수 학 강 의 록 제 12 권

Applications of Spectral Geometry to Geometry and Topology

PETER B. GILKEY

서 울 대 학 교
수학연구소 · 대역해석학 연구센터

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Peter B. Gilkey, University of Oregon, Eugene Oregon 97403, U.S.A.

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0. INTRODUCTION

These notes were prepared in conjunction with a series of lectures given at Seoul National University in July 1993. They are intended as a brief introduction to a subject which has been very important over the past 10 years and which is still very much alive today i.e. the application of techniques of global analysis to both differential geometry and topology. To illustrate the sort of applications possible, we shall use heat equation methods to prove the Gauss Bonnet theorem, the Hirzebruch signature theorem, and Milnor's theorem that non-singular algebra structures exist on R^n only for $n = 1, 2, 4, 8$. There are many other applications.

We begin in Chapter I by reviewing the analytic preliminaries we shall need. §1.1 deals with the spectral theory of elliptic self-adjoint operators, and in particular those of Dirac and Laplace type. We also discuss spherical harmonics as these provide a convenient set of examples. In §1.2, we turn to the heat equation asymptotics, the zeta function, and the eta function. We discuss the relationship between these invariants and compute some specific examples. We conclude Chapter I in §1.3 by discussing the formula for the index of an elliptic operator in terms of heat equation asymptotics found by Bott and by introducing both the de Rham and the Hirzebruch signature complexes.

We turn our attention in Chapter II to differential geometry. In §2.1, we discuss the characteristic classes in terms of curvature. We define the Euler polynomial, the Chern character, and the Hirzebruch L polynomial; these play a crucial role in the index theorem. In §2.2, we discuss invariance theory; invariance theory to build a link between the invariants of the heat equation and the characteristic classes. We discuss the Weyl theorem on the invariants of the orthogonal group. We state some formulas for the invariants of the heat equation on differential forms derived by Patodi to illustrate some of the concepts involved. We conclude with a technical result from invariance theory giving a diagonal form. In §2.3, we compute some normalizing constants by evaluating the Euler form on even dimensional spheres and the Hirzebruch L polynomial on products of complex projective spaces to show that the Euler characteristic and signature result in these examples.

We conclude in Chapter III by discussing some topological applications. §3.1 deals with the Gauss-Bonnet theorem and §3.2 with the Hirzebruch signature formula. We use the invariants of the heat equation discussed in Chapter I to provide us with local formulas for the index of the respective elliptic complexes. The invariance theory of §2.2 is then used to identify these invariants with the appropriate characteristic classes of §2.1. In §3.3, we change focus slightly and use the eta invariant rather than the index to define a topological invariant in K and to prove Milnor's theorem. We conclude in §3.4 by giving a brief introduction to the Lefschetz fixed point formulas.

At the end of these notes, we provide a brief list of some of the most important references in this subject; we refer to the bibliography of Professor Schroeder which will be part of the second edition of Gilkey's book (to appear fall 93 with CRC press) for a more complete set of references.

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1. ANALYTIC RESULTS

§1.1 Spectral theory

Let M be a compact Riemannian manifold without boundary. Let

$$x = (x^1, ..., x^m) \tag{1.1.1}$$

be a system of local coordinates on M. Let

$$\partial_i := \frac{\partial}{\partial x^i} \text{ and } dx^i \tag{1.1.2}$$

be the coordinate frames for the tangent and cotangent bundles of M. We adopt the Einstein convention and sum over repeated indices. Thus, for example, the metric g can be expanded locally in the form:

$$g = g_{ij} dx^i \circ dx^j. \tag{1.1.3}$$

Let g^{ij} be the inverse matrix; these are the components of the dual metric on the cotangent bundle T^*M. The Riemannian volume element takes the form locally:

$$|\mathrm{dvol}\,| = \sqrt{\det(g_{ij})}|dx^1 \cdots dx^m|. \tag{1.1.4}$$

We use the absolute value signs to emphasize this is a measure not a differential form as this will be important later.

Let $\mathfrak{V}^{\mathrm{U}}(M)$ be the set of smooth complex vector bundles over M which are equipped with a positive definite Hermitian inner product. If $V \in \mathfrak{V}^{\mathrm{U}}(M)$, let $C^\infty(V)$ be the space of smooth sections to V and let $\mathrm{End}\,(V)$ be the bundle of endomorphisms of V. Let α be a multi-index and let ∂_x^α denote multiple partial differentiation. If D is a k^{th} order partial differential operator on $C^\infty(V)$, we may decompose D locally in the form:

$$D = \Sigma_{|\alpha| \le k} p_\alpha(x) \partial_x^\alpha. \tag{1.1.5}$$

The leading symbol of D is defined to be

$$\sigma_L(D) := (\sqrt{-1})^k \Sigma_{|\alpha|=k} p_\alpha(x)\xi^\alpha \tag{1.1.6}$$

where we formally replace differentiation by function multiplication. This is invariantly defined on T^*M.

A second order partial differential operator D on $C^\infty(V)$ is said to be of **Laplace type** if the leading symbol of D is given by the metric tensor. This means that locally D has the form:

$$D = -(g^{ij} I_V \partial_i \partial_j + A^i \partial_i + B) \tag{1.1.7}$$

where A^i, $B \in C^\infty \mathrm{End}(V)$. Let $\mathfrak{L}(V)$ be the set of all self-adjoint operators of Laplace type.

The following is a brief summary of the spectral theory of such operators. Choose a connection ∇ on V. We use the Levi-Civita connection and ∇ to covariantly differentiate tensors of all types. If $\phi \in C^\infty(V)$, let $\nabla^k \phi \in \otimes^k T^*M \otimes V$ and let

$$||\phi||_{\infty,k} = \sup{}_{x \in M} \Sigma_{j \le k} ||\nabla^j \phi(x)||; \tag{1.1.8}$$

since M is compact, different metrics on M and different connections on V define equivalent norms.

Theorem 1.1.1: *Let $D \in \mathfrak{L}(V)$ be self-adjoint. There exists a complete orthonormal basis $\{\phi_\nu\}$ for $L^2(V)$ so that:*

(a) *The $\phi_\nu \in C^\infty(V)$ and $D\phi_\nu = \lambda_\nu \phi_\nu$.*

(b) *For any $k \in \mathbf{N}$, there exists $C(k)$ and $\ell(k)$ so that*

$$||\phi||_{\infty,k} \le C_k (1 + |\lambda_\nu|)^{\ell(k)}.$$

(c) *Only a finite number of eigenvalues are negative. Order $\lambda_1 \le \lambda_2 \le \ldots$ There exists a positive constant C so*

$$\lim_{n \to \infty} n^{-2/m} \lambda_n = C.$$

(d) *Decompose $\phi \in L^2(V) = \Sigma_\nu c_\nu \phi_\nu$ where $c_\nu = (\phi, \phi_\nu)_{L^2}$ are the Fourier coefficients. Then*

$$\phi \in C^\infty(V) \;\Leftrightarrow\; \Sigma_\nu \nu^k |c_\nu| < \infty \;\forall k \in \mathbf{N}.$$

Let $D = -\partial_\theta^2$ on S^1. Then $\{e^{in\theta}, n^2\}_{n\in\mathbf{Z}}$ is the spectral resolution of D. The expansion $\phi = \Sigma_n c_n e^{in\theta}$ is the usual expansion in terms of Fourier series. More generally, spherical harmonics give the decomposition of the Laplacian Δ_θ on the unit sphere S^m of $\mathbf{R}^{m+1}$. Let

$$S(m+1,j) = \{f \in \mathbf{C}[x^1,...,x^{m+1}] : f(t\vec{x}) = t^j f(\vec{x}) \text{ for } t \in \mathbf{C}\} \quad (1.1.9)$$

be the vector space of polynomials in the $\{x^i\}$ variables which are homogeneous of degree j. Let $\Delta_e = -\partial_1^2 - ... - \partial_{m+1}^2$ be the Euclidean Laplacian. Let

$$H(m,j) = \{f \in S(m+1,j) : \Delta_e f = 0\} \quad (1.1.10)$$

be the subspace of harmonic polynomials; identify a harmonic polynomial with its restriction to S^m. Let $r = |x|^2 = x_1^2 + ... + x_{m+1}^2$.

Theorem 1.1.2:

(a) $\dim\{S(m+1,j)\} = \binom{m+j}{m}$.

(b) $S(m+1,j) = r^2 S(m+1,j-2) \oplus H(m,j)$.

(c) $\dim\{H(m,j)\} = \binom{m+j}{m} - \binom{m+j-2}{m}$.

(d) $\{j(j+m-1), H(m,j)\}_{j=0}^\infty$ *is the spectral resolution of* Δ_θ.

Remark: If $m = 1$, let $z = x_1 + ix_2 \in S(2,1)$. Then

$$H(1,j) = \begin{cases} \text{Span}\,(1) & \text{if } j = 0, \\ \text{Span}\,(z^j, \bar{z}^j) & \text{if } j > 0. \end{cases} \quad (1.1.11)$$

Consequently the spectral resolution $-\partial_\theta^2$ is the Fourier series decomposition discussed above:

$$L^2(S^1) = \oplus_j e^{ij\theta} \cdot \mathbf{C}. \quad (1.1.12)$$

Proof: We prove (a) by induction using the following relationships:

$$\begin{aligned} &S(m+1,j) = x_{m+1} \cdot S(m+1,j-1) \oplus S(m,j), \\ &\dim\{S(m+1,j)\} = \dim\{S(m+1,j-1)\} + \dim\{S(m,j)\}, \\ &\dim\{S(m+1,0)\} = 1, \text{ and } \dim\{S(1,j)\} = 1. \end{aligned}$$

If $p = \Sigma_\alpha p_\alpha x^\alpha \in S(m+1,j)$, let

$$P(p) := \Sigma_\alpha p_\alpha \partial_\alpha. \quad (1.1.13)$$

Define a positive definite symmetric bilinear inner product $\langle\cdot,\cdot\rangle$ on the space of homogeneous polynomials $S(m+1,j)$ by:

$$\langle p,q\rangle := P(p)(\bar{q}) = \Sigma_{\alpha,\beta}\, p_\alpha \partial_\alpha\{\bar{q}_\beta x^\beta\} = \Sigma_\alpha \alpha! p_\alpha \bar{q}_\alpha. \tag{1.1.14}$$

Let $p \in S(m+1,j-2)$ and $q \in S(m+1,j)$. Since $P(r^2) = -\Delta_e$,

$$-\langle p, \Delta_e q\rangle = \langle r^2 p, q\rangle. \tag{1.1.15}$$

Multiplication by r^2 is injective. Since $\operatorname{coker}(r^2) = \mathfrak{N}(\Delta_e)$, (b) and (c) follow.

We have identified a harmonic function with its restriction to S^m. Let

$$\mathcal{A} = \Sigma_j H(m,j) \subset C^\infty(S^m) \tag{1.1.16}$$

be the subspace generated by the $H(m,j)$. Since $r^2|_{S^m} = 1$, we use (b) to see:

$$\begin{aligned} &\Sigma_{\nu\le 2j} H(m,\nu) = \{S(m+1,2j) + S(m+1,2j-1)\}|_{S^m} \\ &\mathcal{A} = \cup_j \{S(m+1,2j) + S(m+1,2j-1)\}|_{S^m}. \end{aligned} \tag{1.1.17}$$

Since

$$S(m+1,j)\cdot S(m+1,k) \subset S(m+1,j+k), \tag{1.1.18}$$

$\mathcal{A}$ is a sub-algebra of $C^\infty(S^m)$. Since $1 \in H(m,0)$, $\mathcal{A}$ is unital. Since the coordinate functions $x^i \in H(m,1)$, $\mathcal{A}$ separates points. Thus by the Stone-Weierstrauss theorem, $\mathcal{A}$ is dense in $C^\infty(S^m)$ so

$$L^2(S^m) = \bar{\mathcal{A}}. \tag{1.1.19}$$

We introduce polar coordinates $x = (r,\theta)$ for $r \in [0,\infty)$ and $\theta \in S^m$ on $\mathbf{R}^{m+1}$ to express the Euclidean Laplacian in the form

$$\Delta_e = -\partial_r^2 - mr^{-1}\partial_r + r^{-2}\Delta_\theta. \tag{1.1.20}$$

If $f \in H(m,j)$, then $\Delta_e(f) = 0$ so (1.1.20) implies

$$\Delta_\theta f(\theta) = j(j+m-1)f(\theta). \tag{1.1.21}$$

Since Δ_θ is self-adjoint, the eigenspaces $E(\lambda,\Delta_\theta$ satisfy

$$E(\lambda,\Delta_\theta) \perp E(\mu,\Delta_\theta) \text{ for } \lambda \ne \mu. \tag{1.1.22}$$

Since

$$H(m,j) \subseteq E(j(j+m-1), \Delta_{S^m}), \tag{1.1.23}$$

$H(m,j)$ and $H(m,k)$ are orthogonal in $L^2(S^m)$ for $j \neq k$. This shows

$$\begin{aligned} L^2(S^m) &= \oplus_j H(m,j) \\ H(m,j) &= E(j(j+m-1), \Delta_{S^m}). \blacksquare \end{aligned} \tag{1.1.24}$$

Let $\Lambda^p M$ be the bundle of p forms on M and let

$$d_p : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M \tag{1.1.25}$$

be exterior differentiation. Let

$$\delta_p : C^\infty \Lambda^{p+1} M \to C^\infty \Lambda^p M \tag{1.1.26}$$

be the dual, interior multiplication. Then the Laplacian

$$\Delta_p = \delta_p d_p + d_{p-1}\delta_{p-1} \in \mathfrak{L}(\Lambda^p M). \tag{1.1.27}$$

Define the de Rham cohomology groups by:

$$H^p(M) := \mathfrak{N}(d_p)/\mathfrak{R}(d_{p-1}). \tag{1.1.28}$$

The de Rham theorem provides an isomorphism between these cohomology groups and the ordinary topological cohomology groups. The Hodge decomposition theorem relates these cohomology groups to spectral theory:

Theorem 1.1.3 (Hodge decomposition theorem):

(a) *If $\phi \in C^\infty(\Lambda^p M)$, then $\phi \in \mathfrak{N}(\Delta_p)$ if and only if $d_p\phi = 0$ and $\delta_{p-1}\phi = 0$. $\mathfrak{N}(\Delta_p)$ is finite dimensional.*

(b) *$C^\infty(\Lambda^p M) = \mathfrak{N}(\Delta_p) \oplus d_{p-1}(C^\infty(\Lambda^{p-1}M)) \oplus \delta_p(C^\infty(\Lambda^{p+1}M))$ is a direct sum decomposition which is orthogonal with respect to the L^2 inner product.*

(b) *The inclusion map is an isomorphism from $\mathfrak{N}(\Delta_p)$ to $H^p(M)$.*

Proof: We use Theorem 1.1.1. Let $\{\phi_\nu, \lambda_\nu\}$ be a spectral resolution of Δ_p. Since $\lim_{n\to\infty} \lambda_n = \infty$, only a finite number of eigenvalues are zero so $\mathfrak{N}(\Delta_p)$ is finite dimensional. Let $\phi \in \mathfrak{N}(\Delta_p)$. We compute:

$$\begin{aligned} 0 =&(\Delta_p\phi, \phi)_{L^2} = (d_{p-1}\delta_{p-1}\phi, \phi)_{L^2} + (\delta_p d_p \phi, \phi)_{L^2} \\ =&(\delta_{p-1}\phi, \delta_{p-1}\phi)_{L^2} + (d_p\phi, d_p\phi)_{L^2}. \end{aligned} \tag{1.1.29}$$

Consequently $\Delta_p\phi = 0$ implies $\delta_{p-1}\phi = 0$ and $d_p\phi = 0$; this proves (a) as the reverse implication is immediate.

Expand $\phi \in C^\infty(\Lambda^p M)$ in the form $\phi = \Sigma_\nu c_\nu \phi_\nu$ for $c_\nu = (\phi, \phi_\nu)_{L^2}$. Let

$$\Phi_0 := \Sigma_{\lambda_\nu = 0} c_\nu \phi_\nu \text{ and } \Phi_1 := \Sigma_{\lambda_\nu \neq 0} c_\nu \lambda_\nu^{-1} \phi_\nu. \tag{1.1.30}$$

Since $\mathfrak{N}(\Delta_p)$ is finite dimensional, the sum defining Φ_0 is finite and hence $\Phi_0 \in C^\infty(\Lambda^p M)$. Since $\phi \in C^\infty(\Lambda^p M)$,

$$\Sigma_\nu \nu^k |c_\nu| < \infty \ \forall k. \tag{1.1.31}$$

The non-zero eigenvalues of Δ_p are uniformly bounded away from zero. Consequently

$$\Sigma_{\lambda_\nu \neq 0} \nu^k |c_\nu| \lambda_\nu^{-1} < \infty \tag{1.1.32}$$

for all k and hence $\Phi_1 \in C^\infty(\Lambda^p M)$. We note:

$$\Phi_0 \in \mathfrak{N}(\Delta_p) \text{ and } \Delta_p \Phi_1 = \Sigma_{\lambda_\nu \neq 0} c_\nu \phi_\nu. \tag{1.1.33}$$

Consequently $\phi = \Phi_0 + \Delta_p \Phi_1$. This shows we may express

$$\begin{aligned} &\phi = \Phi_0 + d_{p-1}(\delta_{p-1}\Phi_1) + \delta_p(d_p\Phi_1) \text{ so} \\ &C^\infty(\Lambda^p) = \mathfrak{N}(\Delta_p) + d_{p-1}C^\infty(\Lambda^{p-1}M) + \delta_p(C^\infty(\Lambda^{p+1}M)). \end{aligned} \tag{1.1.34}$$

We complete the proof of (b) by showing this is an orthogonal direct sum decomposition:

$$\begin{aligned} &(\Phi_0, d_{p-1}\psi_{p-1})_{L^2} = (\delta_{p-1}\Phi_0, \psi_{p-1})_{L^2} = 0, \\ &(\Phi_0, \delta_p\psi_{p+1})_{L^2} = (d_p\Phi_0, \psi_{p+1})_{L^2} = 0, \\ &(d_{p-1}\psi_{p-1}, \delta_p\psi_{p+1})_{L^2} = (d_p d_{p-1}\psi_{p-1}, \psi_{p+1})_{L^2} = 0. \end{aligned} \tag{1.1.35}$$

If $\phi \in \mathfrak{N}(\Delta_p)$, then $d_p\phi = 0$ so $\phi \in \mathfrak{N}(d_p)$ and

$$\phi \to [\phi] \in H^p(M) \tag{1.1.36}$$

is a well defined map from $\mathfrak{N}(\Delta_p)$ to $H^p(M)$. If $\phi = d_{p-1}\psi_{p-1}$, then

$$\phi \in \mathfrak{N}(\Delta_p) \cap \mathfrak{R}(d_{p-1}) = \{0\}; \tag{1.1.37}$$

this shows this correspondence is 1-1. Let $\psi \in \mathfrak{N}(d_p)$. We expand

$$\psi = \Phi_0 + d_{p-1}(\psi_{p-1}) + \delta_p(\psi_{p+1}). \tag{1.1.38}$$

We complete the proof by showing $\delta_p(\psi_{p+1}) = 0$ so $[\psi] = [\Phi_0]$ and the correspondence is onto:

$$\begin{aligned} 0 =&(d_p\psi, \psi_{p+1})_{L^2} \\ =&(d_p\Phi_0, \psi_{p+1})_{L^2} + (d_p d_{p-1}\psi_{p-1}, \psi_{p+1})_{L^2} \\ &+ (d_p\delta_p\psi_{p+1}, \psi_{p+1})_{L^2} \\ =&0 + 0 + (\delta_p\psi_{p+1}, \delta_p\psi_{p+1})_{L^2}. \blacksquare \end{aligned} \tag{1.1.39}$$

We conclude this subsection with a brief description of the Hodge $\star$ operator. Let $\omega_k \cdot \theta_k$ be the inner product on $\Lambda^k M$ arising from the Riemannian metric. Define the Hodge $\star$ operator

$$\star_k \in C^\infty(\mathrm{Hom}\,(\Lambda^k M, \Lambda^{m-k} M)) \text{ by } \omega_1 \wedge \star\omega_2 = (\omega_1 \cdot \omega_2)\mathrm{dvol}. \tag{1.1.40}$$

The following is well known.

Lemma 1.1.4:

(a) $\star_{m-k}\star_k = (-1)^{k(m-k)}$.

(b) $\delta_k = (-1)^{mk+1} \star_{m-k} d_{m-k-1} \star_{k+1}$.

(c) $\star\Delta_p = \Delta_{m-p} \star$.

Remark: $\star_k$ defines an isomorphism from $\mathfrak{N}(\Delta_k)$ to $\mathfrak{N}(\Delta_{m-k})$ which is the realization of Poincaré duality.

§1.2 Heat equation, zeta, and eta functions

Let $D \in \mathfrak{L}(V)$. The heat equation is the system of equations for $t > 0$:

(1) $(\partial_t + D)h(x,t) = 0$. (Evolution equation)

(2) $\lim_{t \to 0} h(x,t) = \phi(x)$. (Initial condition)

Let $\{\phi_\nu, \lambda_\nu\}$ be a spectral resolution of D. Decompose $\phi \in L^2(V)$ in a Fourier series:

$$\phi(x) = \Sigma_\nu c_\nu \phi_\nu(x) \text{ for } c_\nu := (\phi, \phi_\nu)_{L^2(V)}. \tag{1.2.1}$$

For $t > 0$, and $x \in M$, define:

$$(e^{-tD}\phi)(x,t) := \Sigma_\nu e^{-t\lambda_\nu} c_\nu \phi_\nu(x). \tag{1.2.2}$$

Since $|c_n| \le ||\phi||_{L^2}$, we may use Theorem 1.1.1 to estimate:

$$||e^{-t\lambda_\nu} c_\nu \phi_\nu||_{\infty,k} \le C(k)||\phi||_{L^2} e^{-t\lambda_\nu}(1 + |\lambda_\nu|)^{\ell(k)}. \tag{1.2.3}$$

We use Theorem 1.1.1 to see there exists $\epsilon > 0$ and ν_0 so

$$\lambda_\nu \ge \nu^\epsilon \text{ for } \nu \ge \nu_0. \tag{1.2.4}$$

The estimate $e^{-t\lambda}\lambda^\ell \le C(\ell)t^{-\ell}e^{-t\lambda/2}$ for $\lambda > 0$ then permits us to estimate that if $\nu \ge \nu_0$,

$$||e^{-t\lambda_\nu} c_\nu \phi_\nu||_{\infty,k} \le C(\ell)t^{-\ell}e^{-t\nu^\epsilon} . \tag{1.2.5}$$

This implies that the series in (1.2.2) converges absolutely to define a C^k function; the convergence is uniform in t if we bound t away from zero. Since this holds for all k,

$$e^{-tD}\phi \in C^\infty(V) \text{ for } t > 0. \tag{1.2.6}$$

It is immediate from the definition that $e^{-tD}\phi$ satisfies the differential equation:

$$(\partial_t + D)(e^{-tD}\phi) = 0. \tag{1.2.7}$$

It is also immediate that if $\phi \in C^\infty$, then $e^{-tD}\phi \to \phi$ in C^∞ as $t \to 0$. Thus (1.2.2) defines the solution to the heat equation.

The operator e^{-tD} is defined by a smooth kernel function. Let

$$K(t,x,y) := \Sigma_n e^{-t\lambda_n}\phi_n(x) \otimes \bar\phi_n(y) \in \mathrm{End}\,(V_y, V_x); \tag{1.2.8}$$

the estimates given above show that this series converges uniformly and absolutely in the $||_{\infty,k}$ norm so K is C^∞ for $t > 0$. It is then immediate from the definition that:

$$\begin{aligned}&\textstyle\int_M K(t,x,y)\phi(y)|\mathrm{dvol}\,(y)| = \Sigma_n e^{-t\lambda_n} c_n \phi_n(x)\\ &= (e^{-tD}\phi)(x).\end{aligned} \tag{1.2.9}$$

Example 1.2.1: Let $D = -\partial_\theta^2$ on $C^\infty(S^1)$. Then

$$K(t,x,y) = \Sigma_n e^{-tn^2} e^{in(x-y)}. \tag{1.2.10}$$

The operator e^{-tD} is of trace class on $L^2(V)$ for $t > 0$;

$$\begin{aligned}\mathrm{Tr}_{L^2}(e^{-tD}) &:= \Sigma_\nu e^{-t\lambda_\nu}\\ &= \textstyle\int_M \mathrm{Tr}_{V_x} K(t,x,x)|\mathrm{dvol}\,(x)|;\end{aligned} \tag{1.2.11}$$

again the estimate $\lambda_\nu \geq \nu^\epsilon$ for $\nu \geq \nu_0$ shows this series converges uniformly if we bound t away from zero. Then

$$\mathrm{Tr}_{L^2}(e^{-tD}) = \textstyle\int_M \mathrm{Tr}_{V_x} K(t,x,x)|\mathrm{dvol}\,(x)|. \tag{1.2.12}$$

The behavior of the trace of the fundamental solution to the heat equation is of particular interest as $t \downarrow 0$.

Theorem 1.2.1: *As $t \downarrow 0$, there is an asymptotic expansion*

$$\mathrm{Tr}_{L^2}(e^{-tD}) \sim \sum_{n=0}^{\infty} a_n(D) t^{(n-m)/2}.$$

The $a_n(D)$ vanish if n is odd. If n is even, there exist local invariants $a_n(x,D)$ of the jets of the total symbol of D so

$$a_n(D) = \textstyle\int_M a_n(x,D)|\mathrm{dvol}\,(x)|.$$

More generally, if $f \in C^\infty(M)$, then

$$\mathrm{Tr}_{L^2}(fe^{-tD}) \sim \sum_{n=0}^{\infty} a_n(f,D) t^{(n-m)/2}$$

where $a_n(f,D) = \int_M f(x) a_n(x,D)|\mathrm{dvol}\,(x)|$.

Although we have chosen to study the heat equation, there is a close relationship with the zeta function. Let $D \in \mathfrak{L}(V)$ be positive semi-definite. Let $\mathfrak{N}(D)$ be the null space of D. We define the zeta function:

$$\zeta(s, D) := \Sigma_{\lambda_\nu > 0} \lambda_\nu^{-s}; \tag{1.2.13}$$

the estimates of Theorem 1.1.1 show this is well defined and holomorphic for $\mathrm{Re}(s) >> 0$.

Example 1.2.2: Let $D = -\partial_\theta^2$ on $C^\infty(S^1)$. Then $\zeta(s, D)$ is the Riemann zeta function

$$\zeta(s, D) = 2 \sum_{n=1}^{\infty} n^{-2s}. \tag{1.2.14}$$

More generally, if Δ_θ is the spherical Laplacian on S^m, we use Theorem 1.1.2 to see:

$$\begin{aligned} &\zeta(s, \Delta_\theta) = \Sigma_{j \geq 1} \{ \tbinom{m+j}{m} - \tbinom{m+j-2}{m} \} \{ j(j + m - 1) \}^{-s} \\ &\zeta(s, \Delta_\theta + \tfrac{1}{2}(m-1)) = \Sigma_{j \geq 0} \{ \tbinom{m+j}{m} - \tbinom{m+j-2}{m} \} \\ &\quad \cdot (j + \tfrac{1}{2}(m-1))^{-2s}. \end{aligned} \tag{1.2.15}$$

We use the Mellin transform to relate the zeta function to the invariants of the heat equation. For $\mathrm{Re}(s) > 0$, we define the gamma function:

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt. \tag{1.2.16}$$

We use the functional equation

$$s\Gamma(s) = \Gamma(s+1) \tag{1.2.17}$$

to extend Γ to a meromorphic function on $\mathbf{C}$ with isolated simple poles at

$$s = 0, \ -1, \ -2, \ .... \tag{1.2.18}$$

There is an infinite product formula for Γ which shows Γ is never zero. Let

$$\begin{aligned} &\tilde{a}_n(D) = \begin{cases} a_n(D) & \text{if } n \neq m, \\ a_m(D) - \dim \mathfrak{N}(D) & \text{if } n = m, \end{cases} \\ &h(t) := \mathrm{Tr}_{L^2}(e^{-tD}) - \dim \mathfrak{N}(D) \\ &\quad \sim \sum_{n=0}^{\infty} a_n(D) t^{(n-m)/2}. \end{aligned} \tag{1.2.19}$$

Theorem 1.2.2: *Let $D \in \mathfrak{L}(V)$ be positive semi-definite.*

(a) *If* $\mathrm{Re}(s) >> 0$, *then* $\Gamma(s)\zeta(s,D) = \int_0^\infty t^{s-1}h(t)dt$.

(b) $\Gamma(s)\zeta(s,D)$ *has a meromorphic extension to* $\mathbf{C}$ *with isolated simple poles at the values* $s = (m-n)/2$ *for* $n \in \mathbf{N}$. *Furthermore,*

$$\mathrm{Res}_{\,s=(m-n)/2}\Gamma(s)\zeta(s,D) = \tilde{a}_n(D).$$

Remark: Let s be a non positive integer. Then $\Gamma(\cdot)$ has a simple pole at s with non zero residue. We use (b) to see ζ is regular at s. In particular, since $\mathrm{Res}_{\,s=0}\Gamma(s) = 1$, ζ is regular at 0 and $\zeta(0,D) = \tilde{a}_m(D)$. All the $\tilde{a}_n$ are locally computable except $n = m$.

Proof: Since we have subtracted off the effect of the 0-spectrum of D, h is exponentially decreasing as $t \to \infty$. Therefore $h(t)t^{s-1}$ is integrable for $\mathrm{Re}(s) > m/2$. We use the following identity to prove (a):

$$\int_0^\infty t^{s-1}e^{-\lambda t}dt = \lambda^{-s}\Gamma(s). \tag{1.2.20}$$

To prove (b), we decompose $\Gamma(s)\zeta(s) = \mathcal{F}_0 + \mathcal{F}_1$ where:

$$\mathcal{F}_0(s) := \int_0^1 t^{s-1}h(t)dt, \quad \text{and} \tag{1.2.21}$$

$$\mathcal{F}_1(s) := \int_1^\infty t^{s-1}h(t)dt. \tag{1.2.22}$$

Since $|h(t)| \le e^{-\delta t}$ for $t \ge 1$, (1.2.22) converges exponentially and $\mathcal{F}_1(s)$ is entire. Given k, we can find $N = N(k)$ so that

$$\begin{aligned} &h(t) = \Sigma_{n \le N(k)}\tilde{a}_n(D)t^{(n-m)/2} + R_N(t), \text{ and} \\ &|R_N(t)| \le c_k t^{-k}. \end{aligned} \tag{1.2.23}$$

Consequently,

$$\begin{aligned} &\mathcal{F}_0(s) = \Sigma_{n \le N}\tilde{a}_n(D)(s + (n-m)/2)^{-1} + \mathcal{E}_N \text{ for} \\ &\mathcal{E}_N := \int_0^1 t^{s-1}R_N(t)dt. \end{aligned} \tag{1.2.24}$$

We use (1.2.23) to see $\mathcal{E}_N(s)$ is holomorphic in s for $\mathrm{Re}(s) > -k$. This gives a a meromorphic extension of ζ to the halfplane $\mathrm{Re}(s) > -k$; we use uniqueness of analytic extensions to see the extension is independent of k. The remaining assertions are immediate. ■

If P is a first order self-adjoint partial differential operator on $C^\infty(V)$, we say P is of Dirac type if P^2 is of Laplace type. This means that $p := \sigma_L(P)$ satisfies:

$$p(x,\xi)^2 = |\xi|^2 \cdot I_V. \tag{1.2.25}$$

Let $\mathfrak{D}(V)$ be the set of such operators. While operators of Laplace type exist on every vector bundle, not every vector bundle admits operators of Dirac type. The operator

$$d + \delta \text{ on } C^\infty(\Lambda M) \tag{1.2.26}$$

is of Diract type; there are many other such operators.

If $P \in \mathfrak{D}(V)$ is an operator of Dirac type, it is possible to define a more subtle invariant which measures the difference between positive and negative spectrum. Let

$$\operatorname{sign}(\lambda) := \begin{cases} 1 & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0, \\ -1 & \text{if } \lambda < 0. \end{cases} \tag{1.2.27}$$

We define:

$$\eta(s,P) := \Sigma_\nu \operatorname{sign}(\lambda_\nu)|\lambda_\nu|^{-s}. \tag{1.2.28}$$

Since $\lambda_\nu^2 \geq \nu^\epsilon$ for $\nu \geq \nu_0$, this series converges absolutely for $\operatorname{Re}(s) >> 0$ to define a holomorphic function of s.

Theorem 1.2.3: *$\Gamma((s+1)/2)\eta(s,P)$ has a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (m+1-n)/2$ for $n \in \mathbf{N}$. All the residues are locally computable. η is regular at $s = 0$. We reduce mod $\mathbf{Z}$ to define:*

$$\eta(P) = \tfrac{1}{2}\{\dim \mathfrak{N}(P) + \eta(s,P)\}|_{s=0} \in \mathbf{R}/\mathbf{Z}.$$

If $P(\epsilon)$ is a smooth 1-parameter family of such operators, then $\partial_\epsilon \eta(P(\epsilon))$ is locally computable. If m is even, $\eta(P(\epsilon))$ is independent of ϵ.

Remark: $\eta(0,P)$ has $2\mathbf{Z}$ discontinuities as eigenvalues cross the origin; consequently it is necessary to reduce mod $\mathbf{Z}$ to ensure η is smooth. This is an essential feature of the eta invariant and one which makes it different from most other analytic invariants; torsion plays a crucial role.

The following construction will be important in §3.3. We begin with

Lemma 1.2.4: *Let $m = 2n \geq 2$. We can find $2^n \times 2^n$ complex self-adjoint matrices e_i for $0 \leq i \leq 2n$ satisfying the Clifford commutation relations $e_i e_j + e_j e_i = 2\delta_{ij}$.*

Proof: If $m = 2$, let

$$e_0^2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e_1^2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_2^2 := \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}. \quad (1.2.29)$$

We prove the general case by induction. Suppose $\{e_0^{m-2}, ..., e_{m-2}^{m-2}\}$ are $2^{n-1} \times 2^{n-1}$ matrices satisfying the conditions of the Lemma. Define

$$\begin{aligned} &e_0^m := e_0^{m-2} \otimes e_0^2, ..., e_{m-2}^m := e_{m-2}^{m-2} \otimes e_0^2, \blacksquare \\ &e_{m-1}^m := e_{m-2}^m \otimes e_1^2, \; e_m^m := e_{m-2}^{m-2} \otimes e_2^2. \end{aligned} \quad (1.2.30)$$

Let $\{e_i\}_{0 \leq i \leq m}$ be as in Lemma 1.2.4. Let $e(x) = \Sigma_j e_j x_j$ and define Q on $C^\infty(\mathbf{R}^{m+1} \times \mathbf{C}^\nu)$ by:

$$Q := \sqrt{-1}\Sigma_j e_j \partial_{x_j}. \quad (1.2.31)$$

Then Q is self-adjoint and of Dirac type since

$$Q^2 = \Delta_e \cdot I_\nu = -(\partial_1^2 + ... + \partial_m^2) I_\nu. \quad (1.2.32)$$

Lemma 1.2.5: *Let $m = 2n \geq 2$ and let $\nu = 2^n$.*

(a) *There exists a first order tangential operator A so*

$$Q = \sqrt{-1}e(\theta)\partial_r + r^{-1}A$$

(b) *A satisfies the identities:*

$$\begin{aligned} &A = A^* + \sqrt{-1}me(\theta), \\ &Ae(\theta) + e(\theta)A = \sqrt{-1}m \cdot I_\nu, \text{ and} \\ &A^2 - \sqrt{-1}e(\theta)A = \Delta_\theta \otimes I_\nu. \end{aligned}$$

(c) *Let $P = \sqrt{-1}e(\theta)A + \frac{1}{2}(m-1)I_\nu$. Then*

  (i) *P is self-adjoint and of Dirac type.*

  (ii) *$P^2 = \{\Delta_\theta + \frac{1}{4}(m-1)^2\} \otimes I_\nu$.*

  (iii) *$\eta(s, P) = \Sigma_{j \geq 0} \nu \binom{m+j-2}{m-2} (j + \frac{1}{2}(m-1))^{-s}$.*

Proof: Let $\{\psi_j\}_{1\le j\le m}$ be local coordinates on S^m and decompose

$$Q = f(r,\theta)\partial_r + \Sigma_j f_j(r,\theta)\partial_{\psi_j} + g(r,\theta). \tag{1.2.33}$$

We define

$$A := \Sigma_j f_j(r,\theta)\partial_{\psi_j} + g(r,\theta). \tag{1.2.34}$$

Let $v \in \mathbf{C}^\nu$ be a constant vector. We compute:

$$\begin{aligned} Q(v) &= g(r,\theta)v = 0 \\ Q(r^2 v) &= 2rf(r,\theta)v = \sqrt{-1}\Sigma_j e_j \partial_{x_j}(r^2 v) = 2r\sqrt{-1}e(\theta)v \\ Q(x_k v) &= \sqrt{-1}e_k v = \sqrt{-1}e(\theta)\partial_r(r\theta_k v) + \Sigma_j f_j(r,\theta)\partial_{\psi_j}(r\theta_k v) \\ &= \{\sqrt{-1}e(\theta)\theta_k + r\Sigma_j f_j(r,\theta)\partial_{\psi_j}(\theta_k)\}v. \end{aligned} \tag{1.2.35}$$

This shows $g(r,\theta) = 0$ and $f(r,\theta) = \sqrt{-1}e(\theta)$. Since $\sqrt{-1}e_k v$ is independent of r, $rf_j(r,\theta)$ is independent of r so f_j is homogeneous of degree -1 in r. This proves (a).

Let $|\mathrm{dvol}_e|$ and $|\mathrm{dvol}_\theta|$ be the Euclidean and spherical volume elements.

$$\begin{aligned} |\mathrm{dvol}_e| &= |r^m dr \mathrm{dvol}_\theta| \\ Q &= \sqrt{-1}e(\theta)\partial_r + r^{-1}A \\ Q^* &= r^{-m}\{\sqrt{-1}e(\theta)\partial_r + r^{-1}A^*\}r^m \\ &= \sqrt{-1}e(\theta)\partial_r + r^{-1}(\sqrt{-1}me(\theta) + A^*). \end{aligned} \tag{1.2.36}$$

Since $Q = Q^*$, $A = A^* + \sqrt{-1}me(\theta)$. We compute

$$\begin{aligned} Q^2 &= \Delta_e \otimes I_\nu = \{-\partial_r^2 - mr^{-1}\partial_r + r^{-2}\Delta_\theta\} \otimes I \\ &= -\partial_r^2 + \sqrt{-1}\{e(\theta)\partial_r \cdot r^{-1}A + r^{-1}Ae(\theta)\partial_r\} + r^{-2}A^2 \\ &= -\partial_r^2 + \sqrt{-1}r^{-1}\{e(\theta)A + Ae(\theta)\}\partial_r - \sqrt{-1}r^{-2}e(\theta)A + r^{-2}A^2. \end{aligned} \tag{1.2.37}$$

We equate radial and spherical derivatives to prove (b).

We show P is self-adjoint by computing:

$$\begin{aligned} P^* &= -\sqrt{-1}A^*e(\theta) + \tfrac{1}{2}(m-1) \\ &= -\sqrt{-1}\{A - \sqrt{-1}me(\theta)\}e(\theta) + \tfrac{1}{2}(m-1) \\ &= -\sqrt{-1}\{Ae(\theta) - \sqrt{-1}m\} + \tfrac{1}{2}(m-1) \end{aligned} \tag{1.2.38}$$

$$= -\sqrt{-1}\{-e(\theta)A\} + \tfrac{1}{2}(m-1) = P.$$

We square P to prove (ii) and show P is of Dirac type:

$$\begin{aligned} P^2 &= -e(\theta)Ae(\theta)A + \sqrt{-1}(m-1)e(\theta)A + \tfrac{1}{4}(m-1)^2 \\ &= -e(\theta)(-e(\theta)A + \sqrt{-1}m)A + \sqrt{-1}(m-1)e(\theta)A + \tfrac{1}{4}(m-1)^2 \\ &= A^2 - \sqrt{-1}e(\theta)A + \tfrac{1}{4}(m-1)^2 \\ &= (\Delta_\theta + \tfrac{1}{4}(m-1)^2) \otimes I_\nu. \end{aligned} \tag{1.2.39}$$

P is the square root of the normalized Laplacian

$$\{\Delta_\theta + \tfrac{1}{4}(m-1)^2\} \otimes I_\nu \tag{1.2.40}$$

so the eigenvalues of P are $\pm\frac{1}{2}(2j+m-1)$ by Theorem 1.1.2; as $m \geq 2$, 0 is not an eigenvalue of P. Since these eigenvalues are distinct for different values of j, the harmonic spaces $H(m,j) \otimes \mathbf{C}^\nu$ are invariant under P. Decompose

$$H(m,j) \otimes \mathbf{C}^\nu = H^+(m,j,\nu) \oplus H^-(m,j,\nu) \tag{1.2.41}$$

into the eigenspaces of P for the eigenvalues $\pm\frac{1}{2}(2j+m-1)$. We must show

$$\dim\ H^+(m,j,\nu) - \dim\ H^-(m,j,\nu) = \nu\binom{m+j-2}{m-2}. \tag{1.2.42}$$

Choose scalar partial differential operators B_{cd} and B_0 so

$$B := \sqrt{-1}e(\theta)A = \Sigma_{c<d}B_{cd}e_c e_d + B_0 I_\nu. \tag{1.2.43}$$

By (b) $\sigma_L(A)$ and hence $\sigma_L(B)$ anti-commute with $e(\theta)$. This shows B_0 is a 0^{th} order operator. If $v \in \mathbf{C}^\nu$, $Qv = \partial_r(v) = 0$ so $A(v) = B(v) = 0$. Since $\{e_c e_d, 1\}$ are linearly independent endomorphisms, $B_0 = 0$. Since

$$4B_{ab} = (e_a B e_b - e_a e_b B - B e_a e_b - e_b B e_a) \tag{1.2.44}$$

we may conclude

$$B_{ab}(H(m,j,\nu)) \subseteq H(m,j,\nu). \tag{1.2.45}$$

Since $Tr(e_a e_b) = 0$ for $a < b$,

$$\mathrm{Tr}_{\,H(m,j,\nu)}B = \Sigma_{a<b}\ \mathrm{Tr}_{\,\mathbf{C}^\nu}(e_a e_b)\mathrm{Tr}_{\,H(m,j)}(B_{ab}) = 0. \tag{1.2.46}$$

Since the eigenvalues of $P = B + \frac{1}{2}(m-1)$ are

$$\pm\tfrac{1}{2}(2j+m-1) \tag{1.2.47}$$

on $H^{\pm}(m,j,\nu)$, we conclude that:

$$\begin{aligned}\operatorname{Tr}_{H(m,j,\nu)} P &= \tfrac{1}{2}\nu(m-1)\dim H(m,j,\nu)\\ &= \tfrac{1}{2}\nu(m-1)\{\tbinom{m+j}{m} - \tbinom{m+j-2}{m}\}\\ &= \tfrac{1}{2}(2j+m-1)\{\dim H^{+}(m,j,\nu)\\ &\qquad - \dim H^{-}(m,j,\nu)\}\end{aligned} \tag{1.2.48}$$

Consequently

$$\begin{aligned}&\dim H^{+}(m,j,\nu) - \dim H^{-}(m,j,\nu)\\ =&\nu(m-1)\{\tbinom{m+j}{m} - \tbinom{m+j-2}{m}\}/(2j+m-1)\\ =&\nu\tbinom{m+j-2}{m-2}. \blacksquare\end{aligned} \tag{1.2.49}$$

§1.3 Index theory

Let $V_i \in \mathfrak{V}^{\mathrm{U}}(M)$ and let $P : C^\infty(V_1) \to C^\infty(V_2)$ be a first order partial differential operator. We say that

$$\mathcal{P} := \{(P, V_1, V_2)\} \tag{1.3.1}$$

is an elliptic complex of Dirac type if the associated second order operators

$$D_1 = P^*P \text{ and } D_2 = PP^* \tag{1.3.2}$$

are of Laplace type. We define

$$\begin{aligned}\operatorname{Index}(\mathcal{P}) :=& \dim \mathfrak{N}(P) - \dim \mathfrak{N}(P^*)\\ =& \dim \mathfrak{N}(D_1) - \dim \mathfrak{N}(D_2);\end{aligned} \tag{1.3.3}$$

this is well defined by Theorem 1.1.1. Let $a_n(\cdot)$ be the invariants of the heat equation defined in Theorem 1.2.1. We define:

$$\begin{aligned}a_n(x, \mathcal{P}) :=& a_n(x, D_1) - a_n(x, D_2)\\ a_n(\mathcal{P}) =& \textstyle\int_M a_n(x, \mathcal{P})|\mathrm{dvol}\,(x)|.\end{aligned} \tag{1.3.4}$$

Theorem 1.3.1:
(a) $a_n(\mathcal{P}) = 0$ *for* $n \neq m$.
(b) $a_m(\mathcal{P}) = \operatorname{Index}(\mathcal{P})$.

Proof: We shall reproduce Bott's original argument since it is extremely elegant. Let $E(\lambda, D_i) \subset C^\infty(V_i)$ be the eigenspaces. Since P and P^* intertwine the D_i,

$$P : E(\lambda, D_1) \to E(\lambda, D_2) \text{ and } P^* : E(\lambda, D_2) \to E(\lambda, D_1). \tag{1.3.5}$$

Since P^*P is multiplication by λ, these maps are isomorphisms for $\lambda \neq 0$. Consequently,

$$\dim\ E(\lambda, D_1) = \dim\ E(\lambda, D_2) \text{ for } \lambda \neq 0, \tag{1.3.6}$$

so that

$$\begin{aligned}&\operatorname{Tr}_{L^2}(e^{-tD_1}) - \operatorname{Tr}_{L^2}(e^{-tD_2})\\ =&\Sigma_\lambda e^{-t\lambda}\{\dim\ E(\lambda, D_1) - \dim\ E(\lambda, D_2)\}\\ =&\dim\ E(0, D_1) - \dim\ E(0, D_2) = \operatorname{Index}(\mathcal{P}).\end{aligned} \tag{1.3.7}$$

On the other hand, we use Theorem 1.2.1 to expand:

$$\begin{aligned}&\mathrm{Tr}_{L^2}(e^{-tD_1}) - \mathrm{Tr}_{L^2}(e^{-tD_2})\\ &\sim\Sigma_{n\geq 0}\{a_n(D_1) - a_n(D_2)\}t^{(n-m)/2}\\ &\sim\Sigma_{n\geq 0}a_n(\mathcal{P})t^{(n-m)/2}. \blacksquare\end{aligned} \tag{1.3.8}$$

Example 1.3.1 (de Rham complex): Let

$$\Lambda^e M := \oplus_p \Lambda^{2p} M \text{ and } \Lambda^o M := \oplus_p \Lambda^{2p+1} \tag{1.3.9}$$

be the bundles of differential forms of even and odd degrees. Let d be exterior differentiation and δ be interior differentiation. We define:

$$\begin{aligned}&P_e = d + \delta : C^\infty \Lambda^e M \to C^\infty \Lambda^o M, \text{ and}\\ &P_o = d + \delta : C^\infty \Lambda^o M \to C^\infty \Lambda^e M.\end{aligned} \tag{1.3.10}$$

Let

$$\mathcal{P}_{deR} := (P_e, \Lambda^e M, \Lambda^o M) \tag{1.3.11}$$

be the de Rham complex. It is immediate that $P_e^* = P_o$ and that

$$D^e := \oplus_p \Delta_{2p} \text{ and } D^o := \oplus_p \Delta_{2p+1} \tag{1.3.12}$$

are of Laplace type.We use the Hodge decomposition theorem to see the index is the Euler Poincare characteristic $\chi(M)$:

$$\begin{aligned}\mathrm{Index}(\mathcal{P}) &= \dim \mathfrak{N}(\Delta^e) - \dim \mathfrak{N}(\Delta^o)\\ &=\Sigma_p(-1)^p \dim \mathfrak{N}(\Delta_p)\\ &=\Sigma_p(-1)^p \dim \mathfrak{N}(H^p(M;\mathbf{C}))\\ &=\chi(M).\end{aligned} \tag{1.3.13}$$

Example 1.3.2 (signature complex): Let M be an oriented manifold of real dimension $m = 4k$. Let $\star_p : \Lambda^p M \to \Lambda^{m-p} M$ be the Hodge operator. Define $\tau_p \in \mathrm{Hom}(\Lambda^p M, \Lambda^{m-p} M)$ by:

$$\tau_p(\omega_p) = (-1)^k (-1)^{p(p-1)/2} \star_p (\omega). \tag{1.3.14}$$

Let $\tau = \oplus_p \tau_p \in \mathrm{Hom}(\Lambda M, \Lambda M)$. We use Lemma 1.1.4 to see that:

$$\tau^2 = 1 \text{ and } (d+\delta)\tau = \tau(d+\delta). \tag{1.3.15}$$

Let $\Lambda^{\pm}M$ be the eigenbundles of τ. Then

$$d+\delta : C^\infty\Lambda^+M \to C^\infty\Lambda^-M. \tag{1.3.16}$$

Define the signature complex by:

$$\begin{aligned}\mathcal{P}_{sign} &:= ((d+\delta), \Lambda^+M, \Lambda^-M);\\ \operatorname{Sign}(M) &:= \operatorname{Index}(\mathcal{P}_{sign}).\end{aligned} \tag{1.3.17}$$

The index of this elliptic complex is called the signature of M. It can be computed topologically. We define a natural symmetric bilinear form $\mathcal{J}$ on the real de Rham cohomology group $H^{2k}(M;\mathbf{R})$ by:

$$\mathcal{J}(\alpha_1, \alpha_2) := \int_M \alpha_1 \wedge \alpha_2. \tag{1.3.18}$$

The index of $\mathcal{J}$ is the number of $+1$ eigenvalues minus the number of -1 eigenvalues when $\mathcal{J}$ is diagonalized over the real numbers $\mathbf{R}$. Then

$$\operatorname{Sign}(M) = \operatorname{Index}(J). \tag{1.3.19}$$

The observation that the index of an elliptic complex of Dirac type is given by a local formula has many important consequences. We present just a few below. Let $M_1\#M_2$ be the connected sum of M_1 and M_2; this is defined by punching out disks in both manifolds and gluing along the common resulting boundaries. Let CP^{2k} be complex projective space of real dimension $m = 4k$.

Theorem 1.3.2: *Let m be even.*

(a) *Let $F \to M_1 \to M_2$ be a finite covering. Then $\chi(M_1) = |F|\chi(M_2)$. Assume further that M_2 is oriented and that $m \equiv 0$ (4). Give M_1 the orientation inherited from M_2. Then* $\operatorname{Sign}(M_1) = |F|\operatorname{Sign}(M_2)$.

(b) *Let $M = M_1\#M_2$. Then $\chi(M) = \chi(M_1)+\chi(M_2)-2$. Assume further that M is oriented and that $m \equiv 0$ (4). Give the M_i the induced orientation. Then* $\operatorname{Sign}(M) = \operatorname{Sign}(M_1) + \operatorname{Sign}(M_2)$.

(c) *Let $F \to CP^{2k} \to M$ be a finite covering. Then $|F| = 1$ so $M = CP^{2k}$.*

Proof: Since local formulas are multiplicative under finite coverings and since both the Euler characteristic and the signature are given by local formulas, (a) follows.

To prove (b), we note that the two disks removed when creating M glue together to form a sphere. We use the additivity of local formulas to show that

$$\begin{aligned}&\chi(M)+\chi(S^m)=\chi(M_1)+\chi(M_2), \text{ and}\\&\operatorname{Sign}(M)+\operatorname{Sign}(S^m)=\operatorname{Sign}(M_1)+\operatorname{Sign}(M_2);\end{aligned} \tag{1.3.20}$$

(b) follows since $\chi(S^m)=2$ and $\operatorname{Sign}(S^m)=0$.

We recall that

$$H^\nu(CP^{2k};\mathbf{C})=\begin{cases}\mathbf{C} & \text{if } \nu\equiv 0(2) \text{ and } if\ 0\leq\nu\leq 4k,\\ 0 & \text{if otherwise.}\end{cases} \tag{1.3.21}$$

Consequently $\chi(CP^{2k})=2k+1$ and $\operatorname{sign}(CP^{2k})=1$ (with the usual orientation). By (a),

$$|F|\chi(M)=2j+1, \tag{1.3.22}$$

so $|F|$ is odd. Consequently, F is orientation preserving and M is orientable. We see that $|F|=1$ from the identity:

$$1=\operatorname{Sign}(\mathrm{CP}^{2j})=|F|\operatorname{Sign}(M).\ ■ \tag{1.3.23}$$

2. DIFFERENTIAL GEOMETRY

§2.1 Characteristic classses

The characteristic classes are topological invariants of a vector bundle which are represented by differential forms. They are defined in terms of the curvature of a connection. Let M be a smooth manifold. We adopt the Einstein convention and sum over repeated indices. We introduce the following notational conventions.

Let $V \in \mathfrak{V}^{\mathrm{U}}(M)$. A local unitary frame $\vec{s} = (s^1, ..., s^k)$ for V is a collection of smooth local sections s^i to V over an open set O which are an orthonormal basis for the fiber of V over every point of O. The total derivative d is not invariantly defined on $C^\infty(V)$ unless V has a given global trivialization. A unitary connection ∇ on $\mathfrak{V}^{\mathrm{U}}(M)$ should be thought of as an invariantly defined total derivative. It is an additional piece of structure that is imposed on V. ∇ is a first order partial differential operator from $C^\infty(V)$ to $C^\infty(T^*M \otimes V)$ such that

$$\begin{aligned} &\nabla(fs_1) = df \otimes s_1 + f\nabla s_1, \text{ and} \\ &(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2) \end{aligned} \tag{2.1.1}$$

for all $f \in C^\infty(M)$ and $s_i \in C^\infty(V)$. Let $\mathfrak{C}^{\mathrm{U}}(V)$ be the set of all unitary connections on V.

Let $\vec{s} = (s^1, ...)$ be a local unitary frame for V. Let the decomposition

$$\nabla s^i = \omega^i_j \otimes s^j \tag{2.1.2}$$

define the connection 1-form $\omega = (\omega^i_j)$ of ∇ relative to $\vec{s}$; ∇ is uniquely determined by the ω^i_j and the Leibnitz property. We emphasize that ω is not invariantly defined and we will use the notation $\omega = \omega(\vec{s})$ when we wish to exhibit the dependence on $\vec{s}$ explicitly. Since ∇ is unitary,

$$\omega + \omega^* = 0. \tag{2.1.3}$$

We can always construct unitary connections locally. Since the convex combination of unitary connections is again a unitary connection, $\mathfrak{C}^{\mathrm{U}}(V)$ is non empty.

If we choose another frame $\vec{t} = (t^1, ..., t^k)$, then we can express

$$t^i = h^i_j s^j \tag{2.1.4}$$

where $h := h^i_j$ is a smooth local section to $\mathrm{End}(V)$. Let h^{-1} be the inverse endomorphism; it is defined by the identity:

$$h^i_j (h^{-1})^j_k = \delta^i_k \tag{2.1.5}$$

where δ is the Kronecker symbol. Let $\nabla t^i = \tilde{\omega}^i_j t^j$. We compute:

$$\begin{aligned} \nabla t^i &= \tilde{\omega}^i_j t^j = \nabla(h^i_k s^k) \\ &= d(h^i_k) \otimes s^k + h^i_k \omega^k_l \otimes s^l \\ &= (dh^i_k (h^{-1})^k_j + h^i_k \omega^k_l (h^{-1})^l_j) \otimes t^j. \end{aligned} \tag{2.1.6}$$

Consequently $\tilde{\omega}$ satisfies the transformation law:

$$\begin{aligned} \tilde{\omega}^i_j &= (dh^i_k (h^{-1})^k_j + h^i_k \omega^k_l (h^{-1})^l_j), \text{ i.e.} \\ \tilde{\omega} &= dh \cdot h^{-1} + h\omega h^{-1}. \end{aligned} \tag{2.1.7}$$

This is, of course, the manner in which the 0^{th} order symbol of a first order partial differential operator transforms.

We extend ∇ to a derivation of $C^\infty(\Lambda M \otimes V)$ as follows. Let $\theta_p \in C^\infty(\Lambda^p M)$, and let $s \in C^\infty(V)$. We define:

$$\nabla(\theta_p \otimes s) := d\theta_p \otimes s + (-1)^p \theta_p \wedge \nabla s \in C^\infty(\Lambda^{p+1} M \otimes V). \tag{2.1.8}$$

Let $f \in C^\infty(M)$, and let $s \in C^\infty(V)$. We show ∇^2 is a 0^{th} order operator by computing:

$$\begin{aligned} \nabla^2(fs) &= \nabla(df \otimes s + f\nabla s) \\ &= d^2 f \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s \\ &= f\nabla^2 s. \end{aligned} \tag{2.1.9}$$

The curvature $\Omega \in C^\infty(\Lambda^2 M \otimes \mathrm{End}(V))$ is defined by:

$$\Omega(x_0)s := \nabla^2(\tilde{s})(x_0) \tag{2.1.10}$$

where $\tilde{s}$ is any extension of $s \in V(x_0)$ to a smooth local section; this is independent of the particular extension chosen. Ω is invariantly defined independent of the choice of the local unitary frame. Let $\Omega^i_j(\vec{s})$ be the components of Ω relative to the local unitary frame $\vec{s}$. Then:

$$\begin{aligned}
&\Omega s^i = \nabla(\omega^i_j \otimes s^j) = d\omega^i_j \otimes s^j - \omega^i_j \wedge \omega^j_k \otimes s_k, \\
&\Omega^i_j(\vec{s}) = d\omega^i_j(\vec{s}) - \omega^i_k(\vec{s}) \wedge \omega^k_j(\vec{s}), \text{ i.e.} \\
&\Omega(\vec{s}) = d\omega(\vec{s}) - \omega(\vec{s}) \wedge \omega(\vec{s}).
\end{aligned} \tag{2.1.11}$$

We use the notation Ω when we are thinking of the curvature as an invariantly defined endomorphism; we use the notation $\Omega(\vec{s})$ when we are thinking of the curvature as a matrix $\Omega^i_j(\vec{s})$ of $2-$forms. Since ∇ is unitary,

$$\Omega + \Omega^* = 0. \tag{2.1.12}$$

Since ∇^2 is a 0^{th} order operator, Ω transforms like a tensor. If $\vec{t} = h\vec{s}$,

$$\Omega^i_j(\vec{t}) = h^i_k \Omega^k_l(\vec{s})(h^{-1})^l_j \text{ i.e. } \Omega(\vec{t}) = h\Omega(\vec{s})h^{-1}. \tag{2.1.13}$$

Lemma 2.1.1: *Let $V \in \mathfrak{V}^{\mathrm{U}}(M)$, let $\nabla \in \mathfrak{C}^{\mathrm{U}}(V)$, and let $x_0 \in M$. There exists a local unitary frame $\vec{t}$ for V so that $\omega(\vec{t}, x_0) = 0$ and $d\Omega(\vec{t}, x_0) = 0$.*

Proof: Let $\omega(\vec{s}) = \omega^i_j(\vec{s})$ be the connection $1-$ form of ∇ relative to a local unitary frame for V defined over a contractible open set O. Choose $h : O \to \mathrm{U}(\cdot)$ smooth so that

$$h(x_0) = I \text{ and } dh(x_0) = -\omega(\vec{s}, x_0). \tag{2.1.14}$$

Let $\vec{t} = h\vec{s}$. We use (2.1.7) and (2.1.11) to see:

$$\begin{aligned}
&\omega(\vec{t}, x_0) = dh(x_0) + \omega(\vec{s}, x_0) = 0 \\
&d\Omega(\vec{t}, x_0) = d(d\omega(\vec{t}) - \omega(\vec{t}) \wedge \omega(\vec{t}))(x_0) \\
&\quad = \{-d\omega(\vec{t}, x_0) \wedge \omega(\vec{t}, x_0) + \omega(\vec{t}, x_0) \wedge d\omega(\vec{t}, x_0)\} = 0. \blacksquare
\end{aligned} \tag{2.1.15}$$

Remark: As the curvature is invariantly defined and does not vanish in general, it is not possible to find a parallel frame s so ω vanishes in a neighborhood of x_0.

We now discuss the Chern-Weyl homomorphism. Let

$$\mathfrak{u}(n) := \{A \in \mathrm{End}(\mathbf{C}^n) : A + A^* = 0\} \tag{2.1.16}$$

be the Lie algebra of the unitary group $\mathrm{U}(n)$. Let $P : \mathfrak{u}(n) \to \mathbf{C}$ be a complex valued polynomial map. Let

$$\mathfrak{I}(\mathrm{U}(n)) = \{P : P(hAh^{-1}) = P(h)\ \forall h \in \mathrm{U}(n),\ \forall A \in \mathfrak{u}\} \tag{2.1.17}$$

be the algebra of invariant polynomials. We decompose

$$\mathfrak{I}(\mathrm{U}(n)) = \oplus_j \mathfrak{I}_j(\mathrm{U}(n)) \tag{2.1.18}$$

as the direct sum of the invariant polynomials which are homogeneous of order j. Let $P \in \mathfrak{I}_j(\mathrm{U})$. Expand

$$P(t_1A_1 + \ldots + t_jA_j)/j! = \Sigma_\alpha t^\alpha P_\alpha(A_1, \ldots, A_j). \tag{2.1.19}$$

Let $\alpha = (1, \ldots, 1)$, and let $P(A_1, \ldots, A_j) := P_\alpha(A_1, \ldots, A_j)$ be the total polarization. This is a symmetric multi-linear function of its arguments such that

$$\begin{aligned} &P(A) = P(A, \ldots, A), \text{ and} \\ &P(hA_1h^{-1}, \ldots, hA_jh^{-1}) = P(A_1, \ldots, A_j). \end{aligned} \tag{2.1.20}$$

For example, the complete polarization of $\mathrm{Tr}(A^3)$ is

$$\tfrac{1}{2}\mathrm{Tr}(A_1A_2A_3 + A_2A_1A_3). \tag{2.1.21}$$

Let $V \in \mathfrak{V}^{\mathrm{U}}(M)$, let $\nabla \in \mathfrak{C}^{\mathrm{U}}(V)$, let $P \in \mathfrak{I}(\mathrm{U})$, and let $\vec{s}$ be a local unitary frame for V. Since forms of even dimensions commute, we substitute to define:

$$P(\Omega(\vec{s})) \in C^\infty(\Lambda M). \tag{2.1.22}$$

If $\vec{t} = h\vec{s}$, then $P(\Omega(\vec{t})) = P(h\Omega(\vec{s})h^{-1}) = P(\Omega(\vec{t}))$ is invariantly defined; we denote this common value by $P(\nabla)$. If $P \in \mathfrak{I}_j(\mathrm{U}(n))$,

$$P(\nabla) \in C^\infty(\Lambda^{2j}M \otimes \mathbf{C}). \tag{2.1.23}$$

Lemma 2.1.2: *Let $V \in \mathfrak{V}^{\mathrm{U}}(M)$, and let $P \in \mathfrak{I}(\mathrm{U}(n))$.*

(a) *If $\nabla \in \mathfrak{C}^{\mathrm{U}}(V)$, then $dP(\nabla) = 0$ so $P(\nabla)$ is closed.*

(b) *If $\nabla_i \in \mathfrak{C}^{\mathrm{U}}(V)$, there exists $TP(\nabla_0, \nabla_1) \in C^\infty \Lambda M$ so*

$$dTP(\nabla_0, \nabla_1) = P(\nabla_1) - P(\nabla_0).$$

(c) *Let $P(V) = [P(\nabla)] \in H^*(M; \mathbf{C})$. This is independent of the connection chosen and of the fiber metric chosen.*

Proof: By decomposing $P = P_0+...+P_\nu$ for $P_j \in \mathfrak{I}_j(\mathrm{U})$, we may suppose without loss of generality that P is homogeneous of degree j. We take the complete polarization to replace P by a multi-linear symmetric invariant function. Fix $x_0 \in M$ and choose a local unitary frame $\vec{s}$ so $\omega(\vec{s}, x_0) = 0$ and $d\Omega(\vec{s}, x_0) = 0$. Then

$$\begin{aligned} dP(\Omega(\vec{s}, x_0)) &= dP(\Omega(\vec{s}), ..., \Omega(\vec{s}))(x_0) \\ &= jP(d\Omega(\vec{s}), \Omega(\vec{s}), ..., \Omega(\vec{s}))(x_0) = 0. \end{aligned} \tag{2.1.24}$$

Since $dP(\Omega(\vec{s}))$ is independent of $\vec{s}$, (a) follows. Let

$$\nabla(\epsilon) = \epsilon\nabla_1 + (1-\epsilon)\nabla_0 \tag{2.1.25}$$

be an affine homotopy in $\mathfrak{C}^{\mathrm{U}}(V)$ between ∇_0 and ∇_1 for $\epsilon \in [0,1]$. The connection $1-$ form of $\nabla(\epsilon)$ is given by:

$$\omega(\epsilon) = \omega_0 + \epsilon\theta \text{ for } \theta = \omega_1 - \omega_0. \tag{2.1.26}$$

We use (2.1.7) to see that in a new frame

$$\begin{aligned} \tilde{\theta} =& \tilde{\omega}_1 - \tilde{\omega}_0 \\ =& (dh \cdot h^{-1} + h\omega_1 h^{-1}) - (dh \cdot h^{-1} + h\omega_0 h^{-1}) \\ =& h\theta h^{-1} \end{aligned} \tag{2.1.27}$$

so θ is invariantly defined and transforms like a tensor. This is, of course, because the difference between two first order operators with the same leading symbol is a 0^{th} order operator. Let $\Omega(\epsilon)$ be the curvature of the connection $\nabla(\epsilon)$. Since θ is a $1-$ form, it commutes with $2-$ forms and we can define

$$P(\theta, \Omega(\epsilon), ..., \Omega(\epsilon)) \in \Lambda^{2j-1}M \tag{2.1.28}$$

by substitution. Since P is invariant, the complete polarization of P is invariant and

$$\begin{aligned} & P(\theta(\vec{t}), \Omega(\epsilon, \vec{t}), ..., \Omega(\epsilon, \vec{t})) \\ =& P(h\theta(\vec{s})h^{-1}, h\Omega(\epsilon, \vec{s})h^{-1}, ..., h\Omega(\epsilon, \vec{s})h^{-1}) \\ =& P(\theta(\vec{s}), \Omega(\epsilon, \vec{s}), ..., \Omega(\epsilon, \vec{s})) \end{aligned} \tag{2.1.29}$$

is invariantly defined and independent of the frame chosen. We compute:

$$\begin{aligned} P(\nabla_1) - P(\nabla_0) =& \int_0^1 \partial_\epsilon P(\nabla(\epsilon)) d\epsilon \\ =& j \int_0^1 P(\dot{\Omega}(\epsilon), \Omega(\epsilon), ..., \Omega(\epsilon)) d\epsilon. \end{aligned} \tag{2.1.30}$$

We define:

$$TP(\nabla_1, \nabla_0) = j \int_0^1 P(\theta, \Omega(\epsilon), ..., \Omega(\epsilon)) d\epsilon \in \Lambda^{2j-1} M. \tag{2.1.31}$$

To complete the proof of the Lemma, it suffices to check

$$dP(\theta, \Omega(\epsilon), ..., \Omega(\epsilon)) = P(\dot{\Omega}(\epsilon), \Omega(\epsilon), ..., \Omega(\epsilon)) \tag{2.1.32}$$

for all $\epsilon \in [0, 1]$. Since both sides of (2.1.32) are invariantly defined, we can choose a local unitary frame to simplify the computation. Let $x_0 \in M$ and $\epsilon_0 \in [0, 1]$. Use Lemma 2.1.1 to choose $\vec{t}$ so relative to this local frame, $\omega(\epsilon_0, \vec{t}, x_0) = 0$ and $d\Omega(\epsilon_0, \vec{t}, x_0) = 0$. We prove (b) by computing:

$$\begin{aligned} \dot{\Omega}(\epsilon_0, \vec{t}, x_0) &= \partial_\epsilon \{ d\omega_0(\vec{t}) + \epsilon d\theta(\vec{t}) - \omega(\epsilon, \vec{t}) \wedge \omega(\epsilon, \vec{t}) \}(\epsilon, x_0) \\ &= d\theta(\vec{t}, x_0) - \dot{\omega}(\epsilon_0, \vec{t}, x_0) \wedge \omega(\epsilon_0, \vec{t}, x_0) \\ &- \omega(\epsilon_0, \vec{t}, x_0) \wedge \dot{\omega}(\epsilon_0, \vec{t}, x_0) = d\theta(\vec{t}, x_0), \end{aligned} \tag{2.1.33}$$

and

$$\begin{aligned} & dP(\theta(\vec{t}), \Omega(\epsilon_0, \vec{t}), ..., \Omega(\epsilon_0, \vec{t}))(x_0) \\ &= P(d\theta(\vec{t}), \Omega(\epsilon_0, \vec{t}, x_0), ..., \Omega(\epsilon_0, \vec{t}, x_0)) \\ &= P(\dot{\Omega}(\epsilon_0, \vec{t}, x_0), \Omega(\epsilon_0, \vec{t}, x_0), ..., \Omega(\epsilon_0, \vec{t}, x_0)). \end{aligned} \tag{2.1.34}$$

We complete the proof by showing $[P(\nabla)]$ is independent of the fiber metrics chosen. Suppose given two fiber metrics $(\cdot, \cdot)_i$. Let $\tilde{V} = V \times [0, 1]$ be the induced bundle over $M \times [0, 1]$; use a partition of unity to define a fiber metric $(\cdot, \cdot)$ on $\tilde{V}$ agreeing with $(\cdot, \cdot)_0$ near $\epsilon = 0$ and with $(\cdot, \cdot)_1$ near $\epsilon = 1$. Let $\tilde{\nabla}$ be a unitary connection for $\tilde{V}$; we restrict $\tilde{\nabla}$ to define unitary connections ∇_i on $M \times \{i\}$ for $i \in \{0, 1\}$. Since $dP(\tilde{\Omega}) = 0$, it follows that the restriction to $M \times \{0\}$ and to $M \times \{1\}$ gives the same element of de Rham cohomology. ■

We can apply functorial constructions to connections. Let $V_i \in \mathfrak{V}^{\mathrm{U}}(M)$ and $\nabla_i \in \mathfrak{C}^{\mathrm{U}}(V_i)$. Let ω_i and Ω_i be the associated connection $1-$forms and curvatures relative to some local unitary frame. Choose the dual frame for the dual bundle V_1^*. Define:

(1) $\nabla_1^* \in \mathfrak{C}^{\mathrm{U}}(V_1^*)$ by:

$$\begin{aligned}&(s_1, \nabla_1^* s_1^*) := d(s_1, s_1^*) - (\nabla_1 s_1, s_1^*),\\ &\omega_{\nabla_1^*} = -\omega_1^t, \quad\text{and } \Omega_{\nabla_1^*} = -\Omega_1^t.\end{aligned} \tag{2.1.35}$$

(2) $\nabla_1 \oplus \nabla_2 \in \mathfrak{C}^{\mathrm{U}}(V_1 \oplus V_2)$ by:

$$\begin{aligned}&(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) := \nabla_1 s_1 \oplus \nabla_2 s_2,\\ &\omega_{(\nabla_1 \oplus \nabla_2)} = \omega_1 \oplus \omega_2, \quad\text{and } \Omega_{(\nabla_1 \oplus \nabla_2)} = \Omega_1 \oplus \Omega_2.\end{aligned} \tag{2.1.36}$$

(3) $\nabla_1 \otimes \nabla_2 \in \mathfrak{C}^{\mathrm{U}}(V_1 \otimes V_2)$ by:

$$\begin{aligned}&(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) := \nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2,\\ &\omega_{(\nabla_1 \otimes \nabla_2)} = \omega_1 \otimes 1 + 1 \otimes \omega_2, \quad\text{and}\\ &\Omega_{(\nabla_1 \otimes \nabla_2)} = \Omega_1 \otimes 1 + 1 \otimes \Omega_2.\end{aligned} \tag{2.1.37}$$

(4) $\nabla \in \mathfrak{C}^{\mathrm{U}}\,\mathrm{Hom}\,(V_1, V_2))$ by:

$$\begin{aligned}&\nabla(E)(s_1) := \nabla_2(E(s_1)) - E(\nabla_1(s_1)),\\ &\omega_\nabla(E) = (\omega_2 E - E\omega_1), \quad\text{and } \Omega_\nabla(E) = \Omega_2 E - E\Omega_1.\end{aligned} \tag{2.1.38}$$

(5) Let $f : N \to M$ be smooth, let $V \in \mathfrak{V}^{\mathrm{U}}(M)$, and let

$$f^*V := \{(x, v) \in N \times V : f(x) = \pi(v)\} \in \mathfrak{V}^{\mathrm{U}}(N) \tag{2.1.39}$$

be the pull-back bundle. If $s \in C^\infty(V)$, let

$$f^*(s)(x) := (x, s(f(x)) \in C^\infty(f^*V) \tag{2.1.40}$$

be the pull-back section. If $\nabla \in \mathfrak{C}^{\mathrm{U}}(V)$, the pull back connection

$$f^*\nabla \in \mathfrak{C}^{\mathrm{U}}(f^*V) \tag{2.1.41}$$

is defined by:

$$\begin{aligned}&(f^*\nabla)(f^*s) := f^*(\nabla s) \text{ for } s \in C^\infty(V),\\ &\omega_{f^*\nabla} = f^*\omega_\nabla, \quad\text{and } \Omega_{f^*\nabla} = f^*\Omega_\nabla.\end{aligned} \tag{2.1.42}$$

Remark: (4) is a special case of (1) and (3) since $\mathrm{Hom}\,(V_1, V_2) = V_1^* \otimes V_2$. In a similar fashion, we can define the induced connection on $\Lambda^p V$ (the bundle of $p-$ forms), on S^pV (the bundle of symmetric $p-$ tensors), and on any other natural bundle.

We define the total Chern polynomial $c(A)$ by:

$$\begin{aligned} c(A) :=& \det(I + i\tfrac{A}{2\pi}) \\ =& 1 + c_1(A) + \ldots + c_k(A). \end{aligned} \tag{2.1.43}$$

The corresponding characteristic classes are called the Chern classes.

Lemma 2.1.3:
(a) $\mathfrak{I}(\mathrm{U}(k)) = \mathbf{C}[c_1, \ldots, c_k]$.
(b) $c(V_1 \oplus V_2) = c(V_1) \wedge c(V_2)$ *and* $c_k(V^*) = (-1)^k c_k(V)$.

Proof: It is immediate that the $c_i(A)$ are invariant polynomials which are homogeneous of degree $2i$. Let $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ be a diagonal matrix. Let $s_j(\vec{\lambda})$ be the elementary symmetric functions;

$$\det(I + A) = \Pi_j(1 + \lambda_j) = s_0(\vec{\lambda}) + \ldots + s_k(\vec{\lambda}). \tag{2.1.44}$$

Let $x_j = i\lambda_j/2\pi$. Then

$$c_j(A) = s_j(\vec{x}). \tag{2.1.45}$$

Let $P \in \mathfrak{I}(\mathrm{U}(k))$. Then $P(A) = P(\vec{x})$ is polynomial in these variables. Since we can permute the eigenvalues of A by conjugating A with a suitable element of $\mathrm{U}(k)$, $P(\vec{x})$ is a symmetric function of $\vec{x}$. The elementary symmetric functions are an algebraic basis for the algebra of all symmetric polynomials so there is a unique polynomial Q so

$$P(A) = Q(c_1, \ldots, c_k)(A). \tag{2.1.46}$$

We have established (2.1.46) for diagonal matrices. Since P is invariant, (2.1.46) holds for diagonalizable matrices. Since the elements of $\mathfrak{u}$ are all diagonalizable, (2.1.46) holds for all A. This shows $\mathfrak{I}(\mathrm{U}(k))$ is generated algebraically by the $\{c_1, \ldots, c_k\}$. Since the c_i correspond to the elementary symmetric functions, there are no relations in this algebra. This proves (a).

The first assertion of (b) is immediate. The curvature of the dual connection ∇^* on V^* is $-\Omega^t$. Consequently

$$\begin{aligned} c(V) &= \det(I + \tfrac{i}{2\pi}\Omega) \\ c(V^*) &= \det(I - \tfrac{i}{2\pi}\Omega^t) = \det(I - \tfrac{i}{2\pi}\Omega). \ \blacksquare \end{aligned} \tag{2.1.47}$$

We define the Chern character by the generating function:

$$\begin{aligned} &\mathrm{ch}\,(A) = \mathrm{Tr}\,(e^{iA/2\pi}) = k + \mathrm{ch}\,_1(A) + ... + \mathrm{ch}\,_k(A) + ...; \\ &\mathrm{ch}\,_\nu(A) = (i/2\pi)^\nu \mathrm{Tr}\,(A^\nu)/\nu!. \end{aligned} \tag{2.1.48}$$

Strictly speaking, $\mathrm{ch}\,(A)$ is an infinite series rather than a polynomial. This causes no additional difficulties since we can always work with the homogeneous pieces. When we substitute the components of the curvature tensor, $\mathrm{Tr}\,(\Omega^j) = 0$ if $2j > m$ so the discussion above applies.

So far, we have discussed complex vector bundles, the real theory is analogous. Let $\mathfrak{o}\,(n)$ be the Lie-algebra of the orthogonal group. We define the Pontrjagin polynomials by:

$$p(A) = \det(I + (2\pi)^{-1}A) = 1 + p_1 + ... \tag{2.1.49}$$

The p_ν are invariant polynomials which are homogeneous of degree 2ν in the components of A. Let $A \in \mathfrak{o}\,(k)$ have eigenvalues $\{\pm i\lambda_1, ..., \pm i\lambda_\ell\}$ if $k = 2\ell$ or $\{0, \pm i\lambda_1, ..., \pm i\lambda_\ell\}$ if $k = 2\ell + 1$. By conjugating A by an appropriate element of $\mathrm{O}\,(k)$, we can put A in the form:

$$A = \begin{pmatrix} 0 & -\lambda_1 & 0 & 0 & .... \\ \lambda_1 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & -\lambda_2 & ... \\ 0 & 0 & \lambda_2 & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}. \tag{2.1.50}$$

If k is odd, the last block will be a 1×1 block with zero. Let $x_j = -\lambda_j/2\pi$. Then the total Pontrjagin class is given by

$$P(A) = \Pi_j(1 + x_j^2) \tag{2.1.51}$$

so the p_j are the elementary symmetric functions in the x_j.

There is one additional characteristic class, called the Euler form, which is important. If k is odd, let $e_k = 0$. If $k = 2j$ is even, we define e_k as follows. Let $A(s_a) = A_{ab}s_b$; we use the metric to raise and lower indices. Let σ_k be the group of permutations of k symbols. Let

$$e_k(A) := (-4\pi)^{-j}\Sigma_{\sigma\in\sigma_k}\ \mathrm{sign}(\sigma)A_{\sigma(1)\sigma(2)}...A_{\sigma(k-1)\sigma(k)}/j!. \tag{2.1.52}$$

We omit the proof of the following result in the interests of brevity as it is analogous to the proof already given of Lemma 2.1.3.

Lemma 2.1.4:

(a) *If* $k = 2j + 1$, *then* $\mathfrak{I}(\mathrm{SO}(k)) = \mathfrak{I}(\mathrm{O}(k)) = \mathbf{C}[p_1, ..., p_j]$.

(b) *If* $k = 2j$, *then* $e_k^2 = \rho_j$, $e_k \in \mathfrak{I}(\mathrm{SO}(k))$, *and*

$$\mathfrak{I}(\mathrm{O}(k)) = \mathbf{C}[p_1, ..., p_j], \text{ and}$$
$$\mathfrak{I}(\mathrm{SO}(k)) = \mathbf{C}[p_1, ..., p_j] \oplus e_k \cdot \mathbf{C}[p_1, ..., p_j].$$

We define the Hirzebruch L-genus using generating functions. Let:

$$\mathrm{L}(A) = \Pi_j \frac{x_j}{\tanh(x_j)} \in \mathfrak{I}(\mathrm{O}(k)). \tag{2.1.53}$$

The function $z/\tanh(z)$ is an even functions of the parameter z so the ambiguity in the choice of the sign of x_j plays no role. We decompose

$$\mathrm{L}(A) = 1 + \mathrm{L}_1(A) + ... \tag{2.1.54}$$

into symmetric functions L_i of degree $2i$ in the normalized eigenvalues of A which are even functions of the x_j. Since the elementary symmetric functions of x_j^2 generate all such symmetric functions,

$$\mathrm{L}_j = \mathrm{L}_j(p_1, ..., p_j) \tag{2.1.55}$$

is expressible in terms of the Pontrjagin polynomials and hence is polynomial in the components of A.

It is immediate from the definition that the Chern character satisfies the identities:

$$\begin{aligned} &\mathrm{ch}_k(V_1 \oplus V_2) = \mathrm{ch}_k(V_1) + \mathrm{ch}_k(V_2), \\ &\mathrm{ch}_k(V_1 \otimes V_2) = \Sigma_{p+q=k} \mathrm{ch}_p(V_1) \wedge \mathrm{ch}_q(V_2), \\ &\mathrm{ch}_k(V^*) = (-1)^k \mathrm{ch}_k(V). \end{aligned} \tag{2.1.56}$$

In other words, ch is a ring homomorphism – i.e.

$$\begin{aligned} &\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2), \text{ and} \\ &\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1) \wedge \mathrm{ch}(V_2). \end{aligned} \tag{2.1.57}$$

We may express these invariant polynomials in terms of the Chern and Pontrjagin polynomials; we list only the first few terms:

$$\begin{aligned} \mathrm{ch} &= k + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \ldots \\ \mathrm{L} &= 1 + \tfrac{1}{3}p_1 + \tfrac{1}{45}(7p_2 - p_1^2) + \ldots \end{aligned} \tag{2.1.58}$$

§2.2 Invariance theory

In §2.2, we discuss Weyl's theorem on invariants of the orthogonal group, give Patodi's theorem, and state some specialized results concerning orthogonal invariance.

Introduce formal variables

$$g_{ij/\alpha} = \partial_x^\alpha g_{ij} \tag{2.2.1}$$

for the partial derivatives of the metric tensor. We will also use the notation $g_{ij/kl\dots}$. We emphasize these variables are not tensorial, but depend on the choice of the coordinate system. We define the order in the jets of the metric by:

$$\operatorname{ord}(g_{ij/\alpha}) = |\alpha|. \tag{2.2.2}$$

Let G be a Riemannian metric on M and let $x_0 \in M$. We say that a system of coordinates X on M is normalized with respect to (G, x_0) if $x_0 = (0, ..., 0)$ is at the center and if:

$$g_{ij}(X,G)(x_0) = \delta_{ij} \text{ and } g_{ij/k}(X,G)(x_0) = 0. \tag{2.2.3}$$

In such coordinates, the value of the curvature tensor at x_0 for the coordinate frame ∂_i is given by:

$$R_{ijkl} = \tfrac{1}{2}(g_{jl/ik} + g_{ik/jl} - g_{il/jk} - g_{jk/il})(X,G)(x_0). \tag{2.2.4}$$

Let

$$\mathfrak{A}_m^g = \mathbf{C}[g_{ij/\alpha}] \text{ for } |\alpha| \geq 2 \tag{2.2.5}$$

be the polynomial algebra in these variables. We restrict to coordinate systems which are normalized with respect to (G, x_0) and consequently we may omit the variables $\{g_{ij}, g_{ij/k}\}$ from consideration. Such coordinate systems always exist; for example we could take geodesic polar coordinates. Let $\mathfrak{A}_{m,n}^g$ be the linear subset of polynomials which are homogeneous of order n in the jets of the metric.

We define $g_{ij/\alpha}(X,G)(x_0)$ and $P(X,G)(x_0)$ by evaluation. We say that $P \in \mathfrak{A}_m^g$ is invariant if

$$P(X,G)(x_0) = P(Y,G)(x_0) \tag{2.2.6}$$

for any coordinate systems X and Y which are normalized with respect to (G, x_0). We denote the common value by $P(G)(x_0)$. Let $\mathfrak{I}^g_m \subset \mathfrak{A}^g_m$ be the ring of all invariant polynomials in the derivatives of the metric which are defined in the category of Riemannian manifolds of dimension m. Let

$$\mathfrak{I}^g_{m,n} \subset \mathfrak{A}^g_{m,n} \tag{2.2.7}$$

be the linear subspace of polynomials which are homogeneous of order n in the jets of the metric.

We shall also need to consider invariants which are form valued. Let

$$P = \Sigma_I P_I dx^I \text{ for } P_I \in \mathfrak{A}^g_m. \tag{2.2.8}$$

Let

$$P(X,G)(x_0) := \Sigma_I P_I(X,G)(x_0) dx^I \in \Lambda^p M(x_0) \tag{2.2.9}$$

be the evaluation. We say that P is invariant if

$$P(X,G)(x_0) = P(Y,G)(x_0) \tag{2.2.10}$$

for any coordinate systems X and Y which are normalized with respect to (G, x_0). We denote the common value by $P(G)(x_0)$.

Let $\mathfrak{I}^g_{m,*,*}$ denote the algebra of all invariant form valued polynomials; let $\mathfrak{I}^g_{m,n,p}$ denote the subspace of p form valued polynomials which are homogeneous of order n in the jets of the metric;

$$\mathfrak{I}^g_{m,n} = \mathfrak{I}^g_{m,n,0}. \tag{2.2.11}$$

We shall always assume $p \leq m$. If A is a monomial, let $c(A, P_I)$ be the coefficient of A in P_I;

$$P = \Sigma_{A,I} c(A, P_I) A dx^I. \tag{2.2.12}$$

We say A is a monomial of P_I if $c(A, P_I) \neq 0$.

Example 2.2.1: The scalar curvature τ belongs to $\mathfrak{I}^g_{m,2,0}$. If the coordinate system is normalized with respect to (G, x_0), then $\tau(G)(x_0)$ is given by the formula:

$$\tau(G)(x_0) = 2\Sigma_{i,j}(g_{ij/ij} - g_{ii/jj})(X,G)(x_0). \tag{2.2.13}$$

The variable $g_{11/22}$ is and the variable $g_{11/11}$ is not a monomial of τ. Similarly, $d\tau \in \mathfrak{I}^g_{m,3,1}$ since

$$d\tau(G)(x_0) = \{\Sigma_{ijk}(g_{ij/ijk} - g_{ii/jjk})dx^k\}(X,G)(x_0). \tag{2.2.14}$$

The definition of ord which we have given is purely algebraic; there is a geometrical characterization that is useful.

Lemma 2.2.1:

(a) *Let* $P \in \mathfrak{I}^g_{m*,p}$. *Then* $P \in \mathfrak{I}^g_{m,n,p}$ *if and only if for all* (G, x_0) *and* $c \neq 0$,

$$P(c^2G)(x_0) = c^{p-n}P(G)(x_0).$$

(b) $\mathfrak{I}^g_{m,*,*} = \oplus_{n,p}\mathfrak{I}^g_{m,n,p}$.

(c) $\mathfrak{I}^g_{m,n,p} = 0$ *for* $n+p$ *odd.*

(d) *If* $0 \neq P \in \mathfrak{I}^g_{m,n,p}$, *there exists* (G, x_0) *so* $P(G)(x_0) \neq 0$.

Proof: Let $P \in \mathfrak{I}^g_{m,*,p}$. Fix $c \neq 0$ and let X be normalized with respect to (G, x_0). Let $Y = cX$. Since $\partial^y_i = c^{-1}\partial^x_i$,

$$\begin{aligned} &G(\partial^y_i, \partial^y_j) = c^{-2}G(\partial^x_i, \partial^x_j), \\ &g_{ij/\alpha}(Y,G) = c^{-2-|\alpha|}g_{ij/\alpha}(X,G), \text{ and} \\ &g_{ij/\alpha}(Y,c^2G) = c^{-|\alpha|}g_{ij/\alpha}(X,G). \end{aligned} \tag{2.2.15}$$

If A is a monomial of some P_I, then A is homogeneous of order n in the jets of the metric so

$$A(Y,c^2G)(x_0) = c^{-n}A(X,G)(x_0). \tag{2.2.16}$$

We note $dy^I = c^p dx^I$. Since Y is normalized with respect to (c^2G, x_0) and X is normalized with respect to (G, x_0), we use (2.2.16) to establish one implication of (a):

$$\begin{aligned} P(c^2G)(x_0) &= \Sigma_{A,I}c(A,P_I)A(Y,c^2G)(x_0)dy^I \\ &= \Sigma_{A,I}c^{p-n}c(A,P_I)A(X,G)(x_0)dx^I \\ &= c^{p-n}P(G)(x_0). \end{aligned} \tag{2.2.17}$$

The other implication of (a) follows similarly.

If P is form valued, we may decompose

$$P = P^{(0)} + ... + P^{(m)} \tag{2.2.18}$$

where each $P^{(p)}$ is p form valued; clearly P is invariant if and only if each $P^{(p)}$ is invariant. Further decompose

$$P^{(p)} = \Sigma_n P_n^{(p)} \tag{2.2.19}$$

where the $P_n^{(p)}$ are homogeneous of order n in the jets of the metric. We use (a) to see the $P_n^{(p)} \in \mathfrak{I}^g_{m,n,p}$ are invariant separately. Consequently:

$$\mathfrak{I}^g_{m,*,*} = \oplus_{n,p} \mathfrak{I}^g_{m,n,p}. \tag{2.2.20}$$

This gives $\mathfrak{I}^g_{m,*,*}$ the structure of a graded algebra and proves (b); we take $c = -1$ to prove (c).

Let $0 \neq P \in \mathfrak{I}^g_{m,n,p}$ be non-zero as a polynomial. We use (b) to assume without loss of generality that P is homogeneous of order n in the jets of the metric and is p form valued. Choose real constants $c^I_{ij/\alpha}$ for $|\alpha| \geq 2$ so $0 \neq P_I(c_{ij/\alpha})$ for some I. Define the germ of a metric G on $\mathbf{R}^m$ by:

$$g_{ij}(x) = \delta_{ij} + \Sigma_{2 \leq |\alpha| \leq n} c_{ij/\alpha} x^\alpha / \alpha!. \tag{2.2.21}$$

Use a partition of unity to extend G to a compact manifold. Then the standard coordinates on $\mathbf{R}^m$ are normalized with respect to G and

$$P(X, G)(0) = \Sigma_J P_J(c_{ij/\alpha}) dx^J \neq 0. \blacksquare \tag{2.2.22}$$

Remark: (d) is the reason we work with the algebra of jets; it is a pure polynomial algebra and there are no relations. If we worked instead with the algebra of covariant derivatives of the curvature tensor, we would be forced to introduce the additional relations which correspond to the Bianchi identities and the covariant derivatives of the Bianchi identities; the resulting algebra would no longer be a pure polynomial algebra. We identify the polynomial and the formula which it defines henceforth.

Lemma 2.2.2: *$a_n(x, \Delta_p)$ defines an element of $\mathfrak{I}^g_{m,n}$.*

Proof: The total symbol of Δ_p is functorial so $a_n(x, \Delta_p)$ is given by a polynomial in the jets of the metric. Since $a_n(x, \Delta_p)$ is invariantly defined, this polynomial is invariant. Let c be constant. Then

$$\Delta_p(c^2 G) = c^{-2}\Delta_p(G). \tag{2.2.23}$$

Let $f \in C^\infty(M)$. By Theorem 1.2.1,

$$\begin{aligned} &\mathrm{Tr}_{L^2}(fe^{-tD}) \sim \Sigma_{n\geq 0} a_n(f, D) t^{(n-m)/2} \text{ where} \\ &a_n(f, D) = \textstyle\int_M f a_n(x, D) |\mathrm{dvol}\,(x)|. \end{aligned} \tag{2.2.24}$$

This specifies the $a_n(x, D)$ uniquely. We use this to compute:

$$\begin{aligned} &\mathrm{Tr}_{L^2}(fe^{-t\Delta_p(c^2G)}) \\ &\sim \Sigma_n t^{(n-m)/2} \textstyle\int_M f a_n(x, \Delta_p)(c^2G)\mathrm{dvol}\,(c^2G) \\ &\sim c^m \Sigma_n t^{(n-m)/2} \textstyle\int_M f a_n(x, \Delta_p)(c^2G)\mathrm{dvol}\,(G) \\ &= \mathrm{Tr}_{L^2}(fe^{-tc^{-2}\Delta_p(G)}) \\ &\sim \Sigma_n t^{(n-m)/2} c^{(m-n)} \textstyle\int_M f a_n(x, \Delta_p)(G)\mathrm{dvol}\,(G). \end{aligned} \tag{2.2.25}$$

We equate coefficients in the asymptotic expansions to see

$$\begin{aligned} &\textstyle\int_M f a_n(x, \Delta_p)(c^2G)\mathrm{dvol}\,(G) \\ &= c^{-n} \textstyle\int_M f a_n(x, \Delta_p)(G)\mathrm{dvol}\,(G). \end{aligned} \tag{2.2.26}$$

Since f was arbitrary,

$$a_n(x, \Delta_p)(c^2G) = c^{-n} a_n(x, \Delta_p)(G). \tag{2.2.27}$$

We use Lemma 2.2.1 to see that a_n is homogeneous of order n in the jets of the metric. ■

Introduce constants:

$$\begin{aligned} &c(m, p) = \binom{m}{p} = \tfrac{m!}{p!(m-p)!}, \\ &c_0(m, p) = c(m, p) - 6c(m-2, p-1), \\ &c_1(m, p) = 5c(m, p) - 60c(m-2, p-1) + 180c(m-4, p-2), \\ &c_2(m, p) = -2c(m, p) + 180c(m-2, p-1) - 720c(m-4, p-2), \\ &c_3(m, p) = 2c(m, p) - 30c(m-2, p-1) + 180c(m-4, p-2). \end{aligned}$$

Set $c(m,p) = c_\nu(m,p) = 0$ for $p < 0$ or $p > m$. The following theorem for $p = 0$ is due to McKean and Singer and to Patodi for general p; we present it without proof.

Theorem 2.2.3:

(a) $a_0(\Delta_p) = (4\pi)^{-m/2}\int_M c(m,p)|\text{dvol}\,|$.

(b) $a_2(\Delta_p) = (4\pi)^{-m/2}6^{-1}\int_M c_0(m,p)\tau|\text{dvol}\,|$.

(c) $a_4(\Delta_p) = (4\pi)^{-m/2}360^{-1}\int_M \{c_1(m,p)\tau^2 + c_2(m,p)\rho^2$
$+c_3(m,p)R^2\}|\text{dvol}\,|$.

We review H. Weyl's theorem on the invariants of the orthogonal group. Let V be a finite dimensional real vector space equipped with a positive definite inner product $(\cdot,\cdot)$. Let $\mathrm{O}(V)$ be the group of linear transformations of V preserving the inner product. Let

$$\otimes^k V = V \otimes ... \otimes V \tag{2.2.28}$$

be the k^{th} tensor product of V. If $g \in \mathrm{O}(V)$, extend g to act orthogonally on $\otimes^k V$ and on $\Lambda^p V$; let $z \to g \cdot z$ denote this action. We let $\mathrm{O}(V)$ act trivially on $\mathbf{C}$.

We say $f \in \mathrm{Hom}(\otimes^k V, \Lambda^p V)$ is equivariant if

$$f(g \cdot z) = g \cdot f(z)\ \forall g \in \mathrm{O}(V) \text{ and } \forall z \in \otimes^k V. \tag{2.2.29}$$

We denote the set of equivariant mappings by:

$$\mathrm{Hom}^{\,\mathrm{O}(V)}(\otimes^k V, \Lambda^p V). \tag{2.2.30}$$

If $p = 0$, the action of $\mathrm{O}(V)$ on the scalars is trivial and f is simply said to be invariant.

We construct such equivariant mappings as follows. Let $\sigma(k)$ be the group of permutations of $\{1,...,k\}$. If $\rho \in \sigma(k)$, let F^ρ act on $\otimes^k V$ by permuting the factors. Let

$$\lambda_p : \otimes^p V \to \Lambda^p V \tag{2.2.31}$$

be the anti-symmetrization; if $p = 0$, λ_p is the identity map. Let

$$g : V \otimes V \to \mathbf{R} \tag{2.2.32}$$

be the inner product and let

$$g^{(j)} = g \otimes .... \otimes g : \otimes^{2j} V \to \mathbf{R}. \tag{2.2.33}$$

If $\rho \in \sigma(k)$ and if $k - p = 2j$ is even, let

$$f^{\rho}_{k,p} = (g^{(j)} \otimes \lambda_p) \circ F^p \in \text{Hom}^{\,O(V)}(\otimes^k V, \Lambda^p V). \tag{2.2.34}$$

We set $f^{\rho}_{k,p} = 0$ if $k < p$ or if $k - p$ is odd.

Theorem 2.2.4 (H. Weyl): $\{f^{\rho}_{k,p}\}_{\rho\in\sigma(k)}$ *spans* $\text{Hom}^{\,O(V)}(\otimes^k V, \Lambda^p V)$.

Example 2.2.2: Let $\{e_i\}$ be an orthonormal basis for V and let $k = 4$. We sum over repeated indices to expand $z \in \otimes^4 V$ in the form:

$$z = a_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l. \tag{2.2.35}$$

Let $p = 0$. We remove duplications to construct the spanning set:

$$g_1(z) = a_{iijj},\ g_2(z) = a_{ijji},\ \text{and } g_3(z) = a_{ijij}. \tag{2.2.36}$$

Here $g_1 = f_{4,0}$ corresponds to the identity permutation, g_2 corresponds to the permutation which interchanges the second and the fourth factors, and g_3 corresponds to the permutation which interchanges the second and third factors. The g_i are linearly independent if $\dim(V) \geq 2$. If $p = 2$, the spanning set becomes:

$$\begin{aligned} &h_1(z) = a_{iijk} e_j \wedge e_k,\ h_2(z) = a_{ijik} e_j \wedge e_k,\\ &h_3(z) = a_{ijki} e_j \wedge e_k,\ h_4(z) = a_{jiki} e_j \wedge e_k,\\ &h_5(z) = a_{jiik} e_j \wedge e_k,\ h_6(z) = a_{jkii} e_j \wedge e_k. \end{aligned} \tag{2.2.37}$$

We use theorem 2.2.4 to give a spanning set for the spaces $\mathfrak{I}^{\,g}_{m,n,p}$:

Lemma 2.2.5: *Consider monomial expressions in the covariant derivatives of the curvature tensor where the order is n and where we contract indices in pairs relative to a local orthonormal frame. Such expressions span $\mathfrak{I}^{\,g}_{m,n}$. There is a similar spanning set for $\mathfrak{I}^{\,g}_{m,n,p}$; instead of fully contracting all indices in pairs, we alternate p of the indices and contract the remaining indices.*

Proof: We refer to the discussion in Atiyah, Bott, and Patodi for further details and only summarize the argument here briefly. In geodesic polar coordinates, all the jets of the metric can be expressed in terms of the covariant derivatives of the curvature tensor. Thus we can regard P as a polynomial in the $\{R_{\dots;\dots}\}$ variables.

The curvature $R \in \otimes^4 TM$ consists of tensors satisfying the 3 relations:

$$\begin{aligned}&R_{ijkl} = R_{klij},\ R_{ijkl} = -R_{jikl}, \text{ and}\\&R_{ijkl} + R_{iklj} + R_{iljk} = 0.\end{aligned} \tag{2.2.38}$$

Similarly $\nabla^k R$ lives in a sub-bundle of $\otimes^{4+k} TM$ which is defined by the higher order Bianchi identities. Consequently P defines an equivariant map

$$P : \mathcal{R} \to \Lambda^p M \tag{2.2.39}$$

where $\mathcal{R}$ is a suitable $\mathrm{O}(m)$ invariant subspace of the direct sum of tensor powers of the tangent bundle. Extend P to be 0 on the orthogonal complement of $\mathcal{R}$. Then P is an equivariant map from the direct sum of tensor powers of the tangent bundle to $\Lambda^p M$. We apply H. Weyl's theorem to each summand to derive Lemma 2.2.5 from Theorem 2.2.4. ■

The following diagonalization Lemma will be extremely useful. As the proof is a bit technical, we shall omit details.

Lemma 2.2.6: *Let* $0 \neq P = \Sigma_I P_I dx^I$ *for* $P_I \in \mathfrak{A}^{g,\nabla,L}_{m,n,p}$ *be invariant. Let*

$$A = g_{i_1 j_1/\alpha_1} \cdots g_{i_\nu j_\nu/\alpha_\nu}$$

be a monomial of P. Then we can find a monomial A_1 *with*

$$A_1 = g_{k_1 l_1/\beta_1} \cdots g_{k_\nu l_\nu/\beta_\nu}$$

so that for $1 \leq \mu \leq \mu$,

$$\begin{aligned}&|\beta_\mu| = |\alpha_\mu|,\ 1 \leq k_\mu \leq j_\mu \leq \mu,\\&\beta_\mu(j) = 0 \ \textit{for}\ j > \nu + \mu.\end{aligned}$$

§2.3 Normalizing constants

We begin this section by computing the volume of even dimensional spheres and by evaluating the Chern character of vector bundles given by Clifford matrices over spheres. A $\mathrm{Clif}(\mathbf{R}^{2j+1})$ module structure on $\mathbf{C}^k$ is a linear map q from $\mathbf{R}^{2j+1}$ to the set of self-adjoint $k \times k$ complex matrices such that

$$q(x)^2 = |x|^2 \cdot I_k. \tag{2.3.1}$$

Expand $x = x_i s^i$ relative to a unitary frame $\vec{s}$ for $\mathbf{C}^k$ and let

$$q(x) = x_i q^i. \tag{2.3.2}$$

Then the $\{q^i\}$ satisfy the Clifford commutation rules:

$$q^i q^j + q^j q^i = 2\delta^{ij}. \tag{2.3.3}$$

Let

$$\pi_{\pm}^q(x) = \tfrac{1}{2}(1 \pm q(x)) \text{ for } |x| = 1 \tag{2.3.4}$$

be orthogonal projection on the ± 1 eigenspaces of $q(x)$, and let

$$\Pi_{\pm}^q = \{(x, v) \in S^m \times \mathbf{C}^k : q(x)v = \pm v\} \tag{2.3.5}$$

be the corresponding eigenbundles. Since $\mathrm{Tr}(q(x)) \in \mathbf{Z}$ is a continuous function, it is constant. It vanishes since $q(-x) = -q(x)$. Thus $\dim(\Pi_{\pm}^q) = \frac{1}{2}k$ is constant so the $\Pi_{\pm}^q$ define smooth complementary sub-bundles;

$$S^m \times \mathbf{C}^k = \Pi_{+}^q \oplus \Pi_{-}^q. \tag{2.3.6}$$

Example 2.3.1: If $m = 2$, we could take:

$$q^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; q^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\text{ and } q^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{2.3.7}$$

to be the Dirac matrices. More generally, we could take the symbol of any operator of Dirac type.

Lemma 2.3.1:

(a) $\mathrm{vol}(S^{2j}) = j!\pi^j 2^{2j+1}/(2j)!$.

(b) *Let q give a* Clif $(\mathbf{R}^{2j+1})$ *module structure to* $\mathbf{C}^k$. *Orient* S^{2j} *so that* $dx^1 \wedge ... \wedge dx^{2j}$ *is the volume form at the north pole* $x = (1, 0, ..., 0)$. *Then*

$$\int_{S^{2j}} \text{ch}_j(\Pi_+^q) = i^j 2^{-j} \text{Tr}\,(q^0 ... q^m).$$

Proof: To prove (a), we use polar coordinates and integrate by parts to see:

$$\begin{aligned}
\pi^j \sqrt{\pi} &= \int_{\mathbf{R}^{2j+1}} e^{-|x|^2} dx &(2.3.8)\\
&= \int_0^\infty \int_{S^{2j}} r^{2j} e^{-r^2} d\theta dr\\
&= \text{vol}(S^{2j}) \int_0^\infty r^{2j} e^{-r^2} dr\\
&= \text{vol}(S^{2j}) \tfrac{2j-1}{2} \tfrac{2j-3}{2} ... \tfrac{1}{2} \int_0^\infty e^{-r^2} dr. &(2.3.9)
\end{aligned}$$

We compare (2.3.8) and (2.3.9) to prove (a).

Let $\Pi_\pm = \Pi_\pm^q$, $\pi_\pm = \pi_\pm^q$, etc. We use the decomposition (2.3.6) to project the flat connection on $S^{2j} \times \mathbf{C}^k$ to define connections $\nabla_\pm$ on $\Pi_\pm$:

$$\nabla_\pm s_\pm = \pi_\pm ds_\pm. \tag{2.3.10}$$

Let $\vec{s}_\pm(x_0)$ be a basis for $\Pi_\pm(x_0)$; we extend this basis to a local frame by defining:

$$\vec{s}_\pm(x) = \pi_\pm(x)\vec{s}_\pm(x_0). \tag{2.3.11}$$

We compute:

$$\begin{aligned}
\nabla_\pm(x)\vec{s}_\pm(x) &= \pi_\pm(x) d\pi_\pm(x) \vec{s}_\pm(x_0),\\
\Omega_\pm(x)\vec{s}_\pm(x) &= \pi_\pm(x) d\pi_\pm(x) d\pi_\pm \vec{s}_\pm(x_0).
\end{aligned} \tag{2.3.12}$$

This shows that:

$$\Omega_\pm = \pi_\pm d\pi_\pm d\pi_\pm \in \text{End}\,(\Pi_\pm). \tag{2.3.13}$$

We use (2.3.13) to compute ch_j. Choose oriented orthonormal coordinates for $\mathbf{R}^{2j+1}$ so that $x_0 = (1, 0, ..., 0)$ is the north pole. We note

$$(q^0 ... q^{2j})^2 = \pm 1 \text{ so } \text{Tr}\,(q^0 ... q^{2j}) \in \mathbf{Z} \cup i\mathbf{Z}. \tag{2.3.14}$$

Consequently, this is continuous and hence constant on S^{2j}; it is invariantly defined and only depends on the orientation of $\mathbf{R}^{2j+1}$. We sum over

$1 \le i \le 2j$ compute:

$$\begin{aligned}
&\mathrm{dvol}\,(x_0) = dx^1 \wedge ... \wedge dx^{2j}, \\
&\pi_+(x_0) = \tfrac{1}{2}(1 + q^0), \\
&d\pi_+(x_0) = \tfrac{1}{2} dx^i \cdot q^i, \\
&\Omega_+(x_0) = \tfrac{1}{2}(1 + q^0)(\tfrac{1}{2} dx^i \cdot q^i)^2, \\
&\Omega_+(x_0)^j = 2^{-2j-1}(2j)!(1 + q^0)(q^1 ... q^{2j})(dx^1 \wedge ... \wedge dx^{2j}) \\
&\quad = 2^{-2j-1}(2j)!(1 + q^0)(q^1 ... q^{2j})\mathrm{dvol}\,.
\end{aligned} \tag{2.3.15}$$

Since q^1 anti-commutes with $q^1 .... q^{2j}$, $\mathrm{Tr}\,(q^1 ... q^{2j}) = 0$. Consequently

$$\mathrm{ch}_j(\nabla_+)(x_0) = (\tfrac{i}{2\pi})^j \tfrac{(2j)!}{j!} 2^{-2j-1} \mathrm{Tr}\,(q^0 ... q^{2j})\mathrm{dvol}\,. \tag{2.3.16}$$

Since x_0 was arbitrary:

$$\textstyle\int_{S^{2j}} \mathrm{ch}_j(V_+) = (\tfrac{i}{2\pi})^j \tfrac{(2j)!}{j!} 2^{-2j-1} \mathrm{Tr}\,(q^0 ... q^{2j}) \ \mathrm{vol}(S^{2j}). \ \blacksquare \tag{2.3.17}$$

Next we study the Euler form on products of two dimensional spheres. Recall that if $m = 2n$, then

$$\begin{aligned}
c_m &= \{(-8\pi)^n n!\}^{-1}, \text{ and} \\
E_m &= c_m \Sigma_{I,J} \epsilon_{I,J} R_{i_1 i_2 j_1 j_2} ... R_{i_{m-1} i_m j_{m-1} j_m}.
\end{aligned} \tag{2.3.18}$$

Lemma 2.3.2: $\int_{S^{2n}} E_{2n} |\mathrm{dvol}\,| = 2.$

Proof: The curvature tensor of the standard sphere is given by

$$R_{ijkl} = (\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}). \tag{2.3.19}$$

Consequently

$$\begin{aligned}
&E_{2n} = c_{2n}(-1)^n 2^n (2n)! = (4\pi)^{-n}(2n)!/n! \\
&\textstyle\int_{S^{2n}} E_{2n} |\mathrm{dvol}\,| = (4\pi)^{-n}(2n)!/n! \cdot \mathrm{vol}(S^{2n}) = 2. \ \blacksquare
\end{aligned} \tag{2.3.20}$$

Let CP^n be complex projective space; this is the set of complex lines in $\mathbf{C}^{n+1}$ and is a holomorphic manifold. The real dimension of CP^n is $2n$. Let L be the tautological line bundle over CP^n;

$$L = \{\langle x \rangle \times z \in \mathrm{CP}^n \times \mathbf{C}^{n+1} : z \in \langle x \rangle\}. \tag{2.3.21}$$

Let L^* be the dual line bundle; this is often called the hyperplane bundle. Let T_c be the holomorphic tangent bundle. The following result is well known.

Lemma 2.3.3: *Let* $x_n = -c_1(L) = c_1(L^*) \in H^2(\mathrm{CP}^n)$.

(a) $\int_{\mathrm{CP}^n} x_n^n = 1$.

(b) $H^*(\mathrm{CP}^n; \mathbf{C}) = \mathbf{C}[x_n]/(x_n^{n+1} = 0)$.

(c) *Let* $i_n : \mathrm{CP}^{n-1} \to \mathrm{CP}^n$. *Then* $i_n^*(x_n) = x_{n-1}$.

(d) $T_c(\mathrm{CP}^n) \oplus 1 \cong (n+1)L^*$.

(e) $c(T_c(\mathrm{CP}^n)) = (1 + x_n)^{n+1}$ *and* $p(T(\mathrm{CP}^n)) = (1 + x_n^2)^{n+1}$.

Remark: Since the cohomology class x_n is stable under pull back, we drop the dependence on the dimension n and let $x \in H^2(\mathrm{CP}^n; \mathbf{C})$ denote this universal class.

Let $m = 2n$, let P be a characteristic class, and let

$$P[M] := \int_M P_n(TM). \tag{2.3.22}$$

The projective spaces form a dual basis to the real characteristic classes. Let $\rho = (i_1, ..., i_j)$ for $i_1 \geq ... \geq i_j > 0$ be a partition of

$$k = |\rho| := i_1 + ... + i_j. \tag{2.3.23}$$

We let $\ell(\rho) = j$ be length of ρ. For example, if $k = 4$, there are 5 partitions:

$$\begin{aligned} &4 = 4,\ 4 = 3 + 1,\ 4 = 2 + 2, \\ &4 = 2 + 1 + 1,\ \text{and } 4 = 1 + 1 + 1 + 1. \end{aligned} \tag{2.3.24}$$

We define classifying manifolds and characteristic classes:

$$\begin{aligned} &\mathrm{CP}(\rho) = \mathrm{CP}^{i_1} \times ... \times \mathrm{CP}^{i_j}, \quad \text{and} \\ &p(\rho) = p_{i_1} ... p_{i_j}. \end{aligned} \tag{2.3.25}$$

Lemma 2.3.4: *Let* $k \in N$. *The matrix* $\{p(\tau)[\mathrm{CP}(2\rho)]\}_{|\rho|=|\tau|=k}$ *is invertible.*

Remark: This means that products of even dimensional projective spaces form a dual basis to $\mathfrak{I}(\mathrm{O})$. In other words, real characteristic classes are completely determined by their values on the appropriate classifying manifolds. This also means that given constants $C_r(\rho)$, there exist unique characteristic polynomials $Q_r \in \mathfrak{I}(\mathrm{O})$ so that

$$Q_r[\mathrm{CP}(2\rho)] = C_r(\rho). \tag{2.3.26}$$

There are corresponding statements in the complex category.

Proof: If $V \in \mathfrak{V}^{\mathbf{R}}M$, define:

$$\mathrm{ch}_\nu^r(V) = \mathrm{ch}_\nu(V \otimes \mathbf{C}). \tag{2.3.27}$$

It is easy to establish that

$$\mathfrak{I}(\mathrm{O}(2k)) = \mathbf{C}[\mathrm{ch}_2^r, ..., \mathrm{ch}_{2k}^r]. \tag{2.3.28}$$

Let

$$C_{\rho\sigma} = \mathrm{ch}(2\tau)[\mathrm{CP}_c(2\rho)]_{|\rho|=|\tau|=k}; \tag{2.3.29}$$

we must show this matrix is non-singular.

The advantage of working with the Chern character rather than with the Pontrjagin classes is that the Chern character is additive with respect to Cartesian product – i.e.

$$\mathrm{ch}_\nu(M^1 \times M^2) = \mathrm{ch}_\nu(M^1) + \mathrm{ch}_\nu(M^2). \tag{2.3.30}$$

Furthermore, $\mathrm{ch}_\nu(M) = 0$ for $\nu > 2\dim(M)$. We note:

$$\mathrm{ch}_{2\tau}[\mathrm{CP}_c(2\rho)] = 0 \tag{2.3.31}$$

if $\ell(\tau) < \ell(\rho)$ or if $\ell(\tau) = \ell(\rho)$ but $\tau \neq \rho$. We define a partial order $\tau < \rho$ if $\ell(\tau) < \ell(\rho)$ and extend this to a total order. Then the matrix defined in (2.3.29) is triangular. We complete the proof by showing the diagonal elements are non-zero.

We use (2.3.30) to see it suffices to show

$$\mathrm{ch}_{2n}[\mathrm{CP}_c^{2n}] \neq 0. \tag{2.3.32}$$

We use the identity

$$(T(\mathrm{CP}^{2n}) \oplus 1^2) \otimes \mathbf{C} = (2n+1)L + (2n+1)L^* \tag{2.3.33}$$

to compute that

$$\mathrm{ch}_{2n}(T(\mathrm{CP}^{2n})) = 2(2n+1)x^{2n}/n!. \blacksquare \qquad (2.3.34)$$

The Hirzebruch L-polynomial was defined using generating functions. The generating functions were chosen so they would be particularly simple on the classifying examples. We recall the definition. Let $A \in \mathfrak{u}(k)$ have eigenvalues $\{\lambda_j\}_{j=1}^k$. Let $x_j = i\lambda_j/2\pi$. Let

$$\mathrm{L}(A) = \Pi_j \frac{x_j}{\tanh(x_j)} \in \mathfrak{I}(\mathrm{U}). \qquad (2.3.35)$$

Let $A \in \mathfrak{o}$ have non-zero eigenvalues $\{\pm i\lambda_j\}_{j=1}^{\ell}$. Let $x_j = -\lambda_j/2\pi$. Let

$$\mathrm{L}(A) = \Pi_j \frac{x_j}{\tanh(x_j)} \in \mathfrak{I}(\mathrm{O}). \qquad (2.3.36)$$

Let $B \in \mathfrak{u}(k)$ and let $B_r \in \mathfrak{o}(2k)$ be the underlying real matrix. Then

$$\mathrm{L}(B) = \mathrm{L}(B_r). \qquad (2.3.37)$$

Lemma 2.3.5: $\mathrm{L}[\mathrm{CP}(2\rho)] = 1\ \forall \rho$.

Proof: The Hirzebruch polynomial is a multiplicative class. Consequently, it suffices to prove Lemma 2.3.5 in the special case that $M = \mathrm{CP}^{2n}$. We must show

$$\mathrm{L}[\mathrm{CP}^{2n}] = 1. \qquad (2.3.38)$$

If V is a complex vector bundle, let V_r be the underlying real vector bundle. Then $\mathrm{L}^c(V) = \mathrm{L}(V_r)$. Since $\{T_c(\mathrm{CP}^{2n})\}_r = T(\mathrm{CP}^{2n})$,

$$\begin{aligned} \mathrm{L}(T(\mathrm{CP}^{2n})) &= \mathrm{L}^c(T_c(\mathrm{CP}^{2n})) = \mathrm{L}^c(T_c(\mathrm{CP}^{2n}) \oplus 1) \\ &= \mathrm{L}^c(L^* \otimes 1^{2n+1}) = \mathrm{L}(x)^{2n+1} \\ &= x^{2n+1}/\tanh(x)^{2n+1}. \end{aligned} \qquad (2.3.39)$$

We must show the coefficient of x^{2n} in this expression is 1; i.e.

$$\mathrm{Res}_{x=0} \tanh(x)^{-2n-1} = 1. \qquad (2.3.40)$$

We compute:

$$\begin{aligned} \tanh(x) &= \tfrac{e^x - e^{-x}}{e^x + e^{-x}}, \\ \tanh(x)^{-1} &= (2x)^{-1}(2 + O(x)) = x^{-1}(1 + O(x)), \end{aligned} \tag{2.3.41}$$

so (2.3.40) holds for $n = 0$. Establishing (2.3.40) directly would be a combinatorial nightmare for larger values of k, so we use instead a standard trick from complex variables. If $g(x)$ is any meromorphic function, then

$$\operatorname{Res}_{x=0} g'(x) = 0. \tag{2.3.42}$$

We set $g(x) = \tanh(x)^{-k}$. Then

$$g'(x) = -k \tanh(x)^{-k-1}(1 - \tanh(x)^2). \tag{2.3.43}$$

Consequently,

$$\operatorname{Res}_{x=0} \tanh(x)^{-k-1} = \operatorname{Res}_{x=0} \tanh(x)^{-k+1}. \ \blacksquare \tag{2.3.44}$$

3. TOPOLOGICAL APPLICATIONS

§3.1 Gauss-Bonnet theorem

We shall give an axiomatic characterization of the Euler integrand and prove the Gauss-Bonnet theorem using heat equation methods. We recall the definition of the Euler integrand. Let R_{ijkl} be the components of the curvature tensor of the Levi-Civita connection relative to a local orthonormal frame for the tangent space; with our sign convention, the scalar curvature $\tau = R_{ijji}$. Let $E_m = 0$ if m is odd. If $m = 2n$ is even define $E_m(g) \in \mathfrak{I}^{\,g}_{m,m}$ by:

$$\begin{aligned} &\epsilon_{I,J} = (e^{i_1} \wedge \ldots \wedge e^{i_m}, e^{j_1} \wedge \ldots \wedge e^{j_m}),\\ &c_m = \{(-8\pi)^n n!\}^{-1}, \text{ and} \\ &E_m = c_m \Sigma_{I,J} \epsilon_{I,J} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{m-1} i_m j_{m-1} j_m}. \end{aligned} \tag{3.1.1}$$

For example:

$$E_2 = (4\pi)^{-1}\tau, \text{ and} \tag{3.1.2}$$

$$E_4 = (32\pi^2)^{-1}\{\tau^2 - 4|\rho|^2 + |R|^2\}. \tag{3.1.3}$$

We use H. Weyl's theorem to see

$$\begin{aligned} &\mathfrak{I}^{\,g}_{m,2} = \text{Span}\,\{\tau\}, \text{ and} \\ &\mathfrak{I}^{\,g}_{m,4} = \text{Span}\,\{\tau^2, \rho^2, \ R^2, \ \Delta\tau\}. \end{aligned} \tag{3.1.4}$$

The scalar curvature τ is non-zero and forms a basis for $\mathfrak{I}^{\,g}_{m,2}$ if $m \geq 2$; the scalar invariants of (3.1.4) are linearly independent and form a basis for $\mathfrak{I}^{\,g}_{m,4}$ if $m \geq 4$. Clearly $\tau = 0$ if $m = 1$ since the metric is flat. If $m = 3$, the invariants of (3.1.4) are linearly dependent; there is a single additional relation which is given by the Euler form:

$$(32\pi^2)^{-1}\{\tau^2 - 4|\rho|^2 + |R|^2\} = 0. \tag{3.1.5}$$

This reflects the fact that the Euler form is an unstable characteristic class; $E_{2n}(G) = 0$ if $m < 2n$ since we must repeat an index in (3.1.1). Similarly $E_m(G) = 0$ if the metric is flat in one direction.

There is a natural restriction map

$$r : \mathfrak{I}^g_{m,n} \to \mathfrak{I}^g_{m-1,n} \tag{3.1.6}$$

which is defined algebraically as follows. Let

$$\deg_k(g_{ij/\alpha}) := \delta_{i,k} + \delta_{j,k} + \alpha(k) \tag{3.1.7}$$

be the number of times the index k appears in the variable $g_{ij/\alpha}$. Let

$$r(g_{ij/\alpha}) = \begin{cases} g_{ij/\alpha} & \text{if } \deg_m(g_{ij/\alpha}) = 0, \\ 0 & \text{if } \deg_m(g_{ij/\alpha}) \neq 0. \end{cases} \tag{3.1.8}$$

Since $r(g_{ij/\alpha})$ does not involve the last index, $r(g_{ij/\alpha}) \in \mathfrak{A}^g_{m-1}$. We extend r to an algebra homomorphism from $\mathfrak{A}^g_m$ to $\mathfrak{A}^g_{m-1}$ which preserves the grading defined by the order in the jets of the metric;

$$r(\mathfrak{A}^g_{m,n}) \subset \mathfrak{A}^g_{m-1,n}. \tag{3.1.9}$$

The map r is the dual of a natural extension map. Let $\tilde{G}$ be a metric on a manifold $\tilde{M}$ of dimension $m-1$. Let

$$i(\tilde{G}) := \tilde{G} + d\theta^2 \text{ on } M := \tilde{M} \times S^1. \tag{3.1.10}$$

Let θ_0 be the base point of the circle. If $\tilde{X}$ are coordinates on $\tilde{M}$ which are normalized with respect to $(\tilde{G}, \tilde{x}_0)$, then the coordinates

$$i(\tilde{X}) := (\tilde{X}, \theta) \tag{3.1.11}$$

are normalized with respect to $(G, x_0 \times \theta_0)$. It is immediate from the definition that:

$$r(P)(\tilde{X}, \tilde{G})(x_0) = P(i(\tilde{X}), i(\tilde{G}))(x_0 \times \theta_0). \tag{3.1.12}$$

What we have done evaluating on a product manifold is to introduce the relation which says the metric is flat in the last coordinate. This shows restriction r is the dual of this natural extension i; consequently rP is invariant if P is invariant; r defines an algebra homomorphism

$$r : \mathfrak{I}^g_m \to \mathfrak{I}^g_{m-1} \tag{3.1.13}$$

which restricts to linear maps

$$r : \mathfrak{I}^{g}_{m,n} \rightarrow \mathfrak{I}^{g}_{m-1,n}. \tag{3.1.14}$$

We define:

$$\mathcal{K}^{r}_{m,n} = \mathfrak{N}(r : \mathfrak{I}^{g}_{m,n} \rightarrow \mathfrak{I}^{g}_{m-1,n}). \tag{3.1.15}$$

Theorem 3.1.1:

(a) $r : \mathfrak{I}^{g}_{m,n} \rightarrow \mathfrak{I}^{g}_{m-1,n}$ *is surjective.*

(b) *The Euler invariant* $E_m \in \mathcal{K}^{r}_{m,m}$.

(c) *If* $n < m$, *then* $\mathcal{K}^{r}_{m,n} = \{0\}$.

(d) $\mathcal{K}^{r}_{m,m} = \operatorname{Span}\{E_m\}$.

Remark: This provides an axiomatic characterization of the Euler integrand. The Euler form is an unstable characteristic class in contrast to the Pontrjagin classes; this characterization captures this property.

Proof: We use a third description of r to prove (a). In a Weyl spanning set, the indices are summed from 1 through m; the restriction is defined by letting the indices range from 1 through $m-1$. Thus $r(R_{ijji}) = R_{ijji}$ is its own restriction in a formal sense; of course $r(R_{ijji}) = 0$ if $m = 2$ since all the jets of the metric vanish on a circle. If we take a Weyl spanning set for $\mathfrak{I}^{g}_{m-1,n}$, we extend these elements to define invariant polynomials in $\mathfrak{I}^{g}_{m,n}$ by extending the range of summation. This proves (a); (b) is immediate if m is odd since $E_m = 0$. If m is even, we defined:

$$E_m(g) = c_m \Sigma_{I,J} \epsilon_{I,J} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{m-1} i_m j_{m-1} j_m}. \tag{3.1.16}$$

Since $\epsilon_{I,J} = 0$ if an index is repeated in either I or in J, both I and J are permutations of $\{1, \ldots, m\}$. Thus in particular, some $i_\nu = m$ if $\epsilon_{I,J} \neq 0$. To define r, we took a product with the circle and hence $R_{ijkl} = 0$ if the index m appears. This proves (b).

We use orthogonal invariance to prove (c) and (d). Let

$$0 \neq P \in \mathcal{K}^{r}_{m,n} = \mathfrak{N}(r : \mathfrak{I}^{g}_{m,n} \rightarrow \mathfrak{I}^{g}_{m-1,n}). \tag{3.1.17}$$

Let

$$A = g_{i_1 j_1/\alpha_1} \ldots g_{i_\ell j_\ell/\alpha_\ell} \tag{3.1.18}$$

be a monomial of P. Since $r(P) = 0$, $\deg_m(A) \neq 0$. Since P is invariant under the action of the coordinate permutations, $\deg_i(A) \neq 0$ for all indices i. Since P is invariant if we change the sign of a coordinate function, $\deg_i(A)$ is even. This shows

$$\deg_i(A) \geq 2 \text{ for } 1 \leq i \leq m. \tag{3.1.19}$$

A total of $2\ell + n$ indices appear in A. Thus

$$2m \leq \Sigma_{1 \leq i \leq m} \deg_i(A) = 2\ell + n. \tag{3.1.20}$$

On the other hand, we normalized the coordinate systems so $|\alpha_\nu| \geq 2$. Thus

$$2\ell \leq \Sigma_{1 \leq \nu \leq \ell} |\alpha_\nu| \leq n. \tag{3.1.21}$$

We use (3.1.20) and (3.1.21) to see:

$$2m \leq 2\ell + n \leq 2n. \tag{3.1.22}$$

Consequently, if $n < m$, then $\mathcal{K}^r_{m,n} = \{0\}$; this proves (c).

Since $0 \neq E_m \in \mathcal{K}^r_{m,m}$, to prove (d), we must show

$$\dim(\mathcal{K}^r_{m,m}) \leq 1. \tag{3.1.23}$$

In the limiting case $n = m$, all the inequalities of the previous paragraph must have been equalities. Thus any monomial A of P can be put in the form:

$$A = g_{i_1 j_1 / p_1 q_1} \cdots g_{i_\ell j_\ell / p_\ell q_\ell} \tag{3.1.24}$$

where $2\ell = m$ and where $\deg_i(A) = 2$ for $1 \leq i \leq m$. We use Lemma 2.2.6 to construct a monomial $\tilde{A}$ of P with

$$i_\nu, j_\nu \leq \nu \quad \text{and } p_\nu, q_\nu \leq \ell + \nu. \tag{3.1.25}$$

This shows $i_1 = j_1 = 1$. Since each index appears exactly twice in $\tilde{A}$, the index 1 appears nowhere else in $\tilde{A}$. Thus $i_2 = j_2 = 2$ and inductively $i_\nu = j_\nu = \nu$. All these indices appear exactly twice so none of the indices from 1 through ℓ appear in the $\{p_\nu, q_\nu\}$. The same argument now shows $p_\nu = q_\nu = \nu + \ell$ so:

$$\tilde{A} = g_{11/\ell+1,\ell+1} \cdots g_{\ell\ell/mm} \tag{3.1.26}$$

is a monomial of P. We summarize. If $0 \neq P \in \mathcal{K}^r_{m,m}$, then $0 \neq c(\tilde{A}, P)$. Since one linear functional separates polynomials in the kernel,

$$\dim(\mathcal{K}^r_{m,m}) \leq 1. \blacksquare \tag{3.1.27}$$

Remark: It is absolutely crucial that we are dealing with a free polynomial algebra at this stage so that $c(A_\ell, P)$ is a well defined linear functional; this argument would fail if we were dealing with the covariant derivatives of the curvature tensor owing to the Bianchi identities.

We can now prove the Gauss-Bonnet theorem. We recall some notation. We use Lemma 2.2.2 to see $a_n(x,\Delta_p) \in \mathfrak{I}^g_{m,n}$. Let

$$a_n(x,d+\delta) := \Sigma_p(-1)^p a_n(x,\Delta_p) \in \mathfrak{I}^g_{m,n}.$$

The index of the de Rham complex is the Euler-Poincaré characteristic $\chi(M)$. We use Theorem 1.3.1 to see:

$$\int_M a_n(x,d+\delta)|\mathrm{dvol}\,| = \begin{cases} 0 & \text{if } n \neq m, \\ \chi(M) & \text{if } n = m. \end{cases} \tag{3.1.28}$$

Theorem 3.1.2:

(a) $a_n(x,d+\delta) = 0$ *if either* m *is odd or if* $n < m$.

(b) $a_m(x,d+\delta) = E_m$ *is the Euler integrand.*

(c) $\chi(M) = \int_M E_m(G)(x)|\mathrm{dvol}\,|$. **(Gauss-Bonnet)**

Proof: Locally, we can always choose an orientation for the tangent bundle TM. Let $\star$ be the Hodge operator. Thus, $\star$ intertwines Δ_p and Δ_{m-p}. Since these two operators are locally isomorphic, their local invariants agree so

$$a_n(x,\Delta_p) = a_n(x,\Delta_{m-p}). \tag{3.1.29}$$

Since the argument is local, (3.1.29) holds even if M does not admit a global orientation. Consequently, we may compute:

$$\begin{aligned} a_n(x,d+\delta) =&\Sigma_p(-1)^p a_n(x,\Delta_p) \\ =&\Sigma_p(-1)^p a_n(x,\Delta_{m-p}) \\ =&(-1)^m \Sigma_p(-1)^{m-p} a_n(x,\Delta_{m-p}) \\ =&(-1)^m a_n(x,d+\delta). \end{aligned} \tag{3.1.30}$$

This proves $a_n(x,d+\delta) = 0$ for m odd. We therefore suppose m even for the remainder of the proof.

Let $M = N \times S^1$ have the product metric. Decompose

$$\Lambda(M) = \Lambda(N) \oplus d\theta \wedge \Lambda(N). \tag{3.1.31}$$

Let $F = \operatorname{ext}^l(d\theta) + \operatorname{int}^l(d\theta) \in \operatorname{End}(\Lambda M)$; if $\omega = \omega_1 + d\theta \wedge \omega_2$, then:

$$F(\omega) = d\theta \wedge \omega_1 + \omega_2. \tag{3.1.32}$$

Since the metric is flat in the S^1 direction, $F\Delta = \Delta F$. If we decompose

$$\Lambda(M) = \Lambda^{ev}(M) \oplus \Lambda^{od}(M) \tag{3.1.33}$$

into the forms of even and odd degree, then F interchanges these two factors. Thus $a_n(x, \Delta_{ev}) = a_n(x, \Delta_{od})$ and

$$a_n(x, d + \delta) = 0 \tag{3.1.34}$$

for such a product metric. This implies $r(a_n(\cdot, d + \delta)) = 0$. Consequently $a_n = 0$ for $n < m$ which completes the proof of (a). Furthermore, there is some universal constant $\tilde{c}(m)$ so that

$$a_m(\cdot, d + \delta) = \tilde{c}(m) E_m. \tag{3.1.35}$$

We use Lemma 2.3.2 to show $\tilde{c}(m) = 1$ by computing:

$$2 = \chi(S^m) = \textstyle\int_{S^m} E_m = 2\tilde{c}(M). \blacksquare \tag{3.1.36}$$

Remark: This result was first established if $m = 2$ by McKean and Singer. The general case was first proved by Patodi who used a complicated cancellation argument very different from the argument we have just given based on invariance theory. This gives a heat equation proof of the Gauss-Bonnet theorem; we refer to Chern for a more geometrical proof. There are many other proofs, of course.

§3.2 Hirzebruch signature theorem

We begin with an axiomatic characterization of the Pontrjagin forms. We adopt the notation of §2.1. Let $\mathfrak{I}^{g}_{m,n,p}$ be the vector space of p form valued polynomials which are homogeneous of order n in the jets of the metric and which are defined in dimension m. If $P \in \mathfrak{I}^{g}_{m,n,p}$, decompose

$$P = \Sigma_{|I|=p} P_I dx^I \text{ for } P_I \in \mathfrak{A}^{g}_{m} = \mathbf{C}[g_{ij/\alpha}] \text{ with } |\alpha| \geq 2. \tag{3.2.1}$$

The Pontrjagin forms are examples of such polynomials. Let R be the curvature of the Levi-Civita connection on the tangent bundle TM;

$$p(R) := \det(I + (2\pi)^{-1}R) = 1 + p_1(R) + \ldots + p_{[m/4]}(R). \tag{3.2.2}$$

Since R is homogeneous of order 2 in the jets of the metric and since R is 2-form valued, $p_\nu \in \mathfrak{I}^{g}_{m,4\nu,4\nu}$. For example:

$$\begin{aligned} p_1 =& (-8\pi^2)^{-1}\mathrm{Tr}\,(R \wedge R) \\ =& (-32\pi^2)^{-1} R_{ijk_1k_2} R_{jik_3k_4} e^{k_1} \wedge e^{k_2} \wedge e^{k_3} \wedge e^{k_4}. \end{aligned} \tag{3.2.3}$$

The following theorem provides an abstract characterization of the algebra generated by the Pontrjagin forms.

Theorem 3.2.1:

(a) $\mathfrak{I}_{m,n,p} = \{0\}$ *for* $n < p$.

(b) $\oplus_p \mathfrak{I}^{g}_{m,p,p}$ *is generated by the Pontrjagin forms of the tangent bundle.*

Proof: Let $0 \neq P = \Sigma_{|I|=p} P_I dx^I \in \mathfrak{I}^{g,\nabla}_{m,n,p}$. Let A be a monomial of some P_I where

$$A = g_{i_1j_1/\alpha_1} \cdots g_{i_\nu j_\nu/\alpha_\nu} \tag{3.2.4}$$

We apply Lemma 2.2.6 to choose a monomial A_1 of some P_J so that

$$\nu(A_1) = \nu,\ |\beta_i| = |\alpha_i|, \text{ and } \deg_k(A_1) = 0 \text{ for } k > 2\nu. \tag{3.2.5}$$

Since P is invariant under the action of hyperplane reflections,

$$\deg_k A_1 + \deg_k J \text{ is even} \tag{3.2.6}$$

for all k. Consequently $\deg_k(J) = 0$ for $k > 2\nu$ so that

$$p \leq 2\nu. \tag{3.2.7}$$

Since $|\alpha_i| \geq 2$ and $|\beta_i| \geq 1$, we may estimate:

$$2\nu \leq |\alpha_1| + ... + |\alpha_\nu| = n. \tag{3.2.8}$$

Consequently $0 \neq P$ implies $p \leq n$ which proves (a).

In the limiting case $n = p$, all the inequalities of the preceeding paragraph must have been equalities. Thus p is even and P is a polynomial in the 2-jets of the metric. We now shift our point of view completely. We choose geodesic polar coordinates centered at x_0; this reduces the structure group from the group of germs of normalized diffeomorphisms to the orthogonal group $\mathrm{O}(m)$. In such coordinate systems, the 2-jets of the metric can be expressed in terms of the curvature tensor R_{ijkl} and vica versa. Consequently, we may regard P as a polynomial which is invariant under the action of $\mathrm{O}(m)$ in the variables

$$\{R_{ijkl}\}. \tag{3.2.9}$$

The group $\mathrm{O}(m)$ acts on the indices $\{i, j, k, l\}$ from the tangent bundle. The algebra in the variables (3.2.9) is no longer free; the Bianchi identities enter.

Let A be a monomial of P where

$$A = R_{i_1 i_2 i_3 i_4} ... R_{i_{4\nu-3} i_{4\nu-2} i_{4\nu-1} i_{4\nu}}. \tag{3.2.10}$$

We apply H. Weyl's Theorem to this setting. A spanning set for the space of p form valued invariants can be formed by alternating $p = 2\nu$ indices and contracting the remaining indices in pairs. By the Bianchi identities, we can not alternate more than 2 indices in any R_{ijkl} variable without getting zero. Thus a counting argument shows we must alternate exactly two indices and contract exactly two indices in each $R_{....}$ variable. We use the Bianchi identities to see:

$$R_{i_1 k_1 i_2 k_2} e^{k_1} \wedge e^{k_2} = \tfrac{1}{2} R_{i_1 i_2 k_1 k_2} e^{k_1} \wedge e^{k_2} \tag{3.2.11}$$

so we may always assume that the last two indices in R are alternated. Let

$$R_{ij} := \tfrac{1}{2} R_{ijk_1 k_2} e^{k_1} \wedge e^{k_2} \tag{3.2.12}$$

be the curavture 2-form of the Levi-Civita connection. Then

$$P = P(R^a_{ij}) \tag{3.2.13}$$

is a polynomial which is homogeneous of degree $\frac{1}{2}p$ in these commuting 2-form valued variables.

The structure group is $O(m)$. The analysis of §2.1 extends at once to this setting to show that the ring of invariant form valued polynomials is generated by the Pontrjagin forms of the tangent bundle. Contracting tangential variables corresponds to taking products of traces of endomorphisms of TM and yields Pontrjagin forms. ■

We now discuss the signature complex. Let

$$(d+\delta) : C^\infty \Lambda M \to C^\infty \Lambda M \tag{3.2.14}$$

be exterior differentiation d plus its adjoint interior differentiation δ as discussed earlier. As discussed earlier, we decompose

$$\Lambda M = \Lambda^+ M \oplus \Lambda^- M \tag{3.2.15}$$

and let

$$(d+\delta)^\pm : C^\infty \Lambda^\pm M \to C^\infty \Lambda^\mp M. \tag{3.2.16}$$

The adjoint of $(d+\delta)^+$ is $(d+\delta)^-$. Let

$$\text{Sign}\,(M) := \text{index}(d+\delta)^+. \tag{3.2.17}$$

We decompose the Laplacian

$$\Delta = \Delta^+ \oplus \Delta^- \tag{3.2.18}$$

where $\Delta^\pm$ are operators of Laplace type on $C^\infty \Lambda^\pm M$. Then by the Hodge decomposition theorem,

$$\text{Sign}\,(M) = \dim \mathfrak{N}(\Delta^+) - \dim \mathfrak{N}(\Delta^-). \tag{3.2.19}$$

Let

$$a_n^s(x, g, \text{orn}) = \{a_n(x, \Delta^+) - a_n(x, \Delta^-)\}\text{dvol}. \tag{3.2.20}$$

By Theorem 1.3.1

$$\int_M a_n^s = \begin{cases} 0 & \text{if } n \neq m, \\ \text{Sign}\,(M) & \text{if } n = m. \end{cases} \tag{3.2.21}$$

The operator underlying operator $(d+\delta)$ on $\Lambda_c M$ is real. If $m \equiv 2$ mod 4, then τ is pure imaginary. Complex conjugation defines an isomorphism

$$\Lambda^+ M \simeq \Lambda^- M \tag{3.2.22}$$

which intertwines Δ^+ with Δ^-. Consequently, if $m \equiv 2$ mod 4,

$$\operatorname{Sign}(M) = 0 \text{ and } a_n^s(x, g, \text{ orn}) = 0. \tag{3.2.23}$$

We therefore assume $m = 4k$ henceforth.

Let $\star$ be the Hodge operator and let $*$ be Clifford multiplication. We identify $\Lambda_c M$ with $\operatorname{Clif}(M)$. Then:

$$\begin{aligned}\tau(e_1 \wedge \dots \wedge e_p) =&(\sqrt{-1})^{m/2}(e_1 * \dots * e_m * e_1 * \dots * e_p)\\ =&(\sqrt{-1})^{m/2}(-1)^{p(2m-p+1)/2} e_{p+1} \wedge \dots \wedge e_m\\ =&(\sqrt{-1})^{m/2}(-1)^{p(p-1)/2} \star_p .\end{aligned} \tag{3.2.24}$$

The spaces $\Lambda^p M \oplus \Lambda^{m-p} M$ are invariant under τ. If $2p \neq m$, the map

$$\omega_p \to \tfrac{1}{2}(\omega_p \pm \tau\omega_p) \tag{3.2.25}$$

defines an isomorphism

$$\Lambda^p M \simeq (\Lambda^p M \oplus \Lambda^{m-p} M)^{\pm} \tag{3.2.26}$$

which intertwines Δ^p and $\Delta^{\pm}$ on $C^\infty((\Lambda^p M \oplus \Lambda^{m-p} M)^{\pm})$. Consequently these terms cancel off in the alternating sum and the only contribution is made in the middle dimension $m = 2p$.

If $m = 4k$ and $p = 2k$, then $\tau = \star$ by (3.2.24). We see that:

$$\begin{aligned}&N(\Delta^{2k}) = N(\Delta^{2k,+}) \oplus N(\Delta^{2k,-})\\ &\operatorname{Sign}(M) = \dim \mathfrak{N}(\Delta^{2k,+}) - \dim \mathfrak{N}(\Delta^{2k,-})\\ &a_n^s = \{a_n(x, \Delta^{2k,+}) - a_n(x, \Delta^{2k,-})\} \text{dvol}.\end{aligned} \tag{3.2.27}$$

Example 3.2.1: Let $M = S^{4k}$ be the sphere. Then $H^{2k}(M; \mathbf{C}) = 0$ so

$$\operatorname{sign}(S^{4k}) = 0. \tag{3.2.28}$$

Example 3.2.2: Let $M = \mathrm{CP}^{2k}$ be complex projective space. Let

$$x \in H^2(M; \mathbf{C}) = \mathbf{C} \tag{3.2.29}$$

be the generator discussed in Lemma 2.3.3. Then if $0 \leq \nu = 2j \leq 4k$,

$$H^\nu(M;\mathbf{C}) = x^j \cdot \mathbf{C}; \tag{3.2.30}$$

$H^\nu(M;\mathbf{C}) = 0$ otherwise. We note x^k generates $H^{2k}(M;\mathbf{C})$ and $(x^k)^2 = x^{2k}$ gives the orientation of $H^{4k}(M;\mathbf{C})$. Consequently

$$\star(x^k) = x^k \text{ and } \operatorname{sign}(\mathrm{CP}^{2k}) = 1. \tag{3.2.31}$$

The signature complex is multiplicative with respect to products. Let M_i be oriented even dimensional manifolds and let $M = M_1 \times M_2$ have the induced orientation. Then

$$\begin{aligned}
\Lambda^+M &= (\Lambda^+M_1 \otimes \Lambda^+M_2) \oplus (\Lambda^-M_1 \otimes \Lambda^-M_2),\\
\Lambda^-M &= (\Lambda^+M_1 \otimes \Lambda^-M_2) \oplus (\Lambda^-M_1 \otimes \Lambda^+M_2),\\
\mathfrak{N}(\Delta_M^+) &= (\mathfrak{N}(\Delta_{M_1}^+) \otimes \mathfrak{N}(\Delta_{M_2}^+)) \oplus (\mathfrak{N}(\Delta_{M_1}^-) \otimes \mathfrak{N}(\Delta_{M_2}^-)),\\
\mathfrak{N}(\Delta_M^-) &= (\mathfrak{N}(\Delta_{M_1}^+) \otimes \mathfrak{N}(\Delta_{M_2}^-)) \oplus (\mathfrak{N}(\Delta_{M_1}^-) \otimes \mathfrak{N}(\Delta_{M_2}^+)).
\end{aligned} \tag{3.2.32}$$

This shows that

$$\operatorname{Sign}(M) = \operatorname{Sign}(M_1)\operatorname{Sign}(M_2). \tag{3.2.33}$$

Example 3.2.3: Let $\rho = (i_1, ..., i_j)$ for $i_1 \geq ... \geq i_j > 0$ and $k = i_1 + ... + i_j$. Let

$$\mathrm{CP}(2\rho) = \mathrm{CP}^{2i_1} \times ... \times \mathrm{CP}^{2i_j}. \tag{3.2.34}$$

Since $\operatorname{sign}(\mathrm{CP}^{2k}) = 1$,

$$\operatorname{Sign}(\mathrm{CP}(2\rho)) = 1. \tag{3.2.35}$$

Theorem 3.2.2: *Let $m = 4k$.*

(a) $a_n^s := (a_n(\cdot, \Delta^+) - a_n(\cdot, \Delta^-))\mathrm{dvol} \in \mathfrak{I}_{m,n,m}^g$.

(b) $a_n^s = 0$ for $n < 4k$.

(c) $a_{4k}^s = L_k$ *is the Hirzebruch polynomial.*

(d) $\operatorname{Sign}(M) = \int_M L_k$. **(Hirzebruch signature formula)**

Proof: The invariant a_n^s is m form valued. If we reverse the orientation, the roles of Δ^+ and Δ^- are interchanged since $\tau(-\mathrm{orn}) = -\tau(\mathrm{orn})$. Similarly, $\mathrm{dvol}(-\mathrm{orn}) = -\mathrm{dvol}(\mathrm{orn})$. Thus a_n^s is an invariantly defined m form which does not depend on the orientation.

We can express τ functorially in terms of the metric tensor. It is worth noting that if we replace the metric G by c^2G for $c > 0$, then the spaces $\Lambda^{\pm}$ are not invariant. On Λ^p, we have

$$\tau(c^2G)(\omega_p) = c^{2p-m}\tau(G)(\omega_p). \tag{3.2.36}$$

However τ is unchanged on Λ^{2k} so the spaces $\Lambda^{2k,\pm}$ are unchanged by rescaling. We note

$$\Delta^{2k,\pm}(c^2G) = c^{-2}\Delta^{2k,\pm}(G). \tag{3.2.37}$$

We use Lemma 2.2.1 to see a_n^s is homogeneous of order n in the jets of the metric.

We use Theorem 3.2.1 to prove (b) and to see that $a_m^s \in \mathfrak{I}_m(\mathrm{O}(m))$ can be expressed in terms of Pontrjagin forms. We use (3.2.21) to see that

$$\int_{\mathrm{CP}(2\rho)} a_m^s = \operatorname{sign}(\mathrm{CP}(2\rho)) = 1. \tag{3.2.38}$$

By Lemma 2.3.5

$$\int_{\mathrm{CP}(2\rho)} L = 1. \tag{3.2.39}$$

Consequently

$$\int_{\mathrm{CP}(2\rho)} \{a_m^s - L\} = 0 \tag{3.2.40}$$

so that $a_m^s = L$ is the Hirzebruch polynomial by Lemma 2.3.4. ■

Remark: This proof is similar to the proof given for the Gauss Bonnet theorem. The heat equation provides a local formula for $\operatorname{Sign}(M)$. The invariance theory of Chapter II identifies this local formula as a characteristic class. We evaluate this local formula on a sufficient number of classifying examples to determine the normalizing constants and to prove $a_m^s = L_k$ is the Hirzebruch polynomial.

Remark: It is possible to generalize the signature complex by taking coefficients in an auxiliary coefficient bundle V. The analysis we have described generalizes easily to that setting to show

$$\operatorname{Sign}(M, V) = \Sigma_{4s+2t=m} \int_M 2^t \operatorname{ch}(V) \wedge L_s(M). \tag{3.2.41}$$

§3.3 Milnor's theorem

We use the eta invariant to give an analytic proof of Milnor's theorem that $\mathbf{R}^n$ admits a non-singular bilinear multiplication only for $n = 1, 2, 4, 8$.

We begin by extending the η invariant discussed in Chapter I to a map in K-theory. Let $P \in \mathfrak{D}(V)$ be an operator of Dirac type. Let p be the leading symbol of P. Let $W \in \mathfrak{V}^{\mathrm{U}}(M)$. Use a partition of unity to construct an operator P_W of Dirac type on $C^\infty(V \otimes W)$ with leading symbol $p \otimes I_V$. This means that if locally

$$P = \Sigma_i \gamma^i \partial_i + a \tag{3.3.1}$$

for $\gamma^i, a \in C^\infty(\mathrm{End}\,(V))$, then locally

$$P_V = \Sigma_i \gamma^i \otimes I_W + b \tag{3.3.2}$$

for $b \in C^\infty(\mathrm{End}\,(V \otimes W))$.

Lemma 3.3.1: *If m is even, then $\eta(P_V) \in \mathbf{R}/\mathbf{Z}$ is independent of the choices made. If $V_1 \simeq V_2$, then $\eta(P_{V_1}) = \eta(P_{V_2})$.*

Proof: Let P_V and $\tilde{P}_V$ be operators of Dirac type on $C^\infty(V \otimes W)$ with the same leading symbol. Then

$$P(\epsilon) := \epsilon P_V + (1 - \epsilon)\tilde{P}_V \tag{3.3.3}$$

is a smooth 1-parameter family of operators of Dirac type. The desired result now follows from Theorem 1.2.3. ■

We study the Hurwitz zeta function. Let a be a complex parameter with $Re(a) > 0$ henceforth. Let $p(j)$ be a polynomial of degree ν. Let

$$\zeta_p(s, z, a) = \Sigma_{j \geq 0}\, z^j p(j)(j + a)^{-s}; \tag{3.3.4}$$

ζ_p is holomorphic for $Re(s) > \nu + 1$ and $|z| \leq 1$.

Lemma 3.3.2: *$\zeta_p(s, z, a)$ is holomorphic for $s \in \mathbf{C}$ and $z \in \mathbf{C} - [1, \infty)$.*

Proof: Let

$$\zeta(s,z,a)=\Sigma_{j\geq 0}z^j(j+a)^{-s}. \tag{3.3.5}$$

correspond to $p=1$. If $c_n=p^{(n)}(-a)/n!$, then:

$$\begin{aligned}p(j)&=\Sigma_n c_n(j+a)^n\\ \zeta_p(s,z,a)&=\Sigma_n c_n\Sigma_j z^j(j+a)^{-(s-n)}\\ &=\Sigma_n c_n\ \zeta(s-n,z,a).\end{aligned} \tag{3.3.6}$$

This shows it suffices to prove Lemma 3.3.2 in the special case $p=1$. Define:

$$\begin{aligned}f(z,a,t)&=(1-ze^{-t})^{-1}e^{-at}\\ \delta(s,z,a)&=\textstyle\int_0^1 f(z,a,t)t^{s-1}dt\\ \delta_1(s,z,a)&=\textstyle\int_1^\infty f(z,a,t)t^{s-1}dt.\end{aligned} \tag{3.3.7}$$

Let $z\in\mathbf{C}-[1,\infty)$. δ_1 is holomophic since f decays exponentially at ∞. As f is holomorphic at $t=0$,

$$\begin{aligned}f(z,a,t)&=\Sigma_{0\leq j\leq n}c_j(z,a)t^j/j!+O(t^{n+1})\\ \delta(s,z,a)&=\Sigma_{0\leq j\leq n}c_j(z,a)(s+j)^{-1}/j!+\epsilon_n(s,z,a).\end{aligned} \tag{3.3.8}$$

The $c_j(z,a)$ are holomorphic. $\epsilon_n(s,z,a)$ is holomorphic if $Re(s)>-n$. We use the Mellin transform discussed in Chapter I:

$$\begin{aligned}\zeta(s,z,a)&=\Gamma(s)^{-1}\Sigma_{j\geq 0}\textstyle\int_0^\infty z^je^{-jt}e^{-at}t^{(s-1)}dt\\ &=\Gamma(s)^{-1}\textstyle\int_0^\infty f(z,a,t)t^{(s-1)}dt\\ &=\Gamma(s)^{-1}\{\delta(s,z,a)+\delta_1(s,z,a)\}.\end{aligned} \tag{3.3.9}$$

The zeros of Γ^{-1} cancel the poles of $(s+j)^{-1}$ so ζ is regular $\forall s\in\mathbf{C}$. ■

Let $m=2n$ and let $RP^m=S^m/\mathbf{Z}_2$ be real projective space. We adopt the notation of Lemmas 1.2.4 and 1.2.5. Let $\nu=2^n$ and let $\{e_i\}_{0\leq i\leq m}$ be $\nu\times\nu$ Clifford matrices. Let $e(x)=\Sigma_j x_j e_j$ and let

$$Q:=\Sigma_j ie_j\partial/\partial x_j \text{ on } C^\infty(\mathbf{R}^{m+1}\times\mathbf{C}^{2^n}). \tag{3.3.10}$$

Decompose $Q=\sqrt{-1}e(\theta)\partial/\partial r+r^{-1}A$. Let $m=2n$ and let

$$P:=\sqrt{-1}e(\theta)A+\tfrac{1}{2}(m-1)I_\nu. \tag{3.3.11}$$

P is a self-adjoint elliptic first order partial differential operator on S^m; if we replace θ by $-\theta$, then Q changes sign so $A(-\theta) = -A(\theta)$. Since $e(-\theta) = -e(\theta)$, P does not change sign under the antipodal map and defines an operator of Dirac type over RP^m. Let $L = S^m \times \mathbf{R}/\mathbf{Z}_2$ where we identify $(s, \lambda) = (-s, -\lambda)$; we may also identify L with the classifying line bundle

$$L = \{(p, x) \in RP^m \times \mathbf{R}^{m+1} : x \in p\}. \tag{3.3.12}$$

We complexify. Since L is flat, P also defines an operator

$$P_L : C^\infty(\nu \cdot L) \to C^\infty(\nu \cdot L) \text{ on } RP^m.$$

Theorem 3.3.3: *Let $m = 2n$. Then $\eta(P_1) - \eta(P_L) = 2^{-n}$.*

Proof: Let $f \in H^\pm(m, j, \nu)$. If j is even, let $\epsilon = 1$ and if j is odd, let $\epsilon = L$. Sections s to ϵ over RP^m correspond to functions g on S^m so $g(x) = (-1)^j g(-x)$. Consequently $f \in C^\infty(\nu \cdot V)$ and $P_V f = \pm\frac{1}{2}(2j + m - 1)$. This provides the complete spectral resolution of P_1 and P_L so

$$\eta(s, P_1) - \eta(s, P_L) = \nu \Sigma_j (-1)^j \binom{m+j-2}{m-2} \{\tfrac{1}{2}(2j + m - 1)\}^{-s}. \tag{3.3.13}$$

Since $\binom{m+j-2}{m-2}$ is a polynomial of degree $m - 2$,

$$\zeta_m(s, z) = \nu \Sigma_j \; z^j \; \binom{m+j-2}{m-2} \; (\tfrac{1}{2}(2j + m - 1))^{-s} \tag{3.3.14}$$

is a Hurwicz zeta function. Since ζ_m is holomorphic in (s, z),

$$\eta(0, P_1) - \eta(0, P_L) = \lim_{z \to -1} \zeta_m(0, z).$$

Differentiate the equation $(1 - z)^{-1} = \Sigma_k \; z^k \;\; m - 2$ times to see:

$$\begin{aligned}
(m-2)!(1-z)^{1-m} &= \Sigma_k k(k-1)...(k-m+3) z^{k-(m-2)} \\
&= \Sigma_j (j+m-2)(j+m-3)...(j+1) z^j \\
\nu(1-z)^{1-m} &= \nu \Sigma_k \tbinom{m+j-2}{m-2} z^j = \zeta_m(0, z) \\
\eta(0, P_1) - \eta(0, P_L) &= \nu 2^{1-m}.
\end{aligned} \tag{3.3.15}$$

Since $\mathfrak{N}(P_1) = \ker(P_L) = \{0\}$,

$$\eta(P_1) - \eta(P_L) = \tfrac{1}{2}\eta(0, P_1) - \tfrac{1}{2}\eta(0, P_L) = \nu 2^{-m} = 2^{-n}. \blacksquare \tag{3.3.16}$$

We can now give a purely analytic proof of Milnor's theorem:

Theorem 3.3.4: *If* $\mathbf{R}^k$ *admits a non-singular multiplication, then*

$$k \in \{1, 2, 4, 8\}.$$

Remark: **R**, **C**, the quaternions **H** and the Cayley numbers provide examples of non-singular multiplications on $\mathbf{R}, \mathbf{R}^2$, $\mathbf{R}^4$, and $\mathbf{R}^8$.

Proof: Let [] be the greatest integer function. Suppose $\exists$ a non-singular bilinear multiplication $f : \mathbf{R}^k \times \mathbf{R}^k \rightarrow \mathbf{R}^k$. Let $f(x, y) = g(x) \cdot y$ define

$$g : \{\mathbf{R}^k - 0\} \rightarrow GL(k, \mathbf{R}); \tag{3.3.17}$$

$g(-y) = -g(y)$. g defines an isomorphism between the bundles $k \cdot 1$ and $k \cdot L$ over RP^{k-1}. Let $n = [(k-1)/2]$ and $m = 2n$. Then $k \cdot 1$ and $k \cdot L$ are isomorphic over RP^m since $RP^m = RP^{k-1}$ or RP^{k-2} as k is odd or even. Thus by Lemma 3.3.1,

$$k(\eta(P_1) - \eta(P_L)) = 0. \tag{3.3.18}$$

Thus by Lemma 3.3.3, $2^{-n}k = 0$ in $\mathbf{R}/\mathbf{Z}$ so 2^n divides k. We check cases. Let $k \neq 1, 2, 4, 8$.

$k = 3$	$n = 1$	2 divides 3	impossible
$k = 5, 6$	$n = 2$	4 divides 5	impossible
$k = 7$	$n = 3$	8 divides 7	impossible.

If $k > 8$, then $2^n > k$ so this case is impossible as well. ∎

§3.4 Lefschetz fixed point formula

We conclude with a brief discussion of the Lefschetz fixed point formulas for the de Rham complex. Let $T : M \to M$ be a smooth map. Let T^* denote the natural action by pullback on $C^\infty \Lambda^p M$. Since

$$d_p T^* = T^* d_p, \tag{3.4.1}$$

T^* induces a natural action on the de Rham cohomology groups. Let

$$\mathcal{L}(T) := \Sigma_p (-1)^p \mathrm{Tr}\,(T^* \text{ on } H^p(M)). \tag{3.4.2}$$

Remark: $\mathcal{L}(T) \in \mathbf{Z}$. If T_i are homotopic maps, then $\mathcal{L}(T_1) = \mathcal{L}(T_2)$; we shall not need these facts in what follows.

Theorem 3.4.1 (Lefschetz): *If T has no fixed points, then $\mathcal{L}(T) = 0$.*

We postpone the proof for the moment to give some applications of this formula; they are all fairly elementary but illustrate the sorts of consequences which follow from this formula. Let

$$\mathrm{CP}^{\,n} := S^{2n+1}/S^1 \tag{3.4.3}$$

denote complex projective space.

Corollary 3.4.2:

(a) *Let $T : \mathrm{CP}^{\,n} \to \mathrm{CP}^{\,n}$ for n even. Then T has a fixed point.*

(b) *Let $F \to \mathrm{CP}^{\,n} \to M$ be a finite cover. Then $|F| = 1$ if n is even and $|F| \le 2$ if n is odd.*

(c) *If G is a compact Lie group of dimension $m \ge 1$, then $\chi(G) = 0$.*

Proof: The de Rham cohomology of projective space takes the form:

$$H^\nu(\mathrm{CP}^{\,n}) = \begin{cases} x^k \cdot \mathbf{R} & \text{if } \nu = 2k \text{ for } 0 \le k \le n, \\ 0 & \text{if otherwise.} \end{cases} \tag{3.4.4}$$

It is a truncated polynomial ring on a two dimensional generator x. Let $T^*(x) = \lambda x$ for $\lambda \in \mathbf{R}$. Since T^* is a ring homomorphism, $T^*(x^n) = T^*(x)^n$ and

$$\mathcal{L}(T) = 1 + \lambda + \ldots + \lambda^n. \tag{3.4.5}$$

We suppose T is without fixed points so $\mathcal{L}(T) = 0$. Then $\lambda \neq 1$. We multiply by $(1-\lambda)$ to see

$$0 = (1-\lambda)\mathcal{L}(T) = (1-\lambda)(1+\lambda+\ldots+\lambda^n) = 1-\lambda^{n+1}. \tag{3.4.6}$$

If n is even, there are no real solutions to (3.4.5). Hence T must have a fixed point if n is even. This proves (a).

Let $F \to \mathrm{CP}^n \to M$ be a finite covering projection. Since CP^n is simply connected,

$$F = \pi_1(M) \tag{3.4.7}$$

acts on CP^n via deck transformations; if $f \in F$ and $f \neq I$, then f is fixed point free. This is impossible for n even. If n is odd, let $f^*x = \lambda(f)x$. Then

$$1 - \lambda(f)^{n+1} = 1. \tag{3.4.8}$$

Since $\lambda(f) \neq 1$, $\lambda(f) = -1$. The map $f \to \lambda(f)$ gives a representation of F to $\mathbf{Z}_2$ which is faithful; (b) now follows.

Let G be a Lie group of dimension $m \geq 1$. Let $g_n \in G$ with $g_n \to I$ and $g_n \neq I$. Let

$$G_n(g) = g_n \cdot g \tag{3.4.9}$$

be left translation in the group. This is fixed point free and hence

$$\mathcal{L}(G_n) = 0. \tag{3.4.10}$$

On the other hand, $g_n \to I$ implies $G_n^* \to I^*$ and hence $\mathcal{L}(G_n) \to \mathcal{L}(I)$; (c) now follows since

$$\mathcal{L}(I) = \chi(M). \blacksquare \tag{3.4.11}$$

The operator $e^{-t\Delta_p}$ is trace class in $L^2(V)$; T^* defines a bounded operator on $L^2(V)$. Thus

$$\mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}) \tag{3.4.12}$$

is well defined for $t > 0$.

Theorem 3.4.2: *Let $T : M \to M$ be a smooth map.*

(a) $\Sigma_p(-1)^p \mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}) = \mathcal{L}(T)$.

(b) *If T is fixed point free,* $\mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p})$ *vanishes to infinte order as* $t \to 0^+$.

Proof: We extend Theorem 1.3.1 to the equivariant setting to prove (a). Let $E(\lambda, \Delta_p)$ be the eigenspaces of the Laplacian Δ_p on $C^\infty \Lambda^p M$. Let $\pi(\lambda, p)$ be orthogonal projection from $L^2(\Lambda^p M)$ to $E(\lambda, \Delta_p)$. Then

$$\Sigma_p(-1)^p \mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}) = \Sigma_\lambda e^{-t\lambda} \Sigma_p (-1)^p \mathrm{Tr}\,(\pi(\lambda, p) T^*). \qquad (3.4.13)$$

The first assertion will then follow from the identity:

$$\Sigma_p \mathrm{Tr}\,(\pi(\lambda, p) T^*) = \begin{cases} \mathcal{L}(T) & \text{if } \lambda = 0, \\ 0 & \text{if } \lambda > 0. \end{cases} \qquad (3.4.14)$$

We use the Hodge decomposition theorm to identify

$$E(0, \Delta_p) = H^p(M); \qquad (3.4.15)$$

it is an easy exercies to verify that

$$\pi(0, p) T^* : E(0, \Delta_p) \to E(0, \Delta_p) \qquad (3.4.16)$$

corresponds to the action of T^* on $H^p(M)$ under this identification so the first assertion of (3.4.14) now follows. To prove (b), we note that

$$d_p T^* = T^* d_p \quad \text{and} \quad d_p \pi(\lambda, p) = \pi(\lambda, p+1) d_p. \qquad (3.4.17)$$

Consequently $\pi(\lambda, p) \circ T^*$ defines a chain map on the chain complex

$$\{E(\lambda, \Delta_p), d\}. \qquad (3.4.18)$$

We observed in the proof of Theorem 1.3.1 that this chain complex was acyclic. Consequently

$$\Sigma_p(-1)^p \mathrm{Tr}\,(\pi(\lambda, p) \circ T^*) = 0. \qquad (3.4.19)$$

The proof of (b) uses some analytic facts concerning the kernel of the heat equation. Let $K(t, x, y, \Delta_p)$ be the fundamental solution of the heat equation;

$$(e^{-t\Delta_p} \phi)(x) = \textstyle\int_M K(t, x, y, \Delta_p) \phi(y) dy. \qquad (3.4.20)$$

Consequently

$$\begin{aligned} (T^* e^{-t\Delta_p} \phi)(x) :=& (dT(x))^* \phi(Tx) \\ =& \textstyle\int_M (dT(x))^* K(t, Tx, y, \Delta_p) \phi(y) dy. \end{aligned} \qquad (3.4.21)$$

From this it follows immediately that

$$\mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}) = \int_M \mathrm{Tr}\,(dT(x)^* K(t, Tx, x, \Delta_p))dx. \tag{3.4.22}$$

The kernel of the heat equation $K(t, x, y, \Delta_p)$ vanishes to infinite order if we bound the distance between x and y away from zero; it is only on the diagonal that there is an asymptotic series. We refer to Gilkey [8] for details. Since T is fixed point free, $Tx \neq x$ by hypothesis. ■

We can now complete the proof of Theorem 3.4.1. Let $t > 0$ and let $T : M \to M$ be fixed point free. Then by (a).

$$\mathcal{L}(T) = \Sigma_p(-1)^p \mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}). \tag{3.4.23}$$

On the other hand, by (b)

$$\mathrm{Tr}_{L^2}(T^* e^{-t\Delta_p}) \leq Ct \tag{3.4.24}$$

since T is fixed point free. Thus

$$|\mathcal{L}(T)| \leq Ct \tag{3.4.25}$$

for any $t \in (0, 1)$. We take the limit as $trightarrow0$ to see $\mathcal{L}(t) = 0$. ■

If T is fixed point free, $\mathcal{L}(T) = 0$. More generally, Theorem 3.4.2 shows that $\mathcal{L}(T)$ is determined only by the behavior of T near the fixed point set. The case of isolated fixed points is particularly simple. We refer to Gilkey [8] for a heat equation proof of the following result as well as other results related to the Lefschetz fixed point formulas.

Theorem 3.4.3 (Lefschetz): *Let $T : M \to M$ be a smooth map with a finite number of fixed points x_i. Assume as a non-degeneracy condition that*

$$\det(I - dT(x_i)) \neq 0$$

for all i. Then $\mathcal{L}(T) = \Sigma_i \mathrm{sign}\,(det(I - dT(x_i))$.

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Peter B Gilkey, Mathematics Department, University of Oregon, Eugene Oregon 97403 USA (gilkey@math.uoregon.edu) July 13 1993