수 학 강 의 록 제 14 권

THE CAUCHY PROBLEM FOR CONVOLUTION EQUATIONS

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Notes of the Series of Lectures
held at the Seoul National University

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펴낸날 : 1994년 4월 20일
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Preface

This monograph on Convolution equations and the Cauchy problem is based on the series of 12 lectures delivered at Seoul National University from January 6 to February 9, 1993.

The series of lectures were organized by Global Analysis Research Center (GARC) under the financial support of Korean Institute of Science and Technology (KIST).

I want to express my deep gratitude to Professor Dohan Kim and his young collaborators for great patience and morale support. I also want to express my deep gratitude to Miss Jin Young Bai who managed to type my horrible hand-written manuscript.

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Seoul, February 10, 1993

Introduction

The lectures I want to give are based on the recent book by Prof. Simon Gindikin and myself. "Distributions and convolution equations". The initial task of our investigations summed up in the book was to give up-to-date interpretation to the classic work by I. G. Petrovskii "On the Cauchy problem for systems of linear partial differential equations for non-analytic functions" of 1938.

Petrovskii considered the Cauchy problem for system of differential operators on functions in $\mathbb{R}^{n+1}$. To shorten the notation I shall formulate his result for a scalar differential operator

$$P(t, D_x, D_t) = D_t^m + \sum_{j=1}^{m} P_j(t, D_x) D_t^{m-j}.$$

Here $t \in \mathbb{R}$ is time and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are the space variables. As usual $D_t = -i\partial/\partial t, D_x = (D_1, \ldots, D_n), D_j = -i\partial/\partial x_j$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, then $|\alpha| = \alpha_1 + \cdots + \alpha_n, D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$.

Following Hadamard Petrovskii says that the problem

$$(1)\ P(t; D_t, D_x)u(x,t) = f(x,t), x \in \mathbb{R}^n, 0 \le t \le T,$$
$$(2)\ D_t^k u(x,0) = \varphi_k(x), \quad k = 0, \ldots, m-1, x \in \mathbb{R}^n$$

is correctly posed if for any right-hand side $\{f, \varphi_o, \ldots, \varphi_{n-1}\}$ bounded together with a finite number of derivatives there exists a unique solution $u(x,t)$ bounded on the interval $0 \le t \le T$ together with the given finite number of derivatives, the operator $\{f, \varphi_o, \ldots, \varphi_{m-1}\} \to u$ being continuous in the corresponding norms.

Petrovskii proved that the problem (1), (2) is well-posed (in the above sense) if and only if the ordinary differential operator $P(t, \frac{1}{i}\frac{d}{dt}, \xi)$ obtained after formal Fourier transform in (1) has the fundamental system of solutions $\{W_j(t,\xi), j = 1, \ldots, m\}$ which admits on the interval $[O, T]$ the estimates with constants increasing non faster than power of $|\xi|$. In the case of constant coefficients Petrovskii condition is equivalent to the algebraic condition for the polynomial (symbol) $P(\tau, \xi)$: $\exists \gamma_0$ such that

$$P(\tau, \xi) \neq 0, \quad Im\tau < \gamma_0, \quad (Re\tau, \xi) \in \mathbb{R}^{n+1}.$$

L. Schwartz showed that within the framework of his distribution theory, the theory of I. G. Petrovskii admit of a natural generalization, namely that the Petrovskii condition is a necessary and sufficient condition for the correctness of the Cauchy problem in the space S' of tempered distributions smooth with respect to time. Schwartz also noticed that it is reasonable to broaden the class of equations (systems) under consideration by replacing differential operators with respect to spatial variables by more general operators of convolution with distributions.

The indicated relationship between the question for the Cauchy problem to be well-posed and the theory of distributions and the passage to convolution equations is of conceptual importance. Proceeding from this relationship we describe the general scheme for delivering results of the type of Petrovskii theorem and present the structures responsible for such results.

We shall encounter a situation which will repeat many times in various circumstances. We start from the linear topological space Φ of C^∞ functions. We describe in detail the scales of spaces of functions and distributions closely related to Φ. In terms of these spaces and operations of inductive and projective limits the description of the space $\mathfrak{C}(\Phi)$ of convolutors on Φ is given. $\mathfrak{C}(\Phi)$ can be transformed into an algebra by introducing the operation of convolution for its elements. If Φ is a "good" space the space $\mathfrak{C}(\Phi)$ admits the full description, and the solvability of the convolution equation

$$A * u = f \qquad u, f \in \Phi, A \in \mathfrak{C}(\Phi)$$

turns out to be equivalent to the existence of a fundamental solution G

$$A * G = \delta(x),$$

which is a convolutor, or in other words A is an invertible element of the algebra $\mathfrak{C}(\Phi)$.

Due to the description of convolutors algebra $\mathfrak{C}(\Phi)$ and, particularly, their Fourier images $F\mathfrak{C}(\Phi)$ the solvability condition becomes effective. It means that the symbol $\hat{A}$ is an invertible element of the algebra $F\mathfrak{C}(\Phi)$ (with respect to multiplication). In the case of a differential operator the Seidenberg-Tarski theorem makes it possible to simplify the conditions since the required algebraic estimates are fulfilled automatically.

In the above described results an important factor manifests itself: the conditions of solvability of convolution equations in the scales and in the "limiting" spaces, say, Φ are equivalent.

In the beginning we shall realize this program for the case when Φ is L. Schwartz space $\mathcal{S}$.

The second more difficult step is connected with the Cauchy problem in Φ with zero Cauchy data. We separate out one of the coordinates, say $t = x_1$, in $\mathbb{R}^n$ and denote by Φ_+ the subspace of Φ consisting of functions vanishing for $t < 0$. We shall thoroughly study scales of such functions and distributions and describe their Fourier transforms (the Paley-Wiener theorems). We define the operation of convolution and convolutor spaces $\mathfrak{C}(\Phi_+)$. Then equation

$$A * u_+ = f_+ \qquad u_+, f_+ \in \Phi_+, A \in \mathfrak{C}(\Phi_+)$$

is solvable if and only if A is an invertible element of the algebra $\mathfrak{C}(\Phi_+)$. Using Fourier-Laplace transform we come to the condition for the symbol of A.

The conjugate space of the space Φ_+ of functions (distributions) with support in the half-space $t > 0$ is the factor space $(\Phi')_\oplus = \Phi'/(\Phi')_-$.

Along with the theory of convolutors and convolution equations in Φ_+ we can construct the conjugate theory for factor spaces $\Phi_\oplus$.

Denote by T_a the translation operator $\varphi(t, y) \longrightarrow \varphi(t + a, y)$. Then $\Phi_a = T_{-a}\Phi_+$ is a subspace of those elements of Φ which are equal to 0 for $t \leq a$.

For $a < b$ we put:

$$\Phi[a, b) = T_{-a}\Phi_+/T_{-b}\Phi_+$$

and we shall study the theory of convolution equations in $\Phi[a, b)$. This theory corresponds to the Cauchy problem in a finite strip $a \leq t <$b with zero Cauchy data for t=a.

To study the Cauchy problem with arbitrary Cauchy data we use following notations. Let $\Phi_{[+]}$ be a space of elements of Φ "cut off" for $t < 0$, i.e.

$$\Phi_{[+]} = \{\theta_+(t)\varphi,\ \varphi \in \Phi, \theta_+(t) = 1 \quad \text{for} \quad t > 0,\ \theta_+(t) = 0 \quad \text{for} \quad t < 0\}.$$

We denote by $\Phi_{[+]}^{\{-\infty\}}$ the space of the elements of the form $Q(D)\varphi_+$, $\varphi_+ \in \Phi_{[+]}$, where $Q(D)$ are arbitrary partial differential operators. The elements of this space are of the form

$$\Phi_{[+]}^{\{-\infty\}} = \left\{ \psi = \sum_{j=0}^{m} \varphi_j(x) D_t^j \delta(t) + \varphi_+, \varphi_+ \in \Phi_{[+]} \right\},$$

and they are smooth functions for $t > 0$. It is possible to describe the convolutors algebra $\mathfrak{C}(\Phi_{[+]}^{\{-\infty\}})$ closely related to pseudodifferential operators with transmission property of Vishik-Eskin and Boutet de Monvel. Then along the same lines as above we can construct the theory of convolution equations in these spaces. This theory, as a particular (and important) case contains classical (nonhomogenuos) Cauchy problem.

As it was mentioned above the lecture notes are based on the monograph [1], there can be found all the references and details of the proofs. Basic facts concerning distributions can be found in [4-6].

Chapter 1. Convolution equations in $S(\mathbb{R}^n)$ and in the related spaces of functions and distributions

§ 1. Preliminaries, Space S

1.1. To construct the space S we introduce spaces $C_{(\ell)}^{(m)} = C_{(\ell)}^{(m)}(\mathbb{R}^n)$ of m-times continuously differentiable functions with a finite norm

$$(1)\qquad |\varphi|_{(\ell)}^{(m)} = \sup_{x\in\mathbb{R}^n, |\alpha|\leq m} (1+|x^2|)^{\ell/2}|D^\alpha \varphi(x)|, \quad m\in\mathbb{Z}_+, \quad l\in\mathbb{R}.$$

$C_{(\ell)}^{(m)}$ is a complete Banach space. For $m=0$ or $l=0$ the corresponding index is dropped (i.e. we write $C_{(\ell)}$ or $C^{(m)}$). It is easy to show that for different values of ℓ the spaces $C_{(\ell)}^{(m)}$ are isomorphic and the isomorphisms are specified by operators of multiplication by a function. We need an elementary

Lemma 1. Let $a(x)$ be an m-times continuously differentiable function satisfying for $|\alpha|\leq m$, the inequalities

$$|D^\alpha a(x)| \leq K_\alpha (1+|x|^2)^{\tau/2}.$$

Then the operator

$$C_{(\ell)}^{(m)} \longrightarrow C_{(l-\tau)}^{(m)} (\varphi \longrightarrow a\varphi)$$

is continuous.

Proof follows immediately from Leibniz' formula.

Proposition. (i) *For any l, $\tau\in\mathbb{R}$ the mapping*

$$(2)\qquad C_{(\ell)}^{(m)} \longrightarrow C_{(\ell-\tau)}^{(m)} (\varphi \longrightarrow (1+|x|^2)^{\ell/2}\varphi)$$

is an isomorphism.

(ii) *The inclusion $\psi\in C_{(\ell)}^{(m)}$ holds if and only if*

$$\psi = (1+|x|^2)^{-\ell/2}\varphi, \quad \varphi\in C^{(m)},$$

and there is a constant $K_{m\ell} > 0$ such that

$$K_{ml}^{-1}|\varphi|^{(m)} \leq |\psi|_{(\ell)}^{(m)} \leq K_{ml}|\varphi|^{(m)}.$$

Proof. (i) Lemma 1 implies the continuity of the operator (2) and of the operator

$$(2') \qquad C_{(l-\tau)}^{(m)} \longrightarrow C_{(\ell)}^{(m)}(\psi \longrightarrow (1+|x|^2)^{-\tau/2}\psi).$$

Since the compositions of (2) and (2′) are identity operators, the proposition is proved.

(ii) follows form (i) for $\tau = \ell$. □

The proposition makes it possible to extend to all the spaces $C_{(\ell)}^{(m)}$ the properties proved for a fixed ℓ, say $\ell = 0$. Obviously, for a function $\varphi \in C_{(\ell)}^{(m)}$ all the norms $|\varphi|_{(\ell')}^{(m')}$ are finite for $m' \leq m$ and $l' \leq l$, and we have

$$|\varphi|_{(\ell')}^{(m')} < |\varphi|_{(\ell)}^{(m)}, \qquad m' \leq m, \quad \ell' \leq \ell.$$

Thus the continuous embeddings

$$i_{m\ell}^{m'\ell'} : C_{(\ell)}^{(m)} \subset C_{(\ell')}^{(m')}, \qquad m' \leq m, \quad l' \leq l$$

take place and we have a scale $\{C_{(\ell)}^{(m)}, i_{lm}^{l'm'}\}$ (see § 3).

Let us discuss the density properties of these embeddings.

Denote by $\mathcal{D}$ the Schwartz' space of C^∞ functions of compact support equipped with the natural topology.

The space $\mathcal{D}$ is nondense in $C_{(\ell)}^{(m)}$. For $l = 0$ this follows from the fact that $1 \in C^{(m)}$ and that for any function $\varphi \in \mathcal{D}$

$$|1-\varphi|^{(m)} \geq \sup_{x\in\mathbb{R}^n} |1-\varphi(x)| \geq 1.$$

Lemma 2. For any $\ell > \ell'$ the space $\mathcal{D}$ is dense in $C_{(\ell)}^{(m)}$ relative to the topology of $C_{(\ell')}^{(m)}$.

Proof. The elements $\varphi \in C_{(\ell)}^{(m)}$ can be approximated in $C_{(\ell')}^{(m)}$ norm $(l' < l)$ by elements of this space having compact support (we can use "cut off" functions). Then we can use the traditional regularization technique.

Denote by $\overset{\circ}{C}{}_{(\ell)}^{(m)}$ the closure of $\mathcal{D}$ in $C_{(\ell)}^{(m)}$ – norm. As we mentioned above, $\overset{\circ}{C}{}_{(\ell)}^{(m)}$ is a proper subspace of $C_{(\ell)}^{(m)}$, and according to lemma 2, we have inclusions

$$\overset{\circ}{C}{}_{(\ell)}^{(m)} \subset C_{(\ell)}^{(m)} \subset \overset{\circ}{C}{}_{(\lambda)}^{(\mu)}, \quad \mu \leq m, \quad \lambda < \ell. \tag{3}$$

1.2. The space $\mathcal{S}$ regarded as a vector space is the intersection of the spaces $C_{(\ell)}^{(m)}$ or according to (3) the subspaces $\overset{\circ}{C}{}_{(\ell)}^{(m)}$:

$$\mathcal{S} = \bigcap_{m \in \mathbb{Z}_+, \ell \in \mathbb{R}} C_{(\ell)}^{(m)} \overset{\text{def}}{=} C_{(\infty)}^{(\infty)}, \quad \mathcal{S} = \cap \overset{\circ}{C}{}_{(\ell)}^{(m)}. \tag{4}$$

The same space is obtained if we confine ourselves to the integral values of ℓ in (4).

Using the system of norms (1) we introduce the structure of a countably normed space in $\mathcal{S}$, i.e. the system of neighborhoods in $\mathcal{S}$ is determined by

$$|\varphi|_{(\ell)}^{(m)} < \varepsilon, \quad m, l \in \mathbb{Z}_+, \quad \varepsilon \in \mathbb{R}_+. \tag{5}$$

The topology generated by these neighborhoods can be interpreted as the topology of the projective limit of the spaces $C_{(\ell)}^{(m)}$ (See § 3).

Using the operators of intersection and union we can introduce a number of vector spaces of functions which are important for our further aims:

$$C_{(\ell)}^{(\infty)} = \bigcap_m C_{(\ell)}^{(m)}, \quad C_{(\infty)}^{(m)} = \bigcap_\ell C_{(\ell)}^{(m)}, \quad C_{(-\infty)}^{(m)} = \bigcup_\ell C_{(\ell)}^{(m)}.$$

We shall turn these spaces into topological linear spaces by endowing them with the topologies of projective and inductive limits. Applying the operations of intersection and union to these topological spaces we form the spaces

$$\mathcal{O} = \bigcap_\ell C_{(\ell)}^{(\infty)}, \quad \mathcal{M} = \bigcap_m C_{(-\infty)}^{(m)}$$

which play an important role in what follows.

The space $\mathcal{O}$ consists of C^∞ functions $\varphi(x)$ whose all derivatives $D^\alpha\varphi(x)$ increase for $|x| \longrightarrow \infty$, not stronger than a fixed power of $|x|$ depending on φ solely. The space $\mathcal{M}$ consists of C^∞ functions whose every derivative is estimated with the aid of a specific power of $|x|$. If it is obvious that

$$\mathcal{O} \subset \mathcal{M}$$

and the inclusion is strict what show the example

$$w(x) = \exp(i(x_1^2 + \cdots + x_n^2)) \in \mathcal{M}, \quad w(x) \notin \mathcal{O}.$$

The definitions of $\mathcal{S}$ and $\mathcal{M}$ immediately imply

Proposition. *(i) $\mathcal{M}$ is a commutative ring relative to multiplication, i.e.*

$$a(x), b(x) \in \mathcal{M} \Longrightarrow a(x)b(x) \in \mathcal{M}.$$

(ii) $\mathcal{S}$ is an ideal of $\mathcal{M}$, i.e. the operation of multiplication is defined:

$$\mathcal{M} \times \mathcal{S} \longrightarrow \mathcal{S}(\{a(x), \varphi(x)\} \longmapsto a(x)\varphi(x)).$$

As usual we define the conjugate space $\mathcal{S}'$ as the space of continuous linear functionals on $\mathcal{S}$ equipped with the strong topology of the conjugate space (See §3)). We denote by (f, φ) the value of $f \in \mathcal{S}'$ on the element $\varphi \in \mathcal{S}$. There exists a canonical embedding of $\mathcal{S}$ into $\mathcal{S}'$

$$\mathcal{S} \longrightarrow \mathcal{S}'(f(x) \longmapsto (f, \varphi) = \int f(x)\varphi(x)dx, \quad \varphi \in \mathcal{S}),$$

so we can understand $\mathcal{S}$ as a subset (dense) in $\mathcal{S}'$.

1.3. Fourier transform in $\mathcal{S}$ is given by the classic formula

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-i\langle x, \xi\rangle)\varphi(x)dx. \tag{6}$$

Since the integral is absolutely convergent and the convergence is retained when $\varphi(x)$ is replaced by $x^\alpha D^\beta \varphi(x)$ for any α and β, the differentiation under the integral sign and integration by parts result in

$$\xi^\alpha \partial^\beta \varphi(\xi) = (2\pi)^{n/2} \int (-1)^{|\beta|} \exp(-i\langle x, \xi\rangle) x^\beta D^\alpha \varphi(x) dx.$$

If $n' > n$, then

$$|\xi^\alpha \partial^\beta \hat{\varphi}(\xi)| \leq (2\pi)^{-n/2} \left(\int (1 + |x|)^{-n'} dx \right) \max(1 + |x|)^{n'} |x^\beta \mathcal{D}^\alpha \varphi(x)|,$$

and we have derived the estimate

$$|\hat{\varphi}|_{(m)}^{(\ell)} \leq K(m, l, n, n')|\varphi|_{(l+n')}^{(m)}. \tag{7}$$

From this estimate follows

Lemma. For any integers m, $\ell > 0$ and $n' > n$ the Fourier operator

$$F : C_{(l+n')}^{(m)} \longrightarrow C_{(m)}^{(\ell)} (\varphi \longmapsto \hat{\varphi})$$

is continuous.

Estimates of the type (7) is natural to call "**Parseval's inequalities**". From these inequalities it follows (for the exact definitions see subsection 3.6) the continuity of the Fourier operator

$$F : \mathcal{S}(\mathbb{R}_x^n) \longrightarrow \mathcal{S}(\mathbb{R}_\xi^n)(\varphi \longmapsto \hat{\varphi}). \tag{8}$$

The continuity of this operator and the reflection operator

$$I : \varphi \longrightarrow \varphi(\varphi(x) \longmapsto \varphi(-x)) \tag{9}$$

implies that the composition of these operators

$$IF : \mathcal{S}(\mathbb{R}_\xi^n) \longrightarrow \mathcal{S}(\varphi(\xi) \longmapsto \psi(-x)) \tag{10}$$

is continuous. As is known the composition of (8) and (10) is an identity operator, i.e. the inversion formula holds

$$\varphi(x) = (2\pi)^{-n/2} \int_{R^n} \exp(i\langle x, \xi \rangle)\hat{\varphi}(\xi) d\xi. \tag{6'}$$

Thus (10) is an the inverse of (8) $IF = F^{-1}$, and the mapping (8) is a topological isomorphism.

Based on conjugacy, the Fourier operator in $\mathcal{S}'$ is defined: for $f \in \mathcal{S}'$ we put

$$(11) \qquad (\hat{f}, I\hat{\varphi}) = (f, \varphi).$$

As has already been said $IFS = \mathcal{S}$, and therefore the functional $\hat{f}$ is defined throughout $\mathcal{S}$.

According classical Parseval's theorem the relation (11) (with, (f, φ) replaced by $\int \dots dx$) holds for any $\varphi, f \in \mathcal{S}$. It follows that the Fourier operator on $\mathcal{S}$ commutes with the canonical embedding $\mathcal{S} \longrightarrow \mathcal{S}'$

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{F} & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{S}' & \xrightarrow[F]{} & \mathcal{S}' \end{array}$$

so the diagram is commutative and we can interpret $\hat{f}$ as the integral (6) and as the Fourier transform in the sense of $\mathcal{S}'$.

If we put $\varphi(x) = \hat{\psi}(x)$ where $\psi(\xi) \in \mathcal{S}$, this results in

$$(11') \qquad (\hat{f}, \varphi) = (f, \hat{\varphi}),$$

i.e.

$$(12) \qquad F : \mathcal{S}' \longrightarrow \mathcal{S}'$$

is adjoint operator of (8) and hence is an isomorphism.

1.4. Pseudo differential operators (PDO) in $\mathcal{S}$.

By PDO meant operators having the form

$$(13) \qquad \begin{aligned} (a(x, D)\varphi)(x) &= (2\pi)^{-n/2} \int \exp(i\langle x, \xi\rangle) a(x, \xi)\hat{\varphi}(\xi) d\xi \\ &= F^{-1}_{\xi \to x} a(x, \xi) F_{x \to \xi} \varphi. \end{aligned}$$

The function $a(x, \xi)$ is called the symbol of the operator. If the symbol is a polynomial in ξ, i.e.

$$a(x, \xi) = \sum a_\alpha(x) \xi^\alpha \varphi(x),$$

then using the inversion formula (6′) we can rewrite (13) in the traditional form

$$(a(x,D)\varphi)(x) = \sum_{\alpha} a_\alpha(x) D^\alpha \varphi(x).$$

If the symbol $a(x,\xi)$ does not depend on x, then the corresponding PDO (13) will be called a PDO with constant coefficients:

$$(a(D)\varphi)(x) = F^{-1}_{\xi\to x} a(\xi) F_{x\to\xi}\varphi. \tag{14}$$

Up to now all definitions were quite formal since we did not indicated to what functions the operators (13) and (14) were applied and why the corresponding integrals made sense.

We now consider the case $\varphi \in \mathcal{S}$ and $a \in \mathcal{M}$. Then (14) is a composition of three continuous operators transforming $\mathcal{S}$ into itself and hence is a continuous operator from $\mathcal{S}$ into $\mathcal{S}$. Under the composition of operators (14) their symbols are multiplied:

$$b(D)(a(D)\varphi) = a(D)(b(D)\varphi) = (ab)(D)\varphi \quad a, b \in \mathcal{M}, \varphi \in \mathcal{S}.$$

In the case of polynomials $a(\xi)$ this is compatible with the definition of differential operators with constant coefficients.

By conjugacy the same is true for pseudodifferential operators acting on $\mathcal{S}'$.

§ 2. The scale of Hilbert spaces associated with $\mathcal{S}'$

In the foregoing section we introduced scales of Banach spaces $C^{(m)}_{(\ell)}$ and, applying the operations of inductive and projective limits to these spaces, constructed Schwartz' spaces $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{M}$. In this section we shall consider scales of Hilbert spaces $H^{(s)}_{(\ell)}$ consisting of functions possessing s derivatives (in the sense of distributions) square integrable with weight $(1+|x|^2)^\ell$. We shall establish inclusion relations for the spaces $C^{(m)}_{(\ell)}$ and $H^{(s)}_{(\ell)}$ (Sobolev imbedding theorems) which will be later used to prove that $\mathcal{S}$ is the projective limit of $H^{(m)}_{(\ell)}$, i.e., $\mathcal{S}$ is countably Hilbert space.

The scales of $H^{(s)}_{(\ell)}$ have a natural extension to fractional and negative values of s, and the spaces $H^{(s)}_{(\ell)}$ and $H^{(-s)}_{(-\ell)}$ are dual. This duality will permit us to establish that $\mathcal{S}'$ can be described as the inductive limit of $H^{(s)}_{(\ell)}$, and the space $\mathcal{O}'$ (conjugate to $\mathcal{O}$) is also can be described in terms of union and intersection of these spaces.

2.1. The space $H_{(\ell)}(\mathbb{R}^n)$ (or, simply, $H_{(\ell)}$) consists of locally square integrable measurable functions for which the norm

$$(1) \qquad ||\psi||_{(\ell)} = \left(\int |\psi(x)|^2(1+|x|^2)^\ell dx\right)^{\frac{1}{2}}$$

is finite.

For $\ell = 0$ we obtain the space $L_2(\mathbb{R}^n)\overset{\text{def}}{=}H\overset{\text{def}}{=}H_{(o)}$. It is obvious that for a fixed ψ the norm (1) is strictly increasing function of ℓ, which implies the embeddings $i^{(\ell')}_{(\ell)} : H_{(\ell)} \longrightarrow H_{(\ell')}, \ell' < \ell$, and we can consider a scale $\{H_{(\ell)}, i^{\ell'}_\ell\}$. Using obvious relation

$$||\psi||_{(\ell)} = ||(1+|x|^2)^{\tau/2}\psi||_{(\ell-\tau)},$$

we readily derive.

Proposition 1. (i) *For any* ℓ *and* τ *the mapping*

$$H_{(\ell)} \longrightarrow H_{(\ell-\tau)}(\psi \longmapsto (1+|x|^2)^{\tau/2}\psi)$$

is an isometric isomorphism.

(ii) *The inclusion* $\psi \in H_{(\ell)}$ *holds if and only if*

$$\psi = (1+|x|^2)^{-\ell/2}\varphi, \quad \varphi \in H \quad \textit{and} \quad ||\psi||_{(\ell)} = ||\varphi||.$$

The proposition makes it possible to extend the properties proved for a fixed ℓ to all spaces $H_{(\ell)}$. In particular the completeness of L_2 implies that all $H_{(\ell)}$ are complete.

Lemma. The space $\mathcal{D}$ (and, consequently the space $\mathcal{S}$) is dense in $H_{(\ell)}$ for $\forall \ell \in \mathbb{R}$.

Proof. For the case $\ell = 0$ the lemma is well-known. Since the operator of multiplication by $(1+|x|^2)^{\ell/2}$ transforms $\mathcal{D}$ and $\mathcal{S}$ into themselves, the general case is reduced to the case $\ell = 0$.

The space $H_{(\ell)}$ becomes a Hilbert space if the Hermitian scalar product

$$[\varphi,\psi]_{(\ell)} = ((1+|x|^2)^{\ell/2}\varphi, \quad (1+|x|^2)^{\ell/2}\,\overline{\psi})$$

is introduced in it. Here

$$(\varphi,\psi) = \int \varphi(x)\psi(x)dx. \tag{2}$$

For $\ell = 0$ the script ℓ will be omitted.

Proposition 2. (i) *The bilinear form* (2) is continued by continuity from $\mathcal{S}\times\mathcal{S}$ to $H_{(\ell)}\times H_{(-\ell)}$.

(ii) The embedding

$$H_{(\ell)} \longrightarrow (H_{(\ell)})'(\varphi \longrightarrow (\varphi,\cdot))$$

induced by the continuation of the form (2) is a canonical isometric isomorphism of $H_{(-\ell)}$ and the Banach conjugate space of $H_{(\ell)}$.

Proof. (i) Follows from Schwartz' inequality

$$|(\varphi,\psi)| = |(1+|x|^2)^{-\ell/2}\varphi, \quad (1+|x|^2)^{\ell/2}\psi)| \;\leq\; ||\varphi||_{(-\ell)}||\psi||_{(\ell)}$$

and the lemma.

(ii) For $\ell = 0$, the assertion coincides with well-known Fisher-Riesz theorem.

For $\ell \neq 0$ it should be taken into consideration that, by the conjugacy, proposition 1 implies that the operator of multiplication by $(1+|x|^2)^{\ell/2}$ determines an isometric isomorphism of $(H_{(\ell)})'$ and H', so $(H_{(\ell)})' = (1+|x|^2)^{\ell/2}H' = (1+|x|^2)^{\ell/2}H = H_{(-\ell)}$.

2.2. The spaces $H^{(s)}(\mathbb{R}^n)$ As was already indicated, the Fourier operator $F:\mathcal{S}' \longrightarrow \mathcal{S}'$ is one-to-one and transforms the subset $\mathcal{S}\subset\mathcal{S}'$ into itself. The reflection operator I possesses similar properties. We have:

$$\mathcal{S}\subset H_{(s)}\subset\mathcal{S}'.$$

Denote by $H^{(s)}(\mathbb{R}^n)$ (or, simply, $H^{(s)}$) the image of $H_{(s)}$ under the operator IF:

$$H^s(\mathbb{R}^n_x) = IFH_{(s)}(\mathbb{R}^n_\xi).$$

Since the composition of F and IF is an identity operator in $\mathcal{S}$, we have

$$FH^{(s)}(\mathbb{R}^n_x) = H_{(s)}(\mathbb{R}^n_\xi).$$

We introduce the norm,

$$||f||^{(s)} = ||Ff||_{(s)} \tag{3}$$

in the space $H^{(s)}$. Thus, $H^{(s)}$ consists of those $f \in \mathcal{S}'$, possessing Fourier transform, which are locally square summable and whose norm (3) is finite. On the other hand, since $\mathcal{S}$ is dense in $H_{(s)}$ and the operators F and I transform $\mathcal{S}$ into itself, $\mathcal{S}$ is dense in $H^{(s)}$, and therefore $H^{(s)}$ can be regarded as the completion of $\mathcal{S}$ relative to the norm (3).

If we substitute $f \in \mathcal{S}$ and $\varphi = \overline{f}$ into the relation

$$(\hat{f}, I\hat{\varphi}) = (f, \varphi)$$

and take into account that $(F\overline{f})(-\xi) = \overline{Ff(\xi)}$, then for $f \in \mathcal{S}$ we obtain Parseval's relation

$$\int |f(x)|^2 dx = \int |\hat{f}(\xi)|^2 d\xi. \tag{4}$$

Hence, for s=0 the norms $H^{(s)}$ and $H_{(s)}$ coincide on $\mathcal{S}$, whence it follows that these spaces coincide. Thus, the notation $H^{(0)} = H$ and $H_{(0)} = H$ is not contradictory. The norm in H will be denoted by $|| \quad ||$.

Recalling the definition of the pseudodifferential operator $(1 + |D|^2)^{s/2}$ we find

$$||f||^{(s)} = ||(1 + |D|^2)^{s/2} f||. \tag{3'}$$

The propositions proved for $H_{(s)}$ spaces imply

Proposition. (i) *For any s and τ the mapping*

$$H^{(s)} \longrightarrow H^{(s-\tau)}(f \longmapsto (1 + |D|^2)^{\tau/2} f)$$

is an isometric isomorphism.

(ii) *The inclusion $f \in H^{(s)}$ takes place if and only if*

$$f = (1 + |D|^2)^{-s/2} g, \quad g \in H, \quad ||f||^{(s)} = ||g||.$$

(iii) *The bilinear form (2) is continued by continuity from $\mathcal{S} \times \mathcal{S}$ to $H^{(s)} \times H^{(-s)}$. The embedding $H^{(s)} \longrightarrow (H^{(s)})'$ induced by this form is an isometric isomorphism of $H^{(-s)}$ and the Banach conjugate space of $H^{(s)}$:*

$$(H^{(s)})' = H^{(-s)}. \tag{5}$$

The space $H^{(s)}$ can be turned into Hilbert space by introducing the Hermitian scalar product

$$[f, g]^{(s)} = [Ff,\ Fg]_{(s)} = \left[(1 + |D|^2)^{s/2} f, \quad (1 + |D|^2)^{s/2} g\right].$$

Remark. For $s > 0$ the representation in proposition (ii) can be rewritten in the form

$$f(x) = \int \mathcal{G}_s(x - y) g(y) dy,$$

where $\mathcal{G}_s$ is so called Bessel potential. This representation is extensively used in the theory of Sobolev's spaces.

Using original definition of $H^{(s)}$ or the proposition (ii) we can easily obtain various properties of the spaces $H^{(s)}$. We shall state them without proofs.

1) Let $m > 0$ be an integer. The inclusion $f \in H^{(s)}$ takes place if and only if $D^\alpha f \in H_{(s-m)}, |\alpha| \leq m$ and

$$||f||^{(s)} \approx \left(\sum_{|\alpha| \leq m} ||D^\alpha f||^{(s-m)^2} \right)^{\frac{1}{2}}.$$

In the case $s = m$ we obtain the following statement.

2) For an integer $m > 0$ the inclusion $f \in H^{(m)}$ holds if and only if $D^\alpha f \in L_2, |\alpha| \leq m$, and

$$||f||^{(m)} \approx \left(\sum_{|\alpha| \leq m} ||D^\alpha f||^2 \right)^{\frac{1}{2}}.$$

3) The inclusion $h \in H^{(s)}$ holds if and only for an integer $m > 0$ h can be represented in the form

$$h = \sum_{|\alpha| \leq m} D^\alpha h_{\alpha m},$$

where

$$h_{m\alpha} \in H^{(s+m)}, \quad ||h_{m\alpha}||^{(s+m)} \leq \text{const } ||h||^{(s)}.$$

In the case $s = -m$ we obtain the following statement.

4) Let $m > 0$ be an integer. The inclusion $f \in H^{(-m)}$ takes place if and only if there are $h_\alpha \in L_2, ||h_\alpha|| \leq \text{const}||h||^{(-m)}$ such that h is represented in the form

$$h = \sum_{|\alpha| \leq m} D^\alpha h_\alpha.$$

2.3. The space $H^{(s)}_{(\ell)}(\mathbb{R}^n)$ In subsections 1 and 2 we considered one-parameter families of spaces where the graduation with respect to smoothness or growth (decrease) at infinity was specified by the operators $(1 + |D|^2)^{s/2}$ and $(1+|x|^2)^{\ell/2}$. Now we are going to include these families in a unified two-parameter scale of spaces with graduation with respect to both the smoothness and growth. Since the operators $(1 + |D|^2)^{s/2}$ and $(1 + |x|^2)^{\ell/2}$ do not commute, their order in the definition of the norm is a priori substantial.

We introduce the space

$$H^{(s)}_{(\ell)}(\mathbb{R}^n) = \{f \in \mathcal{S}', (1 + |x|^2)^{\frac{\ell}{2}} f \in H^{(s)}\} \tag{6}$$

and supply it with norm

$$||f||^{(s)}_{(\ell)} = ||(1 + |x|^2)^{\ell/2} f||^{(s)} = ||(1 + |D|^2)^{s/2}(1 + |x|^2)^{\ell/2} f||\,; \tag{7}$$

we similarly consider the space

$$'H^{(s)}_{(\ell)}(\mathbb{R}^n) = \left\{f \in \mathcal{S}', (1 + |D|^2)^{s/2} f \in H_{(\ell)}\right\} \tag{6'}$$

and introduce the norm

$$'||f||^{(s)}_{(\ell)} = ||(1 + |D|^2)^{s/2} f||_{(\ell)} = ||(1 + |x|^2)^{\ell/2}(1 + |D|^2)^{s/2} f||. \tag{7'}$$

Since the operators $(1+|D|^2)^{s/2}$ and $(1+|x|^2)^{\ell/2}$ transform $\mathcal{S}$ and $\mathcal{S}'$ into themselves, the continuous dense embeddings

$$\mathcal{S} \subset H^{(s)}_{(\ell)} \subset \mathcal{S}', \quad \mathcal{S} \subset {}'H^{(s)}_{(\ell)} \subset \mathcal{S}'$$

hold, which make it possible to interpret the spaces $H^{(s)}_{(\ell)}$, ${}'H^{(s)}_{(\ell)}$ as completion of $\mathcal{S}$ relative to norms (6) and (6′), respectively. In trivial manner these spaces can be turned into Hilbert spaces.

Form the results above follows

Proposition 1. (i) *For any* s, ℓ, τ *the mappings*

$$H^{(s)}_{(\ell)} \to H^{(s)}_{(\ell-\tau)}(f \mapsto (1+|x|^2)^{\tau/2} f), \; {}'H^{(s)}_{(\ell)} \to {}'H^{(s-\tau)}_{(\ell)}(f \to (1+|D|^2)^{\tau/2} f)$$

are isometric isomorphisms.

(ii) *The inclusions* $f \in H^{(s)}_{(\ell)}$, ${}'H^{(s)}_{(\ell)}$ *take place if and only if*

$$f = (1+|x|^2)^{-\ell/2}(1+|D|^2)^{-s/2} g, \quad ||f||^{(s)}_{(\ell)} = ||g||,$$

and, respectively,

$$f = (1+|D|^2)^{-s/2}(1+|x|^2)^{-\ell/2} h, \quad {}'||f||^{(s)}_{(\ell)} = ||h||.$$

(iii) *The bilinear form (2) is continued by continuity to* $E^{(s)}_{(\ell)} \times E^{(-s)}_{(-\ell)}, (E = H, {}'H)$ *and induces canonical isometric isomorphisms:*

$$\left(H^{(-s)}_{(-\ell)}\right)' = H^{(-s)}_{(-\ell)}, \quad \left({}'H^{(s)}_{(\ell)}\right)' = {}'H^{(-s)}_{(-\ell)}. \tag{8}$$

Since the Fourier operator transforms $(1+|D|^2)^{s/2}$ *and* $(1+|x|^2)^{s/2}$ *into each other, we have isometric isomorphisms:*

$$FH^{(s)}_{(\ell)} = {}'H^{(\ell)}_{(s)}, \quad F'H^{(s)}_{(\ell)} = H^{(\ell)}_{(s)}. \tag{9}$$

Proposition 2. *The norms (7) and (7′) are equivalent for any s, ℓ, i.e., there is a constant $K = k(s, \ell)$ such that*

$$K^{-1}||f||^{(s)}_{(\ell)} \leq {}'||f||^{(s)}_{(\ell)} \leq K||f||^{(s)}_{(\ell)}, \quad \forall f \in \mathcal{S}. \tag{10}$$

As $\mathcal{S}$ is dense in the spaces $H^{(s)}_{(\ell)}$ and ${}'H^{(s)}_{(\ell)}$ from (10) readily follows that $H^{(s)}_{(\ell)} = {}'H^{(s)}_{(\ell)}$. In what follows the spaces above will be denoted by $H^{(s)}_{(\ell)}$; however, different symbols will be retained for their norms (7) and (7′).

Corollary 1. $FH^{(s)}_{(\ell)} = H^{(s)}_{(\ell)}$ $\forall s, \ell \in \mathbb{R}$.
The proof follows from relations (9).

Corollary 2. The embeddings take place

$$H^{(s)}_{(\ell)} \subset H^{(s')}_{(\ell')} \quad s \geq s', \ell \geq \ell'.$$

Remark. The inequalities (and the corresponding embeddings) $||f||^{(s')}_{(\ell)} \leq ||f||^{(s)}_{(\ell)}$, ${}'||f||^{(s)}_{(\ell')} \leq ||f||^{(s)}_{(\ell)}$ are trivial consequences of definitions (6), (6′). The inequalities

$$||f||^{(s)}_{(\ell')} \leq const\ ||f||^{(s)}_{(\ell)}, \quad {}'||f||^{(s')}_{(\ell)} \leq const\ {}'||f||^{(s)}_{(\ell)}$$

are meaningful, since the first is equivalent to the boundedness of the operator of multiplication by $(1+|x|^2)^{-\varepsilon}\varepsilon > 0$ in any $H^{(s)}$, and the second is equivalent to the boundedness of the PDO $(1 + |D|^2)^{-\varepsilon}, \varepsilon > 0$ in any $H_{(\ell)}$.

For arbitrary s, ℓ the proof of the proposition is based on the calculus of PDO. But for our near aims we can restrict ourselves to the case where s, ℓ are even numbers of any sign.

Proof of Proposition 2. Further we shall assume without a special stipulation, that numbers s and ℓ take even integral values of any sign. By virtue of proposition 1 (ii) the left-hand side in inequality (10) is equivalent to the boundedness of the operator.

$$A_{s\ell} = (1 + |D|^2)^{s/2}(1 + |x|^2)^{\ell/2}(1 + |D|^2)^{-s/2}(1 + |x|^2)^{-\ell/2} : H \longrightarrow H, \tag{11}$$

and the right-hand inequality is equivalent to the boundedness of

(12) $B_{s\ell} = (1+|x|^2)^{\ell/2}(1+|D|^2)^{s/2}(1+|x|^2)^{-\ell/2}(1+|D|^2)^{-s/2} : H \longrightarrow H.$

These operators satisfy the conjugacy relations(with respect to the scalar product in H)

$$A_{s\ell} = (B_{-s,-\ell})', \quad B_{s\ell} = (A_{-s,-\ell})' \tag{13}$$

and the duality relations with respect to the Fourier operators (which follow from the definition of PDO):

$$FA_{s\ell} = B_{\ell s}, \quad FB_{\ell s} = A_{s\ell}. \tag{14}$$

By virtue of (13), (14), if we show that $A_{s\ell} \in \mathcal{L}(H,H)$, this will prove that $B_{-s,-\ell}, B_{\ell s}$ and $A_{-\ell,-s} \in \mathcal{L}(H,H)$. Similarly, from the inclusion $B_{s\ell} \in \mathcal{L}(H,H)$ it follows that $A_{-s,-\ell}; A_{\ell s}; B_{-\ell,-s} \in \mathcal{L}(H,H)$.

By virtue of what has been said, to prove the boundedness of the operators (11), (12) for any s and ℓ, it is sufficient to verify that

$$B_{s\ell} \in \mathcal{L}(H,H) \ \ s \geq 0, \quad -\infty < \ell < \infty, \tag{15}$$

$$A_{s\ell} \in \mathcal{L}(H,H), \quad s \geq 0, \quad \ell \leq 0. \tag{16}$$

As s is even, $(1+|D|^2)^{s/2}$ is a differential operator, we denote it by $P(D)$. Using Leibniz' formula (written in Hörmander's form):

$$P(D)(uv) = \sum \frac{1}{\alpha!} D^\alpha u P^\alpha(D) v, \qquad P^{(\alpha)}(\xi) = \partial^\alpha P(\xi)$$

we can rewrite $B_{s\ell}$ for $s \geq 0$ in the form:

$$B_{s\ell} v = \sum_\alpha \left[(1+|x|^2)^{\ell/2} \frac{1}{\alpha!} D^\alpha (1+|x|^2)^{-\ell/2} \right] \left[\frac{P^{(\alpha)}(D)}{P(D)} \right] v.$$

Since for any τ and α

$$(1+|x|^2)^{\tau/2} |D^\alpha (1+|x|^2)^{-\tau/2}| \leq C(\alpha,\tau,n),$$

the operators in square brackets are bounded in L_2, which implies (13).

The proof of (16) is based on a version of the commutation formula for PDO.

Lemma. If $\lambda(\xi) \in \mathcal{M}$ and $Q(x)$ is a polynomial, then

$$\lambda(D)(Qv) = \sum \frac{1}{\alpha!} Q^{(\alpha)}(x) \lambda^{(\alpha)}(D)v.$$

(As $Q(x)$ is a polynomial, the right-hand sum may contain only finite number of terms.)

2.4. Sobolev embedding theorems for the spaces $C_{(\ell)}^{(m)}$ and $H_{(\ell)}^{(s)}$

Now we shall prove

Theorem. *For any integer $m > 0, \ell \in \mathbb{R}$, and $\forall \kappa > \tau/2$ the following embeddings (with topology) hold:*

$$C_{(\ell+\kappa)}^{(m)} \subset H_{(\ell)}^{(m)}, \tag{17}$$

$$H_{(\ell)}^{(m+\kappa)} \subset C_{(\ell)}^{(m)}. \tag{18}$$

Remarks. 1) As was already mentioned above, for the spaces $C_{(\ell)}^{(m)}$ and $H_{(\ell)}^{(m)}$ there exists a canonical identification with some subsets in $\mathcal{S}'$. Therefore, (17), (18) are in fact reduced to the following two assertions:

(i) the left-hand spaces in (17), (18) are contained, in set-theoretical sense, in the right-hand ones;

(ii) the right-hand spaces in (17), (18) induce topologies on the left-hand spaces which are weaker than the original topologies.

2) It is useful to note that for the Banach spaces (17) and (18) the assertion (ii) is a direct consequence of (i). Indeed, let E and F be, respectively, the left-hand and the right-hand spaces in (17), (18) and let $J : E \longrightarrow F$ be the embedding operator associating with each element $\varphi \in E$ that very element regarded, however, as an element of F. The continuity of the embedding

in the space of distributions $J' : E \longrightarrow \mathcal{S}'$ implies the closedness of this operator. Since it is defined throughout E, by the closed graph theorem, the operator J is continuous, i.e., (ii) holds.

3) Instead of (18) we shall prove in fact a somewhat stronger assertion

$$H^{(m+\kappa)}_{(\ell)} \subset \overset{\circ}{C}{}^{(m)}_{(\ell)}. \tag{18$_\circ$}$$

So left-hand sides in ($18_\circ$), (17) are dense in right-hand ones.

The proof of the theorem. The embeddings ($18_\circ$), (17) are automatic consequences of the inequalities

$$||\varphi||^{(m)}_{(\ell)} \leq K_{m\ell\kappa}|\varphi|^{(m)}_{(\ell+\kappa)} \quad \forall\varphi \in C^{(m)}_{(\ell+\kappa)}, \tag{19}$$

$$|\varphi|^{(m)}_{(\ell)} \leq K'_{m\ell\kappa}||\varphi||^{(m+\kappa)}_{(\ell)} \quad \forall\varphi \in \mathcal{S}. \tag{20}$$

The inequality (19) is proved quite simply. We start from an obvious estimate

$$|g(x)| \leq (1+|x|^2)^{-\kappa}|g|_{(\kappa)}.$$

If $\kappa > n/2$, we can take L_2-norms of both sides and obtain:

$$||g|| \leq K_\kappa |g|_{(\kappa)}, \quad \kappa > n/2.$$

Substitute the function $(1+|x|^2)^{\ell/2}D^\alpha f(x)$, $|\alpha| \leq m$ for g; this yields

$$||D^\alpha f||_{(\ell)} \leq K_\kappa |D^\alpha f|_{(\ell+\kappa)} \leq K_\kappa |f|^{(m)}_{(\ell+\kappa)}.$$

Summing these inequalities over $\alpha, |\alpha| \leq m$ and using property 2) in subsection 2.2 we derive (19).

We now prove (20). According to Fourier's inversion formula, for $\varphi(x) \in \mathcal{S}$ we have

$$|D^\alpha\varphi(x)| \leq (2\pi)^{-n/2}\left|\int \exp(i\langle x,\xi\rangle)\xi^\alpha\hat{\varphi}(\xi)d\xi\right|$$

$$\leq (2\pi)^{-n/2}\left[\int |\xi^\alpha|^2(1+|\xi|^2)^{-m-\kappa}d\xi\right]^{1/2}\left[\int (1+|\xi|^2)^{m+\kappa}|\hat{\varphi}(\xi)|^2 d\xi\right]^{1/2}.$$

For $|\alpha| \leq m$ and $\kappa > n/2$ the first integral on the right-hand side is convergent, whence

$$|D^\alpha \varphi(x)| \leq K_{m,\kappa} ||\varphi||^{(m+\kappa)}.$$

Replacing φ by $(1+|x|^2)^{\ell/2} f$, we obtain the inequality

$$\sup_{|\alpha|\leq m,\ x\in\mathbb{R}^n} |D^\alpha(1+|x|^2)^{\ell/2} f| \leq \text{const } ||f||_{(\ell)}^{(m+\kappa)}.$$

Using Leibniz' formula it is easy to prove that left-hand side norm is equivalent to $|\ \ |_{(\ell)}^{(m)}$-norm.

Remark. The embeddings in the theorem imply corresponding embeddings for the Banach conjugate spaces. Using (8) we obtain the inclusions:

$$H_{(-\ell+\kappa)}^{(-m)} \subset \left(C_{(\ell)}^{(m)}\right)' \subset H_{(-\ell)}^{(-m-\kappa)}, \qquad \kappa > n/2. \tag{21}$$

The embeddings (17) and (18) show that the space $\mathcal{S} = \bigcap C_{(\ell)}^{(m)}$ coincide with the intersection

$$\mathcal{S} = H_{(\infty)}^{(\infty)} = \bigcap H_{(\ell)}^{(s)},$$

and according to embeddings (21) $\mathcal{S}'$ can be realized as a union of $H_{(\ell)}^{(s)}$ spaces:

$$\mathcal{S}' = H_{(-\infty)}^{(-\infty)} = \bigcup H_{(\ell)}^{(s)}.$$

These relations are isomorphisms of linear spaces. As for the equivalence of topologies, this is a rather delicate question. In the following section we shall give a short exposition of the theory of linear topological spaces and the theory of continuous linear operators in them. Then at the end of the section we shall return to the spaces $\mathcal{S}, \mathcal{S}', \mathcal{O}, \mathcal{O}'$.

Appendix to §2

Here we present a new embedding theorem recently found by Jaeyoung Chung, Soon-Yeong Chung and Dohan Kim. We give the proof of this theorem using their original idea and technique developed above.

Theorem. *For arbitrary $s, \ell \in \mathbb{R}$ the embedding takes place*

$$H^{(2s)} \cap H_{(2\ell)} \subset H^{(s)}_{(\ell)}. \tag{1}$$

Corollary. If $s \geq 2\kappa > n$, then the embedding takes place

$$H^{(s)} \cap H_{(\ell)} \subset C^{(\frac{s}{2}-\kappa)}_{(\frac{\ell}{2})}. \tag{2}$$

Proof. Embedding (2) follows from (1) and (18).

Proof of the theorem. The left-hand side space in (1) can be turned into complete Banach space supplied by a natural norm of the intersection of Banach spaces:

$$||f, H^{(s)} \cap H_{(\ell)}|| = ||f||^{(s)} + ||f||_{(\ell)}.$$

The space $\mathcal{S}$ is dense in $H^{(s)} \cap H_{(\ell)}$. So to prove the inclusion (1) it is sufficient to prove the inequality:

$$||u||^{(s)}_{(\ell)} \leq \text{ const } (||u||^{(2s)} + ||u||_{(2\ell)}) \quad \forall u \in \mathcal{S}. \tag{3}$$

To prove this inequality in complete form we need some calculus of PDO. But, as in subsection 2.3 more elementary proof can be given in the case when either s is natural or $\ell/2$ is natural. Indeed:

$$\begin{aligned}\left(||u||^{(s)}_{(\ell)}\right)^2 &= \left((1+|D|^2)^{s/2}(1+|x|^2)^{\ell/2}u, (1+|D|^2)^{s/2}(1+|x|^2)^{\ell/2}\bar{u}\right) \\ &= \left((1+|D|^2)^{s}(1+|x|^2)^{\ell/2}u, (1+|x|^2)^{\ell/2}\bar{u}\right).\end{aligned}$$

Using commutation relation for PDO we can write (compare proposition 2 in subsection 2.3), with $P(\xi) = (1+|\xi|^2)^s$:

$$(1+|D|^2)^s \left[(1+|x|^2)^{\ell/2}u\right] = \sum \frac{1}{\alpha!} D^\alpha (1+|x|^2)^{\ell/2} P^{(\alpha)}(D)u.$$

In our conditions the sum contains only finite number of terms. Then the right-hand side of our relation takes form

$$\begin{aligned}&\sum \frac{1}{\alpha!}\left(P^{(\alpha)}u, D^\alpha(1+|x|^2)^{\ell/2}(1+|x|^2)^{\ell/2}\bar{u}\right) \\ &\leq \sum \frac{C_\alpha}{\alpha!}||u||^{(2s-|\alpha|)}||u||_{(2\ell-|\alpha|)} \leq const||u||^{(2s)}||u||_{(2\ell)}.\end{aligned}$$

So we proved that

$$\left(||u||_{(\ell)}^{(s)}\right)^2 \leq \text{ const } ||u||^{(2s)}||u||_{(2\ell)}.$$

Inequality (3) is the consequence of this inequality.

As $H^{(s)} \cap H_{(\ell)} \subset H^{(s')} \cap H_{(\ell')}$, $s \geq s'$, $\ell \geq \ell'$ we can consider the inductive limit $H^{(\infty)} \cap H_{(\infty)}$.

From (1) it follows that

$$H_{(2\ell)}^{(2s)} \subset H^{(2s)} \cap H_{(2\ell)} \subset H_{(\ell)}^{(s)}.$$

Then for the inductive limits we obtain

$$\mathcal{S} = H^{(\infty)} \cap H_{(\infty)}.$$

In the paper of J. Chung, S.-Y. Chung and D. Kim, it is also shown that

$$\mathcal{S} = C^{(\infty)} \cap C_{(\infty)}.$$

§3. Scales of topological linear spaces and their inductive and projective limits

For the sake of convenience of references in this section we first of all make a number of well-known facts of the theory of topological linear spaces (TLS) used in these lectures, these facts are to be found in any text-book devoted to TLS. The presentation of the theory of linear operators in scales and in the limiting spaces of the scales is of a less standard character. As will be shown the operators on the scales generate the operators on the limiting spaces. The converse as a rule is not true. We separate a subclass of regular operators admitting such an extension. One of the main purposes of this section is to show that in the case of the limits of reflexive Banach spaces all continuous operators are regular. In the second part of section the main results of the first part will be applied to the spaces we are interested in: $\mathcal{S}, \mathcal{S}', \mathcal{O}, \mathcal{O}'$.

3.1. Fundamental notions in the theory of TLS

A vector space E (over a field of complex or real numbers) is called a topological linear space if E is endowed with topology in which the operations of addition $\{x, y\} \longmapsto x + y$ and multiplication by scalar $\{x, \alpha\} \longmapsto \alpha x$ are continuous.

From this it follows that a topology in TLS is determined completely by setting a system of neighborhoods of zero.

Generally, a given space can be equipped with different topologies compatible with vector structure. Let there be two topologies on E determined by systems of neighborhoods $\mathcal{T}_1$ and $\mathcal{T}_2$. If each of the neighborhoods of zero $V \in \mathcal{T}_2$ belongs simultaneously to $\mathcal{T}_1$, we say that the topology determined by the system of neighborhoods $\mathcal{T}_1$ or, simply the topology $\mathcal{T}_1$ is stronger that topology $\mathcal{T}_2$, (or $\mathcal{T}_2$ weaker than $\mathcal{T}_1$). If the topology $\mathcal{T}_2$ is simultaneously weaker and stronger than the topology $\mathcal{T}_1$, the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be equivalent.

The setting of a system of neighborhoods of zero in a vector space determines convergence (sequential topology) in it. Namely, a sequence converges in E ($e_n \longrightarrow e$) if for any neighborhood U of zero there is such number N(U) that $e_n - e \in U$ for $n \geq N(U)$.

A set $B \subset E$ is said to be bounded if for any neighborhood U of zero such that $U \subset E$ and there is a number $N(U)$ such that $B \subset \lambda U$ for all $|\lambda| > N(U)$ (i.e., B is contained by any neighborhood of zero). This definition can also be restated thus: for any sequence $\{e_n\} \subset B$ and any sequence λ_n of positive numbers, $\lambda_n \longrightarrow 0$, the sequence $\lambda_n e_n$ converges to zero in E.

A TLS is said to be locally convex if each neighborhood of zero contains a convex subset. We shall consider only locally convex TLS, as a rule we shall not stipulate it.

A system $\mathcal{T}$ of neighborhoods of zero is called a neighborhood base (at zero) of a TLS E if, given any neighborhood U of zero such that $U \subset E$, there is a neighborhood $V \subset \mathcal{T}$ such that $V \subset U$. If a TLS has a countable neighborhood base it is said to satisfy the first axiom of countability.

Let E and F be two LTS. A linear operator $A : E \longrightarrow F$ is said to be continuous if for any neighborhood V of zero such that $V \subset F$ there exists a neighborhood of zero $U \in E$ such that $AU \subset V$. The family of all continuous linear operators from E into F will be denoted $\mathcal{L}(E, F)$.

It follows from the above definitions that if $A \in \mathcal{L}(E, F)$ and $e_n \longrightarrow e$ in E, then $Ae_n \longrightarrow Ae$ in F.

This fact and the description of bounded sets in terms of convergent sequences imply that each continuous operator $A : E \longrightarrow F$ transforms bounded sets in E into bounded sets in F; such operators are called bounded.

For the space E satisfying the first axiom of countability the converse assertion is also true: each bounded operator $A : E \longrightarrow F$ is continuous.

The conjugate space E' of continuous linear functionals on E is a special case of the space $\mathcal{L}(E, F)$ (corresponding to $F = \mathbb{C}, \mathbb{R}$). Of course, all that has been said about continuous and bounded operators relates to this special case as well.

If B is a set in E, then by its polar is meant the subset of E'

$$B^\circ = \{\ell' \in E', |(\ell', \ell)| < 1 \quad \forall \ell \in B\}.$$

Where (ℓ', ℓ) denotes the value of the functional $\ell' \in E$ on the element $\ell \in E$. The topology in E' is determined with the aid of polars of a family of sets in E.

Taking the polars of all bounded sets $B \subset E$ as neighborhoods of zero in E' we obtain the so-called strong topology in E'. Forming a system of neighborhood of zero in E consisting of the polars of any finite subsets in E we construct the weak topology in E'.

We note that the strong convergence $\ell'_j \longrightarrow 0$ in E' implies uniform convergence of number sequences $(\ell'_j, \ell) \longrightarrow 0$ on any bounded set $B \subset E$. The weak convergence $\ell'_j \longrightarrow 0$ implies the convergence $(\ell'_j, \ell) \longrightarrow 0$ for any $\ell \in E$.

The conjugate space E' will be endowed with strong topology provided that the contrary is not stipulated.

3.2. Scales of TLS

The notion of a scale plays an important role in our lectures. By a scale $\mathbb{E} = \{E_\alpha, i_\alpha^\beta\}$ will be meant a system consisting of a family of TLS (as a rule Banach or Hilbert spaces) E_α, parametrized by points α belonging to a partially ordered set K (we shall be interested in case $K = \mathbb{R}$ or $\mathbb{R}^2$ or K is a cone in $\mathbb{R}^j$, j=1, 2), and a family of continuous mappings:

$$i_\alpha^\beta : E_\alpha \longrightarrow E_\beta, \quad \beta < \alpha$$

defined for any $\beta < \alpha$ and satisfying the following natural conditions:

$$(a)\ i_\beta^\gamma\ i_\alpha^\beta = i_\alpha^\gamma,$$
$$(b)\ i_\alpha^\alpha = \text{id (identity operator)}.$$

3.3. Inductive limits of TLS

We consider a scale $\mathbb{E} = \{E_\alpha, i_\alpha^\beta\}$, $\alpha \in K$ and we suppose that the partially ordered set K satisfies following condition:

$$(1) \qquad \forall \alpha,\ \beta \in K \quad \exists \gamma \in K \quad \text{such that} \quad \gamma < \alpha,\ \gamma < \beta.$$

The scales with this property will be called inductive scales. The main examples of K satisfying (1) are $K = \{\alpha \in \mathbb{R},\ \alpha \leq a\}$ and $K = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2,\ \alpha_1 \leq a_1, \alpha_2 \leq a_2\}$.

With an inductive scale we can associate its inductive limit whose definition reads thus. Consider the union of all spaces belonging to the scale: $\bigcup_{\alpha \in K} E_\alpha$.

Two elements $e_\alpha \in E_\alpha$ and $e_\beta \in E_\beta$ of this union are said to be equivalent, $e_\alpha \sim e_\beta$ if $\exists \gamma < \alpha, \beta$ such that $i_\alpha^\gamma e_\alpha = i_\beta^\gamma e_\beta$. So we have a partition of the union consisting of classes of equivalent elements. The set of equivalence classes is denoted $E_{-\infty}$. For the classes of equivalent elements we define in a natural manner operations of addition and multiplication by a scalar, i.e., $E_{-\infty}$ is supplied with a structure of a vector space.

A linear operation associating with each element $e_\alpha \in E_\alpha$ the equivalence class it belongs to will be denoted i_α (it would be more correct to write $i_\alpha^{-\infty}$)

$$(2) \qquad E_\alpha \longrightarrow E_{-\infty} \quad (e \longmapsto i_\alpha e).$$

Let us introduce in $E_{-\infty}$ the strongest locally convex topology in which all canonical morphisms (2) are continuous. According to this definition, by a neighborhood of zero in $E_{-\infty}$ will be meant any set containing a convex set V whose full inverse images i_α^{-1} V are neighborhoods of zero in E_α for all $\alpha \in K$. The topology will be called inductive and the space $E_{-\infty}$ will be called inductive limit of the (inductive) scale $\mathbb{E} = \{E_\alpha, i_\alpha^\beta\}$.

A set $B \subset E_\infty$ is said to be regularly bounded if there are α and a bounded set $B_\alpha \subset E_\alpha$ such that $i_\alpha(B_\alpha) = B$. There exist examples of inductive limits having bounded sets which are not regularly bounded. Due to Makarov is the following

Definition. The inductive limit $E_{-\infty}$ of a scale $\{E_\alpha, i_\alpha^\beta\}$ is said to be regular if each bounded set in $E_{-\infty}$ is regularly bounded.

Theorem (Makarov). *If $\{E_\alpha, i_\alpha^\beta\}$ is a scale of reflexive Banach spaces then the inductive limit $E_{-\infty}$ is regular, moreover the space $E_{-\infty}$ is reflexive.*

In our lectures we shall deal only with those scales where the morphisms $i_\alpha^\beta : E_\alpha \longrightarrow E_\beta$ are injective, i.e., are embeddings. In this case E_α can be identified with its image $i_\alpha^\beta E_\alpha \subset E_\beta, \beta < \alpha$, i.e., E_α can be regarded as a subspace (as a rule not closed) in E_β, and the space $E_{-\infty}$ is identified with the set-theoretical union of the spaces E_α, i.e., $E_{-\infty} = \bigcup_{\alpha \in K} E_\alpha$. Such inductive limits are called inner. In them a subset $U \subset E_{-\infty}$ is a neighborhood if and only if the intersections $U_\alpha = U \cap E_\alpha$ are neighborhoods in E_α, $\forall \alpha \in K$. A subset $B \subset E_{-\infty}$ is regularly bounded if $B \subset E_\alpha$ for some α and B is bounded in E_α.

3.4. Projective limits of TLS

A scale $\mathbb{E} = \{E_\alpha, i_\alpha^\beta\}$, $\alpha \in K$ is called projective if the partially ordered set K satisfies the following condition (compare (1)):

$$\forall \alpha, \beta \in K \quad \exists \gamma \in K \quad \gamma > \alpha, \ \gamma > \beta. \tag{3}$$

The main examples are $K = \{\alpha \in \mathbb{R}, \alpha \geq a\}$ or $K = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2, \alpha_1 \geq a_1, \gamma_2 \geq a_2\}$.

By a thread we shall mean any sequence

$$e = \{e_\alpha \in E_\alpha, \alpha \in K\}$$

whose elements satisfy the following condition: if $\alpha > \beta$, then $e_\beta = i_\alpha^\beta e_\alpha$. In other words for any two elements e_α and e_β of a thread there is an element $e_\gamma, \gamma \geq \alpha, \gamma \geq \beta$ (here we use (3)) such that $e_\alpha = i_\gamma^\alpha e_\gamma, e_\beta = i_\gamma^\beta e_\gamma$.

The set of threads will be denoted E_∞; this set can be supplied, in a natural way, with a structure of a vector space. A linear operation associating with each thread its element e_α will be denoted as i^α (it would be more correct to write i_∞^α). The morphism

$$i^\alpha : E_\infty \longrightarrow E_\alpha \tag{4}$$

is a canonical mapping of E_∞ into E_α.

We introduce in E_∞ the weakest locally convex topology in which all the morphisms (4) are continuous. This topology is called projective and the space E_∞ with this topology is called the projective limit of the scale $\mathbb{E}$. According to the above definition, by a neighborhood of zero in E_∞ is meant any set containing a set of the form $(i^\alpha)^{-1}$ (U_α), where U_α is a neighborhood of zero in E_α. A set $B \subset E_\infty$ is said to be bounded if $(i^\alpha)^{-1}$ B is a bounded subset of E_α for any $\alpha \in K$.

When the mappings i^β_α are embeddings, we identify E_α and $i^\beta_\alpha E_\alpha$. In this case E_∞ can be identified with the intersection $\bigcap_{\alpha\in K} E_\alpha$ and the neighborhoods in E_∞ are sets having the form $U_\alpha \cap E_\infty$, where U_α is a neighborhood in E_α. The boundedness $B \subset E_\infty$ means that B is bounded on any E_α.

3.5. Duality between projective and inductive limits

Let $\mathbb{E} = \{E_\alpha, i^\beta_\alpha\}$ be a projective scale of TLS and let E_∞ be its projective limit. The system of conjugate spaces E'_β and dual mappings $j^\alpha_\beta = (i^\beta_\alpha)^*$: $E'_\beta \longrightarrow E'_\alpha$ forms an inductive scale $\mathbb{E}' = \{E'_\beta, j^\alpha_\beta\}$ and we can introduce the inductive limit of this scale E'_∞. Then the isomorphism of vector spaces takes place

$$(E_\infty)^* = E'_{-\infty}. \tag{5}$$

In the same manner, if the scale $\mathbb{E}$ is inductive we can define the inductive limit $E_{-\infty}$, then we can consider the projective limit E'_∞, and the isomorphism of vector spaces takes place:

$$(E_{-\infty})^* = E'_\infty. \tag{6}$$

Now the left-hand sides of (5), (6) can be endowed with the strong topology of the conjugate space, and the right-hand sides are equipped with the topology of inductive (projective) limit. We say that these isomorphisms are topological if these topologies are equivalent. In general, without additional conditions on the scale $\mathbb{E}$ the topological isomorphisms do not take place.

Theorem. *Let $\mathbb{E}$ be a scale of reflexive Banach spaces. Then*

(i) *the inductive limit $E_{-\infty}$ is reflexive and the isomorphism (topological) (6) takes place;*

(ii) *the projective limit* E_∞ *is reflexive and the isomorphism (topological) (5) takes place.*

Proof. (i) The topology of $(E_{-\infty})^*$ is given by polars of arbitrary bounded sets in $E_{-\infty}$. From above given definitions of projective limits follows that the topology in E'_∞ is given by the polars of regularly bounded sets (see the Definition in subsection 3.3). As was mentioned above a priori the set of bounded sets is wider than the set of regularly bounded sets, so the left-hand topology in (6) is stronger than the topology of E'_∞. If $\mathbb{E}$ is a scale of reflexive Banach spaces then according theorem 3.3 all the bounded sets of $E_{-\infty}$ are regularly bounded, which implies the desired assertion.

(ii) Consider the scale $\mathbb{E}' = \left\{E'_\alpha, i^\beta_\alpha\right\}$ of the conjugate spaces, and let $E'_{-\infty}$ be its inductive limit. By virtue of the reflexivity of the spaces E_α, the scale of conjugate spaces of the spaces in the scale $\mathbb{E}'$ coincides with the original scale $\mathbb{E}$. If E_∞ is the projective limit of the scale $\mathbb{E}$, then applying the above proved proposition (i) to E' we obtain

$$E_\infty = \left(E'_{-\infty}\right)^*,$$

whence

$$(E_\infty)^* = \left(\left(E'_{-\infty}\right)^*\right)^*.$$

According to theorem 3.3 the space E'_∞ is a reflexive space, and we obtain the topological isomorphism (5).

3.6. Operators in the scales and their inductive and projective limits

Let $\mathbb{E} = \{E_\alpha, i^\beta_\alpha\}, \alpha \in K$ and $\mathbb{F} = \{F_\gamma, i^\delta_\gamma\}, \gamma \in K'$, be two scales of TLS. By a compatible family of linear operators from $\mathbb{E}$ to $\mathbb{F}$ meant a system $\mathfrak{A} = \{A^\gamma_\alpha, \Sigma\}$ consisting of set $\Sigma \subset K \times K'$ of pairs of indices (α, γ) and the corresponding linear operators

$$A^\gamma_\alpha : E_\alpha \longrightarrow F_\gamma \quad (\alpha, \gamma) \in \Sigma \tag{7}$$

such that following natural conditions hold:

(I) if the set Σ contains a pair of indices (β, γ), it also contains the pairs (α, γ), $\alpha > \beta$ and (β, δ), $\delta < \gamma$;

(II) the operators (7) commute with the mappings $i_\alpha^\beta, j_\gamma^\delta$, i.e., the diagram

$$\begin{array}{ccc} E_\alpha & \xrightarrow{A_\alpha^\gamma} & F_\gamma \\ {\scriptstyle i_\alpha^\beta}\downarrow & & \downarrow{\scriptstyle j_\gamma^\delta} \\ E_\beta & \xrightarrow{A_\gamma^\delta} & F_\delta \end{array}$$

$$(\alpha,\gamma)\in\Sigma \qquad \beta\le\alpha,\ \delta\le\gamma$$

is commutative.

Given three scales and two compatible systems of operators in them we can naturally define the composition of these systems.

1) Let $\mathbb{E}$ and $\mathbb{F}$ be two inductive scales. By a continuous operator from inductive scale $\mathbb{E}$ into inductive scale $\mathbb{F}$,

$$A : \mathbb{E} \longrightarrow \mathbb{F} \tag{8}$$

is meant a compatible family of operators (7) satisfying the following condition:

$\forall\alpha \quad \exists\gamma = \gamma(\alpha)$ such that operator $A_\alpha^\gamma \in \mathcal{L}(E_\alpha, F_\gamma)$ is defined.

Operators (7) are called components of the operator (8).

Proposition 1. *Each continuous operator (8) from inductive scale $\mathbb{E}$ into inductive scale $\mathbb{F}$ induce an operator on the inductive limits*

$$A : E_{-\infty} \longrightarrow F_{-\infty}. \tag{9}$$

This operator is continuous and $\forall\alpha \ \exists\gamma = \gamma(\alpha)$ such that

$$j_\gamma\, A_\alpha^\gamma e = A i_\alpha e \qquad \forall e \in E_\alpha. \tag{9'}$$

Here j_γ, i_α are the morphisms (2).

The proof is straightforward.

Operator (9) is called *regular* if it is induced by a continuous operator (8). The family of regular operators is denoted $\mathcal{L}_{\text{reg}}(E_{-\infty}, F_{-\infty})$.

2) Now let $\mathbb{E}$ and $\mathbb{F}$ be two projective scales. In this case by a continuous operator (8) is meant a compatible family of operators (7) satisfying following condition :

$\forall\gamma \ \exists\alpha = \alpha(\gamma)$ such that the component $A_\alpha^\gamma \in \mathcal{L}(E_\alpha, F_\gamma)$ exists.

Proposition 2. *Each continuous operator (8) from a projective scale $\mathbb{E}$ into a projective scale $\mathbb{F}$ induces an operator*

$$A : E_\infty \longrightarrow F_\infty. \tag{10}$$

This operator is continuous and $\forall \gamma \in K'$ there is $\alpha = \alpha(\gamma) \in K$ such that

$$A^\gamma_\alpha i^\alpha e = j^\gamma A e \qquad \forall e \in E_\infty. \tag{10'}$$

Here i^α an j^γ are the morphisms (4).

As above, operators (10) induced by continuous operators (8) are called *regular*, their family is denoted $\mathcal{L}_{\text{reg}}(E_\infty, F_\infty)$.

3) We are also interested in the case when the scale $\mathbb{E}$ in (8) is a projective scale and the scale $\mathbb{F}$ is an inductive scale. In this case by a continuous operator (8) is meant any nonvoid compatible family (7) of operators $A^\gamma_\alpha \in \mathcal{L}(E_\alpha, F_\gamma)$. This definition implies that there are α and γ such that the component A^γ_α exists.

Proposition 3. *Each continuous operator (8) from a projective scale $\mathbb{E}$ into inductive scale $\mathbb{F}$ induces a continuous operator A from the projective limit into the inductive limit*

$$A : E_\infty \longrightarrow F_{-\infty}, \tag{11}$$

and for some α and γ the relations

$$Ae = j_\gamma \, A^\gamma_\alpha \, i^\alpha e, \quad \forall e \in E_\infty \tag{11'}$$

hold.

The set of operators arising in Proposition 3 is denoted $\mathcal{L}_{\text{reg}}\ (E_\infty, F_{-\infty})$.

4) We can also consider continuous operators (8) from an inductive scale into a projective scale. By such an operator is meant a compatible family (7) satisfying the following condition: for any α and γ there exists a continuous component A^γ_α. Each continuous operator from an inductive scale into a projective scale induces a continuous operator

$$(12) \qquad A : E_{-\infty} \longrightarrow F_{\infty}$$

which is natural to be defined as a regular operator. It can be easily understood that in this case all continuous operators (12) are regular.

Propositions 1, 2, 3 directly follow from the definitions and in some sense are trivial. The nontrivial question is: when all continuous operators (9), (10), (11) are regular. Before formulating sufficient conditions which provide this property we shall suppose in advance that in our scales all the morphisms (2), (4) are injections and operators (4) have dense images.

Theorem 1. *Let $E_{-\infty}$ and $F_{-\infty}$ be the inductive limits of inductive scales and the following conditions hold:*

(i) $\mathbb{F}$ *is a scale of reflexive Banach spaces;*

(ii) $\mathbb{E}$ *is a scale of spaces satisfying the first axiom of countability.*

Then each continuous operator (9) is regular.

Theorem 2. *Let E_{∞} and F_{∞} be projective limits of projective scales and* $\mathbb{F}$ *is a scale of Banach spaces. Then each operator (10) is regular.*

Theorem 3. *Let* $\mathbb{E}$ *be a projective scale of Banach spaces and let* $\mathbb{F}$ *be an inductive scale of reflexive Banach spaces. Then each continuous operator (11) is regular.*

3.7. $\mathcal{S}$ as a countably Hilbert space

In sections 1, 2 we constructed the scale $\mathbb{C} = \left\{ C_{(\ell)}^{(m)}, i_{m\ell}^{m'\ell'} \right\}$ of Banach spaces and the scale of Hilbert spaces $\mathbb{H} = \left\{ H_{(\ell)}^{(s)}, i_{s\ell}^{s'\ell'} \right\}$. These scales are projective because the set K of pairs of indices is $\mathbb{R} \times \mathbb{Z}_+$ in the first case and $\mathbb{R}^2$ in the second case. According to the theory above we can construct projective limits $C_{(\infty)}^{(\infty)}$ and $H_{(\infty)}^{(\infty)}$ of these scales. From the embeddings established in §2 it follows that the sets of elements of the spaces $C_{(\infty)}^{(\infty)}$ and $H_{(\infty)}^{(\infty)}$ coincide and the systems of norms $|\ |_{(\ell)}^{(m)}$ and $||\ ||_{(\ell)}^{(m)}$ determine systems of neighborhoods contained in each other. Thus

$$\mathcal{S} = C_{(\infty)}^{(\infty)} = H_{(\infty)}^{(\infty)}$$

is a countably Hilbert space.

According to the dualities

$$\left(H_{(\ell)}^{(s)}\right)' = H_{(-\ell)}^{(-s)}, \tag{13}$$

$\mathcal{S}$ is a projective limit of the scale of reflexive Banach spaces and we can apply Theorem 2 from previous subsection to linear operators in $\mathcal{S}$. According to this theorem if the operator

$$A : \mathcal{S} \longrightarrow \mathcal{S} \tag{14}$$

is continuous then $\forall s, \ell$ there exist s', ℓ' and the operator $A_{s\ell}^{s'\ell'} \in \mathcal{L}\left(H_{(\ell')}^{(s')}, H_{(\ell)}^{(s)}\right)$ such that the relation (10′) takes place:

$$A_{s'\ell'}^{s\ell} i^{s'\ell'} \varphi = i^{s\ell} A\varphi, \qquad \forall \varphi \in \mathcal{S}.$$

Here $i^{s\ell}$ are the morphisms (4) for $E_\alpha = H_{(\ell)}^{(s)}$ and $E_\infty = \mathcal{S}$. As $\mathcal{S}$ is dense in $H_{(\ell')}^{(s')}$ the operator (14) can be extended to a continuous operator $A_{s'\ell'}^{s\ell} \in \mathcal{L}\left(H_{(\ell')}^{(s')}, H_{(\ell)}^{(s)}\right)$. So we have proved the

Theorem. *A linear operator (14) is continuous if and only if* $\forall s, \ell$, $\exists s', \ell'$ *such that inequality*

$$||A\varphi||_{(\ell)}^{(s)} \le K_{s\ell} ||\varphi||_{(\ell')}^{(s')}, \quad \forall \varphi \in \mathcal{S}$$

holds.

3.8. $\mathcal{S}'$ as a regular inductive limit

According to the duality (13) an inductive limit

$$H_{(-\infty)}^{(-\infty)} = \bigcup_{s,\ell} H_{(\ell)}^{(s)}$$

of the scale $\mathbb{H}$ is regular and according to Theorem 3.5 the topological isomorphism

$$S' = H^{(-\infty)}_{(-\infty)} \tag{15}$$

holds. In other words the topology of the strong conjugate space to $\mathcal{S}$ coincides with the topology of a regular inductive limit $H^{(-\infty)}_{(-\infty)}$. From this follows the description of the elements of $\mathcal{S}'$.

Proposition. *For any $f \in \mathcal{S}'$ there are $\lambda \in \mathbb{R}, k \in \mathbb{Z}_+$ and $f_\circ \in C_{(\lambda)}$ such that $f = (1+|D|^2)^k f_\circ$.*

Proof. If $f \in \mathcal{S}'$ then $f \in H^{(\delta)}_{(\lambda)}$ for some δ, λ and f can be represented in the form: $f = (1+|D|^2)^{-(\delta-\delta_\circ)/2} f_\circ$, $f_\circ \in H^{(\delta_\circ)}_{(\lambda)}$. If $\delta_\circ > n/2$, then $f_\circ \in C_{(\lambda)}$. The number $-(\delta - \delta_\circ)/2$ can be regarded as being positive integer (if otherwise, we decrease δ so that condition is fulfilled).

From theorem 2 in subsection 3.6 follows the

Theorem. *A linear operator $A : \mathcal{S}' \longrightarrow \mathcal{S}'$ is continuous if and only if $\forall s, \ell, \exists s', \ell'$ such that*

$$||Af||^{(s')}_{(\ell')} \le K_{s\ell} ||f||^{(s)}_{(\ell)}, \quad \forall f \in H^{(s)}_{(\ell)}.$$

3.9. The spaces $\mathcal{O}, \mathcal{O}'$ and $\mathcal{M}$

Making use of the operations of inductive and projective limits we can construct spaces $H^{(\pm\infty)}_{(\ell)}$ and $H^{(s)}_{(\pm\infty)}$. They will be projective or regular inductive limits and the dualities (13) are also valid when s or ℓ is $\pm\infty$.

Using natural embeddings

$$i^{\ell'}_\ell : H^{(\pm\infty)}_{(\ell)} \longrightarrow H^{(\pm\infty)}_{(\ell')}, \quad \ell > \ell',$$

we can introduce the new scales $\left\{H^{(\infty)}_{(\ell)}, i^{\ell'}_\ell\right\}, \left\{H^{(-\infty)}_{(\ell)}, i^{\ell'}_\ell\right\}$ and define their inductive and projective limits:

$$\mathcal{O} = \bigcup_\ell H^{(\infty)}_{(\ell)}, \quad \mathcal{O}' = \bigcap_\ell H^{(-\infty)}_{(\ell)}. \tag{16}$$

The space $\mathcal{O}'$ as a vector space can be identified with the space of linear functionals on $\mathcal{O}$. Denote by O* the space conjugate to $\mathcal{O}$ and endowed with the strong topology of the conjugate space. It is an open question whether $\mathcal{O}$ is a regular inductive limit. If the answer to this question is negative, then the topology of $\mathcal{O}^*$ is stronger than the natural topology of $\mathcal{O}'$ (see the proof of Theorem 3.5).

We also do not know whether there exist continuous operators from $\mathcal{O}$ ($\mathcal{O}'$) into $\mathcal{O}$ ($\mathcal{O}'$) which are not regular.

According to the definition of subsection 3.6 an operator

$$A : \mathcal{O} \longrightarrow \mathcal{O} \tag{17}$$

is called *regular* if $\forall \ell$, $\exists \ell'$ and the operator $A_\ell^{\ell'} \in \mathcal{L}\left(H_{(\ell)}^{(\infty)}, H_{(\ell')}^{(\infty)}\right)$ such that the restriction of A on $H_{(\ell)}^{(\infty)}$ coincides with $A_\ell^{\ell'}$. As $H_{(\ell)}^{(\infty)}$ is a projective limit of reflexive Banach spaces $H_{(\ell)}^{(s)}$, then $\forall s$, $\exists s'$ such that operator $A_\ell^{\ell'}$ can be extended as a continuous operator from $H_{(\ell)}^{(s')}$ into $H_{(\ell')}^{(s)}$. In other words we have proved

Proposition 1. *Operator (17) is regular if and only if* $\forall \ell$, $\exists \ell'$ $\forall s$ $\exists s'$ *such that*

$$||A\varphi||_{(\ell')}^{(s)} \leq const\, ||\varphi||_{(\ell)}^{(s')}, \quad \forall \varphi \in H_{(\ell)}^{(\infty)}.$$

Along the same lines can be treated regular operators

$$A : \mathcal{O}' \longrightarrow \mathcal{O}'. \tag{18}$$

According to the definition of regular operators in projective limits $\forall \ell$ $\exists \ell'$ such that A is a restriction to $H_{(\ell')}^{(-\infty)}$ of an operator $A_{(\ell')}^{(\ell)} \in \mathcal{L}\left(H_{(\ell')}^{(-\infty)}, H_{(\ell)}^{(-\infty)}\right)$. As the inductive limits $H_{(\lambda)}^{(-\infty)}$, $\lambda = \ell$, ℓ' are regular inductive limits and the operator $A_{\ell'}^{\ell}$ is regular, so $\forall s$ $\exists s'$ such that the restriction of $A_{\ell'}^{\ell}$ to $H_{(\ell')}^{(s)}$ is a continuous operator into $H_{(\ell)}^{(s')}$. In other words we have proved

Proposition 2. *Operator (18) is regular if and only if* $\forall \ell\ \exists \ell'\ \forall s\ \exists s'$ *such that*

$$||A\varphi||_{(\ell)}^{(s')} \leq const||\varphi||_{(\ell')}^{(s)}, \quad \forall \varphi \in \mathcal{S}.$$

§4. Convolution in spaces of smooth functions and tempered distributions

4.1. convolution of continuous functions If f and g are continuous functions and the expressions $f(x-y)g(y)$ and $f(y)g(x-y)$ regarded as functions of y are absolutely integrable for each $x \in \mathbb{R}^n$, then the classical operation of convolution

$$(1) \qquad (f * g)(x) = \int f(x-y)g(y)dy = \int f(y)g(x-y)dy$$

is defined for them, and $f * g = g * f$. First of all we refine the estimates of the convolution (1) for the scale $\left\{C_{(\ell)}^{(m)}\right\}$. In particular, we shall present sufficient conditions on the numbers m_j, ℓ_j, $j = 1, 2, 3$, such that

$$C_{(\ell_1)}^{(m_1)} * E_{(\ell_2)}^{(m_2)} \subset E_{(\ell_3)}^{(m_3)} \quad E = C, H.$$

We start with an elementary but key

Lemma. Let $\ell > |\lambda| + n$. Then the estimates take place:

$$(2) \qquad |f * g|_{(\lambda)} \leq \text{const}|f|_{(\ell)}|g|_{(\lambda)},$$

$$(3) \qquad ||f * g||_{(\lambda)} \leq \text{const}|f|_{(\ell)}||g||_{(\lambda)}.$$

Proof. As $|f(z)| < (1 + |z|^2)^{-\ell/2}|f|_{(\ell)}$ and the same estimate takes place for g with ℓ replaced by λ, we have

$$(f * g)(x) \leq |f|_{(\ell)}|g|_{(\lambda)} \int (1 + |x-y|^2)^{-\lambda/2}(1 + |y|^2)^{-\ell/2}dy.$$

According to an elementary estimate

$$(1+|x-y|^2)^{-\lambda/2} \le 2^{|\lambda|/2}(1+|y|^2)^{|\lambda|/2}(1+|x|^2)^{-\lambda/2}$$

the integral in the right-hand side can be estimated from above by the constant $(1+|x|^2)^{-\lambda/2}$, so we have estimate (2).

To prove (3) we put $g(y) = (1+|y|^2)^{-\lambda/2} g_\circ(y), g_\circ(y) \in L_2$, then

$$(1+|x|^2)^{\lambda/2}|(f*g)(x)| \le |f|_{(\ell)} \int K(x,y)|g_\circ(y)|dy,$$

where

$$\begin{aligned} K(x,y) &= (1+|x|^2)^{\lambda/2}(1+|y|^2)^{-\lambda/2}(1+|x-y|^2)^{-\ell/2} \\ &\le 2^{|\lambda|/2}(1+|x-y|^2)^{-\ell+|\lambda|} \overset{\text{def}}{=} h(x-y), \; h \in L_1. \end{aligned}$$

According to Young inequality

$$\begin{aligned} ||f*g||_{(\lambda)} &= ||(1+|x|^2)^{\lambda/2}(f*g)|| \le |f|_{(\ell)}||h*g_\circ|| \le \text{const}|f|_{(\ell)}||g_\circ|| \\ &= \text{const}|f|_{(\ell)}||g||_{(\lambda)}. \end{aligned}$$

If $f \in C^{(k)}_{(\ell)}, g \in C^{(m)}_{(\lambda)}$ and $\ell > |\lambda| + n$, then applying differential operators to (1), differentiating under the integral sign and integrating by parts we find

$$(4) \qquad D^{\alpha+\beta}(f*g)(x) = \left((D^\alpha f) * \left(D^\beta g\right)\right) \quad |\alpha| \le k, \quad |\beta| \le m.$$

So we proved

Proposition. *Let $f \in C^{(k)}_{(\ell)}, g \in C^{(m)}_{(\lambda)}, H^{(m)}_{(\lambda)}$. Then the convolution $f*g = g*f \in C^{(m+k)}_{(\lambda)}, H^{(m+k)}_{(\lambda)}$ exists for $\ell > |\lambda| + n$ and relation (4) takes place.*

From estimates (2), (3) easily follows continuity of the operator

$$C^{(m)}_{(\lambda)}, H^{(m)}_{(\lambda)} \longrightarrow C^{(m+k)}_{(\lambda)}, H^{(m+k)}_{(\lambda)} \quad (g \longmapsto f*g),$$

where $f \in C^{(k)}_{(\ell)}$ and $\ell > |\lambda| + n$.

Following inclusions directly follow from the proposition:

(5) $$\mathcal{S} * \mathcal{S} \subset \mathcal{S},$$

(6) $$\mathcal{S} * \mathcal{O} \subset \mathcal{S},$$

(7) $$\mathcal{S} * H^{(\infty)}_{(\lambda)} \subset H^{(\infty)}_{(\lambda)}, \quad \mathcal{S} * C^{(\infty)}_{(\lambda)} \subset C^{(\infty)}_{(\lambda)}.$$

4.2. Convolution of tempered distributions

Proposition 1. *The operation of convolution defined originally on a dense subset $\mathcal{S} \times \mathcal{O} \subset \mathcal{O}' \times \mathcal{S}'$ (see(6)) is continued by continuity to a mapping of $\mathcal{O}' \times \mathcal{S}'$ into $\mathcal{S}'$. The resulting operator commutes with differentiation, i.e., (4) holds for all α and β.*

We first of all give a constructive definition of convolution for $f \in \mathcal{O}'$ and $g \in \mathcal{S}'$. This definition is faced on the description of elements of $\mathcal{S}'$ and $\mathcal{O}'$.

Lemma. (i) If $f \in \mathcal{S}'$ then $\exists \lambda \in \mathbb{R},\ m \in \mathbb{Z}_+$ such that

$$f = (1 + |D|^2)^m f_\circ, \quad f_\circ \in C_{(\lambda)}.$$

(ii) If $f \in \mathcal{O}'$ then $\forall \ell \in \mathbb{R}, \quad \exists k = k(\ell) \in \mathbb{Z}_+$ such that

$$f = (1 + |D|^2)^k f_\ell, \quad f_\ell \in C_{(\ell)}.$$

Now we pose for $f \in \mathcal{O}'$ and $g \in \mathcal{S}'$

(8) $$f * g = (1 + |D|^2)^{k+m}(f_\ell * g_\circ), \quad \ell > |\lambda| + n.$$

It can be checked that the definition is correct, i.e., the right-hand side of (8) depends on f, g but does not depend on the choice of numbers k and m. In the case of regular functions ($k = m = 0$) the convolution (8) coincides with the classical convolution (1).

We now show that

(9) $$f_j \to f \text{ in } \mathcal{O}', g_j \to g \text{ in } \mathcal{S}' \Rightarrow f_j * g_j \to \text{ in } \mathcal{S}'.$$

It follows from the definitions of convergence in $\mathcal{S}'$ and $\mathcal{O}'$ that there are such λ, m, ℓ and $k = k(\ell)$ that

$$g_{\circ j} = (1 + |D|^2)^{-m} g_j \to (1 + |D|^2)^{-m} g = g_\circ \text{ in } C_{(\lambda)},$$
$$f_{\ell j} = (1 + |D|^2)^{-k(\ell)} f_j \to (1 + |D|^2)^{-k(\ell)} f = f_{\ell o} \text{ in } C_{(\ell)}.$$

Since k, m are integers and $\ell > |\lambda| + n$, then

$$f_j * g_j = (1 + |D|^2)^{k+m} (f_{\ell j} * g_{oj}) \to (1 + |D|^2)^{k+m} (f_\ell * g_o) \text{in } \mathcal{S}'.$$

The direct corollary of proposition 1 and the lemma is

Proposition 2. *Following inclusions hold:*

$$\mathcal{O}' * \mathcal{S} \subset \mathcal{S}, \quad \mathcal{S} = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}', C^{(\infty)}_{(\ell)}, H^{(\pm\infty)}_{(\ell)} \quad \forall \ell \in \mathbb{R}, \tag{10}$$

$$\mathcal{S}' * \mathcal{S} \subset \mathcal{O}. \tag{11}$$

In conclusion we note that the convolution of tempered distributions possesses the properties of associativity and commutativity:

$$f * (g * \varphi) = g * (f * \varphi) = (f * g) * \varphi, \quad f \in \mathcal{S}', g, \ \varphi \in \mathcal{O}'. \tag{12}$$

To prove these relations we approximate f, g, φ by smooth functions and use proposition 1.

4.3. Operators of convolution with distributions belonging to $\mathcal{O}'$

By virtue of the results of the foregoing section following statements take place.

Proposition 1. *The operator*

$$\text{con}_f : \Phi \longrightarrow \Phi \quad (\varphi \longmapsto f * \varphi), \quad f \in \mathcal{O}'$$

is continuous for $\Phi = \mathcal{S}, \mathcal{S}', H^{(\pm\infty)}_{(\ell)}$ *and* $C^{(\infty)}_{(\ell)}$ *and is regular for* $\Phi = \mathcal{O}, \mathcal{O}'$.

Proposition 2. *(i) $\mathcal{O}'$ is a commutative algebra relative to the convolution,*

(ii) $\mathcal{S}$ is an ideal of the algebra $\mathcal{O}'$,

(iii) $f \longmapsto \mathrm{con}_f : \Phi \to \Phi$ is a mapping of the algebra $\mathcal{O}'$ into the algebra $\mathcal{L}(\Phi, \Phi)$ for $\Phi = \mathcal{S}, \mathcal{S}'$ and into the algebra $\mathcal{L}_{\mathrm{reg}}(\Phi, \Phi)$ for $\Phi = \mathcal{O}, \mathcal{O}'$.

In the classical case there is well-known relationship between the convolution and the Fourier operator. Namely, if $f, g \in L_1$, then

$$(F(f * g))(\xi) = (2\pi)^{n/2}(Ff)(\xi)(Fg)(\xi). \tag{13}$$

We shall elucidate the proof of this relation. Since $f * g \in L_1$, the left-hand side of (13) involves an absolutely convergent integral

$$(2\pi)^{-n/2} \int \exp(-i\langle x, \xi \rangle) \left(\int f(x-y)g(y)dy \right) dx.$$

Since the integrals with respect to x and y are absolutely convergent, the order of integration can be interchanged. Hence, the above integral is equal to

$$\int \exp(-i\langle x, \xi \rangle)g(y) \ (2\pi)^{-n/2} \int \exp(-i\langle x-y, \xi \rangle)f(x-y)dx \ dy$$

i.e., (13) holds.

The relation (13) also remain valid when $f, g \in \mathcal{O}'$. Indeed, let us define $f * g$ with the aid of (8), where m and k so large, that for some $\ell > n$ we have $f_\circ, g_\circ \in C_{(\ell)} \subset L_1$. Then

$$(2\pi)^{-n/2}(F(f * g))(\xi) = \left(1 + |\xi|^2\right)^{k+m} \hat{f}_\circ(\xi)\hat{g}_\circ(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Proposition 3. (i) *The relation $F\mathcal{O}' = \mathcal{M}$ is an isomorphism of algebras.*

(ii) *The operator $\mathrm{con}_f : \Phi \longrightarrow \Phi, \Phi = \mathcal{S}, \mathcal{O}'$ is a pseudodifferential operator with the symbol $(2\pi)^{n/2}\hat{f}(\xi)$, i.e.*

$$\mathrm{con}_f = (2\pi)^{n/2}\hat{f}(D) \quad \forall f \in \mathcal{O}'.$$

By the definition of the Dirac's delta function $\delta(x)$:

$$(\delta, \varphi) = \varphi(0) = (2\pi)^{-n/2} \int \hat{\varphi}(\xi)d\xi = (2\pi)^{-n/2}(1, \hat{\varphi}),$$

i.e., $\hat{\delta} = (2\pi)^{-n/2}$, whence $\hat{\delta} \in \mathcal{M}$, i.e., $\delta(x) \in \mathcal{O}'$. It follows from (13) that $\delta(x)$ is the unity of the ring $\mathcal{O}'$:

$$f * \delta = \delta * f = f, \quad \forall f \in \mathcal{O}'. \tag{14}$$

4.4. Some additional remarks

With each vector $h \in \mathbb{R}^n$ we associate the translation operator

$$(T_h\varphi)(x) = \varphi(x+h).$$

It can be easily proved that the operator

$$\Phi \longrightarrow \Phi \quad (\varphi(x) \longmapsto \varphi(x+h))$$

is continuous for $\Phi = C^{(m)}_{(\ell)}, C^{(\infty)}_{(\ell)}, \mathcal{S}$ and regular for $\Phi = \mathcal{O}$. From the definition of the Fourier operator it follows that

$$(F(T_h\varphi))(\xi) = \exp(i\langle h, \xi\rangle)\hat{\varphi}(\xi), \quad \forall \varphi \in \mathcal{S},$$

and the operator T_h on $\mathcal{S}$ coincide with the pseudodifferential operator $\mathrm{con}_{\delta(x-h)}$.

For functions belonging to $\mathcal{S}$ we have

$$(T_h\varphi, \psi) = \int \varphi(x+h)\psi(x)dx = \int \varphi(x)\psi(x-h)dx = (\varphi, T_{-h}\psi).$$

Proceeding from this relation we can define the translation operator on $\mathcal{S}'$, it is a pseudodifferential operator.

Using the translation operator, we can rewrite the right-hand integral (1) as

$$\int f(y)(IT_xg)(y)dy,$$

i.e., the convolution (1) can be defined by means of the relation

$$(f * \varphi)(x) = (f, IT_x\varphi). \tag{15}$$

If $\varphi \in \Phi = \mathcal{S}, \mathcal{O}$, then $IT_x\varphi \in \Phi$, and the right-hand side of (15) makes sense for any distribution $f \in \Phi'$.

We can prove this formula approximating the distribution $f \in \varphi'$ by the functions from $\mathcal{S}$ and using the continuity arguments.

§5. Convolution equations

5.1. Convolution operators

An operator $A : \Phi \longrightarrow \Phi$, continuous for $\Phi = \mathcal{S}, \mathcal{S}'$ and regular for $\Phi = \mathcal{O}, \mathcal{O}'$ is called a convolution operator, if it commutes with translations.

Theorem. *Let $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$. For each convolution operator $A : \Phi \longrightarrow \Phi$ there is a distribution $f \in \mathcal{O}'$ such that*

$$A\varphi = \mathrm{con}_f\varphi = f * \varphi, \quad \forall\varphi \in \Phi. \tag{1}$$

In partucilar, each convolution operator on $\mathcal{S}$ and $\mathcal{O}$ can be represented in the form

$$(A\varphi)(x) = (f, IT_x\varphi), \quad \forall\varphi \in \mathcal{S}, \mathcal{O}, \ f \in \mathcal{O}'. \tag{2}$$

We begin with proving representation (2).

Proposition 1. *Let Φ be a space of smooth functions invariant with respect to translations and reflections. If the topology in Φ is stronger than the topology of pointwise convergence, then each continuous operator A from Φ into Φ, commuting with translations is of the form (2).*

Proof. We associate with A a family of linear functionals:

$$(f_x, IT_x\varphi) = A\varphi(x). \tag{2'}$$

Since the operators I and T_x perform one-to-one mappings of Φ into themselves, the functional (2′) is defined throughout the space Φ. The proposition will be proved if we show that

(i) f_x is a continuous linear functional, i.e., $f_x \in \Phi'$;

(ii) the functional f_x does not depend on $x : f_x = f \in \Phi'$.

To prove (i) we note that A is continuous operator and topology in Φ is stronger than the topology of pointwise convergence.

(ii) follows from the commutability of A with translations. Indeed, for $\psi \in \Phi$ we have

$$(f_x, \psi) = (f_x, IT_xT_{-x}I\psi) \overset{\text{def}}{=} (AT_{-x}I\psi)(x)$$
$$= T_{-x}(AI\psi)(x) = (AI\psi)(O) \overset{\text{def}}{=} (f, \psi).$$

As regular operators are continuous, in the case of $\Phi = \mathcal{O}$ we have already proved that every continuous operator commutable with translations is a convolution operator and for this operator the representation (1) takes place.

In the case of $\Phi = \mathcal{S}$ the theorem reduces to the following assertion

$$\{f \in \mathcal{S}', f * \varphi \in \mathcal{S}\} \Longrightarrow f \in \mathcal{O}'. \tag{3}$$

Formula (2) makes it possible to introduce the convolution between $f \in \Phi'$ and $\varphi \in \Phi$. A distribution $f \in \Phi'$ is called a convolutor if $f*\varphi \in \Phi, \quad \forall\varphi \in \Phi$. The set of convolutors will be denoted $\mathfrak{C}(\Phi)$. We have already proved that $\mathfrak{C}(\mathcal{O}) = \mathcal{O}'$. In previous section we proved that $\mathcal{O}' \subset \mathfrak{C}(\mathcal{S})$. So we have to show that

$$\mathfrak{C}(\mathcal{S}) \subset \mathcal{O}'. \tag{4}$$

This statement and other statements of the theorem follows from

Proposition 2. (i) *Each convolution operator*

$$A_\circ : \Phi \longrightarrow \Phi \quad (\Phi = \mathcal{S}, \mathcal{O}) \tag{5}$$

is continued by continuity to a convolution operator

$$A : \Psi \longrightarrow \Psi \quad (\Psi = \mathcal{O}', \mathcal{S}'). \tag{6}$$

(ii) *Let (6) be a convolution operator. Then its restriction (5) to subspace $\Phi = \mathcal{S}, \mathcal{O}$ is a convolution operator on Φ.*

Before proving the proposition we complete the proof of the theorem.

Let $A_\circ : \Phi \longrightarrow \Phi$ be a convolution operator. Then (proposition 1) $A\varphi = f * \varphi, f \in \mathcal{S}'$ and (proposition 2(i)) $f * \varphi \in \mathcal{O}', \quad \forall \varphi \in \mathcal{O}'$. Since $\delta(x) \in \mathcal{O}'$, we have $f = f * \delta \in \mathcal{O}'$.

If $A : \mathcal{O}' \longrightarrow \mathcal{O}'$ is a convolution operator, then the restriction of A to $\mathcal{S}$ is also a convolution operator, i.e., $A\varphi = f * \varphi, \quad \forall \varphi \in \mathcal{S}$ where $f \in \mathcal{O}'$. By virtue of the properties of the operator con_f, which we know from §4 the continuous extension of this operator to $\mathcal{O}'$ has the form (1).

As was already mentioned, the representation (1) for $\Phi = \mathcal{O}$ follows from proposition 1. Therefore, proposition 2 implies (1) for $\Phi = \mathcal{S}'$.

In view of propositions 1 and 2, it is natural to put

$$\mathfrak{C}(\mathcal{O}') = \mathfrak{C}(\mathcal{S}), \;\; \mathfrak{C}(\mathcal{S}') = \mathfrak{C}(\mathcal{O}),$$

and statement of the theorem results in a chain of relations

$$\mathfrak{C}(\mathcal{S}) = \mathfrak{C}(\mathcal{O}) = \mathfrak{C}(\mathcal{S}') = \mathfrak{C}(\mathcal{O}') = \mathcal{O}'. \tag{7}$$

The proof of Proposition 2. (i) If $\Phi = \mathcal{S}$, then according to proposition 3.7 the continuity of $A = \text{con}_f, f \in \mathcal{S}'$, implies that $\forall s, \ell \; \exists s' = s'(s,\ell), \ell' = \ell'(s,\ell)$ such that

$$\|f * \varphi\|_{(\ell)}^{(s)} \leq \text{const}\|\varphi\|_{(\ell')}^{(s')}. \tag{8}$$

We shall prove that there exist such functions $\sigma(\ell), \lambda(\ell)$, that we can chose

$$s' = s + \sigma(\ell), \;\; \ell' = \lambda(\ell). \tag{9}$$

Then replacing in (6) s by $s - \sigma(\ell)$ we obtain the estimate

$$\|f * \varphi\|_{(\ell)}^{(s')} \leq \text{const}\|\varphi\|_{(\ell')}^{(s)}. \tag{8'}$$

According to section 3.9 this estimate means that the operator con_f can be extended to a regular operator from $\mathcal{O}'$ into $\mathcal{O}'$.

If $\Phi = \mathcal{O}$, then $f \in \mathcal{O}'$, and the required assertion follows from the results of section 4.3.

The estimate (6) for $s = 0$ means that

$$\|f * \varphi\|_{(\ell)} \leq \quad \text{const} \quad \|\varphi\|_{(\lambda(\ell))}^{(\sigma(\ell))}.$$

Replacing φ by $(1+|D|^2)^{s/2}\varphi$ and taking into consideration the equivalence of norms (2.7) and (2.7′) we find

$$\begin{aligned}||f*(1+|D|^2)^{s/2}\varphi||_{(\ell)} &\le \text{const}\,||(1+|D|^2)^{s/2}\varphi||^{(\sigma(\ell))}_{(\lambda(\ell))}\\ &\le \text{const}'||(1+|D|^2)^{s/2}\varphi||^{(\sigma(\ell))}_{(\lambda(\ell))} = \text{const}'||\varphi||^{(s+\sigma(\ell))}_{(\lambda(\ell)} = \text{const}\,||\varphi||^{(s+\sigma(\ell))}_{(\lambda(\ell))}.\end{aligned}$$

As $(1+|\xi|^2) \in \mathcal{M}$ the corresponding pseudodifferential operator is an operator of convolution with a distribution from $\mathcal{O}'$ and commutes with $\text{con}_f, f \in \mathcal{S}'$. Then the left-hand side can be estimated from below by

$$||(1+|D|^2)^{s/2}f*\varphi||_{(\ell)} = '||f*\varphi||^{(s)}_{(\ell)} \ge \text{const}\,||f*\varphi||^{(s)}_{(\ell)}.$$

(ii) We first of all show that a convolution operator (6) commutes with pseudodifferential operator $(1+|D|^2)^k, k = \pm1, \pm2, \cdots$. To prove this we note that commutability of A with translations T_h implies commutability with finite-difference operators $|h|^{-1}(T_h - 1)$. Passing to limit for $|h| \longrightarrow 0$, we see that $AD_j = D_jA, j = 1, \dots, n$, whence it follows that A is commuting with any differential operator with constant coefficients, and, in particular commute with $(1+|D|^2)^k$ for integral values of $k > 0$. Since the operators $(1+|D|^2)^{-k}, k \in \mathbb{Z}_+$, are inverse to $(1+|D|^2)^k$, they also commute with A.

The regularity of (6) for $\Psi = \mathcal{O}'$ implies that $\forall\ell\ \exists\ell'\ \forall s'$ such that

$$||A\varphi||^{(s')}_{(\ell)} \le \text{const}\,||\varphi||^{(s)}_{(\ell')}.$$

Using the same arguments as in (i) we prove (9). Then replacing by $s-\sigma(\ell)$ we prove continuity of the restriction of A to $\mathcal{S}$. The same way is proved that for $\Psi = \mathcal{S}'$ the restriction of (6) to $\mathcal{O}$ is a regular operator.

As a consequence of the theorem, we can give a description of multipliers on $\mathcal{S}$ and $\mathcal{S}'$.

A continuous function $a(x)$ is called a multiplier on a space Ψ if $a\psi \in \Psi$ for all $\psi \in \Psi$. The multipliers form a commutative algebra which we denote $\mathfrak{M}(\Psi)$. Now we shall prove that

$$\mathfrak{M}(\mathcal{S}) = \mathfrak{M}(\mathcal{S}') = \mathcal{M}. \tag{10}$$

As $\mathcal{M} \subset \mathfrak{M}(\mathcal{S})$ (see subsection 1.1), it is sufficient to show that

$$\mathfrak{M}(\mathcal{S}) \subset \mathcal{M}.$$

If $a(\xi) \in \mathfrak{M}(\mathcal{S})$, then the pseudodifferential operator $a(D)$ is defined throughout $\mathcal{S}$, transforms $\mathcal{S}$ into itself and commutes with translations. By the closed graph theorem this operator is continuous, i.e., it is a convolution operator. According to the theorem, there is $f \in \mathcal{O}'$ such that $a(D)\varphi = f * \varphi \ \forall \varphi \in \mathcal{S}$. It follows that $a(\xi)\hat{\varphi}(\xi) = \hat{f}(\xi)\hat{\varphi}(\xi)$. Since $\hat{\varphi} \in \mathcal{S}$ is arbitrary, we have $a(\xi) = \hat{f}(\xi) \in \mathcal{M}$.

5.2. Convolution equations

Theorem. *Let $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ and $A \in \mathfrak{C}(\Phi) = \mathcal{O}'$. The following two conditions are equivalent.*

(I) *For any $f \in \Phi$ the convolution equation*

$$A * u = f \tag{11}$$

possesses a unique solution $u \in \Phi$.

(II) *Equation (11) has a fundamental solution $G \in \mathfrak{C}(\Phi) = \mathcal{O}'$*

$$A * G = G * A = \delta(x). \tag{12}$$

Proof. (I)⇒(II). For $\Phi = \mathcal{O}'$ this assertion is a tautology since $\delta(x) \in \mathcal{O}'$. In the case $\Phi = \mathcal{S}, \mathcal{S}', \mathcal{O}$ according to the Banach inverse operator theorem, which holds for Fréchet spaces and their inductive limits, the condition (I) is equivalent to the existence of a continuous operator

$$(\mathrm{con}_A)^{-1} : \Phi \longrightarrow \Phi, \Phi = \mathcal{S}, \mathcal{S}', \mathcal{O}. \tag{13}$$

Since the operator con_A commutes with translations, the operator (13) possesses the same property. Consequently $(\mathrm{con}_A)^{-1}$ is a convolution operator on $\mathcal{S}, \mathcal{S}'$ and $\mathcal{O}$. By theorem 5.1 there is $G \in \mathcal{O}'$ such that $(\mathrm{con}_A)^{-1} = \mathrm{con}_G$.

By the definition of an inverse operator

$$(A * (G * f))(x) = (G * (A * f))(x) = f(x), \ \ \forall f \in \mathcal{S}. \tag{14}$$

Note that $(F * \varphi)(x) = (F, IT_x\varphi)$ and $(F * \varphi)(0) = (F, I\varphi)$. Putting x=0 in (14) and using commutativity of convolution of elements of $\mathcal{O}'$ and $\mathcal{S}$ we obtain:

$$(A * G, If) = (G, If) = f(0). \ \ \forall \varphi \in \mathcal{S},$$

i.e., (12) is fulfilled.

(II)$\Rightarrow$(I). If $G \in \mathcal{O}'$ satisfies (12), then $G * f$ is a solution to Equation (11) since $A * (G * f) = (A * G) * f = \delta * f = f$. Further, if $A * u = 0$, then $0 = G * (A * u) = (G * A) * u = \delta * u = u$, i.e., Equation (13) possesses no more than one solution.

The operator $\text{con}_A, A \in \mathcal{O}'$, has a symbol $\hat{A}(\xi) \in \mathcal{M}$, and (II) is equivalent to the following condition

(II′) There are constants $c > 0$ and μ such that

$$|A(\xi)| > C(1 + |\xi|)^{\mu} \quad \forall \xi \in \mathbb{R}^n. \tag{15}$$

Indeed, as $F\mathcal{O}' = \mathcal{M}$, the condition (II) is equivalent to

$$\hat{A}^{-1}(\xi) \in \mathcal{M}. \tag{16}$$

Lemma. *For a function $\hat{A}(\xi) \in \mathcal{M}$ the inclusion (16) takes place if and only if condition (15) is fulfilled.*

Proof. The necessity is obvious and the sufficiency follows from the chain rule

$$D^{\alpha} \hat{A}^{-1}(\xi) = \sum C_{\alpha\gamma^1 \dots \gamma^k} D^{\gamma^1} \hat{A}(\xi) \cdots D^{\gamma^k} \hat{A}(\xi) \hat{A}^{-k-1}(\xi),$$

where $\gamma^1, \dots, \gamma^k$ are multiindices, $\gamma^1 + \cdots + \gamma^k = \alpha$ and $C_{\alpha\gamma^1 \dots \gamma^k}$ are constants.

4.3. Differential equations with constant coefficients in $\mathbb{R}^n$

From (15) we derive a weaker condition

$$\hat{A}(\xi) \neq 0, \;\; \xi \in \mathbb{R}^n, \tag{17}$$

which is necessary for equation (11) to have unique solution. Generally, this condition is not sufficient, since a symbol $\hat{A}(\xi) \in \mathcal{M}$, satisfying (17) may tend to zero stronger than any power of $|\xi|$ for $|\xi| \to \infty$. However, if con_A is a differential operation, i.e., $\hat{A}(\xi) = P(\xi)$, then according to the Seidenberg-Tarski theorem $\hat{A}(\xi) \to 0$ for $|\xi| \to \infty$ not stronger than some power $|\xi|$, i.e., in view of (17) the condition (II′) holds. We have thus proved the following.

Theorem. *A differential equation with constant coefficients*

$$P(D)u = f$$

possesses a unique solution in $\mathcal{S}, \mathcal{O}, \mathcal{S}'$ *and* $\mathcal{O}'$ *if and only if its symbol* $P(\xi)$ *is nonzero throughout* $\mathbb{R}^n$.

4.4. Some versions of theorems 4.2 and 4.3

In the conditions of theorem 4.2 an operator

$$\text{(18)} \qquad \operatorname{con}_A : \Phi \longrightarrow \Phi, \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$$

possesses an inverse operator which is continuous for $\Phi = \mathcal{S}, \mathcal{S}'$ and regular for $\Phi = \mathcal{O}, \mathcal{O}'$. Interpreting the conditions of regularity of these operators (continuous operators on $\mathcal{S}$ and $\mathcal{S}'$ are regular) we conclude, that in the set of equivalent conditions of Theorem 4.2 can be included following conditions:

(I_a) $\forall s, \ell\ \exists s', \ell'$ such that $\forall f \in E^{(s')}_{(\ell')}$ (13) possesses a unique solution $u \in E^{(s)}_{(\ell)}$, $E = H, C$.

(I_b) $\forall s, \ell\ \exists s', \ell'$ such that $\forall f \in E^{(s)}_{(\ell)}$ (13) possesses a unique solution $u \in E^{(s')}_{(\ell')}, E = H, C$.

(I_c) $\forall \ell\ \exists \ell'\ \forall s\ \exists s'$ such that $\forall f \in E^{(s')}_{(\ell)}$ (13) possesses a unique solution $u \in E^{(s)}_{(\ell')}, E = H, C$.

(I_d) $\forall \ell\ \exists \ell'\ \forall s\ \exists s'$ such that $\forall f \in E^{(s)}_{(\ell')}$ (13) possesses a unique solution $u \in E^{(s')}_{(\ell)}, E = H, C$.

In other words, the solvability of (11) in "limiting"spaces induct the solvability in Hölder and Hilbert scales, introduced in §1, 2.

The commutability of the operator (18), $A \in \mathcal{O}'$ with the family of pseudodifferential operators $(1 + |D|^2)^{s/2}$ implies that if assertions (I_a) and (I_d) hold for all $\ell \in \mathbb{R}$ and some $s = \bar{s}$, then they hold for all s, and s' and ℓ' are expressed in terms of s and ℓ by (9).

Thus, the following conditions can be included in the set of equivalent conditions of Theorem 4.2:

(I_e) $\forall \ell\ \exists \sigma(\ell), \lambda(\ell)$ such that $\forall f \in H^{(s+(\ell)}_{(\lambda(\ell))}$ (13) possesses a unique solution $u \in H^{(s)}_{(\ell)}$.

(I_f) $\forall \ell\ \exists \sigma(\ell), \lambda(\ell)$ such that $\forall f \in H^{(s)}_{(\ell)}$ (13) possesses a unique solution $u \in H^{(s-\sigma(\ell))}_{(\lambda(\ell))}$.

Finally, the continuity of an operator $\mathrm{con}_g, G \in \mathcal{O}'$ in any $H^{(\infty)}_{(\ell)}$ implies a condition stronger than $(I_a) - (I_f)$.

(I_k) $\exists \ell\ \exists \sigma(\ell)$ such that $\forall f \in H^{(s+\sigma(\ell))}_{(\ell)}$ or $C^{(s+\sigma(\ell))}_{(s)}$ (13) possesses a unique solution $u \in H^{(s)}_{(\ell)}$ or $u \in C^{(s)}_{(\ell)}$.

In other words, if $A \in \mathcal{O}'$, then the solution of (13) decrease (increase) at infinity with the same rate as the right-side, and the difference of smoothness of the solution and the right-side depends only on the behavior of the solution at infinity.

Chapter 2. Convolution equations in the weighted spaces. Cauchy problem

§1. General remarks on the scales of weighted spaces and the theory of convolution in them

If $\mu(x)$ is a nonnegative function on $\mathbb{R}^n$, we can associate with it a scale of the spaces $C^{(m)}_{(\ell)\mu}$ consisting of m times continuously differentiable functions with a finite norm

$$(1) \qquad |\varphi|^{(m)}_{(\ell)\mu} = \sup_{x\in\mathbb{R}^n, |\alpha|\leq m} \mu(x)(1+|x|^2)^{\ell/2}|D^\alpha\varphi(x)|.$$

Using the operations of projective and inductive limits we define the spaces $C^{(\infty)}_{(\ell)\mu}, C^{(m)}_{(\infty)\mu}, \mathcal{S}_\mu, \mathcal{O}_\mu$:

$$(2) \qquad \mathcal{S}_\mu = C^{(\infty)}_{(\infty)\mu} = \bigcap C^{(m)}_{(\ell)\mu}, \quad \mathcal{O}_\mu = \bigcup_\ell C^{(\infty)}_{(\ell)\mu}.$$

Along with (1) we can introduce the Hilbert norms

$$(3) \qquad ||\varphi||^{(m)}_{(\ell)\mu} = \left(\sum_{|\alpha|\leq\mu} ||\mu D^\alpha\varphi||^2_{(\ell)}\right)^{1/2},$$

and the corresponding (Hilbert) spaces $H^{(m)}_{(\ell)\mu}$, their projective limit $H^{(\infty)}_{(\infty)\mu} = \bigcap_{m,\ell} H^{(m)}_{(\ell)\mu}$ not necessarily coinciding a priori with $\mathcal{S}_\mu$.

We assume that the weight function $\mu(x)$ in (1) can take on the values $+\infty$ and 0 on an open set.

In the first case the elements of the original space must vanish on this set and in the other case we identify the functions differing on the set where $\mu = 0$. Thus, the subspaces Φ_+ and the corresponding factor spaces $\Phi_\oplus$ can be interpreted as the spaces Φ_μ corresponding to the respective weights

$$(4) \qquad \mu(x) = \begin{cases} 1, & t = x_1 \geq 0, \\ +\infty, & t < 0 \end{cases}, \quad \mu(x) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0 \end{cases}.$$

To extend the theory developed in the first chapter we have to investigate the following problems.

(I) Sobolev type embedding theorems:

$$C^{(m)}_{(\ell+\kappa),\mu} \subset H^{(m)}_{(\ell)\mu} \subset C^{(m-\kappa')}_{(\ell),\mu}, \quad \kappa, \kappa' > n/2. \tag{5}$$

These embeddings imply that

$$\mathcal{S}_\mu = \bigcap_{m,\ell} H^{(s)}_{(\ell)\mu} = H^{(\infty)}_{(\infty)\mu}, \quad \mathcal{O}_\mu = \bigcup_\ell \bigcap_m H^{(s)}_{(\ell)\mu}.$$

In general, the right embeddings are not true for an arbitrary weight μ, so it is an interesting and difficult problem to describe such weights. We shall restrict ourselves with the simplest examples of weights for which (5) is true.

(II) The problem of "extension" of the scale $H^{(s)}_{(\ell)\mu}$ to any real $s \in \mathbb{R}$. Then we can define the limiting spaces

$$(\mathcal{S}')_\mu = \bigcup_{s,\ell} H^{(s)}_{(\ell)\mu}, \quad (\mathcal{O}')_\mu = \bigcap_\ell \bigcup_s H^{(s)}_{(\ell)\mu}. \tag{6}$$

Here the left-hand sides should be understood as formal symbols. However, they have to be interpreted as conjugate spaces, and we came to the third step of our program.

(III) Duality between the scales and limiting spaces corresponding to the weights $\mu(x)$ and $\mu^{-1}(x)$.

The duality between "zero" spaces:

$$(H_\mu)' = H_{1/\mu}$$

is a direct corollary of Riesz-Fisher theorem. We have to extend the scales $H^{(s)}_{(\ell)1/\mu}$ to arbitrary $s \in \mathbb{R}$ and prove the dualities

$$\left(H^{(s)}_{(\ell)\mu}\right)' = H^{(-s)}_{(-\ell)1/\mu}, \quad \left(H^{(s)}_{(\ell)1/\mu}\right)' = H^{(-s)}_{(-\ell_\mu)}. \tag{7}$$

From dualities (7) follows that the scales $\left\{H^{(s)}_{(\ell)\mu}\right\}$ and $\left\{H^{(s)}_{(\ell)1/\mu}\right\}$ are the scales of reflexive Banach (Hilbert) spaces, so the natural dualities between projective and inductive limits of these scales takes place. So the relations

$$(\mathcal{S}_\mu)' = (\mathcal{S}')_{1/\mu}, \quad (\mathcal{S}_{1/\mu})' = (\mathcal{S}')_\mu . \tag{8}$$

are topological isomorphisms, and the relations

$$(9) \qquad (\mathcal{O}_\mu)' = (\mathcal{O}')_{1/\mu}, \quad (\mathcal{O}_{1/\mu})' = (\mathcal{O}')_\mu$$

are only the isomorphisms of the vector spaces, the topology of the right-hand sides being weaker than the topology of left-hand sides.

(IV) Convolution, convolution operators and convolutors on the spaces $\Phi_\mu, \Phi_{1/\mu}$, $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$. If we are able to define the space $\mathfrak{C}(\Phi_\mu), \mathfrak{C}(\Phi_{1/\mu})$ we came to the problem of description of these eight spaces. In reality, we have to describe only two spaces: $\mathfrak{C}(\mathcal{S}_\mu), \mathfrak{C}(\mathcal{S}_{1/\mu})$. In the interesting cases, as in chapter 1, convolution operators on $\mathcal{S}_\mu, \mathcal{S}_{1/\mu}, \mathcal{O}_\mu, \mathcal{O}_{1/\mu}$ can be continued by continuity to the spaces (respectfully) $(\mathcal{O}')_\mu, (\mathcal{O}')_{1(\mu)}, (\mathcal{S}')_\mu, (\mathcal{S}')_\mu$. In other words

$$(10) \qquad \mathfrak{C}(\Phi_\mu) = \mathfrak{C}(\Psi_\mu), \mathfrak{C}(\Phi_{1/\mu}) = \mathfrak{C}(\Psi_{1/\mu}),$$

$$\Phi = \mathcal{S}, \mathcal{O}, \quad \Psi = \mathcal{O}', \mathcal{S}'.$$

From the duality relations between the scales corresponding to μ and $1/\mu$ can be deduced natural dualities for the convolutor spaces:

$$(11) \qquad \mathfrak{C}(\Phi_\mu) = I\mathfrak{C}((\Phi')_{1/\mu}), \quad \mathfrak{C}(\Phi_{1/\mu}) = I\mathfrak{C}((\Phi')_\mu), \quad \Phi = \mathcal{S}, \mathcal{O}.$$

Finally, from (10) and (11) follows that,

$$(12) \qquad \mathfrak{C}(\mathcal{S}_\mu) = \mathfrak{C}((\mathcal{O}')_\mu) = I\mathfrak{C}(\mathcal{O}_{1/\mu}) = I\mathfrak{C}((\mathcal{S}')_{1/\mu}),$$

$$(12') \qquad \mathfrak{C}(\mathcal{S}_{1/\mu}) = \mathfrak{C}((\mathcal{O}')_{1/\mu}) = I\mathfrak{C}(\mathcal{O}_\mu) = I\mathfrak{C}((\mathcal{S}')_\mu).$$

(V) Convolution equation

$$(13) \qquad A * u = f, \quad u, f \in \Phi_\mu, \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}', \quad A \in \mathfrak{C}(\Phi_\mu).$$

The main result of this theory is the equivalence of unique solvability of (13) and invertability of A in the algebra (with respect to convolution) $\mathfrak{C}(\Phi_\mu)$.

(VI) "Explicit" conditions of solvability. In many interesting cases Fourier transform exists in the convolutor spaces $\mathfrak{C}(\mathcal{S}_\mu), \mathfrak{C}(\mathcal{S}_{1/\mu})$ even if in the original spaces $\Phi_\mu, \Phi_{1/\mu}$ it can fail to exist. Then we can define the symbol $\hat{A}$

of our convolution operator. So the conditions of solvability in the spaces $\Phi_\mu, \Phi_{1/\mu}, \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ are reduced to one of two conditions:

$$\hat{A}^{-1} \in F\mathfrak{C}(\mathcal{S}_\mu), \quad \hat{A}^{-1} \in F\mathfrak{C}(\mathcal{S}_{1/\mu}). \tag{14}$$

The first is the condition of solvability in $\mathcal{S}_\mu, (\mathcal{O}')_\mu, (\mathcal{S}')_{1/\mu}, \mathcal{O}_{1/\mu}$, and the second is the corresponding condition for $\mathcal{S}_{1/\mu}, (\mathcal{O}')_{1/\mu}, (\mathcal{S}')_\mu, \mathcal{O}_\mu$.

Along the same lines as in the case of the weight (4) corresponding to the spaces of type Φ_+ and $\Phi_\oplus = \Phi/\Phi_-$, we can consider the theory of $\Phi_\mu[a,b)$ spaces and the theory of $\Phi_{\mu[+]}^{\{-\infty\}}$ and convolutors on them, this theory permits to obtain exact results on the solvability of inhomogeneous Cauchy problem in the spaces with the weight μ.

Remark. Let the variables $x \in \mathbb{R}^n$ be split into two groups: $x = (x', x''), x' \in \mathbb{R}^m, x'' \in \mathbb{R}^{n-m}$ and suppose that our weight is a product of weights depending only on x' or x'':

$$\mu(x) = \mu_1(x')\mu_2(x'').$$

In this case the functions $\varphi(x', x'') \in \Phi_\mu(\mathbb{R}^n), \Phi = \mathcal{S}, \mathcal{O}$ are elements of $\Phi_{\mu_1}(\mathbb{R}^m)$ for x'' fixed and elements of $\Phi_{\mu_2}(\mathbb{R}^{n-m})$ for x' fixed. In other words the space Φ_μ can be interpreted as tensor product $\Phi_{\mu_1}(\mathbb{R}^m) \oplus \Phi_{\mu_2}(\mathbb{R}^{n-m})$. In the category of spaces considered in these lectures the operation of tensor multiplication is commutable with the mapping $\Phi \longmapsto \mathfrak{C}(\Phi)$, i.e.,

$$\mathfrak{C}(\Phi_{\mu_1} \oplus \Phi_{\mu_2}) = \mathfrak{C}(\Phi_{\mu_2}) \oplus \mathfrak{C}(\Phi_{\mu_2}). \tag{15}$$

The spaces $\mathfrak{C}(\Phi_{\mu_j})$, as a rule, are obtained under a complex combination of operations of union and intersection of Banach spaces of distributions, so the tensor product of such spaces is far from an unambiguous notion, and therefore an intuitive understanding of the right-hand side of (15) as a space of functions $f(x', x'')$ belonging to $\mathfrak{C}(\Phi_{\mu_1})$ as functions of x' and to $\mathfrak{C}(\Phi_{\mu_2})$ as functions x'' should be supplemented with a formal (and often rather cumbersome) definition. In complicated situations (15) allows one to "guess" the structure of the space $\mathfrak{C}(\Phi_\mu), \mu = \mu_1\mu_2$, knowing the structure of the spaces $\mathfrak{C}(\Phi_{\mu j}), j = 1, 2$. After this the corresponding relations should be justified rigorously.

§2. The spaces $\mathcal{S}_{[\omega]}$ and the related scales

If Φ is a space of functions or distributions in $\mathbb{R}^n$ and $\omega \in \mathbb{R}^n$, then we shall denote by $\Phi_{[\omega]}$ the space of functions (distributions), having the form

$$\Phi_{[\omega]} = \{u = \exp\langle -\omega, x\rangle \varphi,\ \varphi \in \Phi\} \tag{1}$$

endowed with the natural topology. In other words $\Phi_{[\omega]}$ is the space Φ_μ of previous section for $\mu = \exp\langle \omega, x\rangle$. In the case of this weight all the results can be easily deduced from the results of Chapter 1. So we shall present the main results without proofs and we shall made some remarks which will be of constant use further.

2.1. If Φ is a normed space, then the natural norm

$$\left|u, \Phi_{[\omega]}\right| = \left|\exp\langle \omega, x\rangle \varphi, \Phi\right| \tag{2}$$

is defined in the space (1).

If Φ' is the conjugate space of Φ, the duality of $(f, \varphi), f \in \Phi', \varphi \in \Phi$ induces the duality of $\Phi_{[\omega]}$ and $(\Phi')_{[-\omega]}$ so that

$$\left(\Phi_{[\omega]}\right)' = (\Phi')_{[-\omega]}. \tag{3}$$

According §1 we can state definitions of the spaces $C^{(m)}_{(\ell)[\omega]}$ and $H^{(m)}_{(\ell)[\omega]}, \mathcal{S}_{[\omega]}, \mathcal{O}_{[\omega]}$. Interpreting the definitions of inductive and projective limits one can easily show that passage to such limit "commutes" with multiplication by exp $\langle \omega, x\rangle$, so that

$$\mathcal{S}_{[\omega]} = \cap C^{(m)}_{(\ell)[\omega]}, \quad \mathcal{O}_{[\omega]} = \bigcup_\ell \bigcap_m C^{(m)}_{(\ell)[\omega]}.$$

From Sobolev embedding theorems for the scales $C^{(m)}_{(\ell)}$, $H^{(m)}_{(\ell)}$ trivially follows corresponding embeddings for $C^{(m)}_{(\ell)[\omega]}$, $H^{(m)}_{(\ell)[\omega]}$, so that

$$\mathcal{S}_{[\omega]} = \cap H^{(m)}_{(\ell)[\omega]}, \quad \mathcal{O}_{[\omega]} = \bigcup_\ell H^{(\infty)}_{(\ell)[\omega]}.$$

The scale $H^{(s)}_{(\ell)[\omega]}$ can be extended from s natural to arbitrary $s \in \mathbb{R}$, we have the dualities

$$\left(H^{(s)}_{(\ell)[w]}\right)' = H^{(s)}_{(-\ell)[-w]}, \tag{3'}$$

and (3) holds for $\Phi = \mathcal{S}$ and $\mathcal{O}$.

2.2. The definition of a convolution of smooth functions implies that if $f * g$ is defined, then so it does for $\exp\langle\omega, x\rangle f$ and $\exp\langle\omega, x\rangle g$, and

$$(4) \qquad (\exp\langle\omega, x\rangle f) * (\exp\langle\omega, x\rangle g) = \exp\langle\omega, x\rangle f * g.$$

It follows from (4) that if some spaces $\Phi_j, j = 1, 2, 3$, are such that

$$(5) \qquad \Phi_1 * \Phi_2 \subset \Phi_3,$$

then the inclusion

$$(5') \qquad \Phi_{1[\omega]} * \Phi_{2[\omega]} \subset \Phi_{3[\omega]}$$

holds.

All the results of §I.5 remain true when $\mathcal{S}, \mathcal{S}', \mathcal{O}, \mathcal{O}'$ are replaced by $\mathcal{S}_{[\omega]}, \dots,$ so that

$$(6) \qquad \mathfrak{C}\left(\mathcal{S}_{[\omega]}\right) = \mathfrak{C}\left(\mathcal{O}_{[\omega]}\right) = \mathfrak{C}\left((\mathcal{O}')_{[\omega]}\right) = \mathfrak{C}\left((\mathcal{S}')_{[\omega]}\right) = \mathcal{O}'_{[\omega]}.$$

In subsequent sections we shall need some estimation for Fourier operators in the spaces (1), and embedding operators, and it will be important for us that the constants in these estimates do not depend on ω. These estimates demands some accuracy in defining Fourier transform, pseudodifferential operators in the spaces (1) and corresponding norms.

2.3. In relation to Fourier operators, we consider complex space $\mathbb{C}^n$ of points $\zeta = \xi + i\omega$ and regard $\mathbb{R}^n$ as the subspace $\{Im\zeta = 0\}$ of $\mathbb{C}^n$. We associate with $\omega \in \mathbb{R}^n$ the complex Fourier (Laplace) operator

$$(7) \qquad \begin{aligned} \hat{\varphi}(\xi + i\omega) &= (F_\omega \varphi)(\xi + i\omega) = F(\exp\langle\omega, x\rangle\varphi)(\xi) \\ &= (2\pi)^{-n/2} \int \exp(-i\langle\xi + i\omega, x\rangle)\varphi(x)\, dx. \end{aligned}$$

For the operator (7) we can write the inversion formula

$$(8) \qquad \varphi(x) = (2\pi)^{-n/2} \int \exp(i\langle\xi + i\omega, x\rangle)\hat{\varphi}(\xi + i\omega)\, d\xi.$$

Parseval's relation is extended in a natural way to operators (7):

$$(9) \qquad (\varphi, \psi) = (F_\omega \varphi, IF_{-\omega}\psi),$$

where the integral on the right-hand side extends over $\mathbb{R}^n + i\omega$.

Regarding the Fourier operator (7) we shall consider the spaces $\Psi^{[\omega]}$ of functions $\psi(\xi)$ on the subspace $Im = \omega$ such that the functions $\psi_\omega(\xi) = \psi(\xi + i\omega)$ belong to Ψ (here ω plays the role of parameter). Accordingly we introduce the spaces $\mathcal{S}^{[\omega]} = C_{(\infty)}^{(\infty)[\omega]}$, $\mathcal{M}^{[\omega]} = \bigcap_m C_{(-\infty)}^{(m)[\omega]}$. The Fourier operator determines the isomorphisms

$$(10) \qquad F_\omega \mathcal{S}_{[\omega]} = \mathcal{S}^{[\omega]}, \quad F_\omega(\mathcal{O}')_{[\omega]} = \mathcal{M}^{[\omega]}.$$

The space $\mathcal{M}^{[w]}$ is the ring of multipliers on the space $\mathcal{S}^{[w]}$:

$$(11) \qquad \mathfrak{M}\left(\mathcal{S}^{[\omega]}\right) = \mathcal{M}^{[\omega]}.$$

Based on Fourier operators (7), (8) we can define pseudodifferential operators on spaces of the type of $\Phi_{[\omega]}$:

(12)

$$\begin{aligned} {}_{[\omega]}a(D)\varphi &= F_\omega^{-1} a(\xi + iw) F_\omega \\ &= (2\pi)^{-n/2} \int \exp\left(i\langle \xi + i\omega, x\rangle a(\xi + i\omega)\hat{\varphi}(\xi + i\omega) d\xi.\right. \end{aligned}$$

2.4. In subsequent sections we shall deal with intersections of spaces $\Phi_{[\omega]}$, corresponding to different $\omega \in \mathbb{R}^n$. If Φ is a space of regular functions, than the fact that function belongs simultaneously to $\Phi_{[\omega_j]}, j = 1, 2$ has a natural meaning. Let Φ be a space of distributions, for instance, let $\Phi = \Psi'$ where Ψ is a space of smooth functions. In this case we shall deal with the spaces of continuous linear functionals on $\Psi_{[-\omega_j]}, j = 1, 2$. For definiteness, assume that these spaces contain $\mathcal{D}$ as a smooth subset. Then we define $\Phi_{[\omega_1]} \cap \Phi_{[\omega_2]}$ as a set of distributions $f \in \mathcal{D}'$ such that they are continued by continuity to both $\Psi_{[-\omega_1]}$ and $\Psi_{[-\omega_2]}$.

Let us discuss the action of pseudodifferential operators on intersections of the spaces $\Phi_{[\omega]}$. There arises a natural question of about the compatibility of operators (12) on the intersection: whether is true that

$$(13) \qquad {}_{[\omega_1]}a(D)\varphi = {}_{[\omega_2]} a(D)\varphi, \quad \forall \varphi \in \Phi_{[\omega_1]} \cup \Phi_{[\omega_2]}.$$

In view of the definition of intersection of the spaces $\Phi_{[\omega]}$ (13) means that

$$(14) \qquad {}_{[-\omega_1]}a(-D)\psi =_{[-\omega_2]} a(-D)\psi, \quad \forall \psi \in \mathcal{D}.$$

For symbols of the general form relations (15) are not necessarily fulfilled. However, (15) are fulfilled under some natural assumptions concerning the holomorphy of the symbol.

We choose the variables so that the vectors ω_1 and ω_2 differ only in the first component i.e., we assume that $\omega_1 = (\gamma', \omega')$ and $\omega_2 = (\gamma'', \omega')$, where $\gamma' \leq \gamma''$. Accordingly we shall write $\xi = (\xi_1, \xi')$.

Note that if $\psi \in \mathcal{D}$, then the Fourier transform $\hat{\psi}(\xi)$ is holomorphic in all variables and decrease stronger than any power of $|Re\xi|$. So if the symbol $a(-\zeta_1, -\xi' - i\omega')$ is holomorphic for $\gamma' < Im\zeta_1 < \gamma''$ and increase non stronger that some power of $|Re\zeta_1| + |\xi'|$, then relation (14) easily follows from Cauchy theorem.

In this case we can omit the subscript ω in notation (12).

2.5. Further we shall need some estimation for Fourier operators and embedding operators for the scales $C^{(m)}_{(\ell)[\omega]}$, $H^{(s)}_{(\ell)[\omega]}$, and it will be important for us that the constants in these estimates do not depend on ω. Such estimates can be obtained not in the norms (2) but in some equivalent norms (where equivalence constants depend on ω). In particular, for integral values m and ℓ we shall consider norms

$$(15) \qquad |\varphi|^{(m)}_{(\ell)[\omega]} = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m} \exp\langle \omega, x\rangle (1 + |x|^2)^{\ell/2} \, |D^\alpha \varphi(x)|,$$

$$(16) \qquad |\varphi|^{(m)[\omega]}_{(\ell)} = \sup_{\xi \in \mathbb{R}^n, |\alpha| \leq m} \left(1 + |\xi + i\omega|^2\right)^{\ell/2} \, |\partial^\alpha \varphi(\xi + i\omega)|\,.$$

Repeating literally the argument of §1.1 (see inequality 1.1.7) we derive Parseval's inequalities for the scale $C^{(m)}_{(\ell)[\omega]}$ with constants independent of ω:

$$|\hat{\varphi}|^{(\ell)[\omega]}_{(m)} \leq \text{const}\, |\varphi|^{(m)}_{(\ell+n')[\omega]} \quad n' > n,$$

$$|\varphi|^{(m)}_{(\ell)[\omega]} \leq \text{const}\, |\hat{\varphi}|^{(\ell)}_{(m+m')[\omega]}, \quad m' > n.$$

We shall consider analogous integrated norms:

$$(17)\qquad ||\varphi||_{(\ell)[\omega]}^{(m)} = \left(\int \exp(2\langle\omega, x\rangle)(1+|x|^2)^\ell \sum_{|\alpha|\leq m} |D^\alpha\varphi(x)|^2\, dx\right)^{\frac{1}{2}},$$

$$(18)\qquad ||\varphi||_{(\ell)}^{(m)[\omega]} = \left(\int (1+|\xi+i\omega|^2)^\ell \sum_{|\alpha|\leq m} |\partial^\alpha\varphi(\xi+i\omega)|^2\, d\xi\right)^{\frac{1}{2}}.$$

Parseval's relation can be written for these norms

$$(19)\qquad \int \left|(\xi+i\omega)^\alpha \partial^\beta \hat{\varphi}(\xi+i\omega)\right|^2 d\xi = \int \exp(2\langle\omega, x\rangle)\left|x^\beta D^\alpha \varphi(x)\right|^2\; dx.$$

For $\beta = 0$ the relations (19) implies that the operator $H_{[\omega]}^{(m)} \longrightarrow H_{(m)}^{[\omega]}$ ($\varphi \longmapsto \hat{\varphi}$) is isometric, i.e.,

$$||\varphi||_{[\omega]}^{(m)} = ||\hat{\varphi}||_{(m)}^{[\omega]}.$$

Replacing the Fourier operator F in the proof of the theorem 1.2.4 by F_ω we find for the norms of embedding operators $C_{(\ell+\kappa)[\omega]}^{(m)} \subset H_{(\ell)[\omega]}^{(m)}$, $H_{(\ell)[\omega]}^{(m+\kappa)} \subset C_{(\ell)[w]}^{(m)}$, $\kappa > n/2$, the estimates independent of ω.

§3. Convolutors and convolution equations in spaces of functions satisfying exponential estimates

In this section we shall realize the above program in the case of the weights

$$(1)\qquad \mu(\Gamma', \Gamma'', x) = \exp\left(\sum_{j=1}^{n}\left(\gamma_i^{''}(x_i)_+ + \gamma_i^{'}(x_i)_-\right)\right),$$

where $\Gamma' = \left(\gamma_1^{'}, \ldots, \gamma_n^{'}\right)$, $\Gamma'' = \left(\gamma_1^{''}, \ldots, \gamma_n^{''}\right)$ are two given vectors and $(x_i)_\pm = x_i$ for $\pm x_i > 0$ and $(x_i)_\pm = 0$ for $\pm x_i < 0$. The corresponding spaces will be denoted as $\Phi_{[\Gamma', \Gamma'']}$, $\Phi = C_{(\ell)}^{(m)}$, $H_{(\ell)}^{(m)}$, etc. The theory of the spaces $\Phi_{[\Gamma', \Gamma'']}$ depends essentially on the inequality relations for the components of the vectors Γ' and Γ''.

As the weight (1) is the product of the weights

$$\exp\left(\gamma_i''(x_i)_+ + \gamma_i'(x_i)_-\right)$$

depending on only one variable, the function spaces corresponding to (1) can be interpreted as tensor product of the spaces on a line. So to simplify the notation we shall study in detail only the spaces $\Phi_{[\Gamma',\Gamma'']}(\mathbb{R})$. At the end we shall make some remarks about the general case.

In what follows we shall use the notions of the intersection and sum of two TLS.

Let there be TLS E_1 and E_2 embedded in TLS E. Denote by $E_1 \cap E_2$ the subspace of elements of E_1, belonging to E_2. The topologies of E_1 and E_2 induce topologies in $E_1 \cap E_2$. In case E_1 and E_2 are Banach spaces, the Banach norm

$$|\varphi, E_1 \cap E_2| = |\varphi, E_1| + |\varphi, E_2|$$

is defined on $E_1 \cap E_2$.

In the same manner we denote by $E_1 + E_2$ the space of sums $\varphi_1 + \varphi_2$, $\varphi_1 \in E_1$, $\varphi_2 \in E_2$, equipped with the norm

$$|\varphi, E_1 + E_2| = \inf_{\varphi_1,\varphi_2,\varphi_1+\varphi_2=\varphi} |\varphi_1, E_1| + |\varphi_2, E_2|\,.$$

If E_1' and E_2' are Banach conjugate spaces to E_1 and E_2 respectively, then there are natural duality relations:

$$(E_1 \cap E_2)' = E_1' + E_2', \quad (E_1 + E_2)' = E_1' \cap E_2'. \tag{2}$$

3.1. The space $\mathcal{S}_{[\gamma',\gamma'']}$ and the related scales (case $\gamma'' > \gamma'$)

3.1.1. We shall study spaces of functions $\Phi_\mu = C^{(m)}_{(\ell)\mu}(\mathbb{R})$, $H^{(m)}_{(\ell)}(\mathbb{R})$ on the line $t \in \mathbb{R}$ corresponding to the weight (1) in the case n=1, i.e.,

$$\mu(\gamma', \gamma'', t) = \exp(\gamma'' t_+ + \gamma' t_-)\,. \tag{1'}$$

These spaces we shall denote $\Phi_{[\gamma',\gamma'']}$, $\Phi = C^{(m)}_{(\ell)}$, $H^{(\ell)}_{(\ell)}$, $\mathcal{S}$, etc. In this subsection we suppose that $\gamma'' > \gamma'$. Then from the elementary inequality

$$\exp(\gamma'' t_+ + \gamma' t_-) \;\le\; \exp(\gamma'' t) + \;\exp(\gamma' t) \;\le\; 2\exp(\gamma'' t_+ + \gamma' t_-)$$

follows the equivalence of the norms $|\varphi|^{(m)}_{(\ell),\mu}$, $||\varphi||^{(m)}_{(\ell)\mu}$ and respectively $|\varphi|^{(m)}_{(\ell)[\gamma']}+$ $|\varphi|^{(m)}_{(\ell)[\gamma'']}$, $||\varphi||^{(m)}_{(\ell)[\gamma']}+||\varphi||^{(m)}_{(\ell)[\gamma'']}$, so

$$(3) \qquad E^{(m)}_{(\ell)[\gamma',\gamma'']}=E^{(m)}_{(\ell)[\gamma']}\bigcap E^{(m)}_{(\ell)[\gamma'']}, E=C^{(m)}_{(\ell)}, H^{(m)}_{(\ell)}, \quad (\gamma''>\gamma').$$

From this description of the spaces $C^{(m)}_{(\ell)[\gamma',\gamma'']}$ and $H^{(m)}_{(\ell)[\gamma',\gamma'']}$ and (1.2.17), (1. 2.18) directly follows Sobolev embedding theorems:

$$C^{(m)}_{(\ell+\kappa)[\gamma',\gamma'']}\subset H^{(m)}_{(\ell)[\gamma',\gamma'']}\subset C^{(m-\kappa')}_{(\ell)[\gamma',\gamma'']}, \quad \kappa,\kappa'>n/2.$$

So the scales $\left\{C^{(m)}_{(\ell)[\gamma',\gamma'']}\right\}$, $\left\{H^{(m)}_{(\ell)[\gamma',\gamma'']}\right\}$ are equivalent and

$$(4) \qquad \mathcal{S}_{[\gamma',\gamma'']}=\bigcap C^{(m)}_{(\ell)[\gamma',\gamma]}=\bigcap H^{(m)}_{(\ell)[\gamma',\gamma'']},$$

$$(5) \qquad \mathcal{O}_{[\gamma',\gamma'']}=\bigcup_{\ell}\bigcap_{m} C^{(m)}_{(\ell)[\gamma',\gamma'']}=\bigcup_{\ell}\bigcap_{m} H^{(m)}_{(\ell)[\gamma',\gamma'']}.$$

Remark. Originally, the left-hand sides should be understood as formal symbols. But according to isomorphisms (3) we have isomorphisms:

$$\mathcal{S}_{[\gamma',\gamma'']}=\mathcal{S}_{[\gamma']}\cap\mathcal{S}_{[\gamma'']}, \quad \mathcal{O}_{[\gamma',\gamma'']}=\mathcal{O}_{[\gamma']}\cap\mathcal{O}_{[\gamma'']}.$$

Moreover, the weight $(1')$ is not smooth in the point $t=0$. We can easily define a weight $\tilde{\mu}(\gamma',\gamma'',t)\in C^{\infty}$ which can be estimated from above and from below by $(1')$. Then the spaces $\Phi_{[\gamma',\gamma'']}$, $\Phi=\mathcal{S}$, $\mathcal{O}$, $C^{(m)}_{(\ell)}$, $H^{(m)}_{(\ell)}$ we can understand as the spaces of such functions $\varphi(t)$, that $\tilde{\mu}(\gamma',\gamma'',t)\,\varphi(t)\in\Phi$.

To describe the Fourier transform in $\mathcal{S}_{[\gamma',\gamma'']}$ it is convenient to use another (equivalent) definition of the space $C^{(m)}_{(\ell)[\gamma',\gamma'']}$.

Proposition. $C^{(m)}_{(\ell)[\gamma',\gamma'']}$ *consists of those and only those elements* φ *of the intersection* $\bigcap C^{(m)}_{(\ell)[\gamma]}$, $\gamma'<\gamma<\gamma''$, *for which the norm (see the definition of norms (1.15))*

$$\sup_{\gamma'<\gamma<\gamma''}|\varphi|^{(m)}_{(\ell)[\gamma]}$$

is finite. This norm is equivalent to the original norm in $C^{(m)}_{(\ell)[\gamma',\gamma'']}$.

3.1.2. 1) We shall describe the Fourier operator in $\mathcal{S}_{[\gamma',\gamma'']}$. To this end we introduce the scale of spaces $C^{(m)[\gamma',\gamma'']}_{(\ell)}$ whose elements are functions $\psi(\tau)$ which are holomorphic in the tube domain

$$T(\gamma',\gamma'') = \{\tau \in \mathbb{C}, \gamma' < \mathrm{Im}\tau < \gamma''\}, \tag{6}$$

2) $\psi_\gamma(\sigma) = \psi(\sigma + i\gamma) \in C^{(m)}_{(\ell)}, \gamma' \leq \gamma \leq \gamma''$ and
3) the norm

$$|\psi|^{(m)[\gamma',\gamma'']}_{(\ell)} = \sup_{\gamma' \leq \gamma \leq \gamma''} |\psi|^{(m)[\gamma]}_{(\ell)}$$

is finite.

Denote by $\mathcal{S}^{[\gamma',\gamma'']}$ the projective limit of the spaces $C^{(m)[\gamma',\gamma'']}_{(\ell)}$.

The Fourier-Laplace operator induces the continuous operators

$$FC^{(m)}_{(\ell+n')[\gamma',\gamma'']} \subset C^{(\ell)[\gamma',\gamma'']}_{(m)} \quad n' > n,$$

$$F^{-1}C^{(\ell)[\gamma',\gamma'']}_{(m+m')} \subset C^{(m)}_{(\ell)[\gamma',\gamma'']}, \quad m' > n,$$

and the corresponding "Parseval's inequalities" are fulfilled. As a consequence we have the isomorphism

$$F\mathcal{S}_{[\gamma',\gamma'']} = \mathcal{S}^{[\gamma',\gamma'']}, \quad \gamma'' > \gamma'. \tag{7}$$

We define the space

$$\mathcal{M}^{[\gamma',\gamma'']} = \bigcap_m \bigcup_\ell C^{(m)[\gamma',\gamma'']}_{(\ell)}. \tag{8}$$

It is clear, that this space is a ring relative to multiplication, and $\mathcal{S}^{[\gamma',\gamma'']}$ is an ideal of this ring. It follows that for $a(\tau) \in \mathcal{M}^{[\gamma',\gamma'']}$ the pseudodifferential operator

$$a(D)\varphi = \exp(-\gamma t)a(D + i\gamma)\exp(\gamma t)\varphi$$

does not depend on the choice of $\gamma \in (\gamma, \gamma'')$ (see subsection 2.4) and transforms $\mathcal{S}_{[\gamma',\gamma'']}$ into itself.

3.1.3. We now proceed to "extend" Hilbert scale $H^{(s)}_{(\ell)[\gamma',\gamma'']}$ to arbitrary $s \in \mathbb{R}$. As a "zeroth space" we take the space $H_{[\gamma',\gamma'']}$ of measurable functions square integrable with weight (1'), i.e., having the finite norm

(9) $$||\varphi||_{[\gamma',\gamma'']} = ||\mu(\gamma',\gamma'',t)\,\varphi||\,.$$

Then $\varphi \in H_{[\gamma',\gamma'']}$ if and only if $\varphi \in H_{[\gamma]}$, $\sigma' < \gamma < \gamma$ and the norm

(10) $$'||\varphi||_{[\gamma',\gamma'']} = \sup_{\gamma'<\gamma<\gamma''} ||\psi||^{[\gamma]}$$

is finite.

Let $H^{[\gamma',\gamma'']}$ denote the space of functions $\psi(\tau)$ holomorphic in the tube domain (6) and such that $\psi_\gamma(\sigma) = \psi(\sigma + i\gamma) \in H$, $\gamma' \leq \gamma \leq \gamma$ and the norm

$$||\psi||^{[\gamma',\gamma'']} = \sup_{\gamma'<\gamma<\gamma''} ||\psi||^{[\gamma]}$$

is finite. According Paley-Wiener theorem

(11) $$FH_{[\gamma',\gamma'']} = H^{[\gamma',\gamma'']},$$

and the Parseval's relation takes place

(12) $$'||f||_{[\gamma',\gamma'']} = ||\hat{f}||^{[\gamma',\gamma'']}.$$

Further, the function $\psi(\tau) \in H^{[\gamma',\gamma'']}$ assume for $\mathrm{I}m\tau \to \gamma'$ *or* γ'' boundary conditions in the sense of $H(\mathbb{R}_\sigma)$, $\sigma = \mathrm{R}e\tau$.

It can be proved, that $\mathcal{S}_{[\gamma',\gamma'']}$ is dense in $H_{[\gamma',\gamma'']}$, so $\mathcal{S}^{[\gamma',\gamma'']}$ is dense in $H^{[\gamma',\gamma'']}$.

We introduce the class of symbols

(13) $$\delta_{s,N} = \left(\tau^2 + N^2\right)^{s/2}.$$

If $N > \max\{|\gamma'|, |\gamma''|\}$, then (13) is a holomorphic function in the tube domain (6), more precisely $\delta_{s,N}(\tau) \in C^{(\infty)[\gamma',\gamma'']}_{(s)}$. Using pseudodifferential operators with symbols (13) we define the space $H^{(s)}_{[\gamma',\gamma'']}$ as the closure of $\mathcal{S}_{[\gamma',\gamma'']}$ in the norm

$$||\delta_{s,N}(D)f||_{[\gamma',\gamma'']} = ||f||^{(s)}_{[\gamma',\gamma'']}.$$

From this follows, that we can define $H^{(s)}_{[\gamma',\gamma'']}$ as the intersection of $H^{(s)}_{[\gamma']}\bigcap H^{(s)}_{[\gamma'']}$ and as the set of those $\varphi\in\bigcap H^{(s)}_{[\gamma]}$, $\gamma'<\gamma<\gamma''$, for which the norm

$$'||f||^{(s)}_{[\gamma',\gamma'']}=\sup_{\gamma'<\gamma<\gamma''}||\delta_s,N(D)f||_{[\gamma]}$$

is finite.

We now define $H^{(s)}_{(\ell)[\gamma',\gamma'']}$ as the closure of $\mathcal{S}_{[\gamma',\gamma'']}$ with respect to the norm

$$||f||^{(s)}_{(\ell)[\gamma',\gamma'']}=\left|\left|(1+|x|^2)^{\ell/2}f\right|\right|^{(s)}_{[\gamma',\gamma'']},$$

i.e.,

$$H^{(s)}_{(\ell)[\gamma',\gamma'']}=(1+|x|^2)^{-\ell/2}H^{(s)}_{[\gamma',\gamma'']}=H^{(s)}_{(\ell)[\gamma']}\bigcap H^{(s)}_{[\gamma'']}.$$

From this fact we can easily obtain that $H^{(s)}_{(\ell)[\gamma'\gamma'']}$ can be defined as $\delta_{-s,N}(D)$ $H_{(\ell)[\gamma',\gamma'']}$ and the corresponding norms are equivalent.

Now we define the spaces

$$(\mathcal{S}')_{[\gamma',\gamma'']}=\bigcup H^{(s)}_{(\ell)[\gamma',\gamma'']}=(\mathcal{S}')_{[\gamma']}\bigcap(\mathcal{S}')_{[\gamma'']},$$

$$(\mathcal{O}')_{[\gamma',\gamma'']}=\bigcup_{\ell}\bigcap_{s}H^{(s)}_{(\ell)[\gamma',\gamma'']}=(\mathcal{O}')_{[\gamma']}\bigcap(\mathcal{O}')_{[\gamma'']}.$$

3.1.4. Now we study the Fourier-Laplace transform in the spaces $\Phi_{[\gamma',\gamma'']}$. The case $\Phi=\mathcal{S}$ was considered above. For natural ℓ we define $H^{(\ell)[\gamma',\gamma'']}_{(s)}$ as the set of functions $\psi(\tau)$, belonging to $H^{[\gamma',\gamma'']}_{(s)}$ with derivatives $\partial^k\psi$, $k=0,\dots,l$. Then

$$FH^{(s)}_{(\ell)[\gamma',\gamma'']}=H^{(\ell)[\gamma',\gamma'']}_{(s)}$$

and

$$F(\mathcal{O}')_{[\gamma',\gamma'']}=\bigcap_{\ell}\bigcup_{s}H^{(\ell)[\gamma',\gamma'']}_{(s)}.$$

We can prove that the scales $H^{(\ell)[\gamma',\gamma'']}_{(s)}$ and $C^{(\ell)[\gamma',\gamma'']}_{(s)}$ are equivalent, which implies the isomorphism

$$F(\mathcal{O}')_{[\gamma',\gamma'']}=\mathcal{M}^{[\gamma',\gamma'']}. \tag{15}$$

The new feature of the spaces $\Phi_{[\gamma',\gamma'']}$ (in comparison to the spaces treated in chapter 1) is contained in the fact, that it is possible to define the Fourier-Laplace transform in the space $(\mathcal{S}')_{[\gamma',\gamma'']}$ (and along the way in $\mathcal{O}_{[\gamma',\gamma'']}$).

To describe the Fourier transforms of $(\mathcal{S}')_{[\gamma',\gamma'']}$we have to extend the scale $C_{(s)}^{(m),[\gamma',\gamma'']}$ to negative integral values of m. Let $C_{(s)}^{(-\ell)[\gamma',\gamma'']}$, $\ell \in \mathbb{Z}_+$ denote the space of functions $\psi(\tau)$ holomorphic in the tube domain $T(\gamma',\gamma'')$ and having singularities on the boundary up to order ℓ; in other words the estimates take place:

$$|\psi(\tau)| < Kd^{-\ell}(\gamma',\gamma'',\mathrm{Im}\tau)(1+|\tau|)^s$$

where

$$d(\gamma',\gamma'',\omega) = \min\{\omega-\gamma',\gamma''-\omega\}.$$

Then

$$F(\varphi')_{[\gamma',\gamma'']} = \mathcal{L}^{[\gamma',\gamma'']} \stackrel{\mathrm{def}}{=} \bigcup_{\ell,s} C_{(s)}^{(-\ell)[\gamma',\gamma'']}. \tag{16}$$

3.2. The space $\mathcal{S}_{[\gamma',\gamma'']}$ and related scales (the case $\gamma' > \gamma''$)

To establish the duality relations for the spaces treated in section 3.1, we must introduce and study the spaces related to the weights (1′) with $\gamma' > \gamma''$.

2.2.1. The scales $C^{(m)}_{(\ell)[\gamma',\gamma'']}$ and $H^{(m)}_{(\ell)[\gamma',\gamma'']}$, $m \in \mathbb{Z}_+$ in this case was in fact defined above. The new fact, arising in this case, is their description in terms of spaces $E^{(m)}_{(\ell)[\gamma']}$, $E^{(m)}_{(\ell)[\gamma'']}$, $E = C, H$.

Proposition. *For $\gamma' > \gamma''$ and $E = C, H$ the space $E^{(m)}_{(\ell)[\gamma',\gamma'']}$ coincides with the linear hull of the spaces $E^{(m)}_{(\ell)[\gamma']}$ and $E^{(m)}_{(\ell)[\gamma'']}$, and the norm in $E^{(m)}_{(\ell)[\gamma',\gamma'']}$ is equivalent to the norm of $E^{(m)}_{(\ell)[\gamma']} + E^{(m)}_{(\ell)[\gamma'']}$.*

Proof. If $\gamma' > \gamma''$, then $\gamma'' t_+ + \gamma' t_- \le \rho t$, $\rho = \gamma'$, γ''. But then the linear hull of $E^{(m)}_{(\ell)[\gamma']} + E^{(m)}_{(\ell)[\gamma'']}$ is contained in $E^{(m)}_{(\ell)[\gamma',\gamma'']}$ the inclusion (with topology) takes place

$$E^{(m)}_{(\ell)[\gamma']} + E^{(m)}_{(\ell)[\gamma'']} \subset E^{(m)}_{(\ell)[\gamma',\gamma'']}, \quad \gamma' > \gamma''. \tag{17}$$

To prove the opposite inclusion we take the function $\chi(t) \in C^\infty(\mathbb{R}), \chi(t) = 1$ for $t \geq 1$ and $\chi(t) = 0$ for $t \leq -1$. Then if $f \in E^{(m)}_{(\ell)}$, then $f = \chi f + (1-\chi) f$ where $\chi f \in E^{(m)}_{(\ell)[\gamma'']}$ and $(1-\chi)f \in E^{(m)}_{(\ell)[\gamma']}$. The embedding of the right-hand side of (17) into left-hand side is a trivial consequence of this partition.

From the proposition directly follows Sobolev embedding theorems for the scales $E^{(m)}_{(\ell)[\gamma',\gamma'']}$, $\gamma' > \gamma''$, so

$$\mathcal{S}_{[\gamma',\gamma'']} = \bigcap C^{(m)}_{(\ell)[\gamma',\gamma'']} = \bigcap H^{(m)}_{(\ell)[\gamma',\gamma'']} = \mathcal{S}_{[\gamma']} + \mathcal{S}_{[\gamma'']}$$

$$\mathcal{O}_{[\gamma',\gamma'']} = \bigcup_\ell \bigcap_m C^{(m)}_{(\ell)[\gamma',\gamma'']} = \bigcup_\ell \bigcap_m H^{(m)}_{(\ell)[\gamma',\gamma]} = \mathcal{O}_{[\gamma']} + \mathcal{O}_{[\gamma'']}.$$

3.2.2. To extend the scale $H^{(s)}_{(\ell)[\gamma',\gamma'']}$, $\gamma' > \gamma''$ from $s \in \mathbb{Z}_+$ to arbitrary $s \in \mathbb{R}$ we shall define pseudodifferential operators in the linear hull of $(\mathcal{S}')_{[\gamma']} + (\mathcal{S}')_{[\gamma'']}$. Let f is an element of this space. Then f can be represented (non uniquely) in the form:

$$f \in (\mathcal{S}')_{[\gamma']} + (\mathcal{S}')_{[\gamma'']}, \ f = f' + f'', \ f' \in (\mathcal{S}')_{[\gamma']}, \ f'' \in (\mathcal{S}')_{[\gamma'']}. \tag{18}$$

Then if $a(\tau) \in \mathcal{M}^{[\gamma'',\gamma']}$, we pose

$$a(D)f =_{[\gamma']} a(D)f' +_{[\gamma'']} a(D)f'', \quad a \in \mathcal{M}^{[\gamma'',\gamma']}, \ \gamma'' < \gamma'. \tag{19}$$

We have to check, that this definition is correct, i.e., do not depend on representation $f = f' + f''$. Indeed, if we have another representation $f = f_1' + f_1''$, then $f' - f_1'' = -f'' + f_2'' \in (\mathcal{S}')_{[\gamma']} \cap (\mathcal{S}')_{[\gamma'']} = \mathcal{S}_{[\gamma'',\gamma']}$. But on this space the operators ${}_{[\gamma']}a(D)$ and ${}_{[\gamma'']}a(D)$ coincide, so the definition (19) is correct.

3.2.3. Using pseudodifferential operators with symbols (13) we define spaces $H^{(s)}_{(\ell)[\gamma',\gamma'']}$ for $\gamma' > \gamma$. Then the obvious duality relation

$$\left(H_{(\ell)[\gamma',\gamma'']}\right) = H_{(-\ell)[-\gamma',-\gamma'']}$$

valid for arbitrary γ' and γ'' can be extended to the duality

$$\left(H^{(s)}_{(\ell)[\gamma',\gamma'']}\right)' = H^{(-s)}_{(-\ell)[-\gamma',-\gamma'']}. \tag{20}$$

From this follow duality relations for limiting spaces:

$$(\Phi_{[\gamma',\gamma'']})' = (\Phi')_{[-\gamma',-\gamma'']}, \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}', \tag{21}$$

in the case $\Phi = \mathcal{S}$ (21) is a topological isomorphism.

3.3. Convolutors and convolution equations in $\Phi_{[\gamma',\gamma'']}$ and $\Phi_{[\gamma',\gamma'']}$ for $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$

3.1. As we mentioned above, if Φ_1, Φ_2, Φ_3 are the space of smooth functions and for this triple of spaces the inclusion

$$\Phi_1 * \Phi_2 \subset \Phi_3$$

takes place, then for an arbitrary $\rho \in \mathbb{R}$,

$$\Phi_{1[\rho]} * \Phi_{2[\rho]} \subset \Phi_{[3]\rho}.$$

Let $\gamma'' > \gamma'$. Using the representations

$$\Phi_{[\gamma',\gamma'']} = \Phi_{[\gamma'']} \cap \Phi_{[\gamma']}, \quad \Phi_{[\gamma'',\gamma']} = \Phi_{[\gamma']} + \Phi_{[\gamma'']},$$

we easily see that

$$\Phi_{1[\gamma',\gamma'']} * \Phi_{2[\gamma',\gamma'']} \subset \Phi_{3[\gamma',\gamma'']},$$

$$\Phi_{1[\gamma',\gamma'']} * \Phi_{2[\gamma'',\gamma']} \subset \Phi_{3[\gamma'',\gamma']}.$$

Pseudodifferential operators with symbols (13) can be applied to extend these inclusion relations to the convolution of distributions as well. We thus derive the following important relations

$$(\mathcal{O}')_{[\gamma',\gamma'']} * \Phi_{[\gamma',\gamma'']} \subset \Phi_{[\gamma',\gamma'']} \quad \Phi = \mathcal{S}, \mathcal{O}', \tag{22}$$

$$(\mathcal{O}')_{[\gamma',\gamma'']} * \Psi_{[\gamma'',\gamma']} \subset \Psi_{[\gamma'',\gamma']} \quad \Psi = \mathcal{O}, \mathcal{S}'. \tag{23}$$

By virtue of (22) $(\mathcal{O}')_{[\gamma',\gamma'']}$ is an algebra with respect to convolution and isomorphism (15) is an isomorphism of algebras.

We have showed that the Fourier operator transforms $(\mathcal{S}')_{[\gamma',\gamma'']}$ into the algebra $\mathcal{L}^{[\gamma',\gamma'']}$. It gives us hope, that $(\mathcal{S}')_{[\gamma',\gamma'']}$ is an algebra with respect to convolution.

The following key lemma takes place.

Lemma. Let $f \in C_{(\ell)[\gamma',\gamma'']}$, $\gamma'' > \gamma'$. Then
(i) $\forall\mu\ \exists\lambda$ such that

$$f * g \in C_{(\mu)[\gamma'',\gamma']} \quad \forall g \in C_{(\lambda)[\gamma'',\gamma']}.$$

(ii) $\forall\lambda\ \exists\mu$ such that

$$f * g \in C_{\mu[\gamma',\gamma'']} \quad \forall g \in C_{(\lambda)[\gamma',\gamma'']}.$$

Proof. (i) According to the hypothesis

$$|f(t)| < \text{const}\,(1+|t|)^{-\ell}\ \exp\,(-\gamma'' t_+ - \gamma' t_-)\,,$$

$$|g(t)| < \text{const}\,(1+|t|)^{-\lambda}\ \exp\,(-\gamma' t_+ - \gamma'' t_-)\,,$$

and the desired assertion reduces to the proof of the fact that the integral

$$I(t) = \int_{-\infty}^{\infty} (1+|\theta|)^{-\ell}(1+|t-\theta|)^{-\lambda}$$
$$\exp\,(\gamma' t_+ + \gamma'' t_- - \gamma''\theta_+ - \gamma'\theta_- - \gamma'(t-\theta)_+ - \gamma''(t-\theta)_-)\,d\theta.$$

increases not stronger than const $(1+t)^{-\mu}$ for $\lambda \geq \lambda(\mu)$. We assume that $t > 0$ (the case $t < 0$ is considered in a similar way). Let us represent the integral as a sum of three integrals

$$I(t) = I_1(t) + I_2(t) + I_3(t) = \int_{-\infty}^{0} d\theta + \int_{0}^{t} d\theta + \int_{t}^{\infty} d\theta.$$

Replacing θ by $-\theta$ and assuming that $\lambda > 0$ we see that

$$I_1(t) = \int_0^\infty (1+\theta)^{-\ell}(1+t+\theta)^{-\lambda}\ dt$$
$$\leq\ (1+t)^{-\mu} \times \int_0^\infty (1+\theta)^{-\ell}(1+t+\theta)^{-\lambda+|\mu|} \leq \text{const}\,(1+t)^{-\mu}$$

on condition that $\lambda > \ell + |\mu| + 1$.

Further, for $\lambda > 0$ we have

$$I_2(t) = \int_0^t (1+\theta)^{-\ell}(1+t-\theta)^{-\lambda} \exp\left(-(\gamma''-\gamma')\,\theta\right) d\theta$$
$$\le (1+t)^{-\lambda} \int_0^\infty (1+\theta)^{-\ell} \exp\left(-(\gamma''-\gamma')\,\theta\right) d\theta = \operatorname{const}(1+t)^{-\mu}$$

if $\lambda > \mu$. Finally, when $\lambda > 0$, then

$$I_3(t) = \int_t^\infty (1+\theta)^{-\ell}(1+\theta-t)^{-\lambda} \exp\left(-(\gamma''-\gamma')\,t\right) d\theta$$
$$\le (1+t)^{\lambda} \exp\left(-(\gamma''-\gamma')\,t\right) \int_0^\infty (1+\theta)^{-\ell-\lambda} \le C_\mu(1+t)^{-\mu}$$

provided that $\lambda > |\ell| + 1$.

(ii) is proved in a similar way.

Using the lemma and the representation of the elements of $(\mathcal{S}')_{[\gamma',\gamma'']}$ as differential operators applied to some elements in $C_{(\ell)[\gamma',\gamma'']}$ we prove that

$$(\mathcal{S}')_{[\gamma',\gamma'']} * \Phi_{[\gamma'',\gamma']} \subset \Phi_{[\gamma'',\gamma']}, \quad \Phi = \mathcal{S}, \mathcal{O}', \tag{24}$$

$$(\mathcal{S}')_{[\gamma',\gamma'']} * \Psi_{[\gamma',\gamma'']} \subset \Psi_{[\gamma',\gamma'']}, \quad \Psi = \mathcal{O}, \mathcal{S}'. \tag{25}$$

In particular, according to (25) $(\mathcal{S}')_{[\gamma',\gamma'']}$, $\gamma'' > \gamma'$ is an algebra relative to the operation of convolution and it is readily verified that (16) is an isomorphism of algebras.

3.2. The spaces $\Phi_{[\gamma',\gamma'']}$ and $\Phi_{[\gamma'',\gamma']}, \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ are invariant with respect to translations and therefore the definition of convolution operators is the same as for the case $\gamma' = \gamma'' = 0$.

Theorem. *Let $\gamma'' > \gamma'$ and let $\Phi = \mathcal{S}(\mathbb{R}), \mathcal{O}'(\mathbb{R})$ and $\Psi = \mathcal{O}(\mathbb{R}), \mathcal{S}'(\mathbb{R})$.*

(i) *For each convolution operator A on $\Phi_{[\gamma',\gamma'']}, \Psi_{[\gamma'',\gamma']}$ there exists a distribution $f \in (\mathcal{O}')_{[\gamma',\gamma'']}$ such that*

$$A\varphi = f * \varphi, \quad \forall \varphi \in \Phi_{[\gamma',\gamma'']}, \quad \Psi_{[\gamma'',\gamma']}. \tag{26}$$

(ii) *For each convolution operator A on $\Phi_{[\gamma'',\gamma']}$ and $\Psi_{[\gamma',\gamma'']}$ there exists a distribution $f \in (\mathcal{S}')_{[\gamma',\gamma'']}$ such that*

$$A\varphi = f * \varphi, \quad \forall\varphi \in \Phi_{[\gamma'',\gamma']}, \quad \Psi_{[\gamma',\gamma'']}. \tag{27}$$

Remarks. 1) The right-hand sides of (26), (27) make sense in view of (22), (23) and (respectively) (24), (25).

2) We can rewrite the statements of the theorem in the following form:

$$\begin{aligned}\mathfrak{C}\left(\mathcal{S}_{[\gamma',\gamma'']}\right) = \mathfrak{C}\left((\mathcal{O}')_{[\gamma',\gamma'']}\right) = \mathfrak{C}\left((\mathcal{O})_{[\gamma'',\gamma']}\right) = \mathfrak{C}\left((\mathcal{S}')_{[\gamma'',\gamma']}\right) \\ = (\mathcal{O}')_{[\gamma',\gamma'']}, \qquad \gamma' < \gamma''.\end{aligned} \tag{28}$$

$$\begin{aligned}\mathfrak{C}\left(\mathcal{S}_{[\gamma'',\gamma']}\right) = \mathfrak{C}\left((\mathcal{O}')_{[\gamma'',\gamma']}\right) = \mathfrak{C}\left(\mathcal{O}_{[\gamma',\gamma'']}\right) = \mathfrak{C}\left((\mathcal{S}')_{[\gamma',\gamma'']}\right) \\ = (\mathcal{S}')_{[\gamma',\gamma'']}, \qquad \gamma' < \gamma''.\end{aligned} \tag{29}$$

Proof of the theorem follows the plan of the proof of theorem 1.5.1.

1) Almost literal repetition of the proof of proposition 1 from subsection 1.5.1 shows that every convolution operator on the space $\Phi_{[\gamma_1,\gamma_2]}$, $\Phi = \mathcal{S}, \mathcal{O}$, γ_1, γ_2 arbitrary, can be represented in the form

$$(A\varphi) = (f * \varphi)(x) \stackrel{\text{def}}{=} (f, IT_x\varphi), \quad \varphi \in \Phi_{[\gamma_1,\gamma_2]} \tag{30}$$

with the distribution.

$$f \in I\left(\Phi_{[\gamma_1,\gamma_2]}\right)' = I(\Phi')_{[-\gamma_1,-\gamma_2]} = (\Phi')_{[\gamma_2,\gamma_1]}. \tag{31}$$

2) Using (30) we can define the spaces $\mathfrak{C}\left(\Phi_{[\gamma_1,\gamma_2]}\right)$ for $\Phi = \mathcal{S}, \mathcal{O}$. From this definition follows, that

$$\mathfrak{C}\left(\mathcal{S}_{[\gamma'',\gamma']}\right) \subset (\mathcal{S}')_{[\gamma',\gamma'']}, \quad \mathfrak{C}\left(\mathcal{O}_{[\gamma'',\gamma']}\right) \subset (\mathcal{O})_{[\gamma',\gamma'']}.$$

As from (24), (23) follows the opposite inclusions, we come to the relations:

$$\mathfrak{C}\left(\mathcal{S}_{[\gamma'',\gamma']}\right) = (\mathcal{S}')_{[\gamma',\gamma'']}, \quad \mathfrak{C}\left(\mathcal{O}_{[\gamma'',\gamma']}\right) = (\mathcal{O}')_{[\gamma',\gamma'']}. \tag{32}$$

3) Repeating the argument of proposition 2 in subsection 1.5.1 we prove that each convolution operator $A_\circ$ on $\Phi_{[\gamma_1,\gamma_2]}$, $\Phi = \mathcal{S}, \mathcal{O}$, can be continued by continuity to a convolution operator $A : \Psi_{[\gamma_1,\gamma_2]} \to \Psi_{[\gamma_1,\gamma_2]}$, $\Psi = \mathcal{O}', \mathcal{S}'$. And the restriction of the convolution operator on $\Psi_{[\gamma_1,\gamma_2]}$ is a convolution operator on $\Phi_{[\gamma_1,\gamma_2]}$.

Now we can define

$$(33) \qquad \mathfrak{C}\left((\mathcal{O}')_{[\gamma_1,\gamma_2]}\right) = \mathfrak{C}\left(\mathcal{S}_{[\gamma_1,\gamma_2]}\right), \quad \mathfrak{C}\left((\mathcal{S}')_{[\gamma_1,\gamma_2]}\right) = \mathfrak{C}\left(\mathcal{O}_{[[\gamma_1,\gamma_2]}\right).$$

4) As the spaces $(\mathcal{O}')_{[\gamma',\gamma'']}$ and $(\mathcal{S}')_{[\gamma',\gamma'']}$ contain $\delta(x)$, then $\mathfrak{C}\left(\Psi_{[\gamma',\gamma'']}\right) \subset \Psi_{[\gamma',\gamma'']}$, $\Psi = \mathcal{O}', \mathcal{S}'$. As the opposite inclusions take place (see (22), (25)), we obtain the relations

$$(34) \qquad \mathfrak{C}\left((\mathcal{O}')_{[\gamma',\gamma'']}\right) = (\mathcal{O}')_{[\gamma',\gamma'']}, \quad \mathfrak{C}\left((\mathcal{S}')_{[\gamma',\gamma'']}\right) = (\mathcal{S}')_{[\gamma',\gamma'']}.$$

Combining these relations with (32), (33) we obtain all relations (28), (29).

3.3. We can now discuss the question of solvability of the convolution equation

$$A * u = f$$

in the spaces $\Phi_{[\gamma',\gamma'']}$ and $\Phi_{[\gamma'',\gamma']}$, where $\gamma'' > \gamma'$ and $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$. As in theorem 5.2 it can be proved that unique solvability of this equation is equivalent to invertibility of A in the space of corresponding convolutors, or to the invertibility of the symbol $\hat{A}(\tau)$ in the space of the Fourier transforms of convolutors. In other words solvability condition in $\mathcal{S}_{[\gamma',\gamma'']}, (\mathcal{O}')_{[\gamma',\gamma'']}, \mathcal{O}_{[\gamma'',\gamma']}, (\mathcal{S}')_{[\gamma'',\gamma']}$ is

$$(35) \qquad \hat{A}^{-1}(\tau) \in \mathcal{M}^{[\gamma',\gamma'']}.$$

This condition is equivalent to an estimate from below of the form:

$$(35') \qquad \left|\hat{A}(\tau)\right| > \text{const}\,(1 + |\tau|)^{\mu}, \quad Im\tau \leq 0.$$

The condition of solvability in $\mathcal{S}_{[\gamma'',\gamma']}, (\mathcal{O}')_{[\gamma'',\gamma']}, \mathcal{O}_{[\gamma',\gamma'']}, (\mathcal{S}')_{[\gamma',\gamma'']}$ is

$$(36) \qquad \hat{A}^{-1}(\tau) \in \mathcal{L}^{[\gamma',\gamma'']}.$$

This condition is equivalent to an estimate of the form:

$$(36') \qquad \left|\hat{A}(\tau)\right| > \text{const}\,(1+|\tau|)^{\mu}\; d^{-\ell}\;(\gamma', \gamma'', \text{Im}\tau) \quad \text{Im}\tau < 0.$$

In the case when A is differential operator, $\hat{A}(\tau) = P(\tau)$, conditions (35′) and (36′) are equivalent (Seidenberg-Tarski theorem) to the conditions:

$$(37) \qquad P(\tau) \neq 0, \quad \text{Im}\tau \leq 0,$$

$$(38) \qquad P(\tau) \neq 0, \quad \text{Im}\tau < 0.$$

So we see, that conditions of solvability of the differential equation $P(D)u = f$ in $\mathcal{S}_{[\gamma'',\gamma']}$, $\mathcal{O}_{[\gamma',\gamma'']}$ are more weak that analogous conditions for $\mathcal{S}_{[\gamma',\gamma'']}$ and $\mathcal{O}_{[\gamma'',\gamma']}$.

Concluding remark. The theory developed in this section can be extended without any difficulties to the scales in $\mathbb{R}^n$ corresponding to the weight (1) in the case $\Gamma'' > \Gamma'$ or $\Gamma > \Gamma''$. The "mixed" case when, say, $\gamma_i^{'} < \gamma_i^{''}$ for $i = 1, \ldots, k < n$ and $\gamma_{k+1}^{'} > \gamma_{k+1}^{''}, \ldots, \gamma_n^{'} > \gamma_n^{''}$ is more difficult. These additional difficulties we shall discuss on the example of scales corresponding to the spaces $\Phi_+(\mathbb{R}^n)$.

§4. Convolution equations in $\mathcal{S}(\mathbb{R}^n)_+$ and in the related spaces of functions and distributions

As we mentioned in Introduction, if in the space of functions or distributions $\Phi(\mathbb{R}^n)$ the notion of support is defined, then we can define the subspace

$$(1) \qquad \Phi_+ = \left\{\varphi \in \Phi,\ \text{supp}\,\varphi \in \mathbb{R}^n_+\right\},$$

where $R^n_+ = \{x \in \mathbb{R}^n,\ x_1 \geq 0\}$. In what follows we separate out one of the coordinates, say $t = x_1$, in $\mathbb{R}^n$, and denote the other coordinates as φ, i.e., $x = (t, y)$, $y = (x_2, \ldots, x_n)$, let $\xi = (\sigma, \eta)$, $\eta = (\xi_2, \ldots, \xi_n)$, be the dual variables relative to the form $\langle x, \xi\rangle = t\sigma + \langle \xi, \eta\rangle$ and let $\tau = \sigma + i\gamma$ be the complex coordinate dual to t.

As it was explained in §1 the spaces of the type (1) can be regarded as a weight space Φ_μ with the weight μ of the form (1.1′).

In the case $\Phi = \mathcal{S}(\mathbb{R}^n)$, $\mathcal{O}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{O}(\mathbb{R}^n)$, $n = 1$ the scales corresponding to (1) are the limiting case of scales $\Phi_{[\gamma',\gamma'']}(\mathbb{R})$, when $\gamma'' = 0$ and $\gamma' = -\infty$, and we need not new ideas to work with them. The same is true in the case $n > 1$. The difficulties arise only in the moment when we try to define the convolutors and are connected with the fact, that Φ_+ is not invariant under arbitrary translation operators. Additional difficulties arise in description of convolutors on $\mathcal{O}_+$. To overcome these difficulties we have to introduce spaces of functions and distributions having different degrees of smoothness and different behavior τ at infinity with respect to different variables and use special kernel theorems.

4.1. The space $\mathcal{S}_+$ and connected scales

4.1.1. Since the notion of support is defined in $\mathcal{D}'$, the general definition of the subspace Φ_+ given above makes sense for $\Phi = \mathcal{S}$, $\mathcal{O}$, $C^{(m)}_{(\ell)}$, $H^{(m)}_{(\ell)}$, $H^{(s)}_{(\ell)}$, $\mathcal{O}'$, $\mathcal{S}' \subset \mathcal{D}'$ and Φ_+ is a closed subspace, i.e.,

$$(1) \qquad \{\varphi_j \in \Phi_+,\ \varphi_j \to \varphi \text{ (in the topology of } \Phi)\} \Longrightarrow \{\operatorname{supp}\varphi \in \mathbb{R}^n_+\},$$

$$\Phi = \mathcal{S},\ \mathcal{O},\ C^{(m)}_{(\ell)},\ H^{(s)}_{(\ell)},\ \mathcal{O}',\ \mathcal{S}'.$$

It is suffices to prove this assertion for the broadest of these spaces, i.e., for $\Phi = \mathcal{S}'$. In this case (1) is equivalent to the trivial assertion

$$\{\varphi_j \in \mathcal{S}',\ (\varphi_j, \psi) = 0,\quad \forall \psi_- \in \mathcal{S}_-,\ \varphi_j \to \varphi \text{ in } \mathcal{S}'\}$$
$$\Rightarrow \{(\varphi, \psi_-) = 0,\quad \forall \psi_- \in \mathcal{S}_-\} \Leftrightarrow \{\operatorname{supp}\varphi \in \mathbb{R}^n_+\}.$$

The embeddings

$$E^{(m)}_{(\ell)} \subset E^{(m')}_{(\ell')},\quad m \geq m',\ \ell \geq \ell',\ E = C, H$$

induce the analogous embeddings for the subspaces

$$E^{(m)}_{(\ell)+} \subset E^{(m')}_{(\ell')+},\quad m \geq m',\ \ell \geq \ell',\ E = C, H,$$

commutable with the injective embeddings $C^{(m)}_{(\ell)+} \to C^{(m)}_{(\ell)}$, $H^{(m)}_{(\ell)+} \to H^{(m)}_{(\ell)}$, etc. This makes it possible to define the scales of spaces $\left\{C^{(m)}_{(\ell)+}\right\}$, and

$\left\{H^{(m)}_{(\ell)+}\right\}$ and their projective limits. The latter are identified in a natural manner with the (closed) subspace $\mathcal{S}_+ \subset \mathcal{S}$,

$$\mathcal{S}_+ = C^{(\infty)}_{(\infty)+} \overset{\text{def}}{=} \cap\, C^{(m)}_{(\ell)+}.$$

In view of embedding theorems

$$\mathcal{S}_+ = H^{(\infty)}_{(\infty)+} \overset{\text{def}}{=} \cap\, H^{(\ell)}_{(\ell)+}.$$

The situation is similar in the case of the space $\mathcal{O}$,

$$\mathcal{O}_+ = \bigcup_{\ell} \bigcap_{m} C^{(m)}_{(\ell)+} = \bigcup_{\ell} \bigcap_{m} H^{(m)}_{(\ell)+}.$$

To study the Fourier operator in $\mathcal{S}_+$ the following statement is of use :

Proposition. *$\varphi \in C^{(m)}_{(\ell)+}$ if and only if $\varphi \in C^{(m)}_{(\ell)[\gamma]}$ for all $\gamma < 0$ and the norm*

(2) $$|\varphi|^{(m)}_{(\ell)+} = \sup_{\gamma<0} |\varphi|^{(m)}_{(\ell)[\gamma]}$$

is finite. Moreover

(2′) $$|\varphi|^{(m)}_{(\ell)+} = |\varphi|^{(m)}_{(\ell)}, \quad \forall \varphi \in C^{(m)}_{(\ell)+}.$$

Proof. If $\varphi \in C^{(m)}_{(\ell)+}$, then the norm $|\varphi|^{(m)}_{(\ell)[\gamma]}$ is finite, increase monotonically with γ, and attains maximum at $\gamma = 0$, i.e., (2′) holds.

On the other hand, if a function has a finite norm (2), it vanishes at $t < 0$. Further, since (2) is finite

$$e^{\gamma t}(1 + |t|^2)^{\ell/2}|D^\alpha \varphi(x)| \leq |\varphi|^{(m)}_{(\ell)+}, \quad |\alpha| \leq m, \ \gamma < 0.$$

This inequality is continued by continuity to $\gamma = 0$, i.e., $\varphi \in C^{(m)}_{(\ell)+}$.

4.1.2. Denote by $C^{(m)+}_{(\ell)}$ the space of functions of the variable τ, η, $\tau = \sigma + i\rho$, $\xi = (\sigma, \eta)$, possessing the following properties (compare $C^{(m)[\gamma',\gamma'']}_{(\ell)}$ for $\gamma' = -\infty$, $\gamma'' = 0$):

1) the functions $\varphi(\tau,\eta) \in C^{(m)+}_{(\ell)}$ are holomorphic with respect to τ for $Im\tau < 0,\ \eta \in \mathbb{R}^{n-1}$.

2) $\varphi_\gamma(\xi) = \varphi(\sigma + i\gamma, \eta) \in C^{(m)}_{(\ell)}$ for $\gamma < 0$ $\left(\text{i.e., } C^{(m)+}_{(\ell)} \subset \bigcap_{\gamma<0} C^{(m)[\gamma]}_{(\ell)}\right)$.

3) The norm

$$|\varphi|^{(m)+}_{(\ell)} = \sup_{\gamma<0} |\varphi|^{(m)[\gamma]}_{(\ell)} = \sup_{\gamma<0,\ \xi\in\mathbb{R}^n,\ |\alpha|<m} \left(1+\gamma^2+|\xi|^2\right)^{\ell/2} |\partial^\alpha \varphi(\sigma+i\gamma,\eta)| \tag{3}$$

is finite.

Denote by $\mathcal{S}^+$ the projective limit of the spaces $C^{(m)+}_{(\ell)}$, i.e.,

$$\mathcal{S}^+ \stackrel{\text{def}}{=} C^{(\infty)+}_{(\infty)} = \bigcap C^{(m)+}_{(\ell)}.$$

Theorem. *The Fourier-Laplace operator*

(4)

$$F : \varphi(t,y) \longmapsto \hat\varphi(\tau,\eta) = (2\pi)^{-n/2} \int_0^\infty \int_{\mathbb{R}^{n-1}} \exp(-it\tau - i\langle y,\eta\rangle)\varphi(t,y)\,dy\,dt$$

is defined for all $\varphi \in \mathcal{S}_+$ and generates a one-to-one and bicontinuous mapping

$$F\mathcal{S}_+ = \mathcal{S}^+. \tag{5}$$

The assertion of the theorem follows from the continuity of operators

$$F : C^{(m)}_{(\ell+n')+} \longrightarrow C^{(\ell)+}_{(m)}, \quad F^{-1} : C^{(\ell)+}_{(m+m')} \longrightarrow C^{(m)}_{(\ell)}, \quad n', m' > n. \tag{6}$$

The proof of the continuity of the first operator reduces to straightforward verification of conditions 1)-3) in the original definition. As for the continuity of the second operator, let us consider inverse Fourier-Laplace operator

$$\hat\varphi(\tau,\eta) \longmapsto \varphi(t,y) = (2\pi)^{-n/2} \int_{Im\tau=\gamma} \exp(it\tau + i\langle y,\eta\rangle)\hat\varphi(\tau,\eta)\mathrm{Re}\tau\, d\eta. \tag{7}$$

Since $\hat{\varphi}(\tau,\eta)$ is holomorphic with respect to τ and decreases sufficiently strong for $\mathrm{Im}\tau =$ const, $|\mathrm{Re}\tau| \to \infty$, Cauchy integral theorem implies that the right-hand side of (7) does not change if the contour of integration with respect to τ, $\mathrm{Im}\tau = \gamma$ is replaced by the line $\mathrm{Im}\tau = \gamma' < 0$.

Thus, the right-hand side of (7) does not depend on the choice of $\gamma < 0$, i.e., (7) is defined correctly. Applying Parseval's inequality (we obtain)

$$|\varphi|^{(m)}_{(\ell)[\gamma]} \leq C \ |\hat{\varphi}|^{(\ell)+}_{(m+m')}.$$

Taking supremum over $\gamma < 0$ and using proposition 3.1.1 we find, that

$$|\varphi|^{(m)}_{(\ell)+} \leq C|\hat{\varphi}|^{(\ell)+}_{(m+m')}.$$

Remark. It can be checked that if $\varphi \in C^{(m)}_{(\ell+n')_+}$, $n' > n$, then $\hat{\varphi}(\tau,\eta)$ assumes boundary values $\hat{\varphi}(\xi)$ for $\mathrm{Im}\tau \to -0$ in the sense of $C^{(\ell)}_{(m)}$. From this follows that function $\psi(\tau,\eta) \in \varphi^+$ assumes boundary conditions for $\mathrm{Im}\tau \to -0$ in the topology of $\mathcal{S}$.

We introduce the space

$$\mathcal{M}^+ = \bigcap_m \bigcup_\ell C^{(m)+}_{(\ell)}.$$

One readily proves that $\mathcal{M}^+$ is a commutative algebra relative to multiplication and $\mathcal{S}^+$ is an ideal of $\mathcal{M}^+$. In other words $a(\tau,\eta) \in \mathcal{M}^+$ are multipliers on $\mathcal{S}^+$ and the Fourier-Laplace operators (4), (7) makes it possible to define correctly pseudodifferential operators

$$(8)\qquad \begin{aligned} a(D)\varphi &= F^{-1}_{(\tau,\eta)\to x}\ a(\tau,\eta)\ F_{x\mapsto(\tau,\eta)}\varphi \\ &= (2\pi)^{-n/2} \int\limits_{Im\tau=\gamma<0} a(\tau,\eta)\exp(it\tau + i\langle y,\eta\rangle)\hat{\varphi}(\tau,\eta)d\xi, \\ & a \in \mu^+,\ \varphi \in \varphi_+. \end{aligned}$$

4.1.3. Now we can define the scale $H^{(s)}_{(\ell)+}$. We shall begin from the case $s = \ell = 0$. In this case $\varphi \in H_+$ if and only if $\exp(\gamma t)\varphi \in H$ for $\gamma < 0$ and the norm

$$||\varphi||_+ = \sup_{\gamma<0} ||\varphi||_{[\gamma]} \stackrel{\text{def}}{=} \sup_{\gamma<0} ||exp(\gamma t)\varphi||$$

is finite. Moreover

$$||\varphi||_+ = ||\varphi||, \quad \forall \varphi \in H_+.$$

Denote by H^+ the space of functions $\psi(\tau, \eta)$ such that

1) they are holomorphic with respect to τ for $\mathrm{Im}\tau < 0$ and almost all $\eta \in \mathbb{R}^{n-1}$,

2) $\psi_\gamma(\xi) = \psi(\sigma + i\gamma, \eta) \in H$ for all $\gamma < 0$, and

3) the norm

$$||\psi||^+ = \sup_{\gamma<0} ||\psi||^{[\gamma]}$$

is finite.

There is classical

Theorem. *(Paley-Wiener). The Fourier-Laplace operator determines the isometric isomorphism*

$$FH_+ = H^+, \quad ||\varphi||_+ = ||\hat{\varphi}||^+. \tag{9}$$

Now we shall consider $H_+^{(s)}$ spaces for arbitrary $s \in \mathbb{R}$.

The theory of the spaces $H^{(s)}$ presented in §I.2 was based on the representation of $H^{(s)}$ as the image of "zeroth" space $H^{(0)} = H$ under the graduating pseudodifferential operator $\delta_{-s}(D)$:

$$H^{(s)} = \delta_{-s}(D)H.$$

The analogous representation takes place for subspaces

$$H_+^{(s)} = \left\{ f \in H^{(s)}, \quad \mathrm{supp}\, f \in \mathbb{R}_+^n \right\},$$

if we replace the operators $\delta_s(D)$ (which do not preserve the support of distribution) by operators $\delta_s^+(D)$ with symbols

$$\delta_s^+(\tau, \eta) = \left(i\tau + \sqrt{1 + |\eta|^2}\right)^s. \tag{10}$$

Then we pose

$$||f||^{(s)} = ||\delta_s^+(D)||.$$

Proposition. (i) *The mapping*

$$H_+^{(s)} \to H_+^{(s-\tau)} \quad (f \mapsto \delta_\tau^+(D)f)$$

is an isometric isomorphism for any s, τ,

(ii) $f \in H_+^{(s)}$ *if and only if* $f = \delta_{-s}^+(D)g,\ g \in H_+,\ ||f||^{(s)} = ||g||$,

(iii) $f \in H_+^{(s)}$ *if and only if* $f \in H_{[\gamma]}^{(s)}$ *for any* $\gamma < 0$ *and the norm*

$$||f||_+^{(s)} = \sup_{\gamma<0} ||\delta_s^+(D)f||_{[\gamma]}$$

is finite. In this case

$$||f||_+^{(s)} = ||f||^{(s)}.$$

Proof. We shall prove only (i). In view of proposition 1.2.2, it is only required to show that the operator $\delta_{-\tau}^+(D)$ preserves support in the sense that

$$\operatorname{supp} \delta_\tau^+(D)f \subset \overline{\mathbb{R}_+^n} \text{ for } f \in H_+^{(s)}.$$

Take $\varphi \in \Phi_-$ and consider

$$\left(\delta_\tau^+(D)f, \varphi\right) \overset{\text{def}}{=} \left(f, \delta_\tau^+(-D)\varphi\right).$$

The operator $\delta_\tau^+(-D)$ transforms $\mathcal{S}_-$ into itself and the right-hand side is equal to 0 since the support of f belongs to $\overline{\mathbb{R}_+^n}$.

Denote by $H_{(s)}^+$ the image of H^+ under operator of multiplication by $\delta_{-s}^+(\tau, \eta)$. Then we have isometric isomorphisms

$$FH_+^{(s)} = H_{(s)}^+, \quad ||\hat{f}||_{(s)}^+ = ||f||_+^{(s)}.$$

If in the theory of section 1.2.3, we replace H by H_+ and pseudodifferential operators $(1 + |D|^2)^{s/2}$ by $\delta_s^+(D)$, we obtain the theory of $H_{(\ell)+}^{(s)}$ spaces.

According to the general convention on notation, $(\mathcal{S}')_+$ consists of distributions $f \in \mathcal{S}'$ whose supports belong to $\overline{\mathbb{R}_+^n}$. On the other hand, we can consider the inductive limit of the scale $\left\{H_{(\ell)+}^{(s)}\right\}$. One can reading show that these spaces coincide:

$$(\mathcal{S}')_+ = \bigcup_{s,\ell} H_{(\ell)+}^{(s)},$$

i.e., in this specific case the operations of passing to the inductive limit and to subspaces of elements with support in $\overline{\mathbb{R}^n_+}$ are commutable.

An analogous situation takes place in the case of the definition of $(\mathcal{O}')_+$:

$$(\mathcal{O}')_+ = \bigcap_{\ell} \bigcup_{s} H^{(s)}_{(\ell)+}.$$

4.1.4. We denote by $H^{(\ell)+}_{(s)}$, where $\ell > 0$ is an integer, the set of functions $\psi(\sigma + i\gamma, \eta)$ belonging to $H^{+}_{(s)}$ together with their all derivatives $\partial^\alpha \psi$, $|\alpha| \leq \ell$ and equipped with natural norm. Then we have isomorphism

$$FH^{(s)}_{(\ell)+} = H^{(\ell)+}_{(s)}, \quad \forall s \in \mathbb{R}, \quad \forall \ell \in \mathbb{Z}_+.$$

These isomorphisms imply that

$$F(\mathcal{O}')_+ = \bigcap_{\ell} \bigcup H^{(\ell)+}_{(s)}.$$

It can be easily shown that there are embeddings

$$C^{(\ell)+}_{(s+\kappa)} \subset H^{(\ell)+}_{(s)} \subset C^{(\ell-\kappa)+}_{(s)}, \quad \kappa > n.$$

From these embeddings follows the equivalence of scales $\left\{C^{(\ell)+}_{(s)}\right\}$, $\left\{H^{(\ell)+}_{(s)}\right\}$ and

(11) $$F\left((\mathcal{O}')_+\right) = \mathcal{M}^+.$$

4.1.5. In previous section in the case of functions of one variable we defined the Fourier-Laplace transform in the spaces $\mathcal{O}(\mathbb{R})_{[\gamma',\gamma'']}$, $(\mathcal{S}'(\mathbb{R}))_{[\gamma',\gamma'']}$, when $\gamma' < \gamma''$. These results are also true in the case $\gamma' = -\infty$.

Proposition 1. *For a function $\psi(\tau)$ in the half-plane $Im\tau < 0$ the following conditions are equivalent.*

(i) *$\psi(\tau)$ is the Fourier-Laplace transform of a function $\psi \in \mathcal{O}_+$, i.e., is represented as an absolutely convergent integral*

$$\psi(\tau) = (2\pi)^{-1/2} \int^{\infty} \exp(-it\tau)\varphi(t)dt, \quad \varphi \in \mathcal{O}_+(\mathbb{R}),\ Im\tau < 0.$$

(ii) *$\psi(\tau)$ is holomorphic for $Im\tau < 0$ and $\exists \ell \geq 0$ such that $\forall m \in \mathbb{Z}_+$,*

$$|\Psi(\tau)| < Cm\ |Im\tau|^{-\ell}\ (1 + |\tau|)^{-m}.$$

Proposition 2. *For a function $\psi(\tau)$ in the half-plane $Im\tau < 0$ the following conditions are equivalent.*

(i) $\psi(\tau)$ is the Fourier-Laplace transform of a distribution $\varphi \in (\mathcal{S}'(\mathbb{R}))_+$, i.e., $\forall\gamma < 0$ the function $\psi(\sigma + i\gamma, \eta)$ is the Fourier transform (in the sense of $\mathcal{S}'$) of the distribution $\exp(\gamma t)\varphi \in (\mathcal{O}'(\mathbb{R}))_+$.

(ii) the function $\psi(\tau)$ is holomorphic for $Im\tau < 0$ and there are $\ell \geq 0$, $c > 0$, and s such that

$$|\psi(\tau)| < C|Im\tau|^{-\ell}\,(1 + |\tau|)^s.$$

We denote by $\mathcal{L}^+(\mathbb{R})$ the space of holomorphic functions, described in proposition 2. For $\gamma'' = 0$, $\gamma' = 0$ it coincides with the space $\mathcal{L}^{[\gamma',\gamma'']}$, considered in §1. So we have an isomorphism:

$$F\,(\mathcal{S}'(\mathbb{R}))_+ = \mathcal{L}^+(\mathbb{R}). \tag{12}$$

4.2. Convolution, convolution operators and convolutors in $\mathcal{S}_+$, $(\mathcal{O}')_+$, homogeneous Cauchy problem in decreasing functions

4.2.1. Let functions $f(t, y)$ and $g(t, y)$ belong to a pair of spaces for which the classical operation of convolution is defined and let $f(t, y) = g(t, y) = 0$ for $t < 0$. Then we have

$$\begin{aligned}(f * g)(t, y) &= \int_0^t \int_{\mathbb{R}^{n-1}} f(t - t', y - y')g(t', y')dy'\,dt' \\ &= \int_0^t \int_{\mathbb{R}^{n-1}} f(t', y')g(t - t', y - y')dy'\,dt',\end{aligned} \tag{13}$$

whence it follows that $(f * g)(t, y) = 0$ for $t < 0$. Here, if $\Phi_i, i = 1, 2, 3$ is a triple of spaces for which classical operation of convolution is defined

$$\Phi_1 * \Phi_2 \subset \Phi_3,$$

then we have inclusion relations for subspaces:

$$\Phi_{1+} * \Phi_{2+} \subset \Phi_{3+}.$$

The above assertions remain true in the case of $\Phi_1 = \mathcal{O}'$, $\Phi_2 = \mathcal{S}'$, and they apply to subspaces of these spaces. To show this it should be noted that convolution of $(\mathcal{O}')_+$ and $(\mathcal{S}')_+$ can be defined with the aid of the pseudodifferential operators $\delta_s^+(D)$ which transforms distributions with supports belonging to $\mathbb{R}_+^n$ into distributions of the same type. We put

$$f * g = \delta_{k+m}^+(D)\left((\delta_{-k}^+(D)f) * (\delta_{-m}^+(D)g)\right). \tag{14}$$

In particular, we obtain that

$$(\mathcal{O}')_+ * \Phi_+ \subset \Phi_+, \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'. \tag{15}$$

From (15) follows that $(\mathcal{O}')_+$ is an algebra with respect to convolution and the isomorphism (11) is an isomorphism of algebras.

The integration in (13) with respect to t extends over a finite interval, and therefore the growth of the functions f and g with respect to t does not affect the convergence of the integral. However, some estimates for the rate of growth with respect to t are needed for convolution to belong to the required space. In particular we can easily deduce from (14) that $(\mathcal{S}'(\mathbb{R}))_+$ is an algebra with respect to convolution:

$$(\mathcal{S}'(\mathbb{R})_+ * \mathcal{S}'(\mathbb{R}))_+ \subset (\mathcal{S}'(\mathbb{R}))_+ \tag{16}$$

and isomorphism (12) is an isomorphism of algebras.

4.2.2. If $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ then for $h = (h_1, \ldots, h_n)$, $T_h\Phi_+ \subset \Phi_+$ if and only if $h_1 \leq 0$, i.e., on Φ_+ only the semigroup of translations T_h, $h \in \overline{\mathbb{R}_-^n}$ is defined. However, it suffices to use the semigroup to construct a meaningful theory of convolution operators in the sense of §1.5.

By a convolution operator $A : \Phi_+ \to \Phi_+$, $\Phi = \mathcal{S}, \mathcal{S}'$, is meant a continuous operator commuting with translations T_h, $h \in \mathbb{R}_-^n$. This definition is retained in the case $\Phi = \mathcal{O}, \mathcal{O}'$ as well; it is only necessary to replace the condition of continuity of the operator by the regularity condition. As any continuous operator on $\mathcal{S}$ and $\mathcal{S}'$ is regular, the convolution operators on $\mathcal{S}_+, \ldots, (\mathcal{O}')_+$ can be defined in unified manner as regular operators commutable with translations T_h, $h \in \overline{\mathbb{R}_-^n}$.

In this section we shall prove

Theorem. (i) *Let* $\Phi = \mathcal{S}_+, (\mathcal{O}')_+$. *Then for each convolution operator* $A : \Phi_+ \to \Phi_+$ *there is a distribution* $f \in (\mathcal{O}')_+$ *such that*

$$A\varphi = \mathrm{con}_f \varphi = f * \varphi, \quad \forall \varphi \in \Phi_+. \tag{17}$$

(ii) *Let n=1 and* $\Phi = \mathcal{O}(\mathbb{R}),\ \mathcal{S}'(\mathbb{R})$. *Then for each convolution operator* $A : \Phi_+ \to \Phi_+$ *there is a distribution* $f \in (\mathcal{S}'(\mathbb{R}))_+$ *such, that (17) holds.*

The assertions of this theorem are proved following the scheme of theorem 1.5.1. On this way arise difficulties of technical character connected with mentioned above fact of invariance of our spaces under only semigroup of translations.

The generalization of (ii) to the case $n > 1$ (as it was mentioned in the beginning of this subsection) demands new ideas and will be given in the next subsection.

4.2.3. As in §1.5 we begin with an analog of proposition 1 in 1.5.1 for convolution operators on $\mathcal{S}_+$ and $\mathcal{O}_+$. To this end we must select a space of distributions which a priori contain those f for which the convolution (1.5.2) make sense for $\varphi \in \mathcal{S}_+, \mathcal{O}_+$ and $x \in \mathbb{R}^n$.

Consider the family of subspaces

$$\Phi\ (-\infty, c] = \{\varphi(t, y) \in \Phi,\ \ \varphi(t, y) = 0 \ \text{ for } \ t \geq 0\}\,, \ \ \Phi = \mathcal{S}, \mathcal{O}.$$

We have the natural embeddings

$$\Phi(-\infty, c] \subset \Phi(-\infty, c'], \quad c < c',$$

the right-hand space inducing a topology in the left-hand space equivalent to the original topology. Consider the inductive limit

$$\Phi_\infty = \bigcup_{c=0}^{\infty} \ \Phi(-\infty, c],$$

Φ_∞ is a strict inductive limit and according to the result by Bourbaki is a regular inductive limit, and, consequently, the conjugate space $(\Phi_\infty)'$ can be identified with the projective limit

$$(\Phi_\infty)' = \bigcap_{c=0}^{\infty} \ (\Phi(-\infty, c])'.$$

However, we shall not use this description of $(\Phi_\infty)'$. Since $\mathcal{D}$ is dense in Φ_∞, the space $(\Phi_\infty)'$ will be interpreted as the set of those $f \in D'$ which are continued by continuity to any space $\Phi(-\infty, c]$, $c \geq 0$.

We note that if $\varphi \in \Phi_+$, then

$$\left(IT_{(t,y)}\varphi\right)(t', y') = \varphi(t - t', y - y') \in \Phi(-\infty, t],$$

and therefore, for $t > 0$ the expression

$$(f * \varphi)(t, y) = \left(f, IT_{(t,y)}\varphi\right) \tag{18}$$

is defined for $\varphi \in \Phi_+$ and $f \in (\Phi_\infty)'$.

Proposition 1. *Let $\Phi = \mathcal{S}, \mathcal{O}$. Then for each convolution operator $A : \Phi_+ \to \Phi_+$ there is a distribution $f \in ((\Phi_\infty)')_+$ such that the operator A is represented in the form (18).*

Proof. Since the topology of $\mathcal{S}_+$, $\mathcal{O}_+$ is stronger than the topology of pointwise convergence, the functionals a_x:

$$(a_x, \varphi) = A\varphi(x), \quad \forall \varphi \in \Phi_+$$

are continuous, i.e., $a_x \in (\Phi_+)'$ and $a_{(t,y)} = 0$ for $t \leq 0$. Put $f_x = IT_x a_x$. Then $f_{(t,y)} \in (\Phi(-\infty, t])'$ and

$$(f_x, IT_x\varphi) = (a_x, \varphi) = (A\varphi)(x).$$

The proposition will be proved if we show that the functionals f_x, $x = (t, y)$ are the restrictions to $(\Phi(-\infty, t])'$ of a universal functional $f \in (\Phi_\infty)'$ with $\operatorname{supp} f \in \overline{\mathbb{R}^n_+}$. The invariance of Φ_+ with respect to the translations $T_{(0,y)}$ and the commutability of A with these translations immediately imply that the functionals $f_{(t,y)}$ do not depend on y, i.e., $f_{(t,y)} = f_{(t,0)} \overset{\text{def}}{=} f_t$.

Let $\psi \in \Phi(-\infty, t]$, i.e., $\psi = IT_t\varphi$, $\varphi \in \Phi_+$. Then, as was already mentioned, $(f_t, \psi) = (A\varphi)(t, 0)$. If $t' > t$, then the functional $f_{t'}$ is defined on a broader space $\Phi(-\infty, t'] \supset \Phi(-\infty, t]$ and

$$\begin{aligned}(f_{t'}, \psi) &= (a_{t'}, T_{-t'}I\psi) = (a_{t'}, T_{-t'+t}\varphi) \\ &= (AT_{-t'+t}\varphi)(t', 0) = T_{-t'+t}(A\varphi)(t', 0) = (A\varphi)(t, 0) = (f_t, \psi).\end{aligned}$$

We have thus shown that $f_t = f$ and $(f, \varphi) = 0$ for $\varphi \in \Phi(-\infty, t]$, $t < 0$. The proposition is proved.

The proposition makes it possible to define spaces of convolutors on $\mathcal{S}_+$ and $\mathcal{O}_+$:

$$\mathfrak{C}(\Phi_+) = \left\{f \in ((\Phi_\infty)')_+ ,\ f * \varphi \in \Phi_+ \quad \forall \varphi \in \Phi_+\right\}, \quad \Phi = \mathcal{S}, \mathcal{O}.$$

Proposition 2. (i) *Each convolution operator* $A_\circ : \Phi_+ \to \Phi_+$, $\Phi = \mathcal{S}, \mathcal{O}$ *is continued by continuity to a convolution operator* $A : \Psi_+ \to \Psi_+$ *where* $\Psi = \mathcal{O}', \mathcal{S}'$.

(ii) *Let* $A : \Psi_+ \to \Psi_+$, $\Psi = \mathcal{O}', \mathcal{S}'$ *be a convolution operator. Then its restriction* $A_\circ$ *to the subspace* $\Phi_+ = \mathcal{S}_+, \mathcal{O}_+$ *is a convolution operator on* Φ_+.

The proof of this proposition (as proposition 2 in 1.5.1) is based on the commutability of A with differential operators, in particular, with the operators $(iD_t + 1)^{k_1} \left(1 + |D_y|^2\right)^{k_2}$, $k_1, k_2 \in \mathbb{Z}_+$. Therefore A commutes with the inverse operators $(iD_t + 1)^{-k_1} \left(1 + |D_y|^2\right)^{-k_2}$. It can be shown that we can graduate $\mathcal{S}_+$ and $\mathcal{O}_+$ with the aid of these operators. Repeating the proof of proposition 2 in 1.5.1 we prove the desired assertion.

According to proposition 2 we can pose

$$\mathfrak{C}\left((\mathcal{O}')_+\right) = \mathfrak{C}(\mathcal{S}_+), \quad \mathfrak{C}\left((\mathcal{S}')_+\right) = \mathfrak{C}(\mathcal{O}_+). \tag{19}$$

4.2.4. Proof of the theorem. The general plan of the proof is the same as in theorem 1.5.1. We extend convolution operators from $\mathcal{S}_+, \mathcal{O}_+$ to the spaces $(\mathcal{O}')_+, (\mathcal{S}')_+$ which contain $\delta(x)$. From this follows that

$$\mathfrak{C}\left(\mathcal{S}_+\right) \subset (\mathcal{O}')_+, \quad \mathfrak{C}(\mathcal{O}_+) \subset (\mathcal{S}')_+. \tag{20}$$

The inclusion opposite to the first inclusion is proved above (see (15)), so we have

$$\mathfrak{C}(C_+) = \mathfrak{C}\left((\mathcal{O}')_+\right) = (\mathcal{O}')_+. \tag{21}$$

The inclusion opposite to second inclusion in (20) is true only in the case $n = 1$ (see (16)), so

$$\mathfrak{C}\left(\mathcal{O}_+(\mathbb{R})\right) = \mathfrak{C}\left(\mathcal{S}'(\mathbb{R})'_+\right) = \left(\mathcal{S}'(\mathbb{R})\right)_+. \tag{22}$$

But technically, the proof of (20) differs from the analogous proof in Chapter 1. There the convolutor a priori belonged to $\mathcal{S}'$, and it was known in advance that the operation of convolution was associative. We now only know that convolutors are distributions belonging to the space $(\mathcal{S}_\infty)'$, and the associativity of the operation of convolution has to be proved directly. Here some additional difficulties appear in relation to the fact that $(\mathcal{S}_\infty)'$ may contain infinite-order elements. The desired assertion about associativity is contained in the following lemma.

Lemma. *Let $A : \mathcal{S}_+ \to \mathcal{S}_+$ be a convolution operator. Then*

$$A(g * \varphi) = (Ag) * \varphi \quad \forall g, \varphi \in \mathcal{S}_+. \tag{23}$$

First of all we shall accurately prove the first inclusion (20), the second is proved the same way. We select a sequence $\delta_j(x) \in \mathcal{S}_+$, $\delta_j(x) \to \delta(x)$ in the topology of $\mathcal{O}'$. By the lemma

$$A(\delta_j * \varphi) = (A\delta_j) * \varphi. \tag{24}$$

According to proposition 2(i) the sequence $\{A\delta_j\}$ converges to some distribution $A\delta \overset{\text{def}}{=} f \in (\mathcal{O}')_+$ in the topology of $\mathcal{O}'$. Since this topology is not weaker than the topology of $\mathcal{S}'$, for $j \to \infty$ the expression

$$(A\delta_j * \varphi)(t, y) = \left(A\delta_j, IT_{(t,y)}\varphi\right)$$

tends to $f * \varphi$ at each point (t, y) for a fixed function $\varphi \in \mathcal{S}_+$. On the other hand, the sequence $\delta_j * \varphi$ converges to $\delta * \varphi = \varphi$ in $\mathcal{S}$ and therefore $A(\delta_j * \varphi) \to A\varphi$ in $\mathcal{S}$, and, all the more, this convergence takes place at each point (t, y). Thus, the passage to limit in (24) results in (17).

Proof of the lemma. To shorten the notation we shall give the proof for $n = 1$. With regard to proposition 1, we must show that

$$(f * (g * \varphi)(t)) = ((f * g) * \varphi)(t), \quad \forall g, \varphi \in \mathcal{S}_+. \tag{25}$$

Take an arbitrary $T > 0$. We shall prove (25) for $t \le T$. In this case f can be replaced by a finite-order distribution $f_T \in (\mathcal{S}')_+$ such that

$$(f * \psi)(t) = (f_T * \psi), \quad \forall \psi \in \mathcal{S}_+, \ \forall t \le T. \tag{26}$$

To prove this we note that if $\psi \in \varphi_+$, then $IT_t\psi = \psi(t - t') = 0$ for $t' > t$, and therefore the left-hand side of (26) does not change if f is replaced by $\chi_T f$ where $\chi_T \in C^\infty$, $\chi_T = 1$ for $t \le T$ and $\chi_T = 0$ for $t \ge T + 1$. The functional $f \in \mathfrak{C}(\mathcal{S}_+) \subset (\Phi_\infty)'$ is defined for those functions belonging to $\mathcal{S}$ whose supports are bounded from the right. Therefore the functional $\chi_T f$ is defined and continuous throughout the space $\mathcal{S}$, i.e., $f_T = \chi_T f \in \mathcal{S}'$, which proves (26).

Replacing ψ in (26) by $g * f$ and, as in Chapter 1 using commutativity of convolution we find

$$(f*(g*\varphi))(t) = (f_T*(g*\varphi)) = ((f_T*g)*\varphi)(t) = \int_0^t (f_T*g)(t-t')\varphi(t')dt'$$

$$= \int_0^t (f*g)(t-t')\varphi(t')dt' = ((f*g)*\varphi)(t).$$

4.2.5. Now we can repeat the argument of theorem 1.5.2 and prove that convolution equation

$$A*u = f, \quad A \in (\mathcal{O}')_+$$

for any $f \in \mathcal{S}_+, (\mathcal{O}')_+$ have the unique solution $u \in \mathcal{S}_+, (\mathcal{O}')_+$ if and only if there exists the fundamental solution $\mathcal{G} \in (\mathcal{O}')_+$. The last condition means that

$$\hat{A}^{-1}(\tau,\eta) \in F(\mathcal{O}')_+ = \mathcal{M}^+,$$

or there are such constants $c > 0$, μ that

$$|\hat{A}(\tau,\eta)| > C(1+|\tau|+|\eta|)^{\mu}, \quad Im\tau \le 0. \tag{27}$$

In the case of differential operators $A = P(D)\delta$ the last condition according to Tarski-Zeidenberg theorem is equivalent to the following condition

$$P(\tau,\eta) \ne 0 \quad Im\tau \le 0, \quad (\mathrm{Re}\tau,\eta) \in \mathbb{R}^n. \tag{28}$$

So we proved :

Theorem. *Differential equation*

$$P(D)u = f$$

possesses a unique solution in $\mathcal{S}_+$ and $(\mathcal{O}')_+$ if and only if condition (28) takes place.

4.3. Kernel theorem, convolutors in $\mathcal{O}_+$ and $(\mathcal{S}')_+$, Cauchy problem in increasing functions

We can interpret the space $\mathcal{O}_+(\mathbb{R}^n_x)$ as a tensor product $\mathcal{O}_+(\mathbb{R}_t)\otimes\mathcal{O}(\mathbb{R}^{n-1}_y)$. Then according to the remark in §1 we have to interpret the space of convolutors on $\mathcal{O}_+$ as a tensor product of the form

$$\mathfrak{C}(\mathcal{O}_+) = \mathfrak{C}(\mathcal{O}_+(\mathbb{R}))\otimes\mathfrak{C}\left(\mathcal{O}\left(\mathbb{R}^{n-1}\right)\right) = (\mathcal{S}'(\mathbb{R}_t))_+\otimes\mathcal{O}'\left(\mathbb{R}^{n-1}_y\right). \tag{29}$$

In other words $\mathfrak{C}(\mathcal{O}_+)$ consists of distributions which are decreasing with respect to y and, in general, may increase as some power of t. So before studying convolutors in $\mathcal{O}_+$ we have to introduce new classes of functions and distributions with different behavior with respect to various variables.

4.3.1. Let the variables $x\in\mathbb{R}^n$ be split in two groups: $x=(x',x'')$, $x'\in\mathbb{R}^m$, $x''\in\mathbb{R}^{n-m}$.

The space $C^{(m_1,m_2)}_{(\ell_1,\ell_2)}$ consists of functions $\varphi(x',x'')$ continuous together with their derivatives $D^\alpha_{x'}$, $D^\beta_{x''}\varphi$, $|\alpha|\le m_1$, $|\beta|\le m_2$ and having a finite norm

$$|\varphi|^{(m_1,m_2)}_{(\ell_1,\ell_2)} = \sup_{\substack{x'\in\mathbb{R}^m,\ x''\in\mathbb{R}^{n-m}\\ |\alpha|\le m_1,\ |\beta|\le m_2}} (1+|x'|^2)^{\ell_1/2}(1+|x''|^2)^{\ell_2/2}\left|D^\alpha_{x'}\,D^\beta_{x''}\,\varphi(x)\right|.$$

These spaces form a 4-parameter scale whose projective limit coincides with $\mathcal{S}$. Similarly we can obtain $\mathcal{O}$ as an inductive limit of projective limits of these spaces.

But this new scale permits to define the new space

$$\mathcal{K} = \bigcup_{\ell_2} C^{(\infty,\,\infty)}_{(\infty,\,\ell_2)} = \bigcup_{\ell_2}\ \bigcap_{m_1,m_2,\ell_1} C^{(m_1,m_2)}_{(\ell_1,\,\ell_2)}. \tag{30}$$

It is obvious that if $\varphi(x',x'')\in\mathcal{K}$ then for fixed x'' this function belongs to $\mathcal{S}(\mathbb{R}^m)$ and for fixed x' it is an element of $\mathcal{O}(\mathbb{R}^{n-m})$. The following embedding with topology takes place

$$\mathcal{S}\subset\mathcal{K}\subset\mathcal{O}.$$

We shall need the space $\mathcal{K}'$ dual to $\mathcal{K}$. To introduce this we have to introduce the scale of Hilbert spaces $H^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}$, equivalent to the scale $C^{(m_1,\,m_2)}_{(\ell_1,\,\ell_2)}$.

We introduce the spaces

$$H_{(\ell_1,\,\ell_2)} = (1+|x'|^2)^{-\ell_1/2}(1+|x''|^2)^{-\ell_2/2}H(\mathbb{R}^n),$$
$$H^{(s_1,\,s_2)} = (1+|D'|^2)^{-s_1/2}(1+|D''|^2)^{-s_2/2}H(\mathbb{R}^n),$$

and equip them with their natural norms. Following the definition of $H^{(s)}_{(\ell)}$ spaces we can introduce

$$H^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)} = \left\{ f \in \varphi',\ (1+|x'|^2)^{-\ell_1/2}(1+|x''|^2)^{\ell_2/2} f \in H^{(s_1,s_2)} \right\}, \tag{31}$$

$$'H^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)} = \left\{ f \in \varphi',\ (1+|D'|^2)^{s_1/2}(1+|D''|^2)^{s_2/2} f \in H_{(\ell_1,\,\ell_2)} \right\} \tag{31'}$$

and endow them with natural norms. Since pseudodifferential operators in variables $x'(x'')$ commute with operators of multiplication by functions of $x''(x')$ the proof of proposition 2 in section 1.2.3 implies the equivalence of the norms and thus coincidence of the spaces (31) and (31′).

Embedding theorems of section 1.2.5 are extended to above spaces and

$$\mathcal{K} = \bigcup_{\ell_2} H^{(\infty,\,\infty)}_{(\infty,\,\ell_2)} = \bigcup_{\ell_2} \bigcap_{s_1,s_2,\ell_1} H^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}. \tag{30'}$$

For our spaces the duality relations hold

$$H^{(s_1,\,s_2)'}_{(\ell_1,\,\ell_2)} = H^{(-s_1,\,-s_2)}_{(-\ell_1,\,-\ell_2)}. \tag{32}$$

In view of this duality the space

$$\mathcal{K}' = \bigcap_{\ell_2} H^{(-\infty,\,-\infty)}_{(-\infty,\,\ell_2)} = \bigcap_{\ell_2} \bigcap_{s_1,\,s_2,\,\ell_1} H^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)} \tag{33}$$

regarded as a vector space coincides with the space of continuous linear functionals on $\mathcal{K}$. As to the topology, we supply $\mathcal{K}'$ with the topology of projective limit of the spaces $H^{(-\infty,\,-\infty)}_{(-\infty,\,\ell_2)}$ (which are regular inductive limits). The distributions belonging to $\mathcal{K}'$ have a finite order, increase in x' not stronger than some power of $|x'|$ and decrease in x'' stronger than any power of $|x''|$. The following embeddings (with the topology) take place

$$\mathcal{O}' \subset \mathcal{K}' \subset \mathcal{S}'.$$

$\mathcal{K}'$ can be thought of as a tensor product

$$\mathcal{K}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^m) \otimes \mathcal{O}'(\mathbb{R}^{n-m}).$$

4.3.2. With each distribution $a \in \mathcal{S}(\mathbb{R}^n)$ we can associate an operator

$$A : \mathcal{S}\left(\mathbb{R}^m_{x'}\right) \longrightarrow \mathcal{S}'\left(\mathbb{R}^m_{x'}\right) \tag{34}$$

determined by

$$(A\varphi,\ \psi) = (a,\ \varphi\psi), \quad \forall \varphi \in \mathcal{S}\left(\mathbb{R}^m_{x'}\right). \tag{35}$$

The left-hand side in (35) involves the value of the functional $A\varphi \in \mathcal{S}'\left(\mathbb{R}^{n-m}\right)$ on the element $\psi \in \mathcal{S}'\left(\mathbb{R}^{n-m}\right)$, and the right-hand side contains the value of the functional $a \in \mathcal{S}'(\mathbb{R}^n)$ on the element

$$\varphi\psi \in \mathcal{S}(\mathbb{R}^m) \times \mathcal{S}(\mathbb{R}^{n-m}) \subset \mathcal{S}(\mathbb{R}^n).$$

Similarly with each $a \in \mathcal{K}'$ we can associate an operator

$$A : \mathcal{S}\left(\mathbb{R}^n_{x'}\right) \longrightarrow \mathcal{O}'\left(\mathbb{R}^{n-m}_{x''}\right), \tag{36}$$

since in this case we can take the functions $\psi \in \mathcal{O}\left(\mathbb{R}^{n-m}_{x''}\right)$ in (35).

Theorem. (i) *If $a \in \mathcal{S}'(\mathbb{R}^n)$ then the operator (34) determined by means of (36) is a regular operator from a projective to an inductive limit, i.e., there are s_1, s_2, ℓ_1, and ℓ_2 such that*

$$||A\varphi||^{(s_2)}_{(\ell_2)} \leq C||\varphi||^{(s_1)}_{(s_1)}. \tag{37}$$

In the case $a \in \mathcal{K}'(\mathbb{R}^n)$ $\forall \ell_2$ $\exists s_1$, s_2, ℓ_2 such that (37) holds.

(ii) *For each continuous operator (34) there is a distribution $a \in \mathcal{S}'(\mathbb{R}^n)$ such that the operator is represented in the form (35).*

If the image of the operator (34) belongs to $\mathcal{O}'\left(\mathbb{R}^{n-m}\right)$, then $a \in \mathcal{K}'(\mathbb{R}^n)$.

Proof. (i) If $a \in \mathcal{S}'$ then $\exists s_1$, s_2, ℓ_1, ℓ_2 such that $a \in H^{(-s_1,\ s_2)}_{(-\ell_1,\ -\ell_2)}$. According to Schwartz inequality

$$\begin{aligned} |(A\varphi, \psi)| &\leq ||a||^{(-s_1,\ s_2)}_{(-\ell_1,\ \ell_2)}\ ||\varphi\psi||^{(s_1,\ -s_2)}_{(\ell_1,\ -\ell_2)} \\ &= ||a||^{(-s_1, s_2)}_{(-\ell_1,\ \ell_2)}\ ||\varphi||^{(s_1)}_{(\ell_1)}\ ||\psi||^{(-s_2)}_{(-\ell_2)}. \end{aligned}$$

According to inverse Schwartz inequality

$$(37') \qquad \|A\varphi\|_{(\ell_2)}^{(s_2)} \leq \|a\|_{(-\ell_1,\,\ell_2)}^{(-s_1,\,s_2)} \|\varphi\|_{(s_2)}^{(s_1)} = \text{const}\,\|\varphi\|_{(s_2)}^{(s_1)}.$$

In case $a \in \mathcal{K}'$, given an arbitrary ℓ_2 there are $s_1,\ s_2$, and ℓ_1 such that (37′) holds.

(ii) According the general theorem 3 in subsection I.3.6 continuity of the operator (34) implies its regularity, i.e., existence $s_1,\ s_2,\ \ell_1$ and ℓ_2 such that (37) is fulfilled. Put

$$A_2\varphi = \left(1 + |D_{x'}|^2\right)^{-\sigma_2/2} A\varphi.$$

If σ_2 is sufficiently large, then A_2 is continuous operator from $H_{(\ell_1)}^{(s_1)}\,(\mathbb{R}^m)$ into $C_{(\ell_2)}\,(\mathbb{R}^{n-m})$, i.e.,

$$(38) \qquad |A_2\varphi|_{(\ell_2)} \leq C_1\|\varphi\|_{(\ell_1)}^{(s_1)}.$$

It follows that $(A_2\varphi)(x'')$ is continuous linear functional on $\mathcal{S}(\mathbb{R}^m)$ for each x'', and, consequently, there is a distribution $a_2(\cdot, x'') \in \mathcal{S}'(\mathbb{R}^m)$ depending on x'' as a parameter such that

$$(A_2\varphi)(x'') = (a_2(\cdot, x''), \varphi)\,.$$

Put

$$a_1(x',\ x'') = \left(1 + |D_{x'}|^2\right)^{-\sigma_1/2} a_2(\cdot,\ x'').$$

Then we have that

$$(39) \qquad a = \left(1 + |D_{x''}|^2\right)^{\sigma_1/2} \left(1 + |D_{x''}|^2\right)^{\sigma_2/2},\ a_1 \in H_{(-\ell_1,\ \ell_2-\kappa)}^{(-\sigma_1,\ -\sigma_2)}.$$

If the image of the operator (34) belongs to $\mathcal{O}'(\mathbb{R}^{n-m})$ then $\forall \ell_2$ it belongs to $H_{(\ell_2)}^{(-\infty)}\,(\mathbb{R}^{n-m})$, and the closedness of graph of A in $\mathcal{S} \times \mathcal{S}'$ implies its closedness in $\mathcal{S} \times H_{(\ell_2)}^{(-\infty)}$. Thus, for any ℓ_2 the operator

$$(40) \qquad A : \mathcal{S}(\mathbb{R}^m) \longrightarrow H_{(\ell_2)}^{(-\infty)}\,(\mathbb{R}^{n-m})$$

is closed. Since $\mathcal{S}$ is a Fréchet space and $H_{(\ell_2)}^{(-\infty)}$ is an inductive limit of Fréchet spaces, by the closed graph theorem, the closedness of the operator (40) implies its continuity.

The continuity of the operator implies its regularity and existence of s_1, s_2 and ℓ_1 such that (37) holds. Therefore, the above argument implies that the operator (36) is represented in the form (35) where for any ℓ_2, a has the form (39) with a fixed $\kappa > (n-m)/2$. Hence, according our definitions, $a \in \mathcal{K}'$.

4.3.3. To study convolutors in $\mathcal{O}_+$ we shall need another version of kernel theorem.

With each distribution $a \in \mathcal{K}'(\mathbb{R}^n)$ we can associate, along with the operator (34′) from $\mathcal{S}\left(\mathbb{R}^m_{x'}\right)$ into $\mathcal{O}'\left(\mathbb{R}^{n-m}_{x''}\right)$, another operator

$$(41) \qquad A : \mathcal{O}\left(\mathbb{R}^{n-m}_{x'}\right) \longrightarrow \mathcal{S}'\left(\mathbb{R}^m_{x'}\right)$$

determined by (cf. (35))

$$(42) \qquad (A\varphi,\ \psi) = (a, \varphi\psi), \quad \varphi \in \mathcal{O}\left(\mathbb{R}^{n-m}_{x''}\right), \quad \psi \in \mathcal{S}\left(\mathbb{R}^m_{x'}\right).$$

The kernel theorem in application to the operator (41) reads:

Theorem. (i) *If $a \in \mathcal{K}'(\mathbb{R}^n)$, then the operator (41) determined with the aid of (42) is regular, i.e., $\forall \ell_2\ \exists\, s_1,\ s_2,\ \ell_1$ such that*

$$||A\varphi||^{(s_1)}_{(\ell_1)} \leq C\ ||\varphi||^{(s_2)}_{(\ell_1)}, \quad \forall\varphi \in H^{(\infty)}_{(\ell_2)}\ \left(\mathbb{R}^{n-m}_{x''}\right).$$

(ii) *For each continuous operator (41) there is a distribution $a \in \mathcal{K}'(\mathbb{R}^n)$ such that the operator is represented in the form (42).*

Proof. (i) If $a \in \mathcal{K}'$ then $\forall \ell_2\ \exists s_1,\ s_2,\ \ell_1$ such that $a \in H^{(s_1,\ -s_2)}_{(\ell_1,\ -\ell_2)}$, and then we can repeat the argument of the theorem above (see item 4.3.2).

(ii) Since $\mathcal{O}$ is an inductive limit of countably normed spaces for which the first axiom of countability holds and $\mathcal{S}'$ is a regular inductive limit according to theorem 3 in subsection 1.3.6 the continuity of the operator (41) implies its regularity, i.e., $\forall \ell_2\ \exists s_1,\ \ell_1$ such that the operator

$$A : H^{(\infty)}_{(\ell_2)}\ \left(\mathbb{R}^{n-m}_{x''}\right) \longrightarrow H^{(s_1)}_{(\ell_1)}\ \left(\mathbb{R}^m_{x'}\right)$$

is continuous. Therefore, there is s_2 such that this operator is continued by continuity to a continuous operator

$$A : H^{(s_2)}_{(\ell_2)}\ \left(\mathbb{R}^{n-m}_{x''}\right) \longrightarrow H^{(s_1)}_{(\ell_1)}\ \left(\mathbb{R}^m_{x'}\right),$$

i.e., (37) is satisfied. It now remains to repeat almost literally the proof of theorem above.

4.3.4. Now we confine ourselves to the case $m = 1$, so $x' = t \in \mathbb{R}$, $x'' = y \in \mathbb{R}^{n-1}$. We can imagine the space $\mathcal{K}'(\mathbb{R}^n)$ as the tensor product $\mathcal{S}'(\mathbb{R}_t) \otimes \mathcal{O}'\left(\mathbb{R}^{n-1}_y\right)$ and $\left(\mathcal{K}'(\mathbb{R}^n)\right)_+$ as $\left(\mathcal{S}'(\mathbb{R}_+)\right)_+ \otimes \mathcal{O}'\left(\mathbb{R}^{n-1}_y\right)$. As the Fourier Laplace transform is defined in $\left(\mathcal{S}'(\mathbb{R})\right)_+$ and $\mathcal{O}'\left(\mathbb{R}^{n-1}\right)$ we can expect that the same is true for the space $(\mathcal{K}')_+$, and on intuitive level we can represent this space as $\mathcal{L}^+(\mathbb{R}) \otimes \mathcal{M}\left(\mathbb{R}^{-1}\right)$.

Denote by $\mathcal{L}^+(\mathbb{R}^n) = \mathcal{L}^+$ the space of functions $\psi(\tau, \eta)$ defined for $Im\tau < 0$, $\eta \in \mathbb{R}^{n-1}$, holomorphic with respect to τ, infinitely differentiable with respect to η and such that $\exists \ell > 0$ such that $\forall \beta\ \exists K_\beta,\ s_\beta$ such that

$$\left|\partial^\beta_\eta \psi(\tau, \eta)\right| < K_\beta |\mathrm{Im}\tau|^{-\ell} \left(1 + |\tau| + |\eta|\right)^{s_\beta}. \tag{43}$$

Theorem. *A function $\psi(\tau, \eta)$ is the Fourier-Laplace transform of a distribution $\varphi \in (\mathcal{K}')_+$ if and only if $\psi \in \mathcal{L}^+$, i.e., the isomorphism*

$$F(\mathcal{K}')_+ = \mathcal{L}^+ \tag{44}$$

takes place.

We shall not give the proof, it uses the ideas of the proofs of analogous results for $\left(\mathcal{S}'(\mathbb{R})\right)_+$ and $\mathcal{O}'\left(\mathbb{R}^{n-1}\right)$. Pseudodifferential operators permit to reduce the theorem to the case of spaces $C^{(0,\, m)}_{(\lambda_1,\, \lambda_2)+}$ of continuous functions. Then we can separately study the behavior of the Fourier transform of the function with respect to t and to y.

4.3.5. Now we shall discuss the convolution with distributions from $(\mathcal{K}')_+$. As we mentioned above, if functions $f(x)$ and $g(x)$ are equal zero for $t < 0$, then the growth of functions with respect to t does not affect the convergence of the integral which defines the convolution. We have the following :

Lemma. Let $f \in C_{(\ell_1,\, \ell_2)}$, $g \in C_{(\lambda_1,\, \lambda_2)}$ where $\ell_2 > |\lambda_2| + n - 1$. Then $f * g \in C_{(\mu,\, \lambda_2)}$ where $\mu \leq -|\ell_1| - |\lambda_1| - 1$.

Representing elements of $(\mathcal{K}')_+$ as differential operators applied to functions from $C_{(\ell_1,\, \ell_2)}$ (ℓ_2 can be chosen arbitrary) we obtain following inclusions:

$$(\mathcal{K}')_+ * \Phi_+ \subset \Phi_+, \quad \Phi = \mathcal{O}, \mathcal{S}', \mathcal{K}'. \tag{45}$$

From (45) for $\Phi = \mathcal{K}'$ it follows that $\mathcal{K}'$ is an algebra with respect to convolution and the isomorphism (44) is an isomorphism of algebras.

4.3.6. Now we can formulate the main result of this section.

Theorem. *Let $\Phi = \mathcal{O}, \mathcal{S}'$. Then for each convolution operator $A : \Phi \to \Phi$ there is a distribution $f \in (\mathcal{K}')_+$ such that*

$$A\varphi = \mathrm{con}_f \varphi = f * \varphi, \quad \forall \varphi \in \Phi_+. \tag{46}$$

Proof. If follows from (45) that

$$(\mathcal{K}')_+ \subset \mathfrak{C}(\Phi_+).$$

So to prove the theorem we must prove the inclusion :

$$\mathfrak{C}(\mathcal{O}_+) \subset (\mathcal{K}')_+. \tag{47}$$

The case $\Phi = \mathcal{S}'$ follows from the relation (.,.).

Let $A : \mathcal{O}(\mathbb{R})_+ \to \mathcal{O}(\mathbb{R}^n)_+$ be a convolution operator. Then for a fixed function $\psi \in \mathcal{O}\left(\mathbb{R}_y^{n-1}\right)$,

$$\mathcal{O}(\mathbb{R}_t) \longrightarrow \mathcal{O}(\mathbb{R}_t)_+ \quad (\varphi(t) \mapsto (A(\varphi\psi))(t,0)).$$

is a convolution operator. As we already proved for convolution operators on $\mathcal{O}(\mathbb{R}_+)_+$ there is a distribution $F_\psi \in (\mathcal{S}'(\mathbb{R}_+))_+$ such that

$$(A(\varphi\psi))(t,0) = (F_\psi * \varphi)(t). \tag{48}$$

Now we consider the linear operator

$$\mathcal{O}\left(\mathbb{R}_y^{n-1}\right) \longrightarrow (\mathcal{S}'(\mathbb{R}_+))_+ \quad (\psi \mapsto F_\psi). \tag{49}$$

Lemma. *Operator (49) is a continuous operator.*

We now apply kernel theorem (see item 4.3.4) to the continuous operator (49). According to the theorem, there is a distribution $a \in \mathcal{K}'$ such that

$$(a, \chi\psi) = (F_\psi, \ \chi), \quad \forall \chi \in \varphi(\mathbb{R}_t).$$

Substitute the function $\chi(s) = IT_t\varphi(s)$, where $\varphi \in \mathcal{S}(\mathbb{R})_+$, into this relation. With regard to (48), this results in

$$(50) \qquad (a, (IT_t\varphi)\,\psi) = (F_\psi,\ IT_t\varphi) = (F_\psi * \varphi)\,(t) = A(\varphi\psi)(t,0).$$

On the other hand, we have also proved that there exists a distribution $f \in ((\mathcal{O}_\infty)')_+$ such, that A is the operator of convolution with this distribution. Therefore

$$(A(\varphi\psi))(t,0) = (f, (IT_t\varphi)\,\psi)$$

comparing this relation with (50), we see that

$$(a, (IT_t\varphi)\,\psi) = (f, (IT_t\varphi)\,\psi)\,, \ \ \forall\psi \in \mathcal{O}\left(\mathbb{R}^{n-1}\right), \quad \forall\varphi \in \mathcal{S}(\mathbb{R})_+.$$

The set of the functions

$$(IT_t\varphi)\,\psi, \quad \varphi \in \mathcal{S}(\mathbb{R})_+, \quad \psi \in \mathcal{O}\left(\mathbb{R}^{n-1}\right), \quad t \geq 0$$

and their linear combinations is dense in the space $\mathcal{K}(\mathbb{R}^n)$, and therefore the functionals $a \in \mathcal{K}'$ are uniquely determined by their values on these functions. We have thus proved that

$$f = a \in \mathcal{K}' \cap ((\mathcal{O}_\infty)')_+ = (\mathcal{K}')_+.$$

Proof of the lemma. By virtue of the closed graph theorem (holding for inductive limits of Fréchet spaces, in particular, for $\mathcal{O}$), it suffices to show that this operator is closed, i.e.,

$$\{\psi_j \to 0 \text{ in } \mathcal{O}\left(\mathbb{R}^{n-1}\right), \quad F_{\psi_j} \to F \text{ in } (\mathcal{S}'(\mathbb{R}))_+\} \Longrightarrow \{F = 0\}.$$

Indeed, if $\varphi(t) \in \mathcal{O}(\mathbb{R}_+)$ and $\psi_j(y) \to 0$ in $\mathcal{O}\left(\mathbb{R}^{n-1}\right)$, $j \to \infty$, then $\varphi(t)\psi_j(y) \to 0)$ in $\mathcal{O}(\mathbb{R}^n)$, and, by the continuity of convolution operator A, we have $(A(\varphi\psi_j))(t,y) \to 0$ in $\mathcal{O}(\mathbb{R}^n)$. According to definition of F_ψ (see (48))

$$(A\,(\varphi\psi_j))(t,0) = \left(F_{\psi_j} * \varphi\right)(t) \longrightarrow 0, \quad j \longrightarrow \infty.$$

Thus, if F is the limit of the sequence F_{ψ_j}, then

$$F * \varphi \equiv 0 \quad \forall\varphi \in \mathcal{O}(\mathbb{R})_+.$$

As in the spaces $\mathcal{S}'(\mathbb{R})_+$, $\mathcal{O}(\mathbb{R})_+$ the Fourier-Laplace transform is defined, passing to Fourier-Laplace transforms we see that $\hat{F}(\tau)\hat{\varphi}(\tau) \equiv 0$ for $\mathrm{Im}\tau < 0$, whence $\hat{F}(\tau) \equiv 0$.

4.4.5. Theorem. *Let $A \in (\mathcal{K}')_+$. Then following conditions are equivalent*

(I) *For any $f \in (\mathcal{S}')_+$ the convolution equation*

$$A * u = f \tag{51}$$

possesses a unique solution $u \in (\mathcal{S}')_+$.

(II) *Equation (51) possesses a fundamental solution $\mathcal{G} \in (\mathcal{K}')_+$, i.e. A is an invertible element of the algebra $(\mathcal{K}')_+$.*

Remark. It should be noted that the theorem above (in contrast to theorem 1.5.2) does not involve the space $\mathcal{O}_+$. This is due to the fact that unique solvability of (51) in the space $\mathcal{O}_+$ implies (by the closed graph theorem) that the operator $\mathrm{con}_A : \mathcal{O}_+ \to \mathcal{O}_+$ possesses a continuous inverse operator commutable with the translations T_h, $h \in \overline{\mathbb{R}^n_-}$. However it is not known whether the operator is regular and, hence, whether it is a convolution operator. Therefore we cannot assert (as in the case of $\mathcal{S}_+$ and $(\mathcal{S}')_+$) that $(\mathrm{con}_A)^{-1} = \mathrm{con}_G$. We note that the condition (II) is sure to be sufficient for the solvability of (51) in $\mathcal{O}_+$.

Since the Fourier operator is defined in $(\mathcal{K}')_+$, the condition (II) is equivalent to the invertibility of the symbol $\hat{A}(\tau, \eta)$ in $(\mathcal{K}')_+$, i.e.,

$$A^{-1}(\tau, \eta) \in \mathcal{L}^+.$$

The last condition is equivalent to the following condition.

(II′) There exist constants $c > 0$, μ and ν such that

$$\left|\hat{A}(\tau, \eta)\right| > C(1 + |\tau| + |\eta|)^{\mu}|\mathrm{Im}\tau|^{\nu}, \quad \mathrm{Im}\tau < 0, \quad \eta \in \mathbb{R}^{n-1}.$$

The condition (II′) implies necessary condition for solvability in $(\mathcal{S}')_+$:

$$\hat{A}(\tau, \eta) \neq 0, \quad \mathrm{Im}\tau < 0, \quad \eta \in \mathbb{R}^{n-1}.$$

In the case of differential operators, i.e., $A = P(D_y, D_+)\delta$ the corresponding condition

$$P(\eta, \tau) \neq 0, \quad \mathrm{Im}\tau < 0, \quad \eta \in \mathbb{R}^{n-1}. \tag{52}$$

is necessary and sufficient for unique solvability in $(\mathcal{S}')_+$ and sufficient condition of solvability in $\mathcal{O}_+$.

§5. Convolution equations in a finite strip

5.1. Scales of spaces in a finite strip

According to the notation in the Introduction $\Phi[a,\infty)$ and $\Phi(-\infty,b]$ will denote the subspaces of Φ consisting of those elements of Φ whose supports belong to the half-spaces $t \geq a$ and $t \leq b$, respectively. As was already noted $\Phi[a,\infty) = T_{-a}\Phi_+$ and $\Phi(-\infty,b] = T_{-b}\Phi_-$ where $T_a = T_{(0,a)}$.

For $a < b$ we put

$$C_{(\ell)}^{(m)}[a,b) = C_{(\ell)}^{(m)}[a,\infty)/C_{(\ell)}^{(m)}[b,\infty) \tag{1}$$

and introduce in these spaces the natural quotient norm

$$|\varphi|_{(\ell)[a,b)}^{(m)} = \inf_{\varphi_-} |\varphi_\circ + \varphi_-|_{(\ell)}^{(m)}$$

where $\varphi_\circ$ is an arbitrary representative of the residue class φ and $\varphi_- \in C_{(\ell)}^{(m)}[b,\infty)$.

The embeddings of subspaces

$$C_{(\ell)}^{(m)}[a,\infty) \longrightarrow C_{(\ell')}^{(m')}[a,\infty), \quad C_{(\ell)}^{(m)}[b,\infty) \longrightarrow C_{(\ell')}^{(m')}[b,\infty)$$

where $\ell \leq \ell'$, $m \leq m'$ induce the embeddings of factor-spaces (1), i.e., the later form a projective scale, and we can consider projective limit $C_{(\infty)}^{(\infty)}[a,b)$. We have

$$\mathcal{S}[a,b) = C_{(\infty)}^{(\infty)}[a,b) \stackrel{\text{def}}{=} \cap\, C_{(\ell)}^{(m)}[a,b), \tag{2}$$

where the left-hand side should be understood as the factor space of $\mathcal{S}[a,\infty)$ relative to the subspace $\mathcal{S}[b,\infty)$. These relations can be derived from the general results on projective limits of Fréchet spaces. In what follows we shall not use this relation and the right-hand side of (2) can be regarded as definition of $\mathcal{S}[a,b)$. A similar situation takes place in the case of the space $\mathcal{O}$

$$\mathcal{O}[a,\ell) = \bigcup_{\ell} \bigcap_{m} C_{(\ell)}^{(m)}[a,b).$$

Proceeding from the Hilbert spaces $H_{(\ell)}^{(s)}$, we put

$$H_{(\ell)}^{(s)}[a,\ell) = H_{(\ell)}^{(s)}[a,\infty)/H_{(\ell)}^{(s)}[b,\infty)$$

and introduce in these spaces the natural quotient norms.

The operator $\delta_\tau^+(D)$ commutes with translation operators T_{-a}, T_{-b} and in a natural way induce the isomorphisms:

$$\delta_\tau^+(D)H_{(\ell)}^{(s)}[c,\infty) = H_{(\ell)}^{(s-\tau)}[c,\infty).$$

These isomorphisms induce the isomorphism of the factor spaces:

$$\delta_\tau^+(D)H_{(\ell)}^{(s)}[a,b) = H_{(\ell)}^{(s-\tau)}[a,b).$$

The embedding of subspaces

$$C_{(\ell+\kappa)}^{(m)}[c,\infty) \subset H_{(\ell)}^{(m)}[c,\infty) \subset C_{(\ell)}^{(m-\kappa)}[c,\infty), \quad c = a,b, \quad \kappa > n/2$$

imply the embeddings of factor spaces

$$C_{(\ell+\kappa)}^{(m)}[a,b) \subset H_{(\ell)}^{(m)}[a,b) \subset C_{(\ell)}^{(m-\kappa)}[a,b),$$

whence if follows that

$$\mathcal{S}[a,b) = \bigcap_{s,\ell} H_{(\ell)}^{(s)}[a,b), \quad \mathcal{O}[a,b) = \bigcup_s \bigcap_s H_{(\ell)}^{(s)}[a,b).$$

We put

$$(3) \qquad (\mathcal{S}')[a,b) = \bigcup H_{(\ell)}^{(s)}[a,b), \quad (\mathcal{O}')[a,b) = \bigcap_\ell \bigcup_s H_{(\ell)}^{(s)}[a,b).$$

We do not verify the correctness of these definitions, i.e., the relations

$$(\Phi')[a,b) = (\Phi')[a,\infty)/(\Phi')[b,\infty), \quad \Phi = \mathcal{S}, \mathcal{O}$$

are not proved, and (3) can be understood as definitions of left hand spaces.

Remark. If we define for $a < b$

$$\Phi(a,b] = \Phi(-\infty,b]/\Phi(-\infty,a].$$

Then in the case $\Phi = C_{(\ell)}^{(m)}, \ H_{(\ell)}^{(s)}$,

$$\Phi(a,b] = I\Phi[-b,-a)$$

and therefore any assertion proved for the spaces $\Phi[a,b)$ is extended in trivial manner to the spaces $\Phi(a,b]$ and vice versa.

The form $\int f(x)\varphi(x)dx$, $\{f,\varphi\}\in\mathcal{S}\times\mathcal{S}$ can be defined correctly on $\mathcal{S}[a,b)\times\mathcal{S}(a,b]$ or $\mathcal{S}(a,b]\times\mathcal{S}[a,b)$. It is continued by continuity to

$$H^{(-s)}_{(-\ell)}[a,b)\times H^{(s)}_{(\ell)}(a,b] \text{ or } H^{(-s)}_{(-\ell)}(a,b]\times H^{(s)}_{(\ell)}[a,b)$$

and induces the duality relations

$$\left(H^{(s)}_{(\ell)}[a,b)\right)' = H^{(-s)}_{(-\ell)}(a,b],\quad \left(H^{(s)}_{(\ell)}(a,b]\right)' = H^{(-s)}_{(-\ell)}[a,b), \tag{4}$$

so the spaces $H^{(s)}_{(\ell)}[a,b)$ and $H^{(s)}_{(\ell)}(a,b]$ are reflexive. Then $\mathcal{S}[a,b)$, $\mathcal{S}(a,b]$ are projective limits of reflexive spaces, $(\mathcal{S}')[a,b)$, $(\mathcal{S}')(a,b]$ are regular inductive limits and the dualities

$$(\mathcal{S}[a,b))' = (\mathcal{S}')(a,b],\quad (\mathcal{S}(a,b])' = (\mathcal{S}')[a,b) \tag{5}$$

are topological isomorphisms, and

$$(\mathcal{O}[a,b))' = (\mathcal{O}')(a,b],\quad (\mathcal{O}(a,b])' = (\mathcal{O}')[a,b) \tag{6}$$

are isomorphisms of vector spaces.

Starting from the spaces $\Phi^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}$, $\Phi=C,H$, corresponding to the partition $\mathbb{R}^n_x=\mathbb{R}_t+\mathbb{R}^{n-1}_y$, we can define in a natural way the spaces $\Phi^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}[a,b)$. We note, that the elements of the spaces $\Phi^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}$ and $\Phi^{(s_1,\,s_2)}_{(\lambda_1,\,\ell_2)}$, $\Phi=C,H$ are of the same degree of smoothness and differ only in the character of growth (decrease) relative to t for $t\to\pm\infty$. It can naturally be expected that this distinction disappears when one passes to a finite strip. Following isomorphism can be easily proved

$$\Phi^{(s_1,\,s_2)}_{(\ell_1,\,\ell_2)}[a,b) = \Phi^{(s_1,\,s_2)}_{(\lambda_1,\,\ell_2)}\quad \forall\ell_1,\ \lambda_1,\quad \Phi=C,H. \tag{7}$$

With regard to the description of spaces $\mathcal{K}'$ and $\mathcal{O}'$ we obtain the following important isomorphism

$$(\mathcal{K}')[a,b) = (\mathcal{O}')[a,b). \tag{8}$$

By analogy with isomorphisms (7) we can obtain the isomorphisms:

$$\Phi[a,b) = \Phi_{[\rho]}[a,\infty)/\Phi_{[\rho]}[b,\infty) \tag{9}$$

which holds for any $\rho \in \mathbb{R}$ and $-\infty < a < b < +\infty$.

5.2. Convolution operators in a finite strip.

From the definition of convolution of regular functions it trivially follows that if for the spaces Φ_i, $i = 1, 2, 3$ the inclusion

$$\Phi_1 * \Phi_2 \subset \Phi_3 \tag{10}$$

takes place, then the inclusion relations

$$\Phi_1[0, \infty) * \Phi_2[c, \infty) \subset \Phi_3[c, \infty)$$

take place. Whence it follows that the operator con_f, $f \in \Phi_1[0, \infty)$ is extended in a natural manner to the factor space $\Phi_2[a, b)$ and we have

$$\Phi_1[0, \infty) * \Phi_2[a, b) \subset \Phi_3[a, b).$$

Further, since according to the definition of convolution

$$\Phi_1[b - a, \infty) * \Phi_2[a, \infty) \subset \Phi_3[b, \infty)$$

the operator con_g, $g \in \Phi_1[a, \infty)$, transforms $\Phi_2[a, b)$ into zero, we come to the relation

$$\Phi_1[0, b - a) * \Phi_2[a, b) \subset \Phi_3[a, b). \tag{11}$$

Using definitions of the spaces of distributions we can easily obtain from (11) the following inclusions

$$\mathcal{O}'[0, b - a) * \Phi[a, b) \subset \Phi[a, b), \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'. \tag{12}$$

If $h \in \mathbb{R}^n$, then the translation operator $T_h : \Phi \to \Phi$, where Φ is one of the function spaces under consideration, induces the isomorphism

$$T_h : \Phi[a, b) \longrightarrow \Phi[a - h_1, b - h_1). \tag{13}$$

If $h_1 \leq 0$, then there exists a natural mapping of the right-hand space (13) into the left-hand space. For the composition of T_h and this mapping we retain the notation T_h, i.e.,

$$T_h \Phi[a, b) \subset \Phi[a, b), \quad h \in \overline{\mathbb{R}^n_-}. \tag{14}$$

By a convolution operator on $\Phi[a, b)$, $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ is meant a regular operator commutable with translations (14). As in section 1.5 we can prove

Proposition 1. *For each convolution operator* $A : \Phi[a,b) \to \Phi[a,b)$, $\Phi = \mathcal{S}, \mathcal{O}$ *there exists a distribution* $f \in (\Phi')(0, b-a]$ *such that*

$$(A\varphi)(t,y) = \big(f, IT_{(t,y)}\varphi\big) = (f * \varphi)(t,y), \quad a \le t \le b, \quad \varphi \in \Phi[a,b).$$

Remark. The space $(\Phi')[0, b-a)$ contains no infinite-order elements (in contrast to $\big((\Phi_\infty)'\big)_+$), which simplifies the description of convolution operators on $\Phi[a,b)$ for $b < +\infty$ in comparison with the analogous problem to $b = +\infty$.

The commutability of the convolution operator A with translations (14) implies its commutability with the pseudodifferential operators $(iD_t + 1)^{k_1}$ $(1 + |D_y|^2)^{k_2}$, k_1, $k_2 \in \mathbb{Z}$, i.e., the graduating operators in the scale of spaces $H^{(s_1,\, s_2)}_{(\ell_1,\, \ell_2)}[a,b)$. Since $\Phi[a,b)$, $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ are the "limiting" spaces of the scale, the proposition below holds.

Proposition 2. *Let* $\Phi = \mathcal{S}, \mathcal{O}$ *and* $\Psi = \mathcal{O}', \mathcal{S}'$. *Then for any* $a < b$ *each convolution operator* $A_\circ : \Phi[a,b) \to \Phi[a,b)$ *is continued by continuity to a convolution operator* $A : \Psi[a,b) \to \Psi[a,b)$. *Conversely, the restriction of a convolution operator* $A : \Psi[a,b) \to \psi[a,b)$ *to the space* $\Phi[a,b)$ *is a convolution operator on this space.*

Proceeding from proposition 1 we can define convolutors in a strip

(15)
$$\mathfrak{C}(\Phi[a,b)) = \{f \in \Phi'[o, b-a),\ f * \varphi \in \Phi[a,b),\ d\forall\varphi \in \Phi[a,b)\},$$
$$\Phi = \mathcal{S}, \mathcal{O}.$$

By virtue of proposition 2,

$$\mathfrak{C}(\mathcal{O}'[a,b)) = \mathfrak{C}(\mathcal{S}[a,b)), \quad \mathfrak{C}(\mathcal{S}'[a,b)) = \mathfrak{C}(\mathcal{O}[a,b)). \tag{16}$$

Theorem. $\forall a, b \in \mathbb{R}$, $a < b$, *we have*

$$\mathfrak{C}(\Phi[a,b)) = \mathcal{O}[o, b-a) \quad \Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'. \tag{17}$$

Proof. According to (12) the right-hand side space (17) is contained in the left-hand side. According to the definition of convolutors (15) for $\Phi = \mathcal{O}$ the opposite inclusion takes place. With regard to second relation (16) it proves (17) for $\Phi = \mathcal{O}, \mathcal{S}'$. With regard to first relation (16) it remains to show that

$$\mathfrak{C}(\mathcal{O}'[a,b)) \subset \mathcal{O}'[o, b-a). \tag{18}$$

From the isomorphisms (13) it follows that

$$\mathfrak{C}(\Phi[a,b)) = \mathfrak{C}(\Phi[a+c, b+c))$$

and therefore it suffices to verify (18) for the case a=0:

$$\mathfrak{C}(\mathcal{O}'[o,b)) \subset \mathcal{O}'[0,b).$$

Now it remains to note that $\delta(x) \in \mathcal{O}'[0,b)$.

Remark. It should be noted that in the case of **a finite strip** all the four spaces of convolutors $\mathfrak{C}(\Phi[a,b))$, $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ are the same and coincide with $\mathcal{O}[o, b-a)$. For $b = +\infty$ these spaces coincide pairwise and the strict inclusion relation takes place

$$(\mathcal{O}')_+ = \mathfrak{C}(\mathcal{S}_+) = \mathfrak{C}((\mathcal{O}')_+) \subset \mathfrak{C}(\mathcal{O}_+) = \mathfrak{C}((\mathcal{S}')_+) = (\mathcal{K}')_+.$$

The distinction between the cases of a finite strip and a half-space is due to the fact that the isomorphism (8) takes place.

5.3. Convolution and differential equations in a finite strip, The Petrovskii condition

Following the argument of theorem 1.5.2 we prove

Theorem 1. *Let $A \in \mathcal{O}'[0,b)$ and let $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$. Then the following conditions are equivalent.*

(I) *The convolution equation*

$$A * u = f \tag{19}$$

is uniquely solvable in $\Phi[a, a+b)$ for any $a \in \mathbb{R}$ (for some $a \in \mathbb{R}$), $b > 0$.

(II) *The distribution A is an invertible element of $(\mathcal{O}')[0,b)$, i.e., there is a distribution $G \in \mathcal{O}'[0,b)$ such that*

$$A * G - \delta(x) = G * a - \delta(x) = 0 \text{ (in the sense of} (\mathcal{O}')[0,b)). \tag{20}$$

The condition (II) (in contrast to analogous conditions above) does not allow one to find effective necessary and sufficient conditions on the symbol guaranteeing the fulfillment of (I). However, (20) makes it possible to derive separately a necessary condition and a sufficient condition for unique solvability of (19). These conditions are similar and coincide in the case of polynomial symbols.

Let $A \in \mathcal{O}'[0,b)$, $b > 0$. Denote by $\hat{A}(\tau,\eta) \in \mathcal{M}^+$ the Fourier-Laplace transform of a distribution belonging to $\mathcal{O}'[0,\infty)$ which is one of the representatives of the residue class A. To each $A \in \mathcal{O}'[0,b)$ there corresponds a class of symbols $\hat{A} \in \mathcal{M}^+$ differing by symbol belonging to $F(T_{-b}(\mathcal{O}')_+) = \exp(-ib\tau)\mathcal{M}^+$.

According to (20) there is a symbol $\hat{G} \in \mathcal{M}^+$ such that

$$\hat{A}(\tau,\eta)\hat{G}(\tau,\eta) - 1 \in \exp(-ib\tau)\mathcal{M}^+.$$

Therefore there are $c > 0$andμ such that

$$|\hat{A}(\tau,\eta)\hat{\mathcal{G}}(\tau,\eta) - 1| < C(1 + |\tau| + |\eta|)^\mu \exp(b\mathrm{Im}\tau), \quad b \geq 0.$$

This inequality implies a necessary condition for solvability of (19) in a finite strip:

there exist constants C_1 and C_2 such that

$$\left\{\hat{A}(\tau,\eta) = 0, \quad \mathrm{Im}\tau \leq 0\right\} \Longrightarrow \{Im\tau > C_1\ell_n(1 + |\tau| + |\eta|) + C_2\}. \tag{21}$$

We now state a sufficient condition for solvability of (19) in a finite strip: there are $c > 0$, μ and $\rho \leq 0$ such that

$$|\hat{A}(\tau,\eta)| > C(1 + |\tau| + |\eta|)^\mu, \quad \mathrm{Im}\tau \leq \rho. \tag{22}$$

The condition (22) is necessary and sufficient for the invertibility of the distribution $F^{-1}\hat{A} \in (\mathcal{O}')_+ \subset (\mathcal{O}')_{[\rho]+}$ in the space $\left((\mathcal{O}')_{[\rho]}\right)_+$. Thus, according to (22), there is $G \in (\mathcal{O}')_{[\rho]+}$ such that $A * G = G * A = \delta(x)$. It now remains to apply isomorphism (9).

Let us discuss relationship between (21) and (22). The condition (22) implies that for some $\rho \leq 0$ the symbol $\hat{A}(\tau,\eta)$ has no zeros in the half-plane $Im\tau \leq \rho$ and admits of an algebraic estimate form below in this half-plane. Under the condition (21) the symbol $\hat{A}(\tau,\eta)$ can have zeros in the lower half-plane, however, with increasing $|\eta|$ the manifold of these zeros moves very slowly away from the manifold $\mathrm{Im}\tau = \mathrm{const}$. In the case of partial differential operators with constant coefficients, i.e. $A = P(D_t,\ D_y)\delta,\ \hat{A} = P(\tau,\eta)$, the condition (22) is equivalent to the requirement that the polynomial $P(\tau,\eta)$ to be nonzero in the closed half-plane $\mathrm{Im}\tau \leq \rho$

$$P(\tau,\eta) \neq 0 \quad \mathrm{Im}\tau \leq \rho, \quad \eta \in \mathbb{R}^{n-1}. \tag{23}$$

On the other hand, by Seidenberg-Tarski theorem the imaginary part of the algebraic function $\tau(\eta)$ (the solution to the algebraic equation $P(\tau,\eta) = 0$) cannot tend to $-\infty$ slower than a certain power of $|\eta|$, and therefore (21) implies that there must be $\rho < 0$ such that (23) holds. We have thus proved

Theorem 2. *The differential equation with constant coefficients*

$$P(D_t,\ D_y)\,u = f$$

possesses a unique solution in the space $\Phi[a,b)$, for $\Phi = \mathcal{S}, \mathcal{O}, \mathcal{S}', \mathcal{O}'$ for any $a < b \in \mathbb{R}$ (or some $a < b$) if and only if there is ρ such that (23) is fulfilled.

Definition. A polynomial $P(\tau,\eta),\ \ \tau \in \mathbb{C},\ \ \eta \in \mathbb{R}^{n-1}$, is said to satisfy the homogeneous Petrovskii condition if (23) holds for some ρ.

The homogeneous Petroviskii condition differs from the classical Petrovskii condition in the absence of the assumption that the polynomial is solved with the respect to the highest power of τ. This condition is connected with non homogeneous Cauchy problem.

§6. Some remarks about non homogeneous Cauchy problem for convolution equations

6.1. By the non homogeneous Cauchy problem, say, in $\mathcal{S}$, for a differential operator

$$P\left(D_t, D_y\right) = \sum_{j=0}^{m} P_j(D_y) D_t^{m-j} \tag{1}$$

is meant the problem of determining a function $u(t,y) \in \mathcal{S}\left(\mathbb{R}^n_+\right)$, satisfying in $\mathbb{R}^n_+$ the differential equation

$$\left(P\left(D_t, D_y\right) u\right)(t,y) = f(t,y), \quad t > 0, \quad y \in \mathbb{R}^{n-1}, \tag{2}$$

with preassigned Cauchy data to order $m-1$

(3) $$\left(D_t^{k-1}u\right)(0,y)=\varphi_k(y),\quad k=1,\dots,m.$$

Here $f\in\mathcal{S}\left(\mathbb{R}^n_+\right)$, and $\varphi_k\in\mathcal{S}\left(\mathbb{R}^{n-1}\right)$, $k=1,\dots,m$.

Remark. We can define spaces $C^{(m)}_{(\ell)}\left(\mathbb{R}^n_+\right)$ consisting of functions $\varphi(x)$ defined and continuous in the closure $\overline{\mathbb{R}^n_+}$, having in $\mathbb{R}^n_+$ continuous derivatives $D^\alpha\varphi$ to the order m extendible to continuous functions in $\mathbb{R}^n_+$, and such that the norm

$$\left|\varphi,\mathbb{R}^n_+\right|^{(m)}_{(\ell)}=\sup_{x\in\mathbb{R}^n_+,\ |\alpha|\le m}\left(1+|x|^2\right)^{\ell/2}\left|D^\alpha\varphi(x)\right|$$

is finite. Then in a natural way we can define $\mathcal{S}\left(\mathbb{R}^n_+\right)$ as the projective limit of this scale and $\mathcal{O}\left(\mathbb{R}^n_+\right)$ as the inductive limit of $C^{(\infty)}_{(\ell)}\left(\mathbb{R}^n_+\right)$. It is possible to prove that these spaces coincide with the spaces of restrictions of functions from $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{O}(\mathbb{R}^n)$ to the half-space $\mathbb{R}^n_+$.

In Chapter 2, by the homogeneous Cauchy problem we meant, somewhat conditionally, the problem of inverting an operator $P(D)$ on $\mathcal{S}_+$. We remind the reader that each function belonging to $\mathcal{S}_+$ has infinite-order zero at $t=0$, and therefore in the case of an equation on $\mathcal{S}_+$ we require that not only the derivatives to order $m-1$ of the solution, but also the subsequent derivatives should vanish at $t=0$ (accordingly the right-hand side must also have an infinite order zero). To discard these excessive requirements it seems reasonable to take, as a solution space, a space of functions having exactly a zero of order $m-1$. It will be more convenient to regard functions in the half-space $\mathbb{R}^n_+$ as functions on $\mathbb{R}^n$ extended as zero to $t<0$. More formally as the space of right-hand sides in (2), instead of the space $\mathcal{S}\left(\mathbb{R}^n_+\right)$, we shall take the space

(4) $$\mathcal{S}_{[+]}=\left\{\varphi_+=\theta_+\varphi,\quad \varphi\in\mathcal{S}(\mathbb{R}^n)\right\},$$

where $\theta_+(t)$ is the characteristic function of the positive half-line. As the solution space in (2) we shall consider the subspace

(4′) $$\mathcal{S}^{(m)}_{[+]}=\left\{\varphi_+\in\mathcal{S}_{[+]},\quad D_t^k\varphi_+(+0)\equiv 0,\quad k=0,\dots,m-1\right\}$$

corresponding a natural m. By the homogeneous Cauchy problem we now mean the problem of inverting the operator

(5) $$P\left(D_t,D_y\right):\mathcal{S}^{(m)}_{[+]}\longrightarrow\mathcal{S}_{[+]}.$$

6.2. This interpretation of the homogeneous Cauchy problem makes it possible to easily pass to the non homogeneous Cauchy problem. The action of the differential operator on $\mathcal{S}_{[+]}$ is not compatible with embedding this space into $(\mathcal{O}')_+$. The images of differential operators in the two definitions differ by a distribution with a support at $t = 0$. The following relation takes place:

$$P(D_t, D_y)(\theta_+ u) - \theta_+ P(D_t, D_y)\, u = \sum_{j=1}^{m} h_j(y) D_t^{j-1} \delta(t) \tag{6}$$

where

$$h_j(y) = -i \sum_{k=0}^{m-j} P_k(D_y) \left(D_t^{m-j-k} u \right)(0, y). \tag{7}$$

Let $u(t,y) \in \mathcal{S}\left(\mathbb{R}^n_+\right)$ be a solution to the problem (2), (3), then the function $u_+ = \theta_+ u$ satisfies the equation in distributions

$$P(D_t, D_y)\, u_+ = f_+ + \sum_{j=1}^{m} h_j(y) D_t^{j-1} \delta(t) \tag{8}$$

where, with regard to (7),

$$h_j(y) = -i \sum_{k=0}^{m-j} P_k(D_y) \varphi_{m-j-k+1}(y) \in \mathcal{S}\left(\mathbb{R}^{n-1}\right), \quad j = 1, \ldots, m. \tag{9}$$

The relations (9) we can understand as a system of differential equations on functions $\varphi_j(y)$. Let us rewrite this system putting $j = m, m-1, \ldots$

$$\begin{aligned} &P_\circ(D_y)\varphi_1 = i h_m, \\ &P_\circ(D_y)\varphi_2 = i h_{m-1} - P_1(D_y)\varphi_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &P_\circ(D_y)\varphi_m = i h_1 - P_1(D_y)\varphi_{m-1} \cdots - P_{m-1}(D_y)\varphi_1. \end{aligned} \tag{10}$$

If

$$P_\circ(\eta) \neq 0 \quad \eta \in \mathbb{R}^{n-1}, \tag{11}$$

then differential operator $P_o(D)$ is invertible in the spaces, $\mathcal{S}\left(\mathbb{R}^{n-1}\right)$, $\mathcal{O}\left(\mathbb{R}^{n-1}\right)$. So given the functions $h_j \in \mathcal{S}, \mathcal{O}$ we can reconstruct uniquely the Cauchy data $\varphi_j \in \mathcal{S}$, $\mathcal{O}$ from (10).

By what has been said, it is advisable to extend the scale of $\mathcal{S}_{[+]}^{(m)}$ to negative exponents by putting

$$\mathcal{S}_{[+]}^{(-m)} = \{\varphi = \varphi_+ + \sum_{j=1}^{m} \varphi_j(y) D_t^{j-1}\delta(t), \quad \varphi_+ \in \mathcal{S}_{[+]},$$
$$\varphi_j \in \mathcal{S}\left(\mathbb{R}^{n-1}\right), \quad j = 1, \ldots, m\}.$$

We can introduce topology in these spaces and define the inductive limit

$$\mathcal{S}_{[+]}^{(-\infty)} = \bigcup_{x \in \mathbb{Z}} \mathcal{S}_{[+]}^{(m)}.$$

Then the non homogeneous Cauchy problem can be interpreted as the problem of inverting the operator

$$(5') \qquad P(D) : \mathcal{S}_{[+]} \longrightarrow \mathcal{S}_{[+]}^{(-m)}.$$

The problems of inverting operators (5) and (5′) are special cases of a formally more general problem of inverting an operator

$$(12) \qquad P(D) : \mathcal{S}_{[+]}^{(-\infty)} \longrightarrow \mathcal{S}_{[+]}^{(-\infty)}.$$

The definitions above are applied in trivial manner to the spaces $\mathcal{O}$ and we can also consider the problem of inverting

$$(13) \qquad P(D) : \mathcal{O}_{[+]}^{(-\infty)} \longrightarrow \mathcal{O}_{[+]}^{(-\infty)}.$$

6.3. The problem of solving a differential equation with constant coefficients in the spaces $\Phi_{[+]}^{(-\infty)}$, $\Phi = \mathcal{S}, \mathcal{O}$ we can include in more general problem of solving convolution equations in these spaces. To this end we must first describe the spaces of convolutors $\mathfrak{C}\left(\Phi_{[+]}^{(-\infty)}\right)$. The latter problem reduces to separating out distributions belonging to $(\mathcal{O}')_+ = \mathfrak{C}(\mathcal{S}_+)$ or $(\mathcal{K}')_+ = \mathfrak{C}(\mathcal{O}_+)$ such that under the convolution with them the smoothness for $t > 0$ of distributions is preserved. It is possible to prove that this is the property of those and only those distributions $A \in (\mathcal{O}')_+$, $(\mathcal{K}')_+$ whose symbols $\hat{A}(\tau, \eta)$ are expandable at infinity with respect to variable τ into asymptotic Laurent's series. Describing the convolutors on these spaces it is possible to prove, following the scheme of these lectures, that the solvability of a convolution equation $A * u = f$ in $\mathcal{S}_{[+]}^{(-\infty)}$ is equivalent to the existence of the fundamental solution $G \in \mathfrak{C}\left(\mathcal{S}_{[+]}^{(-\infty)}\right)$ of the equation. In the case of differential operators the condition of invertibility of $P(D)$ in $\mathcal{S}_+$ is supplemented with the condition (11). If we shall treat analogous problem on a finite interval of time, i.e. the problem of invertibility of $P(D)$ in the spaces

$$\mathcal{S}_{[+]}^{(-\infty)} / \mathcal{S}[b, \infty), \quad b > 0$$

then homogeneous Petrovskii condition will be supplemented by (11).

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