수 학 강 의 록 제 11 권

Topological Methods for Asymmetric Boundary Value Problems

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PREFACE

The topic of these notes is a review of the sort of results that one can obtain with the relatively elementary methods of nonlinear topological analysis. Although some of the results can be improved by variational methods, many are at the frontier of mathematical knowledge at present. I hope that collecting this set in one place has the effect of inspiring students of degree theory to work on concrete problems in nonlinear partial differential equations.

I wish to thank all those who made the seminars at GARC possible. In particular, I would like to thank professors Jongsik Kim, Dohan Kim, Sung Ki Kim, and Q-Heung Choi for their separate roles in the seminar. Dongwoo Sheen's perceptive comments during the seminar have added to this presentation.

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Chapter I

Degree theory and the semilinear Dirichlet problem

I.1 Introduction

The purpose of this series of lectures is to explain recent progress on a classical problem of nonlinear mathematics, and to show how this material leads to new insights into applied mathematics and engineering.

We shall confine ourselves primarily to the use of degree theory, although variational methods also play an important role in the study of nonlinear boundary value problems.

The first result we know of is the result of Picard on the nonlinear boundary value problem

$$\begin{aligned} u'' + f(x,u) &= 0 \quad \text{in } (0,1), \\ u(0) = u(1) &= 0. \end{aligned} \tag{I.1}$$

Picard rewrite this equation as a nonlinear integral equation

$$u(x) = \int_0^1 g(x,s) f(s,u(s)) ds \tag{I.2}$$

with the usual Green function, and was able to show that the iterative scheme

$$u^{n+1}(x) = \int_0^1 g(x,s) f(s,u^n(s)) ds$$

converges to a solution of (I.1) under the additional assumption that $\left|\frac{\partial f}{\partial u}(x,s)\right| < 8$ for all x, s.

This is now an easy exercise for a beginning analysis class since one merely shows that $|\int_0^1 g(x,s)ds| \leq \frac{1}{8}$ and therefore the right hand side of (I.2) defines a contraction on $C(0,1)$.

It is a little-known fact that it was this result which led Picard to his celebrated method of successive approximations, a method which most people now associate with the initial value problem.

It is also typical of the history of this problem that it should lead to new interesting methods with wide spread application. Indeed, the contraction mapping theorem can fairly be said to have its roots in this first result.

It is clear that some limitation on the nonlinearity f is required since in the case $f(x,u) = u - \sin \pi x$, the equation (I.1) has no solution.

The correct limitation was eventually realized for the more general corresponding partial differential equation

$$\begin{array}{rcll} \Delta u + f(u) & = & h(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega \end{array} \tag{I.3}$$

with Ω a bounded (nice) region in $\mathbf{R}^n$, and f a continuously differentiable function. We shall not state Hammerstein's Theorems in their most general form but shall content ourselves with a version that fits the context of these notes.

I.2 Some non-resonance results

We begin with a uniqueness theorem, for the partial differential equation

$$\begin{array}{rcll} \Delta u + f(u) & = & h(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \tag{I.4}$$

THEOREM I.1 (HAMMERSTEIN'S THEOREM) *If there exist real numbers μ and ϵ with $\mu > 0, \epsilon > 0$, and $-\mu + \epsilon < f'(s) < \lambda_1 - \epsilon$ for all s, then equation (I.4) has a unique solution for all $h \in L^2(\Omega)$.*

Proof. The proof is by now a fairly standard contraction mapping argument. We let $\gamma = (\lambda_1 - \mu)/2$, and $g(u) = f(u) - \gamma u$ and we rewrite equation (I.4) as

$$-\Delta u - \gamma u = f(u) - \gamma u - h(x) \equiv N(u). \tag{I.5}$$

where $N : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a nonlinear map taking the function $u(x)$ into $f(u(x)) - \gamma u(x) - h(x)$. We note that

$$\begin{aligned} \|N(u) - N(v)\|_{L^2(\Omega)} &= \int_Q |f(u) - f(v) - \gamma(u-v)|^2 \\ &= \int |g'(\theta(x))|^2 |u-v|^2 \end{aligned}$$

be the Intermediate Value Theorem.

Since $-u + \epsilon < f'(u) < \lambda_1 - \epsilon$, we have

$$\frac{-\lambda_1}{2} - \frac{\mu}{2} + \epsilon < g'(u) < \frac{\lambda_1}{2} + \frac{\mu}{2} - \epsilon.$$

Representing the linear unbounded operator $(-\Delta - \gamma I)$ as a matrix with respect to the orthonormal basis $\{\phi_i\}$ on $L^2(\Omega)$, we get the unbounded matrix

$$\begin{bmatrix} \frac{\lambda_1+\mu}{2} & 0 & 0 \\ 0 & \lambda_2 - \frac{\lambda_1}{2} + \frac{\mu}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and of course $(-\Delta - \gamma I)^{-1}$ is obtained by inverting the diagonal elements of the matrix. Since the operator norm of a diagonal matrix is given by the supremum of the absolute value of the diagonal entries, it follows that

$$\|(-\delta - \gamma I)^{-1}\| = \left(\frac{\lambda_1}{2} + \frac{\mu}{2}\right)^{-1}.$$

Now, if we rewrite equation (I.5) as

$$u = (-\Delta - \gamma I)^{-1}(g(u) - h(x)),$$

we can observe that the right hand side

$$T(u) = (-\Delta - \gamma I)^{-1}(g(u) - h(x)) \tag{I.6}$$

satisfies

$$\|T(u) - T(v)\| = \|(-\Delta - \gamma I)^{-1}(g(u) - g(v))\|$$

and thus

$$\|T(u) - T(v)\| \leq \left(\frac{\lambda_1}{2} + \frac{\mu}{2}\right)^{-1} \left(\frac{\lambda_1}{2} + \frac{\mu}{2} - \epsilon\right) \|u - v\|$$

and thus T is a contraction on $L^2(\Omega)$. Therefore it has a unique fixed point in $L^2(\Omega)$ and the equivalent equation (I.4) has a unique solution in $L^2(\Omega)$. ∎

We remark that in its way this theorem is very precise. If the condition $f'(u) < \lambda_1$ is relaxed, uniqueness may fail. Let $F(u) = \lambda_1 u$ on $(-1, +1)$, $f(u) = \lambda_1$, $u \geq 1$ and $f(u) = -\lambda_1$, $u \leq -1$ and let $h(x) = 0$. Then $u = \epsilon\phi_1$ are solutions for all $\epsilon < 1/(\sup|\phi|)$.

It is also worth remembering that at least in certain circumstances, if $f'(\pm\infty) = \lambda_1$ then existence may fail. Consider the case were

$$f(u) = \lambda_1 u + \arctan u.$$

Then equation (I.4) becomes

$$\Delta(u) + \lambda_1 u - \arctan u = h(x) \tag{I.7}$$

and if we multiply across by ϕ_1 and integrate, we obtain

$$(\Delta u + \lambda_1 u, \phi_1) - \int (\arctan u)\phi_1 = \int h(x)\phi_1.$$

The first inner product vanishes since $\Delta\phi_1 + \lambda_1\phi_1 = 0$ and the second integral can be estimated from above by $\frac{\pi}{2}\int\phi_1$ and below by $-\frac{\pi}{2}\int\phi_1$. Thus, in this case, we obtain for (I.7) to have a solution, a **necessary** condition is that $-\frac{\pi}{2}\int\phi_1 < \int h\phi_1 < \frac{\pi}{2}\int\phi_1$ and therefore, solutions may fail to exist. We now turn to the situation where existence remains although uniqueness fails.

THEOREM I.2 *Suppose* $\lim_{s\to+\infty}\frac{f(s)}{s} = B$, *and* $\lim_{s\to-\infty}\frac{f(s)}{s} = A$, *where* $-\infty < A, B < \lambda_1$. *Then* (I.4) *has at least one solution.*

Proof. Again, write (I.4) as

$$u = (-\Delta - \gamma I)^{-1}(f(u) - \gamma u - h(x)).$$

Let $f(u) = Bu^+ - Au^- + f_1(u) + f_2(u)$ where $|f_1(u)| < \tilde{\epsilon}|u|$ (we choose $\tilde{\epsilon} > 0$ later) and there exists $c_4 > 0$ so that $|f_2(u)| \le c_4$ for all u. Choose $\mu > 0$ so that $-\mu < A, B < \lambda_1$ and choose $\gamma = (\lambda_1 + \mu)/2$ as before. Let $g_0(u) = Bu^+ - Au^- - \gamma u$ and, as before we write (I.4) as

$$u = (-\Delta - \gamma I)^{-1}(g_0(u) + f_1(u) + f_2(u) - h(x)).$$

Again, as in the previous theorem, observe that

$$\|(-\Delta - \gamma I)^{-1}g_0(u)\| \le c_1\|u\| \text{ for } c_1 < 1,$$

$$\|(-\Delta - \gamma I)^{-1}f_1(u)\| \le \left(\frac{\lambda_1 + \mu}{2}\right)\tilde{\epsilon}\|u\|$$

and

$$|(-\Delta - \gamma I)^{-1}f_2(u)\| \le \left(\frac{\lambda_1 + \mu}{2}\right)c_4 = c_2.$$

Letting $\epsilon = \tilde{\epsilon}\frac{\lambda_1 + \mu}{2}$, and $c_3 = \|(-\Delta - \gamma I)^{-1} h(x)\|$, we obtain that

$$\|(-\Delta - \gamma I)^{-1}(f(u) - \gamma u + h(x))\| \leq (c_1 + \epsilon)\|u\| + c_2 + c_3,$$

and we can choose $\tilde{\epsilon}$ so that $c_1 + \epsilon < 1$. Now choose R so large that

$$(c_1 + \epsilon)R + c_2 + c_3 < R.$$

Then the map

$$Tu = (-\Delta - \gamma I)^{-1}(f(u) - \gamma u - h(x))$$

satisfies

$$\|Tu\| \leq R \quad \text{if} \quad \|u\| = R$$

and thus maps the ball of radius R into itself. Since T is compact we obtain by Schauder fixed point theorem that it must have a fixed point and equivalently (I.4) must have at least one solution. ∎

The reader may ask if the requirement that A and B be finite is essential. There are answers. If one considers only (I.4), then we can use the maximum principle if h is bounded to obtain a priori pointwise bounds, then solve a modified problem which satisfies the same a priori bounds. However, if we consider more general self-adjoint operators for which there is no maximum principle, then one must use an $L^1(\Omega)$ argument to prove the existence of weak solutions, for which regularity is not known [McK-R].

Our first two theorems said that the nonlinearity f (actually its derivative) remained below the first eigenvalue, either always, as in Theorem I.1 or at infinity as in Theorem I.2. The following two theorems, due to Dolph, provide a natural extension.

THEOREM I.3 *Suppose there exists* $\epsilon > 0$ *so that* $\lambda_n + \epsilon < f'(u) < \lambda_{n+1} - \epsilon$ *for all* u*. Then equation (I.4) has exactly one solution for all* $h \in L^2(\Omega)$.

Proof. Let $\gamma = \frac{\lambda_n + \lambda_{n+1}}{2}$. As before, write (I.4) as

$$u = (-\Delta - \gamma I)^{-1}(f(u) - \gamma u - h(x))$$

and let $g(u) = f(u) - \gamma u$. Observe that $|g'(u)| \leq \frac{\lambda_{n+1} + \lambda_n}{2} - \epsilon$ and, by the same reasoning as before, that

$$(-\Delta - \gamma I)^{-1} = \frac{2}{\lambda_{n+1} - \lambda_n}.$$

We can conclude, as before, that the nonlinear map on $L^2(\Omega)$ given by

$$Tu \equiv (-\Delta - \gamma I)^{-1}(g(u) - h(x))$$

must be a contraction and thus has a unique fixed point. Therefore the equivalent equation (I.4) must have a unique solution. ∎

THEOREM I.4 *Let* $\lim_{s \to +\infty} \frac{f(s)}{s} = B$, $\lim_{s \to +\infty} \frac{f(s)}{s} = A$ *and* $\lambda_n < A, B < \lambda_{n+1}$. *Then equation (I.4) has at least one solution.*

Proof. This is left as an exercise for the reader. ∎

On a separate problem, there have been some papers dealing with solving (I.4) with the hypotheses of Theorem I.1 under the assumption that $h(x) \in L^1(\Omega)$. Can one do the same thing for Theorem I.3?

The above results all share a curious feature; they all show the existence and (sometimes) uniqueness by using the fact that the equations can be written as a perturbation of a linear equation. The equation may be nonlinear but it is close to the linear equation

$$\begin{aligned} \Delta u + \gamma u &= h(x) && \text{in} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

and this is what guarantees solutions. In the next section, we begin the study of what happens when we can no longer say the solution is close to an invertible linear problem.

I.3 The piecewise linear nonlinearity crosses the first eigenvalue

In this section, we begin our study of the case where the interval $(f'(-\infty), f'(+\infty))$ contains at least one eigenvalue. To get some feeling for this situation we study a deceptively simple equation

$$\begin{aligned} \Delta u + bu^+ - au^- &= s\phi_1 && \text{in} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega. \end{aligned} \tag{I.8}$$

We have already seen that if $a, b < \lambda_1$ or if $\lambda_n < a, b < \lambda_{n+1}$, then (I.8) has exactly one solution for all real s, and indeed we can write it down. For example if $a, b < \lambda_1$, we can verify that if $s > 0$, then $u = \frac{s\phi_1}{b - \lambda_1}$ is a solution, whereas if $s < 0$, then $u = \frac{s\phi_1}{a - \lambda_1}$ is.

Now, let us see what happens if $a < \lambda_1 < b$.

THEOREM I.5 *If $a < \lambda_1 < b < \lambda_2$, then equation* (I.8) *has*

exactly 2 solutions	*if*	$s > 0$,
exactly 1 solution	*if*	$s = 0$,
no solution	*if*	$s < 0$.

Proof. We consider the three cases separately.

a) $s < 0$.

Write (I.8) as

$$(\Delta + \lambda_1)u + (b - \lambda_1)u^+ - (a - \lambda_1)u^- = s\phi_1.$$

Multiply across by $\phi_1(x)$ and integrate over Ω. Note that $((\Delta+\lambda_1)u, \phi_1)=0$. We obtain that

$$\int((b - \lambda_1)u^+ - (a - \lambda_1)u^-)\phi_1 = s\int \phi_1^2 = s. \tag{I.9}$$

But $(b-\lambda_1)u^+ - (a-\lambda_1)u^-$ is greater than or equal to zero and strictly greater than zero if u is. Also, $\phi_1 > 0$ in Ω. Therefore the left hand side of (I.9) is always greater than or equal to zero, and there are no solutions of (I.8) if $s < 0$. Also, if $s = 0$, then the only possibility is that $u \equiv 0$.

b) $s > 0$.

Let P be orthogonal projection in $L^2(\Omega)$ onto the subspace spanned by ϕ_1. Thus $Pu = (\int_\Omega u\phi_1)\phi_1$ for all $u \in L^2(\Omega)$. Let $u = v + w$, $v = Pu$, $w = (I - P)u$. Since the operator P commutes with Δ, we have that equation I.5 is equivalent to the pair of equations

$$\begin{array}{llcl} \text{(a)} & \Delta w + (I - P)(b(v + w)^+ - a(v + w)^-) & = & 0, \\ \text{(b)} & \Delta v + P(b(v + w)^+ - a(v + w)^-) & = & s\phi_1. \end{array} \tag{I.10}$$

First observe that for fixed v, the equation (I.10.a) has a unique solution $w(v)$. Write $g_v(w) = b(v + w)^+ - a(v + w)^-$ and observe that $g' \leq b$. Now suppose we had two solutions w_1, w_2 for some fixed v. Then

$$\Delta(w_1 - w_2) + (I - P)(g_v(w_1) - g_v(w_2)) = 0. \tag{I.11}$$

Taking the inner product of (I.11) with $w_1 - w_2$ we obtain that

$$-\int |\nabla(w_1 - w_2)|^2 + \int (g_v(w_1) - g_v(w_2))(w_1 - w_2) = 0.$$

Now, for any $w \in L^2$, $w = \sum a_n \phi_n$ and $\int w^2 = \sum a_n^2 \int |\nabla \phi_n|^2 = \sum_{n=1}^{\infty} \lambda_n a_n^2$. If $w \perp \phi_1$, then $\int |\nabla w|^2 = \sum_{n=2}^{\infty} \lambda_n a_n^2 \geq \lambda_2 \sum_{n=2}^{\infty} a_n^2 = \lambda_2 \int |w|^2$. Thus

$$\int |\nabla(w_1 - w_2)|^2 \geq \lambda_2 \|w_1 - w_2\|^2 \text{ and } |g_v(w_1) - g_v(w_2)|^2 \leq b|w_1 - w_2|.$$

Therefore

$$\begin{aligned} \lambda_2 \|w_1 - w_2\|^2 &\leq \int |\nabla(w_1 - w_2)|^2 \\ &= \int |g_v(w_1) - g_v(w_2)||w_1 - w_2| \leq b\|w_1 - w_2\|^2. \end{aligned}$$

But $b < \lambda_2$, and this implies that $w_1 = w_2$. Thus we have proved that every solution of (I.10.a) is unique. Now note that $w = 0$ is a solution of (I.10.a) for any $v \in PH$, $v > 0$ or $v < 0$ everywhere in Ω. Thus if $v > 0$, $bv^+ - av^- = bv$ and if $v < 0$, $bv^+ - av^- = av$. In either case we have $(I - P)(bv^+ - av^-) = 0$ and thus $w = 0$ satisfies

$$\Delta 0 + (I - P)(bv^+ - av^-) = 0.$$

Thus equation (I.8) is reduced to

$$\Delta v + P(bv^+ - av^-) = s\phi_1$$

where $v = c\phi_1$, $c \in \mathbf{R}$.

Case 1. $c > 0$. In this case, we have

$$-\lambda_1 c + bc = s, \qquad c = \frac{s}{b - \lambda_1}.$$

Case 2. $c < 0$. In this case, we have

$$-\lambda_1 c + ac = s, \qquad c = \frac{s}{a - \lambda_1},$$

and therefore (I.8) has exactly two solutions. This concludes the proof of Theorem I.5. ■

Theorem I.5 raises many more questions then it answers. One is, what happens if the equation is not piecewise linear but only satisfies $A < \lambda_1 < B$, where $A = \lim_{s \to -\infty} \frac{f(s)}{s}$, $B = \lim_{s \to +\infty} \frac{f(s)}{s}$.

A second natural question is, what exactly is the role of λ_2. The *existence* of two solutions to (I.8) for $s > 0$ only depended on $a < \lambda_1 < b$. On the other hand, the statement that the obvious solutions were the only ones possible depended heavily on the fact that $b < \lambda_2$.

The first of these questions is the one that will occupy us for the remainder of this chapter.

I.4 The nonlinearity crosses the first eigenvalue

Frequently in this area, it is the case that one can easily prove something for the piecewise linear case, whereas obtaining the same result for the nonlinear equation requires more detail. The following results are a good example of this. Throughout this section, we will study the equation

$$\begin{aligned} \Delta u + f(u) &= t\phi_1 + h(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{I.12}$$

assuming that $h \in C^\alpha(\Omega)$ and $\partial\Omega$ is at least C^2. We shall write $A = \lim_{s\to-\infty} \frac{f(s)}{s}$, and $B = \lim_{s\to+\infty} \frac{f(s)}{s}$ and we shall be considering the case where $A < \lambda_1 < B$.

We shall use the techniques of upper and lower solutions. We shall say that an *upper solution* $\overline{u}$ to the equation

$$\Delta u + F(x,u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

satisfies

$$\Delta \overline{u} + F(x,\overline{u}) < 0 \quad \text{in } \Omega, \qquad u \geq 0 \quad \text{on } \partial\Omega$$

and a **lower solution** $\underline{u}$ satisfies

$$\Delta \underline{u} + F(x,\underline{u}) > 0 \quad \text{in } \Omega, \qquad u \leq 0 \quad \text{on } \partial\Omega.$$

We write $L = -\Delta$, with Dirichlet boundary conditions.

LEMMA I.1 *For fixed s, we can always find a lower solution to equation (I.12) if $A < \lambda_1 < B$.*

Proof. Choose $\epsilon = \frac{\lambda_1 - A}{2}$ and choose $M \in \mathbf{R}$ so that $f = f_0 + f_1$ where $f_0 \geq (A+\epsilon)u$ for all $u \leq 0$ and $f_1 > M$ for $u \leq 0$ (This is possible since $\lim_{s\to-\infty} \frac{f(s)}{s} = A$). Let w be the solution of

$$\Delta w = -M + h \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega$$

and let $\underline{u} = w + c\phi_1$, where c will be chosen later, in such a way that $\underline{u} \leq 0$. Then

$$\Delta\underline{u} + f(\underline{u}) - s\phi_1 - h(x) \geq \left(\frac{\lambda_1 + A}{3}\right) w - \left(\frac{\lambda_1 - A}{2}\right) c\phi_1 - s\phi_1 > 0,$$

if c is chosen sufficiently large and negative. This proves the lemma. ∎

One important step in the proof was the choice of c so that $c\phi_1 + w$ could be made negative. This uses heavily the fact that Ω is C^2. We are not sure if this would work on a square, for example.

LEMMA I.2 *For given $h(x)$, there exists s_0 such that if $s > s_0$, then equation (I.12) has an upper solution.*

Proof. Choose w so that

$$\Delta w = h(x) - f(0) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

Then if s is so large that $s\phi_1 > f(w) - f(0)$, we have

$$\Delta w + f(w) - f(0) - s\phi_1 - h(x) + f(0) < 0$$

so we can take $\overline{u} = w$. ∎

Lemma I.1 and I.2 combined show that equation (I.12) has both an upper and lower solution for large s.

LEMMA I.3 *There exists a real number $S(h)$ so that if $s < S(h)$, equation (I.12) has no solution.*

Proof. Let $s_0 = \min(f(x) - \lambda_1 x)$. This minimum exists since $A < \lambda_1 < B$. Now write equation (I.12) as

$$(\Delta + \lambda_1)u + f(u) - \lambda_1 x = s\phi_1 + h(x).$$

Multiply by ϕ_1 and integrate. We obtain

$$s_0 \int \phi_1 \leq \int (f(u) - \lambda_1 u)\phi_1 = s.$$

Thus if $s < s_0 \int \phi_1$, equation (I.12) has no solution. ∎

LEMMA I.4 *For fixed s (or for s in a bounded interval) there is an a priori bound in $C^{1+\alpha}(\Omega)$ on solution of* (I.12).

Proof. We prove that there exists $R(\mu)$ such that if $h \in C^\alpha(\overline{\Omega})$ and $\sup|h| \leq \mu$, then $\|u\|_{1+\alpha} \leq R(\mu)$, where $\|u\|$ denotes the $C^{1+\alpha}$- norm of u. Suppose not! Then there exists a sequence u_n, h_n $\sup|h_n| \leq \mu$, $\|u_n\| \to \infty$ so that

$$Lu_n + f(u_n) = s\phi_1 + h_n.$$

Let $w_n = \frac{u_n}{\|u_n\|}$. Then w_n are bounded in $C^{1+\alpha}$, and there exists a subsequence (still called w_n) such that $w_n \to w$ strongly in C^α, and

$$Lw_n + \frac{1}{\|u_n\|} f(\|u_n\| w_n) = \frac{h_n + s\phi_1}{\|u_n\|}. \tag{I.13}$$

It is easy to see that

$$\frac{1}{\|u_n\|} f(\|u_n\| w) \longrightarrow Bw^+ - Aw^- \qquad \text{in } C^\alpha(\Omega)$$

and thus by virtue of (I.13), $w_n \longrightarrow w$ in $C^{2+\alpha}$ and

$$\begin{aligned} \Delta w + Bw^+ - Aw^- &= 0 \qquad \text{in} \quad \Omega, \\ w &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

By Theorem I.5, this implies that $w = 0$. But this contradicts the fact that $\|w_n\| = 1$ and $w_n \longrightarrow w$ in $C^{1+\alpha}(\Omega)$ since we already know $w_n \longrightarrow w$ in $C^{2+\alpha}(\Omega)$. ■

We now prove an abstract theorem about upper and lower solutions and Leray-Schauder degree theory.

LEMMA I.5 *Let v, w satisfy $v, w \in C^{2+\alpha}(\Omega)$*

(i) $Lv > F(x,v)$,

(ii) $Lw < F(x,w)$,

(iii) $v \leq w$ in Ω, $v = w = 0$ on $\partial\Omega$.

Then $v < w$ in Ω, and in the Banach space

$$E = \{u \in C^{1+\alpha}(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$$

there exists $R > 0$ such that the Leray-Schauder degree

$$\deg(u - L^{-1}F(x,u), \mathrm{int}K, 0) = 1$$

where $K = \{u \in E \mid v(x) \leq u(x) \leq w(x), \|u\| \leq R\}$.

Proof. Let

$$\tilde{F}(x,u) = \begin{cases} F(x, w(x)), & \text{if } u(x) \geq w(x), \\ F(x, u(x)), & \text{if } v(x) \leq u(x) \leq w(x), \\ F(x, v(x)), & \text{if } u(x) \leq v(x). \end{cases}$$

Let $u - L^{-1}\tilde{F}(x,u) = 0$. Suppose $u(x) > w(x)$. Let x_0 be the maximum of $u - w$. Then

$$0 = \Delta u(x_0) + F(x, u(x_0)) = \Delta(u-w)(x_0) + \Delta w + F(x_0, w(x_0)).$$

But $\Delta(u-w)(x_0) \leq 0$ and $\Delta w + F(x_0, w(x_0)) < 0$ which is a contradiction. It is then an easy consequence of the strong maximum principle that

$$v(x) < u(x) < w(x) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial w}{\partial n} \quad \text{on } \partial\Omega.$$

This is necessary to show that the set K has interior.

Now, since the operator $u \longrightarrow L^{-1}\tilde{F}(x,u)$ is absolutely bounded, we can choose Ω so that it maps B_R into itself, and thus

$$d(u - L^{-1}\tilde{F}(x,u), B_R, 0) = 1.$$

However by the maximum principle, if $uL^{-1}\tilde{F}(x,u)$, then $u \in \operatorname{int} K$. This implies that $\deg(u - L^{-1}\tilde{F}(x,u), B_R \backslash \overline{K}, 0) = 0$, and finally this implies that $\deg(u - L^{-1}\tilde{F}, \operatorname{int} K, 0) = 1$. Since $F(x,u) = \tilde{F}(x,u)$ on $\overline{K}$, the degrees will be the same on K, and the lemma follows. ∎

LEMMA I.6 *If u_1 is a solution of equation (I.12) for $t = t_1$, then u_1 is an upper solution of (I.12) for $t > t_1$.*

Proof.

$$\Delta u_1 + f(u_1) - t\phi_1 - h(x) = (t_1 - t)\phi_1 < 0$$

be the strong positivity of ϕ_1. ∎

We now have a local result on the degree (roughly speaking) on a region between an upper and a lower solution. Our next task in studying (I.12) is to calculate the degree on a sufficiently big ball.

LEMMA I.7 *Given any* t_0*, there exists* $R_0(t_0)$ *such that* $R > R_0$, $t < t_0$ *implies*

$$\deg(u - L^{-1}(h + t\phi_1 - f(u)), B_R, 0) = 0.$$

Proof. The proof uses lemmas I.3 and I.4. Consider the homotopy

$$G(t,u) = u - L^{-1}(h_1 + t\phi_1 - f(u)). \tag{I.14}$$

Choose R_0 so large that (I.14) has no zero in B_{R_0} for $t = -T$ (we know that this can be done by Lemma I.3) and so large that (I.14) has no zeros on ∂B_{R_0} for $-T \le t \le t_0$ (which can be done by the a priori bound in Lemma I.4). Since $\deg(G(-T,u), B_{R_0}, 0) = 0$ as we know there are no zeros of $G(-T,u)$, and by homotopy,

$$\deg(G(-T,u), B_{R_0}, 0) = \deg(G(T,u), B_{R_0}, 0)$$

for all $-T \le t \le t_0$, the lemma follows. ■

We have now assembled the major ingredients of the main theorem of this section.

THEOREM I.6 *If* $\lim_{s\to-\infty} \frac{f(s)}{s} = A$, $\lim_{s\to+\infty} f(s) = B$, *and* $-\infty < A < \lambda_1 < B < \infty$, *then there exists* $t_0(h)$ *such that the equation*

$$\begin{aligned} \Delta u + f(u) &= t\phi_1 + h \quad &\text{in } \Omega, \\ u &= 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{I.12}$$

has

$$\begin{array}{lll} \textit{no solutions} & \textit{if} & t < t_0, \\ \textit{at least one solution} & \textit{if} & t = t_0, \\ \textit{at least two solutions} & \textit{if} & t > t_0. \end{array}$$

Proof. The proof is a series of observations of the preceding lemmas. If t is sufficiently large and positive, we can find upper and lower solutions and hence a region K where the degree is $+1$. Lemma I.6 shows that the set of t for which we can find upper solutions is an unbounded interval $t > t_0$. Lemma I.1 shows that this interval is not the whole real line since we can find t large and negative for which no solution (and therefore no upper solution) exists. Thus we can always find $t_0(h)$ such that if $t > t_0(h)$ upper solutions exist, and if $t < t_0(h)$ then upper solutions do not exist.

Therefore for any $t > t_0$, we find a region K which the degree of $u - L^{-1}F(x,u)$ is $+1$. We then find a ball containing K on which the degree is zero by Lemma I.6. We know there are no solutions on ∂K and thus the degree on $B_R \backslash \overline{K}$ is -1. Thus for $t > t_0$, we can conclude that there are always at least two solutions.

It is now an easy argument in compactness to show there is one solution at $t = t_0$. Choose $t_n \longrightarrow t_0$, u_n solutions of

$$\Delta u_n + f(u_n) = t_n \phi_1 + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

and observe by the *a priori* bounds that there exists a subsequence (still called u_n) which converge to u. One can then conclude that u is a solution of (I.12) with $t = t_0$. This concludes the proof of the theorem.∎

We shall have some historical remarks on this theorem at the end of the chapter. But we should remark that some of the hypotheses can be weakened. In [K-W], it was observed that $A = -\infty$ is permitted, and in [A-H],[D], it was observed that $A = +\infty$ is permitted with certain restrictions on the growth rate of f at plus infinity. In [A-H], the restriction was that $f(s) \sim S^{\left(\frac{N+1}{N-1}\right)}$ at plus infinity. The latest, and presumably optimal result from [Sol] and using variational methods is that $f(s)$ can have growth rate almost $S^{\left(\frac{N+2}{N-2}\right)}$. This is a common restriction when using variational methods.

I.5 At most two solutions - the nonlinear case

In the last section, we saw that $A < \lambda_1 < B$ was sufficient to produce at least two solutions. It is natural to ask if in any circumstances this is exact, since we have seen that it can be exact in the piecewise linear case. In order to obtain an easy result of this type, we recall some results on eigenvalues comparison theorems. In Chapter III, we return to this subject with some more advanced techniques.

Consider the eigenvalue problem

$$\begin{aligned} \Delta u + p(x)u + \gamma u &= 0 \qquad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned}$$

where p is Holder continuous, γ is a real parameter. By uniform ellipticity, there are eigenvalues $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \longrightarrow +\infty$. By smoothness of p, Ω the associated eigenfunctions belong to $C^{2+\alpha}(\Omega)$. By the strong maximum principle, γ_i is simple and the associated eigenfunction never vanishes.

If we consider a second eigenvalues problem

$$\Delta u + \tilde{p}(x)u + \tilde{\gamma}(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

then $\tilde{p} \leq p \Longrightarrow \gamma_k \leq \tilde{\gamma}_k$.

Now consider the equation

$$\Delta u + F(x,u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{I.15}$$

LEMMA I.8 *If* $\frac{\partial F}{\partial u} < \lambda_2$ *on* $(-\infty,\infty) \times \Omega$, *then any two solutions of* (I.12) *must satisfy* $w(x) - v(x) \neq 0$ *in* Ω.

Proof. Let $z = v - w$. Then z satisfies

$$\Delta z + q(x)z = 0$$

where $q(x) = \int_0^1 \frac{\partial F}{\partial u}(x, w + s(v(x) - w(x)))dx$. There exists c so that $q \leq c < \lambda_2$. Let $\tilde{\gamma}, \gamma$ denote the eigenvalues of

$$\begin{array}{lrcl} \text{(a)} & \Delta u + q(x)u + \tilde{\gamma} u & = & 0, \\ \text{(b)} & \Delta u + cu + \gamma u & = & 0. \end{array} \tag{I.13}$$

The eigenvalues of (b) are $(\lambda_n - c)$ so $\gamma_2 > 0$. But $\tilde{\gamma}_2 \geq \gamma_2$, which implies that $\tilde{\gamma}_1 = 0$, since the fact that z is a solution of (b) with $\gamma = 0$ means that $\gamma_i = 0$ for some i. If z is the first eigenfunction, then z must be non-vanishing, which proves the lemma. ∎

LEMMA I.9 *If* $\frac{\partial F}{\partial u}(x,u) < \lambda_2$ *for all* $(x,u) \in (-\infty,\infty) \times \Omega$, *and if* $\frac{\partial F}{\partial u}$ *is strictly increasing (or decreasing) in* u, *then equation* (I.12) *has* **at most two** *solutions.*

Proof. Suppose there exists at least three solutions, u_1, u_2, u_3. Then by the previous lemma these solutions must be ordered so we assume that $u_1(x) < u_2(x) < u_3(x)$ in Ω. Now let $y = u_3 - u_1$, $z = u_2 - u_1$. Then just as before, y and z must satisfy.

$$\begin{array}{lll} \text{(a)} & \Delta y + Q(x)y = 0 & Q = \int_0^1 \frac{\partial F}{\partial u}(s, u_1 + s(u_3 - u_1))ds, \\ \text{(b)} & \Delta z + q(x)z = 0 & q = \int_0^1 \frac{\partial F}{\partial u}(s, u_1 + s(u_2 - u_1))ds. \end{array} \tag{I.14}$$

Now the assumption that $\dfrac{\partial f}{\partial u}$ is strictly increasing, together with the ordering assumption on the u_i's says that $q(x) \le Q(x)$.

Multiplying (a) by y, (b) by z and integrating, using the boundary conditions, we obtain

$$0 = \int (y\Delta z - z\Delta y) = -\int yz(q - Q) > 0$$

which is a contradiction. Thus there are at most two solutions to equation (I.12). ■

Thus we have the following theorem, by combining Lemma I.8 with Theorem I.6.

THEOREM I.7 *If $f'(-\infty) = A, f'(+\infty) = B$, where $A < \lambda_1 < B < \lambda_2$ and $f''(s) > 0$, then there exists $t_0(h)$ such that if $t > t_0(h)$ then the equation*

$$\Delta u + f(u) = t\phi_1 + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

has exactly two solutions.

We remark that it is possible to show that if $t = t_0(h)$ then the equation has exactly one solution. One way is to use a global Lyapunov-Schmidt argument as in section 2, and show that the function defined on span $\{\phi_1\}$ must have a unique minimum.

In later chapters we shall return to the question of finding exact numbers of solutions. Here we pose a question : Is there any generalization of Lemma I.7 and I.8? For example, is it possible that if $f'' > 0$ and $\frac{\partial f}{\partial u} < \lambda_3$ then there are **at most four** solutions? Or possibly if $f''' > 0$ and $\frac{\partial f}{\partial u} < \lambda_2$, then there are at most three solutions? There are few results in this area.

I.6 More than two solutions

In the last section, we considered the equation

$$\Delta u + f(u) = t\phi_1 + h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{I.12}$$

under the assumption that $f'(-\infty) = a < \lambda_1 < b = f'(+\infty) < \lambda_2$, and $f''(s) > 0$ for all s. We showed that in this case, for large t, the equation always had **exactly two solutions.** We also know that if $a < \lambda_1 < b$ then there are always at least two solutions. In this section, we use elementary degree theoretic methods to show that if $b > \lambda_2$ then we may expect more solutions at least for t large. The idea of this main result of this section is conceptually simple, as can be seen best in the case of the piecewise linear equation

$$\Delta u + bu^+ - au^- = t\phi_1 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Here we can exhibit two solutions

$$u_2 = \left(\frac{t}{b-\lambda_1}\right)\phi_1 \text{ and } u_1 = \left(\frac{t}{a-\lambda_1}\right)\phi_1.$$

Each of these solutions ought have a local Leray-Schauder degree associated with them. How does one calculate the Leray-Schauder degree of the solution to the linear equation

$$\Delta u_2 + bu_2 = \phi_1\,?$$

One simply writes it as

$$u_2 - (-\Delta)^{-1}(bu_2 - \phi_1) = 0$$

and counts the number of negative eigenvalues of $I-(-\Delta)^{-1}bI$ which, if $\lambda_2 < b < \lambda_3$ is two, all other eigenvalues $\left\{1 - \frac{b}{\lambda_i}\right\}$ being positive. Thus, roughly, we ought to say that for any finite dimensional approximation, the sign of the determinant ought to be positive and thus the local degree ought to be $+1$. A similar line of reasoning gives that the local degree of

$$\Delta u_1 + au_1 = \phi_1$$

should be $+1$ since all eigenvalues will be positive. Now, if there is any justice, since the solutions of the piecewise linear problems are really solutions of the linear problem, we should conclude that the degree of the nonlinear map ought to be the same as the "linearized" problem and thus we ought to have two small non-intersecting balls with degree $+1$. Since the degree on an arbitrarily large ball is 0, by excision we ought to have that the degree on the big ball minus the two small balls is -2. This should guarantee the existence of at least three, and generically four solutions.

The remainder of this section is an effort to make this line of reasoning precise. The difficulties are two-fold : First we must deal with the non-linear and not piecewise-linear equation and it is not clear what should replace u_1 and u_2. Second, linearizing things is not that easy if we do it in $L^2(\Omega)$. This section shows one way round this difficulty in $L^2(\Omega)$. In Chapter III, we show another way to attack it.

Our next theorem is that there exist solutions of the semilinear problem close to the u_1 and u_2 of the piecewise linear problem. As usual, λ_i are the eigenvalues of the Laplacian with Dirichlet boundary conditions, each repeated as often as its multiplicity. Obviously, we can rewrite

$$\Delta u + f(u) = t\phi_1 + h(x) \text{ as } u - (-\Delta)^{-1}(f(u) - s\phi_1 - h(x)) = 0.$$

THEOREM I.8 *Let f be Lipschitzian with Lipschitz constant L. Let $\lim_{s \to +\infty} \frac{f(s)}{s} = b$, where $\lambda_n < b < \lambda_{n+1}$. Then there exists $\epsilon > 0$, $s_0 > 0$ such that the Leray-Schauder degree*

$$\deg(u - (-\Delta)^{-1}(f(u) - s\phi_1 - h), B_{s\epsilon}(S\theta), 0) = (-1)^n,$$

where $\theta = (\Delta + b)^{-1}\phi_1 = \frac{\phi_1}{b - \lambda_1}$.

Proof. The proof depends on an elementary equicontinuity lemma.

LEMMA I.10 *Let K be a compact set in $L^2(\Omega)$. Let $\phi > 0$ a.e. Then there exists a modulus of continuity $\delta : \mathbf{R} \longrightarrow \mathbf{R}$ depending only on K and ϕ such that*

$$\|(|\psi| - \frac{\phi}{\eta})^+\| \leq \delta(\eta) \quad \textit{for all } \psi \in K.$$

Proof. For any $\psi \in K$, let $\psi_\eta = |\psi| - \frac{\phi}{\eta}$. Since $0 \le \psi_\eta \le \psi$, and since $\psi_\eta(x) \longrightarrow 0$ as $\eta \longrightarrow 0$ a.e., it follows that $\|\psi_\eta\| \longrightarrow 0$ for all $\psi \in K$. We shall show that given $\epsilon > 0$, there exists $\alpha > 0$ such that if $\psi \in K$, then $\|\psi_\eta\| \le 2\epsilon$ for all $\eta \in [0, \alpha]$. Choose $\{\psi_i,\ i = 1, \cdots, N\}$ as an ϵ-net for K. Choose α so that $\|(\psi_i)_\alpha\| < \epsilon$ for $i = 1, \ldots, N$. Then for any $\psi \in K$, there exists $\psi_K, \delta, \|\delta\| < \epsilon$ such that $\psi = \psi_K + \delta$. Since $(a+b)^+ \le a^+ + b^+$, we have $\|\psi_\alpha\| \le (\psi_K)_\alpha + |\delta|$ and therefore $\|\psi_\eta\| \le \|\psi_\alpha\| + \|\delta\| \le 2\epsilon$. This concludes the proof of the lemma. ∎

Proof of Theorem

Let $R = (\Delta + b)^{-1}$, and let $g(u) = f(u) - bu$. Then we can rewrite (I.9), after multiplying across by regrouping and R, as

$$u = t\theta + Rh + Rg(u) \equiv Su.$$

Since f was Lipschitz, we can choose $A > 0$ so that $|g(u) - g(v)| \le A|u - v|$. By earlier calculations on the norms of diagonal operators, we can see that if $\beta = \min(b - \lambda_n, \lambda_{n+1} - b)$, then the operator norm of R, $\|R\|$ equals $\frac{1}{\beta}$. Let B be the unit ball in $L^2(\Omega)$ and let $K = \overline{R(B)}$. Let $\delta : \mathbf{R} \longrightarrow \mathbf{R}$ be the modulus of continuity of Lemma I.10 corresponding to θ and K, and of course $\delta(\eta) \longrightarrow 0$ as $\eta \longrightarrow 0$. Choose $\epsilon > 0$, $q < 0$ so that

$$\delta(\epsilon(A+B)) \le \frac{\beta}{4A(A+B)}, \quad (\|\theta\| + \epsilon)q < \epsilon\frac{\beta}{4}, \tag{I.15}$$

that is, choose $\epsilon > 0$ to satisfy the first inequality and choose $q > 0$ to satisfy the second. The function g can be written $g = g_0 + g_1 + g_2$ where the g_i satisfy (i) $|g_0(u)| \le q|u|$, (ii) g_1 is bounded, and (iii) $g_2 = 0$ for $u \ge 0$, $|g_2(u) - g_2(v)| \le A|u - v|$. Now let $u \in t\theta + t\epsilon\overline{B}$. From (i) above we have $\|g_0(u)\| \le qt(\|\theta\| + \epsilon)$, since $\|u\| \le t(\|\theta\| + \epsilon)$. We also have

$\|h\| + \|g_1(u)\| \le \frac{1}{4}t\epsilon\beta$ for $t \ge t_0$ if t_0 is sufficiently large. And finally, we have $\|g_2(u)\| \le A\|u^-\|$. Now since $u = t\theta + t\epsilon v$ for some $v \in \overline{B}$, we have $\|u^-\| = \|(t\theta + t\epsilon v)^-\| \le At\epsilon + \frac{1}{4}t\epsilon\beta + qt(\|\theta\| + \epsilon)$. But we choose $q(\|\theta\| + \epsilon) < \frac{\epsilon\beta}{4}$ so we have $\|h + g(u)\| \le t\epsilon(A + \beta)$.

Since $K = \overline{R(B)}$, $Su = t\theta + Rh + Rg(u)$ can be written $Su = t\theta + t\epsilon(A + \beta)\psi$ for some $\psi \in K$. Since if u is a solution of (I.9), then $u = Su$ and the previous lemma implies

$$\begin{aligned}\|u^-\| = t\|(\theta + \epsilon(A + \beta)\psi)^-\| &\le t\epsilon(A + \beta)\delta(\epsilon(A + \beta)) \\ &\le \frac{t\beta\epsilon}{4A}\end{aligned}$$

and we conclude that if u satisfies (I.9), and $u \in t\theta + t\epsilon B$, then $\|g_2(u)\| \le A\|u^-\| \le \frac{t\beta\epsilon}{4}$. Now recalling that $\|h + g(u)\| \le \|g_2(u)\| + \frac{1}{2}t\epsilon\beta$, we can conclude that if u is a solution of (I.9), $u \in t\theta + t\epsilon B$, we have

$$\|R(h + g(u))\| \le \frac{3}{4}t\epsilon.$$

Thus any solution of (I.9) satisfying $u \in t\theta + t\epsilon B$ must belong to $t\theta + \frac{3}{4}t\epsilon B$, since $u = t\theta + R(h + g(u))$. This statement works equally well if we replace $g(u)$ by $\lambda(h + g(u))$, $0 \le \lambda \le 1$. Thus the equations

$$u = (-\Delta)^{-1}(bu - \lambda(h + g(u)) - s\phi_1)$$

have no solution on the boundary of the ball $B_{t\epsilon}(t\theta)$ for $0 \le \lambda \le 1$. Therefore the degree

$$\deg(u - (-\Delta)^{-1}(bu - (h + g(u)) - t\phi_1, B_{t\epsilon}(t\theta), 0)$$

is defined and independent of λ for $0 \le \lambda \le 1$. For $\lambda = 0$, we have

$$\deg(u - (-\Delta)^{-1}(bu - t\phi_1), B_{t\epsilon}(t\theta), 0) = (-1)^n.$$

This is true since if we take finite dimensional approximations on the spaces $P_m H = \text{span}\{\phi_1 \cdots \phi_m\}$, then for $m \geq n+1$, we have that the finite dimensional degree, the sign of the determinant of the linear map is precisely the minus one to the power of number of negative eigenvalues of $I - b(-\Delta)^{-1}$, which is n. This concludes the proof of the theorem.∎

COROLLARY TO THEOREM I.8. *If* $\lim_{s \to -\infty} \frac{f(s)}{s} = a$, *there exists* $t_0 > 0, \epsilon > 0$ *so that*

$$\deg(u - (-\Delta)^{-1}(f(u) - h - s\phi_1), B_{t\epsilon}(t\overline{\theta}), 0) = 1,$$

where $\overline{\theta} = \dfrac{\phi_1}{a - \lambda_1}$.

Now we need a couple of lemmas which are close to those of section 3, the only difference being that we are in $L^2(\Omega)$ instead of $C^\alpha(\overline{\Omega})$ and thus, Sobolev estimates are used instead of Schauder estimates. The proofs are left as an exercise for the reader ($L^2(\Omega)$ estimates will also be given in Chapter IV).

LEMMA I.11 *Assume that* $|f(u)| \leq a + c|u|$ *and* $f(u) - \lambda_1 u \geq \epsilon|u| - b$, *and that* $\|h\| \leq r$, $s \leq s_0$ *where* $a, b, \epsilon, c, r, s_0$ *are positive constants. Then there exists* C, *depending only on* $a, b, \epsilon, c, r, s_0$, *so that any solution of* (I.9) *satisfies* $\|u\| \leq C$.

LEMMA I.12 *There exists* R_0 *depending only on* $a, b, c, r, \epsilon, t_0$ *so that*

$$\deg(u - (-\Delta)^{-1}(f(u) - h - t\phi_1), B_R(0), 0) = 0$$

for all $R \geq R_0$ *and* $t \leq t_0$.

We can now state our main theorem on the existence of more than two solutions.

THEOREM I.9 *Assume* $h \in L^2(\Omega)$ *and* $\lim_{s \to -\infty} \frac{f(s)}{s} = a$, $\lim_{s \to +\infty} \frac{f(s)}{s} = b$, *where* $a < \lambda_1$, $\lambda_n < b < \lambda_{n+1}$. *Then there exists* t_0 *so that if* $t \geq t_0$, *equation* (I.9) *has at least two solutions if* n *is odd and at least three solutions if* n *is even.*

Proof. Theorem I.8 guarantees the existence of two balls $B_1 = B_{\epsilon t}(t\theta)$ and $B_2 = B_{\epsilon t}(t\overline{\theta})$, such that

$$\deg(u - (-\Delta)^{-1}(f(u) - t\phi_1 - h(x)), B_1, 0) = (-1)^n,$$

$$\deg(u - (-\Delta)^{-1}(f(u) - t\phi_1 - h), B_2, 0) = +1.$$

Moreover, if $\epsilon < \min\{(\lambda_1 - a)^{-1}, (b - \lambda_1)^{-1}\}$, then they are disjoint. This gives two solutions. If n is even, choose B_R so large that B_R contains B_1 and B_2 and the degree is zero on B_R. Since our map has no zeros on $\partial B_1 \cup \partial B_2$, by excision, we have the degree on $B_R - (\overline{B}_1 \cup \overline{B}_2)$ is -2. This proves the existence of at least three solutions, and concludes the proof of the theorem. ∎

We conclude this theorem with some remarks. It is worth looking briefly at the generality of Theorem I.9.

We studied an equation on $L^2(\Omega)$ of the type

$$Lu + f(u) = t\phi_1 + h$$

under the assumptions that L is a invertible self-adjoint operator with compact inverse, with eigenvalues $\lambda_i \longrightarrow +\infty$ and with a positive first eigenfunction. No assumptions on smoothness were made on ϕ_1, h or f. These facts will be worth noting in Chapter II, when we tackle more general classes of operators. Although we can see many improvements on Theorem I.9 for an elliptic operator with Dirichlet boundary conditions and some regularity assumptions, using variational methods and eigenvalue comparison theorems, Theorem I.9 remains unimproved in this generality in the more general operator theoretic case, except when $\lambda_2 < b < \lambda_3$, this problem remains open.

This theorem led the authors to speculate that if $\lambda_n < b < \lambda_{n+1}$, then for sufficiently large t, equation (I.9) has at least $2n$ solutions. This proved to be true in the O.D.E. case, but not true in general.

Also still open is the case where $f'(+\infty) = +\infty$, (presumably with growth less than the critical exponent). Kwong [KWO] has shown that in the O.D.E. case, the number of solutions increases without bound as $s \longrightarrow +\infty$.

Questions raised by Theorem I.9

Theorem I.9 raises many questions, some of which remain unanswered today. For example, the theorem shows that there are at least three solutions if $\lambda_n < b < \lambda_{n+1}$ and n **is even.** Surely, we would be led to conjecture, these solutions also exist if n is odd and greater than 1.

This was solved quickly in the elliptic case, but in the more general, non-selfadjoint case, it remains unproven.

The theorem also shows that for "most" h, (using Sard's Theorem) there are at least four solutions if $\lambda_{2n} < b < \lambda_{2n+1}$. Some results on this are available in the elliptic (variational) setting, but in the P.D.E.

case, to the best of our knowledge, it is not even known if there are at least three solutions for large s.

I.7 Historical remarks

Theorem I.1 and I.2 are in Hammerstein [H], who was the first to see the importance of the first eigenvalue, although earlier work goes back to Picard in the 1890's.

Theorem I.3 and I.4 are due to Dolph [D]. The first result on crossing the first eigenvalue was in [A-P]. The first to see the importance of decomposing the right hand side into $t\phi_1 + h$ appear to have been, independently, Kazdan and Warner [K-W], and Berger and Podalak [B-P]. Kazdan and Warner also were first to notice that the methods of upper and lower solutions could be used here. The combined use of degree theory and upper and lower solutions, as used in Theorem I.6 was due independently to Amann and Hess [A-H], and Dancer [Da].

The first version of Theorem I.8 showing the existence of at least three solutions for large t was given in $C^{1+\alpha}$ by the authors [L-Mcl]. Later, with a view to obtaining extensions to the case of non-compact inverse and the non-selfadjoint case, a proof in $L^2(\Omega)$ was given by McKenna and Walter [Mc-W].

Concluding Remarks

Although, as I have repeatedly emphasized, this chapter was about the second order elliptic problem, the methods were operator-theoretic, and could possibly generalize to some of the linear operators treated in Chapter II.

Chapter II

More general operators

The purpose of this chapter is to extend the last result of Chapter I to other equations where the linear part is not just the Laplacian. We shall treat three separate cases, first a semilinear heat equation, then a semilinear string equation, and finally a vibrating beam equation. The third is the most interesting.

II.1 Periodic solutions of a semilinear equation with source terms

We look for weak solutions of the equation

$$u_t = Lu + f(u) - s\phi_1 - h(x,t) \quad \text{in } \Omega \times \mathbf{R} \tag{II.1}$$

which vanish on $\partial\Omega$ and are periodic in t with period T.

We assume that the eigenfunctions ϕ_i of L are an orthonormal basis for $L^2(\Omega)$ with eigenvalues $-\lambda_i, \lambda_1 > 0, \lambda_i \longrightarrow +\infty$, and that $\phi_1(x) > 0, x \in \Omega$. These are the assumptions of this Chapter. We shall work with the complex Hilbert space $H_T^* = L^2(\Omega \times (0,T))$, equipped with the usual inner product

$$< v, w >^* = \int_0^{2\pi} \int_\Omega v(x,t)\overline{w}(x,t)dxdt$$

and norm $\|v\| =< v, v >^{*\frac{1}{2}}$. Later we shall switch to the real subspace H_T. The functions $\phi_{mn} = \dfrac{\phi_n(x)e^{imt}}{\sqrt{2\pi}}$, $n \geq 1, m = 0, \pm 1, \pm 2, \ldots$ are a complete orthonormal basis for H^*. Let $\sum^*$ denote sums over the indices m, n. Every $v \in H^*$ has a Fourier expansion

$$v = \sum\nolimits^* v_{mn}\phi_{mn},$$

with $\sum |v_{mn}|^2 = \|v\|^2$, $v_{mn} =< v, \phi_{mn} >^*$. A weak solution to the boundary value problem (II.1) is, by definition, a function $u \in H$ satisfying $Lu \in H$, i.e. $\sum^* |u_{mn}|^2(m^2 + \lambda_n^2) < \infty$ satisfying (II.1) in H.

For real $\alpha \neq \lambda_n$, the operator $R = (L + \alpha - D_t)^{-1}$ denoted by

$$u = Rh \longleftrightarrow u_{mn} = \frac{h_{mn}}{-\lambda_n + \alpha - in}$$

is a compact linear operator on H^* and the operator norm of R, $\|R\| = \frac{1}{|\alpha - \lambda_n|}$, where λ_n is an eigenvalue of $-L$ closest to α.

From now on, we restrict ourselves to the real subspace H and observe that it is invariant under R.

Our first theorem is a non-self-adjoint version of Theorem I.8.

THEOREM II.1 *Assume that f' is bounded, that $f'(+\infty) = \alpha$ satisfies $\lambda_n < \alpha < \lambda_{n+1}$ and that $h \in H$. Then there exists $s_0 > 0$, $\epsilon > 0$ such that the Leray-Schauder degree*

$$\deg(u - (-L + D_t)^{-1}(f(u) - s\phi_1 - h), B^*_{s\epsilon}(s\theta), 0) = (-1)^n \quad \text{(II.2)}$$

for $s \geq s_0$. Here B^*_r denotes a ball of radius r in H and

$$\theta = -(-L - \alpha + D_t)^{-1}\phi_1 = \frac{\phi_1}{\alpha - \lambda_1}.$$

REMARKS. Before we begin the outline of the proof, we shall provide a little motivation. The idea here is that, since the operator is **real**, complex eigenvalues occur in conjugate pairs. Now when we evaluate the sign of the determinant, as in Chapter I, we shall have a number of real negative eigenvalue, plus an infinite number of real positive eigenvalues, plus an infinite number of conjugate pairs of complex eigenvalues. However, the determinant of two complex conjugate eigenfunctions is always positive, so we do not expect the complex eigenvalues to affect the computation of the degree.

Proof. The first part of the proof, where we show there are no solutions on the boundary of the ball in question is the same as that of

Theorem I.8. we shall just indicate the changes, so we can be sure the degree is defined.

Let R be the operator $(-L-\alpha+D_t)^{-1}$, let $A=(D_t-L)^{-1}$, and let $g(u)=\alpha u - f(u)$. Then the periodic problem (II.1) is equivalent to

$$u = s\theta + Rh + Rg(u) \equiv Su. \tag{II.3}$$

Let B^* be the open unit ball in H, let $K=\overline{R(B^*)}$. It follows that any solution $u \in s\theta + s \in \overline{B}^*$, of (II.3) belongs to $s\theta + \frac{3}{4}s\epsilon\overline{B}^*$ and this holds when $h+g(u)$ is replaced by $\lambda(h+g(u))$, $0 \le \lambda \le 1$. Solutions of the corresponding equation (II.3) are solutions of

$$u = A(-s\phi_1 + \alpha u\lambda(h+g(u)))$$

and it follows that if $G = B^*_{s\epsilon}(s\theta_1)$,

$$\deg(u - A(-s\phi_1 + \alpha_u - (h+g(u))), G, 0) = \deg(u - A(\alpha u - s\phi_1), G, 0).$$

Now by substituting $v = u - s\theta$, and using $u - A(\alpha u - s\phi_1) = u - s\theta + \alpha(Au - s\theta)$, we observe that

$$\deg(u - A(\alpha u - s\phi_1), G, 0) = \deg(v - \alpha Av, s\epsilon B^*, 0).$$

Thus, to prove the theorem, we have to show that this degree is $(-1)^n$. To do this, we calculate the degree on finite dimensional subspaces which we now choose. The functions

$$\begin{aligned}
\phi_{on} &= \frac{1}{\sqrt{2\pi}}\phi_n(x) \\
\phi^c_{mn} &= \frac{1}{\sqrt{\pi}}\phi_n(x)\cos mt \qquad m = 1,2,3\ldots \\
\phi^s_{mn} &= \frac{1}{\sqrt{\pi}}\phi_n(x)\sin mt
\end{aligned}$$

form a real orthonormal basis for H. If $h \in H$, then $h = \sum h_{mn}\phi_{mn}$ in H^* and h can be expanded in terms of $\phi_{on}, h^c_{mn}, h^s_{mn}$, with the identities

$$\|(A - PA)h\|) = \Sigma \frac{1}{\lambda_n^2 + m^2}(|h_{mn}|^2 + |h_{-m,n}|^2)$$

it follows that

$$\|A - PA\|^2 \leq \min_{m,n>b} \frac{1}{\lambda_n^2 + m^2} \leq \max\left(\frac{1}{p+1}, \frac{1}{\lambda_{p+1}}\right)$$

and by the definition of degree

$$\deg(v - \alpha PAv, s\epsilon B^*, 0) = \deg(v - \alpha Av, s\epsilon B^*, 0)$$

for large p, since the operator PA is of finite rank, with its range contained in PH.

Taking the functions $\phi_{on}, \phi^c_{mn}, \phi^s_{mn}, 1 \leq m, n \leq p$, as a basis H_p, the equation $v + \alpha PAv$ becomes a matrix equation on the space H_p, of the form

$$(I + \alpha C)x = 0 \quad \text{for} \quad x \in \mathbf{R}^q, \qquad q = p(2p+1)$$

where Iq is the identity matrix of rank q, C is a $q \times q$ block diagonal matrix $C = \text{diag}(C_1, \cdots, C_p)$ and each C_n is a $2p+1$ by $2p+1$ block diagonal matrix given by

$$C_n = \text{diag}\left(-\frac{1}{\lambda_n}, A_{1n}, \cdots, A_{pn}\right)$$

Now let $D = I + \alpha C = \text{diag}(D_1, \cdots, D_n)$ where

$$D_n = \text{diag}\left(1 - \frac{\alpha}{\lambda_n}, I_2 - \alpha A_{1n}, \cdots, I_2 - \alpha A_{pn}\right).$$

Since $\det D_n = \left(1 - \frac{\alpha}{\lambda_n}\right) a_{1n} \cdots a_{pn}$ where $\det(I_2 - \alpha A_{mn}) = a_{mn} > 0$, we finally get for large p that

$$\text{sign } \det D = \text{ sign } \left(1 - \frac{\alpha}{\lambda_1}\right) \cdots \left(1 - \frac{\alpha}{\lambda_p}\right) = (-1)^n.$$

Recalling that $\lambda_n < \alpha < \lambda_{n+1}$. Since sign $\det D$ is equal to $\deg(v + \alpha P v, s\epsilon B^*, 0)$ for large p, the theorem is proved by letting $p \longrightarrow +\infty$.

COROLLARY II.1 *If f' is bounded, and $\overline{\alpha} = f'(-\infty) < \lambda_1$, then there exist positive constants s_0, ϵ such that*

$$\deg(u - (D_t - L)^{-1}(f(u) - s\phi_1 - h), B^*_{s\epsilon}(s\overline{\theta}), 0) = 1$$

for $s \geq s_0$, where $\overline{\theta} = \dfrac{\phi_1}{\overline{\alpha} - \lambda_1} < 0$.

LEMMA II.1 *Assume that $|f(u)| \leq a + c|u|$, $f'(-\infty)$, $f'(+\infty)$ exist, that $f(u) - \lambda_1 u \geq \epsilon|u| - b$, and that $h \in H$ satisfies $\|h\| \leq r$ where a, b, c, r, ϵ are positive constants. Then there exists C depending only on a, b, c, r, ϵ such that*

$$D_t u = Lu + f(u) - s\phi_1 - h$$

$$u(x, t + 2\pi) = u(x, t)$$

satisfies $\|u\| \leq c$.

Proof. Suppose not. Then there exist u_n with $\|u_n\| \longrightarrow \infty$ which satisfy the equation. Now let $v_n = \dfrac{u_n}{\|u_n\|}$, and v_n satisfies

$$D_t v_n = Lv_n + \frac{1}{\|u_n\|} f(\|u_n\| v_n) - h_n(x, t).$$

Since $f_n(u) - \lambda_1 u \geq \epsilon|u| - b$, we can conclude, by multiplying across by ϕ_1 and integrating, that

$$< D_t u_n - Lu_n - \lambda_1 u_n, \phi_1 > = < f(u_n) - \lambda_1 u_n, \phi_1 > - < h_n, \phi_1 >$$

and thus

$$\begin{aligned} 0 &\geq \int (\epsilon|u_n| - b)\phi_1 - \|h_n\| \\ &\geq \epsilon \int u\phi_1 - b\int \phi_1 - r. \end{aligned}$$

from which we conclude that if $u_n = c_n\phi_1 + x_n$, then the c_n's are bounded. Now,

$$v_n = (D_t - \Delta)^{-1}\left(\frac{1}{\|u_n\|}F(\|u_n\|v_n) - \frac{h_n}{\|u_n\|}\right)$$

and one can check that the v_n's are precompact in H since, by virtue of $|f(u)| \leq a + c|u|$, we have that $\frac{1}{\|u_n\|}(f(\|u_n\|v_n) - h_n)$ is bounded and $(D_t - \Delta)^{-1}$ is a compact operator. Therefore, there exists a convergent subsequence, still called v_n, converging to v. Since $v_n = \frac{1}{\|u_n\|}(c_n\phi_1 + x_n)$ and the c_n's are bounded, it follows that $v \bot \phi_1$. Since $f(s) = f'(+\infty)s^+ - f'(-\infty)s^- + f_1(s)$ where $\frac{f_1(s)}{s} \longrightarrow 0$ as $s \longrightarrow +\infty$, we have that

$$\frac{1}{\|u_n\|}(f(\|u_n\|v) - h_n) \longrightarrow f'(+\infty)v^+ - f'(-\infty)v^-$$

and

$$(D_t - L)v = f'(+\infty)v^+ - f'(-\infty)v^-$$

or

$$(D_t - L - \lambda_1)v = (f'(+\infty) - \lambda_1)v^+ - (f'(-\infty) - \lambda_1)v^-.$$

Since $(f'(+\infty)-\lambda_1)v^+ - (f'(-\infty)-\lambda_1)v_1 \geq \epsilon|v|$ after multiplying across by ϕ_1 and integrating by parts, we obtain a contradiction. ∎

LEMMA II.2 *Let $s_1 \in R$ under the assumptions of the preceding lemma, there exists $C_1 > 0$, depending on s_1 and the constants of* Lemma II.1, *such that*

$$\deg(u - (D_t - Lu)^{-1}(f(u) - (h + s\phi_1)), B^*_\beta(0), 0) = 0$$

for $s \leq s_1$ and $\beta > c_1$.

The proof of Lemma II.2 is the same as those for the self-adjoint case, as done in Chapter I. There are no solutions on the boundary of the ball for $s \leq s_1$, by the previous lemma. Therefore, by homotopy, the degree is the same for all $s \leq s_1$, and since it must be zero for large negative s, it must be zero for all $s \leq s_1$.

We have now assembled all the ingredients for our first existence theorem.

THEOREM II.2 *Let $h \in H^*$. Assume f' is bounded, $f'(-\infty) < \lambda_1, \lambda_n < f'(+\infty) < \lambda_{n+1}$. Then there exists s_0 so that if $s \geq s_0$, the equation* (II.1) *has at least two 2π-periodic solutions if n is even, and at least three if n is odd.*

The proof is by now obvious. The degree on a large ball is zero. By Theorem II.1, we can find a ball near $\overline{\theta}$, on which the degree of the map

$$u - (D_t - L)^{-1}(f(u) - (s\phi_1 + h(x)))$$

is 1, and a ball near θ, on which the degree is $(-1)^n$. Now, if these are both in a ball on which the degree is zero, we have two solutions if n is odd, and three if n is even. This concludes the proof.

We conclude with some open problems.

PROBLEM. 1. Can we remove the assumption that n is odd in Theorem II.2 and still get at least three solutions?

PROBLEM. 2. Can we get **four** solutions? The answer is yes if T is sufficiently small and $n = 2$, but is now known otherwise.

PROBLEM. 3. What happens if $f'(+\infty) = +\infty$. Do there exist at least two solutions? How about more then two?

PROBLEM. 4. What about more general periodic-parabolic linear operators? These are known to have a positive first eigenvalue, cite Hess, Lazer.

II.2 Periodic solutions of semilinear wave equations

The purpose of this section is to study the existence of 2π-periodic solutions of the equation

$$\begin{aligned} -u_{tt} + u_{xx} + f(u) &= s \sin x + h(x,t) \\ u(0,t) &= u(\pi,t) = 0 \end{aligned} \tag{II.4}$$

where we assume that $\frac{f(s)}{s} \longrightarrow a(b)$, $0 < a < 1(3 < b)$ as $a \longrightarrow -\infty(b \longrightarrow \infty)$.

As usual in this problems [B-N], we assume that f is monotone increasing. The difficulty which arises in this type of problem is that the linear operator does not posses a compact resolvent, and thus some additional work is necessary before classical Leray-Schauder degree theory can be used. With apologies to the reader, we will prove do this work in an abstract setting. The reason for this will be clear at the end of the chapter.

Thus we let $H = L_2([0,\pi] \times [-\pi,\pi])$, and assume that $D(L) \subseteq H$, $L : D(L) \longrightarrow H$; and that L is self-adjoint. We assume that $-L$ possesses the eigenvalues $\{\lambda_i, i \in \mathbf{Z}\}$ and that $\lambda_0 = 0, \lambda_i \leq \lambda_j$ if $i \leq j$, and $\lambda_i \to +\infty$ as $i \longrightarrow +\infty$ and $\lambda_i \longrightarrow -\infty$ as $i \longrightarrow -\infty$. We assume that λ_0 is of infinite multiplicity and that every other eigenvalue is of finite multiplicity. Finally we assume that λ_1 is simple, that the eigenfunction corresponding to λ_1 is $\phi_1(x)$ and satisfies $\phi_1(x) > 0$ in $(0,\pi)$, and that if ϕ_n is another eigenfunction of L, then there exists $\epsilon_n > 0$ such that $\phi_1(x) \pm \epsilon_n \phi_n(x,t) > 0$ in $(0,\pi) \times (-\pi,\pi)$. (We interject a comment at this stage; the reader should check these "abstract hypotheses" against the example in the beginning of the section. Also, the assumption that ϕ_1 depends only on x is not necessary, and it would be enough to allow it to depend on x, t and to require that $\phi_1(x,t) > 0,\ x \in (0,\pi),\ t \in (-\pi,\pi)$).

Now we prove some results on the existence of 2π-periodic solutions of the equation

$$\begin{aligned} -u_{tt} + u_{xx} + f(u) &= H(x,t) = s\sin x + h(x,t), \\ u(0,t) &= u(\pi,t) = 0. \end{aligned} \tag{II.4}$$

We study the abstract operator equation

$$Lu + f(u) = s\phi_1 + h(x,t) = H(x,t). \qquad \text{(II.4.a)}$$

Let H_1 be the direct sum of the eigenspace corresponding to the eigenvalues λ_i, $i \leq 0$. Note that since λ_i are the eigenvalues of $-L$, it follows that L is monotone on the invariant subspace H_1. Let P be orthogonal projection onto H_1. Let $H_2 = (I - P)H$. We write (II.4) as

$$Lu + f(u) = s \sin x + h(x,t)$$

with the additional assumption

$$\begin{array}{ll} \text{(a)} & h(x,t) \in PH. \\ \text{(b)} & \infty > M \geq f'(u) \geq \alpha_1 > 0 \text{ for some } \alpha, M \\ \text{(c)} & \text{If } g(x) \in L_2(0,\pi) \text{ then } g(x) \in PH. \end{array} \qquad \text{(II.5)}$$

Let $w = (I - P)u$, $v = Pu$, and write the problem as

$$\begin{array}{ll} \text{(a)} & Lw + (I-P)f(v+w) = 0 \\ \text{(b)} & Lv + Pf(v+w) = s \sin x + h(x,t) \end{array} \qquad \text{(II.6)}$$

Now fix v and regard (II.6.a) as an equation in $w \in (I - P)H$. Assumption (II.5.b) guarantees that for each v, equation (II.6.b) has a unique solution $w(v)$. This can be proved by the methods of Chapter I, Theorem 1, and the contraction mapping theorem, or by monotone operator theory. Furthermore $w(v)$ depends continuously on v, with Lipschitz constant depending on α and M. Thus we have reduced the problem

$$Lu + f(u) = s\phi_1 + h(x,t)$$

on the space H, the problem

$$Lv + Pf(v + w(v)) = s \sin x + h(x,t). \qquad \text{(II.7)}$$

THEOREM II.3 *If* $\lim_{u\to+\infty}\frac{f(u)}{u} = b$ *and* $\lim_{u\to-\infty}\frac{f(u)}{u} = a$ *and* $\lambda_i < a, b < \lambda_{i+1}$ *for some* $i \geq 0$*, then equation* (II.4.a) *has a solution for any right handside.*

Proof. It is easy to see by the methods of Chapter I that there exists $B, C, D,\ B < 1$ such that if $\alpha = \frac{\lambda_i + \lambda_{i+1}}{2}$, then

$$\|(-L-\alpha I)^{-1}(f(u) - \alpha u - H(x,t))\| \leq B\|u\|^2 + C\|u\| + D.$$

We can write (II.7) as

$$v = (-L-\alpha I)^{-1}P(f(v+w(v)) - \alpha(v+w(v) - H(x,t)).$$

Recall $Pw(v) = 0$. This gives

$$\|(-L-\alpha I)^{-1}(f(v+w(v)) - \alpha u)\|^2 \leq B^2(\|v\|^2 + \|w(v)\|^2) + c_1\|v\| + c_2$$

for some $c_1, c_2, \ldots$ (recall w is Lipschitz continuous in $v, w(0) = 0$). Write $E = -L - \alpha I$, write $g(u) = f(u) - \alpha u$. Then the fact that w satisfies (II.6.a) means that

$$w(v) = E(I-P)g(v+w(v))$$

and thus

$$\begin{aligned}\|E^{-1}g(v+w(v))\|^2 &= \|E^{-1}Pg(v+w(v))\|^2 + \|E^{-1}(I-P)g(v+w(v))\|^2\\ &= \|w(v)\|^2 + \|E^{-1}(I-P)g(v+w(v))\|^2\\ &\leq B^2(\|v\|^2 + \|w(v)\|^2) + c_1\|v\| + c_2.\end{aligned}$$

From this, we can conclude that for $B < 1$,

$$\|(-L-\alpha I)^{-1}P(f(v+w(v)) - \alpha(v+w(v)) - H(x,t))\|^2 \leq B\|v\|^2 + c_1\|v\| + c_3$$

for a new c_3. Thus, choosing R sufficiently large that

$$BR^2 + c_1 R + c_3 < R^2,$$

we can conclude that the ball of radius R in H_1 is mapped into itself by the map

$$v \longrightarrow (-L - \alpha I)^{-1} P(f(v + w(v)) - \alpha v - H(x,t)). \qquad \text{(II.8)}$$

Furthermore, since $\lambda_1 \longrightarrow +\infty$ are the eigenvalues of $-L$ on the space H_1, we see that the operator $(-L-\alpha I)^{-1}$ is *compact.* Thus, by Schauder fixed point theorem we can conclude that the map in (II.8) has a fixed point and, equivalently, that (II.4.a) has a solution for all $H(x,t)$. This proves the theorem. ∎

REMARKS. It is sufficient for the existence of all solutions that $f' \geq 0$, instead of $f' \geq \alpha > 0$. However the proof then becomes technically messy, as the map $v \longrightarrow w(v)$ is no longer single valued, but in fact set-valued. The map $v \longrightarrow Pf(v + w(v))$ is continuous in the sense of taking strongly convergent sequences $\{v_n\}$ into weakly convergent sequences. This, combined with the compactness of $(-L - \alpha I)^{-1}$ is enough to do the job. For details, see [McK 1]. We also remark that if for some $i, \lambda_i < f'(s) < \lambda_{i+1}$ for all s, then (II.4.a) has a unique solution for all right hand sides. Now we go on to the case where $0 < a < \lambda_1$, $\lambda_i < b < \lambda_{i+1}$.

THEOREM II.4 *Assume that* (II.6) *is satisfied and that* $\lambda_k < b < \lambda_{k+1}$, $k \geq 1$. *Then there exists* s_0 *such that for* $s \geq s_0$, *and* $0 < \epsilon \leq \epsilon_0$

(ϵ_0 depending on s) the Leray-Schauder degree

$$deg(v - L^{-1}(s\phi_1 + h - P(f(v + w(v)))), B_{s\epsilon}(s\tilde{\phi}), 0) = (-1)^k$$

where $b = \lim_{s \to +\infty} \frac{f(s)}{s}$, $\tilde{\phi} = (L+b)^{-1}\phi_1 = \frac{\phi_1}{b - \lambda_1} > 0$.

REMARK. It ought to be clear from the context that the ball $B_{s\epsilon}(s\tilde{\phi})$ is in the space H_1.

Proof. Much of this proof is very similar to that of Theorem I.8 of the first chapter. We assume that v satisfies (II.7) or equivalently

$$v = L^{-1}(s\phi_1 + h - Pf(v + w(v)))$$

then writing $R = (L+b)^{-1}$ and $g(u) = bu - f(u)$, it follows that

$$v = s\tilde{\phi} + Rh + RPg(v + w(v)).$$

Now let $v = s\tilde{\phi} + s\epsilon\overline{v}$, $\|v\| \leq 1$. Write $g = g_0 + g_1 + g_2$ exactly as in the first chapter, $w(s\tilde{\phi}) = 0$ since $Pf(s\phi) = 0$. Hence

$$v + w(v) = s\tilde{\phi} + s\epsilon\overline{v} + w(s\tilde{\phi} + s\epsilon\overline{v}) - w(s\tilde{\phi})$$

and so $\|v + w(v) - s\tilde{\phi}\| \leq s\epsilon(1 + r)$, where r is the Lipschitz constant for the function $w(v)$, which the reader may check is $\frac{M}{\alpha_1}$, where M and α_1 are given in (II.5). It follows that

$$\|(v + w(v))^-\| \leq s\epsilon(1 + r),$$

and

$$\|h + (g_0 + g_1)(v + w(v))\| \leq \frac{1}{2}s\epsilon\beta$$

$$\|g_2(v + w(v))\| \le As\epsilon(1 + r)$$

for large s, where $\beta = \|R\|$, and

$$\|h + g(v + w(v))\| < s\epsilon A' \quad \text{for} \quad s \ge s_0,$$

where $A' = (1 + r)(A + \beta)$. Now suppose $v = s\tilde{\phi} + s\epsilon\overline{B}$ is a solution of

$$v = s\tilde{\phi} + Rh + RPg(v + w(v)).$$

Then $v = s\tilde{\phi} + s\epsilon A'\psi$, $\psi \in \overline{R(B)}$. This is not exactly what we need in order to apply the method of Chapter I. What we need, in order to estimate $\|(v+w(v))^-\|$, and declare it small, is that $v+w(v) = s\tilde{\phi}+s\epsilon X$ for $X \in K$ for a suitable choice of a compact set K. Then we can get a suitable modulus of continuity for K. It is intuitively clear that such a K must exist, since $w(v)$ is a Lipschitz map from the compact set $s\tilde{\phi} + s\epsilon A'R(B)$ and $w(s\phi) = 0$.

Finding K requires a little care however. Consider $w(s\tilde{\phi}+\delta\psi)$, $\psi \in R(\overline{B})$, with $\delta \longrightarrow 0+$. It follows from the Lipschitz condition, and from $w(s\tilde{\phi}) = 0$ that

$$\|w(s\tilde{\phi} + \delta\psi)\| \le r\delta\|\psi\|.$$

Let w_0 be the solution of

$$Lw_0 + (I - P)f'(s\tilde{\phi})(\psi + w_0) = 0. \tag{II.9}$$

(Why must w_0 exist?) If $w(s\tilde{\phi}+\delta w)$ is written in the form $w(s\tilde{\phi}+\delta w) = \delta(w_0 + w_1)$, then

$$\delta L(w_0 + w_1) + (I - P)f(s\tilde{\phi} + \delta(\psi + w_0 + w_1)) = 0.$$

Now

$$\begin{aligned} & f(s\tilde{\phi}+\delta(\psi+w_0+w_1))-\delta f'(s\tilde{\phi})(\psi+w_0)-f(s\tilde{\phi}) \\ = \;& f(s\tilde{\phi}+\delta(\psi+w_0+w_1))-f(s\tilde{\phi}+\delta(\psi+w_0))+f(s\tilde{\phi}+\delta(\psi+w_0)) \\ & -\delta f'(s\tilde{\phi})(\psi+w_0)-f(s\tilde{\phi}) \\ = \;& F_1\delta w_1+F_2\delta(\psi+w_0) \end{aligned}$$

where $F_1 = F_1(x,\delta) \geq \alpha_1$ and

$$f_2(x,\delta)=\int_0^1 [f'(s\tilde{\phi}+\delta\tau(\psi+w_0))-f'(s\phi)]d\tau \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0.$$

Hence

$$\delta L w_1+(I-P)f(s\tilde{\phi}+\delta(\psi+w_0+w_1))-\delta(I-P)f'(s\tilde{\phi})(\psi+w_0)=0.$$

Multiply the last equation by w_1 and observe that $(w_1, Lw_1) \geq 0$. We get that

$$\delta(F_1 w_1+F_2(\psi+w_0),w_1) \leq 0$$

or

$$\alpha_1\|w_1\| \leq \|F_2(\psi+w_0)\| \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0+.$$

Thus $\frac{1}{\delta}w(s\tilde{\phi}+\delta\psi) \longrightarrow w_0$ as $\delta \longrightarrow 0+$ and the solution $w_0 = w_0(\psi)$ of (II.9) satisfies $\|w_0(\psi)-w_0(\psi')\| \leq r\|\psi-\psi'\|$. Thus the set $\{w_0(\psi) : \psi \in R(\overline{B})\}$ is precompact and finally we have that

$$K_0=\{\frac{1}{\delta}w(s\tilde{\phi}+\delta\psi) : 0<\delta<\delta_0, \quad \psi \in R(B)\}$$

is precompact. Now we can say that there exists a compact set $K \subset H$ such that for any $v = s\tilde{\phi}+s\epsilon A'\psi$, $\psi \in R(\overline{B})$, we have

$$v+w(v)=s\phi+s\epsilon X, \longleftrightarrow x \in K \qquad \text{(II.10)}$$

for $0 \le \epsilon \le 1$, with $s \ge s_0$ fixed. If $\delta(\eta)$ is the modulus of continuity corresponding to K, then

$$\|(v + w(v))^-\| \le s\|(\phi + \epsilon X)^-\| \le s\epsilon\delta(\epsilon) < \frac{s\epsilon\beta}{4A'}, \quad 0 < \epsilon \le \epsilon_0,$$

if ϵ_0 is chosen so small that

$$s\epsilon\delta(\epsilon) \le s\epsilon\frac{\beta}{4A'} \quad \text{for} \quad 0 < \epsilon \le \epsilon_0.$$

Thus

$$\|g_2(v + w(v))\| \le A\|(v + w(v))^-\| \le \frac{\beta s\epsilon}{4}$$

and therefore

$$\|RP(h + g(u))\| \le \frac{3}{4}s\epsilon.$$

Thus

$$v = s\phi + Rh + RPg(v + w(v)), \quad v \in s\tilde{\phi} + s\epsilon B$$

implies $v \in s\tilde{\phi} + \frac{3}{4}s\epsilon B$ and thus

$$v \longrightarrow v - L^{-1}(s\phi_1 + h - Pf(v + w(v)))$$

has no zeros on the boundary of $B_{s\epsilon}(s\tilde{\phi})$. This applies, a fortiori, if g and h are replaced by λg and λh, $0 \le \lambda \le 1$.

Thus, to prove the theorem, we need only consider

$$deg(u - L^{-1}(s\phi_1 - bu), B_{s\epsilon}(s\phi), 0)$$

and it is clear, by the arguments of Chapter I, that this degree is $(-1)^k$ if $\lambda_k < b < \lambda_{k+1}$. ∎

COROLLARY II.2 *If we replace the assumption* $\lambda_k < b = \lim_{s\to+\infty} \frac{f(s)}{s} < \lambda_{k+1}$ *by the assumption* $0 < b < \lambda_1$, *and all other assumptions are satisfied, then there exists* $s_0 > 0$ *such that for* $s \geq s_0$, *and* ϵ *small,*

$$deg(v - L^{-1}(s\phi_1 + h - Pf(v + w(v))), B_{s\epsilon}(s\tilde{\phi}), 0) = 1$$

where the meanings are the same as in the theorem.

Proof. Exactly as in the theorem. ∎

LEMMA II.3 *Under the hypothesis* $f(u) \geq \lambda_1 u + \epsilon|u|$ *and* (II.6), *there exists* C *depending only on* α_1, M, r, *such that any solution to*

$$Lu + f(u) = h$$

with $h \in H$ *and* $\|h\| \leq r$ *satisfies* $\|u\| \leq C$.

Proof. By now this should be fairly familiar, so we present an outline. Suppose $\|u_n\| \longrightarrow +\infty$, u_n satisfies

$$Lu_n + f(u_n) = h_n.$$

Let $u_n = c_n\phi_1 + x_n$, where $x_n \perp \phi_1$ and conclude, multiplying across by ϕ_1 and integrating, that c_n's are bounded. Let $p_n = \|u_n\|$, $\overline{u}_n = \frac{u_n}{p_n}$. Then

$$L\overline{u}_n + \frac{1}{p_n} f(p_n \overline{u}_n) = \frac{h}{p_n}.$$

Observe that $\|\overline{u}_n\| = 1$ and if $\overline{u}_n = v_n + w(v_n)$, then

$$v_n = L^{-1}(I - P)\left(\frac{h}{p_n} \frac{f(p_n\overline{u_n})}{p_n}\right).$$

By compactness, we may conclude that v_n and thus $\overline{u}_n$ converge in H. Let $\overline{u} = \lim \overline{u}_n$, and $\overline{f} = \lim \frac{f(p_n\overline{u}_n)}{p_n}$. We know that (a) $\overline{u}$ is perpendicular to ϕ_1 since the ϕ_1 component of $\overline{u}_n$ tended to zero and (b) that $\overline{u}$ satisfies

$$L\overline{u} + \overline{f} = (L\overline{u} + \lambda_1\overline{u}) + \overline{f} - \lambda_1\overline{u} = 0.$$

However, $L\overline{u} + \lambda_1\overline{u}$ must be perpendicular to ϕ_1 and $\overline{f} - \lambda_1 u \geq \epsilon|\overline{u}|$, so $\overline{f} - \lambda_1\overline{u}$ cannot be perpendicular to ϕ_1 unless $\overline{u} = 0$. This contradicts the fact that $\|\overline{u}\| = 1$. Thus we have a contradiction, and the theorem is proved. ■

Now that we have an a priori bound, we can prove, by observing that there are no solutions of (II.7) for large negative s and we obtain

LEMMA II.4 *Let $s \in \mathbf{R}$. Under the assumptions of* Theorem II.4 *and* Lemma II.3, *there exists a constant C (depending on s and the constants of* Lemma II.3*) such that*

$$\deg(v - L^{-1}(s\phi_1 + h - Pf(v + w(v))), B_\beta(0), 0) = 0$$

for $\beta > C$.

Proof. Exactly the same as in Chapter I. ■

Finally we put all these lemmas together again as in Chapter I, to obtain the main theorem of this section.

THEOREM II.5 *Under the assumptions of* Theorem II.4 *on f and h, and assuming that* $0 < a = \lim_{s\to-\infty} \frac{f(s)}{s} < \lambda_1$ *and* $\lambda_k < b = \lim_{s\to-\infty} \frac{f(s)}{s} <$

λ_{k+1}, $k \geq 1$, *then there exists* $s_0 > 0$ *such that for* $s \geq s_0$, *the equation* (II.4.a) *has at least two solutions if k is even and at least three solutions if k is odd.*

Proof. The proof is the same as the one given in Chapter I. We have two solutions $\frac{\phi_1}{b-\lambda_1}$ and $\frac{\phi_1}{a-\lambda_1}$ and the degree on a small ball about each of them is $(-1)^k$ and $+1$ respectively. This gives two solutions. Now choose a big ball containing both so that the degree on the big ball is zero. Excise the two smaller balls. The degree on the remaining region is -2 if k is even. This gives the third solution and proves the theorem.

REMARKS. We are not sure if the requirement $h \in (I-P)H$ is necessary certainly it simplified the calculations. Nor do we know whether the theorem can be proved if one allows $\alpha_1 = 0$. From the remark at the end of the previous section, one would think there is some hope of remaining the strict monotonicity requirement, which has not been needed in the literature [B-N], [McK], [Wi].

The big question is – can one obtain four solutions, as Hofer did in the elliptic case, by use of the mountain pass theorem? This result depended heavily on regularity. A simpler question is, can one obtain at least three solutions when k is odd, as Solimini did [Sol] for the elliptic problem.

One can use critical point theory in this situation as did Brezis, with the Ekeland-Lasry dual formulation, to obtain a variational problem on PH. One could hope to show that, in this context, the mountain pass was degenerated or of the wrong Morse index.

Now we study the equation (II.4). In this case, all our hypotheses on L are satisfied and $\phi_1 = \sin x$, $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 5$ etc. If $h(x,t)$ is even in t on $[-\pi, \pi]$, then (II.4) has at least three solutions for large s, if $0 < f'(-\infty) < \lambda_1$, $\lambda_2 < f'(+\infty) < \lambda_3$. This is done by restricting ourselves to the smaller symmetric subspace, in which λ_2 has multiplicity one. It also has three solutions if $\lambda_3 < f'(+\infty) < \lambda_4$ since in the big space, λ_2 has multiplicity 2 and λ_3 has multiplicity 1.

(There is a slight confusion here in that $\lambda_k < f'(+\infty) < \lambda_{k+1}$ means in the theorem that we are counting eigenvalues as often as the multiplicities, whereas, when we say $\lambda_2 = 3$, $\lambda_3 = 4$, λ_2 is only counted once. If the reader has followed this far, we feel sure he or she is not thrown off by this minor point.) Physically, the equation

$$-u_{tt} + u_{xx} + bu^+ - au^- = s \sin x,$$

$$u(0,t) = u(\pi,t) = 0,$$

$$u(x, t + 2\pi) = u(x,t)$$

corresponds to a string with stretching creating a restoring force and with a repulsive force proportional to the distance from equilibrium. This is decidedly non-physical. If we consider the case $a, b < 0$ then we have a string with two restoring forces, one that of stretching and the other an asymmetric spring restoration term, with the force proportional to the distance from rest. The $s \sin x$ term is just a loading. In this case, one can prove that if $-1 < b < 0$, and $-1 - \epsilon < a < -1$, then for s large positive the solution is unique, whereas for s large negative, there are three solutions, two of which are oscillatory. The technique is via reduction, by the contraction fixed point theorem, to the space spanned by $\{\sin x, \cos mt\}$ and, as we see in Chapter II, a careful study of the two dimensional map.

II.3 A symmetry theorem and some applications

In this section, we prepare for the study of the suspension bridge equation by proving a symmetry theorem which is of independent interest. This theorem is, (as far as we know), the most general application of the basis idea of Theorem I.5.

Let H be the Hilbert space $L^2(\Omega)$. We assume that the operator L is self-adjoint, $\mathcal{D}(L) \subseteq H$ and $L : \mathcal{D}(L) \longrightarrow H$.

A1 We assume L possesses an invariant subspace $H_1 \subseteq H$, that is $L(H_1) \subseteq H$, $L(H_1^\perp) \subseteq H_1^\perp$.

Let σ denote the spectrum of L, let σ_1 denote the spectrum of L restricted to H_1 and let σ_2 denote the spectrum of L restricted to H_2. Of course $\sigma = \sigma_1 \cup \sigma_2$.

We require $\dfrac{\partial f}{\partial u}(x, u)$ to be piecewise smooth in u. Let $[a, b]$ be an interval which contains the range of $\dfrac{\partial f}{\partial u}(u, x)$ for all $u \in (-\infty, \infty)$ and all $x \in \Omega$.

We make the following assumptions on the nonlinear operator $u \longrightarrow f(u(\cdot), \cdot)$.

A2 a and b are finite and the intersection of $[a,b]$ with σ_2 is empty.

A3 The operator $u \longrightarrow f(u(\cdot), \cdot)$ maps the space H_1 into itself.

THEOREM II.6 *Under the assumptions* **A1**, **A2**, **A3**, *every solution of*

$$Lu = f(u) \tag{II.11}$$

belongs to H_1.

REMARK. As we shall see, in the applications sections, the space H_1 will usually be defined to be a space of functions satisfying a symmetry property. The fact that it is invariant under f reflects the fact that f also has these symmetries. The conclusion then is, (with the appropriate restrictions on f), that all solutions also have these symmetries.

Proof of Theorem II.6. Let P be orthogonal projection on H_1. Then equation (II.11) is equivalent to the pair of equations (we suppress the x term in f).

$$(I-P)Lu = (I-P)f(u), \tag{II.12}$$

$$PLu = Pf(u), \tag{II.13}$$

which we rewrite, letting $v = Pu$, $w = (I-P)u$, as

$$\begin{array}{ll} \text{(i)} & Lw = (I-P)f(v+w), \\ \text{(ii)} & Lu = Pf(v+w). \end{array} \tag{II.14}$$

(Note that the assumption H_1 guarantee that P commutes with L.) Rewrite (II.14.i) as

$$(L-\gamma)w = (I-P)(f(v+w)-\gamma(v+w))$$

where $\gamma = \frac{a+b}{2}$.

Note that, by the assumptions on $\frac{\partial f}{\partial u}$, the function $g(u) = f(u) - \gamma u$ must be globally Lipschitz with Lipschitz constant $M \leq \frac{b-a}{2}$. Therefore we can conclude that

$$\|f(v+w_1)-\gamma(v+w_1)-(f(v+w_2)-\gamma(v+w_2))\| \leq \frac{b-a}{2}\|w_1-w_2\|$$

for all $w \in (I-P)H$.

Also, if $L_2 = L(I-P)$ is regarded as an operator on $H_2 = (I-P)H$, then since the spectrum of L_2 is outside the interval $[a,b]$, it follows that the operator norm $\|(L_2-\gamma I)^{-1}\|$ is strictly less than $\left(\frac{b-a}{2}\right)^{-1}$.

Thus the map

$$w \longrightarrow (L_2-\gamma I)^{-1}(I-P)(f(v+w)-\gamma(v+w))$$

is a contraction on the space H_2, and we can conclude that for each fixed v, there exists a unique $w(v)$ which satisfies (II.14.i).

Now observe that $w(v) = 0$ satisfies (II.14.i), for any choice of v. This is true since $(I-P)f(v) = 0$ since $f(v) \in H_1$ if $v \in H_1$. Therefore, every solution of (II.11) must be a solution of the equation

$$Lv = PNv, \quad \text{where } Nv = f(v), \quad \text{and } v \in H_1$$

from which we conclude that all solutions must be in H_1. This concludes the proof of the theorem. ∎

REMARKS. The same method proves a similar theorem for normal operators. We replace assumption **A1** by

A1′ The closed disk, centered at $\left(\frac{a+b}{2}, 0\right)$ with radius $\frac{b-a}{2}$, does not intersect the spectrum of the operator L restricted to the invariant subspace H_2.

A2′ $H_1 = H_2^{\perp}$ is invariant under L and the nonlinear map $u \longrightarrow f(u)$. These assumptions are enough to ensure that the theorem is true if L is normal.

We now proceed to some simple applications.

Some immediate corollaries.

a) Even-odd symmetries.

We consider first the ordinary differential equations

$$u'' + f(u) = h(x), \qquad u'(0) = u'(\pi) = 0. \tag{II.15}$$

EXAMPLE 1. If $h(x) \equiv c$ (a constant) and $f'(s) < 1$ for all s, then the only solutions of (II.15) are constants.

Proof. Take $PH = H_1$ to be the constant functions. ∎

EXAMPLE 2. If $h(x) = h\left(\frac{\pi}{2} - x\right)$ on $\left(0, \frac{\pi}{2}\right)$ and $(2n-1)^2 < f'(u) < (2n+1)^2$, then all solutions u of (II.15) satisfy $u(x) = u\left(\frac{\pi}{2} - x\right)$ for $x \in \left(0, \frac{\pi}{2}\right)$.

Proof. Take $H_1 = \text{span}\{1, \cos 2x, \cos 4x, \ldots\}$. This space is invariant under f and the spectrum of L_2 is $\{1, 3^2, 9^2, \ldots\}$. Clearly H_1 and H_2 are invariant under L and H_1 is invariant under any map of the form $u \longrightarrow f(u)$ (or $u \longrightarrow f(u,x)$ with $f\left(u, \frac{\pi}{2} - x\right) = f(u,x)$ for all $x \in \Omega,\ u \in (-\infty, \infty)$). ∎

EXAMPLE 3. If $h(x) = -h\left(\frac{\pi}{2} - x\right)$ on $\left(0, \frac{\pi}{2}\right)$ and $(2n)^2 < f'(u) < (2n+2)^2$ and $f(u)$ is odd about $\frac{\pi}{2}$, then all solutions of (II.15) satisfy $u(x) = -u\left(\frac{\pi}{2} - x\right)$.

Proof. Take $H_1 = \text{span}\{\cos x, \cos 3x, \ldots\}$. Of course this is invariant under L and f, provided that f is odd in u (and x). ∎

Examples 2 and 3 have the obvious analogues for the Dirichlet problem.

II.4 The suspension bridge equation

In this section, we develop some of the ideas from the previous section for a different type of equation, that of a non-linearly supported vibrating beam.

We consider a one-dimensional beam of length L suspended by cables. When the cables are stretched, there is a restoring force which is assumed to be proportional to the amount of the stretching (Hooke's law). But when the beam moves in the opposite direction, then there is no restoring force exerted on it.

If $u(x,t)$ denotes the displacement in the downward direction at position x and time t, then a simplified model is given by the equations

$$\begin{aligned} u_{tt} + K_1 u_{xxxx} + K_2 u^+ &= W(x) + \epsilon f(x,t), \\ u(0,t) = u(L,t) = 0, \quad u_{xx}(0,t) &= u_{xx}(L,t) = 0. \end{aligned} \tag{II.16}$$

Here $W(x)$ is the weight per unit length at x, and $f(x,t)$ is an externally imposed periodic function. In [A2] these equations were considered under the assumption that $W(x) = W_0 \sin \frac{\pi x}{L}$, which allowed the partial differential equation to be reduced by separation of variables to an ordinary differential equation. It was then shown that, if K_2 is large, then large numbers of highly oscillatory solutions could exist.

The purpose of this paper is to study periodic solutions of Problem (II.16) under the more realistic assumption that the weight per unit length is constant, $W(x) = W_0$, $0 < x < L$.

By obvious changes of variables, Problem (II.16) can be reduced to

$$\begin{aligned} u_{tt} + u_{xxxx} + bu^+ &= 1 + \epsilon h(x,t) \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbf{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) &= u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0. \end{aligned} \tag{II.17}$$

We shall assume that h is even in x and t and periodic with period π, and we shall look for π-periodic solutions of (II.17). Our methods apply to more general restoring forces (instead of bu^+) and weight functions (instead of w =const.), but we shall not elaborate on this.

Part of our analysis concerns the steady-state case of (II.17). A positive force c produces a steady-state deflection $w(x)$ satisfying

$$w^{(4)} + bw^+ = c \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$w\left(\pm\frac{\pi}{2}\right) = w''\left(\pm\frac{\pi}{2}\right) = 0.$$

Physical intuition suggests that w is positive, as the beam deflects under the load, and one would also expect that the externally imposed

π-periodic force $\epsilon h(x,t)$ produces small oscillations of the order of magnitude ϵ around the steady-state solution. We shall demonstrate that for certain ranges of b additional highly oscillatory π-periodic solutions, which change sign, also exist.

The mathematical situation is now somewhat similar to that of the semilinear wave equation. There are two principal differences. First, because of the additional symmetries, the inverse of the linear operator is now compact. Second, the nonlinearity is not crossing the eigenvalue with the positive eigenfunction $(-1, \sin \pi x)$ so we have to use a different technique to obtain an *a priori* bound. The reader will be struck by how similar the rest of the proof is.

Let L be the differential operator

$$Lu = u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u = u(x,t)$

$$\begin{gathered} Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbf{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi) \end{gathered} \tag{II.18}$$

(the last line indicates that u is even in t and x and π-periodic in t) has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\lambda_{mn} = (2n+1)^4 - 4m^2, \quad \phi_{mn} = \cos 2mt \cdot \cos(2n+1)x \quad (m, n = 0, 1, 2, \ldots).$$

We remark that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17. \tag{II.19}$$

The normalized eigenfunctions are denoted by

$$\theta_{mn} = \frac{\phi_{mn}}{\|\phi_{mn}\|}, \quad \text{where } \|\phi_{mn}\| = \frac{\pi}{2} \quad \text{for } m > 0, \quad \|\phi_{0n}\| = \frac{\pi}{\sqrt{2}}.$$

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H the Hilbert space defined by

$$H = \{u \in L_2(Q) : u \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\theta_{mn}\}$ is an orthonormal base in H.

We consider weak solutions of problems of the type

$$\begin{aligned} u_{tt} + u_{xxxx} &= f(u, x, t) \quad \text{in} \quad \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbf{R}, \\ u\left(\pm\tfrac{\pi}{2}, t\right) &= u_{xx}\left(\pm\tfrac{\pi}{2}, t\right) = 0, \end{aligned} \tag{II.20}$$

$$u \text{ even and } \pi\text{-periodic in } t \text{ and even in } x.$$

A weak solution of (II.20), which is also called a solution in H, is of the form

$$u = \sum c_{mn}\theta_{mn} \quad \text{with} \quad Lu = \sum \lambda_{mn}c_{mn}\theta_{mn} \in H,$$

i.e., with $\sum c_{mn}^2\lambda_{mn}^2 < \infty$, which implies $u \in H$. Our functions f will be such that $u \in H$ implies $f(u, x, t) \in H$. For simplicity of notation, a weak solution of (II.20) is characterized by

$$Lu = f(u, x, t) \quad \text{in} \quad H. \tag{II.20$'$}$$

Our main result is the following:

THEOREM II.7 *Let $h \in H$, $\|h\| = 1$ and $3 < b < 15$. Then there exists $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ the equation*

$$Lu + bu^+ = 1 + \epsilon h \quad \text{in} \quad H \tag{II.21}$$

has at least two solutions.

First we use the symmetry theorem of Section 3 to obtain an *a priori* bound.

LEMMA II.5 *For $-1 < b < 15$ the problem*

$$Lu + bu^+ = 0 \quad \text{in} \quad H \tag{II.22}$$

has only the trivial solution $u = 0$.

Proof. The space $H_1 = \text{span}\{\cos x \cos 2mt; m \geq 0\}$ is invariant under L and under the map $u \mapsto bu^+$. The spectrum σ_1 of L restricted to H_1 contains $\lambda_{10} = -3$ and does not contain any other point in the interval $(-15, 1)$. The spectrum σ_2 of L restricted to $H_2 = H_1^\perp$ does not intersect the interval $(-15, 1)$. From Theorem II.7, we conclude that any solution of (II.22) belongs to H_1, *i.e.*, it is of the form $y(t)\cos x$, where y satisfies

$$y'' + by^+ + y = 0.$$

Any nontrivial periodic solution of this equation is periodic with period

$$\frac{\pi}{\sqrt{b+1}} + \frac{\pi}{1} \neq \pi.$$

This shows that there is no nontrivial solution of (II.22). ∎

The next lemma establishes *a priori* bounds for solutions of (II.21).

LEMMA II.6 *Let $h \in H$ with $\|h\| = 1$ and $\alpha > 0$ be given. There exists $R_0 > 0$ (depending only on h and α) such that for all b with $-1 + \alpha \leq b \leq 15 - \alpha$ and all $\epsilon \in [-1, 1]$, the solutions u of* (II.21) *satisfy $\|u\| < R_0$.*

Proof. If not, then there exists a sequence (b_n, ϵ_n, u_n) with $b_n \in [\alpha - 1, 15 - \alpha]$, $|\epsilon_n| \leq 1$, $\|u_n\| \longrightarrow \infty$ such that

$$u_n = L^{-1}(1 - b_n u_n^+ + \epsilon_n h).$$

The functions $w_n = \frac{u_n}{\|u_n\|}$ satisfy the equation

$$w_n = L^{-1}\left(\frac{1}{\|u_n\|} - b_n w_n^+ + \frac{\epsilon_n}{\|u_n\|}h\right).$$

Now L^{-1} is a compact operator. Therefore we may assume that $w_n \longrightarrow w_0$ and $b_n \longrightarrow b_0 \in (-1, 15)$. Since $\|w_n\| = 1$ it follows that $\|w_0\| = 1$ and

$$w_0 = L^{-1}(-b_0 w_0^+) \quad \text{or} \quad Lw_0 + b_0 w_0^+ = 0 \quad \text{in } H.$$

This contradicts Lemma II.5 and proves the lemma. ■

LEMMA II.7 *Under the assumptions and with the notations of* Lemma II.6,

$$d_{LS}(u - L^{-1}(1 - bu^+ + \epsilon h), B_R, 0) = 1$$

for all $R \geq R_0$, where d_{LS} denotes the Leray-Schauder degree.

Proof. If $b = 0$, we have

$$d_{LS}(u - L^{-1}(1 + \epsilon h), B_R, 0) = 1,$$

since the map is simply a translation of the identity and since $\|L^{-1}(1 + \epsilon h)\| < R_0$ by Lemma II.6. ■

The general result follows in the usual way by invariance under homotopy, since all solutions are in the open ball B_{R_0}.

Now we turn attention to the steady-state solutions of equation (II.21). We need the following somewhat surprising result of Schröder [Sch].

THEOREM II.8 *The Green's function for the boundary value problem*

$$y^{(4)} + by = f(x) \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

is nonnegative if and only if $-1 < b \le c_0 = \frac{4\kappa^4}{\pi^4}$, *where* κ *is the smallest positive zero of the function* $\tan x - \tanh x$. *We have* $\kappa = 3.9266$ *and* $c_0 = 9.762$.

This result implies that for a positive right hand side f the solution y is nonnegative if $b \le c_0$, while for $b > c_0$ there are positive functions f for which the solution takes also negative values. But for constant f we have the following result.

LEMMA II.8 *For all* $b > -1$, *the unique solution* y *of the boundary value problem*

$$y^{(4)} + by = 1 \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

is positive in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *and even in* x.

Proof. For $b \leq c_0$ the result follows from Theorem II.8. We give a proof which is valid for $b \geq 4$. It is convenient to write $b = 4\beta^4$, $a = \frac{\pi}{2}\beta$ and to introduce a function

$$z(x) = \frac{1}{4\beta^4} - y\left(\frac{x}{\beta}\right).$$

The function y is a solution of (II.24) if and only if z satisfies

$$z^{(4)} + 4z = 0 \quad \text{in} \quad |x| \leq a, \qquad z(\pm a) = \frac{1}{4\beta^4}, \qquad z''(\pm a) = 0 \quad (\text{II.24}')$$

and the inequality $y > 0$ is equivalent to $z(x) < z(\pm a)$ for $|x| < a$. The solution z of (II.24′) is explicitly given (up to a positive constant) by

$$z(x) = sSs_0S_0 + cCc_0C_0,$$

where

$$s = \sin x,\ S = \sinh x,\ c = \cos x,\ C = \cosh x,\ s_0 = \sin a,\ S_0 = \sinh a, \ldots.$$

We assume that $a \geq \frac{\pi}{2}$ (that is $b \geq 4$). For $0 \leq x \leq a - \frac{\pi}{2}$ we have $z(a) \geq S_0^2$ and $z(x) \leq CC_0$ since $|ss_0| + |cc_0| = |\cos(a \pm x)| \leq 1$; hence the inequality $z(x) < z(a)$ follows from

$$CC_0 < S_0^2 \iff 1 + CC_0 < C_0^2 \iff \frac{1}{C_0} < C_0 - C.$$

This is certainly true since $C_0 - C \geq \cosh\frac{\pi}{2} - 1 > 1.5$.

In order to prove that $z(x) < z(a)$ holds also for $a - \frac{\pi}{2} < x < a$, we show that z' is positive in this interval. We get

$$\begin{aligned}
z' &= (cS + sC)s_0S_0 + (-sC + cS)c_0C_0 \\
&= SS_0(cs_0 - sc_0 + ss_0 + cc_0) + Se_0cc_0 + S_0ess_0 - sc_0(eS_0 + e_0S + ee_0) \\
&= SS_0(\sin(a - x) + \cos(a - x)) + Se_0(c - s)c_0 + S_0e(s_0 - c_0)s - e_0esc_0 \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

Here we have used the notation $e = e^{-x}$, $e_0 = e^{-a}$ and the equations $S + e = C$, $S_0 + e_0 = C_0$. For $0 < a - x < \frac{\pi}{2}$ the first term A_1 is $> SS_0$. Furthermore

$$\begin{aligned} A_3 &\geq 0 \quad \text{for } \tfrac{\pi}{2} \leq a \leq \pi, \\ |A_3| &\leq \sqrt{2} e S_0 \leq 0.294 S_0 \quad \text{for } a \geq \pi, \\ |A_2| &\leq e_0 S \leq 0.208 \quad \text{for } \tfrac{\pi}{2} \leq a \leq \tfrac{3}{4}\pi, \\ A_2 &\geq 0 \quad \text{for } \tfrac{3}{4}\pi \leq a \leq \tfrac{5}{4}\pi, \\ |A_2| &\leq \sqrt{2} e_0 S \leq 0.0285 \quad \text{for } a \geq \tfrac{5}{4}\pi, \\ A_4 &\geq 0 \quad \text{for } \tfrac{\pi}{2} \leq a \leq \pi, \\ |A_4| &\leq e e_0 \leq 0.01 \quad \text{for } a \geq \pi. \end{aligned}$$

These inequalities show that indeed $z' > 0$ for $a - x \leq \frac{\pi}{2}$, $a \geq \frac{\pi}{2}$. Hence Lemma II.8 is proved. ■

REMARK. The case $0 \leq b < 4$ can be dealt with directly by using the decomposition

$$y^{(4)} + 4\beta^4 y = (D^2 + \sqrt{2}\beta D + 2\beta^2)(D^2 - \sqrt{2}\beta D + 2\beta^2)y.$$

It is easy to see, by calculating the first eigenvalues of $y'' \pm \sqrt{2}\beta y' + 2\beta^2 y = 0$ with Dirichlet boundary conditions, that the inverse of each of these operators remains positive if $0 \leq \beta < 1$, that is, if $0 \leq b < 4$. The case $-1 < b \leq 0$ is simple because the inverse of $M = D^4$ with the given boundary values is a positive operator, hence the right hand side of the equivalent equation $y = M^{-1}(1 - by)$ is increasing and contractive.

An easy consequence of Lemma II.8 is

LEMMA II.9 *For $-1 < b$ the boundary value problem*

$$y^{(4)} + by^+ = 1 \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0 \qquad \text{(II.25)}$$

has a unique solution y, which is even and positive and satisfies

$$y'\left(-\frac{\pi}{2}\right) > 0 \quad \textit{and} \quad y'\left(\frac{\pi}{2}\right) < 0.$$

Proof. The solution y of the linear problem (II.24) is positive, hence it is also a solution of (II.25). Uniqueness follows from the contraction principle in the following familiar way. The eigenvalues of $My = \lambda y$, where $M = D^4$ with the boundary conditions given in (II.24), are all ≥ 1. Hence, for any $c < 1, \|(M-c)^{-1}\| = \frac{1}{1-c}$. Any problem $My = f(y,x,t)$ with $c \leq f_y \leq 1-\epsilon$ has a unique solution, since solutions y are characterized by $y = (M-c)^{-1}[f(y,x,t) - cy]$, where the right hand side is Lipschitz continuous with a Lipschitz constant $\leq \frac{1-\epsilon-c}{1-c} < 1$. ∎

In passing from the unperturbed equation ($\epsilon = 0$) to the perturbed equation (II.21), the following lemma is needed, which was first used in [L-McK7].

LEMMA II.10 *Let K be a compact set in $L_2 = L_2(\Omega)$, and let $\phi \in L_2$ be positive almost everywhere. Then there exists a modulus of continuity δ depending only on K and ϕ such that*

$$\|(\eta|\psi| - \phi)^+\| \leq \eta\delta(\eta) \quad \textit{for } \eta > 0 \quad \textit{and} \quad \psi \in K.$$

The following short proof of this lemma was suggested by R. Lemmert of Karlsruhe. The function $f_n : K \longrightarrow \mathbf{R}$ given by

$$f_n(\psi) = \|(|\psi| - n\phi)^+\| \quad (\psi \in K; n = 1, 2, 3, \ldots)$$

are continuous, and the sequence (f_n) is decreasing with $\lim f_n(\psi) = 0$ (pointwise) for every $\psi \in K$. It follows from Dini's theorem that the convergence is uniform. Therefore the function $\delta(\eta)$ defined by

$$\delta(\eta) = \max_{\psi \in K} \left\| \left(|\psi| - \frac{\phi}{\eta} \right)^+ \right\|$$

is increasing in η; furthermore $\delta\left(\frac{1}{n}\right) \longrightarrow 0$, which shows that δ is a modulus of continuity.

Now we come to the key step in the proof of our theorem. The reader will notice that it is very similar to the proof of Theorem I.8.

LEMMA II.11 *Assume that* $3 < b < 15$. *Then there exist* $\gamma > 0, \epsilon_0 > 0$ *such that*

$$d_{LS}(u - L^{-1}(1 - bu^+ + \epsilon h), B_\gamma(y), 0) = -1$$

for $|\epsilon| < \epsilon_0$, *where* y *is the unique positive solution of* (II.25).

Proof. Let K be the closure of $L^{-1}(B)$, where B is the closed unit ball in H. Clearly K is compact. Let $\delta(\eta)$ be the modulus of continuity corresponding to K and y (see the preceding lemma). We note that $\|L^{-1}\| = 1$. Let u be a solution of (II.21). Using the notation $u = y + \phi$ and $\|\phi\| = \gamma$, we see easily that

$$L\phi = \epsilon h + by - b(y + \phi)^+ = \epsilon h - b\phi - b(y + \phi)^-; \qquad \text{(II.26)}$$

here we have used the identity $u = u^+ - u^-$. It follows that

$$\phi \in (\epsilon_0 + 2b\gamma)K \quad \text{for} \quad |\epsilon| < \epsilon_0. \qquad \text{(II.27)}$$

We assume that $\epsilon_0 \leq \gamma$. Then $\psi = \frac{\phi}{\gamma}$ has the properties $\|\psi\| = 1$ and $\psi \in (2b+1)K$. Since ψ is in a compact set and different from zero and since $-b$ is not an eigenvalue of L, we get

$$\inf_{\psi} \|\psi + L^{-1}b\psi\| = \alpha > 0;$$

hence $\|\phi + L^{-1}b\phi\| \geq \alpha\gamma$. It follows from (II.26) that

$$\phi + L^{-1}b\phi = L^{-1}(\epsilon h - b(y+\phi)^-). \tag{II.28}$$

Since $\omega \in K$ satisfies $\|(\eta\omega - y)^+\| \leq \eta\delta(\eta)$, we get from (II.27)

$$\|(\phi + y)^-\| = \|(-\phi - y)^+\| \leq (\epsilon_0 + 2b\gamma)\delta(\epsilon_0 2b\gamma).$$

Denoting the two sides of equation (II.28) by S_l and S_r and keeping in mind that $\|L^{-1}\| = 1$, we get for $\epsilon_0 \leq \gamma \min\left(1, \frac{\alpha}{2}\right)$

$$\|S_l\| \geq \alpha\gamma \quad \text{and} \quad \|S_r\| \leq \frac{1}{2}\alpha\gamma + (2b\gamma + \frac{1}{2}\alpha\gamma)\delta(2b\gamma + \frac{1}{2}\alpha\gamma).$$

Now we choose $\gamma > 0$ so small that the right-hand side is $< \alpha\gamma$. It follows that for this value of γ there is no solution of (II.21) of the form $u = y + \phi$ with $\|\phi\| = \gamma$.

The same conclusion holds for solutions $u = y + \phi$ of the equation

$$Lu + bu = 1 + \lambda(\epsilon h - bu^-), \quad \text{where } 0 \leq \lambda \leq 1.$$

Here, $\lambda = 1$ gives equation (II.21), while for arbitrary λ the function $\phi = u - y$ satisfies (II.28) with a factor λ on the right side. Hence we have the same conclusion: There is no solution $u = y + \phi$ with $\|\phi\| = \gamma$. Since the degree is invariant under a homotopy, we get

$$d_{LS}(u - L^{-1}(1 - bu^+ + \epsilon h), B_\gamma(y), 0) = d_{LS}(u - L^{-1}(1 - bu), B_\gamma(y), 0).$$

The equation $u - L^{-1}(1 - bu) = 0$ has the unique solution $u = y$, and the degree on the right-hand side is equal to

$$d_{LS}(u + L^{-1}bu, B_\gamma(0), 0).$$

The eigenvalues ρ of the operator $u + L^{-1}bu$ are connected with the eigenvalues λ of L by

$$u + L^{-1}bu = \rho u \Longleftrightarrow Lu = \frac{b}{\rho - 1}u$$

or $\rho = 1 + \frac{b}{\lambda}$. It follows from (II.19) that there is just one negative eigenvalue ρ which corresponds to $\lambda_{10} = -3$. Thus the usual method of approximating on finite-dimensional subspaces spanned by eigenvectors with dimension going to infinity (see [N]) shows that the desired degree is -1, as asserted in Lemma II.11.

Proof of Theorem II.8. Equation (II.21) can be written in the form

$$Su := u - L^{-1}(1 - bu^+ + \epsilon h) = 0.$$

The degree of Su on a large ball of radius $R > R_0$ is $+1$ by Lemma II.7. We know from Lemma II.11 that the degree on the ball $B_\gamma(y)$ is -1. Choosing $R > R_0$ so large that $B_\gamma(y) \subset B_R$, we can conclude that

$$d_{LS}(Su, B_R \backslash B_\gamma(y), 0) = 2.$$

Therefore, equation (II.21) has at least two solutions, one in $B_\gamma(y)$ and one in $B_R \backslash B_\gamma(y)$. This concludes the proof of Theorem II.18. ∎

It is natural to ask if the phenomenon of multiple solutions continues to occur for large values of b. The next theorem gives a partial answer to this question.

THEOREM II.9 *Suppose the eigenvalue* λ_{m0} *is simple and* λ' *and* λ'' *are the nearest eigenvalues to the left and right of* λ_{m0}, *respectively. Then equation* (II.21) *has at least two solutions for* b *in* (λ', λ_{m0}), *or it has at least two solutions for* b *in* $(\lambda_{m0}, \lambda'')$.

Proof. The proof of Theorem II.9 runs parallel to that of Theorem II.8. Lemma II.5 and Lemma II.6 hold for $\lambda' < b < \lambda''$, and the proofs carry over. Instead of Lemma II.7 we can conclude, by virtue of the a priori bound, that the degree given there is defined and constant for $\lambda' < b < \lambda''$. The reasoning of Lemma II.11 is valid in each of the intervals (λ', λ_{m0}) and $(\lambda_{m0}, \lambda'')$. The degree is $+1$ on one of these intervals and -1 in the other interval. Hence, for b in one of these intervals, we have the same situation as in the proof of Theorem III.6, namely different degrees on a large ball and a small ball contained in the large ball. ∎

It is still an *open problem* as to whether at least two solutions exist for $b > 15$. One can also ask whether *more* solutions exists, if $b > 15$.

We remark that considerably more is known in the case that the load is not constant, but is distributed as $W(x) = W_0 \cos x$. In particular, many additional solutions are obtained for all positive values of b. In addition, stability results are established in [L-McK6].

It would be interesting to include viscous damping in equation (II.16). The methods of this paper do not seem to apply to this case as H is not invariant under change of the damping term. Clearly, some additional hypotheses on $h(x,t)$ are required.

We also remark that the linear problem

$$\begin{aligned} \Delta^2 u + bu &= f(x) && \text{in} \quad \Omega, \\ u &= \Delta u = 0 && \text{on} \quad \partial\Omega \end{aligned}$$

needs more study. Namely, we should be able to estimate the value $c(\Omega)$ with the property that for $0 \leq b < c(\Omega)$ the inverse operator is positivity preserving (we remind that in the one-dimensional case Schröder's Theorem II.8 gives a complete answer to this question).

In [McK-W1], it was *conjectured* that this constant $c(\Omega)$ is largest among all regions of a given volume, when Ω is a ball.

Chapter III

The contraction mapping theorem and finite dimensional problems

In this chapter, we begin the study of the proof of the existence of at least four solutions, and if $f'(+\infty) > \lambda_3$, of more than four solutions. The flavor of this chapter is fairly abstract, insofar as we shall be making few assumptions on the operator A in the equation

$$Au + f(u) = t\phi_1 + h, \qquad h \in L^2(\Omega). \tag{III.1}$$

Of course the reader will think of the Laplacian, with Dirichlet boundary conditions on a bounded region in $\mathbf{R}^n$. Throughout the chapter, A is a operator from $L^2(\Omega) \longrightarrow L^2(\Omega)$ with compact inverse, with eigenvalues $-\lambda_i$, each repeated as often as its multiplicity, with $0 < \lambda_1 < \lambda_2 \cdots \leq \lambda_i \leq \cdots \longrightarrow +\infty$. We assume the first eigenfunction ϕ_1 satisfies $\phi_1(x) > 0$ for all $x \in \Omega$ and that there exists ϵ_0 so that $\phi_1(x) \pm \epsilon_0\phi_2(x) \geq 0, \ \ \forall x \in \Omega$.

The plan of the chapter is as follows. In section 1, we prove the existence of at least four solutions to the piecewise linear equation. In section 2, we extend the results to the nonlinear equation, proving that if $f'(-\infty) > \lambda_1$, $\lambda_2 < f'(+\infty) < \lambda_3$, then for large t, (III.1) has at least four solutions. In section 3, we consider the case where the multiplicity of λ_3 in odd, and produce five (and generically six solutions).

III.1 The piecewise linear case

THEOREM III.1 *If* $t > 0,\ a < \lambda_1,\ \lambda_2 < b < \lambda_3$, *then the equation*

$$Au + bu^+ - au^- = t\phi_1 \tag{III.2}$$

has at least four solutions.

Proof. The general plan of the proof is somewhat similar to that of Theorem I.5. We use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $L^2(\Omega)$ to a finite dimensional one. Last time it was the one dimensional subspace spanned by ϕ_1. This time it is the two dimensional subspace spanned by $\{\phi_1, \phi_2\}$.

Thus if $g \in L^2(\Omega)$, then $g = \sum_{n=1}^{\infty} c_n\phi_n$, where $\int |g|^2 = \sum c_n^2$. Let $Pg = c_1\phi_1 + c_2\phi_2$, so that P is orthogonal projection on the space spanned by ϕ_1, ϕ_2, and of course, since $Ag = -\sum \lambda_n c_n \phi_n$, we have that P and $I-P$ commute with A. Thus, letting $u = Pu+(I-P)u = v+w$, where $v = Pu,\ w = (I-P)u$, we have that (III.2) is equivalent to

$$\begin{array}{lrcl} \text{(i)} & Aw + (I-P)(b(v+w)^+ - a(v+w)^-) & = & 0, \\ \text{(ii)} & Av + P(b(v+w)^+ - a(v+w)^-) & = & t\phi_1. \end{array} \tag{III.3}$$

We look on this as a system of two equations in the two unknowns v and w.

Let us show that for fixed v, (III.3.i) has a unique solution $w = \theta(v)$, and that, furthermore, $\theta(v)$ is continuous in terms of v. This step is similar to the proof of Theorem I.1, and the reader should notice that an abstract version of (I.1) would prove this.

Choose $\mu > 0, \epsilon > 0$ so that $-\mu + \epsilon < a, b < \lambda_3 - \epsilon$. Let $\gamma = \frac{\lambda_3 - \mu}{2}$. Write (III.3.i) as

$$(A-\gamma)w = -(I-P)(b(v+w)^+ - a(v+w)^- - \gamma(v+w))$$

or equivalently

$$w = (A-\gamma)^{-1}(I-P)g_v(w),$$

where

$$g_v(w) = b(v+w)^+ - a(v+w)^- - \gamma(v+w).$$

An elementary calculation gives that

$$|g_v(w_1) - g_v(w_2)| \leq \left(\frac{\lambda_3+\mu}{2} - \epsilon\right) |w_1 - w_2|$$

since $|a-\gamma|, |b-\gamma| \leq \left(\frac{\lambda_3+\mu}{2} - \epsilon\right)$. Thus we can conclude that, if $\| \ \|$ is the norm in $L^2(\Omega)$, $\|g_v(w_1) - g_v(w_2)\| \leq \left(\frac{\lambda_3+\mu}{2} - \epsilon\right) \|w_1 - w_2\|$ as in the proof of Theorem I.1, it is easy to check that $\|(A-\gamma)^{-1}(I-P)\| = \frac{2}{\lambda_3+\mu}$, since the operator $(A-\gamma)^{-1}(I-P)$ has eigenvalues $0, 0, (-\lambda_3-\mu)^{-1}, (-\lambda_4-\mu)^{-1}$ and the largest of these is, in absolute value $(\lambda_3+\mu)^{-1}$. Therefore, for each fixed $v \in PH$, we have that the map $T_v : (I-P)H \longrightarrow (I-P)H$ given by

$$T_v(w) = -(A-\gamma)^{-1}(I-P)(b(v+w)^+ - a(v+w)^- - \gamma(v+w))$$

satisfies

$$\|T_v(w_1) - T_v(w_2)\| \leq \left(\frac{\lambda_3+\mu}{2}\right)^{-1} \left(\frac{\lambda_3+\mu}{2} - \epsilon\right) \|w_1 - w_2\|$$

and thus is a contraction, with constant $1 - \frac{2\epsilon}{\lambda_3+\mu}$. Thus there exists a unique fixed point $\theta(v)$ satisfying (III.3.i). It is left as an exercise to verify that $\theta(v)$ is Lipschitz continuous, with constant $\frac{c}{1-c}$.

Therefore we have reduced the Problem III.2 to the study of an equivalent problem

$$Av + P(b(v+\theta(v))^{+} - a(v+\theta(v))^{-}) = s\phi_1 \qquad \text{(III.4)}$$

defined on the two dimensional subspace PH spanned by $\{\phi_1, \phi_2\}$.

While one feels instinctively that (III.4) ought to be easier to solve, there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some very important v's.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (III.3.i) reduces to

$$A0 + (I-P)(bv^{+} - av^{-}) = 0$$

which is satisfied because $v^{+} = v$, $v^{-} = 0$ and $(I-P)v = 0$, since $v \in PH$.

Since $v = c_1\phi_1 + c_2\phi_2$, there exists a cone C_1 defined by $c_1 \geq 0, |c_2| \leq \epsilon_0 c_1$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_2, $c_1 \leq 0$, $|c_2| \leq \epsilon_0 |c_1|$ so that $v \leq 0$ for all $v \in C_2$.

Thus, we do not know $\theta(v)$ for all $v \in PH$, but we know $w \equiv 0$ for $v \in C_1 \cup C_2$, and we need to study the map

$$v \longrightarrow \Phi(v) = Av + P(b(v+\theta(v))^{+} - a(v+\theta(v))^{-}).$$

First we consider the image of the cone C_1. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\begin{aligned}\Phi(v) &= -\lambda_1 c_1\phi_1 - \lambda_2 c_2\phi_2 + b(c_1\phi_1 + c_2\phi_2) \\ &= (b-\lambda_1)c_1\phi_1 + (b-\lambda_2)c_2\phi_2.\end{aligned}$$

Thus the images of the rays $c_1\phi_1 \pm \epsilon_0 c_1\phi_2$ can be explicitly calculated and they are

$$(b-\lambda_1)c_1\phi_1 \pm (b-\lambda_2)\epsilon_0 c_1\phi_2,$$

or in other words the rays

$$d_1\phi_1 \pm \epsilon_0\left(\frac{b-\lambda_2}{b-\lambda_1}\right) d_1\phi_1.$$

Thus Φ maps C_1 into the cone

$$D_1 = \left\{ d_1\phi_1 + d_2\phi_2,\ d_1 \geq 0,\ |d_2| \leq \epsilon_0\left(\frac{b-\lambda_2}{b-\lambda_1}\right)\right\}$$

and we can represent this in figure 1.

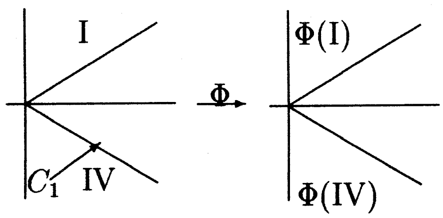

Figure 1.

Similarly for C_2 we can explicitly calculate the image under Φ. If $c \leq 0$,

$$\Phi(c_1\phi_1 \pm \epsilon_0 c_1\phi_2) = (a-\lambda_1)c_1\phi_1 \pm \epsilon_0(a-\lambda_2)c_1\phi_2$$

and we have figure 2.

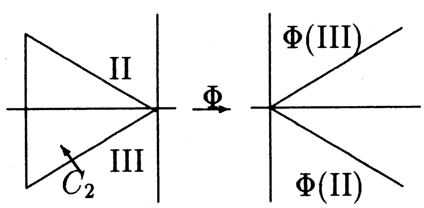

Figure 2.

Thus, $\Phi(v) = t\phi_1$ has one solution in each of the cones C_1, C_2, namely $\frac{t\phi_1}{b-\lambda_1}, \frac{t\phi_1}{a-\lambda_1}$. At this stage we need a lemma.

LEMMA III.1 *There exists $d > 0$ so that*

$$(\Phi(c_1\phi_1 + c_2\phi_2), \phi_1) \geq d|c_2|. \tag{III.5}$$

Proof. Let us write $f(u) = bu^+ - au^-$ for brevity. Then

$$\Phi(c_1\phi_1 + c_2\phi_2) = A(c_1\phi_1 + c_2\phi_2) + P(f(c_1\phi_1 + c_2\phi_2 + \theta(c_1, c_2)).$$

So if $u = c_1\phi_1 + c_2\phi_2 + \theta(c_1, c_2)$, then

$$(\Phi(c_1\phi_1 + c_2\phi_2), \phi_1) = ((A + \lambda_1)(c_1\phi_1 + c_2\phi_2), \phi_1) + (f(u) - \lambda_1 u, \phi_1).$$

The first term is zero because $(A + \lambda_1)\phi_1 = 0$ and A is self-adjoint. The second term satisfies $f(u) - \lambda_1 u \geq \gamma|u|$, where $\gamma = \min\{b - \lambda_1, \lambda_1 - a\} > 0$. Therefore $(\Phi(c_1\phi_1 + c_2\phi_2),\ \phi_1) \geq \gamma \int |u|\phi_1$. Now there exists $d > 0$ so that $\gamma\phi_1 \geq d|\phi_2|$ and therefore

$$\gamma \int |u|\phi_1| \geq d \int |u||\phi_2| \geq d| \int u\phi_2| = d|(u, \phi_2)|$$

which concludes the proof of the lemma. ∎

We are now in a position to describe the behavior of Φ in the complement of the two cases C_1 and C_2. Let us consider the image under Φ of $c_1\phi_1 + c_2\phi_2$ with $c_2 \geq \epsilon|c_1|$, $c_2 = k$ for some $k > 0$.

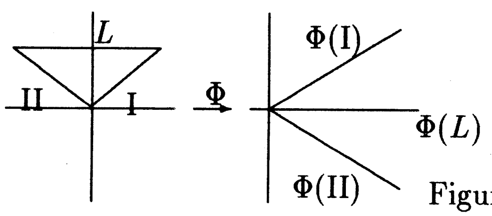

Figure 3.

The lemma tells us that the image $\Phi(L)$ of $c_2 = k, |c_1| \leq \frac{1}{\epsilon}k$ must lie to the right of the line $c_1 = dk$, and must therefore cross the positive ϕ_1 axis in the image space.

Thus we have shown that if $u = c_1\phi_1 + k\phi_2 + \theta(c_1, k)$, $k > 0$, $|c_1| \leq \left(\frac{k}{\epsilon}\right)$, then u satisfies, for some c_1, $Au + bu^+ - au^- = s\phi_1$ for some $s > dk$ and k positive. Letting $\tilde{u} = \left(\frac{t}{s}\right) u$, we see that $\tilde{u}$ satisfies

$$A\tilde{u} + b\tilde{u} - a\tilde{u} = t\phi_1.$$

Similarly one shows the existence of another solution $\underset{\sim}{u}$ satisfying $A\underset{\sim}{u} + b\underset{\sim}{u}^+ - a\underset{\sim}{u}^- = t\phi$, with $(u, \phi_2) < 0$. Thus we have four solutions, one in each of the four cones, which C_1, C_2 divide the ϕ_1, ϕ_2 plane into. This concludes the proof of Theorem III.1. ■

REMARKS. Later, we will see more specialized and technically sophisticated proofs of four solutions, one of which gives *exactly four*, another which works for all $b \geq \lambda_2$, not just $\lambda_2 < b < \lambda_3$. However, this remains the most elementary and general. The proof depended very heavily on the fact that we reduced the equation to a two-dimensional equation. It is natural to suspect that some version of this method would work if $a < \lambda_1, \lambda_3 < b < \lambda_4$, and we reduced the problem to one on the space spanned by $\{\phi_1, \phi_2, \phi_3\}$. We would again know Φ explicitly on two cones, but then the complement would no longer be disconnected. Perhaps instead of considering images of rays in the complement of the cone, one should consider images of planes. If $\lambda_2 = \lambda_3$, we may only get four solutions for a slightly perturbed problem, but in the case of simple eigenvalues, it would be natural to expect six.

III.2 At least four solutions - the nonlinear case

In the last section, we depended very heavily on the exact form of the piecewise nonlinearity. In this section, we return to the study of equation (I.1) for large t. We would like to say that, in some sense, Theorem III.1 was stable under perturbation, which is after all, the relationship of the nonlinear equation with large t to the piecewise linear. If we were content to consider the case where $f'(s) < \lambda_3$ for all s, then the task would be fairly simple. We would show that the reduced two-dimensional picture remained largely unchanged as c_1 and c_2 were made very large. However, we regard that restriction as somewhat artificial, restrictions on the derivative being only used previously for obtaining upper bounds in the number of solutions.

On the other hand, if we abandon the restriction $f' < \lambda_3$, then we cannot just consider the two dimensional reduction any since no longer would there be a unique solution to equation (III.3.i) and thus no reduced problem.

Our plan is the following; we first convert the two dimensional statements in section 1 into degree theoretic statements in the space $L^2(\Omega)$, and then show that these results can be perturbed to give the result for the nonlinear equation with large t.

Our first lemma is a degree theoretic interpretation of Theorem III.1.

Recall that ϕ_1, ϕ_2 satisfy $\phi_1(x) - \epsilon_0|\phi_2(x)| \geq 0$ for all $x \in \Omega$. Also recall that if $\Phi : PH \longrightarrow PH$ is defined in Theorem III.1 then there exists $d > 0$ satisfying the conditions of Lemma III.1.

The map $\Phi : PH \longrightarrow PH$ takes the value ϕ_1, once in each of the four different regions of the plain. The next lemma gives information on the degree of the map in these regions.

We define $F : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ by

$$F(s_1, s_2) = (t_1, t_2) \text{ if } v = s_1\phi_1 + s_2\phi_2, \ \Phi(v) = t_1\phi_1 + t_2\phi_2.$$

LEMMA III.2 *Let $p = (1,0)$. Let r be so large that $r > 1$, $r(b-\lambda_1) > 1$, $r(\lambda_1 - a) > 1$, $r\epsilon_0 > 1$ and $rd\epsilon_0 > 1$, where d and ϵ_0 come from section 1. Let*

$$\begin{aligned} D_1 &= \{(s_1, s_2) | 0 < s_1 < r; |s_2| < \epsilon_0\epsilon_1\} \\ D_2 &= \{(s_1, s_2) | \, |s_1| \le r, \epsilon |s_1| < s_2 < \epsilon_0 r\} \\ D_3 &= \{(s_1, s_2) | -r < s_1 < 0, |s_2| < \epsilon |s_1|\} \\ D_4 &= \{(s_1, s_2) | \, |s_1| \le r, \ -\epsilon_0 r < -\epsilon_0 |s_1|\} \end{aligned} \tag{III.6}$$

If deg(F, D_k, p) denotes the Brouwer degree of F with respect to D_k and p for $1 \le k \le 4$, then $d(F, D_k, p)$ is defined for $1 \le k \le 4$ and

$$\deg(F, D_k, p) = (-1)^{k+1}.$$

Proof.

First consider D_1. If $(s_1, s_2) \in \overline{D}$ and $v = s_1\phi_1 + s_2\phi_2$, then $\theta(v) = 0$. On D_1, the map $F(s_1, s_2)$ is given by $F(s_1, s_2) = ((b-\lambda_1)s_1, (b-\lambda_2)s_2)$. Since $1 < r(b - \lambda_1)$ the equation $F(s_1, s_2) = p$ has the unique solution $(s_1, s_2) = ((b - \lambda_1)^{-1}, 0)$. Since the determinant of the linear diagonal map is positive, we have

$$\deg(F, D_1, p) = 1.$$

In the case of D_3, we have the diagonal map with two negative entries, $(a - \lambda_1), (a - \lambda_2)$ and the determinant is also positive near the unique solution in this region given by $((a-\lambda_1)^{-1}, 0)$, so again $\deg(F, D_3, p) = 1$.

Now consider D_2. The boundary of D_2 consists of three line segments;

(i) a ray in the first quadrant R_1.

(ii) a ray in the second quadrant R_2.

(iii) a line segment L of $s_2 = \epsilon_0 r$, parallel to the s_1 axis.

As we observed in the proof of Theorem III.1, the image of R_1 under F will be a straight line segment in the fourth quadrant, the image of R_2 will be a straight line segment in the fourth quadrant and the image of L will be to the right of the line $s_1 = 1$, by virtue of the requirement.

Now consider the linear map $u \longrightarrow Bu$, where B is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The image of R_1 under B, BR_1, will be a straight line in the first quadrant, so if $0 \leq \lambda \leq 1$, we have

$$\lambda Bs + (1-\lambda)F(s) \neq p, \qquad s = (s_1, s_2) \in R_1.$$

The image of the ray R_2 under B is in the fourth quadrant and again we have

$$\lambda Bs + (1-\lambda)F(s) \neq p, \qquad s = (s_1, s_2) \in R_2.$$

Finally, if $s \in L$, then $s_2 = \epsilon_0 r > 1$ so

$$Bs \in \{(s_1, s_2) | s_1 > 1\}$$

and thus $\lambda Bs + (1-\lambda)F(s) \neq p$ for $s \in L$. By the usual homotopy argument,

$$\deg(F, D_2, p) = \deg(B, D_2, p).$$

But we know that $Bs - p$ has exactly one zero in D_2 and the sign of the determinant of B is -1. Thus

$$\deg(F, D_2, p) = -1.$$

The proof for D_4 is similar so we leave it as an exercise for the reader.

■

Using the definition of the degree of a mapping on an arbitrary finite dimensional space, we obtain, letting $V = PH$;

LEMMA III.3 *If for* $1 \leq k \leq 4$,

$$U_k = \{v \in V \,|\, v = s_1\phi_1 + s_2\phi_2, \quad (s_1, s_2) \in D_k\}$$

and $T : V \longrightarrow V$ *is defined by*

$$Tv = PA^{-1}(b(v + \theta(v))^+ - a(v + \theta(v))^-)$$

then

$$\deg\left(I + T, U_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1}.$$

We have now calculated the degree of the two dimensional map on the various regions. But we remind ourselves that the two dimensional map is obtained from the infinite dimensional map by using the contraction fixed point theorem. Our aim now is to perturb the equation

$$Au + bu^+ - au^- = s\phi_1.$$

To do this, and arrive at the full non-linearity f equation, we could proceed in two ways. We could restrict the class of f under discussion so that they satisfied $f' \leq \lambda_3 - \epsilon$. Then each perturbed problem could be reduced to a two dimensional problem which could be viewed, for large s, as a perturbation of the piecewise linear problem.

The reader will soon see that this would be extremely restrictive. What we do instead is to deduce, from our knowledge of the two dimensional degree, a result on the degree of the associated map on the infinite dimensional space. This can then be perturbed by small perturbations, which perturbations need only be continuous.

Let $Nu = A^{-1}(bu^+ - au^-)$.

LEMMA III.4 *Let U_k, $1 \leq k \leq 4$, and T be as in the preceding lemma. If $r_2 > 0$ is sufficiently large, and for $1 \leq k \leq 4$*

$$Y_k = \{u \in L^2(\Omega) | Pu \in U_k, \ \|(I-P)u\| < r_2\}, \tag{III.7}$$

then the Leray-Schauder degree $d\left(I + N, Y_k, -\frac{\phi_1}{\lambda_1}\right)$ is defined and

$$d\left(I + N, Y_k, -\frac{\phi_1}{\lambda_1}\right) = d(I + T, U_k, \frac{\phi_1}{\lambda_1}) = (-1)^{k+1}.$$

Proof. The proof of this lemma comes in several steps. First, we observe that there exists $r_1 > 0$ such that if $v \in \overline{U}_k$, $1 \leq k \leq 4$, and $w = (1-s)(I-P)N(v+w)$, then $\|w\| < r_1$. This is because, as already observed, the map $w \longrightarrow (1-s)(I-P)N(v+w)$ is a contraction on $(I-P)H$, for s, $0 \leq s \leq 1$. Now choose $r_2 > r_1$, and Y_k as defined in (III.7), for some fixed k. Define $h_1 : Y_k \times [0,1] \longrightarrow L^2$ by

$$h_1(u,s) = (I-P)N(v+w) + PN(v+w+s(\theta(v)-w)),$$

where $v = Pu,\ w = (I - P)u$. We obtain

$$u + h_1(u,s) \neq -\frac{\phi_1}{\lambda_1} \quad \text{for} \quad (u,s) \in \partial Y_k \times [0,1].$$

There are two possibilities to consider in $u \in \partial Y_k$. One is that $u = v+w$ with $v \in \partial U_k$, $\|w\| < r_2$, $s \in [0,1]$, and $u + h_1(u,s) = -\frac{\phi_1}{\lambda_1}$. In this case,

$$w + (I - P)N(v + w) = 0$$

and

$$v + PN(v + w + s(\theta(v) - w)) = -\frac{\phi_1}{\lambda_1}.$$

The first of these implies $w = \theta(v)$, and the second implies $v + PN(v + \theta(v)) = v + N(v) = -\frac{\phi_1}{\lambda_1}$, which contradicts the fact that $v \in \partial U_k$.

Now suppose $v \in U_k$, $w \in (I - P)H$, $\|w\| = r_2$. If $0 \leq s \leq 1$ and $u + h_1(u,s) = -\frac{\phi_1}{\lambda_1}$, then

$$w + (I - P)N(v + w) = 0,$$

so $w = \theta(v)$ and $\|w\| \leq r_1 < r_2$, which is a contradiction. This shows that $u + h_1(u,s) \neq \frac{\phi_1}{\lambda_1}$ for all $(u,s) \in \partial Y_k \times [0,1]$, and it follows by homotopy invariance of degree that

$$d\left(I + N, Y_k, -\frac{\phi_1}{\lambda_1}\right) = d\left(I + h_1(\cdot,1), Y_k, -\frac{\phi_1}{\lambda_1}\right).$$

Now let $h_2 : Y_k \times [0,1] \longrightarrow L^2(\Omega)$ be defined by

$$h_2(u,s) = (1 - s)(I - P)N(u) + PN(v + \theta(v)), \quad v = Pu.$$

If $v \in \partial U_k$, $w \in (I-P)H$, $0 \leq s \leq 1$, $u = v+w$, and $u+h_2(u,s) = \frac{\phi_1}{\lambda_1}$, then

$$v + T(v) = v + PN(v + \theta(v)) = P(u + h_2(u,s)) = -\frac{\phi_1}{\lambda_1},$$

which contradicts the fact that there are no solutions if $v \in \partial U_k$. If $u = v + w,\ v \in U_k,\ w \in (I-P)H,\ \|w\| = r_2$, then

$$0 = (I-P)(u + h_2(u,s)) = w + (1-s)(I-P)N(v+w),$$

which would imply that $\|w\| < r_2$, which is a contradiction. Therefore, $u + h_2(u,s) \neq \frac{\phi_1}{\lambda_1}$, for $(u,s) \in \partial Y_k \times [0,1]$. Since $h_1(u,1) = h_2(u,0)$, we infer by homotopy invariance that

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d(I + h_2(\cdot,1), Y_k, -\frac{\phi_1}{\lambda_1}).$$

Let B be the open ball of radius r_2 in $(I-P)H$. If $u \in \overline{Y}_k$, $v = Pu, w = (I-P)u$, then $u + h_2(u,1) = v + PN(v + \theta(v)) + w$.

Thus we see that the map $u \longrightarrow u + h_2(u,1)$ is uncoupled on $PH \oplus (I-P)H$ and is the identity on $(I-P)H$. Therefore by the product property of degree,

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d\left(I + T, U_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1}. \tag{III.8}$$

This concludes the proof of Lemma III.4. ∎

REMARKS. What we have just proved can be put into an abstract context. Assume one has an operator equation

$$Lu + N(u) = 0 \tag{III.9}$$

on a Hilbert space H. Assume that there exists P, commuting with L, so that

$$H = PH \oplus (I-P)H, \quad u \in H, \quad v = Pu, \quad w = (I-P)u,$$

and (III.9) is equivalent to

$$\begin{array}{lrcl} \text{i)} & Lw + (I-P)N(v+w) & = & 0 \\ \text{ii)} & Lv + PN(v+w) & = & 0. \end{array} \tag{III.10}$$

Assume that for fixed v, (III.10.i) may be solved uniquely and continuously for $w = \theta(v)$ and that for bounded v, there exists an *a priori* bound for $\theta(v)$. Then

LEMMA III.5 (THE PRISM LEMMA) *Given a bounded region $U \subseteq PH$ such that*

$$v + L^{-1}PN(v + \theta(v)) = 0 \tag{III.11}$$

has no solution on ∂U, and $r > 0$ so that $v \in \overline{U}$, $Lw + (1-s)N(v+w) = 0$, $0 \leq s \leq 1$, imply $\|w\| < r$, then if $Y = \{u : Pu \in U, \|(I-P)u\| \leq r\}$, we have

$$d(w + L^{-1}PN(v + \theta(v)), U, 0) = d(u + L^{-1}N(u), Y, 0).$$

Finally, having proved Lemma III.4, we are in a position to produce solutions to the semilinear problem

$$Au + f(u) = s\phi_1 + h_1(x) \tag{III.12}$$

instead of the piecewise linear one,

$$Au + bu^+ - au^- = \phi_1, \quad a < \lambda_1, \quad \lambda_1 < b < \lambda_2. \tag{III.2}$$

Then, as before, if $f_1(u) = bu^+ - au^-$, we have

$$f(\zeta) = f_1(\zeta) + f_0(\zeta), \quad \text{with} \lim_{|\zeta| \to \infty} \frac{f_0(\zeta)}{\zeta} = 0. \tag{III.12.a}$$

We rewrite (III.12) as

$$Az + f_1(z) + \frac{f_0(sz)}{s} = \phi_1(x) + \frac{h(x)}{s}. \tag{III.12.b}$$

In view of (III.13), we consider (III.14) as a perturbation of (III.2). Let

$$N_s(z) = A^{-1}\left(f_1(z) + \frac{f_0(sz)}{s} - \frac{h}{s}\right)$$

and let

$$N(z) = A^{-1}(f_1(z)).$$

Then it is easy to verify that

$$\lim_{s\to\infty} \|N(z) - N_s(z)\| = 0$$

uniformly for z in bounded subsets of $L^2(\Omega)$.

Finally, we have everything in place to prove the main result of this lecture.

THEOREM III.2 *Let f satisfy* (III.12.a). *There exists s_0 so that $s \geq s_0$ implies that*

$$Au + f(u) = s\phi_1(x) + h(x)$$

has at least four solutions.

Proof. We have established that

$$z + N(z) = -\frac{\phi_1}{\lambda_1} \quad \text{for all } z \in \partial Y_k, \quad 1 \leq k \leq 4.$$

Since ∂Y_k is closed and bounded, and N is continuous and compact, there exists $\eta > 0$ such that

$$\left\| z + N(z) + \frac{\phi_1}{\lambda_1} \right\| \geq \eta \quad \text{if} \quad \eta \in \partial Y_k.$$

Now choose s_0 so that

$$\|N_s(z) - N(z)\| < \frac{\eta}{2} \quad \text{for } z \in \partial Y_k, \quad 1 \leq k \leq 4.$$

Then

$$\left\| z + N(z) + (1-\lambda)(N_s(z) - N(z)) + \frac{\phi_1}{\lambda_1} \right\| \geq \frac{\eta}{2}$$

for $0 \leq \lambda \leq 1$, from which we conclude

$$d\left(I + N_s, Y_k, -\frac{\phi_1}{\lambda_1}\right) = d\left(I + N, Y_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1}, \quad 1 \leq k \leq 4.$$

This proved the theorem, since we have at least one solution in Y_k, $1 \leq k \leq 4$. ■

III.3 Existence of more than four solutions

The plan to demonstrate the existence of more than four solutions is as follows:

We have already produced four regions of non-zero degree if $a < \lambda_1, \lambda_2 < b < \lambda_3$. We show that these regions have the same degree even if $b = \lambda_3$. We show that there are no solutions on the boundary of our regions Y_i, even if $b = \lambda_3$. This allows us to shows that there are no solutions if $\lambda_3 < b < \lambda_3 + \epsilon$. This in itself would only demonstrate the existence of four solutions. However, the solution $\frac{\phi_1}{b - \lambda_1}$ is now contained in a small ball on which the topological degree is -1 which is inside a prism Y_1 with topological degree $+1$. This means that Y_1

must contain at least two and generically three solutions and thus the total number of solutions will be at least five and generically six.

To carry out this plan, we need some technical lemmas.

LEMMA III.6 *If* $\lambda_1 < b' < b''$, *there exists* r_3 *so that if* $b' \leq b \leq b''$ *and* $Au + bu^+ - au^- = \phi_1$, *then* $\|u\| < r_3$.

Proof. Left as an exercise to the reader. It can be done by contradiction, as in the earlier estimate. It can also be done by splitting the equation into PH and $(I - P)H$, where P is a projection onto $\{\phi_1 \cdots \phi_n\}$ with $b' < \lambda_n$. Then one equation is coercive on $(I - P)H$ and this gives a bound in terms of Pu. This proves that

$$\|Pu\| \longrightarrow +\infty \text{ implies } (Au + bu^+ - au^-, \quad \phi_1) \longrightarrow +\infty.$$

The next lemma is just a "uniform" restatement of the previous Lemma II.4.

LEMMA III.7 *Let* r *satisfy* $r > 1$, $1 < (\lambda_2 - \lambda_1)r$, $1 < (\lambda_1 - a)r$, $1 < \epsilon_0 r$, *and* $1 < \epsilon_0 dr$. *Let* D_k, U_k *be as before (*Lemma III.2, III.3*). If* r_3 *is chosen so that* $Au + bu^+ - au^- = \phi_1$ *and* $\lambda_2 \leq b \leq \lambda_3$ *imply* $\|u\| \leq r_3$, *and* $Z_k = \{u \in L^2(\Omega) | Pu \in U_k, \ \|(I - P)u\| < r_3\}$ *for* $k = 1, \ldots, 4$ *and* $K : L^2(\Omega) \times [\lambda_2, \infty) \longrightarrow L^2(\Omega)$ *is defined by* $K(u, b) = A^{-1}[bu^+ - au^-]$, *then*

$$d\left(I + K(\cdot, b), Z_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1} \textit{ for } \lambda_2 < b < \lambda_3$$

and $k = 1, \ldots, 4$.

Proof. This is just the observation that we can choose the region Z_k in such a way that as long as $\lambda_2 < b < \lambda_3$, the estimates for the proof of Lemma III.4 still hold. We have the degree defined for $\lambda_2 < b < \lambda_3$. The next task is to show that the degrees remain defined if $b = \lambda_3$. To do this, we must show that

$$Au + bu^+ - au^- = \phi_1$$

has no solution on the boundaries of Z_k, $k = 1, \ldots, 4$.

Note that there are no solutions if $\|(I-P)u\| = r_3$. Similarly, there are no solutions if $|(u, \phi_2)| = \epsilon d$, since then

$$(Au + bu^+ - au^-, \phi_1) > 1$$

by Lemma III.1.

Therefore if there are solutions, they must be of the form

$$u = c_1\phi_1 + c_2\phi_2 + w$$

with $c_2 = \pm\epsilon_0 c_1$.

Again, there are no solutions of this form with $w = 0$, since in this case $u \geq 0$ or $u \leq 0$ and we have solutions to the linear equation. Now let us suppose

$$u = v + w, \qquad v = c_1\phi_1 \pm \epsilon_0 c_1 \phi_2,$$

$$Au + bu^+ - au^- = \phi_1.$$

Then taking projections, we have, for any γ,

$$(A+\gamma)w + (I-P)(f(v+w) - \gamma(v+w) - f(v) - \gamma v) = 0.$$

Recall $f(v)$ is equal to either bv or av depending on whether c_1 is positive or negative. Now choose $\gamma = a$, and we have

$$(A+\gamma)w + (I-P)(\lambda_3 - a)[(v+w)^+ - v^+] = 0.$$

But $\|(A+\gamma)w\| \geq (\lambda_3 - a)\|w\|$ and $\|(v+w)^+ - v^+\| < \|w\|$ unless $v+w \geq 0,\ v \geq 0$. Now unless $v+w \geq 0$ and $v \geq 0$ we have a contradiction, since

$$\begin{aligned}(\lambda_3 - a)\|w\| &\leq \|(A+\gamma)w\| \\ &= (\lambda_3 - a)\|(I-P)((v+w)^+ - v^+)\| \\ &< (\lambda_3 - a)\|w\|.\end{aligned}$$

On the other hand, if $v+w \geq 0,\ v \geq 0$, then taking P of $Au + f(u) = \phi_1$, we get $f(u) = bu$, and hence

$$Av + bv = \phi_1,$$

and we know this has no solution for $v = c_1\phi_1 \pm \epsilon_0 c_1 \phi_2$. We have now proved the following.

LEMMA III.8 *If Z_k are as defined in* Lemma III.7, *then we have*

$$d\left(I + K(\cdot, \lambda_3), Z_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1}.$$

Thus the equation

$$Au + \lambda_3 u^+ - au^- = \phi_1$$

has at least four solution. Furthermore, by the same reasoning as before, we have $\|z + K(z, \lambda_3)\| \geq \delta > 0$ for all $z \in \partial Z_k$, $k = 1, \ldots, 4$. This has no solutions on ∂Z_k, $k = 1, \ldots, 4$, if $\lambda_3 \le b \le \lambda_3 + \epsilon$. From this we conclude that

$$d\left(I + K(\cdot, b), Z_k, -\frac{\phi_1}{\lambda_1}\right) = (-1)^{k+1}$$

for $\lambda_3 \le b \le \lambda_3 + \epsilon$. However, by Theorem I.8 of Chapter I, we may now choose a small ball $\tilde{B}$ with center $\frac{\phi_1}{b - \lambda_1}$ so that

$$d\left(I + K(\cdot, b), \tilde{B}, -\frac{\phi_1}{\lambda_1}\right) = -1,$$

if the multiplicity of λ_3 is odd. This ball can be chosen properly contained in Z_1 and with no zeros of $I + K(\cdot, b) + \frac{\phi_1}{\lambda_1}$ on the boundary. This gives, by excision that the degree

$$d\left(I + K(\cdot, b), Z_1 \tilde{B}, -\frac{\phi_1}{\lambda_1}\right) = 2,$$

since

$$d\left(I + K(\cdot, b), Z_1, -\frac{\phi_1}{\lambda_1}\right) = +1.$$

This allows us to conclude that Z_1 must contain at least two solutions and that the equation

$$Au + bu^+ - au^- = \phi_1$$

has at least five solutions for $\lambda_3 < b < \lambda_3 + \epsilon$. Put in the usual perturbation argument with $\frac{f_0(s)}{s} \longrightarrow 0$ as $|s| \longrightarrow +\infty$, and we have proved;

THEOREM III.3 *Let λ_3 be of odd multiplicity. Then there exists $\epsilon >$ 0, $t_0 > 0$ so that if $a < \lambda_1$, $\lambda_3 < b < \lambda_3 + \epsilon$ and $t \ge t_0$,*

$$\lim_{s \to +\infty} \frac{f(s)}{s} = b, \qquad \lim_{s \to -\infty} \frac{f(s)}{s} = a,$$

then the equation

$$Au + f(u) = t\phi_1 + h(x)$$

has at least five (and generically six) solutions.

In concluding this section, we ought remark that we are left in the same situation that we were in at the end of Chapter I. There, we knew there were at least three and generically four solutions for large t, if $f'(-\infty) < \lambda_1$, $\lambda_2 < f'(+\infty) < \lambda_3$. Now we know there are at least four always.

We know that if $f'(-\infty) < \lambda_1$, $\lambda_3 < f'(+\infty) < \lambda_3+\epsilon$, then there are at least five and **generically six** solutions if λ_3 is of odd multiplicity. If λ_3 has multiplicity one, Solimini was able to show there are exactly six solutions.

III.4 More general operators

To emphasize the generality of the method of this chapter, we apply these methods a non variational problem, already studied in Chapter II.

Here we look for T-periodic solutions of

$$D_t u = Lu + f(u) - s\phi_1 - h(x,t) \tag{III.13}$$

which we consider as a perturbations of the problem

$$D_t u = Lu + bu^+ - au^- - s\phi_1. \tag{III.14}$$

The object is to reduce the study of (III.13) to a map on a two-dimensional subspace spanned by $\phi_1(x)$ and $\phi_2(x)$ with ϕ an $\epsilon_0 > 0$ so that $\phi_1(x) \pm \epsilon_0\phi_2(x) > 0$, all $x \in \Omega$.

To do this, we need to introduce the orthogonal projection P onto the subspace spanned by $\phi_{01}(x)$, $\phi_{02}(x)$ and the orthogonal projection $I - P$ onto the space spanned by ϕ_{0n}, $n \geq 3$ and ϕ^c_{mn}, ϕ^s_{mn}, $m \geq 1$.

As before, (III.13) is equivalent to, writing $w = (I - P)u, v = Pu$,

$$\begin{array}{llll} \text{(i)} & w & = & (D_t - L)^{-1}(I - P)f_1(v + w) \\ \text{(ii)} & D_t v & = & L_v + Pf_1(v + w) \end{array} \qquad \text{(III.15)}$$

The key ingredient here is that for fixed $v \in PH$, equation (III.15.i) should have a **unique** solution $w = \theta(v)$ in $(I - P)H$. To ensure this, we must have an additional assumption on T. If σ is the spectrum of $D_t - L$, then recall that $\sigma = \left\{\lambda_n \pm \frac{i2\pi m}{T},\ n \geq 1,\ m \geq 0\right\}$ and when it comes to estimating $\|(D_t - L - \alpha I)^{-1}(I - P)\|$ we observe, since $D_t - L$ is normal that

$$\begin{aligned} & \|(D_t - L - \alpha I)^{-1}(I - P)\| \\ = \ & \operatorname{dist}\left(\alpha, \left\{\left(\lambda_n \pm \frac{i2\pi m}{T} - \alpha\right)^{-1} \mid \ m \geq 0,\ n \geq 2\right\}\right). \end{aligned}$$

Thus for (III.5.i) to have a unique solution, we require that there exist α so that the map

$$w \longrightarrow (D_t - L - \alpha I)^{-1}(I - P)(b(v + w)^+ - a(v + w)^- - \alpha(v + w))$$

be a contraction. For this to occur, we must require that there exist a circle C of radius r and center α such that C contains the points on the real line $a, b, \lambda_1, \lambda_2$ and does not contain any other point of the spectrum of $D_t - L$.

Once this detail is handled, all the other calculations of Chapter III go through in a routine manner, in particular the calculations of degree and the use of the Prism Lemma and one has the following:

THEOREM III.4 *Let $h \in H$. Assume $f'(-\infty) = a$, $f'(+\infty) = b$ exist and are finite. There exists T_0 such that if $T < T_0$ and $a < \lambda_1$, $\lambda_2 < b \leq \lambda_3$ where λ_2 is simple, then the equation*

$$\begin{aligned} D_t u &= Lu + f(u) - s\phi_1 - h(x,t) \\ u(x, t+T) &= u(x,T) \end{aligned}$$

has at least four solutions for sufficiently large s.

Moreover, there exists ϵ such that if λ_3 is of odd multiplicity and $\lambda_3 < b < \lambda_3 + \epsilon$, then the equation has at least five (and generically six) solutions for ϵ sufficiently large s.

We remark that there is still a lot missing from the non-self adjoint operator. Surely the restriction that T is sufficiently small is not necessary. One can also apply these methods to the semilinear string equation to get **at least five solutions** (I think). This has not been done.

Can these finite dimensional methods be used to generate four (or more) solutions when $b > \lambda_3$? This seems hard.

Can we get *exactly* four solutions if $b < \lambda_3$?

III.5 Nonlinearities crossing higher eigenvalues

The reader will have noted that in Chapter I, we started with Hammerstein's and Dolph's theorems, when the nonlinearity remained strictly below the first eigenvalue, and strictly between two eigenvalues. We have spent some time exploring the case where the first eigenvalue and the first two or three eigenvalues are crossed. In this section, we consider the case of crossing other eigenvalues than the first. It would be surprising if there were not some analogous results in this case, and there are.

Throughout this section we shall continue to study the operator equation

$$Au + bu^+ - au^- = h \quad \text{in} \quad L^2(\Omega)$$

where A is self-adjoint, has eigenvalues $-\lambda_i$, $\lambda_i \longrightarrow +\infty$, $\lambda_1 > 0$, the eigenfunction ϕ_1 associated with λ_1 is strictly positive in Ω, all ϕ_i's are regular.

One important point : In the following, we drop our practice of counting each eigenvalue as often as its multiplicity, and *count each distinct eigenvalue only once.*

Note that all eigenfunctions corresponding to eigenvalues other than λ_1 must change sign in Ω.

LEMMA III.9 *Given* $a > 0$, $\lambda_n < a < \lambda_{n+1}$, *there exists* $b_1 = b_1(a)$

satisfying $\lambda_{n+1} < b < \lambda_{n+2}$*, and a constant* $k < 1$ *such that if* $\lambda_{n+1} < b \leq b_1$*,* $F : L^2(\Omega) \longrightarrow L^2(\Omega)$ *is given by*

$$F(u) = bu^+ - au^- \qquad \text{and} \quad c = \frac{\lambda_{n+1} + a}{2},$$

then

$$\|(-A - cI)^{-1}(F(u) - cu)\| \leq k\|u\| \tag{III.16}$$

for all $u \in L^2(\Omega)$*.*

Proof. Let $H = L^2(\Omega)$. Let P be orthogonal projection of H onto the finite dimensional subspace of H spanned by the eigenfunctions of $-A$ corresponding to λ_{n+1}.

Since a non-zero element of PH is continuous and changes sign on Ω, there exists $\gamma > 0$ so that

$$\gamma^2 = \max\{\int_\Omega (\theta^+)^2 dx | \theta \in PH, \ \|\theta\| = 1\} < 1.$$

Since the eigenvalues of the compact self-adjoint operator

$$(-A - c)^{-1}(I - P) : L^2(\Omega) \longrightarrow L^2(\Omega)$$

are $(\lambda_m - c)^{-1}$ with $m \neq n + 1$, if $k_2 = \max\{(c - \lambda_n)^{-1}, \ (\lambda_{n+2} - c)^{-1}\}$, then

$$\|(-A - c)^{-1}(I - P)\| = k_2. \tag{III.17}$$

Similarly, $\|(-A - c)^{-1}P\| = (\lambda_{n+1} - c)^{-1} \equiv k_1$ and, of course, $k_2 < k_1$.

Let $\lambda_{n+1} < b < \lambda_{n+2}$, and we claim that if $k^2 = k^2(b) = \{\gamma^2 k_1^2 + (1 - \gamma^2)k_2^2\}(b - c)^2$, then (III.16) holds for all $u \in L^2(\Omega)$. To prove this, note that since $b - c > \lambda_{n+1} - c = c - a$.

$$(c - a)|u| \leq (b - c)u^+ + (c - a)u^- \leq (b - c)|u| \quad \text{a.e. on } \Omega$$

for all $u \in L^2(\Omega)$. Therefore if $u \in L^2(\Omega)$, then

$$\|F(u) - cu\|^2 \leq (b - c)\|u\|^2. \tag{III.18}$$

Let $v_1 = P(F(u) - cu)$, $v_2 = (I - P)(F(u) - cu)$ and let x be given by $\|v_2\| = x(b - c)\|u\|$. From (III.18), we see that

$$\|v_1\|^2 \leq (b - c)\|u\|^2 - \|v_2\|^2 = (1 - x^2)(b - c)^2\|u\|^2$$

and therefore

$$\begin{aligned} \|(-A - c)^{-1}(F(u) - cu)\| &= \|(-A - c)^{-1}v_1\|^2 + \|(A - c)^{-1}v_2\|^2 \\ &\leq (k_1^2(1 - x^2) + k_2^2 x^2)(b - c)^2\|u\|^2. \end{aligned}$$

If $x^2 \geq 1 - \gamma^2$, then since

$$\begin{aligned} k_1 > k_2,\ k_1^2(1 - x^2) + k_2^2 x^2 &= k_1^2 + x^2(k_2^2 - k_1^2) \\ &\leq k_1^2 + (1 - \gamma^2)(k_2^2 - k_1^2) \\ &= \gamma^2 k_1^2 + (1 - \gamma^2)k_2^2. \end{aligned}$$

Thus (III.16) holds with $k = k(b)$. **If** $x^2 \leq 1 - \gamma^2$, then noting that $F(u) - cu \geq 0$, we have

$$\begin{aligned} \|v_1\|^2 &= \int (F(u) - cu)v_1 \leq \int (F(u) - cu)v_1^+ \\ &\leq \left(\int (F(u) - cu)^2\right)^{\frac{1}{2}}\left(\int (v^+)^2\right)^{\frac{1}{2}} \\ &\leq (b - c)\|u\|\gamma\|v_1\|. \end{aligned}$$

Therefore $\|v_1\|^2 \leq (b - c)^2\gamma^2\|u\|^2$ and so

$$\begin{aligned} \|(-A - c)^{-1}(F(u) - cu)\|^2 &\leq k_1^2\|v_1\|^2 + k_2^2\|v_2\|^2 \\ &\leq (k_1^2\gamma^2 + (1 - \gamma^2)k_2^2)(b - c)^2\|u\|^2. \end{aligned}$$

so again (III.16) holds.

Note that as $b \downarrow \lambda_{n+1}$, we have

$$k(b) \longrightarrow \gamma^2 + (1-\gamma^2)k_2^2(\lambda_{n+1} - c)^2.$$

Since $k_2^2(\lambda_{n+1} - c)^2 < 1$, we have $\gamma^2 + (1-\gamma^2)k_2^2(\lambda_{n+1} - c)^2 < 1$ and from thus we can infer the existence of a number b_1 in $(\lambda_{n+1}, \lambda_{n+2})$ such that $k(b) < 1$ for $\lambda_{n+1} < b \leq b_1$. This proves Lemma III.9. ∎

LEMMA III.10 *Let k^* satisfy $k < k^* < 1$. Given $h \in L^2(\Omega)$, there exists $R_0 = R_0(h) > 0$ such that if $\|u\| \geq R_0$, then*

$$\|(-A-c)^{-1}(F(u) - cu - h)\| \leq k^*\|u\|. \tag{III.19}$$

Consequently, if $R \geq R_0$, $B_R = \{u \in L^2(\Omega) |\, \|u\| \leq R\}$ and $N : L^2(\Omega) \longrightarrow L^2(\Omega)$ is the compact operator defined by $Nu = (-A - c)^{-1}(F(u) - cu - h)$ then

$$\deg(I - N, B_R, 0) = 1.$$

Proof. From (III.15), we can check that if $R_0 = \dfrac{\|(-A-c)^{-1}h\|}{k^* - k}$, then the first assertion of the lemma is true. If $0 \leq s \leq 1$, and $\|u\| \geq R_0$, then

$$\|u - sN(u)\| \geq (1 - k^*s)\|u\| > 0.$$

Thus by homotopy, $\deg(I - N, B_R, 0) = \deg(I, B_R, 0) = 1$. This proves the second assertion of the lemma. ∎

As in earlier sections of this chapter, we shall be primarily concerned with the equation

$$Au + bu^+ - au^- = t\phi_1. \tag{III.20}$$

If $t > 0$, this equation has as a solution $Z = \frac{t\phi_1}{b - \lambda_1}$.

LEMMA III.11 *If λ_{n+1} is an eigenvalue of multiplicity $k \geq 1$, then there exits $r > 0$ so that if*

$$B_r(Z) = \{u \in L^2(\Omega) | \, \|u - Z\| \leq r\}$$

and if

$$Tu = (-A - c)^{-1}((b - c)u^+ - (a - c)u^- - t\phi_1)$$

with c as above, then

$$\deg(u - Tu, B_r(Z), 0) = (-1)^k.$$

Proof. The proof is almost identical to that of Theorem I.5 in Chapter I. One simply show that $I - T$ is homotopic on a suitably chosen ball, to $I - (-A - c)^{-1}(bI)$ and proceeds to note that this linear operator has exactly k eigenvalues less than zero. ∎

LEMMA III.12 *If λ_{n+1} is an eigenvalue of odd multiplicity, $\lambda_n < a < \lambda_{n+1}$ and $b \in (\lambda_{n+1}, b_1(a))$ then the boundary value problem*

$$Au + bu^+ - au^- = \phi_1$$

has at least two solutions.

Proof. Choose a ball of radius r so that

$$\deg(I - T, B_r, 0) = (-1)^k = -1$$

where T is as in the preceding lemma. Now choose R sufficiently large that

$$\deg(I - T, B_R, 0) = 1$$

and B_R properly contains B_r. Then $\deg(I-T, B_R \backslash \overline{B}_r, 0) = +2$ and we can conclude the existence of at least two solutions in B_R. This proves the lemma. ∎

Note that if $t = 0$, the *a priori* estimate of (III.15) shows that $u = 0$ is the unique solution of (III.19). It is a nice measure-theory exercise to show that the same remains true if $t < 0$. Thus Lemma III.12 shows that if λ_{n+1} is of odd multiplicity, then for $t \leq 0$ we have exactly one solution, and for $t > 0$, we have at least two and generically three solutions. It is reasonable to suspect that one ought to be able to prove at least three solutions. Now we use the Prism Lemma of the last section to do this in the case of a simple eigenvalue. This time, we decompose the space $H = L^2(\Omega)$ into the orthogonal complements PH and $(I-P)H$, where PH is the span of ϕ_{n+1}, the eigenfunction corresponding to λ_{n+1}. Recall that if a, b satisfy $\lambda_{n+1} < a,\ b < \lambda_{n+1}$, then we can decompose the equation (III.10) $Au + bu^+ - au^- = t\phi_1$ or the equivalent equation

$$u = (-A - c)^{-1}((b-c)u^+ - (a-c)u^- - t\phi_1) \equiv Tu$$

as the pair of equations

$$\begin{array}{lrcl} \text{(i)} & (I-P)u & = & (I-P)Tu \\ \text{(ii)} & Pu & = & PTu. \end{array} \qquad \text{(III.21)}$$

If we write $u = v + w$, where $v = s\phi_{n+1}, w \perp \phi_{n+1}$, then for each fixed v (or s) the equation (III.21.i) $w = (I-P)T(s\phi_{n+1} + w)$ has a unique solution $w^*(t)$, by the contraction mapping theorem (This bears the same relationship to the decomposition in Section 1 of this chapter that Dolph's theorem bears to Hammerstein's theorem).

Thus equation (III.20) is equivalent to $v = PT(v + w^*(v))$ or equivalently, after taking inner products with ϕ_{n+1}

$$\Phi(s) = s - (T(s\phi_{n+1} + w^*(s)), \phi_{n+1}) = 0. \tag{III.22}$$

Now we are in the position of the Prism Lemma of Section 2, except that in this case, the space is one dimensional. This will give us more information.

Let us first restate the Prism Lemma in the form which we intend to use it here.

LEMMA III.13 *Let* $s_1, s_2 \in \mathbf{R}$ *such that* $\Phi(s_1) \neq 0, \Phi(s_2) \neq 0$. *Then there exits* r_2 *sufficiently large that if*

$$D = \{u \in L^2(\Omega) | Pu = s\phi_{n+1},\ s \in (s_1, s_2),\ \|(I - P)u\| < r_2\}$$

then the Leray-Schauder degree $\deg(I - T, D, 0)$ *is defined and*

$$\deg(I - T, D, 0) = \begin{cases} 0 & \text{if } \Phi(s_1)\Phi(s_2) > 0 \\ 1 & \text{if } \Phi(s_1) < 0,\ \Phi(s_2) > 0 \\ -1 & \text{if } \Phi(s_1) > 0,\ \Phi(s_2) < 0 \end{cases}.$$

Proof. The proof is virtually the same as in Section 2. The additional factor is that you can calculate the degree of a map on an interval by the signs of the function at the two end points. ∎

The next lemma takes advantage of the one dimensional character of the map Φ. Recall that in Lemma III.12, we had a ball with topological degree -1 inside a ball with topological degree $+1$. In a one dimensional space, this would be enough to guarantee the existence of three solutions, since its graph would have to appear as in Figure 4.

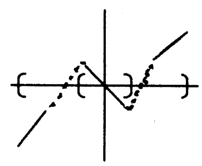

Figure 4.

LEMMA III.14 *With the assumptions of* Lemma III.12, and the additional assumption that λ_{n+1} is simple, there exits real numbers τ_i, $i = 1, \ldots, 4$ and $r_2 > 0$ such that $\tau_i < \tau_{i+1}$, $i = 1, 2, 3$, and if

$$D_k = \{u \in L^2(\Omega) | Pu = s\phi_{n+1},\ s \in (\tau_k, \tau_{k+1}),\ \|(I-P)u\| < r_2\}$$

then

$$\deg(I - T, D_k, 0) = (-1)^{k+1}, \qquad k = 1, 2, 3.$$

Proof. According to Lemma III.10, there exists R so that all solutions of $u - Tu = 0$ satisfy $\|u\| \leq R$ and $\deg(I - T, B_R, 0) = 1$. Let r_2, τ_1, τ_4 be chosen so that if

$$D = \{u \in L^2(\Omega) | Pu = s\phi_{n+1},\ \tau_1 < s < \tau_4,\ \|(I-P)u\| < r_2\},$$

then $B_R < D$, $\|w^*(s)\| \leq r_2$, all $s \in (\tau_1, \tau_4)$. By excision,

$$\deg(I - T, D, 0) = \deg(I - T, B_R, 0) = 0.$$

Therefore, by Lemma III.13, we must have that $\Phi(\tau_1) < 0$ and $\Phi(\tau_4) > 0$. Since $(\phi_1, \phi_{n+1}) = 0$ and $Z = \dfrac{t\phi_1}{b - \lambda_1}$, we see that $Z = (I - P)Z$, and therefore $Z = (I - P)T(Z)$. By the definition of w, it follows that $W^*(0) = Z$ and $\Phi(0) = 0$. According to Lemma III.11, Z is an isolated solution of $u - T(u) = 0$. Since $s\phi_{n+1} + w^*(s)$ is a solution of $u - T(u) = 0$ if and only if $\Phi(s) = 0$, it follows that $s = 0$ is an isolated zero of $\Phi(s)$.

Now we use the Prism Lemma again. If

$$D_2 = \{u \in L^2(\Omega) | Pu = s\phi_{n+1},\ \tau_2 < s < \tau_3,\ \|(I - P)u\| \leq r_2\},$$

then since Z is the only solution of $u - T(u) = 0$ in D_2 and the Leray-Schauder index of $u - T(u) = 0$ at Z with respect to zero is -1 we have $\deg(I - T, D_2, 0) = -1$. Therefore, according to the Prism Lemma III.13, we have

$$\Phi(\tau_2) > 0, \qquad \Phi(\tau_3) < 0.$$

If D_1 and D_3 are defined as in the statement of Lemma III.14 we must have that $\deg(I - T, D_k, 0) = (-1)^{k+1}$, and this proves the lemma. ∎

Thus we have proved the following :

THEOREM III.5 *Given $a \in (\lambda_n, \lambda_{n+1})$ there exists $b_1 \in (\lambda_{n+1}, \lambda_{n+2})$ such that if λ_{n+1} has odd multiplicity, then the equation*

$$Au + bu^+ - au^- = t\phi_1 \qquad \text{(III.2)}$$

has at least one solution for t negative and at least two solutions for t positive.

Moreover, if $t > 0$, and if λ_{n+1} has multiplicity one, then (III.2) has at least three solutions.

We remark that it is possible, by the work in this section, to give constructive estimates on b_1.

It is also possible to show that for negative t, the equation has exactly one solution. Note that Lemma III.19 shows that for $t = 0$, $u = 0$ is the unique solution. This method, with some complications, will also give exactly one solution for t negative.

Theorem III.5 can also be extended to the semilinear equation, where the conclusions would hold for t *large* negative and *large* positive instead of merely positive or negative.

One can also do a two dimensional reduction to the space spanned by ϕ_1 and ϕ_{n+1}, and do a winding number proof as in Section 2. This would be fun, but seems to give no additional information beyond that given by the methods if this section.

III.6 Some geometric interpretations

Let us discuss briefly a geometric interpretation of the results of this chapter if $f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2$, then for s large positive equation (III.1) has two positive solutions, and for s large negative, it has no solutions. This is vaguely suggestive of a simple fold on the Banach space, and indeed this was proved in [B-C], with the additional assumptions $0 < f'(-\infty) < \lambda_1$ and $f'(s) > 0$. How then can we interpret Theorem III.1 and the zero-four phenomenon? We know the reduced two dimensional equation looks like Figure 3.

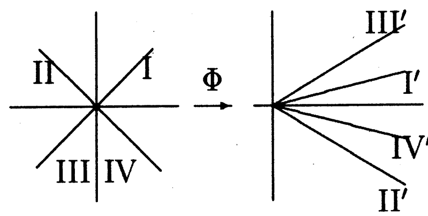

Figure 5.

And this might suggest a double fold. However, this would not fit in with the one-three phenomenon as we cross one higher eigenvalue.

The one-three phenomenon is more suggestive of a cusp, of a singularity that appears as in Figure 6.

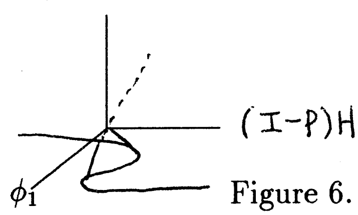

Figure 6.

This conjecture actually gives a unified description of the zero-four and one-three solutions. When we cross λ_1, a fold is created. When we cross λ_2 a cusp is created. The map in Figure 5 is merely the composition of a cusp where R_2 goes into the third quadrant and R_3 goes into the second quadrant, followed by a fold, which maps them both into the right half plane.

Of course all this is speculation. None of these theorems have been proved.

III.7 Historical remarks

The proof of at least four solutions in the case $f'(+\infty) > \lambda_2$ was first given using variational methods by Hofer [Ho]. There it was assumed that $f'(-\infty) < \lambda_1$, $\lambda_2 < f'(+\infty)$, $f'(+\infty) \neq \lambda_n$ for any n. Later but independently, the authors proved the existence of at least four solutions if $\lambda_2 < f'(+\infty) \leq \lambda_3$ and at least five (generically six) if $\lambda_3 < f'(+\infty) < \lambda_3 + \epsilon$. Then Solimini [Sol] showed, first that one could remove the restriction $f'(+\infty) \neq \lambda_n$ from Hofer and established, under hypotheses that will be elaborated in Chapter III, that if $\lambda_2 < f'(+\infty) \leq \lambda_3$, there were exactly four solutions, and if $\lambda_3 < f'(+\infty) < \lambda_3 + \epsilon$, there were exactly six (assuming λ_2 and λ_3 are both simple). Dancer [DAN1], later gave another proof of the existence of at least four solutions in the Hofer situation.

III.8 Results on a floating beam equation

The purpose of this section is to describe a study of the effect of nonlinear oscillations in floating beams. For more details see [L-McK9]. In naval architecture, a ship is frequently modeled as a floating beam. However, the nonlinear effects that occur when the ship is partly out of the water, (known in the literature as 'bottom slamming') or partly submerged (known as wetting) have not been the subject of much study. Thus, one would expect the predictions of linear theory to be quite accurate for small oscillations but not necessarily for large ones.

Although no catastrophic failures of ships have been directly attributed to large-scale oscillations of ships, the authors have come across at least one case in the literature which we feel can only be explained by the presence of large-scale flexings. This is the case of the destruction of the U.S.S. *Orion.*

As described in [BMR], on 2 December 1925 this ship was steaming out of the Chesapeake Capes, very nearly head on into a winds of force 9 which has been blowing onshore for at least 24 hours. The draft of the ship was 29.8 feet forward and 31.2 feet aft. The charted depth in the area being crossed by the ship was at least 35 feet. The waves were reported by the ship personnel to be 150 feet long and 10 feet high. Within 10 minutes of the time that the upset course was taken, the ship began to experience 'excessive vibration'. Although the record is not clear, the ship was undoubtedly pounding its forefoot on the bottom at this time. Following every impact, the hull would vibrate for a few seconds, probably until just before the next impact. *Although slowed to half speed*, the 'vibrations' continued to from 20 to 40 minutes.... The ship was turned around in the sea and taken back to port, where its bottom was found to be pushed upwards and fractured. The inner bottom, for a distance of from about 25 to 130 feet abaft the stem, was found to be buckled and distorted.

Despite the reported small size of the waves relative to the ship, the bow apparently pounded on the sea bed intermittently for about 20 minutes. The final damage was so severe that the ship was unloaded, decommissioned at once, and never again put back into military service.

Given that this represents a forced oscillation problem, it seems clear that flexing of the ship must have played a role in this curious oscillation that seemed to favor one end of the ship. It would be difficult to explain what sort of forcing term could have created this effect in the linear theory.

The other thing that seems clear from the admittedly sketchy accounts that are available to us is that the *frequency* of the forcing term cannot have been as important as one would expect it to be if one was seeking to explain this phenomenon as one of linear resonance. The first action taken by the ship's crew was to reduce speed by half, which would have the effect of reducing the frequency of the forcing term. (This is standard seafaring practice when dealing with any threatening rigid body motion of the ship.)

In this section, we consider a nonlinear model, which takes into account that the ship may be partly lifting out of the water or partly submerged but not both. We show that this causes large amplitude oscillations that would not be predicted by the linear theory. These oscillations occur in a wide range of frequencies, and frequently several different periodic oscillations can coexist for the same forcing term. In this situation, whether the ship goes into large or small oscillations depends on the initial conditions. Furthermore, numerical results indicate that the magnitude of the oscillations increases as one decreases the frequency of the forcing term. It is also a striking feature of this model that asymmetric solutions, with oscillations favoring one end are predicted in the presence of almost symmetric forcing data.

The authors also feel that this may be an explanation for catastrophic failures such as that of the wreck of the S.S. Edmund Fitzgerald, usually explained by the failure of the hatches in the presence of an (otherwise unobserved) 'wall of water'. It is easy to imagine that a series of large waves would push the ship into a large flexing motion which would have precisely the effect of popping the hatches at the center of the ship.

First consider a rectangular block of cross-section A floating in water. Assume that the difference in density between the block and water is ρ. Let $U(t)$ denote the depth of the bottom of the block as it floats. Assume that the block is floating high in the water, so that it may lift out of the water but is never submerged. Then the force pushing the block up in the water is given by $\rho U(t)A$ if U is positive and zero if U is negative. Thus, if there is an external forcing term, as well as the force of gravity, the equation satisfied by the block is given by

$$\frac{\partial^2 U}{\partial t^2} = c - bU^+ + f(t) \qquad \text{(III.22)}$$

for suitable constants b and c.

Now, if we consider the case, not of a floating block, but a floating beam of length L, it is clear that the equation will be of the form

$$U_{tt} + U_{xxxx} + \delta U_t + aU^+ = c + f(x,t) \qquad \text{(III.23)}$$

where now, $U = U(x,t)$, where $0 \leq x \leq L$ and where the ends of the beam satisfy free-end boundary conditions, i.e.,

$$U_{xx}(0,t) = U_{xx}(L,t) = U_{xxx}(0,t) = U_{xxx}(L,t) = 0.$$

The constant a is a measure of the cross-section of the beam, and the constant δ represents the viscous damping in the beam.

Now, the purpose of this paper is to examine periodic solutions of this equation, subject to the free-end boundary conditions. In order to approach this problem, we must first gather some information on the operator $LU = U_{tt} + U_{xxxx}$ with these boundary conditions. First, we consider the case where the interval is $(0, \pi)$.

In turn, to do this, we must first understand the ordinary differential operator

$$\mathcal{L}y = y^{(iv)}, \quad y^{(2)} = y^{(3)} = 0, \quad x = 0, \pi \tag{III.24}$$

with the additional symmetry condition that U is symmetric about $\frac{\pi}{2}$. This operator is a self-adjoint operator with an infinite sequence of eigenvalues λ_i, and their associated eigenvectors ϕ_i.

The functions ϕ_i are given by normalizing multiples of

$$\hat{\phi}_i = a_i \cos\left(\nu_i\left(x - \frac{\pi}{2}\right)\right) + b_i \cosh\left(\nu_i\left(x - \frac{\pi}{2}\right)\right) \tag{III.25}$$

where the ν_i are the successive zeros of $\tanh\left(\frac{\nu\pi}{2}\right) + \tan\left(\frac{\nu\pi}{2}\right)$ and $a(i) = \cosh\left(\frac{\nu_i\pi}{2}\right)$ and $b(i) = \cos\left(\frac{\nu_i\pi}{2}\right)$. The corresponding λ are given by $\lambda_i = \nu_i^4$, $i > 0$. Of course, $\lambda_0 = 0$, with the corresponding $\phi_0 = 1$. (Recall that the eigenfunction $\left(x - \frac{\pi}{2}\right)$ is ruled out by the symmetry condition). The functions $\{\phi_i, \ i \geq 0\}$ are an orthonormal basis for the Hilbert space $\mathcal{H} = L^2(0, \pi) \cap \left\{\text{functions even about } \frac{\pi}{2}\right\}$.

If we are interested in periodic solutions of periodic 2π of the partial differential equation, we will look in the space $H = \mathcal{H}_x \oplus L_t^2(0, 2\pi) \cap \{\text{functions even in } t\}$. By our earlier remarks, we have an orthonormal basis of this space given by $\Phi_{m,n} = \phi_m(x)\cos nt$, $m, n \geq 0$, with associated eigenvalues $\Lambda_{m,n}$.

The unbounded operator L is a self-adjoint operator on the space H and the functions $\Phi_{m,n}(x, t) = \phi_m(x)\cos nt$, $m, n \geq 0$ are eigenfunctions of L with eigenvalues $\Lambda_{m,n} = -n^2 + \lambda_m$. Note that an easy calculation gives that the ν_i rapidly approach $2m - \frac{1}{2}$, $m = 1, 2, \ldots$ and thus the $\Lambda_{m,n}$ rapidly approach $\left(2\left(m - \frac{1}{4}\right)\right)^4 - n^2$.

Now, if we are interested in the problem on the x-interval $\left(0, \frac{\pi}{q}\right)$, we will have the corresponding Hilbert Space H_q with the eigenfunctions (still denoted $\Phi_{m,n}$) given by $\Phi_{m,n} = \phi_m(qx) \cos nt$, $m, n \geq 0$ with associated spectrum $n^2 - \left(2\left(m - \frac{1}{4}\right)q\right)^4$ (modulo the first few eigenvalues which are calculated by a secant method).

Thus, we can conclude that if the interval which we are considering is $\left(0, \frac{\pi}{q}\right)$ where q is a rational number with an odd exponent, on the orthogonal complement of the subspace spanned by the constants, (which is the eigenspace corresponding to the eigenvalue zero), the operator L has a bounded, compact inverse and for various values of q, we can explicitly calculate the first few negative eigenvalues of L. Because the eigenvalues of L go to infinity in modulus rapidly, we need only calculate the first few values, if we are interested, (as we shall be in the next section) in the first and second negative eigenvalues.

It is instructive to find the first and second negative values for a few values of q. For example, if $q = 1$, then the first negative eigenvalue of L corresponds to $\nu_1 = 1.50561869$ and $n = 3$. This is a flexing motion about the center of the beam of period $\frac{2\pi}{3}$ and thus if we are using a forcing term of period π, this would correspond to a *superharmonic* solution. On the other hand, if we consider the case $q = 0.75$ (the longer boat), then the first negative eigenvalue corresponds to $\nu_2 = 3.50001049$ and $n = 7$. This is a multi-noded motion of short period. The second eigenvalue is -2.37405777.

Again, lengthening the beam still further $\left(q = \frac{2}{3}\right)$ leads to a first negative eigenvalue -2.98493242 whose motion is the same as the case

$q = 1$.

To summarize, the spectrum of the operator L is extremely sensitive to length. We can say that if the length is an rational multiple of the period π, there are many situations where we get a first and second negative eigenvalue separated from each other by a large interval. This is precisely the situation which we shall study by theoretical methods in the next section.

Some comments about the model are in order at this stage. If we assume that there is no forcing term, f, then equation (III.23) has a unique steady state solution, $u(x,t) \equiv \frac{c}{A}$ and this is globally attracting. If f is small and the solution u of (III.23) is of the same order of magnitude as f (i.e., we are not in a situation of linear resonance) then to solve (III.23) we need only solve the linear equation

$$U_{tt} + U_{xxxx} + \delta U_t + aU = c + f(x,t). \tag{III.26}$$

Therefore, we can find periodic solutions of (III.23) simply by finding the well-understood solutions of the linear equation. These solutions will be (away from resonance) of the form $\frac{c}{A} + \hat{u}$ where $\hat{u}$ is of the same order of magnitude as the forcing term $f(x,t)$. This is what we would call the intuitively obvious solution...a small forcing term results in a small perturbation about equilibrium.

However, if the forcing is small, then one can ask two related nonlinear questions : first, is this the only periodic motion and if other periodic solutions exist, are they stable?

Second, one should also seek to answer the question of what will happen if the forcing term f becomes large enough to force the naturally occurring linear solution out of the positive range, and thus cease to be a true solution of the actual nonlinear problem. We can give quantitative and qualitative answers to the 'small forcing' problem in certain ranges of the parameters but only present some numerical results about the second.

The equation assumes that no nonlinear effects occur until the ship actually lifts out of the water and that the motion is purely linear until this happens. If in fact, this would only be the case if the beam had a rectangular cross-section. If the cross-section of the boat was triangular, then the restoring force due to flotation would be proportional to $((u(x,t)^+)^2$, and therefore the nonlinear effects would occur immediately. This is more likely to occur at the front of the rear of the ship and would add to the likelihood of oscillation with favors one end occurring, if it exists.

We now prove a theorem which uses these methods.

Let Ω be a bounded domain in $\mathbf{R}^n$, let H denote a real Hilbert space which is a closed subspace of $L^2(\Omega)$ where we denote the usual L^2-inner product by $(\ ,\ )$. Let $L : D(L) \subseteq H \longrightarrow H$ be a self-adjoint operator. We shall discuss an abstract operator equation

$$Lu + bu^+ = c\psi_0 \tag{III.27}$$

which we shall later relate to the differential equation.

We make the following assumptions :

A. 1

$$\dim\ker(L) = 1 \textit{ and } \ker(L) = \{s\psi_0 | s \in \mathbf{R}\} \tag{III.28}$$

A. 2 *$b > 0, c > 0$ and there exists $\psi_1 \in H$ with ψ_1 not $\equiv 0$ such that $L\psi_1 = -\alpha\psi_1$ where $0 < \alpha < b$;*

A. 3 *There exists a number* $d > 0$ *such that if* $|s| \leq d$*, then for all* $x \in \Omega$

$$\psi_0 + s\psi_1 > 0;$$

A. 4. *There exists numbers* r_1 *and* r_2 *with*

$$r_1 < -b < 0 < r_2 \tag{III.29}$$

such that if V *is the two-dimensional subspace of* H *spanned by* ψ_0 *and* ψ_1*,* L_1 *is the restriction of* L *to the invariant subspace* $V^\perp$ *of* H *and* σ_1 *denotes the spectrum of* L_1*, then*

$$\sigma_1 \subset (-\infty, r_1) \cup (r_2, \infty).$$

Using assumptions (**A. 1**) and (**A. 3**), we find by inspection the obvious solution $u = \frac{c\psi_0}{b}$ of (III.27). The following result gives us two more solutions which are less obvious.

THEOREM III.6 *There exist solutions* u_1 *and* u_2 *of* (III.27) *such that* $(u_1, \psi_1) > 0$*,* $(u_2, \psi_2) < 0$.

Proof. The proof will be based on two lemmas. First, using global Lyapunov-Schmidt method we shall reduce the study of (III.27) to the geometrical analysis of a mapping of the two-dimensional space V into itself. For this reason, we shall refer to V as the **Reductive base space**, (RBS). Let $P : H \longrightarrow V$ denote orthogonal projection.

LEMMA III.15 *Given $v \in V$, there exists a unique $w \in W \equiv V^{\perp}$ such that*

$$Lw + (I - P)b(v + w)^{+} = 0. \qquad \text{(III.30)}$$

If we denote w by $\theta(v)$, then the mapping $\theta : V \longrightarrow W$ is Lipschitzian.

Proof. If $w \in W$, then (III.30) is equivalent to

$$\left(L_1 + \frac{b}{2}\right) w = -(I - P)\left\{b(v + w)^{+} - \frac{b}{2}(v + w)\right\}. \qquad \text{(III.31)}$$

Since the spectrum of $L_1 + \frac{b}{2}$ is contained in $\left(-\infty, r_1 + \frac{b}{2}\right) \cup \left(r_2 + \frac{b}{2}, \infty\right)$, it follows from (A. 4) that $L_1 + \left(\frac{b}{2}\right) I$ is invertible and

$$\left\|\left(L_1 + \frac{b}{2}\right)^{-1}\right\| \le \max\left\{\left|r_1 + \frac{b}{2}\right|^{-1}, \left|r_2 + \frac{b}{2}\right|^{-1}\right\} < \frac{2}{b}.$$

Therefore (III.31) is equivalent to

$$w = -\left(L_1 + \frac{b}{2}I\right)^{-1}(I - P)bN(v + w) \equiv K_v(w),$$

where $N : H \longrightarrow H$ is defined by $N(u)(x) = \frac{b|u(x)|}{2}$. It follows that $K_v : W \longrightarrow W$ is Lipschitzian with Lipschitz constant $c = \frac{b}{2}\left\|\left(L_1 + \frac{b}{2}I\right)^{-1}\right\| < 1$. Therefore, there exists a unique solution of (III.30), which we denote by $\theta(v)$.

A standard argument shows that for all $v_1, v_2 \in V$ we have

$$\|\theta(v_1) - \theta(v_2)\| \le \frac{c}{1 - c}\|v_1 - v_2\|$$

and the proof is complete. ∎

If u is a solution of (III.30) and $Pu = v$ and $(I - P)u = w$, then it follows by application of $(I - P)$ to (III.26) that (III.28) holds and thus, $w = \theta(v)$. Applying P to (III.26) gives

$$Lv + Pb(v + \theta(v))^+ = c\psi_0. \tag{III.32}$$

Conversely, if (III.31) holds, then $u = v + \theta(v)$ is a solution of (III.27).

We now make a geometric study after mapping F of the RBS into itself where F is defined by

$$F(v) = Lv + Pb(v + \theta(v))^+. \tag{III.33}$$

We consider the sectors $\mathcal{S} = \{s\psi_0 + t\psi_1 | s \geq 0,\ |t| \leq ds\}$, and $-\mathcal{S} = \{s\psi_0 + t\psi_1 | s \leq 0,\ |t| \leq ds\}$, in the RBS. According to (**A.3**), if $v \in \mathcal{S}$, then $v(x) \geq 0$ for all $x \in \Omega$.

LEMMA III.16 *If* $v \in \mathcal{S}$ *and* $v = s\psi_0 + t\psi_1$, *then* $\theta(v) = 0$ *and* $F(v) = bs\psi_0 + (b - \alpha)t\psi_1$. *If* $v \in -\mathcal{S}$ *and* $v = s\psi_0 + t\psi_1$, *then* $\theta(v) = 0$ *and* $F(v) = -\alpha t\psi_1$.

Proof. If $v \in \mathcal{S}$, then $v \geq 0$ on Ω and therefore

$$(I - P)bv^+ = (I - P)bv = 0.$$

Therefore, $w = 0$ is a solution of (III.30), so, by uniqueness, $\theta(v) = 0$. It follows that if $v = s\psi_0 + t\psi_1$, then

$$F(v) = Lv + Pbv = Lv + bv = bs\psi_0 + (b - \alpha)t\psi_1.$$

Similarly, if $v \in -\mathcal{S}$, then $v(x) \leq 0$ on Ω, so $v^+ = 0$, from which it follows that $w = 0$ is a solution of (III.30). Therefore $\theta(v) = 0$ so

$F(v) = Lv = -\alpha t\psi_1$. This proves the lemma. ∎

To carry out the proof of the theorem, we analyze what the mapping F does to the two straight-line segments

$$\Gamma_1 = \left\{ s\psi_0 + \psi_1 | -\frac{1}{d} \le s \le \frac{1}{d} \right\}$$

and

$$\Gamma_2 = \left\{ s\psi_0 - \psi_1 | -\frac{1}{d} \le s \le \frac{1}{d} \right\}.$$

Let $g_k : \left(-\frac{1}{d}, \frac{1}{d}\right) \longrightarrow \mathbf{R}$, $k = 1, 2$ be defined by

$$g_1(s) = (\psi_1, F(s\psi_0 + \psi_1)),$$

$$g_2(s) = (\psi_1, F(s\psi_0 - \psi_1)).$$

Since both $-\frac{1}{d}\psi_0 + \psi_1$ and $-\frac{1}{d}\psi_0 - \psi_1$ belong to $-\mathcal{S}$, it follows from Lemma III. 14 that $g_1\left(\frac{1}{d}\right) = (\psi_1, -\alpha\psi_1) < 0$ and $g_2\left(\frac{1}{d}\right) = (\psi_1, \alpha\psi_1) > 0$.

Similarly, since both $\frac{1}{d}\psi_0 + \psi_1$ and $\frac{1}{d}\psi_0 - \psi_1$ belong to S, Lemma III.14 implies that $g_1\left(\frac{1}{d}\right) = (\psi_1, \frac{b}{d}\psi_0 + (b-\alpha)\psi_1) = (b-\alpha)(\psi_1, \psi_1) > 0$ and $g_2\left(\frac{1}{d}\right) = -(b-\alpha)(\psi_1, \psi_1) < 0$.

By continuity, we infer the existence of numbers s_1 and s_2 in the interval $\left(-\frac{1}{d}, \frac{1}{d}\right)$ such that $g_1(s_1) = g_2(s_2) = 0$.

This means that $F(s_1\psi_0 + \psi_1) = \gamma_1\psi_0$ and $F(s_1\psi_0 - \psi_1) = \gamma_2\psi_0$ for certain numbers γ_1 and γ_2. We assert that both γ_1 and γ_2 are strictly positive. In fact, we have

$$\gamma_1(\psi_0, \psi_0) = (\psi_0, Lv_1) + b(\psi_0, (v_1 + \theta(v_1))^+) = b(\psi_0, (v_1 + \theta(v_1))^+).$$

Since $\psi_0 > 0$ and $(v_1 + \theta(v_1))^+(x) \ge 0$ for $x \in \Omega$, it follows that $\gamma_1 > 0$, unless $(v_1 + \theta(v))^+(x) \equiv 0$. But, if $(v_1 + \theta(v_1))^+(x) \equiv 0$ then $\gamma_1 = 0$, and

$$\alpha\psi_1 = L(s_1\psi_0 + \psi_1) = F(s_1\psi_0 + \psi_1) = 0,$$

which is a contradiction. This shows that $\gamma_1 > 0$ and a similar argument shows that $\gamma_2 > 0$.

To finish the proof of the theorem, we note that as a consequence of the positive homogeneity of degree one of the function $\zeta \longrightarrow \zeta^+$, we can conclude that if $v \in V$, $w \in W$ and (III.30) holds, then (III.30) will remain true if v is replaced by kv and kw for any $k > 0$. This means that for $v \in V$, $\theta(kv) = k\theta(v)$, $k > 0$. Since this implies that $F(kv) = kF(v)$ for all $v \in V$ and $k > 0$, it follows that if we set $\hat{v}_1 = \frac{c}{\gamma_1}(s_1\psi_0 + \psi_1)$ and $\hat{v}_2 = \frac{c}{\gamma_2}(s_1\psi_0 - \psi_1)$, then $F(\hat{v}_1) = F(\hat{v}_2) = c\psi_0$.

From previous considerations, we see that if $u_1 = \hat{v}_1 + \theta(\hat{v}_1)$, $u_2 = \hat{v}_2 + \theta(\hat{v}_2)$, then both u_1 and u_2 are solutions of (III.27). Since $(u_1, \psi_1) > 0$ and $(u_2, \psi_1) < 0$, the proof is complete. ∎

Now we show how the abstract theorem of the previous subsection can be applied to the periodic flexing of the floating beam. We shall be considering the equation

$$U_{tt} + U_{xxxx} + aU^+ = c \qquad \text{(III.34)}$$

on the interval $(0, L)$ with the free-end boundary conditions described in the second section.

Case 1. If, for example, we take L to be of length π, then the reductive base space will be spanned by the constants (which are the ψ_0) and the function ψ_1 will be given $\phi_1(x)\cos(3t)$, with corresponding eigenvalue -3.86122036. The next negative eigenvalue corresponds to $\phi_1(x)\cos(4t)$, and is -10.86122040. If we define L to be the operator $Lu = u_{tt} + u_{xxxx}$ defined on the space H = closure span of $\{\phi_m(x)\cos nt,\ m \geq 1, n \geq 0\}$.

Thus, we are looking for periodic solutions of (III.34) which are even in t, even about the mid-point of the beam, and are periodic of period 2π. Therefore, what the main theorem of the previous section shows is that if $3.86122036 < \alpha < 10.86122040$, equation (III.34) has at least two large amplitude oscillatory solutions with a non-zero component of period $\frac{2\pi}{3}$. It is a standard result that if $0 < a < 3.86122036$, the equation (III.34) has at most one solution, the obvious $u \equiv \frac{c}{a}$. Thus, when a crosses the eigenvalue -3.86122036, we create additional solutions for a wide range of values of a.

Case 2. Now, if we increase the length of the beam to $\frac{3\pi}{2}$, the first negative eigenvalue is -2.98493242 and the second is -6.35766602. The range of a for which the solutions exist may have become smaller.

Case 3. If we consider the interval to be of length $\frac{7\pi}{6}$, then the absolute value of the first negative eigenvalue is 1.22621465 and solutions are known to exits if a exceeds this value and remains lower that 6.22621441. We should emphasize at this point that what we know for certain is that solutions definitely exist if $1.22621465 < a < 6.22621441$, and definitely do not exist if a is less than 1.22621465 (and, of course, greater than zero). It is a limitation of the method of proof that we can say nothing about the case where a is greater that 6.22621441.

Case 4. The case of the interval being of length $\frac{13\pi}{3}$ is interesting. In this case, the first negative eigenvalue is -0.02660656 and thus if a is greater than -0.02660656, and less than 0.57441235, we know that large-scale flexing solutions exist. Furthermore, the proof actually shows that they have a component of the form $\lambda_4(x)\cos(3t)$. This is a higher-frequency multi-noded oscillation and is presumably more dangerous than the simpler motions described earlier.

We conclude this section with some remarks. Although the theorem only deals with the **unforced** equation, it is possible to use the same methods to prove a similar theorem with a small forcing term of the form $\epsilon f(x,t)$, so long as the forcing term satisfies the same symmetry restrictions that were used in the theorem. Notice that our methods give no information on the case where the periodic is not a rational multiple of the length.

Numerical computation suggests that the solutions are indeed stable. However, thus seems to be difficult to prove. Numerical solutions indicate that there are solutions which favor one end. Can one prove that these exist ? How about for larger values of b ? Can one find more solutions ?

Chapter IV

Directions for further research and study

We remark that these lectures have focused exclusively on degree theoretic methods for nonlinear boundary value problems.

We point out here several other methods which can be used and also several other problems.

(i) Variational methods.

These have been used by Solimini [SOL], Hofer [HOF] and others on the problem

$$\begin{aligned} \Delta u + f(u) &= s\phi_1 + h(x) \quad \text{in } \Omega, \\ u &= 0 \end{aligned}$$

but less commonly on other types of semilinear equations. Hofer got four solutions if $b > \lambda_2$, $b \neq \lambda_k$ by showing that (a) there was a mountain pass solution and (b) that the mountain pass solution has degree -1.

(ii) Ordinary differential equation methods.

These have been used in [L-McK1] to show at least $2n$ solutions for the boundary value problem in one dimension, via shooting, and also crossing higher eigenvalues [L-McK4]. They have also been used to study the periodic boundary value problem case, and to obtain results an stability, see [L-McK5].

A subset of O.D.E. methods are the methods of global bifurcation, which have been used [RUF] to get a different proof of $2n$ solutions and crossing higher eigenvalues. Two recent results are of interest in 1992, [KWO] who gets unbounded numbers of solutions, if $f'(+\infty) = \infty$, and [CDS] who get exactly $2n$ solutions for the piecewise linear O.D.E.

(iii) Maximum principle methods.

These can be used for the Laplacian with Dirichlet boundary conditions to get results on **exact** number of solutions (e.g. exactly four of $a < \lambda_1$, $\lambda_2 < b < \lambda_3$). See [SOL] and [L-McK4] for more details.

(iv)Numerical studies.

Numerical methods arise in two ways. First, can one use standard nonlinear methods to give more information on the structure of the solution set of some of these problems? A start on this way made in [CJM] using continuation methods.

Second, can one develop new numerical max-min methods to **find** the additional oscillatory solutions where existence is proved in this paper. Progress in this direction has been made recently by [CM] for the elliptic case and [CMR] for the hyperbolic case. A large amount of work remains to be done in this area.

(v) Problems in $\mathbf{R}^n$.

This is an intriguing area in which the only information comes from calculus. Suppose we look for travelling wave solutions of

$$u_{tt} + u_{xxxx} + bu^+ = 1 \qquad \text{on } (-\infty, \infty).$$

Then we must look four solutions of the form $1 + y(x - ct)$ and get

$$y^{(4)} + c^2 y'' + (1 + y)^+ - 1 = 0.$$

In [McK-W3], it was shown, via calculus, that this problem appears to have many solutions $y(x)$ which go to zero exponentially fast as $|x| \longrightarrow +\infty$, if $0 < c < \sqrt{2}$. How can this result be generalized ? Can one prove that there are an infinite number of solutions (ignoring translations)?

Can one generalize this result to the case where $f(u)$ is "like" $(1 + y)^+ - 1$, (or perturb this result in any way)?

Can one generalize this result to equations on $\mathbf{R}^N$? In this case, it is not clear what the appropriate restriction on C is. Can there ANY other results on $\mathbf{R}^N$? Can one establish *stability* of the travelling waves established in [McK-W3] ?

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