by $(\cdot,\cdot)$, $|\cdot|$ respectively. The operator $A$ ( corresponding to the Stokes operator with space periodic boundary condition) is a linear self-adjoint unbounded positive operator with domain $D(A) \subset H$. $B(u) = B(u,u)$ where $B(u,u)$ is defined by

$$(B(u,u),w) = \sum_{j,k=1}^{3} \int_Q u_j \frac{\partial u_k}{\partial x_j} w_k dx$$

For any $m > 0$ we denote $V_m = D(A^{\frac{m}{2}})$ where $D(A^{\frac{m}{2}})$ consists of functions $u$:

$$u(x) = \sum_{k \in Z^3} u_k e^{2\pi i k\cdot x/L}, u_k \in \mathbb{C}^3, u_{-k} = \overline{u_k}, u_0 = 0, \tag{11}$$

with

$$k \cdot u_k = 0 \tag{12}$$

$$\sum_{k \in Z^3} (\frac{2\pi}{L})^{2m} |k|^{2m} |u_k|^2 = |A^{\frac{m}{2}} u|^2 < \infty \tag{13}$$

Thus $H = V_0$, and, as usual, we denote $V = V_1$. We will use the notation $|A^{\frac{m}{2}} u| = |u|_m$ with the roman letters and arabic numbers for the subscripts; in particular, for the scalar product and norm in $V$ we will use $((\cdot,\cdot)), \|\cdot\|(=$

$|\cdot|_1)$. For $\tau, s > 0$ given the Gevrey class $D(e^{\tau A^s})$ is a set of functions $u$ satisfying (11)-(12) and

$$\sum_{k\in Z^3} e^{2\tau(\frac{2\pi}{L})^{2s}|k|^{2s}}|u_k|^2 = |e^{\tau A^s}| < \infty \tag{14}$$

In particular for the scalar product and the norm in $D(e^{\tau A^{\frac{1}{2}}})$ we use $(\cdot,\cdot)_\tau, |\cdot|_\tau$, while for $D(A^{\frac{1}{2}}e^{\tau A^{\frac{1}{2}}})$ we use $((\cdot,\cdot))_\tau, \|\cdot\|_\tau$ with greek letters only for the subscripts. It is easy to check the (compact) imbedding ,$D(e^{\sigma A^{\frac{1}{2}}}) \subset V_m$, with

$$|u|_m \le c_{m,\sigma}|u|_\sigma \tag{15}$$

for any $\sigma > 0$ and $m \ge 0$. For the combinations we use $D(A^{\frac{m}{2}}e^{\tau A^{\frac{1}{2}}}) = V_{m,\tau}$ and $|A^{\frac{m}{2}}e^{\tau A^{\frac{1}{2}}}| = |u|_{m,\tau}$ respectively.

Now let us recall some facts on the weak and the strong solutions to (9) - (10). Below $T$ is an arbitrarily given positive number. Firstly, the well-known Leray-Hopf theorem asserts that for $f \in L^\infty(0,T;H)$ and $u_0 \in H$ given, there exists at least one (weak) solution $u$ of (9)-(10), such that

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{16}$$

and

$$\frac{du}{dt} \in L^{\frac{4}{3}}(0,T;V') \tag{17}$$

where $V'$ denotes the dual of $V$. For the strong solutions we have the following: Let $t \ge 0$ be given, and $u(t) \in V$ then there exists unique (strong) solution $u$ of (9)-(10) such that

$$C([t,T_1(t)];V) \cap L^2(t,T_1(t);D(A)) \tag{18}$$

and

$$\frac{du}{dt} \in L^2(t,T_1(t);H) \tag{19}$$

whrere $T_1(\cdot)$ is given, throghout this paper, by

$$T_1(t) = \frac{c_3}{(1+\|u(t)\|^2)^2} \tag{20}$$

with constant $c_3$ depending on $f, \nu, Q$. The following a priori inequality for the strong solutions, which can be derived from (9) easily (see (3.27) pp.20 of [T2]), is also very useful for us:

$$\frac{d}{dt}\|u(t)\|^2 \leq c_4(1 + \|u(t)\|^2)^3 \tag{21}$$

where the constant $c_3$ depends on the data $f$, $Q$ and $\nu$. We say that an interval $(t_l, t_r) \subset R^+$ is maximal $H^m-$regularity interval for solution $u$ if

$$u \in C((t_l, t_r); H^m(Q)) \text{ and} \tag{22}$$

and there is no greater interval than $(t_l, t_r)$ containing this interval and having the property (22). From the definition itself and the local in time (regular) existence property of strong solutions one can show easily (see [C,F] or [T2]) that

$$\lim_{t \to t_r - 0} \sup |u(t)|_m = +\infty \tag{23}$$

We denote

$$O_m = \{t \in (0, \infty), \exists \epsilon > 0 \text{ such that } u \in C((t - \epsilon, t + \epsilon); H^m)\}$$

i.e. $O_m$ is the maximal open set on which $H^m$-regularity holds. The following lemma was established in [F,G,T].

**Lemma 1.** *Let $u$ be the weak solutions of the Navier-Stokes equations with $u_0 \in H$ and $f \in L^\infty(0, T; V_{m-1})$ ,then we have the following:*

$$O_1 = O_r, \quad r = 1, 2, \cdots, m \tag{24}$$

*and*

$$[0, T] \setminus O_1 \quad \textit{has Lebesgue measure } 0 \tag{25}$$

Concerning local in time regularization effect for the strong solutions of the Navier-Stokes equations we have the following recent results due to C. Foias and R. Temam(see Theorem 1.1 pp.361 [F,T]).

**Lemma 2.** *Let $u(t) \in V$ and $f \in L^\infty(0,\infty; D(e^{\sigma A^{\frac{1}{2}}}))$ for some $\sigma > 0$, and $u \in C([t, t+T_1(t)]; V)$ with $T_1(t)$ given in (20) be the associated strong solution to (9)-(10). Then, the strong solution satisfies: The mapping $s \rightarrow A^{\frac{1}{2}} e^{\psi(s-t)A^{\frac{1}{2}}} u(s)$ from $(t, t+T_1(t))$ into $H$ is continuous and*

$$\|e^{\psi(s-t)A^{\frac{1}{2}}} u(s)\|^2 \leq c_5(1 + \|u(t)\|^2) \tag{26}$$

*for all $s$ satisfying*

$$t \leq s \leq t + T_1(t) \tag{27}$$

*where the function $\psi(\cdot)$ is defined*

$$\psi(s) = min\{\sigma, s\} \tag{28}$$

*and, the constant $c_5$ depending on the data $\nu, f$ and $Q$.*

We will use the function $\psi(\cdot)$ defined above throughout this paper. Below we state the temporal anayticity result for the strong solutions, but holding also for the weak solutions for the set of time in which $u(t) \in V$ [F,T]. This was obtained for the complexified equations of (9)-(10). By this complexification we mean we consider complex time $\xi \in \mathbb{C}$, the complexified space $H_\mathbb{C}$ with the corresponding scalar product and norm, the extended operators $A, B$ , and the corresponding complexified spaces $D(A^{\frac{m}{2}})$'s and $D(e^{\tau A^s})$'s.

**Lemma 3.** *Let $u(t) \in V$ and $f \in D(e^{\sigma A^{\frac{1}{2}}})$ for some $\sigma > 0$, and $u(\xi)$ be the associated strong solution to (9)-(10). Then, the strong solution satisfies: The mapping $\xi \rightarrow e^{\psi(Re\xi - t)A^{\frac{1}{2}}} u(\xi)$ from $\Delta_t$ into $V_\mathbb{C}$ is analytic and*

$$\sup_{\xi \in \Delta_t} \|e^{\psi(Re\xi - t)A^{\frac{1}{2}}} u(\xi)\|^2 \leq 2(1 + \|u(t)\|^2) \tag{29}$$

*where the function $\psi(\cdot)$ is defined in (28). and $\Delta_t \subset \mathbb{C}$ is given by*

$$\Delta_t = \{\xi = t + s_1 e^{i\theta}, \ \ 0 \leq s_1 \ T_1(t), \frac{\sqrt{2}}{2} \leq \cos\theta \leq 1\} \tag{30}$$

## 3. The Main Results

In this section we present our main theorems and their corollaries; the proofs of those are omitted here, which could be found in [Ch1],[Ch2]. The first theorem below can be viewed as an integral version of the lemma 2.

**Theorem 1.** *Let us assume* $u_0 \in H$, $f \in L^\infty(0,T;D(A^{\frac{1}{2}}e^{\sigma A^{\frac{1}{2}}}))$ *for some* $\sigma > 0$ *and that* $u$ *is a weak solution of the Navier-Stokes equations. Then* $u$ *satisfies*

$$u \in L^2(0,T;D(e^{\phi(t)A^{\frac{1}{2}}})), \quad \int_0^T \|e^{\phi(t)A^{\frac{1}{2}}}u(t)\|^2 dt \le C_1(1+T) \tag{31}$$

*for some function* $\phi(t)$, *continuous on* $[0,T]$ *and*

$$\phi(t) > 0 \ \ for \ almost \ every \ t \in [0,T] \tag{32}$$

*In (31) the constant* $C_1$ *depends on the data* $\nu, f, u_0, Q$.

*Remark.* Specifically the function $\phi(t)$ constructed in the proof of the theorem is the following:

$$\phi(t) = \psi(t-\tau(t)) = \min\{\sigma, t-\tau(t)\} \tag{33}$$

where the function $\tau(\cdot)$ is defined by its inverse $\tau^{-1}(\cdot)$,

$$\tau^{-1}(t) = \begin{cases} t + \min\{t - l(t), cT_1(t)\} t \in O_1 \\ \qquad\qquad\qquad\qquad tt \in [0,T] \setminus O_1 \end{cases} \tag{34}$$

where $l(t)$ is the left end point of the maximal interval of $H^1$-regularity containing $t$ and $c$ is a chosen constant. Actually, we have $\{t \in [0,T] : \phi(t) > 0\} = O_1$, the complement of which has Lebesgue measure 0 by the lemma 1. For further detailed description of the function $\tau^{-1}(\cdot)$ (hence, $\phi(\cdot)$) see the next section. We note (32), which makes the theorem 3 below an improvement of the corresponding Foias-Guillopé-Temam's. The above theorem 1 itself can be viewed as an improvement of the Leray-Hopf's estimate (5).

Note the above proposition implies immediately that

$$u(t) \in D(A^{\frac{1}{2}}e^{\phi(t)A^{\frac{1}{2}}}) \ \text{ with } \phi(t) > 0 \tag{35}$$

for almost every $t \in [0,T]$. In virtue of the injection relation (15), (35) implies, in turn,

$$u(t) \in V_m \quad \text{for any } m \geq 0 \tag{36}$$

for almost every $t \in [0,T]$, Thus the theorem 1 already implies a regularity in [F,G,T], which is described at the introduction of this paper.

The following theorem is a generalized version of the lemma 2 , which states that the strong solution becomes further regular locally in time if the forcing is more regular.

**Theorem 2.** *For any given $t > 0$ assume $u(t) \in V$ and $f \in L^{\infty}_{loc}(0,\infty; D(A^{\frac{m-1}{2}} e^{\sigma A^{\frac{1}{2}}}))$ for some $\sigma \geq 0$ and $m \geq 1$, then the associated strong solution $u$, defined on $[t, t+T_1(t)]$ satisfies: The mapping $s \to A^{\frac{m-1}{2}} e^{\psi(s-t)A^{\frac{1}{2}}} u(s)$ from $(t, t+T_1(t))$ into $H$ is continuous.*

The following theorem can be viewd as an integral version of the above one.

**Theorem 3.** *Let us assume $u_0 \in H$, $f \in L^{\infty}(0,T; D(A^{\frac{m-1}{2}} e^{\sigma A^{\frac{1}{2}}}))$ for some $\sigma \geq 0$ and $m \geq 1$ and that $u$ is a weak solution of the Navier-Stokes equations. Then $u$ satisfies*

$$u \in L^{\alpha_r}(0,T; D(A^{\frac{r}{2}} e^{\phi(t)A^{\frac{1}{2}}})), \quad \int_0^T |A^{\frac{r}{2}} e^{\phi(t)A^{\frac{1}{2}}} u(t)|^{\alpha_r} dt \leq C_r(1+T) \tag{37}$$

*for all $r = 1, 2, \cdots, m+1$, where the constant $C_r$ depends on the data $\nu, f, u_0, Q$ and $r$, and $\phi(t)$ is the same function as in the theorem 1 and $\alpha_r$ is given by*

$$\alpha_r = \frac{2}{2r-1} \tag{38}$$

*Remark.* For $m = 1$, $r = 1$ the above theorem reduces to the theorem 1. On the other hand, for formal substitution of $\sigma = 0$, the theorem reduces to the Foias-Guillopé-Temam's estimate (6), since $\phi(t) \equiv 0$ in this case.

Similarly to the corollary 1 in [F,G,T] we can deduce the following:

**Corollary 1.** *Let us assume $u_0 \in H$, $f \in L^\infty(0,T;D(e^{\sigma A^{\frac{1}{2}}}))$ for some $\sigma > 0$ and that $u$ is a weak solution of the Navier-Stokes equations. Then $u$ satisfies*

$$\int_0^T |e^{\phi(t)A^{\frac{1}{2}}}u(t)|_{L^\infty(Q)}dt \le C(1+T) \tag{39}$$

*for the same function $\phi(t)$ as above, where the constant $C$ depends on the data $u_0, \nu, f, Q$.*

Using the local in time analyticity result of strong solutions in a periodic domain we obtain the following another generalization of theorem 1.

**Theorem 4.** *Let us assume $u_0 \in H$, $f \in D(e^{\sigma A^{\frac{1}{2}}})$ for some $\sigma > 0$ and that $u$ is a weak solution of the Navier-Stokes equations (9)-(10). Then $u$ satisfies*

$$\int_0^T \|(\frac{d}{dt})^k\{e^{\psi(t-\tau(t))A^{\frac{1}{2}}}u(t)\}\|^2(t-\tau(t))^{2k}dt \le C_k(1+T) \tag{40}$$

*for all integer $k \ge 0$, where the function $\tau(\cdot)$ is defined by its inverse as in (34). In (40) the constant $C_k$ depends. on the data $\nu, f, u_0, Q$.*

The following is an immediate result from the above theorem:

**Corollary 2.** *Let $u_0$ and $f$ satisfy the same assumptions as in the theorem 4 and $u$ be the corresponding weak solutions to the 3-D Navier-Stokes equations in the periodic domain $Q$. Then, $u$ satisfies*

$$(t-\tau(t))e^{\psi(t-\tau(t))A^{\frac{1}{2}}}u(t) \in C([0,\infty);V) \tag{41}$$

*with the same function $\tau$ as in the theorem 4.*

Using the local in time analyticity result in more general boundary conditions and forcing we can obtain:

**Theorem 5.** *Let us assume $u_0 \in H$, $f \in H$ and that $u$ is a weak solution of the Navier-Stokes equations in a periodic, or bounded domain $\Omega$ with $\partial\Omega \in C^2$. Then $u$ satisfies*

$$\int_0^T \|(\frac{d}{dt})^k u(t)\|^2(t-\tau(t))^{2k}dt \le C'_k(1+T) \tag{42}$$

*for all integer $k \ge 0$, where the constant $C'_k$ depends on the data $\nu, f, u_0, Q$.*

Again, by the essentially same arguments leading to the corollary 1 we obtain:

**Corollary 3.** *Let $u_0$ and $f$ satisfy the same assumptions as in the theorem 5 and $u$ be the corresponding weak solutions to the 3-D Navier-Stokes equations in the periodic, or bounded domain $\Omega$ with $\partial\Omega \in C^2$. Then, $u$ satisfies*

$$(t - \tau(t))u(t) \in C([0, \infty); V) \tag{43}$$

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# FREE BOUNDARY OF EVOLUTIONARY P-LAPLACIAN FUNCTIONS

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## 1. INTRODUCTION

We study the free boundary problems involving the evolutionary p-Laplace equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = u_t - \Delta_p u = 0,\ p > 2 \tag{1}$$

in $\mathbf{R}^n \times (0,\infty)$, $n \geq 1$, with a nonnegative initial datum

$$u(x,0) = u_0(x)$$

of compact support.

Since we are assuming $p > 2$, the equation (1) is degenerate when $\nabla u = 0$. Hence we need to consider a weak solution. A function $u(x,t)$ satisfies the followings:

For any $T > 0$,

$$\int_0^T \int_{\mathbf{R}^n} u^2(x,t) + |\nabla u|^p \ dxdt < \infty$$

and

$$\int_0^T \int_{\mathbf{R}^n} u\frac{\partial \phi}{\partial t} - |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \ dxdt + \int_{\mathbf{R}^n} u_0(x)\phi(x,0)\ dx \ = 0$$

for any continuously differentiable function $\phi$ with compact support in $\mathbf{R}^n \times [0,T)$. The unique solvability of our Cauchy problem in $\mathbf{R}^n \times (0,T)$ follows from Theorem 1 and 4 in [12].

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For the porous medium equation

$$v_t - \Delta(v^m) = 0,\ m > 1$$

various results for the interface are known. Assuming the initial datum $v_0$ has compact support, the interface consists of two parts -a moving boundary and a nonmoving boundary - and the support $\Omega(t) = \{x \in \mathbf{R}^n : v(x,t) > 0\}$ is monotonically increasing. Furthermore, the regularity of interface has been investigated by many authors(see [1], [5], [6], [7], [16] and etc.).

Here, following a similar argument to [5] and [7] we study the interface questions for the evolutionary p-Laplace equation (1). We define

$$\Lambda = \{(x,t) \in \mathbf{R}^n \times [0,\infty) : u(x,t) > 0\}$$

$$\Omega(t) = \{x \in \mathbf{R}^n : u(x,t) > 0\}$$

$$\Gamma(t) = \text{ the boundary of } \Omega(t)$$

and

$$\Gamma = \cup_{t\geq 0}\Gamma(t).$$

Hence $\Gamma(0)$ is the boundary of $\{x \in \mathbf{R}^n : u_0 > 0\}$.

In section 2 we consider the initial behaviour of the interface. When the initial datum $u_0$ satisfies

$$u_0(x) \geq c\,[\mathrm{dist}(x)]^{\frac{p}{p-2}}\,, \tag{2}$$

where $\mathrm{dist}(x) = \mathrm{distance}(x, \Gamma(0))$, then $\Omega(t)$ is strictly increasing initially. On the other hand we show that (2) is sufficient for strict monotonicity of the set $\Omega(t)$. In fact we show that if there is a supporting hyper-plane $P$ at $x_0 \in \Gamma(0)$ and

$$u_0(x) \leq c\,[\mathrm{dist}(x)]^{\gamma}\,,\ \gamma > \frac{p}{p-2},$$

then there is a positive time $\tau > 0$ such that

$$u(x_0, t) = 0,\ \text{for all } 0 < t < \tau$$

and hence the interface does not move at $x_0$ for a short period. There are corresponding results for porous medium equations(see [16] for one dimension and [5] for higher dimension. Finally we find an integral expression which describes the pointwise behaviour of the interface. In fact we use the Harnack type inequality and an integral estimate by DiBenedetto and Herrero([12]) and obtain a necessary and sufficient condition for moving point.

In section 3 from the Harnack principle and the maximum principle we prove Hölder regularity of the interface. First we show that the interface consists of two parts- a moving part $\Gamma_1$ and a nonmoving part $\Gamma_2$. In particular, if the interface $\Gamma$ contains a vertical line segment $\sigma = \{(x,t) : x = x_0, t_0 < t < t_1\}, 0 < t_0 < t_1$, then the entire segment $\{(x,t) : x = x_0, 0 < t < t_1\}$ belongs to $\Gamma$. Following an iteration method we show that if $(x_0, t_0) \in \Gamma$ does not lie on a vertical line segment belonging to $\Gamma$, then $\Gamma \cap \{t = \tau\}$ increases at a rate $\geq (\tau - t_0)^\mu$ for $\tau > t_0, \tau - t_0$ small, $|x - x_0|$ small, for some $\mu > 1$. Solutions of porous medium equations show the same behaviour(see Theorem 3.2 in [5]). Therefore the Hölder continuity of the interface follows. Here, the Harnack principle is a main tool in showing the above results.

## 2. THE BEHAVIOUR OF THE INTERFACE AT $t = 0$

In this section we study the initial growth rate of the interface. We show that $\Gamma(t)$ is increasing, if the initial datum near $\Gamma(0)$ satisfies a certain integral criterion.

Following an argument similar to Knerr([16]), it is shown that the Hölder exponent of $\nabla u$ is critical in the behaviour of interface. Suppose $x_0$ is on $\Gamma(0)$. Constructing a sequence of subsolutions near interface, we prove that if

$$u_0(x) \geq [\text{dist}(x)]^{\frac{\gamma}{p-2}}$$

for some $\gamma < p$, then the interface is moving near $x_0$. Otherwise, the interface does not move for a short period.

**Lemma 1.** *Let $\Gamma(0)$ be of $C^2$. Suppose that $B_R(0) \subset \Omega(0)$ and $\overline{B_R(0)} \cap \Gamma(0) = \{x_0\}$. Furthermore, we assume that*

$$u_0(x) \geq [dist(x)]^{\frac{\gamma}{p-2}} \tag{3}$$

*for some $\gamma < p$. Then we have*

$$u(x_0, t) > 0$$

*for all $t > 0$.*

*proof.* Without loss of generality we may assume that

$$p - 1 < \gamma < p.$$

Let $g(r) = (R-r)^\eta$, where $\eta = \frac{\gamma}{p-1}$. Note that $\eta > 1$ and this is crucial for the following argument. Since we are assuming

$$\overline{B_R(0)} \cap \Gamma(0) = \{x_0\},$$

we see that

$$|x_0| = R.$$

Now we choose $a$ sufficiently close to $R$ such that

$$\frac{n-1+\frac{1}{2}(\eta-1)(p-1)}{n-1+(\eta-1)(p-1)}R < a < R.$$

We find $s$ satisfying

$$g'(s) = -\eta(R-s)^{\eta-1} = \frac{1}{2}\frac{g(a)}{a-R}$$

and set $\tilde{x}$ by

$$\tilde{x} = s\frac{x_0}{|x_0|}.$$

Since $g$ is a convex function and $g'(R) = 0$, it is always possible to choose such $s$. We set $s_1 = s$. Define recursively

$$s_{k+1} = \frac{s_k + R}{2}$$

for $k = 1, 2, 3, \ldots$ and

$$\tilde{x}_k = s_k\frac{x_0}{|x_0|}.$$

We define a supporting cone $T_k$ at $(\tilde{x}_k, g(|\tilde{x}_k|))$ as

$$T_k(x) = g(s_k) + \eta(R-s_k)^{\eta-1}(s_k - |x|).$$

We see that

$$T_k(0) = g(s_k) + \eta(R-s_k)^{\eta-1}s_k.$$

Moreover when

$$|x| = s_k + \frac{g(s_k)}{\eta(R-s_k)^{\eta-1}} = s_k + \frac{1}{\eta}(R-s_k) \equiv \tilde{s}_k,$$

we find that

$$T_k(x) = 0.$$

Since $g(s)$ is convex near $R$, we conclude that

$$T_k(x) \le g(|x|)$$

for all $x \in B_R(0)$. We also observe that $T_k(0)$ decreases monotonically as $k$ goes to $\infty$.

Now we are ready to construct comparison functions which are subsolutions to (1). Since the maximum principle holds for solutions of (1), we can assume

$$u_0(x) = u(x,0) = \left[(R - |x|)^+\right]^\eta$$

without loss of generality. From a result of Lieberman(see [19]), if $\nabla u_0$ is Hölder continuous, then $\nabla u$ is Hölder continuous up to $t = 0$. Hence it follows that

$$u_t(x,0) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

$$= \eta^{p-1}\left(R - |x|\right)^{(\eta-1)(p-1)-1}\left[(\eta-1)(p-1) - \frac{n-1}{|x|}\left(R - |x|\right)\right] \ge 0,$$

if

$$\frac{n-1}{(\eta-1)(p-1)+n-1}R \le |x| \le R.$$

Recall, from the choice of $a$.

$$\frac{n-1}{(\eta-1)(p-1)+n-1}R \le a \le R.$$

So we see that there exists $\tau > 0$ such that

$$u_t(\bar{x}, t) > 0, \text{ for all } t \in [0, \tau),$$

where $\bar{x} = a\frac{x_0}{|x_0|}$.

Now we consider a family of functions $\{f_k(x,t)\}$ given by

$$f_k(x,t) = \alpha_k^p t + \alpha_k(\tilde{s}_k - |x|), \text{ if } |x| < \tilde{s}_k + \alpha_k^{p-1}t$$

and

$$f_k(x,t) = 0, \text{ otherwise,}$$

where $\alpha_k = \eta(R - s_k)^{\eta-1}$. Note that $f_k(x,0) = T_k(x)$. We define

$$u_k(x,t) = [f_k(x,t)]^q,$$

where $q = \frac{p-1}{p-2}$. Then after suitable calculation we have that

$$(u_k(x,t))_t = q\alpha_k^p \left[\alpha_k^p t + \alpha_k(\tilde{s}_k - |x|)\right]^{q-1}$$

and

$$\operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right) = q^p \alpha_k^p \left[\alpha_k^p t + \alpha_k(\tilde{s}_k - |x|)\right]^{q-1}$$
$$-q^{p-1}\alpha_k^{p-1}\left[\alpha_k^p t + \alpha_k(\tilde{s}_k - |x|)\right]^q \frac{n-1}{|x|}.$$

Thus we have

$$(u_k(x,t))_t - \operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right)$$
$$= \alpha_k^p \left[\alpha_k^p t + \alpha_k(\tilde{s}_k - |x|)\right]^{q-1} \cdot$$
$$\left(q - q^p + q^{p-1}\frac{n-1}{\alpha_k|x|}\left[\alpha_k^p t + \alpha_k(\tilde{s}_k - |x|)\right]\right).$$

Therefore if

$$q - q^p + q^{p-1}\frac{n-1}{|x|}\left[\alpha_k^{p-1}t + (\tilde{s}_k - |x|)\right] \le 0,$$

that is,

$$|x| \ge \frac{q^{p-1}(n-1)(\alpha_k^{p-1}t + \tilde{s}_k)}{q^p + q^{p-1}(n-1) - q},$$

then $u_k$ is a subsolution to (1) and

$$(u_k(x,t))_t - \operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right) \le 0.$$

We note that

$$u_k(x_0,t) = 0, \text{ for all } 0 \le t \le \tau_k$$

with

$$\tau_k = \frac{1}{\alpha_k^{p-1}}(R - \tilde{s}_k).$$

Hence we find that

$$\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \frac{1}{\alpha_k^{p-1}}(R-\tilde{s}_k) \leq \lim_{k\to\infty} \frac{R-\tilde{s}_k}{[\eta(R-\tilde{s}_k)^{\eta-1}]^{p-1}}$$
$$= \lim_{k\to\infty} \frac{1}{\eta^{p-1}}(R-\tilde{s}_k)^{p-\eta(p-1)} = 0,$$

since

$$R > \tilde{s}_k \geq s_k \text{ and } p - \eta(p-1) > 0.$$

From a direct computation it is rather simple to see that

$$f_k(\bar{x}, \tau_k) = \alpha_k(R - |\tilde{x}|) = \alpha_k(R-a)$$

and since $\alpha_k < \alpha_1$ for all $k \geq 2$, it follows that from the choice of $s_1$

$$f_k(\bar{x}, \tau_k) < \alpha_1(R-a) = \frac{1}{2}g(a) = \frac{1}{2}g(|\bar{x}|).$$

Now recall

$$q = \frac{p-1}{p-2} > 1$$

and

$$u_t(\bar{x}, t) > 0, \text{ for all } 0 < t < \tau.$$

Therefore we conclude that for $R \leq 1$

$$u_k^{\frac{1}{q}}(\bar{x}, t) = f_k(\bar{x}, t) \leq g(|\bar{x}|) \leq u^{\frac{1}{q}}(\bar{x}, t)$$

for all $0 \leq t < \tau$, since

$$u_t(\bar{x}, t) > 0, \text{ for all } 0 < t < \tau.$$

Recall $u_k$ is a subsolution to (1). Consequently from the comparison principle

$$u_k^{\frac{1}{q}}(x_0, t) = f_k(x_0, t) \leq u^{\frac{1}{q}}(x_0, t)$$

for all $\tau_k < t < \tau$ and since

$$\lim_{k\to 0} \tau_k = 0,$$

we conclude that

$$u(x_0, t) > 0$$

for all $0 < t < \tau$ .

Now we prove a lemma which is a converse to Lemma 1. The growth condition (2) is almost sufficient for showing that the interface is moving at $(x_0, 0)$.

**Lemma 2.** *Suppose that $x_0 \in \Gamma(0)$ and there is a supporting hyper-plane $P$ in $\mathbf{R}^n$ such that $x_0 \in P$ and $\Omega(0)$ lies completely in one side of $P$. If*

$$u_0(x) \leq [dist(x, \Gamma(0))]^{\frac{p}{p-2}}, \tag{4}$$

*then there is a small positive constant $\tau > 0$ such that*

$$u(x_0, t) = 0$$

*for all $0 \leq t \leq \tau$.*

*proof.* We find a supersolution bounding $u$. Since the partial differential equation (1) is invariant under translation and rotation for the space variable $x$, we assume that $x_0$ is origin, $P = \{x \in \mathbf{R}^n : x_n = 0\}$ and $\Omega(0)$ is contained in the upper half space. We notate $x = (x', x_n)$ and hence $x' = (x_1, x_2, ..., x_{n-1})$. Since $\Omega(0)$ lies in the upper half space, we have for each $x \in \Omega(0)$

$$\text{dist}(x, \Gamma(0)) \leq x_n$$

and from the assumption (4)

$$u_0(x) \leq x_n^{\frac{p}{p-2}}.$$

We choose $M$ so that

$$M \geq u(x,t)$$

for $0 \leq t \leq T_0$, where $T_0$ is a fixed positive time. We define $d = \text{diam}(\Omega(0))$, then $u_0(x', x_n) = 0$ on $\partial B'_d(x') \times (0, d)$ and $u_0(x', 0) = 0$, where $B'_d(x')$ is the $n-1$ dimensional ball centered at $x'$ with radius $d$. By direct computation we have

$$w = b\left(\frac{x_n^p}{T-t}\right)^{\frac{1}{p-2}}$$

is a solution to (1), where

$$b = \left(\frac{p-2}{p}\right)^q \frac{1}{[2(p-1)]^{p-2}}.$$

Fix $\delta > 0$. Now we take $T$ so small that

$$b\left(\frac{\delta^p}{T}\right)^{\frac{1}{p-2}} \geq 2M.$$

Hence we obtain

$$w(x,0) \geq u_0(x)$$

and

$$w(x,t) \geq M$$

for all $(x,t) \in B_d'(x') \times \{\delta\} \times (0,T)$. Therefore from the comparison principle we conclude that

$$w(x,t) \geq u(x,t)$$

for all $(x,t) \in B_d'(x') \times (0,\delta) \times (0,T)$ and in particular

$$w(0,t) = u(0,t) = 0$$

for all $0 \leq t < T$.

Now we find an integral expression which describes the initial behaviour of the interface. From the Harnack type inequalities we prove the following theorem which implies Lemma 1 and Lemma 2.

**Theorem 1.** *Define*

$$I(x) = \sup_R R^{-n-\frac{p}{p-2}} \int_{B_R(x)} u_0(y)dy.$$

*Given $x \in \mathbf{R}^n$ we have $u(x,t) > 0$ for all $t > 0$ if and only if $I(x) = \infty$, that is,*

$$\cap_{t>0}\Omega(t) = \{x : I(x) = \infty\}.$$

*Moreover there exists a constant $c = c(n,p) > 0$ such that $u(x,t) = 0$ for every $(x,t)$ such that*

$$0 < t < cI^{2-p}(x).$$

*proof.* Suppose $I(x) = \infty$. From the Harnack principle(see Corollary 1 in [9]) we have that

$$(5) \quad R^{-n-\frac{p}{p-2}} \int_{B_R(x)} u_0(x)dx \leq c\left(t^{-\frac{1}{p-2}} + t^{\frac{n}{p}} R^{-n-\frac{p}{p-2}} u(x,t)^{\frac{n(p-2)+p}{p}}\right).$$

Hence if $u(x,t) = 0$ for some $t > 0$, then

$$I(x) \leq ct^{-\frac{1}{p-2}}$$

and this contradicts $I(x) = \infty$.

Now we assume $I(x) < \infty$. From Theorem 1 in [12] we know that

$$\sup_{B_\rho} u(x,t) \le ct^{-\frac{n}{\kappa}} \rho^{\frac{p}{p-2}} (I_R(x))^{\frac{p}{\kappa}}$$

for all $\rho > R > 0$ and $0 < t < T(R)$, where

$$I_R(x) = \sup_{\rho > R} \rho^{-n-\frac{p}{p-2}} \int_{B_\rho(x)} u_0(y)dy$$

and

$$T(R) = c\,[I(x)]^{-(p-2)}.$$

Taking $R \to 0$ we are set.

## 3. Hölder regularity of the interface

We show the interface is a Hölder continuous graph as a function of $x$. The Harnack principle, which is proved by DiBenedetto([9]), is a main tool.

**Theorem 2.** *(Harnack principle by DiBenedetto) Let $u$ be a nonnegative weak solution of (1). Let $(x_0, t_0) \in \mathbf{R}^n \times (0,T)$ and $B_R(x_0)$ be the ball of radius $R$ centered at $x_0$. We assume $u(x_0,t_0) > 0$. Then there are constants $c_0$ and $c_1$ depending only on $n$ and $p$ such that*

$$u(x_0,t_0) \le c_0 \inf_{x \in B_R(x_0)} u(x, t_0 + \theta), \tag{6}$$

*where*

$$\theta = \frac{c_1 R^p}{[u(x_0,t_0)]^{p-2}},$$

*provided $t_0 \ge \theta$.*

For the proof of Harnack principle, the fundamental solution $\Phi_{k,\rho}$ to (1) plays a central role(see [9]):

$$\Phi_{k,\rho}(x,t;\bar{x},\bar{t}) = k\rho^n S(t)^{-\frac{n}{\kappa}} \left[1 - \left(\frac{|x-\bar{x}|}{S(t)^{\frac{1}{\kappa}}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}},$$

where

$$S(t) = (\gamma_0(n,p)k^{p-2}\rho^{n(p-2)}(t-\bar{t}) + \rho^\kappa),\ t \ge \bar{t},$$

$$\gamma_0(n,p) = \kappa \left(\frac{p}{p-2}\right)^{p-1}, \quad \kappa = n(p-2)+p.$$

Considering the above fundamental solutions, we can show that if $\overline{\Omega(0)}$ is compact, then $\overline{\Omega(t)}$ is compact for all $t \geq 0$. Moreover an integral estimate independent scaling follows from the Harnack principle and we omit the proof(compare with Corollary 1 in [9]).

**Lemma 3.** *For all $R, \theta > 0$ such that $Q_{2R}(\theta) \subset \mathbf{R}^n \times (0,\infty)$ there holds*
(7)

$$\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u^p(x,t_0)dx \leq B\left(\left(\frac{R^p}{\theta}\right)^{\frac{p}{p-2}} + \left(\frac{\theta}{R^p}\right)^n \left[\inf_{x\in B_R(x_0)} u(x_0,t_0+\theta)\right]^\kappa\right)$$

*for some positive constant $B$ depending only on $n$ and $p$, where the cylinder $Q_{2R}(\theta)$ is defined by*

$$Q_{2R}(\theta) = B_{2R}(x_0) \times (t_0-\theta, t_0+\theta),$$

*and*

$$\kappa = n(p-2)+p.$$

For the Harnack principle the condition that $p > 2$ is rather critical. As the following example shows. Let $\frac{2n}{n+2} < m < 2$ and $u$ be the solution to a Dirichlet problem

$$u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = 0 \text{ in } B_R(0) \times (0,\infty)$$

with initial boundary condition $u(x,0) = u_0(x) \geq 0$ for all $x \in B_R(0)$ and lateral boundary condition $u(x,t) = 0$ for all $(x,t) \in \partial B_R(0) \times (0,\infty)$. Then there is a finite time $T$ depending on $||u_0||_{L^2}$ such that

$$u(x,t) = 0, \text{ for all } (x,t) \in B_R(0) \times (T,\infty).$$

So we can not expect Harnack principle of the form (6).

The monotonicity of the interface follows immediately from the Harnack principle.

**Theorem 3.** *$\Omega(t)$ is monotonically increasing, that is,*

$$\Omega(t_1) \subset \Omega(t_2), \text{ if } 0 < t_1 \le t_2.$$

*proof.* Let $x_0 \in \Omega(t_1)$, then

$$u(x_0, t_1) > 0$$

and there exists a small ball $B_{R_0}(x_0) \subset \Omega(t_1)$. Define

$$\theta_0 = \frac{c_1 R_0^p}{[u(x_0, t_1)]^{p-2}}$$

and we assume $R_0$ is sufficiently small so that

$$t_1 \ge \theta_0.$$

Hence from the Harnack principle we have

$$u(x_0, t_1) \le c_0 \inf_{x \in B_R(x_0)} u(x, t_1 + \theta)$$

for all $R < R_0$, where

$$\theta = \frac{c_1 R^p}{[u(x_0, t_1)]^{p-2}}.$$

Now if

$$t_2 < t_1 + \theta_0,$$

taking $R$ sufficiently small we have

$$u(x_0, t_1) \le c_0 u(x, t_1 + h) \tag{8}$$

for all $0 < h < \theta_0$. We observe that since $p > 2$, $\theta_0$ goes to $\infty$ as $u(x_0, t_1)$ goes to 0. By the maximum principle $u$ is bounded and $\theta_0 > \varepsilon$ for some fixed positive number $\varepsilon$. Therefore if

$$t_2 \ge t_1 + \theta_0,$$

we can iterate (8) and we obtain

$$u(x_0, t_1) \le c_0^k u(x, t_2)$$

for some $k$. Therefore

$$u(x_0, t_2) > 0$$

and $x_0 \in \Omega(t_2)$.

Indeed following an argument of Benilan and Crandall([3] and [11]), Theorem 3 can be proved without referring to the Harnack principle. We can show that the unique solution $v$ with initial datum $v(x,0) = \lambda^{\frac{1}{p-2}} u_0(x), \lambda > 0$ is given by

$$v(x,t) = \lambda^{\frac{1}{p-2}} u(x, \lambda t).$$

If $\lambda > 1$, then $v(x,0) \geq u(x,0)$. Hence from the comparison principle we have $u(x,t) \leq v(x,t)$ for all $(x,t) \in \mathbf{R}^n \times (0,\infty)$. Choosing $\lambda = 1 + \frac{h}{t}$ for small positive number $h$, we obtain

$$\begin{aligned} u(x,t+h) - u(x,t) = u(x,\lambda t) - u(x,t) &= \lambda^{\frac{1}{2-p}} v(x,t) - u(x,t) \\ &\geq \left(\lambda^{\frac{1}{2-p}} - 1\right) u(x,t). \end{aligned}$$

Dividing by $h$ and sending $h$ to 0, we conclude that

$$u_t \geq -\frac{1}{p-2} \frac{u}{t}.$$

and this implies Theorem 3.

Now considering Theorem 1 we can not expect that $\Omega(t)$ is strictly increasing. Indeed, if the interface contains a vertical segment, this segment goes all the way down to $t = 0$(see [5] and [11]). This phenomena appears for the porous medium equations too. Define a cylinder $Q_R^h(x,t)$ by

$$Q_R^h(x,t) = B_R(x) \times (t, t+h).$$

Following a Moser type iteration method we have a local maximum principle.

**Lemma 4.** *Suppose $u(x,t_0) = 0$ for all $x \in B_{R_0}(x_0)$. Then we have*

$$\sup_{Q^h_{\frac{R_0}{2}}(x_0,t_0)} u \leq c \left(\frac{h}{R_0^p}\right)^{\frac{1}{2}} \left(\frac{1}{|Q^h_{R_0}(x_0,t_0)|} \int_{Q^h_{R_0}(x_0,t_0)} u^p \, dz\right)^{\frac{1}{2}} \tag{9}$$

*for some $c$ depending only on $n$ and $p$.*

Since the proof of the lemma 4 follows from the assumption $u(x,t_0) = 0$ for all $x \in B_{R_0}(x_0)$ and a standard iteration, we omit the proof.

Under the assumption of Lemma 4 that $u(x,t_0) \equiv 0$ in $B_{R_0}(x_0)$, it is shown that if the input of the total mass is small, the speed of propagation of the mass is small. Here the Harnack principle is a main tool.

**Lemma 5.** *Suppose that $u(x,t_0)\equiv 0$ in $B_R(x_0)$. Let $h<\frac{t_0}{2}$. There exists a large constant c such that if*

$$\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u^p(x,t_0+h)\,dx \le \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}},$$

*then*

$$u(x,t)\equiv 0$$

*in $B_{\frac{R}{4}}(x_0)\times(t_0,t_0+h)$.*

*proof.* From the maximum principle(see Lemma 4) we have

$$\sup_{Q^h_{\frac{R}{2}}} u \le c\left(\frac{h}{R^p}\right)^{\frac{1}{2}} \sup_{Q^h_R} u^{\frac{p}{2}}.$$

Let $x\in B_{\frac{R}{4}}(x_0)$, then $B_{\frac{R}{4}}(x)\subset B_{\frac{R}{2}}(x_0)$. Set $R_k=\frac{R}{2^k}, k=1,2,3,...,$ then we have

$$M_{k+1}\le c\left(2^{kp}\frac{h}{R^p}\right)^{\frac{1}{2}} M_k^{\frac{p}{2}},$$

where $M_k=\sup_{Q^h_{\frac{R}{2^k}}} u$. Since we are assuming $\frac{p}{2}>1$, we obtain

$$M_k\to 0, \text{ if } M_1<\frac{1}{c\left(\frac{h}{R^p}\right)^{\frac{1}{p-2}}}$$

for some $c$. In other word if

$$\sup_{Q^h_{\frac{R}{2}}} u \le \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{1}{p-2}} \tag{10}$$

for some large $c$, then

$$u(x,t)=0$$

for all $t_0<t<t_0+h$ and all $x\in B_{\frac{R}{4}}(x_0)$.

Now we show (10) is true if

$$\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u^p(x,t_0+h)\,dx \le \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}},$$

for some large $c$. From the Harnack principle we get

$$u(x,t) \le cu(x,t_0+h) \tag{11}$$

for all $t_0 < t < t_0 + h$ and all $x \in B_R$. Considering the maximum principle (9) and the Harnack principle (11) we have

$$\sup_{Q^h_{\frac{R}{2}}} u \le c\left(\frac{h}{R^p}\right)^{\frac{1}{2}} \left(\frac{1}{|Q^h_R|}\int_{Q^h_R} u^p\, dz\right)^{\frac{1}{2}}$$

$$\le c\left(\frac{h}{R^p}\right)^{\frac{1}{2}} \left[\frac{1}{h}\int_{t_0}^{t_0+h} dt \frac{1}{|B_R|}\int_{B_R} u^p(x,t)\, dx\right]^{\frac{1}{2}}$$

$$\le c\left(\frac{h}{R^p}\right)^{\frac{1}{2}} \left[\frac{1}{|B_R|}\int_{B_R} u^p(x,t_0+h)\, dx\right]^{\frac{1}{2}}$$

for some $c$. Hence if

$$\frac{1}{|B_R|}\int_{B_R} u^p(x,t_0+h)\, dx \le \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}}$$

for some large $c$, we obtain

$$M_1 = \sup_{Q^h_{\frac{R}{2}}} u \le c\left(\frac{h}{R^p}\right)^{\frac{1}{2}} \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{p}{2(p-2)}}$$

$$\le \frac{1}{c}\left(\frac{R^p}{h}\right)^{\frac{1}{p-2}}.$$

This implies

$$u(x,t) = 0$$

for all $(x,t) \in B_{\frac{R}{4}} \times (t_0, t_0+h)$ and we completes the proof.

As in the case of porous medium equations we prove that the interface consists of moving part and nonmoving part. We prove this by contradiction. We refer ([5]) for the porous medium equations.

**Lemma 6.** $\Gamma_1 \cup \Gamma_2 = \Gamma$

*proof.* Suppose the assertion is not true. Then for some $(x_0, t_0) \in \Gamma$, there exists $t_1$ and $t_2$ with $0 < t_1 < t_2 < t_0$ such that

$$u(x, t_1) = 0, \text{ for } x \in B_R(x_0)$$

for some $R > 0$ and

$$\sup_{B_{\frac{R}{2}}(x_0)} u(x, t_2) > 0.$$

Furthermore without loss of generality we may assume that

$$s = \frac{t_0 - t_2}{t_2 - t_1}$$

is sufficiently large. Hence from Lemma 5 we conclude that

$$\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u^p(x, t_2)\, dx \geq \frac{1}{c} \left( \frac{R^p}{t_2 - t_1} \right)^{\frac{p}{p-2}}$$

and

$$\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u^p(x, t_2)\, dx \geq \frac{1}{c} \left( \frac{t_0 - t_2}{t_2 - t_1} \right)^{\frac{p}{p-2}} \left( \frac{R^p}{t_0 - t_2} \right)^{\frac{p}{p-2}}.$$

Now we recall Lemma 3 and obtain

$$\frac{1}{c} \left( \frac{t_0 - t_2}{t_2 - t_1} \right)^{\frac{p}{p-2}} \left( \frac{R^p}{t_0 - t_2} \right)^{\frac{p}{p-2}} \leq \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u^p(x, t_2)\, dx$$

$$\leq c \left[ \left( \frac{R^p}{t_0 - t_2} \right)^{\frac{p}{p-2}} + \left( \frac{t_0 - t_2}{R^p} \right)^n u(x_0, t_0)^\kappa \right].$$

Thus if $\frac{t_0 - t_2}{t_2 - t_1}$ is large enough, then have

$$\left( \frac{t_0 - t_2}{R^p} \right)^n u(x_0, t_0)^\kappa \geq c \left( \frac{R^p}{t_0 - t_2} \right)^{\frac{p}{p-2}} > 0$$

and this contradicts the fact that $(x_0, t_0) \in \Gamma$ and $u(x_0, t_0) = 0$.

**Lemma 7.** *$\Gamma_1$ is relatively open in $\Gamma$ and $\Gamma_2$ is relatively closed in $\Gamma$.*

*proof.* We need only to show that $\Gamma_2$ is close. Let $(x_0, t_0)$ be a limit point of $\Gamma_2$, then there is a sequence of points $(x_k, t_k) \in \Gamma_2$ such that

$$(x_k, t_k) \to (x_0, t_0).$$

Since $x_k \in \Gamma(0)$, we note that $x_0 \in \Gamma(0)$. From Lemma 6 we know that $\Gamma_1 \cup \Gamma_2 = \Gamma$. Therefore we conclude that

$$(x_0, t_0) \in \Gamma_2.$$

Now we prove that the rate of the growth of $\Gamma_1$ is Hölder continuous.

**Theorem 4.** *Suppose that $(x_0, t_0) \in \Gamma_1$, that is, the vertical segment does not contain any point of $\Gamma$. Here we assume $t_0$ is a certain positive time. Then there exist constants $c, h$ and $\alpha$ such that*

$$u(x,t) = 0, \text{ for } t_0 - h \le t \le t_0 \text{ and } |x - x_0| \le c(t_0 - t)^\alpha$$

*and*

$$u(x,t) > 0, \text{ for } t_0 < t \le t_0 + h \text{ and } |x - x_0| \le c(t_0 - t)^\alpha.$$

*proof.* Let $t_1 < t_0$ be fixed and $h = t_0 - t_1$. We know that, from Lemma 6, there exists $R$ such that $B_R(x_0) \cap \Omega(t_1) = \emptyset$, that is,

$$u(x, t_1) = 0, \text{ for all } x \in B_R(x_0).$$

Let $t = t_1 + \delta h$, where $\delta$ is fixed later. From Lemma 5, we see that if

$$\operatorname{dist}(x_0, \Omega(t)) < dR,$$

then

$$\frac{1}{|B_{(1-d)R}(x_0)|} \int_{B_{(1-d)R}(x_0)} u^p(x,t)\, dx > \frac{1}{c} \left[ \frac{(1-d)^p R^p}{\delta h} \right]^{\frac{p}{p-2}},$$

where $d < \frac{1}{4}$ is a small number fixed later. Thus we obtain

$$\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u^p(x,t)\, dx > \frac{1}{c} \frac{(1-d)^n (1-d)^{\frac{p^2}{p-2}}}{\delta^{\frac{p}{p-2}}} \left[ \frac{R^p}{h} \right]^{\frac{p}{p-2}}.$$

Again as in the proof of Lemma 6 we have

$$\frac{1}{c}\frac{(1-d)^n(1-d)^{\frac{p^2}{p-2}}}{\delta^{\frac{p}{p-2}}}\left[\frac{R^p}{h}\right]^{\frac{p}{p-2}} \le \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u^p(x,t)\,dx$$

$$\le B\left[\left(\frac{R^p}{t_0-t}\right)^{\frac{p}{p-2}} + \left(\frac{t_0-t}{R^p}\right)^n u(x_0,t_0)^\kappa\right]$$

$$\le B\left[\left(\frac{R^p}{(1-\delta)h}\right)^{\frac{p}{p-2}} + \left(\frac{(1-\delta)h}{R^p}\right)^n u(x_0,t_0)^\kappa\right],$$

where $B$ is the constant appearing in (7). Hence we obtain

$$\left[\frac{1}{c}\frac{(1-d)^n(1-d)^{\frac{p^2}{p-2}}}{\delta^{\frac{p}{p-2}}} - \frac{B}{(1-\delta)^{\frac{p}{p-2}}}\right]\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}}$$

$$\le B\left(\frac{(1-\delta)h}{R^p}\right)^n u(x_0,t_0)^\kappa.$$

On the other hand if $\delta$ is small and $d$ is near 0, then

$$\frac{1}{c}\frac{(1-d)^n(1-d)^{\frac{p^2}{p-2}}}{\delta^{\frac{p}{p-2}}} - \frac{B}{(1-\delta)^{\frac{p}{p-2}}} > 0$$

and this contradicts the fact that

$$u(x_0,t_0) = 0.$$

Thus we have

$$\operatorname{dist}(x_0,\Gamma(t)) \ge dR.$$

We set $d=(1-\delta)^\alpha$. Hence we have

$$\operatorname{dist}(x_0,\Gamma(t_0-(1-\delta)h)) \ge (1-\delta)^\alpha R.$$

Repeating the above process with $t_1=t$, we obtain

$$\operatorname{dist}\left(x_0,\Gamma(t_0-(1-\delta)^2h)\right) \ge (1-\delta)^{2\alpha} R.$$

In a similar way we can iterate the above process for all $k \geq 1$ and we conclude that

$$\text{dist}\left(x_0, \Gamma(t_0 - (1-\delta)^k h)\right) \geq (1-\delta)^{k\alpha} R$$

for all $k$. Varying $h$ we conclude that

$$\text{dist}\left(x_0, \Gamma(t)\right) \geq \left(\frac{t_0 - t}{h}\right)^{\alpha} R$$

and this completes the proof for the first claim. The second claim can be proved in the same way.

From Theorem 1 we know that if

$$u_0(x) \geq [\text{dist}(x)]^{\frac{\gamma}{p-2}}, \ \gamma < p$$

for all $x \in \Omega(0)$, then $\Gamma = \Gamma_1$. Moreover Theorem 4 implies that the interface is given by a function

$$t = S(x)$$

and if $S(x_0) \geq \eta_0$, for some fixed $\eta_0 > 0$, then

$$|S(x) - S(x_0)| \leq c|x - x_0|^{\frac{1}{\alpha}}.$$

for some $c$ depending on $\eta_0$. Hence $\Gamma(t+h)$ is contained in the $\left(ch^{\frac{1}{\alpha}}\right)$ neighborhood of $\Gamma(t)$ for $h < 1$. Now we find a bound for the Hölder exponent of the expansion rate of the interface.

**Theorem 5.** *For any $\eta_0 > 0$ there exists a positive constant $c$ depending only on $p, n, \eta_0$ such that for any $t > \eta_0, 0 < \delta < 1$*

*$\Gamma(t+h)$ is contained in the $\left(ch^{\frac{1}{p}}\right)$ neighborhood of $\Gamma(t)$.*

*proof.* Suppose that $u(x_0, t_0) = 0$ and $\text{dist}\left(x_0, \Gamma(t_0)\right) = a$. Let

$$v(x,t) = \lambda\left([\alpha^p(t - t_0) + \alpha(|x - x_0| - b)]^+\right)^q,$$

where $q = \frac{p-1}{p-2}$ and $\alpha$ and $b$ are decided later. Observe that

$$(q-1)(p-1) = q.$$

From a direct computation we obtain

$$v_t - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)$$

$$= \lambda q \alpha^p \left([\alpha^p(t-t_0) + \alpha(|x-x_0|-b)]^+\right)^{\frac{1}{p-2}} \cdot$$

$$\left(1 - \lambda^{p-2}q^{p-1} - \lambda^{p-2}q^{p-2}\frac{(n-1)}{\alpha|x-x_0|}\left[\alpha^p(t-t_0)+\alpha(|x-x_0|-b)\right]^+\right).$$

Thus if

$$1 - \lambda^{p-2}q^{p-1} - \lambda^{p-2}q^{p-1}\frac{(n-1)}{|x-x_0|}\left[\alpha^{p-1}(t-t_0)+(|x-x_0|-b)\right]^+ \geq 0,$$

that is,

$$\lambda^{p-2}q^{p-1} + \lambda^{p-2}q^{p-1}\frac{(n-1)}{|x-\ {}_0|}\left[\alpha^{p-1}(t-t_0)+(|x-x_0|-b)\right]^+ \leq 1, \tag{12}$$

then $v$ is a supersolution.

Now we take $\alpha$ and $b$ such that

$$\alpha^q(a-b)^q = \frac{M}{\lambda},$$

where

$$M = \sup u.$$

With this choice of $\alpha, \lambda$ and $b$ we find that $v(x,t) \geq u(x,t)$ for all $x$, $|x-x_0| = a$ and $t_0 \leq t \leq t_1$, where $t_1$ is a certain fixed time. By the usual comparison principle we see that

$$u(x,t) \leq v(x,t)$$

for all $x \in B_a(x_0)$ and $t_0 \leq t \leq t_1$ Note that the interface of $v(x, t_0+h)$ is decided by

$$|x-x_0| = b - \alpha^{p-1}h.$$

We take $\alpha$ satisfying

$$\frac{1}{\alpha}\left(\frac{M}{\lambda}\right)^{\frac{1}{q}} = \alpha^{p-1}h$$

and hence

$$\alpha = \left(\frac{1}{h}\right)^{\frac{1}{p}}\left(\frac{M}{\lambda}\right)^{\frac{1}{pq}}.$$

Therefore the interface of $v(x, t_0 + h)$ is

$$|x - x_0| = b - \alpha^{p-1}h = a - \frac{1}{\alpha}\left(\frac{M}{\lambda}\right)^{\frac{1}{q}} - \alpha^{p-1}h$$

$$= a - 2\left(\frac{M}{\lambda}\right)^{\frac{p-1}{pq}} h^{\frac{1}{p}}.$$

Taking $\lambda$ so small that

$$\lambda^{p-2}q^{p-1} + \lambda^{p-2}q^{p-1}\frac{(n-1)}{|x-x_0|}\left[\alpha^{p-1}(t-t_0) + (|x-x_0| - b)\right]^+ \leq 1,$$

we see that $\Gamma(t_0 + h)$ is contained in the $ch^{\frac{1}{p}}$ neighborhood of $\Gamma(t_0)$.

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POSTECH

# ON THE VARIATIONAL INEQUALITIES FOR CERTAIN CONVEX FUNCTION CLASSES

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ABSTRACT. The existence and the $C^{1,\alpha}$ regularity of the weak soluti on to the variational inequality

$$-(a_i(x,u,\nabla u))_{x_i} - (g_i(x,u))_{x_i} + b(x,u,\nabla u) \geq 0$$

with respect to a closed convex function class is proved. For the r egularity, we used the fact that the regularity for the viscosity s olutions to the Hamilton-Jacobi equations implies the $C^{1,\alpha}$ interior regularity of the solution to the bilateral obstacle pr oblem which in turn gives that of the solution to the variational i nequality.

## 1. INTRODUCTION

In this paper we consider quasilinear variational inequalities for certain convex function classes. We show the existence and $C^{1,\alpha}$ regularity for the solutions of nonlinear variational inequalities with some general constraints on functions and their gradients. Such a variational inequality arises in elasto-plasticity and optimal control problems. As a canonical example we might consider a minimization problem such that

$$\min \int_\Omega |\nabla u|^2 - fu \; dx$$

with respect to a function class $K = \{u \in W_0^{1,2}(\Omega) + u_0 : G(\nabla u) \leq 0\}$, where $G$ is a convex function.

Brezis and Stampacchia[Br1] considered the case $G(A) = |A|^2 - 1$ and proved $W^{2,p}$ regularity for solutions. We also recall that Gerhardt proved $W^{2,p}$ regularity for a quasilinear operator. In [Evans] Evans studied the

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problem of solving a linear second order elliptic variational inequality with a function class $K = \{|\nabla u| \leq g\}$ for some smooth function $g$. He proved $W^{2,p}$ regularity for solutions and $W^{2,\infty}$ regularity for restricted cases. His result for $W^{2,\infty}$ regularity was extended later by Wiegner[Wieg]. On the other hand Ishii and Koike[Ish] considered the existence and uniqueness of the solutions of the variational inequalities of the forms which are considered by Evans. Caffarelli and Riviere[Caff] proved $W^{2,\infty}$ regularity for elasto-plastic problem such as the canonical example using apriori estimate on the free boundary. Finally Choe[Choe] showed $W^{2,\infty}$ regularity for a quasilinear operator without nonhomogeneous term under some general setting on the constraint.

Now we state the problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^2$ boundary. Let

$$G(A) : \mathbb{R}^n \to \mathbb{R}$$

be a $C^2$ convex function and strictly convex on $A$ such that

$$[G_{A_i}(A_1) - G_{A_i}(A_2)]\,[A_{1,i} - A_{2,i}] \geq c|A_1 - A_2|^2 \tag{1}$$

for all $A_1, A_2$ and for some positive constant $c$. Let $u_0$ be a $C^2(\overline{\Omega})$ function and

$$G(\nabla u_0(x)) \leq 0$$

for all $x \in \overline{\Omega}$. We define a closed convex function class $K$ by

$$K = \{v \in W_0^{1,2}(\Omega) + u_0 : G(\nabla v) \leq 0\}$$

that is nonempty since $u_0 \in K$.

Suppose that $\{a_i, i = 1, ..., n\}$ are functions

$$a_i(x, u, A) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \; i = 1, ..., n$$

satisfying

i) $a_i(x, u, A)$ are Hölder continuous in $x$ for all $(u, A)$, that is,

$$|a_i(x, u, A) - a_i(y, u, A)| \leq c|x - y|^\alpha$$

for all $x, y \in \Omega$, for some $\alpha > 0$, $c$ and for all $(u, A) \in \mathbb{R} \times \mathbb{R}^n$

ii) $a_i(x, u, A)$ are Hölder continuous in $u$ for all $A$ and for all $x \in \Omega$, that is,

$$|a_i(x, u, A) - a_i(x, v, A)| \leq c|u - v|^\alpha$$

for some $\alpha > 0$, for some $c$, for all $A \in \mathbb{R}^n$, and for all $u, v \in \mathbb{R}$ and for all $x \in \Omega$

iii) $a_i(x, u, A)$ are $C^1$ function in $A \in \mathbb{R}^n$ for all $u \in \mathbb{R}$ and for all $x \in \Omega$ with the ellipticity condition

$$a_{i,A_j}(x, u, A)\xi_i\xi_j \geq \lambda|\xi|^2$$

for some positive constant $\lambda$ and for all $x \in \Omega$, for all $u \in \mathbb{R}$ and for all $A, \xi \in \mathbb{R}^n$.

Note that we don't assume any growth condition on $a_i$ as $|(u, A)|$ goes to $\infty$.

Suppose that

$$g_i(x, u) : \Omega \times \mathbb{R} \to \mathbb{R}, \ i = 1, 2, ..., n$$

are Hölder continuous such that

$$|g_i(x, u) - g_i(y, v)| \leq c(|x - y|^\alpha + |u - v|^\alpha), \ i = 1, ..., n.$$

for some $c$ and for some $\alpha > 0$.

Also suppose that

$$b(x, u, A) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

is bounded if $|(u, A)|$ is bounded, that is, if $|(u, A)| \leq M$, then there is a constant $c(M)$ such that

$$|b(x, u, A)| \leq c(M)$$

for almost all $x \in \Omega$. We say that $u \in K$ is a weak solution to

$$-(a_i(x, u, \nabla u))_{x_i} - (g_i(x, u))_{x_i} + b(x, u, \nabla u) \geq 0 \tag{2}$$

if $u$ satisfies

$$\int_\Omega a_i(x, u, \nabla u)(v - u)_{x_i} + g_i(x)(v - u)_{x_i} + b(x, u, \nabla u)(v - u)dx \geq 0$$

for all $v \in K$.

The following theorem is our main result in this paper.

**Theorem 1.** *There exists a weak solution $u \in K$ to* (2). *Furthermore*

$$u \in C^{1,\alpha}(\overline{\Omega})$$

*for some $\alpha > 0$.*

We describe the outline of the proof. For the proof of $C^{1,\alpha}$ regularity in theorem 1, we employ a comparison method suitable for using Campanato space. To show the interior regularity we consider a comparison function which is a solution to a very nice differential operator with the same type of constraint. The existence and uniqueness property of the comparison function follows from the monotone operator theory. For the regularity of the comparison function we consider a bilateral obstacle problem where the obstacles are defined by the solutions of the vanishing viscosity equations. Sending the viscosity term to zero, we conclude that the bilateral obstacles converge uniformly to the viscosity solutions to certain Hamilton-Jacobi equations. In fact, the Perron process for the viscosity solutions to Hamilton-Jacobi equations, discovered by Ishii[Ishii], characterizes the upper and lower envelopes for the function class $K$. Furthermore, the semiconcavity and semiconvexity regularity for the viscosity solutions to Hamilton-Jacobi equations is translated to $C^{1,\alpha}$ regularity in the interior to the solutions of the bilateral obstacle problems. We then use a maximum principle to show that the solution to the bilateral obstacle problem, where obstacles are characterized by the viscosity solutions to certain Hamilton-Jacobi equations, is the solution to the variational inequality with a nice differential operator. Now the usual comparison argument shows that the solution to (2) is $C^{1,\alpha}$ in the interior.

Near the boundary, similarly, we follow a comparison argument in which comparison functions come from the variational inequalities with a nice differential operator. We show by the maximum principle that the solution to the variational inequality with respect to $K$ is the solution to the bilateral obstacle problems. For the regularity of the comparison functions, we use the fact that near the boundary, the viscosity solution to Hamilton-Jacobi equation can be characterized using the characteristic method if the boundary and the boundary data are smooth enough, that is $C^2$. Hence $C^2$ regularity for the viscosity solutions near the boundary follows immediately. Consequently we can show again the solution to the bilateral obstacle problem is a $C^{1,\alpha}$ function near the boundary. Hence we can proceed to show that the solution $u$ has a Campanato type growth condition near the boundary by the usual comparison argument.

Once we have a priori $C^{1,\alpha}(\overline{\Omega})$ regularity, the existence result follows from Leray-Schauder's fixed point theorem.

The following symbols will be used.

$$x_0 : \text{ a generic point}$$

$$|E| : \text{ the Lebesgue measure of } E$$

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

$$(w)_{R,x_0} = \frac{1}{|B_R(x_0) \cap \Omega|} \int_{B_R(x_0)\cap\Omega} w dx$$

$$W^{1,p}(\Omega) : \text{ the Sobolev space with } L^p \text{ norm}$$

$$W_0^{1,p}(\Omega) : \text{ the closure of } C_0^\infty(\Omega) \text{ in } W^{1,p}(\Omega)$$

$$\|w\|_L : \text{ the } L\text{-norm of } w \text{ in } \Omega$$

$$w_\eta : \text{ the directional derivative of } w \text{ along } \eta$$

If there is no confusion, we drop out the generic point $x_0$ in various expressions. As usual the double indices mean summation up to $n$.

## 2. Interior regularity

In this section we use the solutions to vanishing viscosity equations to approximate the bilateral obstacle problems, where obstacles are defined using the viscosity solutions to vanishing viscosity equations. Indeed, sending the viscosity term to zero we prove the local $C^{1,\alpha}$ regularity when obstacles are solutions to certain Hamilton-Jacobi equations.

Let $B_R \subset \Omega$ and $w_0 \in W^{1,\infty}(B_R)$. Moreover assume that $G(\nabla w_0) \leq 0$. Suppose $w = w^{+,\mu,\varepsilon}$ is the unique solution to the vanishing viscosity equation

$$L^{+,\mu,\varepsilon}(w) = -\varepsilon\Delta w + G(\nabla w) = 0 \tag{3}$$

with $w = w_0 + \mu$ on $\partial B_R$ for some small positive $\mu$ and $\varepsilon$. The existence and uniqueness of such solutions to vanishing viscosity equations are known by a result of Lions(see [Lions2]).

**Lemma 1.** *There exists a unique solution $w$ to (3) such that $w$ is Lipschitz with*

$$||w||_{W^{1,\infty}} \leq c$$

*for some $c$ independent of $\mu$ and $\varepsilon$. Furthermore for all $x \in B_{(1-\delta)R}$, $w$ is semiconcave and*

$$\frac{\partial^2 w}{\partial \zeta^2}(x) \leq c, \text{ for all } |\zeta| \leq 1 \tag{4}$$

*for some $c$ independent of $\mu$ and $\varepsilon$.*

For the proof of lemma 1 we refer to theorem 2.2 in [Lion1]. Observe that the strict convexity condition

$$G_{A_i A_j}(A)\xi_i\xi_j \geq c|\xi|^2$$

for some $c > 0$ and for all $\xi \in \mathbb{R}^n$ is needed to assure the semiconcavity result (4) for the viscosity solutions. Let $w^{+,\mu,0}$ be the viscosity solution to

$$G(\nabla w^{+,\mu,0}) = 0$$

with the boundary condition $w^{+,\mu,0} = w_0 + \mu$ on $\partial B_R$.

The existence of such a viscosity solution $w^{+,\mu,0}$ can be proved by the Perron process[Ishii] or sending $\varepsilon$ to zero in (3). When $\min G < 0$, we have a comparison principle proved by Ishii(see theorem 1 in [Iishii]) and the uniqueness follows immediately. Otherwise, that is, $\min G = 0$, then the uniqueness follows from the fact that the minimizer of $G$ is unique. The following lemma is in [Lion1].

**Lemma 2.** *As $\varepsilon$ goes to zero, $w^{+,\mu,\varepsilon}$ converges to $w^{+,\mu,0}$ uniformly in $C(\overline{B_{(1-\delta)R}})$ for each $\delta > 0$ and $w^{+,\mu,0} \geq v$ for all viscosity subsolutions $v$ such that*

$$G(\nabla v) \leq 0$$

*with $v = w_0 + \mu$ on $\partial B_R$.*

Similarly, we can find $w^{-,\mu,\varepsilon}$ as the solution to the vanishing viscosity equation

$$-\varepsilon \Delta w - G(\nabla w) = 0$$

where $w = w_0 - \mu$ on $\partial B_R$ for each positive $\mu$ and $\varepsilon$. Hence we get the following lemma.

**Lemma 3.** *As $\varepsilon$ goes to zero, $w^{-,\mu,\varepsilon}$ converges uniformly in $C(\overline{B_{(1-\delta)R}})$ for each $\delta > 0$ to the viscosity solution $w^{-,\mu,0}$ of*

$$-G(\nabla w^{-,\mu,0}) = 0$$

*with $w^{-,\mu,0} = w_0 - \mu$ on $\partial B_R$ and*

$$||w^{-,\mu,\varepsilon}||_{W^{1,\infty}} \leq c$$

*for some $c$ independent of $\varepsilon$ and $\mu$. Furthermore, for all $x \in B_{(1-\delta)R}$ $w$ is semiconvex*

$$\frac{\partial^2 w^{-,\mu,\varepsilon}}{\partial \zeta^2}(x) \geq c, \text{ for all } |\zeta| \leq 1$$

*for some $c$ independent of $\mu$ and $\varepsilon$.*

Since $w^{-,\mu,0}$ is Lipschitz and $-G(\nabla w^{-,\mu,0}) = 0$ in the viscosity sense, we see that $G(\nabla w^{-,\mu,0}) = 0$ a.e. and hence by the convexity of $G$ that $G(\nabla w^{-,\mu,0}) \leq 0$ in the viscosity sense. By the comparison argument as in lemma 2 we see that each given $\mu > 0$,

$$w^{-,\mu,\varepsilon} < w^{+,\mu,\varepsilon} \text{ and } w^{-,\mu,0} < w^{+,\mu,0}$$

if $\varepsilon$ is sufficiently small compared to $\mu$.

Now we consider a bilateral obstacle problem. Let us define the function class $J_R^{\mu,\varepsilon}$ by

$$J_R^{\mu,\varepsilon} = \{v \in W_0^{1,2}(B_R) + w_0 : w^{-,\mu,\varepsilon} \leq v \leq w^{+,\mu,\varepsilon}\}$$

for each given positive $\mu$ and $\varepsilon$. Since $w^{-,\mu,\varepsilon}$ and $w^{+,\mu,\varepsilon}$ converge uniformly to $w^{-,\mu,0}$ and $w^{+,\mu,0}$ respectively, then for given $\mu > 0$

$$w^{-,\mu,\varepsilon} < w^{+,\mu,\varepsilon}$$

for sufficiently small $\varepsilon > 0$ compared to $\mu$. We note that $w_0 \in J_R^{\mu,\varepsilon}$.

For notational convenience, we write

$$-(a_i(\nabla w))_{x_i} = -(a_i(x_0, w_0(x_0), \nabla w))_{x_i}$$

and assume that $w = w^{\mu,\varepsilon} \in J_R^{\mu,\varepsilon}$ be the solution to

$$-(a_i(\nabla w))_{x_i} \geq 0$$

with respect to the function class $J_R^{\mu,\varepsilon}$. The following lemma is essential in our comparison argument.

**Lemma 4.** *If $B_R \subset \Omega$ and $\rho \leq \frac{R}{4}$, then the estimate*

$$(5) \qquad \int_{B_\rho} |\nabla w - (\nabla w)_\rho|^2 dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w - (\nabla w)_R|^2 dx + cR^{n+\sigma}$$

*holds for some $c$ independent of $\mu$ and $\varepsilon$ and for all $\sigma \in (0,2)$. In particular, $\nabla w$ is locally Hölder continuous in $B_R$ with*

$$||w||_{C^{1,\alpha}_{loc}} \leq c$$

*for some $c$ independent of $\mu$ and $\varepsilon$.*

*proof.* We follow a penalization method. Let $\beta_1$ be a nondecreasing smooth function such that

$$0 < \beta_1(t) < 5t, \text{ if } t > 0$$

$$\beta_1(t) = 0, \text{ if } t \leq 0$$

$$\beta_1'(t) \geq 0$$

$$\beta_1''(t) \geq 0.$$

Similarly we define $\beta_2$ by a nonincreasing smooth function satisfying

$$5t < \beta_2(t) < 0, \text{ if } t < 0$$

$$\beta_2(t) = 0, \text{ if } t \geq 0$$

$$\beta_2'(t) \geq 0$$

$$\beta_2''(t) \leq 0.$$

We approximate our problem. Fix $p \in \mathbb{R}^n$ so that $G(p) \leq 0$ and let $0 < \theta < 1$. Define $w_\theta(x) = \theta p \cdot x + w_0((1-\theta)x)$. Then $w_\theta$ is defined in $B_{\frac{R}{1-\theta}}$ and satisfies $G(\nabla w_\theta) \leq 0$ a.e. Also, $w_\theta \to w_0$ uniformly in $B_R$ as $\theta \to 0$. Moreover we approximate $w_\theta$ by a smooth function using the mollification technique and we denote the smooth function again $w_\theta$. Let $v = v_\theta^{\mu,\varepsilon,\tau} \in W_0^{1,2}(B_R) + w_\theta$ be the solution to the penalized equation

$$(6) \qquad L^{\mu,\varepsilon,\tau}(v) = -(a_i(\nabla v))_{x_i} + \frac{\beta_1(v - w^{+,\mu,\varepsilon})}{\tau} + \frac{\beta_2(v - w^{-,\mu,\varepsilon})}{\tau} = 0$$

for a given small positive number $\tau$. Existence and uniqueness can be proved from the monotone operator theory(see [Har1]). Since all the following estimate is independent of $\theta$, we omit $\theta$ in the various expressions from now on. Taking some large number $c$ so that

$$||w^{+,\mu,\varepsilon}||_{W^{1,\infty}(B_R)} + ||w^{-,\mu,\varepsilon}||_{W^{1,\infty}(B_R)} \leq c$$

as a supersolution to the operator $L^{\mu,\varepsilon,\tau}$, we can prove that $v$ is bounded from above. In a similar way $v$ can be shown to be bounded from below. Hence we conclude that

$$(7) \qquad ||v||_{L^\infty} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$.

Next we estimate the Lipschitz norm of $v$. Let $\rho = \text{dist}(x, \partial B_R)$ and $\phi^+ = w_0 + \nu\rho - \nu\rho^{\frac{3}{2}}$ for some large $\nu$. We know already that

$$|\nabla\rho| = 1 \text{ and } |\nabla^2\rho| \leq c$$

near the boundary of $B_R$. Therefore we see that

$$\phi^+ = v \text{ on } \partial B_R \text{ and } \phi^+ \geq v \text{ on } \partial B_{(1-\delta)R}$$

for some small $\delta$ when $\nu$ is sufficiently large. Since for some large $\nu$

$$L^{\mu,\varepsilon,\tau}(\phi^+) \geq 0,$$

that is, $\phi^+$ is a supersolution to $L^{\mu,\varepsilon,\tau}$, we conclude that

$$v \leq \phi^+.$$

Similarly we have

$$v \geq \phi^-$$

for some $\phi^-$ and we see that

$$||\nabla v||_{L^\infty(\partial B_R)} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$. Since $L^{\mu,\varepsilon,\tau}$ is a monotone operator, then $|\nabla v|^2$ satisfies the weak maximum principle. Hence it follows that

$$(8) \qquad ||\nabla v||_{L^\infty(B_R)} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$.

Now we apply the $L^p$-theory for quasilinear elliptic equation. First we prove that

$$||\frac{\beta_1}{\tau}||_{L^p} + ||\frac{\beta_2}{\tau}||_{L^p} \leq c \tag{9}$$

for all $p$ and for some $c$ independent of $\mu, \varepsilon, \tau$ and $p$. With this $L^p$ estimate on $\frac{\beta_1}{\tau}$ and $\frac{\beta_2}{\tau}$, we conclude that $v$ is in $W^{2,p}_{loc}(B_R)$ and $\nabla v$ is in $C^{1,\alpha}$ for all $\alpha \in (0,1)$ with Hölder norm independent of $\mu, \varepsilon$ and $\tau$. To see (9), let us choose a nonnegative smooth cutoff function $\eta$ so that

$$\eta = 1 \text{ in } B_{(1-\delta)R},$$

$$\eta = 0 \text{ in } \partial B_{(1-\delta/2)R}$$

and

$$|\nabla \eta| \leq \frac{c}{\delta R}$$

for some $c$ and appropriate $\delta$.

Applying $\left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p$ as a test function to (6), we get

(10)

$$\int a_i(\nabla v) \left[ \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \right]_{x_i} dx + \int \left(\frac{\beta_1}{\tau}\right)^p \eta^p dx + \int \left(\frac{\beta_2}{\tau}\right) \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \, dx = 0.$$

Note that

$$\left(\frac{\beta_1}{\tau}(t)\right) \left(\frac{\beta_2}{\tau}(t)\right) = 0$$

for all $t$. Subtracting

$$\int a_i(\nabla w^{+,\mu,\varepsilon}) \left[ \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \right]_{x_i} dx$$

from the both sides of (10), we have

$$(11) \quad (p-1) \int \left[a_i(\nabla v) - a_i(\nabla w^{+,\mu,\varepsilon})\right] (v - w^{+,\mu,\varepsilon})_{x_i} \left(\frac{\beta_1}{\tau}\right)^{p-2} \frac{\beta_1'}{\tau} \eta^p dx$$

$$+p \int \left[a_i(\nabla v) - a_i(\nabla w^{+,\mu,\varepsilon})\right] \eta_{x_i} \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^{p-1} \, dx + \int \left(\frac{\beta_1}{\tau}\right)^p \eta^p \, dx$$

$$= \int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon}) w^{+,\mu,\varepsilon}_{x_i x_j} \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \, dx.$$

From the ellipticity condition for $a_i$ we see that

$$\int \left[a_i(\nabla v) - a_i(\nabla w^{+,\mu,\varepsilon})\right] (v - w^{+,\mu,\varepsilon})_{x_i} \left(\frac{\beta_1}{\tau}\right)^{p-2} \frac{\beta_1'}{\tau} \eta^p \, dx \geq 0.$$

Therefore we have

(12)
$$\int \left(\frac{\beta_1}{\tau}\right)^p \eta^p \, dx \leq \frac{c}{\delta R} \int \left(||\nabla v||_{L^\infty} + ||\nabla w^{+,\mu,\varepsilon}||_{L^\infty}\right) \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^{p-1} \, dx$$

$$+ \int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon}) w^{+,\mu,\varepsilon}_{x_i x_j} \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \, dx.$$

Since $w^{+,\mu,\varepsilon}$ is semiconcave(see lemma 1), we have

$$w^{+,\mu,\varepsilon}_{x_i x_j} \xi_i \xi_j \leq c|\xi|^2$$

for some $c$ independent of $\mu$ and $\varepsilon$. Moreover, $a_{i,A_j}$ is a positive definite matrix. Therefore,

$$a_{i,A_j}(\nabla w^{+,\mu,\varepsilon}) w^{+,\mu,\varepsilon}_{x_i x_j} \leq c_1$$

for some $c_1$ independent of $\mu$ and $\varepsilon$. It follows that

$$(13) \qquad \int a_{i,A_j}(\nabla w^{+,\mu,\varepsilon}) w^{+,\mu,\varepsilon}_{x_i x_j} \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \, dx$$

$$\leq \int c_1 \left(\frac{\beta_1}{\tau}\right)^{p-1} \eta^p \, dx$$

$$\leq \frac{1}{4} \int \left(\frac{\beta_1}{\tau}\right)^p \eta^p dx + cR^n$$

for some $c$. Using Young's inequality on the first term of the right hand side of (12) and the estimate (13) we have

$$||\frac{\beta_1}{\tau}||_{L^p(B_{(1-\delta)R})} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$. Similarly we also have

$$||\frac{\beta_2}{\tau}||_{L^p(B_{(1-\delta)R})} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$. This proves (9).

From the classical $L^p$ theory for quasilinear equations we conclude that

$$v^{\mu,\varepsilon,\tau} \in W^{2,p}_{loc}$$

and

$$||v^{\mu,\varepsilon,\tau}||_{W^{2,p}_{loc}} \leq c$$

for some $c$ independent of $\mu, \varepsilon$ and $\tau$. Once we have $v^{\mu,\varepsilon,\tau} \in W^{2,p}_{loc}$, we see immediately the following Campanato growth condition for $v$

$$\int_{B_\rho} |\nabla v^{\mu,\varepsilon,\tau} - (\nabla v^{\mu,\varepsilon,\tau})_\rho|^2 \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla v^{\mu,\varepsilon,\tau} - (\nabla v^{\mu,\varepsilon,\tau})_R|^2 \, dx + cR^{n+\sigma}$$

for all $\sigma \in (0,2)$, for all $\rho \leq \frac{R}{2}$ and for some $c$ independent of $\mu, \varepsilon$ and $\tau$.

Sending $\tau$ to zero and using Minty's lemma ([Chipot]), we conclude that the unique solution to (6) is in $W^{2,p}_{loc}$ and satisfies the Campanato type growth condition

$$\int_{B_\rho} |\nabla w^{\mu,\varepsilon} - (\nabla w^{\mu,\varepsilon})_\rho|^2 \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w^{\mu,\varepsilon} - (\nabla w^{\mu,\varepsilon})_R|^2 \, dx + cR^{n+\sigma}$$

for all $\sigma \in (0,2)$, for all $\rho \leq \frac{R}{2}$ and for some $c$ independent of $\mu$ and $\varepsilon$.

Sending $\varepsilon$ to zero we have that the unique solution $w^{\mu,0}$ to the variational inequality

$$-(a_i(\nabla w^{\mu,0}))_{x_i} \geq 0 \tag{14}$$

with respect to $J^{\mu,0}_R$ is in $W^{2,p}_{loc}$ and satisfies the Campanato type growth condition (5).

Set $w^+$ as the viscosity solutions to Hamilton-Jacobi equation

$$G(\nabla w^+) = 0, \; w^+ = w_0 \text{ on } \partial B_R$$

and $w^-$ as the viscosity solution to

$$-G(\nabla w^-) = 0 \; w^- = w_0 \text{ on } \partial B_R.$$

Also set $J_R = \{v \in W_0^{1,\infty}(B_R) + w_0 : w^- \leq v \leq w^+\}$. Suppose $w$ is the unique solution to the variational inequality

$$-(a_i(\nabla w))_{x_i} \geq 0 \tag{15}$$

with respect to $J_R$. Then it is easy to see that

$$w^{\mu,0} \rightharpoonup w \text{ weakly in } W^{2,p}$$

and

$$w^{\mu,0} \to w \text{ strongly in } W^{1,p}$$

for all $p$ as $\mu$ goes to zero. Hence we have the following lemma.

**Lemma 5.** *If $w$ is the unique solution to* (15) *with respect to $J_R$, then $w$ satisfies*

$$\int_{B_\rho} |\nabla w - (\nabla w)_\rho|^2 \, dx \leq c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R} |\nabla w - (\nabla w)_R|^2 \, dx + cR^{n+\sigma} \tag{16}$$

*for all $\sigma \in (0,2)$, for all $\rho \leq \frac{R}{2}$ and for some $c$ independent of $\rho$ and $R$.*

Since the constant $c$ in lemma 5 goes to $\infty$ as $\sigma$ goes to 2, we don't have $C^{1,1}_{loc}$ regularity for $w$. But we recall that Choe[Choe1] proved $C^{1,1}_{loc}$ regularity for homogeneous variational inequalities of the form (15) with gradient constraints employing the truncation idea of DeGiorgi.

Now we want to show that the solution $w$ to the variational inequality (15) with respect to $J_R$ is indeed the unique solution to the variational inequality (15) with respect to a function class $K_R$, where $K_R$ is defined by

$$K_R = \{v \in W_0^{1,\infty}(B_R) + w_0 : G(\nabla v) \leq 0\}.$$

We note that $K_R \subset J_R$. Hence if we show that $w \in K_R$, that is, $G(\nabla w) \leq 0$, then $w$ is the solution to the variational inequality (15) with respect to $K_R$.

We define contact sets $I_R^-$ and $I_R^+$ by

$$I_R^- = \{x \in B_R : w(x) = w^-(x)\}$$

and

$$I_R^+ = \{x \in B_R : w(x) = w^+(x)\}.$$

We also define $I_R$ by

$$I_R = I_R^- \cup I_R^+.$$

Since $G$ is a $C^2$ convex function, we have the following maximum principle.

**Lemma 6.** *We have that*

$$\max_{B_R \setminus I_R} G(\nabla w) \leq 0. \tag{17}$$

Note that $w$ satisfies the strict elliptic equations

$$-(a_i(\nabla w))_{x_i} = 0$$

in $B_R \setminus I_R$. Hence we see that that $G(\nabla w)$ is a subsolution to

$$-(a_{i,A_j}(\nabla w)G_{x_j})_{x_i} \leq 0$$

in $B_R \setminus I_R$. We omit this rather a direct computation.

As in the proof of lemma 4 we regularize $w_0$ by $w_\theta$ and the using the regularity result in section 3 we can assume that $w_\theta$ is differentiable on $\partial B_{\frac{R}{1-\theta}}$ and $G(\nabla w_\theta)$ is continuous in $\bar{B}_{\frac{R}{1-\theta}}$. Since all the estimate is independent of $\theta$, we omit $\theta$ in various expressions. Since $w^+(x) = w^-(x) = w(x)$ for all $x \in \partial B_R$ and $w^-(x) \leq w(x) \leq w^+(x)$ for all $x \in B_R$, we have that

$$\frac{\partial w^+(x)}{\partial \eta} \leq \frac{\partial w(x)}{\partial \eta} \leq \frac{\partial w^-(x)}{\partial \eta}$$

for all $x \in \partial B_R$, where $\eta$ is the outward normal vector at $x \in \partial B_R$. Since

$$\frac{\partial w^+(x)}{\partial \tau} = \frac{\partial w^-(x)}{\partial \tau} = \frac{\partial w(x)}{\partial \tau}$$

for all tangent vector $\tau$ at $x \in \partial B_R$, we have for each $x \in \partial B_R$

$$\nabla w(x) = t\nabla w^+(x) + (1-t)\nabla w^-(x)$$

for some $t \in [0,1]$. Since $G$ is convex, we obtain

$$G(\nabla w(x)) \leq tG(\nabla w^+(x)) + (1-t)G(\nabla w^-(x)) \leq 0$$

for all $x \in \partial B_R$.

Now we show that

$$G(\nabla w(x)) \leq 0$$

on $\partial I_R$. Recall that $w^+$(resp. $w^-$) is semiconcave(resp. semiconvex). Then we find that for each $x \in I_R^+ \cap B_R$(resp. $x \in I_R^- \cap B_R$) $w^+$(resp. $w^-$) is differentiable. For instance, if $x \in I_R^+ \cap B_R$, then $w^+$ is superdifferentiable since it is semiconcave, and also subdifferentiable since $w$ is in $C^1$ and $w^+ - w$ attains a minimum. Once we know the differentiablity we see that $G(\nabla w) \leq 0$ on $I_R \cap B_R$, which is enough to apply the maximum principle. Indeed, if $x \in I_R^+ \cap B_R$, then $G(\nabla w^+(x)) = G(\nabla w(x)) = 0$ since $w^+$ is a viscosity solution. Similarly we see that $G(\nabla w^-(x)) = G(\nabla w(x)) = 0$ for $x \in I_R^- \cap B_R$.

## 3. Boundary regularity for simple case

In this section we show that a Campanato type growth condition holds for solutions near the boundary for simple case.

Let $x_0 \in \partial\Omega$ and $\partial\Omega$ be $C^3$. Also let $w_0$ be a Lipschitz function in $B_R \cap \Omega$ and a $C^2$ function on $\partial\Omega \cap B_R$. From the Perron process we know that the viscosity solution $w^+$ to Hamilton-Jacobi equation

$$G(\nabla w^+) = 0,\ w^+ = w_0 \text{ on } \partial(\Omega \cap B_R) \tag{18}$$

can be characterized by

$$w^+(x) = \sup\{v(x) : v = w_0 \text{ on } \partial(\Omega \cap B_R),$$

$$v \text{ is a viscosity subsolution of } G(\nabla v) = 0\}.$$

For all subsolutions $v$ of $G(\nabla v) = 0$ with $v \le w_0$ on $\partial(B_R \cap \Omega)$, it holds that

$$w^+ \ge v.$$

Similarly we find a viscosity solution $w^-$ to

$$-G(\nabla w^-) = 0,\ w^- = w_0 \text{ on } \partial(B_R \cap \Omega)$$

and for all supersolutions $v$ of $-G(\nabla v) = 0$ with $v \ge w_0$ on $\partial(B_R \cap \Omega)$, it holds that

$$w^- \le v.$$

When $\min G = 0$ the $C^2$ regularity of $w^+$ and $w^-$ near $\partial\Omega$ is trivial. When $\min G < 0$, near $\partial\Omega$ we can compute $w^+$ by the method of characteristics and hence $w^+$ is $C^2(\overline{\Omega_\delta})$ for some small $\delta$(see the appendix and lemma 2.2 in [Flem]), where $\Omega_\delta$ is defined by $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$.

Now we prove a lemma which describes the size of the oscillation of the solution $w$ near the boundary to the variational inequality

$$-(a_i(x_0, w_0(x_0), \nabla w))_{x_i} \ge 0 \tag{19}$$

with respect to $K_{R,x_0} = \{v \in W_0^{1,2}(B_R(x_0) \cap \Omega) + w_0 : G(\nabla v) \le 0\}$.

**Lemma 7.** *If $\rho \le \frac{R}{2}$, then $w$ satisfies*

$$(20)\quad \int_{B_\rho \cap \Omega} |\nabla w - (\nabla w)_\rho|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla w - (\nabla w)_R|^2 dx + cR^{n+\sigma}$$

*for some $c$ and all $\sigma \in (0,1)$.*

*proof.* We consider a bilateral obstacle problem as in section 2. We drop out the generic point $x_0$ in $a_i$ as follows

$$a_i(A) = a_i(x_0, w_0(x_0), A).$$

Define a function class $J_R$ by

$$J_R = \{v \in W_0^{1,2}(B_R \cap \Omega) + w_0 : w^- \le v \le w^+\}.$$

Let $v \in J_R$ be the unique solution to the variational inequality

$$(21)\quad -(a_i(\nabla v))_{x_i} \ge 0$$

with respect to $J_R$. The existence and uniqueness for solutions to (21) also follow from the monotone operator theory.

Let $v^-$ be the solution to

$$(22)\quad -(a_i(\nabla v^-))_{x_i} = -(a_i(\nabla w^-))_{x_i}$$

and suppose that $v - w_0 \in W_0^{1,2}(B_\delta \cap \Omega)$. Note that $w_0$ is $C^2$ near $B_\delta \cap \partial\Omega$ and $w^-$ is $C^2(\overline{B_\delta \cap \Omega})$ for some small $\delta$ which we determine later. Since $v^- = w_0 \ge w^-$ on $\partial(B_\delta \cap \Omega)$, it follows from the maximum principle that

$$v^- \ge w^-$$

in $B_\delta \cap \Omega$. Since $w^- \in C^2(\overline{B_\delta \cap \Omega})$, we see that for small $\rho \le \frac{\delta}{2}$,

$$\int_{B_\rho \cap \Omega} |\nabla v^- - (\nabla v^-)_\rho|^2 dx \le c\left(\frac{\rho}{\delta}\right)^{n+\sigma} \int_{B_\delta} |\nabla v^- - (\nabla v^-)_\delta|^2 dx + c\delta^{n+\sigma}$$

for some $c$ and for all $\sigma \in (0,2)$. Now it is evident that $\bar{v} := v^- \wedge w^+ \in J_\delta$ and is an admissible competing function to (21). Hence we have

$$(23)\quad \int_{B_\delta \cap \Omega} |\nabla v - \nabla v^-|^2 dx \le c\int_{B_\delta \cap \Omega} \left[a_i(\nabla v) - a_i(\nabla v^-)\right](v - v^-)_{x_i} dx$$

$$= c\int_{B_\delta\cap\Omega} a_i(\nabla v)(v - v^- \wedge w^+)_{x_i}dx + c\int_{B_\delta\cap\Omega} a_i(\nabla v)(v^- \wedge w^+ - v^-)_{x_i}dx$$

$$- \int_{B_\delta\cap\Omega} a_i(\nabla v^-)(v - v^-)_{x_i}dx$$

$$=: I + II + III.$$

We note that since $v^- \wedge w^+$ is an admissible competing function in $J_\delta$,

$$I \leq 0. \tag{24}$$

As in lemma 3, it can be shown that

$$||\nabla v||_{L^\infty(B_\delta\cap\Omega)} \leq c$$

for some $c$. Thus we see that

$$II \leq c\int_{B_\delta\cap\Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|dx \tag{25}$$

$$\leq c\delta^{\frac{n}{2}}\left(\int_{B_\delta\cap\Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^2\, dx\right)^{\frac{1}{2}}$$

for some $c$. Since $v^- \wedge w^+ - v^- \in W_0^{1,2}(B_\delta \cap \Omega)$, we have

$$\int_{B_\delta\cap\Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^2 dx$$

$$\leq c\int_{B_\delta\cap\Omega} \left[a_i(\nabla v^-) - a_i(\nabla w^+)\right](v^- - v^- \wedge w^+)_{x_i}dx$$

$$= c\int_{B_\delta\cap\Omega} \left[a_i(\nabla w^-) - a_i(\nabla w^-(x_0))\right](v^- - v^- \wedge w^+)_{x_i}dx$$

$$-c\int_{B_\delta\cap\Omega} \left[a_i(\nabla w^+) - a_i(\nabla w^+(x_0))\right](v^- - v^- \wedge w^+)_{x_i}dx$$

for some $c$. We note that since $w^-$ and $w^+$ are $C^2$ near $\partial\Omega$,

$$|a_i(\nabla w^-) - a_i(\nabla w^-(x_0))|,\ |a_i(\nabla w^-) - a_i(\nabla w^+(x_0))| \leq c\delta$$

for all $x \in B_\delta \cap \Omega$. Hence using Young's inequality we have

$$\int_{B_\delta \cap \Omega} |\nabla(v^- \wedge w^+) - \nabla v^-|^2 dx \leq c\delta^{n+2}$$

and

$$II \leq c\delta^{n+1} \tag{26}$$

for some $c$. Similarly,

$$\begin{aligned} III &= -c \int_{B_\delta \cap \Omega} a_i(\nabla v^-)(v - v^-)_{x_i} dx \\ &= -c \int_{B_\delta \cap \Omega} \left[a_i(\nabla w^-) - a_i(\nabla w^-(x_0))\right] (v - v^-)_{x_i} dx \\ &\leq \frac{1}{4} \int_{B_\delta \cap \Omega} |\nabla v - \nabla v^-|^2 dx + c\delta^{n+2} \end{aligned} \tag{27}$$

for some $c$.

Combining (23) through (27), we conclude that

$$\int_{B_\delta \cap \Omega} |\nabla v - \nabla v^-|^2 dx \leq c\delta^{n+1} \tag{28}$$

for some $c$.

Now we apply a comparison argument to estimate the oscillation of $\nabla v$. For each small $\rho < \frac{\delta}{2}$, we have

$$\begin{aligned} &\int_{B_\rho \cap \Omega} |\nabla v - (\nabla v)_\rho|^2 dx \\ &\leq c \int_{B_\rho \cap \Omega} |\nabla v^- - (\nabla v^-)_\rho|^2 dx + c \int_{B_\rho \cap \Omega} |\nabla v - \nabla v^-|^2 dx \end{aligned} \tag{29}$$

for some $c$. Since $v^-$ is a solution to (22) and satisfies a Campanato type growth condition for $\nabla v^-$, then we estimate the first term of the right hand side of (29) as follows:

$$\int_{B_\rho \cap \Omega} |\nabla v^- - (\nabla v^-)_\rho|^2 dx \tag{30}$$

$$\leq c\left(\frac{\rho}{\delta}\right)^{n+\sigma}\int_{B_\delta\cap\Omega}|\nabla v-(\nabla v)_\delta|^2dx+c\int_{B_\delta\cap\Omega}|\nabla v-\nabla v^-|^2dx+c\delta^{n+\sigma}$$

for some $c$. Furthermore using the estimate (28) we have

$$\int_{B_\delta\cap\Omega}|\nabla v-\nabla v^-|^2dx\leq c\delta^{n+1} \tag{31}$$

for some $c$. Therefore combining (29), (30) and (31), it follows that $\nabla v$ is Hölder continuous in $\overline{B_{\frac{\delta}{2}}\cap\Omega}$ and satisfies

$$\int_{B_\rho\cap\Omega}|\nabla v-(\nabla v)_\rho|^2dx \tag{32}$$

$$\leq c\left(\frac{\rho}{\delta}\right)^{n+\sigma}\int_{B_\delta\cap\Omega}|\nabla v-(\nabla v)_\delta|^2dx+c\delta^{n+\sigma}$$

for all $\rho\leq\frac{\delta}{2}$ and for some $c$.

Since the viscosity solution $w^+$ and $w^-$ can be derived from the method of characteristics if $\partial\Omega$ is smooth enough, e.g., $C^3$, we can prove that the viscosity solution $w^\pm$ to Hamilton-Jacobi equation

$$\pm G(\nabla w^\pm)=0$$

with $w^\pm=w_0$ on $\partial\Omega$ are $C^2(\overline{B_R\cap\Omega})$ for $R<\delta$.

Note that $K_R\subset J_R$. So if we show that

$$v\in K_R,$$

that is,

$$G(\nabla v)\leq 0$$

for all $x\in B_\delta\cap\Omega$, then $v$ is the unique solution to the variational inequality (19) with respect to $K_R$. Thus $w=v$ and we conclude that $\nabla w$ is Hölder continuous up to $B_{\frac{\delta}{2}}\cap\partial\Omega$ and satisfies the Campanato growth condition (20).

Here we use the maximum principle again for $G(\nabla v)$. From a direct computation we see that $G(\nabla v)$ is a subsolution to a strictly elliptic equation and

$$-(a_{i,A_j}(\nabla v)G_{x_j})_{x_i}\leq 0$$

and $G$ satisfies the maximum principle. Let $I_R^-$ and $I_R^+$ be the contact set such that

$$I_R^- = \{x \in B_R \cap \Omega : v(x) = w^-(x)\} \text{ and } I_R^+ = \{x \in B_R \cap \Omega : v(x) = w^+(x)\}.$$

Consequently from the maximum principle it follows that $\max G(\nabla v)$ is attained on $\partial\left(B_R \cap \Omega \setminus (I_R^- \cup I_R^+)\right)$ and this in turn gives

$$G(\nabla v) \leq 0.$$

Therefore we conclude that

$$v \in K_R.$$

## 4. REGULARITY

In this section we work under the full generality. We prove that the solution $u$ to the variational inequality (2) with respect to $K$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$. We exploit the perturbation techniques using the interior and boundary regularity results for the simple case from sections 2 and 3.

We approximate our differential operator. Since the function class is bounded in $W^{1,\infty}$, there exists some large number $M$ such that

$$|v(x)| + |\nabla v(x)| \leq M$$

for all $x \in \Omega$ and $v \in K$. Hence we have

$$(v(x), \nabla v(x)) \in B_M(0) \subset \mathbb{R}^{n+1}$$

for all $x \in \Omega$. We can find functions

$$\bar{a}_i(x, v, A) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \; i = 1, ..., n$$

and

$$\bar{b}(x, v, A) \to \mathbb{R}$$

such that

$$\bar{a}_i(x, v, A) = a_i(x, v, A), \; i = 1, ..., n$$

and

$$\bar{b}(x, v, A) = b(x, v, A)$$

for all $(v, A) \in B_{2M} \subset \mathbb{R}^{n+1}$, and $\bar{a}_i$ and $\bar{b}$ satisfy

$$c_1(M)|\xi|^2 \le \bar{a}_{i,A_j}(x, v, A)\xi_i\xi_j \le c_2(M)|\xi|^2$$

and

$$\bar{b}(x, v, A) \le c(M)$$

for some $c_1(M) > 0$ and for all $(x, v, A) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. For notational simplicity we write $a_i$ and $b$ instead of $\bar{a}_i$ and $\bar{b}$.

Let $u \in K$ be the solution to

$$-(a_i(x, u, \nabla u))_{x_i} - (g_i(x, u))_{x_i} + b(x, u, \nabla u) \ge 0 \tag{33}$$

with respect to $K$. The following lemma is our main result in this section.

**Lemma 8.** *Fix $\alpha$ the minimum of $\frac{1}{2}$ and the $\alpha$'s in the assumptions i) and ii) in the introduction. Suppose $x_0 \in \overline{\Omega}$ and $\rho \le \frac{R}{2}$. Then for all $\sigma \in (0, 2\alpha)$, $\nabla u$ satisfies*

$$\int_{B_\rho \cap \Omega} |\nabla u - (\nabla u)_\rho|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla u - (\nabla u)_R|^2 dx + cR^{n+\sigma} \tag{34}$$

*for some $c$ depending on $u$ only through $M$. Consequently, $u \in C^{1,\alpha}(\overline{\Omega})$.*

*proof.* Let $x_0 \in \overline{\Omega}$ and $K_R$ be the function class with domain in $B_R \cap \Omega$ such that $K_R = \{v \in W_0^{1,\infty} + u : G(\nabla v) \le 0\}$. Let $\bar{u} \in K_R$ be the solution to the frozen coefficient variational inequality

$$-\; a_i(x_0, u(x_0), \nabla \bar{u}))_{x_i} \ge 0$$

with respect to $K_R$. From sections 2 and 3, $\bar{u}$ satisfies a Campanato type growth condition

$$\int_{B_\rho \cap \Omega} |\nabla \bar{u} - (\nabla \bar{u})_\rho|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+\sigma} \int_{B_R \cap \Omega} |\nabla \bar{u} - (\nabla \bar{u})_R|^2 dx + cR^{n+\sigma} \tag{35}$$

for some $c$, for all $\sigma \in (0, 2\alpha)$ and for all $\rho \le \frac{R}{2}$.

Therefore we have

(36)

$$\int_{B_\rho \cap \Omega} |\nabla u - (\nabla u)_\rho|^2 dx \le 2\int_{B_\rho \cap \Omega} |\nabla \bar{u} - (\nabla \bar{u})_\rho|^2 dx + 8\int_{B_\rho \cap \Omega} |\nabla \bar{u} - \nabla u|^2 dx$$

$$\leq c\left(\frac{\rho}{R}\right)^{n+\sigma}\int_{B_R\cap\Omega}|\nabla u-(\nabla u)_R|^2dx+cR^{n+\sigma}$$
$$+c\int_{B_R\cap\Omega}|\nabla u-\nabla\bar{u}|^2dx$$

for some $c$ and for all $\sigma\in(0,2\alpha)$. Now we see that $\bar{u}\in K_R$ and is an admissible competing function to (33). Hence we have

(37)
$$\int_{B_R\cap\Omega}a_i(x,u,\nabla u)(\bar{u}-u)_{x_i}dx+\int_{B_R\cap\Omega}\left[g_i(x,u)-g_i(x_0,u(x_0))\right](\bar{u}-u)_{x_i}dx$$
$$+\int_{B_R\cap\Omega}b(x,u,\nabla u)(\bar{u}-u)dx\geq 0$$

and

$$\int_{B_R\cap\Omega}a_i(x_0,u(x_0),\nabla\bar{u})(u-\bar{u})_{x_i}dx\geq 0. \tag{38}$$

Subtracting (38) from (37) we have

$$\int_{B_R\cap\Omega}\left[a_i(x_0,u(x_0),\nabla\bar{u})-a_i(x,u,\nabla u)\right](\bar{u}-u)_{x_i}dx \tag{39}$$
$$\leq\int_{B_R\cap\Omega}\left[g_i(x,u)-g_i(x_0,u(x_0))\right](\bar{u}-u)_{x_i}dx+\int_{B_R\cap\Omega}b(x,u,\nabla u)(\bar{u}-u)dx.$$

The left hand side of (39) can be written as

$$\int_{B_R\cap\Omega}\left[a_i(x_0,u(x_0),\nabla\bar{u})-a_i(x,u,\nabla u)\right](\bar{u}-u)_{x_i}dx$$
$$=\int_{B_R\cap\Omega}\left[a_i(x_0,u(x_0),\nabla\bar{u})-a_i(x_0,u(x_0),\nabla u)\right](\bar{u}-u)_{x_i}dx$$
$$+\int_{B_R\cap\Omega}\left[a_i(x_0,u(x_0),\nabla u)-a_i(x_0,u,\nabla u)\right](\bar{u}-u)_{x_i}dx$$
$$+\int_{B_R\cap\Omega}\left[a_i(x_0,u,\nabla u)-a_i(x,u,\nabla u)\right](\bar{u}-u)_{x_i}dx$$
$$=:\ I\ +\ II\ +\ III.$$

Using the ellipticity condition we estimate

$$\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx \le cI \tag{40}$$

for some $c$. From Hölder continuity of $a_i$ with respect to $x$ we have

$$\begin{aligned} |III| &\le c\int_{B_R\cap\Omega} R^\alpha |\nabla\bar{u} - \nabla u| dx \\ &\le \frac{1}{8}\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx + cR^{n+2\alpha} \end{aligned} \tag{41}$$

for some $c$. From Young's inequality we also have

$$\begin{aligned} II &= \int_{B_R\cap\Omega} [a_i(x_0, u(x_0), \nabla u) - a_i(x, u, \nabla u)]\,(\bar{u} - u)_{x_i} dx \\ &\le c\int_{B_R\cap\Omega} |u(x) - u(x_0)|^\alpha |\nabla\bar{u} - \nabla u| dx \\ &\le c\int_{B_R\cap\Omega} R^\alpha |\nabla\bar{u} - \nabla u| dx \\ &\le \frac{1}{8}\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx + cR^{n+2\alpha}, \end{aligned} \tag{42}$$

where we used the fact that $u$ is Lipschitz continuous and

$$|u(x) - u(x_0)| \le c|x - x_0|.$$

Finally using Poincaré's inequality we estimate the right hand side of (39) as follows:

$$\int_{B_R\cap\Omega} [g_i(x,u) - g_i(x_0, u(x_0))]\,(\bar{u} - u) dx \le \frac{1}{8}\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx + cR^{n+2\alpha}$$

and

$$\int_{B_R\cap\Omega} b(x, u, \nabla u)(\bar{u} - u) dx \le \frac{1}{8}\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx + cR^{n+2\alpha}.$$

Thus combining all these together we have that

$$\int_{B_R\cap\Omega} |\nabla\bar{u} - \nabla u|^2 dx \le cR^{n+2\alpha} \tag{43}$$

for some $c$. Therefore using the estimate (43) on (36) we conclude that

$$\int_{B_\rho\cap\Omega} |\nabla u - (\nabla u)_\rho|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+2\alpha} \int_{B_R\cap\Omega} |\nabla u - (\nabla u)_R|^2 dx + cR^{n+2\alpha}$$

for some $c$ and this completes the proof.

## 5. Existence

We employ Leray-Schauder's fixed point theorem to show the existence of the solution to

$$L(u) = -(a_i(x,u,\nabla u))_{x_i} - (g_i(x,u))_{x_i} + b(x,u,\nabla u) \geq 0 \tag{44}$$

with respect to $K = \{v \in W_0^{1,\infty}(\Omega) + u_0 : G(\nabla u) \leq 0\}$.

We define a compact map $T : K \to K$. Let $v \in K$ and $u = T(v)$ be the solution to the variational inequality

$$L(v,u) = -(a_i(x,v,\nabla u))_{x_i} - (g_i(x,v))_{x_i} + b(x,v,\nabla v) \geq 0 \tag{45}$$

with respect to $K$. We note that $K$ is a bounded close convex subset of $W_0^{1,2}(\Omega) + u_0$. Moreover for each fixed $v \in K$, $L(v,u)$ is strictly monotone as an operator of $u$. Therefore from the theorem 1.1 in [Har1] we see that there is a unique solution $u = T(v) \in K$ to (45) and hence $T$ is well defined.

From the $C^{1,\alpha}(\overline{\Omega})$ regularity result in section 4 we have that for each $v \in K$

$$u = T(v) \in C^{1,\alpha}(\overline{\Omega}).$$

for some fixed $\alpha > 0$. Moreover the $C^{1,\alpha}$ norm of $u$ is bounded by some fixed number $M$ indepedent of $v \in K$, $C^{1,\alpha}$ norm depends on $v$ only through the upper bound of the Lipschitz norm of $v$. We note that the space $C^{1,\alpha}(\overline{\Omega})$ is compactly imbedded in the space of Lipschitz functions $Lip(\overline{\Omega})$. Hence the image of $K$ under the map $T$ is a precompact subset of $K$. Therefore from Leray-Schauder's fixed point theorem we conclude that there is a fixed point $u$ for $T$ such that

$$u = T(u)$$

and $u$ is a $C^{1,\alpha}(\overline{\Omega})$ solution to (44) with respect to $K$.

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# AN APPLICATION OF A VARIATIONAL REDUCTION METHOD TO A NONLINEAR WAVE EQUATION

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## Introduction

In this paper we investigate the existence of solutions $u(x,t)$ for a piecewise-linear pertubation $au^+ - bu^-$ of the 1-dimesional wave operator $u_{tt} - u_{xx}$ under Dirichlet boundary condition on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and periodic condition on the variable $t$

$$u_{tt} - u_{xx} + au^+ - bu^- = f(x,t) \text{ in } (c,d) \times \mathbb{R}$$
$$u(c,t) = u(d,t) = 0 \tag{0.1}$$
$$u(x,t+T) = u(x,t),$$

where the period T is given.

We assume that the period T is a rational multiple of the length $(d - c)$ of the $x$-interval where problem (0.1) posed (As is well-known, serious difficulties of a number theoritical nature arise when that is not the case). For simplicity, only the case $T = \pi$ will be considered. By obvious changes of variables, problem (0.1) can be reduced to

$$u_{tt} - u_{xx} + au^+ - bu^- = f(x,t) \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$
$$u(\pm\frac{\pi}{2}, t) = 0 \tag{0.2}$$
$$u(x,t+\pi) = u(x,t)$$

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Here $au^+$ is an upward restoring force and $bu^-$ a downward restoring force. We shall assume that $f$ is even in $x$ and periodic in $t$ with period $\pi$, and we shall look for $\pi$-periodic solutions of (0.2).

The existence of multiple solutions of elliptic boundary value problems with nonlinearities crossing multiple eigenvalues was shown by a variational reduction method in Lazer and McKenna [4]. Also the existence of multiple solutions of a nonlinear suspension bridge equation was shown by the same method in Choi, Jung, and McKenna [3].

In this paper we shall use the same method to show the existence of multiple solutions of a nonlinear wave equation (0.2).

In Section 1, we show that only the trivial solution exists for the homogeneous problem in the Banach space $H$ spanned by eigenfunctions. We also prove the continuity and Fréchet differentiability of the corresponding functional to (0.2).

In Section 2, we show the existence of multiple solutions of equation (0.2) when the forcing term is supposed to be a multiple $s\phi_{00}(s \neq 0, s \in \mathbb{R})$ of the first (positive) eigenfunction $\phi_{00} = \frac{\sqrt{2}}{\pi}\cos x$ and $-1 < b < 3 < a < 7$,

$$u_{tt} - u_{xx} + au^+ - bu^- = s\phi_{00} \quad \text{in } H. \tag{0.3}$$

Here $a$ and $b$ satisfy the condition

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} > 1.$$

The main result is the following :

(i) For $s > 0$, (0.3) has at least three solutions, one of which is a positive solution.

(ii) For $s < 0$, (0.3) has at least one sulutions, one of which is a negative solution.

## 1. The Banach space spanned by eigenfunctions

In this section we investigate the properties of the Banach space spanned by the eigenfunctions of the wave operator.

Let $L$ be the wave operator, in $\mathbb{R}^2$,

$$Lu = u_{tt} - u_{xx}.$$

When $u$ is even in $x$ and periodic in t with period $\pi$, the eigenvalue problem for $u(x,t)$,

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \tag{1.1}$$

$$u(\pm\frac{\pi}{2}, t) = 0,$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions $\phi_{mn}$, $\psi_{mn}$ $(m, n \leq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n+1)x & &\text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x & &\text{for } m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos(2n+1)x & &\text{for } m > 0, n \geq 0. \end{aligned}$$

Let $n$ be fixed and define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n + 1, \tag{1.2}$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n - 3. \tag{1.3}$$

Letting $n \longrightarrow \infty$, we obtain that $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$. Hence, it is easy to check that the only eigenvalues in the interval $(-15, 9)$ are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let $Q$ be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $H_0$ the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in $H_0$. Let us denote an element $u$, in $H_0$, as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace $H$ of $H_0$ as follows

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all $m, n$, we have that

(i) $|||u||| \geq \|u\|$, where $\|u\|$ denotes the $L^2$ norm of $u$,
(ii) $\|u\| = 0$ if and only if $|||u||| = 0$,
(iii) $Lu \in H$ implies $u \in H$.

We note that 1 belongs to $H_0$, but does not belong to $H$. Hence we can see that the space $H$ is a proper subspace of $H_0$. The following lemma is very important in this paper.

**Lemma 1.1.** *Let $c$ be not an eigenvalue of $L$ and let $u \in H_0$. Then we have $(L-c)^{-1}u \in H$.*

*Proof.* Suppose that $c$ is not an eigenvalue of $L$. When $n$ is fixed, $\lambda_n^+$ and $\lambda_n^-$ were defined in (1.2) and (1.3):

$$\lambda_n^+ = 4n + 1,$$

$$\lambda_n^- = -4n - 3.$$

We see that $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$ as $n \longrightarrow \infty$. Hence the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite, where $\lambda_{mn}$ is an eigenvalue of $L$. Let

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}).$$

Then

$$(L-c)^{-1}u = \sum (\frac{1}{\lambda_{mn}+c} h_{mn}\psi_{mn} + \frac{1}{\lambda_{mn}+c} k_{mn}\psi_{mn}).$$

Hence we have the inequality

$$\begin{aligned} |||(L-c)^{-1}u||| &= \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}+c)^2}(h_{mn}^2 + k_{mn}^2) \\ &\leq C \sum (h_{mn}^2 + k_{mn}^2) \end{aligned}$$

for some $C$, which means that

$$|||(L-c)^{-1}u||| \leq C_1 \|u\|, \quad C_1 = \sqrt{C}.$$ □

With the above lemma 1.1, we can obtain the folloing lemma.

**Lemma 1.2.** *Let $f(x,t) \in H_0$. Let $a$ and $b$ be not eigenvalues of $L$. Then all the solutions in $H_0$ of*

$$Lu + au^+ - bu^- = f(x,t) \quad \text{in} \quad H_0$$

*belong to $H$.*

Let $\mu_1$ and $\mu_2$ be two successive eigenvalues of $L$. Then we have the uniqueness theorem.

**Theorem 1.1.** *Let $f(x,t) \in H_0$ and $-\mu_2 < a, b < -\mu_1$. Then the equation*

$$Lu + au^+ - bu^- = f(x,t) \tag{1.4}$$

*has a unique solution in $H_0$. Furthermore equation (1.4) has a unique solution in $H$.*

*Proof.* Let $f(x,t) \in H_0$ and $-\mu_2 < a, b < -\mu_1$. Let $\delta = -\frac{1}{2}(\mu_1 + \mu_2)$. The equation (1.4) is equivalent to

$$u = (L+\delta)^{-1}[(\delta - a)u^+ - (\delta - b)u^- + f(x,t)],$$

where $(L+\delta)^{-1}$ is a compact, self-adjoint, linear map from $H_0$ into $H_0$ with morm $\frac{2}{\mu_2 - \mu_1}$. We note that

$$\begin{aligned} \|(\delta - a)(u_2^+ - u_1^+) - (\delta - b)(u_2^- - u_1^-)\| &\le \max\{|\delta - a|, |\delta - b|\}\|u_2 - u_1\| \\ &< \frac{1}{2}(\mu_2 - \mu_1)\|u_2 - u_1\|. \end{aligned}$$

It follows that the right hand side of (1.4) defines a Lipschitz mapping of $H_0$ into $H_0$ with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$ of (1.4).

On the other hand, by Lemma 1.2, if $f(x,t) \in H_0$ then we know that the solution of (1.4) belongs to $H$. □

We now state a symmetry theorem which was proved in Lazer and McKenna [5].

**Theorem A.** *Assume that* $L : \mathcal{D}(L) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ *is a linear self-adjoint operator which posseses two closed invariant subspaces* $H_1$ *and* $H_2 = H_1^\perp$. *Let* $\sigma$ *denote the spectrum of* $L$ *and* $\sigma_i$ *the spectrum of* $L|H_i$ $(i = 1,2;\ \sigma = \sigma_1 \cup \sigma_2)$. *Let* $\frac{\partial f}{\partial u}(u,x) \equiv f_u$ *be piecewise smooth and assume that* $f_u \in [a,b]$ *for all* $u \in \mathbb{R}$ *and* $x \in \Omega$.

*If* $[a,b] \cap \sigma_2 = \phi$ *and if the Nemytzki operator* $U \mapsto Fu = f(u(x,x)$ *maps* $H_1$ *into itself, then every solution of*

$$Lu = f(u,x) \qquad \text{in} \qquad L^2(\Omega)$$

*is in* $H_1$.

With the Theorem A, we have the following theorem, which is useful in the later.

**Theorem 1.2.** *Let* $-1 < a, b < 7$. *We assume that*

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} \neq 1. \tag{1.5}$$

*Then the equation*

$$Lu + au^+ - bu^- = 0 \qquad \text{in} \qquad H_0 \tag{1.6}$$

*has only the trivial solution* $u \equiv 0$.

*Proof.* The space $H_1 = \text{span}\{\cos x \cos 2mt : m \geq 0\}$ is invariant under $L$ and under the map $u \mapsto au^+ - bu^-$. The spectrum $\sigma_1$ of $L$ restricted to $H_1$ contains $\lambda_{10} = -3$ and does not contain any other point in the interval $(-7,1)$. The spectrum $\sigma_2$ of $L$ restricted to $H_2 = H_1^\perp$ does not intersect the interval $(-7,1)$. From Theorem A, we conclude that any solution of (1.6) belongs to $H_1$, i.e., it is of the form $y(t)\cos x$, where $y$ satisfies

$$y'' + y + ay^+ - by^- = 0.$$

Any nontrivial periodic solution of this equation is periodic with period

$$\frac{\pi}{\sqrt{a+1}} + \frac{\pi}{\sqrt{b+1}} \neq \pi.$$

This shows that there is no nontrivial solution of (1.6).

The condition (1.5) is essential. When the equation

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} = 1$$

holds, we can construct a nontrivial solution $u_0$ of (1.6) and any $ku_0 (k > 0)$ becomes a nontrivial solution of (1.6).

In this paper we investigate nonlinear oscillations in the wave of a string, $s \in \mathbb{R}$,

$$Lu + au^+ - bu^- = s\phi_{00} \quad \text{in} \quad H. \tag{1.7}$$

Let us define the functional on $H$, $s \in \mathbb{R}$,

$$F_{a,b}(u,s) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_x|^2) + \frac{a}{2}|u^+|^2 + \frac{b}{2}|u^-|^2 - s\phi_{00}u]dtdx. \tag{1.8}$$

For simplicity we shall write $F = F_{a,b}$ when $a$ and $b$ are fixed. Then $F$ is well-defined in $H$. The solutions of (1.7) coincide with the critical points of $F(u,s)$.

**Proposition 1.1.** *Let $a$ and $b$ be fixed. For $s \in \mathbb{R}$, $F(u,s) = F_{a,b}(u,s)$ is continuous and Fréchet differentiable in $H$.*

*Proof.* Let $u$ be in $H$. For $s \in \mathbb{R}$, to prove the continuity of $F(u,s)$, we consider

$$\begin{aligned} &F(u+v,s) - F(u,s) \\ &= \int_Q [u(v_{tt} - v_{xx}) + \frac{1}{2}v(v_{tt} - v_{xx})]dxdt \\ &+ \int_Q [\frac{a}{2}(|(u+v)^+|^2 - |u^+|^2) + \frac{b}{2}(|(u+v)^-|^2 - |u^-|^2) - s\phi_{00}v]dxdt. \end{aligned}$$

Let $u = \sum(h_{mn}\phi_{mn} + k_{mn}\psi_{mn})$, $v = \sum(\tilde{h}_{mn}\phi_{mn} + \tilde{k}_{mn}\psi_{mn})$. Then we have

$$|\int u(v_{tt} - v_{xx})dxdt| = |\sum \lambda_{mn}(h_{mn}\tilde{h}_{mn} + k_{mn}\tilde{k}_{mn})| \le |||u||| \cdot |||v|||,$$

$$|\int \frac{1}{2}v(v_{tt} - v_{xx})dxdt| = |\sum \lambda_{mn}(\tilde{h}^2_{mn} + \tilde{k}^2_{mn})| \le |||v|||^2.$$

On the other hand,

$$||(u+v)^+|^2 - |u^+|^2| \le 2u^+|v| + |v|^2,$$
$$||(u+v)^-|^2 - |u^-|^2| \le 2u^-|v| + |v|^2,$$

and hence we have

$$|\int(|(u+v)^+|^2 - |u^+|^2)dxdt| \le 2\|u^+\|\|v\| + \|v\|^2 \le 2|||u|||\cdot|||v||| + |||v|||^2,$$

$$|\int(|(u+v)^-|^2 - |u^-|^2)dxdt| \le 2\|u^-\|\|v\| + \|v\|^2 \le 2|||u|||\cdot|||v||| + |||v|||^2.$$

With the above results, we see that $F(u,s)$ is continuous at $u$.

Now we prove that $F(u,s)$ is Fréchet differentiable at $u \in H$ with

$$DF(u,s)v = \int(Lu + au^+ - bu^- - s\phi_{00})vdxdt.$$

To prove the above equation, it is enough to compute the following :

$$\begin{aligned}
&|F(u+v,s) - F(u,s) - DF(u,s)v| \\
=&|\int \frac{1}{2}vLv\,dxdt + \frac{a}{2}\int(|(u+v)^+|^2 - |u^+|^2 - 2u^+v)dxdt \\
&+\frac{b}{2}\int(|(u+v)^-|^2 - |u^-|^2 + 2u^-v)dxdt| \\
\le&\frac{1}{2}|||v|||^2 + \frac{|a|}{2}\int v^2\,dxdt + \frac{|b|}{2}\int v^2\,dxdt \\
\le&\frac{1}{2}(1+|a|+|b|)|||v|||^2,
\end{aligned}$$

since

$$0 \le |(u+v)^+|^2 - |u^+|^2 - 2u^+v \le |v|^2$$

and

$$0 \le |(u+v)^-|^2 - |u^-|^2 + 2u^-v \le |v|^2. \qquad \square$$

## 2. The nonlinearity $-(au^+ - bu^-)$ crosses the first negative eigenvalue $\lambda_{10}$

In this section we investigate the existence of multiple solutions of equation (0.2) when the forcing term is supposed to be a multiple $s\phi_{00}(s \neq 0, s \in \mathbb{R})$ of the first (positive) eigenfunction $\phi_{00}$ and $-1 < b < 3 < a < 7$,

$$Lu + au^+ - bu^- = s\phi_{00} \qquad \text{in} \quad H. \tag{2.1}$$

Hereafter, in this section we assume that $a$ and $b$ satisfy the condition

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} > 1. \tag{2.2}$$

Let $V$ be the 2 dimensional subspace of $H$ which is the closure of the span of the eigenfunctions $\phi_{10}$ and $\psi_{10}$, both of which have the same eigenvalue $\lambda_{10} = -3$. Then $|||v||| = \sqrt{3}\|v\|$ for $v \in V$. Let $W$ be the orthogonal complement of $V$ in $H$.

Now we state the main result in this paper.

**Theorem 2.1.** *Let $-1 < b < 3 < a < 7$ with $\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} > 1$ and $s \in \mathbb{R}$. Then we have the followings.*

(i) *The equation (2.1) has at least three solutions for $s > 0$, one of which"(ii)" is a positive solution.*

(iii) *The equation (2.1) has at least one solution for $s < 0$, one of which is a negative solution.*

For the proof of Theorem 2.1, we need several lemmas which we first prove.

**Lemma 2.1.** *Let $s \in \mathbb{R}$. Then we have :*

(i) *If $s > 0$, then the equation (2.1) has a positive solution.*

(ii) *If $s < 0$, then the equation (2.1) has a negative solution.*

*Proof.* (1) For $s > 0$, the positive solution of (2.1) is $u = \frac{s}{a+1}\phi_{00}$.
(2) For $s < 0$, the negative solution of (2.1) is $u = \frac{s}{b+1}\phi_{00}$. □

Next we shall use a variational reduction method to apply the mountain pass theorem.

Let $P : H \longrightarrow V$ denote the orthogonal projection of $H$ onto $V$ and $I - P : H \longrightarrow W$ denote that of $H$ onto $W$, where $V$ and $W$ are defined in the beginning of this section.

**Lemma 2.2.** *Let $-1 < b < 3 < a < 7$ and let $v \in V$ be given. Then we have :*
*(i) There exists a unique solution $z \in W$ of the equation*

$$Lz + (I - P)[a(v+z)^+ - b(v+z)^- - s\phi_{00}] = 0 \quad \text{in } W. \tag{2.3}$$

*If for fixed $s \in \mathbb{R}$ we put $z = \theta(v,s)$, then $\theta$ is continuous on $V$ and we have*

$$DF(v + \theta(v,s), s)(w) = 0 \quad \text{for all } w \in W.$$

*In particular $\theta$ satisfies a uniform Lipschitz condition in $v$ with respect to the $L^2$ norm (also the norm $|||\cdot|||$).*
*(ii) If $\tilde{F} : V \longrightarrow \mathbb{R}$ is defined by $\tilde{F}(v,s) = F(v + \theta(v,s), s)$, then $\tilde{F}$ has a continuous Fréchet derivative $D\tilde{F}$ with respect to $V$ and*

$$D\tilde{F}(v,s)(h) = DF(v + \theta(v,s), s)(h) \quad \text{for all } h \in V.$$

*If $v_0$ is a critical point of $\tilde{F}$, then $v_0 + \theta(v_0)$ is a solution of (2.1) and conversely every solution of (2.1) is of this form.*

*Proof.* (i) Let $-1 < a < 3 < b < 7$, $\delta = 3$, and $g(\zeta) = a\zeta^+ - b\zeta^-$. If $g_1(\zeta) = g(\zeta) - \delta\zeta$, the equation (2.3) is equivalent to

$$z = (L+\delta)^{-1}(I-P)[-g_1(v+z) + s\phi_{00}]. \tag{2.4}$$

Since $(L+\delta)^{-1}(I-P)$ is a self-adjoint, compact, linear map from $(I-P)H$ into itself, the eigenvalues of $(L+\delta)^{-1}(I-P)$ in $W$ are $(\lambda_{mn}+\delta)^{-1}$, where $\lambda_{mn} > 1$ or $\lambda_{mn} \le -7$. Therefore its $L_2$ norm is $\frac{1}{4}$. Since

$$|g_1(\zeta_2) - g_1(\zeta_1)| \le \max\{|a-\delta|, |\delta - b|\}|\zeta_2 - \zeta_1| < 4|\zeta_2 - \zeta_1|,$$

it follows that the right hand side of (2.4) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)H_0$ into itselt with Lipchitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I-P)H_0$ (also $z \in (I-P)H$) which satisfies (2.4). Since the constant $\delta$ does not depend on $v$, it follows from standard arguments that if $\theta(v)$ denotes the unique $z \in (I-P)H$ which solves (2.4) then $\theta$ is continuous. In fact, if $z_1 = \theta(v_1, s)$ and $z_2 = \theta(v_2, s)$, then we have

$$\begin{aligned}\|z_1 - z_2\| &= \|(L+\delta)^{-1}(I-P)(-g_1(v_1+z_1) + g_1(v_2+z_2))\| \\ &\le \gamma\|(v_1+z_1) - (v_2+z_2)\| \\ &\le \gamma(\|v_1 - v_2\| + \|z_1 - z_2\|).\end{aligned}$$

Hence

$$\|z_1 - z_2\| \le c\|v_1 - v_2\|, \qquad c = \frac{\gamma}{1-\gamma}.$$

With this inequality we have

$$\begin{aligned}|||z_1 - z_2||| &= |||(L+\delta)^{-1}(I-P)[-g_1(v_1+z_1) + g_1(v_2+z_2)]||| \\ &\le \frac{1}{\sqrt{2}}\|(I-P)[-g_1(v_1+z_1) + g_2(v_2+z_2)]\| \\ &\le 2(\|z_1 - z_2\| + \|v_1 - v_2\|) \\ &\le \frac{2}{\sqrt{3}}(c+1)|||v_1 - v_2|||.\end{aligned}$$

Let $v \in V$ and set $z = \theta(v,s)$. If $w \in W$, then from (2.4) we see that

$$\int_Q [-z_t w_t + z_x w_x + a(v+z)^+ w - b(v+z)^- w - s\phi_{00} w] dt dx = 0.$$

Since

$$\int_Q v_t w_t = 0 \qquad \text{and} \qquad \int_Q v_x w_x = 0,$$

we have

$$DF(v + \theta(v,s), s)(w) = 0 \qquad \text{for } w \in W. \tag{2.5}$$

(ii) Let $W_1$ be the subspace of $H$ which is the closure of the span of functions $\phi_{mn}$ and $\psi_{mn}$ whose eigenvalues are $\lambda_{mn} \le -7$ and let $W_2$ be the subspace of $H$ which is the closure of the span of function $\phi_{mn}$ and $\psi_{mn}$ whose eigenvalues are $\lambda_{mn} \ge 1$. Let $v \in V$ and consider the function $h : W_1 \times W_2 \longrightarrow \mathbb{R}$ defined by

$$h(w_1, w_2) = F(v + w_1 + w_2, s).$$

Then the function $h$ has continuous partial Fréchet derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$D_i h(w_1, w_2)(y_i) = DF(v + w_1 + w_2)(y_i) \qquad \text{for} \quad y_i \in W_i, \quad i = 1, 2.$$

Therefore, if $\theta(v,s) = \theta_1(v,s) + \theta_2(v,s)$ with $\theta_i(v,s) \in W_i$, $i = 1,2$, it follows from (2.5) that

$$D_i h(\theta_1(v,s), \theta_2(v,s)) = 0, \qquad i = 1, 2. \tag{2.6}$$

If $w_2$ and $y_2$ are in $W_2$ and $w_1 \in W_1$, then

$$\begin{aligned}&[D_2h(w_1,w_2) - D_2h(w_1,y_2)](w_2-y_2)\\ =&[DF(v+w_1+w_2,s) - DF(v+w_1+y_2,s)](w_2-y_2)\\ =&\int_Q [-|(w_2-y_2)_t|^2 + |(w_2-y_2)_x|^2\\ &\quad + (g(v+w_1+w_2) - g(v+w_1+y_2))(w_2-y_2)]dtdx,\end{aligned}$$

where $g(\zeta) = a\zeta^+ - b\zeta^-$.

Since $(g(\zeta_2) - g(\zeta_1))(\zeta_2-\zeta_1) \geq b(\zeta_2-\zeta_1)^2$ for all $\zeta_1$ and $\zeta_2$, and

$$\int_Q [-|(w_2-y_2)_t|^2 + |(w_2-y_2)_x|^2]dtdx = |||w_2-y_2|||^2,$$

it follows that

$$(D_2h(w_1,w_2) - D_2h(w_1,y_2))(w_2-y_2) \geq (1+\min\{b,0\})|||w_2-y_2|||^2.$$

Therefore $h$ is strictly convex with respect to the second variable, since $1+b > 0$. Similarly, using the fact that $(g(\xi_2)-g(\xi_1))(\xi_2-\xi_1) \leq a(\xi_2-\xi_1)^2$, we see that if $w_1$ and $y_1$ are in $W_1$ and $w_2 \in W_2$, then

$$\begin{aligned}(D_1h(w_1,w_2) - D_1h(y_1,w_2))(w_1-y_1) &\leq -|||w_1-y_1|||^2 + a\|w_1-y_1\|\\ &\leq (-1+\frac{a}{7})|||w_1-y_1|||^2,\end{aligned}$$

where $-1+\frac{a}{7} < 0$. Therefore, $h$ is strictly concave with respect to the first variable. From (2.6) it follows that

$$F(v+\theta_1(v,w)+\theta_2(v,s),s) \leq F(v+\theta_1(v,s)+y_2,s)$$

for $y_2 \in W_2$ with equality if and only if $y_2 = \theta_2(v,s)$.

Since $h$ is strictly concave(convex) with respect to its first(second) variable, Theorem 2.3 of [1] implies that $\tilde{F}$ is $C^1$ with respect to $v$ and

$$D\tilde{F}(v,s)(h) = DF(v+\theta(v,s),s)(h), \qquad h \in V. \tag{2.7}$$

Suppose that $DF(v_0+\theta(v_0,s),s)(v) = 0$ for all $v \in V$. Since (2.5) holds for all $w \in W$ and $H$ is the direct sum of $V$ and $W$, it follows that $DF(v_0 + \theta(v_0,s),s) = 0$ in $H$. Therefore, $u = v_0 + \theta(v_0,s)$ is a solution of (2.1).

Conversely, our reasoning shows that if $u$ is a solution of (2.1) and $v = Pu$, then $D\tilde{F}(v,s) = 0$ in $V$. □

Let $-1 < b < 3 < a < 7$. From Lemma 2.1 we see that for $s > 0$, (2.1) has a positive solution $u_0 = \frac{s}{a+1}\phi_{00} \in W$, and that for $s < 0$, (2.1) has a negative solution $u_1 = \frac{s}{b+1}\phi_{00} \in W$. By the above Lemma 2.2, $u_0$ and $u_1$ can be written as $u_j = v_j + \theta(v_j, s)$, $v_j \in V$ $(j = 0,1)$. Since the solutions $u_0$ and $u_1$ belong to $W$, $v_0 = v_1 = 0$. Therefore we have $u_j = 0 + \theta(0,s)$.

**Lemma 2.3.** *Let $-1 < b < 3 < a < 7$. Then we have the following.*

(i) *For $s > 0$, there exists a small open neighborhood $B$ of 0 in $V$ such that $v = 0$ is a strict local minimum of $\tilde{F}(v,s)$.*

(ii) *For $s < 0$, there exists a small open neighborhood $B_1$ of 0 in $V$ such that $v = 0$ is a strict local maximum of $\tilde{F}(v,s)$.*

*Proof.* (i) Let $s > 0$. Since the positive solution of (2.1) is $u_0 = 0 + \theta(0,s)$ and $I + \theta$, where $I$ is an identity map on $V$, is continuous on $V$, it follows that there exists a small open neighborhood $B$ of 0 in $V$ such that if $v \in B$ then $v + \theta(v,s) > 0$. Here $\theta(v,s) \equiv \theta(0,s)$ in $B$. Therefore, if $v \in B$, then for $z = \theta(v,s)$ we have

$$\begin{aligned}\tilde{F}(v,s) &= F(v+z,s)\\ &= \int_Q [\frac{1}{2}(-|(v+z)_t|^2 + |(v+z)_x|^2) + \frac{a}{2}|(v+z)|^2 - s\phi_{00}(v+z)]dtdx\\ &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_x|^2) + \frac{a}{2}v^2]dtdx + C,\end{aligned}$$

where

$$\begin{aligned}C &= \int_Q [\frac{1}{2}(-|z_t|^2 + |z_x|^2) + \frac{a}{2}z^2 - s\phi_{00}z]dtdx\\ &= F(z,s) \equiv \tilde{F}(0,s).\end{aligned}$$

Each $v \in V$ has the form $v = c_{10}\phi_{10} + c'_{10}\psi_{10}$, where the eigenvalues of $\phi_{10}$ and $\psi_{10}$ is the same integer $\lambda_{10} = -3$. Therefore we have, in $B$,

$$\begin{aligned}\tilde{F}(v,s) - \tilde{F}(0,s) &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_x|^2) + \frac{a}{2}v^2]dtdx\\ &= \frac{1}{2}(-3+a)\int_Q v^2 dtdx.\end{aligned}$$

Since $3 < a < 7$, $v = 0$ is a strict local point of minimum of $\tilde{F}$.

(ii) Let $s < 0$. Since the negative solution of (2.1) is $u_1 = 0 + \theta(0,s)$ and $I + \theta$ is continuous, there exists a small neighborhood $B_1$ of 0 in $V$ such that if $v \in B_1$ then $v + \theta(v,s) < 0$. Here $\theta(v,s) \equiv \theta(0,s)$ in $B_1$. Hence, if $v \in B_1$, then for $z = \theta(v,s)$ we have

$$\begin{aligned}\tilde{F}(v,s) &= F(v+z,s)\\ &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_x|^2) + \frac{b}{2}v^2]dtdx + C,\end{aligned}$$

where $C = F(z,s) \equiv \tilde{F}(0,s)$. Therefore we have, in $B$,

$$\tilde{F}(v,s) - \tilde{F}(0,s) = \frac{1}{2}(-3+b)\int_Q v^2 dtdx.$$

Since $-1 < b < 3$, $v = 0$ is a strict local point of maximum of $\tilde{F}$. □

**Lemma 2.4.** *Let $s \in \mathbb{R}$ be fixed. Let $-1 < b < 3 < a < 7$ and assume that condition (2.2) holds. Then the functional $\tilde{F}(v,s)$, defined on $V$, satisfies the Palais-Smale condition : Any sequence $\{v_n\} \subset V$ for which $\tilde{F}(v_n,s)$ is bounded and $D\tilde{F}(v_n,s) \longrightarrow 0$ posseses a convergent subsequence.*

*Proof.* Suppose that $\tilde{F}(v_n,s)$ is bounded and $D\tilde{F}(v_n,s) \longrightarrow 0$ in $V$ a sequnce. Then, since $V$ is 2 dimensional and spanned by the smooth functions $\phi_{10}$ and $\psi_{10}$, we have, with $u_n = v_n + \theta(v_n,s)$,

$$Lu_n + au_n^+ - bu_n^- = DF(u_n,s) + s\phi_{00} \qquad \text{in } H.$$

Assuming [P.S.] condition does not hold, that is $\|v_n\| \longrightarrow \infty$, we see that $\|u_n\| \longrightarrow +\infty$. Dividing by $\|u_n\|$ and taking $w_n = \|u_n\|^{-1}u_n$ we have

$$Lw_n + aw_n^+ - bw_n^- = \|u_n\|^{-1}(DF(u_n,s) + s\phi_{00}). \tag{2.8}$$

Since $DF(u_n,s) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|u_n\| \longrightarrow \infty$, the right hand side of (2.8) converges to 0 in $L^2(Q)$ as $n \longrightarrow \infty$. Moreover (2.8) shows that $\|Lw_n\|$ is bounded. Since $L^{-1}$ is a compact operator, passing to a subsequence we get that $w_n \longrightarrow w_0$ in $H_0$. Since $\|w_n\| = 1$ for all $n = 1,2,\cdots$, it follows that $\|w_0\| = 1$. Taking the limit of both sides of (2.8), we find

$$Lw_0 + aw_0^+ - bw_0^- = 0$$

**Lemma 2.6.** *Let $-1 < a, b < 7$. Then we have :*

(i) $\tilde{F}^*_{a,0}(v) < 0$ *for all* $v \in V$ *with* $v \neq 0$.

(ii) $\tilde{F}^*_{0,b}(v) < 0$ *for all* $v \in V$ *with* $v \neq 0$.

*Moreover if we let $-1 < b < 3 < a < 7$, satisfying the condition (2.2), then we have*

(iii) $\tilde{F}^*_{a,b}(v) < 0$ *for all* $v \in V$ *with* $v \neq 0$.

*Proof.* (i) To prove (i) , it suffices to show that $\tilde{F}^*_{a,0}(v)$ does not satisfy the following cases :

(a) $\tilde{F}^*_{a,0}(v) \geq 0$ and $\tilde{F}^*_{a,0}(v_0) = 0$ for some $v_0 \in V$ with $v_0 \neq 0$.

(b) $\tilde{F}^*_{a,0}(v) \leq 0$ and $\tilde{F}^*_{a,0}(v_1) = 0$ for some $v_1 \in V$ with $v_1 \neq 0$.

(c) $\tilde{F}^*_{a,0}(v) > 0$ for all $v \in V$ with $v \neq 0$.

(d) There exist $v_1$ and $v_2$ in $V$ such that $\tilde{F}^*_{a,0}(v_1) < 0$ and $\tilde{F}^*_{a,0}(v_2) > 0$.

Suppose that (i) holds. It follows that $\tilde{F}^*_{a,0}(v)$ has an absolute minimum at $v_0$ and hence $D\tilde{F}^*_{a,0}(v_0) = 0$. Therefore, by Lemma 2.2, $u_0 = v_0 + \theta^*(v_0)$ is a nontrivial solution of the equation $Lu + au^+ - bu^- = 0$ in $H$, which is a contradiction. A similar argument shows that it is impossible that (ii) holds.

Suppose that (c) holds. Then there exists $t_0 \in (0,1)$ such that for all $t \leq t_0$

$$t\tilde{F}^*_{a,0}(v) + (1-t)\tilde{F}^*_{0,0}(v) \leq 0 \qquad \text{for all} \quad v \neq 0.$$

We note that there exists $v_0 \neq 0$ and $t(\leq t_0)$ such that $t\tilde{F}^*_{a,0}(v_0) + (1-t)\tilde{F}^*_{0,0}(v_0) = 0$. Let $t_1$ be the greatest number such that

$$t\tilde{F}^*_{a,0}(v_0) + (1-t)\tilde{F}^*_{0,0}(v_0) = 0$$

for some $v_0 \neq 0$ and $t$. Then $0 < t_1 \leq t_0$. Since $t_1\tilde{F}^*_{a,0}(v) + (1-t_1)\tilde{F}^*_{0,0}(v) \leq 0$ for all $v \neq 0$ and hence $v_0$ is a point of maximum of $t_1\tilde{F}^*_{a,0}(v) + (1-t_1)\tilde{F}^*_{0,0}(v)$, we have

$$D[t_1\tilde{F}^*_{a,0}(v_0) + (1-t_1)\tilde{F}^*_{0,0}(v_0)] = 0.$$

Let $v \in V$ be given and $0 < t_1 < 1$. Let $\theta^*_{t_1}(v)$ be the unique solution of the equation

$$Lz + (I-P)(t_1 a(v+z)^+) = 0 \qquad \text{in} \quad W.$$

We note that we can obtain the same results as Lemma 2.2 if we replace $\theta(v,s)$ and $\tilde{F}^*_{a,b}(v,s)$ by $\theta^*_{t_1}(v)$ and $t_1\tilde{F}^*_{a,0}(v) + (1-t_1)\tilde{F}^*_{0,0}(v)$. Therefore, it follows that $v_0 + \theta^*_{t_1}(v_0)$ is a nontirivial solution of the equation

$$t_1(Lu + au^+) + (1-t_1)Lu = 0 \qquad \text{in} \quad H,$$

with $\|w_0\| \neq 0$. This contradicts to the fact that the equation

$$Lu + au^+ - bu^- = 0 \qquad \text{in } H_0$$

has only the trivial solution. □

We now define the functional on $H$

$$F^*(u) = F(u,0) = \int_Q \frac{1}{2}[(-|u_t|^2 + |u_x|^2) + \frac{a}{2}|u^+|^2 + \frac{b}{2}|u^-|^2]dtdx.$$

The critical points of $F^*(u)$ coincide with solutions of the equation

$$Lu + au^+ - bu^- = 0 \qquad \text{in } H. \tag{2.9}$$

Let $-1 < b < 3 < a < 7$ with the condition (2.2). Then the above equation (2.9) has only the trivial solution and hence $F^*(u)$ has only one critical point $u = 0$.

Given $v \in V$, let $\theta^*(v) = \theta(v, 0) \in W$ be the unique solution of the equation

$$Lz + (I - P)[a(v + z)^+ - b(v + z)^-] = 0 \qquad \text{in } W.$$

Let us define the reduced functional $\tilde{F}^*(v)$ on $V$, by $F^*(v + \theta^*(v))$. We note that we can obtain the same result as Lemma 2.2 when we replace $\theta(v,s)$ and $\tilde{F}(v + \theta(v,s))$ by $\theta^*(v)$ and $\tilde{F}^*(v)$. We also note that $\tilde{F}^*(v)$ has only one critical point, $v = 0$.

**Lemma 2.5.** *For $c > 0$, $\tilde{F}^*(cv) = c^2\tilde{F}^*(v)$.*

*Proof.* If $v \in V$ and $z \in W$ satisfy

$$Lz + (I - P)[a(v + z)^+ - b(v + z)^-] = 0 \qquad \text{in } W,$$

then, for $c > 0$,

$$L(cz) + (I - P)[a(cv + cz)^+ - b(cv + cz)^-] = 0 \qquad \text{in } W.$$

Therefore $\theta^*(cv) = c\theta^*(v)$ for $c > 0$. From the definition of $F^*(u)$ we see that

$$F^*(cu) = c^2 F^*(u) \qquad \text{for } \ u \in H \text{ and } c > 0.$$

Hence, for $v \in H$ and $c > 0$,

$$\tilde{F}^*(cv) = F^*(cv + \theta^*(cv)) = c^2 F^*(v + \theta^*(v)) = c^2\tilde{F}^*(v). \qquad \square$$

Now we remember the notation $F_{a,b}$, which was defined in the equation (1.8). Until now, the notations $F$, $F^*$, and $\tilde{F}^*$ denote $F_{a,b}$, $F^*_{a,b}$, and $\tilde{F}^*_{a,b}$, respectively. In the following lemma we use the latter notations.

that is,

$$Lu + t_1 au^+ = 0 \quad \text{in} \quad H,$$

which contradicts to the fact that the above equation has only the trivial solution because $-1 < t_1 a < 7$.

A similar argument shows that it is impossible that (d) holds. This proves (i). The proof of (ii) is similar to that of (i).

(iii) Let $-1 < b < 3 < a < 7$ with the condition (2.2). To prove (iii), It suffices to show that $\tilde{F}^*_{a,b}(v)$ does not satisfy (a)-(d) replaced $\tilde{F}^*_{a,0}$ by $\tilde{F}^*_{a,b}$. The proofs of (a) and (b) for $\tilde{F}^*_{a,b}$ are similar to those of (a) and (b) for $\tilde{F}^*_{a,0}$.

Suppose that $\tilde{F}^*_{a,b}$ satisfies (c), that is, $\tilde{F}^*_{a,b}(v) > 0$ for all $v \neq 0$. Then there exists $t_0 \in (0,1)$ such that for all $t \leq t_0$

$$t\tilde{F}^*_{a,b}(v) + (1-t)\tilde{F}^*_{a,0}(v) \leq 0 \qquad \text{for all} \quad v \neq 0.$$

We note that there exists $v_0 \neq 0$ and $t(\leq t_0)$ such that

$$t\tilde{F}^*_{a,b}(v_0) + (1-t)\tilde{F}^*_{a,0}(v_0) = 0.$$

Let $t_2$ be the greatest number such that

$$t\tilde{F}^*_{a,b}(v_0) + (1-t)\tilde{F}^*_{a,0}(v_0) = 0$$

for some $v_0 \neq 0$. Then $0 < t_2 \leq t_0$. Since $t_2\tilde{F}^*_{a,b}(v) + (1-t_2)\tilde{F}^*_{a,0}(v) \leq 0$ for all $v \neq 0$ and hence $v_0$ is a point of maximum of

$$t_2\tilde{F}^*_{a,b}(v) + (1-t_2)\tilde{F}^*_{a,0}(v),$$

we have

$$D\left[t_1\tilde{F}^*_{a,b}(v_0) + (1-t_1)\tilde{F}^*_{a,0}(v_0)\right] = 0.$$

Let $v \in V$ be given and $0 < t_2 < 1$. Let $\theta^*_{t_2}(v)$ be the unique solution of the equation

$$Lz + (I-P)(a(v+z)^+ - t_2 b(v+z)^-) = 0 \quad \text{in} \quad W.$$

Then $v_0 + \theta^*_{t_2}(v_0)$ is a nontrivial solution of the equation

$$t_2(Lu + au^+ - bu^-) + (1-t_2)(Lu + au^+) = 0 \quad \text{in} \quad H,$$

that is,

$$Lu + au^+ - t_2bu^- = 0 \quad \text{in } H,$$

which is a contradicts the fact that the above equation has only the trivial solution because

$$\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{t_2b+1}} > 1.$$

Similarly we can prove that $\tilde{F}^*_{a,b}(v)$ does not satisfy :

(d) There exist $v_1$ and $v_2$ in $V$ such that $\tilde{F}^*_{a,b}(v_1) < 0$ and $\tilde{F}^*_{a,b}(v_2) > 0$. □

**Lemma 2.7.** *Let $-1 < b < 3 < a < 7$ satisfy the condition (2.2). Then we have $\tilde{F}(v,s) \longrightarrow -\infty$ as $\|v\| \longrightarrow \infty$. Here we note that $|||v||| = \sqrt{3}\|v\|$.*

*Proof.* We showed in Lemma 2.6 that $\tilde{F}^*(v) < 0$ for all $v \neq 0$. Suppose that it is not true that $\tilde{F}(v,s) \longrightarrow -\infty$ as $\|v\| \longrightarrow \infty$. This means that there exists a sequence $\{v_n\}_1^\infty$ in $V$ and a number $M < 0$ such that $\|v_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\tilde{F}(v_n, s) \geq M$.

For given $v_n \in V$ let $w_n = \theta(v_n)$ be the unique solution of the equation

$$Lw + (I-P)(a(v_n + w_n)^+ - b(v_n + w_n)^- - s\phi_{00}) = 0 \quad \text{in } W.$$

According to Lemma 2.2 we have that for some constant $k$

$$\|\theta(v_n) - \theta(0)\| \leq k\|v_n\|, \quad \text{or} \quad |||\theta(v_n) - \theta(0)||| \leq k|||v_n|||.$$

From this we see that the sequence $\left\{\frac{w_n + v_n}{\|v_n\|}\right\}$ is bonded in $H$. Let $z_n = v_n + w_n$, $v_n^* = \frac{v_n}{\|v_n\|}$, $w_n^* = \frac{w_n}{\|v_n\|}$ and $z_n^* = v_n^* + w_n^*$ for $n \geq 1$. For $w_n = \theta(v_n)$ and $w_n^* = \frac{w_n}{\|v_n\|}$, we have

$$w_n^* = L^{-1}(I-P)\left(a\frac{(v_n + w_n)^+}{\|v_n\|} - b\frac{(v_n + w_n)^-}{\|v_n\|} - s\frac{\phi_{00}}{\|v_n\|}\right) \quad \text{in } W.$$

Since $\left\{\frac{w_n + v_n}{\|v_n\|}\right\}$ is bounded and $\frac{\phi_{00}}{\|v_n\|} \longrightarrow 0$ as $\|v_n\| \longrightarrow \infty$, it follows that

$$a\frac{(v_n + w_n)^+}{\|v_n\|} - b\frac{(v_n + w_n)^-}{\|v_n\|} - s\frac{\phi_{00}}{\|v_n\|}$$

is bounded in $H$. Since $L^{-1}$ is a compact operator, passing to a subsequence we get that $w_n^*$ converge to $w^*$ in $W$. Since $V$ is a 2-dimensional space, we may assume that $\{v_n^*\}_1^\infty$ converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, we can assume that $\{z_n^*\}_1^\infty$ converges to an element $z^*$ in $H$.

On the other hand, since $\tilde{F}(v_n, s) \geq M$ for all $n$, we have, for all $n$,

$$\int_Q \left[\frac{1}{2}Lz_n \cdot z_n + \frac{a}{2}|z_n^+|^2 + \frac{b}{2}|z_n^-|^2 - s\phi_{00}z_n\right] dtdx \geq M.$$

Dividing the above inequality by $\|v_n\|^2$, we obtain

$$\int_Q \Big[\frac{1}{2}(-|(z_n^*)_t|^2 + |(z_n^*)_x|^2) + \frac{a}{2}|(z_n^*)^+|^2 + \frac{b}{2}|(z_n^*)^-|^2 -s\phi_{00}\frac{z_n^*}{\|v_n\|}\Big] dtdx \geq \frac{M}{\|v_n\|^2}. \tag{2.10}$$

From the definition of $w_n = \theta(v_n)$, it follows that for any $y \in W$ and $n \geq 1$

$$\int_Q [-(z_n)_t y_t + (z_n)_x y_x + az_n^+ y - bz_n^- y - s\phi_{00}y] dtdx = 0. \tag{2.11}$$

If we set $y = w_n$ in (2.11) and devide by $\|v_n\|^2$, then we have

$$\int_Q [-|(w_n^*)_t|^2 + |(w_n^*)_x|^2 + (a(z_n^*)^+ - b(z_n^*)^- - s\frac{\phi_{00}}{\|v_n\|})w_n^*] dtdx = 0. \tag{2.12}$$

for all $n \geq 1$. Let $y \in W$ be arbitrary. Dividing (2.11) by $\|v_n\|$ and letting $n \longrightarrow \infty$, we obtain

$$\int_Q [-(z^*)_t y_t + (z^*)_x y_x + a(z^*)^+ y - b(z^*)^- y] dtdx = 0. \tag{2.13}$$

We see that (2.13) can be written in the form $D\tilde{F}_{a,b}^*(v^* + w^*)(y) = 0$ for all $y \in W$. Hence by $w^* = \theta(v^*)$. Letting $n \longrightarrow \infty$ in (2.12), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_Q (-|(w_n^*)_t|^2 + |(w_n^*)_x|^2) dtdx \\
&= - \lim_{n\to\infty} \int_Q (a(z_n^*)^+ - b(z_n^*)^- - s\frac{\phi_{00}}{\|v_n\|})w_n^*\, dtdx \\
&= - \int_Q [a(z^*)^+ - b(z^*)^-] w^*\, dtdx \\
&= \int_Q [-(z^*)_t(w^*)_t + (z^*)_x(w^*)_x]\, dtdx \\
&= \int_Q [-|(w^*)_t|^2 + |(w^*)_x|^2]\, dtdx,
\end{aligned}$$

where we have used (2.13). Hence

$$\lim_{n\to\infty}\int_Q [-|(z_n^*)_t|^2 + |(z_n^*)_x|^2]\, dtdx = \int_Q [-|(z^*)_t|^2 + |(z^*)_x|^2]\, dtdx.$$

Letting $n \to \infty$ in (2.10), we obtain

$$\tilde{F}^*_{a,b}(v^*) = \int_Q \left[\frac{1}{2}(-|(z^*)_t|^2 + (z^*)_x|^2) + \frac{a}{2}|(z^*)^+|^2 + \frac{b}{2}|(z^*)^-|^2\right] dtdx \geq 0.$$

Since $\|v^*\| = 1$, this contradicts to the fact that $\tilde{F}^*(v) < 0$ for all $v \neq 0$. This proves that $\tilde{F}(v,s) \to -\infty$ as $\|v\| \to \infty$. □

We now state the deformation lemma, which is useful in the critical point theory [6].

**Lemma 2.8.** *Let $E$ be a real Banach space and $I \in C^1(E,\mathbb{R})$. Suppose $I$ satisfies Palais-Smale condition. Let $N$ be a given neighborhood of the set $K_c$ of the critical points of $I$ at a given level $c$. Then there exists $\epsilon > 0$, as small as we want, and a deformation $\eta : [0,1]\times E \longrightarrow E$ such that, denoting by $A_b$ the set $\{x \in E : I(x) \leq b\}$,*

*(a)* $\eta(0,x) = x \quad \forall x \in E,$

*(b)* $\eta(t,x) = x \quad \forall x \in A_{c-2\epsilon} \cup (E\backslash A_{c+2\epsilon}), \quad \forall t \in [0,1],$

*(c)* $\eta(1,\cdot)(A_{c+\epsilon}\backslash N) \subset A_{c-\epsilon}.$

We now prove our main result in this section, with the aid of several lemmas.

*Proof of Theorem 2.1.* The statement (ii) is valid. Hence it suffices to prove (i). Let $-1 < b < 3 < a < 7$, satisfying the condition (2.2). Let $s > 0$. By Lemma 2.3, there exists a small open neighborhood $B$ of 0 in $V$ such that in $B$, $v = 0$ is a strict local point of minimum of $\tilde{F}$. Since $\tilde{F} \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$ (Lemma 2.7) and $\tilde{F} \in C^1(V,R)$ satisfies Palais-Smale condition, $\max_{v\in V}\tilde{F}(v)$ exists and is a critical value of $\tilde{F}$. Hence there exists a critical point $v_0$ of $\tilde{F}$ such that

$$\tilde{F}(v_0,s) = \max_{v\in V}\tilde{F}(v,s).$$

Let $C$ be an open neighborhood of $v_0$ in $V$ such that $B \cap C = \phi$. Since $\tilde{F}(v,s) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$, we can choose $v_1 \in V\backslash(B\cup C)$ such that

$\tilde{F}(v_1, s) < \tilde{F}(0, s)$. Let $\Gamma$ be the set of all paths in $V$ joining 0 and $v_1$. We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}(v, s).$$

Let $\Gamma' = \{\gamma \in \Gamma : \gamma \cap C = \phi\}$ and

$$c' = \inf_{\gamma \in \Gamma'} \sup_{\gamma} \tilde{F}(v, s).$$

The fact that in $B$, $v = 0$ is strict local point of minimum of $\tilde{F}(v, s)$, the fact that $\tilde{F}(v) \longrightarrow -\infty$ as $|||v||| \longrightarrow \infty$, the fact that $\tilde{F}$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem (cf. [1]) imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}(v, s)$$

is a critical value of $\tilde{F}$.

First we prove that if $\tilde{F}(v_0, s) = c$, then there exists a critical point $v$ of $\tilde{F}$ at level $c$ such that $v \neq v_0$ (of course $v \neq 0$ since $c \neq \tilde{F}(0)$).

We claim that if $\tilde{F}(v_0, s) = c$, then $c = c'$. In fact, since $\Gamma' \subset \Gamma$, $c \leq c'$. On the other hand, $c' \leq c$ since $c$ is the maximum value of $\tilde{F}$. Hence $c = c'$. Suppose by contradiction $K_c = \{v_0\}$. By the above claim $c = c'$. Let us fix $\epsilon$, $\eta$ as in Lemma 2.8 with $E = V$, $I = \tilde{F}$, $c = c$, $N = C$ and taking $\epsilon < \frac{1}{2}(c - \tilde{F}(0, s))$. Taking $\gamma \in \Gamma'$ such that $\sup_{\gamma} \tilde{F} \leq c$. From Lemma 2.8 $\eta(1, \cdot) \circ \gamma \in \Gamma$ and

$$\sup \tilde{F}(\eta(1, \cdot) \circ \gamma) \leq c - \epsilon < c,$$

which is a contradiction. Therefore, there exists a critical point $v$ of $\tilde{F}$ at level $c$ such that $v \neq v_0$, 0, which means that the equation (2.1) has at least 3 solutions.

Finally, if $\tilde{F}(v_0, s) \neq c$, then there exists a critical point $v$ of $\tilde{F}$ at level $c$ such that $v \neq v_0$, 0 (since $c \neq \tilde{F}(v_0, s)$ and $c > \tilde{F}(0, s)$). Therefore, in case $\tilde{F}(v_0, s) \neq c$, the equation (2.1) has also at least 3 solutions. □

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# A CHARACTERIZATION FOR FOURIER (ULTRA-)HYPERFUNCTIONS

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ABSTRACT. The space of test functions for Fourier hyperfunctions is characterized by two conditions $\sup|\varphi(x)|\exp k|x| < \infty$ and $\sup|\hat{\varphi}(\xi)|\exp h|\xi| < \infty$ for some $h, k > 0$. Also, the space of test functions for Fourier ultra-hyperfunctions is characterized in a similar fashion. Combining these results and the new characterization of Schwartz space in [1] we can easily compare important spaces $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{S}$ which are both invariant under Fourier transformations.

The purpose of this talk is to give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions.

In [6], K. W. Kim, S. Y. Chung and D. Kim introduce the real version of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows,

$$\mathcal{F} = \{\varphi \in C^\infty \mid \sup_{\alpha,x} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \quad \text{for some} \quad k,\, h > 0\}.$$

They also show the equivalence of the above definition and Sato-Kawai's original definition in complex form.

Also, in [1] J. Chung, S. Y. Chung and D. Kim give new characterization of the Schwartz space $\mathcal{S}$, i.e., show that for $\varphi \in C^\infty$ the following are equivalent:

(1) $\varphi \in \mathcal{S}$;
(2) $\sup|x^\alpha \varphi(x)| < \infty$, $\sup|\partial^\beta \varphi(x)| < \infty$ for all multi-indices $\alpha$ and $\beta$;
(3) $\sup|x^\alpha \varphi(x)| < \infty$, $\sup|\xi^\beta \hat{\varphi}(\xi)| < \infty$ for all multi-indices $\alpha$ and $\beta$.

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In a similar fashion as above we will give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as the main theorem in this paper which says that for $\varphi \in C^\infty$ the following are equivalent:

(1) $\varphi \in \mathcal{F}$;

(2) $\sup |\varphi(x)| \exp k|x| < \infty$, $\sup |\hat{\varphi}(\xi)| \exp h|\xi| < \infty$ for some $h$, $k > 0$.

Observing the above growth conditions we can easily see that the space $\mathcal{F}$ which is invariant under the Fourier transformation is much smaller than Schwartz space $\mathcal{S}$. Since an element in the strong dual $\mathcal{F}'$ of the space $\mathcal{F}$ is called a Fourier hyperfunction, the space $\mathcal{F}'$ of Fourier hyperfunctions which is also invariant under the Fourier transformation is much bigger than the space $\mathcal{S}'$ of tempered distributions.

The complete version of this talk will be published in [2].

We need the following characterization to compare the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions with the Schwartz space.

**Theorem 1 [1].** *The Schwartz space can be characterized by the following two conditions*

$$\sup |x^\alpha \varphi(x)| < \infty,$$

$$\sup |\xi^\beta \hat{\varphi}(\xi)| < \infty$$

*for all multi-indices $\alpha$ and $\beta$.*

Now, we are going to introduce the original complex version and new real definition of test functions for the Fourier hyperfunctions as in [6], and state their equivalence.

**Definition 2 [7].** A real valued function $\varphi$ is in $\mathcal{F}$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and if there are positive constants $h$ and $k$ such that

$$|\varphi|_{k,h} = \sup_{\alpha,x} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|}\alpha!} \exp k|x| < \infty.$$

**Definition 3.** We denote by $\mathcal{F}'$ the strong dual space of $\mathcal{F}$ and call its elements *Fourier hyperfunctions.*

Now we shall give new characterization of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions which is the main result in this paper.

First, we state

**Theorem 4.** *The following conditions for $\varphi \in C^\infty$ are equivalent:*

(i) *There are positive constants $k$ and $h$ such that*

$$\sup_{\alpha,x} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty.$$

(ii) *There are positive constants $C$, $k$ and $h$ such that*

$$\sup_x |\varphi(x)| \exp k|x| < \infty,$$
$$\sup_x |\partial^\alpha \varphi(x)| \leq C h^{|\alpha|}\alpha!.$$

(iii) *There are positive constants $k$ and $h$ such that*

$$\sup_x |\varphi(x)| \exp k|x| < \infty,$$
$$\sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty.$$

Now we can rephrase Theorem 4 as follows.

**Theorem 5.** *The space $\mathcal{F}$ of test functions for the Fourier hyperfunctions consists of all locally integrable functions such that for some $h,\ k > 0$*

$$\sup_x |\varphi(x)| \exp k|x| < \infty,$$
$$\sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty.$$

Similarly, the space $\mathcal{G}$ of test functions for the Fourier ultra-hyperfunctions can be obtained as follows.

**Theorem 6.** *The space $\mathcal{G}$ of test functions for the Fourier ultra-hyperfunctions consists of all locally integrable functions such that for every $h,\ k > 0$*

$$\sup_x |\varphi(x)| \exp k|x| < \infty,$$
$$\sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty.$$

**Remark.** Combining Theorem 2 on the Schwartz space $\mathcal{S}$, Theorem 5 on the space $\mathcal{F}$ Theorem 6 on the space $\mathcal{G}$ we can easily compare the spaces $\mathcal{S}$, $\mathcal{F}$ and $\mathcal{G}$ all of which are invariant under the Fourier transformations as follows:

(i) The space $\mathcal{S}$ consists of all locally integrable functions $\varphi$ such that $\varphi$ itself and its Fourier transform $\hat{\varphi}$ are both rapidly decreasing.

(ii) The space $\mathcal{F}$ consists of all locally integrable functions $\varphi$ such that $\varphi$ itself and its Fourier transform $\hat{\varphi}$ are both exponentially decreasing.

(iii) The space $\mathcal{G}$ consists of all locally integrable functions $\varphi$ such that $\varphi$ itself and its Fourier transform $\hat{\varphi}$ are both super-exponentially decreasing.

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# AN INVERSE PROBLEM FOR THE HEAT EQUATION

SOON-YEONG CHUNG

ABSTRACT. If $U(x,t)$ is a heat solution satisfying

$$\int |\partial^\alpha U(x,t)|^p dx < M, \quad |\alpha| \le s, \quad t > 0, \quad p > 1$$

then its initial value $U(x,0^+)$ belongs to $W^{p,s}$, which shows the regularity of the initial state. Also, the integral representation of the solutions of the heat equation and a structure theorem for Sobolev spaces are given.

## 0. Introduction

In the theory of partial differential equations with given initial values and boundary values one usually investigates to examine the well-posedness. This problem is called the direct problem in our view point. This theory is strong enough for us to determine the situation anywhere and anytime provided that the data are actually given. However, in many cases the data are not completely known for us. Then in those situations arise the new problem to determine the unknown data by taking other conditions for the solution, which is called the inverse problem.

In this paper we discuss the very simple direct and inverse problems for the heat equation $(\partial_t - \Delta)U(x,t) = 0$ with the initial data. The main theorem states that if $U(x,t)$ is a heat solution satisfying

$$\int |\partial^\alpha U(x,t)|^p dx < M$$

*Key words and phrases.* Sobolev space, heat equation.
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for $0 < t < T$, $|\alpha| \leq s$ and $p > 1$ then its initial value $U(x, 0^+)$ must belong to the Sobolev space $W^{p,s}$ (see Theorem 2.4). Thus in view of Sobolev imbedding theorem we can obtain the regularity of the initial condition by considering the growth of solution. As a corollary of this result we a structure theorem for the Sobolev spaces $W^{p,s}$. The complete version of this paper with the detailed proofs will be published elsewhere.

## §1. A Direct problem

We recall the definition of Sobolev spaces. Let $s$ be a nonnegative integer and let $1 \leq p < +\infty$.

**Definition 1.1.** We denote by $W^{p,s}$ the space of all distributions $u$ such that

$$\partial^\alpha u \in L^p, \quad |\alpha| \leq s$$

equipped with the norm

$$\|u\|_{p,s} = \left[ \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_p^p \right]^{1/p}$$

where $\|\cdot\|_p$ denotes $L^p$-norm on $\mathbb{R}^n$.

Let $E(x,t)$ be the heat kernel defined by

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right), & t > 0 \\ 0, & t \leq 0. \end{cases}$$

First, we present a direct problem which is, in fact, an initial value problem for the heat equation with initial data in $W^{p,s}$.

**Theorem 1.2.** *Suppose that $T > 0$, $S \geq 0$ and $1 \leq p < +\infty$. Then for every $u \in W^{p,s}$ $U(x,t) = E * u$ is well defined and a $C^\infty$ function in $\mathbb{R}^n \times (0,T)$ satisfying that*

(1.1) $(\partial_t - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$

(1.2) *There exists a constant $M > 0$ such that*

$$\int |\partial^\alpha U(x,t)|^p dx < M, \quad 0 < t < T, \ |\alpha| \leq s$$

(1.3) $U(x,t) \to u$ *in* $W^{p,s}$ *as* $t \to 0$, *where $*$ denotes the convolution with respect to the space variable $x$.*

## §2. Inverse problems

Here we restate a uniqueness theorem for the heat equation in simple form which is very useful later

**Theorem 2.1.** ([F], Theorem 1.16) *Let $U(x,t)$ be a continuous function on $\mathbb{R}^n \times [0,T)$ with the following property*

(i) $(\partial_t - \Delta)u(x,t) = 0$ *in* $\mathbb{R}^n \times (0,T)$

(ii) $\int_0^T \int_{\mathbb{R}^n} |u(x,t)| e^{-k|x|^2} dx\, dt < +\infty$ *for some* $k > 0$.

*Then $u(x,0) = 0$ implies that $u(x,t) \equiv 0$ in $\mathbb{R}^n \times [0,T)$.*

Actually the inverse problem given here is nothing but a converse part of Theorem 1.2. But that result will give many meaningful information as corollaries.

Now we are in a position to state and prove the main theorem in this paper.

**Theorem 2.2.** *Suppose that $U(x,t)$ is a $C^\infty$ function in $\mathbb{R}^n \times (0,T)$ satisfying*

(2.1) $(\partial_t - \Delta)U(x,t) = 0, \ (x,t) \in \mathbb{R}^n \times (0,T)$

(2.2) *there exists a constant $M > 0$ such that*

$$\int |\partial^\alpha U(x,t)|^p dx < M, \ 0 < t < T, \ |\alpha| \le s,$$

*for $T > 0$, $s \ge 0$ and $1 < p < \infty$. Then the initial value $U(x,0^+)$ exists in $W^{p,s}$ where the limit $U(x,0^+) = \lim_{t \to 0^+} U(x,t)$ is taken in the topology of $W^{p,s}$. Furthermore, $U(x,t)$ can be uniquely expressed by $U(x,t) = U(x,0+) * E$.*

**Remark.** If $p = 1$ then this theorem may not be true. To see this consider the heat kernel $E(x,t)$. This satisfies all the conditions but $E(x,0^+)$ becomes $\delta(x)$ which does not belong to $L^1$.

Now we will give some corollaries of the above result. Using the Sobolev imbedding theorem we can directly obtain;

**Corollary 2.3.** *If $s > \frac{n}{2} + k$ and if $U(x,t)$ is a $C^\infty$ function in $\mathbb{R}^n \times (0,T)$ satisfying*

$$(\partial_t - \Delta)U(x,t) = 0, \quad 0 < t < T$$

*and*

$$\int |\partial^\alpha U(x,t)|^2 dx < M, \quad 0 < t < T$$

*then the initial value belongs to $C^k(\mathbb{R}^n)$, i.e., $k$-times differentiable function in $\mathbb{R}^n$.*

The following presents a structure theorem of Sobolev spaces.

**Corollary 2.4.** *If $1 < p < \infty$ and $s > 0$ then every $u \in W^{p,s}$ can be written of the form*

$$u(x) = \Delta g(x) + h(x)$$

*where $g$ is a continuous function and $h(x)$ is a real analytic function on $\mathbb{R}^n$ and $\Delta g, h \in W^{p,s}$.*

**Corollary 2.5.** *$U(x,t)$ is a heat solution satisfying that*

$$\int |U(x,t)|^2 dx < M, \quad 0 < t < T$$

*if and only if there exists a function $f \in L^2$ such that*

$$U(x,t) = \int e^{ixy - ty^2} f(y) dy, \quad 0 < t < T.$$

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# REGULARITY FOR DIFFRACTION PROBLEMS

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## 1. INTRODUCTION

In this note, we will describe regularity for solution of the diffraction problems of the system of elasticity in domain consisting two heterogeneous media with non-smooth interface of these media. The idea of this paper is the almost same as the our previous paper(see [E,S:1]) but the novelty in this paper is the complicated computational scheme to carry out the idea of the previous paper.

Let $\Omega$ be a bounded domain and let $D$ be a Lipschitz domain and $D \subset \bar{D} \subset \Omega \subset R^n$.

We will consider for weak solutions of the system of elasticity

(1.1)

$$\text{for each } r = 1, .., n, \quad \sum_{i,j,s=1}^{n} \frac{\partial}{\partial x_i}\left(\left(a_{ij}^{rs}\chi_D + b_{ij}^{rs}\chi_{D^c}\right)\frac{\partial}{\partial x_j}u^s\right) = 0 \quad \text{in } \Omega$$

where

$$\sum_{i,j,s=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{rs}\frac{\partial}{\partial x_j}u^s\right) = \text{ the r-th component of } \mu\Delta\vec{u} + (\lambda+\mu)\nabla div\vec{u},$$

$$\sum_{i,j,s=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}^{rs}\frac{\partial}{\partial x_j}u^s\right) = \text{ the r-th component of } \tilde{\mu}\Delta\vec{u} + (\tilde{\lambda}+\tilde{\mu})\nabla div\vec{u}.$$

We assume that the Lamé constants $\lambda, \mu, \tilde{\lambda}, \tilde{\mu}$ satisfy

$$\mu > 0, \tilde{\mu} > 0, \lambda > \frac{-2\mu}{n} \text{ and } \tilde{\lambda} > \frac{-2\tilde{\mu}}{n}.$$

By a weak solution $\vec{u} = (u_1,..,u_n) \in W^{1,2}(\Omega)$ of the system of elasticity (1.1), we mean ,for all $\vec{\phi} = (\phi_1,..,\phi_n) \in C_0^\infty(\Omega)$,

$$\int_D \mu(\nabla\vec{u} + \nabla\vec{u}^t)\nabla\vec{\phi} + \lambda div(\vec{u})div(\vec{\phi})dX$$
$$+ \int_{\Omega\setminus\bar{D}} \tilde{\mu}(\nabla\vec{u} + \nabla\vec{u}^t)\nabla\vec{\phi} + \tilde{\lambda}div(\vec{u})div(\vec{\phi})dX = 0$$

where $(\nabla\vec{u} + \nabla\vec{u}^t)\nabla\vec{\phi} = \sum_{i,j=1}^n (D_iu^j + D_ju^i)D_i\phi^j$.

We wish to investigate the behavior of $\nabla\vec{u}$ near the interface $\partial D$ and understand the meaning of certain compatibility condition(or transmission condition) on the interface $\partial D$.

There is a large list of works where the regularity of solutions to this type of equations is studied [L,R,U], but in them the boundary of $D$ is required to be sufficiently smooth so that the boundary can be flattened. Of course, in these cases the regularity of the solution is much better; for instance, when $\partial D$ is locally the graph of a function in the class $W^{2,p}$ , the class of all $L^p$ functions with first and second derivatives ( in the distribution sense) belonging to $L^p$, and $p > n$ the gradient of $u$ is Hölder continous up to the boundary from either side of $D$ (See [L,R,U]). But these method does not work even for $C^1$ interface.

In [B,F,I], when $n = 2$ and $D$ is polygon, H. Bellout, A. Friedman, and Isakov studied a weak solution $u$ of the divergence form operator

$$div\,(((k-1)\chi_D + 1)\nabla u) = 0 \quad \text{in } \Omega \tag{1.2}$$

to apply stability for Inverse Problem. They studied the behavior of $u$ near a coner $X_0 \in \partial D$ of $D$ and show that $|u(X) - u(X_0)| \le C|X - X_0|^{\frac{1}{2}+\alpha}$ where $\alpha > 0$.

In [E,F,V], L. Escauriaza, E. Fabes and G. Verchota showed that the weak solution of (1.2) satisfied the estimate (1.3) below for any Lipschitz domain $D$ and any dimension $n$.

In [E,S:1], we considered a weak-solution $u$ of the elliptic equation:

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) = 0 \quad \text{in } \Omega \subset R^n$$

where

$$a_{ij}(x) = a_{ij}^1\chi_D + a_{ij}^2\chi_{\bar{D}^c},$$
$(a_{ij}^1)$ and $(a_{ij}^2)$ are positive definite constant matrices .

Under the condition that $(a^1_{ij}) - (a^2_{ij})$ is a semi-positive definite matrix (or semi-negative definite matrix), we show that the trace of $\nabla u$ on $\partial D$, the boundary of $D$, makes sense, and in fact

$$\int_{\partial D} |(\nabla u)^*|^2 d\sigma < \infty \tag{1.3}$$

where $()^*$ denotes the nontangential maximal function (see body of paper for the relevant definition). In [E,F,V], to obtain (1.3) they reduced the problem to showing that if $|\lambda| > \frac{1}{2}$ and $\lambda$ is real,

$$\lambda I - K^* : L^2(\partial D) \to L^2(\partial D) \quad \text{is invertible,}$$
$$\text{where} \quad K^*(f)(P) = p.v. \frac{1}{\omega_n} \int_{\partial D} \frac{\langle P-Q, N(P)\rangle}{|P-Q|^n} f(Q) d\sigma(Q),$$

$N(P)$ denotes the outer unit normal vector at the point P on $\partial D, \omega_n$ the surface area of the unit sphere in $R^n$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product on $R^n$. As a consequence, they obtained a representation of $u$ in a neighborhood of $D$ as the sum of a Single Layer Potential and of a Newtonian Potential.

For the general case we may assume $(a^1_{ij}) = A$ is a diagonal matrix and $(a^2_{ij}) = I$ is the identity matrix. Formally, through a careful integration by parts, the transmission conditions

$$\begin{cases} u^+ = u^- & \text{on } \partial D, \\ \langle A\nabla u^+, N\rangle = \langle \nabla u^-, N\rangle & \text{on } \partial D \end{cases}$$

together with the condition that $A - I$ ( or $I - A$) is a strictly positive matrix, allow us to bound $\int_{\partial D} |\nabla u|^2 d\sigma$ by a constant times $\int_\Omega |\nabla u|^2 dx$. Here the superscripts $+$ and $-$ indicate limits inside $D$ and outside $\bar{D}$ respectively. This formal reasoning was used to invert a system of operators which gives us a representation formula for the weak solution $u$ as a sum of two different single layer potentials over $\partial D$. In particular, we showed that when $S$ and $\tilde{S}$ denote respectively the Single Layer Potentials(see section 2 for the similar definition) of the constant coefficient operators (i.e.$a^1_{ij}D_{ij}$ and $a^2_{ij}D_{ij}$) in the interior and exterior of $D$ in (1.2), the mapping

$$\begin{aligned} &L^2(\partial D) \times L^2(\partial D) \to L^2_1(\partial D) \times L^2(\partial D) \\ &(f,g) \to \left(S(f) - \tilde{S}(g), \langle A^1 N, \nabla S(f)^+\rangle - \langle A^2 N, \nabla \tilde{S}(g)^-\rangle\right), \end{aligned} \tag{1.4}$$

is invertible, where $A^1 \equiv (a^1_{ij}), A^2 \equiv (a^2_{ij})$. Here $L^2_1(\partial D)$ denotes the set of all $L^2(\partial D)$ functions with first derivatives in $L^2(\partial D)$. As in [E,F,V], this representation of the solution gave us the desired estimates and the concrete meaning of the transmission conditions. In the the case when $A - I$ is semi-positive definite matrix (or semi-negative definite matrix), the integration by parts argument mentioned above only controle

$$||(A-I)\nabla u||_{L^2(\partial D)}.$$

But we again controled the $L^2(\partial D)$ norm of the full derivatives of $u$ by using the singular integral estimate

$$||\nabla\Gamma * (\nabla \cdot ((A-I)\nabla u))\chi_D)||_{L^2(\partial D)} < C||(A-I)\nabla u||_{L^2(\partial D)}.$$

Here $\Gamma$ is the fundamental solution of $\Delta$ and $*$ denote the convolution. The method to obtain the above singular integral estimate is similar to the one used in [D-K-V;1].

But, unfortuately, our method does not work all the cases, that is, we imposed the semi-positive(or semi-negative) restriction on $A^1 - A^2$.

**Problem 1.** *Can we show the estimate (1.3) for solutions of the diffraction problem (1.2) for any positive matrices $A^1$ and $A^2$?*

When $A^1 - A^2$ is not semi-positve and not semi-negative, we met severe difficulty to prove the closed range of the operator in (1.4). Recently, in [E,S:2], we gave a partial answer under severe geometric restriction on $D$.

In the above system (1.1), Frenando[F] studied the behavior of the solution near a singular point in $R^2$ using Mellin transform.

In [E,S:1], under the conditions $\lambda \leq \tilde{\lambda}$ and $\mu \leq \tilde{\mu}$ we show the similar estimate as (1.3).

A more careful analysis shows the following :

**Theorem 1.** *Let $D$ be a bounded Lipschitz domain with $D \subset \bar{D} \subset \Omega$. Let $\vec{u} = (u_1, .., u_n)$ be in $[W^{1,2}(\Omega)]^n$ be a weak solution of the system (1.1). Then, if*

$$2\mu + k\lambda > 2\tilde{\mu} + k\tilde{\lambda} \text{ and } \mu > \tilde{\mu} \text{ for all } k = 1, .., n$$

$$(\text{or } 2\mu + k\lambda < 2\tilde{\mu} + k\tilde{\lambda} \text{ and } \mu < \tilde{\mu} \text{ for all } k = 1, .., n),$$

*there is constant $C$ depending on the Lipschitz character of D and the Lamé constants $\mu, \tilde{\mu}, \lambda, \tilde{\lambda}$ such that*

$$||(\nabla \vec{u}^{\pm})^*||_{L^2(\partial D)} \leq C||\vec{u}||_{W^{1,2}(B)}.$$

In this note, we will explain the idea of the proof of Theorem 1.

## 2. Preliminary

The letters $X, Y$ will denote points in $R^n$, and the letter $P, Q$ will denote points of the boundary of a domain. Derivatives $\frac{\partial}{\partial X_j}$ will often be written $D_j$. We will use the index summation convention of repeated indices. Given a $n \times n$ matrix $A = (a_{ij})$ and vectors $\xi = (\xi_1, ..\xi_n), \eta = (\eta_1, .., \eta_n)$, we can then write

$$\langle A\xi, \eta\rangle = a_{ij}\xi_j\eta_i.$$

An open ball of radius $r$ centered at X will be denoted as $B_r(X)$ and when the center of the ball is the origin we will simply use the notation $B_r$.

*Definition.* A bounded open connected domain $D \subset R^n$ is called a Lipschitz domain if for each $P \in \partial D$ there is an open, right circular, double truncated cylinder $Z(P, r)$ centered at $P$, with radius equal to $r$, whose bottom and top are at a positive distance( usually a multiple of $r$) from $\partial D$, such that there is a coordinate system $(x, s), x \in R^{n-1}, s \in R$, with $s$-axis containing the axis of $Z$ and a Lipschitz function $\phi : R^{n-1} \to R$ such that $Z \cap D = \{(x, s) \in R^{n-1} \times R : s > \phi(x)\} \cap Z$, and $Z \cap \partial D = \{(x, s) \in R^{n-1} \times R : s = \phi(x)\} \cap Z$.

If $u$ is a function defined on a neighborhood of $D$ we define the interior and exterior nontangential maximal fuctions of $u$ at $P \in \partial D$ as

$$(u^+)^*(P) = Sup\{|u(X)| : X \in \gamma_+(P)\}$$
$$\text{and} \quad (u^-)^*(P) = Sup\{|u(X)| : X \in \gamma_-(P)\}$$

where

$$\gamma_+(P) = \{X \in D : d(X, P) < \frac{3}{2}d(X, \partial D), \quad d(X, \partial D) < r\}$$

and

$$\gamma_-(P) = \{X \in R^n \setminus D : d(X, P) < \frac{3}{2}d(X, \partial D), \text{ and } d(X, \partial D) < r\},$$

and $r$ is chosen so that the sets are strictly contained in the interior of $D$ and the interior of $\Omega \setminus \bar{D}$ (here $\Omega$ is in the introduction).

$$\mu\Delta\vec{u} + (\lambda + \mu)\nabla(div\vec{u}) = \vec{0}$$

with Lamé constant

$$\mu > 0 \text{ and } \lambda > -\frac{2\mu}{n},$$

the corresponding Single Layer Potential is given by $\mathcal{S}(\vec{f}) = (\mathcal{S}(\vec{f})_1, .., \mathcal{S}(\vec{f})_n)$, where

$$\mathcal{S}(\vec{f})_i = \int_{\partial D} \Gamma^{ij}(X-Q) f_j(Q) d\sigma(Q) \quad \text{for } X \in R^n, i = 1, .., n$$

$\vec{f} = (f_1, .., f_n) \in [L^p(\partial D)]^n$, and $(\Gamma^{ij}(X))$ is the Kelvin matrix or the fundamental solution (see [Ku])

$$\Gamma^{ij}(X) = \frac{A}{\omega(2-n)} \delta_{ij} |X|^{2-n} + \frac{B}{\omega} X_i X_j |X|^{-n}$$

where

$$A = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda}\right) \text{ and } B = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda}\right).$$

As before, the following properties are well known:

$$\mu \Delta \mathcal{S}(\vec{f}) + (\lambda + \mu) \nabla (div \mathcal{S}(\vec{f})) = \vec{0} \quad \text{in } R^n \setminus \partial D,$$

$$\mathcal{S}(\vec{f}) \text{ is continuous across } \partial D,$$

and from the result in [C,Mc,Me] we have

$$(2.1) \quad \|(\nabla \mathcal{S}(\vec{f})^+)^*\|_{L^p(\partial D)} + \|(\nabla \mathcal{S}(\vec{f})^-)^*\|_{L^p(\partial D)} \le C \|\vec{f}\|_{L^p(\partial D)} \quad 1 < p < \infty$$

where $C$ depends on the Lipschitz character of $D, \lambda$, and $\mu$. Here $\nabla \vec{u}$ is the matrix $(D_i u^j)$. As before, the standard arguments yield the trace formula

$$\lim_{\substack{X \in \gamma^{\pm}(P) \\ X \to P}} D_i \mathcal{S}(\vec{f})_j(X) = \pm \left\{ \frac{1}{2\mu} N_i f_j(P) - B N_i N_j \langle N, \vec{f}(P) \rangle \right\} + p.v. \int_{\partial D} D_i \Gamma^{ik}(P-Q) f_k(Q) d\sigma(Q).$$

We will write the coefficient matrix $(a_{ij}^{rs})$ associated to the system $\mu \Delta \vec{u} + (\lambda + \mu) \nabla (div \vec{u})$ as

$$a_{ij}^{rs} = \mu \delta_{ij} (\delta_{rs} + \delta_{is} \delta_{jr}) + \lambda \delta_{js} \delta_{ri}.$$

Therefore

$$a_{ij}^{rs} = a_{ji}^{sr}$$
$$a_{ij}^{rs}\frac{\partial u^s}{\partial X_j}\frac{\partial u^r}{\partial X_i} = \frac{\mu}{2}|\nabla\vec{u} + \nabla\vec{u}^t|^2 + \lambda(div\vec{u})^2.$$

(This shows we do not have the strong ellipticity condition, that is, $a_{ij}^{rs}\xi_i^r\xi_j^s > c\sum_{lk}|\xi_l^k|^2$.) Here $\nabla u^t$ denote the transpose of the $n \times n$ matrix $\nabla u$. The coefficients satisfy the Legendre-Hadamard condition

$$a_{ij}^{rs}\xi_i\xi_j z^r z^s > c|\vec{\xi}|^2|\vec{z}|^2 \text{for all nonzero } \vec{\xi}, \vec{z} \in R^n,$$

where $c$ is a positive constant. The conormal derivative $\frac{\partial}{\partial \nu}$ on the boundary is given by

$$\frac{\partial \vec{u}}{\partial \nu} = \mu(\nabla\vec{u} + \nabla\vec{u}^t)N + \lambda div(\vec{u})N.$$

Therefore

$$\frac{\partial}{\partial \nu}\mathcal{S}(\vec{f})^{\pm}(P) = \pm\frac{1}{2}\vec{f}(P) + \mathcal{K}^*(\vec{f})(P) \tag{2.2}$$

where $\mathcal{K}^*$ is a bounded singular integral operator on $L^p(\partial D)$, $1 < p < \infty$ by the result of [C,Mc,Me].

Let $\Psi$ denote the space of vector valued function on $R^n$ satisfying the equations $D_i\psi^j + D_j\psi^i = 0 \quad 0 \le i, j \le n$. Define

$$L^2_\Psi(\partial D) = \{\vec{f} \in [L^2(\partial D)]^n : \int_{\partial D} \vec{f}\cdot\vec{\psi}d\sigma = 0 \quad \text{for all } \vec{\psi} \in \Psi\}.$$

**Theorem 2.1 [D,K,V;2].** *Let $D$ be a bounded Lipschitz domain with connected boundary. Then*

$$(i)\ \mathcal{S} : [L^2(\partial D)]^n \to [L^2_1(\partial D)]^n,$$
$$(ii)\ -\frac{1}{2}I + \mathcal{K}^* : L^2_\Psi(\partial D) \to L^2_\Psi(\partial D),$$
$$(iii)\ \frac{1}{2}I + \mathcal{K}^* : L^2(\partial D) \to L^2(\partial D)$$

*are all invertible.*

## 3. Transmission Problems in Elasticity

Throughout this paper, $S$ and $\tilde{S}$ denote respectively the Single Layer Potential of the operators $\mu\Delta + (\lambda+\mu)\nabla div$ and $\tilde{\mu}\Delta + (\tilde{\lambda}+\tilde{\mu})\nabla div$, that is,

$$S(\vec{f})^r(X) = \int_{\partial D} \Gamma^{rs}(X-Q) f^s(Q) d\sigma(Q) \quad \text{and}$$
$$\tilde{S}(\vec{f})^r(X) = \int_{\partial D} \tilde{\Gamma}^{rs}(X-Q) f^s(Q) d\sigma(Q)$$

where $\Gamma$ and $\tilde{\Gamma}$ are respectively the Kelvin matrices or fundamental solutions of the above operators. $\frac{\partial}{\partial\nu}$ and $\frac{\partial}{\partial\tilde{\nu}}$ denote the corresponding conormal derivatives ( also called traction) to the operator in the interior and exterior of $D$ so that

$$\frac{\partial \vec{u}^+}{\partial\nu} = \lambda(div\vec{u}^+)N + \mu(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})N$$
$$\frac{\partial \vec{u}^-}{\partial\tilde{\nu}} = \tilde{\lambda}(div\vec{u}^-)N + \tilde{\mu}(\nabla\vec{u}^- + \nabla\vec{u}^{-t})N.$$

We shall also assume, for simplicity, that the domain $\Omega$ in (1.1) is the unit ball $B$ and $D \subset B_{\frac{1}{2}}$

**Theorem 2.** *If $D, \mu, \lambda, \tilde{\mu}$, and $\tilde{\lambda}$ satisfy the assumptions of Theorem 1 , then the mapping*

$$\mathcal{T} : [L^2(\partial D)]^n \times [L^2(\partial D)]^n \to [L^2_1(\partial D)]^n \times [L^2(\partial D)]^n \quad \textit{defined by}$$
$$\mathcal{T}(\vec{f},\vec{g}) = \left( S(\vec{f}) - \tilde{S}(\vec{g}), \frac{\partial}{\partial\nu} S(\vec{f})^+ - \frac{\partial}{\partial\tilde{\nu}} \tilde{S}(\vec{g})^- \right)$$

*is an invertible operator.*

For simplicity we assume $n \geq 3$ leaving the details when $n = 2$ for the reader. Now, we will prove Theorem 2 assuming that Theorem 1 has been already proved.

*Proof of Theorem 1.* We take a cut-off function $\phi \in C_0^\infty(B_{\frac{7}{8}})$ with $\phi = 1$ on $B_{\frac{3}{4}}$ and define

$$\vec{h} \stackrel{\text{def}}{=} \{\tilde{\mu}\Delta(\vec{u}\phi) + (\tilde{\lambda}+\tilde{\mu})\nabla div(\vec{u}\phi)\}\chi_{R^n \setminus \bar{D}}.$$

As before it is easy to see that $\vec{h} \in [C_0^\infty(B_{\frac{7}{8}} \setminus B_{\frac{3}{4}})]^n$ and $\|\vec{h}\|_{L^2(R^n)} \leq C\|\vec{u}\|_{W^{1,2}(B)}$. Then

$$\tilde{\Gamma}(\vec{h}) \stackrel{\text{def}}{=} \int_{R^n \setminus D} \tilde{\Gamma}(X-Y)\vec{h}(Y)dY.$$

is a smooth function in the neighborhood of $D$. From Theorem 2 we can choose $\vec{f}, \vec{g} \in [L^2(\partial D)]^n$ such that

$$S(\vec{f}) - \tilde{S}(\vec{g}) = -\tilde{\Gamma}(\vec{h}) \quad \text{on } \partial D$$

$$\text{and} \quad \frac{\partial}{\partial \nu} S(\vec{f})^+ - \frac{\partial}{\partial \tilde{\nu}} \tilde{S}(\vec{g})^- = -\frac{\partial}{\partial \tilde{\nu}} \tilde{\Gamma}(\vec{h}) \quad \text{on } \partial D.$$

We introduce a function $\vec{w}$ defined as

$$\vec{w} = \begin{cases} \vec{u}\phi - \tilde{\Gamma}(\vec{h}) + \tilde{S}(\vec{g}) & \text{in } R^n \setminus \bar{D}. \\ \vec{u} + S(\vec{f}) & \text{in } D. \end{cases}$$

It is easy to check that $\vec{w} \in [W^{1,2}_{loc}(R^n)]^n$, and for all $\vec{\varphi} \in [C_0^\infty(R^n)]^n$

$$\int_D \mu(\nabla\vec{w} + \nabla\vec{w}^t)\nabla\vec{\varphi} + \lambda div(\vec{w})div(\vec{\varphi})dX$$
$$+ \int_{\Omega\setminus\bar{D}} \tilde{\mu}(\nabla\vec{w} + \nabla\vec{w}^t)\nabla\vec{\varphi} + \tilde{\lambda} div(\vec{w})div(\vec{\varphi})dX = 0.$$

We will show now that $\vec{w} = \vec{0}$. Let $\eta \in C_0^\infty(B_{2r})$ be a cut-off function with $\eta \equiv 1$ in $B_r$ and $\|\nabla\eta\|_\infty \leq \frac{3}{r}$. We will denote the Fourier transform of $v$ by $\mathcal{F}(v)$. We have

$$\begin{aligned}
\int_{B_r} |\nabla\vec{w}|^2 dX \leq & C\int_{R^n} |\nabla(\vec{w}\eta)|^2 dX \\
\leq & C\int_{R^n} |\mathcal{F}(\nabla(\vec{w}\eta))|^2 dX \\
\leq & C\int_{R^n} \frac{\mu}{2}|\mathcal{F}(\nabla(\vec{w}\eta) + \nabla(\vec{w}\eta^t))|^2 + \lambda(\mathcal{F}(div(\vec{w}\eta)))^2 dX \\
\leq & C\int_{B_{2r}} \frac{\mu}{2}|\nabla(\vec{w}\eta) + \nabla(\vec{w}\eta^t)|^2 + \lambda(div(\vec{w}\eta))^2 dX \\
\leq & C\int_{B_{2r}} \frac{\mu\chi_D + \tilde{\mu}\chi_{R^n\setminus D}}{2}|\nabla(\vec{w}\eta) + \nabla(\vec{w}\eta^t)|^2 \\
& + (\lambda\chi_D + \tilde{\lambda}\chi_{R^n\setminus D})(div(\vec{u}\eta))^2 dX
\end{aligned}$$

The third inequality comes from the Legendre-Hadamard condition (see section 1) and and the fifth ineqality comes from the fact that

$$\lambda(div\vec{w})^2 + \frac{\mu}{2}|\nabla\vec{w} + \nabla\vec{w}^t|^2 \approx |\nabla\vec{w} + \nabla\vec{w}^t|^2 \approx \tilde{\lambda}(div\vec{w})^2 + \frac{\tilde{\mu}}{2}|\nabla\vec{w} + \nabla\vec{w}^t|^2 \tag{3.1}$$

where by $W \approx V$ we mean that there are positive constants $c$ and $C$ such that $cV \leq W \leq CV$. By standard arguments (see [G;chapter3,proposition 2.1]) we have

$$\int_{B_r} |\nabla\vec{w}|^2 dx \quad \leq \quad C \int_{B_{2r}\setminus B_r} |\nabla\vec{w}|^2 dx.$$

Since $|\nabla\vec{w}(X)| = O\left(|X|^{1-n}\right)$ at large $|X|$ we can see that $\int_{B_r} |\nabla\vec{u}|^2 dx < \infty$ for all $r > 0$. By using the standard hole-filling method [G] and letting $r \to \infty$ we get the inequalty

$$\int_{R^n} |\nabla\vec{w}|^2 dx \leq \vartheta \int_{R^n} |\nabla\vec{w}|^2 dx \quad \text{where} \quad 0 < \vartheta < 1.$$

Therefore $|\nabla\vec{w}| = 0$ on $R^n$ and when $n \geq 3$ the behavior of $\vec{w}$ at infinity shows that $\vec{w}$ is identically zero on $R^n$. We obtain thus the representation formula for $\vec{u}$ in the neighborhood of $D$, that combined with the estimate $\|\nabla\tilde{\Gamma}(\vec{h})\|_{L^\infty(B_{\frac{1}{2}})} \leq C\|\vec{u}\|_{W^{1,2}(B)}$ proves theorem 1.

**Lemma 3.** *Under the assumption of theorem 2, given $\vec{f}$ and $\vec{g} \in [L^2(\partial D)]^n$ the function*

$$\vec{u} \stackrel{\text{def}}{=} \begin{cases} \vec{u}^+ \stackrel{\text{def}}{=} S(\vec{f}) & \text{in } D \\ \vec{u}^- \stackrel{\text{def}}{=} \tilde{S}(\vec{g}) & \text{in } R^n \setminus \bar{D}. \end{cases} \tag{3.2}$$

*satisfies the estimate*

$$\|\nabla\vec{u}^+\|_{L^2(\partial D)} + \|\nabla\vec{u}^-\|_{L^2(\partial D)} \leq C\{\|\vec{h}_1\|_{L_1^2(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)} + \|\nabla\vec{u}\|_{L^2(D)} + \|\nabla\vec{u}\|_{L^2(B\setminus D)}\} \tag{3.3}$$

*where*

$$\vec{h}_1 = \vec{u}^+ - \vec{u}^- \quad \text{and } \vec{h}_2 = \frac{\partial}{\partial\nu}\vec{u}^+ - \frac{\partial}{\partial\tilde{\nu}}\vec{u}^- \quad \text{on } \partial D.$$

*Sketch of proof of Lemma 3.* To prove (3.3), we will use the following formulas. Recall from Sec 1 that $\{N, T^1, .., T^{n-1}\}$ is a orthonormal basis associated to some point $P \in \partial D$, where $T^1, .., T^{n-1}$ are $n-1$ tangential vectors.

(3.5)

$$
\begin{aligned}
|(\nabla\vec{v} + \nabla\vec{v}^t)|^2 &= |(\nabla\vec{v} + \nabla\vec{v}^t)N|^2 + \sum_{l=1}^{n-1} |(\nabla\vec{v} + \nabla\vec{v}^t)T^l|^2 \\
&= \langle(\nabla\vec{v} + \nabla\vec{v}^t)N, N\rangle^2 + 2\sum_{l=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, T^l\rangle^2 \\
&\quad + \sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)T^l, T^k\rangle^2
\end{aligned}
$$

(3.6) $$div\vec{v} = \frac{1}{2}\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, N\rangle + \frac{1}{2}\sum_{l=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)T^l, T^l\rangle$$

(3.7)

$$\langle\frac{\partial\vec{v}}{\partial\nu}, N\rangle = (\mu + \frac{\lambda}{2})\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, N\rangle + \sum_{l=1}^{n-1}\frac{\lambda}{2}\langle(\nabla\vec{v} + \nabla\vec{v}^t)T^l, T^l\rangle$$

(3.8)

$$\langle\frac{\partial\vec{v}}{\partial\nu}, T^l\rangle = \mu\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, T^l\rangle \quad \text{for all } l = 1, .., n-1$$

From the above identities we have the identity

$$
\begin{aligned}
&\lambda(div\vec{v})^2 + \frac{\mu}{2}|(\nabla\vec{v} + \nabla\vec{v}^t)|^2 \\
&= \lambda\{\frac{1}{2}\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, N\rangle + \frac{1}{2}\sum_{l=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)T^l, T^l\rangle\} \\
&\quad + \frac{\mu}{2}\{\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, N\rangle^2 + 2\sum_{l=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)N, T^l\rangle^2 \\
&\qquad + \sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\langle(\nabla\vec{v} + \nabla\vec{v}^t)T^l, T^k\rangle^2\} \qquad (\text{ see } (3.5) \text{ and } (3.6)\ )
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\mu+\lambda}\langle\frac{\partial\vec{v}}{\partial\nu},N\rangle^2 + \frac{1}{\mu}\langle\frac{\partial\vec{v}}{\partial\nu},T^l\rangle^2 \\
&+ \sum_{l=1}^{n-1}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\langle(\nabla\vec{v}+\nabla\vec{v}^t)T^l,T^l\rangle^2 \\
(3.9)\quad &+ \mu\sum_{1\le l<k\le n-1}\langle(\nabla\vec{v}+\nabla\vec{v}^t)T^l,T^k\rangle^2 \\
&+ 2\lambda\sum_{1\le l<k\le n-1}\langle(\nabla\vec{v}+\nabla\vec{v}^t)T^l,T^l\rangle\langle(\nabla\vec{v}+\nabla\vec{v}^t)T^k,T^k\rangle
\end{aligned}
$$

Let $\vec{\beta} \in C_0^\infty(B)$ be a vector field such that $\langle\vec{\beta}(Q),N(Q)\rangle > c > 0$ for $Q \in \partial D$. Since $\vec{u}^+$ is a solution of the system

$$\mu\Delta\vec{u} + (\lambda+\mu)\nabla div\vec{u} = 0 \quad \text{in } D,$$

the Rellich-Necas identity ( see [N],[D-K-V,1]) gives

$$
\begin{aligned}
(3.10)\quad &\int_{\partial D}\langle\vec{\beta},N\rangle\{\lambda(div\vec{u}^+)^2 + \mu|\nabla\vec{u}^+ + \nabla u^{+t}|^2\}d\sigma \\
&= 2\int_{\partial D}\langle\beta_i D_i\vec{u}^+,\frac{\partial\vec{u}^+}{\partial\nu}\rangle d\sigma + O\left(\|\nabla u^+\|^2_{L^2(D)}\right).
\end{aligned}
$$

We have the identity

(3.11)

$$
\begin{aligned}
\langle\beta_i D_i\vec{u}^+,\frac{\partial\vec{u}^+}{\partial\nu}\rangle &= \langle\vec{\beta},N\rangle\langle\nabla\vec{u}^+\frac{\partial\vec{u}^+}{\partial\nu},N\rangle + \sum_{l=1}^{n-1}\langle\vec{\beta},T^l\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},\frac{\partial\vec{u}^+}{\partial T^l}\rangle \\
&= \langle\vec{\beta},N\rangle\{\frac{1}{2}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})N,N\rangle + \\
&+ \sum_{l=1}^{n-1}\langle\nabla\vec{u}^+T^l,N\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle\} + \sum_{l=1}^{n-1}\langle\vec{\beta},T^l\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},\frac{\partial\vec{u}^+}{\partial T^l}\rangle.
\end{aligned}
$$

Substituting (3.9) and (3.11) in (3.10) we get the formula

$$
\begin{aligned}
&\int_{\partial D}\langle\vec{\beta},N\rangle\Big\{\frac{1}{2\mu+\lambda}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle^2+\frac{1}{\mu}\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle^2\\
&\qquad+\frac{\mu\lambda}{2(2\mu+\lambda)}\left(\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})T^l,T^l\rangle\right)^2\\
&\qquad+\frac{\mu}{2}\sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})T^l,T^k\rangle^2\\
(3.12)\qquad&\qquad-\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})N,N\rangle\\
&\qquad-2\sum_{l=1}^{n-1}\langle\nabla\vec{u}^+T^l,N\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle\Big\}d\sigma\\
&=2\int_{\partial D}\sum_{l=1}^{n-1}\langle\vec{\beta},T^l\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},\frac{\partial\vec{u}^+}{\partial T^l}\rangle d\sigma+O\left(\|\nabla u^+\|^2_{L^2(D)}\right).
\end{aligned}
$$

From (3.7) we have the identity

$$
\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})N,N\rangle=\frac{2}{2\mu+\lambda}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle-\frac{\lambda}{2\mu+\lambda}\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})T^l,T^l\rangle
$$

and therefore we obtain the identity

$$
\begin{aligned}
(3.13)\qquad&\frac{1}{2\mu+\lambda}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle^2-\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})N,N\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle\\
&=\frac{\lambda}{2\mu+\lambda}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^++\nabla\vec{u}^{+t})T^l,T^l\rangle-\frac{1}{2\mu+\lambda}\langle\frac{\partial\vec{u}^+}{\partial\nu},N\rangle^2.
\end{aligned}
$$

From (3.8) we have the identity

$$
\begin{aligned}
(3.14)\qquad&\frac{1}{\mu}\sum_{l=1}^{n-1}\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle^2-2\sum_{l=1}^{n-1}\langle\nabla\vec{u}N,T^l\rangle\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle\\
&\qquad=\sum_{l=1}^{n-1}\langle\frac{\partial\vec{u}^+}{\partial\nu},T^l\rangle\left(\langle\nabla\vec{u}^tN,T\rangle-\langle\nabla\vec{u}N,T^l\rangle\right)
\end{aligned}
$$

Substituting (3.13) and (3.14) in (3.12) we obtain

$$
\begin{aligned}
&\int_{\partial D}\langle\vec{\beta},N\rangle\{-\frac{1}{2\mu+\lambda}\langle\frac{\partial\vec{u}^{+}}{\partial\nu},N\rangle^{2}\\
&\qquad+\frac{\lambda}{2\mu+\lambda}\langle\frac{\partial\vec{u}^{+}}{\partial\nu},N\rangle\left(\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^{+}+\nabla\vec{u}^{+t})T^{l},T^{l}\rangle\right)\\
&\qquad+\frac{\mu\lambda}{2(2\mu+\lambda)}\left(\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^{+}+\nabla\vec{u}^{+t})T^{l},T^{l}\rangle\right)^{2}\\
(3.15)\quad&\qquad+\frac{\mu}{2}\sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\langle(\nabla\vec{u}^{+}+\nabla\vec{u}^{+t})T^{l},T^{k}\rangle^{2}\\
&\qquad+\sum_{l=1}^{n-1}\langle\frac{\partial\vec{u}^{+}}{\partial\nu},T^{l}\rangle\left(\langle\nabla\vec{u}^{+}N,T^{l}\rangle-\langle\nabla\vec{u}^{+t}N.T^{l}\rangle\right)\}d\sigma\\
&=2\int_{\partial D}\sum_{l=1}^{n-1}\langle\vec{\beta},T^{l}\rangle\langle\frac{\partial\vec{u}^{+}}{\partial\nu},\frac{\partial\vec{u}^{+}}{\partial T^{l}}\rangle d\sigma+O\left(\|\nabla u^{+}\|^{2}_{L^{2}(D)}\right).
\end{aligned}
$$

The same argument leads to the identity corresponding to (3.15) but with $\vec{u}^{+},\mu,\lambda$, and $\frac{\partial}{\partial\nu}$ replaced repectively by $\vec{u}^{-},\tilde{\mu},\tilde{\lambda}$, and $\frac{\partial}{\partial\tilde{\nu}}$ and we obtain

$$
\begin{aligned}
&\int_{\partial D}\langle\vec{\beta},N\rangle\{-\frac{1}{2\tilde{\mu}+\tilde{\lambda}}\langle\frac{\partial\vec{u}^{-}}{\partial\tilde{\nu}},N\rangle^{2}\\
&\qquad+\frac{\tilde{\lambda}}{2\tilde{\mu}+\tilde{\lambda}}\langle\frac{\partial\vec{u}^{-}}{\partial\tilde{\nu}},N\rangle\left(\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^{+}+\nabla\vec{u}^{+t})T^{l},T^{l}\rangle\right)\\
&\qquad+\frac{\tilde{\mu}\tilde{\lambda}}{2(2\tilde{\mu}+\tilde{\lambda})}\left(\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^{-}-\nabla\vec{u}^{-t})T^{l},T^{l}\rangle\right)^{2}\\
(3.16\text{-}1)\quad&\qquad+\frac{\tilde{\mu}}{2}\sum_{l=1}^{n-1}\sum_{k=1}^{n-1}\langle(\nabla\vec{u}^{-}-\nabla\vec{u}^{-t})T^{l},T^{k}\rangle^{2}\\
&\qquad+\sum_{l=1}^{n-1}\langle\frac{\partial\vec{u}^{-}}{\partial\tilde{\nu}},T^{l}\rangle\left(\langle\nabla\vec{u}^{-}N,T^{l}\rangle-\langle\nabla\vec{u}^{-t}N.T^{l}\rangle\right)\}d\sigma\\
&=2\int_{\partial D}\sum_{l=1}^{n-1}\langle\vec{\beta},T^{l}\rangle\langle\frac{\partial\vec{u}^{-}}{\partial\nu},\frac{\partial\vec{u}^{-}}{\partial T^{l}}\rangle d\sigma+O\left(\|\nabla u^{-}\|^{2}_{L^{2}(D)}\right).
\end{aligned}
$$

From (3.8) and the transmission conditions $\vec{h}_1 = \vec{u}^+ - \vec{u}^-$, and $\vec{h}_2 = \frac{\partial \vec{u}^+}{\partial \nu} - \frac{\partial \vec{u}^-}{\partial \tilde{\nu}}$ on $\partial D$, we have

$$
\begin{aligned}
&\int_{\partial D} \langle \frac{\partial \vec{u}^-}{\partial \tilde{\nu}}, T^l \rangle \left( \langle \nabla \vec{u}^- N, T^l \rangle - \langle \nabla \vec{u}^{-,t} N, T^l \rangle \right) d\sigma \\
&= \int_{\partial D} \langle \frac{\partial \vec{u}^-}{\partial \tilde{\nu}}, T^l \rangle \left( 2\langle \nabla \vec{u}^- N, T^l \rangle - \langle (\nabla \vec{u}^- + \nabla \vec{u}^{-,t} N, T^l \rangle \right) d\sigma \\
&= \int_{\partial D} \langle \frac{\partial \vec{u}^-}{\partial \tilde{\nu}}, T^l \rangle \left( 2\langle \nabla \vec{u}^- N, T^l \rangle - \frac{1}{\tilde{\mu}} \langle \frac{\partial \vec{u}^-}{\partial \tilde{\nu}}, T^l \rangle \right) d\sigma \\
&= \int_{\partial D} \langle \frac{\partial \vec{u}^+}{\partial \nu}, T^l \rangle \left( 2\langle \nabla \vec{u}^+ N, T^l \rangle - \frac{1}{\tilde{\mu}} \langle \frac{\partial \vec{u}^+}{\partial \nu}, T^l \rangle \right) d\sigma \\
&\qquad + O\left( \|\vec{h}_1\|^2_{L^2_1(\partial D)} + \|\vec{h}_2\|^2_{L^2(\partial D)} \right) \\
&\qquad + O\left( \|\nabla \vec{u}^+\|_{L^2(\partial D)} \{ \|\vec{h}_1\|_{L^2_1(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)} \} \right) \\
&= \int_{\partial D} \langle \frac{\partial \vec{u}^+}{\partial \nu}, T^l \rangle \left( \frac{2\tilde{\mu} - \mu}{\tilde{\mu}} \langle \nabla \vec{u}^+ N, T^l \rangle - \frac{\mu}{\tilde{\mu}} \langle \nabla \vec{u}^{+t} N, T^l \rangle \right) d\sigma \\
&\qquad + O\left( \|\vec{h}_1\|^2_{L^2_1(\partial D)} + \|\vec{h}_2\|^2_{L^2(\partial D)} \right) \\
&\qquad + O\left( \|\nabla \vec{u}^+\|_{L^2(\partial D)} \{ \|\vec{h}_1\|_{L^2_1(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)} \} \right) .
\end{aligned}
$$

Substituting the last identity and the transmission conditions in (3.16-1) we obtain

(3.16-2)

$$\begin{aligned}
\int_{\partial D} \langle \vec{\beta}, N\rangle \{ -\frac{1}{2\tilde{\mu}+\tilde{\lambda}} \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, N\rangle^2 \\
+ \frac{\tilde{\lambda}}{2\tilde{\mu}+\tilde{\lambda}} \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, N\rangle \left( \sum_{l=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^l \rangle \right) \\
+ \frac{\tilde{\mu}\tilde{\lambda}}{2(2\tilde{\mu}+\tilde{\lambda})} \left( \sum_{l=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^l \rangle \right)^2 \\
+ \frac{\tilde{\mu}}{2} \sum_{l=1}^{n-1} \sum_{k=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^k \rangle^2 \\
+ \sum_{l=1}^{n-1} \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, T^l \rangle \left( \frac{2\tilde{\mu}-\mu}{\tilde{\mu}} \langle \nabla \vec{u}^{+} N, T^l \rangle - \frac{\mu}{\tilde{\mu}} \langle \nabla \vec{u}^{+t} N, T^l \rangle \right) \} d\sigma \\
= 2 \int_{\partial D} \sum_{l=1}^{n-1} \langle \vec{\beta}, T^l \rangle \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, \frac{\partial \vec{u}^{+}}{\partial T^l} \rangle d\sigma + O\left( \|\nabla \vec{u}^{-}\|^2_{L^2(B\setminus D)} \right) \\
+ O\left( \|\vec{h}_1\|^2_{L^2_1(\partial D)} + \|\vec{h}_2\|^2_{L^2(\partial D)} \right) \\
+ O\left( \|\nabla \vec{u}^{+}\|_{L^2(\partial D)} \{ \|\vec{h}_1\|_{L^2_1(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)} \} \right).
\end{aligned}$$

Subtracting (3.16-2) from (3.15) we have

$$\begin{aligned}
\int_{\partial D} \langle \vec{\beta}, N\rangle \{ \left( -\frac{1}{2\mu+\lambda} + \frac{1}{2\tilde{\mu}+\tilde{\lambda}} \right) \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, N\rangle^2 \\
+ \left( \frac{\lambda}{2\mu+\lambda} - \frac{\tilde{\lambda}}{2\tilde{\mu}+\tilde{\lambda}} \right) \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, N\rangle \left( \sum_{l=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^l \rangle \right) \\
+ \left( \frac{\mu\lambda}{2(2\mu+\lambda)} - \frac{\tilde{\mu}\tilde{\lambda}}{2(2\tilde{\mu}+\tilde{\lambda})} \right) \left( \sum_{l=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^l \rangle \right)^2 \\
+ \left( \frac{\mu}{2} - \frac{\tilde{\mu}}{2} \right) \sum_{l=1}^{n-1} \sum_{k=1}^{n-1} \langle (\nabla \vec{u}^{+} + \nabla \vec{u}^{+t}) T^l, T^k \rangle^2 \\
- \sum_{l=1}^{n-1} \langle \frac{\partial \vec{u}^{+}}{\partial \nu}, T^l \rangle \left( \left( \frac{2\tilde{\mu}-\mu}{\tilde{\mu}} - 1 \right) \langle \nabla \vec{u}^{+} N, T^l \rangle - \left( \frac{\mu}{\tilde{\mu}} - 1 \right) \langle \nabla \vec{u}^{+t} N, T^l \rangle \right) \} d\sigma
\end{aligned}$$

$$=O\left(\|\vec{h}_1\|^2_{L^2_1(\partial D)} + \|\vec{h}_2\|^2_{L^2(\partial D)} + \|\nabla\vec{u}^+\|_{L^2(\partial D)}\{\|\vec{h}_1\|_{L^2_1(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)}\}\right)$$
$$+\,O\left(\|\nabla\vec{u}^+\|^2_{L^2(D)} + \|\nabla\vec{u}^-\|^2_{L^2(B\setminus D)}\right).$$

Using the identities (3.7) and (3.8) the right side of the above identity is equal to

$$\begin{aligned}
&A\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})N, N\rangle^2 + B\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})T^l, T^l\rangle^2 \\
&+C\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})N, N\rangle\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})T^l, T^l\rangle \\
&+D\sum_{1\le l<k\le n-1}\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})T^l, T^l\rangle\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})T^k, T^k\rangle \\
&+(\mu-\tilde{\mu})\sum_{1\le l<k\le n-1}\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})T^l, T^k\rangle^2 \\
&+\frac{\mu(\mu-\tilde{\mu})}{\tilde{\mu}}\sum_{l=1}^{n-1}\langle(\nabla\vec{u}^+ + \nabla\vec{u}^{+t})N, T^l\rangle^2
\end{aligned}\tag{3.17}$$

where

$$A = \frac{(2\mu+\lambda)(2\mu+\lambda-2\tilde{\mu}-\tilde{\lambda})}{4(2\tilde{\mu}+\tilde{\lambda})}$$

$$B = \frac{\lambda^2 - \lambda\tilde{\lambda} + 2\mu\tilde{\lambda} + 4\mu\tilde{\mu} - 4\tilde{\mu}\tilde{\lambda} - \tilde{\mu}^2 + 2\tilde{\mu}\lambda}{4(2\tilde{\mu}+\tilde{\lambda})}$$

$$C = \frac{(\lambda-\tilde{\lambda})(2\mu+\lambda)}{2(2\tilde{\mu}+\tilde{\lambda})}$$

$$D = \begin{cases} \dfrac{(\lambda-\tilde{\lambda})(2\tilde{\mu}+\lambda)}{2(2\tilde{\mu}+\tilde{\lambda})} & \text{when } n\ge 3 \\ 0 \quad \text{when } n=2 \end{cases}$$

We will prove that the $n \times n$ matrix

$$M = (M_{ij}) \overset{\text{def}}{=} \begin{pmatrix} A & \frac{C}{2} & \frac{C}{2} & & \dots & \frac{C}{2} \\ \frac{C}{2} & B & \frac{D}{2} & & \dots & \frac{D}{2} \\ \frac{C}{2} & \frac{D}{2} & B & \frac{D}{2} & \dots & \frac{D}{2} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & B & \frac{D}{2} \\ \frac{C}{2} & \frac{D}{2} & & \dots & \frac{D}{2} & B \end{pmatrix}$$

is positive (or negative) definite. Define the $k \times k$ matrix

$$M_k \equiv \begin{pmatrix} M_{11} & \dots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \dots & M_{kk} \end{pmatrix}$$

where the $M_{ij}$ entries are as above. An elementary computation shows then that for $2 \leq k \leq n$

$$\det M_k = \begin{vmatrix} A & \frac{C}{2} & 0 & 0 & & \dots & 0 \\ \frac{C}{2} & B & -1 & 0 & & \dots & 0 \\ \frac{C}{2} & \frac{D}{2} & 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & & & \\ \frac{C}{2} & \frac{D}{2} & 0 & \dots & & 1 & -1 \\ \frac{C}{2} & \frac{D}{2} & 0 & \dots & & 0 & 1 \end{vmatrix} (B - \frac{D}{2})^{k-2}$$

$$= \left(AB + \frac{k-2}{2}AD - (k-1)\frac{C^2}{4}\right)\left(B - \frac{D}{2}\right)^{k-2}$$

$$= \frac{(\mu - \tilde{\mu})(2\mu + \lambda)(2\mu + k\lambda - 2\tilde{\mu} - k\tilde{\lambda})}{8(2\tilde{\mu} + \tilde{\lambda})}\left(\frac{\mu - \tilde{\mu}}{2}\right)^{k-2}$$

From the well-known basic theorem of Linear Algebra([Ho, page 328]), it follows that $M$ is positive definite matrix (or neagtive definite) under the assumptions

$$2\mu + k\lambda > 2\tilde{\mu} + k\tilde{\lambda} \text{ and } \mu > \tilde{\mu} \text{ for all } k = 1, .., n$$
$$(\text{or } 2\mu + k\lambda < 2\tilde{\mu} + k\tilde{\lambda} \text{ and } \mu < \tilde{\mu} \text{ for all } k = 1, .., n).$$

Therefore, from (3.17) and (3.5) we obtain the estimates

$$(3.18)\quad \|\nabla\vec{u}^{+}+\nabla\vec{u}^{+t}\|^2_{L^2(\partial D)}\le C\{\|\vec{h}_1\|^2_{L^2_1(\partial D)}+\|\vec{h}_2\|^2_{L^2(\partial D)}$$
$$+\|\nabla\vec{u}^{+}\|_{L^2(\partial D)}\{\|\vec{h}_1\|_{L^2_1(\partial D)}+\|\vec{h}_2\|_{L^2(\partial D)}\}+\|\nabla\vec{u}^{+}\|^2_{L^2(D)}+\|\nabla\vec{u}^{-}\|^2_{L^2(B\setminus D)}\}.$$

At this point we recall the following form of the Korn type boundary inequality(see [D,K,V:2])

**Theorem 4[D,K,V:2].** *Let $\vec{u}$ be the function in lemma 3. There is a constant $C$ depending on the Lipschitz character of $D, \mu,$ and $\lambda$ such that*

$$\|\nabla\vec{u}\|_{L^2(\partial D)}\le C\{\|\nabla\vec{u}+\nabla\vec{u}^t\|_{L^2(\partial D)}+\|\nabla\vec{u}\|_{L^2(D)}\}.$$

From this theorem, the transmission conditions, and (3.18) we obtain (3.3).

**Lemma 5.** *Under the assumptions of Lemma 3, given $\vec{f}$ and $\vec{g}\in[L^2(\partial D)]^n$ we have the estimate*

$$(3.20)\quad \|\vec{f}\|_{L^2(\partial D)}+\|\vec{g}\|_{L^2(\partial D)}\le C\{\|S(\vec{f})-\tilde{S}(\vec{g})\|_{L^2_1(\partial D)}$$
$$+\|\frac{\partial}{\partial\nu}S(\vec{f})^{+}-\frac{\partial}{\partial\tilde{\nu}}\tilde{S}(\vec{g})^{-}\|_{L^2(\partial D)}+|L_{+}(\vec{f})|+|L_{-}(\vec{g})|\}$$

*where $L_+$ and $L_-$ denote bounded linear functions from $L^2(\partial D)$ to $R^n$ whose norms depend on the Lipschitz character of $D$ and the measures in $R^n$ of $D$ and $B\setminus D$.*

*Proof.* Let

$$B_1=\frac{1}{|D|}\int_D\frac{\nabla\vec{u}-\nabla\vec{u}^t}{2}$$

and

$$B_1=\frac{1}{|B\setminus D|}\int_{B\setminus D}\frac{\nabla\vec{u}-\nabla\vec{u}^t}{2}$$

where $|E|$ is the measure of $E$ in $R^n$.

By the Korn inequality (see [N] or Lemma 1.18 in [D,K,V:2]),

$$\int_D|\nabla(\vec{u}(X)-B_1X)|^2dX\le C\int_D|\nabla(\vec{u}(X)-B_1X)+\nabla(\vec{u}(X)-B_1X)^t|^2dX$$

and

$$\int_{B\setminus\bar{D}} |\nabla(\vec{u}(X) - B_2X)|^2 dX$$
$$\leq C \int_{B\setminus\bar{D}} |\nabla(\vec{u}(X) - B_2X) + \nabla(\vec{u}(X) - B_1X)^t|^2 dX$$

so that from (3.3) we have

$$\begin{aligned} \|\nabla\vec{u}^+\|_{L^2(\partial D)} + \|\nabla\vec{u}^-\|_{L^2(\partial D)} \leq C\{&\|\vec{h}_1\|_{L^2_1(\partial D)} + \|\vec{h}_2\|_{L^2(\partial D)} \\ &+ \|\nabla\vec{u} + \nabla\vec{u}^t\|_{L^2(D)} + \|\nabla\vec{u} + \nabla\vec{u}^t\|_{L^2(B\setminus D)} \\ &+ \int_D |B_1X|^2 dX + \int_{B\setminus D} |B_2X|^2 dX\}. \end{aligned} \tag{3.21}$$

On the other hand, from [D,K,V:2] there is a constant $C$ depending on the Lipschitz charactor of $D$ and the measures in $R^n$ of $D$ and $B \setminus D$ such that

$$\begin{aligned} \|\vec{f}\|_{L^2(\partial D)} &\leq C\{\|\nabla\vec{u}^+\|_{L^2(\partial D)} + |L_+(\vec{f})|\} \\ \|\vec{g}\|_{L^2(\partial D)} &\leq C\{\|\nabla\vec{u}^-\|_{L^2(\partial D)} + |L_-(\vec{g})|\}. \end{aligned} \tag{3.22}$$

From the identities

$$\int_D \frac{\mu}{2}|\nabla S(\vec{f}) + \nabla S(\vec{f})^t|^2 + \lambda\left(divS(\vec{f})\right)^2 dX = \int_{\partial D} S(\vec{f})\frac{\partial}{\partial\nu}S(\vec{f})^+ d\sigma$$

and

$$\int_{B_R\setminus D} \frac{\tilde{\mu}}{2}|\nabla\tilde{S}(\vec{g}) + \nabla\tilde{S}(\vec{g})^t|^2 + \tilde{\lambda}\left(div\tilde{S}(\vec{g})\right)^2 dX = -\int_{\partial D} \tilde{S}(\vec{g})\frac{\partial}{\partial\tilde{\nu}}\tilde{S}(\vec{g})^- d\sigma$$
$$+O(\int_{\partial D_R} |\nabla\tilde{S}(\vec{g})||\tilde{S}(\vec{g})|d\sigma).$$

and Poincaré inequality we have

$$\begin{aligned} \|\nabla\vec{u} + \nabla\vec{u}^t\|^2_{L^2(D)} &\leq C\|\nabla\vec{u}^+\|^2_{L^2(\partial D)} \\ \|\nabla\vec{u} + \nabla\vec{u}^t\|^2_{L^2(B\setminus D)} &\leq C\{\|\nabla\vec{u}^+\|^2_{L^2(\partial D)} + |\int_{\partial D} \vec{u}d\sigma|^2 \end{aligned} \tag{3.23}$$

Inequality (3.20) follows from (3.21),(3,22), and (3.23).

**Theorem 5.** *Let $n \geq 3$, $D$ a smooth domain with a connected boundary and $D \subset B_{\frac{1}{2}}$. For a given $\vec{h} \in [C_0^\infty(R^n)]^n$ there is a solution $\vec{u}$ of the transmission problem*

$$
(3.24)\qquad
\begin{aligned}
&\mu \Delta \vec{u} + (\lambda + \mu)\nabla div \vec{u} = \vec{0} \quad \text{in } D \\
&\tilde{\mu} \Delta \vec{u} + (\tilde{\lambda} + \tilde{\mu})\nabla div \vec{u} = \vec{0} \quad \text{in } R^n \setminus D \\
&\vec{u}^+ - \vec{u}^- = \vec{0} \quad \text{on } \partial D \\
&\frac{\partial \vec{u}^+}{\partial \nu} - \frac{\partial \vec{u}^-}{\partial \tilde{\nu}} = \vec{h} \quad \text{on } \partial D \\
&\|(\nabla \vec{u}^+)^*\|_{L^2(\partial D)} + \|(\nabla \vec{u}^-)^*\|_{L^2(\partial D)} < \infty \\
&|X||\nabla \vec{u}(X)| + |\vec{u}(X)| = O\left(|X|^{2-n}\right) \quad \text{at infinity.}
\end{aligned}
$$

*Moreover*

$$
\vec{u} = \begin{cases} S(\vec{f}) & \text{in } D \\ \tilde{S}(\vec{g}) & \text{in } R^n \setminus D \end{cases} \qquad \text{where } \mathcal{T}(\vec{f}, \vec{g}) = (\vec{0}, \vec{h}).
$$

The proof of the above theorem is standard.

*Proof of Theorem 2.* Given $\vec{f}$ and $\vec{g} \in [L^2(\partial D)]^n$, we will use the same notations in Lemma 3, that is

$$
\vec{u} \stackrel{\text{def}}{=} \begin{cases} \vec{u}^+ \stackrel{\text{def}}{=} S(\vec{f}) & \text{in } D \\ \vec{u}^- \stackrel{\text{def}}{=} \tilde{S}(\vec{g}) & \text{in } R^n \setminus D. \end{cases}
$$

To prove that $\mathcal{T}$ is one to one, let $\mathcal{T}(\vec{f}, \vec{g}) = (\vec{0}, \vec{0})$. From the identities (3.23) we can see that $\vec{u} \in [W^{1,2}_{loc}(R^n)]^n, \vec{u}$ is a weak-solution of the elliptic system (1.1) in the entire domain $R^n$. As in the proof of theorem 1, $\vec{u} = \vec{0}$ on $R^n$. Therefore from the jump relations of the traction, $\vec{f} = \vec{g} = \vec{0}$ on $\partial D$.

From the estimate (3.2) and the fact that the operator in the the theorem 2 is invertible when $D$ is smooth, it follows that $\mathcal{T}$ has closed and dense range. We will leave these details as an exercise for the reader.

We can state the following theorem:

**Theorem 6.** *If $D, \mu, \lambda, \tilde{\mu}$, and $\tilde{\lambda}$ satisfy the assumption of Theorem 3.2, then there exists $\epsilon > 0$ such that the mapping*

$$
\mathcal{T} : [L^p(\partial D)]^n \times [L^p(\partial D)]^n \to [L^p_1(\partial D)]^n \times [L^p(\partial D)]^n \quad \text{defined by}
$$

$$
\mathcal{T}(\vec{f}, \vec{g}) = \left( S(\vec{f}) - \tilde{S}(\vec{g}), \frac{\partial}{\partial \nu} S(\vec{f})^+ - \frac{\partial}{\partial \tilde{\nu}} \tilde{S}(\vec{g})^- \right)
$$

*is an invertible operator for $2 - \epsilon \leq p \leq 2 + \epsilon$.*

From G. David and S. Semmes's $L^2$-booster theorem there is also an $\epsilon = \epsilon(D)$ and a constant C, so that when $\vec{u}$ is as in the theorem 1 the following estimate holds

$$\|(\nabla \vec{u}^{\pm})^*\|_{L^{2+\epsilon}(\partial D)} \leq C\|\vec{u}\|_{W^{1,2}(B)}.$$

From the Sobolev imbedding theorem and Theorem 6, we can obtain the following 3-dimensional result.

**Theorem 7.** *Let $n = 3$. If $\vec{u}$ is as in Theorem 1, then*

$$\vec{u} \in [C^{0,\delta}(\bar{D})]^n \cap [C^{0,\delta}(B_{\frac{3}{4}} \setminus D)]^n \cap [C(\bar{B}_{\frac{3}{4}})]^n$$

*for a small $\delta > 0$.*

**Problem 2.** *In the case where sign$(\mu - \tilde{\mu}) \neq$ sign$(\lambda - \tilde{\lambda})$, the result of Theorem 1 is still open problem.*

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# REGULARITY FOR WEAK SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. We show continuity and the property $|\nabla u|^2 \in K_n^{loc}(\Omega)$ for weak solutions $u$ of general uniformly elliptic partial differential equation with complex coefficients: $Lu \equiv -div(A(x)\nabla u) + \mathbf{c}(x) \cdot \nabla u + V(x)u = f$ under the assumptions $|V|, |\mathbf{c}|^2, f \in K_n^{loc}(\Omega)$. Moreover, when $\mathbf{c}, V$ are real-valued we show Harnack's inequality for nonnegative weak solutions of $Lu = 0$ under the same assumption on $\mathbf{c}, V$.

We also show local boundedness for weak solutions $u \in H^1_{loc}(\Omega)$ of $Tu \equiv -(\nabla - i\mathbf{b}(x))^2u + V(x)u = f$ in $\Omega$ under the assumptions $|V|, |\mathbf{b}|^2, f \in K_n^{loc}(\Omega)$.

## 1. INTRODUCTION AND MAIN RESULTS

We consider the following elliptic equation of second order:

$$(1) \quad Lu \equiv -div(A(x)\nabla u(x)) + \mathbf{c}(x) \cdot \nabla u(x) + V(x)u(x) = f \quad in \quad \Omega.$$

Here $A(x) = (a_{ij}(x))$ is real-valued, $\mathbf{c}$ and $V$ are complex-valued and $A(x)$ satisfies

$$(2)\ a_{ij}(x) = a_{ji}(x), \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad x \in \Omega, \quad \xi \in \mathbf{R}^n$$

for some $\lambda \in (0, 1]$. We assume

$$(A) \qquad |V|, |\mathbf{c}|^2, f \in K_n^{loc}(\Omega).$$

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Here we say $V \in K_n^{\rm loc}(\Omega)$ if $\lim_{r\to 0}\eta(V;r;\Omega_1) = 0$ for each compact subdomain $\Omega_1$ with $\overline{\Omega_1} \subset \Omega$, where

$$\eta(f;r) = \sup_{x\in\mathbf{R}^n}\int_{B_r(x)}\frac{|f(y)|}{|x-y|^{n-2}}\,dy \tag{3}$$

and $\eta(f;r;G) = \eta(f\chi_G;r)$ and $\chi_G$ is the characteristic function of $G$ and $B_r(x) = \{y \in \mathbf{R}^n; |x-y| < r\}$ for $r > 0$. We say $u \in H^1_{\rm loc}(\Omega) = \{u \in L^2_{\rm loc}(\Omega); \nabla u \in L^2_{\rm loc}(\Omega)\}$ is a weak solution of (1) in $\Omega$, if $u$ satisfies

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j \phi + \mathbf{c}\cdot\nabla\phi + + V u\phi\,dx = \int_\Omega f\phi\,dx \tag{4}$$

for every $\phi \in C_o^\infty(\Omega)$.

Thoughout this paper we denote by $C(n,\lambda,\eta)$ the constant depends only on $n$, $\lambda$, and the modulus of functions $\eta(|V|;\cdot)$ and $\eta(\mathbf{c}|^2;\cdot)$. Since we are only concerned with local properties of weak solutions, we may assume that $\Omega$ is bounded and $\eta(|V|;r;\Omega)$, $\eta(|\mathbf{c}|^2;r;\Omega)$, and $\eta(f;r;\Omega)$ tend to zero as $r \to 0$. For simplicity we use the notaion $\eta(g;r) = \eta(g;r;\Omega)$.

**Theorem 1.** *Suppose (2) and* ASSUMPTION (A) *and let $u$ be a weak solution of (1). Then $u$ is continuous in $\Omega$ and there exist constants $r_o = r_o(n,\lambda,\eta), C = C(n,\lambda,\eta) > 0$ and non-decreasing functions $\omega(s)$ and $\omega_f(s)$ satisfying $\lim_{s\to 0}\omega(s) = 0$, $\lim_{s\to 0}\omega_f(s) = 0$ such that*

$$\begin{aligned}|u(x) - u(x_o)| &\le C\omega\left(\frac{|x-x_o|}{r}\right)\|u\|_{L^\infty(B_{5r}(x_o))}\\ &\quad + C\omega_f\left(\frac{|x-x_o|}{r}\right)\end{aligned} \tag{5}$$

*for every $0 < r < r_o$ with $B_{8r}(x_o) \subset \Omega$. Moreover $|\nabla u|^2 \in K_n^{\rm loc}(\Omega)$ holds.*

**Theorem 2.** *Suppose* ASSUMPTION *(A) and $\mathbf{c}$, $V$ are real-valued. Then for nonnegative weak solution of $Lu = 0$ there exist constants $C, r_o > 0$ such that*

$$\max_{B_r} u \le C\min_{B_r} u \tag{6}$$

*for* $0 < r < r_o$ *with* $B_{4r} \subset \Omega$.

We also consider the Schrödinder equation with singular magnetic fields:

$$Tu \equiv -(\nabla - i\mathbf{b}(x))^2 u(x) + V(x)u(x) = f \quad in \quad \Omega, \tag{7}$$

where $i = \sqrt{-1}$, $\mathbf{b}(x) = (b_j(x))_{j=1}^n$ is real-valued and $V(x)$ is complex-valued. If we apply Theorem 1 to this Scrödinger equation we must impose $div\ \mathbf{b} \in K_n^{loc}(\Omega)$. However we can show local boundedness of weak solution of (7) without this additional condition. We say $u \in H^1_{loc}(\Omega; \mathbf{C}) = \{u \in L^2_{loc}(\Omega; \mathbf{C}); \nabla u \in L^2_{loc}(\Omega; \mathbf{C}^n)\}$ is a weak solution of (7) in $\Omega$, if $u$ satisfies

$$\int_\Omega (\nabla u - i\mathbf{b}u) \cdot \overline{(\nabla \phi - i\mathbf{b}\phi)} + Vu\overline{\phi}\, dx = \int_\Omega f\overline{\phi}\, dx \tag{8}$$

for every $\phi \in C_o^\infty(\Omega; \mathbf{C})$, where $\overline{\phi}$ is the complex conjugate of $\phi$. We also write $H^1_{loc} = H^1_{loc}(\Omega; \mathbf{C})$ for simplicity. We denote by $V_R$ the real part of $V$ and use $(V_R)^- = \max(-V_R, 0)$. We also use $⨍_A f\, dx = \frac{1}{|A|}\int_A f(x)\, dx$, where $|A|$ is the Lebesgue measure of $A$. For local boundedness of weak solution of (7) we have

**Theorem 3.** *Suppose* $|V|, |\mathbf{b}|^2, f \in K_n^{loc}(\Omega)$ *and let* $u$ *be a weak solution of (7). Then* $u \in L^\infty_{loc}(\Omega)$ *and there exist constants* $r_o = r_o(n,\eta), C = C(n,\eta) > 0$ *such that*

$$\|u\|_{L^\infty(B_{r/2}(x_o))} \le C\left(⨍_{B_r(x_o)} |u|^2\, dx\right)^{1/2} + C\eta(f; 2r) \tag{9}$$

*for every* $0 < r < r_o$ *with* $B_{2r}(x_o) \subset \Omega$, *where* $r_o$ *depends on* $n$, $p$, $\eta(|V|;\cdot;\Omega)$ *and* $\eta(|\mathbf{b}|^2;\cdot;\Omega)$ *and* $C$ *only on* $n$, $p$ *and* $\eta((V_R)^-;\cdot;\Omega)$. *When* $f = 0$, *we have*

$$\|u\|_{L^\infty(B_{r/2}(x_o))} \le C\left(⨍_{B_r(x_o)} |u|^p\, dx\right)^{1/p} \tag{10}$$

*for every* $0 < p < +\infty$.

Since $-(\nabla - i\mathbf{b})^2 u = -\Delta u + 2i\mathbf{b}(x)\cdot\nabla u + i\ div\mathbf{b}(x)u + |\mathbf{b}(x)|^2 u$, as an immediate consequence of Theorem 1 we have

**Corollary 4.** *Suppose $V, |\mathbf{b}|^2, div\mathbf{b}, f \in K_n^{loc}(\Omega)$ and let $u$ be a weak solution of (7). Then $u$ is continuous in $\Omega$ and $|\nabla u|^2 \in K_n^{loc}(\Omega)$.*

*Remark 1.* If we assume somewhat stronger condition $V, |\mathbf{c}|^2, f \in K_{n,\delta}^{loc}(\Omega)$ for some $\delta > 0$, then Theorem 1 yields Hölder continuity for weak solutions of (1). Here we say $g \in K_{n,\delta}^{loc}(\Omega)$ if

$$\lim_{r\to 0} \sup_{x\in\mathbf{R}^n} \int_{B_r(x)\cap\Omega'} \frac{|g(y)|}{|x-y|^{n-2+\delta}}\, dy = 0 \tag{11}$$

for every compcat subdomain $\Omega'$ of $\Omega$. We remark that $K_{n,\delta}^{loc}(\Omega) \subset K_n^{loc}(\Omega)$ for $\delta > 0$ and that if $g \in L_{loc}^p(\Omega)$ for some $p > n/2$, then $g \in K_{n,\delta}^{loc}(\Omega)$ for some $\delta n > 0$.

Theorem 1, 2 and Theorem 3 generalize the previous results of [LU], [AS], [CFG], [Si], [HK]. The property $|\nabla u|^2 \in K_n^{loc}(\Omega)$ in the statement of Theorem 1 was first shown by Donig [Do] for weak solutions $u$ of $Lu = -\Delta u + Vu = 0, V \in K_n^{loc}(\Omega)$ and was generalized to general elliptic equations (1) in [Ku1,2].

*Remark 2.* If $\mathbf{b} \in C_{loc}^1(\Omega)$ and $V, f \in K_n^{loc}(\Omega)$, one can show local boundedness of a distributional solution of (7), that is $u, Vu \in L_{loc}^1(\Omega)$ and $u$ satisfies (7) in the distributional sense. Because we can use Kato's inequality directly (see [Hi]). We prove Theorem 3 by using Kato's inequality, but for an approximated solution. Since a local bounded distributional solution $u$ belongs to $H_{loc}^1(\Omega)$ (see e.g. [HS,Lemma 2.2]), $u$ is a weak solution. Applying Corollary 4 we can conclude that $u$ is continuous and $|\nabla u|^2 \in K_n^{loc}(\Omega)$ even for a distributional solution of (7).

*Remark 3.* In [HS] Hinz and Stolz proved the local boundedness of distributional solution $u$ of (7) with $u \in L_{loc}^2(\Omega)$ and $\nabla u \in L_{loc}^{4/3}(\Omega)$ under the assumptions $V \in L_{loc}^1(\Omega), (V)_R^- \in K_n^{loc}(\Omega)$ and $|\mathbf{b}|^2, div\mathbf{b} \in L_{loc}^2(\Omega)$. But we do not know continuity of solutions under this conditions.

**Example 1.** We cannot expect in general Hölder continuity under ASSUMPTION (A). Let $u(x) = 1/(\log\frac{1}{r})^\alpha, \alpha > 0, r = |x|$. Then $u$ is a weak solution of $-\Delta u + Vu = f$ in $B_1(O)$ with $V = \frac{\alpha(\alpha+1)}{r^2(\log\frac{1}{r})^2} \in K_n^{loc}(B_1(O))$, $f = \frac{(n-2)\alpha}{r^2(\log\frac{1}{r})^{1+\alpha}} \in K_n^{loc}(B_1(O))$. $u$ is continous but not Hölder continuous.

**Example 2.** Let $V(x) = \frac{1}{r^2(\log\frac{1}{r})^2}$ and $\mathbf{b}(x) = b(r)\frac{x}{r}, r = |x|$ with $b(r) = \frac{1}{r\log\frac{1}{r}}$. Then $V, |\mathbf{b}|^2 \in K_n^{loc}(B_1(O))$. Hence Theorem 1 implies continuity of weak solution of $-div(A(x)\nabla u)+\mathbf{b}\cdot\nabla u+Vu = f \in K_n^{loc}(B_1(O))$, Theorem 3 implies local boundedness of weak solution of $Tu = f$. However since $div\ \mathbf{b} \notin K_n^{loc}(B_1(O))$, $div\ \mathbf{b} = b_r + b\frac{n-1}{r} = \frac{n-2}{r^2\log\frac{1}{r}} - \frac{1}{r^2(\log\frac{1}{r})^2}$, we cannot apply Corollary 4. Note that since $|\mathbf{b}| = |b(r)| \notin K_{n+1}^{loc}(B_1(O))$, the result of [CZ] cannot be applied.

We prove Theorem 1 by using some global integrability of the Green function of $L$ ( (32) ) and the mollified Green function of $L_0 = -div(A(x)\ \nabla$ ) (Theorem 4, Lemma 4 below ) as in [Ku1,2] (cf. [CFG]). This paper is organized as follows. In section 2 we present several lemmas which play an important role in our proof. In section 3 we give the sketch of the proof of Theorem 1 and 3. For the proof of Theorem 2 and the details of the proof of Theorem 1 and 3, see [Ku1,2].

Finally we give comments on different approach on this regularity problem. There exists an approach due to Simader, which is simple in the sense of using the Green function of the principal part $L_0 = -div(A(x)\nabla)$, but only give partial results on our problem in the case $\mathbf{b} \not\equiv 0$. For this approarch see [Ku2, Appendix]. For a probablistic approach see [AS], [CFZ], [CZ]. Especially Cranston and Zhao [CZ] proved Harnack's inequality for $L = -\Delta + \mathbf{b}\cdot\nabla + V$ under $V, |\mathbf{b}|^2 \in K_n^{loc}(\Omega)$ and an additional assumption $|\mathbf{b}| \in K_{n+1}^{loc}(\Omega)$ (see Example 2).

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## 2. Some Lemmas

**2.1 Approximation.** Let $j(x) \in C_o^\infty(\mathbf{R}^n)$ be a function satisfying $0 \leq j(x) \leq 1, j(x) \equiv 0$ for $|x| \geq 1$, and $\int j(x)\,dx = 1$. For $\epsilon > 0$ let $j_\epsilon(x) = \epsilon^{-n}j(\epsilon^{-1}x)$ and let $g^\epsilon(x) = j_\epsilon * g = \int j_\epsilon(x-y)g(y)\,dy$ for $g \in L^1_{loc}(\mathbf{R}^n)$. Let $u$ be a weak solution of

$$(12) \qquad Lu \equiv -div(A(x)\nabla u + \mathbf{b}(x)u) + \mathbf{c}(x)\cdot\nabla u + V(x)u = f$$

on $\Omega$ and consider the smooth solution of

$$(13) \quad L^\epsilon v \equiv -div(A^\epsilon(x)\nabla v + \mathbf{b}^\epsilon(x)v) + \mathbf{c}^\epsilon(x)\cdot\nabla v + V^\epsilon(x)v = f^\epsilon \text{ in } B$$

and $v-u \in H^1_o(B)$. We say $g \in L^1_{\mathrm{loc}}(\Omega)$ satisfies the condition $(\delta)$ on $B$, if

$$\sup_{x\in 2B}\int_{2B}\frac{|g(y)|}{|x-y|^{n-2}}\,dy \le \frac{\delta}{2} \tag{14}$$

for a ball $B$ with $\overline{2B}\subset\Omega$, where we denote by $2B$ the ball concentric with $B$ but with the radius two times as large.

**Lemma 1.** *There exists a constant $\delta=\delta_1(n,\lambda)$ such that if $V, |\mathbf{b}|^2, |\mathbf{c}|^2$ and $f$ satisfy the condition $(\delta)$ on $B$, then there exists a unique solution $u_\epsilon \in C^\infty_{\mathrm{loc}}(B)$ of (13) such that*

$$\|\nabla u_\epsilon - \nabla u\|_{L^2(B)} + \|u_\epsilon - u\|_{L^2(B)} \to 0 \quad (\epsilon \to 0). \tag{15}$$

*Proof.* See [Ku1, Lemma 2.1] □

Note that, for $A^\epsilon, g^\epsilon$, we have

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a^\epsilon_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \quad \eta(g^\epsilon;r)\le\eta(g;r)+\eta(g;\epsilon).$$

The second inequality is due to Simon [Sim,page 455]. This implies

$$\eta(|\mathbf{b}^\epsilon|^2;r)\le\eta(|\mathbf{b}|^2;r)+\eta(|\mathbf{b}|^2;\epsilon),$$

$$\eta(g^\epsilon;r;B_r)\le\eta((g\chi_{B_{2r}})^\epsilon;r;B_r)\le 2\eta(g;r;B_{2r})$$

for $0<\epsilon\le r$, $g\in K^{\mathrm{loc}}_n(\Omega)$ and a ball $B_{2r}\subset\Omega$.

**2.2 Weighted Norm Inequality.** We note the following inequality. For $g\in K^{\mathrm{loc}}_n(\Omega)$ we have

$$\int_{B_r}|g|v^2\,dx \le C_0\eta(g;r;B_r)\left(\int_{B_r}|\nabla v|^2\,dx+\frac{1}{r^2}\int_{B_r}v^2\,dx\right) \tag{16}$$

for $v\in H^1(B_r)$ and $B_r\subset\Omega$, where $C_0$ is a constant depending only on $n$. (It is sufficient to our argument to use a classical inequality in [HK] which is weaker than (16).) The inequality (16) is due to [FGL, Lemma 2.1].

**2.3 Scaling property.** We also note that the following scaling property. Set $u_r(x) = u(rx)$, $A_r(x) = A(rx)$, $\mathbf{b}_r(x) = r\mathbf{b}(rx)$, $\mathbf{c}_r(x) = r\mathbf{c}(rx)$, and $V_r(x) = r^2V(rx)$. If $u$ is a solution of (12) in $B_r \subset \Omega$, then $u_r$ is a solution of $-div(A_r(x)\nabla u_r + \mathbf{b}_r u_r) + \mathbf{c}_r \cdot \nabla u_r + V_r u_r = 0$ in $B_1$. Moreover, $\mathbf{c}_r$ and $V_r$ ( we omit $\mathbf{b}_r$ ) satisfy

(17)

$$\begin{aligned}\sup_{x\in B_1}\int_{B_1}\frac{|V_r(y)|+|\mathbf{c}_r(y)|^2}{|x-y|^{n-2}}\,dy &\le \sup_{x\in B_r}\int_{B_r}\frac{|V_r(z)|+|\mathbf{c}_r(z)|^2}{|w-z|^{n-2}}\,dz\\ &\le \sup_{w\in\mathbf{R}^n}\int_{B_{2r}(w)}\frac{(|V(z)|+|\mathbf{c}|^2(z))\chi_\Omega(z)}{|w-z|^{n-2}}\,dz\\ &= \eta(V;2r)+\eta(|\mathbf{c}|^2;2r).\end{aligned}$$

Therefore, Lemma 1 says that there exists $r_o = r_o(n,\lambda,\eta) > 0$ such that the solution $u$ of (12) can be approximated by the smooth solution $u_\epsilon$ of the approximated equation (13) for every $B = B_r$ with $0 < r < r_o$ and $B_{2r} \subset \Omega$.

**2.4 Caccioppoli-type Inequality.** The following Caccioppoli-type inequality holds for weak solutions of (12).

**Lemma 2.** *Suppose that $V, |\mathbf{b}|^2, |\mathbf{c}|^2, f \in K_n^{\mathrm{loc}}(\Omega)$. Let $u$ be a weak solution of $Lu = f$ on $\Omega$ For $0 < s < t$ with $B_t \subset \Omega$, there exists a constant $C = C(n,\lambda,\eta,\Omega) > 0$ such that*

$$\int_{B_s}|\nabla u|^2\,dx \le \frac{C}{(t-s)^2}\int_{B_t}u^2\,dx + C\int_{B_t}|f|\,dx. \tag{18}$$

*Proof.* The proof is standard. For the details see [Ku1, Lemma 2.1]. □

The following lemma is applied for the Green function of $L$ and is used essentially in the proof of Theorem 1.

**Lemma 3.** *Suppose $f \equiv 0$. and the same assumption as in Lemma 2. Then we have*

$$\left(\fint_{B_{r/2}}u^2\,dx\right)^{1/2} \le C(n,\lambda,\eta)\fint_{B_r}|u|\,dx \tag{19}$$

*for every $B_{2r} \subset \Omega$.*

*Proof.* For the case $f = 0$, Lemma 2 yields

$$\int_{B_s}|\nabla u|^2\,dx \le \frac{C}{(t-s)^2}\int_{B_t}u^2\,dx \tag{20}$$

for $0 < s < t$ with $B_t \subset \Omega$. Once we obtain this estimate we can prove Lemma 3 in the same way as in [CFG], by using Sobolev's inequality. □

**2.5 Mollified Green Function.** To show local boundeness for weak solutions of (1) we must control the term $\int |\nabla u(y)|^2/|x-y|^{n-2}\,dy$. It turns out that the mollified Green function technique yields desired estimates for our purpose. Recall the definition of the mollified Green function $G^\sigma(x,y)$ for $L_0 = -div(A(x)\nabla)$ and its properties. For $y \in D$, $D$ is a domain, the mollified Green function for $L_0$ on $D$ is defined by

$$\int_D \sum_{i,j=1}^n a_{ij}\partial_i\phi\partial_j G^\sigma(x,y)\,dx = ⨍_{B_\sigma(y)} \phi\,dx \tag{21}$$

for all $\phi \in H^1_o(D)$. The mollified Green function has the following properties

$$G^\sigma(\cdot,y) \in H^1_o(D) \cap L^\infty(D) \quad (y \in D); G^\sigma(x,y) \to G(x,y) \ (\sigma \to 0, x \neq y)$$
$$0 \leq G^\sigma(x,y) \leq K(n,\lambda)|x-y|^{2-n} \text{ for } x,y \in D, 0 < \sigma < dist(y,\partial D),$$

where $G(x,y)$ is the Green function for $L_0$ on $D$. $G$ has the following well-known properties:

$$0 \leq G(x,y) \leq K(n,\lambda)|x-y|^{2-n} \quad \text{for} \quad x,y \in D,$$
$$K(n,\lambda)|x-y|^{2-n} \leq G(x,y) \quad \text{for} \quad |x-y| \leq (3/4)dist(y,\partial D).$$

For the proof of these properties, see [GW].

**2.6 Estimate of Green Function for $L$.** Next theorem implies existence and global integral estimate (see (32)) of the Green function for $L = -div(A(x)\nabla) + \mathbf{c}\cdot\nabla + V$.

**Theorem 4.** *Let $B = B_2(O)$ and $n/2 < p < +\infty$. Suppose that $V, |\mathbf{c}|^2$ satisfy the condition ($\delta$) for sufficiently small $\delta = \delta_2(n,\lambda) > 0$ on $B$. Then there exists $C = C(n,p,\lambda) > 0$ such that*

$$\|u\|_{L^\infty(B)} \leq C\|f\|_{L^p(B)} \tag{22}$$

*for any weak solutions $u$ of $Lu = f$ with $u = 0$ on $\partial B$.*

*Proof.* It is sufficient to show the same estimate for the approximated solution $u = u_\epsilon$ of $L^\epsilon u_\epsilon = f^\epsilon$ by Lemma 1. The idea is, for $y \in B$, to take $\phi = uG^\sigma_\epsilon(\cdot,y)$ as a test function of

$$\int_B < A\partial u, \partial\phi > + \mathbf{c}\cdot\nabla u\phi + Vu\phi\,dx = \int_B f\phi\,dx, \quad \phi \in H^1_o(B)$$

where $G^\sigma_\epsilon(\cdot,y)$ is the mollified Green function of $L^\epsilon_0 \equiv -div(A^\epsilon(x)\nabla)$ on $B$. For the details see [Ku1, Theorem 3.1]. □

**2.7 Property $|\nabla u|^2 \in K_n^{loc}(\Omega)$.** We use the notation $\theta(r) = \eta(|V|;r) + \eta(|\mathbf{c}|^2;r)$, $\theta(r;D) = \eta(|V|;r;D)+\eta(|\mathbf{c}|^2;r;D)$, $\theta_0(r) = \eta((V_R)^-;r)+\eta(|\mathbf{c}|^2;r)$, $\theta_0(r;D) = \eta((V_R)^-;r;D) + \eta(|\mathbf{c}|^2;r;D)$. $osc_B f = \sup_{x,y\in B}|f(x) - f(y)|$. $\sigma_{B,r} f = \sup_{x,y\in B,|x-y|<r}|f(x) - f(y)|$. The following lemma plays an important role in the proof of continuity of weak solutions. Let $G(y,x)$ be the Green function of $L_0$ on $B_r(x_o) \subset \Omega$. Then we have

**Lemma 4.** *Let $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be a weak solution of (1) and let $B_{2r}(x_o) \subset \Omega$. Then there exists a constant $C = C(n,\lambda) > 0$ such that*

$$\int_{B_r(x_o)} G(y,x)|\nabla u(y)|^2\,dy \le C osc_{B_r(x_o)}|u|^2 + C\theta_0(2r;B_r(x_o))\|u\|^2_{L^\infty(B_r(x_o))} + C\eta(f;2r;B_r(x_o))\|u\|_{L^\infty(B_r(x_o))} \tag{23}$$

*for $x \in B_r(x_o)$.*

*Proof.* The idea is the same one as in the proof of Theorem 4, but we need a modification because we do not have boundary condition in this case. For the details see [Ku1; Lemma 3.1]. □

Take $x = x_o$ in Lemma 4 and note $G(y,x_o) \ge K(n,\lambda)|y - x_o|^{2-n}$ for $|y - x_o| \le (1/2)\ dist(x_o, \partial B_r(x_o)) = r/2$. Then we obtain

$$\int_{B_{r/2}(x_o)} \frac{|\nabla u(y)|^2}{|y-x_o|^{n-2}}\,dy \le C osc_{B_r(x_o)}|u|^2 + C\theta_0(2r;B_r(x_o))\|u\|^2_{L^\infty(B_r(x_o))}) + C\eta(f;2r;B_r(x_o))\|u\|_{L^\infty(B_r(x_o))}$$

for $B_{2r}(x_o) \subset \Omega$. Once we establish continuity, this estimate yields the property $|\nabla u|^2 \in K_n^{loc}(\Omega)$.

**Theorem 5.** *Suppose Suppose* ASSUMPTION *(A). Let $u \in H^1_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ be a weak solution of (1) and let $\Omega_1 \subset \Omega$ be a compact set. Then $|\nabla u|^2 \in K_n^{loc}(\Omega)$ and there exists a constant $C$ such that*

$$\begin{aligned}\eta(|\nabla u|^2;r;\Omega_1) \le C\theta_0(4r)\|u\|_{L^\infty(\Omega_2)} + \sigma_{\Omega_2,4r}(u))\|u\|_{L^\infty(\Omega_2)} \\ + C\eta(f;4r;\Omega_2)\|u\|_{L^\infty(\Omega_2)}\end{aligned} \tag{24}$$

*for $\Omega_1 \subset \Omega_2 \subset \Omega$ and for sufficiently small $r > 0$.*

*Proof.* See the proof of Theorem 5.2 of [Ku1]. □

In section 3 we will show local boundedness of weak solutions (Theorem 6). Hence Theorem 5 implies that every weak solution $u$ of (1) satishies $|\nabla u|^2 \in K_n^{loc}(\Omega)$.

## 3. Sketch of the Proof

We give the sketch of the proof of Theorem 1 and 3. First we show local boundedness of weak solution of (1).

**Theorem 6.** *Suppose $V, |\mathbf{c}|^2, f \in K_n^{loc}(\Omega)$ and let $u$ be a weak solution of (1). Then there exist constants $r_o = r_o(n, \lambda, \eta), C = C(n, p, \lambda, \eta) > 0$ such that*

$$\sup_{B_{r/2}} |u| \leq C(\fint_{B_r} |u|^2 \, dx)^{1/2} + C\eta(f; 2r) \tag{25}$$

*for every $0 < p < +\infty$ and $0 < r < r_o$ with $B_{4r} \subset \Omega$. When $f = 0$,*

$$\sup_{B_{r/2}} |u| \leq C(n, p, \lambda, \eta)(\fint_{B_r} |u|^p \, dx)^{1/p} \tag{26}$$

*holds for every $0 < p < +\infty$ and $0 < r < r_o$ with $B_{4r} \subset \Omega$.*

*Proof.* For the sake of simplicity we prove only in the case $f \equiv 0$. By the scaling argument in section 2 it suffices, under the condition ($\delta$) on $B_2(O)$ for $V, |\mathbf{c}|^2$, to prove

$$\sup_{B_{1/2}} |u| \leq C\left(\fint_{B_1} |u|^p \, dx\right)^{1/p} \tag{27}$$

for $0 < p \leq 2$. The case $p > 2$ follows from the case $p = 2$ and Hölder's inequality. To show (27) it is enough to prove

$$\|u\|_{L^\infty(B_s)} \leq \frac{C}{(t-s)^{2+n/2}} \left(\int_{B_t} u^2 \, dy\right)^{1/2} \tag{28}$$

for any $1/2 \leq s < t \leq 1$. Because, once we establish (28), the argument of [FS] yield (27) and complete the proof. Hence we prove (28) for a weak solution $u$ of (1). By Lemma 1 it also suffices to show (28) for the solution $u_\epsilon$ of the approximated equation $L^\epsilon v = 0$. Let $G_\epsilon$ be the Green function of $L^\epsilon$

on $B = B_2(O)$. Then, by omiting superscript or subscript $\epsilon$ for $G_\epsilon, u_\epsilon, \mathbf{c}^\epsilon, V^\epsilon$, we have

$$\int_B < A\partial G(\cdot, x), \partial\varphi > + G(\cdot, x)\mathbf{c}(y) \cdot \nabla\varphi + VG(\cdot, x)\varphi \, dy = \varphi(x) \tag{29}$$

for every $\varphi \in C_o^\infty(B)$ ( see e.g., [St]). Take $\phi(x)$ such that $1/2 \leq s < t \leq 1$, $\phi \in C_o^\infty(B_{t-(t-s)/4})$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{(t+s)/2}$, $|\nabla\phi| \leq C/(t-s)$. Substituting $\varphi = u\phi$ into (29), noting $u = u_\epsilon \in C^\infty(B)$ by the regularity theorem, we obtain, for $x \in B_t$,

$$\begin{aligned} u(x)\phi(x) = &\int_B (< A\partial G(\cdot, x), \partial\phi > u - < A\partial\phi, \partial u > G(\cdot, x)) \, dy \\ &+ \int_B G(\cdot, x) u\mathbf{c} \cdot \nabla\phi \, dy. \end{aligned} \tag{30}$$

Here we used the fact that $u$ is the solution of $Lu = 0$. We denote by $J_1$ ($J_2$) the first term (the second term ) in the right-hand side of (30), respectively. For $J_1$, as in [CFG]( see also [Gu]), we obtain

(31)

$$\begin{aligned} |J_1| \leq &\frac{C}{t-s}\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla_y G(y,x)|^2 \, dy\right)^{1/2} \left(\int_{B_t} u^2 \, dy\right)^{1/2} \\ &+ \frac{C}{t-s}\left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |G(y,x)|^2 \, dy\right)^{1/2} \\ &\times \left(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\nabla u|^2 \, dy\right)^{1/2} \\ &\leq \frac{C}{(t-s)^{2+n/2}}\left(\int_{B_t} u^2 \, dy\right)^{1/2}. \end{aligned}$$

In the last inequality of (31), we used Lemma 2, Lemma 3 and Theorem 4 ( the estimate (32) below). Note that $L_* G(\cdot, x) = 0$ on $B \setminus \{x\}$ for the adjoint operator $L_*$ of $L$ and that Theorem 4 and the duality argument yield

$$\sup_{x \in B}\left(\int_B |G(y,x)|^q \, dy\right)^{1/q} < +\infty \tag{32}$$

for $1 < q < n/(n-2)$. For $J_2$ we have

$$
\begin{aligned}
|J_2| &\le \frac{C}{t-s}\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |G(y,x)||\mathbf{c}(y)||u(y)|\,dy \\
&\le \frac{C}{t-s}\Big(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |G(y,x)|^2\,dy\Big)^{1/2} \qquad (33)\\
&\times\Big(\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\mathbf{c}(y)|^2u^2(y)\,dy\Big)^{1/2}.
\end{aligned}
$$

By Lemma 2 and the weighted norm inequality (16),

$$
\begin{aligned}
&\int_{B_{t-(t-s)/4}\setminus B_{(t+s)/2}} |\mathbf{c}|^2u^2(y)\,dy \le \int_{B_{t-(t-s)/4}} |\mathbf{c}|^2u^2(y)\,dy \\
&\le C_0\eta(|\mathbf{b}|^2;1;B_1)\Big(\int_{B_{t-(t-s)/4}} |\nabla u|^2\,dy + \frac{1}{(t-\frac{t-s}{4})^2}\int_{B_{t-(t-s)/4}} u^2\,dy\Big) \\
&\le C_0\eta(|\mathbf{c}|^2;1;B_1)\frac{1}{(t-s)^2}\int_{B_t} u^2\,dy.
\end{aligned}
$$

Hence we obtain

$$
\|u\|_{L^\infty(B_s)} \le \frac{C}{(t-s)^{2+n/2}}\Big(\int_{B_t} u^2\,dy\Big)^{1/2} \qquad (34)
$$

for any $1/2 \le s < t \le 1$ and for the solution $u = u_\epsilon$ of the approximated equation. This completes the proof. □

*Proof of Theorem 1.* By Lemma 4, we can prove Theorem 1 exactly in the same way as in [Theorem 5.1, Ku1]. We give the sketch of the proof.

Let $\phi \in C_o^\infty(B_{2r}(x_o))$ satisfy $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{3r/2}(x_o)$, and $|\nabla\phi| \le C/r$. Let $u$ be a solution of $L^\epsilon u = f^\epsilon$ on $B_{4r}(x_o)$, and let $\Gamma(x,y)$ be the fundamental solution of $L_0^\epsilon$. Notice that there exists a constant $C$, independent of $\epsilon$, such that $|\Gamma(x,y)| \le C(n,\lambda)|x-y|^{-(n-2)}$, $x, y \in \mathbf{R}^n$. We omit the superscript $\epsilon$ for $A^\epsilon, \mathbf{c}^\epsilon, V^\epsilon, f^\epsilon$ in the following argument. Since $u = u_\epsilon \in C^\infty$, we have

$$
\begin{aligned}
u(x)\phi(x) &= -\int \Gamma(x,\cdot)\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j(u\phi))\,dy \\
&= -\int \Gamma(x,\cdot)\Big((\mathbf{c}\cdot\nabla u + Vu - f)\phi \qquad (35)\\
&+ < A\partial\phi, \partial u >\Big)\,dy + \int < A\partial\Gamma(x,\cdot), \partial\phi > u\,dy.
\end{aligned}
$$

Therefore, for $x \in B_r(x_o)$,

(36)

$$u(x) - u(x_o) = \int (\Gamma(x_o,\cdot) - \Gamma(x,\cdot))\mathbf{c} \cdot \nabla u \phi \, dy$$
$$+ \int (\Gamma(x_o,\cdot) - \Gamma(x,\cdot)) V u \phi \, dy - \int (\Gamma(x,\cdot) - \Gamma(x_o,\cdot)) < A\partial\phi, \partial u > dy$$
$$+ \int < A(\partial\Gamma(x,\cdot) - \partial\Gamma(x_o,\cdot)), \partial\phi > u \, dy + \int (\Gamma(x,\cdot) - \Gamma(x_o,\cdot)) f \phi \, dy$$
$$= I_c + I_V + II + III + I_f.$$

For $I_V, II, III, I_f$ we can estimate as in [CFG] and obtain for some $\alpha(n,\lambda)$ and $C(n,\lambda)$

$$|II| + |III| \le C\Big(\frac{|x - x_o|}{r}\Big)^{\alpha} \Big(\fint_{B_{3r}} u^2 \, dx\Big)^{1/2},$$
$$|I_V| \le C\Big( (\frac{|x - x_o|}{r})^{\alpha/2} \eta(V; 4r) + \eta(|V|; 4r^{1/2}|x - x_o|^{1/2}) \Big)$$
$$\times \|u\|_{L^\infty(B_{2r}(x_o))},$$
$$|I_f| \le C\Big( (\frac{|x - x_o|}{r})^{\alpha/2} \eta(f; 4r) + \eta(|f|; 4r^{1/2}|x - x_o|^{1/2}) \Big).$$

For $I_c$ as in [Ku1] we obatin for $C = C(\lambda)$

$$|I_c| \le \frac{C}{N^\alpha} \eta(|\mathbf{c}|^2; 4r; B_{4r}(x_o))^{1/2} (1 + \theta(8r; B_{4r}(x_o)))^{1/2} \|u\|_{L^\infty(B_{4r}(x_o))}$$
$$+ \eta(|\mathbf{c}|^2; 4r^{1/2}|x - x_o|^{1/2}; B_{5r}(x_o))^{1/2}$$
$$\times (1 + \theta(8r^{1/2}|x - x_o|^{1/2}; B_{5r}(x_o)))^{1/2} \|u\|_{L^\infty(B_{5r}(x_o))}$$
$$+ \frac{C}{N^\alpha} \eta(|\mathbf{c}|^2; 4r; B_{4r}(x_o))^{1/2} \eta(f; 8r; B_{4r}(x_o))$$
$$+ \eta(|\mathbf{c}|^2; 4r^{1/2}|x - x_o|^{1/2}; B_{5r}(x_o))^{1/2}$$
$$\times \eta(f; 4r^{1/2}|x - x_o|^{1/2}; B_{5r}(x_o))$$

for $N = 2(r/|x - x_o|)^{1/2}$. Thus we obtain the desired estimate for the solution $u = u_\epsilon$ of the approximated equation. By Lemma 1 we can conclude the same estimate for weak solution $u$ of (1) for sufficiently small $0 < r < r_o = r_o(n, \lambda, \eta)$. Thus we complete the proof. □

*Proof of Theorem 3.* We show that there exist constants $r_o = r_o(n,\eta) > 0$, $C = C(n,\eta) > 0$ such that

$$\|u\|_{L^\infty(B_{r/2})} \leq C(\fint_{B_r} |u|^2\,dx)^{1/2} + C\eta(f;2r) \tag{37}$$

for every $0 < r < r_o$. When $f = 0$ we show that there exist constants $r_o, C > 0$ such that

$$\|u\|^2_{L^\infty(B_{sr})} \leq \frac{C}{r^n(t-s)^n}\int_{B_{tr}} |u|^2 \tag{38}$$

for every $0 < r < r_o$ and $0 < s < t < 1$. Once we obtain (38) we can obtain the estimate

$$\|u\|_{L^\infty(B_{r/2})} \leq C(\fint_{B_r} |u|^p\,dx)^{1/p} \tag{39}$$

for every $0 < p < +\infty$ and $0 < r < r_o$ with $B_{2r} \subset \Omega$, by Lemma 5 in [HK].

By Lemma 1 it suffices to establish (37) for the approximated solution $u_\epsilon$ of $T^\epsilon u_\epsilon = f^\epsilon$. We first show that there exists a constant $C > 0$ independent of $\epsilon$ such that

$$|u_\epsilon(y)| \leq C\left(\fint_{B_r(y)} |u_\epsilon|^2\,dx\right)^{1/2} + C\eta(f;2r) \tag{40}$$

for $0 < r < r_o$ with $B_{2r}(y) \subset \Omega$. Since $\mathbf{b}^\epsilon, u_\epsilon \in C^\infty_{\text{loc}}(B_{2r}(y))$, Kato's inequality implies

$$\begin{aligned} \Delta|u_\epsilon| &\geq \text{Re}\big(sign(\overline{u_\epsilon})(\nabla - i\mathbf{b}^\epsilon)^2 u_\epsilon\big)\\ &= \text{Re}\big(sign(\overline{u_\epsilon})(V^\epsilon u_\epsilon - f^\epsilon)\big)\\ &\geq V^\epsilon_R|u_\epsilon| - |f^\epsilon|\\ &\geq -(V^\epsilon)^-_R|u_\epsilon| - |f^\epsilon| \end{aligned} \tag{41}$$

in the distributional sense in $B_{2r}(y)$. Then by the argument as in the proof of Theorem 1 in [HK] (see also [H1]) we can obtain (40). Here we used $\eta((V^\epsilon)^-_R; r) \leq 2\eta(V^-_R; 2r)$ for $\epsilon < r$ and Lemma 2. As in [HK,page 128] (40) implies the desired estimate (37) for $u_\epsilon$ and hence also for weak solutions $u$. □

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# ON SOME IMPROPERLY POSED ESTIMATE OF THE CAUCHY PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

Kazuya Hayasida

We consider the Cauchy problem for degenerate quasilinear elliptic equations and degenerate elliptic Monge-Ampère equations. It is well-known that this problem is not well-posed even for linear elliptic equations. But an estimate holds for a class of them. Such an estimate is said to be an "improperly posed estimate", which is close to Hadamard's three circles inequality (see [3] and [5]). It is stronger than the unique continuation property. Kazdan [4] proposed the question : Does the strong unique continuation hold for p-harmonic function ? Generally, it is negative for solutions of degenerate quasilinear elliptic equations (see [7]).

Here we treat the two nonlinear operators

$$L_p(u) = \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p-2} \, \partial_{x_i} u \right), \quad p \geqq 2,$$

$$M(u) = \partial_x^2 u \; \partial_y^2 u - (\partial_x \partial_y u)^2 .$$

The operator $L_p$ appears in [6]. If $M(u) \geqq 0$, the operator $M$ is degenerate elliptic.

We write $x = (x_1, \cdots, x_N), x' = (x_1, \cdots, x_{N-1})$ and $y = x_N$. The origin in $R^N$ is denoted by $O$. Let $D$ be a domain in $R^N$ such that $D$ is in the half space $\{y > 0\}$. Let $\Gamma$ be an open subset of $\partial D$ with $\Gamma \ni O$. Let $\Gamma$ be of class $C^1$.

We assume that there is a positive number $a < 1/2$ such that for any $c$ with $0 < c \leqq a, D \cap \{y < c\}$ is connected and

$$\partial(D \cap \{y < c\}) = \{O\} \cup (\Gamma \cap \{0 < y < c\}) \\ \cup (\bar{D} \cap \{y = c\}).$$

This means that $D$ is strictly convex at $O$. We fix such a positive number $a$. From now on we write $D_c = D \cap \{y < c\}$ and $\Gamma_c = \Gamma \cap \{y < c\}$, and $\ell_c = \overline{D} \cap \{y = c\}$.

First we have

**Theorem 1.** *Let u belong to* $c^{1,\alpha}\left(\bar{D}_a\right)$ *for* $\alpha$ *with* $1/2 < \alpha \leqq 1$. *Let*

$$|L_p(u)| \leqq K|u|^{p-1} \quad \text{in } D_a$$

*for a constant K. Then, if*

$$\int_{\Gamma_a} (|u|^p + |\nabla u|^p) \quad dS \leqq \varepsilon,$$
$$\int_{\ell_a} (|u|^p + |\nabla u|^p) \quad dS \leqq M$$

*and* $\varepsilon \exp\left(2^{p-1}K\right), \varepsilon \exp\left(p/2\right) \leqq a^{2p} M$, *it holds that*

$$\int_{D_{a/2}} (|u|^p + |\nabla u|^p)\, dx \leqq C(1+K)^2 \varepsilon^{a/2} M^{(2-a)/2},$$

*where C is a positive constant depending only on p.*

Next we consider the degenerate elliptic Monge-Ampère equation, when $N = 2$. We define

$$||u||_{\infty,D} = \sup_{D_c} |u|, \quad \langle u \rangle_{\infty,\Gamma_c} = \sup_{\Gamma_c} |u|,$$

and

$$\langle u \rangle_{\infty,\ell_c} = \sup_{\ell_c} |u|.$$

**Theorem 2.** *Let u belong to $C^2(\bar{D}_a)$ and let u satisfy $M(u) \geqq 0$ in $D_a$. Then, if*

$$\langle u\rangle_{\infty,\Gamma_a} + \langle \partial_x u\rangle_{\infty,\Gamma_a} + \langle \partial_x\partial_y u\rangle_{\infty,\Gamma_a} + \langle \partial_y^2 u\rangle_{\infty,\Gamma_a} \leqq \varepsilon$$
$$\langle u\rangle_{\infty,\ell_a} + \langle \partial_x u\rangle_{\infty,\ell_a} + \langle \partial_x\partial_y u\rangle_{\infty,\ell_a} + \langle \partial_y^2 u\rangle_{\infty,\ell_a} \leqq M$$

*and $\varepsilon\ e^{8/3} \leqq M$, it holds that*

$$||u||_{\infty,D_{a/2}} + ||\partial_x u||_{\infty,D_{a/2}} \leqq Ca^{-2}\varepsilon^{a/2}M^{(2-a)/2},$$

*where C is a positive constant independent of $\varepsilon, M$ and $a$.*

The precise proof of Theorem 1 is given in [2]. The special case was proved in [1].

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# REMARKS ON THE GLOBALLY REGULAR SOLUTIONS OF SEMILINEAR WAVE EQUATIONS WITH A CRITICAL NONLINEARITY

JONGSIK KIM[1] AND CHOON-HO LEE[2]

ABSTRACT. We prove that the semilinear wave equation with a critical nonlinearity has a globally regular solution provided that it has a locally regular solution

## Introduction

In this paper we study the existence of a globally regular solution of the semilinear wave equation with a critical nonlinearity

(0.1) $$u_{tt} - \Delta u + u^3 = 0,$$

where $u(x,t) : R^4 \times R \to R$ is a function of four space variables and time. In order to solve (0.1) one has to prescribe initial data at a fixed time $t = 0$, i.e.

(0.2) $$u(x,0) = u_0(x), u_t(x,0) = u_1(x).$$

The equation (0.1) is a special case of a more general set of model equations

(0.3) $$u_{tt} - \Delta u + |u|^{p-1}u = 0,$$

where $u(x,t) : R^n \times R \to R$ is a function.

In case $n = 3$ and $p < 5$, Jörgens[5] proved in 1961 that the nonlinear wave equation (0.3) with initial data

(0.4) $$u(x,0) = u_0(x) \in C^3(R^3),\ u_t(x,0) = u_1(x) \in C^2(R^3)$$

has a globally unique $C^2$ solution. In case $n = 3$ and $p = 5$(critical power), Rauch[7] in 1981 first proved the existence of a global $C^2$ solution provided the initial energy is small enough. In 1988 Struwe [9][10] proved the existence of a radially symmetric global $C^2$ solution provided the initial data is radially symmetric. Finally, Grillakis[2] in 1990 was able to remove symmetric assumption in Struwe's result. In case $4 \le n \le 7$ and $p = (n+2)/(n-2)$, several authors proved the globally regular solution of (0.3) with smooth initial data (see [3],[4],[6],and [8]).

In this paper we shall give the sketch of proof of

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**Theorem 0.1.** *Let $u_0 \in C^4(R^4), u_1 \in C^3(R^4)$ be arbitrary initial data. If $u \in C^2(R^4 \times [0,T))$ is a solution of (0.1) and (0.2), then there exists a solution $u \in C^2(R^4 \times [0,\infty))$ to the Cauchy problem (0.1) and (0.2).*

The proof is divided into several parts. In Section 1, we shall establish an integral representation of the solution of a semilinear wave equation. In Section 2, using the Hardy type inequality we prove the existence of a global $C^2$ solution with small initial data. In Section 3, we apply the identities to drive the several estimates of solutions. In Section 4, we shall prove the existence of a global $C^2$ solution with arbitrary initial data. In Section 5, we shall introduce the results of (0.3) in the higher dimensional case.

We shall use the following notations: Let $z = (x,t)$ denote a point in the space -time $R^4 \times R$. Given $z_0 = (x_0, t_0)$, let

$$K(z_0) = \{z = (x,t) : \ |x - x_0| \le t_0 - t\}$$

be the forward(backward) light cone with vertex at $z_0$,

$$M(z_0) = \{z = (x,t) : \ |x - x_0| = t_0 - t\}$$

its mantle, and

$$D(t, z_0) = \{z = (x,t) \in K(z_0)\} \quad (t \text{ fixed})$$

its time-like sections. If $z_0 = (0,0)$, $z_0$ will be omitted. For any space-time region $Q \subset R^4 \times R$ and $T < S$, we let

$$Q_T^S = \{z = (x,t) \in Q : \ T \le t \le S\}$$

the truncated region. Hence, for instance, we have

$$\partial K_t^s = D(s) \cup D(t) \cup M_t^s.$$

If $s = \infty$ or $t = -\infty$, it will be omitted. For $x_0 \in R^4$, let

$$B_R(x_0) = \{x \in R^4 : \ |x - x_0| \le R\}$$

with boundary

$$S_R(x_0) = \{x \in R^4 : \ |x - x_0| = R\}.$$

## 1. Integral Representation

In this section we shall give an integral representation of a solution of the semilinear wave equation

$$(1.1)\qquad u_{tt} - \Delta u + f(u) = 0 \qquad \text{in } R^n \times R$$

with initial data

$$(1.2)\qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x).$$

Assume that $u$ is a solution belonging to $C^2(R^n \times [0,T))$ of (1.1) and (1.2). Let $x_0$ and $x$ be points in $R^n$. Let $y = x - x_0$ where $x_0$ is a fixed point and $x$ is a variable. Define the functions $[u]$ as

$$[u] = u(x, t - |y|).$$

Then

$$\nabla[u] = [\nabla u] - \hat{y}[u_t],$$

$$\Delta[u] = [\Delta u] - 2[\nabla u_t]\cdot \hat{y} + [u_{tt}] - \frac{n-1}{|y|}[u_t],$$

$$\nabla[u_t] = [\nabla u_t] - [u_{tt}]\cdot \hat{y},$$

where $\hat{y} = \frac{y}{|y|}$ is the unit vector of $y$. Eliminating $[\nabla u_t]$ from the above, we have

$$(1.3)\qquad \Delta[u] + 2\hat{y}\cdot\nabla[u_t] + \frac{n-1}{|y|}[u_t] = [\Delta u] - [u_{tt}] = [f(u)].$$

Multiply (1.3) by $\frac{1}{|y|^{n-2}}$ to get the identity

$$(1.4)\ \nabla\cdot\{\frac{1}{|y|^{n-2}}[\nabla u] + \frac{y}{|y|^{n-1}}[u_t] + \frac{n-2}{|y|^n}y[u]\} + \frac{n-3}{|y|^{n-1}}[u_t] = \frac{1}{|y|^{n-2}}[f(u)].$$

Take $z_0 = (x_0, t_0)$ such that $|x_0| \le t_0$ and $t_0 < T$ and integrate (1.4) inside the domain $\Lambda$ bounded by the surfaces $S_\epsilon = \{|y| = \epsilon\}, S = \{|y| = t_0\}$. Then

$$\int_\Lambda \nabla\cdot\{\frac{1}{|y|^{n-2}}[\nabla u] + \frac{y}{|y|^{n-1}}[u_t] + \frac{n-2}{|y|^n}y[u]\}\,dy$$
$$= \int_\Lambda \{-\frac{n-3}{|y|^{n-1}}[u_t] + \frac{1}{|y|^{n-2}}[f(u)]\}\,dy.$$

The divergence theorem gives

$$\int_{|y|=t_0} \frac{1}{|y|^{n-1}}\{\hat{y}\cdot\nabla u(x,0)+u_t(x,0)+\frac{n-2}{|y|}u(x,0)\}\,do$$
$$-\int_{|y|=\epsilon} \frac{1}{|y|^{n-1}}\{\hat{y}\cdot\nabla u(x,t_0-\epsilon)+u_t(x,t_0-\epsilon)+\frac{n-2}{|y|}u(x,t_0-\epsilon)\}\,do$$
$$=\int_{\epsilon<|y|<t_0}\{-\frac{n-3}{|y|^{n-1}}u_t(x,t_0-|y|)+\frac{1}{|y|^{n-2}}f(u)(x,t_0-|y|)\}\,dy.$$

By letting $\epsilon \to 0$ we have

(1.5)
$$\int_{|y|=t_0} \frac{1}{|y|^{n-1}}\{\nabla u(x,0)\cdot\hat{y}+u_t(x,0)+\frac{n-2}{|y|}u(x,0)\}\,do-(n-2)\omega_n u(x_0,t_0)$$
$$=\int_{|y|<t_0}\{-\frac{n-3}{|y|^{n-1}}u_t+\frac{1}{|y|^{n-2}}f(u)\}\,dy.$$

Thus we have

$$\begin{aligned} u(x_0,t_0)=&\frac{1}{(n-2)\omega_n}\int_{|y|=t_0}\frac{1}{|y|^{n-1}}\{\nabla u_0\cdot\hat{y}+u_1+\frac{n-2}{|y|}u_0\}\,do\\ &+\frac{1}{(n-2)\omega_n}\int_{|y|<t_0}\frac{n-3}{|y|^{n-1}}u_t(x,t_0-|y|)\,dy\\ &-\frac{1}{(n-2)\omega_n}\int_{|y|<t_0}\frac{1}{|y|^{n-2}}f(u)(x,t_0-|y|)\,dy\\ =&\,u_L(x_0,t_0)+u_N(x_0,t_0), \end{aligned}$$

where the linear part of $u(x_0,t_0)$ is given by

(1.6)
$$\begin{aligned} u_L(x_0,t_0)=&\frac{1}{(n-2)\omega_n}\int_{|y|=t_0}\frac{1}{|y|^{n-1}}\{\nabla u_0\cdot\hat{y}+u_1+\frac{n-2}{|y|}u_0\}\,do\\ &+\frac{1}{(n-2)\omega_n}\int_{|y|<t_0}\frac{u_t(x,t_0-|y|)}{|y|^{n-1}}\,dy \end{aligned}$$

and the nonlinear part of $u(x_0,t_0)$ is given by

$$u_N(x_0, t_0) = -\frac{1}{(n-2)\omega_n} \int_{|y|<t_0} \frac{f(u)(x, t_0 - |y|)}{|y|^{n-2}} \, dy. \tag{1.7}$$

Let $z_0 = (x_0, t_0)$ and $z = (x, t)$ for $z \in M_0^{t_0}(z_0) = \{(x,t) : \ |x - x_0| = t_0 - t, \ 0 \le t \le t_0\}$. Then $z - z_0 = (y, |y|)$ and

$$u_N(x_0, t_0) = -\frac{(\sqrt{2})^{n-3}}{(n-2)\omega_n} \int_{M_0^{t_0}(z_0)} \frac{f(u)(z)}{|z - z_0|^{n-2}} \, do \tag{1.8}$$

Thus we have proved the

**Theorem 1.1.** *Let $u \in C^2(R^n \times [0,T))$ be a solution of (1.1) and (1.2). Then for every $z_0 \in K_0^T = \{(x,t) | |x| \le T - t, 0 < t \le T\}$, $u$ satisfies the integral equation*

$$u(z_0) = u_L(z_0) + u_N(z_0), \tag{1.9}$$

*where $u_L(z_0)$ and $u_N(z_0)$ are given by (1.6) and (1.8).*

## 2. Globally Regular Solutions for the Small Initial Data

In this section we shall prove the existence of globally regular solutions of semilinear wave equations with small initial data. Given a function $u$ on a cone $K(z_0)$ we denote its energy by

$$e(u) = \frac{1}{2}(|u_t|^2 + |\nabla u|^2) + \frac{1}{4}u^4$$

and by

$$E(u : D(t : z_0)) = \int_{D(t:z_0)} e(u) \, dx$$

its energy on the space-like section $D(t : z_0)$. Let $x = y + x_0$. We denote by

$$d_{z_0}(u) = \frac{1}{2}|\hat{y} u_t - \nabla u|^2 + \frac{1}{4}u^4$$

the energy density of u tangent to $M(z_0)$. The following Hardy's inequalities are useful to prove the regular solutions of semilinear wave equations.

**Lemma 2.1.** *Suppose $u \in L^4(B_R)$ possesses a weak radial derivative $u_r = \hat{x} \cdot \nabla u \in L^2(B_R)$. Then with an constant $C_0$ independent on $\rho$ and $R$ for all $0 \le \rho < R$ the following holds:*

$$\frac{3}{4} \int_{B_R \setminus B_\rho} \frac{|u(x)|^2}{|x|^2}\, dx \le \int_{B_R \setminus B_\rho} |u_r|^2\, dx + \frac{1}{2R} \int_{S_R} |u|^2\, do. \tag{2.1}$$

$$\int_{B_R} \frac{|u(x)|^2}{|x|^2}\, dx \le C_0 \left\{ \int_{B_R} |u_r|^2\, dx + \left( \int_{B_R} u^4\, dx \right)^{1/2} \right\} \tag{2.2}$$

(2.3)
$$\int_{S_R} u^3\, do \le C_0 \left\{ \left( \int_{B_R} u^4\, dx \right)^{1/2} \left( \int_{B_R} u_r^2\, dx \right)^{1/2} + \left( \int_{B_R} u^4\, dx \right)^{3/4} \right\}$$

*Proof.* See [6].

Note that if $u = u(x,t)$ is a solution of (1.1), then $u(x,-t)$is also a solution of (1.1). Since the semilinear wave equation is conformally invariant, the solution is translation invariant in $t$.

Let $\bar{z} = (\bar{x}, \bar{t})$ be given and suppose $u$ is a $C^2$-solution of (0.1) on the deleted backward light cone $K_0'(\bar{z}) = K_0(\bar{z}) \setminus \{\bar{z}\}$. In order to prove that $u$ can be extended to a global solution of (0.1) and (0.2), it suffices to show that for any $\bar{z}$ as above

$$\bar{m} = limsup_{\substack{z_0 \to \bar{z} \\ z_0 \in K(\bar{z}), z_0 \ne \bar{z}}} |u(z_0)| < \infty.$$

We may assume that $\bar{m} = sup_{K_0(\bar{z})} |u|$.

**Lemma 2.2.** *Let $F(u) = \int_0^u f(t)\, dt$ and assume $F(u) \ge 0$ for all $u \in R$. Suppose $u \in C^2(K_0'(\bar{z}))$ solve (1.1) and (1.2). Then for any $0 \le t < s < \bar{t}$ there holds*

$$E(u : D(s, \bar{z})) + \frac{1}{\sqrt{2}} \int_{M_t^s(\bar{z})} d_{\bar{z}}(u)\, do = E(u : D(t : \bar{z})) \le E_0,$$

where $E(u, D(s, \bar{z})) = \int \frac{1}{2}(|u_t|^2 + |\nabla u|^2) + F(u)\, dx$ and $d_{\bar{z}}(u) = \frac{1}{2}|\hat{y}u_t - \nabla u|^2 + F(u)$.

By Lemma 2.2, for any fixed $\bar{z}$ the energy $E(u : D(s, \bar{z}))$ is a monotone decreasing function of $s \in [0, \bar{t})$ and hence converges to a limit as $s \nearrow \bar{t}$. It follows that

$$\int_{M_t^s(\bar{z})} d_{\bar{z}}(u)\, do \to 0 \qquad \text{as } s, t \nearrow \bar{t} \tag{2.4}$$

In Section 1, we had a decomposition of the solution of (0.1) and (0.2) as

$$u = u_L + u_N,$$

where $y = x - x_0$, and $u_L$ and $u_N$ are defined as in (1.6) and (1.8) respectively. Since we are interested in points $z_0$ such that $|u(z_0)| \to \bar{m}$ as $z_0 \to \bar{z}$, we need only consider points $z_0$ satisfying $|u(z_0)| = max_{K_0(z_0)}|u| = m_0$. Thus, and splitting integration over $M_0^T(z_0)$ and $M_T(z_0)$ for suitable $T$, from Hölder's inequality we obtain

(2.5)

$$m_0 = |u(z_0)| \le C + \frac{m_0}{\sqrt{2}\omega_4}\int_{M_T(z_0)} \frac{u^2(z)}{|z-z_0|^2}\,do + \frac{1}{\sqrt{2}\omega_4}\int_{M_0^T(z_0)} \frac{u^3(z)}{|z-z_0|^2}\,do.$$

By Lemma 2.2 the last term is bounded by $C|t_0 - T|^{-1}E_0^{\frac{3}{4}}$. Thus to establish our main result, it suffices to show that for any $\bar{z} = (\bar{x}, \bar{t})$ there exists $T < \bar{t}$ such that

$$\limsup_{\substack{z_0 \to \bar{z} \\ z_0 \in K(\bar{z})}} \int_{M_T(\bar{z})} \frac{u^2(z)}{|z-z_0|^2}\,do < \sqrt{2}\omega_4. \tag{2.6}$$

This observation and Hardy's inequality gives

**Theorem 2.3.** *If $u \in C^2([0,T) \times R^4)$ is a solution of (0.1) and (0.2), then there exists a constant $\epsilon_0 > 0$ such that for any $u_0 \in C^4(R^4), u_1 \in C^3(R^4)$ with*

$$E_0 = \int_{R^4} \Big(\frac{1}{2}(|u_1|^2 + |\nabla u_0|^2) + \frac{1}{4}|u_0|^4\Big)dx < \epsilon_0,$$

*(0.1) and (0.2) admit a global $C^2$ solution.*

*Proof.* Let $v(y) = u(x_0 + y, t_0 - |y|)$. Then by Lemma 2.1 we have

(2.7)

$$\begin{aligned}
\int_{M_T(z_0)} \frac{|u|^2(z)}{|z-z_0|^2}\,do &= \frac{1}{\sqrt{2}}\int_{B_{t_0-T}(0)} \frac{|v(y)|^2}{|y|^2}\,dy \\
&\le C\int_{B_{t_0-T}(0)} |\nabla v|^2\,dy + C\left(\int_{B_{t_0-T}(0)} |u|^4\,dy\right)^{1/2} \\
&\le C\int_{M_T(z_0)} d_{z_0}(u)\,do + C\left(\int_{M_T(z_0)} d_{z_0}(u)\,do\right)^{1/2} \\
&\le C(E_0 + E_0^{1/2}).
\end{aligned}$$

Letting T=0, the theorem holds from (2.5). ////

Since $t = 0$ no longer plays a distinguished role in the following, we may shift coordinates so that $\bar{z} = (0,0)$ and thus in the sequel we may assume that $u$ is a $C^2$ solution of (0.1) on $K_{t_1} \backslash \{(0,0)\}$ for some $t_1 < 0$.

## 3. Some Estimates for the Large Initial Data

In this section, we introduce the multiplier $tu_t + x \cdot \nabla u + \frac{3}{2}u$ to drive the following identity

$$\partial_t Q_d - \mathrm{div} P_d + R_d = 0, \tag{3.1}$$

where

$$\begin{aligned} Q_d &= te(u) + x \cdot \vec{p}(u) + \frac{3}{2}uu_t \\ &= \frac{1}{4}(t-r)(u_t - u_r)^2 + \frac{1}{4}(t+r)(u_t + u_r)^2 \\ &+ \frac{1}{2}t|\nabla u - u_r \hat{x}|^2 + \frac{1}{4}tu^4 + \frac{3}{2}uu_t \\ &= Q_0 + \frac{3}{2}uu_t, \\ P_d &= t\vec{p}(u) + xl(u) + (x \cdot \nabla u)\nabla u + \frac{3}{2}u\nabla u, \\ R_d &= \frac{1}{4}u^4. \end{aligned}$$

The identity (3.1) is equivalent to the identity

$$\begin{aligned} & t\left\{ \frac{d}{dt}(e(u) + \frac{x}{t} \cdot \vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2) \right. \\ & \left. -\mathrm{div}(\vec{p}(u) + \frac{x}{t}l(u) + \frac{1}{t}(x \cdot \nabla u)\nabla u + \frac{3}{2t}u\nabla u) \right\} \\ & + e(u) + \frac{1}{t}x \cdot \vec{p}(u) + \frac{3}{2t^2}u^2 + R_d = 0. \end{aligned} \tag{3.2}$$

**Lemma 3.1.** *There exists a sequence of numbers $t_l \nearrow 0$ such that*

$$\frac{1}{|t_l|}\int_{D(t_l)} uu_t \, dx \le o(1), \tag{3.3}$$

*where $o(1) \to 0$ as $l \to \infty$.*

**Lemma 3.2.** *For any $l \in N$ there holds*

$$\frac{1}{4|t_l|}\int_{K_{t_l}} |u|^4\,dxdt + \int_{D(t_l)} \{e(u) + \frac{x}{t}\cdot \vec{p}(u)\}\,dx \le o(1) \to 0 \tag{3.4}$$

*as $l \to \infty$.*

**Lemma 3.3.** *There exists a sequence of numbers $\bar{t}_l \nearrow 0$ such that the conclusion of Lemma 3.1 holds for $(\bar{t}_l)$ which in addition we have*

$$2 \le \frac{\bar{t}_l}{\bar{t}_{l+1}} \le 4 \tag{3.5}$$

*for all $l \in N$.*

In the sequel to simplify notation we shall assume that $t_l = \bar{t}_l$ for all $l$, initially.

## æ 4. Globally Regular Solutions for the General Data

In this section we shall prove the Theorem 0.1. Fix $z_0 = (x_0, t_0) \in K\backslash\{0\}$ arbitrary. Let $y = x - x_0, \hat{y} = \frac{y}{|y|}, \hat{x} = \frac{x}{|x|}$. Divide (3.2) by $t$ and then for $s > t_0$ integrate over $K_{t_l}^s\backslash K(z_0)$ to obtain the relation

$$\begin{aligned}
0 = &\int_{D(s)} \{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2\}\,dx \\
&- \int_{D(t_l)\backslash D(t_l:z_0)} \{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2\}\,dx \\
&+ \frac{1}{\sqrt{2}}\int_{M_{t_l}^s} \{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2 - \hat{x}\cdot P\}\,do \\
&- \frac{1}{\sqrt{2}}\int_{M_{t_l}(z_0)} \{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t}uu_t + \frac{3}{4t^2}u^2 - \hat{y}\cdot P\}\,do \\
&+ \int_{K_{t_l}^s\backslash K(z_0)} \frac{1}{t}\{e(u) + \frac{1}{t}x\cdot\vec{p}(u) + \frac{3}{2t^2}u^2 + R_d\}\,dxdt \\
= &\, I + II + III + IV + V,
\end{aligned}$$

where $P = \vec{p}(u) + \frac{x}{t}l(u) + (\frac{1}{t}x\cdot\nabla u)\nabla u + \frac{3}{2t}u\nabla u = \frac{1}{t}P_d$.

By Hölder's inequality and Lemma 3.2 the first term $I \to 0$ if we choose $s = t_k$ with $k \to \infty$. Similarly, $II \to 0$ if $l \to \infty$. By Lemma 3.2 also $III \to 0$ as $l \to \infty$. Finally $V \leq 0$. Thus we obtain the estimate for any $z_0 \in K \backslash \{0\}$.

$$\int_{M_{t_l}(z_0)} \{e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{y} \cdot P\}\, do \leq o(1) \to 0 \tag{4.1}$$

as $l \to \infty$, with error term $o(1)$ independent of $z_0$.

In order to bound (2.6) we shall use (4.1). Let $r = |x|$; then we may rewrite

$$\begin{aligned} A &:= e(u) + \frac{1}{t} x \cdot \vec{p}(u) + \frac{3}{2t} u u_t + \frac{3}{4t^2} u^2 - \hat{y} \cdot P \\ &= \frac{1}{2}(1 - \frac{r}{t} \hat{x} \cdot \hat{y}) |u_t|^2 + (1 + \frac{r}{t} \hat{x} \cdot \hat{y})(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4) \\ &+ \frac{3}{2t}(u_t - \hat{y} \cdot \nabla u) u + \frac{r}{t}(u_t - \hat{y} \cdot \nabla u) \hat{x} \cdot \nabla u - u_t \hat{y} \cdot \nabla u + \frac{3}{4t^2} u^2. \end{aligned}$$

Introducing $u_\sigma = \hat{y} \cdot \nabla u, \alpha = \hat{x} - \hat{y}(\hat{y} \cdot \hat{x}), |\alpha| u_\alpha = \alpha \cdot \nabla u, \Omega u = \nabla u - \hat{y} u_\sigma$, we have

$$\begin{aligned} A &:= \frac{1}{2}(1 - \frac{r}{t} \hat{x} \cdot \hat{y})(u_t - u_\sigma)^2 + (1 + \frac{r}{t} \hat{x} \cdot \hat{y})(\frac{1}{2} |\Omega u|^2 + \frac{1}{4} |u|^4) \\ &+ \frac{3}{2t}(u_t - u_\sigma) u + \frac{r}{t} |\alpha| u_\alpha (u_t - u_\sigma) + \frac{3}{4t^2} u^2 \end{aligned}$$

Now let $\hat{x} \cdot \hat{y} = \cos\delta$, $|\alpha| = \sin\delta$ and let $u_\rho = \frac{1}{\sqrt{2}}(u_t - u_\sigma)$. Then we have

$$\begin{aligned} A &= (1 - \frac{r}{t} \cos\delta) |u_\rho|^2 + (1 + \frac{r}{t} \cos\delta)(\frac{1}{2} |\Omega u|^2 + \frac{1}{4} |u|^4) \\ &+ \frac{r}{t} \sqrt{2} |\sin\delta| u_\rho u_\alpha + \frac{3}{\sqrt{2} t} u u_\rho + \frac{3}{4t^2} u^2 \\ &= A_0 + \frac{3}{\sqrt{2t}} u u_\rho + \frac{3}{4t^2} u^2. \end{aligned} \tag{4.2}$$

Note that if we estimate $|u_\alpha| \leq |\Omega u|$, then we have

(4.3)

$$\begin{aligned} A_0 &\geq (1 - \frac{r}{t} \cos\delta) |u_\rho|^2 + (1 + \frac{r}{t} \cos\delta)(\frac{1}{2} |u_\alpha|^2 + \frac{1}{4} |u|^4) \\ &+ \frac{r}{t} \sqrt{2} |\sin\delta| u_\rho u_\alpha \\ &= (1 + \frac{r}{t})(|u_\rho|^2 + \frac{1}{2} |u_\alpha|^2) - \frac{r}{2t}(\sqrt{2}\sqrt{1 + \cos\delta}\, u_\rho - \sqrt{1 - \cos\delta}\, u_\alpha)^2 \\ &+ \frac{1}{4}(1 + \frac{r}{t} \cos\delta) |u|^4 \geq 0 \end{aligned}$$

on $M_{t_l}(z_0)$.

Now for any $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ such that for any $z_0 \in K$ and any $z \in M^{Ct_0}(z_0)$ we may estimate

$$-\frac{r}{t}\sqrt{2}|\sin\delta| \leq \epsilon, \qquad -\frac{r}{t}\cos\delta \geq \frac{1}{2}.$$

In fact, for $z = (x,t) \in M^{Ct_0}(z_0)$ we have

$$\big||x| - |y|\big| \leq |y - x| = |x_0| \leq |t_0| \leq \frac{|t - t_0|}{C - 1} = \frac{|y|}{C - 1}.$$

Hence

$$\hat{x}\cdot\hat{y} = \cos\delta \geq 1 - |\hat{y} - \hat{x}| \geq 1 - 2\frac{|x_0|}{|y|} \geq 1 - \frac{2}{C - 1}$$

while

$$1 \geq -\frac{r}{t} = \frac{|y|}{|t - t_0|}\frac{|t - t_0|}{|t|}\frac{|x|}{|y|} \geq (1 - \frac{1}{C})(1 - \frac{1}{C - 1}).$$

This yields the following estimate.

**Lemma 4.1.** *For any $\epsilon > 0$, any $z_0 \in K$, letting $C = C(\epsilon)$ be determined as above for $t_k \leq Ct_0$ we have*

$$\int_{M_{t_l}^{t_k}(z_0)} A\, do \geq \frac{1}{2}\int_{M_{t_l}^{t_k}(z_0)} |u_\rho|^2\, do - \epsilon E_0$$

Note that $u_\rho$ may be interpreted as a tangential derivative along $M(z_0)$. In fact, let $\Phi$ be the map

$$\Phi : y \to (x_0 + y, t_0 - |y|) \tag{4.4}$$

and let

$$v(y) = u(\Phi(y)) \tag{4.5}$$

wherever the latter is defined. Then the radial derivative $v_s$ of $v$ is given by

$$v_s = \hat{y}\cdot\nabla v = u_\sigma - u_t = -\sqrt{2}u_\rho. \tag{4.6}$$

**Lemma 4.2.** *For any $z_0 \in K$ and any $C \geq 0$ there holds*

$$\int_{M_{(1+C)t_0}(z_0)} \frac{u_\rho u}{t}\, do \geq (1 + log(1+C))o(1),$$

*where $o(1) \to 0$ if $(1+C)t_0 \geq t_l$ and $l \to \infty$.*

Combing Lemma 4.1 and Lemma 4.2 it follows that for any $\epsilon > 0$, if we choose $t_k \leq C(\epsilon)t_0 < t_{k+1}$, we can estimate

$$\begin{aligned} o(1) &\geq \int_{M_{t_l}(z_0)} A\, do \\ &\geq \frac{1}{2}\int_{M_{t_l}^{t_k}(z_0)} |u_\rho|^2\, do - \epsilon E_0 \\ &\quad + \int_{M_{t_l}(z_0)} A_0\, do - o(1)\left(1 + log(1 + C(\epsilon))\right), \end{aligned} \tag{4.7}$$

where $o(1) \to 0$ as $l \to \infty$. To estimate $A_0$ on $M_{t_k}(z_0)$ now introduce the new angle $\delta_0$, where $|x_0| = r_0, \hat{x}_0 = \dfrac{1}{r_0}x_0, \hat{x}_0 \cdot \hat{y} = \cos\delta_0$. Again let $y = x - x_0$ and $|y| = \sigma = |t - t_0|$. With this notation

$$\begin{aligned} r\hat{x}\cdot\hat{y} = x\cdot\hat{y} &= y\cdot\hat{y} + x_0\cdot\hat{y} \\ &= \sigma + r_0\cos\delta_0, \end{aligned}$$

$$\begin{aligned} |\alpha| &= \left|\frac{x - (x\cdot\hat{y})\hat{y}}{r}\right| = \left|\frac{x_0 - (x_0\cdot\hat{y})\hat{y}}{r}\right| \\ &= \frac{r_0}{r}|\sin\delta_0|. \end{aligned}$$

Hence

(4.8)

$$\begin{aligned} A_0 = &\left(1 - \frac{\sigma}{t} - \frac{r_0}{t}\cos\delta_0\right)|u_\rho|^2 + \left(1 + \frac{\sigma}{t} + \frac{r_0}{t}\cos\delta_0\right)\left(\frac{1}{2}|\Omega u|^2 + \frac{1}{4}|u|^4\right) \\ &+ \frac{r_0}{t}\sqrt{2}|\sin\delta|u_\rho u_\alpha. \end{aligned}$$

Estimating $|\Omega u| \geq |u_\alpha|$ as before, we have

(4.9)
$$A_0 \geq \left(2 - \frac{t_0 - r_0}{t}\right)|u_\rho|^2 - \frac{r_0}{2t}\left(\sqrt{2}\sqrt{1+\cos\delta_0}u_\rho - \sqrt{1-\cos\delta_0}u_\alpha\right)^2$$
$$+ \frac{t_0}{2t}\left(1 + \frac{r_0}{t_0}\right)|u_\alpha|^2 + \frac{t_0}{4t}\left(1 + \frac{r_0}{t_0}\cos\delta_0\right)|u|^4.$$

Note that all the latter terms are nonnegative for $z \in M(z_0), z_0 \in K$. Since $r_0 \leq |t_0|$ in (4.9), for $t \leq 2t_0$ we have $A_0 \geq |u_\rho|^2$. Moreover, given, $0 < \epsilon < 1, z_0 \in K$, let $t_m \leq 2t_0 < t_{m+1}$ and set

$$\Gamma = \Gamma(\epsilon : z_0) = \{z \in M_{t_m}(z_0) : |\delta_0| \leq \epsilon^{1/4}\}$$
$$\Delta = \Delta(\epsilon : z_0) = M_{t_m}(z_0) \backslash \Gamma.$$

Note that by (4.8) on $\Gamma$ we have an estimate

$$A_0 \geq |u_\rho|^2 - \sqrt{2}\epsilon^{1/4}|u_\rho u_\alpha|$$
$$\geq |u_\rho|^2 - \sqrt{2}\epsilon^{1/4} d_{z_0}(u)$$

while, by (4.9), on $\Delta$ we have

$$A_0 \geq \frac{t_0}{4t}(1 + \frac{r_0}{t_0}\cos\delta_0)|u|^4$$
$$\geq \frac{1}{32}(1 - (1 - \frac{\epsilon^{1/2}}{2} + \epsilon))|u|^4$$
$$\geq \frac{\epsilon^{1/2}}{32}|u|^4 - \epsilon d_{z_0}(u).$$

Combining (4.7) and Lemma 4.1, we thus obtain

(4.10)
$$\int_\Gamma |u_\rho|^2\, do \leq \int_{M_{t_k}(z_0)} A_0\, do + \sqrt{2}\epsilon^{1/4} E_0$$
$$\leq (\epsilon + \sqrt{2}\epsilon^{1/4})E_0 + o(1)\left(1 + log(1 + C(\epsilon))\right),$$

$$\text{(4.11)} \qquad \frac{\epsilon^{1/2}}{32}\int_\Delta |u|^4\, do \leq \int_{M_{t_k}(z_0)} A_0\, do + \epsilon E_0$$
$$\leq 2\epsilon E_0 + o(1)\left(1 + log(1 + C(\epsilon))\right),$$

(4.12)
$$\int_{M_{t_l}^{t_m}(z_0)} |u_\rho|^2 \, do \le \int_{M_{t_k}^{t_m}(z_0)} A_0 \, do + \int_{M_{t_k}^{t_m}(z_0)} |u_\rho|^2 \, do$$
$$\le 3\epsilon E_0 + o(1) log(1 + C(\epsilon)),$$

where $o(1) \to 0$ as $l \to \infty$, we may assume that $t_l \le t_k \le t_m$.

*Proof of Theorem 0.1.* Given $\epsilon > 0$, we split the integral in (1.9) and use Hölder's inequality as follows

$$\int_{M_{t_l}} \frac{|u|^2}{|z - z_0|^2} \, do \le \int_\Gamma \frac{|u|^2}{|z - z_0|^2} \, do + \int_\Delta \frac{|u|^2}{|z - z_0|^2} \, do + \int_{M_{t_l}^{t_m}} \frac{|u|^2}{|z - z_0|^2} \, do.$$

By Lemma 2.1 and (4.10)

$$\int_\Gamma \frac{|u|^2}{|z - z_0|^2} \, do \le \frac{4}{3} \int_\Gamma |u_\rho|^2 \, do + \frac{1}{6} |t_m - t_0|^{-1} \int_{\partial D(t_m : z_0)} |u|^2 \, do$$
$$\le \frac{4}{3} (\epsilon + \sqrt{2}\epsilon^{1/4}) E_0 + o(1) \left(1 + log(1 + C(\epsilon))\right)$$
$$+ C \left( \int_{\partial D(t_m : z_0)} |u|^3 \, do \right)^{2/3} .$$

By Lemma 2.1 and Lemma 3.2

$$\int_{\partial D(t_m : z_0)} |u|^3 \, do$$
$$\le C \left\{ \left( \int_{D(t_m)} |u|^4 \, dx \right)^{1/2} \left( \int_{D(t_m)} |\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{D(t_m)} |u|^4 \, dx \right)^{3/4} \right\}$$
$$\le C \left\{ \left( \int_{D(t_m)} |\nabla u|^2 \, dx \right)^{1/2} + \left( \int_{D(t_m)} |u|^4 \, dx \right)^{3/2} \right\} \left( \int_{D(t_m)} |u|^4 \, dx \right)^{1/2}$$
$$\le C(E_0) o(1),$$

where $o(1) \to 0$ as $m \ge l$ tend to infinity. Similarly, by Lemma 2.1, Lemma 3.2 and (4.12)

$$\int_{M_{t_l}^{t_m}} \frac{|u|^2}{|z - z_0|^2} \, do \le \frac{4}{3} \int_{M_{t_l}^{t_m}(z_0)} |u_\rho|^2 \, do + \frac{1}{6} |t_m - t_0|^{-1} \int_{\partial D(t_l : z_0)} |u|^2 \, do$$
$$\le 4\epsilon E_0 + o(1) log(1 + C(\epsilon)) + o(1) C(E_0).$$

Finally, by (4.11),

$$\int_\Delta \frac{|u|^2}{|z-z_0|^2}\,do \le \left(\int_\Delta \frac{|u|^{4/3}}{|z-z_0|^{8/3}}\,do\right)^{3/4}\left(\int_\Delta |u|^4\,do\right)^{1/4}$$
$$= 64\epsilon^{1/2}\left(\int_\Delta \frac{|u|^{4/3}}{|z-z_0|^{8/3}}\,do\right)^{3/4} E_0 + o(1)\epsilon^{1/2}\left(1+log(1+C(\epsilon))\right).$$

Hence, if we first choose $\epsilon > 0$ sufficiently small and then choose $l \in N$ sufficiently large, then the integral

$$\int_{M_{t_l}(z_0)} \frac{|u|^2}{|z-z_0|^2}\,do$$

can be made as small as we please. //////

## 5. The Higher Dimensional Case

In this section we shall introduce the recent results for

(5.1) $$u_{tt} - \Delta u + |u|^{p-1}u = 0 \quad \text{in } R^n \times R$$

with the initial data

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x).$$

A solution $u$ of (5.1) is regular if $u$ is locally bounded and as smooth as the initial data permit for general semilinear equation (1.1) involving nonlinearity $f$ all of where derivatives are bounded.

**Theorem 5.1 (Brenner-von Wahl).** *Let $1 < p < \frac{n+2}{n-2}$.If $n \le 9$ ,then (5.1) has a global classical solution.*

*Proof.* See [1].

**Theorem 5.2 (Grillakis, Shatah-Struwe).** *Let $p = \frac{n+2}{n-2}$. Suppose $u$ is a solution of (5.1) with smooth initial data. If $n \le 7$, then $u$ is regular.*

The proof of Theorem 5.2 bases on the energy inequality, the estimate of

$$\int_{D(t;0)} |u|^{\frac{2n}{n-2}}\,dx$$

and the estimate of Strichartz for the inhomogenous wave equations, for details see [4],[8].

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# MULTIPLICITY RESULTS FOR DOUBLE-PERIODIC SOLUTIONS OF NONLINEAR DISSIPATIVE HYPERBOLIC EQUATIONS

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ABSTRACT. Amprosetti-Prodi type multiplicity results for double-periodic solutions for nonlinear disspative hyperbolic equations are treated

## 1 Introduction

Let $Z$ and $R$ be the set of all integers and real numbers, respectively and let $\Omega = [0, 2\pi] \times [0, 2\pi]$.

Let $L^1(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are Lebesgue lntegrable over $\Omega$ with usual norm $\|\cdot\|_{L^1}$. Let $L^2(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are Lebesgue square integrable over $\Omega$ with usual inner product $(,)$ and usual norm $\|\cdot\|_{L^2}$ and let $L^\infty(\Omega)$ be the space of measurable real-valued functions $u : \Omega \to R$ which are essentially bounded with usual essential norm $\|\cdot\|_{L^\infty}$.

Let $C^k(\Omega)$ be the space of all continous functions $u : \Omega \to R$ such that the partial derivatives up to other $k$ with respect to both variables are continuous on $\Omega$, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|_\infty$ and we write $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.

Let $W^{k.2}(\Omega)$ be the Sobolev space of all function $u : \Omega \to R$ in $L^2(\Omega)$ such that all partial distributional derivatives up to $k$ belongs $L^2(\Omega)$ with the usual Sobolev norm.

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The purpose of this work is to investigate the weakend Ambrosetti-Prodi (briefly WAP) type and Ambrosetti-Prodi (briefly AP) type multiplicity results (see [1]) for weak double-periodic solutions of the nonlinear dissipative hyperbolic equations of the form

$$(1.1) \qquad \beta u_t + u_{tt} - u_{xx} + g(t,x,u) = s + h(t,x) \quad \text{in} \quad \Omega$$

where $\beta(\neq 0) \in R$, $u = u(t,x)$, $h \in L^2(\Omega)$,
$g : \Omega \times R \to R$ is a caratheodory function and $s$ is a real parameter.
A *weak double−periodic solution* of (1.1) will be $u \in L^2(\Omega)$ such that

$$(1.2) \qquad (u, -\beta v_t + v_{tt} - v_{xx}) + (g(\cdot,\cdot,u)\cdot v) = (s+h, v)$$

for every $v \in L^2(\Omega)$ satisfying boundary conditions

$$v(t,o) - v(t,2\pi) = v_x(t,o) - v_x(t,2\pi), t \in [0,2\pi]$$

$$v(0,x) - v(2\pi,x) = v_t(0,x) - v_t(2\pi,x), t \in [0,2\pi].$$

Let us remark that a necessary condition for (1,2) to have meaning is that $g$ be such that $g(\cdot,\cdot,u) \in L^2(\Omega)$ when $u \in L^2(\Omega)$.
Besides, $g$ is a Caratheodory function; i.e., $g(\cdot,\cdot,u)$ is measurable on $\Omega$ for each $u \in R$ and $g(t,x,\cdot)$ is continuous on $R$ a.e. on $\Omega$.
We assume the following.
(H1) There exist $a \in L^\infty(\Omega)$ and $b \in L^2(\Omega)$ such that

$$|g(t,x,u)| \leq a(t,x)|u| + b(t,x) \quad \text{a.e.} \quad \text{on} \quad \Omega.$$

Fabry, Mawhin and Nkashama, and Mawhin have shown AP type mutiplicity results for periodic solutions of forced second order ODE in [3] and in [11] respectively. Ding and mawhin treat WAP andAP type multiplicity results for periodic solution of higher order (in fact more general operator is considered) ODE in [2]. Ramos and Sanchez and Ramos deal with WAP and AP,and WAP multiplicity results for periodic solutions of higher order ODE in [12], and in [13] respectively. Lee treats WAP an AP results for periodic solutions of proof is related to that of [2]. However we use the specific properties of the periodic problem for (1,1) in the obtention for the required a priori bound.

## 2 Preliminary results

Now consider the equation

$$(2.1)\qquad \beta u_t + u_{tt} - u_{xx} = h(t,x), \beta \neq 0 \quad \text{and} \quad u = u(t,x)$$

$$u(t,x) = \sum_{(l,m)\in Z\times Z} u_{lm} \exp[i(lt+mx)]$$

$$h(t,x) = \sum_{(l,m)\in Z\times Z} h_{lm} \exp[i(lt+mx)]$$

with $\bar{u}_{lm} = u_{-l-m}$ and $\bar{h}_{lm} = h_{-l-m}$ since $u$ and $h$ are real.

**Lemma 2.1.** *$u \in L^2(\Omega)$ is a weak solution of (2.1) if and only if ,for all $(l,m) \in Z \times Z$,*

$$[\beta li + (m^2 - l^2)]u_{lm} = h_{lm}.$$

Let $DomL = \{u \in L^2(\Omega) : \sum_{(l,m)\in Z\times Z}[\beta^2 l^2 + (m^2-l^2)^2]|u_{lm}|^2 < \infty\}$.
Define an operator $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ by

$$(Lu)(t,x) = \sum_{(l,m)\in Z\times Z} [\beta li + (m^2-l^2)]u_{lm} \exp[i(lt+mx)].$$

Then DomL is dense in $L^2(\Omega)$, $KerL = R$

$$ImL = \{h \in L^2(\Omega) : \int\int_\Omega h(t,x)\,dt\,dx = 0\}$$

ImL is closed, and

$$[KerL]^{-1} = ImL.$$

Moreover, $L^2(\Omega) = KerL \bigoplus ImL$. Consider a continuous projection

$$P : L^2(\Omega) \to L^2(\Omega) \quad \text{such that} \quad ImL = KerP.$$

Then $L^2(\Omega) = KerL \bigoplus KerP$. We consider another continuous projection $Q : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Qh)(t,x) = \frac{1}{4\pi^2}\int\int_\Omega h(t,x)\,dt\,dx.$$

Then we have $L^2(\Omega) = ImQ \bigoplus ImL$, $KerQ = ImL$, and $L^2(\Omega)/ImL$ is isomorphic to $ImQ$.

Since $dim[L^2(\Omega)/ImL] = dim[ImQ] = dim[KerL] = 1$, we have an isomorphism $J : ImQ \to KerL$ and $L$ is a Fredholm mapping of index 0. Moreover, we have easily the following lemma.

**Lemma 2.2.** *$L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ is a closed operatert.*

If $h \in L^2(\Omega)$, then u is a weak solution of (2.1) if and only if $u \in DomL$, $Lu = h$. $L$ is not bijective but the restriction

$$L|_{DomL \bigcap ImL} : ImL \bigcap DomL \to ImL$$

is bijective, so we can define a right inverse

$$K^R = [L|_{DomL \bigcap ImL}]^{-1} : ImL \to ImL \bigcap DomL$$

and

$$(K^R h)(t,x) = \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} h_{lm} \exp[i(lt + mx)].$$

We have the following lemma.

**Lemma 2.3.** *$DomL \bigcap ImL = K^R[ImL] \subseteq W^{1.2}(\Omega) \bigcap C(\Omega) \bigcap ImL$ and $K^R[W^{k.2} \bigcap ImL] \subseteq W^{k+1.2} \bigcap ImL$, $k = 0, 1, 2, 3, \ldots$ . Moreover, if $h \in ImL$, then $\|K^R h\|_{W^{1.2}} \leq C_1 \|h\|$ for some $C_1 > O$ independently of $h$.*

*proof.* See [4], [10] and [7].

Since

$$(K^R h)(t,x) = \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - l^2)]^{-1} h_{lm} \exp[i(lt + mx)].$$

We can represent $K^R$ as a convolution product

$$(K^R h)(t,x) = (K * h)(t,x) = \iint_\Omega K(t-s, x-y) h(s,x) dsdy$$

where $K(t,x) : \frac{1}{4\pi^2} \sum_{\substack{(l,m)\in Z\times Z \\ (l,m)\neq(0,0)}} [\beta li + (m^2 - L^2)]^{-1} \exp[i(lt + mx)]$.

We have the following lemma.

**Lemma 2.4..** *the operator $K^R : ImL \to C(\Omega)$ is compact. If $h \in ImL$, then $\|K^R h\|_\infty \le C_2\|h\|_{L^2}$ for some constant $C_2 > 0$ independently of $h$.*

*proof.* See [4], [10] and [7].

Now we can extend $K^R$ to $L^1(\Omega)$ by defining $\bar{K}^R : L^1(\Omega) \to L^2(\Omega)$ by the formular

$$(\bar{K}^R h)(t,x) = \iint_\Omega K(t-s, x-y)h(s,y)dsdy \quad \text{for} \quad h \in L^1(\Omega).$$

Then, by Hölder's inequality and Fubini's theorem, we have the following lemma.

**Lemma 2.5.** $\|\bar{K}^R\|_{L_2} \le \|K\|_{L_2}\|h\|_{L_1}$.

*proof.* See [7]

## 3 (WAP) Type Multiplicty Result

Let us consider the following double-periodic boundary value problem

$(3.1^\lambda_s)$
$$\beta u_t + u_{tt} - u_{xx} + \lambda g(t,x,u) = \lambda s + \lambda h(t,x), \lambda \in [0,1].$$

Let $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined as befor and define a substitution operator $N^\lambda_s : L^2(\Omega) \to L^2(\Omega)$

$$(N^\lambda_s)(t,x) = \lambda g(t,x,u) - \lambda s - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. By krasnosel'skii's results, $N^\lambda_s$ is continuous and bounded. Let $G$ be any open bounded subset of $L^2(\Omega)$, then $QN : \bar{G} \to L^2(\Omega)$ is bounded and $K^R(I-Q) : \bar{G} \to L^2(\Omega)$ is compact and continuous. Thus, $N^\lambda_s$ is $L$-compact on $\bar{G}$. The coincidence degree $D_L(L + N^\lambda_s, G)$ is well-defined and constant in $\lambda$ if $Lu + N^\lambda_s \ne 0$ for $\lambda \in [0,1]$, $s \in \mathbf{R}$ and $u \in DomL \bigcap \partial G$. It is easy to check that $(u,\lambda)$ is a weak double-periodic solution of $(3.1^\lambda_s)$ if and only if $u \in DomL$ and

$$(3.2^\lambda_s) \qquad Lu + N^\lambda_s u = 0.$$

Here we assume the following.

$$(H2) \qquad g(t,x,u) \ge 0 \quad \text{on} \quad \Omega \times \mathbf{R},$$

$$(H3) \qquad lim_{|u|\to\infty} g(t,x,u) = +\infty \quad \text{uniformly} \quad \text{on} \quad \Omega.$$

**Lemma 3.1.** *If* $(H1)$ *and* $(H2)$ *are satisfied, then, for each* $s^* \in \mathbf{R}$ , *there exists* $M(s^*) > 0$ *such that*

$$\|\tilde{u}\|_{L^2} \leq M(s^*)$$

*holds for each possible Weak double-periodic solution* $u = \bar{u} + \tilde{u}$, *with* $\bar{u} \in KerL$ *and* $\tilde{u} \in ImL$, *of* $(3.1_s^\lambda)$ *where* $s \leq s^*$, $\lambda \in [0,1]$.

*proof.* Let $(u, \lambda)$ be any weak double-periodic solution of $(3.1_s^\lambda)$. Then $(u, \lambda)$ is a solution of $(3.2_s^\lambda)$ where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$.

By applying $\bar{K}^R$ on the both sides of equation $(3.2_s^\lambda)$,we have, since $\bar{K}^R_{|ImL} = K^R$,

$$\tilde{u} = -\lambda \bar{K}^R N_s^\lambda u = \lambda \bar{K}^R[-g(\cdot,\cdot,u) + s + h(\cdot,\cdot)].$$

Hence, by Lemma 2.5,

$$\|\bar{u}\|_{L^2} \leq \|K\|_{L^2}[\|g(\cdot,\cdot,u)\|_{L^1} + 4\pi^2|s| + \|h\|_{L^1}].$$

By taking the inner product with 1 on the both sides of $(3.2_s^\lambda)$, since $1 \in kerL$, we have

$$\iint_\Omega g(t,x,u(t,x))dtdx = 4\pi^2 s + \iint_\Omega h(t,x)dtdx.$$

Hence by Lemma 2.5 and $(H2)$, we have $4\|g(\cdot,\cdot,u)\|_{L^1} \leq 4\pi^2|s^*| + \|h\|_{L^1}$. Therefore, we have

$$\|\tilde{u}\|_{L^2} \leq 2\|K\|_{L^2}[4\pi^2 s^* + \|h\|_{L^1}] \equiv M(s^*).$$

The proof is complete.

**Lemma 3.2.** *If* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied, then, for each* $s^* \in \mathbf{R}$; *there exists* $\gamma(s^*)$ *such that*

$$|\bar{u}| \leq \gamma(s^*)$$

*holds for each possible weak double-periodic solution* $u = \bar{u} + \tilde{u}$, *with* $\bar{u} \in KerL$ *and* $\tilde{u} \in ImL$, *of* $(3.1_s^\lambda)$ *where* $s \leq s^*$ *and* $\lambda \in [0,1]$.

*proof.* Suppose there exist a sequence of constants $\{s_n\}$ with $s_n \leq s^*$ and the corresponding weak double-periodic solutions $\{(u_{m,n}, \lambda_{m,n})\}$ of $(3.1_s^{\lambda_{m,n}})$ with $\{|\bar{u}_{m,n}|\}$ is unbounded. Then $(u_{m,n}, \lambda_{m,n})$ is a solution of $(3.2_{s_n}^{\lambda_{m,n}})$ where $u_{m,n} = \bar{u}_{m,n} + \tilde{u}_{m,n}$ with $\bar{u}_{m,n} \in KerL$ and $\tilde{u}_{m,n} \in ImL$. We may choose a subsequence, say again $\{\bar{u}_{m,n}\}$ such that $|\bar{u}_{m,n}| \to +\infty$ as $m, n \to$

$+\infty$. Now suppose that $\bar{u}_{m,n} \to +\infty$ as $m, n \to +\infty$. Let $M_0 > 2\pi M(s^*)$ where $M(s^*)$ is given in Lemma 3.1 and let $\Omega_{m,n} = \{(t,x)|\tilde{u}_{m,n}(t,x) \leq -\frac{M_0}{4\pi^2}\}$.
Then

$$\begin{aligned} 2\pi M(s^*) &\geq \iint_\Omega |\tilde{u}_{m,n}(t,x)|dtdx \\ &\geq \iint_{\Omega_{m,n}} |\bar{u}_{m,n}(t,x)|dtdx \\ &\geq [\frac{M_0}{4\pi^2}]\text{measure}[\Omega_{m,n}]. \end{aligned}$$

Therefore, measure $[\Omega_{m,n}] \leq 4\pi^2\frac{2\pi M(s^*)}{M_0}$ and hence measure $[\Omega - \Omega_{m,n}] =$ measure $\{(t,x)|u_{m,n}(t,x) > -\frac{M_0}{4\pi^2}\} \geq 4\pi^2[1 - \frac{2\pi M(s^*)}{M_0}] > 0$.

Since $lim_{|u|\to+\infty} g(t,x,u) = +\infty$ uniformly on $\Omega$, there exists $C(s^*) > 0$ such that

$$g(t,x,u) > (4\pi^2 s^* + \iint_\Omega h(t,x)dtdx)/\text{measure}[\Omega - \Omega_{m,n}]$$

for all $m, n$ if $|u| \geq C(s^*)$.

Since $\bar{u}_{m,n} \to +\infty$, there exists $N_1 > 0$, $N_2 > 0$ such that $\bar{u}_{m,n} \geq \frac{M_0}{4\pi^2} + C(s^*)$ if $m \geq N_1$, $n \geq N_2$.

Hence, for $(t,x) \in \Omega - \Omega_{m,n}$ and $m \geq N_1$, $n \geq N_2$, we have

$$u_{m,n}(t,x) = \bar{u}_{m,n} + \tilde{u}_{m,n}(t,x) \geq C(s^*).$$

Thus, for $m \geq N_1$, $n \geq N_2$, we have

$$\iint_{\Omega-\Omega_{m,n}} g(t,x,u_{m,n}(t,x))dtdx > 4\pi^2 s^* + \iint_\Omega h(t,x)dtdx.$$

On the other hand, by taking the inner product with 1 on the both sides of $(3.1_{s_n}^{\lambda_{m,n}})$, we have

$$\iint_\Omega g(t,x,u_{m,n}(t,x))dtdx \leq 4\pi^2 s^* + \iint_\Omega h(t,x)dtdx.$$

Therfore, for $m \geq N_1$, $n \geq N_2$, by $(H2)$,

$$\begin{aligned} 4\pi^2 s^* + \iint_\Omega h(t,x)dtdx &\geq \iint_\Omega g(t,x,u_{m,n}(t,x))dtdx \\ &\geq \iint_{\Omega-\Omega_{m,n}} g(t,x,u_{m,n}(t,x))dtdx \\ &> 4\pi^2 s^* + \iint_\Omega h(t,x)dtdx \end{aligned}$$

which is impossible.

Similary, we can lead another contradition.
The proof is complete.

**Lemma 3.3.** *If $(H1)$, $(H2)$ and $(H3)$ are satisfied, then, for each $s^* \in \mathbf{R}$, we can find an open bounded set $G(s^*)$ in $L^2(\Omega)$ such that $G \supseteq G(s^*)$ we have*

$$D^L(L + N_s^1, G) = 0 \quad \textit{for all} \quad s \leq s^*.$$

*proof.* Since $lim_{|u|\to+\infty} g(t,x,u) = +\infty$, for each $s^* \in \mathbf{R}$ there exists $\tilde{\gamma}^* > 0$ such that $g(t,x,u) \geq s^* + \frac{1}{4\pi^2}\iint_\Omega h(t,x)dtdx$ if $|u| \geq \tilde{\gamma}^*$. Let $G(s^*) = \{u \in L^2(\Omega) : |\bar{u}| < \tilde{\gamma}, \|\tilde{u}\|_{L^2} < \tilde{M}\}$ where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, and $\tilde{\gamma} > \max\{\gamma(s^*), \tilde{\gamma}\}$, $\tilde{M} > M(s^*)$, and $\gamma(s^*)$ and $M(s^*)$ are given in Lemma 3.1 and Lemma 3.2 .

Let $s_0 = \min_{\substack{(t,x)\in\Omega \\ u\in\mathbf{R}}} g(t,x,u) - \frac{1}{4\pi^2}\iint_\Omega h(t,x)dtdx$.
If $(3.2_{\bar{s}}^\lambda)$ has a solution $u$ for some $\bar{s} \in \mathbf{R}$ and $\lambda \in ]0,1]$, then, by taking the inner product with 1 on the both sides of equation $(3.2_{\bar{s}}^\lambda)$, we have

$$s_0 \leq \frac{1}{4\pi^2}[\iint_\Omega g(t,x,u(t,x))dtdx - \iint_\Omega h(t,x)dtdx] \leq \bar{s}.$$

Thus $(3.1_{\bar{s}}^\lambda)$ has no solution for $\bar{s} < s_0$. Hence , we have

$$D_L(L + N_{\bar{s}}^1, G) = 0 \quad \text{for} \quad \bar{s} < s_0.$$

Choose $\bar{s} < s_0$ and define $F : (D(L) \bigcap \bar{G}) \times [0,1] \to L^2(\Omega)$ by

$$F(u,\mu) = Lu + N_{1-\mu)\bar{s}+\mu s}(u) \quad \text{for} \quad s \leq s^*.$$

Then, by Lemma 3.1 and Lemma 3.2, we have $0 \notin F(D(L)\bigcap \partial G) \times [0,1])$ for $s \leq s^*$. By the homotopy inveriance of degree, we have

$$\begin{aligned} D_L(L+N_s^1, G) &= D_L(F(\cdot,1), G) \\ &= D_L(F(\cdot,0), G) \\ &= D_L(L+N_{\bar{s}}^1, G) \\ &= 0 \quad \text{for all} \quad s \leq s^* \end{aligned}$$

and the proof is complete.

**Lemma 3.4.** *If* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied, then there exist* $s_1 \geq s_0$ *such that, for each* $s^* > s_1$, *we can find an open bounded set* $\Delta(G(s^*))$ *in* $L^2(\Omega)$ *on which*

$$|D_L(L+N_s^1, \Delta(G(s^*))| = 1$$

*for all* $s_1 < s \leq s^*$.

*proof.* Let $(t_0, x_0, u_0) \in \Omega \times \mathbf{R}$ such that

$$g(t_0, x_0, u_0) = \min_{\substack{(t,x)\in\Omega \\ u\in\mathbf{R}}} g(t,x,u)$$

and let $s_1 = \max_{\substack{(t,x)\in\Omega \\ u\in[u_0-M, u_0+\tilde{M}]}} g(t,x,u) - \frac{1}{4\pi^2}\iint_\Omega h(t,x)dtdx.$

Let $\Delta(G(s^*)) = \{u \in L^2(\Omega) : u_0 < \bar{u} < \tilde{\gamma}, \|\tilde{u}\|_{L^2} < \tilde{M}\}$ where $\tilde{\gamma}$ and $\tilde{M}$ are given Lemma 3.1 and Lemma 3.3. if $s > s_1$, $\lambda \in ]0,1]$ then $(u, \lambda)$ is a possible solution of $(3.2_s^\lambda)$ such that $u \in \partial\Delta(G(s^*))$, then by Lemma 3.1 and Lemma 3.2, we haxe necessary $\bar{u} = u_0$ and

$$u_o - \tilde{M} < u(t,x) = \bar{u} + \tilde{u}(t,x) < u_0 + \tilde{M}$$

for all $(t,x) \in \Omega$.
By taking the inner product with 1 on the both sides of $(3.2_s^\lambda)$, we have

$$\iint_\Omega g(t,x,u(t,x))dtdx = s + \iint_\Omega h(t,x)dtdx.$$

But

$$s_1 \geq \frac{1}{4\pi^2}[\iint_\Omega g(t,x,u(t,x))dtdx - \iint_\Omega h(t,x)dtdx] = s$$

which is impossible. Thus, for $s > s_1$ and $\lambda \in ]0,1]$,

$$D_L(L + N_s^1, \Delta(G(s^*)))$$

is well defined, and

$$D_L(L + N_s^1, \Delta(G(s^*))) = D_B(JQN_s\lambda, \Delta(G(s^*)) \bigcap KerL, 0)$$

where $JQN_s^\lambda : L^2(\Omega) \to KerL$ is an operator defined by

$$(JQN_s^\lambda u)(t,x) = \frac{1}{4\pi^2}[\iint_\Omega g(t,x,u(t,x))dtdx - \iint_\Omega h(t,x)dtdx] - s$$

(see :[5]) and $D_B$ denotes the Brouwer degree.

Thus, for $s_1 < s \le s^*$, we have $JQN_s^1|_KerL(u_0) \le s_1 - s < 0$ and, by definition of $\tilde{\gamma}$, $JQN_s^1|_{KerL}(\tilde{\gamma}) > 0$.
Therfore

$$\begin{aligned} |D_L(L + N_s^1, \Delta(G(s^*)0)| &= |D_B(JQN_s^1, \Delta(G(s^*)) \bigcap KerL)| \\ &= 1 \end{aligned}$$

and the proof is complete.

**Theorem 1.** *Assume* $(H1)$, $(H2)$ *and* $(H3)$. *Then there exist real numbers* $s_0 \le s_1$ *such that*

$(i)$ (1.1) *has no weak double-periodic solution for* $s < s_0$.

$(ii)$ (1.1) *has at least one weak double-periodic solution for* $s = s_1$.

$(iii)$ (1.1) *has at least two weak double-periodic solutions for* $s > s_1$.

*proof.* Let $s_0$ and $s_1$ be constants defined in Lemma 3.3 and Lemma 3.4. Part $(i)$ has been proved in Lemma 3.3. For part $(iii)$, if $s > s_1$, then we can choose $G \supset (\neq)\Delta(G(s))$, where $G$ and $\Delta(G(s))$ are defined in Lemma 3.3 and Lemma 3.4, respectively. By the additivity of degree, we have

$$0 = D_L(L + N_s^1, G) = D_L(L + N_s^1, G) + D_L(L + N_s^1, G \backslash \Delta(G(s)))$$

and hence, by Lemma 3.4,

$$|D_L(L + N_s^1, G \backslash \overline{\Delta(G(s))})| = 1.$$

Therefore, (1.1) has one weak double-periodic solution in $\Delta(G(s))$ and one in $G\backslash\overline{\Delta(G(s))}$. For part $(ii)$, let $\{s_{(n)}\}$ be a sequence in $\mathbf{R}$ with $s_{(1)} > s_{(2)} >$

$\cdots > s_1$ such that $s_{(n)} \to s_1$ and let $\{u_n\}$ be the corresponding sequence of weak double-periodic solutions of (1.1). Then $u_n$ is a solution to $(3.2_s^1)$ and $u_n = \bar{u}_n + \tilde{u}_n$ with $u_n \in KerL$ and $\tilde{u}_n \in ImL$. By Lemma 3.2, we have a subsequence $\{\bar{u}_{n_k}\}$ of $\{\bar{u}_n\}$ which converge to some $\bar{u}$ in $\mathbf{R}$. On the other hend, by $(H1)$, Lemma 3.1 and Lemma 3.2, we can easily see that $\{Lu_{n_k}\}$ is a bounded sequence in $ImL \subseteq L^2(\Omega)$. Since $K^R : ImL \to C(\Omega)$ is a compact operator and $\tilde{u}_{n_k} = K^R(Lu_{n_k})$, we have a subsequence say again $\{\bar{u}_{n_k}\}$ which converges to some $\tilde{u}$ in $C(\Omega)$. Therefore, we have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges to $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$.

Since $L$ is a closed operator, $u \in DomL$ and $u$ is a solution of $(3.2_s^1)$. Thus $u$ is a weak double-periodic solution of (1.1) for $s = s_1$. This complete our proof.

## 4 (AP) Type Multiplicity Result

Let us consider the following double-periodic boundary value problom

$$(4.1_s^\lambda) \qquad \beta u_t + u_{tt} - u_{xx} + \lambda g(u) = \lambda s + \lambda h(t,x), \lambda \in [0.1]$$

where $g : \mathbf{R} \to \mathbf{R}$ is continus and $h \in ImL$.

Let $L : DomL \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined us before and define the substitution operator defined

$$(N_s^\lambda)(t,x) = \lambda g(u(t,x)) - \lambda s - \lambda h(t,x)$$

for $u \in L^2(\Omega)$ and $(t,x) \in \Omega$. Then $N_s^\lambda$ is L-compact on $\bar{G}$ for any open bounded subset of $L^2(\Omega)$, and $u$ is a weak double-periodic solution to $(4.1_s^\lambda)$ if and only if $u \in DomL$ and

$$(4.2_s^\lambda) \qquad Lu + N_s^\lambda u = 0.$$

Here we assume the following.

$(H3)$ $\lim_{|u| \to +\infty} g(x) = +\infty$,

$(H4)$ there exists $0 < \alpha < 1$ such that

$$|g(u) - g(v)| \le \frac{\alpha}{2\pi C_2}|u - v| \quad \text{for all} \quad u, v \in \mathbf{R},$$

where $C_2$ is a constant defined in Lemma 2.4.

**Lemma 4.1.** *If $(H1)$ is satisfied, then for any $s \in \mathbf{R}$, there exists $M > 0$ which is independent of $s$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

*holds for each possible weak double-periodic solution $u = \bar{u} + \tilde{u}$, with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, of $(4.1_s^\lambda)$ where $\lambda \in [0.1]$.*

*proof.* Let $(u, \lambda)$ be any weak double-periodic solution of $(4.1_s^\lambda)$. Then $(u, \lambda)$ is a solution of $(4.2_s^\lambda)$ where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$.

By taking the inner product with $\tilde{u}_t$ on the both sides of $(4.2_s^\lambda)$, we have

$$(L\tilde{u}, \tilde{u}_t) + \lambda \iint_\Omega g(u)\tilde{u}_t dtdx = \lambda \iint_\Omega h(t,x)\tilde{u}_t dtdx.$$

Since $L\tilde{u} \in L^2(\Omega)$, ther exists a sequence $\{\tilde{y}_n\}$ in $C^\infty(\Omega)\bigcap ImL$ such that $\tilde{y}_n \to L\tilde{u}$ in $C^2(\Omega)$ as $n \to +\infty$.

Let $\tilde{u}_n = K^R\tilde{y}_n$. By Lemma 2.3 and the sobolev embedding theorem ; i.e., $W^{j+2.2}(\Omega) \hookrightarrow C^i(\Omega)$, $j = 0,1,2,\ldots$, $\tilde{u}_n \in C^\infty(\Omega)\bigcap ImL$. Since $K^R$ is cotinuous from $L^2(\Omega)$ into each of $W^{1.2}(\Omega)$ and $C(\Omega)$, we have that $\tilde{u}_n \to K^R(L\tilde{u})$ in each of those spaces as $n \to +\infty$.

Thus $\tilde{u}_{n_t} \to \tilde{u}$ in $L^2(\Omega)$. Intagration of these smooth functions, using the boundar conditions, show that for each $n = 1, 2, 3, \ldots$,

$$(Lu_n, \tilde{u}_{n_t}) = \beta\|\tilde{u}_{n_t}\|_{L^2}^2.$$

Letting $n \to +\infty$, we have $(L\tilde{u}_t, \tilde{u}_t) = \beta\|\tilde{u}_t\|_{L^2}^2$. Moreover, since, for each $n$, the periodicity of $\tilde{u}_n(t,x)$ in $t$ inplies $(g(u_n), \tilde{u}_{n_t}) = 0$, we have $(g(u), \tilde{u}_t) = 0$.

Hence, we have

$$\beta\|\tilde{u}_t\|_{L^2}^2 = \lambda(h, \tilde{u}_t)$$

and

$$\|\tilde{u}_t\|_{L^2}^2 \leq \frac{1}{|\beta|}\|h\|_{L^2}.$$

But since $\|\tilde{u}\|_{L^2} \leq \|\tilde{u}_t\|_{L^2}^2$ for all $\tilde{u} \in DomL\bigcap ImL$, we have

$$\|\tilde{u}\|_{L^2} \leq \frac{1}{|\beta|}\|h\|_{L^2}.$$

The proof is complete.

**Theorem 2.** *Assume* $(H1)$, $(H3)$, *and* $(H4)$. *Then there exists real number* $s_1$ *such that*

(*i*) $(4.1_1^1)$ *has no weak double-periodic solution for* $s < s_1$,

(*ii*) $(4.1_1^1)$ *has at least one weak double-periodic solution for* $s = s_1$,

(*iii*) $(4.1_1^1)$ *has at least two weak double-periodic solutions for* $s > s_1$.

*proof.* By a usual Lyapunov-Schmit argument in [9], $(4.2_s^\lambda)$ is equivalent to

$$L\tilde{u} + (I - Q)g(\bar{u} + \tilde{u}) + h \tag{4.3}$$

$$Qg(\bar{u} + \tilde{u}) = s \tag{$4.4_s^\lambda$}$$

where $u = \bar{u} + \tilde{u}$ with $\bar{u} \in KerL$ and $\tilde{u} \in ImL$, and $Q$ is the continuous projection defined in section 2. For fixed $\bar{u} \in \mathbf{R}$, consider the equation (4.3). Define an operator $N : L^2(\Omega) \to ImL$ by

$$(Nu)(t,x) = -(I - Q)g(\bar{u} + \tilde{u}(t,x)) + h(t,x).$$

Then $N$ is continuous and maps bounded sets into bounded sets.Since the inclusion mapping $i : C(\Omega) \to L^2(\Omega)$ is continuous, the right inverse $K^R$ : $ImL \to L^2(\Omega)$ is compact. Hence $K^R N : L^2(\Omega) \to L^2(\Omega)$ is completely continuous and (4.3) is equivalent to

$$\tilde{u} = \mu K^R N \tilde{u}, \quad \mu \in [0,1]$$

are bounded in $L^2(\Omega)$ independently of $\mu \in [0,1]$.

Thus, by Leray-schauder's theory, (4.3) has at least one solution $\tilde{u}$ for each $\bar{u} \in \mathbf{R}$. Such a solution is unique. Indead, if $\tilde{u}_1$ and $\tilde{u}_2$ are two different solutions with $\bar{u}$, then

$$L(\tilde{u}_1 - \tilde{u}_2) + (I - Q)[g(\bar{u} + \tilde{u}_1) - g(\bar{u} + \tilde{u}_2)] = 0.$$

Applying $K^R$ on the both sides of the above equation, we have; by Lemma 2.4 and $(H4)$,

$$\|\tilde{u}_1 - \tilde{u}_2\|_\infty \le \alpha \|\tilde{u}_1 - \tilde{u}_2\|_\infty$$

which is impossible since $0 < \alpha < 1$. Thus $\tilde{u}_1 = \tilde{u}_2$.

Denot this unique solution of (4.3) by $V(\bar{u})$,

then $V : \mathbf{R} \to C(\Omega) \bigcap ImL$. If $\bar{u}, \bar{u}_0 \in \mathbf{R}$, then

$$L[V(\bar{u}) - V(\bar{u}_0)] + (I - Q)[g(\bar{u} + V(\bar{u})) - g(\bar{u}_0 + V(\bar{u}_0))] = 0.$$

By Lemma 2.4 and $(H4)$, we have

$$\|V(\bar{u}) - V(\bar{u}_0)\|_{L^\infty} \leq \frac{\alpha}{1-\alpha}|\bar{u} - \bar{u}_0|.$$

Thus $V$ is continuous.

By Lemma 4.1, $\|V(\bar{u})\|_{L^2} \leq M$ for all $\bar{u} \in \mathbf{R}$. Let $\Omega_0 = \{(t,x) : |V(\bar{u})(t,x)| \geq \frac{1+M}{2\pi}\}$, then

$$M^2 \geq \iint -\Omega |V(\bar{u})(t,x)|^2 dtdx \geq [\frac{1+M}{2\pi}]^2 \text{measure}[\Omega_0].$$

Thus measure$[\Omega_0] \leq 4\pi^2[\frac{1+M}{2\pi}]^2$.

Let $\Omega_1 = \Omega - \Omega_0 = \{(t,x) : |V(\bar{u})(t,x)| \leq \frac{1+M}{2\pi}\}$, then measure$[\Omega_1] \geq 4\pi^2[1 - \frac{M}{1+M}]^2 > 0$.

Thus

$$\begin{aligned}\iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx &\geq \iint_\Omega [g(\bar{u} + V(\bar{u})(t,x)) - \beta]dtdx + 4\pi^2\beta \\ &\geq \iint_{\Omega_1} [g(\bar{u} + V(\bar{u})(t,x)) - \beta]dtdx + 4\pi^2\beta\end{aligned}$$

where $\beta = \min_{u \in \mathbf{R}} g(u)$.

Therefore, by $(H3)$, $\iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx \to +\infty$ as $|\bar{u}| \to +\infty$.

Define $G : \mathbf{R} \to \mathbf{R}$ by

$$G(\bar{u}) = Qg(\bar{u} + V(\bar{u})) = \frac{1}{4\pi^2}\iint_\Omega g(\bar{u} + V(\bar{u})(t,x))dtdx,$$

$G$ is continuous by $(H4)$ and the continuous of $V$ and $G(\bar{u}) \to +\infty$ as $|\bar{u}| \to +\infty$.

Equation $(4.1_s^\lambda)$ is then reduced to the scalar equation in $\bar{u}$;

$$G(\bar{u}) = Qg(\bar{u} + V(\bar{u})) = s. \tag{4.5}$$

Let $s_1 = \min_{u \in \mathbf{R}} G(\bar{u})$, then $ImG = [s_1, +\infty[$.

If $s < s_1$, clearly (4.5) has no solution.

If $G(\bar{u}_0) = s_1$, we easily prove, by the intermidiate value theorem, that, for each $s > s_1$, (4.5) has one solution in $]-\infty, \bar{u}_0[$ and one in $]\bar{u}_0, +\infty[$.

This complete the proof.

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# SCATTERING FOR SEMILINEAR WAVE EQUATIONS WITH SMALL DATA IN TWO SPACE DIMENTIONS

KIYOSHI MOCHIZUKI

In this talk we are concerned with a small data scattering for the semilinear wave equation

$$\partial_t^2 u - \Delta u = f(u) \quad \text{in} \quad (x,t) \in \mathbf{R}^N \times \mathbf{R}, \tag{1}$$

where $f(u)$ represents a nonlinear term $\gamma|u|^\rho$ or $\gamma|u|^{\rho-1}u$ with $\rho > 1$ and $\gamma \in \mathbf{R} \setminus \{0\}$. In the latter case the energy

$$\frac{1}{2}\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\} - \frac{\gamma}{\rho+1}\|u(t)\|_{L^{\rho+1}}^{\rho+1}$$

is independent of $t$ as long as a good solution exists. However, it becomes indefinite if $\gamma > 0$, and we can not in general expect the existence of global solutions. The scattering theory is based on the global existence of solutions. So, it is necessary to check that under what conditions on $\rho$ we can expect global solutions.

In case $N = 3$, John [5] considered (1) with $f(u) = |u|^\rho$ assuming that the initial data

$$u(x,0) = \varphi(x), \ \partial_t u(x,0) = \psi(x) \tag{2}$$

are smooth and compactly supported functions. He proved the following:

(i) If $\rho > \rho(3) = 1 + \sqrt{2}$, then the problem (1), (2) has a global $C^2$−solution for sufficiently small initial data. (ii) If $\rho < \rho(3)$, then we can not expect global $C^2$−solutions no matter how small initial data are.

The results were extended to $N = 2$ by Glassey [3], [4], and the blow-up part (ii) is extended to $N \geq 4$ by Sideris [13]. Schaeffer [12] showed in case $N = 2, 3$ that for the critical value $\rho = \rho(N)$, the blow-up occurs. Moreover, Zhou [17] recently showed (i) for $N = 4$ in the frame work of strong solutions. In these results $\rho(N)$ is defined as the positive root of

$$(N-1)\rho^2 - (N+1)\rho - 2 = 0,$$

that is,

$$\rho(N) = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}.$$

As I shall explain below, the scattering theory treats the Cauchy problem with initial data at $t=-\infty$. Thus, it is not natural to restrict ourselves to the problem with compactly supported initial data. We assume

$$\sum_{j=0}^{3} |\nabla^j \varphi(x)| + \sum_{j=0}^{2} |\nabla^j \psi(x)| \le \epsilon(1+|x|)^{-k}, \tag{3}$$

where $\epsilon > 0, k > 1$. With this condition, Asakura [2] proved the following results in case $N=3$.

(i) If $\rho > \rho(3)$ and $k \ge \frac{\rho+1}{\rho-1}$, then (1), (2) has a global $C^2$−solution provided that $\epsilon > 0$ in (3) is sufficiently small. (ii) If $\rho > \rho(3)$ and $k < \frac{\rho+1}{\rho-1}$, then we can not expect global $C^2$−solutions no matter how small $\epsilon$ are.

Asakura's results are extended to $N=2$ by Agemi-Takamura [1], Kubota [7] and Tsutaya [15].

Now, the scattering theory compares solutions of (1) with those of the free equation

$$\partial_t^2 u_0 - \Delta u_0 = 0 \quad \text{in} \quad (x,t) \in \mathbf{R}^N \times \mathbf{R} \tag{4}$$

near $t = \pm\infty$. The comparison will be done in the energy space with norm

$$\|u(t)\|_e^2 = \frac{1}{2}\{\|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2\}. \tag{5}$$

More precisely, we start with a solution $u_0^-(t)$ of (4) with sufficiently small initial data

$$u_0^-(x,0) = \varphi^-(x), \quad \partial_t u_0^-(x,0) = \psi^-(x) \tag{6}$$

satisfying (3). Then we construct a global solution $u(t)$ of (1) behaving like $u_0^-(t)$ near $t=-\infty$:

$$\|u(t) - u_0^-(t)\|_e \to 0 \text{ as } t \to -\infty. \tag{7}$$

Moreover, there exists another free solution $u_0^+(t)$ of (4) such that

$$\|u(t) - u_0^+(t)\|_e \to 0 \text{ as } t \to +\infty. \tag{8}$$

Thus, the scattering operator $S : u_0^-(t) \to u_0^+(t)$ is shown to exist on a dense set of a neghborhood of 0 in the energy space.

The existence of the scattering operator has been proved by Strauss [14], Klainerman [6] and Mochizuki-Motai [9],[10] in the general $N(\geq 2)$ dimensional problem. These works are based on the so called $L^p$ estimates of the fundamental solution of the free equation and the Sobolev embedding theorem, and require a stronger restriction on $\rho$. For example, it is assumed in [10] that

$$\rho > \frac{N^2+3N-2+\sqrt{(N^2+3N-2)^2-8N(N-1)}}{2N(N-1)} \equiv \tilde{\rho}(N).$$

We have $\rho(N) < \tilde{\rho}(N)$, and their values for lower dimensions are as follows:

| $N$ | $\rho(N)$ | $\tilde{\rho}(N)$ |
|---|---|---|
| 2 | $\frac{3+\sqrt{17}}{2} = 3.561\cdots$ | $2+\sqrt{3} = 3.732\cdots$ |
| 3 | $1+\sqrt{2} = 2.414\cdots$ | $\frac{4+\sqrt{13}}{3} = 2.535\cdots$ |
| 4 | 2 | $\frac{13+\sqrt{145}}{12} = 2.089\cdots$ |

The gap between $\rho(N)$ and $\tilde{\rho}(N)$ is covered by Pecher [11] in case $N = 3$. He showed the existence of the above mentioned scattering operator for $\rho > \rho(3)$ based on a weighted norm in space and time originally introduced by John [5]. Our aim is to obtain the corresponding sharp result in case $N = 2$.

Our problem (1), (7) will be reduced to the integral equation

$$u(x,t) = u_0^-(x,t) + \int_{-\infty}^{t} \frac{t-\tau}{2\pi} d\tau \int_{|\xi|<1} \frac{f(u(x+(t-\tau)\xi,\tau))}{\sqrt{1-|\xi|^2}} d\xi. \qquad (9)$$

To show the global solvability of (9), we also use the weighted norm of space and time. Note that in the 3-dimensional problem the right side integral becomes simpler, and one can estimate it by almost the same method as in the case of Cauchy problem at time 0. In our 2-dimensional problem, however, to add an extra estimate which is not used in the Cauchy problem is really necessary.

The details of the proof can bee seen in the recently published paper Kubota-Mochizuki [8]. So, in the following we shall only explain how to obtain a good estimate for the nonlinear term. Note that similar results have been obtained independently by Tsutaya [16], where the basic estimate is proved by a different way.

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First we shall estimate the free solution

$$u_0^-(x,t)=\partial_t\left(\frac{t}{2\pi}\int_{|\xi|<1}\frac{\varphi^-(x+t\xi)}{\sqrt{1-|\xi|^2}}d\xi\right)+\frac{t}{2\pi}\int_{|\xi|<1}\frac{\psi^-(x+t\xi)}{\sqrt{1-|\xi|^2}}d\xi \tag{10}$$

of (4), (6). In the following, we require that the initial data satisfies (3), and for the sake of simplicity, we choose $k>2$ in (3).

**Proposition 1.** *There exists a constant $C>0$ depending on $k$ such that*

$$\sum_{j=0}^{2}|\nabla^j u_0^-(x,t)|\le C\epsilon(1+r+|t|)^{-1/2}(1+|r-|t||)^{-1/2}. \tag{11}$$

*where $r=|x|$.*

*Proof.* We shall only estimate the term

$$v(x,t)\equiv\frac{t}{2\pi}\int_{|\xi|<1}\frac{\psi(x+t\xi)}{\sqrt{1-|\xi|}^2}d\xi=\frac{1}{2\pi}\int_{|\xi|<t}\frac{\psi(x+\xi)}{\sqrt{t^2-|\xi|^2}}d\xi.$$

In the last integral we use the polar coordinates $\{\eta,\theta\}$, $\eta=|\xi|$, and put

$$\lambda=\sqrt{r^2+\eta^2+2r\eta\sin\theta}.$$

Then by means of (3),

$$\begin{aligned}|v(x,t)| &\le \frac{\epsilon}{\pi}\int_0^t\eta\{t^2-\eta^2\}^{-1/2}d\eta\int_{-\pi/2}^{\pi/2}\left(1+\sqrt{r^2+\eta^2+2r\eta\sin\theta}\right)^{-k}d\theta\\ &= \frac{2\epsilon}{\pi}\int_0^t\eta\{t^2-\eta^2\}^{-1/2}d\eta\int_{|r-\eta|}^{r+\eta}\lambda(1+\lambda)^{-k}h(\lambda,\eta,r)d\lambda,\end{aligned}$$

where

$$\begin{aligned}h(\lambda,\eta,r) &= \{(r+\eta)^2-\lambda^2\}^{-1/2}\{\lambda^2-(r-\eta)^2\}^{-1/2}\\ &= \{(\lambda+r)^2-\eta^2\}^{-1/2}\{\eta^2-(\lambda-r)^2\}^{-1/2}.\end{aligned}$$

Changing the order of integrations, we have

$$|v(x,t)|\le\frac{2\epsilon}{\pi}\int_{|t-r|}^{t+r}(1+\lambda)^{-k+1}d\lambda\int_{|\lambda-r|}^{t}\eta\{t^2-\eta^2\}^{-1/2}h(\lambda,\eta,r)d\eta \tag{12}$$

$$+\frac{2\epsilon}{\pi}\int_0^{\max\{t-r,0\}}(1+\lambda)^{-k+1}d\lambda\int_{|\lambda-r|}^{\lambda+r}\eta\{t^2-\eta^2\}^{-1/2}h(\lambda,\eta,r)d\eta$$

$$\leq \frac{2\epsilon}{\pi}\int_{|t-r|}^{t+r}(1+\lambda)^{-k+1}\{(\lambda+r)^2-t^2\}^{-1/2}d\lambda$$
$$+\frac{2\epsilon}{\pi}\int_0^{\min\{t-r,0\}}(1+\lambda)^{-k+1}\{t^2-(\lambda+r)^2\}^{-1/2}d\lambda.$$

Since $k-1>1$, the desired estimate easily follows from the last inequality. □

Note here that we have used the following lemma to show the last inequality of (12).

**Lemma 1.** *Let $a<b<c$. Then*

$$\int_a^b(\sigma-a)^{-1/2}(b-\sigma)^{-1/2}(c-\sigma)^{-1/2}d\sigma\leq\pi(c-b)^{-1/2}.$$

Next we shall estimate the nonlinear term

$$L(f(u))=\frac{1}{2\pi}\int_{-\infty}^{t}d\tau\int_{|\xi|<t-\tau}\frac{f(u)(x+\xi,\tau)}{\sqrt{(t-\tau)^2-|\xi|^2}}d\xi, \tag{13}$$

where $f(u)=\gamma|u|^{\rho-1}$ or $=\gamma|u|^{\rho}$ with $\gamma\in\mathbf{R}\setminus\{0\}$ and $\rho>\rho(2)$.

**Proposition 2.** *Assume*

$$|u(x,t)|\leq M(1+r+|t|)^{-1/2}(1+|r-|t||)^{-\nu} \tag{14}$$

*for some $M>0$ and $\frac{1}{\rho}<\nu<\frac{\rho-3}{2}$. Then there exists a $C>0$ depending on $\nu$ such that*

$$|L(f(u))(x,t)|\leq C|\gamma|M^{\rho}(1+r+|t|)^{-1/2}(1+|r-|t||)^{-\nu}. \tag{15}$$

**Remark.** The inequality $\rho>\rho(2)$ imples that

$$2>\frac{\rho+1}{\rho-1}\quad\text{and}\quad\frac{1}{\rho}<\frac{\rho-3}{2}.$$

Noting (14), we put

$$\omega(\lambda,\tau)=|\gamma|M^{\rho}(1+\lambda+|\tau|)^{-\rho/2}(1+|\lambda-|\tau||)^{-\rho\nu}. \tag{16}$$

Then since

$$|f(u)(x+\xi,\tau)|\leq\omega(|x+\xi|,\tau),$$

we have as in the above proof,

$$|L(f(u))| \leq \frac{2}{\pi} \int_{-\infty}^{t} d\tau \tag{17}$$

$$\times \left[ \int_{|t-\tau-r|}^{t-\tau+r} \lambda\omega(\lambda,\tau)d\lambda \int_{|\lambda-r|}^{t-\tau} \eta\{(t-\tau)^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta \right.$$

$$\left. + \int_0^{\max\{t-\tau-r,0\}} \lambda\omega(\lambda,\tau)d\lambda \int_{|\lambda-r|}^{\lambda+r} \eta\{(t-\tau)^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta \right]$$

$$\equiv I_1 + I_2.$$

In the following we shall estimate the first term $I_1$ in case $t > 0$. The case $t < 0$ is easier to estimate (for we are not necessary to use Lemma 2 given below), and the second term $I_2$ can be similarly estimated.

As is already used in the above proof, we have from Lemma 1,

$$\int_{|\lambda-r|}^{t-\tau} \eta\{t^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta \leq \frac{\pi}{2}\{(\lambda+t)^2-(t-\tau)^2\}^{-1/2}. \tag{18}$$

To proceed into the proof, we need one more inequality

$$\int_{|\lambda-r|}^{t-\tau} \eta\{t^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta \tag{19}$$

$$\leq C_1(1+\lambda)^{\delta}\{1+(\lambda+r-t+\tau)^{-\delta}\}\{(t-\tau)^2-(\lambda-r)^2\}^{-1/2},$$

which is resulted from the following lemma. Here $\delta$ is any positive number and $C_1 > 1$.

**Lemma 2.** *Let $a < b < c$. Then*

$$\int_a^b (\sigma-a)^{-1/2}(b-\sigma)^{-1/2}(c-\sigma)^{-1/2}d\sigma$$

$$\leq \sqrt{2}\left\{\pi + \log\frac{4(c-a)}{c-b}\right\}(b-a)^{-1/2}.$$

*Proof of Proposition 2.* We devide $I_1$ into two terms:

$$I_1 = \frac{2}{\pi}\left(\int_{-\infty}^{0} + \int_0^t\right) d\tau \Big[\cdots\Big] \equiv J_1 + J_2,$$

and apply (18) and (19) to $J_2$ and $J_1$, respectively. Then

$$J_2 \leq \int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda\omega(\lambda,\tau)\{(\lambda+r)^2-(t-\tau)^2\}^{-1/2}d\lambda,$$

$$J_1 \le C_1 \int_{-\infty}^{0} d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda\omega(\lambda,\tau)(1+\lambda)^{\delta}$$
$$\times\{1+(\lambda+r-t+\tau)^{-\delta}\}\{(t-\tau)^2-(\lambda+r)^2\}^{-1/2}d\lambda.$$

We choose the new independent variables

$$\alpha = \lambda - \tau \quad \text{and} \quad \beta = \lambda + \tau,$$

Then since $\lambda + |\tau| = \beta$ (or $\alpha$) and $\lambda - |\tau| = \alpha$ (or $\beta$) in the expression of $J_2$ (or $J_1$), noting (16), we obtain

$$J_2 \le \frac{|\gamma| M^\rho}{2} \int_{|t-r|}^{t+r} (1+\beta)^{-\rho/2}(\beta+r-t)^{-1/2} d\beta \tag{20}$$
$$\times \int_{r-t}^{\beta} (1+|\alpha|)^{-\rho\nu}(\alpha+r+t)^{-1/2} d\alpha,$$

$$J_1 \le \frac{C_1|\gamma|M^\rho}{2\pi} \int_{t-r}^{t+r} (1+|\beta|)^{-\rho\nu}(t+r-\beta)^{-1/2}\{1+(\beta-t+r)^{-\delta}\} d\beta \tag{21}$$
$$\times \int_{\max\{r-t,\beta\}}^{\infty} (1+\alpha)^{-\rho/2+1+\delta}(\alpha-r+t)^{-1/2} d\alpha.$$

In (20) the integral for $\alpha$ is estimated by $C(1+t+r)^{-1/2}$ since $\rho\nu > 1$. On the other hand, the integral for $\beta$ is estimated by $C(1+|t-r|)^{-\nu}$ since $\nu < (\rho-3)/2$. Similarly, in (21) we choose

$$0 < \delta < \min\left\{\frac{1}{2}, \frac{\rho-3}{2} - \nu\right\}.$$

Then the integrals for $\beta$ and $\alpha$ are estimated by the terms $C(1+t+r)^{-1/2}$ and $C(1+|t-r|)^{-\nu}$, respectively.

The inequality (15) for $I_1$ is thus proved when $t > 0$. □

Now, we introduce a weighted norm in space and time

$$\|v\|_W = \sup_{(x,t)} [(1+|t|+r)^{1/2}(1+||t|-r|)^{\nu}|v(x,t)|]$$

and define a Banach space

$$Y = \{v|\ \nabla^j v(x,t) \in C(\mathbf{R}^2 \times \mathbf{R}), \|\nabla^j v\|_W < \infty\ (0 \le j \le 2)\}.$$

Since $\nu < 1/2$, we see that the free solution $u_0^-(x,t)$ belongs to $Y$ and

$$\|\nabla^j u_0^-\|_W \le C\epsilon,\ 0 \le j \le 2. \tag{22}$$

On the other hand, Proposition 2 shows that

$$\|L(f(u))\|_W \leq C|\gamma|\|u\|_W^\rho \tag{23}$$

for any $u \in C(\mathbf{R}^2 \times \mathbf{R})$ such that $\|u\|_W < \infty$.

With these inequalites, choosing $\epsilon > 0$ sufficiently small, we can follow a method of successive approximation (already used by John [5]) to establish the unique existence of solutions in $Y$ of the integral equation (9). Moreover, asymptotics as $t \to \infty$ of the solution $u(t)$ can be derived since we have $f(v) \in L^1(\mathbf{R};\mathbf{R}^2)$ for $v \in Y$.

Our results are summarized in the following theorem.

**Theorem.** *Let $\rho, k, \nu$ be as given above, and assume $\{\varphi^-, \psi^-\}$ satisfy* (3).

(i) *There exists an $\epsilon_0 > 0$ depending on the above parameters and $\gamma$ such that the integral equation* (9) *has a unique solution $u \in Y$ provided $0 < \epsilon \leq \epsilon_0$ in* (3).

(ii) *This $u = u(x,t)$ is a classical global solution of the wave equation* (1), *and we have*

$$\|u(t) - u_0^-(t)\|_e \leq C\|u\|_W(1+|t|)^{-\rho+3} \text{ as } t \to -\infty \tag{24}$$

(iii) *If we define*

$$u_0^+(x,t) = u(x,t) - \int_t^\infty \frac{\tau - t}{2\pi} d\tau \int_{|\xi|<1} \frac{f(u)(x+(\tau-t)\xi,\tau)}{\sqrt{1-|\xi|}} d\xi, \tag{25}$$

*then $u_0^+ \in Y$ and satisfies the free wave equation* (4). *Furthermore,*

$$\|u(t) - u_0^+(t)\|_e \leq C\|u\|_W(1+t)^{-\rho+3} \text{ as } t \to +\infty. \tag{26}$$

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# THE INTERMEDIATE SOLUTION OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

BONGSOO KO

## 1. Introduction

We study the existence of an intermediate solution of nonlinear elliptic boundary value problems $(BVP)$ of the form

$$(BVP) \qquad \begin{cases} \Delta u = f(x, u, \nabla u), & \text{in } \Omega \\ Bu(x) = \phi(x), & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbf{R}^n$, $n \geq 1$, and $\partial\Omega \in C^{2,\alpha}$, $(0 < \alpha < 1)$, $\Delta$ is the Laplacian operator, $\nabla u = (D_1 u, D_2 u, \cdots, D_n u)$ denotes the gradient of u and

$$Bu(x) = p(x)u(x) + q(x)\frac{du}{d\nu}(x),$$

where $\dfrac{du}{d\nu}$ denotes the outward normal derivative of $u$ on $\partial\Omega$.

Suppose now, that $\bar{v}$, $\hat{v}$ and $\bar{w}$, $\hat{w}$ are two pairs of subsolutions and supersolutions in the class $C^2(\bar{\Omega})$ or in the usual Sobolev space $W^{2,p}(\Omega)$, $p > n$ of $(BVP)$ such that $\bar{v}(x) \leq \hat{v}(x)$, $\bar{w}(x) \leq \hat{w}(x)$, $\bar{v}(x) \leq \hat{w}(x)$ for all $x \in \bar{\Omega}$ and $\hat{v}(x_0) < \bar{w}(x_0)$ for some $x_0 \in \bar{\Omega}$. Then there is a solution in the order interval $[\bar{v}, \hat{v}] = \{ u \in C(\bar{\Omega}) : \bar{v}(x) \leq u(x) \leq \hat{v}(x), x \in \bar{\Omega} \}$ and a solution in $[\bar{w}, \hat{w}]$. And furthermore Amann [1] or Amann and Crandall [3] showed that there exists an intermediate solution in the set $[\bar{v}, \hat{w}] \setminus ([\bar{v}, \hat{v}] \cup [\bar{w}, \hat{w}])$ under additional conditions.

The existence of a solution given a pair of quasisubsolution and quasisupersolution of $(BVP)$, $\bar{v}$ and $\hat{v}$, with $\bar{v}(x) \leq \hat{v}(x)$ for all $x \in \bar{\Omega}$, is well known (see [9]). Since these functions may have singular points in the interior of $\Omega$, there arises the question, does there also exist an intermediate solution if there are pairs of quasisubsolutions and quasisupersolutions as in the preceding paragraph ?

Suppose now, in addition, that $f$ is independent of $\nabla u$ and that there are pairs of quasisubsolutions and quasisupersolutions as in the preceding paragraph. Ko [6] proved the existence of an intermediate solution in $[\bar{v}, \hat{w}] \setminus ([\bar{v}, \hat{v}] \cup [\bar{w}, \hat{w}])$ under additional conditions. Hence there also arises the question, does there exist an intermediate solution if $f$ depends nonlinearly on $\nabla u$ ?

The author is able to solve the above problem which is the existence of an intermediate solution for $(BVP)$ using Maximum Principles and the theorem on existence of several fixed points (see pp241, [4]). The multiplicity result is a generalization of Theorem(1.6) in [1] or Theorem 2 in [3].

Throughout this paper we assume that $p,\ q \in C^{1,\alpha}(\partial\Omega)$ are nonnegative real valued functions which either $q(x) = 0$ for all $x \in \partial\Omega$ or $q(x) > 0$ for all $x \in \partial\Omega$, and $f$ satisfies the following conditions:

$0 < \alpha < 1$,

(1) $f : \bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n \to \mathbf{R}$ is a $\alpha$-Hölder continuous function, such that $f(\cdot, \xi, \eta) \in C^\alpha(\bar{\Omega})$ and such that $\dfrac{\partial f}{\partial \xi}$ and $\dfrac{\partial f}{\partial \eta}$ are continuous where $(x, \xi, \eta)$ denotes a generic point of $\bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$ and $\mathbf{I}$ is a fixed bounded and closed interval in $\mathbf{R}$.

(2) There exists a continuous function $c : \mathbf{R}_+ \to \mathbf{R}_+ = [0, \infty)$ such that

$$|f(x, \xi, \eta)| \le c(\rho)(1 + |\eta|^2)$$

for every $\rho \ge 0$ and $(x, \xi, \eta) \in \bar{\Omega} \times [-\rho, \rho] \times \mathbf{R}^n$.

(3) $\phi \in C^{2,\alpha}(\bar{\Omega})$ and for the Dirichlet problem case, $\phi(x) \in \mathbf{I}$ for all $x \in \partial\Omega$.

By a solution of $(BVP)$ we mean a function $u : \bar{\Omega} \to \mathbf{I}$ such that $u \in C^2(\bar{\Omega})$ and $u$ satisfies $(BVP)$ pointwise.

## 2. Main Results

First of all, we state definitions of a quasisubsolution and a quasisupersolution of $(BVP)$.

**Definitions.** A function $w : \bar{\Omega} \to \mathbf{R}$ is a quasisupersolution of $(BVP)$ in $\bar{\Omega}$, if for any $x_0 \in \bar{\Omega}$, there exist a neighborhood $N$ of $x_0$ and a finite number of functions $w_k \in C^2(N)$, $k = 1, 2, \cdots, p$ such that

$$(I) \qquad w(x) = \min_{1 \le k \le p} w_k(x),$$

for all $x \in N$, where $p$ may depend on $x_0$, and

$$(II) \qquad \Delta w_k(x) \leq f(x, w_k(x), \nabla w_k(x)),$$

for all $x \in N \cap \Omega$ and $k = 1, 2, \cdots, p$. Furthermore, if $x_0 \in \partial\Omega$,

$$(III) \qquad p(x_0)w_k(x_0) + q(x_0)\frac{dw_k}{d\nu}(x_0) \geq \phi(x_0),$$

for all $k$.

A quasisubsolution $w : \bar{\Omega} \to \mathbf{R}$ is defined similarly, replacing min by max in $(I)$ and reversing the inequalities $(II)$ and $(III)$.

To state the theorem for the existence of an intermediate solution of $(BVP)$, we need the following notations: Let $u$, $v : \bar{\Omega} \to \mathbf{R}$ be functions. Then we write $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \bar{\Omega}$, and $u < v$ if $u \leq v$ but $u \neq v$. By $[u, v]$ we mean the order interval between $u$ and $v$, that is,

$$[u, v] = \{w : \bar{\Omega} \to \mathbf{R} : u \leq w \leq v\}.$$

**Theorem 1.** *Let $f$ satisfy (1) and (2) and $\phi$ satisfy (3). Suppose that $\bar{v}_j$ is a quasisubsolution and $\hat{v}_j$ is a quasisupersolution of $(BVP)$ for $j = 1, 2$ such that $\bar{v}_1 \leq \hat{v}_1$, $\bar{v}_2 \leq \hat{v}_2$, $\bar{v}_1 \leq \hat{v}_2$ and $\hat{v}_1(x_0) < \bar{v}_2(x_0)$ for some $x_0 \in \bar{\Omega}$. Assume moreover that $\hat{v}_1$ and $\bar{v}_2$ are not solutions of $(BVP)$ and $[\bar{v}_1(x), \hat{v}_2(x)] \subset \mathbf{I}$ for all $x \in \bar{\Omega}$. Then $(BVP)$ has at least three distinct solutions $u_j$ such that $\bar{v}_1 \leq u_1 < u_0 < u_2 \leq \hat{v}_2$, $u_j \in [\bar{v}_j, \hat{v}_j]$ for $j = 1, 2$ and $u_0 \in [\bar{v}_1, \hat{v}_2] \setminus ([\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2])$.*

Theorem 1 is a generalization of Theorem(1.6) in [1] or Theorem 2 in [3] and follows at once from the next proposition.

**Proposition.** *Let the hypothese of Theorem 1 hold and let $h : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $h(x) = (h_1(x), h_2(x), \cdots, h_n(x))$ and bounded and of class $C^1$ such that each partial derivatives for $h_i$ is bounded on $\mathbf{R}^n$. Then the following elliptic boundary value problem*

$$(BVP_h) \qquad \begin{cases} \Delta u = f(x, u, h(\nabla u)) & in \quad \Omega \\ Bu = \phi \quad on \quad \partial\Omega \end{cases}$$

*has at least three distinct solutions* $u_0$, $u_1$, $u_2$ *such that* $\bar{v}_1 \le u_1 < u_0 < u_2 \le \hat{v}_2$, $u_j \in [\bar{v}_j, \hat{v}_j]$ *for* $j = 1, 2$, *and* $u_0 \in [\bar{v}_1, \hat{v}_2] \setminus ([\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2])$.

To prove Proposition, we first convert $(BVP_h)$ into an operator equation. Choose $\lambda > 0$ large enough so that $\dfrac{\partial f}{\partial \xi}(x, \xi, h(\eta)) + \lambda > 0$ for all $(x, \xi, \eta) \in \bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$. For any $g$ which belongs to the following set

$$g \in [\bar{v}_1, \hat{v}_2] \cap C^{\alpha}(\bar{\Omega})$$

and we also assume that $\lambda$ satisfies

$$f(x, c_1, h(0)) + \lambda(c_1 - g(x)) < 0 < f(x, c_2, h(0)) + \lambda(c_2 - g(x))$$

for some constants $c_1$, $c_2$ with $c_1 < 0 < c_2$ and for all $x \in \bar{\Omega}$. Furthermore, if $Bu = u$, then $\phi : \bar{\Omega} \to [c_1, c_2]$, and if not, $p(x)c_1 \le \phi(x) \le p(x)c_2$ for all $x \in \partial\Omega$. Then it is known that there is a solution $u \in C^{2,\alpha}(\bar{\Omega})$ of the following boundary value problem

$$\begin{cases} \Delta u = f(x, u, h(\nabla u)) + \lambda(u - g) & \text{in} \quad \Omega \\ Bu = \phi \quad \text{on} \quad \partial\Omega. \end{cases}$$

This solution is denoted by $u = Tg$ below.

**Lemma 1.** *A function* $u \in [\bar{v}_1, \hat{v}_2] \cap C^{\alpha}(\bar{\Omega})$ *is a solution of* $(BVP_h)$ *if and only if* $u = Tu$.

**Lemma 2.** *Let* $\bar{v}_j$ *and* $\hat{v}_j$ *be a quasisubsolution and a quasisupersolution of* $(BVP_h)$, *respectively. Then*

$$\bar{v}_j \le T\bar{v}_j \quad \text{and} \quad T\hat{v}_j \le \hat{v}_j\,.$$

*Proof.* To show $T\hat{v}_j(x) \le \hat{v}_j(x)$ for all $x \in \bar{\Omega}$, suppose that there is a point $x_0 \in \bar{\Omega}$ such that $\hat{v}_j(x_0) < T\hat{v}_j(x_0)$. Let $a = T\hat{v}_j(\hat{x}) - \hat{v}_j(\hat{x})$ be positive maximum value of $T\hat{v}_j - \hat{v}_j$. Then there exists a neighborhood $U_{\hat{x}}$ of $\hat{x}$ such that $\hat{x} \in U_{\hat{x}}$ and $0 < T\hat{v}_j(x) - \hat{v}_j(x) \le a$ for all $x \in U_{\hat{x}}$.

Case 1. $\hat{x} \in \Omega$.

By the definition of a quasisupersolution, there exist a neighborhood $N_{\hat{x}}$ and a finite number of functions $w_k \in C^2(N_{\hat{x}})$, $k = 1, 2, \cdots, p$ such that $\hat{x} \in N_{\hat{x}} \subset U_{\hat{x}}$ and

$$\hat{v}_j(x) = \min_{1 \le k \le p} w_k(x)$$

for all $x \in N_{\hat{x}} \cap \Omega$. Let $\hat{v}_j(\hat{x}) = w_k(\hat{x})$ for some k. Then $0 < T\hat{v}_j(x) - w_k(x) \leq a$ for all $x \in N_{\hat{x}} \cap \Omega$. Since

$$0 < T\hat{v}_j(x) - w_k(x) \leq T\hat{v}_j(x) - \hat{v}_j(x) \leq a$$

for all $x \in N_{\hat{x}} \cap \Omega$, so

$$0 < T\hat{v}_j(x) - w_k(x) \leq T\hat{v}_j(\hat{x}) - w_k(\hat{x}) = a$$

for all $x \in N_{\hat{x}} \cap \Omega$. Hence $T\hat{v}_j - w_k$ has the positive maximum value $a$ at $\hat{x}$ in the neighborhood $N_{\hat{x}} \cap \Omega$.

On the other hand, in $N_{\hat{x}} \cap \Omega$, by Mean Value Theorem,

$$\begin{aligned}
&\Delta(w_k - T\hat{v}_j)(x) \\
&\quad \leq f(x, w_k(x), h(\nabla w_k(x))) - f(x, T\hat{v}_j(x), h(\nabla T\hat{v}_j(x))) \\
&\quad - \lambda(T\hat{v}_j - \hat{v}_j)(x) \\
&\quad = [f_\xi(x, \xi^*(x), h(\nabla w_k(x))) + \lambda](w_k - T\hat{v}_j)(x) \\
&\quad + f_\eta(x, T\hat{v}_j(x), h(\eta^*(x))) \cdot dh \cdot \nabla(w_k - T\hat{v}_j)(x) .
\end{aligned}$$

where $\xi^*(x)$ lies between $w_k(x)$ and $T\hat{v}_j(x)$, $h(\eta^*(x))$ lies on the line segment joining $h(\nabla w_k(x))$ and $h(\nabla T\hat{v}_j(x))$, and

$$dh = dh(p_1^*, p_2^*, \cdots, p_n^*) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(p_1^*) & \cdots & \frac{\partial h_1}{\partial x_n}(p_1^*) \\ \frac{\partial h_2}{\partial x_1}(p_2^*) & \cdots & \frac{\partial h_2}{\partial x_n}(p_2^*) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial h_n}{\partial x_1}(p_n^*) & \cdots & \frac{\partial h_n}{\partial x_n}(p_n^*) \end{pmatrix}$$

and the points $p_1^*, p_2^*, \cdots, p_n^*$ lie on the line segment joining $\nabla w_k(x)$ and $\nabla T\hat{v}_j(x)$.

Since $f_\eta \cdot dh$ is bounded on $\bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$, we can choose a bounded function $b : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$f_\eta(x, T\hat{v}_j(x), h(\eta^*(x))) \cdot dh \cdot \nabla(w_k - T\hat{v}_j)(x) \leq b(x) \cdot \nabla(w_k - T\hat{v}_j)(x)$$

for all $x \in N_{\hat{x}} \cap \Omega$. Hence, on $N_{\hat{x}} \cap \Omega$,

$$\Delta(w_k - T\hat{v}_j)(x) - b(x) \cdot \nabla(w_k - T\hat{v}_j)(x) \leq 0 .$$

By Maximum Principles, $T\hat{v}_j(x) - w_k(x) = a$ for all $x \in N_{\hat{x}} \cap \Omega$, whence $T\hat{v}_j(x) = \hat{v}_j(x) + a$ for all $x \in N_{\hat{x}} \cap \Omega$. By the continuation of the method on the boundary of $N_{\hat{x}} \cap \Omega$, we can conclude that $T\hat{v}_j(x) = \hat{v}_j(x) + a$ for all $x \in \bar{\Omega}$. And so, for any $x \in \Omega$,

$$\begin{aligned}
&\Delta\hat{v}_j(x) \\
&= \Delta T\hat{v}_j(x) \\
&= f(x, \hat{v}_j(x), h(\nabla\hat{v}_j(x))) + (f_\xi(x, \xi^*(x), h(\nabla\hat{v}_j(x))) + \lambda)a,
\end{aligned}$$

where $\xi^*(x)$ lies between $\hat{v}_j(x)$ and $\hat{v}_j(x) + a$. Hence, for any $x \in \Omega$,

$$\Delta\hat{v}_j(x) \geq f(x, \hat{v}_j(x), h(\nabla\hat{v}_j(x))) .$$

Since $\Delta\hat{v}_j(x) \leq f(x, \hat{v}_j(x), h(\nabla\hat{v}_j(x))$ locally in $\Omega$, so

$$[f_\xi(x, \xi^*(x), h(\nabla\hat{v}_j(x))) + \lambda]a \leq 0 .$$

This leads to a contradiction for $a > 0$.

Case 2. $\hat{x} \in \partial\Omega$.

Since $T\hat{v}_j(\hat{x}) = w_k(\hat{x}) + a$ and $0 < T\hat{v}_j(x) - w_k(x) < a$ for all $x \in N_{\hat{x}} \cap \Omega$, so

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} \geq \frac{dw_k(\hat{x})}{d\nu} .$$

If $p(\hat{x}) > 0$, then

$$\begin{aligned}
\phi(\hat{x}) &= p(\hat{x})T\hat{v}_j(\hat{x}) + q(\hat{x})\frac{dT\hat{v}_j(\hat{x})}{d\nu} \\
&\geq p(\hat{x})[w_k(\hat{x}) + a] + q(\hat{x})\frac{dw_k(\hat{x})}{d\nu} \\
&\geq \phi(\hat{x}) + p(\hat{x})a .
\end{aligned}$$

This leads to a contradiction for $p(\hat{x})a > 0$.

Let $p(\hat{x}) = 0$. Then $q(\hat{x}) > 0$. If

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} > \frac{dw_k(\hat{x})}{d\nu} ,$$

then

$$\phi(\hat{x}) = q(\hat{x})\frac{dT\hat{v}_j(\hat{x})}{d\nu} > q(\hat{x})\frac{dw_k(\hat{x})}{d\nu} \geq \phi(\hat{x}) .$$

This also leads to a contradiction. Let

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} = \frac{dw_k(\hat{x})}{d\nu}.$$

For all $x \in N_{\hat{x}} \cap \Omega$,

$$\begin{aligned}&\Delta(T\hat{v}_j - w_k - a)(x)\\ &\geq f(x, T\hat{v}_j(x), h(\nabla T\hat{v}_j(x))) - f(x, w_k(x), h(\nabla w_k(x)))\\ &+ \lambda(T\hat{v}_j - \hat{v}_j)(x).\end{aligned}$$

By the Mean Value Theorem and choosing suitable bounded function $b : \mathbf{R}^n \to \mathbf{R}^n$ as before, we can also show that

$$\Delta(T\hat{v}_j - w_k - a)(x) - b(x) \cdot \nabla(T\hat{v}_j - w_k - a)(x) \geq 0.$$

Since $T\hat{v}_j - w_k - a$ has the zero maximum value of the boundary point $\hat{x}$ in $N_{\hat{x}} \cap \Omega$, by Maximum Principles, for all $x \in N_{\hat{x}} \cap \Omega$,

$$T\hat{v}_j(x) - w_k(x) = a.$$

This implies that $T\hat{v}_j - w_k$ has the positive maximum value $a$ at an interior point of $\Omega$. By Case 1, this also leads to a contradiction. Therefore, $T\hat{v}_j \leq \hat{v}_j$.

Similarly, we can show that $T\bar{v}_j \geq \bar{v}_j$. ■

**Lemma 3.** *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisupersolution of $(BVP_h)$, respectively. Then $T$ is an increasing operator from $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ into itself, i.e. if $u \leq v$, then $Tu \leq Tv$.*

*Proof.* Since $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ is a bounded interval in $C(\bar{\Omega})$, if we choose two constants $c_1$ and $c_2$ such that $c_1 < 0 < c_2$, $c_1 - g < 0$, $c_2 - g > 0$ for all $g \in [\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$, then there is $u \in [c_1, c_2] \cap C^\alpha(\bar{\Omega})$ such that $u = Tg$.

We first show that $T$ is well-defined on $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$. Let $u = Tg$ and $v = Tg$. Then

$$\Delta(u-v)(x) = f(x, u(x), h(\nabla u(x))) - f(x, v(x), h(\nabla v(x))) + \lambda(u-v)(x).$$

If we choose some bounded functions $b_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, 2$ so that

$$\Delta(u-v)(x) - b_1(x) \cdot \nabla(u-v)(x) \leq 0,$$

and

$$\Delta(u-v)(x) - b_2(x)\cdot\nabla(u-v)(x) \geq 0\,,$$

for all $x \in \bar{\Omega}$, then by Maximum Principles, $u = v$.

We secondly prove that $T$ is increasing on $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$. Let $g_1, g_2 \in [\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ and $g_1 \leq g_2$. Then

$$\begin{aligned}
&\Delta(Tg_2 - Tg_1)(x) \\
&= [f(x, Tg_2(x), h(\nabla Tg_2(x))) - f(x, Tg_2(x), h(\nabla Tg_1(x)))] \\
&+ [f(x, Tg_2(x), h(\nabla Tg_1(x))) - f(x, Tg_1(x), h(\nabla Tg_1(x)))] \\
&+ \lambda(Tg_2 - Tg_1)(x) + \lambda(g_1 - g_2)(x)\,.
\end{aligned}$$

We then choose two bounded functions $b : \mathbf{R}^n \to \mathbf{R}^n$ and $\beta : \mathbf{R}^n \to \mathbf{R}$ such that $-\lambda < \beta(x) < \lambda$ for all $x \in \bar{\Omega}$ and

$$\begin{aligned}
&\Delta(Tg_2 - Tg_1)(x) - b(x)\cdot\nabla(Tg_2 - Tg_1)(x) \\
&- (\beta(x) + \lambda)(Tg_2 - Tg_1)(x) \leq \lambda(g_1 - g_2)(x) \leq 0
\end{aligned}$$

for all $x \in \bar{\Omega}$. By Maximum Principles, $(Tg_2 - Tg_1)(x) \geq 0$, $x \in \bar{\Omega}$.

By Lemma 2 and that $T$ is increasing, we note that $u = Tg \in [\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$. ■

**Lemma 4.** *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisupersolution of $(BVP_h)$, respectively. Then $T$ is continuous and compact from $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ into itself.*

*Proof.* We first show that $T$ is continuous. Consider a sequence $\{g_n\}$ in $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ and suppose $\lim_{n\to\infty} g_n = g$ in $[\bar{v}_1, \hat{v}_2] \cap C^\alpha(\bar{\Omega})$ and

$$\lim_{n\to\infty} Tg_n = y$$

in $C^{2,\alpha}(\bar{\Omega})$. Then $\lim_{n\to\infty} \Delta Tg_n = \Delta y$ and $\lim_{n\to\infty} \nabla g_n = \nabla y$ in $C(\bar{\Omega})$. Hence

$$\Delta y(x) = f(x, y(x), h(\nabla y(x))) + \lambda(y - g)(x)$$

for all $x \in \Omega$ and $By(x) = \phi(x)$ for all $x \in \partial\Omega$. By the uniquenees of solutions corresponding $g$, $Tg = y$. By the Closed Graph Theorem, $T$ is continuous on $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$.

We note that $C^{2,\alpha}(\bar{\Omega})$ is compactly embedded in $C^\alpha(\bar{\Omega})$. Hence $T$ is compact on $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$. ■

To extend the operator $T$ to $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ continuously, we will use the following theorem. It can be found in Amann and Crandall [3].

**Theorem 2.** *Let $f$ satisfy $(f2)$. Then there is an increasing function $\gamma : [0,\infty) \to [0,\infty)$ such that if $u$ is a solution of $(BVPh)$ then*

$$\|u\|_{W^{2,p}(\Omega)} \leq \gamma(\|u\|_{C(\bar{\Omega})}) .$$

*Moreover, $\gamma$ depends only on $\Delta$, $B$, $\Omega$, $n$, $p$, and $c$.*

Since $C^{\alpha}(\bar{\Omega})$ is dense in $C(\bar{\Omega})$ and $T$ is a continuous increasing compact operator from $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ into itself, we can extend $T$ to $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ continuously and compactly. To show that this is possible, let $u \in [\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$. Then there exists a monotone sequence $\{u_n\}$ in $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ so that $u_n \to u$ in $C(\bar{\Omega})$ as $n \to \infty$. Since $\{Tu_n\}$ is bounded in $C(\bar{\Omega})$, by Theorem 2, $\{Tu_n\}$ is bounded in $W^{2,p}(\Omega)$, and if $p > n$, then $\{Tu_n\}$ is bounded in $C^{1,\alpha}(\bar{\Omega})$. By the Mean Value Theorem, $\{Tu_n\}$ is equicontinuous on $C(\bar{\Omega})$. By Ascoli-Azela Theorem, $\{Tu_n\}$ has a convergent subsequence in $C(\bar{\Omega})$. Since $\{Tu_n\}$ is monotone, we can define $Tu$ by

$$Tu = \lim_{n\to\infty} Tu_n .$$

Since $Tu_n$ is bounded in $C^{1,\alpha}(\bar{\Omega})$, so $Tu \in C^{\alpha}(\bar{\Omega})$. Therefore, we view $T$ as a continuous extension to an operator (denoted again by $T$) mapping $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$. Since the imbedding of $C^{\alpha}(\bar{\Omega})$ in $C(\bar{\Omega})$ is compact it follows that the operator $T$ maps $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ compactly into $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$.

To complete the proof of Proposition, we need the special ordered Banach space $C_e(\bar{\Omega})$ whose positive cone is normal and has nonempty interior. In defining $C_e(\bar{\Omega})$, $e \in C(\bar{\Omega})$, $e(x) \geq 0$ for all $x \in \bar{\Omega}$, $e(x) \neq 0$. Let $C_e(\bar{\Omega})$ be the set of all functions $u \in C(\bar{\Omega})$ so that

$$-ce(x) \leq u(x) \leq ce(x)$$

for some constant $c \geq 0$ and for all $x \in \bar{\Omega}$. If $u \in C_e(\bar{\Omega})$, we define the norm

$$\|u\|_e = \inf\{c > 0 : -ce(x) \leq u(x) \leq ce(x),\ x \in \bar{\Omega}\} .$$

It can be shown that the Minkowski functional $\|\cdot\|_e$ is a norm on $C_e(\bar{\Omega})$. Furtheremore, $C_e(\bar{\Omega})$ is a Banach space with respect to the norm.(see [2])

Now we state the theorem which will be use in proving the existence of an intermediate solution of $(BVP_h)$ for Dirichlet boundary condition. The main idea of the proof for the following theorem can be found in Amann [2].

**Theorem 3.** *Let $e$ be the unique solution of the boundary value problem*

$$\begin{cases} \Delta e(x) = -1, & x \in \Omega \\ e(x) = 0, & x \in \partial\Omega \end{cases}$$

*and $T$ be the operator induced by the boundary value problem*

$$\begin{cases} \Delta Tu = f(x, Tu, h(\nabla Tu)) + \lambda(Tu - u), \; x \in \Omega \\ Tu(x) = 0, \quad x \in \partial\Omega \end{cases}$$

*with a quasisubsolution $\bar{v}_j$ and a quasisupersolution $\hat{v}_j$ of $(BVP_h)$ so that $\bar{v}_j < \hat{v}_j$. Then $C_e(\bar{\Omega})$ is continuously imbedded in $C(\bar{\Omega})$ and $T$ is a compact operator from $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into $C_e(\bar{\Omega})$.*

*Proof.* By the previous statements, $T$ maps $[\bar{v}_j, \hat{v}_j] \cap \{u \in C(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ compactly into $[\bar{v}_j, \hat{v}_j] \cap \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. Therefore it suffices to show that $\{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ is continuously imbedded in $C_e(\bar{\Omega})$. We follows the proof in [2, Theorem 4.2]. Since, by the Maximum Principle, on every compact subset of $\Omega$, $e$ is bounded below by a positive constant and since, for every $x \in \partial\Omega$, $\frac{\partial e}{\partial \nu} < 0$, it follows by continuity that, for every $u \in C_0^1(\bar{\Omega})$, there exist $\alpha$, $\beta > 0$ with

$$-\alpha e \le u \le \beta e,$$

i.e. $C_0^1(\bar{\Omega})$ is a subset of $C_e(\bar{\Omega})$. Since convergence in the norm of $C_e(\bar{\Omega})$ as well as in the norm of $C_0^1(\bar{\Omega})$ implies pointwise convergence, it is easily seen that the injective map from $C_0^1(\bar{\Omega})$ into $C_e(\bar{\Omega})$ is a closed linear operator. Hence, by the Closed Graph Theorem, $C_0^1(\bar{\Omega})$ is continuously imbedded in $C_e(\bar{\Omega})$ and the statement follows. ■

Now, we obtain the conclusion.

**Theorem 4.** *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisupersolution) of $(BVP_h)$, respectively. Then the operator $T$ induced by the problem $(BVP_h)$ is continuous, increasing and compact from $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into itself.*

Finally, to prove Proposition, we will use the following theorem of existence of several fixed points, and we can find it in Deimling [4] or Amann [2].

**Theorem 5.** *Let $X$ be a Banach space: $S \subset X$ a retract and $T : S \to S$ compact; $S_1$, $S_2$ nonempty disjoint retracts of $S$; $E_j \subset S_j$ open in $S$ for $j = 1, 2$. Suppose that $T(S_j) \subset S_j$ and Fix($T$) $\cap (S_j \setminus E_j) = \emptyset$ for $j = 1, 2$, where Fix($T$) $= \{u \in S : Tu = u\}$. Then $T$ has fixed points $u_j \in E_j$ and a third fixed point $u_0 \in S \setminus (S_1 \cup S_2)$.*

**Proof of Proposition.**

Case 1. $q(x) > 0$ for all $x \in \partial\Omega$.

Let

$$O_1 = \{u \in C(\bar{\Omega}) : u(x) < \hat{v}_1(x),\, x \in \bar{\Omega}\},$$

and

$$O_2 = \{u \in C(\bar{\Omega}) : u(x) > \bar{v}_2(x),\, x \in \bar{\Omega}\},$$

$S = [\bar{v}_1, \hat{v}_2] \cap C(\bar{\Omega})$, $S_1 = [\bar{v}_1, \hat{v}_1] \cap C(\bar{\Omega})$, $S_2 = [\bar{v}_2, \hat{v}_2] \cap C(\bar{\Omega})$, $E_1 = S \cap O_1$, and $E_2 = S \cap O_2$. Then $E_1$ and $E_2$ are open in $S$. From Lemma 2,3,4, and Theorem 4, $T : S \to S$ is compact. Clearly, $S_1$ and $S_2$ are disjoint retracts of $S$, $E_j \subset S_j$, $T(S_j) \subset S_j$ for $j = 1, 2$. To show that Fix($T$) $\cap (S_j \setminus E_j) = \emptyset$, we assume that there is $u \in$ Fix($T$) $\cap (S_j \setminus E_j)$ for some $j$. Then $u \in S_j \setminus E_j$ and $Tu = u$.

Let $j = 1$. We note that $u$ is a solution of $(BVP_h)$. Since $u \in S_1 \setminus E_1$, so $\bar{v}_1 \le u \le \hat{v}_1$ and there is a point $x_0 \in \bar{\Omega}$ such that $u(x_0) = \hat{v}_1(x_0)$. By the definition of a quasisupersolution, let

$$\hat{v}_1(x) = \min_{1 \le k \le p} w_k(x)$$

on some neighborhood $U_{x_0}$ of $x_0$ and let $u(x_0) = w_k(x_0)$ for some $k$. For all $x \in U_{x_0} \cap \Omega$, we can show that

$$\begin{aligned} &\Delta(u - w_k)(x) - b(x) \cdot \nabla(u - w_k)(x) - \lambda(u - w_k)(x) \\ &\ge [-G_\xi(x, \xi^*(x), h(\nabla u(x))) + \lambda](w_k - u)(x) \ge 0, \end{aligned}$$

where $b : \mathbf{R}^n \to \mathbf{R}^n$ is some bounded function. Since $\hat{v}_1$ is not a solution of $(BVP_h)$, by Maximum Principles, $u(x) = w_k(x)$ for all $x \in U_{x_0} \cap \Omega$ and $x_0 \in \partial\Omega$. Since $u - w_k$ has a zero maximum value at the boundary point $x_0$, either $u(x) = w_k(x)$ and $x \in U_{x_0} \cap \bar{\Omega}$ or $\frac{dw_k}{d\nu}(x_0) < \frac{du}{d\nu}(x_0)$. We note that both cases lead to a contradiction. Consequently, Fix($T$) $\cap (S_1 \setminus E_1) = \emptyset$.

Similarly, we can show that $\text{Fix}(T)\cap(S_2\setminus E_2) = \emptyset$. Therefore, by Theorem 5, $T$ has at least three distinct fixed points $u_0$, $u_1$, $u_2$ such that $u_j \in [\bar{v}_j, \hat{v}_j]$ for $j = 1,2$, and especially note that

$$u_0 \in [\bar{v}_1, \hat{v}_2] \setminus [\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2] .$$

Case 2. $q(x) = 0$ for all $x \in \partial\Omega$.

We assume that the Dirichlet boundary condition for $(BVP_h)$, i.e. $Bu = u = \phi = 0$ on $\partial\Omega$. Let

$$S = C_e(\bar{\Omega}) \cap [\bar{v}_1, \hat{v}_2]$$

and

$$S_j = C_e(\bar{\Omega}) \cap [\bar{v}_j, \hat{v}_j]$$

for $j = 1,2$. We note that $T : S \to S$ is compact; $S_j \subset S$ and nonempty; $T(S) \subset S$, $T(S_j) \subset S_j$ for $j = 1,2$. Since $S$, $S_1$ and $S_2$ are convex in $C_e(\bar{\Omega})$, these are retracts of $S$, and clearly $S_1 \cap S_2 = \emptyset$. Let

$$E_1 = S_1 \cap \{u \in C_e(\bar{\Omega}) : u(x) < \hat{v}_1(x), x \in \Omega\}$$

and

$$E_2 = S_2 \cap \{u \in C_e(\bar{\Omega}) : u(x) > \bar{v}_2(x), x \in \Omega\} .$$

We show that $E_1$ and $E_2$ are open in $S$. Let $v \in E_1$. Then $v(x) < \hat{v}_1(x)$ for all $x \in \Omega$, and there is constant $c \geq 0$ such that

$$-ce(x) \leq v(x) \leq ce(x)$$

for all $x \in \bar{\Omega}$. Then we can choose $\beta \geq 0$ so that $\beta \leq c$ and $v(x) + \beta e(x) < \hat{v}_1(x)$ for all $x \in \Omega$. Let $B(v,\beta)$ be the open ball in $S$ with respect to the norm $\|\cdot\|_e$, with center $v$ and radius $\beta$. Then for any $u \in B(v,\beta)$,

$$-\beta e(x) \leq u(x) - v(x) \leq \beta e(x)$$

for all $x \in \bar{\Omega}$. Hence $u(x) \leq \beta e(x) + v(x) < \hat{v}_1(x)$ for all $x \in \Omega$. Hence $u \in E_1$. Therefore, $B(v,\beta) \subset E_1$.

Similarly, we can show that $E_2$ is also open in $S$.

Next, we show that $\text{Fix}(T) \cap (S_j \setminus E_j) = \emptyset$, $j = 1,2$. Suppose that there is $u \in \text{Fix}(T) \cap (S_j \setminus E_j)$ for some $j$. Then $u \in S_j \setminus E_j$ and $Tu = u$.

Let $j = 1$. We note that $u$ is a solution of $(BVP_h)$. Since $u \in S_1 \setminus E_1$, $\bar{v}_1 \leq u \leq \hat{v}_1$ and there is a point $x_0 \in \Omega$ such that $u(x_0) = \hat{v}_1(x_0)$. By Maximum Principles and the definition of a quasisupersolution of $(BVP_h)$, we can show that there is a neighborhood $U_{x_0}$ of $x_0$ such that $u(x) = \hat{v}_1(x)$ for all $x \in U_{x_0} \cap \Omega$. By the continuation of this method on the boundary of the neighborhood $U_{x_0} \cap \Omega$, we can conclude that $u(x) = \hat{v}_1(x)$ for all $x \in \bar{\Omega}$. This implies that $\hat{v}_1$ is a solution of the $(BVP_h)$. This leads to a contradiction because $\hat{v}_1$ is not a solution of $(BVP_h)$.

Similarly, we can prove that $\mathrm{Fix}(T) \cap (S_2 \setminus E_2) = \emptyset$. Therefore, $T$ satisfies all conditions of Theorem 5. So $T$ has at least three distinct fixed points $u_0$, $u_1$, $u_2$ such that $u_j \in [\bar{v}_j, \hat{v}_j]$, $j = 1, 2$, and note that

$$u_0 \in [\bar{v}_1, \hat{v}_2] \setminus [\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2]. \quad \blacksquare$$

To prove the main theorem, we will use the following well known theorem and it can be found in [7]:

**Theorem 6.** *Let $f$ satisfy the condition $(f2)$. For every constants $P > 0$ there exists a constant $Q > 0$ such that: if $u$ is a solution of*

$$\Delta u = f(x, u, \nabla u), \quad x \in \Omega,$$

*$u \in C^2(\bar{\Omega})$, $|u(x)| \leq P$ for all $x \in \bar{\Omega}$, then $|\nabla u(x)| \leq Q$ for all $x \in \bar{\Omega}$. The constant $Q$ only depends on $P$ and the bounding function $c$.*

**Proof of Theorem 1.**

Since we seek solutions of $(BVP_h)$ on the order interval $[\bar{v}_1, \hat{v}_2] \cap C(\bar{\Omega})$, we can choose $Q_0 > 0$ such that if $u$ is a solution of $\Delta u = f(x, u, \nabla u)$, for all $x \in \Omega$ and $\bar{v}_1 \leq u \leq \hat{v}_2$, then $|\nabla u(x)| \leq Q_0$ for all $x \in \bar{\Omega}$. Since $\bar{\Omega}$ is compact, we let

$$\bar{Q}_j = \sup_{x \in \bar{\Omega}} \{\text{any directional derivatives of } \bar{v}_j \text{ at } x\} < \infty$$

and

$$\hat{Q}_j = \sup_{x \in \bar{\Omega}} \{\text{any directional derivatives of } \hat{v}_j \text{ at } x\} < \infty$$

for $j = 1, 2$. Furthermore, let $Q = \max\{Q_0, \bar{Q}_1, \hat{Q}_1, \bar{Q}_2, \hat{Q}_2\}$. Then we choose a bounded smooth function $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $h(\eta) = \eta$ if $|\eta| < Q + 1$ and its differential $dh$ is bounded on $\mathbf{R}^n$. To get the main result, we solve the following boundary value problem

$$\begin{cases} \Delta u = f(x, u, h(\nabla u)), & x \in \Omega \\ Bu(x) = \phi(x), & x \in \partial\Omega. \end{cases}$$

Hence, Proposition implies the proof. ■

*Remark.* The above theorem is valid if we replace $\Delta$ by a uniformly elliptic operator

$$L = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(x) D^{ij} + \sum_{i=1}^{n} A_i(x) D^i + A_0(x),$$

where the coefficients of $L$ and $B$ are smooth.

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# FOURIER SPANNING DIMENSION OF ATTRACTORS FOR THE 2D NAVIER-STOKES EQUATIONS

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## 1. Introduction.

In the construction of inertial manifolds for dissipative partial differential equations, it is usual to construct as an invariant graph from lower-dimensional Fourier modes to higher modes. But, as far as we are concerned in 2D Navier-Stokes equations, the existence of such a graph is still unsolved.

As an attempt to this goal, we here show that the global attractor for the 2D periodic Navier-Stokes equations is actually a part of a graph from low modes to higher modes. In particular, this result implies that the orthogonal projection $P_N$ is injective on the attractor, which was conjected by C. Foias and R. Temam in 1979 (see Foias and Temam (1979) and Foias (1980)).

## 2. Notations and the Main Result.

Suppose $\Omega = (0, L_1) \times (0, L_2)$. We consider the Navier-Stokes equations of viscous incompressible fluid with space periodic boundary condition:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in} \quad \Omega \tag{2.1}$$

$$\nabla \cdot u = 0. \tag{2.2}$$

The unknown functions are $u = (u_1, u_2) = u(x,t), p = p(x,t)$. The volume forces $f = f(x)$ are given and $\nu > 0$ is the kinematic viscosity. We may assume that $f, u$ and $p$ are $\Omega$-periodic. For simplicity, we also assume that the average value of $f, u$ on $\Omega$ is zero:

$$\int_\Omega f(x)dx = 0, \qquad \int_\Omega u(x,t)dx = 0, \qquad \forall t \geq 0. \tag{2.3}$$

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As usual (2.1)-(2.3) are equvalent to an abstract evolution equation for $u$ (see Temam (1988)):

$$\frac{du}{dt} + \nu Au = R(u) \tag{2.4}$$

in a Hilbert space $H$ that is a closed subspace of $L^2(\Omega; \mathbf{R}^2)$. The operator $A$ corresponds to Stokes operator with space periodic boundary condition and it is a linear self-adjoint unbounded positive operator in $H$ with domain $D(A) \subset H$. In fact, if we write

$$V = \{u \in H^1_{per}(\Omega); \text{div } u = 0, \int_\Omega u dx = 0\},$$

then

$$D(A) = H^2_{per}(\Omega; \mathbf{R}^2) \cap V.$$

See Constantin and Foias (1988) and Temam (1983, 1988). Finally $R(u)$ is defined by

$$(R(u), v) = -\int_\Omega ((u \cdot \nabla)v)u dx + \int_\Omega f(x) v dx \quad \text{for} \quad v \in V. \tag{2.5}$$

When an initial-value problem is considered, the equation (2.4) is equipped with

$$u(0) = u_0 \qquad \text{for} \qquad u_0 \in H. \tag{2.6}$$

The existence of the global attractor $\mathcal{A}$ for 2D Navier-Stokes equations is well known, see for example Hale (1988), Temam (1988).

By introducing a nonlinear change of variables

$$J(u) = (u, v, z, w),$$

where

$$\begin{aligned} u &= (u_1, u_2) \\ v &= (v_1, v_2) = \frac{\partial u}{\partial x} \\ z &= (v_3, v_4) = \frac{\partial u}{\partial y} \\ w &= (w_1, w_2, w_3) = u \otimes u \stackrel{\text{def}}{=} (u_1^2, 2u_1u_2, u_2^2), \end{aligned}$$

we can prove that if $u(t)$ is a solution of (2.4), then $(u(t), v(t), z(t), w(t)) = J(u(t))$ is a solution of the following reaction diffusion system;

$$
(2.7)\qquad
\begin{aligned}
u_t &= -\nu Au - \mathcal{P}\begin{bmatrix} w_{1x} + \frac{1}{2}w_{2y} \\ \frac{1}{2}w_{2x} + w_{3y} \end{bmatrix} + f \\
v_t &= -\nu Av - \mathcal{P}\begin{bmatrix} w_{1xx} + \frac{1}{2}w_{2yx} \\ \frac{1}{2}w_{2xx} + w_{3yx} \end{bmatrix} + f_x \\
z_t &= -\nu Az - \mathcal{P}\begin{bmatrix} w_{1xy} + \frac{1}{2}w_{2yy} \\ \frac{1}{2}w_{2xy} + w_{3yy} \end{bmatrix} + f_y \\
w_t &= -\nu Bw - \nu \begin{bmatrix} 2v_1^2 + 2v_3^2 \\ 4v_1v2 + 4v_3v_4 \\ 2v_2^2 + 2v_4^2 \end{bmatrix} + 2u_1 \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix} + 2u_2 \begin{bmatrix} 0 \\ \xi_1 \\ \xi_2 \end{bmatrix},
\end{aligned}
$$

where $\mathcal{P}$ is the Leray projection, $B$ is the operator on $L^2(\Omega; \mathbf{R}^3)$ induced by $-\Delta$ with periodic boundary condition on $\Omega$ and

$$
\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = -\mathcal{P}\begin{bmatrix} u_1v_1 + u_2v_3 \\ u_1v_2 + u_2v_4 \end{bmatrix} + f.
$$

We remark that all of $u, v, z, w$ satisfy periodic boundary condition and in addition, $u, v, z$ satisfy average free and divergence free condition. Conversely we see that if $(u, v, z, w)(t)$ is a solution of (2.7) with initial condition $(u, v, z, w)(0) = J(u_0)$, then from the uniqueness of solution, $u(t)$ is a solution of (2.4). (See Proposition 3.8, Kwak (1991).) In particular, the image of $\mathcal{A}$ under the imbedding $J$ is a bounded invariant set of (2.7).

We are now led to the study of reaction diffusion system (2.7). Let $\mathbf{U} = (u, v, z, w)^t$ and we define an operator $\mathbf{A}$ on the space $D(A) \times D(A) \times D(A) \times D(B)$ by

$$
\mathbf{AU} = \begin{bmatrix} Au + \frac{1}{\nu}\mathcal{P}\begin{bmatrix} w_{1x} + \frac{1}{2}w_{2y} \\ \frac{1}{2}w_{2x} + w_{3y} \end{bmatrix} \\ Av + \frac{1}{\nu}\mathcal{P}\begin{bmatrix} w_{1xx} + \frac{1}{2}w_{2yx} \\ \frac{1}{2}w_{2xx} + w_{3yx} \end{bmatrix} \\ Az + \frac{1}{\nu}\mathcal{P}\begin{bmatrix} w_{1xy} + \frac{1}{2}w_{2yy} \\ \frac{1}{2}w_{2xy} + w_{3yy} \end{bmatrix} \\ Bw \end{bmatrix}.
$$

We also define

$$\tilde{\mathbf{F}} = \begin{bmatrix} f \\ f_x \\ f_y \\ -\nu \begin{bmatrix} 2v_1^2 + 2v_3^2 \\ 4v_1v_2 + 4v_3v_4 \\ 2v_2^2 + 2v_4^2 \end{bmatrix} + 2u_1 \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix} + 2u_2 \begin{bmatrix} 0 \\ \xi_1 \\ \xi_2 \end{bmatrix} \end{bmatrix}.$$

Then the equation (2.7) can be rewritten in the form

$$\frac{d\mathbf{U}}{dt} = -\nu \mathbf{A}\mathbf{U} + \tilde{\mathbf{F}}(\mathbf{U}). \tag{2.8}$$

The operator $\mathbf{A}$ is not self-adjoint but it is sectorial, i.e., $-\mathbf{A}$ generates an analytic semigroup on $\mathbf{H} = H \times H \times H \times L^2(\Omega; \mathbf{R}^3)$ as shown in Proposition 3.6, Kwak (1991). (See also Henry (1981).)

In general, the nonlinearity $\tilde{\mathbf{F}}(\mathbf{U})$ may not admit the dissip ative dynamical system, but since we are interested in the dynamics near the invariant set $J(\mathcal{A})$, we truncate nonlinear term and consider a modified equation which provides the same long-time dynamics near $J(\mathcal{A})$ but different for large norm in $D(\mathbf{A})$.

Let us assume $|\mathbf{A}\mathbf{U}| \leq \frac{\rho_1}{2}$ for $\mathbf{U} \in J(\mathcal{A})$ and the modified equation of (2.8) is

$$\frac{d\mathbf{U}}{dt} = -\nu \mathbf{A}\mathbf{U} + \mathbf{F}(\mathbf{U}), \tag{2.9}$$

where

$$\mathbf{F}(\mathbf{U}) = \phi\left(\frac{|\mathbf{A}\mathbf{U}|^2}{\rho_1^2}\right)\tilde{\mathbf{F}}(\mathbf{U}),$$

and $\phi : \mathbf{R}_+ \to [0,1]$ is a smooth monotone function such that $\phi(s) = 1$ for $0 \leq s \leq 1$, $\phi(s) = 0$ for $s \geq 2$, and $|\phi'(s)| \leq 2$.

We now note that $\mathbf{A}$ has compact resolvent and its spectrum $\sigma(\mathbf{A})$ consists of countable number of eigenvalues with no finite accumulation points and each with finite multiplicity and finite positive index. In fact

$$\sigma(\mathbf{A}) = \{\mu_{m,n}; \mu_{m,n} = 4\pi^2\left(\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2}\right), m \geq 0, n \geq 0\}.$$

Let $\sigma(\mathbf{A}) = \{\lambda_n; \lambda_n < \lambda_{n+1}, n = 0, 1, 2 \cdots\}$ and let $P$ be the projection associated with the first $(N+1)$ eigenvalues of $\mathbf{A}$ and $Q = I - P$. Then we easily see that

$$P(\mathbf{H}) \subset D(\mathbf{A}), \quad \mathbf{H} = P(\mathbf{H}) \oplus Q(\mathbf{H}) \tag{2.10}$$

and $P(\mathbf{H}), Q(\mathbf{H})$ is invariant under $\mathbf{A}$. We denote the projection of $H$ associated with the first $N$ eigenvalues of the stokes operator $A$ by $P_N$ and $Q_N = I - P_N$.

The inertial manifold for the equation (2.9) is constructed as a graph from $P(\mathbf{H})$ to $Q(\mathbf{H}) \cap D(\mathbf{A})$.

**Proposition 2.1.** *If $\mathcal{A}$ is the global attractor of (2.4), then $J(\mathcal{A})$ is contained in the global attractor of (2.9). Moreover if we assume that $(\frac{L_1}{L_2})^2$ is a rational number, then for any constant $\nu, K_1, l$ we can find $N \geq 1$ so that there exists an inertial manifold $\mathcal{M}$ = Graph $\Phi$ for the equation (2.9), where $\Phi$ is a mapping from $P(\mathbf{H})$ to $Q(\mathbf{H}) \cap D(\mathbf{A})$ such that*

(1) *$\Phi$ has a bounded support in $P(\mathbf{H})$;*
(2) $|\mathbf{A}\Phi(p)| \leq \frac{K_1}{\lambda_{N+1}}$ *for all* $p \in P\mathbf{H}$;
(3) $|\mathbf{A}\Phi(p_1) - \mathbf{A}\Phi(p_2)| \leq l|\mathbf{A}p_1 - \mathbf{A}p_2|$ *for all* $p_1, p_2 \in P(\mathbf{H})$.

*Proof.* We refer to Theorem 4.8, Foias, Sell, and Temam (1988) and Proposition 5.6, Kwak (1991) and Theorem 3.7, Sell and You (1990) for the complete proof. □

The existence of an absorbing ball of (2.4),

$$\mathcal{B} = \{u \in D(A^{\frac{3}{2}}); |A^{\frac{3}{2}}u| \leq \rho_2\},$$

for some $\rho_2 > 0$, is proved in Temam (1983). Without loss of generality, we may assume that

$$|\mathbf{AU}| \leq \rho_1 \qquad \text{for} \qquad \mathbf{U} \in J(\mathcal{B}).$$

We notice that $J(\mathcal{B})$ is invariant under the flow of (2.9) and since $\mathcal{M}$ itself is invariant,

$$S = \mathcal{M} \cap J(\mathcal{B})$$

is invariant under the flow of (2.9). In particular $J(\mathcal{A}) \subset S$ and $\mathcal{A} \subset J^{-1}(S)$.

Now we can state our main result.

**Theorem 2.2.** *Under the hypothesis stated above, there exists $N \geq 1$ such that $J^{-1}(S) =$ Graph $\Theta$, where $\Theta$ is a Lipschtz mapping from a bounded subset of $P_N H$ to $Q_N H \cap D(A)$. In particular the orthogonal projection $P_N$ is injective on $J^{-1}(S)$.*

## 3. Proof of Theorem 2.2.

We first note that there exists $\rho_3 > 0$ such that

$$|u|_{L^\infty} \leq \rho_3 \quad \text{for} \quad u \in \mathcal{B}, \tag{3.1}$$

thanks to the imbedding $D(A^{\frac{3}{2}}) \subset L^\infty(\Omega; \mathbf{R}^2)$. Moreover if $u \in J^{-1}(S)$, then $J(u) \in S$. Let $J(u) = \mathbf{U} = (u, v, z, w)$, then

$$u = p_1 + q_1, \quad v = p_2 + q_2, \quad z = p_3 + q_3, \quad w = p_4 + q_4$$

and since $J(u) \in \mathcal{M}$,

$$q_i = \Phi_i(p_1, p_2, p_3, p_4), \quad 1 \leq i \leq 4, \tag{3.2}$$

where $P\mathbf{U} = (p_1, p_2, p_3, p_4)$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$.

Now since $v = u_x, z = u_y$, we easily see that $p_2 = p_{1x}, p_3 = p_{1y}$, i.e., two components $\{p_2, p_3\}$ are uniquely determined by $p_1$. All we have to do is to show that given $p_1$, $p_4$ is also uniquely determined.

It is clear that $p_4 = P_B w$, where $P_B$ is the projection associated with the first (N+1) eigenvalues of $B$, and since $w = u \otimes u$ on $S$, (3.2) implies

$$q_1 = \Phi_1(p_1, p_{1x}, p_{1y}, P_B(p_1 + q_1) \otimes (p_1 + q_1)) \tag{3.3}$$

or

$$q_1 = \psi(p_1, q_1). \tag{3.4}$$

We claim that for $p_1$ in $P_N J^{-1}(S)$,

$$q_1 \to \psi(p_1, q_1) \text{is a strict contraction in a ball in} \quad H \tag{3.5}$$

so that (3.4) can be uniquely resolved into

$$q_1 = \Theta(p_1). \tag{3.6}$$

The proof of (3.5) is as follows. Let $q_1, \tilde{q_1} \in Q_N H$ and let $u = p_1 + q_1, \tilde{u} = p_1 + \tilde{q_1}, \tilde{w} = \tilde{u} \otimes \tilde{u}$ and $\tilde{p_4} = \tilde{w}$. Then

$$
\begin{aligned}
(3.7) \qquad & |\psi(p_1, q_1) - \psi(p_1, \tilde{q_1})|_{L^2} \\
&= |\Phi_1(p_1, p_{1x}, p_{1y}, p_4) - \Phi_1(p_1, p_{1x}, p_{1y}, \tilde{p_4})|_{L^2} \\
&\leq \frac{1}{\lambda_{N+1}} |A\Phi_1(p_1, p_{1x}, p_{1y}, p_4) - A\Phi_1(p_1, p_{1x}, p_{1y}, \tilde{p_4})|_{L^2} \\
&\leq \frac{Cl}{\lambda_{N+1}} |A(p_4 - \tilde{p_4})|_{L^2} \qquad \text{(from Proposition 2.1)} \\
&\leq \frac{Cl\lambda_N}{\lambda_{N+1}} |p_4 - \tilde{p_4}|_{L^2} \\
&\leq CL|w - \tilde{w}|_{L^2}.
\end{aligned}
$$

Moreover

$$w - \tilde{w} = u \otimes u - \tilde{u} \otimes \tilde{u} = (u - \tilde{u}) \otimes u + \tilde{u} \otimes (u - \tilde{u})$$

and

$$
\begin{aligned}
|(u - \tilde{u}) \otimes u| &\leq \sqrt{\frac{3}{2}} |u|_{L^\infty} |u - \tilde{u}|_{L^2} \\
|\tilde{u} \otimes (u - \tilde{u})| &\leq \sqrt{\frac{3}{2}} |\tilde{u}|_{L^\infty} |u - \tilde{u}|_{L^2}.
\end{aligned}
$$

Thus we obtain from (3.1) that

$$(3.8) \qquad |w - \tilde{w}|_{L^2} \leq \sqrt{6}\rho_3 |u - \tilde{u}|_{L^2}.$$

From (3.7) and (3.8), we finally have

$$|\psi(p_1, q_1) - \psi(p_1, \tilde{q_1}|_{L^2} \leq Cl\sqrt{6}\rho_3 |q_1 - \tilde{q_1}|_{L^2}.$$

Hence (3.5) holds provided $Cl\sqrt{6}\rho_3 < 1$. This completes all the proof.

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# SINGULAR SOLUTIONS TO A NONLINEAR PSEUDOFFIFFERENTIAL EQUATION ARISING IN FLUID DYNAMICS

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1. The lecture is devoted to some mathematical problems arising in the study of inviscid incompressible flows with concentrated vorticity. Such flows are potential in the compliment to a small (with respect to the volume) domain where the vorticity is considerable, so its integral is not small. As the mathematical model of such flow one can consider solutions to (nonlinear) equations of hydrodynamics with vorticity function being a distribution. The support of this distribution is a finite set or a line. The problems of this type arise in some applications and there are quite a lot of results obtained by means of computation, but there are not many rigorous results. In the lecture we shall concentrate our attention on mathematical statements of these problems, some of them contain open questions.

2. We shall study below the two-dimensional flows of an ideal fluid. These flows are described by two complonents of velocity $u(x) = \{u_1(x),\ u_2(x)\}$, $x \in \mathbb{R}^2$ and the pressure $p(x)$. These functions satisfy the system of Euler equations

$$\frac{\partial u}{\partial t} + u_1 \frac{\partial u}{\partial x_1} + u_2 \frac{\partial u}{\partial x_2} - grad\ p = 0,$$

and the conservation of mass equation

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0.$$

If we consider $u(x)$ as a vector in $\mathbb{R}^3$, then $rotu(x)$ has only one nonzero component $\omega = \partial u_2/\partial x_1 - \partial u_2/\partial x_2$ called the vorticity function, or shortly, the vorticity. Applying rot operation to the Euler equations we obtain the evolution of vorticity equation

$$\frac{\partial \omega}{\partial t} + u_1 \frac{\partial \omega}{\partial x_1} + u_2 \frac{\partial \omega}{\partial x_2} = 0.$$

For a flow in a simply-connected domain, say $\mathbb{R}^2$, we can introduce the stream fuction $\psi(x)$, so that $u_1 = \partial\psi/\partial x_2$, $u_2 = -\partial\psi/\partial x_1$ and the equation above can be rewritten in the form

$$\frac{\partial \omega}{\partial t} + \{\psi, \omega\} = 0, \ \{\psi, \omega\} = \frac{\partial \psi}{\partial x_2}\,\frac{\partial \omega}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\,\frac{\partial \omega}{\partial x_2}. \tag{1}$$

Inserting the stream function in the mass conservation equation we obtain

$$-\Delta\psi = \omega. \tag{2}$$

The solution $\psi$ to Poisson equation (2) has the form

$$\psi(x) = (G * \omega)(x) = \int_{\mathbb{R}^2} G(x-y)\omega(y)\,dy, \quad G(x) = \frac{1}{2\pi} ln|x|. \tag{3}$$

Inserting from (3) into (1), we obtain a nonlinear pseudodifferential equation for the vorticity function $\omega$:

$$\frac{\partial w}{\partial t} + \{G * \omega, \omega\} = 0. \tag{4}$$

We shall treat the Cauchy problem for this equation, ppreseribing a initial distribution of the vorticity

$$\omega(x, o) = \omega_o(x). \tag{5}$$

For the problem (4), (5) there exist unique solvability theorems in the classes of smooth bounded functions (T. Kato) and in Sobolev spaces (V. Youdovich)

*Remark.* If $\omega \equiv 0$, then $\psi$ is a harmonic function, the conjugate harmonic function $\varphi$ is called the potential of the flow so that $u_1 = \partial\varphi/\partial x_1$, $u_2 = \partial\varphi/\partial x_2$. Such flows are called potential. If $\omega = 0$ in some region $D \subset \mathbb{R}^2$, then we say the flow is potential in $D$.

We are interested in the case of $\omega_\circ(x)$ being a distribution of one of the following forms:

$$\omega_\circ(x) = \sum \gamma_j \delta(x - x^{(j)}), \tag{6}$$

$\delta(x)$ is the Dirac function, $x^{(1)}, \ldots, x^{(J)}$ are some points on the plane,

$$\omega_\circ(x) = \int \gamma(s)\delta(x - x(s))ds, \tag{7}$$

where $x(s) = \{x_1(s), x_2(s)\}, x \in \mathbb{R}^2$, is a smooth curve on the plane,

$$w_\circ(x) = \gamma\delta(x) + a(x), \tag{8}$$

where $a(x)$ is a smooth function.

The initial condition (6) corresponds to the case of a finite number of isolated vortexes. This problem, in fact was investigated in the classical works of Kirchhoff. Condition (7) is a continual analog of (6), such conditions arise when we treat the flows with tangential discontinuities of velocities (see below). Condition (8) corresponds to the problems of interaction of an isolated vortex with the distributed vorticity. Such problems arise in investigation of a hurricane moving over the ocean surface. Unfortunately, the lack of space do not permit us to dwell on this problem.

3. As equation (4) has no mechanism of dissipation, the smoothness of the solutions of the problem (4), (5) do not improve in the course of evolution. So when the initial condition is of the form (6)-(8), the solution must be a distribution for $t > 0$. But in this case we have to explain in what sense our distribution is a solution to the nonlinear equation (4).

A distribution $u \in \mathcal{D}'$ is called a solution of the nonlinear equation L(u)=0, if there exists a sequence $u_\varepsilon$ of smooth functions such that

$$u_\varepsilon \to u \quad \text{for } \varepsilon \to 0, \tag{i}$$

$$L(u_\varepsilon) \to 0 \quad \text{for } \varepsilon \to 0, \tag{ii}$$

where the convergence is understood in the sense of the theory of distributions.

In the nonstationary case we suppose that the distribution $u(x,t)$ depends smoothly on the parameter $t$ and

(iii) $$u_\varepsilon(x,0) \to u_o(x), \ \varepsilon \to 0,$$

where $u_0(x)$ is the initial distribution.

4. On the heuristic level the problem (4), (6) must have the "exact" solution of the form

(9) $$\omega(x,t) = \sum_{j=1}^{j} \gamma_j \delta(x - x^{(j)}(t)),$$

where $x^{(j)}(t) = \left\{x_1^{(j)}(t), x_2^{(j)}(t)\right\}, j = 1, \dots, J$ are solutions of the Cauchy problem for Kirchhoff's system of ordinary differential equations.

(10)
$$\frac{dx_1^{(j)}}{dt} = -\sum_{k\neq j} \gamma_k \frac{\partial G}{\partial x_2^j}\left(x^{(j)} - x^{(k)}\right); \quad \frac{dx_2^{(j)}}{dt} = \sum_{k\neq j} \gamma_k \frac{\partial G}{\partial x_1^{(j)}}\left(x^{(j)} - x^{(k)}\right)$$

with initial conditions:

(11) $$x^{(j)}(0) = x^{(j)}, j = 1, \dots, J.$$

Introducing the Hamiltonian

$$H\left(x^{(1)}, \dots, x^{(j)}\right) = \frac{1}{2} \sum\sum_{\gamma\neq k} \gamma_j \gamma_k G\left(x^{(j)} - x^{(k)}\right)$$

we can rewrite the system (10) in the form

$$\gamma_j \frac{dx_1^{(j)}}{dt} = -\frac{\partial H}{\partial x_2^{(j)}}, \quad \gamma_j \frac{dx_2^{(j)}}{dt} = \frac{\partial H}{\partial x_1^{(j)}}.$$

The distribution defined by (9), (10), (11) is a solution of the problem (4), (6) in the sense of the previous section. To check it we introduce a "cut off" function $\chi(s) \in C^\infty, \chi(s) \geq 0, \chi(s) = 1$ for $s \geq 1$ and $\chi(s) = 0$ for $s \leq 1/2$.

We pose $G_\varepsilon(x) = G(x)\chi(|x|/\varepsilon)$. Then the approximate solution $\omega_\varepsilon$ of (4), (6) can be defined by the relation

$$\omega_\varepsilon(x,t) = \sum \gamma_j \Delta G_\varepsilon(x - x^j(t)). \tag{9'}$$

It is easy to verify that $G_\varepsilon(x)$ and $\Delta G_\varepsilon(x)$ weakly converge to $G(x)$ and $\delta(x)$ respectively, condition (ii) for (9′) can also be verified.

5. As the initial condition (7) is a continual analong of (6) it is reasonable to seek solution of (4), (7) in the form

$$\omega(x,t) = \int \gamma(s)\delta(x - x(s,t))ds. \tag{12}$$

where $x(t,s)$ is a solution of the system of integral differential equations obtained from the system (10) when the number 2J of equations tends to infinity.

$$\frac{\partial x_1}{\partial t} = -\int \frac{x_2(s,t) - x_2(\sigma,t)}{|x(s,t) - x(\sigma,t)|^2}\,\gamma(\sigma)d\sigma, \tag{13}$$

$$\frac{\partial x_2}{\partial t} = \int \frac{x_1(s,t) - x_1(\sigma,t)}{|x(s,t) - x(\sigma,t)|^2}\,\gamma(\sigma)d\sigma. \tag{13'}$$

These equations are complimented by the Cauchy data

$$x(s,0) = x(s).$$

The integrals in (13), (13′) are understood in the sense of the Cauchy principal value.

To justify this solution we consider like in the case of isolated vortexes an approximate solution

$$\omega_\varepsilon(x,t) = \int \gamma(s)\Delta G_\varepsilon(x - x(s,t))ds.$$

The verification of condition (ii) in this case is less straightforward than above, but it can be done along the same lines. The system (13), (13′) can be treated as an infinite dimensional Hamiltonian system, if for the Hamiltonian we take the functional

$$H(x_1(s,t), x_2(s,t)) = (4\pi)^{-1} \iint \gamma(s)\gamma(\sigma)\ ln\ |x(s,t) - x(\sigma,t)|\, ds\, d\sigma.$$

Then we can rewrite (13), (13′) in the form

$$\gamma(s)\ \frac{\partial x_1}{\partial t} = -\frac{\delta H}{\delta x_2}, \quad \gamma(s)\ \frac{\partial x_2}{\partial t} = \frac{\delta H}{\partial x_1},$$

where $\delta \mathrm{H}/\delta x_j, j = 1,2$ denotes the variational derivative of H.

6. If we introduce the complex function $z(s,t) = x_1(s,t)_+ + ix_2(s,t)$, then the system (13), (13′) in the terms of this function can be rewritten as a single complex equation

$$\frac{\partial \overline{z}(s,t)}{\partial t} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\gamma(\sigma)d\sigma}{z(s,t) - z(\sigma,t)} = 0. \tag{14}$$

This equation was introduced by G. Birkhoff as the equation discribing the evolution of the tangential discontinuity. The mechanical problem can be formulated in the following way. We consider a flow which is potential in the complement of a curve $z = z(s,t)$, and on this curve the free boundary conditions are satisfied. The last means that on both sides of the curve the projections to the normal to this curve of the velocities of the fluid and the velocity of the curve itself coincide, and the limiting values of the pressure (or Cauchy-Lagrange integrals) also coincide. If we denot by $\varphi(x,t)$ the potential of the flow and represent this function in the form

$$\varphi(x,t) = Re\ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma(s)\ ln\ (z - z(s,t))ds,$$

then (14) will be the equation of the unknown interface $z = z(s,t)$. Moreover on the curve the relation

$$\frac{\partial \varphi^+}{\partial \tau} - \frac{\partial \varphi^-}{\partial \tau} = \frac{\gamma(s)}{|z_s(s,t)|}$$

holds. Here $\tau$ is the tangent to the interface. From this relation follows that the vorticity density $\gamma(s)$ is determined uniquely by jumps of the tangential velocity on the interface.

7. Let us consider a flow with constant velocity $(U, O)$ for $x_2 > 0$ and $(-\,-U, O)$ for $x_2 < 0$. Then there exists a stationary solution of equation (14):

$$z(s,t) = s, \quad \gamma(s) = 2U, \tag{15}$$

corresponding to this flow. The problem of stability of this flow is a classical problem investigated by Kelvin and Helmholtz.

If we replace in (15) $z$ by $z + \varepsilon w$, differentiate with respect to $\varepsilon$ and pose $\varepsilon = 0$, we obtain the linear equation for small perturbations

$$\frac{\partial \overline{w}}{\partial t} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(s,t) - w(\sigma,t)}{(z(s,t) - z(\sigma,t))^2}\,\gamma(\sigma)\,d\sigma = 0. \tag{16}$$

It can be shown that the second term in the left hand side is a pseudodiferential operator with the symbol

$$\frac{\gamma(s)}{z^2(s,t)}|\xi| + \left(\frac{\gamma'(s)}{z_s^2(s,t)} - \frac{\gamma(s)z_{ss}(s,t)}{z_s^3(s,t)}\right)\ sgn\ \xi + o(1), \ \ |\xi| \to \infty.$$

Writing the system (16) in the real form we obtain the linear elliptic system of pseudodifferential equations. From this it follows that, in general, the Cauchy problem for this system is well-posed only in the spaces of analytic functions. The same is, probably, true for the original nonlinear equation (14).

If we consider small perturbations of the solution (15), we obtain the system with constant coefficients:

$$\frac{\partial u}{\partial t} + U|D|v = 0, \quad \frac{\partial v}{\partial t} + U|D|u = 0, \tag{17}$$

here $|D| = \left(-\partial^2/\partial s^2\right)^{1/2} = H\partial/\partial s$, where $H$ is the Hilbert transform. In the periodical case the solution of the Cauchy problem can be easily written down in the form of $u + iv = \sum c_k(t)\exp(iks)$, and from the explicit formula for the amplitudes $c_k(\mathrm{t})$ it follows, that they grow as $\exp|Uk|t$. In other words the exponential instability takes place. This instability is the cause of considerable difficulties arising in numerical analysis of flows with tangential discontinuities. One can overcome these difficulties by adding the regularizing terms in the equation. In particular, it is possible to add in (14) the term arising from surface tension on the interface. Then the equation will have the form

$$\frac{\partial \bar{z}}{\partial t} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\gamma(\sigma)d\sigma}{z(s,t) - z(\sigma,t)} + \frac{\alpha}{\gamma(s)}\frac{\partial}{\partial s}\left(\frac{\bar{z}_s}{|z_s|}\right) = 0. \tag{18}$$

If we consider the linearization of (18) on the solution (15), we obtain the system of the form

$$\frac{\partial u}{\partial t} + U|D|v + B\,\frac{\partial^2 v}{\partial s^2} = 0, \quad \frac{\partial v}{\partial t} + U|D|u = 0 \tag{19}$$

The symbol of this is of the form:

$$A = \left\|\begin{matrix} i\tau & U|\xi| - B\xi^2 \\ U|\xi| & i\tau \end{matrix}\right\|, \quad -detA = \tau^2 + U^2\xi^2 - UB|\xi|^3.$$

If $UB > 0$, then the Petrovskii condition for the symbol $det\, A(\tau, \xi)$ is satisfied and the Cauchy problem for this system is well-posed in the spaces of functions of finite smoothness (i.e. is well-posed in Hadamard's sense). Numerous calculations made by the author showed the usefulness of this regularization. If we linearize (18) on an arbitrary solution $z(s,t)$, we obtain the system of the form (19), but generally with variable coefficients. It is interesting to investigate whether the Cauchy problem is well posed for this system in the sense of Hadamard.

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# ON THE RELATION BETWEEN SOLVABILITY AND A RESTRICTED HYPOELLIPTICITY OF CONVOLUTION EQUATION

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In this paper we survey the equivalence between the solvability and the restricted hypoellipticity of convolution equations in various distribution spaces.

In the space $\mathcal{D}'$ of distributions on $\mathbf{R}^n$, L. Ehrenpreis[5] proves that, for $S$ in the space $\mathcal{E}'$ of distributions having compact support, the followings are equivalent;

(a) There is positive constants $A, C$ such that

$$\sup_{\substack{|x-\xi|<A\log(1+|\xi|)\\ x\in\mathbf{R}^n}} |\hat{S}(x)| \geq C(1+|\xi|)^{-A}, \qquad \xi\in\mathbf{R}^n$$

(b) $S * \mathcal{D}' = \mathcal{D}'$

(c$'$) Every entire function $G$ satisfying $\hat{S}G \in \hat{\mathcal{D}}$ belongs to $\hat{\mathcal{D}}$

In fact, the condition (c$'$) can be replaced by the following condition, which we will see;

(c) Every $u$ in $\mathcal{E}'$ satisfying $S * u \in \mathcal{D}$ belongs to $\mathcal{D}$

Moreover, S. Sznajder and Z. Zielezny[14] show the similar results in the space $\mathcal{K}_1'$ as follows : for $S$ in the space $\mathcal{O}_c'(\mathcal{K}_1', \mathcal{K}_1')$ of convolution operators in $\mathcal{K}_1'$, the followings are equivalent;

(a)$_1$ There exist positive constants $N, r, C$ such that

$$\sup_{z\in\mathbf{C}^n,\,|z|\leq r} |\hat{S}(\xi+z)| \geq C(1+|\xi|)^{-N}, \quad \xi\in\mathbf{R}^n$$

(b)$_1$ $S * \mathcal{K}_1' = \mathcal{K}_1'$

(c)$_1$ If $u \in \mathcal{O}_c'(\mathcal{K}_1', \mathcal{K}_1')$ and $S * u \in \mathcal{K}_1$, then $u \in \mathcal{K}_1$

The condition(c) in $\mathcal{D}'$ is just the hypoellipticity of $S$ in the space $\mathcal{E}'$. This is the reason why we call this kind of condition as a restricted hypoellipticity.

We now state our first theorem, which include part of the Ehrenpreis' results as a special case generated by $\omega(\xi) = \log(1 + |\xi|)$.

**Theorem 1.** For a convolution operator $S$ in $\mathcal{E}'_\omega$, the following statements are equivalent;

$(\mathrm{a})_\omega$ There exist positive constants $A, C$ such that

$$\sup_{\substack{|x-\xi|\leq A\omega(\xi)\\ x\in\mathbf{R}^n}} |\hat{S}(x)| \geq Ce^{-A\omega(\xi)}, \quad \xi \in \mathbf{R}^n$$

$(\mathrm{b})_\omega$ $S * \mathcal{D}'_\omega = \mathcal{D}'_\omega$
$(\mathrm{c})_\omega$ If $u \in \mathcal{E}'_\omega$ and $S * u \in \mathcal{D}_\omega$, then $u \in \mathcal{D}_\omega$.

The equivalence of the conditions $(\mathrm{a})_\omega$ and $(\mathrm{b})_\omega$ is proved by S. Abdullah in [1]. The space $\mathcal{D}'_\omega$ is the space of Beurling's generalized distributions on $\mathbf{R}^n$ generated by the weight function $\omega(\xi)$ which will be explained later. When $\omega(\xi) = \log(1 + |\xi|)$, $\mathcal{D}'_\omega$ is the space of classical distributions on $\mathbf{R}^n$ and our result is the same as the Ehrenpreis' first three, but our proof is different from his.

Before presenting the idea of the proof we briefly introduce the generalized distribution spaces and their properties which we need in this paper. We denote by $\mathcal{M}_c$ the set of all continuous real valued functions $\omega$ on $\mathbf{R}^n$ satisfying the following conditions;

($\alpha$) $0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \quad \xi, \eta \in \mathbf{R}^n$
($\beta$) $\int_{\mathbf{R}^n} \omega(\xi)(1 + |\xi|)^{-(n+1)} d\xi < \infty$
($\gamma$) $\omega(\xi) \geq a + b\log(1 + |\xi|)$ for some constants $a$ and $b > 0$
($\delta$) $\omega(\xi) = \sigma(|\xi|)$ for an increasing concave function $\sigma$ on $[0, \infty)$

Throughout this paper $\omega$ represents an element in $\mathcal{M}_c$. Let $\mathcal{D}_\omega$ be the space of $\phi$ in $L^1(\mathbf{R}^n)$ which has compact support, equipped with the inductive limit topology of Frechét spaces $\mathcal{D}_\omega(K)$ induced by semi-norms

$$\|\phi\|_\lambda^{(\omega)} = \int_{\mathbf{R}^n} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} < \infty \text{ for every } \lambda > 0$$

where $K$'s are compact in $\mathbf{R}^n$. We denote by $\mathcal{E}_\omega$ the space of complex valued functions $\psi$ on $\mathbf{R}^n$,equipped with the topology induced by semi-norms $\|\phi\psi\|_\lambda$ for every $\phi \in \mathcal{D}_\omega$ and $\lambda > 0$. The dual space $\mathcal{D}'_\omega$ is denoted by the space of continuous linear functionals on $\mathcal{D}_\omega$, whose elements are called by the generalized distributions on $\mathbf{R}^n$ and $\mathcal{E}'_\omega$ the set of generalized distributions whose support are compact in $\mathbf{R}^n$. According to this definition $\mathcal{D}_\omega$ is $\mathcal{D}$ when $\omega(\xi) = \log(1 + |\xi|)$ and $\mathcal{E}'_\omega * \mathcal{D}'_\omega \subset \mathcal{D}'_\omega$ which is defined by $\langle S * u, \phi\rangle = \langle u, \check{S} * \phi\rangle$for $u \in \mathcal{D}'_\omega$ and $\phi \in \mathcal{D}_\omega$. For the further details we refer to [3].

*Sketch of the Proof of Theorem 1.* $(\mathrm{b})_\omega \Rightarrow (\mathrm{c})_\omega$ : If $T = \check{S} \in \mathcal{E}'_\omega$, then $S* : \mathcal{D}'_\omega \to \mathcal{D}'_\omega$ is the transpose of $T* : \mathcal{D}_\omega \to \mathcal{D}_\omega$ and so the condition (b) is the same that $T*$ is an isomorphism of $\mathcal{D}_\omega$ onto $T * \mathcal{D}_\omega$. In particular, the inverse mapping $T * \psi \to \psi$ must be continuous.

Suppose now that $S * u = \phi$, equivalently $T * \check{u} = (-1)^n\check{\phi}$. Applying the approximation identity $\{\psi_k\}$, the sequence $T * (\check{u} * \psi_k) = (-1)^n\check{\phi} * \psi_k$ converges to $(-1)^n\check{\phi}$ in $\mathcal{D}_\omega$ and, from the continuity of the inverse mapping $\check{u} * \psi_k$ has to converge in $\mathcal{D}_\omega$. But this limit has to be $\check{u}$ in $\mathcal{D}_\omega$ because $\check{u} * \psi_k \to \check{u}$ in $\mathcal{D}'_\omega$.

$(\mathrm{c})_\omega \Rightarrow (\mathrm{a})_\omega$ : Let $K = \operatorname{supp}(S) + \overline{B(0,1)}$ and let $\mathcal{F}$ be the space of all continuous functions $u$ on $K$ such that $S * u \in \mathcal{D}_\omega(K)$. Then $\mathcal{F}$ is Frechét space equipped with the semi-norms

$$\|u\|_k = \|u\|_\infty + \|S * u\|_k^{(\omega)}, \quad k = 1, 2, \ldots .$$

Furthermore, let $\mathcal{G}$ be the space of all functions $u \in C^1(K^0) \cap C(K)$ such that

$$\|u\| = \sup_{\substack{x\in K\\ |\alpha|\le 1}} |D^\alpha u(x)| < \infty$$

and $\mathcal{G}$ is a Banach space under this norm.

Assumption $(\mathrm{c})_\omega$ implies that $\mathcal{F} \subset \mathcal{G}$ and the natural mapping is closed and so continuous. Consequently, there exist an integer $m$ and a constant $C_1 > 0$ such that

$$\|u\| \le C_1 \left(\|u\|_\infty + \|S * u\|_k^{(\omega)}\right)$$

for all $u \in \mathcal{F}$, which gives

$$(*) \qquad \begin{aligned} \|u\| - C_1\|u\|_\infty &\le C_1\|S * u\|_k^{(\omega)} \\ &\le C_2 \sup_{\mathbf{R}^n} e^{(k+\frac{n}{b})\omega(\xi)}|\hat{u}(\xi)|. \end{aligned}$$

Suppose now the condition (a) is not satisfied. Then there is a sequence $\{\xi_j\}$ such that $|\xi_j| \to \infty$ and

$$\sup_{|x-\xi_j|\le j\omega(\xi_j)} |\hat{S}(x)| < e^{-j\omega(\xi_j)}$$

Choose $\phi \in \mathcal{D}_\omega$ such that $\phi \ge 0$, $\operatorname{supp}\phi \subset \overline{B(0,1)}$ and $\hat{\phi}(0) = 1$. Let $k_j =$ the Gaussian integer of $\omega(\xi_j)$ and define

$$\phi_j(x) = e^{i\langle x,\xi_j\rangle} \underbrace{\phi_{k_j} * \cdots * \phi_{k_j}}_{k_j\text{-times}}$$

where $\phi_{k_j}(x) = k_j^n \phi(k_j x)$. Substituting $\{\phi_j\}$ into $(*)$ and estimating both sides, we arrive a contradiction. The detailed proof will appear in a different paper.

**Corollary 2.** Every hypoelliptic convolution operator in $\mathcal{D}'_\omega$ is solvable.

We now consider the same problem in the space $\mathcal{K}'_M$ of distributions on $\mathbf{R}^n$ which grow no faster than $e^{M(kx)}$ for some $k > 0$.

**Theorem 3.** Let $S$ be in the space $\mathcal{O}'_c(\mathcal{K}'_M, \mathcal{K}'_M)$ of convolution operators in $\mathcal{K}'_M$ and $\hat{S}$ be its Fourier-Laplace transform of $S$. Then the following statements are equivalent;

$(\mathrm{a})_M$ There exist positive constants $A, C$ and a positive integer $N$ such that

$$\sup_{\substack{|z-\xi|\le A\Omega^{-1}[\log(2+|\xi|)] \\ z\in\mathbf{C}^n}} |\hat{S}(z)| \ge C(1+|\xi|)^{-N}, \quad \xi \in \mathbf{R}^n$$

$(\mathrm{b})_M$ $S * \mathcal{K}'_M = \mathcal{K}'_M$

$(\mathrm{c})_M$ If $u \in \mathcal{O}'_c(\mathcal{K}'_M, \mathcal{K}'_M)$ and $S * u \in \mathcal{K}_M$, then $u \in \mathcal{K}_M$.

The equivalence of (a) and (b) is also proved by S. Sznajer and Z. Zielezny[13] for $M(x) = |x|^p$ and then S. Abdullah[2] for general $M$. Before discussing the idea of the proof we briefly introduce the spaces and their properties which appear in our theorem.

Let $\mu(\xi)$ $(0 \le \xi \le \infty)$ be a continuous function on $\mathbf{R}^+$ such that $\mu(0) = 0$, $\mu(\infty) = \infty$. Then we define

$$M(x) = \begin{cases} \int_0^x \mu(\xi)\, d\xi, & x \ge 0 \\ M(x) = M(-x), & x < 0 \end{cases}$$

and on $\mathbf{R}^n$, $M(x)$ is defined by $M(x_1) + \cdots + M(x_n)$.

The space $\mathcal{K}_M$ of all $C^\infty$-functions $\phi$ in $\mathbf{R}^n$ such that

$$\nu_k(\phi) = \sup_{\substack{x \in \mathbf{R}^n \\ |\alpha| \leq k}} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \quad k = 0, 1, 2, \ldots$$

, equipped the topology induced by these semi-norms, is a Frechét space. The dual $\mathcal{K}'_M$ is the space of continuous linear functionals on $\mathcal{K}_M$ endowed with the topology of uniform convergence on all bounded sets. If $u \in \mathcal{K}'_M$ and $\phi \in \mathcal{K}_M$, then the convolution $u * \phi$ is a $C^\infty$-function defined by $u * \phi(x) = \langle u_y, \phi(x-y) \rangle$.

The space $\mathcal{O}'_c(\mathcal{K}'_M, \mathcal{K}'_M)$ of convolution operators in $\mathcal{K}'_M$, consists of distributions $S \in \mathcal{K}'_M$ satisfying one of the following equivalent conditions; $S * \mathcal{K}'_M \subset \mathcal{K}'_M$ or $S * \mathcal{K}_M \subset \mathcal{K}_M$, where $\langle S * u, \phi \rangle$ is defined by $\langle u, \check{S} * \phi \rangle$ for every $u \in \mathcal{K}'_M$ and $\phi \in \mathcal{K}_M$. For further details we refer to [6].

The proof of theorem 3 can be done by the same spirit as that of theorem 1. The detailed proof will appear in [8] for $M(x) = |x|^p, p > 1$ and [9] for general $M(x)$.

We present some open problems which is related to the solvability of convolution equations;

(1) S. Sznajder and Z. Zielezny[14] show the equivalence of $(a)_s$ and $(c)_s$ in the tempered distribution space, but that of $(b)_s$ in this space is still open.

(2) L. Ehrenpreis[5] show the equivalence of $S * \mathcal{D}' = \mathcal{D}'$ , $S * \mathcal{D}' \supset \mathcal{D}$ and $S * \mathcal{E}' = \mathcal{E}$. Can we generalize these equivalence in various distribution spaces?

(3) C. C. Chou[4] proved the same equivalence of $(a)_u$ and $(b)_u$ corresponding to the ultra distributions defined by Roumieu[11]. Can we have the same equivalence condition $(c)_u$ for this case?

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# A VISCOSITY SOLUTION APPROACH TO FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We are concerned with viscosity solutions for fully nonlinear second-order degenerate elliptic PDEs;

$$F(x, u, Du(x), D^2u(x)) = 0 \text{ for } x \in \Omega.$$

We present a general approach for uniqueness and existence of viscosity solutions so that we can apply our method to various applications; e.g. impulse control, piece-wise deterministic control, switching games, etc.

## 1. INTRODUCTION

In this paper we consider the following scalar functional differential equation

$$F(x, u, Du(x), D^2u(x)) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

where $\Omega$ is a bounded subset of $\mathbf{R}^n$, $u : \overline{\Omega} \to \mathbf{R}$ is the unknown function and $F : \Omega \times C(\overline{\Omega}) \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is a given continuous function, where $\mathbf{S}^n$ denotes the set of all real symmetric matrices of order $n$. We remark that, in order to apply our theory to some systems arising in switching games etc. (see [I-K2]), we will not suppose that $\Omega$ is open.

The aim here is to present a brief survey of the uniqueness and existence results for continuous viscosity solutions of (1).

For the details and for an extension to the evolution problem in this direction we refer to [I-K1]. We also refer to [C-I-L] for the standard notation in the viscosity solution theory.

## 1. DEFINITION

We shall give our definition of viscosity solutions of (1).

**Definition 1.** A function $u \in C(\overline{\Omega})$ is said to be a viscosity subsolution (resp., supersolution) of (1) if the following inequality holds,

$$F(x, u, D\phi(x), D^2\phi(x)) \leq 0 \quad \text{(resp., } \geq 0),$$

whenever $u - \phi$ attains its maximum (resp., minimum) over $\Omega$ at $x \in \Omega$ for some $\phi \in C^2(\Omega)$.

If $u$ is both a viscosity sub- and supersolution of (1), then it is said to be a viscosity solution of (1).

The term "viscosity" will be omitted in our presentation here since we only discuss viscosity sub-, super- and solutions in what follows.

Throughout this paper we suppose the continuity on $F$;

$$\text{(A1)} \qquad F \in C(\Omega \times C(\overline{\Omega}) \times \mathbf{R}^n \times \mathbf{S}^n)$$

We will occasionally use the following fact:

**Proposition.** *Let $u \in C(\overline{\Omega})$ and let (A1) hold. Then $u$ is a subsolution (resp., a supersolution) of (1) if and only if*

$$F(x, u, p, X) \leq 0 \quad \textit{for all} \quad x \in \Omega,\ (p, X) \in \overline{J}_{\Omega}^{2,+} u(x)$$

*(resp.,*

$$F(x, u, p, X) \geq 0 \quad \textit{for all} \quad x \in \Omega,\ (p, X) \in \overline{J}_{\Omega}^{2,-} u(x)).$$

## 3. UNIQUENESS

In this section we shall present a uniqueness theorem for solutions of (1) under the Dirichlet boundary conditions.

First we give the definition of a monotonicity of $F$.

**Definition 2.** (cf. [I-K1]) $F : \Omega \times C(\overline{\Omega}) \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is said to be monotone if there exists a function $\omega_0 \in C([0, \infty))$ satisfying the following properties:

$$\text{(i)} \qquad \omega_0(r) > 0 \quad \text{for} \quad r > 0$$

and, for all $(p, X) \in \mathbf{R}^n \times \mathbf{S}^n$,

$$\text{(ii)} \quad \{F(x, u, p, X) - F(x, v, p, X)\}\operatorname{sgn}(u(x) - v(x)) \geq \omega_0(\|u - v\|_{C(\overline{\Omega})})$$

provided that $u$, $v \in C(\overline{\Omega})$ and $|u(x) - v(x)| = \|u - v\|_{C(\overline{\Omega})} > 0$ for some $x \in \Omega$.

We shall suppose a modified assumption in the standard theory of viscosity solutions.

$(A2)$ For each $u \in C(\overline{\Omega})$, there is a continuous function $\omega_1 : [0, \infty) \to [0, \infty)$ with $\omega_1(0) = 0$ such that if $X,\ Y \in \mathbf{S}^n$ and $\gamma > 0$ satisfying

$$-3\gamma \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\gamma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{2}$$

then

$$F(y, u, \gamma(x-y), -Y) - F(x, u, \gamma(x-y), X) \leq \omega_1(\gamma|x-y|^2 + \frac{1}{\gamma})$$

for all $(x, y) \in \Omega \times \Omega$.

We note that various kinds of second-order degenerate elliptic partial differential operators satisfy (A2). For this we refer to [C-I-L].

We also assume a uniform continuity of $F$ in the variable $u$.

$(A3)$ For each $u \in C(\overline{\Omega})$, there is a continuous function $\omega_2 : [0, \infty) \to [0, \infty)$ with $\omega_2(0) = 0$ such that

$$|F(x, u, p, X) - F(x, v, p, X)| \leq \omega_2(\|u - v\|_{C(\overline{\Omega})})$$

for all $(x, v, p, X) \in \Omega \times C(\overline{\Omega}) \times \mathbf{R}^n \times \mathbf{S}^n$.

**Theorem 1.** *Let $F$ satisfy (A1) – (A3) and be monotone. Let $u,\ v \in C(\overline{\Omega})$ be solutions of (1). If $u = v$ on $\overline{\Omega} \backslash \Omega$, then $u = v$ in $\overline{\Omega}$.*

*Proof.* The proof is by contradiction: Suppose that $\|u - v\|_{C(\overline{\Omega})} \equiv \theta > 0$, then we will get a contradiction.

Put $\Phi(x, y) = |u(x) - v(y)| - |x-y|^2/(2\epsilon)$ for $\epsilon > 0$, and let $(x_\epsilon, y_\epsilon) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of $\Phi$ over $\overline{\Omega} \times \overline{\Omega}$;

$$\Phi(x_\epsilon, y_\epsilon) = \max_{x, y \in \overline{\Omega}} \Phi(x, y).$$

We may assume without loss of generality that $u(x_\epsilon) \geq v(y_\epsilon)$. It is then well-known (see, e.g., [C-I-L]) that

$$\lim_{\epsilon \to 0} \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} = 0. \tag{3}$$

It is clear that $u(x_\epsilon) - v(y_\epsilon) \geq \theta$. From these it is easily observed that $x_\epsilon, y_\epsilon \in \Omega$ for a small $\epsilon > 0$.

Now, we use a well-known fact (see, e.g., [C-I-L]) that there exist $X$ and $Y \in \mathbf{S}^n$ such that

$$(\frac{x_\epsilon - y_\epsilon}{\epsilon}, X) \in \overline{J}_\Omega^{2,+} u(x_\epsilon),\ (\frac{x_\epsilon - y_\epsilon}{\epsilon}, -Y) \in \overline{J}_\Omega^{2,-} v(y_\epsilon),$$

and

$$-\frac{3}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Then, since $u$ and $v$ are solutions of (1), it follows that

$$0 \ge F(x_\epsilon, u, p, X) - F(y_\epsilon, v, p, -Y). \tag{4}$$

Here and henceforth we denote $p = (x_\epsilon - y_\epsilon)/\epsilon$. Furthermore, using (A2), we obtain

$$F(y_\epsilon, u, p, -Y) - F(x_\epsilon, u, p, X) \le \omega_1(\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} + \epsilon). \tag{5}$$

We shall denote the extension of $u$ to $\mathbf{R}^n$ by $u$ again. Define $u_\epsilon \in C(\overline{\Omega})$ by

$$u_\epsilon(x) = (u(x + x_\epsilon - y_\epsilon) \wedge v(x) + \theta_\epsilon) \vee u(x),$$

where $\theta_\epsilon = u(x_\epsilon) - v(y_\epsilon)$. Let $\omega_u$ be a modulus of continuity of $u$. Since $\theta_\epsilon \ge \theta$, it follows that

$$u_\epsilon(x) \ge u(x) \ge v(x) - \theta_\epsilon \quad \text{for all} \quad x \in \overline{\Omega}$$

and that

$$u_\epsilon(x) \le v(x) + \theta_\epsilon \wedge u(x) + \omega_u(|x_\epsilon - y_\epsilon|) \quad \text{for all} \quad x \in \overline{\Omega}.$$

Moreover, observe that $u_\epsilon(y_\epsilon) = u(x_\epsilon) = v(y_\epsilon) + \theta_\epsilon$. From these, we conclude that $\|u_\epsilon - u\|_{C(\overline{\Omega})} \le \omega_u(|x_\epsilon - y_\epsilon|)$ and $(u_\epsilon - v)(y_\epsilon) = \|u_\epsilon - v\|_{C(\overline{\Omega})}$.

By the monotonicity of $F$, we have

$$F(y_\epsilon, u_\epsilon, p, -Y) - F(y_\epsilon, v, p, -Y) \ge \omega_0(\theta_\epsilon).$$

Using this inequality and assumption (A3), from (4) and (5), we have

$$0 \ge \omega_0(\theta_\epsilon) - \omega_1(\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} + \epsilon) - \omega_2(\|u - u_\epsilon\|_{C(\overline{\Omega})}).$$

According to (3), this is a contradiction for sufficiently small $\epsilon > 0$. □

## 4. EXISTENCE

In this section we shall show the existence of a solution of (1). For simplicity, in this section, we shall treat the following functional differential equation instead of (1):

$$\lambda u(x) + G(x, u, Du(x), D^2u(x)) = 0. \tag{6}$$

In order to apply Perron's method for the existence result to our functional equation (6), we will assume the existence of sub- and supersolutions which have extra properties.

**Definition 3.** A couple of functions $(\phi, \psi) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is said to be a pair of a subsolution and a supersolution of (6) in the strong sense if the following properties hold:

(*i*) $\phi \leq \psi$ in $\overline{\Omega}$ and $\phi = \psi$ on $\overline{\Omega}\backslash\Omega$.

(*ii*) For each $w \in C(\overline{\Omega})$ with $\phi \leq w \leq \psi$ in $\overline{\Omega}$, $\phi$ and $\psi$ are, respectively, a subsolution and a supersolution of

$$\lambda u(x) + G(x, w, Du(x), D^2u(x)) = 0 \quad \text{in} \quad \Omega. \tag{7}$$

*Remark.* It is worthy of noting that if whenever $u, v \in C(\overline{\Omega})$ and $u \leq v$ in $\overline{\Omega}$, then

$$G(x, u, p, X) \geq G(x, v, p, X) \quad \text{for all} \quad (x, p, X) \in \Omega \times \times \mathbf{R}^n \times \mathbf{S}^n, \tag{8}$$

and if $\phi$ and $\psi$ are a subsolution and a supersolution of (6), respectively, then the pair $(\phi, \psi)$ is a pair of a subsolution and a supersolution of (6) in the strong sense. In fact, (8) holds in many applications mentioned in the abstract.

We will denote the diameter of $\Omega$ by $d$. We will also use the following notation: For $\phi \in C(\overline{\Omega})$,

$$\omega_\phi(r) = \max\{|\phi(x) - \phi(y)| \mid x, y \in \overline{\Omega} \text{ and } |x - y| \leq r\}.$$

We shall suppose the following hypothesis for our existence result:

($A4$) There is a pair, $(\phi, \psi)$, of a subsolution and a supersolution in the strong sense and a nondecreasing concave function $\hat{\omega} \in C([0, \infty))$ with $\hat{\omega}(0) = 0$ such that

(*i*) $\max\{\omega_\phi, \omega_\psi\} \leq \hat{\omega}$ in $[0, d]$, and (*ii*) if $u \in \{w \in C(\overline{\Omega}) \mid \phi \le w \le \psi \text{ in } \overline{\Omega}, \omega_w \le \hat{\omega} \text{ in } [0,d]\}$, if $x, y \in \Omega$ and $x \ne y$, if $r \in \mathbf{R}$ satisfies $\phi(y) \le r \le \psi(y)$, if $q \in \overline{D^-}\hat{\omega}(r)$ and if $X, Y \in \mathbf{S}^n$ satisfy (2) with $\gamma = q/|x-y|$, then

$$G(y, u, \frac{q}{|x-y|}(x-y), -Y) - G(x, u, \frac{q}{|x-y|}(x-y), X) \le 0. \tag{9}$$

Here $\overline{D^-}\hat{\omega}(r)$ is the closure of $\{p \in \mathbf{R} \mid \hat{\omega}(r+s) \le \hat{\omega}(r) + sp, \forall s \ge 0\}$, which is called the subdifferential of $\hat{\omega}$ at $r$.

**Theorem 2.** *Let $G$ satisfy (A1), (A2) and (A4) and $\Omega$ be locally compact. Then, there is a solution $u \in C(\overline{\Omega})$ of (6) such that $\phi \le u \le \psi$ in $\overline{\Omega}$.*

*Sketch of proof of Theorem 2.* Let $\phi, \psi$ and $\hat{\omega}$ be functions from (A4). We shall define $K \equiv \{w \in C(\overline{\Omega}) \mid \phi \le w \le \psi \text{ in } \overline{\Omega}, \omega_w \le \hat{\omega} \text{ in } [0,d]\}$.

Fix $w \in K$. Noting that $\Omega$ is locally compact and using Perron's method and Theorem 1, we find that there is a unique solution $u \in C(\overline{\Omega})$ of (7) satisfying $\phi \le u \le \psi$ in $\overline{\Omega}$. Define the mapping $S : K \to C(\overline{\Omega})$ by associating to each $w \in K$ the solution $u$ of (7) with $\phi \le u \le \psi$ in $\overline{\Omega}$. By standard stability results for viscosity solutions, we see from (A1) that this mapping $S$ is continuous.

Now, we shall show that $S(K) \subset K$. To this end, we only need to show that $\omega_{S(w)} \le \hat{\omega}$ in $[0,d]$. Fix $w \in K$, and set $v = S(w)$.

Suppose that $\max_{x,y\in\overline{\Omega}}(|v(x) - v(y)| - \hat{\omega}(|x-y|)) > 0$, and will get a contradiction.

Let $(\hat{x}, \hat{y}) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of the function $|v(x) - v(y)| - \hat{\omega}(|x-y|)$ on $\overline{\Omega} \times \overline{\Omega}$. We may assume that $v(\hat{x}) > v(\hat{y})$. Clearly, $\hat{x} \ne \hat{y}$. Since $\phi \le v \le \psi$ in $\overline{\Omega}$ and $\phi = \psi$ on $\overline{\Omega}\backslash\Omega$, it can happen that neither $\hat{x} \in \overline{\Omega}\backslash\Omega$ nor $\hat{y} \in \overline{\Omega}\backslash\Omega$. Set $\hat{r} = |\hat{x} - \hat{y}| \in (0,d]$.

Now we need the following lemma:

**Lemma.** (*Lemma 4.2. in* [I-K1]) *Let $\omega : [0,\infty) \to \mathbf{R}$ be a nonnegative concave function. Then, $\overline{D^-}\hat{\omega}(r) \ne \emptyset$ for all $r > 0$.*

Set $\hat{r} = |\hat{x} - \hat{y}|$ and choose $q \in \overline{D^-}\omega(\hat{r})$. We thus observe that $v(x) - v(y) - q(|x-y| - \hat{r}) - \hat{\omega}(\hat{r})$ attains its maximum over $\overline{\Omega} \times \overline{\Omega}$ at $(\hat{x}, \hat{y})$. Moreover, if we set $\Psi(x,y) = q(|x-y| - \hat{r}) + \omega(\hat{r})$, then we easily see that $D\Psi(\hat{x}, \hat{y}) = (q/\hat{r})(\hat{x} - \hat{y}, \hat{y} - \hat{x})$ and

$$D^2\Psi(\hat{x}, \hat{y}) \le \frac{q}{\hat{r}} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

By a general result concerning the maximum principle (e.g. [C-I-L]), we find that there are matrices $X$ and $Y \in \mathbf{S}^n$ such that

$$(\frac{q}{\hat{r}}(\hat{x}-\hat{y}), X) \in \overline{J}_{\Omega}^{2,+} v(\hat{x}),$$

$$(\frac{q}{\hat{r}}(\hat{x}-\hat{y}), -Y) \in \overline{J}_{\Omega}^{2,-} v(\hat{y}),$$

and

$$\frac{3q}{\hat{r}}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{3q}{\hat{r}}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Now, using that $v$ is a solution of (7) and assumption (A4) together with Lemma 3, we see that

$$0 \geq G(\hat{x}, w, \frac{q}{\hat{r}}(\hat{x}-\hat{y}), X) - G(\hat{y}, w, \frac{q}{\hat{r}}(\hat{x}-\hat{y}), -Y) + \lambda(v(\hat{x}) - v(\hat{y}))$$

$$> G(\hat{x}, w, \frac{q}{\hat{r}}(\hat{x}-\hat{y}), X) - G(\hat{y}, w, \frac{q}{\hat{r}}(\hat{x}-\hat{y}), -Y) \geq 0.$$

This is a contradiction. Thus, we see $v \in K$.

Finally, applyimg Schauder's fixed point theorem to $S : K \to K$, we conclude that there is a function $u \in K$ such that $S(u) = u$. This $u$ is a solution of (6) and satisfies $\phi \leq u \leq \psi$ in $\overline{\Omega}$. □

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# MOVEMENT OF HOT SPOTS IN $\mathbb{R}^N$

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ABSTRACT. We consider the initial-boundary value problems of the heat equation over unbounded domains in $\mathbb{R}^N$, and study the movement of hot spots of the solution for bounded nonnegative initial data having compact support.

## §1. Introduction.

This note is a summary of my recent work with Jimbo [5].

In [3] Chavel & Karp studied the heat equation in Riemannian manifolds and obtained some asymptotic properties of solutions concerning the movement of hot spots. At each time a hot spot is a point where the solution attains its maximum. They characterized the limit of the hot spots of the solution as the time goes to infinity for several kinds of manifolds. Particularly in the case of the Euclidean space $\mathbb{R}^N$, a sharp result was given there. They considered the initial value problem of the heat equation for $u(t,x)$:

$$\partial_t u = \Delta u \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N \quad \text{and} \quad u(0,x)=\varphi(x) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\varphi$ is an initial datum. They showed that if $\varphi$ is a nonzero bounded nonnegative function having compact support, then the set of hot spots:

$$H(t)=\left\{x\in\mathbb{R}^N; u(t,x)=\max_{y\in\mathbb{R}^N} u(t,y)\right\} \tag{1.2}$$

is contained in the closed convex hull of the support of $\varphi$ for any $t>0$, and as $t\to\infty$ it tends to the Euclidean center of mass of $\varphi$ ( see [3, Theorem 1, p. 274] ). By a little more argument in addition to their proof, one can prove that $H(t)$ consists of one point after a finite time. Actually one can see this by calculating the Hessian of the explicit representation of the solution:

$$u(t,x)=(4\pi t)^{-\frac{N}{2}}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy. \tag{1.3}$$

Our purpose is to consider a similar problem on unbounded domains in $\mathbb{R}^N$ to see how the movement of hot spots is influenced by the existence of boundary and boundary condition.

On the other hand, if one considers the initial-boundary value problem over bounded domain in $\mathbb{R}^N$, by the eigenfunction expansion one knows that under the homogeneous Dirichlet boundary condition or under the homogeneous Robin boundary condition with a constant coefficient the shape of the solution approaches that of the first eigenfunction as $t \to \infty$ and under the homogeneous Neumann boundary condition it approaches the shape of the second eigenfunction provided the second coefficient of the expansion is not zero( see [6, 7, 11] ). In the case of the homogeneous Neumann condition, Rauch's observation is that the hot spots move to the boundary as $t \to \infty$ (see [6, pp. 45–46]). In the case of the homogeneous Dirichlet or Robin condition one can say that the hot spots go away from the boundary. For example, consider the case that $N = 1$ and the domain is an interval. Then, under the homogeneous Dirichlet condition or under the homogeneous Robin condition with a constant coefficient the set of hot spots consists of one point after a finite time and it tends to the center of the interval as $t \to \infty$ , and under the homogeneous Neumann condition it consists of one of the boundary points after a finite time provided the second coefficient of the expansion is not zero.

In this note we consider the initial-boundary value problems of the heat equation over *unbounded* domains in $\mathbb{R}^N$ and study the movement of hot spots of solutions. Precisely, let $\Omega$ be an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and consider the problem:

$$(1.4) \qquad \begin{cases} \partial_t u = \Delta u \ \text{ in } \ (0,\infty)\times\Omega, \\ u(0,x) = \varphi(x) \ \text{ in } \Omega, \\ \text{and the boundary condition (BC),} \end{cases}$$

where $\varphi$ is a nonzero nonnegative bounded function and the support of $\varphi$, say $S_\varphi$, is a compact set contained in $\overline{\Omega}$, and where (BC) is one of the following:

$$\text{(D)} \qquad u = 0 \ \text{ on } \ (0,\infty)\times\partial\Omega,$$

$$\text{(N)} \qquad \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ (0,\infty)\times\partial\Omega,$$

$$\text{(R)} \qquad \frac{\partial u}{\partial \nu} + \beta u = 0 \ \text{ on } \ (0,\infty)\times\partial\Omega,$$

where $\nu$ is the exterior unit normal vector to $\partial\Omega$ and $\beta$ is a positive constant. Let $u(t,x)$ be a solution to these problems. We define the set of hot spots $H(t)$ of $u(t,x)$ by replacing $\mathbb{R}^N$ by $\overline{\Omega}$ in (1.2), and let $C(t)$ be the set of critical points with respect to the space variable $x$:

$$C(t) = \left\{x \in \overline{\Omega}; \nabla_x u(t,x) = 0\right\}. \tag{1.5}$$

Note that $H(t) \subset C(t)$. The first example of unbounded domain is the half space $\mathbb{R}^N_+ = \{x = (x_1, ..., x_N) \in \mathbb{R}^N; x_N > 0\}$. By using the explicit representation of the solutions we obtain

**Theorem 1.** *Let $\Omega = \mathbb{R}^N_+$. Consider the boundary conditions (D), (N), or (R). Then, if $T > 0$ is sufficiently large, $C(t)(= H(t))$ consists of only one point, say $x(t) = (x_1(t), ..., x_N(t))$, for any $t > T$. Furthermore,*

$$x_N(t) \times (2t)^{-\frac{1}{2}} \to 1 \quad \text{as } t \to \infty, \text{when (BC) = (D) or (R).} \tag{1.6}$$

*and*

$$x_N(t) = 0 \quad \text{for any } t > T, \text{when (BC) = (N).} \tag{1.7}$$

*Also, for $1 \leqq j \leqq N-1$, as $t \to \infty$*

$$x_j(t) \to \begin{cases} \int y_j y_N \varphi(y)dy / \int y_N \varphi(y)dy, \\ \int y_j \varphi(y)dy / \int \varphi(y)dy, \\ \int y_j \left(y_N + \beta^{-1}\right) \varphi(y)dy / \int \left(y_N + \beta^{-1}\right) \varphi(y)dy, \end{cases}$$

$$\text{when} \begin{cases} (BC) = (D), \\ (BC) = (N), \\ (BC) = (R), \end{cases} \quad \text{respectively.}$$

Namely, the half space is a good example to show that the hot spots go to the boundary under the Neumann condition and they go away from the boundary under the Dirichlet or Robin condition.

The second example of unbounded domain is the exterior domain of a ball. In this case it is not easy to know the Green's function precisely as in the cases of $\mathbb{R}^N, \mathbb{R}^N_+$. For example, it is difficult to know the sign of the differential of the Green's function. However, if the initial datum is radially symmetric with respect to the center of the ball, the problem is reduced to the one-dimensional parabolic boundary value problem.

**Theorem 2.** *Let* $\Omega = \{x \in \mathbb{R}^N; |x| > 1\}$, *and let* $\varphi = \varphi(r)$, *where* $r = |x|$. *Consider the boundary conditions (D), (N), or (R). Then, if* $T > 0$ *is sufficiently large,*

$$H(t) = C(t) = \{x \in \mathbb{R}^N; |x| = r(t)\} \quad \text{for any } t > T, \tag{1.8}$$

*for some smooth function* $r(t) \geqq 1$. *Furthermore, when (BC) = (D) or = (R),* $\limsup_{t\to\infty} r(t) = \infty$, *and when (BC) = (N),* $r(t) = 1$ *for any* $t > T$ *and*

$$C(t) \subset \{x \in \mathbb{R}^N; |x| < \sup\{|y|; y \in S_\varphi\}\} \quad \text{for any } t > 0, \tag{1.9}$$

*where* $S_\varphi$ *is the support of* $\varphi$.

*Remark.* Since the solution is radially symmetric with respect to the origin, let $u = u(t,r)$ be the solution. When $N = 3$, the function $ru(t,r)$ satisfies the one-dimensional heat equation and one can calculate the solution $u(t,r)$ explicitly. Therefore, we see that if $N = 3$ and $(BC) = (D)$, then $r(t) \times (2t)^{-\frac{1}{3}} \to 1$ as $t \to \infty$ and there exists $T > 0$ satisfying $\frac{dr}{dt}(t) > 0$ for any $t > T$.

The last theorem gives estimates of the size of $C(t)$ when $\Omega$ is a general exterior domain in $\mathbb{R}^N$.

**Theorem 3.** *Let* $\Omega$ *be the exterior domain of a bounded smooth domain in* $\mathbb{R}^N$. *Consider the boundary conditions (D), (N), or (R). For any* $\varphi$ *there exists a positive constant* $K$ *satisfying*

$$C(t) \subset \left\{x \in \overline{\Omega}; |x| \leqq K(\sqrt{t|\log t|} + 1)\right\} \quad \text{for any } t > 0. \tag{1.10}$$

*Especially, when* $\Omega = \{x \in \mathbb{R}^N; |x| > 1\}$ *and (BC) = (N), the above estimate becomes*

$$C(t) \subset \left\{x \in \overline{\Omega}; |x| \leqq K(\sqrt{t} + 1)\right\} \quad \text{for any } t > 0. \tag{1.11}$$

*Remark.* In [3, p.285] the similar estimates of the hot spots in complete Riemannian manifolds *without boundary* were obtained with the help of a Harnack inequality of Li & Yau[8].

## §2. On the proofs of the theorems.

In Theorem 1 to prove the uniqueness of the hot spot after a finite time we need

**Lemma 1.** *Let $u$ be the solution. Then there exists $T > 0$ satisfying:*
*If $\nabla_x u(t,x) = 0$ with $t > T$, then $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)\right)_{1 \leqq i,j \leqq N}$ is negative definite.*

The proof of Theorem 3 is due to *the moving plane method* (see [4, 9] or [6, p. 100]). Choose $R > 0$ sufficiently large to get

$$B_R(0) \supset \mathbb{R}^N \setminus \Omega, \tag{2.1}$$

where $B_R(0)$ denotes an open ball in $\mathbb{R}^N$ centered at the origin with radius $R$. Let $u$ be the solution under one of the boundary conditions (D), (N), or (R). To apply *the moving plane method* we need the following key lemma.

**Lemma 2.** *If $C > R$ is sufficiently large, then*

$$\sup_{|x| \geqq C\left(\sqrt{t|\log t|}+1\right)} u(t,x) < \inf_{|x|=R} u(t,x) \quad \textit{for any } t > 0,$$

By using this lemma let us prove (1.10). Let $x = (x', x_N)$ and $x' = (x_1, \ldots, x_{N-1})$. For $\lambda \in \mathbb{R}$ define the function $v_\lambda$ by

$$v_\lambda(t,x) = u(t,x',x_N) - u(t,x',2\lambda - x_N). \tag{2.2}$$

For any $\lambda \geqq C$ consider the domain $D_\lambda$ defined by

$$D_\lambda = \left(\mathbb{R}^N \setminus \overline{B_R(0)}\right) \cap \{x_N < \lambda\}. \tag{2.3}$$

Then $v_\lambda$ satisfies

$$\begin{cases} \partial_t v_\lambda = \Delta v_\lambda & \text{in } (0,\infty) \times D_\lambda, \\ v_\lambda(0,x) \geqq 0 & \text{in } D_\lambda, \\ v_\lambda = 0 & \text{on } (0,\infty) \times \{x_N = \lambda\}. \end{cases} \tag{2.4}$$

Furthermore it follows from Lemma 2 that there exists $T_\lambda > 0$ satisfying

$$v_\lambda > 0 \quad \text{on } (0, T_\lambda) \times \partial B_R(0). \tag{2.5}$$

Hence by the strong maximum principle we get

$$\begin{cases} v_\lambda > 0 \text{ in } (0, T_\lambda) \times D_\lambda, \\ 0 > \left.\frac{\partial v_\lambda}{\partial x_N}\right|_{x_N = \lambda} = 2\frac{\partial u}{\partial x_N}(t, x', \lambda) \text{ for } (t, x') \in (0, T_\lambda) \times \mathbb{R}^{N-1}. \end{cases} \tag{2.6}$$

Since the heat equation is invariant under the rotation of the coordinates, we can adjust the positive $x_N$-axis to any direction. Therefore, in view of Lemma 2, we obtain (1.10). Also we note that (1.11) is a consequence of a similar lemma to Lemma 2.

In Theorem 2 to prove that after a finite time $C(t)$ consists of only one sphere we employ the results of Angenent[1] and Angenent & Fiedler[2] and the idea of Ni & Sacks[10]. Owing to the results of [1] and [2] we can see the path of the bottom of the valley of the temperature precisely provided the problem is one-dimensional. The idea of Ni & Sacks[10] is simple, that is, if a valley of the temperature continues to exist, then the temperature at the bottom of the valley never decreases and this contradicts the decay of the temperature. Using these facts and Theorem 3, we prove that after a finite time $C(t)$ consists of only one sphere. Furthermore by using the comparison principle we see that $\limsup_{t\to\infty} r(t) = \infty$ when (BC) = (D) or = (R). By using *the moving plane method* we can prove (1.9) when (BC) = (N), that is, in this case we can convey *the plane* from the neighborhood of the infinity to the extremal position.

For the details we refer to [5].

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**REMARKS ON THE BLOWUP SHAPE FOR A SEMILINEAR PARABOLIC EQUATION**

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# 1 Introduction

We discuss the parabolic equation

$$u_t - \Delta u = e^u \quad \text{in} \quad Q = B \times (0,T) \tag{1}$$

with the boundary condition

$$u|_{\partial B} = 0 \tag{2}$$

and the initial condition

$$u|_{t=0} = u_0(x) \geq 0. \tag{3}$$

Here, $B = B_R(0) = \{|x| < R\}$ denotes an n-dimensional ball, $u_0 = u_0(|x|) \in C^2(B) \cap C^0(\overline{B})$ a radially symmetric function satisfying

$$-\Delta u_0 \leq \not\equiv e^{u_0}, \quad u_{0r} \leq \not\equiv 0 \quad \text{in } B \tag{4}$$

and $T > 0$ the blowup time:

$$\|u(\cdot,t)\|_{L^\infty} \to +\infty \quad \text{as} \uparrow T \tag{5}$$

The condition (4) is equivalent to suppose that

$$u = u(|x|,t), \ u_t > 0, \ -\Delta u < e^u \quad \text{in } Q \tag{6}$$

and

$$u_r < 0 \quad \text{in } Q \setminus \{x = 0\} \tag{7}$$

for the solution $u$ (c.f. [10]).

In this situation, the blowup shape

$$u(|x|,T) = 2\log\frac{1}{|x|} + \log\log\frac{1}{|x|} + \log 8 + o(1) \quad (|x| \ll 1) \tag{8}$$

is expected by several authors (c.f. [9]). For instance, [8] proved the fact for $B$ replaced by a convex domain $\Omega$. Then the blowup shape (8) holds if $x = 0$ is a blowup point and $u_0(x)$ belongs to an open set in $C^0(\overline{\Omega})$. However, it is not cirtain that the set contains radial functions when $\Omega = B$ and $u_0 = u_0(|x|)$.

About the upper estimate we have a result of [4] as

$$u(|x|, T) \leq 2\log\frac{1}{|x|} + \log\log\frac{1}{|x|} + O(1) \quad (|x| \ll 1). \tag{9}$$

On the other hand the lower estimate

$$u(|x|, T) \geq 2\log\frac{1}{|x|} + \log 2(n-2) \quad (|x| \ll 1) \tag{10}$$

was proven by [5] for $n \geq 3$, and [16] for $n = 2$.

What we prove in this note is the following.

**Main Theorem** Let $n \geq 2$, $u_0 \in C^4(B)$ and

$$\Delta^2 u_0 + e^{u_0}|\nabla u_0|^2 \geq (\Delta u_0)^2. \tag{11}$$

Then, for any $K > 0$ we have

$$u(|x|, T) \geq 2\log\frac{1}{|x|} + K \quad (|x| \ll 1). \tag{12}$$

Here, the condition (11) has to be discussed in connection with (5). It actually holds for $u_0 = u^*$, solving

$$-\Delta u^* = \lambda e^{u^*} \quad \text{in } B \tag{13}$$

with

$$u^* = 0 \quad \text{on } \partial B, \tag{14}$$

where $0 < \lambda < 1$. In the case of $n \leq 9$, we have an appropriate ball $B = B_R(0)$ such that a nonminimal solution $u^{**}$ of (13) with (14) exists for $\lambda = 1$. In fact, $R > 0$ is taken to be arbitrarily large if $n = 2$. For these facts we refer to [12], [17], and [20].

In this situation, it holds that

$$u^{**} \geq \underline{u}, \tag{15}$$

where $\underline{u}$ denotes the minimal solution for (13) with (14) and $\lambda = 1$. Therefore, if $0 < \lambda < 1$ is sufficiently close to 1, the estimate

$$u^* > \underline{u} \quad \text{in } B \tag{16}$$

holds. Now the theorem [11] indicates the blowup in finite or infinite time for the parabolic equation (1) with (2) and (3) for this $u_0 = u^*$. This consideration supports the possibility for the condition (11) to be compatible with (5).

# 2 Strategy for the proof

Uniform estimates on the parabolic region of Giga-Kohn ([13] [14] [15]) have been established by [5], [18] and [7]. However, the estimate (12) which we are going to prove is uniform and does not follow from them by themselves.

We employ the argument of [16] to prove the following lemma.

**Lemma 1** *Suppose the existence of $\phi_0(|x|) \in C^2(B)$, $r_0 \in (0,1]$ and $t_0 \in [0,T)$ such that*

$$-\Delta\phi_0 \geq 0 \quad \text{in } B, \tag{17}$$

*and*

$$J \equiv e^{-u}u_t \geq 1 - e^{-\phi_0} \quad \text{in } |x_0| \leq r_0,\ t_0 \leq t < T. \tag{18}$$

*Suppose furthermore that*

$$\phi_{0r} \geq 0 \quad \text{in } B \tag{19}$$

*if $n \geq 3$. Then it holds that*

$$u(|x|,T) \geq 2\log\frac{1}{|x|} + \phi_0(|x|) + \log 2 \quad (|x| \ll 1). \tag{20}$$

Unfortunately, continuity of $J(x,t)$ around $(x,t) = (0,T)$ cannot be proven (c.f. [6]). However the requirements of the above lemma are assured in the followng way.

**Lemma 2** *The function $J(x,t)$ is monotone increasing in $t$ at each $x \in B$, provided that the condition (11) holds.*

**Lemma 3** *We have*

$$J|_{x=0} \to 1 \quad \text{as } t \uparrow T. \tag{21}$$

Lemma 2 follows from a standard argument of comparison due to [10], while Lemma 3 is a consequence of the asymtotic analysis on the parabolic region of Gigq-Kohn, by [5] and [18].

To prove the main theorem, we just take $\phi_0 \equiv K > 0$, an arbitrary large constant. Then the condition (18) follows from Lemmas 2 and 3.

# 3 Itoh's principle and generalizations

The proof of Lemma 1 is based on two inequalities, Bandle's mean value theorem and Bol's isoperimetric inequality.

**Proposition 4 (Bandle)** *Let $B = B_R(0) \subset \Re^2$ be a ball and $p \in C^2_+(B)$ satisfy the differential inequality*

$$-\Delta \log p \leq p \quad \textit{in } B \tag{22}$$

*with*

$$m = \int_B p dx < 8\pi. \tag{23}$$

*Then it holds that*

$$p(0) \leq 8R^{-2} m(8\pi - m)^{-1}. \tag{24}$$

Here and henceforce, $C^2_+(B)$ denotes the set of positive $C^2$- functions on $B$. In fact, the above assertion is equivalent to corollary 1.1 of [1]. See also [19]. Applying it on the ball $B_{R-|x|}(x)$, we obtain the following.

**Corollary 5** *Under the assumption (22) with (23) we have*

$$p(x) \leq 8\left(R - |x|\right)^{-2} m\left(8\pi - m\right)^{-1} \quad (x \in B). \tag{25}$$

On the other hand, Bol's inequality is nothing but an isoperimetric inequality on the surface $\mathcal{M} = \left(B, p^{1/2} ds\right)$ with the Gaussian curvature $K = -\Delta p / 2p \leq 1/2$, where $ds^2 = dx_1^2 + dx_2^2$ for $x = (x_1, x_2) \in B$.

**Proposition 6 (Bol)** *Under the assumption (22) with (23) we have*

$$\ell^2 \equiv \left(\int_{\partial B} p^{1/2} ds\right)^2 \geq \frac{1}{2} m(8\pi - m). \tag{26}$$

Analytic proof is performed by [2].

Combining these two inequalities, we reach the following conclusion.

**Proposition 7** *Under the assumptions (22), $\ell \leq 2\sqrt{2}\pi$, and $m \leq 4\pi$ we have the estimate*

$$m \leq 4\pi \left(1 - \sqrt{1 - j^2}\right) \tag{27}$$

*and*

$$p(x) \leq 8\left(R - |x|\right)^{-2} j^2 \left(1 + \sqrt{1 - j^2}\right)^{-2} \quad (x \in B), \tag{28}$$

*where $j = \ell / (2\sqrt{2}\pi)$.*

*Proof:* Because of (26), $m \leq 4\pi$, and $\ell \leq 2\sqrt{2}\pi$ we have $m \leq m_-$, where $m_-$ denotes the smaller solution for

$$M^2 - 8\pi M + \ell^2 = 0.$$

Namely, $m_- = 4\pi\left(1 - \sqrt{1-j^2}\right) \leq 4\pi$ and (27) follows. Now (28) holds by (25). □

The following assertion is an immediate consequence, but plays the key role in our argument. We call it Itoh's principle.

**Corollary 8 (Itoh)** *The assumptions (22), $\ell < 2\sqrt{2}\pi$ and $m \leq 4\pi$ deduce that $m < 4\pi$.*

Let $\{p_t\}_{0\leq t<T}$ be a blowingup family of positive functions on a domain $\Omega \subset \Re^n$:

$$\|p_t\|_{L^\infty} \to +\infty \quad \text{as } t \uparrow T. \tag{29}$$

Then the blowup set

$$\mathcal{S}\{p_t\}_{0\leq t<T} = \{x \in \overline{\Omega} \mid p_{t_k}(x_k) \to +\infty \ \text{ with some}$$
$$x_k \to x \ \text{ and } \ t_k \uparrow T \ \text{ as } \ k \to +\infty\} \tag{30}$$

is defined and

$$\mathcal{S}_I\{p_t\}_{0\leq t<T} = \mathcal{S}\{p_t\}_{0\leq t<T} \cap \Omega \tag{31}$$

denotes the interior blowup set. The following theorem is due to T. Itoh.

**Theorem 9** *If $\{p_t\}_{0\leq t<T} \subset C^2_+(\Omega) \cap C^0(\overline{\Omega})$ satisfies (29),*

$$t \mapsto p_t \in C^0(\overline{\Omega}) \text{ continuous, monotone increasing} \tag{32}$$

*and*

$$-\Delta \log p_t \leq p_t \quad \text{in } \Omega \ \text{ for } \ 0 \leq t < T, \tag{33}$$

*then any $x_0 \in \mathcal{S}_I\{p_t\}_{0\leq t<T}$ enjoys the property that*

$$\int_{B_r(x_0)} p_T^{1/2} ds \geq 2\sqrt{2}\pi \quad (0 < r \ll 1), \tag{34}$$

*where*

$$p_T(x) = \lim_{t\uparrow T} p_t(x) \in (0, +\infty] \quad (x \in \Omega). \tag{35}$$

*Proof:* If (34) is false, there exists a sequence $r_k \downarrow 0$ such that

$$\int_{\partial B_{r_k}(x_0)} p_T^{1/2} ds < 2\sqrt{2}\pi \quad (k = 1, 2, \cdots).$$

In particular for some $r > 0$ sufficiently small holds that

$$m_0 < 4\pi \quad \text{and} \quad \ell_T < 2\sqrt{2}\pi, \tag{36}$$

where

$$m_t = \int_{B_r(x_0)} p_t dx \quad \text{and} \quad \ell_t = \int_{\partial B_r(x_0)} p_t^{1/2} ds. \tag{37}$$

From the monotonicity follows $\ell_t < 2\sqrt{2}\pi$ $(0 \leq t < T)$, while $t \mapsto m_t$ is continuous. Therefore, Itoh's principle Corollary 8 works and $m_t < 4\pi$ remains to hold for $0 \leq t < T$, so that

$$p_t(x) \leq 8\left(r - |x - x_0|\right)^{-2} j_t^2 \left(1 + \sqrt{1 - j_t^2}\right)^{-2} \tag{38}$$

for $x \in B_r(x_0)$, $0 \leq t < T$, and $j_t = \ell_t/(2\sqrt{2}\pi)$. This means that $x_0 \notin \mathcal{S}_I \{p_t\}_{0 \leq t < T}$, a contradiction. □

We can reproduce Itoh's result referred to in the introduction.

**Corollary 10** *Let $B = B_R(0) \subset \Re^2$ and $u(|x|, t) \in C^2(B) \cap C^0(\overline{B})$ $(0 \leq t < T)$ have the property that*

$$t \mapsto u(\cdot, t) \in C^0(\overline{B}) \text{ continuous, monotone increasing} \tag{39}$$

$$u(0, t) \to +\infty \quad \text{as } t \uparrow T \tag{40}$$

*and*

$$-\Delta u \leq e^u \quad \text{in } Q \equiv B \times (0, T). \tag{41}$$

*Then it holds that*

$$u(|x|, T) \geq 2 \log \frac{1}{|x|} + \log 2 \quad (0 < |x| \ll 1). \tag{42}$$

*Proof:* We can apply the previous theorem for $p_t = e^{u(\cdot, t)}$, $\Omega = B$, and $x_0 = 0$. The conclusion (34) implies that

$$p(|x|, T) \geq 2/|x|^2 \quad (0 < |x| \ll 1),$$

or (42). □

**Corollary 11** *Let $B = B_R(0) \subset \Re^n$ with $n \geq 3$ and $u = u(|x|, t) \in C^2(B) \cap C^0(\overline{B})$ $(0 \leq t < T)$ have the property that*

$$u_r \leq 0 \quad \text{in } Q \tag{43}$$

*besides (39)-(41). Then the conclusion (42) follows.*

We go back to the two dimensional ball $B = B_R(0) \subset \Re^2$. For each $\varepsilon > 0$ sufficiently small, let a family $\{p_t^\varepsilon\}_{0 \leq t < T} \subset C_+^2(B) \cap C^0(\overline{B})$ be given with the properties that

$$t \mapsto p_t^\varepsilon \in C^0(\overline{B}) \;\; \text{continuous, monotone increasing,} \tag{44}$$

$$p_t^\varepsilon(0) \to +\infty \quad \text{as } t \uparrow T, \tag{45}$$

$$-\Delta \log p_t^\varepsilon \leq p_t^\varepsilon \;\; \text{in } B_{R_\varepsilon}(0) \;\; \text{for } t_\varepsilon \leq t < T, \tag{46}$$

where $R_\varepsilon \downarrow 0$ and $t_\varepsilon \uparrow T$ as $\varepsilon \downarrow 0$. Let another family $\{p_t^\circ\}_{0 \leq t < T} \subset C_+^0(B^\circ)$ be given for $B^\circ = B \setminus \{0\}$ with the property that

$$p_t^\varepsilon(x) \to p_t^\circ(x) \;\; \text{as } \varepsilon \downarrow 0 \;\; \text{for } x \in B^* \text{ and } 0 \leq t < T. \tag{47}$$

In this case the mapping

$$t \mapsto p_t^\circ(x) \quad (x \in B^\circ) \tag{48}$$

is monotone increasing, and we have the limitting fucntion

$$p_T^\circ(x) = \lim_{t \uparrow T} p_t^\circ(x) \in (0, +\infty] \quad (x \in B^\circ). \tag{49}$$

We suppose, furthermore, that

$$\sup_{x \in B^\circ, 0 \leq t < T} |x|^2 \left| p_t^\varepsilon(x) - p_t^\circ(x) \right| \to 0 \;\; \text{as } \varepsilon \downarrow 0. \tag{50}$$

**Lemma 12** *Under those situations we have*

$$\int_{\partial B_r(0)} p_T^{\circ 1/2} ds \geq 2\sqrt{2}\pi \quad (0 < r \ll 1). \tag{51}$$

*Proof:* Applying Theorem 9 directly deduces that

$$\int_{\partial B_r(0)} p_T^{\varepsilon 1/2} ds \geq 2\sqrt{2}\pi \quad (0 < r \ll 1). \tag{52}$$

However, the effective range of $r$ depends on $\varepsilon$ and (51) does not follow.

First we note an inequality implied by (50):

$$\sup_{0 \leq r < R} \left| \int_{\partial B_r(0)} p_T^{\varepsilon 1/2} ds - \int_{\partial B_r(0)} p_T^{\circ 1/2} ds \right| \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{53}$$

In fact, the left-hand side is dominated by

$$\int_0^{2\pi} \sup_{0 \leq r < R} r \left| p_T^{\varepsilon} \left( re^{\imath\theta} \right)^{1/2} - p_T^{\circ} \left( re^{\imath\theta} \right)^{1/2} \right| d\theta,$$

where (50) can be utilized.

If (51) is false, there exists a sequence $r_k \downarrow 0$ such that

$$\int_{\partial B_{r_k}(0)} p_T^{\circ 1/2} < 2\sqrt{2}\pi \quad (k = 1, 2, \cdots). \tag{54}$$

From the inequality (53) follows that

$$\int_{\partial B_{r_k}(0)} p_T^{\varepsilon 1/2} ds < 2\sqrt{2}\pi \quad (k = 1, 2, \cdots) \tag{55}$$

for $\varepsilon > 0$ sufficiently small. For such $\varepsilon$, we have an $r \in (0, R_\varepsilon)$ sufficiently small, with the property that

$$\int_{\partial B_r(0)} p_T^{\varepsilon 1/2} ds < 2\sqrt{2}\pi \tag{56}$$

and

$$\int_{B_r(0)} p_{t_\varepsilon}^{\varepsilon} dx < 4\pi. \tag{57}$$

Now, we repeat the argument of Theorem 9 on $B = B_{R_\varepsilon}(0)$ for $\{p_t^\varepsilon\}_{t_\varepsilon \leq t < T}$ to deduce that $0 \notin \mathcal{S}_I \{p_t^\varepsilon\}_{t_\varepsilon \leq t < T}$, a contradiction. □.

Once Lemma 12 has been proven, the following assertions follow similarly.

**Corollary 13** *Let* $\{v_t^\varepsilon(|x|)\}_{0 \leq t < T} \subset C^2(B) \cap C^0(\overline{B})$ *be a family on* $B = B_R(0) \subset \Re^2$ *for each* $\varepsilon > 0$ *sufficiently small, with the property that*

$$t \mapsto v_t^\varepsilon(|x|) \in C^0(\overline{B}) \text{ continuous, monotone increasing} \tag{58}$$

$$v_t^\varepsilon(0) \to +\infty \quad as\ t \uparrow T \tag{59}$$

$$-\Delta v_t^\varepsilon \le e^{v_t^\varepsilon} \quad in\ B_{R_\varepsilon}(0)\ for\ t_\varepsilon \le t < T, \tag{60}$$

*where* $R_\varepsilon \downarrow 0$ *and* $t_\varepsilon \uparrow T$ *as* $\varepsilon \downarrow 0$. *Let another family* $\{v_t^\circ(|x|)\}_{0\le t<T} \subset C^0(B^\circ)$ *be given on* $B^\circ = B \setminus \{0\}$ *with the property that*

$$v_t^\varepsilon(x) \to v_t^\circ(x) \quad as\ \varepsilon \downarrow 0\ for\ x \in B^\circ,\ 0 \le t < T \tag{61}$$

$$\sup_{x\in B^\circ, 0\le t<T} |x|^2 \exp v_t^\circ(|x|) \left| \exp\left\{v_t^\varepsilon(|x|) - v_t^0(|x|)\right\} - 1 \right| \to 0. \tag{62}$$

*Then we obtain*

$$v^\circ(|x|, T) \ge 2\log\frac{1}{|x|} + \log 2 \quad (0 < |x| \ll 1), \tag{63}$$

*where*

$$v^\circ(|x|, T) = \lim_{t\uparrow T} v^\circ(|x|, t) \quad (x \in B^\circ). \tag{64}$$

**Corollary 14** *Similar conclusion holds on n-dimensional ball* $B = B_R(0)$ *with* $n \ge 3$, *if the assumption*

$$(v_t^\varepsilon)_r \le 0 \quad in\ B\ for\ 0 \le t < T \tag{65}$$

*is added in the previous corollary.*

# 4 Proof of the Main Theorem

We have only to prove Lemmas 1-3.

*Proof of Lemma 1:* For $v_t^\varepsilon = v_t^\circ = u(|x|, t) - \phi_0(|x|)$, we examine the assumptions of Corollary 14. In fact we have

$$\begin{aligned} -\Delta v_t^\varepsilon &= -\Delta u + \Delta\phi_0 \\ &\le -\Delta u = -e^u\left(e^{-u}u_t - 1 + e^{-\phi_0}\right) + e^{u-\phi_0} \\ &\le e^{u-\phi_0} = e^{v_t^\varepsilon} \quad \text{in } |x| \le r_0,\ r_0 \le t < T. \end{aligned} \tag{66}$$

and also

$$(v_t^\varepsilon)_r = u_r - \phi_{0r} \le 0 \quad \text{in } Q. \tag{67}$$

The conclusion (63) means (20). □

*Proof of Lemma 2:* From the equation (1) follows that

$$u_{tt} - \Delta u_t = e^u u_t \quad \text{in } Q \tag{68}$$

and

$$u_{ttt} - \Delta u_{tt} = e^u \left( u_t^2 + u_{tt} \right) \quad \text{in } Q. \tag{69}$$

Therefore, we have for

$$I = u_{tt} - u_t^2 \tag{70}$$

that

$$\begin{aligned} I_t - \Delta I &= u_{ttt} - 2u_t u_{ttt} - \left( \Delta u_{tt} - 2 \left| \nabla u_t \right|^2 - 2u_t \Delta u_t \right) \\ &= e^u \left( u_{tt} - u_t^2 \right) + 2 \left| \nabla u_t \right|^2 \geq e^u I \quad \text{in } Q \end{aligned} \tag{71}$$

as well as

$$I|_{\partial B} = 0. \tag{72}$$

Now the condition (11) is equivalent to

$$I|_{t=0} \geq 0. \tag{73}$$

Hence

$$I \geq 0 \quad \text{in } Q \tag{74}$$

follows, which is nothing but

$$J_t \geq 0 \quad \text{in } Q \tag{75}$$

for the function $J(x,t)$ defined in (18). □

*Proof of Lemma 3:* In the case of $n \geq 3$, we can utilize the result of [5]. The function $w(y,\sigma)$ defined through

$$u(|x|,t) = w\left( x(T-t)^{-1/2}, \log \frac{T}{T-t} \right) - \log(T-t) \tag{76}$$

enjoys the property that

$$w \to 0 \ \text{ loc. unif. in } y \in \Re^n \text{ as } \sigma \to +\infty. \tag{77}$$

Since $w(y,\sigma)$ satisfies the parabolic equation

$$w_\sigma = \Delta w - \frac{1}{2} y \cdot \nabla w + \left( e^w - 1 \right), \tag{78}$$

a standard argument implies that

$$w_\rho, w_\sigma \to 0 \text{ loc. unif. in } u \in \Re^n \text{ as } \sigma \to +\infty, \tag{79}$$

where $\rho = |y|$.

These asymptotics deduce that

$$\begin{aligned} u(0,t) + \log(T-t) &= w\left(0, \log \frac{T}{T-t}\right) \\ &\to 0 \quad \text{as } t \uparrow T \end{aligned} \tag{80}$$

and

$$\begin{aligned} (T-t)u_t(0,t) &= \frac{1}{2}\rho w_\rho \left(0, \log \frac{T}{T-t}\right) + w_\sigma \left(0, \log \frac{T}{T-t}\right) + 1 \\ &\to 1 \quad \text{as } t \uparrow T, \end{aligned} \tag{81}$$

respectively. Therefore,

$$J = e^{-u} u_t \to 1 \quad \text{as } t \uparrow T \tag{82}$$

follows.

For the case $n = 2$ we make use of the result of [18] instead.

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# SECTOR THEORY

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## 1. Introduction

Sector theory was introduced by R. Longo([Lo2]) and the index theory was introduced by V.F.R.Jones([Jo1]) in case of finite factors and by H. Kosaki ([Ko1]) in case of infinite factors. Sector theory is a useful tool for the classification of subfactors of type III factors with finite index ([Iz1], [Iz2], [IK], [Ko3], [KS], [Lo1]) and for the characterization of strongly outer actions ([Ka1], [Ko4]). In this paper we will review basic concepts of the sector theory, the index theory via sector approach, and Fusion rules. Throughout this paper $M$ is a type III factor.

## 2. Preliminaries on sectors

Let $M$ be a type III factor. A Hilbert space $H$ is called an $M-M$ bimodule (or $M-M$ correspondence) if and only if $H$ is an $M-M$ bimodule and the left and right actions of $M$ are $\sigma$ weakly continuous.

Let $L^2(M)$ be the standard Hilbert space of $M$ and $J$ be a modular conjugation. Then $L^2(M)$ is an $M-M$ correspondence by

$$x \cdot \xi \cdot y = xJy^*J\xi \qquad \text{for} \quad \xi \in L^2(M), \quad x, y \in M$$

For $\rho \in \mathrm{End}(M)$, the class of unital nomal *-endomorphisms, define an $M-M$ correspondence $H_\rho = L^2(M)$ by

$$x \cdot \xi \cdot y = \rho(x)Jy^*J\xi \qquad \text{for} \quad \xi \in L^2(M), \quad x, y \in M$$

**Theorem 1.** *(A.Connes) Any $M - M$ bimodule $H$ is unitarily equivalent to $H_\rho$ for some $\rho \in End(M)$.*

*Proof.* Since $M$ is of type III, $M$ right action on $H$ is spatially isomorphic with $M$ right action on $L^2(M)$. So there exists a surjective isometry $v$ : $H \to L^2(M)$ such that $U\xi \cdot m = U(\xi \cdot m)$ for $m \in M, \xi \in H$, i.e. $\pi_r^{L^2}(M)U = U\pi_r^H(M)$. Thus we get

$$\pi_l^H(M) \subset \pi_r^H(M)' = (U^*\pi_r^{L^2}(M)U) = U^*\pi_r^{L^2}(M)'U = U^*\pi_l^{L^2}(M)U.$$

So that $U\pi_l^H(M)U^* \subset \pi_l^{L^2}(M) = M$. Set $\rho(m) = U\pi_l^H(m)U^* \in M$, then $\rho \in \mathrm{End}(M)$and $H \cong H_\rho$ via $U$.

By lemma we can see that the study of $M - M$ bimodule is the same with the study of endomorphisms on $M$.

Let $\rho, \zeta \in \mathrm{End}(M)$ and suppose that $H_\rho$ and $H_\zeta$ are unitarily equivalent as $M - M$ bimodules. Then there exists a untary$U : L^2(M) \to L^2(M)$ such that $U(m_1 \cdot \xi \cdot m_2) = m_1 \cdot U\xi \cdot m_2$, i.e.$U(\rho(m_1)Jm_2^*J\xi) = \zeta(m_1)Jm_2^*JU\xi$.Set $m_1 = 1$ then $UJm_2^*J\xi = Jm_2JU\xi$ for $\xi \in L^2(M)$. So $U \in (JMJ)' = M$. Set $M_2 = 1$ then $U\rho(m_1)\xi = \zeta(m_1)U\xi$ for $\xi \in L^2(M), m_1 \in M$. Hence $\zeta(m) = U\rho(m)U^*$ for $m \in M$, thus $\zeta = \mathrm{Ad}U \cdot \rho$.

**Definition 1.** For $\rho_1, \rho_2 \in \mathrm{End}(M)$ define $\rho_1 \sim \rho_2$,unitary equivanent, if and only if $\rho_2 = \mathrm{Ad}U \circ \rho_1$ for some unitary $U \in M$. We denote by $\mathrm{Sec}(M)$ the quotient of $\mathrm{End}(M)$ by unitary equivalence,and we call elements in $\mathrm{Sec}(M)$ *sectors* of $M$.

We can see that the study of unitary equivalence classes of $M - M$ bimodules is same with the study of the sectors. The class $[\rho]$ ,$\rho \in \mathrm{End}(M)$,will be denoted by $\rho$ sometimes.

There are two basic operations for bimodules, say the relative tensor product and the contragredient bimodule. For $M - M$ bimodules $R_1, R_2$ the relative tensor product is $R_1 \otimes_M R_2$, tensor product as $M$ modules. And $H_{\rho_1} \otimes_M H_{\rho_2} = H_{\rho_1\rho_2}$ (See [Sa]). For $M - M$ bimodule $R$ the contragradient bimodule is $\overline{R} = \{\ \overline{\xi} \mid \xi \in R\}$ via

$$\lambda\overline{\xi} = \overline{\overline{\lambda}\xi}, \quad (\overline{\xi}, \overline{\zeta}) = \overline{(\xi, \zeta)}, \quad m_1 \cdot \overline{\xi} \cdot m_2 = \overline{m_2^*\xi m_1^*}$$

for $\lambda \in \mathbb{C}, \xi, \zeta \in R, m_1, m_2 \in M$.

Let $M \supset N$ be a properly infinite von Neumann algebras on $L^2(M)$, then there exists a common cyclic separating vector $\xi_0$ for $M$ and $N$ ([DM]). Let $J_{N,\xi_0}, J_{M,\xi_0}$ be the modular conjugations of $N, M$ respectively on $L^2(M)$. Set $\gamma = \mathrm{Ad}J_{N,\xi_0}J_{M,\xi_0}$ then $\gamma : M \to N$ and we say that $\gamma$ is the canonocal endomorphism of $M \supset N$, here $\gamma$ does not depend upon the choice of $\xi_0$ up to an inner perturbation (See [Lo1],[Lo2]). And also we can see that $M \supset N \supset \gamma(M)$ is a downward basic construction.

For $\rho \in \mathrm{End}(M)$, we find $\gamma = \gamma_\rho$, the canonical endomorphism of $M \supset \rho(M)$. Then we can find an endomorphism $\overline{\rho}$ on $M$ such that $\rho\overline{\rho} = \gamma$. We say that $[\overline{\rho}]$ is the conjugate sector of $[\rho]$. In this case $\overline{H}_\rho = H_{\overline{\rho}}$ (See [Lo2])

## 3. Irreducibility and irreducible decomposition of sectors.

**Definition 2.** We say that $\rho \in \mathrm{End}(M)$ is *irreducible* if and only if $M \cap \rho(M)' = \mathbb{C}1$

For $\rho_1, \rho_2 \in \mathrm{End}(M)$ choose partial isometries $v_1, v_2 \in M$ such that $v_1v_1^* + v_2v_2^* = 1$. Define

$$\rho(m) = v_1\rho_1(m)v_1^* + v_2\rho_2(m)v_2^* \quad \text{for } m \in M$$
$$[\rho] = [\rho_1] \oplus [\rho_2].$$

Then the definition of $[\rho]$ does not depend on the choice of $v_i$'s.

Let $\rho \in \mathrm{End}(M)$ suct that $M \supset \rho(M)$ is of finite index. Then $M \cap \rho(M)'$ is a finite dimensional algebra ([Ko1], [Ko2]). Let $\{p_j\}_{j=1,\cdots,n}$ be a partition of 1 consisting of minimal projections in $M \cap \rho(M)'$. Choose partial isometries $v_i \in M, i = 1, \cdots, n$, such that $v_iv_i^* = p_i$. Set $\rho_i(m) = v_i^*\rho_i(m)v_i$ then $\rho_i \in \mathrm{End}(M)$ is irreducible and we have the irreducible decomposition of $[\rho]$ as follows

$$[\rho] = [\rho_1] \oplus \cdots \oplus [\rho_n].$$

This decomposition does not depend upon the choice of $p_i$'s and $v_i$'s ([Lo2]).

**Definition 3.** For $\rho \in \mathrm{End}(M)$, we define the *statistical dimension* $d\rho$ of $\rho$ as follows([Lo1], [Lo3]);

$$d\rho = (\text{the minimal index of } M \supset \rho(M))^{1/2}.$$

**Theorem 2.** *(Longo[Lo1], [Lo3] ) Statistical dimensions satisfies the followings;*

(i) $d\rho \in \{2\cos\frac{\pi}{n} \mid n = 3, 4, \cdots\} \cup [4, \infty]$,
(ii) $d\rho = 1$ *if and only if* $\rho \in Aut(M)$,
(iii) $d\rho = d\overline{\rho}$,
(iv) $d(\rho_1\rho_2) = d\rho_1 d\rho_2$,
(v) $d(\sum_{i=1}^{n} \oplus[\rho_i] = \sum_{i=1}^{n} d\rho_i$,

Next theorem says the characterization of the conjugate sector which is an important tool for the Fusion rules. This theorem is due to R. Longo[Lo2] and M. Izumi[Iz3].

**Theorem 3.** *(Characterization of the conjugate sector) Let* $\rho_1, \rho_2$ *be irreducible sectors with* $d\rho_1, d\rho_2 < \infty$, *then the fololowings are equivalent.*

(i) $[id] \prec [\rho_1\rho_2]$,
(ii) $[id] \prec [\rho_2\rho_1]$,
(iii) $[\rho_1] = \overline{[\rho_2]}$.

*In this case* $[\rho_1\rho_2]$ *contains* $[id]$ *with multiplicity one.*

**Corollary 4.** *Let* $\rho, \sigma$ *be sectors such that* $\rho$ *is irreducible and* $d\rho, d\sigma < \infty$. *Then the followings are equivalent.*

(i) $[\rho] \prec [\sigma]$,
(ii) $[id] \prec [\overline{\rho}\sigma]$,
(iii) $[id] \prec [\sigma\overline{\rho}]$.

*Proof.* Let $[\sigma] = [\sigma_1] \oplus \cdots \oplus [\sigma_n]$ be the irreducible decomposition of $[\sigma]$. Assume (i), then $[\rho] = [\sigma_i]$ for some $i$. So that by the theorem we get

$$[id] \prec [\overline{\rho}\sigma_i], [id] \prec [\sigma_i\overline{\rho}],$$

hence we have (ii),(iii)

Now we show (ii)→(i) ((iii)→(i) is similar). By (ii) we have $[id] \prec [\overline{\rho}\sigma_i]$ for some $i$, hence by the theorem 3 $[\rho] = [\sigma_i]$. So that $[\rho] \prec [\sigma]$.

**Theorem 5.** *(Frobenius reciprocity) Let* $\rho_1, \rho_2, \rho_3$ *be irreducible sectors with finite statistical dimensions. Then* $\rho_1\rho_2 \succ \rho_3$ *if and only if* $\rho_3\overline{\rho}_2 \succ \rho_1$.

*Proof.* $\rho_1\rho_2 \succ \rho_3 \iff \rho_1\rho_2\overline{\rho}_3 \succ id \iff \rho_3\overline{\rho}_2\overline{\rho}_1 \succ id \iff \rho_3\overline{\rho}_2 \succ \rho_1$.

## 4. Index theory via sector approach.

Let $M \supset N$ be factor-subfactor pair of finite index, and let

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

be the tower obtained by iterating the basic construction([Jo]). Then we get a chain of finite dimensional algebras;

$$N \cap N' \subset M \cap N' \subset M_1 \cap N' \subset M_2 \cap N' \subset \cdots$$

**Definition 4.** ([GHJ]) The *principal graph* of $M \supset N$ is the bipartite multigraph constructed as follows: On the Bratteli diagram of the derived tower delete on each floor the vertices belonging to the old stuff, and the edges emanating from these vertices.

Dual principal graph of $M \supset N$ is defined similarly using the Bratteli diagram of the following chain of finite dimensional algebras:

$$M \cap M' \subset M_1 \cap M' \subset M_2 \cap M' \subset \cdots$$

**Example 1.** Suppose that the Bratteli diagram of $N \cap N' \subset M \cap N' \subset M_1 \cap N' \subset M_2 \cap N' \subset \cdots$ is given as follows,

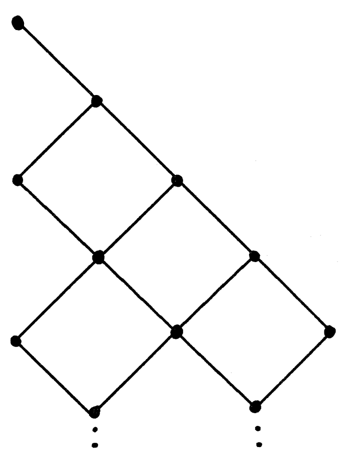

Figure 1.

then the principal graph of $M \supset N$ is $D_6$ ;

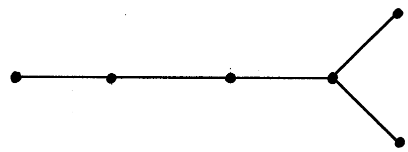

Figure 2.

If $[M : N] < 4$ then the principal dual graph is isomorphic to the dual principal graph (See [Oc]). In this case the principal graph of $M \supset N$is one of $A_n (n \geq 2), D_n (n \geq 2), E_6, E_8$ (See [GHJ],[Oc],[Iz3]).Principal graphs and dual principal graphs are good invariance for studying subfactors ([KS], [Ko3], [Ko4], [Iz1], [Iz2], [IK], [Ka1], [Ka2] and many others)

Let $N \subset M \subset M_1 \subset M_2 \subset \cdots$ be the tower for $M \supset N$, then

$$M_1 = J_M N' J_M = J_M J_N N J_N J_M = \mathrm{Ad}(J_M J_N)(N),$$
$$M_2 = J_{M_1} M' J_{M_1} = J_{M_1} J_M M J_M J_{M_1} = \mathrm{Ad}(J_{M_1} J_M)(M).$$

Let $\xi_0$ be a common cyclic separating vector for $M \supset N$ and let $J_M, J_N$ be modular conjugations such that $J_M(\xi_0) = J_N(\xi_0) = \xi_0$, then $\xi_0$ is also a cyclic separating vector for $M_1 \subset B(L^2(M))$. Thus $J_{M_1} = J_M J_N J_M$, hence we get $M_2 = J_{M_1} J_M M J_M J_{M_1} = \mathrm{Ad}(J_M J_N)(N)$. Similarly we have the following basic extensions ;

$$\begin{aligned} N \subset M \subset M_1 = \mathrm{Ad}(J_M J_N)(N) \subset M_2 = \mathrm{Ad}(J_M J_N)(M) \\ \subset M_3 = \mathrm{Ad}(J_M J_N)^2(N) \subset M_4 = \mathrm{Ad}(J_M J_N)^2(M) \\ \vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}$$

We see that $M \supset N \supset \gamma(M)$ is a downward basic construction. Now

$$\begin{aligned} M_1 \supset N \supset J_N M_1' J_N &= J_N J_{M_1} M_1 J_{M_1} J_N \\ &= J_N J_M J_N J_M \mathrm{Ad}(J_M J_N)(N) J_M J_N J_M J_N \\ &= \mathrm{Ad}(J_N J_M)(N) = \gamma(N) \\ M_2 \supset N \supset J_N M_2' J_N &= J_N J_{M_2} M_2 J_{M_2} J_N \\ &= J_N J_{M_1} J_M J_{M_1} M_2 J_{M_1} J_M J_{M_1} J_N \\ &= J_N J_M J_N J_M J_N J_M M_2 J_M J_N J_M J_N J_M J_N \\ &= J_N J_M J_N J_M J_N J_M \mathrm{Ad}(J_M J_N)(M) J_M J_N J_M J_N J_M J_N \\ &= J_N J_M J_N J_M M J_M J_N J_M J_N \\ &= \mathrm{Ad}(J_N J_M)^2(M) = \gamma^2(M) \end{aligned}$$

So that the iterating downward basic construction , called the tunnel for $M \supset N$, is as follows:

$$M \supset N \supset \gamma(M) \supset \gamma(N) \supset \gamma^2(M) \supset \gamma^2(N) \supset \gamma^3(M) \supset \gamma^3(M) \supset \cdots$$

We write as $\gamma(M) = N_1, \gamma(N) = N_2, \gamma^2(M) = N_3, \gamma^2(N) = N_4, \gamma^3(M) = N_5, \cdots$. If $\rho \in \mathrm{End}(M), N = \rho(M)$ then $\gamma = \rho\overline{\rho}$. So that the tunnel for $M \supset \gamma(M)$ is given as follows;

$$M \supset \rho(M) \supset \rho\overline{\rho}(M) \supset \rho\overline{\rho}\rho(M) \supset \rho\overline{\rho}\rho\overline{\rho}(M) \supset \rho\overline{\rho}\rho\overline{\rho}\rho(M) \supset \cdots$$

Now we will cosider the irreducible decompositions of $\rho, \rho\overline{\rho}, \rho\overline{\rho}\rho, \cdots$. Since $M_1 = \mathrm{Ad}(J_M J_N)(N)$ ,we have $\mathrm{Ad}J_M(M \cap \rho(M)') = M' \cap (\mathrm{Ad}J_M(N)') = M' \cap (\mathrm{Ad}(J_M J_N)(N)) = M' \cap M_1$. Similary we get the followings ;

$$\begin{aligned}
M \cap \rho(M)' &= M \cap N' = \mathrm{Ad}J_M(M_1 \cap M') \\
M \cap \rho\overline{\rho}(M)' &= M \cap N_1' = \mathrm{Ad}J_M(M_2 \cap M') \\
M \cap \rho\overline{\rho}\rho(M)' &= M \cap N_2' = \mathrm{Ad}J_M(M_3 \cap M') \\
M \cap \rho\overline{\rho}\rho\overline{\rho}(M)' &= M \cap N_3' = \mathrm{Ad}J_M(M_4 \cap M') \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

Next we will cosider the irreducible decompositions of $\overline{\rho}\rho, \overline{\rho}\rho\overline{\rho}, \overline{\rho}\rho\overline{\rho}\rho, \cdots$. Since $M \cap \overline{\rho}(M)' \cong \rho(M) \cap \rho\overline{\rho}(M)' = N \cap N_1'$ and $\mathrm{Ad}J_N(N \cap N_1') = N' \cap \mathrm{Ad}J_N\gamma(M)' = N' \cap \mathrm{Ad}J_N\mathrm{Ad}(J_N J_M)(M)' = N' \cap M$, similarly we get the followings;

$$\begin{aligned}
M \cap \overline{\rho}(M)' &\cong \rho(M) \cap \rho\overline{\rho}(M)' = N \cap N_1' = \mathrm{Ad}J_N(M \cap N') \\
M \cap \overline{\rho}\rho(M)' &\cong \rho(M) \cap \rho\overline{\rho}\rho(M)' = N \cap N_2' = \mathrm{Ad}J_N(M_1 \cap N') \\
M \cap \overline{\rho}\rho\overline{\rho}(M)' &\cong \rho(M) \cap \rho\overline{\rho}\rho\overline{\rho}(M)' = N \cap N_3' = \mathrm{Ad}J_N(M_2 \cap N') \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

So that the dual principal graph of $M \supset \rho(M)$ corresponds to the irreducible decompositions of $\rho, \rho\overline{\rho}, \rho\overline{\rho}\rho, \cdots$ and the dual principal graph of $M \supset \rho(M)$ corresponds to the irreducible decomposition of $\overline{\rho}\rho, \overline{\rho}\rho\overline{\rho}, \overline{\rho}\rho\overline{\rho}\rho, \cdots$.

**Example 2.** Suppose that the principle graph of $M \supset \rho(M)$ is $E_6$, then the diagram of the irreducible decompositions of $id, \overline{\rho}, \overline{\rho}\rho, \overline{\rho}\rho\overline{\rho}, \cdots$ is given as follows;

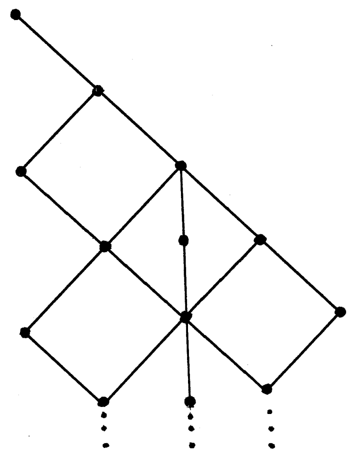


Figure 3.

By the characterization of conjugate sectors, we get $\overline{\rho}\rho \succ id$. Then $\overline{\rho}\rho = id \oplus \alpha$ for some irreducible sector $\alpha$ by the diagram. Since $\overline{\rho}\rho\overline{\rho} = \overline{\rho} \oplus \alpha\overline{\rho}$ and $\overline{\rho}$ is irreducible the left one in level $\overline{\rho}\rho$ is $id$ and the right one is $\alpha$ and $\alpha\overline{\rho}$ is decomposed as sum of three irreducible sectors. By the Frobenius reciprocity $\overline{\rho}\rho \succ \alpha$ implies $\alpha\overline{\rho} \succ \overline{\rho}$. So that $\alpha\overline{\rho} = \overline{\rho} \oplus \beta \oplus \delta$ and $\overline{\rho}\rho\overline{\rho} = 2\overline{\rho} \oplus \beta \oplus \delta$ for some irreducible sectors $\beta, \delta$. In diagram we let the center one be $\beta$ , right one be $\delta$ in level $\overline{\rho}\rho\overline{\rho}$.

Now $\overline{\rho}\rho\overline{\rho}\rho = 2\overline{\rho}\rho \oplus \beta\rho \oplus \delta\rho = 2id \oplus 2\alpha \oplus \beta\rho \oplus \delta\rho$. Since $\alpha\overline{\rho} = \overline{\rho} \oplus \beta \oplus \delta$ , we get $\beta\rho \succ \alpha, \delta\rho \succ \alpha$ by the Frobenius riciprocity. So that by the diagram we get the followings;

$$\beta\rho = \alpha, \ \delta\rho = \alpha \oplus \epsilon, \ \overline{\rho}\rho\overline{\rho}\rho = 2id \oplus 4\alpha \oplus \epsilon$$

for some irreducible sector $\epsilon$. And

$$\overline{\rho}\rho\overline{\rho}\rho\overline{\rho} = 2\overline{\rho} \oplus 4\alpha\overline{\rho} \oplus \epsilon\overline{\rho} = 2\overline{\rho} \oplus 4(\overline{\rho} \oplus \beta \oplus \delta) \oplus \epsilon\overline{\rho} = 6\overline{\rho} \oplus 4\beta \oplus 4\delta \oplus \epsilon\overline{\rho}$$

By the diagram $\epsilon\overline{\rho}$ is irreducible and by $\delta\rho \succ \epsilon$ we get $\epsilon\overline{\rho} \succ \delta$. So $\epsilon\overline{\rho} = \delta$ and $\overline{\rho}\rho\overline{\rho}\rho\overline{\rho} = 6\overline{\rho} \oplus 4\beta \oplus 5\delta$ and we get the following diagram of fusion rules in case of $E_6$

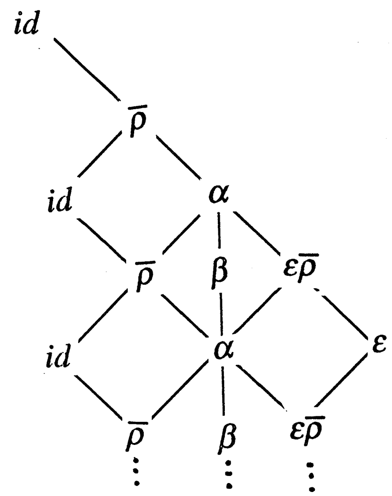


Figure 4.

**Example 3.** Suppose that the dual principle graph of $M \supset \rho(M)$ is $D_6$, then the diagram of the irreducible decompositions of $id, \rho, \rho\overline{\rho}, \rho\overline{\rho}\rho, \cdots$ is given as follows;

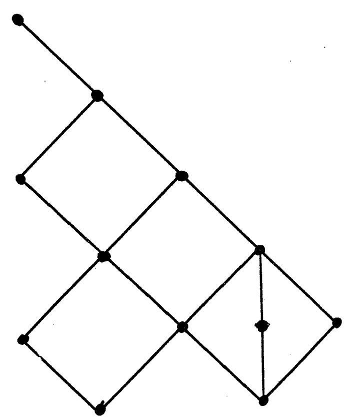

Figure 5.

By the similar method of Example 1 we can get the following diagram of fusion rules.

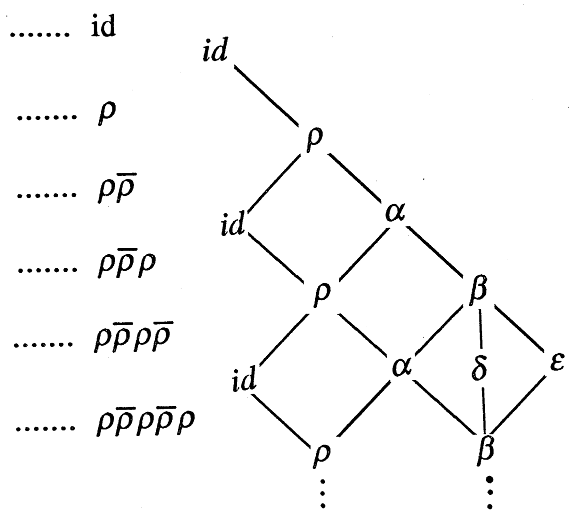


Figure 6.

**Theorem 6.** *(Peron-Frobenius Theorem)([GHJ])For a matrix with non negative entries there exists unique eigen value which is maximum in modulus,this eigen value is nonnegative and of multiplicity one, such eigenvalue is called PF-elgenvalue.*

*In this case we can choose an eigenvector with positive entries for the PF-eigenvalue.*

**Example 4.** Suppose that the principle graph of $M \supset \rho(M)$ is $A_4$, then by the similar method of Example 1 we get the fusion rule as follows;

$$1 \cdot \rho = \rho, \rho \cdot \overline{\rho} = 1 \oplus \alpha, \alpha \cdot \rho = \rho \oplus \beta, \beta \cdot \overline{\rho} = \alpha$$

By theorem 2, we get

$$d1 \cdot d\rho = d\rho, d\rho \cdot d\rho = d1 \oplus d\alpha, d\alpha \cdot d\rho = d\rho \oplus d\beta, d\beta \cdot d\rho = d\alpha$$

So that

$$A \begin{bmatrix} d1 \\ d\rho \\ d\alpha \\ d\beta \end{bmatrix} = d\rho \begin{bmatrix} d1 \\ d\rho \\ d\alpha \\ d\beta \end{bmatrix}$$

where $A$ is the incident matrix of $A_4$. Hence $d\rho$ is the PF-eigenvalue of the incident matrix $A$.

**Theorem 7.** *Let $\rho \in End(M)$ and $M \supset \rho(M)$ be a factor-subfactor pair with finite index. Then $d\rho$ is the PF-eigenvalue of the incident matrix of the principal graph of $M \supset \rho(M)$.*

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