수 학 강 의 록　　　제 13 권

Geometry of Bounded Domains and the Scaling Techniques in Several Complex Variables

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Notes of the Series of Lecture
held at the Seoul National University

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펴낸날 : 1993년 8월 10일
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Contents

Preface

This lecture note is based upon the series of ten lectures I presented as part of the program of the Global Analysis Research Center at Seoul National University, Seoul, Korea in May 1993. Since the audience already had certain backgrounds in Differential Geometry and Several Complex Variables, this exposition starts rather abruptly skimming and skipping through the basics in the first chapter.

In the second chapter, which is the main part of this lecture notes as well as the lectures, the techniques of "scaling methods" in several complex variables and the important results thereof concerning the domains with noncompact automorphism groups are introduced. First, Wong's Theorem and Rosay's generalization receive an alternative proof by scaling method in the beginning sections. The scaling methods have advantages in many places over the classical normal family arguments in actual applications, even if the scaling method itself can be viewed as a version of the classical normal family arguments. The second part of the chapter is devoted to the theorem of Bedford and Pinchuk which characterizes the domains in $\mathbb{C}^2$ with a finite type boundary which possess noncompact automorphism groups. The use of scaling and the holomorphic vector field actions is demonstrated and explained. The final part of the chapter is on the theorem characterizing the convex domains with piecewise Levi flat boundaries possessing noncompact automorphism groups proven by the author. In doing this, I present another version of scaling method initiated by S. Frankel along the way.

The final chapter introduces several sketches of the applications of the scaling methods without the presence of any noncompact automorphism group actions on the domain in question. First item presented is Pinchuk's generalization of the proper mapping theorem of H. Alexander, which demonstrates the application of a scaling method without any noncompact automorphism group actions. Then a very recent theorem by the author in collaboration with J. Yu which generalizes the well-known theorem of Klembeck as well as an earlier theorem by the author on the asymptotic behavior of the holomorphic sectional curvatures of the Bergman metric is explained. There are other important applications of scaling in part of the arguments in the recent articles by Barrett and McNeal to mention only a few. However, time did not permit me to introduce any of these important and interesting ideas and results at this point.

In writing this notes, I would like to thank Professors Hyung-In Choi, Jongsik Kim, Hong-Jong Kim, Hyuk Kim and all the other faculty members of the Department of Mathematics in Seoul National University for inviting me for the lecture series at the Global Analysis Research Center. It is my pleasure to acknowledge my indebtedness to Steve Krantz for his encouragements, for many enlightening discussions and for numerous valuable suggestions throughout the writing of this manuscript. Large part of this lecture notes has been written while I was visiting Washington University in St. Louis in the Spring of 1993. I would like to express my gratitude to its hospitality. Finally, I would like to thank my brother Sang-Tae Kim for his emotional support and constant encouragements throughout my studies, without which I cannot imagine myself even beginning any career of mine in Mathematics.

The author's research is supported in part by a research grant from the National Science Foundation of the United States of America.

CHAPTER I

Basic concepts

1. Some Notations

Let us denote by $\mathbb{C}$ the set of all complex numbers. Let $i = \sqrt{-1}$ as usual. Write

$$\mathbb{C}^n = \{(z_1, \dots, z_n) \,|\, z_j \in \mathbb{C},\ \forall j = 1, \dots, n\}$$

With notations $z_j = x_j + i\, y_j$, $(x_j, y_j \in \mathbb{R})$ $(j = 1, \dots, n)$, we define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

for each $j = 1, \dots, n$.

Furthermore, for each j we set

$$dz_j = dx_j + idy_j$$
$$d\bar{z}_j = dx_j - idy_j$$

Notice that all the vector fields and the differential forms introduced in this way can be understood rigorously as elements of the complexified vector fields and differential forms. From such understanding, following the standard way of introducing the wedge product it is easy to see that

$$(d\bar{z}_1 \wedge dz_1) \wedge \dots (d\bar{z}_n \wedge dz_n) = (2i)^n (dx_1 \wedge dy_1) \wedge \dots (dx_n \wedge dy_n).$$

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a continuously differentiable function, then we define

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$
$$\overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

As usual the standard way of generlization lets us to define the operators ∂ and $\overline{\partial}$ acting on the higher degree differential forms. It follows then that $d = \partial + \overline{\partial}$ and that $\partial\partial = 0 = \overline{\partial}\overline{\partial}$.

Finally, we denote by

$$B^n(p,r) = \{z \in \mathbb{C}^n \mid |z - p| < r\}$$
$$D^n(p,r) = \{z \in \mathbb{C}^n \mid |z_1 - p_1| < r, \dots , |z_n - p_n| < r\}$$

where

$$z = (z_1, \dots , z_n)$$
$$p = (p_1, \dots , p_n)$$
$$|z - p|^2 = |z_1 - p_1|^2 + \dots + |z_n - p_n|^2$$

We call $B^n(p,r)$ the *open ball in* $\mathbb{C}^n$ *centered at* p *with radius* r, whereas $D^n(p,r)$ is called the *polydisk with radius* r *centered at* p.

2. Holomorphic functions

DEFINITION 2.1. Let Ω be an open subset of $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is called *complex analytic* [*analytic*, or *holomorphic*] if it is continuously differentiable and satisfies the Cauchy-Riemann equation $\overline{\partial} f = 0$ on Ω.

Notice that the identity $\overline{\partial} f = 0$ on Ω implies that f is holomorphic in each variable separately. There is a deep (but seldom used) theorem by Hartogs which states

THEOREM 2.1 (HARTOGS). *Let Ω be a domain in $\mathbb{C}^n$. If a function $f : \Omega \to \mathbb{C}$ is holomorphic in each variable separately at every point, then f is continuous.*

Note that the real C^∞ analog of this theorem is false.

EXERCISE 2.1. Construct a discontinuous function which is differentiable indefinitely many times in each variable separately.

It is relatively easy to see that the continuous functions holomorphic in each variable separately are in fact continuously differentiable from the following:

THEOREM 2.2 (CAUCHY'S INTEGRAL FORMULA). *Let $f : D^n(p,r) \to \mathbb{C}^n$ be a continuous function holomorphic in each variable separately. Then*

$$f(z_1,\dots,z_n) =$$
$$\frac{1}{(2\pi i)^n}\int_{|\zeta_1-p_1|=r}\cdots\int_{|\zeta_n-p_n|=r}\frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)}dz_n\cdots dz_1$$

for any $z = (z_1,\dots,z_n) \in D^n(p,r)$.

This theorem is a direct generalization of Cauchy's integral formula in one variable. Observe that the "differentiation under the integral sign" implies

COROLLARY 2.3. *Let Ω be a domain in $\mathbb{C}^n$. If a continuous function $f : \Omega \to \mathbb{C}$ is holomorphic in each variable separately, then f is C^∞ smooth.*

Another obvious imitation of the one-variable argument yields

COROLLARY 2.4. *Any holomorphic function on a domain in $\mathbb{C}^n$ is locally a power series that converges absolutely and uniformly.*

Notice that the integral representation above is always holomorphic even if the function f is not necessarily holomorphic as long as the differentiation under the integral sign is permitted. (Although, in such a case the integral does not necessarily coincide with the original function.) Hence, we get another consequence of the Cauchy integral formula:

COROLLARY 2.5. *If a sequence of holomorphic functions on a domain converges uniformly to a function on every compact subsets, then the limit function is also holomorphic.*

Cauchy's integral formula also provides

THEOREM 2.6 (THE CAUCHY ESTIMATES). *Let K be a compact subset of a bounded domain Ω in $\mathbb{C}^n$, and let $f : \Omega \to \mathbb{C}$ a holomorphic function. Then*

$$\sup_{z\in K}\left|\left(\frac{\partial}{\partial z}\right)^A f(z)\right| \le C_1 \cdot \sup_{\zeta\in\Omega}|f(\zeta)|$$

for some constant C_1 depending only on K and Ω and the length $|A| = |a_1| + \ldots + |a_n|$ of the multi-index $A = (a_1,\dots,a_n)$. Moreover,

$$\sup_{z\in K}\left|\left(\frac{\partial}{\partial z}\right)^A f(z)\right| \le C_2 \cdot \left(\int_\Omega |f(z)|^2 d\mu(z)\right)^{\frac{1}{2}}$$

for some constant $C_2 > 0$ depending only on the multi-index $A = (a_1, \ldots, a_n)$, the compact set K and the domain Ω. In the above, $d\mu$ denotes the standard volume form (or, equivalently, the Lebesgue measure) of $\mathbb{C}^n$ and

$$\left(\frac{\partial}{\partial z}\right)^A = \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{a_n}.$$

PROOF. Exercise. The first inequality is easy to derive. For the second, use polar coordinates in each variable. □

An immediate consequence of the above is that the L^2 convergence of a sequence of square integrable holomorphic functions necessarily yields the uniform convergence of all derivatives on all compact subsets. Thus, the resulting limit is also holomorphic.

3. Domains of Holomorphy

One of the many surprises in the study of the theory of analytic functions in several complex variables in contrast to the one variable case is the following theorem due to Hartogs:

THEOREM 3.1 (HARTOGS). *Let Ω be a domain in $\mathbb{C}^n$ and let K be a compact subset of Ω with $\Omega \setminus K$ is connected. Then, for any holomorphic function $f : \Omega \setminus K \to \mathbb{C}$, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F = f$ on $\Omega \setminus K$.*

This theorem leads us to consider which domains should be the natural domains for the analytic functions of several complex variables. Hence we arrive at the following definition.

DEFINITION 3.1 (DOMAINS OF HOLOMORPHY). An open set Ω in $\mathbb{C}^n$ is called a *region of holomorphy*, if there do not exist nonempty connected open sets $U_1 \not\subset \Omega$ and $U_2 \subset \Omega \cap U_1$ such that for every holomorphic function $g : U_2 \to \mathbb{C}$ there exists a holomorphic function $\tilde{g} : U_1 \to \mathbb{C}$ satisfying $\tilde{g}|_{U_2} = g$.

Such a complicated definition is indeed necessary due to the existence of examples such as the "snake domain biting its own tail." Thanks to the solution of the Levi problem, we now have many useful equivalent definitions for domains of holomorphy which we will introduce from the following sections.

4. Pluriharmonicity

We begin this section with the following:

DEFINITION 4.1. Let Ω be a domain in $\mathbb{C}^n$. An uppersemicontinuous function $\varphi : \Omega \to \mathbb{R}$ is called *plurisubharmonic* on D if $\varphi \circ h$ is subharmonic on the unit disk $B = B^1 = B^1(0;1)$ for every linear holomorphic mapping $h : B \to \Omega$ of the form $a + b \cdot z = (a_1 + b_1 z, \ldots, a_n + b_n z)$. A function ψ is called *plurisuperharmonic* if $-\psi$ is plurisubharmonic. A function is called *pluriharmonic* if it is both plurisubharmonic and plurisuperharmonic.

Several facts are known about this definition.

REMARK 4.1. All pluriharmonic [plurisubharmonic or plurisuperharmonic, respectively] functions are harmonic [subharmonic or superharmonic, respectively]. But the converse is not in general true.

REMARK 4.2. The real part of every holomorphic function is pluriharmonic. And, every pluriharmonic function is locally the real part of a holomorphic function. The first assertion is easy to check. The second follows from for instance the following arguments: First, notice that every harmonic function is C^∞ smooth due to Weyl's Lemma. Then a smooth function u is pluriharmonic if and only if $\partial\overline{\partial}u = 0$. Now, let $\alpha = i(\partial - \overline{\partial})u$. Then, $d\alpha = (\partial + \overline{\partial})\alpha = 0$. So the Poincaré Lemma implies that α is locally d-exact. Namely, in a small neighborhood there exists a smooth real-valued function v such that $\alpha = dv = \partial v + \overline{\partial} v$. Comparing types, one then gets $-i\overline{\partial}u = \overline{\partial}v$. Hence, $u - iv$ is holomorphic.

From now on, we will mention mostly the plurisubharmonicity. For the sake of brevity, we will denote this property by "psh" instead of "plurisubharmonic."

REMARK 4.3. Plurisubharmonicity is preserved under holomorphic mappings in the following sense: Let $u : \Omega \to \mathbb{R}$ be psh. Then for any holomorphic mapping $h : G \to \Omega$, the composite $u \circ h$ is psh. This property is in general false for the usual (sub/super) harmonic functions in complex dimensions higher than one.

REMARK 4.4. Let $\varphi : \Omega \to \mathbb{R}$ be C^2 smooth. Then, φ is psh on Ω if and only if $\varphi \circ h$ is subharmonic for any holomorphic function $h : B \to \Omega$.

REMARK 4.5. Let f be a holomorphic function defined on a domain G in $\mathbb{C}^n$. Then, $\log|f|$, $\log(1 + |f|^p)$, and $|f|^p$ $(p > 0)$ are psh.

5. Convexity and Pseudoconvexity

In the light of Hartogs' extension theorem (Theorem 3.1 above) and the definition of domains of holomorphy, one realizes that some notions including convexity should play an important role in the theory of holomorphic functions in several variables. In this section, we will simply introduce various types of convexity and pseudoconvexity.

DEFINITION 5.1 (GEOMETRIC CONVEXITY). A subset K of $\mathbb{R}^n$ is called *convex*, if for every pair of points $p, q \in K$ the straight line segment joining p, q belongs to K. A point in a convex set is called *extreme* if it is not an interior point of any line segment contained in the convex set. A convex set is called *strictly convex* if every boundary point is extreme.

There are at least two different ways of re-interpreting these concepts.

5.1. Convexity with respect to a class of functions.

PROPOSITION 5.1. *A subset K of $\mathbb{R}^n$ is convex if and only if*

$$\{x \in \mathbb{R}^n \,|\, \ell(x) \leq \sup_{y \in K} \ell(y) \text{ for any first order polynomial } \ell(v) = av + b\}$$

is always contained in the closure of K.

PROOF. Exercise. □

This triggers the following definition of convexity with respect to a certain class of functions:

DEFINITION 5.2. Let a domain Ω be given in $\mathbb{C}^n$ and let K be a compact subset of Ω. Let $\mathcal{F}$ be a given class of real valued functions. Then the *hull* of K in Ω with respect to $\mathcal{F}$ is given by

$$\hat{K}_{\mathcal{F}} = \{z \in \mathbb{C}^n \,|\, u(z) \leq \sup_{\zeta \in K} u(\zeta) \text{ for all } u \in \mathcal{F}\}.$$

We say Ω is *convex with respect to $\mathcal{F}$* if the hull $\hat{K}_{\mathcal{F}}$ is compact for every compact subset K of Ω.

Most common notions of much interest in several complex variables are:

- the convexity with respect to the set of the absolute values of the holomorphic polynomials, which is called the *polynomial convexity*, and
- the convexity with respect to the set of the absolute values of the holomorphic functions on the given domain, which is called the *holomorphic convexity*.

EXERCISE 5.1. Show that an open set is geometrically convex if and only if it is convex with respect to the real-valued linear functions.

EXERCISE 5.2. Show that every open set is convex with respect to the class of real-valued continuous functions.

EXERCISE 5.3. Show that every open set in $\mathbb{C}$ is holomorphically convex. Moreover, construct an example of an open set in $\mathbb{C}^2$ that is not holomorphically convex. Also, give an example of a domain in $\mathbb{C}$ that is <u>not</u> polynomially convex.

5.2. Pseudoconvexity. Another way of considering convexity is along the Hessians of the boundary surface. For such consideration, it is natural to assume C^2 smoothness of the boundary of the domain in question. It turns out that in the study of the theory of holomorphic functions in several variables, the convexity concept along complex tangential directions of the boundary of the domain is relevant as in the following.

Let Ω be a domain in $\mathbb{C}^n$ defined by the inequality

$$\rho(z) = \rho(z_1, \dots, z_n) < 0$$

for a twice differentiable real-valued function defined on $\mathbb{C}^n$ satisfying:

- The boundary $\partial\Omega$ of Ω is defined by $\rho(z) = 0$.
- The gradient vector $\partial\rho = (\partial\rho/\partial z_1, \dots \partial\rho/\partial z_n)$ is never zero at any point of $\partial\Omega$.
- $\mathbb{C}^n \setminus \overline{\Omega} = \{z \in \mathbb{C}^n \mid \rho(z) > 0\}$.

In such a case, ρ is called a *defining function* of Ω. Notice that by the Implicit Function Theorem the boundary of such domain Ω is a C^2 smooth hypersurface in $\mathbb{C}^n$. Now we are ready to give the definition of Levi pseudoconvexity.

DEFINITION 5.3. Let Ω and ρ be as above. A boundary point $p \in \partial\Omega$ is called a *(Levi) pseudoconvex point* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0 \tag{1}$$

for all $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0. \tag{2}$$

If the strict inequality holds in (1) for every nonzero vector $w \in \mathbb{C}^n$ satisfying (2), then the boundary point p is called strongly pseudoconvex. The complex hessian form of ρ in (1) is called the *Levi form* of ρ. The vectors w satisfying (2) are called *complex tangent vectors* to $\partial\Omega$ at p. The domain Ω is called *Levi pseudoconvex* if it is exhausted by the domains whose every boundary point is Levi pseudoconvex. Ω is called *strongly pseudoconvex*, if every boundary point is strongly pseudoconvex.

EXERCISE 5.4. Show that a strongly pseudoconvex domain admits a defining function, say r, such that for some $C > 0$

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k > C|w|^2, \forall 0 \neq w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

Such a function is called *strictly plurisubharmonic.*

EXERCISE 5.5. Suppose that there exists a holomorphic function f from the unit disk in $\mathbb{C}$ into the closure of a strongly pseudoconvex domain in such a way that the origin is mapped to a boundary point of the domain. Then show that f is constant.

One of the big theorems in Several Complex Variables is the solution of the Levi problem which is

THEOREM 5.2 (LEVI PROBLEM). *Every domain of holomorphy in $\mathbb{C}^n$ is Levi pseudoconvex.*

This highly nontrivial problem was solved by K. Oka, H. Bremerman, F. Norguet and others. There are many other useful approaches to the problem and generalizations by many authors. Many important problems related to the Levi problem are still open.

In fact, there are many important and useful conditions that are equivalent to the Levi pseudoconvexity. For a detailed exposition, we would like to refer the readers to the book by S. Krantz ([51]). One fact to mention however is that the Levi pseudoconvexity is equivalent to the holomorphic convexity defined above.

6. Peak points and peak functions

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and let $p \in \partial\Omega$. Then p is called a *peak point* if there exists a continuous function $f : \overline{\Omega} \to \mathbb{C}$ which is holomorphic on Ω such that $f(p) = 1$ and $|f(z)| < 1 \; \forall z \in \overline{\Omega} \setminus \{p\}$. Such a function f is called a *peak function* of Ω at p. Sometimes, it is enough to consider the existence of the peak functions defined only on a neighborhood of the boundary point in consideration. In such cases, such functions are called *local peak function* and the boundary points that admit local peak functions are called the *local peak points.*

There are many subtle and important problems involving the peak functions. Problems concerning peaking functions are of great interests. However, we will discuss only a few simple facts on the local peak functions in this section.

PROPOSITION 6.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $p \in \partial\Omega$ be a local peak point. Suppose that $f : B \to \overline{\Omega}$ is a holomorphic function from the unit disk in $\mathbb{C}$ to the closure of Ω such that $f(0) = p$. Then $f(z) = p \; \forall z \in B$.*

PROOF. Let U be an open ball centered at p with a small radius such that there exists a local peak function h at p defined on $U \cap \overline{\Omega}$ such that $h(p) = 1$ and $|h(z)| < 1$ for all $z \in U \cap \overline{\Omega} \setminus \{p\}$. By continuity, there exists a small open disk D containing the origin in the unit disk B such that $f(D) \subset U \cap \overline{\Omega}$. Then $h \circ f : D \to \mathbb{C}$ is a continuous function which is holomorphic except possibly at the origin. By a removable singularity theorem in one complex variable, $h \circ f$ is indeed holomorphic on entire D. Then the Maximum Principle implies that $h \circ f$ is identically equal to 1. This implies then that every component of f is constant on an open set. Then the theorem of analytic continuation in one complex variable yields the result. □

EXERCISE 6.1. Show that every boundary point of the unit ball in $\mathbb{C}^n$ are peak points. Generalize this to any strictly convex domains.

EXERCISE 6.2. Find the set of all the peak points of the polydisk

$$D^n(1) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| < 1, \ldots, |z_n| < 1\}$$

There are standard techniques of extending local peak functions to a global peak functions by using the L^2 estimate of the $\overline{\partial}$ operator, when the domains in consideration are special. However, we do not include any further details in this note except for providing a few references on the peak functions in the bibliography.

7. Order of contact and Finite type

One of the remarkable recent progress in the study of boundary geometry of the hypersurfaces in $\mathbb{C}^n$ in relation to many geometric and function theoretic problems is the notion of finite type discovered by J. D'Angelo [16]. Again avoiding intricate details we will simply introduce the most simple definition followed by a very basic discussion.

Let M be a real hypersurface in $\mathbb{C}^n$ and let $p \in M$. In a neighborhood of p in $\mathbb{C}^n$, one can find a real-valued function ρ such that M is defined by the equation $\rho = 0$ and such that the gradient $\partial\rho$ is never zero on M. Consider a germ of an analytic variety defined by a holomorphic mapping

$$\psi : U \to \mathbb{C}^n$$

such that $\psi(0) = p$, where U is the open unit disk in $\mathbb{C}$ centered at the origin. Denote by $v_q(f)$ the order of vanishing of the function f at q. Let

$$\tau(M, p, \psi) = \frac{v_0(\rho \circ \psi)}{v_0(\psi)}.$$

Then, the *type* of M at p is defined by

$$\tau(M, p) = \sup \tau(M, p, \psi)$$

where the supremum is taken over all the germs of analytic varieties $\psi : U \to \mathbb{C}^n$ with $\psi(0) = p$. If $\tau(M, p)$ is finite, then p is called a *point of finite type.*

It turns out that the strongly pseudoconvex boundary point is of type 2. If the Levi form of the hypersurface vanishes identically at every point along all the complex tangent vectors of M, then such a surface is called *Levi flat.* Notice that every point on a Levi flat surface is a point of infinite type. See the exercise below.

EXERCISE 7.1. Let X be a vector field which is complex tangent to a real hypersurface M in $\mathbb{C}^n$ with a defining function ρ. Let $\bar{X}$ denote the complex conjugate of X. Then verify first that $\bar{X}$ is also tangent to M. Now, let u be the tangent vector field to M such that $\sqrt{-1}\, u$ is normal to M. Then compute the real valued function λ on M satisfying

$$[X, \bar{X}]_p = \sqrt{-1}\, \lambda(p) u_p$$

modulo the space of complex tangent vectors to M at p. Here the bracket operation is the standard Lie bracket operation of $\mathbb{C}^n$.

EXERCISE 7.2 (CONTINUED). If the surface M is Levi flat, show that there exists a nontrivial complex analytic variety contained in the surface.

EXERCISE 7.3. Show that real hypersurface does not admit any nontrivial analytic variety in it passing through a point of finite type.

EXERCISE 7.4. Find examples of various finite type points on various real hypersurfaces in $\mathbb{C}^2$.

For many useful and important details concerning the notion of finite type, we refer to [16].

8. Automorphism groups of bounded domains

Let Ω be a domain in $\mathbb{C}^n$. By an *automorphism* of Ω we mean a biholomorphic self-mapping of Ω. The set of all automorphisms of Ω is called the *automorphism group* of Ω and is denoted by Aut D, which is naturally equipped with the law of composition of mappings.

For Ω is bounded, H. Cartan [14] proved that Aut Ω is a finite dimensional (real) Lie group with respect to the topology of uniform convergence on compact subsets of Ω. H. Cartan also proved the following theorem which is, in particular, very effective in computing the automorphism groups of the ball, the polydisks, and some other types of circular domains.

THEOREM 8.1 (H. CARTAN'S THEOREM I). *Let D be a bounded domain in $\mathbb{C}^n$ for $n \geq 1$. Let $p \in D$. If $f : D \to D$ is a holomorphic mapping such that*

$$f(p) = p, \text{ and } df(p) = I$$

then $f(z) = z$ for all $z \in D$. Here, I denotes the identity matrix.

PROOF. This theorem follows from the iteration of f, and the Cauchy estimates. Assume without loss of generalities that p is the origin. Since $f(0) = 0$ and $df(0) = I$, the Taylor expansion of the k-th component of f at $p = 0$ is given by

$$f_k(z_1, \ldots, z_n) = z_k + P(z_1, \ldots, z_n) + \text{higher order terms}$$

where P denotes the first non-vanishing (if any) homogeneous holomorphic polynomial beyond the linear terms. If f is not the identity map, choose k such that P above is the lowest degree, say m, polynomial beyond the linear terms among the Taylor expansions at p of all the components of f. Now notice that the k-th component of the N-th iterate $f^{\circ N} = f \circ f \circ \ldots \circ f$ of f is

$$z_k + N \cdot P(z_1, \ldots, z_n) + \text{higher order terms}$$

This implies that there exists a multi-index $(a_1, \ldots, a_n)$ with $m = |a_1| + \ldots + |a_n|$ such that

$$\lim_{N \to \infty} \left| \frac{\partial^m f^{\circ N}}{\partial z_1^{a_1} \cdots \partial z_n^{a_n}}(0) \right| = \infty$$

On the other hand, the range of $f^{\circ N}$ for every N is the domain Ω which is bounded. Therefore, the supremum of the absolute value of $f^{\circ N}$ on Ω is bounded by a constant independent of N. Which means that all the mixed partial derivatives of order m at the origin of all $f^{\circ N}$ has to be uniformly bounded for all N by Cauchy estimates. Apparently, this causes a contradiction. Therefore, f has to be the identity mapping. □

A modification of this brilliant proof yields even more surprising results as in what follows. Denote by

$$r_\theta(z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n).$$

Then, a domain D is called *circular* if $r_\theta \in \operatorname{Aut} D$ for all $\theta \in \mathbb{R}$. Now we state the following

THEOREM 8.2 (H. CARTAN'S THEOREM II). *Let D_1 and D_2 be circular domains in $\mathbb{C}^n$ that contain the origin inside. Let $f : D_1 \to D_2$ be a biholomorphic mapping such that $f(0) = 0$. Then, f is a complex linear mapping of $\mathbb{C}^n$.*

PROOF. Exercise. Consider $F = f^{-1} \circ r_{-\theta} \circ f \circ r_\theta$ and apply Cartan's Theorem I to this function. Then compare the Taylor coefficients of the mappings $r_\theta \circ f$ and $f \circ r_\theta$. (See [55] for a detailed proof.) □

EXERCISE 8.1. Use the above theorems to find an upper bound of the dimension of the automorphism group of a bounded domain.

EXERCISE 8.2. Using the above theorems, compute the automorphism group of the unit ball and the polydisks in $\mathbb{C}^n$. For the case of the unit ball centered at 0, use the unitary maps and the Möbius transformations of the form

$$(z_1, \ldots, z_n) \mapsto \left(\frac{z_1 + a}{1 + \bar{a} z_1}, \frac{\sqrt{1 - |a|^2}\, z_2}{1 + \bar{a} z_1}, \ldots, \frac{\sqrt{1 - |a|^2}\, z_n}{1 + \bar{a} z_1} \right)$$

As the readers may have notices already, the theorems above are generalizations of classical Schwarz's Lemma in one complex variable. There are various other types of generalizations of Schwarz's Lemma in various different settings. For our purposes, we will simply introduce the following simple case of a theorem of H. Wu ([69]):

THEOREM 8.3 (WU). *Let Ω be a bounded domain and let $p \in \Omega$. Suppose that $f : \Omega \to \Omega$ is a holomorphic mapping such that $f(p) = p$. Then, the following hold:*

(1) *Every eigenvalue of the holomorphic Jacobian matrix $J_{\mathbb{C}} f(p)$ has absolute value not bigger than one.*
(2) $\det J_{\mathbb{C}} f(p) = 1$ *if and only if* $f \in \operatorname{Aut} \Omega$.

PROOF. Exercise. Use the scheme of the proof of Cartan's Theorem I and consider the Jordan canonical form of the holomorphic Jacobian matrix. □

THEOREM 8.4 (H. CARTAN). *Show that the isotropy subgroup, say G_p, of the automorphism group $G = \operatorname{Aut}\Omega$ at p defined by*

$$G_p = \{f \in \operatorname{Aut}\Omega \mid f(p) = p\}$$

is compact with respect to the topology of uniform convergence on compact subsets, provided that Ω is a bounded domain in $\mathbb{C}^n$.

PROOF. Exercise. □

PROPOSITION 8.5. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then $\operatorname{Aut}\Omega$ is noncompact with respect to the topology of uniform convergence on compact subsets if and only if there exist a point $p \in \Omega$ and a sequence $\{f_j\}_{j=1}^{\infty} \subset \operatorname{Aut}\Omega$ such that the sequence $\{f_j(p)\}_{j=1}^{\infty}$ accumulates at a boundary point of Ω.*

PROOF. If such a sequence of automorphisms $\{f_j\}_{j=1}^{\infty}$ and an interior point $p \in \Omega$ exist as in the statement of the proposition, choose a subsequence of $\{f_j\}_{j=1}^{\infty}$ that converges uniformly on compact subsets to a holomorphic function, say f, using the usual normal family argument. Then $f(\Omega) \subset \overline{\Omega}$ and $f(p) \in \partial\Omega$. This implies that f is not an automorphism of Ω. Therefore, $\operatorname{Aut}\Omega$ is not compact.

Conversely, assume that there does not exist any orbit of a point by a sequence of automorphisms accumulating at a boundary point. Then for every $q \in \Omega$, $G(q) := \{f(q) \mid f \in \operatorname{Aut}\Omega\}$ is compact. We prove first the sequential compactness of the automorphism group. Fix the point q, and consider an arbitrary sequence $\{f_j\}_j \subset \operatorname{Aut}\Omega$. By compactness of $G(q)$, we may choose a subsequence which we will also denote by $\{f_j\}_j$ such that $f_j(q) = y_j \to y \in G(q) \subset \Omega$ as $j \to \infty$. Since Ω is bounded, by using Cauchy estimates and a standard normal family argument and by choosing a subsequence again, we may assume that f_j converges to a holomorphic function f. Now, consider the subsequence chosen last time and consider the sequence of inverse mappings f_j^{-1}. Again, one can easily extract a subsequence to show that the chosen subsequence of mappings f_j^{-1} converges to a holomorphic mapping $g : \Omega \to \overline{\Omega}$. This implies that the determinant of the holomorphic Jacobian matrices of f_j^{-1} is uniformly bounded on compact subsets. This yields in turn that the determinant of the holomorphic Jacobian of f is never zero on Ω. Hence f is locally one to one, and hence $f(\Omega) \subset \Omega$. Again using uniform convergence on compact subsets with derivative estimates, it is not hard to show that f is indeed globally one-to-one. Do the same for g. Then it is simple to see that f and g are inverse to each other. Hence, we have established the sequential compactness of $\operatorname{Aut}\Omega$.

To finish we will simply provide the metric on $G = \operatorname{Aut}\Omega$ which induces the topology of uniform convergence on compact subsets. Let $\{K_j \mid j = 1, 2, \dots\}$ be a compact exhaustion of Ω. Then we simply define the metric on G by

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \cdot \frac{\sup_{z\in K_j} |f(z) - g(z)|}{1 + \sup_{z\in K_j} |f(z) - g(z)|}$$

for $f, g \in G$. □

More on automorphism groups will be discussed in the later chapters.

9. Finslerian Invariant Metrics

9.1. The Carathéodory metric. In this section, we introduce the (pseudo) distances which makes all holomorphic mappings distance-decreasing. We start with

DEFINITION 9.1. Let $B = \{z \in \mathbb{C} \mid |z| < 1\}$. Then, we define the *Poincaré (infinitesimal) metric* on the tangent bundle of B by

$$ds_B^2 = \frac{dz \otimes d\bar{z}}{(1 - |z|^2)^2}.$$

This notation simply means that for a piecewise C^1 smooth curve $\gamma : [0,1] \to B$, the length $L(\gamma)$ of γ is given by

$$L(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt.$$

Accordingly, the *Poincaré (integrated) distance* is defined by

$$\rho_B(p,q) = \inf_{\gamma} L(\gamma)$$

where the infimum is taken over all possible piecewise C^1 smooth curves in B joining p and q.

The fact that the holomorphic mappings from B to B are distance decreasing with respect to the Poincaré metric follows from

THEOREM 9.1 (SCHWARZ-PICK LEMMA). *Let $f : B \to B$ be a holomorphic mapping. Then*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

for all $z \in B$.

PROOF. Exercise: First compose with appropriate Möbius transforms so that the origin is preserved. Then use the classical Schwarz lemma. □

REMARK 9.1. The real part of the Poincaré infinitesimal metric is a well-defined Riemannian metric on B. The open unit disk B equipped with the Poincaré distance is a complete metric space. Moreover, the Gauss curvature of the Poincaré metric is constant -4.

REMARK 9.2. Distance decreasing property implies that all the biholomorphic self-mappings of the disk B are isometries with respect to the Poincaré metric and distance.

EXERCISE 9.1. Compute an explicit form of the Poincaré distance on B. First realize what the distance minimizing curves are, either by computation or by using some facts on geodesics in Riemannian manifolds.

The next step is to implant the Poincaré metric and distance to an arbitrary domains [1] in $\mathbb{C}^n$. For convenience, we will denote by $H(D, G)$ the set of holomorphic functions from the domain D into the domain G in what follows.

DEFINITION 9.2. Let Ω be a domain in $\mathbb{C}^n$. Then the *Carathéodory distance* is defined by

$$\rho_C^\Omega(p,q) = \sup\{\rho_B(f(p), f(q)) \mid f \in H(\Omega, B)\}.$$

Likewise, the *Carathéodory metric* is defined by

$$F_C^\Omega(p,\xi) = \sup\{|df(p)\xi| \mid f \in H(\Omega, B)\}$$

It is known that the integrated distance of the infinitesimal Carathéodory metric is not in general the same as the Carathéodory distance defined above.

EXERCISE 9.2. Show that all the holomorphic mappings are distance decreasing with respect to the Carathéodory metric and distances.

9.2. The Kobayashi-Royden metric. Now we introduce the Kobayashi-Royden metric and distance. First, define by, on a domain Ω in $\mathbb{C}^n$,

$$\delta_\Omega(p,q) = \inf\{\rho_B(a,b) \mid f(a) = p, f(b) = q \text{ for some } f \in H(B,\Omega)\}.$$

In general, this definition does not yield a pseudo-distance, because the triangle inequality fails.[2] Therefore, one needs the following definition by S. Kobayashi ([43], [45]):

[1] In fact one can introduce the invariant metrics for the complex manifolds exactly in the same way. However, we restrict ourselves to the case of domains in $\mathbb{C}^n$.

[2] For the bounded convex domains δ above does become a metric. ([52])

DEFINITION 9.3. Let Ω be a domain in $\mathbb{C}^n$. Then the Kobayashi distance on Ω is given by

$$\rho_K^\Omega(p,q) = \inf \sum \delta_\Omega(z_j, w_j)$$

where the infimum is taken over all the finite collection of points

$$(z_1, w_1), \ldots, (z_N, w_N) \in \Omega \times \Omega$$

and the mappings $f_1, \ldots, f_N \in H(B, \Omega)$ such that

$$f(z_1) = p, f(w_N) = q, f(w_j) = f(z_{j+1}), \forall j = 1, \ldots, N-1.$$

EXERCISE 9.3. Show that the Kobayashi distance satisfies the triangle inequality.

The infinitesimal version of the Kobayashi distance is defined as follows:

DEFINITION 9.4. Let Ω be a domain in $\mathbb{C}^n$. Then the *Kobayashi-Royden* metric is defined by

$$F_K^\Omega(p,\xi) = \inf\{|t| \mid f(0) = p, df(0)t = \xi \text{ for some } f \in H(B,\Omega)\}$$

EXERCISE 9.4. Show that all the holomorphic mappings are distance decreasing with respect to the Kobayashi-Royden metric/distance.

The integrated metric of the infinitesimal Kobayashi-Royden metric is indeed the Kobayashi distance. This was proven by H.L. Royden in 1970. See [64].

REMARK 9.3. The metrics and distances introduced above are not in general positive definite. However, for the bounded domains, they are always positive definite. (Why?)

REMARK 9.4. Unlike Riemannian metrics, neither the Carathéodory nor the Kobayashi metrics is in general an inner product metric. Rather, they only define a norm on the tangent spaces. Such metrics in general are called Finslerian metrics. It is known that among the Finslerian metrics that make the holomorphic mappings distance-decreasing, the Kobayashi metric is the largest while the Carathéodory metric is the smallest.

9.3. The Sibony metric. Now we introduce the metric introduced by N. Sibony in [66]. The definition is as follows.

Let Ω be a domain in $\mathbb{C}^n$, and let $p \in \Omega$. Let $\mathcal{S}_p$ be the family of C^2 functions u defined on Ω such that

- $0 \le u \le 1$, and $u(p) = 0$; and
- $\log u$ is psh on Ω.

Then the *Sibony metric* is the pseudometric on the tangent bundle of Ω defined by

$$F_S^\Omega(p, \xi) = \sup_{u \in \mathcal{S}_p} \sqrt{\sum_{j,k=1}^{n} \frac{\partial^2 u(p)}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k}.$$

This metric enjoys many useful and flexible features including the distance decreasing properties of holomorphic mappings. The distance decreasing property implies immediately that the Sibony metric is smaller than the Kobayashi-Royden metric. Hence this metric is in particular good to produce significant lower bound estimates for the Kobayashi metric when there is a good psh function on the domain for instance as in the following proposition.

PROPOSITION 9.2. *Let Ω be a domain in $\mathbb{C}^n$. If there is a psh function $u : \Omega \to \mathbb{R}$ bounded from above which is strictly psh in a neighborhood U of a point p, then there exist a constant $\delta > 0$ and a neighborhood $V \subset U$ of p such that*

$$F_S^\Omega(q, \xi) \ge \delta |\xi|,$$

for all $q \in V$ and $\xi \in T_q\Omega$.

REMARK 9.5. The Carathéodory, Kobayashi and Sibony metrics can be easily defined on complex manifolds without changing the definitions. Among them, the Sibony metric seems most flexible to deal with, since the psh functions are rather easier to handle than the holomorphic functions.

10. The Bergman Kernel Function and the Bergman Metric

10.1. The Bergman kernel function. Chronologically, the first invariant metric (in the sense that the biholomorphic mappings are isometric with respect to it) discovered is the *Bergman metric* which is discovered by Stefan Bergman in 1928. (See for instance [9] and its references.) We will introduce the definition and some basic properties of this important metric in this section.

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $L^2(\Omega)$ be the space of square integrable complex valued functions on Ω. Then we first define the *Bergman space*

$$\mathcal{A}^2(\Omega) := \{f \in L^2(\Omega) \mid f \text{ holomorphic}\}.$$

By the Cauchy estimates, the Bergman space $\mathcal{A}^2(\Omega)$ is a closed subspace of the separable Hilbert space $L^2(\Omega)$, with respect to the L^2 inner product. So we may choose an orthonormal basis system $\{\varphi_j \mid j = 1, 2, \dots\}$. Now the *Bergman kernel function* $K : \Omega \times \Omega \to \mathbb{C}$ is defined by

$$K_\Omega(z; \zeta) := \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(\zeta)}.$$

We will also introduce a different definition which is more canonical in the sense that it is *a priori* independent of the choice of the orthonormal basis systems of the Bergman space.

Let $z \in \Omega$, and let $\Phi_z : \mathcal{A}^2(\Omega) \to \mathbb{C}$ be the point-evaluation at z defined by $\Phi_z(f) = f(z)$ for every $f \in \mathcal{A}^2(\Omega)$. By the Riesz Representation Theorem, there exists a holomorphic function $k_z^\Omega \in \mathcal{A}^2(\Omega)$ such that

$$f(z) = \int_\Omega f(\zeta)\overline{k_z^\Omega(\zeta)}\, d\mu(\zeta), \ \forall f \in \mathcal{A}^2(\Omega),$$

where $d\mu$ denotes the Euclidean volume form (or, equivalently, the standard Lebesgue measure) of $\mathbb{C}^n$.

PROPOSITION 10.1. *On a bounded domain Ω in $\mathbb{C}^n$, $K_\Omega(z, \zeta)$ converges absolutely and uniformly on compact subsets and in fact $K_\Omega(z, \zeta) = \overline{k_z^\Omega(\zeta)}$. In particular, the definition of the Bergman kernel function is independent of the choice of the orthonormal basis of $\mathcal{A}^2(\Omega)$.*

COROLLARY 10.2. *For a bounded domain Ω in $\mathbb{C}^n$,*

$$K_\Omega(z, \zeta) = \overline{K_\Omega(\zeta, z)} \text{ for all } (z, \zeta) \in \Omega \times \Omega.$$

COROLLARY 10.3. *For a bounded domain Ω in $\mathbb{C}^n$,*

$$f(z) = \int_\Omega K(z, \zeta) f(\zeta)\, d\mu$$

for all $f \in \mathcal{A}^2(\Omega)$.

EXERCISE 10.1. Compute the Bergman kernel function of the unit ball.

EXERCISE 10.2. Prove that the Bergman kernel function of a product domain is the product of the Bergman kernel functions of the components.

Denote by $J_\mathbb{C} f(z)$ the holomorphic Jacobian matrix of f whose jk-th entry is $(\partial f_j / \partial z_k)(z)$, where f_j represents the j-th component of the mapping f. Then we have

PROPOSITION 10.4. *Let $f : \Omega \to G$ be a biholomorphic mappings of between two bounded domains in $\mathbb{C}^n$. Then,*

$$K_G(f(z), f(\zeta)) = \det(J_{\mathbb{C}}f(z)) \cdot K_\Omega(z, \zeta) \cdot \overline{\det(J_{\mathbb{C}}f(\zeta))}$$

for $(z, \zeta) \in \Omega \times \Omega$.

10.2. The Bergman metric. Let Ω be a bounded domain in $\mathbb{C}^n$. Then it is easy to see that one can choose an orthonormal system by applying the Gramm-Schmidt process with respect to the L^2 inner product starting with the constant function 1. This in particular implies that $K_\Omega(z, z) > 0$ for all $z \in \Omega$. We will fix the domain Ω momentarily and denote simply by $K = K_\Omega$. Then we define the $(1,1)$-tensor as follows

$$g = g^\Omega = g_{j\bar{k}} dz^j \otimes d\bar{z}^k = \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \bar{z}_k} dz^j \otimes d\bar{z}^k. \tag{3}$$

Now we present

PROPOSITION 10.5. *The Hermitian $(1,1)$-tensor g in (3) above is positive definite.*

PROOF. This proof is entirely due to Bergman himself. Fix $z_0 \in \Omega$ and a nonzero tangent vector $u = u^1\frac{\partial}{\partial z_1} + \ldots + u^n\frac{\partial}{\partial z_n}$ at z_0. Then We will choose a special orthonormal basis system $\{\varphi_j\}_{j=0}^\infty$ for $A^2(\Omega)$ and show that $g(u, u) > 0$. Since the Bergman kernel function is independent of the choice of the orthonormal basis system for the Bergman space and since z_0 and u are chosen arbitrarily, this will show the positive definiteness of the tensor $g = g^\Omega$.

First choose $\varphi_0 \in A^2(\Omega)$ such that

- $\varphi_0(z_0)$ is a positive real value;
- $||\varphi_0||_{L^2} = 1$; and
- $f(z_0) < \varphi_0(z_0)$ whenever $f \in A^2(\Omega)$ satisfies the first two conditions φ_0 does.

Why would such φ_0 exist? Notice that for any $f \in A^2$ satisfying the first two conditions above φ_0 satisfies, one gets

$$\begin{aligned} f(z_0) &= \int_\Omega K(z_0, \zeta) f(\zeta)\, d\mu(\zeta) \\ &= \left| \int_\Omega K(z_0, \zeta) f(\zeta)\, d\mu(\zeta) \right|, \end{aligned}$$

since $f(z_0) > 0$;

$$\leq \left(\int_\Omega |K(z_0, \zeta|^2\, d\mu(\zeta) \right)^{\frac{1}{2}} \cdot 1,$$

by Hölder inequality and by $\|f\|_{L^2} = 1$;

$$= \left(\int_\Omega K(z_0, \zeta) K(\zeta, z_0)\, d\mu \right)^{\frac{1}{2}}$$
$$= K(z_0, z_0)^{\frac{1}{2}}.$$

This shows that $\varphi_0(z) = K(z, z_0)/\sqrt{K(z_0, z_0)}$. (*Verify!*)

Now we find the next member of the orthonormal basis system. First realize that the orthogonal complement of φ_0 in $A^2(\Omega$ consists of $f \in A^2(\Omega)$ such that $f(z_0) = 0$. (*Verify!*) Now we choose $\varphi_1 \in (\varphi_0)^\perp$ such that

- $\varphi_1(z_0) = 0, \|\varphi_1\|_{L^2} = 1$, and $d\varphi_1(z_0)u$ is a positive real value; and
- $df(z_0)u \leq d\varphi_1(z_0)u$ whenever $f \in A^2(\Omega)$ satisfies the first condition φ_1 does.

Proving the existence of such φ_1 could be slightly more technical, but it follows the similar ideas. So we omit the proof here. (First do it with $u = (0, \dots, 1, \dots, 0)$ and then think about creating one for an arbitrary u.)

Then consider the orthogonal complement in the Bergman space of two elements φ_0 and φ_1. (What is the characterization of this new orthogonal complement?) Simply choosing any orthonormal system there, complete the orthonormal system for the Bergman space $A^2(\Omega)$ so that

$$K(z, z) = \sum_{j=0}^{\infty} \varphi_j(z) \overline{\varphi_j(z)},$$

where $d\varphi_j(z_0)u = 0$ for all $j \geq 2$. Then by a simple calculation, one sees easily that $g(u, u) > 0$, as desired. (*Exercise.* Fill in the details.) □

DEFINITION 10.1. The positive definite Hermitian tensor g^Ω in the above is called the *Bergman metric* of Ω.

PROPOSITION 10.6. *Let $f : \Omega \to G$ be a biholomorphic mapping of bounded domains. Then $f^* g^G = g^\Omega$, i.e., $g^G_{f(z)}(df(z)X, df(z)Y) = g^\Omega_z(X, Y)$, for any $X, Y \in T_z\Omega$.*

This immediately implies

COROLLARY 10.7. *Let d_Ω and d_G denote the integrated Bergman distances of the domains Ω and G. Then any biholomorphic mapping from Ω onto G is indeed distance-preserving.*

REMARK 10.1. However, the Bergman metric/distance behaves rather irregularly with the general holomorphic mappings. Being a Kähler metric, the Bergman metric opens an important connection between Several Complex Variables and Riemannian Geometry, even though it does not conform with the general holomorphic mappings as nicely as the invariant Finslerian metrics such as the Carathéodory and Kobayashi metrics.

REMARK 10.2. There are very few known examples of invariant Kähler metrics. In fact, at least at this writing, we are not aware of any discovery of invariant Kähler metrics other than the Bergman metric and the Kähler-Einstein metric. In this lecture notes, we will concentrate on the Bergman metric when we lead into geometric discussions.

REMARK 10.3. Certain important facts concerning completeness, curvature behaviors, approximate expressions of the Bergman metrics are known. And yet the larger part of the features is yet to be discovered. We will discuss on this subject in the later chapters.

CHAPTER II

Domains with Noncompact Automorphism Groups

1. Perturbation of the domains and automorphism groups

One of the striking differences between the complex one dimension and the higher complex dimensions is the failure of the Riemann mapping theorem in complex dimension two or higher. The first example given by Poincaré is that there is no biholomorphic mapping between the ball in $\mathbb{C}^2$ and the bidisk in $\mathbb{C}^2$. Of course, the bidisk is the complex two dimensional polydisk which is the product of two unit open disks in $\mathbb{C}$. More recently, D. Burns, S. Shnider and R. Wells ([12]) proved

THEOREM 1.1 (D. BURNS, S. SHNIDER, R. WELLS). *The domains which are obtained by perturbing the boundary of the unit open ball in $\mathbb{C}^n, n \geq 2$ in C^∞ sense yields an infinite dimensional family of holomorphically distinct bounded strongly pseudoconvex domains. Moreover, an arbitrary dimensional family of such distinct domains can be chosen such that their automorphism groups are trivial, consisting of the identity alone.*

Keeping in mind that these domains are all diffeomorphic to the ball up to the boundaries, this theorem indicates not only that the Riemann mapping theorem in classical sense fails in an essential way, but also that the bounded domains with compact automorphism groups should be generic. On the other hand the last remark is only philosophical in the light of the above theorem of Burns, Shnider and Wells, because in the family of domains produced from perturbing the unit ball may contain some other domains with larger automorphism groups. For instance, the domain

$$\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 + \epsilon|w|^4 < 1\}$$

is close to the unit ball in $\mathbb{C}^2$ *in C^∞ sense* (which we will define in the following paragraph) for $\epsilon > 0$ sufficiently small. But, notice that this domain contains enough rotations in its automorphism group.

R.E. Greene and S.G. Krantz ([25]) later obtained a finer version of the above mentioned theorem of Burns-Shnider-Wells. To state their theorem precisely, as well as to make the terminologies in the preceding theorem, we introduce the concept of C^∞ perturbation of the boundary of the domains.

Let ρ_0 be a C^∞ smooth defining function for a bounded domain $\Omega_0 \in \mathbb{C}^n$. A domain $\Omega \in \mathbb{C}^n$ is called ϵ-close to Ω_0 in C^∞ sense if there exists a C^∞ smooth diffeomorphism $F : \overline{\Omega}_0 \to \overline{\Omega}$ such that

$$\|F - I\|_{C^k} < \epsilon$$

for every $k = 0, 1, 2, \ldots$, where the $\| \ \|_{C^k}$ denotes the maximum taken over all the supremum norms of all the mixed partials of order k of all the components of the given mapping. As usual I denotes the identity mapping.

EXERCISE 1.1. Let G_0 be a strongly pseudoconvex domain in $\mathbb{C}^n$. Show that there exists $\epsilon > 0$ such that any bounded domain with a C^∞ boundary that is ϵ-close to G_0 is strongly pseudoconvex.

Now we state

THEOREM 1.2 (GREENE-KRANTZ). *Let B^n be the unit open ball in $\mathbb{C}^n$, where $n \geq 2$. Then there exists $\epsilon > 0$ such that, for any Ω that is ϵ-close to B^n in C^∞ sense, either*

(1) *Ω is biholomorphic to B^n, or*
(2) Aut Ω *is compact, has a common fixed point, and admits a faithful unitary representation by a biholomorphism $\Phi : \Omega \to G \subset \mathbb{C}^n$.*

These theorems suggest in principle that the bounded domains admitting a noncompact automorphism group must be very rare. Hence it is natural to ask a question

> *Which bounded domains in $\mathbb{C}^n$ admit noncompact automorphism groups?*

The current chapter is devoted to this question. Before we begin to discuss any further details, we list in the below several known examples of bounded domains with noncompact automorphism groups.

- The bounded symmetric domains including the ball and the polydisks.
- Thullen domains of type

$$\{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + |z_2|^{r_2} + \ldots + |z_n|^{r_n} < 1\}.$$

- The examples by P. Griffiths: There exists a bounded domain in $\mathbb{C}^2$ which covers a compact variety but the automorphism group is completely discrete.
- An example by Greene and Krantz: There exists a bounded domain in $\mathbb{C}^n$ for every $n \geq 2$ whose automorphism group is isomorphic to $\mathbb{Z}$. This domain has a smooth boundary except at one point where the automorphism group accumulates.

2. Wong-Rosay Theorem

2.1. Statement and Background. In the standard realization table by E. Cartan ([13]) of the bounded symmetric domains, it is somewhat simple to realize that the open unit ball is the only kind of domains that admit globally smooth boundary. One obvious thing to try may be an effort to find a new holomorphic embedding of bounded symmetric domains other than the ball to produce a model that has globally smooth a boundary. The following theorem by B. Wong ([68]) and J.P. Rosay ([63]) demonstrates that such an attempt cannot be successful. This indicates another aspect of "rigidity" of the domains that admit noncompact automorphism groups, or the "persistence" of the singularities in the boundary in complex dimensions higher than one.

THEOREM 2.1 (WONG 1977; ROSAY 1979). *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 1$, which admits a point $q \in \Omega$ and a sequence of automorphisms $\{f_j\} \subset \operatorname{Aut}\Omega$ such that the sequence of points $f_j(q)$ accumulates at a boundary point $p \in \partial\Omega$ at which $\partial\Omega$ is C^2 smooth strongly pseudoconvex. Then Ω is biholomorphic to the open unit ball.*

This in particular implies

THEOREM 2.2 (WONG). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^2 smooth strongly pseudoconvex boundary. Then* $\operatorname{Aut}\Omega$ *is noncompact, if and only if it is biholomorphic to the open unit ball in $\mathbb{C}^n$.*

Historically speaking, B. Wong proved the preceding theorem in 1977. For the C^∞ smooth strongly pseudoconvex boundary case, P. Klembeck ([42]) presented about at the same time a completely different proof which is much more differential geometric. Then in 1979, J.P. Rosay strengthened the theorem and the proof of B. Wong and presented the full version of Theorem 2.1 stated above. Rosay's contribution is indeed an improvement because it implies the following

COROLLARY 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^2 smooth boundary. If it covers holomorphically a compact complex variety, then Ω is biholomorphic to the open unit ball.*

At this point, a theorem by S. Frankel ([22]) should be mentioned. The statement is

THEOREM 2.4 (S. FRANKEL). *If a bounded convex domain in $\mathbb{C}^n$ covers holomorphically a compact complex manifold, it is bounded symmetric.*

This theorem is unique in the sense that boundary smoothness is not at all assumed except for the convexity. On the other hand, the assumption on the automorphism groups is much stronger than mere noncompactness, because the hypothesis of the preceding theorem allows the automorphism orbit accumulate at every boundary point. Therefore, it does not fit into the frame of this lecture note, and hence we will simply refer the readers to [22] for further details. However, one of the methods Frankel introduces in the above mentioned paper is the "convex scaling technique" which plays a key role in proving the Product Domain Theorem in the later sections of this chapter. This method will be presented in detail.

2.2. Proof of Wong-Rosay Theorem by Scaling. In this section, we are going to introduce the scaling technique which was initiated by S. Pinchuk in the late 1970's ([60]). In doing so, we prove Wong-Rosay Theorem by Scaling Technique. In earlier papers of Pinchuk, this was called "the stretching coordinates." Similar, but different both in techniques and results, ideas precede Pinchuk's technique in many other branches of mathematics.

We clearly point out that the proof and the idea that follow are mainly due to Pinchuk. The original proof by Wong (which is later improved by Rosay) is completely different from the arguments we introduce in the below. However, we take such direction because of two reasons: (1) To demonstrate the idea of the Pinchuk scaling technique; (2) To connect a bridge to the weakly pseudoconvex cases generalizing Wong-Rosay Theorem to Bedford-Pinchuk Theorem, which is the main content of the next section.

Preparation for Scaling

To prove the Wong-Rosay Theorem, let us start with the bounded domain Ω with a boundary point $p \in \partial\Omega$ such that

(i) $\partial\Omega$ is C^2 strongly pseudoconvex at p; and
(ii) there exist a point $q \in \Omega$ and a sequence $\{\varphi_j\}_{j=1}^{\infty} \in \operatorname{Aut}\Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$.

Passing to a quadratic polynomial biholomorphic mapping of $\mathbb{C}^2$ mapping p to the origin after choosing an appropriate defining function of Ω, we may assume that p is the origin and that there exists an open ball U centered at the origin such that

$U \cap \Omega$ is in fact strongly convex. (See for instance Narasimhan's Lemma in p. 128–129 of [51].) Then by a complex linear change of coordinates, and by shrinking U to a smaller ball if necessary, the set $U \cap \Omega$ is represented by the inequality

$$\rho(z) = \Re e\, z_1 + |'z|^2 + o(|\Im m\, z_1| + |'z|^2) < 0 \tag{4}$$

where

$$\Re e\, \alpha = \frac{\alpha + \bar{\alpha}}{2}$$
$$\Im m\, \alpha = \frac{\alpha - \bar{\alpha}}{2i}$$

for any complex number α, and where

$$'z = (z_2, \ldots, z_n)$$
$$|'z|^2 = |z_2|^2 + \ldots + |z_n|^2$$

Now we prove the following localization lemma:

LEMMA 2.5. *Let Ω, p, q and φ_j be as above. Then for any neighborhood V of p and for any compact subset K of Ω there exists $N > 0$ such that*

$$\varphi_j(K) \subset V \cap \Omega.$$

Furthermore, for any $v \in \mathbb{C}^n$,

$$\lim_{j \to \infty} |d\varphi(q)v| = 0.$$

PROOF. Notice that the sequence $\{\varphi_j\}$ is uniformly bounded, since $\varphi_j(\Omega) = \Omega$ and Ω is bounded. Therefore, every subsequence of $\{\varphi_j\}$ admits a subsequence which converges uniformly on compact subsets to a holomorphic function, say $\varphi : \Omega \to \overline{\Omega}$, with $\varphi(q) = p \in \partial\Omega$. Since the strongly pseudoconvex boundary point does not admit any nontrivial analytic variety passing through it (why?), $\varphi(z) = p$ for all $z \in \Omega$. Since this convergence is uniform on K in particular, the first assertion follows rather easily.

For the second assertion, introduce a fixed number $r > 0$ such that $h(z) = q + rzv \in \Omega$ for all $z \in \mathbb{C}$ with $|z| \leq 1$. Then apply the normal family argument to $\varphi_j \circ h : B \to \Omega$ as above. (Here, as before, B denotes the open unit disk in $\mathbb{C}$.) If there exists a subsequence $\{\varphi_{j_k}\}_k$ such that

$$|d\varphi_{j_k}(q)v| \geq \delta > 0 \tag{5}$$

for some fixed $\delta > 0$ independent of k. Then choose a subsequence again so that a subsequence of $\varphi_{j_k} \circ h$ converges to a mapping, $G : B \to \overline{\Omega}$ with $G(0) = p$. By the standard normal family argument, $G'(0) \neq 0$. This means that there exists a nontrivial analytic curve contained in $\overline{\Omega}$ passing through p. As above this is impossible, and so the proof is complete. □

REMARK 2.1. The assertions above as well as the proof are valid for the boundary point other than strongly pseudoconvex points. As one sees from the proof, non-existence of the non-trivial analytic varieties in the closure of the domain passing through the boundary point in consideration is sufficient for the conclusion and the proof.

Pinchuk's Scaling Method

Continuing from the preceding section and the Wong-Rosay Theorem, let us denote by

$$p_\nu = \varphi_\nu(q), \text{ for each } \nu = 1, 2, \ldots$$

Recall that $p_\nu \to p = 0$ as $\nu \to \infty$ and that the domain Ω in consideration is defined, at least near $p = 0$, by the inequality (4). Write $p_\nu = (p_1^\nu, \ldots, p_n^\nu)$. Then, for each ν, consider the biholomorphic change of complex coordinates of $\mathbb{C}^n$ by

$$\begin{cases} \hat{z}_1 = e^{i\theta_\nu} z_1 - p_\nu^* - \sum_{\ell=2}^n a_\ell (z_\ell - p_\ell^\nu) \\ \hat{z}_\ell = z_\ell - p_\ell^\nu \qquad \text{for } \ell = 2, \ldots, n \end{cases} \tag{6}$$

where $p_\nu^* \in \mathbb{C}$ and $a_\ell \in \mathbb{C}$ are chosen so that in the coordinates $(\hat{z}_1, \ldots, \hat{z}_n)$ we have

- the point $(0, \ldots, 0) \in \partial\Omega$,
- p_ν is given by $(-\epsilon_\nu, 0, \ldots, 0)$ for some $\epsilon_\nu > 0$ for every ν; and
- the tangent plane to $\partial\Omega$ at $(0, \ldots, 0)$ is given by $\Re e\, z_1 = 0$.

Now the defining function (4) in new coordinates is given by

$$\hat{\rho}_\nu(\hat{z}) = \hat{c}_\nu \Re e \left(\hat{z}_1 + \sum_{\ell=1}^n A_\ell^\nu \hat{z}_\ell^2 \right) + \sum_{k,\ell=1}^n B_{k\ell}^\nu \hat{z}_k \bar{\hat{z}}_\ell + E_\nu(\hat{z}) \tag{7}$$

where $E_\nu(\hat{z}) = o(|\Im m\, \hat{z}_1| + |'\hat{z}|^2)$, and where the coefficients of the quadratic terms converge to the corresponding quadratic terms of the defining function ρ in (4) as $\nu \to \infty$. Moreover, $c_\nu \to 1$ as $\nu \to \infty$. Let D_ν be the intersection of U (as in the paragraph of (4) and the domain defined by $\hat{\rho}_\nu < 0$ above.

Now we finally introduce the Pinchuk's scaling sequence

$$\begin{cases} \tilde{z}_1 = \hat{z}_1 / \epsilon_j \\ \tilde{z}_\ell = \hat{z}_\ell / \sqrt{\epsilon_j} \quad \text{for } \ell = 2, \ldots, n \end{cases} \tag{8}$$

Let us denote by $L(\hat{z}) = \tilde{z}$. Clearly, L is a complex linear isomorphism of $\mathbb{C}^n$, and hence in particular is a biholomorphic mapping. Since $\epsilon_\nu \to 0$ as $\nu \to \infty$, the defining function of the domain $L(D_\nu)$ may be normalized as

$$(9) \qquad \rho_\nu(z) = c_\nu \Re e \left(z_1 + \sum A_\ell^\nu z_\ell^2\right) + \sum B_{k\ell}^\nu z_k \bar{z}_\ell + \epsilon_\nu^{-1} E_\nu(\epsilon_\nu z_1, \sqrt{\epsilon_\nu}\,'z) < 0.$$

Form the above, it is clear that the limiting defining function as $\nu \to \infty$ is defined by

$$(10) \qquad \hat{\rho}(z) = \Re e\, z_1 + |'z|^2 < 0.$$

It is not hard to see that in fact the domains $L(D_j)$ converges to the domain $\hat{\Omega}$ defined by (10) above, in the sense of local Hausdorff set convergence. Notice that $\hat{\Omega}$ is biholomorphic to the open unit ball.

Now here is the crucial step. Combining all the complex coordinate changes above in this section as well as in the previous section, we arrive at an injective holomorphic mapping for each ν, say,

$$g_\nu : U \cap \Omega \to L(D_\nu).$$

Let K be a compact subset of $\hat{\Omega}$. Then there exists $N > 0$ such that $K \subset\subset L(D_\nu)$ for all $\nu > N$. So we consider the sequence of injective holomorphic mappings

$$G_\nu := \varphi_\nu^{-1} \circ g_\nu^{-1} : K \to \Omega.$$

Since Ω is bounded and since K is an arbitrary compact subset of $\hat{\Omega}$, a subsequential limit will yield a holomorphic mapping, say $G : \hat{\Omega} \to \Omega$. Now we present

LEMMA 2.6. *$G : \hat{\Omega} \to \Omega$ is a biholomorphism.*

Notice that this lemma implies the Wong-Rosay Theorem.

PROOF. Observe that $\det dG_\nu$ is never zero at any point of K since G_ν is one-to-one and holomorphic. Therefore, Hurwitz's Theorem implies that $\det dG$ is either identically zero or nowhere zero on $\hat{\Omega}$. Let $q_0 = (-1, 0, \dots, 0)$. By choices of our D_ν, it is simple to observe that the Kobayashi metrics satisfy the following uniform estimate

$$F_K^{L(D_\nu)}(q_0, \xi) \geq \delta|\xi|$$

for all $\xi \in \mathbb{C}^n$ for some $\delta > 0$ independent of ν. This implies that $\det dG(q_0) \neq 0$. Notice that $\det dG$ is a holomorphic function which is at the same time the limit of nowhere vanishing holomorphic functions which converges uniformly on compact subsets. Applying Hurwitz's Theorem, we can deduce easily that $\det dG$ is nowhere zero. So $G : \hat{\Omega} \to \Omega$ is locally one-to-one. From the uniform convergence on compact subsets, it is easy to see that G is globally one-to-one.

It remains to show that G is onto. To see this, we remark that $G_\nu(q_0) = q$ for all ν in our construction. Now, observe that the Kobayashi distance of Ω is Cauchy complete. (Exercise.) By choosing U above small, realize that all the domains $L(D_\nu)$ stays inside a fixed domain defined by $\Re e\, z_1 + r|'z|^2 < 0$. And hence by the distance decreasing property, the image of any fixed compact set under $g_\nu \circ \varphi_\nu$ is indeed uniformly bounded for all ν. Then it is an easy exercise to construct an inverse holomorphic mapping of G from Ω to $\hat{\Omega}$. Therefore, the proof is complete. □

EXERCISE 2.1 (HURWITZ'S THEOREM). Let Ω be a domain in $\mathbb{C}^n$ and let $f_j : \Omega \to \mathbb{C}$ form a sequence of holomorphic functions that converges to $f : \Omega \to \mathbb{C}$ uniformly on compact subsets. Assume further that none of the functions f_j vanishes anywhere on Ω. Show that either f is nowhere zero or it is identically zero.

REMARK 2.2. If one knows beforehand that the point sequence $\varphi_j(q)$ approaches p inside a cone with vertex at p contained in Ω, then the adjustment such as (6) would be totally unnecessary.

REMARK 2.3. The above proof is much more descriptive than the original proof of Wong which uses the quotient of the Carathéodory-Eisenmann volume and the Kobayashi-Royden volume.

REMARK 2.4. There are other proofs of the Wong-Rosay theorem. We will introduce a Differential Geometric proof in the final chapter following the discussions of the Bergman curvatures.

3. Finite Type Cases

Consider now the following domains, usually known as the *Thullen domains*

$$E_{2m} := \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^{2m} < 1\} \tag{11}$$

for $m = 1, 2, \ldots$. The boundary of this domain is not strongly pseudoconvex everywhere. In fact, the Levi form of the defining function degenerates along the curve defined by the equations $w = 0, |z| = 1$. Furthermore, this domain admits the automorphisms of the following form

$$(z, w) \mapsto \left(\frac{z + a}{1 + \bar{a}z}, \left(\frac{\sqrt{1 - |a|^2}}{1 + \bar{a}z} \right)^{\frac{1}{m}} \right)$$

for any branch of the m-th root in the expression. Therefore, the automorphism group is noncompact. It is also known that this domain is not homogeneous.

Hence, Wong-Rosay Theorem of the preceding section yields that the orbit of a point of E_{2m} can only accumulate at the weakly pseudoconvex points.

If one considers the bounded domains in $\mathbb{C}^n$ with a globally C^∞ smooth boundary, then the Thullen domains E_{2m} above are the only examples that are known by far. Therefore, a natural question to ask is if the Thullen domains constitute the complete list of the bounded domains in $\mathbb{C}^n$ with a C^∞ smooth boundary that admit a noncompact automorphism group.

3.1. Characterization of the Thullen Domains. The first result in this direction is the following theorem of R.E. Greene and S.G. Krantz ([26]) later improved by K.T. Kim ([35]) to the present form:

THEOREM 3.1 (GREENE-KRANTZ (1987), KIM (1989)). *Let Ω be a bounded domain in $\mathbb{C}^2$ with a boundary point $p \in \partial\Omega$ such that*

(i) *there exist a point $q \in \Omega$ and a sequence of automorphisms φ_j such that $\lim_{j\to\infty} \varphi_j(q) = p$;*
(ii) *there exist neighborhoods U of p in $\mathbb{C}^2$ and V of $(1,0)$ in $\mathbb{C}^2$ and a diffeomorphism $F : U \cap \overline{\Omega} \to V \cap \overline{E_{2m}}$ which is holomorphic on $U \cap \Omega$ and satisfies $F(U \cap \partial\Omega) = V \cap \partial E_{2m}$ and $F(p) = (1,0)$.*

Then, Ω is biholomorphic to E_{2m}.

Original theorem of Greene and Krantz was proven under the extra hypothesis that $\partial\Omega$ is globally C^3 smooth. Kim gave a new proof using the scaling technique (modified Frankel scaling) and also removed the global boundary regularity assumption. Then A. Kodama, in [48] and other papers, obtained various generalization for the generalized Thullen domains which may have singularities in the boundary, by using Pinchuk's scaling method combined with other techniques. After the original version of the above theorem of Greene-Krantz was presented, about the best result one can hope for in complex dimension two was presented by Bedford and Pinchuk. From the following sections, we would like to present the theorem of Bedford and Pinchuk.

3.2. Statement of Bedford-Pinchuk Theorem. Even if there are more recent papers of Bedford and Pinchuk which contain further generalizations to higher dimensions, we would like to restrict our attention, in this note, to the complex dimension two. For the readers who would like to know more details and further developments in this direction, we refer them to [6] and [7].

THEOREM 3.2 (BEDFORD-PINCHUK). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with C^∞ smooth boundary such that every boundary point is of finite type. If the automorphism group of Ω is noncompact, then Ω is biholomorphic to a Thullen domain E_{2m} for some positive integer m.*

Even if the above mentioned theorem of Greene-Krantz is *not* a consequence of this theorem (because of the global finite type assumption), it appeals to us that this theorem of Bedford and Pinchuk deserves more attention. Therefore, we would like to discuss in detail the proof of this theorem in the following sections.

REMARK 3.1. The first version of the theorem above was initially proven with the stronger assumption that the boundary of Ω is real analytic. An argument needed to establish the biholomorphic equivalence between the bounded domain and the scaled domain (See the proof below, please) was not given in [5], and it was pointed out by Berteloot and Cœuré and proven in [10]. Another proof is given by Bedford and Pinchuk in [6]. S. Bell and D. Catlin (unpublished) discovered how to generalize the result to C^∞ smooth finite type boundary cases. Lemma 3.9 in this note (which is Lemma 6 of [6]) was essentially from the ideas of Bell and Catlin. Most recently, Bedford and Pinchuk obtained a higher dimensional generalization for bounded convex domains with finite type boundaries ([7]).

REMARK 3.2. The final problem to solve is *how to localize* the hypothesis in the Bedford-Pinchuk theorem. Namely, one would hope to prove the same conclusion under the hypothesis that there exists an automorphism orbit accumulating at a point of finite type without assuming the global finite type condition on the boundary. At this writing, this problem is yet to be answered.

In what follows, we present the proof of the Bedford-Pinchuk Theorem in several steps.

3.3. Preliminary Scaling. Keeping in mind the scaling arguments we presented in the proof of the Wong-Rosay theorem in the preceding section, we will also perform the scaling at the orbit accumulation point.

First we normalize the defining function. By a linear change of coordinates, let us assume that

- $p_\infty = (0,0) \in \partial\Omega$, and
- there exist $q \in \Omega$ and $\{\varphi_j\} \subset \operatorname{Aut}\Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p_\infty$.

and the tangent plane to $\partial\Omega$ at the origin is $\{\Im m\, w = 0\}$ in $\mathbb{C}^2$. Therefore, near the origin, the defining function of Ω may be given by

$$r(z,w) = v + a(z,\bar{z},u) < 0$$

where $u = \Re e\, w$ and where $v = \Im m\, w$. Now, we may write

$$a(z,\bar{z},u) = a_0(z,\bar{z}) + O(|uz|).$$

Notice that $a_0(z,\bar{z})$ cannot vanish to infinite order at $z = 0$, since every boundary point of Ω is of finite type. In fact, if the origin p_∞ is the point of type m, then up to a holomorphic change of local coordinates, we may assume that a_0 must vanish to order μ, and for any holomorphic coordinate change a_0 cannot vanish to order any higher than μ. (Such an argument is in general false in higher dimensions.) Let us assume that a_0 vanishes to order μ at the origin. Then, let

$$\psi(z,\bar{z}) = \sum_{k+\ell=\mu} a_{k\ell} z^k \bar{z}^\ell.$$

Then not all the coefficients of the non-harmonic monomials in the expression of ψ can be zero. For, otherwise, the type of the boundary at the origin cannot be μ. Finally, Ω being pseudoconvex, μ must be even. (Exercise: *Verify!*) Therefore, we have the following new (local) defining function ρ of Ω near the origin:

$$\rho(z,w) = v + \sum_{\ell=1}^{2m} a_\ell z^\ell \bar{z}^{2m-\ell} + o(|u| + |uz| + |z|^{2m}). \tag{12}$$

In the above, observe that $\bar{a}_\ell = a_{2m-\ell}$ and not all the values of a_ℓ are zero.

Now we will perform the first scaling. Let

$$p_\nu = \varphi_\nu(q) = (z_\nu, w_\nu).$$

Introduce the new holomorphic coordinates of $\mathbb{C}^2$ by

$$\begin{cases} \hat{z} = z - z_\nu \\ \hat{w} = e^{i\theta_\nu} w - w_\nu^* - b_\nu(z - z_\nu) \end{cases} \tag{13}$$

where $w_\nu^*, b_\nu \in \mathbb{C}$ and $\theta_\nu \in \mathbb{R}$ are chosen such that in the new coordinate system

- $(0,0) \in \partial\Omega$,
- $p_\nu = (-i\epsilon_\nu, 0)$ for some $\epsilon_\nu > 0$, and
- the tangent to $\partial\Omega$ at $(0,0)$ is $\{\Im m\, \hat{w} = 0\}$.

This can be achieved as follows: First, choose w_ν^* so that $(0,0) \in \partial\Omega$, then choose b_ν such that the real line segment joining $w - w_\nu^* - b_\nu(z - z_\nu)$ to the origin is complex perpendicular to $\hat{z}$-plane. Then choose the correct rotation factor (which may change the values of w_ν^* and b_ν, respectively) so that all three conditions are satisfied.

The (local) defining inequality of Ω near $(0,0)$ in the new coordinates is

$$c_\nu \hat{v} + \sum_{k=2}^{2m} \hat{\psi}_{k,\nu}(\hat{z}, \bar{\hat{z}}) + E_\nu(\hat{u}, \hat{z}) < 0$$

where:

- $\hat{w} = \hat{u} + i\hat{v}$,
- $E_\nu(\hat{u}, \hat{z}) = o(|\hat{u}| + |\hat{u}\hat{z}| + |\hat{z}|^{2m})$, and
- $\hat{\psi}_k$ is a polynomial in $z, \bar{z}$ with homogeneous degree k, for each k.

Introduce the "stretching coordinates"

$$(14) \qquad \begin{cases} z = \hat{z}/\delta_\nu \\ w = \hat{w}/\epsilon_\nu \end{cases}$$

For each ν, consider the function

$$(15) \qquad \rho_\nu(z,w) = c_\nu \Im m\, w + \frac{1}{\epsilon_\nu}\sum_{k=2}^{2m} \delta_\nu^k \hat{\psi}_{k,\nu}(z,\bar{z}) + \frac{1}{\epsilon_\nu} E_\nu(\epsilon_\nu \Re e\, w, \delta_\nu z)$$

Choose δ_ν, for each ν, such that the largest absolute value of the coefficients of the monomials in the expression of $\sum \epsilon_\nu^{-1}\delta_\nu^k \hat{\psi}_{k,\nu}$ is 1.

Notice that by our choice of coordinate change in (13) above, $\hat{\psi}_{2m,\nu}$ converges to ψ_{2m} uniformly on compact subsets, as $\nu \to \infty$. Consequently,

$$\sup_\nu \frac{\delta_\nu^{2m}}{\epsilon_\nu} < \infty.$$

Due to the growth condition on E_ν, passing to a subsequence if necessary, we may conclude that ρ_ν converges to

$$(16) \qquad \rho = \Im m\, w + P(z,\bar{z}),$$

uniformly in C^∞ norm on compact subsets of $\mathbb{C}^2$, where P is a polynomial in $z, \bar{z}$ of degree at most $2m$. (Exercise: *Verify!*)

At this point, we define by

$$D_\nu = \{(z,w) \in \mathbb{C}^2 \mid \rho_\nu(z,w) < 0, |z| < \nu, |w| < \nu\}$$

for each ν.

Now we need the following technical lemma.

LEMMA 3.3. *In the above, P is subharmonic such that $\partial^2 P/\partial z \partial \bar{z}$ does not vanish identically. Moreover, D admits a peak function at infinity, i.e. there exists a holomorphic function $f : D \to \mathbb{C}$ such that $|f| < 1$ at every point of D and such that* $\lim_{D \ni p \to \infty} f(p) = 1$.

PROOF. Being the uniform limit of psh functions ρ_ν in C^∞ topology on compact subsets, ρ is psh. Therefore, P is subharmonic. Since P has no pure harmonic terms, $\partial^2 P/\partial z \partial \bar{z}$ cannot vanish identically. Now, notice that the degree of P has to be even since P is subharmonic.

Let us denote by P_{2m} the top degree homogeneous part of P. The existence of the peak function at infinity follows from the construction of peaking functions by Bedford and Fornaess ([4]) for the domain

$$\tilde{D} := \{(z,w) \in \mathbb{C}^2 \mid \Im m\, w + P_{2m}(z,\bar{z}) < \epsilon(|w| + |z|^{2m}) + C\}$$

for a sufficiently small $\epsilon > 0$ and an arbitrary $C > 0$. Notice that for a given $\epsilon > 0$, $D \subset \tilde{D}$ if C is large enough. Let us focus on $\tilde{D}$. By a linear change of coordinates, we may assume that $C = 0$. According to [4], there exists a holomorphic function $h : \tilde{D} \to \mathbb{C}$ satisfying

- For some constants $0 < c < C$, $c(|z|^{2m} + |w|) \leq |h(z,w)| \leq C(|z|^{2m} + |w|)$, for all $(z,w) \in \tilde{D}$;
- There exists an integer $N > 1$ such that a branch of $\sqrt[N]{h}$ exists and $|\operatorname{Arg} \sqrt[N]{h}| \leq \pi/4$; and
- $\exp(-\sqrt[N]{h})$ is holomorphic on $\tilde{D}$.

Notice that $\exp(\sqrt[N]{h})$ is then the peak function at infinity for both $\tilde{D}$. Changing the coordinates back to the original, one gets the second assertion. □

We are now at the final phase of the preliminary scaling. For each ν, consider the mapping $G_\nu : \mathbb{C}^2 \to \mathbb{C}^2$ which is the composition of the linear biholomorphic mappings introduced in the form of change of the local coordinate systems (13) followed by (14) in the above.

Let K be a compact subset of D. Notice that there exists $N > 0$ such that $K \subset D_\nu$ for every $\nu > N$. Now for such ν, define $g_\nu : K \to \Omega$ by

$$g_\nu = \varphi_\nu^{-1} \circ G_\nu^{-1}.$$

This mapping is well-defined if one chooses a larger value for N. Notice that $g_\nu(-i,0) = q$ for all ν by construction. Since Ω is bounded, passing to a subsequence, we may assume that g_ν converges to a holomorphic mapping $g : D \to \overline{\Omega}$ uniformly on compact subsets of D. Obviously, $g(-i,0) = q$. Now we conclude the preliminary scaling with the following

LEMMA 3.4. *$g(D) = \Omega$ and $g : D \to \Omega$ is a biholomorphic mapping.*

PROOF. Note that $g(-i,0) = q$. Hence, by maximum principle, $g(D) \subset \Omega$. By [18], there exists a smooth function α with $\alpha(0) = 0$ such that $\tilde{r} = -e^{\alpha}(-r)^{\delta}$ is a psh exhaustion function for Ω, where r is the defining function of Ω we began with. Then

$$\tilde{\rho}_\nu = -e^{\alpha \circ g_\nu}(-r \circ g_\nu)^{\delta} \epsilon_\nu^{-\delta} = -e^{\alpha \circ g_\nu}(-\rho_\nu)^{\delta}$$

is psh. Moreover, $\tilde{\rho}_\nu$ converges uniformly in C^∞ topology on compact subsets to $\tilde{\rho} := -(-\rho)^{\delta}$ as $\nu \to \infty$. Now choose a point $q_0 \in D$ such that $g(q_0) \in \Omega$ and $e^{\tilde{\rho}}$

is strictly psh at q_0. For ν sufficiently large, $e^{\tilde{\rho}_\nu}$ is uniformly psh at q_0. Hence, we get a uniform lower bound for the Sibony metric

$$F_S^{D_\nu}(q_0, \xi) \geq c|\xi|, \text{ for all } \xi \in \mathbb{C}^n$$

for some $c > 0$ independent of ν. Due to the distance decreasing property, and the fact that $F_S^{D_\nu}$ is strictly positive definite, if follows that $dg(q_0)$ is nonsingular. Since every g_ν is one-to-one and since $g_\nu \to g$ uniformly on compact subsets as $\nu \to \infty$, g is (globally) one-to-one. Namely, $g : D \to g(D)$ is a biholomorphic mapping.

Now we show that $g(D) = \Omega$. First, choose small neighborhoods B_0 of q_0 and B_1 of $g(q_0)$ such that $g_\nu(B_0) \subset B_1$ for ν large. We choose $s > 0$ such that $\tilde{\rho} < -s$ on $\overline{B_0}$ (and thus the same inequality holds for $\tilde{\rho}_\nu$ for large ν). Now let $h : \Omega \setminus \overline{B_0} \to \mathbb{R}$ be the harmonic function such that $h = -c$ on ∂B_0 and $h = 0$ on $\partial\Omega$. By the Hopf lemma, there exists a constant $\epsilon > 0$ such that $-\epsilon \operatorname{dist}(p, \partial\Omega) > h(p)$ for $p \in \Omega$. For an arbitrary $\epsilon' > 0$, choose a compact subset K of Ω such that $h > -\epsilon'$ outside K. Hence, for ν sufficiently large, we have

$$-\epsilon \operatorname{dist}(p, \partial\Omega) > h > \tilde{\rho}_\nu(g_\nu^{-1}(p)) - \epsilon'$$

for $p \in K$, since the last item above is subharmonic with a smaller boundary data. If $p \in g(D)$, we may pass to the limit as $\nu \to \infty$ and obtain the same inequality for $\tilde{\rho}$ with $\epsilon' = 0$. On the other hand, the defining function ρ for D satisfies that for a constant $R > 0$ there exists $\eta > 0$ such that

$$-\eta \operatorname{dist}(q, \partial D) < \rho(q), \text{ for all } q \in D \text{ with } |q| < R.$$

We conclude then, by a simple calculation, that for $q \in D$ with $|q| < R$,

$$\epsilon^{1/\delta}\eta \operatorname{dist}(g(q), \partial\Omega)^{1/\delta} \leq \operatorname{dist}(q, \partial D). \tag{17}$$

This can be roughly summarized that g maps the boundary of D to the boundary of Ω. Now, we would like to finish the proof of the lemma. Suppose g is not onto. Then, for any $p_0 \in \Omega \cap \partial(g(D))$, the preceding estimate implies that

$$\lim_{p \to p_0} g^{-1}(p) = \infty$$

where $p \in g(D)$. Let f be the peak function for D at infinity constructed in Lemma 3.3 above, then

$$f \circ g^{-1}(p) \to 1$$

as $p \to p_0$, $p \in g(D)$. By Rado's Theorem (See [30]), $f \circ g^{-1}$ extends holomorphically onto Ω if we set $f \circ g^{-1}$ to be identically 1 on $\Omega \setminus g(D)$. However, since $|f \circ g^{-1}| < 1$ on $g(D)$, the extended function has modulus at most 1 on Ω. This implies that $\Omega \setminus g(D)$ is empty, by the Maximum Principle. This complete the proof. □

Notice that the assertion of the Bedford-Pinchuk theorem does not follow immediately from this lemma. It is mainly because it is not obvious at this point why the scaled model has to be biholomorphically equivalent to one of the Thullen domains.

To achieve such a goal, Bedford and Pinchuk study a new automorphism orbit which is obtained as a result of the preliminary scaling. In the following section, we will observe that at the boundary point of Ω at which the new automorphism group accumulates, one obtains a noncompact holomorphic vector field action in the boundary of Ω which results in that the defining function of Ω at the above mentioned orbit accumulation boundary point is has the lower order terms that resemble the defining function of a Thullen domain.

It is still a step away from the final conclusion. But, investigating closely the behavior of the above mentioned automorphism orbit, Bedford and Pinchuk rescale the domain repeating the same procedure we introduced in this section and gets the desired result.

So, we now introduce the new automorphism orbit and the induced holomorphic tangent vector field action on $\overline{\Omega}$ in the following section.

3.4. Parabolic flow. Notice that on the scaled domain D there exists a noncompact one parameter group action by translations

$$L_t : (z, w) \mapsto (z, w + t)$$

for all $t \in \mathbb{R}$. The biholomorphism $g : D \to \Omega$ then yields a one parameter family of holomorphic automorphisms

$$h_t := g \circ L_t \circ g^{-1}.$$

LEMMA 3.5. *The group $\mathcal{H} = \{h_t \,|\, t \in \mathbb{R}\}$ is parabolic, i.e. there exists a point $p \in \Omega$ such that*

$$\lim_{t\to-\infty} h_t(q) = p = \lim_{t\to\infty} h_t(q), \text{ for every } q \in \Omega.$$

In the sequel, p is called the parabolic point.

PROOF. By the localization lemma (Lemma 2.5 in page 33), it is enough to show that

$$\lim_{t\to-\infty} f(0, -i + t) = \lim_{t\to\infty} f(0, -i + t).$$

By the lower bound estimate for the Kobayashi metric in [4] and [15], and the obvious estimate of the Kobayashi metric on the plane $z = 0$ in D, we have

$$F_K^D((0, w), (0, \xi)) \leq c_1 |X| / |\Im m\, w| \tag{18}$$

$$F_K^\Omega(p, Y) \geq c_2 |Y| (\text{dist}(z, \partial\Omega))^{-\epsilon} \tag{19}$$

for some positive constants c_1, c_2 and ϵ. Moreover, due to the existence of the holomorphic function $h : D \to \mathbb{C}$ such that

- h extends continuously across the boundary of D;
- $c_3(|z|^{2m} + |w|)^{1/N} \leq |g(z,w)| \leq c_4(|z|^{2m} + |w|)^{1/N}$ on D; and
- $\operatorname{Arg} g(z,w) \in [-\pi/4, \pi/4]$.

Now consider the function

$$\varphi(z,w) = \left|\frac{g(z,w)-1}{g(z,w)+1}\right|^2 - 1$$

which is negative psh on D satisfying the estimate

$$|\varphi(0,w)| \sim |w|^{-1/N} \tag{20}$$

on D. (See the proof of Lemma 3.4.) Applying the Hopf lemma to $\varphi \circ g^{-1} : \Omega \to \mathbb{C}$, one gets

$$|\varphi \circ g^{-1}(p)| \geq c_5 \operatorname{dist}(p, \partial\Omega).$$

Combining this with (20) above, we get

$$\operatorname{dist}(g(0,w), \partial\Omega) \leq c_6 |w|^{-1/N}. \tag{21}$$

From (18), (19), (21), and the invariance of the Kobayashi metric under biholomorphisms, one gets

$$\left|\frac{\partial g_\ell}{\partial \zeta_k}(0,w)\right| \leq c|\Im m\, w|^{-1}|w|^{-\epsilon/N} \tag{22}$$

for all $\ell, k = 1, 2$ where $(\zeta_1, \zeta_2) = (z, w)$. The convergence of the improper integral $\int_1^\infty x^{-1-\epsilon/N} dx$ implies that there exists a point $p_\infty \in \partial\Omega$ such that

$$\lim_{t\to\infty} g(0, -i-it) = p_\infty.$$

Finally, the derivative estimate in (22) implies that

$$\begin{aligned} |g(0,-i\pm t) - g(0,-i-it)| &\leq c\int_0^t \frac{ds}{(1+s)t^{\epsilon/N}} \\ &= ct^{-\epsilon/N}\ln(1+t) \to 0 \text{ as } t \to \infty. \end{aligned}$$

This finishes the proof. □

Notice that together with the estimate (17) in page 42, the lemma above implies that the biholomorphism $g : D \to \Omega$ extends to a homeomorphism between $\overline{D} \cup \{\infty\}$ and $\overline{\Omega}$.

Parabolic Vector Field Action

By [10] and [18], every automorphism of Ω extends to a diffeomorphism of $\overline{\Omega}$. Therefore, it make sense to define the holomorphic vector fields $H = (H_1, H_2)$ acting on $\overline{\Omega}$ induced by $\mathcal{H}$ in the sense that

$$H(p) = \frac{dh_t(p)}{dt}\Big|_{t=0}.$$

Note that $H(p_\infty) = 0$, by construction. Also by definition, $\Re e\, H$ is tangent to the boundary $\partial\Omega$ of the bounded domain Ω. Hence, for any $\zeta \in \partial\Omega$,

$$\Re e \left(\frac{\partial\rho(\zeta,\bar\zeta)}{\partial\zeta_1} H_1(\zeta) + \frac{\partial\rho(\zeta,\bar\zeta)}{\partial\zeta_2} H_2(\zeta) \right) = 0 \tag{23}$$

where ρ is a defining function of Ω. In what follows, without loss of generality we set $p_\infty = 0$. Expand ρ near 0 as in (12) in page 39:

$$\rho = \Im m\, w + \psi(z,\bar z) + o(|w| + |wz| + |z|^{2m}), \tag{24}$$

where ψ is a subharmonic polynomial with the homogeneous degree $2m$ which does not have any pure harmonic terms. Then we observe the series of lemmas in the following. We do not include the proofs of the first three lemmas here and simply refer to pages 146–147 of [5].

LEMMA 3.6. *For all z we have*

$$\Re e\,(z\psi(z,\bar z)) \equiv m\psi(z,\bar z).$$

Furthermore, if $\Im m\,(z\psi(z,\bar z)) \equiv a\psi(z,\bar z)$, then $a = 0$, and $\psi(z,\bar z) = c|z|^{2m}$.

PROOF. This follows from elementary computations on coefficients. We leave the details as an exercise. □

LEMMA 3.7. *Suppose that $\Im m\, w = -\psi(z,\bar z)$ and that*

$$\Re e \left(azw^{k-1}\frac{\partial\psi}{\partial z} + bw^k \right) = 0,$$

where the complex numbers a and b are not zero at the same time. Then, k is either 1 or 2; and

(a) *If $k = 1$, then $b = 2m\Re e\, a$; and if $a \neq 0$ then $\psi(z,\bar z) = c|z|^{2m}$.*
(b) *If $k = 2$, then $\psi(z,\bar z) = c|z|^{2m}$, $b = ma$ and both a and b are purely imaginary.*

PROOF. Exercise. □

Now assign weight 1 to the variable z and weight $2m$ to w.

LEMMA 3.8. *Let $Q_1(z,w)$ and $Q_2(z,w)$ be weighted homogeneous polynomials with weights q and $q+2m-1$ respectively. Suppose that*

$$\Re e\left(Q_1(z,w)\frac{\partial\psi}{\partial z}(z,\bar{z})+Q_2(z,w)\right)=0$$

holds for $\Im m\, w=-\psi(z,\bar{z})$. Then q must be either 1 or $2m+1$, and

(a) *If $q=1$, then $Q_1=az$ and $Q_2=2mbw$ with $b=\Re e\, a$ for some $a\in\mathbb{C}$. Further, if $\Im m\, a\neq 0$, then $\psi(z,\bar{z})=c|z|^{2m}$.*
(b) *If $q=2m+1$, then $Q_1=i\alpha zw$ and $Q_2=im\alpha w^2$ for some $\alpha\in\mathbb{R}$, and $\psi(z,\bar{z})=c|z|^{2m}$.*

PROOF. See Lemma 3.3 in page 146 of [5]. □

LEMMA 3.9. *The vector field H vanishes to finite order at the boundary point $p_\infty=g(\infty)$.*

PROOF. Recall that for a smooth real-valued test function φ on Ω, we have

$$(\Re e\, H)(\varphi)|_{p=g(q)}=\frac{d(\phi\circ g(q_1+t,q_2))}{dt}\Big|_{t=0} \tag{25}$$

where $q=(q_1,q_2)\in D$. It is shown in the last inequality in the proof of Lemma 3.5 in page 43 that there exists $\epsilon>0$ such that

$$|g(q_0+(t,0))-p_\infty|\leq C|t|^{-\epsilon} \tag{26}$$

holds for $q_0\in D$ and for $|t|$ large. Observe that $\epsilon>0$ can be chosen that $t^\epsilon|g(q_0+(t,0))|\to 0$ as $t\to\infty$. Also observe that at a point $p=g(q)\in\Omega$, we have

$$|\Re e\, H(p)|=\left|\frac{dg(q+(t,0))}{dt}\Big|_{t=0}\right| \tag{27}$$

To prove the lemma, let us set $p_\infty=0$ and choose $\delta>0$ such that $1-\delta>2^{-\epsilon}$. Let us denote by $g(t)=g(q+(t,0))$ in the rest of the proof. Then since $t^\epsilon|g(t)|\to 0$ as $t\to\infty$, there exists a sequence $t_j\nearrow\infty$ such that

$$|t_j^\epsilon g(t_j)|\geq(1-\delta)t^\epsilon|g(t)| \tag{28}$$

for all $t\geq t_j$. Thus,

$$|g(t_j)-g(2t_j)|\geq(1-\delta)^{-1}2^{-\epsilon}|g(t_j)|,\ \forall j.$$

By the Mean Value Theorem, there exists ξ with $t_j\leq\xi\leq 2t_j$ such that

$$|t_jg'(\xi)|\geq c|g(t_j)|\geq c2^{-\epsilon}|g(\xi)| \tag{29}$$

where the last inequality follows from (28), and where $c=(1-\delta)^{-1}2^{-\epsilon}$. From (26), we have $t_j\leq\xi\leq|Cg(\xi)|^{-1/\epsilon}$. Then it follows from (29) that $|g'(\xi)|\geq C_1|g(\xi)|^{1+1/\epsilon}$

for some constant $C_1 > 0$. Therefore, from (27), $\Re e\, H$ can only vanish to finite order at $p_\infty = 0$. $\square$

As remarked in [6], this somewhat crucial lemma was first observed by S. Bell and D. Catlin. In the original setting that $\partial\Omega$ is real analytic, the same conclusion was proven in Lemma 3.4 of [5].

LEMMA 3.10. *The holomorphic vector field $H = (H_1, H_2)$ is given by one of the following, in a neighborhood of $p_\infty = (0,0)$:*
(a) $H_1 = (\alpha + i\beta)z + \dots$ *and* $H_2 = 2m\alpha w + \dots$, *where the dots represent the terms of higher weight,* $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$, *and if* $\beta \neq 0$ *then* $\psi(z, \bar z) = c|z|^{2m}$;
(b) $H_1 = i\alpha z w + \dots$ *and* $H_2 = im\alpha w^2 + \dots$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *and* $\psi(z, \bar z) = c|z|^{2m}$.

PROOF. Form the preceding lemma, H vanishes to finite order at $p_\infty = (0,0)$. Thus, we may set

$$H_\mu = Q_\mu(z, w) + \dots \quad (\mu = 1, 2)$$

where Q_1 and Q_2 are weighted homogeneous polynomials with weights q and $q + 2m - 1$ respectively. Such restriction on weights follow from the fact that $\Re e\, H$ is tangent to $\partial\Omega$ admitting the defining function $\rho = \Im m\, w + \psi(z, \bar z) + \dots$. Again, the dots represent the terms with larger weights, and neither Q_1 nor Q_2 is identically zero. Then comparing the terms with smallest weights, the arguments that proved the preceding analogous lemmas without larger weight terms apply here and yield the desired conclusion. $\square$

PROPOSITION 3.11. *$\psi(z, \bar z) = c|z|^{2m}$ where ψ is the one in (24) of 45.*

PROOF. If ψ does not have the indicated form, then the above lemma implies that $H_1 = \alpha z + \gamma w + \dots$ and $H_2(z, w) = 2m\alpha w + \dots$, where $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Perform the change of coordinates

$$z = \delta \tilde z, w = \delta^{2m} \tilde w.$$

Then the vector field H in new coordinates may be written as $\tilde H = (\tilde H_1, \tilde H_2)$ where

$$\tilde H_1 = \delta^{-1} H_1(\delta \tilde z, \delta^{2m} \tilde w) = \alpha \tilde z + \gamma \delta^{2m-1} \tilde w + \dots$$
$$\tilde H_2 = \delta^{-2m} H_2(\delta \tilde z, \delta^{2m} \tilde w) + 2m\alpha \tilde w + \dots$$

Then consider the Euclidean scalar product of the vectors $\tilde H$ and $(\tilde z, \tilde w)$ which is

$$\alpha(|\tilde z|^2 + 2m|\tilde w|^2) + 2\Re e\, (\gamma \delta^{2m-1} \bar{\tilde z} \tilde w) + \text{ higher order terms.} \tag{30}$$

For a small value of $\delta > 0$ the sign of this product is either identically negative or identically positive near the origin depending upon the sign of α. This means that the flow of H is either attracting or repelling, accordingly. However, this contradicts the fact that the origin is a parabolic point of H-flow. Hence the conclusion follows. □

COROLLARY 3.12. *The scaled domain D scaled at the parabolic point by the parabolic orbit above is convex.*

PROOF. It follows from that the local Hausdorff set limits of a sequence of convex sets is convex. □

Final Rescaling by Parabolic Orbits

Before we perform the final scaling by parabolic orbit, we need to know that the parabolic orbit is "not too tangential" to the boundary. From the arguments above, we now have

- $0 \in \partial\Omega$ is the parabolic point, associated with the parabolic group $\mathcal{H}$.
- $g : D \to \Omega$ is the biholomorphic mapping from the scaled domain D to Ω.
- $D := \{(z,w) \in \mathbb{C}^2 \,|\, \Im m\, w + P(z,\bar{z}) < 0\}$ is convex, where $P(z,\bar{z})$ is a positive polynomial of degree $2m$.

Passing to the mapping $(z,w) \mapsto (z, -1/w)$, the domain D takes the form

$$\{(z,w) \in \mathbb{C}^2 \,|\, \Im m\, w + |w|^2 P(z,\bar{z}) < 0\}$$

which we will henceforth denote by D^*.

The parabolic point $0 \in \Omega$ is now corresponding to the complex line $\{(z,0) \in \mathbb{C}^2 \,|\, z \in \mathbb{C}\} \subset \partial D^*$, and the preimages of the parabolic orbits under the transform g are now the circles given by

$$\begin{aligned} z &= a_1, \\ w &= -\frac{1}{ia_2 + t}, \quad -\infty < t < \infty. \end{aligned}$$

Since $P(z,\bar{z}) > 0$ due to convexity of D, we have $D^* \subset \{\Im m\, w < 0\}$. The function $v(\zeta_1, \zeta_2) := \Im m\, \zeta_2$ is negative psh on D^*. We also have $|v(0,\zeta)| = \operatorname{dist}((0,\zeta), \partial D^*)$ for $(0,\zeta) \in D^*$. Applying the Hopf lemma to $v \circ g^{-1}$ defined on Ω, one gets the estimate

$$\operatorname{dist}(f(p), \partial\Omega) \le c_1 \operatorname{dist}(p, \partial D^*) \tag{31}$$

for the points $p \in D^*$ of the form $p = (0,\zeta)$.

As before, denote by ρ the defining function of Ω. Let

$$L_1(p) = \left(\frac{\partial\rho}{\partial w}(p), -\frac{\partial\rho}{\partial z}(p)\right), \quad L_2(p) = \left(\frac{\partial\rho}{\partial \bar z}(p), \frac{\partial\rho}{\partial \bar w}(p)\right)$$

denote the "tangential" and "normal" vector fields to the boundary $\partial\Omega$. Since the defining function ρ is defined on a neighborhood of the closure of the domain Ω, the vector field $X = a_1L_1 + a_2L_2$ is well defined in a neighborhood of the boundary of Ω. By [15], in a neighborhood of 0, we have the estimate on Kobayashi metric

$$F_K^{\Omega}(p, X) \geq c_2\left(\frac{|a_1|}{|\rho(p)|^{1/2m}} + \frac{|a_2|}{|\rho(p)|}\right) \tag{32}$$

Denote by $\tilde g(z, w) = g(z, -1/w)$. Combining (31) and (32) with standard techniques, one easily obtains

LEMMA 3.13. *There exists a neighborhood V containing the origin in $\mathbb{C}^2$ such that the biholomorphic mapping $\tilde g : D^* \to \Omega$ extends to $V \cap \overline{D^*}$ as a mapping of Hölder class $1/2m$.*

Notice that we may choose a defining function $\tilde\rho$ of Ω so that it is psh near 0 by Corollary 3.12. Hence, applying the Hopf lemma to $\tilde\rho \circ \tilde g$ in $V \cap D^*$, we have the estimate

$$\operatorname{dist}(g(p), \partial\Omega) \geq c_3 \operatorname{dist}(p, \partial D^*) \tag{33}$$

where $\tilde g = (\tilde g_1, \tilde g_2)$. Then we have

LEMMA 3.14. *Let $m > 1$ in the expansion (35) of the defining function of Ω at the origin, which is a parabolic point. Then there exist constants $c_4 > 0$ and $\delta_0 > 0$ such that for all $\delta_0 > \delta > 0$ we have*

$$|\tilde g_2(0, -i\delta)| \leq c_4\delta.$$

PROOF. From the estimate (32) and by the definition of X, we have

$$\left|\frac{\partial\tilde g}{\partial w}(p)\right| \leq c_5\left(1 + \left|\frac{\partial\rho}{\partial z}(\tilde g(p))\right| \left|\frac{1}{(\rho\circ\tilde g(p))^{(2m-1)/2m}}\right|\right) \tag{34}$$

where ρ is the defining function of Ω.

Since we are interested in the behavior of the parabolic orbit in Ω near the origin p_∞, we may freely assume that $\rho = \tilde\rho$ and it takes the expansion

$$\rho(z, w) = \Im m\, w + c|z|^{2m} + \text{terms with larger weights} \tag{35}$$

as in (30). Since

$$\left|\frac{\partial \rho}{\partial z}(z,w)\right| = O(|z|^{2m-1} + |\Im m\, w|^2),$$

and since $\tilde{g}(0,-i\delta) = O(\delta^{1/2m})$, from (31) and (33) one gets

$$\left|\frac{\partial \tilde{g}_2}{\partial w}(0,-\delta)\right| = O(\delta^{(1-m)/m}),$$

and so $\tilde{g}_2(0,-i\delta) = O(\delta^{1/m})$. Substitute this into (34) and repeat the same procedure. Then one gets

$$\left|\frac{\partial \tilde{g}_2}{\partial w}(0,-i\delta)\right| = O(\delta^{(3-2m)/2m}),$$

i.e. $\tilde{g}_2(0,-i\delta) = O(\delta^{3/2m})$. Repeating this a finitely many times, one eventually gets $\tilde{g}_2(0,-i\delta) = O(\delta)$, as desired. □

We are arriving at the final phase of the proof of the Bedford-Pinchuk theorem. First of all, if Ω is strongly pseudoconvex at the parabolic point, then the conclusion of the Bedford-Pinchuk theorem follows from the Wong-Rosay Theorem. Thus, we may assume in the following that the parabolic point is a weakly pseudoconvex boundary point.

Now consider the defining function ρ of Ω with the expansion (35) above. Since $\Im m\, w < 0$ for $(z,w) \in \Omega$ near the origin, Lemma 3.14 above allows us to use the Julia lemma (See [65]) for the function $\zeta = \tilde{g}_2(0,w)$. The Julia lemma implies in particular that for any constant $K > 0$, there exists $\epsilon > 0$ such that the image $\gamma(t)$ of the circle $t \mapsto (0, -1/(-i\epsilon + t))$ under $\tilde{g}_2$ lies in the disk $\Im m\, w + K|w|^2 < 0$. Notice that the image $\gamma(t)$ in Ω is the orbit of the parabolic group $\mathcal{H}$ at a point in Ω. In fact, $h_t(\gamma(0)) = \gamma(t)$ for all $t \in \mathbb{R}$.

Denote by $\gamma = (\gamma_1, \gamma_2)$, and we begin the final rescaling with the following notations:

- $\delta = |\Im m\, \gamma_2(t)|$
- $s_2 = \Re e\, \gamma_2(t)$
- $s_1 = \gamma_1(t)$

In the above, we observe that by the choice of ϵ we have

$$s_2^2/\delta \le 1/K$$

and by (35) we get

$$|s_1|^{2m} \le 4\delta.$$

Now we apply the scaling method as in the earlier sections. Let us consider the transformation $(\zeta, \xi) = S_t(z, w)$ of $\mathbb{C}^2$ defined by

$$\begin{cases} \zeta = z / \sqrt[2m]{\delta} \\ \xi = (w - s_2)/\delta \end{cases}$$

The composition $S_t \circ h_t$ maps the domain Ω biholomorphically to its image defined by the defining function given by

$$\rho_t(\zeta, \xi) = \frac{1}{\delta} \rho \circ S_t^{-1}(\zeta, \xi).$$

Investigating the convergence of these defining functions as $t \to \infty$, one gets that ρ_t converges to the expression

$$\hat{\rho}(\zeta, \xi) = \Im m\, \xi + |\zeta|^{2m} + c$$

for some real constant c, by the estimates on s_1, s_2 by δ in the above. Denote by

$$\hat{E}_{2m} = \{(\zeta, \xi) \in \mathbb{C}^2 \mid \hat{\rho}(\zeta, \xi) < 0\}.$$

This domain is biholomorphic to the Thullen domain E_{2m} by a linear fractional transformation. (Exercise: *Verify!*) Repeating the arguments of the preceding section on preliminary scaling, it is easy to see that a subsequential limit of the family $\{S_t \circ h_t\}_t$ as $t \to \infty$ will yield a biholomorphic mapping from Ω to $\hat{E}_{2m}$. Therefore, the conclusion of Theorem 3.2 follows.

PROBLEM 3.1. For a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^∞ boundary. Is it true that up to a holomorphic change of coordinates the domain in a neighborhood of any boundary point at which an automorphism orbit accumulates is indeed convex? It is proven by Greene and Krantz ([27]) that for a bounded domain in $\mathbb{C}^n$ with a C^2 smooth boundary, the boundary is pseudoconvex in a neighborhood of every orbit accumulation point.

PROBLEM 3.2. Can one generalize the arguments above to higher dimensions? If Ω is bounded *convex* with a C^∞ smooth finite type boundary, then the optimal generalization is known ([7]). Can one find a way of handling the non-convex cases?

PROBLEM 3.3. Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^∞ smooth boundary. Prove that every orbit accumulation boundary point (if any) is a point of finite type. This is a conjecture by Greene and Krantz. Some partial answers are known. ([28], [40]) However, this problem is still open in its full generalities.

4. Product Domain Theorem

In the preceding sections, we discussed the bounded domains in $\mathbb{C}^n$ with a noncompact automorphism group when the boundary surface is smooth and not Levi flat at least at the orbit accumulation points. In this section, we will discuss the case when the boundary of the domain is either singular or Levi flat at the orbit accumulation point. This section is from the author's recent paper [38].

4.1. Statement of the Product Domain Theorem. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a C^∞ smooth function such that

$$\nabla\rho\ (p) \neq 0, \forall p \text{ with } \rho(p) = 0.$$

The Implicit Function Theorem implies that the set $\Sigma_\rho := \{p \in \mathbb{C}^n : \rho(p) = 0\}$ is then a C^∞ smooth real hypersurface of $\mathbb{C}^n$. Call such a hypersurface Σ_ρ *Levi flat* at $p \in \Sigma_\rho$ if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k = 0$$

for any $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ satisfying

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) \cdot w_j = 0.$$

Moreover, Σ_ρ is called *Levi flat*, if it is Levi flat at every point.

Let D be a bounded domain in $\mathbb{C}^n$. Its boundary ∂D is said to be *piecewise C^∞ smooth* if

$$D = \{z \in \mathbb{C}^n \mid \rho_1(z) < 0, \ldots, \rho_k(z) < 0\}$$

for some C^∞ smooth functions $\rho_1, \ldots, \rho_k$ satisfying the condition that, for any possible choice of mutually distinct indices $i_1, ..., i_\ell$,

$$d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_\ell}\ (p) \neq 0$$

whenever $\rho_{i_1}(p) = \cdots = \rho_{i_\ell}(p) = 0$. Moreover, ∂D is said to be *piecewise Levi flat* if the hypersurface $\{p \in \mathbb{C}^n \mid \rho_{i_m}(p) = 0\}$ is Levi flat, for every $m = 1, ..., k$.

The first main result of this section is the following characterization of the bi-disk in $\mathbb{C}^2$ by its automorphism group among bounded convex domains in $\mathbb{C}^2$ with a piecewise C^∞ smooth Levi flat boundary.

THEOREM 4.1 (K.T. KIM). *Let $D \subset \mathbb{C}^2$ be a bounded convex domain with a piecewise C^∞ smooth Levi flat boundary. If* Aut D *is non-compact, then D is biholomorphic to the bi-disk in $\mathbb{C}^2$.*

Note that an attempt to generalize directly this statement to the polydisks of higher dimensions is not possible, since the assumption of noncompactness of the automorphism group of G is not strong enough in case the boundary is only piecewise Levi flat, as one may see in the examples such as the product of the open unit disk in $\mathbb{C}$ and an arbitrary domain in $\mathbb{C}^{n-1}$ with a piecewise Levi flat boundary. The best one can hope for with such a boundary may be the following.

THEOREM 4.2 (K.T. KIM). *Let $D \subset \mathbb{C}^n$ be a bounded convex domain with a piecewise C^∞ smooth Levi flat boundary. If* $\operatorname{Aut} D$ *is non-compact, then D is biholomorphic to a domain that is the product of the unit disk in $\mathbb{C}$ and a convex domain in $\mathbb{C}^{n-1}$.*

Partial results can be easily deduced from the theorems presented in the earlier papers such as S. Frankel [22], A. Kodama [48], [49], [50] and S. Pinčuk [61] for the domains that are essentially homogeneous or admit a special strictly pseudoconvex boundary point where a certain non-compact orbit (with a certain behavior control) of the automorphism group accumulates. Major differences of the above theorems are twofold. First, there is no special restriction on the boundary points being smooth or being singular at which the orbits of the automorphism group accumulate. Second, the assumption of non-compactness of the automorphism group is in most cases much weaker than being homogeneous or covering a quotient with a finite volume with respect to an intrinsic Kähler metric. On the other hand the convexity assumption in the statement of the theorms above is somewhat strong. It may be possible to relax the convexity condition slightly, but some kind of such a global restriction on the domain seems inevitable, since there is no hope of localizing the arguments in case the boundary is Levi flat.

Needless to say, the theorems above are in good contrast with the Wong-Rosay theorem and the Bedford-Pinchuk theorem introduced in the preceding sections.

4.2. Convex Scaling Technique. In this section, we introduce another scaling technique which was initiated by Sidney Frankel around 1986 ([22]). This technique, when modified properly, is also a powerful technique comparable to Pinchuk's scaling method. Frankel's scaling has a certain technical merit for the proof of the theorems above, and hence we will introduce the technique here, rather briefly.

Let D be a bounded convex domain in $\mathbb{C}^n$ whose boundary ∂D is piecewise Levi flat. Call a boundary point $p \in \partial D$ *singular*, it ∂D is not smooth at p (*regular*, if ∂D is smooth at p, respectively). Denote by $S_{\partial D}$ the set of all singular boundary point of D. Also define $R_{\partial D} = \partial D \setminus S_{\partial D}$.

Moreover, throughout this section without exception, we assume that there are a point $q \in D$ and a sequence $\{f_j\} \subset \operatorname{Aut} D$ such that

$$\lim_{j\to\infty} f_j(q) = p \in \partial D.$$

Now we introduce the convex scaling method initiated by S. Frankel.

THEOREM 4.3 (S. FRANKEL). *Let D be a bounded convex domain in $\mathbb{C}^n$ and let $q \in D$ and $f_j \in \operatorname{Aut} D$ be such that $f_j(q) \to \partial D$ as $j \to \infty$. Then, the sequence of holomorphic mappings from D into $\mathbb{C}^n$ defined by*

$$\omega_j(z) = [\partial f_j(q)]^{-1}(f_j(z) - f_j(q))$$

is a normal family, every subsequential limit of which is a holomorphic embedding of D into $\mathbb{C}^n$. Here, $\partial f_j(q)$ denotes the holomorphic Jacobian matrix of f_j at q.

PROOF. Consider the map

$$F_j : D \times D \to D$$

defined by

$$F_j(z,\zeta) = \omega_j^{-1}\left(\frac{\omega_j(z) + \omega_j(\zeta)}{2}\right)$$

Notice that this mapping is well-defined.

Since D is bounded and since $F_j(q,q) = q$ for all j, $\{F_j\}_j$ is a compact normal family. Therefore, for each compact subset K of D, all the derivatives of F_j of a given multi-index are uniformly bounded on $K \times K$. Now, from the definition of F_j we have

$$\omega_j \circ F_j(z,\zeta) = \frac{\omega_j(z) + \omega_j(\zeta)}{2}.$$

In the following we drop the index j momentarily and indicate the derivatives by subscripts. We consider the second order derivatives in short-hand notations:

$$\omega'' \cdot F_z \cdot F_z + \omega' F_{zz} = \frac{1}{2}\omega'' \tag{36}$$

$$\omega'' \cdot F_z \cdot F_\zeta + \omega' F_{z\zeta} = 0 \tag{37}$$

Notice that by the symmetry in the definition of F, $F_z = F_\zeta$ at a point of type $(p,p) \in D \times D$. Hence, at $(p,p) \in D \times D$ one gets by subtracting the second expression from the first

$$\omega'' = 2(F_{zz} - F_{z\zeta}) \cdot \omega'$$

Now, this gives rise to the estimate on each compact set K of D such as

$$\|\omega''\| \leq C_K \|\omega'\|$$

where $C_K > 0$ is a constant independent of the index j while it is clearly depending upon K and the sequence of the automorphisms. Now using the fact that $d\omega(q) = I$ and using the comparison theory of ordinary differential equations, one obtains that ω' is locally bounded (uniformly on the index j which has been suppressed). Finally, using the fact that $\omega(q) = 0$ always, one gets the conclusion that ω_j's form a compact normal family.

So, we assume, by choosing a subsequence, that ω_j converges to the mapping $\omega : D \to \mathbb{C}^n$, and the sets $\omega_j(D)$ converges in the sense of local Hausdorff set convergence. (Blaschke selection, for instance. Notice that $\omega_j(D)$ is convex for all j.) Observe that $\det \partial\omega_j$ is never zero on D, since ω_j is a one-to-one mapping onto $\omega_j(D)$. Therefore, by Hurwitz's theorem, either $\det \partial\omega$ is never zero on D, or it is identically zero. Since $\partial\omega_j(q) = I$ for all j, $\partial\omega_j$ is nonsingular at every point of D. Hence ω is locally one-to-one. Using the uniform convergence on compact subsets of D, a standard argument shows that ω is globally one-to-one.

Finally, the convexity makes it easier to show that $\omega(D) = \lim_{j\to\infty} \omega_j(D)$. We leave this final part as an exercise to the reader. □

The following modification observed earlier by the author (cf. Lemma A, p. 143 of [34]) is also useful.

PROPOSITION 4.4. *Let D, p, q, f_j be as before. Assume further that there is no non-constant analytic set at p contained in ∂D. Then, by choosing a subsequence of f_j if necessary, we have a sequence $\{p_j\}$ of the boundary points of D converging to p such that the family of holomorphic mappings on D into $\mathbb{C}^n$ defined by*

$$\sigma_j(z) = [\partial f_j(q)]^{-1}(f_j(z) - p_j),$$

any subsequential limit of which is a holomorphic embedding of D into $\mathbb{C}^n$. Moreover, it follows that

$$\lim_{j\to\infty} \|\partial f_j(q)\| = 0. \tag{38}$$

REMARK 4.1. Notice that with one extra assumption in its hypothesis, the scaling effect by σ_j's is more explicit in the modified version *not because it scales linearly near the fixed boundary point p but because it shows that every vector scales to an infinite length by $[\partial f_j(q)]^{-1}$.*

PROOF. (Sketch) The second assertion follows from Lemma 2.5. Therefore, every vector scales to infinite speed by $[\partial f_j(q)]^{-1}$. By Theorem 4.3 above, we also know that $\lim_{j\to\infty} \omega_j(D)$ cannot contain a complex line, because it must be biholomorphic to the bounded domain D. Now suppose that for every sequence of boundary points s_j of ∂D we have

$$\lim_{j\to\infty}[\partial f_j(q)]^{-1}(s_j - f_j(q)) = \infty. \tag{39}$$

Let s_j be a boundary point that is closest to $f_j(q)$. Choose the affine complex line L_j containing the line segment joining s_j and $f_j(q)$. Then consider the sequence of sets $[\partial f_j(q)]^{-1}(L\cap D - f_j(q))$. Note that the complex line $[\partial f_j(q)]^{-1}(L - f_j(q))$ always passes through the origin for every j. Therefore, the assumption (39) above then implies that $\lim_{j\to\infty}[\partial f_j(q)]^{-1}(L\cap D - f_j(q))$ is in fact a complex line, which is contained in the scaled limit $\lim_{j\to\infty}\omega_j(D)$. This is not possible. Therefore, there exists a sequence of boundary points p_j, say, of D such that

$$\lim_{j\to\infty}[\partial f_j(q)]^{-1}(p_j - f_j(q))$$

is bounded. Then the conclusion of the proposition follows from the triangle inequality and a standard normal family argument. □

4.3. Proof of the Product Domain Theorem. We first proceed in complex dimension 2 in detail. Then, at the end, we show that the same technique and the proof can be generalized to establish the proof in the higher dimensions.

Scaling at a Singular Boundary Point

Let D be a bounded convex domain in $\mathbb{C}^2$ with a piecewise Levi flat boundary from now on. Let $p \in S_{\partial D}$. Note that there is no non-constant analytic set at p contained in the closure of D. To see this, let U be an open neighborhood such that

$$U\cap D = \{(z_1,z_2)\in\mathbb{C}^2 \,:\, \rho_1 < 0, \rho_2 < 0\}$$

where

$$d\rho_1 \wedge d\rho_2\,(x) \neq 0 \quad\text{whenever}\quad \rho_1\,(x) = \rho_2\,(x) = 0, \forall x\in U.$$

In particular, the vectors $\nabla\rho_1(p)$ and $\nabla\rho_2(p)$ are linearly independent over $\mathbb{C}$. Let us assume, without loss of generality, that p is the origin in $\mathbb{C}^2$. Then by a complex linear change of coordinates, it can be easily arranged so that

$$U\cap D = \{(z_1,z_2)\in\mathbb{C}^2 \,:\, \Im m\, z_1 > \psi_1(z_1,z_2), \Im m\, z_2 > \psi_2(z_1,z_2)\}$$

where ψ_1 and ψ_2 are smooth convex positive real-valued functions such that the normal vectors at $p=(0,0)$ to the hypersurfaces $\{\Im m\, z_1 - \psi_1 = 0\}$ and $\{\Im m\, z_2 - \psi_2 = 0\}$ are parallel to the $\Im m\, z$ and $\Im m\, w$ axes, respectively. Then, it is easy to see that there exists a strictly psh function

$$h(z) = \left|\frac{i - z_1}{i+z_1}\right|^2 + \left|\frac{i-z_2}{i+z_2}\right|^2 - 1$$

satisfying

$$h(z) < 1 \ \forall z \in \overline{D} \setminus \{0\}, \text{ and } h(0) = 1.$$

Hence, there is no non-constant analytic set at p in $\overline{D}$.

Therefore, we can apply Proposition 4.4 to prove Theorem 4.1 in case

$$\lim_{j\to\infty} f_j(q) = p \in S_{\partial D}.$$

Denote by

$$\partial f_j(q) = \begin{pmatrix} a^j_{11} & a^j_{12} \\ a^j_{21} & a^j_{22} \end{pmatrix} \tag{40}$$

for each $j = 1, 2, \dots$, then it follows that

$$\lim_{j\to\infty} a^j_{k\ell} = 0, \forall k, \ell = 1, 2. \tag{41}$$

With our previous setting with $p = 0$ and ψ_1, ψ_2, the defining inequalities for the domain D can be written as follows:

$$\Im m\,(z_1) > h_1(z_2) + O(z_1^2, z_1 z_2)$$
$$\Im m\,(z_2) > h_2(z_1) + O(z_1 z_2, z_2^2)$$

Since D is convex Levi flat at every regular boundary point, all the analytic subsets of the boundary ∂D is trivial or Euclidean flat (see the lemma below) and consequently both h_1 and h_2 are linear. Now we apply Proposition 4.4 here. First, let r be an arbitrary positive number. Let $p_j = (p^j_1, p^j_2)$. Then due to (41) above, there exists an integer j_0 such that for any $j \geq j_0$ the scaled domain $\sigma_j(D) \cap B_r(0)$ inside the open ball $B_r(0)$ is represented by the following two inequalities

$$\begin{aligned} \Im m\,(a^j_{11}\zeta_1 + a^j_{12}\zeta_2 + p^j_1) &> h_1(a^j_{21}\zeta_1 + a^j_{22}\zeta_2 + p^j_2) \\ &\quad + O\Big((a^j_{11}\zeta_1 + a^j_{12}\zeta_2 + p^j_1)^2, (a^j_{11}\zeta_1 + a^j_{12}\zeta_2 + p^j_1)(a^j_{21}\zeta_1 + a^j_{22}\zeta_2 + p^j_2)\Big) \end{aligned} \tag{42}$$

$$\begin{aligned} \Im m\,(a^j_{21}\zeta_1 + a^j_{22}\zeta_2 + p^j_2) &> h_2(a^j_{11}\zeta_1 + a^j_{12}\zeta_2 + p^j_1) \\ &\quad + O\Big((a^j_{11}\zeta_1 + a^j_{12}\zeta_2 + p^j_1)(a^j_{21}\zeta_1 + a^j_{22}\zeta_2 + p^j_2), (a^j_{21}\zeta_1 + a^j_{22}\zeta_2 + p^j_2)^2\Big). \end{aligned} \tag{43}$$

Without loss of generality, by extracting a subsequence from $\{f_j\}$ if necessary, we may assume that

$$\left| \lim_{j\to\infty} \frac{a^j_{12}}{a^j_{11}} \right| \leq C < \infty. \tag{44}$$

Then divide by $|a_{11}^j|$ the inequalities of (42) and (43) above, and consider their limits. By Proposition 4.4, a subsequence of this will converge to a domain biholomorphic to D, in the sense of local Hausdorff set convergence. To have (42) and (43) above yield a subsequential limit of $\{\sigma_j(D)\}$ in the sense of local Hausdorff distance so that neither is its image contained in a lower dimensional subset of $\mathbb{C}^2$ or it contains a complex line, we end up with the limit described by

$$\Im m\,(A_{11}\zeta_1 + A_{12}\zeta_2) > \Im m\,(C_{21}\zeta_1 + C_{22}\zeta_2) \tag{45}$$

$$\Im m\,(A_{21}\zeta_1 + A_{22}\zeta_2) > \Im m\,(C_{11}\zeta_1 + C_{12}\zeta_2) \tag{46}$$

for some constants $A_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$. Then the domain

$$\sigma(D) = \lim_{j\to\infty} \sigma_j(D)$$

is defined by the inequalities

$$\Im m\,(B_{11}\zeta_1 + B_{12}\zeta_2) > 0 \tag{47}$$

$$\Im m\,(B_{21}\zeta_2 + B_{22}\zeta_2) > 0 \tag{48}$$

for some constants $B_{k\ell} \in \mathbb{C}$, $k, \ell = 1, 2$. Since $\sigma(D)$ is biholomorphic to D by Proposition 4.4, it cannot contain a complex line. Therefore, it follows that

$$\det \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \neq 0 \tag{49}$$

This shows in turn that $\sigma(D)$ is (Hence D is also) biholomorphic to the domain

$$H = \{(z_1, z_2) \in \mathbb{C}^2 : \Im m\, z_1 > 0, \Im m\, z_2 > 0\}, \tag{50}$$

which was desired. (This argument is related to earlier work of Pinčuk ([60], [61]) and Kodama [49] in its idea, but different in its techniques.)

However, as one notices, our proof of Theorem 4.1 is indeed not yet complete, since the case that the automorphism orbits accumulate only at the regular (Levi flat) boundary points has not been treated. Hence we continue our argument in the following section.

Scaling at a Levi flat Point

Let $D \subset \mathbb{C}^2$ be as in the preceding arguments, except

$$p = \lim_{j\to\infty} f_j(q) \in R_{\partial D}.$$

Then the boundary ∂D is Levi flat at p. To treat this case, we first make a few easy observations that will be used later.

It is well-known that a hypersurface, say S, is Levi flat, if and only if the complex tangent vector fields on S form an integrable distribution, in the sense of Frobenius. Thus, S is foliated by complex analytic curves. In particular, at any regular boundary point $x \in R_{\partial D}$, there is a smooth analytic curve V_x through x contained in $R_{\partial D}$. Then, by convexity of D and the Maximum Principle, we get

LEMMA 4.5. *For any $x \in R_{\partial D}$, the analytic curve V_x through x contained in $R_{\partial D}$ is Euclidean flat.*

Notice that, unfortunately, Proposition 4.4 cannot be applied in this case, since $\lim f_j(q)$ is a Levi flat boundary point. So we analyze the effect of the scaling method of Theorem 4.3 so that we may come up with a technique effective enough and comparable to Proposition 4.4 and finish the proof of Theorem 4.1. So, we observe

LEMMA 4.6. *There exists a constant $\delta > 0$ and a unit vector $w \in \mathbb{C}^2$ such that*

$$\|\partial f_j(q)w\| \geq \delta, \ \forall j = 1, 2, \dots .$$

by choosing a subsequence of $\{f_j\}$, if necessary.

PROOF. Let V_x be the analytic curve through $x \in R_{\partial D}$, contained in ∂D, where x is very close to p. Note that Frobenius' Theorem also yields that the dependence of V_x upon x is C^∞ smooth. Therefore, there exists an open neighborhood G of p such that, for each $x \in G \cap \partial D$, V_x is represented by a smooth family of holomorphic maps

$$h_x : \Delta \to \partial D \subset \mathbb{C}^2$$

satisfying

$$h_x(0) = x, \ h_x(\Delta) \subset \partial D, \ |h'_x(0)| \geq \delta_1 > 0 \tag{51}$$

for some constant $\delta_1 > 0$. Using Lemma 4.5 and the convexity of D, we infer that, by shrinking G if necessary, for any $y \in G \cap D$ there exists a holomorphic map

$$h_y : \Delta \to D \subset \mathbb{C}^2$$

such that

$$h_y(0) = y, \ |h'_y(0)| \geq \delta_1/2 > 0. \tag{52}$$

Thus, there exists an integer $j_0 > 0$ such that $f_j^{-1} \circ h_{p_j}$ maps the unit disk into D for all $j \geq j_0$, where $p_j = f_j(q)$. Since D is also hyperbolic in the sense of Kobayashi ([44]), there is a constant $C > 0$ such that

$$\left| \left(f_j^{-1} \circ h_{p_j} \right)' (0) \right| \leq C, \ \forall j \geq j_0.$$

Thus, let

$$w_j := \frac{[\partial f_j(q)]^{-1} h'_{p_j}(0)}{\|[\partial f_j(q)]^{-1} h'_{p_j}(0)\|}.$$

It follows then that

$$\|\partial f_j(q) w_j\| \geq \frac{1}{C} \|h'_{p_j}(0)\| \geq \frac{\delta_1}{2C}. \tag{53}$$

By choosing a subsequence, we may assume that the sequence $\{w_j\}$ converges, and denote by $w = \lim w_j$. Then, using the Kobayashi hyperbolicity of D again, it is easy to show that at each q, the sequence $\{\|\partial f_j(q)\|\}$ is bounded. Thus, it follows that there exists a constant $\delta > 0$ such that

$$\|\partial f_j(q) w\| \geq \delta, \ \forall j \geq j_0. \tag{54}$$

□

We also observe the following

LEMMA 4.7. *Let $u = \sqrt{-1}\,\vec{n} \in \mathbb{C}^2$, where $\vec{n}$ denotes the outward unit normal vector to ∂D at p above. Then for the sequence of 2 by 2 matrices $\partial f_j(q)$, we get*

$$\lim_{j \to \infty} \|[\partial f_j(q)]^{-1} u\| = \infty.$$

Needless to say, in the above lemma, one can make better sense if one translates u to a vector with its initial point at $f_j(q)$, for each j, by the Euclidean parallel translation. However, we are safe even if we understand $\partial f_j(q)$ as a 2×2 matrix, for each j, with respect to the standard complex vector space basis for $\mathbb{C}^2$.

PROOF. The proof of Lemma 4.7 also follows easily by a normal family argument. Suppose that there is a subsequence of $\{f_j\}$, which we again call $\{f_j\}$ by an abuse of notations, satisfying

$$\|[\partial f_j(q)]^{-1} u\| \leq C$$

for some constant $C > 0$ independent of j. For convenience, denote by

$$v_j := [\partial f_j(q)]^{-1} u.$$

Now, consider complex analytic disks

$$h_j : \lambda \mapsto \lambda \epsilon v_j + q \, : \Delta \to D$$

where $\epsilon > 0$ is chosen independent of j to guarantee that the image of each h_j is contained in D. Then as before, we obtain an analytic disk

$$\eta := \lim_{j \to \infty} f_j \circ h_j$$

contained in ∂D, satisfying $\eta(0) = p$ and $\eta'(0) = u$, which is not allowed. □

Therefore, if we choose, for each $j = 1, 2, \ldots$, the eigenvalue ℓ_j of $[\partial f_j(q)]^{-1}$ that is of the larger absolute value than the other, we have

$$\lim_{j\to\infty} \ell_j = \infty.$$

Let us denote by L_j the complex eigenspace of $[\partial f_j(q)]^{-1}$ corresponding to ℓ_j, for each j. By Lemma 4.6, Lemma 4.7 and the preceding observation, for large j's, L_j is complex one dimensional. For our scaling purposes, by L_j we mean the affine hyperplane $L_j + f_j(q)$ in $\mathbb{C}^2$ from now on. Again, choosing a subsequence if necessary, we may assume that the sequence $\{L_j\}$ converges in the local Hausdorff sense. Write $L = \lim L_j$. Again, as before, it is easy to see that L cannot be tangential to the boundary ∂D of D.

Notice that, with the notations of Theorem 4.3, the proof of Theorem 4.1 is complete as soon as we show, for instance, that the domain

$$\begin{aligned}\omega(D) &= \lim_{j\to\infty} \omega_j(D) \\ &= \lim_{j\to\infty} [\partial f_j(q)]^{-1}(D - f_j(q))\end{aligned}$$

is biholomorphic to the bidisk.

To show this, we now concentrate on the effect of scaling by $[\partial f_j(q)]^{-1}$ on $L_j \cap D$ with the origin at $f_j(q)$. First of all, the set-limit

$$\lim_{j\to\infty} [\partial f_j(q)]^{-1}(L_j \cap D - f_j(q)) \tag{55}$$

is clearly contained in $\omega(D)$ above. (The limit exists, by taking a subsequence if necessary, since all the sets in the sequence are convex.) Moreover, note that there exist an open neighborhood W of p and an integer j_0 such that

$$\{s_j \in \partial(L_j \cap D) \mid \operatorname{dist}(s_j, f_j(q)) = \operatorname{dist}(f_j(q), \partial(L_j \cap D))\} \cap W \tag{56}$$

is non-empty for all $j \geq j_0$ and such that $\operatorname{dist}(\overline{W}, S_{\partial D}) > 0$.

Note then that the sequence $\{\ell_j \operatorname{dist}(s_j, f_j(q))\}$ has to be bounded. Otherwise, $\omega(D)$ will contain a complex line, which is not allowed because D is bounded. Hence, again extracting a subsequence if necessary, we get that the set

$$\begin{aligned}&\lim_{j\to\infty} (L_j \cap D - f_j(q)) \\ &\qquad = \lim_{j\to\infty} [\partial f_j(q)]^{-1}(L_j \cap D - s_j) + \lim_{j\to\infty} \ell_j(s_j - f_j(q))\end{aligned}$$

is in fact an open half plane in a complex affine hyperplane in $\mathbb{C}^2$. Moreover, we obtain another scaling sequence

$$\tau_j(z) := [\partial f_j(q)]^{-1}(f_j(z) - s_j) \tag{57}$$

which will be used in place of ω_j from now on for the rest of the proof of Theorem 4.1. We will also denote by $\tau = \lim \tau_j$. For convenience, we write $\hat{D} = \tau(D)$. It is obvious that the proof will be complete as soon as we show that $\hat{D}$ is biholomorphic to the bidisk. So we define

$$H := \lim_{j\to\infty} [\partial f_j(q)]^{-1}(L_j \cap D - s_j)$$

which is again a half plane contained in the closure $\overline{\tau(D)}$ of $\tau(D)$. Clearly, $H \subset \hat{D}$ and $\partial H \subset \partial\hat{D}$. Fix a point $\hat{o} \in H$. Then we claim that, for any boundary point $\hat{p} \in \hat{D} \subset \mathbb{C}^2$, the half plane defined by

$$\begin{aligned} H_{\hat{p}} &= H + (\hat{p} - \hat{o}) \\ &= \{x \in \mathbb{C}^2 : x - \hat{p} + \hat{o} \in H\} \end{aligned} \tag{58}$$

is properly contained in $\hat{D}$, meaning that

$$H_{\hat{p}} \subset \hat{D} \quad \text{and} \quad \partial H_{\hat{p}} \subset \partial\hat{D}. \tag{59}$$

Justification of this claim is fairly simple and based only on the convexity of $\hat{D}$. Moreover, it is also not hard to see that H is not parallel to any real two dimensional subspace of the tangent plane.

Notice that the claim above implies only that the domain $\hat{D}$ is set theoretically a product of a real straight line and a convex domain in $\mathbb{R}^3$. Since it is obviously not yet enough to conclude that $\hat{D}$ (and hence D also, biholomorphically) is a product of two convex domains in $\mathbb{C}$, we need investigate further.

So we come back to consider the complex direction that stays bounded under the linear scaling sequence $\{[\partial f_j(q)]^{-1}\}$. We observed in an earlier step that, for each $s_j \in \partial D$, there exists a Euclidean flat analytic set $V_{s_j} \subset \partial D$ which is in turn obviously convex (and hence also flat). Then we also have that the set

$$\hat{V} := \lim_{j\to\infty} [\partial f_j(q)]^{-1}(V_{s_j} - s_j) \subset \partial\hat{D} \tag{60}$$

exists, by taking a subsequence again if necessary, and is also a convex analytic subset of $\partial\hat{D}$ in $\mathbb{C}^2$.

Since both $\hat{V}$ and H are open subsets of two independent $\mathbb{C}$-linear subspaces of $\mathbb{C}^2$, the vector sum

$$\hat{V} + H = \{x + y \,:\, x \in \hat{V}, y \in H\}$$

is an open subset of $\hat{D}$. We claim that $\hat{D} = \hat{V} + H$. To see this, let $x \in \hat{D}$. Then by the construction of $\hat{D}$, there exists a sequence $\{y_j\} \subset D$ such that

$$x = \lim_{j\to\infty} [\partial f_j(q)]^{-1}(y_j - s_j).$$

Since V_{s_j} and L_j span $\mathbb{C}^2$, we can write

$$y_j - s_j = u_j + c_j b_j$$

for some $u_j \in L_j$ and $b_j \in V_{s_j} - s_j$, and $c_j \in \mathbb{C}$. Then it follows that

$$\begin{aligned} x &= \lim_{j\to\infty} [\partial f_j(q)]^{-1}(u_j + c_j b_j) \\ &= \lim_{j\to\infty} [\partial f_j(q)]^{-1}(u_j) + c_j \cdot \lim_{j\to\infty} [\partial f_j(q)]^{-1}(b_j) \\ &\in (\lim_{j\to\infty} c_j) \cdot \hat{V} + H \end{aligned} \tag{61}$$

The second identity follows due to the boundedness of the second limit. To be precise, one needs to choose a subsequence again if necessary. Moreover, in the above, we have $\lim_{j\to\infty}[\partial f_j(q)]^{-1}(u_j)$ bounded also. This implies in particular that $u_j \to 0$ as j tends to ∞. Then, due to convexity of D, we must have that, for any $\epsilon > 0$, there exists $j_0 > 0$ such that $c_j b_j \in (1+\epsilon)(V_{s_j} - s_j)$ for all $j > j_0$. Therefore

$$x \in (1+\epsilon)\hat{V} + H.$$

Since ϵ is arbitrary, we get $x \in \hat{V} + H$.

Now we have established that $\hat{D}$ is a product of two convex plane domains that are also Kobayashi hyperbolic. By Riemann Mapping Theorem, $\hat{D}$ is, and hence D is also, biholomorphic to the bi-disk in $\mathbb{C}^2$. Now, the proof of Theorem 4.1 is complete.

REMARK 4.2. $\hat{V}$ and H are obtained by separating the directions of scaling by a bounded sequence and the directions by an unbounded scaling sequence, respectively.

Higher Dimensional Cases

Now we observe that almost all the techniques we have used in the preceding section can be generalized to the higher dimensional cases. Let D be a bounded convex domain in $\mathbb{C}^n$ with a piecewise Levi flat boundary given by

$$D = \{z \in \mathbb{C}^n \, : \, \rho_1(z) < 0, ..., \rho_k(z) < 0\}.$$

Suppose, under the hypothesis of Theorem 4.2, that the limit boundary point of the sequence $\{f_j(q)\}$, for some $\{f_j\} \subset \operatorname{Aut} D$ and some $q \in D$, is a regular boundary point. Then we see first of all, the piecewise Levi flat condition on the boundary ∂D of the domain $D \subset \mathbb{C}^n$ ensures the fact that ∂D is (at least locally) foliated by real codimension one complex analytic subsets (hence of complex dimension $n-1$). As in the previous sections, because D is also convex, all the (complex analytic) leaves are Euclidean flat, by Maximum principle. Of course, Lemma 4.7 continues to be valid. Then all the arguments continue to be valid except that the complex analytic flat set $\hat{V}$ is now complex $(n-1)$ dimensional. Therefore, D becomes biholomorphically equivalent to $\hat{V}$, a convex domain in $\mathbb{C}^{n-1}$, and H, the upper half plane in $\mathbb{C}$.

There are some difficulties in the scaling at the singular boundary points, as there may exist nontrivial analytic subsets passing through them. If the point $p = \lim_{j\to\infty} f_j(q) \in \partial D$ is such that

$$\begin{aligned} &\rho_{i_1}(p) = \cdots = \rho_{i_n}(p) = 0 \text{ and} \\ &d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_n}(p) \neq 0, \end{aligned}$$

then the techniques for **scaling at a singular boundary point** in dimension two through a line by line imitation in this n-dimensional case, since there is no non-trivial analytic subset at p contained in ∂D. (And hence, the technique of Proposition 4.4.) In this case, we obtain even better a conclusion that D is biholomorphic to the n-dimensional polydisk.

For the remaining cases, let m be an integer satisfying $1 < m < n$ and consider the case described by

$$\begin{aligned} &\rho_{i_1}(p) = \cdots = \rho_{i_m}(p) = 0, \\ &d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_m}(p) \neq 0, \text{ and} \\ &\rho_\ell(p) \neq 0 \ \forall \ell \notin \{i_1, ..., i_j\}. \end{aligned}$$

Let us start with describing the complex analytic set in the boundary of D passing through p. For each $j \in \{1, ..., m\}$, denote by X_j the (complex $n-1$ dimensional) analytic set through p contained in the real hypersurface

$$\Sigma_j = \{z \in \mathbb{C}^n : \rho_j(z) = 0\}.$$

For convenience, assume without loss of generality that $i_j = j$, for each $j = 1, ..., m$. Again, the convexity of D forces that Σ_j be Euclidean flat. Thus, in particular, the maximal analytic set through p contained in ∂D is, near p, the complex $n-m$ dimensional flat set

$$X(p) := X_1 \cap \cdots \cap X_m.$$

Now we begin analyzing the effect of the scaling of Theorem A in this setting.

Let $X_0 = X(p) - p$. Then there exists $\epsilon > 0$ such that, for any j, the complex analytic disk $h_j : \Delta \to \mathbb{C}^n$ defined by

$$h_j(z) = zv + f_j(q)$$

satisfies $h_j(\Delta) \subset D$, whenever $v \in X_0$ and $\|v\| < \epsilon$. Then, the vectors

$$\xi_j := (f^{-1} \circ h_j)'(0)$$

satisfy

$$\frac{1}{C} \leq \|\xi_j\| \leq C$$

for some constant $C > 0$ independent of j. Taking a subsequence if necessary, let

$$w(v) = \lim_{j\to\infty} \frac{\xi_j}{\|\xi_j\|}.$$

Choose an orthonormal basis $v_1, \ldots, v_{n-m}$ for the span of X_0, and denote by

$$w_\alpha = w(v_\alpha), \text{ for } \alpha = 1, 2, \ldots, n-m.$$

We then claim that

LEMMA 4.8. *The vectors $w_1, \ldots, w_{n-m}$ are linearly independent over $\mathbb{C}$.*

PROOF. Let $a_1, \ldots, a_{n-m} \in \mathbb{C}$ be such that

$$a_1 w_1 + \cdots + a_{n-m} w_{n-m} = 0.$$

Then

$$\begin{aligned} 0 = \sum_{\ell=1}^{n-m} a_\ell w_\ell &= \sum_{\ell=1}^{n-m} a_\ell \lim_{j\to\infty} \frac{(f_j^{-1} \circ h_j)'(0)}{\|(f_j^{-1} \circ h_j)'(0)\|} \\ &= \sum_{\ell=1}^{n-m} a_\ell \lim_{j\to\infty} \frac{1}{\|\partial f_j(q)^{-1} v_\ell\|} \cdot \partial f_j(q)^{-1} v_\ell \\ &= \lim_{j\to\infty} f_j(q)^{-1} \left[\sum_{\ell=1}^{n-m} \frac{a_\ell}{\|\partial f_j(q)^{-1} v_\ell\|} \cdot v_\ell \right]. \end{aligned}$$

Since $\{\|\partial f_j(q)\|\}_j$ is bounded, since $\{\|\partial f_j(q)^{-1}\|\}_j$ is bounded when restricted to the span of X_0, and since $v_1, \ldots, v_{n-m}$ are linearly independent, we may conclude that $a_\ell = 0$ for all $\ell = 1, \ldots, n-m$. This completes the proof. □

Thus, we have proven that the scaled domain

$$\hat{D} = \lim_{j\to\infty} \omega_j(D)$$

(where ω_j's are as in Theorem 4.3) contains in its boundary the convex complex analytic subset

$$\hat{X} = \lim_{j\to\infty} [\partial f_j(q)]^{-1}(X_0) \tag{62}$$

of complex dimension $n - m$.

Now we consider the behavior of the sequence $\{\partial f_j(q)^{-1}\}$ along the directions that are tangential but not complex tangential to the boundary of D. Let ν_k be a unit normal vector to Σ_k at p, for each $k = 1, \ldots, m$. Define

$$u_k = \sqrt{-1}\, \nu_k, \ (k =, \ldots, m).$$

Then again u_k's may be tangential to ∂D, but there is no non-trivial analytic subset at p in the direction of u_k. Thus by a usual normal family argument one can easily show that

$$\lim_{j\to\infty} \|\partial_j(q)^{-1} u_k\| = \infty, \ (k = 1, \ldots, m).$$

Define

$$W := \mathbb{C}u_1 \oplus \cdots \oplus \mathbb{C}u_m$$

and denote by

$$P_j := (W + f_j(q)) \cap D$$

for each j, then

$$\lim_{j\to\infty} [\partial f_j(q)]^{-1}(P_j - f_j(q))$$

exists, up to subsequences, as a convex subset of $\mathbb{C}^n$. Imitating the arguments in (40) through (50) above, one can deduce that this limit set is equivalent to the set

$$H^m = \{(z_1, ..., z_n) \in \mathbb{C}^n : \Im m\, z_1 > 0, \ldots, \Im m\, z_m > 0, \quad \text{and } z_{m+1} = \cdots = z_n = 0\}$$

via a complex Euclidean rigid motion in $\mathbb{C}^n$. Then the remaining arguments are a line by line imitation of those in (55) through (61) in the case of complex two dimension, which implies in turn that D is biholomorphic to $\hat{X} \times H^m$ which is in fact the product of the m-dimensional polydisk and a $(n-m)$-dimensional convex domain. Therefore, Theorem 4.2 follows. In fact, we have proven slightly stronger a conclusion than Theorem 4.2 as follows.

THEOREM 4.9. *Let $D \subset \mathbb{C}^n$ be a bounded convex domain with a piecewise Levi flat boundary. If an orbit of* Aut D *is accumulating at a boundary point, say p, and if p is a boundary point such that*

$$\begin{aligned}&\rho_{i_1}(p) = \cdots = \rho_{i_m} = 0,\\ &d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_m}(p) \neq 0, \text{ and}\\ &\rho_\ell(p) \neq 0 \;\forall \ell \notin \{i_1, ..., i_j\}\end{aligned}$$

for some $m \in \{1, \ldots, n\}$. Then D is biholomorphic to the product $\Delta^m \times D'$ of the complex m-dimensional polydisk and a convex domain of complex dimension $n - m$.

REMARK 4.3. There is a way of relaxing the convexity assumption in the Product Domain Theorem above (an unpublished note by the author). However, a general theorem without convexity is yet to be discovered.

CHAPTER III

Further Scaling with Stability

As final subjects of the lecture notes, we introduce the scaling technique without assuming the existence of the noncompact automorphism orbit which used to play an important role in the scaling methods in the preceding sections. This particular scheme, when accompanied with holomorphic invariants that are stable in the interior under the perturbation of the boundary of the bounded domain in consideration, serves as an effective way of computing the boundary behavior of the complex analytic invariants. We will introduce such a scheme initiated by the author in [37] around 1989, which was originally inspired by Pinchuk's generalization of well-known Alexander's proper mapping theorem in [61] which introduces a version of scaling method without noncompact automorphism orbits accumulating at a boundary point.

Following the chronological order, we begin with the ideas of Pinchuk. Then we will introduce the modified scaling scheme of the author demonstrating the sketch of the ideas of the proof of several different generalization of Klembeck's theorem on the boundary behavior of the holomorphic curvatures of the Bergman metric.

1. Pinchuk's Generalization of The Proper Mapping Theorem

Philosophically speaking, the effect of the scaling technique which magnifies (or, blows up) successively the infinitesimal data in the boundary geometry to the global boundary geometry of much simpler a domain usually called the *model* domain. Changing the viewpoint, the scaling of a domain is the same as the *scaling of the identity mapping* of the domain in consideration.

To illustrate this viewpoint, consider the unit open ball B^2 in $\mathbb{C}^2$, and the identity mapping $I : B^2 \to B^2$. For each j, let $a_j = 1 - 1/j$ and let

$$\varphi_j(z,w) = \left(\frac{z - a_j}{1 - a_j z}, \frac{\sqrt{1 - a_j^2}\, w}{1 - a_j z}\right)$$

Consider also the linear mapping L_j of $\mathbb{C}^2$ defined by

$$L_j(z,w) = \left(\frac{z-1}{1-a_j^2}, \frac{w}{\sqrt{1-a_j^2}}\right)$$

for each $j = 1, 2, \ldots$. Then the *scaling of the identity mapping* by φ_j and L_j which is the sequence

$$\omega_j := L_j \circ I \circ \phi_j : B^2 \to \mathbb{C}^2$$

that in turn gives the scaling we discussed in the preceding section.

Scaling of a Mapping

As the readers might have noticed already, it is possible to perform the scaling of mappings in much more general situations. For instance, consider

(†) *Let Ω and G be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ and let $f : \Omega \to G$ be a holomorphic mapping which admits a sequence of points $p_j \in \Omega$ such that $p_j \to p \in \partial\Omega$ and $q_j := f(p_j) \to q \in \partial G$ as $j \to \infty$.*

Now we discuss what the scaling of f by the sequences p_j and q_j.

At the moment, we assume for the sake of simplicity that

- $p = q =$ the origin in $\mathbb{C}^n$.
- Ω near $p = (0, \ldots, 0)$ is defined by the inequality

$$\Im m\, z_1 + |'z|^2 + o(|'z|^2 + |z_1|) < 0.$$

- G near the origin q is defined by

$$\Im m\, w_1 + |'w|^2 + o(|'w|^2 + |w_1|) < 0.$$

- $p_j = (-i\alpha_j, 0, \ldots, 0)$, $q_j = (-i\beta_j, 0, \ldots, 0)$, for some positive real numbers α_j, β_j for each j.

Then, consider for each j the linear mappings

$$\Phi_j(z_1, \ldots, z_n) := (\alpha_j z_1, \sqrt{\alpha_j}\,'z)$$

$$\Psi_j(w_1, \ldots, w_n) := (\beta_j z_1, \sqrt{\beta_j}\,'w$$

and the holomorphic mapping

$$\hat{f}_j := \Psi_j^{-1} \circ f \Phi_j : \Psi_j^{-1}(\Omega) \to \Phi_j(G)$$

First of all, notice that in the sense of local Hausdorff set convergence we have

$$\lim_{j\to\infty} \Phi_j^{-1}(\Omega) = \mathcal{R}^n = \lim_{j\to\infty} \Psi_j^{-1}(G)$$

where

$$\mathcal{R}^n = \{(z_1,'z) \in \mathbb{C}^n \mid \Re e\, z_1 + |'z|^2 < 0\}.$$

By a standard normal family argument, the holomoprhic mappings $\hat{f}_j$ converge to a holomorphic mapping, say,

$$\hat{f} : \mathcal{R}^n \to \mathcal{R}^n,$$

with $\hat{f}(-i, 0, \ldots, 0) = (-i, 0, \ldots, 0)$. Passing to a linear fractional transformation, $\hat{f}$ gives rise to a holomorphic self-mapping $\tilde{f}$ of the unit open ball in $\mathbb{C}^n$ preserving the origin.

At a glance, the mapping $\tilde{f}$ does not have much reason to have any relations with the original holomorphic mapping f. However, the following is a good start toward seeing that certain properties are preserved throughout the "direct" scaling process even without the presence of the noncompact automorphism orbits.

EXERCISE 1.1. Show that $\tilde{f}$ is proper, if f is proper.

In the above, by a *proper* mapping we mean a continuous mapping such that the preimage of every compact subset of the co-domain in compact.

This case may appear to restricted. However, without such strong restrictions we started with in the direct scaling above, the same (!) conclusion can be obtained with an arbitrary sequence as in (†) by applying Pinchuk's stretching coordinates as introduced in the Section 2 of Chapter II. In this general case, we give

EXERCISE 1.2. Do the same exercise as the above with the general sequences.

Pinchuk's Generalization of The Alexander Proper Mapping Theorem

Such a cicle of ideas result in the theorem of Pinchuk ([61]) which is a generalization of the Proper Mapping Theorem of H. Alexander ([1]). The statements of the theorems are:

THEOREM 1.1 (ALEXANDER). *Every proper holomorphic self-mapping of the open unit ball in* $\mathbb{C}^n$, $n \geq 1$, *is an automorphism.*

THEOREM 1.2 (PINCHUK). *Every proper holomorphic mapping from a bounded strongly pseudoconvex domain with a* C^2 *smooth boundary to another is in fact a biholomorphic mapping.*

In the above, by a *proper mapping* we simply mean a continuous mapping which possesses the property that the preimage of every compact set is compact.

The proof of Alexander's theorem does not use scaling at all. The role of scaling in Pinchuk's theorem is to reduce the proof to the case of Alexander's Theorem. While referring to [60] for any details, we will sketch the ideas in the proof very briefly.

First, one proves that the proper holomorphic mapping is locally one-to-one. Assuming the contrary, one looks at the variety along which the determinant of the holomorphic Jacobian matrix of the proper holomorphic mapping vanishes. Hartogs' extension theorem implies that the variety must meet the boundary of the domain. So choose a sequence on the variety that converges to a boundary point. Then using this sequence, the image sequence under the proper mapping and using the strong pseudoconvexity, we scale the image domain, the proper mapping and the domain of definition simultaneously. The resulting subsequential limit (produced by Montel's theorem) is then a proper holomorphic self-mapping of the open unit ball. Moreover, the way we scale gives a nontrivial variety along which the determinant of the holomorphic Jacobian of the new limit proper holomorphic mapping. This contradicts Alexander's theorem, and hence we can conclude that the original proper holomorphic mapping must be locally one-to-one.

Then finally one argues that the proper holomorphic mapping that is locally one-to-one is indeed a covering map. Appealing to the continuous extension to the boundary, one obtains global injectivity as desired.

2. Asymptotic Behavior of the Bergman Curvature

Another merit of the scaling technique in comparison to the other methods involving analytic invariants is that the requirements on the regularity of the boundary surface of the bounded domains in consideration is quite weak. This point yields new results in Bergman geometry as in [37], [39], [41], [54] and others.

Boundary Behavior Problem

Let us now consider how to obtain information on the boundary behavior of the holomorphic curvature of the Bergman metric of a bounded domain Ω in $\mathbb{C}^n$. More precisely, the general problem is

PROBLEM 2.1. Let Ω be a bounded domain in $\mathbb{C}^n$, and let p_j form a sequence in Ω such that $\lim_{j\to\infty} p_j = p \in \partial\Omega$. Now choose a nonzero vector $\xi_j \in \mathbb{C}^n = T_{p_j}\Omega$ for each j. Then describe the behavior of the sequence of the holomorphic (sectional) curvature of the Bergman metric of Ω at p_j in the direction ξ as $j \to \infty$.

The first result in this direction is due to P. Klembeck [42] which states

THEOREM 2.1 (KLEMBECK). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^∞ smooth strongly pseudoconvex boundary, and let $\{p_j\} \subset \Omega$ be a sequence accumulating at a boundary point. Also let ξ_j be a nonzero vector in $\mathbb{C}^n = T_{p_j}\Omega$ for each j. Then the holomorphic sectional curvature of the Bergman metric of Ω at p_j in the direction ξ_j converges to $-1/(n+1)$ as j tends to ∞.*

This theorem is based upon the celebrated theorem of C. Fefferman which is better known as the *asymptotic expansion of the Bergman kernel function* on the bounded domain with C^∞ smooth strongly pseudoconvex domains in $\mathbb{C}^n$. We state a simpler version of the theorem here which is sufficient for this exposition. For more details and the full version, see [20].

THEOREM 2.2 (FEFFERMAN). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^∞ smooth strongly pseudoconvex boundary. Then there exists a neighborhood U of the boundary $\partial\Omega$ of Ω such that there exist C^∞ smooth functions $\Phi, \Psi : U \cap \overline{\Omega} \to \mathbb{C}$ such that the Bergman kernel function of Ω satisfies*

$$K_\Omega(z,z) = \Phi(z) \cdot \{\text{dist}(z,\partial\Omega)^{-(n+1)} + \Psi(z) \cdot \log \text{dist}(z,\partial\Omega)\}, \ \forall z \in U \cap \Omega$$

where Φ is never zero on $\partial\Omega$.

It is possible to localize the theorem of Klembeck using pseudo-local estimates of the $\overline{\partial}$ operator. This results in that the same conclusion as in Klembeck's theorem is valid if one only assumes that Ω is pseudoconvex, that $\partial\Omega$ is C^∞ smooth, and that $\partial\Omega$ is strongly pseudoconvex at the boundary point at which the reference points p_j approach as $j \to \infty$.

One notices however that C^∞ smoothness is not the natural regularity condition for such a problem, since the minimum regularity condition required for the strong speudoconvexity is C^2. On the other hand, as long as one stays in the circle of ideas depending upon the pseudo-local estimates for the $\overline{\partial}$ and/or the asymptotic expansion formula of the Bergman kernel function, it seems impossible, at least at this writing, to obtain the natural C^2 version of the theorem of Klembeck due to the essential difficulty that both Fefferman's formula and the pseudo-local estimate need C^∞ regularity of the boundary of the domain in question. Successful generalizations of either Fefferman's expansion or the pseudo-local estimates to the C^2 strongly pseudoconvex cases are yet to be discovered if at all possible.

Therefore, the technique we introduce in what follows is indeed new in the sense that both the ideas and the techniques do not depend upon either one of the methods used in Klembeck's work. The following method is in fact much simpler as well. We first introduce the results, and then the ideas of the proof.

THEOREM 2.3 (KIM-YU). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let p be a boundary point of Ω at which the boundary $\partial\Omega$ is C^2 smooth and strongly pseudoconvex. Then for any sequence of points $p_j \in \Omega$ tending to p as $j \to \infty$ and for any sequence of holomorphic sections Π_j in the tangent space of Ω at p_j for each j, the sequence of the holomorphic curvature of the Bergman metric at p_j in the direction Π_j converges to the constant $-4/(n+1)$.*

Several new results follows from this theorem. For details and some of the applications, see [41].

Let us now discuss first how one can use the scaling technique to obtain such a result.

Conversion of Boundary Problem to Interior Problem

Here, we will demonstrate that one can convert the problem on the boundary behavior [or equivalently, the asymptotic behavior] of the holomorphic (sectional) curvature of the Bergman metric to an *interior stability problem* of the Bergman kernel function under a suitable perturbation of the boundary. Such a frame of ideas was introduced first by the author around 1989 (See [37]) as follows:

Let Ω be a bounded domain in $\mathbb{C}^n$, let p_j form a sequence in Ω converging to a boundary point $p \in \partial\Omega$, and let $\xi_j \in \mathbb{C}^n = T_{p_j}\Omega$ for each j. Assume without loss of generality that p is the origin. Then find a suitable sequence of complex linear isomorphisms $L_j : \mathbb{C}^n \to \mathbb{C}^n$ such that

(A) The open sets $L_j(\Omega)$ converge to an open set, say $\hat{\Omega}$, in an appropriate sense such as the defining functions of $L_j(\Omega)$ converging locally uniformly to the defining function of $\hat{\Omega}$ as $j \to \infty$, for instance;
(B) The Bergman curvature of $\hat{\Omega}$ is known;
(C) There exists a compact subset F of $\hat{\Omega}$ which stays inside $L_j(\Omega)$ uniformly bounded away from the boundary for all large j; and
(D) On the compact set $F \times F$, the Bergman kernel functions $K_{L_j(\Omega)}(z,\zeta)$ of the domains $L_j(\Omega)$ and $K_{\hat{\Omega}}(z,\zeta)$ of the domain $\hat{\Omega}$ satisfy that

$$\lim_{j\to\infty} \sup_{F\times F} \left| K_{\hat{\Omega}}(z,\zeta) - K_{L_j(\Omega)}(z,\zeta) \right| = 0$$

Notice that the Bergman kernel function $K(z,\bar{\zeta})$ is holomorphic in both z and ζ variables. Therefore, *if all four conditions above hold*, then (D) and the Cauchy estimates imply immediately that the partial derivatives of every order of the Bergman kernel function converges uniformly on compact subsets of $F \times F$. Thus the interior stability of the holomorphic curvatures in ξ directions of the Bergman metric follows. More precisely, let $S_G(x,\xi)$ denote the holomorphic (sectional) curvature of the Bergman metric of the bounded domain G at x in the holomorphic direction generated by the tangent vector ξ to G at x. Choosing a subsequence, assume that

$$\lim_{j\to\infty} dL_j(\xi)/\|dL_j(\xi_j)\| = \hat{\xi}.$$

Then we get

$$\begin{aligned} S_\Omega(p_j,\xi_j) &= S_{L_j(\Omega)}(L_j(p_j), dL_j(\xi_j)) \\ &\to S_{\hat{\Omega}}(q,\hat{\xi}), \text{ as } j\to\infty. \end{aligned}$$

Consequently, one obtains the limiting behavior of the holomorphic curvature tensor S_Ω along $(p_j,\xi_j) \in T\Omega$ in terms of the interior curvature behavior of the Bergman metric of the domain $\hat{\Omega}$.

More Discussions and Details

If the boundary of the domain Ω in question is C^2 smooth strongly pseudoconvex at $p \in \partial\Omega$ in the preceding section, and if Ω is pseudoconvex, then after some modifications the above scheme in (A) through (D) is indeed valid with $\hat{\Omega}$ being biholomorphic to the unit ball. While referring to [41] for any detailed arguments, we will discuss the ideas, difficulties and appropriate techniques in what follows.

The most serious difficulty of all in this process is proving (D). With a general pseudoconvex domain Ω with a C^2 strongly pseudoconvex boundary point p as in the above, it is not clear if (D) can hold in general. But if Ω can be replaced by a very small piece of Ω near p, then (D) holds due to rather simple arguments. (See for instance [37] and [39].) Thus, an effective localization of the Bergman curvature is necessary to get around such difficulties. Indeed, in [41] a sharp estimate of the localization of the Bergman curvature is proven, based upon the sharp estimates for the localization of the minimum integrals introduced by S. Bergman. We point out that for such estimates, the L^2 estimates of the $\bar{\partial}$ operator by Hörmander with appropriate weights is enough as clearly remarked in [41]. Hence, no regularity assumption beyond the existence of the local peaking function at the boundary point p in consideration is necessary for the localization in this arguments.

Combining this with the scaling scheme in (A) to (D) the theorem by Kim and Yu follows immediately. We choose not to include any further details in this notes.

REMARK 2.1. The scheme in (A) through (D) above together with the aforementioned localization arguments should be applicable to much broader a collection of pseudoconvex domains. For essentially locally strictly/strongly convex domains, even with the presence of certain singular boundary points, the asymptotic behavior of the holomorphic curvature of the Bergman metric is understood to a considerable extent. On the other hand, weakly pseudoconvex finite type domains, verifying (B) seems very difficult in contrast to the strongly pseudoconvex cases. At this writing, we are not aware of any effective way of circomventing such difficulties.

In the light of the above dicussions, it may be fair to pose the following interesting problem:

PROBLEM 2.2. Compute the (holomorphic) curvature tensor of the Siegel domains.

Only a very simple cases has been treated. See [37].

3. Final Remarks

Although simple in its ideas, the scaling technique seems prosper in the geometric theory of Several Complex Variables. Different, yet equally or even more powerful, scaling schemes can be found in the recent papers by J. McNeal [54] and other papers of his, and work of D. Barrett [2] concerning another important aspects of the Bergman kernel function. As far as the Bergman kernel function itself is concerned, McNeal's results seem most general and powerful for the essentially convex domains with finite type boundaries.

We refrain ourselves from introducing any further topics and details here.

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