Introduction to Differentiable Manifolds

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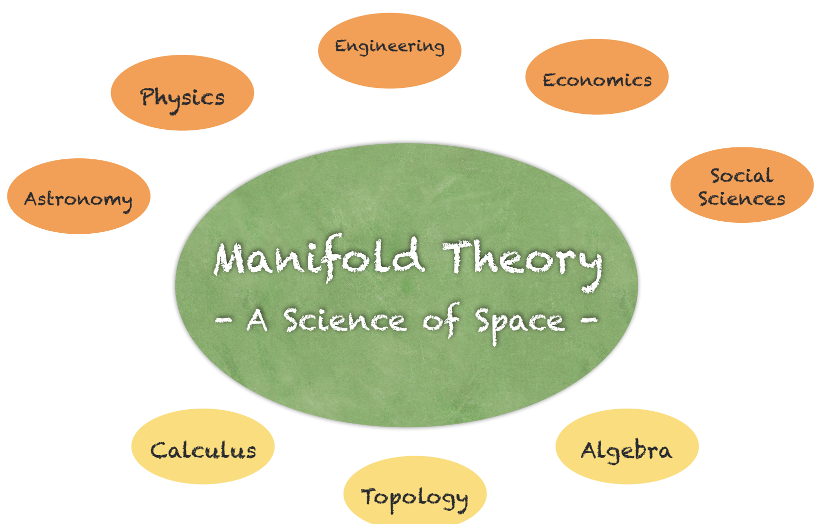
Chapter 1

Topological Manifolds



What is a space?
What are model spaces?

A differentiable manifold is a topological manifold
with a differentiable structure.
What is a topological manifold?

1.1 Locally Euclidean Spaces

A topological space M is said to be **locally Euclidean**[1] if it is locally homeomorphic to a Euclidean space, i.e., for any point p in M, there exists a neighborhood which is homeomorphic to $\mathbf{E}^n$ some nonnegative integer n. This integer n is called the **dimension** of M at p.

Theorem 1.1.0.1 (Invariance of Dimension) *At each point of a locally Euclidean space M, the dimension is well-defined.*[2]

Proof. Let $x : U \to \mathbb{R}^n$ and $y : V \to \mathbb{R}^m$ be two homeomorphisms for some neighborhoods U and V of p. We may assume that U and V are open neighborhoods of p. Then[3]

$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

is a homeomorphism between a non-empty open subset of $\mathbb{R}^m$ and an open subset of $\mathbb{R}^n$. Thus by (a corollary of) the Invariance of Domain (see Appendix), $n = m$. □

For a locally Euclidean space M, let $\dim(M, p)$ be the dimension of M at p. Then the map

$$\dim(M, \ ) : M \to \mathbb{Z}, \quad p \mapsto \dim(M, p)$$

is locally constant. The **dimension** of M is

$$\dim M := \sup_{p \in M} \dim(M, p).$$

1.1.0.2 Pure dimensional Space

We say that M is **of pure dimension** if every component of M has the same dimension.

Note that pure dimensional locally Euclidean spaces are of finite dimension.

1.1.0.3 Infinite dimensional Space

The "disjoint union" $\coprod_{n=0}^{\infty} \mathbb{R}^n$ is locally Euclidean and of infinite dimension.

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[1]Although *Euclidean spaces* have more structures, we take only topological properties of them and follow the traditional terminology. A metric space is a *Euclidean space* if it is isometric to a Cartesian space $\mathbb{R}^n$ for some nonnegative integer n. All n-dimensional Euclidean spaces are isometric to each other, and they are all denoted by $\mathbb{E}^n$. The Cartesian space $\mathbb{R}^n$ is an example of an n-dimensional Euclidean space.

[2]Since our main interests are smooth manifolds, Invariance of Dimension follows easily from Linear Algebra: *If an open subset of $\mathbb{R}^n$ is diffeomorphic to an open subset of $\mathbb{R}^m$, then $n = m$.*

[3]The *composition* $g \circ f$ of maps $f : A \to B$ and $g : C \to D$ is defined on $f^{-1}(B \cap C)$.

1.1.0.4 Zero-dimensional Spaces

Note that $\mathbb{R}^0 = \{0\}$. If the dimension of M at p is zero, then p is an isolated point of M, and the trivial map $\{p\} \to \{0\}$ is the homeomorphism.

A topological space is a locally Euclidean space of dimension zero if and only if it is a discrete space.

1.1.0.5

The next easy theorem is often useful.

Theorem 1.1.0.6 *Let n be a nonnegative integer and let M be a set. Suppose $\mathcal{A}$ is a collection of maps*

$$x : U_x \to \mathbb{R}^n \tag{1.1}$$

such that

(i) *each U_x is nonempty and $M = \bigcup\{U_x \mid x \in \mathcal{A}\}$*

(ii) *for any $x \in \mathcal{A}$, x is injective and its image $x(U_x)$ is an open subset of $\mathbb{R}^n$*

(iii) *for any $x, y \in \mathcal{A}$, the composition*

$$x \circ y^{-1} : y(U_x \cap U_y) \to x(U_x \cap U_y)$$

is a homeomorphism between open subsets of $\mathbb{R}^n$.

Then there exists a unique topology on M such that $\{U_x \mid x \in \mathcal{A}\}$ is an open cover of M and the map (1.1) is a coordinate system for each $x \in \mathcal{A}$.

Moreover, this topology is Hausdorff if and only if for any distinct points $p \in U_x$ and $q \in U_y$, there exist open subsets $V_x \subset x(U_x)$ and $V_y \subset y(U_y)$ such that

$$x(p) \in V_x, \quad y(q) \in V_y, \quad x^{-1}(V_x) \cap y^{-1}(V_y) = \varnothing.$$

Proof. Let a subset S of M be open if and only if $x(S \cap U_x)$ is open in $\mathbb{R}^n$ for every $x \in \mathcal{A}$. $\square$

1.1.1 The Line with two origins

Let $\mathbb{R}_* := \mathbb{R} - \{0\}$ and

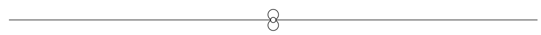
$$\mathbb{R}^+ := \mathbb{R} \times \{1\}, \quad \mathbb{R}^+_* := \mathbb{R}_* \times \{1\}, \quad \mathbb{R}^- := \mathbb{R} \times \{-1\}, \quad \mathbb{R}^-_* := \mathbb{R}_* \times \{-1\}.$$

Let $\mathbb{R}^+ \cup_f \mathbb{R}^-$ be the quotient space of $\mathbb{R}^+ \cup \mathbb{R}^-$ by the identification map[4]

$$f : \mathbb{R}^+_* \to \mathbb{R}^-_*, \quad (x, 1) \mapsto (x, -1).$$

Then $\mathbb{R}^+ \cup_f \mathbb{R}^-$ is locally $\mathbb{R}^1$, but not Hausdorff.

[4] For topological spaces X, Y, and a continuous map f from a subset A of X into Y, the *space $X \cup_f Y$ obtained by attaching X to Y by f* is the quotient space of the disjoint union $X \coprod Y$, where every point in A is identified with its image. See Appendix "General Topology" for the basic notions of topology.



The line with two origins

1.1.2 Exercises

1. Show that each point in a locally Euclidean space of pure dimension n has a neighborhood homeomorphic to $\mathbb{R}^n$.

2. Show that a one-to-one continuous map between locally Euclidean spaces of the same pure dimension is an open map.

3. Show that every locally Euclidean space M is locally connected[5] and hence each (connected) component of M is (closed and) open. An open subset of a locally Euclidean space is also locally Euclidean. In particular, each component of a locally Euclidean space is also locally Euclidean.

4. Show that a locally Euclidean space is locally compact[6].

[5] A topological space is *locally connected* if any neighborhood of an arbitrary point contains a connected neighborhood.

[6] A topological space is *locally compact* if any point has a compact neighborhood. For a Hausdorff space X, the following conditions are equivalent [Dugundji, p.238]:

(a) X is locally compact.

(b) for any point p in X and an open neighborhood U of p, there exists a compact neighborhood K of p contained in U.

(c) For any compact set K contained in an open set U of X, there exists a compact neighborhood C of K contained in U.

(d) X has a basis consisting of relatively compact open subsets.

Euclid of Alexandria by Justus van Gent, 15th century

1.2 Topological Manifolds

Theorem 1.2.0.1 *Let M be a locally Euclidean Hausdorff topological space. Then the following are equivalent.*

(i) *M is metrizable.*

(ii) *M is paracompact.*

(iii) *Every component of M is σ-compact.*[7]

(iv) *Every component of M is a Lindelöf space.*[8]

(v) *Every component of M is second countable.*[9]

Proof. (i) $\Rightarrow$ (ii). This follows from a result of A. H. Stone (1948) which says that *every metric space is paracompact* [Dugundji, p.186], [Kelley, p.160], [Marsden et al], [Munkres 2000], [Willard, p.147].

(ii) $\Rightarrow$ (iii). Note that every closed subspace of a paracompact space is again paracompact. Thus each component of M is paracompact. Now it suffices to show that any connected, locally compact paracompact space X is σ-compact.

Let $\mathcal{U}$ be an open cover of X such that the closure $\overline{U}$ is compact for every $U \in \mathcal{U}$. Such a cover exists, since X is locally compact. Now since X is paracompact, we may assume that $\mathcal{U}$ is locally finite.

Let W_0 be any nonempty set in $\mathcal{U}$. We construct subsets W_k, $k = 1, 2, \ldots$, of X inductively as follows:

$$W_k := \bigcup \{U \in \mathcal{U} \mid U \cap W_{k-1} \neq \varnothing\}.$$

Obviously, W_k's are open and $W_0 \subset W_1 \subset W_2 \subset \ldots$. We claim that $X = \bigcup_{k=1}^{\infty} W_k$. Since X is connected, it suffices to show that $\cup W_k$ is closed. (Clearly, it is open and nonempty.) Let p be a boundary point of $\cup W_k$ in X. Then $p \in U$ for some $U \in \mathcal{U}$ and hence $U \cap W_k \neq \varnothing$ for some k. This implies that $p \in W_{k+1}$. Thus $\cup W_k$ is closed, and hence $X = \cup W_k = \cup \overline{W_k}$.

Now we claim that $\overline{W_k}$ is compact, by induction on k. Of course, $\overline{W_0}$ is compact. Suppose that $\overline{W_{k-1}}$ is compact. Then W_{k-1} intersects with only a finite number of U's in $\mathcal{U}$ and hence W_k has a compact closure. Moreover, $\overline{W_k} \subset W_{k+1}$. This proves the claim.

(iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v).

[7] A topological space is *σ-compact* if it is a countable union of compact subsets.

[8] A topological space is called a *Lindelöf space* if every open cover has a countable subcover. cf. [Matsushima]

[9] A topological space is *second countable* if it has a *countable basis* for the topology. Recall that a collection $\mathcal{B}$ of open subsets is a *basis* for the topology if and only if each neighborhood of a point p contains $B \in \mathcal{B}$ such that $p \in B$ i.e., every open subset is the union of a subcollection of $\mathcal{B}$. Note that a subspace of a second countable space is second countable.

Note that locally Euclidean space is locally second countable[10], and any subspace of locally second countable space is also locally second countable.

Let X be a component of M. It is locally second countable, and hence each point p in X has a second countable neighborhood U_p.

If X is compact, then there exists a finite number of points $p_1, \dots, p_k$ in X such that $X = U_{p_1} \cup \cdots \cup U_{p_k}$. Thus X is second countable.

If X is a countable union of compact sets $X_1, X_2 \dots$, then each X_i is second countable and hence X is second countable.

(v) $\Rightarrow$ (i). Note that a locally compact Hausdorff space is, obviously, regular.[11] In particular, any locally Euclidean space is regular. Recall Urysohn's Metrization Theorem [Dugundji, p.195], [Willard, p.166]: *A regular and second countable space is metrizable*.

Thus each component of a locally Euclidean space is metrizable, and hence every locally Euclidean space is metrizable.[12] □

1.2.1 Topological Manifolds

A locally Euclidean Hausdorff space of pure dimension is called a **topological manifold** if it satisfies one of the above equivalent conditions.[13]

Zero-dimensional topological manifolds are discrete spaces.

A **curve** is a manifold of dimension 1.

A **surface** is a manifold of dimension 2.

In the next chapter, we will introduce a **differentiable manifold** which will be simply called a **manifold**.

We need the Hausdorff condition for a manifold because we want the **uniqueness of the solution** of an ODE.

1.2.1.1 Compact Exhaustion

In the proof of the theorem (1.2.0.1) we have actually shown that a connected topological manifold has a compact exhaustion.[14] In particular, a locally com-

[10]A topological space is *locally second countable* if any neighborhood of an arbitrary point contains a second countable neighborhood.

[11]A topological space X is *regular* if for any closed set A in X and a point $x \in X - A$ there exist disjoint open subsets U and V of X with $x \in U$ and $A \subset V$.

[12]For any metric d on a set X, the new metric $d/(1+d)$ gives the same topology as d. Thus there is a metric d on M such that on each component $d < 1$ and $d(p,q) = 1$ if p and q are not in the same component.

[13]Some authors require the second countable condition. I prefer to include every discrete space as a zero-dimensional space.

[14]A sequence $(C_k)_{k=1}^{\infty}$ of compact subsets of a topological space X is called a *compact exhaustion* if

$$X = \bigcup_{k=1}^{\infty} C_k \quad \text{and} \quad \forall k (C_k \subset \mathrm{Int}(C_{k+1})).$$

pact space is σ-compact if and only if it has a compact exhaustion [Dugundji, p. 241].

Theorem 1.2.1.2 Every second countable topological manifold M is homeomorphic to a subset of $\mathbb{R}^N$ for some positive integer N.

Proof. We will prove the case when M is compact.[15] Take a finite open cover $\mathcal{U} := \{U_1, \ldots, U_k\}$ of M where each U_i has a topological embedding[16] $x_i : U_i \to \mathbb{R}^n$. Let $\{\rho_1, \ldots, \rho_k\}$ be a partition of unity subordinate to the cover $\mathcal{U}$ (indexwise). Then for each i, define $\tilde{x}_i : M \to \mathbb{R}^n$ by

$$\tilde{x}_i(p) := \begin{cases} \rho_i x_i & \text{if } p \in U_i \\ \mathbf{0} & \text{if } p \in M - \operatorname{supp} \rho_i. \end{cases}$$

Now let

$$f := (\tilde{x}_1, \ldots, \tilde{x}_k, \rho_1, \ldots, \rho_k) : M \to \mathbb{R}^N \qquad (N = k(n+1)).$$

Then f is continuous.

We claim that f is one-to-one. Let p be a point in M. Since $\rho_1(p) + \cdots + \rho_k(p) = 1$, there exists $j \in \{1, \ldots, k\}$ such that $\rho_j(p) \neq 0$. Then $p \in U_j$. If $q \in M$ and $f(p) = f(q)$, then $\rho_j(q) \neq 0$ and hence q is also in U_j. Thus $\rho_j(p)x_j(p) = \tilde{x}_j(p) = \tilde{x}_j(q) = \rho_j(q)x_j(q) = \rho_j(p)x_j(q)$, and hence $x_j(p) = x_j(q)$, which implies $p = q$. Thus f is one-to-one.

Since M is compact, f is an embedding.[17] □

1.2.2 Fundamental Group

The first fundamental group of a connected manifold is countable [John Lee, 2011]

1.2.3 Infinite dimensional manifolds

Infinite dimensional topological spaces, e.g., Hilbert spaces, Banach spaces, Frechét spaces, are all interesting model spaces for infinite dimensional manifolds, although we do not consider them here. See [Lang], for instance.

[15] For general case, see [Munkers, 2000].

[16] A *topological embedding* is a map which is homeomorphic to its image.

[17] A one-to-one continuous map from a compact space into a Hausdorff space is an embedding.

1.2.4 Exercises

1. Is the set $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ a topological manifold?

2. Show that $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset \mathbb{R}$ is not a manifold.

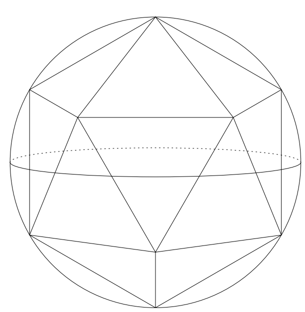
3. Let X be the set of all lines in the plane $\mathbb{R}^2$. Make X into a connected topological manifold. What is the dimension of X? Is X homeomorphic to an open Möbius band?

4. Let $P(n)$ be the set of all $n \times n$ real positive definite (symmetric) matrices. Show that, with natural topology on $P(n)$, it is homeomorphic to the Euclidean space of dimension $n(n+1)/2$.

5. Let X be the collection of all quadratic polynomials
$$q(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2 + 2(b_1x + b_2y) + c$$
with real coefficients. Identify two quadratic polynomials q_1 and q_2 if there is a **Euclidean motion** i.e., an isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $q_1 = T^*q_2$. This gives an equivalence relation on X and let M be the quotient space. Is M a topological manifold? (Hint: To each quadratic polynomial q assign a matrix $Q = \begin{pmatrix} a_{20} & a_{11} & b_1 \\ a_{11} & a_{02} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$. Show that
$$\det Q, \quad \det \begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix}, \quad \operatorname{tr} \begin{pmatrix} a_{20} & a_{11} \\ a_{11} & a_{02} \end{pmatrix}$$
are complete invariants.)

6. What kind of topological spaces do you get by attaching following spaces?
   (i) $\mathbb{R}$ and $\mathbb{R}$ with $f(a) = 1/a$.
   (ii) $\mathbb{R}^2$ and $\mathbb{R}^2$ with $f(a,b) = (a, 1/b)$.
   (iii) $\mathbb{R}^2$ and $\mathbb{R}^2$ with $f(a,b) = (1/a, 1/b)$.
   (iv) $\mathbb{R}^2$ and $\mathbb{R}^2$ with $f(a,b) = (1/a, b/a)$.

7. Let M be the collection of all icosahedra inscribed in a given sphere in $\mathbb{R}^3$. What is the dimension of M?



Chapter 2

Differentiable Manifolds

Über die Hypothesen welche der Geometrie zugrunde liegen.
(On the hypotheses that lie at the foundation of Geometry)

June 10, 1854, G. F. B. Riemann

Bernhard Riemann, 1822—1866

A differentiable manifold is a space where one can talk about tangent vectors, or the velocities of moving particles. One can do calculus on differentiable (usually non linear) manifolds.

If we want to talk about the acceleration of a particle in a space, forces, and curvatures, then we need more than a smooth structure, e.g., a parallelism or a connection.

2.1 Coordinate Systems

What is the **coordinates** of a point in a space?

Given a topological space M and a point p in M, a (local) **coordinate system**[1] (or a **coordinate map**) at p is a one-to-one continuous map

$$x = (x^1, \dots, x^n) : U \to \mathbb{R}^n$$

of some open neighborhood U of p onto an open subset of $\mathbb{R}^n$ for some non-negative integer n, such that the inverse map $x^{-1} : x(U) \to U$ is also continuous.[2]

The ordered collection $(x^1(p), \dots, x^n(p))$ of numbers is called the **coordinates** of p with respect to the coordinate system x.

This open set U is called a **coordinate neighborhood** of p, or just a **coordinate domain**.

If the domain of a coordinate system is the whole space M, then x is called a **global** coordinate system. The identity map

$$x : \mathbb{R}^n \to \mathbb{R}^n$$

is an example of a global coordinate system.

A topological manifold is a Hausdorff topological space, each of its components can be covered by countably many coordinate domains.

The inverse map of a coordinate map is called a **parametrization**.

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[1]The concept of coordinates is due to R. Descartes and P. Fermat, or due to earlier mathematical painters in the middle age. The name *coordinates* is due to G. Leibniz [Coxeter].

[2]A map $f : X \to Y$ between two topological spaces is called a *topological embedding*[3] if it is a homeomorphism onto its image, i.e., f is a continuous one-to-one map such that the inverse $f^{-1} : f(X) \to X$ is also continuous. Thus a coordinate system is a topological embedding onto an open subset of a Cartesian space.

2.2 Differentiable Structures

Let M be a topological manifold of dimension n.

2.2.1 Definitions

1. A set $\mathcal{A} = \{x : U_x \to \mathbb{R}^n\}$ of coordinate systems on M is called a smooth (or differentiable) **atlas** if

   (i) $M = \bigcup_{x \in \mathcal{A}} U_x$, and

   (ii) for each $x, y \in \mathcal{A}$, the (coordintae) **transition map**, i.e., the composition[4]
   $$x \circ y^{-1}$$
   is a smooth ($\mathcal{C}^\infty$) map (between ope subsets of $\mathbb{R}^n$).

   A coordinate system in a smooth atlas is also called a smooth **chart**.

2. Two smooth atlases are **equivalent** if their union is also a smooth atlas.

   It should be easy to check that this relation is an equivalence relation.

   From now on an **atlas** means a smooth atlas.

3. An atlas $\mathcal{A}$ is said to be **maximal** if it is not contained properly in any other atlas, i.e., if y is a local coordinate system on M such that
   $$y \circ x^{-1}, \qquad x \circ y^{-1}$$
   are smooth for any $x \in \mathcal{A}$, then $y \in \mathcal{A}$.

   Note that any atlas $\mathcal{A}$ is contained in a unique maximal atlas $\overline{\mathcal{A}}$, the **completion** of $\mathcal{A}$. In fact, $\overline{\mathcal{A}}$ is the union of all atlases which are equivalent to $\mathcal{A}$.

4. A **differentiable** (or **smooth**) **structure** on M is a maximal atlas on it. Or equivalently, we may say that a differentiable structure on M is an equivalence class of atlases.

   A topological manifold M together with a smooth atlas $\mathcal{A}$ has a unique differentiable structure which contains $\mathcal{A}$.

5. A topological manifold together with a differentiable structure is called a **differentiable** (or **smooth**) manifold.

   Thus if $\mathcal{A}$ and $\mathcal{A}'$ are two differentiable atlases on a topological manifold M, then
   $$(M, \mathcal{A}) = (M, \mathcal{A}')$$
   if and only if the completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ is the same as the completion $\overline{\mathcal{A}'}$ of $\mathcal{A}'$.

[4] This composition is a homeomorphism between open subsets of $\mathbb{R}^n$.

We often write a smooth manifold as M instead of $(M, \mathcal{A})$ when there is no confusion.

From now on **manifolds** mean smooth manifolds.

2.2.1.1

If a topological manifold M has a global chart $x : M \to \mathbb{R}^n$, then the singleton $\mathcal{A} = \{x\}$ is a smooth atlas, since the only transition map

$$x \circ x^{-1}$$

is the identity map.

Thus any global coordinate system on a topological space defines a smooth structure.

2.2.2 $\mathbb{R}^n$

The set $\{\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n\}$ consisting of the identity map on $\mathbb{R}^n$ is an atlas (of class $\mathcal{C}^\omega$).

This atlas gives the **standard** differentiable structure on $\mathbb{R}^n$.

2.2.2.1 Finite dimensional vector spaces

Let V be a finite dimensional real vector space.[5] For any basis $\mathbf{b} = (b_1, \ldots, b_n)$ of V, let $\mathbf{b}^* := (b_1^*, \ldots, b_n^*)$ be the dual basis of $\mathbf{b}$. Then the isomorphism

$$\mathbf{b}^* : V \to \mathbb{R}^n. \tag{2.1}$$

induces a topology on V, and V is homeomorphic to $\mathbb{R}^n$. This topology is independent of $\mathbf{b}$.[6]

The isomorphism (2.1) also defines a smooth structure on V, which is independent of the choice of basis, and hence it is a smooth manifold.

For instance, the complex n-space $\mathbb{C}^n$ and the quaternion n-space $\mathbb{H}^n$ are smooth manifolds.

For positive integers m and n, and a ring R, let $M_{m\times n}(R)$ be the space of $m \times n$ matrices with entries in R. Then for $R = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $M_{m\times n}(R)$ is a vector space over $\mathbb{R}$ of dimension $mn, 2mn, 4mn$, respectively.[7]

[5] E.g., the space of all polynomials in one-variable with real coefficients of degree less than n.

[6] In fact, any finite dimensional real vector space V has a unique *vector topology*, i.e., a Hausdorff topology which makes the addition and the scalar multiplication continuous [Rudin, p.16], [Dieudonné, Vol. 2, (12.13.2.ii)].

[7] We can identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$; $(z_1 + jw_1, \ldots, z_n + jw_n) \mapsto (z_1, w_1, \ldots, z_n, w_n)$. Then we have an inclusion

$$i : M_{m\times n}(\mathbb{H}) \hookrightarrow M_{2m\times 2n}(\mathbb{C}).$$

For instance, if $z, w \in \mathbb{C}$, then $z + jw \in \mathbb{H}$ corresponds to $\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. The image

2.2.3 Affine Spaces

An **affine space modeled on a vector space** $\vec{\mathbb{A}}$ is a set $\mathbb{A}$ together with a map

$$\mathbb{A} \times \mathbb{A} \to \vec{\mathbb{A}}, \quad (a, b) \mapsto \overrightarrow{ab}$$

such that

1. for any $a, b, c \in \mathbb{A}$, $\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac}$
2. for any $a \in \mathbb{A}$, the map

$$O_a : \mathbb{A} \to \vec{\mathbb{A}}, \quad b \mapsto \overrightarrow{ab} \tag{2.2}$$

   is a bijection.

In other words, an affine space is set $\mathbb{A}$ together with a free transitive action of a vector space $\vec{\mathbb{A}}$ (as an additive group) on it.

Thus for any $a, b \in \mathbb{A}$, there exists a unique vector $\overrightarrow{ab} \in V$ such that $b = a + \overrightarrow{ab}$ or $b - a = \overrightarrow{ab}$. Each point a in $\mathbb{A}$ defines a **vectorization** (2.2). The topologies on $\mathbb{A}$ induced by vectorizations are all the same and hence every finite dimensional affine spaces are manifolds.

A **Euclidean space** is an affine space whose associated vector space is equipped with an inner product.

2.2.4 Open subsets

Any open subset of a manifold is a manifold.

Since each component of a manifold is open (and closed), it is a manifold. Thus a manifold is a 'disjoint union' of connected manifolds.

For instance, if $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then the vector space $M_n(F)$ of $n \times n$ square matrices with entries in F is a smooth manifold in the canonical way. The group $\mathrm{GL}(n, F)$ of invertible $n \times n$ matrices is an open subset of $M_n(F)$ and hence it is a smooth manifold of dimension $\dim_{\mathbb{R}}(F) \times n^2$. They are called the **general linear groups**.

The group $\mathrm{GL}^+(n, \mathbb{R})$ of all $n \times n$ real matrices with positive determinant is a component of $\mathrm{GL}(n, \mathbb{R})$. $\mathrm{GL}(n, \mathbb{R})$ has two components and each component is a smooth manifold.

Let k, n be integers such that $1 \le k \le n$.

Let $F_k(\mathbb{R}^n)$ be the collection of all k-tuple

$$(v_1, \dots, v_k)$$

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of the inclusion i is

$$\{X \in M_{2m \times 2n}(\mathbb{C}) \mid J_m \bar{X} = X J_n\},$$

where J_k is the direct sum of k copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

of linearly independent vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$. Then we have a canonical inclusion

$$F_k(\mathbb{R}^n) \hookrightarrow M_{n\times k}(\mathbb{R}) =: M_{n\times k}.$$

Then a matrix $X \in M_{n\times k}$ is in $F_k(\mathbb{R}^n)$ if and only if

$$\sum_J \left(\det(X_J)\right)^2 \neq 0$$

where X_J denotes the J-th $k \times k$-minor of X for $J = (j_1, \ldots, j_k)$, $1 \leq j_1 < \cdots < j_k \leq n$. Thus $F_k(\mathbb{R}^n)$ is an open subset of $M_{n\times k}$ and hence $F_k(\mathbb{R}^n)$ is a smooth manifold.

In general, if V is an n-dimensional vector space (over $\mathbb{R}$), then the collection $F_k(V)$ of all k-tuples of linearly independent vectors in V is an open subset of $\prod^k V := \underbrace{V \times \cdots \times V}_{n}$, since there is a continuous (multilinear) map

$$\prod^k V \to \bigwedge^k V$$

and $F_k(V)$ consists of the inverse images of nonzero elements.

2.2.5 Diffeomorphic Structures

Let $\mathcal{A}$ be a smooth atlas on a topological manifold M. If M' is a topological manifold and $h : M' \to M$ is a homeomorphism, then

$$h^*\mathcal{A} := \{x \circ h \mid x \in \mathcal{A}\}$$

is a smooth atlas on M'.

We say that two smooth structures $\mathcal{A}$ and $\mathcal{A}'$ on topological manifolds M and M', respectively, are **diffeomorphic** if there exists a homeomorphism $h : M' \to M$ such that $h^*\mathcal{A} = \mathcal{A}'$ (or equivalently, $h^*\mathcal{A} \subset \mathcal{A}'$).

For instance, if $\mathcal{A}$ is the standard smooth structure on $\mathbb{R}$ defined by the identity map $x : \mathbb{R} \to \mathbb{R}$, then the homeomorphism $x^3 : \mathbb{R} \to \mathbb{R}$ defines a new smooth structure $\mathcal{A}'$ on $\mathbb{R}$, which is diffeomorphic to the standard one. Note that $\mathcal{A}'$ does not contain the identity map $x : \mathbb{R} \to \mathbb{R}$.

2.2.5.1 Remark

If $n \neq 4$, then $\mathbb{R}^n$ has a unique differentiable structure up to diffeomorphisms. But $\mathbb{R}^4$ has uncountably many non diffeomorphic differentiable structures [Tau].

2.2.6 Exercises

1. For each homeomorphism $h : \mathbb{R} \to \mathbb{R}$, let $\mathbb{R}_h$ be the real line $\mathbb{R}$ together with the maximal $\mathcal{C}^\infty$ atlas $\mathcal{A}_h$ containing h. Show that $\mathbb{R}_h$ is diffeomorphic

to the standard $\mathbb{R}$. Show that $\mathcal{A}_h$ is not equal to the standard $\mathcal{C}^\infty$ structure of $\mathbb{R}$ unless h is a diffeomorphism. How many different $\mathcal{C}^\infty$ structures are there on a topological space $\mathbb{R}$?

2. If $x : X \to Y$ is a map, then

$$x^* : \mathrm{Map}(Y, \mathbb{R}) \to \mathrm{Map}(X, \mathbb{R})$$

is the pull-back map: $x^*(f) = f \circ x$ for any $f \in \mathrm{Map}(Y, \mathbb{R})$.

Let n be a positive integer. Show that a connected Hausdorff space M together with a sheaf $\mathcal{F}$ of (real valued partial) functions on M is a smooth manifold of dimension n if and only if there exists a countable atlas $\mathcal{A}$ such that for any chart $x : U \to \mathbb{R}^n$ in $\mathcal{A}$ and any open set $V \subset U$,

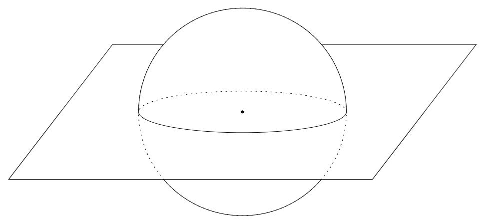
$$\mathcal{F}(V) = x^*(\mathcal{C}^\infty(x(V))).$$

2.3 Spheres

The n-**sphere**

$$\mathbf{S}^n := \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1^2 + \cdots + a_{n+1}^2 = 1\}$$

is a compact connected manifold[8] of dimension n.



For $i = 1, \ldots, n+1$, let

$$U_i^+ := \{(a_1, \ldots, a_{n+1}) \in \mathbf{S}^n \mid a_i > 0\}, \quad U_i^- := \{(a_1, \ldots, a_{n+1}) \in \mathbf{S}^n \mid a_i < 0\}.$$

Then these $2(n+1)$ open subsets[9] cover $\mathbf{S}^n$. Let x_i^+ be the restriction to U_i^+ of the projection map

$$\pi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad (a_1, \ldots, a_{n+1}) \mapsto (a_1, \ldots, \widehat{a_i}, \ldots, a_{n+1})$$

where hatted component means the deleted one. Then the image of x_i^+ is the standard open unit ball

$$\mathbf{B}^n := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 < 1\}$$

in $\mathbb{R}^n$. Similarly let x_i^- be the restriction of π_i to U_i^-. Then we have a smooth atlas

$$\mathcal{A} := \{x_1^+, x_1^-, \ldots, x_{n+1}^+, x_{n+1}^-\}.$$

for the sphere $\mathbf{S}^n$.

Stereographic projections define anther atlas $\mathcal{A}'$ on the sphere. Let $p_+ := (0, \ldots, 0, 1)$ and $p_- := (0, \ldots, 0, -1)$ be the **north** and the **south** poles of $\mathbf{S}^n$. Then the **stereographic projection**

$$x_+ : \mathbf{S}^n - \{p_+\} \to \mathbb{R}^n$$

[8] Any subspace S of a metric space (X, d) is a metric space. But the *inner metric* on S is more interesting than the induced metric. The distance between two points p and q of S with respect to the inner metric is the infimum of the length of paths in S joining p and q (for simplicity, we assume that S is path connected). The length of a path $\gamma : [0,1] \to S$ is the supremum of $\sum_{i=1}^k d(\gamma(t_{i-1}), \gamma(t_i))$ for all partitions $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval [0,1].

[9] For each $p \in \mathbf{S}^n$, the hemisphere $U_p := \{q \in \mathbf{S}^n \mid p \cdot q > 0\}$ is an open neighborhood of p. These open subsets cover the sphere. Note that $-U_p = U_{-p}$ and $U_p \cap U_{-p} = \varnothing$.

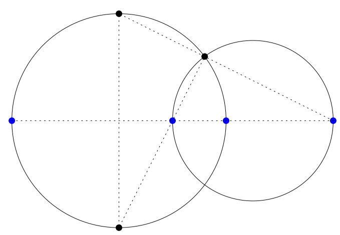
of $\mathbf{S}^n$ from the north pole is given by

$$x_+(a_1, \ldots, a_{n+1}) := \frac{1}{1 - a_{n+1}}\,(a_1, \ldots, a_n).$$

This map is obviously continuous and its inverse is

$$(a_1, \ldots, a_{n+1}) = x_+^{-1}(r_1, \ldots, r_n) = \frac{1}{|r|^2 + 1}\,\left(2r_1, \ldots, 2r_n, |r|^2 - 1\right)$$

where $|r| = (r_1^2 + \cdots + r_n^2)^{\frac{1}{2}}$.



Similarly, we have the stereographic projection

$$x_- : \mathbf{S}^n - \{p_-\} \to \mathbb{R}^n$$

of $\mathbf{S}^n$ from the south pole. Note that if $\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the reflexion through the hyperplane $\mathbb{R}^n = \{x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$, then

$$x_- = x_+ \circ \rho \qquad \text{on} \quad \mathbf{S}^n - \{p_-\}.$$

Thus

$$x_-(a_1, \ldots, a_{n+1}) = \frac{1}{1 + a_{n+1}}\,(a_1, \ldots, a_n)$$

and $x_-^{-1} = \rho \circ x_+^{-1}$, i.e.,

$$(a_1, \ldots, a_{n+1}) = x_-^{-1}(r_1, \ldots, r_n) = \frac{1}{|r|^2 + 1}\,\left(2r_1, \ldots, 2r_n, 1 - |r|^2\right).$$

Note that

$$x_- \circ x_+^{-1}(r) = \frac{r}{|r|^2} =: I(r), \qquad r \in \mathbb{R}^n_* := \mathbb{R}^n - \{0\},$$

where $I : \mathbb{R}^n_* \to \mathbb{R}^n_*$ is the **inversion** along the unit sphere $\mathbf{S}^{n-1}$. Thus $x_- = I \circ x_+$ on $\mathbf{S}^n - \{p_+, p_-\}$. We have an atlas

$$\mathcal{A}' := \{x_+, x_-\}$$

on $\mathbf{S}^n$.

Two atlases $\mathcal{A}, \mathcal{A}'$ on $\mathbf{S}^n$ are compatible and they define the same smooth structure on $\mathbf{S}^n$.

2.3.0.1 Exercises

1. Show that the inversion is **orientation reversing**, i.e., it has the negative Jacobian determinant.

2. Show that $\mathbf{S}^n$ is connected.

3. Show that

$$C := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \max\{|a_1|, \ldots, |a_n|\} = 1\}$$

   is homeomorphic to the sphere $\mathbf{S}^{n-1}$, and hence it has a smooth structure. But C is NOT a **smooth submanifold** of $\mathbb{R}^n$.

4. Show that there is a set-theoretic bijection between $\mathbf{S}^n$ and $\mathbb{R}$ and hence $\mathbb{R}$ has a topology and smooth structure which is diffeomorphic to $\mathbf{S}^n$. Of course, we are not interested in this smooth structure.

2.4 Differentiable Maps

What is the use of a differentiable structure?

Since we have 'objects', i.e., (differentiable) manifolds, now it is time to introduce 'morphisms' between them.

A **continuous map** $\varphi : M \to N$ between manifolds is said to be **differentiable** ($\mathcal{C}^\infty$, or **smooth**), if for any charts[10] x of M and y of N, the composition

$$y \circ \varphi \circ x^{-1},$$

which is defined on open subset of a Euclidean space with values in a Euclidean space, is $\mathcal{C}^\infty$.

The identity maps are smooth. The composition of two smooth maps are smooth. In particular, the restriction of a smooth map to an open subset is also smooth.

Diffeomorphisms are defined in a standard way. This concept is the same as the one we have introduced already. If M and N are diffeomorphic, then we write it as

$$M \simeq N.$$

Each chart on a smooth manifold is a smooth **embedding**, i.e., a diffeomorphism onto its image.

The set of self-diffeomorphisms of M is a group, denoted by Diff(M).

Note that a function $f : M \to \mathbb{R}$ is differentiable if and only if for any (local) chart x of M, the composition $f \circ x^{-1}$ is differentiable. The algebra of smooth functions on M is denoted by $\mathcal{C}^\infty(M)$ and it is a subalgebra of $\mathcal{C}^0(M)$:

$$\mathcal{C}^\infty(M) \subset \mathcal{C}^0(M).$$

If $\varphi : M \to N$ is smooth, we have the **pull-back** homomorphism

$$\varphi^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M), \quad f \mapsto f \circ \varphi.$$

2.4.0.1 Exercises

1. Show that the exponential map

$$e : \mathbb{R} \to \mathbf{S}^1, \quad \theta \mapsto (\cos\theta, \sin\theta)$$

   is a smooth map. Show that it is a **local diffeomorphism**.

2. Show that the inclusion map $\mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.

3. Show that if $\varphi : M \to N$ and $\psi : N \to L$ are smooth maps, then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* : \mathcal{C}^\infty(L) \to \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M).$$

[10] From now on, a chart on a $\mathcal{C}^\infty$ manifold M is a chart in the given differentiable structure of M.

4. Let $\varphi : M \to N$ be an arbitrary map between smooth manifolds. Let $\{U_\alpha\}$ be an open cover of M such that the **restriction map** $\varphi|U_\alpha$ is smooth for each α. Show that φ is smooth.

2.5 Projective Spaces

2.5.1 Real Projective Spaces

The real projective space $\mathbf{P}^n := \mathbf{P}^n(\mathbb{R})$ is, by definition, the space of all lines in $\mathbb{R}^{n+1}$ through the origin.[11] Thus it is the quotient of $\mathbb{R}^{n+1}_* := \mathbb{R}^{n+1} - \{0\}$ by the action of non zero scalars $\mathbb{R}_* := \mathbb{R} - \{0\}$. Let

$$\tilde{\pi} : \mathbb{R}^{n+1}_* \to \mathbf{P}^n \tag{2.3}$$

be the quotient map.[12]

To construct coordinate systems, let

$$[a_1, \dots, a_{n+1}] := \pi(a_1, \dots, a_{n+1}) \in \mathbf{P}^n$$

be the line through the point $(a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}_*$ and the origin in $\mathbb{R}^{n+1}$. Now for $i = 1, \dots, n+1$, let

$$U_i = \{[a_1, \dots, a_{n+1}] \in \mathbf{P}^n \mid a_i \neq 0\}.$$

Then $\{U_i : i = 1, \dots n+1\}$ is an open cover of $\mathbf{P}^n$ and the coordinate systems

$$x_i : U_i \to \mathbb{R}^n, \quad [a_1, \dots, a_{n+1}] \mapsto \left(\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_{n+1}}{a_i}\right)$$

form a smooth atlas.

Note that each $x_i : U_i \to \mathbb{R}^n$ is a homeomorphism and

$$\mathbf{P}^n - U_{n+1} = \{[a_1, \dots, a_n, 0] \in \mathbf{P}^n\} = \mathbf{P}^{n-1}.$$

This is often denoted by

$$\mathbf{P}^n = \mathbf{A}^n \cup \mathbf{P}^{n-1}.$$

Thus $\mathbf{P}^n$ is an n-manifold.[13]

The restriction of the smooth map (2.3) to the unit sphere $\mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}_*$ is surjective, and hence $\mathbf{P}^n$ is compact and connected. Since two antipodal points in $\mathbf{S}^n$ maps to the same point in $\mathbf{P}^n$, $\mathbf{P}^n$ is diffeomorphic to the quotient $\mathbf{S}^n/\mathbb{Z}_2$.

We may regard $\mathbf{S}^n$ as the collection of oriented lines through the origin (or the collection of all directions) in $\mathbb{R}^{n+1}$. Then the quotient map

$$\pi : \mathbf{S}^n \to \mathbf{P}^n$$

means that each line has two orientations.

---

[11] We may regard $\mathbf{P}^n$ as the set of equivalence classes of lines in $\mathbb{R}^{n+1}$, where two lines are equivalent if and only if they are *parallel*. For a vector space V, $\mathbf{P}(V)$ denotes the set of all 1-dimensional subspaces in V. Thus $\mathbf{P}^n(\mathbb{R}) = \mathbf{P}(\mathbb{R}^{n+1})$. Note that $\mathbf{P}^0$ is a singleton.

[12] The angles between two lines define a metric topology on the projective space. Thus projective spaces are Hausdorff spaces.

[13] An *n-manifold* is a manifold of dimension n.

2.5.1.1 Exercises

1. Show that the map
$$\mathbf{S}^1 \to \mathbf{S}^1, \quad z \mapsto z^2$$
descends to a diffeomorphism $\mathbf{P}^1 \simeq \mathbf{S}^1$.

2. The dual projective n-space $\mathbf{P}^{*n}$ is the collection of all linear subspaces of codimension 1 in $\mathbb{R}^{n+1}$. Show that
$$\mathbf{P}^n \simeq \mathbf{P}^{*n}.$$

2.5.2 Complex Projective Spaces

The complex projective space $\mathbf{P}^n(\mathbb{C})$ is the space of all complex lines through the origin in $\mathbb{C}^{n+1}$. Thus $\mathbf{P}^n(\mathbb{C}) = \mathbb{C}_*^{n+1}/\mathbb{C}_*$, where $\mathbb{C}_*^{n+1} = \mathbb{C}^{n+1} - \{0\}$ and $\mathbb{C}_*$ denotes the group of non zero complex numbers. If

$$\pi : \mathbb{C}_*^{n+1} \to \mathbf{P}^n(\mathbb{C})$$

denotes the canonical projection, then the sphere

$$\mathbf{S}^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}$$

projects onto $\mathbf{P}^n(\mathbb{C})$ and hence $\mathbf{P}^n(\mathbb{C})$ is compact and connected. There is a circle action on $\mathbf{S}^{2n+1}$ given by

$$e^{i\theta} : (z_1, \ldots, z_{n+1}) \mapsto (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$$

and hence
$$\mathbf{P}^n(\mathbb{C}) \simeq \mathbf{S}^{2n+1}/\mathbf{S}^1.$$

As before, the subsets

$$U_i := \{[z_1, \ldots, z_{n+1}] \in \mathbf{P}^n(\mathbb{C}) : z_i \neq 0\}$$

are homeomorphic to $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, for each $i = 1, \ldots, n+1$, and hence $\mathbf{P}^n(\mathbb{C})$ is a $2n$-manifold(or a complex n-manifold).

$\mathbf{P}^1(\mathbb{C})$ is also called the Riemann sphere.

2.5.2.1 Exercise

Note that $\mathbf{S}^2 \subset \mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$. Show that the map

$$\mathbf{P}^1(\mathbb{C}) \to \mathbf{S}^2, \quad [z_1, z_2] \mapsto \left(\frac{2z_1\overline{z_2}}{|z_1|^2 + |z_2|^2}, \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} \right)$$

is a diffeomorphism.

2.5.3 Quaternion Projective Spaces

The quaternion projective space $\mathbf{P}^n(\mathbb{H})$ is the space of all quaternion lines through the origin in $\mathbb{H}^{n+1}$, where $\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\} = \{z + wj : z, w \in \mathbb{C}\}$ is the ring of quaternions. It is a compact connected manifold of dimension $4n$. Since the quaternions are not commutative, we have to fix either a right or a left action.

2.5.3.1 Exercise

Show that the $\mathbf{P}^1(\mathbb{H})$ is diffeomorphic to $\mathbf{S}^4$.

2.6 Products

The product $M \times N$ of two smooth manifolds M and N is a smooth manifold of dimension $n + m$, where $n = \dim M$ and $m = \dim N$.

If $\mathcal{A}$ is the maximal atlas of M and $\mathcal{B}$ is the maximal atlas of N, then

$$\mathcal{A} \times \mathcal{B} = \{x \times y \mid x \in \mathcal{A},\ y \in \mathcal{B}\}$$

defines the smooth structure on $M \times N$.

The projection maps

$$\mathrm{pr}_1 : M \times N \to M, \qquad \mathrm{pr}_2 : M \times N \to N$$

are smooth.

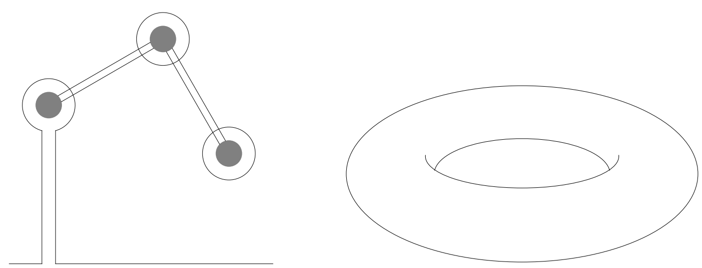
If L is a manifold, then a map $\varphi : L \to M \times N$ is smooth if and only if $\mathrm{pr}_i \circ \varphi$ is smooth for $i = 1, 2$.

2.6.1 Tori

The n-dimensional torus

$$\mathbf{T}^n := \underbrace{\mathbf{S}^1 \times \cdots \times \mathbf{S}^1}_{n}$$

is an n-dimensional compact connected manifold.



2.6.1.1

In [Nelson, p.33], the configuration space of a car moving on a plane is identified with $\mathbb{R}^2 \times \mathbf{T}^2$.

2.6.2 Polar Coordinates

The product

$$\mathbb{R}_+ \times \mathbf{S}^{n-1}$$

is a smooth n-manifold, where $\mathbb{R}_+$ is the space of positive real numbers. This manifold has a global coordinate system

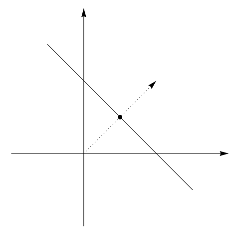
$$x : \mathbb{R}_+ \times \mathbf{S}^{n-1} \to \mathbb{R}^n_* \subset \mathbb{R}^n, \quad (r, z) \mapsto rz.$$

where $\mathbb{R}^n_* = \mathbb{R}^n - \{0\}$.

2.6.3 Lines in the Plane

Let M be set of all oriented lines in $\mathbb{R}^2$. Then M is diffeomorphic to the cylinder

$$\mathbb{R} \times \mathbf{S}^1 \simeq M, \quad (r, e^{i\theta}) \mapsto x\cos\theta + y\sin\theta = r.$$



Now the map

$$\alpha : \mathbb{R} \times \mathbf{S}^1 \to \mathbb{R} \times \mathbf{S}^1, \quad (r, e^{i\theta}) \mapsto (-r, -e^{i\theta})$$

is an involution, i.e., $\alpha^2 = \mathrm{id}$, and the quotient

$$\ddot{M} := (\mathbb{R} \times \mathbf{S}^1)/\alpha$$

is the space of all (un-oriented) lines in the plane, which is diffeomorphic to the Möbius band.

Note that each line in $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$ defines a plane through the origin in $\mathbb{R}^3$ and hence

$$\mathbf{P}^2 - \{\bullet\}$$

is a Möbius band.

$$\begin{array}{ccc} \mathbb{R}^n \times \mathbf{S}^n & \longrightarrow & (\mathbb{R}^n \times \mathbf{S}^n)/\mathbb{Z}_2 \\ \big\downarrow & & \big\downarrow \\ \mathbf{S}^n & \xrightarrow[\pi]{} & \mathbf{P}^n \end{array}$$

2.7 Submanifolds of Euclidean Spaces

Let m, n be nonnegative integers. A subset M in $\mathbb{R}^m$ is a callled a (smooth) **submanifold** of dimension n in $\mathbb{R}^m$ if for any point $p \in M$, there exist

(i) an open neighborhood W of p in $\mathbb{R}^m$,

(ii) an open set V in $\mathbb{R}^n$,

(iii) a $\mathcal{C}^\infty$ map $h : V \to W \subset \mathbb{R}^m$ such that $h(V) = W \cap M$,

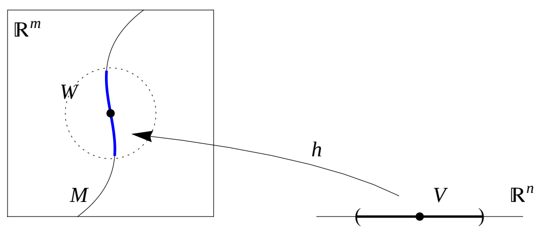
which satisfies

(iv) h is an **immersion**, i.e., for any $q \in V$, the vectors

$$D_1 h(q) := \frac{\partial h}{\partial x^1}(q), \quad \dots , \quad D_n h(q) := \frac{\partial h}{\partial x^n}(q)$$

in $\mathbb{R}^m$ are linearly independent,[14] and

(v) h is (one-to-one and) homeomorphic to its image.



The homeomorphism h is called a local **parametrization** of M, and its inverse h^{-1} is a local **coordinate system** or a **chart** for M.

Theorem 2.7.0.1 *A submanifold of a Euclidean space is a smooth manifold.*

Proof. Since M is a second countable Hausdorff topological manifold, we have to show that the coordinate transition maps are smooth. Let $p \in M$ and let x, y are two coordinate maps in a neighborhood of p such that x^{-1} and y^{-1} are smooth immersions. Since $y^{-1} : \mathbb{R}^n \dashrightarrow \mathbb{R}^m$ is smooth, for any projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$, the composition $\pi \circ y^{-1}$ is smooth.

Note that, from the inverse function theorem (or, inverse mapping theorem), there exists a projection map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ such that the map

$$\pi \circ x^{-1} : \mathbb{R}^n \dashrightarrow \mathbb{R}^m \to \mathbb{R}^n$$

is a local diffeomorphism at $x(p)$. Thus the composition

$$x \circ y^{-1} = (\pi \circ x^{-1})^{-1} \circ (\pi \circ y^{-1})$$

[14] In other words, the Jacobian matrix of h at q has rank n. The map h which satisfies the conditions (iv) and (v) is called an (smooth) *embedding*.

is a local diffeomorphism at $y(p)$. □

In fact, Whitney's embedding theorem says that every (abstract) second countable manifold has an embedding in some Euclidean space.

2.7.1 Graphs

Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be smooth. Then the graph of f

$$\Gamma := \{(x, y) \in U \times \mathbb{R} \mid f(x) = y\} \subset \mathbb{R}^{n+1}$$

is a smooth n-manifold, since the parametrization

$$h : U \to \Gamma \subset \mathbb{R}^{n+1}, \quad \mathbf{t} \mapsto (\mathbf{t}, f(\mathbf{t}))$$

is a one-to-one smooth map such that the vectors

$$D_1 h(\mathbf{t}),\ \ldots,\ D_n h(\mathbf{t})$$

in $\mathbb{R}^{n+1}$ are linearly independent for every $\mathbf{t} \in U \subset \mathbb{R}^n$, and its inverse (i.e., the global coordinate system)

$$h^{-1} : \Gamma \to U \hookrightarrow \mathbb{R}^n$$

is continuous, which is the restriction of the canonical projection map $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$.

Note that the inclusion

$$\Gamma \hookrightarrow \mathbb{R}^{n+1}$$

is smooth.

2.7.1.1 Exercise

If $\varphi : M \to N$ be a smooth map, then its graph

$$\Gamma := \{(x, y) \in M \times N \mid y = \varphi(x)\} \subset M \times N$$

is smooth manifold diffeomorphic to M, and the inclusion map $\Gamma \hookrightarrow M \times N$ is smooth.

2.7.2 Regular Level Sets in a Euclidean space

Let U be an open set in $\mathbb{R}^n$ and let

$$f : U \to \mathbb{R}$$

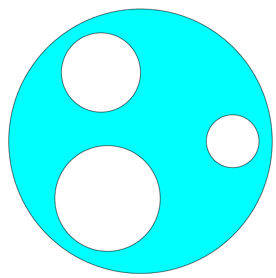
be a smooth function such that for any p in the zero locus $Z(f)$ of f, $\operatorname{grad} f(p) \neq \mathbf{0}$. (In this case, we say that the value 0 of f is a regular value of f.) Then $Z(f)$

is a smooth **hypersurface**, i.e., a submanifold of codimension 1, because it s locally a graph of a smooth map by the implicit function theorem.

Here is a concrete example. For a positive integer g, let

$$D_1 := D_1(p_1, r_1), \quad \ldots, \quad D_g := D(p_g, r_g)$$

be disjoint closed disks in $\mathbb{R}^2$ centered at the points $p_1, \ldots, p_g$ with radii $r_1, \ldots, r_g$, respectively. Let $D_0 := D(p_0, r_0)$ be a closed disk in the plane which contains the previous disks $D_1, \ldots, D_g$ in its interior.



the region with $f(\mathbf{x}) \geq 0$

Let
$$f(\mathbf{x}) := (r_0^2 - |\mathbf{x} - p_0|^2)(|\mathbf{x} - p_1|^2 - r_1^2) \cdots (|\mathbf{x} - p_g|^2 - r_g^2)$$
for $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Then

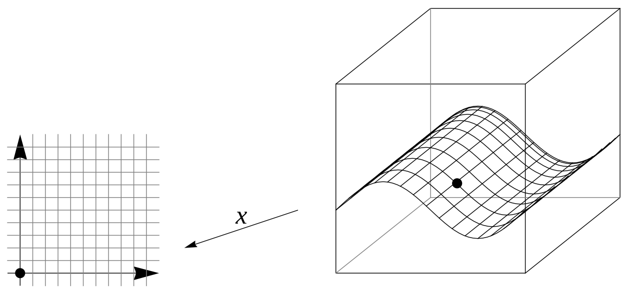
- $f(\mathbf{x}) > 0$ if and only if $\mathbf{x}$ is in the interior of the large disc D_0 and is in the exterior of the small discs D_k for $k = 1, \ldots, g$.
- $f(\mathbf{x}) < 0$ if and only if either $\mathbf{x}$ is in the exterior of the disc D_0 or in the interior of one of the discs $D_1, \ldots, D_g$.
- $f(\mathbf{x}) = 0$ if and only if $\mathbf{x}$ lies on one of the boundaries of $D_0, D_1, \ldots, D_g$. In this case $\operatorname{grad} f(\mathbf{x}) \neq \mathbf{0}$.

Now let
$$F(x, y, z) := z^2 - f(x, y) \qquad (x, y, z) \in \mathbb{R}^3.$$
Then 0 is a regular value of F and the zero-level set Z of F is a compact connected orientable[15] surface of genus g.

[15] cf. (3.6.2), (6.1.4.2)

2.8 Partial Derivatives

Let $x = (x^1, \ldots, x^n)$ be a coordinate system at a point p of a manifold M, and let f be a smooth function defined on M (or defined in an open neighborhood of p).



Then the (ordinary) i-th partial derivative of the function $f \circ x^{-1}$ (defined on an open subset of $\mathbb{R}^n$) at a point $x(p)$ is denoted by

$$\frac{\partial f}{\partial x^i}(p), \quad \left.\frac{\partial f}{\partial x^i}\right|_p, \quad \text{or} \quad \left.\frac{\partial}{\partial x^i}\right|_p f$$

for $i = 1, 2, \ldots, n$. Thus if $e_1, \ldots, e_n$ denotes the standard basis for $\mathbb{R}^n$ and $c_i(t) := x^{-1}(x(p) + te_i)$ is the i-th coordinate curves, for $t \in \mathbb{R}$ with small $|t|$, then

$$\frac{\partial f}{\partial x^i}(p) = D_i(f \circ x^{-1})(x(p)) = \left.\frac{d}{dt}\right|_0 (f \circ x^{-1})(x(p) + te_i) = \left.\frac{d}{dt}\right|_0 f(c_i(t)).$$

For instance,

$$\frac{\partial x^i}{\partial x^j}(p) = \delta^i_j := \begin{cases} 1 & (i = j) \\ 0 & (i \neq 0). \end{cases}$$

If $y = (y^1, \ldots, y^n)$ is another coordinate system at p, then the **chain rule** says that

$$\left.\frac{\partial f}{\partial y^j}\right|_p = \sum_{i=1}^n \left.\frac{\partial f}{\partial x^i}\right|_p \left.\frac{\partial x^i}{\partial y^j}\right|_p \qquad (j = 1, \ldots, n).$$

Note that the matrices $\left(\left.\frac{\partial x^i}{\partial y^j}\right|_p \right)$ and $\left(\left.\frac{\partial y^i}{\partial x^j}\right|_p \right)$ are the inverses of each other.

2.8.1 Polar Coordinates

In $\mathbb{R}^2$, let (x, y) be the standard "identity coordinate system". Let $\mathbb{R}^2_* := \mathbb{R}^2 - \{0\}$. Then the **polar coordinate system** on $\mathbb{R}^2_*$ is a partial map

$$u := (r, \theta) : \mathbb{R}^2_* \dashrightarrow \mathbb{R}_+ \times \mathbb{R}$$

such that $\epsilon \circ u(z) = (|z|, z/|z|)$, where $\epsilon(a, t) = \big(a, e^{it}\big)$.

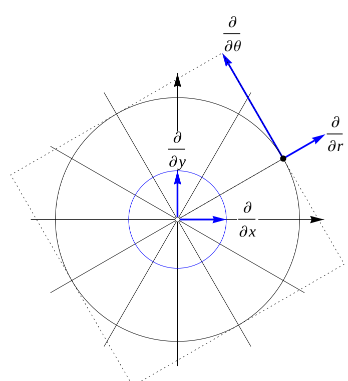
$$\begin{tikzcd}
 & \mathbb{R}_+ \times \mathbb{R} \arrow[d, "\epsilon"] \\
\mathbb{R}^2_* \arrow[ur, dotted, "u"] \arrow[r, "\simeq"'] & \mathbb{R}_+ \times \mathbf{S}^1
\end{tikzcd}$$

Now we have a relation between the functions on $\mathbb{R}^2_*$:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then we have

$$\begin{cases} \dfrac{\partial}{\partial r} &= \cos\theta\dfrac{\partial}{\partial x} + \sin\theta\dfrac{\partial}{\partial y} \\[2ex] \dfrac{\partial}{\partial \theta} &= r\left(-\sin\theta\dfrac{\partial}{\partial x} + \cos\theta\dfrac{\partial}{\partial y}\right). \end{cases} \tag{2.4}$$



Although the function θ is not globally defined (as a continuous function) on $\mathbb{R}^2_*$, $\frac{\partial}{\partial\theta}$ is globally defined.

2.8.2 Chain Rule

Let $\varphi : M \to N$ be a smooth map and let $g : N \to \mathbb{R}$ be a smooth function. Let $x = (x^1, \ldots, x^n)$ be a chart at a point p of M, and let $y = (y^1, \ldots, y^m)$ be a chart at $\varphi(p)$ of N. Then

$$\left.\frac{\partial(g\circ\varphi)}{\partial x^j}\right|_p = \sum_{i=1}^m \left.\frac{\partial g}{\partial y^i}\right|_{\varphi(p)} \left.\frac{\partial \varphi^i}{\partial x^j}\right|_p ,$$

where $\varphi^i := y^i \circ \varphi$.

2.8.3 Rank of a map

If $\varphi : M \to N$ is a smooth map between manifolds, $x = (x^1, \dots, x^n)$ is a coordinate system at $p \in M$, $y = (y^1, \dots, y^m)$ is a coordinate system at $\varphi(p) \in N$ and $\varphi^i = y^i \circ \varphi$, then the rank of the matrix

$$\left(\frac{\partial \varphi^i}{\partial x^j}(p)\right)$$

is independent of the choice of the chart x and y, and hence we get a notion of **rank** of a smooth map. The rank of φ at a point p will be denoted by $\operatorname{rank}\varphi(p)$. We have the map

$$\operatorname{rank}\varphi : M \to \mathbb{Z}.$$

Note that given a point $p \in M$, there exists a neighborhood U of p such that for any $q \in U$,

$$\operatorname{rank}\varphi(q) \geq \operatorname{rank}\varphi(p).$$

Thus $\operatorname{rank}\varphi$ is lower semi-continuous.[16]

2.8.4 Exercises

1. (Spherical Coordinates) In $\mathbb{R}^3$, let (x, y, z) be the identity coordinates. Then the spherical coordinates (ρ, ϕ, θ) are given by

$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \tan\theta = \frac{y}{x} \end{cases}$$

   Determine a region where (ρ, ϕ, θ) becomes a diffeomorphism. Compute $(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta})$ in terms of $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

2. Let k be a real number and let f be a real valued C^1 function defined on $\mathbb{R}^n - \{\mathbf{0}\}$ such that

$$f(t\mathbf{x}) = t^k f(\mathbf{x})$$

   for any positive real number t and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}} - \{\mathbf{0}\}$.

[16] A map f from a topological space M to $\mathbb{R}$ is *lower semi-continuous* if for any $t \in \mathbb{R}$, the set $\{p \in M \mid f(p) \leq t\}$ is closed (or, equivalently, the set $\{p \in M \mid f(p) > t\}$ is open) in M.

(a) Show that if $t > 0$, then $\frac{\partial f}{\partial x^i}(t\mathbf{x}) = t^{k-1}\frac{\partial f}{\partial x^i}(\mathbf{x})$ for all $i = 1, \ldots, n$.

(b) Show **Euler's identity**

$$x^1\frac{\partial f}{\partial x^1} + \cdots + x^n\frac{\partial f}{\partial x^n} = kf.$$

(c) For $(y^2, \ldots, y^n) \in \mathbb{R}^{n-1}$, let

$$f_1(y^2, \ldots, y^n) := f(1, y^2, \ldots, y^n).$$

Show that if $f_1(p) = 0$ for some $p \in \mathbb{R}^{n-1}$, then

$$\left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)(1, p) = \mathbf{0}$$

if and only if

$$\left(\frac{\partial f_1}{\partial y^2}, \ldots, \frac{\partial f_1}{\partial y^n}\right)(p) = \mathbf{0}.$$

2.8.5 Immersions and submersions

Let $\varphi : M \to N$ be a smooth map. Then φ is called

(i) an **immersion**[17] at $p \in M$, if $\operatorname{rank}\varphi(p) = \dim M$, or equivalently, the vectors

$$\begin{pmatrix} \frac{\partial\varphi^1}{\partial x^1}(p) \\ \vdots \\ \frac{\partial\varphi^m}{\partial x^1}(p)\end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} \frac{\partial\varphi^1}{\partial x^n}(p) \\ \vdots \\ \frac{\partial\varphi^m}{\partial x^n}(p)\end{pmatrix}$$

are linearly independent, for any (or for some) local coordinate systems $x = (x^1, \ldots, x^n)$ at p in M and $y = (y^1, \ldots, y^m)$ at $\varphi(p)$ in N.

(ii) a **submersion**[18] at $p \in M$, if $\operatorname{rank}\varphi(p) = \dim N$, i.e., the vectors

$$\left(\frac{\partial\varphi^1}{\partial x^1}(p), \ldots, \frac{\partial\varphi^1}{\partial x^n}(p)\right), \quad \ldots, \quad \left(\frac{\partial\varphi^m}{\partial x^1}(p), \ldots, \frac{\partial\varphi^m}{\partial x^n}(p)\right)$$

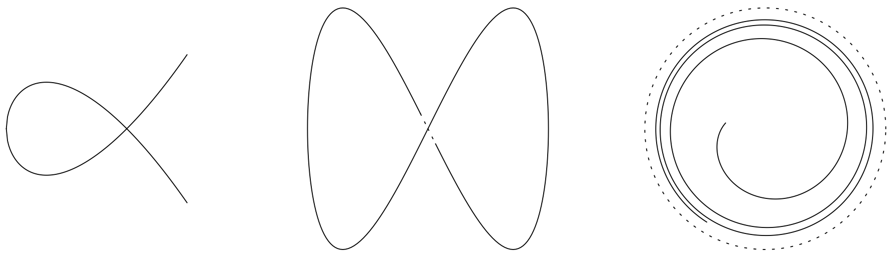
are linearly independent, with respect to some (or any) local coordinate systems $x = (x^1, \ldots, x^n)$ for M and $y = (y^1, \ldots, y^m)$ for N.

(iii) **étale** at $p \in M$, if $\operatorname{rank}\varphi(p) = \dim M = \dim N$,

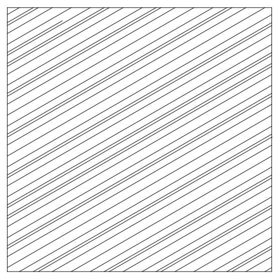
(iv) an **embedding** if φ is an immersion (at every point of M) and homeomorphic to its image.

[17] We will see soon that an immersion is a map whose derivative is injective at the given point.

[18] We will see soon that a submersion is a map whose derivative is surjective at the given point.



`immersion,` `one-to-one immersion,` `embedding`



`A` *`skew`* `(or` *`irrational`*`) line immersed in a torus`

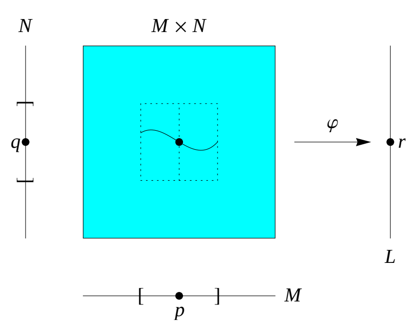
Note that if φ is an immersion (resp. submersion) at p, then it is an immersion (resp. submersion) in a neighborhood of p.

Theorem 2.8.5.1 (Inverse Function Theorem) *If $\varphi : M \to N$ is étale at p, then φ is a local diffeomorphism at p.*

Theorem 2.8.5.2 (Implicit Function Theorem) *Let $\varphi : M \times N \to L$ be $\mathcal{C}^\infty$, $(p, q) \in M \times N$ and $r = \varphi(p, q)$. If*

$$\varphi_p : N \to L, \qquad y \mapsto \varphi(p, y)$$

is étale at $y = q$,[19] then there exist a neighborhood U_p of p in M, a neighborhood V_q of q in N, and a $\mathcal{C}^\infty$ map $\psi : U_p \to V_q$ such that, for $(x, y) \in U_p \times V_q$, $\varphi(x, y) = r$ if and only if $y = \psi(x)$.



[19] Thus we must have $\dim N = \dim L$.

The following theorem may be found in many standard texts.

Theorem 2.8.5.3 (Rank Theorem) *Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map with $\varphi(0) = 0$. If $\operatorname{rank}(\varphi, 0) = k$, then there exist a chart x centered*[20] *at $0 \in \mathbb{R}^m$ and a chart y centered at $0 \in \mathbb{R}^n$ such that*

$$y^i \circ \varphi(x^1, \dots, x^m) = x^i \quad \text{for } 1 \le i \le k.$$

If $\operatorname{rank} \varphi = k$ in a neighborhood of $0 \in \mathbb{R}^m$, then we can further assume that

$$y^j \circ \varphi(x^1, \dots, x^m) = 0 \quad \text{for } k < j \le n.$$

2.8.6 Exercises

1. (Steiner's **Roman Surface**) Show that the map

   $$f : \mathbf{P}^2 \to \mathbb{R}^4, \quad (x : y : z) \mapsto (yz, zx, xz, x^2 + 2y^2 + 3y^2)/(x^2 + y^2 + z^2)$$

   is an embedding.

2. An immersion is locally an embedding.

3. If $\varphi : M \to N$ is a one-to-one immersion and U is a relatively compact[21] open subset of M, then the restriction map

   $$\varphi \restriction U \to N$$

   is an embedding.

4. A closed[22] one-to-one immersion is an embedding.

5. Show that *a continuous map from a compact space into a Hausdorff space is a closed map*.

6. A map $f : M \to N$ is said to be **proper** if the preimage of a compact set is compact. Show that *a continuous proper map from a topological space into a manifold is a closed map*.

7. Show that submersions are open maps.

8. Let $f_1, \dots, f_k : M \to \mathbb{R}$ be smooth functions such that $\operatorname{rank}(f_1, \dots, f_k) = k$ at some point $p \in M$. Show that there exist smooth functions $f_{k+1}, \dots, f_n : M \to \mathbb{R}$ such that $(f_1, \dots, f_k, f_{k+1}, \dots, f_n)$ is a coordinate system in a neighborhood of p.

9. Let L, M, N be smooth manifolds, and let $g : M \to N$ an immersion. Show that if $f : L \to M$ is a continuous map such that $g \circ f : L \to N$ is smooth, then f is smooth.

[20] A chart x at p is said to be *centered at p* if $x(p) = 0$.
[21] A subset of a topological space is *relatively compact* if its closure is compact.
[22] A map is *closed* if the image of a closed set is closed.

2.9 Submanifolds

Theorem 2.9.0.1 *Let S be a subset of an n-manifold M, and let k be a positive integer less than or equal to n. Then for any $p \in S$, the following three conditions are equivalent:*

(i) there exists an open neighborhood U (in M) such that $S \cap U$ is the common zero set of some smooth functions $f^1, \ldots, f^k$ on U with $\operatorname{rank}(f^1, \ldots, f^k) \equiv k$.

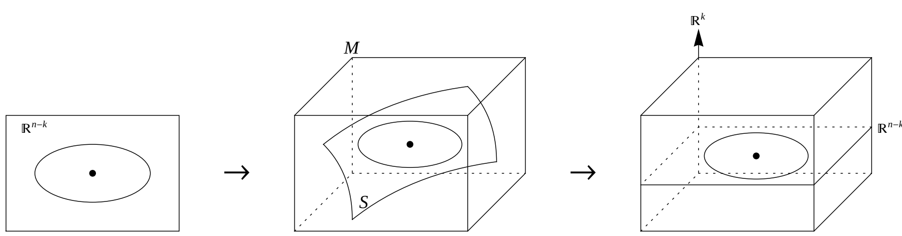
(ii) there exists an open neighborhood U of p in M and a coordinate system $x : U \to \mathbb{R}^n$ such that

$$x(U \cap S) = x(U) \cap \mathbb{R}^{n-k}.$$

(iii) there exists an open neighborhood U of p in M, an open set V in $\mathbb{R}^{n-k}$, and a homeomorphism $h : V \to U \cap S$ such that the map

$$h : V \to U \cap S \hookrightarrow U$$

is an immersion.



A subset S of an n-manifold M satisfying one of the above equivalent conditions is called a (**regular** or **embedded**) **submanifold**[23] of M of *codimension* k.

Submanifolds of codimension 1 are called **hypersuface**s.
Submanifolds of codimension n are discrete subsets.
Open subsets on a manifold are submanifolds of codimension 0.

A submanifold S of an n-manifold M of codimension k is a (smooth) manifold of dimension $n-k$. There exists a canonical differentiable structure on S. For instance, in the case of (i) in the above theorem, implicit function theorem supplies the charts.

Submanifolds are always assumed to be equipped with this differentiable structure. Note that the inclusion map

$$\operatorname{inc} : S \hookrightarrow M$$

[23]Sometimes *one-to-one immersions* are called submanifolds. This convention is suitable for the Frobenius theorem for involutive distributions, and for the Lie Theory. Note that an immersion is locally an embedding. For other types of submanifolds see [Sharpe] or [Jeffrey Lee].

is $\mathcal{C}^\infty$. The differentiable structure on S is the unique differentiable structure satisfying the following property: *For any smooth manifold N, a map $\varphi : N \to S$ is smooth if and only if the composition*

$$\mathrm{inc} \circ \varphi : N \to S \hookrightarrow M$$

is smooth.

2.9.0.2 Closed submanifolds

A submanifold S of M is **proper** if the inclusion map (2.9) is proper. *A submanifold is proper if and only if it is a closed submanifold.*

Note that if we replace the condition `for any $p \in S$' in the theorem 2.9.0.1 with `for any $p \in M$', then we have a closed submanifold.

2.9.0.3 Exercises

1. Submanifolds are locally closed.[24]
2. Show that $\mathbf{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$. The differentiable structure as a submanifold coincides with the standard one.
3. Show that $M := \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^6 = 1\}$ is diffeomorphic to the unit circle $\mathbf{S}^1$.
4. Show that the manifold given by $x^2 + y^4 + z^6 = 1$ is diffeomorphic to the sphere given by $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.
5. Let p be a point in a $\mathcal{C}^\infty$ manifold M. Take an open neighborhood U of p and a diffeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\varphi(p) = 0$. Replace U with

   $$\widehat{\mathbb{R}^n} := \{(x, y) \in \mathbb{R}^n \times \mathbf{P}(\mathbb{R}^n) : x \in y\}.$$

   This way, by gluing $\widehat{\mathbb{R}^n}$ to $M - \{p\}$, we obtain a new manifold $\hat{M}$, called the **blowing up** of M at p. Discuss a differentiable structure on $\hat{M}$. Show that the choice of φ and p does not alter the smooth structure (up to diffeomorphism) if M is connected.

2.10 Regular Values and Critical Values

Let $\varphi : M \to N$ be a smooth map. A point $p \in M$ is called a **critical point** of φ if $\operatorname{rank} \varphi(p) < \dim N$. Otherwise, p is called a **regular point** of φ.

[24] A subset S of a topological space X is said to be *locally closed at* $p \in S$ if there is a neighborhood U of p in X such that $S \cap U$ is a closed subset of U. S is said to be *locally closed in* X if it is locally closed at each $p \in S$. It is easy to see that $S \subset X$ is locally closed if and only if S is the intersection of an open subset and a closed subset of X if and only if S is open in its closure $\overline{S}$ in X.

Thus p is a regular point of φ if and only if φ is a submersion at p.

The image of a critical point is called a **critical value**.

A point q in N is called a **regular value** of φ if it is not a critical value of φ. In particular, if q is not in the image of φ, then it is a regular value of φ.

Theorem 2.10.0.1 (Regular Value Theorem) *Let q be a regular value of a map $\varphi : M \to N$. Then $\varphi^{-1}(q)$ is a submanifold of M of codimension $\dim N$, unless it is empty.*

Theorem 2.10.0.2 (Constant Rank Theorem) *If a smooth map $\varphi : M \to N$ has the constant rank k, then each level set of φ is a submanifold of M of codimension k.*

2.10.0.3 A cubic plane curve

Given real numbers a and b, consider the polynomial

$$f(x) := x^3 + ax + b.$$

Note that $f(x)$ has a multiple root if and only if $4a^3 + 27b^2 = 0$. We will assume that

$$4a^3 + 27b^2 \neq 0.$$

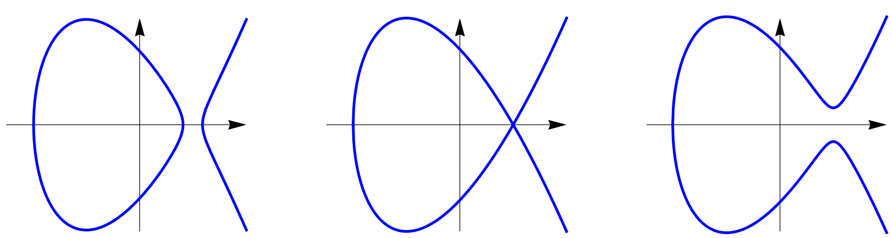
Then

$$f(x_0) = 0 \quad \Rightarrow \quad f'(x_0) \neq 0.$$

On $\mathbb{R}^2$ consider the function

$$F(x, y) = f(x) - y^2.$$

Then the zero set $E := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$ is a smooth curve in $\mathbb{R}^2$.



a=−3, b= 1.9 $\qquad$ a=−3, b= 2 $\qquad$ a=−3, b= 2.1

Cuves $y^2 = x^3 + ax + b$

2.10.0.4 Exercise

Given real numbers a, b, c, find the critical points of the map

$$f : \mathbf{S}^2 \to \mathbb{R}, \qquad (x, y, z) \mapsto ax^2 + by^2 + cz^2.$$

2.11 Lie groups

A smooth manifold G together with a group structure is called a **Lie group** if the multiplication map

$$\mu : G \times G \to G, \quad (g_1 g_2) \mapsto g_1 g_2$$

is smooth.[25]

Sophus Lie (1842—1899)

[25]A topological space G together with a group structure is a *topological group* if the multiplication map and the inversion map

$$\mu : G \times G \to G, \qquad \iota : G \to G$$

are both continuous. The 5-th Hilbert problem was the following: *Is any locally Euclidean topological group admits a smooth structure?* This problem was answered affirmatively: *Any locally Euclidean topological group admits a unique smooth structure for which the multiplication map is smooth.* Moreover, *any continuous group homomorphism between two Lie groups is smooth.*

Note that for $a, b \in G$, the left and right multiplication maps

$$L_a : G \to G,\ g \mapsto ag, \qquad R_b : G \to G,\ g \mapsto gb$$

are commuting diffeomorphisms of G. Note that

$$L_{ab} = L_a \circ L_b, \qquad R_{ab} = R_b \circ R_a$$

for any $a, b \in G$.

Theorem 2.11.0.1 *For a Lie group G, the inversion map*

$$i : G \to G, \quad i(g) = g^{-1}$$

is a diffeomorphism.

Proof. Since the left (and right) multiplication maps are smooth, it suffices to show that i is smooth on a neighborhood of the identity element e of G.[26]

Consider the set

$$S := \{(x, y) \in G \times G \mid xy = e\}.$$

Then $(e, e) \in S$. The 'derivative ' (or the Jacobian matrix with respect to a chart) of the identity map

$$G \hookrightarrow G \times G \xrightarrow{\mu} G, \quad y \mapsto (e, y) \mapsto \mu(e, y) = y$$

at the identity element e is invertible and hence by the implicit function theorem, there exist open neighborhoods U and V of e such that the inversion $i : U \to V$ is smooth. □

A map between two Lie groups is called a **Lie group homomorphism** if it is a smooth group homomorphism.

2.11.1 Examples

2.11.1.1 General Linear Groups

Let $\mathfrak{gl}(n, \mathbb{R}) = \prod^n \mathbb{R}^n := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n}$ be the vector space of $n \times n$ real matrices.

Then the map

$$\det : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R} \tag{2.5}$$

is obviously $\mathcal{C}^\infty$. The **general linear group**

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid \det A \neq 0\}$$

[26] For any $a \in G$, $i = R_a \circ i \circ L_a = L_a \circ i \circ R_a$.

of rank n is an open subset of $\mathfrak{gl}(n,\mathbb{R})$ and hence is a $\mathcal{C}^\infty$ manifold. The multiplication map

$$\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \qquad (X,Y) \mapsto XY$$

is obviously $\mathcal{C}^\infty$ and hence $\mathrm{GL}(n,\mathbb{R})$ is a Lie group.

The general linear group $\mathrm{GL}(n,\mathbb{R})$ has two components. The identity component, i.e., the component containing the identity element, consists of matrices with positive determinants.

2.11.1.2 Special Linear Groups

Now we compute the **derivative** of the determinant map (2.5) at $A = (a_1,\ldots,a_n) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ **in the direction** $B = (b_1,\ldots,b_n) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$;

$$\begin{aligned} D_B \det(A) &:= \tfrac{d}{dt}\Big|_0 \det(A+tB) = \tfrac{d}{dt}\Big|_0 \det(a_1+tb_1,\ldots,a_n+tb_n) \\ &= \det(b_1,a_2,\ldots,a_n) + \det(a_1,b_2,a_3,\ldots,a_n) + \cdots + \det(a_1,\ldots,a_{n-1},b_n). \end{aligned}$$

If $A = (e_1,\ldots,e_n) = 1_n$, the identity matrix, then $D_B \det(1_n) = \mathrm{tr}(B)$, i.e.,

$$D\det(1_n) = \mathrm{tr} : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}.$$

Thus the determinant map (2.5) is regular at 1_n.

If A is any point in $\mathrm{GL}(n,\mathbb{R})$, then

$$\begin{aligned} D_B \det(A) &= \frac{d}{dt}\Big|_0 \det(A+tB) = \frac{d}{dt}\Big|_0 \det(1_n + tBA^{-1}) \cdot \det A \\ &= \mathrm{tr}(BA^{-1}) \cdot \det A. \end{aligned}$$

Thus any nonsingular matrix is a regular point of $\det$. (In fact, $f : G \to H$ is a group homomorphism, then $L_{f(g)} \circ f \circ L_g$ for any $g \in G$, Thus if f is smooth at the identity element, then f is smooth everywhere.)

Now $0 \in \mathbb{R}$ is the only critical value of $\det$ and hence the **special linear group**

$$\mathrm{SL}(n,\mathbb{R}) := \det{}^{-1}(1)$$

of rank n is a submanifold of $\mathrm{GL}(n,\mathbb{R})$.

The multiplication map

$$\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$$

is smooth, since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) & \longrightarrow & \mathrm{SL}(n,\mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) & \underset{\mu}{\longrightarrow} & \mathrm{GL}(n,\mathbb{R}) \end{array}$$

Thus $\mathrm{SL}(n,\mathbb{R})$ is a Lie group of dimension n^2-1.

Special linear groups are connected.

Exercise. Let
$$H := \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$
Show that H is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ isomorphic to $\mathbb{R}$.

2.11.1.3 Orthogonal Groups

We now show that the orthogonal group[27]
$$\mathrm{O}(n) := \{A \in \mathfrak{gl}(n,\mathbb{R}) : A^{\mathrm{t}} A = 1_n\}$$
is a Lie group of dimension $\frac{1}{2}(n-1)n$.

For this, define a map
$$f : \mathfrak{gl}(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R}) := \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^{\mathrm{t}} = A\}$$
by $f(A) = A^{\mathrm{t}} A$. Then $\mathrm{Sym}(n,\mathbb{R})$ is a linear space of dimension $\frac{n(n+1)}{2}$ and
$$D_B f(A) = \frac{d}{dt}\bigg|_0 f(A+tB) = B^{\mathrm{t}} A + A^{\mathrm{t}} B.$$
If $A \in \mathrm{O}(n)$, then
$$Df(A) : \mathfrak{gl}(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R})$$
is surjective, for if $C \in \mathrm{Sym}(n,\mathbb{R})$, then with $B = \frac{1}{2}AC$
$$D_B f(A) = B^{\mathrm{t}} A + A^{\mathrm{t}} B = \frac{1}{2}\{(AC)^{\mathrm{t}} A + A^{\mathrm{t}}(AC)\} = C.$$
Thus $A \in \mathrm{O}(n)$ is regular and hence $\mathrm{O}(n) = f^{-1}(1_n)$ is a $\mathcal{C}^\infty$ submanifold of $\mathfrak{gl}(n,\mathbb{R})$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. The multiplication map is $\mathcal{C}^\infty$ as before and hence $\mathrm{O}(n)$ is a Lie group. Since $\mathrm{O}(n) \subset \mathbf{S}^{n-1} \times \cdots \times \mathbf{S}^{n-1}$, $\mathrm{O}(n)$ is compact.

Orthogonal group $O(n)$ has two components. the special orthogonal group
$$\mathrm{SO}(n) := O(n) \cap \mathrm{SL}(n,\mathbb{R})$$
is the identity component.

2.11.2 Exercises

1. Show that for any $x \in M_n(\mathbb{R})$,
$$\det(e^x) = e^{\operatorname{tr} x}.$$
Is this identity also true for complex square matrices?

[27] A^{t} denotes the transpose of A.

2. Show that the derivative of the determinant map $\det : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}$ at matrix A in the direction V is
$$\operatorname{tr}(V \operatorname{Adj}(A)),$$
where $\operatorname{Adj}(A)$ denotes the **adjoint matrix** of A.

3. Find the derivative of the inverse map $i : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ at a point $A \in \mathrm{GL}(n, \mathbb{R})$ in the direction $V \in \mathfrak{gl}(n, \mathbb{R})$.

4. [Tau2] Note that the space of 2×2 real matrices is isomorphic to $\mathbb{R}^2 \times \mathbb{R}^2$:
$$f : \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathrm{M}_2(\mathbb{R}), \qquad ((x, y), (u, v)) \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} -u & v \\ v & u \end{pmatrix}.$$
Then $f(x, y, u, v)$ is of determinant 1 if and only if $x^2 + y^2 = u^2 + v^2 + 1$. Show that $\mathrm{SL}(2, \mathbb{R})$ is diffeomorphic to $\mathbf{S}^1 \times \mathbb{R}^2$.

5. Show that the general linear groups
$$\mathrm{GL}(n, \mathbb{C}), \quad \mathrm{GL}(n, \mathbb{H})$$
are all Lie groups.

6. For a complex matrix X, let X^* be the conjugate-transpose of X. Show that the **unitary group**
$$\mathrm{U}(n) := \{X \in \mathrm{M}_n(\mathbb{C}) \mid X^* X = 1_n\}$$
is a Lie group of dimension n^2, and the **special unitary group**
$$\mathrm{SU}(n) := U(n) \cap SL(n, \mathbb{C})$$
is a Lie groups of dimension $n^2 - 1$.

7. For a quaternion matrix X, let X^* be the conjugate-transpose of X. Show that the (compact) **symplectic group**
$$\mathrm{Sp}(n) := \{X \in \mathrm{M}_n(\mathbb{H}) \mid X^* X = 1_n\}$$
is a Lie group of dimension $2n^2 + n$.

8. Show that $\mathbf{S}^3$ is diffeomorphic to $\mathrm{SU}(2)$.

9. Show that $\mathrm{SO}(3)$, the group of rotations in $\mathbb{R}^3$, is diffeomorphic to the projective 3-space $\mathbf{P}^3(\mathbb{R})$.

10. On $\mathbf{S}^3 \times \mathbf{S}^3$, consider the identification $(x, y) \sim (-x, -y)$. Show that the quotient space $(\mathbf{S}^3 \times \mathbf{S}^3)/\sim$ is diffeomorphic to $\mathrm{SO}(4)$.

11. For a Lie group G, let G° denote the component of G containing the neutral element $e \in G$. Show that G° is a normal subgroup of G and G/G° is a discrete Lie group.

2.11.3 Lie Subgroups

Let G be a Lie group. An abstract subgroup H of G is called a **Lie subgroup** of G if there exists a topology and a smooth structure such that the inclusion map $H \hookrightarrow G$ is an immersion and multiplication map $H \times H \to H$ is smooth.

Note that once a topology on H is fixed, there exists at most one smooth structure on H.

If H is an abstract subgroup of G and is an embedded submanifold of G, then H is a Lie subgroup. In this case, H is called an **embedded Lie subgroup** of G.

2.11.4 Closed Subgroups

Note that any embedded Lie subgroup of a Lie group is a closed subset.

Theorem 2.11.4.1 (Cartan, von Neumann) *Let H be a closed subgroup of a Lie group G. Then*

(i) H is a submanifold of G, and is a properly embedded Lie subgroup of G.

(ii) The quotient space G/H (of left cosets of H in G) has a unique smooth structure such that for any smooth manifold N, a map $\varphi : G/H \to N$ is smooth if and only if the composition $\varphi \circ \pi : G \to G/H \to N$ is smooth, where $\pi : G \to G/H$ is the quotient map.

(iii) There is a canonical left action of G on G/H.

2.11.5 Action

Given a smooth manifold M, let $\mathrm{Diff}(M)$ be the group of all self-diffeomorphisms of M.

An **action** (or a **left action**) of a Lie group G on a smooth manifold M is a group homomorphism

$$\rho : G \to \mathrm{Diff}(M)$$

such that the induced map

$$G \times M \to M, \qquad (g, p) \mapsto (\rho(g))(p) =: gp$$

is smooth.

Given an action of G on M, and a point p in M, by definition,

- the **orbit** of p is
$$\mathrm{Orb}(p) := \{gp \mid g \in G\} \subset M$$

- the **stabilizer** (or **isotropy group**) of p is
$$G_p := \{g \in G \mid gp = p\},$$
which is a closed subgroup of G.

An action of G on a manifold M is said to be **proper** if the map

$$G \times M \to M \times M, \quad (g, m) \mapsto (gm, m)$$

is a proper map.[28]

Proposition 2.11.5.1 *An action $\mu : G \times M \to M$ is proper if and only if for any compact subset K of M*

$$G_K := \{g \in G \mid gK \cap K \neq \varnothing\}$$

is compact. In this case, for any point p in M, the isotropy group $G_p := \{g \in G \mid gp = p\}$ is compact.

Note that any action of a compact group is proper.

If G is discrete, then an action is proper if and only if for any points p and q in M, there exist neighborhoods U of p and V of q such that $\{g \in G \mid gU \cap V \neq \varnothing\}$ is finite [Jeffrey Lee, p. 231].

Theorem 2.11.5.2 *Let G be a Lie group which acts on a smooth manifold M smoothly. If the action is free and proper, then there exists a unique smooth structure on the orbit space $G\backslash M$ such that the map $\pi : M \to G\backslash M$ is a submersion. In this case, π is a fiber bundle map with the fiber G.*

For a proof, cf. [Jeffrey Lee, p.236], [Dieudonné, 16.10.3], [Gallier].

Theorem 2.11.5.3 *Let G be a discrete group of diffeomorphisms on a smooth manifold M. If the action is free and proper, then there exists a unique smooth structure on the orbit space $G\backslash M$ such that the map $\pi : M \to G\backslash M$ is a covering map.*

cf. [Gallot et al., p.30] or [Thurston, p.155].

A smooth action of G on M is **properly discontinuous** if for any p in M, there exists a neighborhood U of p such that $U \cap gU = \varnothing$ for any nontrivial $g \in G$ [do Carmo]. Such an action is free.

Theorem 2.11.5.4 *If G acts on M properly discontinuously, then The quotient $G\backslash M$ is a (Hausdorff paracompact smooth) manifold and the map $M \to G\backslash M$ is a regular covering.*

2.11.6 Homogeneous spaces

The action is said to be **transitive** if for any pair (p, q) of points in M, there exists $g \in G$ such that $gp = q$. Thus the action is transitive if and only if the orbit of a point is the whole space. In this case, M is said to be **homogeneous**.[29]

[28] In particular, the action map $G \times M \to M$ is proper.

[29] Though this notion is much older, the term has been coined by Elie Cartan [Freudenthal], [Cartan, 1894].

Theorem 2.11.6.1 *Let G be a Lie group, which acts transitively on M. Then there exists a G-equivariant diffeomorphism*

$$M \simeq G/H$$

for some closed subgroup H of G.

2.12 Covering Spaces

An action G on M is said to be **proper** if the map

$$G \times M \to M \times M, \quad (g,p) \mapsto (gp, p)$$

is proper.

An action G on M is **free** if the isotropy group G_p is trivial for any $p \in M$.

Theorem 2.12.0.1 *If G is a discrete group acting smoothly, freely and properly on a smooth manifold M, then the quotient map*

$$\pi : M \to M/G$$

is a covering map and M/G has a unique smooth structure such that π is smooth.

2.12.0.2 Tori

Consider the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Then the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to $\mathbf{T}^n$. In fact the continuous, open, surjective group homomorphism

$$\mathbb{R}^n \to \mathbf{T}^n, \quad (r_1, \ldots, r_n) \mapsto (\exp(2\pi i r_1), \ldots, \exp(2\pi i r_n))$$

has the kernel $\mathbb{Z}^n$ which induces a homeomorphism

$$\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbf{T}^n.$$

2.12.0.3 Lens Spaces

Note that the odd-dimensional sphere $\mathbf{S}^{2n-1}$ may be regarded as the unit sphere in $\mathbb{C}^n$:

$$\mathbf{S}^{2n-1} = \{(z_1, \ldots, z_n) \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

Let q be an integer bigger than 1,

$$\omega := e^{2\pi i/q}$$

and let $p_1, \ldots, p_n$ be positive integers relatively prime to q. Consider the map

$$f : \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}, \qquad (z_1, \ldots, z_n) \mapsto (\omega^{p_1} z_1, \ldots, \omega^{p_n} z_n)\,.$$

Then

$$f^q = \mathrm{id}$$

and for each $i \in \{1, \ldots, q-1\}$, f^i has no fixed points. Thus we have a free action of the cyclic group (f) on $\mathbf{S}^{2n-1}$ and the quotient space

$$L(q : p_1, \ldots, p_n) := \mathbf{S}^{2n-1}/(f)$$

is called a **lens space**.

2.13 Stiefel Manifolds

For integers k and n such that $1 \le k \le n$, an ordered k-tuple $\mathbf{v} := (v_1, \ldots, v_k)$ of pairwise orthogonal unit vectors in $\mathbb{R}^n$ is called an orthonormal k-**frame** on $\mathbb{R}^n$. The **Stiefel manifold**[30] $S(n, k)$ is the set of all orthonormal k-frames in $\mathbb{R}^n$. We may regard $S(n, k)$ as a subspace of $M_{n,k}$, the space of all $n \times k$ real matrices. Thus

$$S(n, k) = \{\mathbf{v} \in M_{n,k} \mid \mathbf{v}^{\mathsf{t}}\mathbf{v} = 1_k\},$$

where 1_k is the $k \times k$ identity matrix. Thus Stiefel manifolds are compact spaces.

Note that $S(1, n)$ is the $(n-1)$-sphere $\mathbf{S}^{n-1}$, and $S(n, n)$ is the **orthogonal group** $\mathrm{O}(n)$.

Let Sym_k be the space of all symmetric $k \times k$ real matrices. Then the derivative of the map

$$f : M_{n,k} \to \mathrm{Sym}_k, \qquad \mathbf{v} \mapsto \mathbf{v}^{\mathsf{t}}\mathbf{v} - 1_k$$

at $\mathbf{v} \in M_{n,k}$ is

$$df_{\mathbf{v}}(w) = \mathbf{v}^{\mathsf{t}}w + w^{\mathsf{t}}\mathbf{v} \qquad (w \in M_{n,k}).$$

Thus if $\mathbf{v} \in S(n, k)$, then for any symmetric $k \times k$ matrix s, $w := \frac{1}{2}\mathbf{v}s \in M_{n,k}$ is a solution of the equation

$$df_{\mathbf{v}}(w) = s.$$

Thus $S(n, k)$ is the regular zero level of f, and hence it is a smooth submanifold manifold $M_{n,k}$ of dimension $nk - \frac{1}{2}k(k+1)$.

[30] Eduard Stiefel, 1909–1978. A Swiss mathematician.

2.14 Grassman Manifolds

Let V be a finite dimensional real vector space and let $\mathrm{Grass}_k(V)$ be the set of all k dimensional subspaces of V. There is a canonical injection of $\mathrm{Grass}_k(V)$ into $\mathbf{P}(\wedge^k V)$, the space of 1-dimensional subspaces of $\wedge^k V$, the k-th exterior power[31] of V. Then there is a unique differentiable structure on $\mathrm{Grass}_k(V)$ such that a function on $\wedge^k V$ is differentiable if and only if its pull-back to $\mathrm{Grass}_k(V)$ is differentiable.[32]

[31] Readers may consult the appendix for the exterior powers.
[32] Hermann Günther Grassmann(1809–1877)

2.15 Existence of Smooth Functions

By a function we mean a real valued map. The set

$$\mathcal{C}^\infty(M)$$

of smooth functions on a manifold M is a commutative algebra over $\mathbb{R}$.[33]

If M_α's are connected components of M, then

$$\mathcal{C}^\infty(M) = \prod_\alpha \mathcal{C}^\infty(M_\alpha).$$

If U is an open subset of M, the restriction $f \restriction U$ of $f \in \mathcal{C}^\infty(M)$ to U is a smooth function on U:[34]

$$\mathrm{rst} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(U).$$

This map is neither injective nor surjective, in general.

Observation 2.15.0.1 *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. If $\{f_\alpha\}$ is a family of functions with $f_\alpha \in \mathcal{C}^\infty(U_\alpha)$ for each index $\alpha \in A$ such that*

$$f_\alpha \restriction (U_\alpha \cap U_\beta) = f_\beta \restriction (U_\alpha \cap U_\beta) \qquad (\alpha, \beta \in A),$$

then there exists a unique $f \in \mathcal{C}^\infty(M)$ with $f \restriction U_\alpha = f_\alpha$ for any index α.

2.15.0.2 Functions with compact support

Let

$$\mathcal{C}_c^\infty(M)$$

be the subalgebra of $\mathcal{C}^\infty(M)$ consisting of functions with compact support. If M is compact, then $\mathcal{C}_c^\infty(M) = \mathcal{C}^\infty(M)$.

If a smooth map $\varphi : M \to N$ is proper, then the pull-back map induces a homomorphism

$$\varphi^* : \mathcal{C}_c^\infty(N) \to \mathcal{C}_c^\infty(M).$$

Observation 2.15.0.3 *For any open set U in M, there is a canonical inclusion*

$$\mathcal{C}_c^\infty(U) \hookrightarrow \mathcal{C}_c^\infty(M)$$

by extending trivially. Its image consists of functions on M whose support is compact and contained in U.

[33]Eventually, we will concern modules over $\mathcal{C}^\infty(M)$.

[34]If $U = \varnothing$, then $\mathcal{C}^\infty(\varnothing) := \{0\}$.

2.15.1 Bump functions

A bump function for sets $K \subset U \subset M$ is a function $f \in \mathcal{C}^\infty(M)$ such that

$$0 \leq f(x) \leq 1, \qquad f \restriction K \equiv 1, \qquad \operatorname{supp} f \subset U.$$

If $K = U = M$, then the bump function is the constant 1 function.

Theorem 2.15.1.1 *Given a manifold M, let K be a closed subset of M, and let U be an open neighborhood of K. Then there exists a bump function f for $K \subset U \subset M$.*

We will prove the theorem in various steps.
First note that the function $\rho : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

is a $\mathcal{C}^\infty$ function.

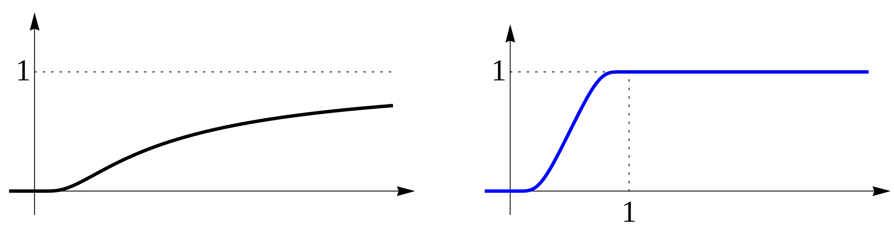


Figure: $y = \rho(x)$ and $y = \tilde{\rho}(x)$

Now the function

$$\tilde{\rho}(x) := \frac{\rho(x)}{\rho(x) + \rho(1-x)}$$

is an increasing function with

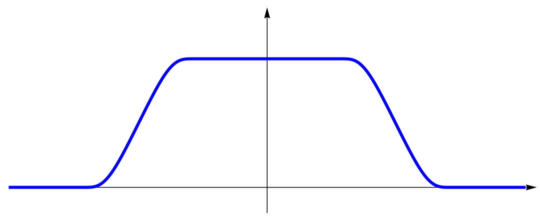
$$\tilde{\rho}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Now if $R > 1$, then the function

$$f(x) := \tilde{\rho}(x)\tilde{\rho}(R + 1 - x)$$

is a bump function which is identically equal to 1 on the interval $[1, R]$ with the support $[0, R + 1]$.

Lemma 2.15.1.2 *For any real numbers r and R with $0 < r < R$, there exists a bump functionn $f : \mathbb{R} \to \mathbb{R}$ for $[-r, r] \subset (-R, R) \subset \mathbb{R}$.*



Let $\mathbf{B}^n(\rho)$ be the open ball in $\mathbb{R}^n$ centered at the origin with radius ρ and let $\bar{\mathbf{B}}^n(\rho)$ be its closure.

Lemma 2.15.1.3 *For any real numbers r and R with $0 < r < R$, there exists a bump functionn $f : \mathbb{R}^n \to \mathbb{R}$ for $\bar{\mathbf{B}}^n(r) \subset \mathbf{B}^n(R) \subset \mathbb{R}^n$.*

Proof of (2.15.1.1). Without loss of generality, we may assume that M is connected.

(Case 1.) Suppose K is compact.

For each point $p \in U$, there exist an open neighborhood U_p of p such that $\overline{U}_p \subset U$ and a bump function $f_p \in \mathcal{C}^\infty(M)$ for $\overline{U}_p \subset U \subset M$.

Since K is compact, there exists a finite number of points, say $p_1, \dots, p_l$ of K such that $\{U_{p_1}, \dots, U_{p_l}\}$ covers K. Now let

$$f = 1 - (1 - f_{p_1}) \cdots (1 - f_{p_l}).$$

Then $f \in \mathcal{C}^\infty(M)$ and

$$0 \le f \le 1, \quad f \restriction K = 1, \quad \operatorname{supp} f \subset U.$$

Thus f is a bump function for $K \subset U \subset M$.

(Case 2.) Now suppose K is non-compact.

Since we are assuming that M is connected, there exists a compact exhaustion $(K_i)_{i=1,2,\dots}$ of M. Then there exists a bump function $f_i \in \mathcal{C}^\infty(M)$ with respect to the subsets

$$(K_{i+1} - \overset{o}{K_i}) \cap K \quad \subset \quad (\overset{o}{K}_{i+2} - K_{i-1}) \cap U$$

for $i = 1, 2, \dots,$ where $K_0 := \varnothing$. Then

$$f := 1 - \prod_{i=1}^{\infty} (1 - f_i)$$

is the desired bump function. □

Corollary 2.15.1.4 *Let U be an open neighborhood of a point p in a smooth manifold M and let $F : U \to N$ be a $\mathcal{C}^\infty$ map. Then there exist an open neighborhood V of p in U and a $\mathcal{C}^\infty$ map $\tilde{F} : M \to N$ such that $F \restriction V = \tilde{F} \restriction V$.*

Proof. We give two proofs. First, take a diffeomorphism $x : V \to \mathbb{R}^n$, where V is an open neighborhood of p such that $\bar{V} \subset U$. Then there exists a bump function $\rho : M \to \mathbb{R}$ for V supported in U. Now

$$\tilde{x} := \begin{cases} \rho\, x & \text{on } U \\ \mathbf{0} & \text{on } M - \operatorname{supp} \rho \end{cases}$$

is a smooth map of M into $\mathbb{R}^n$. Then

$$\tilde{F} := F \circ x^{-1} \circ \tilde{x} : M \to \mathbb{R}^n \to U \to N$$

is a smooth map such that $\tilde{F} \restriction = F \restriction V$.

Here is another proof. Let $y : W \to \mathbb{R}^m$ be a diffeomorphism, where W is an open neighborhood of $F(p)$ in N. Then we have a smooth map

$$y \circ F : F^{-1}(W) \cap U \to \mathbb{R}^m.$$

Then there exists a smooth map $G : M \to \mathbb{R}^m$ which is identically equal to $y \circ F$ in a neighborhood V of p. Now $\tilde{F} := y^{-1} \circ G$ is the desired map. □

2.15.2 Whitney Embedding Theorem

Theorem 2.15.2.1 (Whitney, 1936, 1944) *Any n-manifold can be embedded in $\mathbb{R}^{2n}$.*

For simplicity,[35] we will show that any compact manifold M has an embedding into some $\mathbb{R}^N$. Let $p \in M$ and let U be an open neighborhood of p with a diffeomorphism $x : U \to \mathbb{R}^n$. Take a $\mathcal{C}^\infty$ function $\rho : M \to \mathbb{R}$ with $\operatorname{supp} \rho \subset U$ and $\rho \equiv 1$ in a small neighborhood of p. Then the map[36]

$$\tilde{x} = \begin{cases} \rho\, x & \text{on } U \\ 0 & \text{on } M - \operatorname{supp} \rho \end{cases}$$

is a $\mathcal{C}^\infty$ map of M into $\mathbb{R}^n$, which is a local diffeomorphism in a neighborhood of p.

Since M is compact, there exist a finite number of points $p_1, \ldots, p_k$ in M and $\mathcal{C}^\infty$ maps $\tilde{x_1}, \ldots, \tilde{x_k} : M \to \mathbb{R}^n$ such that each $\tilde{x_i}$ is a local diffeomorphism in a neighborhood V_i of p_i, for $i = 1, \ldots, k$, and $V_1 \cup \cdots \cup V_k = M$. Shrink[37] $\{V_1, \ldots, V_k\}$ to open sets $\{W_1, \ldots, W_k\}$ so that

$$\overline{W}_1 \subset V_1,\ \ldots,\ \overline{W}_k \subset V_k, \qquad M = W_1 \cup \cdots \cup W_k.$$

[35] For the detail, see [Boothby], [Bröcker and K. Jänich], [Hirsh] or [Munkers (1966), p.18].
[36] Note the inclusion $\mathcal{C}_c^\infty(U, \mathbb{R}^n) \hookrightarrow \mathcal{C}^\infty(M, \mathbb{R}^n)$.
[37] Compact Hausdorff spaces are normal.

Now take bump functions $\rho_1, \dots, \rho_k$ on M with $\operatorname{supp} \rho_i \subset V_i$ and $\rho_i \restriction W_i = 1$. Then the map

$$\varphi := (\rho_1, \dots, \rho_k, \tilde{x_1}, \dots, \tilde{x_k}) : M \to \mathbb{R}^k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{k+kn}$$

is a desired $\mathcal{C}^\infty$ map.

To check that φ is one-to-one, suppose $\varphi(p) = \varphi(q)$ for some $p, q \in M$. Since $p \in W_i \subset V_i$ for some i, $1 = \rho_i(p) = \rho_i(q)$. Thus $q \in V_i$. Thus $\tilde{x}_i(p) = \tilde{x}_i(q)$ and hence $p = q$.

The map φ is clearly an immersion.

Since M is compact, φ is an embedding. □

In fact, John Forbes Nash (1928−2015) proved (1954, 1956) that every second countable Riemannian manifold has an isometric embedding into a Euclidean manifold.

2.15.3 Partition of Unity

On a manifold M, a collection $\{\rho_\alpha \mid \alpha \in A\}$ of $\mathcal{C}^\infty$ functions on M is called a smooth **partition of unity**, if

(i) for any $\alpha \in A$, $0 \le \rho_\alpha \le 1$.

(ii) $\{\operatorname{supp} \rho_\alpha \mid \alpha \in A\}$ is locally finite.

(iii) $\sum_\alpha \rho_\alpha = 1$.

For an open cover $\{U_\alpha\}$ of M, a $\mathcal{C}^\infty$ partition of unity $\{\rho_\alpha\}$ (with the same index set) is said to be **subordinate to** $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for every α.

Lemma 2.15.3.1 (Shrinking Lemma) *If $\{U_\alpha\}_{\alpha\in A}$ is a locally finite open cover of a manifold M, then there exists an open cover $\{V_\alpha\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for each index α.*

Proof. We may also assume that $U_\alpha \neq \varnothing$ for all $\alpha \in A$. We may also assume that M is connected. Then M is σ-compact. Thus for any compact set K in M, there are only finitely many α's such that $K \cap U_\alpha \neq \varnothing$, and hence A is a countable set. Arrange the indices so that $A = \{1, 2, 3, \dots\}$. Then

$$C_1 := M - (U_2 \cup U_3 \cup \dots) \subset U_1.$$

Thus there exists an open set V_1 such that $C_1 \ll V_1 \ll U_1$. Now $\{V_1, U_2, U_3, \dots\}$ is an open cover of M. Let

$$C_2 := M - (V_1 \cup U_3 \cup U_4 \cup \dots) \subset U_2.$$

Then there exists an open set V_2 such that $C_2 \ll V_2 \ll U_2$. Now $\{V_1, V_2, U_3, \dots\}$ is an open cover of M. In this way we have a desired open cover $\{V_\alpha\}$ of M. □

Theorem 2.15.3.2 *For any open cover $\{U_\alpha\}$ of a $\mathcal{C}^\infty$ manifold M, there exists a $\mathcal{C}^\infty$ partition $\{\rho_\alpha\}$ of unity subordinate to $\{U_\alpha\}$.*

Proof. (Case 1.) Suppose that $\{U_\alpha\}$ is locally finite. Shrink $\{U_\alpha\}$ to an open cover $\{V_\alpha\}$. Then by the theorem (2.15.1.1), there exist a bump function $f_\alpha \in \mathcal{C}^\infty(M)$ for $V_\alpha \subset U_\alpha$. Now let

$$\rho_\alpha := \frac{f_\alpha}{\sum_\beta f_\beta}.$$

(Case 2.) In general, $\{U_\alpha\}_{\alpha \in A}$ has a locally finite open refinement $\{V_\lambda\}_{\lambda \in \Lambda}$, i.e., there exists a map $i : \Lambda \to A$ such that $V_\lambda \subset U_{i(\lambda)}$. Let $\{\phi_\lambda\}$ be a partition of unity subordinate to $\{V_\lambda\}$. Now for each $\alpha \in A$ define

$$\rho_\alpha = \sum_{\lambda \in i^{-1}(\alpha)} \phi_\lambda.$$

By convention, empty sum is equal to 0. Now

$$\operatorname{supp} \rho_\alpha \subset \bigcup_{\lambda \in i^{-1}(\alpha)} \operatorname{supp} \phi_\lambda \subset \bigcup_{\lambda \in i^{-1}(\alpha)} V_\lambda \subset U_\alpha,$$

$\{\operatorname{supp} \rho_\alpha\}$ is locally finite,[38] and $\sum \rho_\alpha = 1$. This completes the proof. □

2.15.4 Exercises

1. Show that $\dim(\mathcal{C}^\infty(M)) = \infty$ if $\dim M \geq 1$.

2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $\mathcal{C}^\infty$-structures on a topological manifold M and let $M_1 = (M, \mathcal{D}_1)$, $M_2 = (M, \mathcal{D}_2)$. Show that $\mathcal{D}_1 = \mathcal{D}_2$ if (and only if) $\mathcal{C}^\infty(M_1) = \mathcal{C}^\infty(M_2)$.

3. Let M and N be smooth manifolds and let $\mathcal{C}^0(M, N)$ (resp. $\mathcal{C}^\infty(M, N)$) be the set of continuous (resp. $\mathcal{C}^\infty$) maps from M to N. Show that $\mathcal{C}^\infty(M, N)$ is a dense subset of $\mathcal{C}^0(M, N)$ with the compact-open topology.

4. Show that a map $\varphi : M \to N$ is $\mathcal{C}^\infty$ if and only if for any open subset V of N and $g \in \mathcal{C}^\infty(V)$, $\varphi^*(g) := g \circ \varphi$ is a $\mathcal{C}^\infty$ function on $\varphi^{-1}(V)$.

5. Let $\mathrm{Diff}(M)$ be the group of all diffeomorphisms of M onto itself. Show that, if M is connected, $\mathrm{Diff}(M)$ acts on M transitively.

[38] For, if $p \in M$, there exists an open neighborhood W of p such that $W \cap V_\lambda = \emptyset$ for all λ exept for a finite number of λ's. Now if $\operatorname{supp} \rho_\alpha \cap W \neq \emptyset$, then $V_\lambda \cap W \neq \emptyset$ for some $\lambda \in i^{-1}(\alpha)$. Thus there are only finite number of α's with $\operatorname{supp} \rho_\alpha \cap W \neq \emptyset$.

6. (Taylor Expansion) Let $f \in \mathcal{C}^\infty(M)$. Show that if $x = (x^1, \ldots, x^n)$ is a chart on M centered at a point $p \in M$, then there exist $g_1, \ldots, g_n \in \mathcal{C}^\infty(M)$ such that
$$f = f(p) + \sum_i x^i \, g_i$$
in a neighborhood of p. In this case, $g_i(p) = \frac{\partial f}{\partial x^i}(p)$.

2.16 Sard's Theorem

A subset S of $\mathbb{R}^n$ is said to be of (Lebesgue) **measure zero** if for any positive real number ϵ, there exists a countable set of n-balls[39] $B_1, B_2, \ldots$ in $\mathbb{R}^n$ such that $S \subset B_1 \cup B_2 \cup \cdots$ and $\operatorname{vol} B_1 + \operatorname{vol} B_2 + \cdots < \epsilon$.

For instance, the set $\mathbb{Q}^n$ of rational points in $\mathbb{R}^n$ is of measure zero.

Proposition 2.16.0.1 *Let A be a measure zero set in $\mathbb{R}^n$. Let U be an open subset of $\mathbb{R}^n$ containing A. If $\phi : U \to \mathbb{R}^n$ is either Lipschitz or $\mathcal{C}^1$, then $\phi(A)$ is of measure zero.*

Proof. Let $\epsilon > 0$ be given and take a sequence $\{B_k : k = 1, 2, \ldots\}$ of n-balls $B_k := \mathbf{B}^n(p_k, r_k)$, with center p_k and radius r_k, such that $A \subset \cup_{k=1}^{\infty} B_k$ and $\sum_k \mathrm{Vol}(B_k) < \epsilon$.

(Lipschitz Case) Let L be the Lipschitz constant of Φ. Then

$$\phi(A \cap B_k) \subset \mathbf{B}^n(\phi(p_k), Lr_k).$$

Now

$$\phi(A) \subset \cup_{k=1}^{\infty} \phi(A \cap B_k) \subset \cup_k \mathbf{B}^n(\phi(p_k), Lr_k)$$

and

$$\sum_k \mathrm{Vol}(\mathbf{B}^n(\phi(p_k), Lr_k)) = \sum_k L^n \,\mathrm{Vol}(B_k) \leq \epsilon L^n.$$

Since $\epsilon > 0$ is arbitrary, $\phi(A)$ has measure zero.

($\mathcal{C}^1$ Case) Note that U is a countable union of convex compact subsets $\{K_\alpha : \alpha = 1, 2, \ldots\}$. Thus ϕ is Lipschitz on K_α and $A \cap K_\alpha$ has measure zero. Hence $\phi(A \cap K_\alpha)$ has measure zero and $\phi(A)$ has measure zero. This completes the proof.cf. [Dubrovin et al.], [Milnor, 1969], [John Lee]. □

Corollary 2.16.0.2 "Measure zero" *is a $\mathcal{C}^1$-diffeomorphism invariant.*

A subset S of a smooth n-manifold M is said to be of (Lebesgue) **measure zero** if for any chart $x : U \to \mathbb{R}^n$, $x(U \cap S)$ is of measure zero in $\mathbb{R}^n$.

Proposition 2.16.0.3 *If $F : M \to N$ is a $\mathcal{C}^1$ map between manifolds of the same dimension, and S is a measure zero set in M, then $F(S)$ is of measure zero.*

Theorem 2.16.0.4 (Morse (1939), Sard(1942)) *Let M be second countable. Let $\phi : M \to N$ be a $\mathcal{C}^k$ map, where $k \geq 1 + \max\{\dim M - \dim N, 0\}$. Then the set of critical values of ϕ is of measure zero.*

[39] Instead of balls we may use boxes. An *n-box* is a product $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ of compact intervals.

For the proof, since the countable union of measure zero sets has measure zero, we may assume that M and N are open subsets of $\mathbb{R}^n_-$. Let C be the set of critical points of ϕ and let

$$C_i = \{p \in M \mid \frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}(p) = 0 \in \mathbb{R}^n,\ \forall\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+,\ |\alpha| \leq i\}$$

for $i = 1, 2, \ldots, k$. Then in [Mil;T], the following steps are proved.
Step 0. The image $\phi(C - C_1)$ has measure zero.
Step 1. The image $\phi(C_1 - C_2)$ has measure zero.
...
Step $k-1$. The image $\phi(C_{k-1} - C_k)$ has measure zero.
Step k. The image $\phi(C_k)$ has measure zero.

We omit the details.

Note that if S is of measure zero in M, then the complement $M - S$ is a dense subset of M.

Corollary 2.16.0.5 ([Brown]) *The set of regular values of a smooth map $\phi : M \to N$ is dense in N, if M has at most countable number of components.*

Corollary 2.16.0.6 *If $\dim M < \dim N$, then there exists no surjective $\mathcal{C}^1$ map from M to N, provided M has at most countable number of components.*

2.17 Morse Functions

Theorem 2.17.0.1 *Morse functions form an open dense subset in $\mathcal{C}^\infty(M)$.*

cf. Milnor, Lectures on the h-cobordism theorem, Princeton Univ. Press, 1965. p.14.

2.18 C^k Differentiable Structures

Let M be a topological n-manifold.

2.18.1 Definitions

1. A set[40] $\mathcal{A}$ of local charts, or local coordinate systems on M is called a (topological) **atlas** if $\{\mathrm{dom}(x) \mid x \in \mathcal{A}\}$ covers M.

   Note that if x and y are local charts of M, then

   $$x \circ y^{-1}$$

   is a homeomorphism between open subsets of $\mathbb{R}^n$.

2. An atlas $\mathcal{A}$ of M is said to be **of class** $\mathcal{C}^k$ for some $k = 0, 1, \ldots, \infty, \omega$, if for any pair x, y of elements in $\mathcal{A}$, the composition

   $$x \circ y^{-1}$$

   is a $\mathcal{C}^k$-diffeomorphism.[41]

   Note that every atlas is automatically of class $\mathcal{C}^0$. Moreover, if $k < l$, then a $\mathcal{C}^l$-atlas, i.e., an atlas of class C^l, is a $\mathcal{C}^k$-atlas.

3. Two $\mathcal{C}^k$-atlases are **equivalent** if their union is also a $\mathcal{C}^k$-atlas.

   It should be easy to check that this relation is an equivalence relation.

4. An atlas $\mathcal{A}$ of class $\mathcal{C}^k$ is **maximal** if it is not contained properly in any other atlas of class $\mathcal{C}^k$, i.e., if $y : U \to \mathbb{R}^n$ is a chart such that $y \circ x^{-1}$ is a $\mathcal{C}^k$-diffeomorphism for any $x \in \mathcal{A}$, then $y \in \mathcal{A}$.

   Note that any atlas $\mathcal{A}$ of class $\mathcal{C}^k$ is contained in a unique maximal atlas $\overline{\mathcal{A}}$ of class $\mathcal{C}^k$. In fact, $\overline{\mathcal{A}}$ is the union of all C^k-atlases which are equivalent to $\mathcal{A}$.

5. A $\mathcal{C}^k$**-structure** on M is a maximal atlas of class $\mathcal{C}^k$. Or equivalently, we may say that a $\mathcal{C}^k$-structure on M is an equivalence class of $\mathcal{C}^k$-atlases.

6. A topological manifold together with a $\mathcal{C}^k$ structure is called a $\mathcal{C}^k$**-manifold**.

A topological manifold M together with an atlas A of class $\mathcal{C}^k$ has a unique $\mathcal{C}^k$-structure which contains $\mathcal{A}$.

A topological space has a canonical **sheaf** $\mathcal{C}^0$ of continuous functions. One can define a $\mathcal{C}^k$-differentiable structure using a subsheaf of $\mathcal{C}^k$-functions [Bredon].

[40] Thus an atlas of M is a subset of all partial homeomorphisms from M to $\mathbb{R}^n$, whose domains are nonempty open subsets of M and whose codomains are nonempty open subsets of $\mathbb{R}^n$.

[41] A map is, by definition, of class $\mathcal{C}^\omega$ if it is an analytic map.

2.18.2 Diffeomorphic Structures

Let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas on a topological manifold M, and let $h : M \to M$ be a homeomorphism. Let

$$h^*\mathcal{A} := \{x \circ h \mid x \in \mathcal{A}\}.$$

Then $h^*\mathcal{A}$ is a C^k-atlas.

We say that two $\mathcal{C}^k$-structures $\mathcal{A}$ and $\mathcal{A}'$ on a topological manifold M are **diffeomorphic** if there exists a homeomorphism $h : M \to M$ such that $\mathcal{A}' = h^*\mathcal{A}$. This relation among $\mathcal{C}^k$-structures is an equivalence relation.

2.18.3 Whitney's Theorem

It is obvious that any $\mathcal{C}^l$-manifold is also $\mathcal{C}^k$ for $k < l$.

H. Whitney(1907--1989) proved that any $\mathcal{C}^1$-atlas on a topological manifold M contains a $\mathcal{C}^\omega$-atlas, which is unique up to diffeomorphism [Hirsch, ch. 2], [Munkres, §4.7--4.9].

Chapter 3

Tangent and Cotangent Vectors

3.1 Tangent Vectors

Let M be a manifold of dimension n and let $p \in M$. We say that two (parametrized) smooth curves $c_1, c_2 : \mathbb{R} \to M$ are **equivalent** at p if and only if

$$c_1(0) = p = c_2(0), \qquad (x \circ c_1)'(0) = (x \circ c_2)'(0) \in \mathbb{R}^n$$

for some (and hence for any) chart x at p. The equivalence class of a curve $c : \mathbb{R} \to M$ with $c(0) = p$ will be denoted by $c'(0)$, and is called a **tangent vector** of M at p. The set

$$TM_p$$

of all tangent vectors of M at p is called the **tangent space** of M at p.[1] Note that if p and q are distinct points of M, then TM_p and TM_q are disjoint.

If $c \in \mathcal{C}^\infty(\mathbb{R}, M)$ and $t_0 \in \mathbb{R}$, then $c'(t_0)$ is the tangent vector of M at $c(t_0)$ defined by the curve $\gamma(t) := c(t + t_0)$.

3.1.0.1 Example

For a finite dimensional real linear space V, the tangent space of V at any point $p \in V$ has a canonical identification with V, or for better with $\{p\} \times V$:

$$TV_p \simeq \{p\} \times V \simeq V.$$

To see this let $e_1, \dots, e_n$ be a basis for V and let $x^1, \dots, x^n$ be the dual basis. Then $x = (x^1, \dots, x^n) : V \to \mathbb{R}^n$ is a (global) chart for V. If $c : \mathbb{R} \to V$ is a curve in V with $c(0) = p$, then

$$c(t) = \sum c^i(t) e_i$$

[1] Many authors use the notation T_pM or M_p. Our notation is the same as the one in [Milnor, 1969] and [Arnold].

where $c^i(t) = x^i(c(t))$. We identify $c'(0)$ with

$$x^{-1}\left(\frac{dc^1}{dt}(0), \ldots, \frac{dc^n}{dt}(0)\right) = \sum \frac{dc^i}{dt}(0)e_i \in V.$$

Note that $c'(0)$ is independent of the choice of basis of V.

3.1.0.2 Exercise

Prove the following statements.

1. If $c_1, c_2 \in \mathcal{C}^\infty(\mathbb{R}, M)$ are the same in a neighborhood of 0, then $c_1'(0) = c_2'(0)$.

2. For a positive real number ϵ, if $c : (-\epsilon, \epsilon) \to M$ is smooth, then there exists $\tilde{c} \in \mathcal{C}^\infty(\mathbb{R}, M)$ such that $\tilde{c} = c$ on $\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$.

3.1.1 Derivations

A **derivation**[2] of $\mathcal{C}^\infty(M)$ **at** p is an $\mathbb{R}$-linear map $\delta : \mathcal{C}^\infty(M) \to \mathbb{R}$ such that

$$\delta(fg) = (\delta f)g(p) + f(p)(\delta g)$$

for any $f, g \in \mathcal{C}^\infty(M)$. The set of all derivations of $\mathcal{C}^\infty(M)$ at p is denoted by $\mathrm{Der}(M, p)$.

Lemma 3.1.1.1 *Let δ be a derivation of M at a point p in M.*

(i) if const *is a constant function,* $\delta(\text{const}) = 0$.

(ii) derivations are local, i.e., if f and g are smooth functions on M which are identical in a neighborhood of p, then $\delta(f) = \delta(g)$.

Proof. (i) is trivial, since $\delta(1) = 0$.

(ii) It suffices to show that if f is identically zero in an open neighborhood U of p, then $\delta(f) = 0$. Take a $\rho \in \mathcal{C}^\infty(M)$ such that $\operatorname{supp}\rho \subset U$ and $\rho \equiv 1$ in a neighborhood of p. Then $\rho f \equiv 0$ and hence

$$0 = \delta(\rho f) = (\delta\rho)f(p) + \rho(p)(\delta f) = \delta(f).$$

□

[2] Let A be a commutative algebra over a field $\mathbb{F}$ and let M be an A-module. Then an $\mathbb{F}$-linear map

$$\delta : A \to M$$

is called a *derivation* if

$$\delta(ab) = (\delta a)b + a(\delta b)$$

for any $a, b \in A$.

Since derivations are local operators,[3] we can define δf for any $f \in \mathcal{C}^\infty(U)$, U being an open neighborhood of p, as follows: Take any $\tilde{f} \in \mathcal{C}^\infty(M)$ such that $\tilde{f} = f$ in a small neighborhood of p. Then

$$\delta f := \delta \tilde{f}$$

is well-defined, i.e., independent of choice of extension.

3.1.1.2 Examples

(i) Let $x = (x^1, \dots, x^n)$ be a coordinate system at p. Then the partial derivative operators

$$\left.\frac{\partial}{\partial x^j}\right|_p : \mathcal{C}^\infty(M) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x^j}(p) \qquad (j = 1, \dots, n)$$

are derivations.

(ii) If $c : \mathbb{R} \to M$ is a curve, we have the pull-back map

$$c^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathbb{R})$$

which is an algebra-homomorphism. Thus the composition of c^* with the derivation

$$\left.\frac{d}{dt}\right|_0 : \mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}$$

of $\mathbb{R}$ at the origin (where $t : \mathbb{R} \to \mathbb{R}$ denotes the identity coordinate map) is a derivation of M at the point $c(0)$.

Now the tangent vector $c'(0)$ at p defines a derivation

$$d_{c'(0)} : \mathcal{C}^\infty(M) \to \mathbb{R}, \quad f \mapsto \left.\frac{d}{dt}\right|_0 (f \circ c)(t).$$

One can easily see that this definition is well-defined.

Theorem 3.1.1.3 *(i) There is a canonical one-to-one correspondence*

$$d : TM_p \to \mathrm{Der}(M, p).$$

(ii) $\mathrm{Der}(M, p)$ *is a vector space of dimension* n*.*

Proof. We prove the second assertion first. Obviously derivations form a linear space. For any chart x at p, it is trivial to see that the derivations

$$\left.\frac{\partial}{\partial x^1}\right|_p, \;\dots,\; \left.\frac{\partial}{\partial x^n}\right|_p$$

[3] Some authors like to use *germs* of functions.

are linearly independent, since $\frac{\partial}{\partial x^j}\big|_p (x^i) = \delta^i_j$.

Now we show that they generate the whole derivations at p. Let $f \in \mathcal{C}^\infty(M)$. Then there exist an open neighborhood U (contained in the domain of x) of p (cf. 2.15.4.6 or Appendix) and smooth functions $h_{ij} \in \mathcal{C}^\infty(U)$ such that

$$f \restriction U = f(p) + \sum_i (x^i - x^i(p)) \left.\frac{\partial}{\partial x^i}\right|_p (f) + \frac{1}{2}\sum_{i,j}(x^i - x^i(p))(x^j - x^j(p))h_{ij}(x).$$

Thus for any derivation δ at p,

$$\delta(f) = \sum_i \delta(x^i) \left.\frac{\partial}{\partial x^i}\right|_p (f).$$

Therefore $\delta = \sum_i \delta(x^i) \left.\frac{\partial}{\partial x^i}\right|_p$.

Now we prove the first assertion. To check the injectivity of d, suppose $d_v = d_{\tilde{v}}$ for $v = c'(0)$ and $\tilde{v} = \tilde{c}'(o)$, where $c, \tilde{c} : \mathbb{R} \to M$ satisfy $c(0) = p = \tilde{c}(0)$. Note that for any chart $x = (x^1, \ldots, x^n)$,

$$d_v(x^i) = \left.\frac{d}{dt}\right|_0 x^i(c(t)) = D(x^i \circ x^{-1})_{x(p)}(x \circ c)'(0) = \operatorname{proj}_i(x \circ c)'(0)$$

for $i = 1, \ldots, n$, and similarly for $\tilde{v}$. Since $d_v = d_{\tilde{v}}$, we have

$$(x \circ c)'(0) = (x \circ \tilde{c})'(0)$$

which means that $v = \tilde{v}$.

To see the surjectivity of d, let $\sum_i a^i \left.\frac{\partial}{\partial x^i}\right|_p$ be a derivation at p. Then the curve

$$t \mapsto x^{-1}\left(x(p) + t(a^1, \ldots, a^n)\right) \in M$$

defined for small $|t|$, can be used to get a tangent vector v so that $d_v = \sum a^i \left.\frac{\partial}{\partial x^i}\right|_p$.

This completes the proof. □

From now on we will identify tangent vectors and derivations. Thus

$$d_v f = v(f)$$

for any tangent vector v and any function f. The set of tangent vectors TM_p is a vector space of dimension n.

If $(x^1, \ldots, x^n)$ is a chart at a point p of M, then we have the **coordinate curves**

$$c_1(t), \quad \ldots, \quad c_n(t)$$

on M, defined for small $|t|$, with $c_1(0) = p$, ..., $c_n(0) = p$. Then

$$c_i{}'(0) = \left.\frac{\partial}{\partial x^i}\right|_p \qquad (i = 1, \ldots, n).$$

3.1.1.4 Classic notion of tangent vectors

In the proof of the above theorem, we have seen that every tangent vector $v \in TM_p$ "is" a linear combination of $\frac{\partial}{\partial x^j}\big|_p$ for any chart x at p. If we take another chart $\tilde{x}$ at p, then

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p$$

where

$$v^i = \sum_j \left.\frac{\partial x^i}{\partial \tilde{x}^j}\right|_p \tilde{v}^j \qquad (i = 1, \ldots, n). \tag{3.1}$$

Thus if $\mathcal{A}$ is a smooth atlas on a manifold M, a tangent vector on M at a point $p \in M$ is an equivalence class of $(x, v) \in \mathcal{A} \times \mathbb{R}^n$, where $(x, v) \sim (\tilde{x}, \tilde{v})$ if and only if they satisfy the relation (3.1).

3.1.2 Exercises

1. Let U be an open subset of a finite dimensional vector space V, and $p \in U$. Show that
$$TU_p \simeq TV_p \simeq V.$$

2. Let $c : (-\epsilon, \epsilon) \to M$ be a curve with $c(0) = p$ and $c'(0) = 0$. Show that
$$c''(0) \in TM_p$$
is well defined.

3. On the unit sphere $\mathbf{S}^2$ in $\mathbb{R}^3$, let $r = \sqrt{x^2 + y^2}$, the distance from a point (x, y, z) on $\mathbf{S}^2$ to the z-axis. We have a parametrization
$$\begin{gathered} \mathbb{R} \times (-\pi, \pi) \to \mathbf{S}^1 \times (-\pi, \pi) \to \mathbf{S}^2 \\ (\theta, \psi) \mapsto ((\cos\theta, \sin\theta), \psi) \mapsto (\cos\theta, \sin\theta, 0)\cos\psi + (0, 0, 1)\sin\psi \end{gathered}$$
Show that
$$\left.\frac{\partial}{\partial\theta}\right|_{(x,y,z)} = \frac{1}{r}(-y, z, 0), \qquad \left.\frac{\partial}{\partial\psi}\right|_{(x,y,z)} = -\frac{z}{r}(x, y, 0) + r(0, 0, 1).$$

3.2 Derivatives of Maps

Let $F : M \to N$ be a smooth map between manifolds. Then the **derivative** of F at $p \in M$ is the linear map

$$TF_p : TM_p \to TN_{F(p)} \tag{3.2}$$

defined by

$$TF_p(c'(0)) = (F \circ c)'(0)$$

where c is a curve in M with $c(0) = p$.

In terms of derivations, if $v \in TM_p$ is a derivation of M at p, then $TF_p(v)$ is a derivation of N at $F(p)$ such that

$$(TF_p(v))g = v(g \circ F), \quad g \in \mathcal{C}^\infty(N).$$

Now it is clear that TF_p is linear.

If $x = (x^1, \ldots, x^n)$ is a chart of M at p and if $y = (y^1, \ldots, y^m)$ is a chart of N at $F(p)$, then $\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\}$ and $\{ \frac{\partial}{\partial y^1}\Big|_{f(p)}, \ldots, \frac{\partial}{\partial y^m}\Big|_{f(p)}\}$ are bases for TM_p and $TN_{F(p)}$, respectively. With respect to these bases, we have

$$TF_p = \left(\frac{\partial F^i}{\partial x^j}\bigg|_p \right)$$

where $F^i = y^i \circ F$.

It is obvious that if $\mathrm{id} : M \to M$ is the identity map, then $T(\mathrm{id})_p$ is the identity map on TM_p and

$$T(G \circ F)_p = TG_{F(p)} \circ TF_p$$

for $F : M \to N$ and $G : N \to L$.

Let U and U' be open subsets of vector spaces V and V', respectively. In Calculus classes, the ordinary derivative of a smooth map

$$F : U \to U'$$

at a point p in U is a linear map

$$DF_p : V \to V'$$

which is 'identical' to the derivative of a function between manifolds. We use the following notations for the derivative of F at a point p:

$$
\begin{tikzcd}
\mathrm{TU}_p \arrow[r, "TF_p"] \arrow[d, "\approx"'] \arrow[ddr, "dF_p"] & \mathrm{TU'}_{F(p)} \arrow[d, "\approx"] \\
\mathrm{TV}_p \arrow[d, "\approx"'] & \mathrm{TV'}_{F(p)} \arrow[d, "\approx"] \\
\mathrm{V} \arrow[r, "DF_p"'] & \mathrm{V'}
\end{tikzcd}
$$

Thus for any smooth function $f : M \to \mathbb{R}$ and a vector $v \in TM_p$, we have

$$df_p(v) = v(f).$$

Note that df_p is an element of the **cotangent space** TM_p^*, i.e., the dual space of TM_p.

3.2.0.1 Exercise

Let p be a point of M. Show that the map

$$d_p : \mathcal{C}^\infty(M) \to TM_p^*, \quad f \mapsto df_p$$

is linear and satisfies the Leibniz rule.

3.3 Regular Points and Critical Points

A map $F : M \to N$ is an **immersion** at $p \in M$ if and only if the map (3.2) is injective. A map $F : M \to N$ is a **submersion** (or **regular**) at $p \in M$ if and only if the map (3.2) is surjective. Non-regular points are **critical points**. A point in the codomain of a function is a **regular value** if it is not a critical point.

Note that a point $p \in M$ is a critical point of a function $f \in \mathcal{C}^\infty(M)$, if

$$df_p : TM_p \to \mathbb{R}$$

is the trivial map.

Note that if p is a local extremum of f, then it is a critical point.

If S is a submanifold of M, then for any $p \in S$, there is a canonical inclusion

$$TS_p \hookrightarrow TM_p.$$

If S is the regular zero locus of a smooth function $f : M \to \mathbb{R}$, then

$$TS_p = \ker(df_p : TM_p \to \mathbb{R}).$$

In general we have the following claim.

Claim 3.3.0.1 *Let $\varphi : M \to N$ be a smooth map and let $q \in \varphi(M) \subset N$ be a regular value of φ so that $S := \varphi^{-1}(q)$ is a submanifold of M. Let $\mathrm{inc} : S \to M$ be the inclusion map. Then for any $p \in S$, the sequence*

$$0 \to TS_p \xrightarrow{T\,\mathrm{inc}_p} TM_p \xrightarrow{T\varphi_p} TN_q \to 0$$

is exact.

Proof. (1) $T\varphi_p$ is surjective, since q is a regular value.

(2) $T\varphi_p \circ T\,\mathrm{inc}_p = T(\varphi \circ \mathrm{inc})_p = T(q) = 0$.

(3) $T\,\mathrm{inc}_p$ is injective: Take a chart $x = (x^1, \dots, x^n)$ of M at p, and a chart $y = (y^1, \dots, y^m)$ at q such that

$$x^1 = \varphi^1 := y^1 \circ \varphi,\ \dots\ ,\ x^m = \varphi^m := y^m \circ \varphi.$$

Then $z = (z^1, \dots, z^{n-m}) = (x^{m+1}, \dots, x^n) \circ \mathrm{inc}$ is a chart of S at p. Now

$$T\,\mathrm{inc}_p\left(\left.\frac{\partial}{\partial z^1}\right|_p\right) = \left.\frac{\partial}{\partial x^{m+1}}\right|_p, \quad \dots, \quad T\,\mathrm{inc}_p\left(\left.\frac{\partial}{\partial z^{n-m}}\right|_p\right) = \left.\frac{\partial}{\partial x^n}\right|_p$$

and hence we are done.

(4) Now exactness is trivial, since $\dim S + \dim N = \dim M$. □

3.3.1 Examples

(i) The sphere $\mathbf{S}^{n-1}$ is defined by the equation $(x^1)^2 + \cdots + (x^n)^2 = 1$ on $\mathbb{R}^n$. Thus the tangent space of $\mathbf{S}^{n-1}$ at a point p consists of vectors in $\mathbb{R}^n$ which are perpendicular to p.

Given $\mathbf{a} := (a_1, \dots, a_n) \neq 0$, consider the function

$$f = a_1 x^1 + \cdots + a_n x^n$$

on $\mathbf{S}^{n-1}$. Then f has two[4] critical points on $\mathbf{S}^{n-1}$. They are $\pm\mathbf{a}/|\mathbf{a}|$.

(ii) We show that the tangent space of $\mathrm{SO}(n)$ at the identity matrix 1_n is the space of all $n \times n$ skew-symmetric real matrices. To see this consider the defining equation

$$\varphi : \mathrm{GL}^+(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R}), \quad A \mapsto A^{\mathfrak{t}} A$$

where $\mathrm{GL}^+(n, \mathbb{R})$ denotes the group of matrices with positive determinant, and $\mathrm{Sym}(n, \mathbb{R})$ denotes the space of all $n \times n$ real symmetric matrices. Then

$$T\varphi_{1_n}(X) = X^{\mathfrak{t}} + X, \quad X \in T\,\mathrm{GL}^+(n, \mathbb{R})_{1_n} = \mathfrak{gl}(n, \mathbb{R}).$$

Thus the kernel of $T\varphi_{1_n}$, which is the tangent space of $\mathrm{SO}(n)$ at 1_n, is the space $\mathfrak{so}(n)$ of skew-symmetric matrices.

3.3.2 Exercises

1. Show that
$$c'(0) = Tc_0\left(\left.\frac{d}{dt}\right|_0\right)$$
for a curve $c : \mathbb{R} \to M$.

2. Given distinct real numbers $a_1, \dots, a_n$, let $f : \mathbf{S}^{n-1} \to \mathbb{R}$ be defined by $f(x_1, \dots, x_n) = \sum_{k=1}^n a_k (x_k)^2$. Find all critical points of f.

   Note that if $A : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is the antipodal map, then $f \circ A = f$ and hence, f descends to a map $\bar{f} : \mathbf{P}^{n-1} \to \mathbb{R}$. Find all critical points of $\bar{f}$.

3. Given distinct real numbers $a_1, \dots, a_n$, find all critical points of
$$f : \mathbf{P}^{n-1}(\mathbb{C}) \to \mathbb{R}, \quad [z_1, \dots, z_n] \mapsto \sum_{k=1}^n a_k |z_k|^2 / (|z_1|^2 + \cdots + |z_n|^2).$$

4. Show that there is no map $f : \mathbf{P}^n(\mathbb{R}) \to \mathbb{R}$ such that $f^{-1}(y) = \mathbf{P}^{n-1}(\mathbb{R})$ for some regular value y of f.

[4] The Euler characteristic of $\mathbf{S}^{n-1}$ is $1 + (-1)^{n-1}$.

5. Let $0 < r < R$ and let T be the torus in $\mathbb{R}^3$ given by the equation

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2.$$

Find the critical points of the function $x \restriction T : T \hookrightarrow \mathbb{R}^3 \xrightarrow{x} \mathbb{R}$.

6. Let $\pi^1 : M \times N \to M$ and $\pi^2 : M \times N \to N$ be the projection maps and let $(p, q) \in M \times N$.

   (i) Show that the map

   $$T\pi^1_{(p,q)} \oplus T\pi^2_{(p,q)} : T(M \times N)_{(p,q)} \to TM_p \oplus TN_q$$

   is an isomorphism.

   (ii) Let $i^1_q : M \to M \times N$ and $i^2_p : N \to M \times N$ be the maps

   $$i^1_q(x) = (x, q), \qquad i^2_p(y) = (p, y)$$

   for $x \in M$ and $y \in N$. Let

   $$T_1F_{(p,q)} := T(F \circ i^1_q)_p : TM_p \to TL_r, \quad T_2F_{(p,q)} := T(F \circ i^2_p)_q : TN_q \to TL_r.$$

   Show that

   $$TF_{(p,q)} = T_1F_{(p,q)} \circ T\pi^1_{(p,q)} + T_2F_{(p,q)} \circ T\pi^2_{(p,q)}.$$

3.4 Morse Functions

If p is a critical point of $f : M \to \mathbb{R}$, then for any chart $(x^1, \ldots, x^n)$ centered at p, there exist functions $h_{ij} = h_{ji}$ such that

$$f = f(p) + \frac{1}{2}\sum_{i,j=1}^{n} h_{ij} x^i x^j$$

in a neighborhood of p. In this case,

$$h_{ij}(p) = \frac{\partial^2 f}{\partial x^i \, \partial x^j}(p).$$

Now p is said to be **non-degenerate** if the **Hessian matrix**

$$Hf_p := \left(\frac{\partial^2 f}{\partial x^i \, \partial x^j}(p)\right)$$

(at a critical point with respecp to a chart x) is non-singular. The number of negative eigenvalues of Hf_p is called the **index** of f at p.[5]

A function whose critical points are all non-degenerate is called a **Morse function**. If f is a Morse function on a compact manifold M, then f has only finitely many critical points. If c_k is the number of critical points of f with index k, then

$$\chi(M) = \sum_{k=0}^{n} (-1)^k c_k = \sum_{p \in \mathrm{Crit}(f)} (-1)^{\mathrm{ind}\, f(p)}$$

is the Euler-Poincaré characteristic of M. In fact, if C^k is the free vector space generated by the critical points of index k of a Morse function f on M, then there exists a **chain complex**

$$\{0\} \to C^0 \to C^1 \to \cdots \to C^n \to \{0\}$$

such that whose cohomology groups are isomorphic to the cohomology groups $H^\bullet(M)$. These cohomology groups will be discussed later. In particular,

$$c_k \geq b^k \qquad (k = 0, \ldots, n)$$

and we have the Morse inequalities:

$$\begin{array}{rcl} c_0 & \geq & b_0 \\ c_0 - c_1 & \geq & b^0 - b^1 \\ & \vdots & \\ c_0 - c_1 + \cdots + (-1)^n c_n & = & b^0 - b^1 + \cdots + (-1)^n b^n \end{array}$$

[5] in this case, $(-1)^{\mathrm{ind}\, f(p)} = \mathrm{ind}\, \nabla f(p)$, the index of the gradient vector field of f with respect to any Riemannian metric on M (cf. 3.9.2).

where $b^k = \dim H^k(M)$ is the k-th **Betti number** of M, which will be explained later.

We have

$$c_M(f) \geq b_M$$

where $c_M(f)$ is the sum of critical points and b_M is the sum of betti numbers.

Morse functions are dense in $\mathcal{C}^\infty(M)$.

3.5 Transversality

Let L be a submanifold of N. Then a map $\varphi : M \to N$ is said to be **transversal** to L if for any $p \in \varphi^{-1}(L)$,

$$d\varphi_p(TM_p) \hookrightarrow TN_{\varphi(p)} \to TN_{\varphi(p)}/TL_{\varphi(p)}$$

is surjective. In this case, if $\varphi(M) \cap L$ is nonempty, then we must have $\dim N \leq \dim M + \dim L$.

Theorem 3.5.0.1 *If $\varphi : M \to N$ is transversal to a codimension k submanifold L of N, then $\varphi^{-1}(L)$ is a codimension k submanifold of M, unless it is empty.*

Proof. Let $p \in \varphi^{-1}(L)$. Then $\varphi(p) \in L$ and hence there exists an open neighborhood U of $\varphi(p)$ and a regular map $F : U \to \mathbb{R}^k$ such that $U \cap L$ is the zero set $Z(F)$ of F. Now $V := \varphi^{-1}(U)$ is an open neighborhood of p and $F \circ \varphi \restriction V \to \mathbb{R}^k$ is a regular map. Moreover $V \cap \varphi^{-1}(L) = Z(F \circ \varphi \restriction V)$. This completes the proof. $\square$

3.6 Orientations

3.6.1 Orientation of Vector Space

Let V be a vector space over $\mathbb{R}$ of finite dimension $n \geq 1$. Two (ordered) bases $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ of V are said to have **the same orientation** if and only if they are in the same connected component of the space $B(V)$ of all basis of V, i.e., there exists $g \in \mathrm{GL}^+(n, \mathbb{R})$ such that $w = vg$. (Note that there is a simple transitive action of $\mathrm{GL}(n, \mathbb{R})$ on $B(V)$.)

A linear isomorphism $l : V \to V$ is said to be **orientation preserving** if for some (and hence for any) basis $v = (v_1, \ldots, v_n)$ of V, $l(v) := (l(v_1), \ldots, l(v_n))$ and v have the same orientation. Thus l preserves the orientation if and only if $\det l > 0$.

A linear map $l : V \to V$ is **orientation reversing** if $\det l < 0$.

3.6.1.1

An **orientation** of a vector space V of dimension $n \geq 1$ is a choice of the equivalence class of a basis on V, where two bases are equivalent if they have the same orientation.

$\mathbb{R}^n$ is always oriented with the **standard orientation**: $(e_1, \ldots, e_n) > 0$.

3.6.2 Orientable Manifolds

3.6.2.1 Orientation Preserving Maps

Let U be an open subset of $\mathbb{R}^n$ for $n \geq 1$. A smooth map

$$F : U \to \mathbb{R}^n$$

is **orientation preserving** at a point $p \in U$ if the Jacobian matrix

$$\left(\frac{\partial F^i}{\partial x^j}(p) \right)_{1 \leq i,j \leq n}$$

of F at p has positive determinant, i.e., the derivative $DF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation preserving isomorphism.

3.6.2.2

An atlas $\mathcal{A}$ on a manifold M of dimension $n \geq 1$ is said to be **oriented** if the coordinate transition map

$$x \circ y^{-1} : y(U_x \cap U_y) \to x(U_x \cap U_y)$$

is orientation preserving for any chart $x : U_x \to \mathbb{R}^n$ and $y : U_y \to \mathbb{R}^n$ in $\mathcal{A}$.

A manifold M is **orientable** if there exists an oriented atlas $\mathcal{A}$ for M. A manifold M is **oriented** if an oriented atlas $\mathcal{A}$ for M is chosen.

Spheres $\mathbf{S}^n$ are orientable.

3.6.2.3

If a manifold M is oriented, then each tangent space TM_p is oriented: If $(x^1, \ldots, x^n)$ is a positively oriented chart of M at p, then

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ \ldots, \ \left.\frac{\partial}{\partial x^n}\right|_p$$

is a positively oriented basis for TM_p.

3.6.2.4

An **orientation** of a zero-dimensional manifold M is a map from M into $\{1, -1\}$.

3.6.2.5 Orientation Preserving Maps

A map $f : M \to N$ between oriented manifolds is said to be **orientation preserving** at a point $p \in M$ if $TF_p : TM_p \to TN_{F(p)}$ is an orientation preserving isomorphism.

The composition of two orientation preserving maps is orientation preserving. The composition of two orientation reversing maps is orientation preserving. The composition of an orientation preserving map and orientation reversing map is orientation reversing.

Any **reflection** map along a hyperplane in a Euclidean space $\mathbb{E}^n$ is an orientation reversing map. Any **central** (or point) **symmetry** in a Euclidean space $\mathbb{E}^n$ preserves the orientation if and only if n is even.

The antipodal map $-I$ on $\mathbf{S}^n$ is orientation preserving if and only if n is odd.

$$\begin{tikzcd}
\mathbf{S}^n \arrow[rr, "-I"] \arrow[dr] & & \mathbf{S}^n \arrow[dl] \\
& \mathbf{P}^n &
\end{tikzcd}$$

Theorem 3.6.2.6 $\mathbf{P}^n$ *is orientable if and only if n is odd.*

3.6.2.7 Exercise

Let p be the north pole on the unit sphere $\mathbf{S}^n := \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid |\mathbf{x}|^2 + t^2 = 1\}$, where $\mathbf{S}^n$ is oriented in the standard way so that $(e_2, \ldots, e_{n+1})$ is the positively oriented basis for the tangent space of $\mathbf{S}^n$ at the east pole e_1. Show that the antipodal map

$$(\mathbf{S}^n - \{p\}) \to \mathbb{R}^n, \qquad (\mathbf{x}, t) \mapsto \frac{\mathbf{x}}{1-t}$$

is orientation preserving if and only if n is odd.

3.6.3 Intersection Numbers

Let M, N be submanifolds of L which intersect transversally. We assume that L, M, N are all oriented. Then the **intersection number** of M and N is

$$M \circ N := \sum_{p \in M \cap N} \operatorname{sgn}(M, N, p)$$

where $\operatorname{sgn}(M, N, p) = 1$ if $TM_p \oplus TN_p$ and TL_p have the same orientation, and $\operatorname{sgn}(M, N, p) = -1$ otherwise.

Then the intersection number is homotopy invariant [Dubrovin et al.].

Theorem 3.6.3.1 *Let M be a compact connected hypersurface in $\mathbb{R}^n$. Then $\mathbb{R}^n - M$ has two components, and M is orientable.*

3.6.3.2 Lefschetz Numbers

Let M be a compact oriented manifold and let $F : M \to M$ be a self map. A fixed point p of F is said to be **non-degenerate** if

$$\det(\mathrm{id} - TF_p) \neq 0$$

i.e., the graph of F intersect with the diagonal $\Delta \subset M \times M$ transverally. In this case the **Lefschetz number** of F is

$$\Lambda_F := \sum_{p \in \mathrm{Fix}(F)} \operatorname{sgn}(F, p).$$

Corollary 3.6.3.3 (Brouwer's fixed point theorem) *Any self map on a closed Euclidean ball has a fixed point.*

Proof. Note that the n-ball is homeomorphic to the lower hemisphere S_- of $\mathbf{S}^n$. We will show that any self map F of S_- has a fixed point. Let $\pi : \mathbf{S}^n \to S_-$ be a map which is identity on S_-. Then the composition

$$\mathbf{S}^n \xrightarrow{\ \pi\ } S_- \xrightarrow{\ F\ } S_- \hookrightarrow \mathbf{S}^n$$

is homotopic to a constant map. Thus it has a fixed point p, which is also a fixed point of F. cf. [Dubrovin et al.]. $\square$

3.7 Tangent Bundle

The **total space** of the tangent bundle of M is the collection

$$TM := \coprod_{p\in M} TM_p$$

of all tangent spaces of M. The **tangent bundle** of M is the total space of the tangent bundle together with the canonical projection map

$$\pi : TM \to M.$$

If $\{x_\alpha : U_\alpha \to \mathbb{R}^n\}$ is an atlas of M, then the map

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \tag{3.3}$$

given by

$$\phi_\alpha\left(\sum_{i=1}^n v^i(p)\left.\frac{\partial}{\partial x_\alpha^i}\right|_p\right) = (p, v^1(p), \ldots, v^n(p))$$

is bijective and if $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ is defined by

$$\phi_\alpha \circ \phi_\beta^{-1}(p,v) = (p, \phi_{\alpha\beta}(p)v),$$

then $\phi_{\alpha\beta} = \left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right)_{1\le i,j\le n}$, the Jacobian matrix of $x_\alpha \circ x_\beta^{-1}$.

Now the next proposition is trivial.[6]

Proposition 3.7.0.1 *There exists a unique topology on the total space of the tangent bundle TM of a manifold M such that for some (and hence for any) atlas $\{x_\alpha : U_\alpha \to \mathbb{R}^n\}$ of M, the map (3.3) is a homeomorphism. With this topology on TM, it is a topological manifold of dimension $2n$. Moreover, there exists a unique differentiable structure and an orientation on TM such that the map (3.3) is an orientation preserving diffeomorphism.*

If $F : M \to N$ is $\mathcal{C}^\infty$, then

$$TF : TM \to TN \tag{3.4}$$

is defined by $(TF)(p,v) := TF_p(v)$ for $(p,v) \in TM$. If $G : N \to L$ is smooth, we have

$$T(G \circ F) = TG \circ TF.$$

If `MFD` is the category of smooth manifolds and smooth maps, we have a functor

$$T : \texttt{MFD} \to \texttt{MFD}.$$

3.7.0.2 Exercise

Show that the map (3.4) is $\mathcal{C}^\infty$.

[6] Let X be an arbitrary set and let $\{X_\alpha\}$ be a collection of subsets of X which covers X. We assume a topology $\mathcal{T}_\alpha$ on X_α for each α such that $\mathcal{T}_\alpha|X_\alpha \cap X_\beta = \mathcal{T}_\beta|X_\alpha \cap X_\beta$ for any α, β. Then there exists a unique topology $\mathcal{T}$ on X such that $\mathcal{T}|X_\alpha = \mathcal{T}_\alpha$ for all α.

3.8 Vector Fields

A smooth section[7] X of $\pi : TM \to M$ is called a **vector field** on M. Thus for each $p \in M$, $X(p)$ or X_p is a tangent vector of M at p for each p in M.

For a chart $x : U \to \mathbb{R}^n$ of M, $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are vector fields on U and any vector field X on U is a linear combination $\sum f^i \frac{\partial}{\partial x^i}$ for some $f^i \in \mathcal{C}^\infty(U)$. In fact, $f^i = X(x^i)$.

Note that, for any smooth vector field X and a smooth function f on M,

$$X(f) : M \to \mathbb{R}, \qquad p \mapsto X_p(f)$$

is a $\mathcal{C}^\infty$ function on M.

The set of all vector fields on M is denoted by

$$\mathfrak{X}(M)$$

which is a module over $\mathcal{C}^\infty(M)$.

Lemma 3.8.0.1 *$\mathfrak{X}(M)$ is isomorphic to* $\mathrm{Der}(\mathcal{C}^\infty(M))$ *of all derivations*[8] *of $\mathcal{C}^\infty(M)$, as $\mathcal{C}^\infty(M)$-modules.*

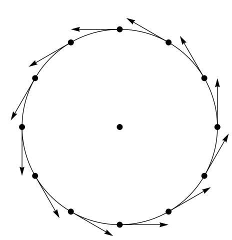
3.8.0.2 Exercises

(i) Show that $\dim \mathfrak{X}(M) = \infty$ if $\dim M \geq 1$.

(ii) Show that for any $v \in TM_p$, there exists a vector field X on M such that $X(p) = v$.

(iii) Let U be an open neighborhood of a point p in a manifold M, and let X be a vector field on U. Show that there exists a vector field $\tilde{X}$ on M such that $X = \tilde{X}$ on some neighborhood of p.

(iv) Show that $T\mathbf{S}^1$ is diffeomorphic to $\mathbf{S} \times \mathbb{R}$.



[7] A *section* of a map $\pi : A \to B$ is a map $s : B \to A$ such that $\pi \circ s = \mathrm{id}_B$.

[8] A *derivation* of a real algebra A is an $\mathbb{R}$-linear map $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$.

3.8.1 Brackets of Vector Fields

Since derivations (of an associative algebra) form a Lie algebra,[9] $\mathfrak{X}(M)$ is a Lie algebra. Thus the (Poisson) bracket $[X, Y]$ of $X, Y \in \mathfrak{X}(M)$ is given by

$$[X, Y](f) = X(Yf) - Y(Xf)$$

for $f \in \mathcal{C}^\infty(M)$.

Note that if $X, Y \in \mathfrak{X}(M)$, then

$$X \circ Y,\ Y \circ X\ : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

are second order differential operator with the same principal symbol so that the commutator $[X, Y]$ is a first order differential operator.

Note that

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$$

for $f, g \in \mathcal{C}^\infty(M)$.

If $(x^1, \ldots, x^n)$ is a chart on M, then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

for all $i, j = 1, \ldots, n$.

3.8.1.1 Exercise

For each $n \times n$ matrix $A = (a^i_j) \in \mathrm{M}_n(\mathbb{R})$, define a vector field X_A on $\mathbb{R}^n$ by

$$X_A(p) = -Ap \in T\mathbb{R}^n_p = \mathbb{R}^n \quad \text{or} \quad X_A = -\sum_{i,j=1}^n a^i_j x^j \frac{\partial}{\partial x^i}.$$

Show that $[X_A, X_B] = X_{[A,B]}$. Thus

$$X : \mathrm{M}_n(\mathbb{R}) \to \mathfrak{X}(\mathbb{R}^n), \qquad A \mapsto X_A$$

is a Lie algebra homomorphism.

3.8.2 Related Vector Fields

A map $F : M \to M'$ induces a map

$$TF : TM \to TM'.$$

But, in general, F does **not** induce a map $F_* : \mathfrak{X}(M) \to \mathfrak{X}(M')$.

We say that a vector field X on M and a vector field X' on M' are F-**related**, if for any $p \in M$, $T\varphi_p(X(p)) = X'(\varphi(p))$.

Note that if $X \in \mathfrak{X}(M)$ and $X' \in \mathfrak{X}(M')$ are F-related, and $Y \in \mathfrak{X}(M)$ and $Y' \in \mathfrak{X}(M')$ are F-related, then $[X, Y]$ and $[X', Y']$ are also F-related.

[9] A linear space L is called a *Lie algebra* if it is equipped with a bilinear pairing $[\ ,\ ] : L \times L \to L$ such that $[l, l] = 0$ and $[l_1, [l_2, l_3]] + [l_2, [l_3, l_1]] + [l_3, [l_1, l_2]] = 0$ (the Jacobi identity) for any $l, l_1, l_2, l_3 \in L$. Any (associative) algebra A together with $[a, b] := ab - ba$, $a, b \in A$, is a Lie algebra. In particular, $\mathfrak{gl}(A) = \mathrm{End}(A)$ is a Lie algebra, and $\mathrm{Der}(A) \subset \mathfrak{gl}(A)$ is a *Lie subalgebra*.

3.8.2.1 Vector Fields and Diffeomorphisms

If $F : M \to N$ is a diffeomorphism, then we have an isomorphism

$$F_* : \mathfrak{X}(M) \simeq \mathfrak{X}(N)$$

given by the commutative diagram for $X \in \mathfrak{X}(M)$:

$$\begin{array}{ccc}
\mathcal{C}^\infty(M) & \xrightarrow{X} & \mathcal{C}^\infty(M) \\
{\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\
\mathcal{C}^\infty(N) & \xrightarrow[F_*(X)]{} & \mathcal{C}^\infty(N)
\end{array}$$

If $G : N \to L$ is a diffeomorphism, then

$$(G \circ F)_* = G_* \circ F_*.$$

3.8.2.2 Exercises

1. Let $F : M \to N$ be a diffeomorphism. Then
   (i) for any $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(N)$, $(F_*X)(f) = X(f \circ F) \circ F^{-1}$.
   (ii) F_* is a Lie algebra homomorphism, i.e.,
   $$F_*[X, Y] = [F_*X, F_*Y]$$
   for any $X, Y \in \mathfrak{X}(M)$.
2. Given a vector field X on $\mathbf{S}^{n-1}$, define a vector field $\tilde{X}$ on $\mathbb{R}^n_*$ by
   $$\tilde{X}(x) := |x|\, X\left(\frac{x}{|x|}\right) \qquad (x \in \mathbb{R}^n_*).$$
   Show that
   $$\left[\tilde{X}, \frac{\mathbf{r}}{r}\right] = 0.$$

3.9 Zeros of Vector Fields

Given a vector field X on M, a point p in M is called a **zero pont**, **singular point**, or **singularity** of X if $X_p = 0$.

3.9.1 Vector Fields on Spheres

On the odd dimensional sphere

$$\mathbf{S}^{2n-1} := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

we have a non-vanishing vector field

$$X_{(z_1,\dots,z_n)} := \mathbf{i}(z_1, \dots, z_n).$$

Conversely, one can show that if $\mathbf{S}^n$ admits a non-vanishing vector field, then n is odd. This can be proved using Brouwer's **degree theory**:[10] If X is a unit vector field on $\mathbf{S}^n$, then

$$F_t : \mathbf{S}^n \to \mathbf{S}^n, \qquad p \mapsto (\cos \pi t)p + (\sin \pi t)X(p)$$

defines a homotopy between the identity map and the antipodal map. Thus $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}$, which means that n is odd.

Here is another proof, using the **Lefschetz fixed point theorem.**[11] If X is a unit vector field on $\mathbf{S}^n$, then

$$h_t(p) := \frac{tp + (1-t)X(p)}{|tp + (1-t)X(p)|}$$

is a homotopy between $X : \mathbf{S}^n \to \mathbf{S}^n$ and $\mathrm{id} : \mathbf{S}^n \to \mathbf{S}^n$. Hence the Lefschetz number of X is

$$\Lambda_X = \Lambda_{\mathrm{id}} = \chi(\mathbf{S}^n) = 1 + (-1)^n.$$

Thus if n is even, then X has a fixed point, which is absurd since $p \perp X(p)$ for every $p \in \mathbf{S}^2$. Thus n must be odd.

In particular, we have the following theorem.[12]

Theorem 3.9.1.1 *Every vector field on an even dimensional sphere has a singularity.*[13]

[10] The degree of a proper map $f : M \to N$ between oriented manifolds is the signed number of points in $f^{-1}(q)$, where $q \in N$ is a regular value of f. The signs are determined by the orientations. Well-definedness of the degree is obtained once we have a theory of integration, which we will discuss later. Note that regular values are dense in N, by Morse-Sard-Brown's theorem [Milnor].

[11] Let M be a compact manifold and let $F : M \to M$ be a continuous map. Then F has a fixed point if its *Lefschetz number*

$$\Lambda_F := \sum (-1)^k \operatorname{tr}(f_* : H_k(M, \mathbb{R}) \to H_k(M, \mathbb{R}))$$

is nontrivial. In fact, if F has a finite number of fixed points, then Λ_F is equal to the sum of the indices of the fixed points of F.

[12] First proved by Poincaré for $n = 2$ [McGrath]. For the Milnor's proof, see [Gallot et al.].

[13] The *Euler characteristic* of the even dimensional sphere is 2.

3.9.2 Index of Vector Fields

If a zero p of a vector field X on M is isolated, then we have the integer, called the **index** of X at p. If p is a nonsingular point of X, then $\operatorname{ind} X(p) = 0$, by definition.

Theorem 3.9.2.1 (Poincaré (1885), Hopf (1926)) *If M is compact, and the singular points of a vector field X on M are isolated, then*

$$\sum_{p \in M} \operatorname{ind} X(p) = \chi(M).$$

Note that if a compact manifold M has a nonvanishing vector field, then $\chi(M) = 0$. A theorem of Hopf says that *if M is a compact connected manifold with $\chi(M) = 0$, then M admits a non-vanishing vector fields.*

3.9.3 Parallelizable Manifolds

An n-manifold M is said to be **parallelizable** if there exist (global) vector fields $X_1, \ldots, X_n$ such that they are linearly independent at each point of M. Thus M is parallelizable if and only if the tangent bundle is **trivial**, i.e., $TM \simeq M \times \mathbb{R}^n$ over M.[14] In this case, $\mathfrak{X}(M) \simeq \mathcal{C}^\infty(M)^n$ as vector spaces over $\mathbb{R}$ (but not necessarily as Lie algebras).

Every Lie group is parallelizable. In particular,

$$\mathbf{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

and

$$\mathbf{S}^3 = \{q \in \mathbb{H} \mid |q| = 1\}$$

are parallelizable, since $\mathbf{S}^n$ is a Lie group if and only if $n = 0, 1, 3$.[15] For instance, the vector fields

$$I_q := iq, \qquad J_q := jq, \qquad K_q := kq \qquad (q \in \mathbf{S}^3)$$

are mutually perpendicular unit vector fields on $\mathbf{S}^3$.

In 1958, Michel Kervaire, and independently by Raoul Bott and John Milnor, proved that $\mathbf{S}^n$ is parallelizable if and only if $n = 1, 3, 7$.

To see that $\mathbf{S}^7$ is parallelizable, **octonions** (or **octaves**) may help us (cf. Appendix):

$$\mathbf{S}^7 = \{u \in \mathbb{O} \mid |u| = 1\}.$$

Given a unit vector u in $\mathbb{O}$, the vectors

$$ue_i \quad (i = 1, \ldots, 7)$$

are mutually perpendicular unit tangent vectors of the sphere $\mathbf{S}^7$ at u. This shows that $\mathbf{S}^7$ is parallelizable.

[14] The meaning of '$\simeq$' will be made clear soon.

[15] If G is a compact connected Lie group with $H^1(G, \mathbb{R}) = \{0\}$, then $H^3(G, \mathbb{R}) \neq \{0\}$.

3.10 Cotangent Vectors

For $f \in \mathcal{C}^\infty(M)$, the *differential*[16] of f at $p \in M$, denoted by df_p, is the collection of all directional derivatives of f at p, i.e., df_p is a (linear) map from TM_p into $\mathbb{R}$ such that

$$df_p(v) := d_v(f), \quad v \in TM_p.$$

Thus we get a map

$$df : M \to TM^*$$

where

$$TM^* := \coprod_{p \in M} TM_p^*$$

is the **cotangent bundle** of M, which is the collection of the dual space TM_p^* of TM_p for all $p \in M$.

Proposition 3.10.0.1 *Let $x : U \to \mathbb{R}^n$ be a chart on M. The for each $p \in U$, $\{dx_p^1, \ldots, dx_p^n\}$ is a basis for TM_p^*. If $f \in \mathcal{C}^\infty(U)$, then*

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

on U.

3.10.1 Exercises

(i) Let $f \in \mathcal{C}^\infty(M)$. Show that $p \in M$ is a critical point of f if and only if $df_p = 0$.

(ii) Show that the cotangent bundle TM^* of M is a smooth manifold in a canonical way.

(iii) On a manifold M and $f \in \mathcal{C}^\infty(M)$, $df = 0$ implies that f is locally constant.

(iv) Show that if f is a smooth function on a compact manifold, then df vanishes at some point.

3.10.2 Differential 1-Forms

A smooth section of the cotangent bundle $TM^* \to M$ is called a **covector field** or a **differential 1-form** on M. The collection of all 1-forms on M will be denoted by

$$\Omega^1(M)$$

[16]More generally, if V is a vector space and $f : M \to V$ is a V-valued $\mathcal{C}^\infty$ map on M, then we have a "V-valued differential" $df_p : TM_p \to V$. For instance if $z : M \to \mathbb{C}$ is a complex valued function with x and y as real and imaginary parts, then

$$dz = dx + idy.$$

which is a $\mathcal{C}^\infty(M)$-module.

Note that if $\omega \in \Omega^1(M)$ and $x : U \to \mathbb{R}^n$ is a chart on M, then $\omega|U = \sum_{i=1}^n f_i dx^i$ for some $f_i \in \mathcal{C}^\infty(U)$.

3.10.2.1 Covector fields on $\mathbb{R}^n$

If x is the standard coordinate system on $\mathbb{R}$, then every 1-form on $\mathbb{R}$ is of the form

$$f(x)\,dx$$

for some function f on $\mathbb{R}$.

Let U be an open subset of $\mathbb{R}^n$. Then with the standard coordinate system, every 1-forms on U is of the form

$$\omega = f_1\,dx^1 + \cdots + f_n\,dx^n$$

for some smooth functions $f_1, \ldots, f_n$ on U.

In general, if M is a parallelizable manifold, then there exist global vector fields $X_1, \ldots, X_n$ on M which are linearly independent everywhere. Now let $\theta 1, \ldots, \theta^n$ be the dual forms. Then every 1-form on M is a linear combination of these. Thus

$$\Omega^1(M) \simeq \mathcal{C}^\infty(M)^n.$$

3.10.2.2

Let

$$\mathfrak{X}(M)^* := \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\mathfrak{X}(M), \mathcal{C}^\infty(M)).$$

Lemma 3.10.2.3 *Let X be a vector field on M which vanishes at some point p in M. If $l \in \mathfrak{X}(M)^*$, then $l(X)_p = 0$.*

Proof. Suppose $X = 0$ in a neighborhood U of p. Then take a bump function f which is identically equal to 1 in a neighborhood of p and supported in U. Then fX is a global vector field on M which is identically equal to 0. Thus $0 = l(fX) = f\,l(X)$. In particular, by applying p, we have $0 = l(X)_p$.

From this observation, we know that if $Y \in \mathfrak{X}(M)$ and $X = Y$ in a neighborhood of p, then $l(X)_p = l(Y)_p$.

Now we consider the original question. Take a local chart $x = (x^1, \ldots, x^n)$ defined on an open neighborhood U of p. Then on U, $X = \sum f^i \frac{\partial}{\partial x^i}$ for some smooth functions f^i on U, which vanish at p. Now extend, for each i, f^i and $\frac{\partial}{\partial x^i}$ to a global function $\tilde{f}^i$ and a global vector field Y_i, respectively, so that they preserve the original near p. Then

$$l(X)_p = l\left(\sum \tilde{f}^i Y_i\right)_p = \sum \tilde{f}^i(p)\, l(Y_i)_p = 0.$$

$\square$

Proposition 3.10.2.4
$$\Omega^1(M) \simeq \mathfrak{X}(M)^*$$
as $\mathcal{C}^\infty(M)$-modules.

Theorem 3.10.2.5 *The differential*
$$d : \mathcal{C}^\infty(M) \to \Omega^1(M)$$
is a derivation.

3.10.3 Canonical 1-form on cotangent bundle

The total space of the cotangent bundle $\pi : TM^* \to M$ has a canoniccal 1-form α which is defined by
$$\alpha(v) := a(\pi_* v)$$
for any tangent vector v of TM^* at $a \in TM^*$. If $x = (x^1, \ldots, x^n)$ is a local coordinate system near $p := \pi(a)$, then $a = \sum_{i=1}^n a_i\, dx^i$ for some scalars a_i and hence obtain a local coordinate system
$$(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)$$
for the cotangent bundle, where $\dot{x}^i$ are local functions on the cotangent bundle given by
$$\dot{x}^i(a) := a\left(\frac{\partial}{\partial x^i}\right) = a_i.$$
(We regard x^i as functions defined on the total space.) Then
$$\alpha = \sum_{i=1}^n \dot{x}^i\, dx^i.$$

3.10.4 Naturality of Differential

A smooth map $F : M \to N$ induces a linear map $TF_p : TM_p \to TN_{F(p)}$ and hence its dual map $TF_p^* : TN_{F(p)}^* \to TM_p^*$. Thus we have a linear map
$$F^* : \Omega^1(N) \to \Omega^1(M)$$
defined by
$$(F^*(\omega))_p(v) := \omega_{F(p)}(TF_p(v))$$
for $\omega \in \Omega^1(N)$ and $v \in TM_p$.[17]

[17] Although we have a smooth map $TF : TM \to TN$ (not a map $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ unless F is a diffeomorphism), we do not have a map $TF^* : TN^* \to TM^*$ unless F is a diffeomorphism.

Proposition 3.10.4.1 *For a smooth map $F : M \to N$,*

$$F^*(f\omega) = (F^*f)(F^*\omega), \quad F^*(df) = d(F^*f)$$

for any $f \in \mathcal{C}^\infty(N)$ and $\omega \in \Omega^1(N)$.

$$\begin{array}{ccc} \mathcal{C}^\infty(M) & \xrightarrow{d} & \Omega^1(M) \\ {\scriptstyle F_*}\uparrow & & \uparrow{\scriptstyle F_*} \\ \mathcal{C}^\infty(N) & \xrightarrow[d]{} & \Omega^1(N) \end{array}$$

3.10.4.2 Exercises

(i) Consider the 1-form

$$\omega := x\,dy - y\,dx$$

on $\mathbb{R}^2$, and the exponential map

$$\epsilon : \mathbb{R} \to \mathbf{S}^1, \quad \theta \mapsto (\cos\theta, \sin\theta).$$

Show that

$$(\mathrm{inc} \circ \epsilon)^*(\omega) = d\theta$$

where inc : $\mathbf{S}^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map.

(ii) Let

$$S_+ := \mathbf{S}^1 - \{(1,0)\}, \qquad S_- := \mathbf{S}^1 - \{(-1,0)\}, \qquad S_0 := S_+ \cap S_-.$$

On $\mathbf{S}^1$, we have “angle functions”

$$\theta_+ : S_+ \to (0, 2\pi), \qquad \theta_- : S_- \to (-\pi, \pi).$$

Show that $d\theta_+ = d\theta_-$ on S_0. Thus $d\theta_+$ and $d\theta_-$ defines a global 1-form σ on $\mathbf{S}^1$. Show that there exists no function θ on $\mathbf{S}^1$ whose differential is equal to σ. Show that if $\mathrm{inc} : \mathbf{S}^1 \hookrightarrow \mathbb{R}^2_*$ denotes the inclusion map, then

$$\sigma = \mathrm{inc}^* \left(\frac{xdy - ydx}{x^2 + y^2} \right).$$

Show that

$$\Omega^1(\mathbf{S}^1) = \mathcal{C}^\infty(\mathbf{S}^1)\sigma.$$

(iii) Consider the map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \quad (r, \theta) \mapsto (x, y) = (r\cos\theta, r\sin\theta).$$

Compute $\varphi^*(x\,dy - y\,dx)$.

(iv) Find a 1-dimensional submanifold S of $\mathbb{R}^2$ such that $\mathrm{inc}^*(xdx + ydy) = 0$, where $\mathrm{inc} : S \to \mathbb{R}^2$ denotes the inclusion map.

3.11 Lie Algebras of Lie Groups

3.11.1 Invariant Vector Fields on Lie Groups

Let G be a Lie group. A vector field X on G is said to be **left invariant** if for any $a \in G$,

$$L_{a*}(X) = X,$$

i.e., $X_g = (TL_a)_g(X_g) = X_{ag}$ for any $g \in G$, i.e., $X_a = (TL_a)_*(X_e)$.

Clearly a left invariant vector field on G is determined by its value at the identity element. Conversely, every left invariant vector field on G is obtained from a vector at the identity element: Let $\mathrm{Lie}(G)$ be the tangent space of G at the identity element e. Then for each $v \in \mathrm{Lie}(G)$, the vector field v^L on G defined by

$$(v^L)_g := (TL_g)_e(v) \qquad (g \in G)$$

is a left invariant smooth[18] vector field on G.

Thus $\mathrm{Lie}(G)$ is isomorphic to the space $\mathfrak{X}^L(G)$ of left invariant vector fields. Since $\mathfrak{X}^L(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$, $\mathrm{Lie}(G)$ becomes THE Lie algebra of G.

If $v_1, \dots, v_n$ is a basis for $\mathrm{Lie}(G)$, then the vector fields

$$v_1^L, \quad \dots, \quad v_n^L$$

are linearly independent everywhere and hence we have the **Maurer-Cartan isomorphism**

$$TG \simeq G \times \mathrm{Lie}(G).$$

Similarly, a vector field X on G is said to be **right invariant** if for any $a \in G$,

$$R_{a*}(X) = X,$$

i.e., $(TR_a)_g(X_g) = X_{ga}$ for any $g \in G$, i.e., $X_a = (TR_a)_*(X_e)$. Thus $\mathrm{Lie}(G)$ is isomorphic to the space $\mathfrak{X}^R(G)$ of right invariant vector fields. $\mathfrak{X}^R(G)$ is also a Lie subalgebra of $\mathfrak{X}(G)$. The inversion map $I : G \to G$ induces linear isomorphism

$$I : \mathfrak{X}^L(G) \simeq \mathfrak{X}^R(G).$$

Note that

$$[I(X), I(Y)] = -I[X, Y].$$

Lemma 3.11.1.1 *Let $f : G \to H$ be a Lie group homomorphism, and let $v \in \mathrm{Lie}(G)$. If V is the left invariant vector field on G with $V_e = v$ and if W is a left invariant vector field on H with $W_e = f_*(v)$, then V and W are f-related.*

Lemma 3.11.1.2 *Let H be a Lie subgroup of a Lie group G. Then $\mathrm{Lie}(H)$ is a Lie subslgebra of $\mathrm{Lie}(G)$.*

---

[18] If $v = c'(0)$ for some smooth curve $c : \mathbb{R} \to G$, then

$$G \times \mathbb{R} \to G \times G \to G, \quad (g, t) \mapsto g \cdot c(t)$$

is smooth and hence v^L is smooth.

3.11.2 The Lie algebra of the general linear group

For $a = (a^i_j) \in \mathfrak{gl}(n, \mathbb{R})$, let $x^i_j(a) = a^i_j$. Then $(x^i_j)_{i \leq i,j \leq n}$ form a coordinate system on $\mathfrak{gl}(n, \mathbb{R})$. Now for $a \in \mathfrak{gl}(n, \mathbb{R}) = T\,\mathrm{GL}(n, \mathbb{R})_{1_n}$, let A be the left invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ with $A(1_n) = a$. Then for $g \in \mathrm{GL}(n, \mathbb{R})$,

$$A(g) = ga$$

or

$$A = \sum_{i,j,k} x^i_j a^j_k \frac{\partial}{\partial x^i_k}.$$

Similarly, if B is the left invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ with $B(1_n) = b = (b^i_j) \in \mathfrak{gl}(n, \mathbb{R}) = T\,\mathrm{GL}(n, \mathbb{R})_{1_n}$, then

$$B = \sum_{i,j,k} x^i_j b^j_k \frac{\partial}{\partial x^i_k}.$$

Now $AB(x^p_q) = A(\sum_l x^p_l b^l_q) = \sum_{l,i} x^p_i a^i_l b^l_q$ and $BA(x^p_q) = \sum_{l,i} x^p_i b^i_l a^l_q$. Thus

$$[A, B](x^p_q) = \sum_{i,l} x^p_i (a^i_l b^l_q - b^i_l a^l_q).$$

This implies that

$$[A, B] = \sum_{p,q,i,l} x^p_i (a^i_l b^l_q - b^i_l a^l_q) \frac{\partial}{\partial x^p_q}$$

and the value of $[A, B]$ at $1_n = (\delta^i_j)$ is

$$\sum_{q,i,l} (a^i_l b^l_q - b^i_l a^l_q) \frac{\partial}{\partial x^i_q}.$$

Thus the Lie bracket $[a, b]$ of $a, b \in \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R}))$ is

$$[a, b] = ab - ba.$$

3.11.2.1 Exercises

1. Check the Lie algebra of the unitary group $\mathrm{U}(n)$.

2. For $a, b \in \mathfrak{gl}(n, \mathbb{R})$, let A, B be the right invariant vector fields on $\mathrm{GL}(n, \mathbb{R})$. Then
$$[A, B]_e = -[a, b].$$

3.11.3 Maurer-Cartan Form

Let G be a Lie group. Then a vector field X on G is said to be **left invariant** if for any $g \in G$, $(L_g)_* X = X$, i.e.,

$$X(gh) = L_{g*}(X(h)) \tag{3.5}$$

for any $g, h \in G$. The above equation implies that a left invariant vector field is completely determined by its value at a single point. Thus we have a linear isomorphism between the tangent space TG_e of G at the neutral element e of G and the linear space of left invariant vector fields on G. Since

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y]$$

for $g \in G$ and $X, Y \in \mathfrak{X}(G)$, the space of left invariant vector fields is a Lie subalgebra of $\mathfrak{X}(G)$. Hence we obtain a Lie algebra structure on TG_e, which is **the Lie algebra** of G and denoted by $\mathrm{Lie}\, G$.

The left translation induces an isomorphism

$$\theta_g : TG_g \simeq \mathrm{Lie}\, G \quad \text{for } g \in G.$$

This Lie algebra valued 1-form θ on G is the (left-invariant) **Maurer-Cartan form** of G.

3.11.3.1 Exercise

Let $e_1, \dots, e_n$ be a basis for TG_e, and let $\epsilon^1, \dots, \epsilon^n$ be the left invariant 1-forms on G such that $\epsilon^i_e(e_j) = \delta^i_j$. Show that

$$\theta = \sum_{i=1}^n \epsilon^i e_i.$$

3.11.4 Serret-Frenet Formula

Let $c : I \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ defined on some open interval $I \subset \mathbb{R}$, and let $e_1, \dots, e_n$ be orthonormal vector fields along[19] c. Thus we have a curve

$$e(t) := (e_1(t), \dots, e_n(t))$$

in the orthogonal group $\mathrm{O}(n)$. The derivative $e'_j(t)$ is a linear combination of $e_1(t), \dots, e_n(t)$;

$$e'_j(t) = \sum_{i=1}^n e_i(t) a^i_j(t)$$

[19]A *vector field X along a smooth map $f : N \to M$* is an assignment

$$N \ni p \mapsto X(p) \in TM_{f(p)}.$$

Locally, for some chart y on M, $X(p) = \sum_i a^i(p) \frac{\partial}{\partial y^i}|_{f(p)}$. Then X is $\mathcal{C}^\infty$ if and only if a^i's are $\mathcal{C}^\infty$.

or $e'(t) = e(t)A(t)$, which is a tangent vector of $\mathrm{O}(n)$ at $e(t)$. Thus

$$e(t)^{-1}e'(t) = A(t)$$

is a tangent vector of $\mathrm{O}(n)$ at the identity element. Thus $A(t)$ is a skew-symmetric matrix.

In fact, if G is a Lie group, I is an open interval and

$$A : I \to \mathrm{Lie}(G)$$

is a piecewise differentiable curve, then there exists a piecewise differentaible curve

$$a : I \to G$$

such that

$$a(t)^{-1}a'(t) = A(t)$$

for all $t \in I$.[20]

3.11.5 Exercises

(i) Consider

$$\begin{aligned} \mathrm{SU}(2) &= \left\{ \hat{q} = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \middle| \ z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\} \\ &\simeq \{ q = z + wj \in \mathbb{H} \mid q\overline{q} = 1 \} \ = \ \mathbf{S}^3. \end{aligned}$$

Note that the map

$$f : \mathbb{C}^2 \to \mathbb{H}, \qquad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto a - \bar{b}j = a - jb$$

is an isomorphism between right $\mathbb{C}$-vector spaces and for any $q \in \mathbf{S}^3$

$$\hat{q}\begin{pmatrix} a \\ b \end{pmatrix} = f^{-1}\left(q\, f\begin{pmatrix} a \\ b \end{pmatrix} \right).$$

Show that

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is a basis for the Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$. Show that

$$[I, J] = 2K, \quad [J, K] = 2I, \quad [K, I] = 2J.$$

[20] cf. Nomizu, p.29

(ii) Note that the tangent space of $\mathbf{S}^3 \subset \mathbb{H}$ at the identity element is

$$\operatorname{im}\mathbb{H} := \{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid x, y, z \in \mathbb{R}\}.$$

Let I, J, K be the left invariant vector fields associated to the tangent vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.[21] Show that

$$[I, J] = 2K, \quad [JK] = 2I, \quad [K, I] = 2I.$$

(iii) Show that $T\mathbf{S}^3$ is diffeomorphic to $\mathbf{S}^3 \times \mathbb{R}^3$. Show that $T\mathbf{P}^3 \simeq \mathbf{P}^3 \times \mathbb{R}^3$.

[21] Thus for any $q \in \mathbf{S}^3$, $I_q = q\mathbf{i}$, $J_q = q\mathbf{j}$, $K_q = q\mathbf{k}$.

3.12 Riemannian Metric

A **Riemannian metric** on a smooth manifold M is an assignment to each point $p \in M$ an inner product

$$g_p = \langle\ ,\ \rangle_p \in (TM_p \otimes TM_p)^* = TM_p^* \otimes TM_p^*$$

such that for any $X, Y \in \mathfrak{X}(M)$, the map

$$\langle X, Y \rangle : M \to \mathbb{R}, \quad p \mapsto \langle X(p), Y(p) \rangle_p$$

is smooth.

Locally, for a coordinate system $x^1, \dots, x^n$, let

$$g_{ij}(p) := \left\langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right\rangle_p, \quad 1 \le i, j \le n.$$

Then g_{ij} is $\mathcal{C}^\infty$ and the matrix $(g_{ij}(p))$ is (symmetric and) positive definite. We also have

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j.$$

Thus a Riemannian metric is a "smooth section of the vector bundle"[22] $TM^* \otimes TM^*$.

3.12.0.1 Submanifolds

Note that a submanifold of a Riemannian manifold inherits a Riemannian metric. E.g., the sphere $\mathbf{S}^n$ has a canonical metric induced from $\mathbb{R}^{n+1}$.

3.12.0.2 Exercise

Describe a Riemannian metric on $\mathbb{R}^n$ induced from the stereographic projection from $\mathbf{S}^n$.

3.12.1 Existence of a Riemannian metric

Locally, there exists a Riemannian metric. Thus we have an open cover $\{U_\alpha\}$ of a manifold M and a Riemannian metric g_α on U_α for each α. Take a $\mathcal{C}^\infty$ partition $\{\rho_\alpha\}$ of unity subordinate to $\{U_\alpha\}$. Then

$$g := \sum_\alpha \rho_\alpha g_\alpha$$

is a Riemannian metric on M. This proves, in fact, that **on any vector bundle, there exists a Euclidean structure.**

[22] This concept will be explained in chapter 5.

3.12.2 Musical Isomorphism

A Riemannian metric induces ``musical isomorphisms"

$$\flat : TM \to TM^*, \qquad \sharp = \flat^{-1} : TM^* \to TM$$
$$\flat : \mathfrak{X}(M) \to \Omega^1(M), \qquad \sharp = \flat^{-1} : \Omega^1(M) \to \mathcal{X}(M).$$

Hence for a vector field X on M, the 1-form $\flat(X)$ is characterized by

$$\flat(X)(Y) = \langle X, Y \rangle, \quad \forall Y \in \mathfrak{X}(M).$$

Locally, if $X = \sum X^i \frac{\partial}{\partial x^i}$, then $\flat(X) = \sum X_i dx^i$, where

$$X_i = \sum_{j=1}^{n} g_{ij} X^j.$$

Similarly, for a 1-form ω on M, a vector field $\sharp\omega$ is characterized by

$$\omega(Y) = \langle \sharp\omega, Y \rangle, \quad \forall Y \in \mathfrak{X}(M).$$

Locally, if $\omega = \sum \omega_i dx^i$, then $\sharp\omega = \sum \omega^i \frac{\partial}{\partial x^i}$, where

$$\omega^i = \sum_{j=1}^{n} g^{ij} \omega_j$$

and (g^{ij}) is the inverse matrix of (g_{ij}).

3.12.2.1 Exercise

Let (V, g) be a finite dimensional inner product space, and let $\flat : V \to V^*$ be the musical isomorphism. Let g^* be the inner product on V^* such that $\flat$ is an isometry. Show that, if $e_1, \dots, e_n$ is a basis for V, $e^1, \dots, e^n$ the dual basis, $g_{ij} = g(e_i, e_j)$ and $g^{ij} = g^*(e^i, e^j)$, then (g^{ij}) is the inverse matrix of (g_{ij}). In particular, if $e_1, \dots, e_n$ is an orthonormal basis for V, then $e^1, \dots, e^n$ is an orthonormal basis for V^*.

3.12.3 Gradient Vector Field

For a smooth function f on M, the gradient of f with respect to the Riemannian metric g is a vector field on M defined by

$$\nabla f := \sharp(df).$$

In other words,

$$g(\mathrm{grad}\, f, X) = df(X)$$

for any $X \in \mathfrak{X}(M)$.

Locally,

$$\nabla f = \sharp\left(\sum \frac{\partial f}{\partial x^i} dx^i\right) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

3.12.3.1 Exercise

Let M be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then M inherits a metric g from $\tilde{g}$. For a tangent vector v of $\tilde{M}$ at a point $p \in M$, let $v^\perp$ be the orthogonal projection of v to the tangent space TM_p of M. Let $\tilde{f} : \tilde{M} \to \mathbb{R}$ be a smooth function and let $f = \tilde{f} \restriction M$. Show that

$$\operatorname{grad} f(p) = (\operatorname{grad} \tilde{f}(p))^\perp.$$

3.12.4 Lagrange Multiplier

Let M be a Riemannian manifold. Let $f : M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $\varphi = (\varphi_1, \ldots, \varphi_k) : M \to \mathbb{R}^k$ be a smooth function woth $o \in \mathbb{R}^k$ as a regular value so that $S := \varphi^{-1}(0)$ is a (non empty) submanifold of M of codimension k. Show that if $p \in S$ is a critical point of the restriction map $f|S : S \to \mathbb{R}$, then $\nabla f(p)$ is a linear combination of $\nabla\varphi_1(p)$, ..., $\nabla\varphi_k(p)$.

3.12.5 Induced Metric

If $\varphi : N \to M$ is an immersion and g is a Riemannian metric on M, then

$$(\varphi^* g)_q(v, w) := g_{\varphi(q)}(T\varphi_q v, T\varphi_q w), \quad v, w \in TN_q$$

defines a Riemannian metric $\varphi^* g$ on N.

If N is equipped with a Riemannian metric h and φ is a diffeomorphism such that $\varphi^* g = h$, then φ is called an **isometry**. Isometires preserve angles.

3.12.5.1 Exercise

On the **upper half plane**

$$\mathbf{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

consider a Riemannian metric

$$g := \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

Note that the special linear group

$$\mathrm{SL}(2, \mathbb{R}) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1 \right\}$$

acts on $\mathbf{H}$:

$$A : z \mapsto \frac{az + b}{cz + d}.$$

Show that each $A \in \mathrm{SL}(2, \mathbb{R})$ acts as an isometry on $\mathbf{H}$.

3.12.6 Length of a curve

Let $\gamma : [a, b] \to M$ be a piecewise $\mathcal{C}^1$ map[23] into a connected Riemannian manifold (M, g). Then the **length** of γ is

$$L(\gamma) := \int_a^b |\gamma'(t)| dt.$$

The **distance** between two points $p, q \in M$ is

$$\mathrm{dist}_g(p, q) := \inf\{L(\gamma) \mid \gamma : [0, 1] \to M, \text{ piecewise } \mathcal{C}^1, \ \gamma(0) = p, \ \gamma(1) = q\}.$$

As an example, let's computer the distance between two points ai, bi in the upper half plane $\mathbf{H}$, where a, b are real numbers with $0 < a < b$.

Let $\gamma(t) = ti$ for $a \le t \le b$. Then $\gamma'(t) = i$ and $|\gamma'(t)|_{\mathbf{H}} = \frac{1}{t}$. Thus

$$L(\gamma) = \int_a^b \frac{dt}{t} = \log \frac{b}{a}.$$

Thus $\mathrm{dist}(ai, bi) \le \log \frac{b}{a}$. Now we claim that γ is the shortest path, i.e.,

$$\mathrm{dist}(ai, bi) = \log \frac{b}{a}.$$

For if

$$\mu(t) = x(t) + y(t)i \qquad (x(t), y(t) \in \mathbb{R})$$

is a new path with $\mu(0) = ai$ and $\mu(1) = bi$, then $\mu'(t) = x'(t) + y'(t)i$ and $|\mu'(t)|_{\mathbf{H}} \ge \frac{|y'(t)|}{y(t)} \ge \frac{y'(t)}{y(t)}$. Thus

$$L(\mu) \ge \int_0^1 \frac{y'(t)}{y(t)} dt = \log y(t)\Big|_0^1 = \log \frac{b}{a}.$$

This shows the claim.

3.12.6.1 Exercise

Show that dist_g is a metric on M and the induced topology on M is equal to the original topology.

3.12.7 Invariant Metrics on Lie Groups

[23] A map $\gamma : [a, b] \to M$ is *piecewise* $\mathcal{C}^1$ if it is continuous and there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that $\gamma|[t_{i-1}, t_i] \to M$ is $\mathcal{C}^1$ for $i = 1, \ldots, k$.

3.13 Complex Manifolds

3.13.1 Holomorphic functions of one-variable

Let $\mathbb{C}$ be the field of complex numbers. With respect to the standard coordinate system $z = x + y\mathbf{i}$, where x and y are the real and imaginary parts of z, respectively, let

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right).$$

A complex valued C^1 function f defined on an open subset U of $\mathbb{C}$ is said to be holomorphic if it satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In this case, we have

$$\frac{\partial f}{\partial z}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

for all $z_0 \in U$.

Theorem 3.13.1.1 (Chain Rule in One Variable) Let $f, g : \mathbb{C} \dashrightarrow \mathbb{C}$ be C^1 maps. We express them with variables so that $w = f(z)$ and $u = g(w)$. Then

$$\frac{\partial u}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial z}, \qquad \frac{\partial u}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}}.$$

Corollary 3.13.1.2 The composition of two holomorphic functions is holomorphic.

3.13.2 Holomorphic functions of several variables

Let n be a positive integer. Then we have the identification

$$\mathbb{R}^n \simeq \mathbb{C}^n, \quad (x^1, y^1, \ldots, x^n, y^n) \mapsto (x^1 + y^1\mathbf{i}, \ldots, x^n + y^n\mathbf{i}).$$

Let $(z^1, \ldots, z^n)$ be the standard, i.e., the identity coordinate system on $\mathbb{C}^n$. We have the corresponding real coordinate system $(x^1, y^1, \ldots, x^n, y^n)$. Then

$$\frac{\partial}{\partial z^\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right), \quad \frac{\partial}{\partial \bar{z}^\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right) \qquad (\alpha = 1, \ldots, n).$$

A complex valued function f defined on an open subset of $\mathbb{C}^n$ is called holomorphic if it satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}^\alpha} = 0 \qquad (\alpha = 1, \ldots, n).$$

A map $f = (f^1, \ldots, f^m) : \mathbb{C}^n \dashrightarrow \mathbb{C}^m$ is holomorphic if component functions $f^1, \ldots, f^m$ are holomorphic.

A bijective map f from an on subset of $\mathbb{C}^n$ onto an open subset of $\mathbb{C}^n$ is called biholomorphic if f^{-1} is also holomorphic.

Theorem 3.13.2.1 (Chain Rule in General)

Corollary 3.13.2.2 *The composition of two holomorphic maps are holomorphic.*

3.13.3 Complex Manifolds

A **holomorphic atlas** on a topological manifold M of dimension $2n$ is a topological atlas $\mathcal{A}$ on M such that for any $z, w \in \mathcal{A}$, the map

$$z \circ w^{-1}$$

is biholomorphic.

A topological $2n$-manifold M together with a maximal holomorphic structure is called a **complex manifold** of (complex) dimension n.

A complex manifold of (complex) dimension 1 is called a **Riemann surface**.

3.13.3.1 Automorphism group

The group of biholomorphic self maps on a complex manifold M is denoted by $\mathrm{Aut}(M)$. Then

$$\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a \in \mathbb{C}^\times,\ b \in \mathbb{C}\}.$$

Let $\mathbb{H}$ be the upper half plane. Then

$$\mathrm{Aut}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} \,\middle|\, a, b, c, d, \in \mathbb{R},\ ad - bc = 1 \right\} =: \mathrm{PSL}(2, \mathbb{R}).$$

3.13.3.2 Complex Projective Spaces

The complex projective line $\mathbf{P}^1(\mathbb{C})$ is a topological sphere, and

$$\mathrm{Aut}(\mathbf{P}^1(\mathbb{C})) = \left\{ \frac{az+b}{cz+d} \,\middle|\, a, b, c, d, \in \mathbb{C},\ ad - bc = 1 \right\} =: \mathrm{PSL}(2, \mathbb{C}).$$

3.13.3.3 Elliptic Curves

Let a and b be complex numbers with

$$4a^3 + 27b^2 \neq 0.$$

Then the curve

$$E := \{(x, y) \in \mathbb{C} \mid x^3 + ax + b - y^2 = 0\} \subset \mathbb{C}^2 \simeq \mathbb{C}^2 \times \{1\} \subset \mathbf{P}^2(\mathbb{C})$$

is a connected Riemann surface, whose compactification

$$\bar{E} := \{[x, y, z] \in \mathbf{P}^2(\mathbb{C}) \mid x^3 + axz^2 + bz^3 - y^2 z = 0\}$$

is a torus. To see this, note that

$$\bar{E} = E \cup \{p_\infty\} \subset \mathbf{P}^2(\mathbb{C})_*$$

where $p_\infty = [0, 1, 0]$ and $\mathbf{P}^2(\mathbb{C})_* := \mathbf{P}^2(\mathbb{C}) - \{[1, 0, 0]\}$. The projection map

$$\pi : \mathbb{C}^2 \to \mathbb{C}, \quad (x, y) \mapsto y$$

extends to the projection

$$\bar{\pi} : \mathbf{P}^2(\mathbb{C})_* \to \mathbf{P}^1(\mathbb{C}), \qquad [x, y, z] \mapsto [y, z].$$

Now consider the surjection

$$\bar{E} \hookrightarrow \mathbf{P}^2(\mathbb{C})_* \to \mathbf{P}^1(\mathbb{C}).$$

$$\begin{array}{ccc} E & \longrightarrow & \bar{E} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbf{P}^1(\mathbb{C}) \end{array}$$

The point-at-infinity $\infty := [1, 0] \in \mathbf{P}^1(\mathbb{C})$ is covered by the point-at-infinity $p_\infty \in \bar{E}$.

If $a \neq 0$, then there are four points y_1, y_2, y_3, y_4 in $\mathbb{C}$ which are doubly covered. All other points in $\mathbb{C}$ are triply covered.

If $a = 0$, then there are two points y_1, y_2 in $\mathbb{C}$ which are simply covered. All other points in $\mathbb{C}$ are triply covered.

In any case, we can see that the Euler characteristic of $\bar{E}$ is

$$\chi(\bar{E}) = 0$$

and hence $\bar{E}$ is a torus.

3.13.4 Almost Complex Structure

Any complex manifold M is a smooth manifold. The tangent bundle TM has the canonical automorphism

$$I : TM \to TM, \qquad I^2 = -\operatorname{id} \tag{3.6}$$

which sends

$$\frac{\partial}{\partial x^\alpha} \mapsto \frac{\partial}{\partial y^\alpha}, \qquad \frac{\partial}{\partial y^\alpha} \mapsto -\frac{\partial}{\partial x^\alpha} \qquad (\alpha = 1, \ldots, n)$$

where $(x^1, y^1, \ldots, x^n, y^n)$ is the real chart corresponding to a holomorphic chart $(z^1, \ldots, z^n)$ on M.

In general, on a smooth $2n$-manifold M, an automorphism I of the tangent bundle satisfying the relation (3.6) is called an **almost complex structure**. A smooth manifold together with an almost complex structure is called an **almost complex manifold**.

An almost complex structure I on M is said to be **integrable** if the **Nijenhuis tensor** $N : TM \otimes TM \to TM$ defined by

$$N(X, Y) := [IX, IY] - I[IX, Y] - I[X, IY] - [X, X], \qquad (X, Y \in \mathcal{X}(M)$$

is identically equal to zero.

Theorem 3.13.4.1 (Newlander and Nirenberg (1956)) *An almost complex manifold (M, I) is a complex manifold if and only if I is integrable.*

3.13.4.2

The 4-sphere $\mathbf{S}^4$ has no almost complex structure [Steenrod].

3.13.4.3 $\mathbf{S}^6$

For $n \neq 6$, the sphere $\mathbf{S}^n$ has a complex structure if and only if $n = 0, 2$. (cf. Chern, p.77)

Let $\mathbb{I}$ be the 7-dimensional space of pure imaginary octonions. Then for the sphere

$$\mathbf{S}^6 := \{u \in \mathbb{I} \mid |u| = 1\}$$

the endomorphsm

$$I : T\mathbf{S}^6 \to T\mathbf{S}^6, \quad I_u(v) := uv$$

is an almost complex structure, which is not integrable.

Chapter 4

Vector Fields and Flows

4.1 Integral Curves

Let X be a smooth vector field[1] on a smooth manifold M and $p \in M$. Then a smooth map $c : I \to M$ from an open interval $I \subset \mathbb{R}$ containing the origin 0 is called an integral curve of X with the initial position p, if $c(0) = p$ and

$$c'(t) = X(c(t))$$

for all $t \in I$. With a local chart $(x^1, \ldots, x^n)$, the above equation is a system of ODE:

$$\frac{dc^i}{dt} = f^i(c^1(t), \ldots, c^n(t)) \qquad (i = 1, \ldots, n)$$

where $c^i = x^i \circ c$ and $X = \sum_i (f^i \circ x) \frac{\partial}{\partial x^i}$.

4.1.0.1 Exercise

Show that if $c(t)$ is an integral curve of X, then $c(-t)$ is an integral curve of $-X$.

4.1.1 Examples

1. Let

$$\mathbf{r} = x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n}$$

be the identity vector field (or, the Euler vector field) on $\mathbb{R}^n$: $\mathbf{r}(x) = x$ for $x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$

$$x_t := e^t x$$

[1] A manifold together with a vector field on it is called a *dynamical system*.

is the integral curve of $\mathbf{r}$ with the initial position x. Show the **Euler relation**: for any homogeneous function f on $\mathbb{R}^n$ of degree k

$$\mathbf{r}(f) = kf.$$

In particular, on $\mathbb{R}^n_*$

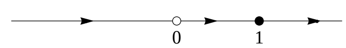
$$\mathbf{r}(r) = r \qquad \text{or} \qquad \frac{\mathbf{r}}{r}(r) = 1$$

where $r(x_1, \ldots, x_n) = \sqrt{x_1^2 + \cdots + x_n^2}$.

2. (Blowing up in finite time) Consider the vector field $X(x) = x^2$ on $\mathbb{R}$.[2] Then

$$x(t) = \frac{1}{1-t}, \qquad -\infty < t < 1$$

is the integral curve of X with the initial position $x(0) = 1$.



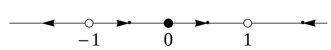
Consider the vector field

$$Y(x) = 1 - x^2$$

on $\mathbb{R}$. Then

$$x(t) = \tanh(t)$$

is the integral curve of Y with the initial position 0.



3. (Complete field) Consider a vector field $X(x, y) = (y, x)$, i.e., $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Then $(\cosh t, \sinh t)$, $t \in \mathbb{R}$, is the integral curve of X with the initial position $(1, 0)$.

4. (Annihilator) Let ω be a 1-form on M, which annihilates a vector field X on M, i.e., $\omega(X) = 0$. Show that if $c : I \to M$ is an integral curve of X, then $c^*\omega = 0$.

5. (Periodic case) Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on the plane $\mathbb{R}^2$. Then $c(t) := (\cos t, \sin t)$, $t \in \mathbb{R}$, is the integral curve with the initial position $(1, 0)$. If we consider an annihilator $\omega = 2(xdx + ydy) = d(r^2)$ of X, then the image of c is a submanifold $i : S \hookrightarrow \mathbb{R}^2$ such that

$$0 = i^*\omega = i^*(d(r^2)) = di^*(r^2).$$

Thus S is a circle $r =$ constant.

[2] We may write $X = x^2\frac{d}{dx}$, if we prefer differential operators.

4.1.1.1 Exercise

On $\mathbb{R}^2 - \{(0,0)\}$, let $X(x,y) = -\frac{(x,y)}{(x^2+y^2)^{3/2}}$. Show that if $c(t) = (x(t), y(t))$ is an integral curve of X, then the ratio $x(t) : y(t)$ is constant. From this observation conclude that

$$c(t) = \Big((1-3t)^{1/3}, 0\Big) \qquad (-\infty < t < 1/3)$$

is the integral curve of X with the initial position $(1,0)$.

Theorem 4.1.1.2 *Let X be a smooth vector field on M. Then*

(1) *For any p in M, there exists an integral curve $c : I \to M$ of X with the initial position p.*

(2) *If $c_1 : I_1 \to M$ and $c_2 : I_2 \to M$ are integral curves of X with the same initial position, then $c_1 = c_2$ on $I_1 \cap I_2$.*

(3) *For any p in M, there exists a unique maximal integral curve*[3] *$c_p : I_p \to M$ of X with the initial position p. In particular, we have Newton's first law:* any integral curve with the nonzero initial velocity has always nonzero velocity.

(4) *Let*
$$\mathcal{I} := \coprod_{p\in M} I_p = \{(p,t) \in M \times \mathbb{R} \mid t \in I_p\}.$$
and
$$\Phi : \mathcal{I} \to M, \qquad (p,t) \mapsto c_p(t).$$
If $s, t \in I_p$, then $s - t \in I_{\Phi(p,t)}$ and
$$\Phi(\Phi(p,t), s-t) = \Phi(p,s).$$

(5) *For each $t \in \mathbb{R}$, let*
$$\mathcal{D}_t = \{p \in M \mid t \in I_p\}.$$
Then
$$\Phi_t : \mathcal{D}_t \to \mathcal{D}_{-t}, \qquad p \mapsto \Phi(p,t) \tag{4.1}$$
is bijective. We have
$$\Phi_t(\mathcal{D}_t \cap \mathcal{D}_{s+t}) = \mathcal{D}_{-t} \cap \mathcal{D}_s, \quad \forall t, s \in \mathbb{R}$$
$\Phi_s \circ \Phi_t = \Phi_{s+t}$ on $\mathcal{D}_{s+t} \cap \mathcal{D}_t$, $\mathcal{D}_0 = M$, $\Phi_0 = \mathrm{id}_M$, and $\Phi_{-t} = (\Phi_t)^{-1}$.

(6) *$\mathcal{I}$ is open in $M \times \mathbb{R}$ and $\Phi : \mathcal{I} \to M$ is smooth.*

(7) *For each $t \in \mathbb{R}$, $\mathcal{D}_t$ is open in M and the map (4.1) is a diffeomorphism.*

[3] I.e., if $c : I \to M$ is an integral curve of X with the initial position p, then $I \subset I_p$ and $c = c_p \restriction I$.

(8) *Let* $I_p := (-t^-(p), t^+(p))$. *Then* $t^+, t^- : M \to (0, \infty]$ *are lower semi continuous.*

The collection

$$\left\{ \Phi_t \;\middle|\; -\sup_{p\in M} t^-(p) < t < \sup_{p\in M} t^+(p) \right\}$$

is called the **local 1-parameter group of local diffeomorphisms of M generated by X**, or simply the **flow** generated by X.[4]

4.1.1.3 Examples

1. (Rotation Fields) On $\mathbb{R}^3$, Let

$$L_z = -y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}.$$

   Then

$$R_z(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   is the 1-parameter family of diffeomorphisms generated by L_z.

2. (Linear Vector Fields) Let $A = (a^i_j)$ be an $n \times n$ real matrix. Define a vector field

$$X_A := \sum_{i,j} a^i_j x^j \frac{\partial}{\partial x^i}$$

   on $\mathbb{R}^n$. Then for each $p \in \mathbb{R}^n$,

$$c(t) := e^{tA}p$$

   is the integral curve of X_A. The map

$$\Phi_t := e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$$

   is a 1-parameter family of diffeomorphisms of $\mathbb{R}^n$.

   Note that for an $n \times n$ matrix B

$$[X_A, X_B] = -X_{[A,B]}$$

   where $[A, B] = AB - BA$.

[4] $\mathfrak{X}(M)$ is the tangent space of $\mathrm{Diff}(M)$ at the identity element.

4.1.1.4 Remark

From the equation

$$\frac{d}{dt}\Phi_t(p) = X_{\Phi_t(p)}$$

we have

$$\frac{d}{dt}(\Phi_t^* f)(p) = \frac{d}{dt} f(\Phi_t(p)) = X_{\Phi_t(p)}(f) = (Xf)(\Phi_t(p)) = (\Phi_t^*(Xf))(p)$$

for $p \in M$ and $f \in \mathcal{C}^\infty(M)$. Thus we have the equation

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ X \ : \ \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M).$$

From this we have, at least formally,

$$\Phi_t^* = \sum_{k=0}^{\infty} \frac{1}{k!}(tX)^k = e^{tX}.$$

Thus

$$f(\Phi_t(p)) = \sum_k \frac{t^k}{k!}(X^k f)(p).$$

For instance, if $X = \frac{d}{dx}$ on $\mathbb{R}$, then

$$f(p+t) = \sum_k \frac{t^k}{k!} f^{(k)}(p)$$

for $f \in \mathcal{C}^\omega(\mathbb{R})$.

4.1.2 Complete Vector Fields

A vector field X on M is said to be *complete* if every maximal integral curve of X with arbitrary initial position is defined for all $t \in \mathbb{R}$. A complete vector field X on M generates a global 1-parameter group $\{\mathrm{Exp}(tX) \mid t \in \mathbb{R}\}$ of diffeomorphisms of M.

Theorem 4.1.2.1 *Every left invariant vector field on a Lie group G is complete. Every right invariant vector field on a Lie group G is complete.*

4.1.2.2 Exercise

(0) Is the vector field $x^2 \frac{d}{dx}$ on $\mathbb{R}$ complete?

(i) Show that for a vector field X on M, if the domain $\mathcal{I} \subset \mathbb{R} \times M$ of the flow Φ of X contains $(-\epsilon, \epsilon) \times M$ for some $\epsilon > 0$, then X is complete.

(ii) Let X be a vector field on a manifold M and let f be a smooth positive function on M. Show that the maximal integral curves $\Phi(\ ,p) : (-t^-(p), t^+(p)) \to M$ and $\tilde{\Phi}(\ ,p) : (-\tilde{t}^-(p), \tilde{t}^+(p)) \to M$ of vector fields X and $\tilde{X} := fX$, respectively, with the initial position $p \in M$ are the reparametrizations of each other. Show that

$$\tilde{t}^+(p) = \int_0^{t^+(p)} \frac{dt}{f(\Phi(t,p))}, \qquad \tilde{t}^-(p) = \int_{-t^-(p)}^0 \frac{dt}{f(\Phi(t,p))}.$$

(iii) Let X be the left invariant vector field on $\mathrm{SO}(n)$ associated to a skew-symmetric matrix $x \in \mathfrak{so}(n)$. Show that

$$\Phi(t,g) = g\exp(tx), \quad (t,g) \in \mathbb{R} \times \mathrm{SO}(n)$$

is the flow on $\mathrm{SO}(n)$ generated by X.

(iv) Let X be a left invariant vector field on a Lie group G with $X_e = x$. Show that

$$\mathrm{Exp}(tX) = R_{\exp(tx)} : G \to G,$$

where $\mathrm{Exp}(tX)$ is the flow on G generated by X.

4.1.2.3

For a vector field X on M, the support of X, denoted by $\mathrm{supp}\, X$, is the closure of $\{p \in M \mid X(p) \neq 0\}$ in M.

Theorem 4.1.2.4 *A vector field with compact support is complete. In particular, every vector field on a compact manifold is complete.*

This theorem follows trivially from the next lemma.

4.1.2.5 Definition

Let $-\infty \le a, b \le \infty$. Then a curve $c : (a,b) \to M$ is said to **tend to infinity as** $t \nearrow b$, if for any compact subset K of M, there exists $t_0 \in (a,b)$ such that $t_0 < t < b$ implies $c(t) \notin K$. In this case we write

$$\lim_{t \nearrow b} c(t) = \infty.$$

Similarly, we have a notion of

$$\lim_{t \searrow a} c(t) = \infty.$$

Lemma 4.1.2.6 *Let X be a vector field on M and let $c : (a,b) \to M$ be a maximal integral curve of X. If $b < \infty$, then*

$$\lim_{t \nearrow b} c(t) = \infty.$$

Proof. If not, then there exist a compact set K in M and an increasing sequence $\{t_k\}$ tends to b such that $c(t_k) \in K$ and $\{c(t_k)\}$ converges, say, to p. Now take an open neighborhood U of p and $\epsilon > 0$ such that $U \times (-\epsilon, \epsilon) \subset \mathcal{I}$. Then, for k large, $c(t_k) \in U$ and hence $c(t_k + \epsilon)$ is defined. This contradicts the maximality. $\square$

Proposition 4.1.2.7 *Let X be a vector field on M. Then there exists a $\mathcal{C}^\infty$ positive function f on M such that fX is complete.*

Proof. We may assume that M is connected. Take a compact exhaustion $\{K_i\}_{i=1}^\infty$ for M, i.e., each K_i is a compact subset of M contained in the interior $\operatorname{Int} K_{i+1}$ of K_{i+1} and $M = \cup K_i$. Note that

$$K_i - \operatorname{Int} K_{i-1} \subset \operatorname{Int} K_{i+1} - K_{i-2}$$

and hence there exists a nonnegative smooth function ρ_i on M such that

$$\rho_i|(K_i - \operatorname{Int} K_{i-1}) = 1, \quad \operatorname{supp} \rho_i \subset \operatorname{Int} K_{i+1} - K_{i-2},$$

where $K_0 = K_{-1} = \emptyset$. Note that, for each i, there exists $\epsilon_i > 0$ such that

$$|t| \le \epsilon_i \Rightarrow \Phi_t(K_i) \subset K_{i+1},$$

where Φ_t denotes the flow of X. Now define

$$f := \sum_{i=1}^\infty \epsilon_i \rho_i.$$

Then f is a smooth positive function on M. Now from the above exercise 5.1.(ii), if $p \in K_i$, then

$$\tilde{t}^+(p) = \int_0^{t^+(p)} \frac{dt}{f(\Phi(t,p))} \ge \int_0^{\epsilon_i} \frac{dt}{\epsilon_i} = 1$$

and

$$\tilde{t}^-(p) = \int_{-t^-(p)}^0 \frac{dt}{f(\Phi(t,p))} \ge \int_{-\epsilon_i}^0 \frac{dt}{\epsilon_i} = 1.$$

Thus $(-1,1) \times M$ is contained in the domain of the flow of fX. Thus again by the exercise 5.1.(i), fX is complete. $\square$

4.1.3 Singularities of Vector Fields

Let X be a vector field on M. Then a point $p \in M$ is called a singularity, singular point, or a fixed point of X if $X(p) = 0$.

Regular (i.e., nonsingular) points of a vector field are not very interesting as we can see in the following proposition.

Proposition 4.1.3.1 *If p is a regular point of a vector field X on M, then there exists a chart $(x^1, \ldots, x^n)$ at p such that $X = \frac{\partial}{\partial x^1}$ in a neighborhood of p.*

Proof. Since the statement is local we may assume that X is a vector field on an open neighborhood U of the origin O in $\mathbb{R}^n$. Without loss of generality we may also assume that $X(O) = (1, 0, \ldots, 0)$. Let Φ_t be the flow of X and define

$$h(t_1, \ldots, t_n) := \Phi_{t_1}(0, t_2, \ldots, t_n)$$

for small $(t_1, \ldots, t_n)$ in U. Then h is a smooth map with $h(O) = O$ and

$$D_1 h(t_1, \ldots, t_n) = X(h(t_1, \ldots, t_n)) \tag{4.2}$$

and

$$D_i h(O) = 1 \qquad (i = 2, \ldots, n).$$

Thus h is a local diffeomorphism at the origin. Let $x = h^{-1}$. Then the equation (4.2) says that $X = \frac{\partial}{\partial x^1}$. $\square$

We will soon see a generalization of this proposition.

If ∇f denotes the gradient field of a function f on a Riemannian manifold M, then a point $p \in M$ is a singularity of ∇f if and only if p is a critical point of f.

Proposition 4.1.3.2 *Let f be a smooth function on a Riemannian manifold M. If $p \in M$ is a regular point of f, then f increases strictly along the integral curve c_p of the gradient field ∇f.*

Proof. Note that

$$\frac{d}{dt} f(c_p(t)) = \left\langle \nabla f(c_p(t)), \frac{d}{dt} c_p(t) \right\rangle = |\nabla f(c_p(t))|^2 \geq 0,$$

where the inequality is strict since p is a regular point of ∇f. $\square$

Proposition 4.1.3.3 *Let M be a compact Riemannian manifold, $f : M \to \mathbb{R}$ be a smooth map whose critical points are isolated, and $\gamma : \mathbb{R} \to M$ be a maximal integral curve of* grad f. *Then $\lim_{t\to\infty} \gamma(t)$ converges to a critical point of f.*

4.1.3.4 Example (Tunnels)

On $\mathbb{R}^3 = \{(x, y, z)\}$, project the vector field $-\frac{\partial}{\partial z}$ onto the sphere $\mathbf{S}^2$ to get a vector field[5]

$$X(x, y, z) = (0, 0, -1) - ((0, 0, -1) \cdot (x, y, z))(x, y, z) = (xz, yz, z^2 - 1)$$

on $\mathbf{S}^2$. Then the integral curve of X satisfies the equation

$$\dot{x} = xz, \quad \dot{y} = yz, \quad \dot{z} = z^2 - 1.$$

[5] This is the gradient vector field of the height function$z$ on $\mathbf{S}^2$.

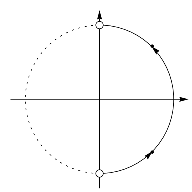
We assume that the initial position is on the equator $z = 0$. Then

$$z(t) = -\tanh t, \qquad -\infty < t < \infty.$$

With $r = \sqrt{x^2 + y^2}$, we have

$$r^2 = 1 - z^2 = \operatorname{sech}^2 t, \qquad \text{i.e.,} \quad r(t) = \operatorname{sech} t.$$

Note that $(x, y, 0)/r$ is constant and the integral curve is the great semi-circle connecting the north and the south poles.



4.2 Lie Derivatives

Let X be a vector field on M and let $\{\Phi_t\}$ be the flow generated by X. The **Lie derivatives** $\mathcal{L}_X(\cdot)$ with respect to X are defined as follows;

If $f \in \mathcal{C}^\infty(M)$, then $\mathcal{L}_X f \in \mathcal{C}^\infty(M)$ is

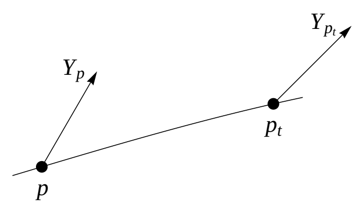
$$\mathcal{L}_X f(p) = \left.\frac{d}{dt}\right|_0 f(\Phi_t p) \qquad (p \in M).$$

If $Y \in \mathfrak{X}(M)$, then $\mathcal{L}_X Y \in \mathfrak{X}(M)$ is

$$(\mathcal{L}_X Y)_p = \left.\frac{d}{dt}\right|_0 (\Phi_{-t})_*(Y_{\Phi_t(p)}) \qquad (p \in M).$$

If $\omega \in \Omega^1(M)$, then $\mathcal{L}_X \omega \in \Omega^1(M)$ is

$$(\mathcal{L}_X \omega)_p = \left.\frac{d}{dt}\right|_0 {\Phi_t}^*(\omega_{\Phi_t(p)}) \qquad (p \in M).$$



In other words,

$$\mathcal{L}_X f = \left.\frac{d}{dt}\right|_0 \Phi_t^* f, \qquad \mathcal{L}_X Y = \left.\frac{d}{dt}\right|_0 (\Phi_{-t})_* Y, \qquad \mathcal{L}_X \omega = \left.\frac{d}{dt}\right|_0 \Phi_t^* \omega.$$

Note that the Lie derivatives

$$\mathcal{C}^\infty(M) \xrightarrow{\mathcal{L}_X} \mathcal{C}^\infty(M), \qquad \mathcal{X}(M) \xrightarrow{\mathcal{L}_X} \mathcal{X}(M), \qquad \Omega^1(M) \xrightarrow{\mathcal{L}_X} \Omega^1(M)$$

are all $\mathbb{R}$-linear. More generally, since a vector field moves every "tensor field" on M, Lie derivative $\mathcal{L}_X$ is defined on the space of tensor fields, which will be discussed in the next chapter.

Proposition 4.2.0.1 *For $f, g \in \mathcal{C}^\infty(M)$, $X, Y, Z \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$, Lie derivatives satisfy the following relations;*

(i) $\mathcal{L}_X f = Xf$

(ii) $\mathcal{L}_X Y = [X, Y]$

(iii) $(\mathcal{L}_X \omega)(Y) = \mathcal{L}_X(\omega Y) - \omega(\mathcal{L}_X Y)$

In particular, we have

(i-1) $\mathcal{L}_X(fg) = (\mathcal{L}_X f)g + f(\mathcal{L}_X g)$

(ii-1) $\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f(\mathcal{L}_X Y)$

(ii-2) $\mathcal{L}_X[Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z], \qquad \mathcal{L}_{[X,Y]}(Z) = [\mathcal{L}_X, \mathcal{L}_Y](Z)$

(iii-1) $\mathcal{L}_X(df) = d(\mathcal{L}_X f), \quad \mathcal{L}_X(f\omega) = (\mathcal{L}_X f)\omega + f\mathcal{L}_X(\omega).$

Proof. (i) is trivial.

(ii) Let $p \in M$. Then there exist a neighborhood U of p and $\epsilon > 0$ such that the flow Φ_t^X of X and the flow Φ_s^Y of Y are defined on U, for $|t| < \epsilon$ and for $|s| < \epsilon$. We will show that for any $f \in \mathcal{C}^\infty(M)$,

$$(\mathcal{L}_X Y)_p f = [X, Y]_p f.$$

Note that

$$\begin{aligned} [X, Y]_p f &= X_p(Yf) - Y_p(Xf) \\ &= \left.\frac{d}{dt}\right|_0 (Yf)(\Phi_t^X(p)) - \left.\frac{d}{ds}\right|_0 (Xf)(\Phi_s^Y(p)) \\ &= \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 f(\Phi_s^Y \Phi_t^X p) - \left.\frac{d}{ds}\right|_0 \left.\frac{d}{dt}\right|_0 f(\Phi_t^X \Phi_s^Y p). \end{aligned}$$

On the other hand

$$\begin{aligned} \mathcal{L}_X(Y)_p f &= \left.\frac{d}{dt}\right|_0 ((\Phi_{-t}^X)_* Y)_p f = \left.\frac{d}{dt}\right|_0 Y_{\Phi_t^X(p)}(f \circ \Phi_{-t}^X) \\ &= \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 f(\Phi_{-t}^X \Phi_s^Y \Phi_t^X p) \\ &= \left.\frac{d}{ds}\right|_0 \left(\left.\frac{d}{dt}\right|_0 f(\Phi_{-t}^X \Phi_s^Y p) + \left.\frac{d}{dt}\right|_0 f(\Phi_s^Y \Phi_t^X p) \right) \end{aligned}$$

Thus $\mathcal{L} = [X, Y]$.

Here is another proof.[6] Take a smooth family of smooth functions g_t on U such that

$$f \circ \Phi_t = f + t g_t$$

for $|t| < \epsilon$. Then $g_0 = Xf$ and hence

$$\begin{aligned} (\mathcal{L}_X Y)(f) &= \left.\frac{d}{dt}\right|_0 ((\Phi_{-t})_* Y)(f) = \left.\frac{d}{dt}\right|_0 (Y(f \circ \Phi_{-t}) \circ \Phi_t) \\ &= \left.\frac{d}{dt}\right|_0 (Y(f - t g_{-t}) \circ \Phi_t) = \left.\frac{d}{dt}\right|_0 Y(f) \circ \Phi_t - Y(g_0) \\ &= X(Y(f)) - Y(X(f)). \end{aligned}$$

(iii) The identities (iii-1) are easy. From these the third identity follows. This completes the proof. □

[6] cf. Spivak

4.2.0.2 Exercise

Show that

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + \omega(X)df.$$

4.2.0.3

Note that if $h(s,u)$ is a smooth function defined for (s,u) near $(0,0)$ such that $D_1^2h(0,0) = 0 = D_2^2h(0,0)$, then $\left(\frac{d}{dt}\right)_0^2 (h(t,t)) = 2\ \frac{\partial}{\partial s}\Big|_0 \frac{\partial}{\partial u}\Big|_0 h(s,u)$.

Theorem 4.2.0.4 *Given a vector field X, Y on M and a point p in M, let*

$$h(t) := \operatorname{Exp}_{-tY} \operatorname{Exp}_{-tX} \operatorname{Exp}_{tY} \operatorname{Exp}_{tX}(p).$$

Then

$$h'(0) = 0, \qquad \frac{1}{2}h''(0) = [X,Y]_p.$$

4.2.0.5

Let $F : M \to N$ be a diffeomorphism and let $\{\Phi_t\}$ be the flow generated by a vector field X on M. Then $\{F \circ \Phi_t \circ F^{-1}\}$ is the flow generated by the vector field F_*X on N.

Theorem 4.2.0.6 *Let X and Y be vector fields on M. Then $[X,Y] = 0$ if and only if $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$ for small $t, s \in \mathbb{R}$.*

Proof. Since $[X,Y] = \mathcal{L}_X Y$, $[X,Y] = 0$ if and only if $(\Phi_t^X)_* Y = Y$ for all t. Since the flow generated by $(\Phi_t^X)_* Y$ is $\Phi_t^X \circ \Phi_s^Y \circ \Phi_{-t}^X$, we are done. □

Corollary 4.2.0.7 *Let $X_1, \ldots, X_k$ be vector fields on M such that $[X_i, X_j] = 0$ for $1 \le i, j \le k$. If $X_1(p), \ldots, X_k(p)$ is linearly independent for some point $p \in M$, then there exists a coordinate system $x = (x^1, \ldots, x^n)$ in a neighborhood U of p such that*

$$X_j \restriction U = \frac{\partial}{\partial x^j}$$

for $1 \le j \le k$.

Proof. Given a point p in M, take vector fields $X_{k+1}, \ldots, X_n$ so that $X_1, \ldots X_k$, ..., X_n is a frame in some neighborhood U of p. For some $\epsilon > 0$, let ϕ_t^i be the flow on U generated by the vector field X_i for $i = 1, \ldots, n$, for $|t| \le \epsilon$. Note that

$$\phi_{t_1}^1, \quad \ldots, \quad \phi_{t_k}^k$$

are commuting vector fields.

Now consider the map

$$f(x_1, \dots, x_n) := \phi^1_{x_1} \circ \phi^2_{x_1} \circ \dots \phi^n_{x_n}(p)$$

for $(x_1, \dots, x_n) \in (-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon)$. Then

$$\frac{\partial}{\partial x_a} f(x_1, \dots, x_n) = X_a(f(x_1, \dots, x_n)) \qquad (a = 1, \dots, k)$$

and

$$\frac{\partial}{\partial x_i} f(0, \dots, 0) = \left.\frac{d}{dt}\right|_0 \phi^i_t(p) = X_i(p) \qquad (i = 1, \dots, n)$$

Thus f is a diffeomorphism in a neighborhood of the origin and hence its inverse map $(x^1, \dots, x^n)$ is a chart such that $X_a = \frac{\partial}{\partial x^a}$ for $a = 1, \dots k$. $\square$

See chapter 8 for the Frobenius Theorem.

4.2.1 Example

In $\mathbb{R}^3 = \{(x, y, z)\}$, let

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Find all $f \in \mathcal{C}^\infty(M)$ such that $X(f) = 0$ and $Y(f) = 0$.

First Solution. Note that X and Y are linearly independent everywhere and $[X, Y] = 0$. Let

$$u = x, \ v = y, \ w = z - xy$$

or

$$x = u, \ y = v, \ z = w + uv.$$

Then

$$\frac{\partial}{\partial u} = X, \ \frac{\partial}{\partial v} = Y, \ \frac{\partial}{\partial w} = \frac{\partial}{\partial z}.$$

Thus $\frac{\partial f}{\partial u} = 0 = \frac{\partial f}{\partial v}$ implies

$$f = g(w) = g(z - xy)$$

for some function g on M.

Second Solution. Note that f is constant along the integral curves of X and Y. The integral curve of X with the initial position (x_0, y_0, z_0) is $(t + x_0, y_0, ty_0 + z_0)$ and the integral curve of Y with the same initial position is $(x_0, t + y_0, tx_0 + z_0)$. These curves form a surface

$$S : z - xy = z_0 - x_0 y_0$$

and X, Y are tangent to S. Thus $f = g(z - xy)$ for some $g \in \mathcal{C}^\infty(M)$.

Third Solution. Let ω be a 1-form ``perpendicular'' to X and Y, e.g.,

$$\omega = -ydx - xdy + dz.$$

(Observe that $(1, 0, y) \times (0, 1, x) = (-y, -x, 1)$.) If $\iota : S \to \mathbb{R}^4$ is the surface (or the integral submanifold) generated by X and Y, then $\iota^*\omega = 0$. But $\omega = d(z - xy)$ and hence

$$S = \{z - xy = \text{const.}\}$$

Chapter 5

Vector Bundles

A (smooth) map $\pi : E \to M$ is a **vector bundle over** M **of rank** r if

(1) each **fiber** $E_p := \pi^{-1}(p)$, $p \in M$, is a (real) vector space of dimension r, and

(2) each point in M has an open neighborhood U and a diffeomorphism

$$\varphi : E|U := \pi^{-1}(U) \to U \times \mathbb{R}^r,$$

called a **local trivialization** or a (**vector**) **bundle chart**, such that $\mathrm{proj}_1 \circ \varphi = \pi$ and $\varphi|E_p \to \{p\} \times \mathbb{R}^r$ is a linear isomorphism.

A collection $\{\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r\}$ of bundle charts is called a **bundle atlas** if $\{U_\alpha\}$ covers M.

5.0.0.1 Exercises

(i) Let $\{\varphi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^r\}$ be a bundle atlas of a vector bundle E over M. Show that the *transition maps* $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ defined by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) := (p, \varphi_{\alpha\beta}(p)v) \quad \text{for } (p, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^r \qquad (2.1)$$

are smooth. Show that

$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

for any indices α, β, γ. In particular,

$$\varphi_{\alpha\alpha} = 1_r, \ \varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}.$$

(ii) Suppose that E is an abstract set together with a map $\pi : E \to M$. Suppose, for some open cover $\{U_\alpha\}$ of M, there exist bijections

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$$

such that $\mathrm{proj}_1 \circ \varphi_\alpha = \pi$, for all α. If there exists a smooth map

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$$

for each indices α, β satisfying (2.1), then there exists a unique topology and a unique differentiable structure on E such that π is smooth and φ_α's are diffeomorphisms. Moreover each fiber of E is a vector space of dimension r and $\pi : E \to M$ is a vector bundle.

(iii) The set of all sections of a vector bundle E over M will be denoted by

$$\Gamma(M, E)$$

Show that $\Gamma(M, E)$ is canonically a module over $\mathcal{C}^\infty(M)$.

5.0.0.2 Examples

(i) The tangent and cotangent bundles of a manifold are vector bundles.

(ii) For a real vector space V, the projection map $M \times V \to M$ is a vector bundle, called a **product bundle**. This bundle will be denoted by V_M. A section of the product bundle $V_M \to M$ is just a V-valued $\mathcal{C}^\infty$ function on M:

$$\Gamma(M, V_M) \simeq \mathcal{C}^\infty(M) \otimes_{\mathbb{R}} V$$

5.1 Vector Bundle Homomorphism

Let $\pi : E \to M$ and $\psi : F \to M$ be vector bundles over the same **base space** M. Then a map $f : E \to F$ between the **total spaces** is called a **vector bundle homomorphism** if $\psi \circ f = \pi$ and the restriction map $f \restriction E_p \to F_p$ is linear for each p in M.

Two (vector bundle) homomorphisms can be composed and there exist identity homomorphisms and isomorphisms.

A vector bundle isomorphic to a product bundle is called a **trivial bundle**.

The tangent bundle of a Lie group is trivial.

5.1.0.1

The tangent bundle of $\mathbf{S}^{2n}$ is non trivial, since every vector field on $\mathbf{S}^{2n}$ has a singularity.

5.1.0.2 Exercises

(i) A vector bundle E is trivial if and only if there exist (global) sections $s_1, \ldots, s_r$ of E such that $s_1(p), \ldots, s_r(p)$ is a basis of E_p for each $p \in M$. Such an r-tuple $(s_1, \ldots, s_r)$ is called a (global) **frame** for E.

(ii) Let $\{\varphi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^r\}$ be a bundle atlas for a vector bundle E over M. Then each φ_α defines a local frame $s^\alpha = (s_1^\alpha, \ldots, s_r^\alpha)$ for E over U_α by pulling back the canonical frame for $U_\alpha \times \mathbb{R}^r \to U_\alpha$. Conversely, any local frame s^α for E over U_α defines a local trivialization φ_α. Show that for any indices α, β,

$$s^\beta = s^\alpha \cdot \varphi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta$$

where $\{\varphi_{\alpha\beta}\}$ are the transition maps for $\{\varphi_\alpha\}$. Note that if ξ is a $\mathcal{C}^\infty$ section of E, then

$$\xi|U_\alpha = \sum_{i=1}^r \xi_\alpha^i s_i^\alpha = s^\alpha \xi_\alpha \tag{4.1.1}$$

for some $\xi_\alpha = \begin{pmatrix} \xi_\alpha^1 \\ \vdots \\ \xi_\alpha^r \end{pmatrix} \in \mathcal{C}^\infty(U_\alpha, \mathbb{R}^r) = \mathcal{C}^\infty(U_\alpha) \otimes \mathbb{R}^r$. Show that for any indices α, β

$$\xi_\alpha = \varphi_{\alpha\beta}\xi_\beta \qquad \text{on } U_\alpha \cap U_\beta. \tag{4.1.2}$$

Conversely, suppose we are given $\xi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathbb{R}^r)$ for each α such that (4.1.2) is true for any indices α, β. Then $\{\xi_\alpha\}$ defines a unique global section ξ for E such that (4.1.1) is true.

(iii) A **Euclidean structure** on a vector bundle E over a M is a smooth assignment to each point $p \in M$ an inner product $\langle\ ,\ \rangle$ on E_p. Show that there exists a Euclidean structure on every vector bundle over a (paracompact) manifold.

5.2 Pull-back Bundle

Let E be a vector bundle over M. For a smooth map $\psi : N \to M$, the **pull back bundle** of E by ψ is the bundle

$$\psi^* E := \{(q, e) \in N \times E \mid e \in E_{\psi(q)}\}$$

over N. If s is a local section of E defined over an open subset U of M, then $\psi^* s$ is a section of $\psi^* E$ defined over the open subset $\psi^{-1}(U)$ of N. In particular, the pullback of a trivial bundle is again trivial. Note that if $\{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})\}$ is a collection of transition maps associated to some bundle atlas for E, then $\psi^* E$ has the transition maps $\{\psi^*\varphi_{\alpha\beta} \mid \psi^{-1}(U_\alpha) \cap \psi^{-1}(U_\beta) \to \mathrm{GL}(r, \mathbb{R})\}$.

5.2.0.1 Example

Let $\mathbf{P}^n$ be the real projective n-space. Then each point $p \in \mathbf{P}^n$ is a line l_p in $\mathbb{R}^{n+1}$. Let

$$\begin{aligned} L_n &:= \{p \times v \in \mathbf{P}^n \times \mathbb{R}^{n+1} \mid v \in l_p\} \\ &= \{(p_1 : \cdots : p_{n+1}) \times (v_1, \ldots, v_{n+1}) \in \mathbf{P}^n \times \mathbb{R}^{n+1} \mid p_i v_j = p_j v_i \quad \forall i, j\}. \end{aligned}$$

Then L_n is called the **universal** or **tautological line bundle** over $\mathbf{P}^n$. The total space L_n is the blowing up of $\mathbb{R}^{n+1}$ at the origin. For $\alpha = 1, \ldots, n+1$, let

$$U_\alpha = \{p = (p_1 : \cdots : p_{n+1}) \in \mathbf{P}^n \mid p_\alpha \neq 0\}.$$

Then $\{U_\alpha\}$ is an open cover of $\mathbf{P}^n$. We have a non vanishing section s_α of L_n defined on U_α given by

$$s_\alpha(p_1 : \cdots : p_{n+1}) = p \times \left(\frac{p_1}{p_\alpha}, \ldots, \frac{p_{n+1}}{p_\alpha}\right).$$

Then the transition maps are given by

$$\varphi_{\alpha\beta}(p) = \frac{p_\alpha}{p_\beta}.$$

Thus a section of L_n is a collection of $\xi_\alpha \in \mathcal{C}^\infty(U_\alpha)$ satisfying

$$\xi_\beta = \xi_\alpha \varphi_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

This means, in particular, that the line bundle L_1 is obtained from two copies of $\mathbb{R} \times \mathbb{R}$ by the gluing process

$$(\mathbb{R} - \{0\}) \times \mathbb{R} \ni (t, v) \mapsto \left(\frac{1}{t}, tv\right) \in (\mathbb{R} - \{0\}) \times \mathbb{R}.$$

Thus the total space L_1 is the "Möbius band".

We now claim that L_n is nontrivial, or equivalently, every section ξ of L_n has a zero. Since L_1 is the pull back of L_n under the canonical embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^n$, it suffices to show that L_1 is nontrivial. Now, if we identify U_1 and U_2 in a standard way, then a section of L_1 is a pair (f_1, f_2) of smooth functions on $\mathbb{R}$ such that

$$f_1(1/t) = t f_2(t) \quad \text{for all } t \in \mathbb{R}_*.$$

If f_1 and f_2 are non vanishing, then their signatures are constant, which contradicts the above identity. Thus every section of L_1 (and hence of L_n) has a zero.

5.3 Operations of vector bundles

5.3.1 Dual bundles

For a vector bundle $E \to M$, we have the **dual bundle** $E^* \to M$.

5.3.2 Whitney Sum

For vector bundles E and F over M, $E \oplus F$ is the "vector bundle" with the fiber

$$(E \oplus F)_p = E_p \oplus F_p,$$

called the **Whitney** (or **direct**) **sum** of E and F.

5.3.3 Tensor Product

For vector bundles E and F over M, we have the tensor product bundle $E \otimes F$ over M.

5.3.4 Exercises

(1) For vector bundles E and F over M, obtain new vector bundles

$$\mathrm{Hom}(E, F), \quad \wedge^k E,$$

where $\wedge^k$ is the k-th wedge power. Show that if $\{s_i\}$ and $\{t_j\}$ are local frames for E and F, respectively, then $\{s_i \otimes t_j\}$ is a local frame for $E \otimes F$. If $\{s_i^*\}$ is the `dual frame' of $\{s_i\}$, then $\{s_i^* \otimes t_j\}$ is a local frame for $\mathrm{Hom}(E, F)$ and $\{s_{i_1} \wedge \cdots \wedge s_{i_k} \mid i_1 < \cdots < i_k\}$ is a local frame for $\wedge^k E$.[1]

(2) For a vector bundle E over M of rank r, the direct sum

$$\wedge^\bullet E := \sum_{k \geq 0}^{r} \wedge^k E$$

is the bundle of exterior algebras of E.[2] For $\xi \in \Gamma(M, \wedge^k E)$ and $\eta \in \Gamma(M, \wedge^l E)$, check the **anti commutativity**

$$\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi \in \mathcal{C}^\infty(M, \wedge^{k+l} E).$$

Thus

$$\Gamma(M, \wedge^\bullet E) = \sum_{k \geq 0} \Gamma(M, \wedge^k E)$$

is a graded anti-commutative algebra.

[1] In fact, for any 'continuous' functor $\mathcal{F}$ form a category $\mathbf{C}$ (e.g., of finite dimensional vector spaces and linear maps) into a category of $\mathbf{D}$ (e.g., of similar sorts) , there exists a corresending vector bundle construction [Atiyah, K-theory]

[2] We use the notation

$$\det E$$

for the line bundle $\wedge^r E$.

(3) Show that any vector bundle E is isomorphic to its double dual E^{**} canonically.

(4) Show that any real vector bundle E is (non canonically) isomorphic to its dual E^*.

(5) We have already seen that

$$\Omega^1(M) \simeq \mathfrak{X}(M)^*$$

which is a special case of

$$\Gamma(M, E^*) \simeq \Gamma(M, E)^* := \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(M, E), \mathcal{C}^\infty(M))$$

for any vector bundle $E \to M$. This isomorphism is again a special case of $\mathcal{C}^\infty(M)$-module isomorphism

$$\Gamma(M, \mathrm{Hom}(E, F)) \simeq \mathrm{Hom}_{\mathcal{C}^\infty(M)}(\Gamma(M, E), \Gamma(M, F))$$

for any vector bundles E and F over M.

5.3.5 Contraction map

The bundles

$$E \otimes E^*, \quad E^* \otimes E, \quad \mathrm{Hom}(E, E)$$

are all isomorphic, and the trace map (or the evaluation map)

$$E^* \otimes E \to \mathbb{R}_M$$

is called the contraction map.

5.3.6 Remark

Let V be a finite dimensional real vector space. Then we identify $\wedge^k(V^*)$ with the space $L^k_{alt}V$ of alternating k-linear forms on V as follows. If $\xi^1, \ldots, \xi^k \in V^*$ and $v_1, \ldots, v_k \in V$, then

$$(\xi^1 \wedge \cdots \wedge \xi^k)(v_1, \ldots, v_k) = \det(\xi^i(v_j)) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\xi^1(v_{\sigma(1)}) \ldots \xi^k(v_{\sigma(k)}), \tag{5.1}$$

where S_k denotes the group of permutations on $\{1, \ldots, k\}$. In particular, if $\xi, \eta \in V^*$, then

$$(\xi \wedge \eta)(v_1, v_2) = \xi(v_1)\eta(v_2) - \xi(v_2)\eta(v_1).$$

5.3.6.1 Exercise

Show that if $\xi \in \wedge^k(V^*)$ and $\eta \in \wedge^l(V^*)$, then under the above identification, we have

$$\begin{aligned} & (\xi \wedge \eta)(v_1, \ldots, v_{k+l}) \\ = & \sum_{\sigma \in S_{k,l}} \operatorname{sgn}(\sigma)\, \xi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\ = & \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \xi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \end{aligned}$$

for $v_1, \ldots, v_{k+l} \in V$, where $S_{k,l}$ denotes the set of all (k, l)-shuffles.[3]

5.3.6.2

For a module $\mathcal{E}$ over a commutative ring R, let

$$\mathcal{A}^k_R(\mathcal{E})$$

be the R-module consisting of all alternating k-linear maps $\mathcal{E} \times \cdots \times \mathcal{E} \to R$. Now if E is a vector bundle over M, then the $\mathcal{C}^\infty(M)$-module isomorphism

$$\Gamma(M, \wedge^k(E^*)) \simeq \Gamma(M, (\wedge^k E)^*) \simeq \mathcal{A}^k_{\mathcal{C}^\infty(M)}(\Gamma(M, E))$$

is given by (5.1) for sections $\xi^1, \ldots, \xi^k$ for E^* and sections $v_1, \ldots, v_k$ for E.

[3] A (k,l)-*shuffle* is a permutation σ of $\{1, 2, \ldots, k+l\}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. Note that the *Laplace expansion* of the determinant of an $n \times n$ matrix $A = (a^i_j)$ with respect to the columns A_J, $J = (j_1, \ldots, j_k) \in \mathcal{I}^n_k := \{(i_1, \ldots, i_k) \mid 1 \le i_1 < \cdots < i_k \le n\}$, is

$$\det A = \sum_{I \in \mathcal{I}^n_k} \operatorname{sgn} \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix} \det A^I_J \cdot \det A^{I^c}_{J^c}$$

where $I^c = (i_{k+1} < \cdots < i_n) \in \mathcal{I}^n_{n-k}$ and $J^c = (j_{k+1} < \cdots < j_n) \in \mathcal{I}^n_{n-k}$ are the complements of I and J in $\{1, 2, \ldots, n\}$, A^I_J are the $k \times k$ minors of A, and $A^{I^c}_{J^c}$ are the $(n-k) \times (n-k)$ minors of A. Note that $\operatorname{sgn}(I\ I^c) = (-1)^{i_1 + \cdots i_k + k(k+1)/2}$ and hence $\operatorname{sgn} \begin{pmatrix} J & J^c \\ I & I^c \end{pmatrix} = (-1)^{j_1 + \cdots j_k + i_1 + \cdots + i_k}$.

5.4 Orientation

A vector bundle E of rank r is *orientable* if there exists a bundle atlas $\{\varphi_\alpha\}$ such that the transition maps $\{\varphi_{\alpha\beta}\}$ have values in $\mathrm{GL}^+(r, \mathbb{R})$. An "oriented atlas " induces an **orientation** on each fiber of E.

5.4.0.1 Remark

Recall that an **orientation** on a finite dimensional real vector space V is an equivalence class of (ordered) basis for V. Two basis $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ for V are **equivalent** if there exists a continuous family $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t))$ of basis for V, $0 \leq t \leq 1$, such that $\mathbf{u}(0) = \mathbf{v}$ and $\mathbf{u}(1) = \mathbf{w}$. Note that, in this case, there exists a unique continuous curve $g(t) \in \mathrm{GL}(n, \mathbb{R})$ such that $\mathbf{u}(t) = \mathbf{v}g(t)$. Since $\det g(0) = \det 1_n = 1 > 0$, $\det g(1) > 0$ by the continuity. Conversely, if $\mathbf{v}$ and $\mathbf{w}$ are basis for V such that $\mathbf{w} = \mathbf{v}g$ for some $g \in \mathrm{GL}^+(n, \mathbb{R})$, then $\mathbf{v}$ and $\mathbf{w}$ are equivalent, since $\mathrm{GL}^+(n, \mathbb{R})$ is connected.

Thus an orientation on an n-dimensional real vector space V is a choice of connected component of $\wedge^n V - \{0\}$.

A linear isomorphism $l : V \to V$ is **orientation preserving** if for some (and hence for any) basis $(v_1, \ldots, v_n)$ of V, $(l(v_1), \ldots, l(v_n))$ is equivalent to $(v_1, \ldots, v_n)$, or equivalently, $\det l > 0$. A $\mathcal{C}^\infty$ map $h : U \to V$ from an open subset U of V is said to be *orientation preserving at* $p \in U$ if the derivative

$$Dh(p) : V \to V$$

is orientation preserving.

The Cartesian space $\mathbb{R}^n$ is always equipped with the `standard orientation'.

Theorem 5.4.0.2 *A real vector bundle E over M is orientable if and only if $\det E$ is a trivial bundle.*

Proof. Let $\{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})\}$ be the collection of transition maps for E. Thus for each index α we have a local frame

$$s^\alpha = (s^\alpha_1, \ldots, s^\alpha_r)$$

for $E \restriction U_\alpha$. These frames satisfy the relation

$$s^\beta = s^\alpha \varphi_{\alpha\beta}$$

on $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then

$$\hat{s}^\alpha := s^\alpha_1 \wedge \cdots \wedge s^\alpha_r$$

is a local frame for $\det E$ over U_α and

$$\hat{s}^\beta = \hat{s}^\alpha \det \varphi_{\alpha\beta}$$

on $U_{\alpha\beta}$.

Now if E is orientable, then one can choose transition maps having positive determinants. Let $\{\rho_\alpha : M \to \mathbb{R}\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$ of M. Consider the global section

$$\xi := \sum_\beta \rho_\beta \hat{s}^\beta$$

of $\det E$. If p is a point in M which lies in U_α for some α, then

$$\xi(p) = \left(\sum_\beta \rho_\beta(p) \det(\varphi_{\alpha\beta}(p)) \right) \hat{s}^\alpha(p) \neq 0.$$

Thus ξ is nonvanishing and hence $\det E$ is trivial.[4]

Conversely, suppose $\det E$ has a nonvanishing section ξ. Then one can choose a local frame s^α such that $\hat{s}^\alpha = \xi$ on U_α. Then all transition maps have determinant 1. □

5.4.0.3 Exercise

Note that the Möbius band $\ddot{M}$ is the quotient of $\mathbf{S}^1 \times \mathbb{R}$ by the involution

$$a : (z, t) \mapsto (-z, -t)$$

Show that $\ddot{M} \to \mathbf{S}^1$ is nontrivial.

5.4.1 Orientable manfolds

A manifold M is **orientable** if its tangent bundle is orientable. Thus on an orientable manifold M, there exists an atlas $\{x_\alpha : U_\alpha \to \mathbb{R}^n\}$ such that the transition maps $x_\alpha \circ x_\beta^{-1}$ have the positive Jacobian determinant for all α and β. A manifold M is **oriented** if an `oriented atlas' for M is specified. Then each tangent space of an oriented manifold is oriented.

A map $\phi : M \to N$ between oriented manifolds of the same dimension is said to be **orientation preserving at** $p \in M$, if the derivative $T\phi_p : TM_p \to TN_{\phi(p)}$ is orientation preserving. Obviously, if ϕ preserves the orientation at p, then it preserves the orientation in a neighborhood of p. The map ϕ is said to be **orientation preserving** if it preserves the orientation at every point of M.

5.4.1.1 Exercise

Let $\phi : M \to N$ and $\psi : N \to L$ be orientation preserving maps between oriented manifolds of the same dimension. Show that $\psi \circ \phi : M \to L$ is also orientation preserving.

[4] One may use a Euclidean structure for E and local orthonormal frames.

5.4.2 Example

(1) The n-sphere $\mathbf{S}^n$ is orientable. The usual orientation on $\mathbf{S}^n$ is given as follows; Note that the tangent space of $\mathbf{S}^n$ at a point $p \in \mathbf{S}^n$ is

$$T\mathbf{S}^n_p = \{v \in \mathbb{R}^{n+1} : v \perp p\}.$$

Then a basis $(v_1, \ldots, v_n)$ for $T\mathbf{S}^n_p$ is `positively oriented' if $(p, v_1, \ldots, v_n)$ is a positively oriented basis for $\mathbb{R}^{n+1}$.

The antipodal map $a : \mathbf{S}^n \to \mathbf{S}^n$ is orientation preserving if and only if n is odd.[5] To see this it suffices to check at one point, say at $p = (1, 0, \ldots, 0)$. Then $v_1 = (0, 1, 0, \ldots, 0), \ \ldots, \ v_n = (0, \ldots, 0, 1)$ is a positively oriented basis for the tangent space of $\mathbf{S}^n$ at p. Now

$$Ta_p(v_1) = (0, -1, 0, \ldots, 0), \ \ldots, \ Ta_p(v_n) = (0, \ldots, 0, -1)$$

is a basis for the tangent space of $\mathbf{S}^n$ at $-p$, which is positively oriented if and only if $(-1)^{n+1} = 1$, i.e., n is odd. Note that the stereographic projection of $\mathbf{S}^n$ from the north pole preserves the orientation if and only if$n$ is odd.

(2) Let $\pi : \mathbf{S}^n \to \mathbf{P}^n$ be the canonical projection of the n-sphere onto the projective n-space. Suppose that $\mathbf{P}^n$ is orientable. Then we may choose an orientation on $\mathbf{P}^n$ so that π is orientation preserving. Let $a : \mathbf{S}^n \to \mathbf{S}^n$ be the antipodal map. Then $\pi = \pi \circ a$ and hence $\pi \circ a$ is orientation preserving. Thus a must preserve the orientation, i.e., n is odd. Thus even dimensional projective spaces are not orientable. Conversely, an orientation on an odd dimensional projective space is obtained from the orientation of the odd sphere, where the antipodal map preserves the orientation.

Note that on a connected orientable manifold there are two different orientations.

5.4.3 Exercise

Does there exist a smooth function $f : \mathbb{R}^3 \dashrightarrow \mathbb{R}$ such that zero is a regular value of f and the zero level surface $Z := f^{-1}(0)$ is diffeomorphic to the Möbius band $\ddot{M}$.

[5] This property is independent of the choice of an orientation of $\mathbf{S}^n$.

5.5 Subbundles and quotient bundles

A subset F of a vector bundle E over M is a **subbundle** of E, if there exists an integer k with $0 \leq k \leq r$ and a vector bundle atlas $\{\varphi_a : E|U_\alpha \simeq U_\alpha \times \mathbb{R}^r\}$ of E such that

$$F \cap (E|U_\alpha) = \varphi_\alpha^{-1}(U_\alpha \times \mathbb{R}^k)$$

where $\mathbb{R}^k \hookrightarrow \mathbb{R}^r$ in standard way.

5.5.1 Exercises

(i) A subbundle F of a vector bundle E is also a vector bundle and the inclusion $F \hookrightarrow E$ is a bundle homomorphism.

(ii) A bundle monomorphism $f : E' \to E$ is an isomorphism onto a subbundle of E.

(iii) A subbundle F of E induces a quotient bundle E/F.

(iv) Let H_n be the subbundle of the product bundle $\mathbf{P}^n \times \mathbb{R}^{n+1} \to \mathbf{P}^n$, which is the orthogonal complement of the tautological line bundle L_n. Thus each fiber of H_n over a point $p \in \mathbf{P}^n$ is the hyperplane in $\mathbb{R}^{n+1}$ perpendicular to the line $l_p \subset \mathbb{R}^{n+1}$. Show that $H_n \simeq L_n \otimes T\mathbf{P}^n$.

5.6 Tensor Fields

For a nonnegative integer k, a **tensor bundle** of a manifold M is a tensor product

$$E := E_1 \otimes \cdots \otimes E_k$$

of vector bundles, where each E_i is either the tangent bundle TM or the cotangent bundle TM^* of M. Thus

$$E \simeq TM^{\otimes p} \otimes TM^{*\otimes q} =: TM^{(p,q)}$$

for some nonnegative integers p and q with $p + q = k$. When $k = 0$, E is the trivial line bundle.

A section of a tensor bundle is called a **tensor field** of M. The tensor product of two tensor fields is again a tensor field.

5.6.1 Lie Derivatives of Tensor Fields

If X is a vector field on M and τ is a tensor field on M, then the flow generated by X defines the *Lie derivative* $\mathcal{L}_X \tau$. It is a tensor field of the `same type' as τ, uniquely characterized by the following property:

$$\begin{aligned}
\mathcal{L}_X(\tau_1 \otimes \tau_2) &= (\mathcal{L}_X \tau_1) \otimes \tau_2 + \tau_1 \otimes (\mathcal{L}_X \tau_2) \\
\mathcal{L}_X(\tau_1 + \tau_2) &= \mathcal{L}_X \tau_1 + \mathcal{L}_X \tau_2 \\
\mathcal{L}_X(C(\tau)) &= C(\mathcal{L}_X(\tau))
\end{aligned}$$

where C is any **contraction** operators. The above properties determine the Lie derivative uniquely once we know its action on functions and vector fields.

We also have

$$\mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}$$

for any vector fields X and Y on M [Gallot et al.].

We will discuss more on the Lie derivatives of differential forms in the next chapter.

Chapter 6

Differential Forms

6.1 Exterior Differential Algebra

Let M be a manifold of dimension n. A section of the k-th wedge power $\wedge^k TM^*$ of the cotangent bundle $TM^* \to M$ is called a **differential form of degree** k, or simply a k-**form** on M. We denote by $\Omega^k(M)$ the $\mathcal{C}^\infty(M)$-module of k-forms on M for $k = 0, 1, \ldots$. Then $\Omega^0(M) = \mathcal{C}^\infty(M)$ and $\Omega^{n+1}(M) = \Omega^{n+2}(M) = \cdots = 0$. If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \in \Omega^{k+l}(M).$$

The graded algebra

$$\Omega^\bullet(M) := \sum_{k \geq 0} \Omega^k(M)$$

is called the **exterior differential algebra** of M.[1]

If $x = (x^1, \ldots, x^n)$ is a local coordinate system on M, then

$$\begin{aligned} \omega &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

for some uniquely determined smooth functions $\omega_{i_1 \ldots i_k}$ which are skew-symmetric with respect to the indices.

6.1.1 Exercise

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that $S := f^{-1}(0)$ is a regular level set. Find a nowhere vanishing 2-form on S.

[1] We will soon introduce the exterior differential operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$. The notation d is due to E. Kähler [Cartan].

6.1.2 Remark

We may regard, as we have seen in the last chapter, a k-form ω on M as an alternating k-linear (over $\mathcal{C}^\infty(M)$) map of $\mathfrak{X}(M)$ into $\mathcal{C}^\infty(M)$;

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M).$$

For instance, if $\theta^1, \dots, \theta^k$ are 1-forms on M and $X_1, \dots, X_k$ are vector fields on M, then

$$(\theta^1 \wedge \cdots \wedge \theta^k)(X_1, \dots, X_k) = \det(\theta^i(X_j)).$$

Thus, if $\eta \in \Omega^l(M)$, then for vector fields $X_1, \dots, X_{k+l}$ on M,

$$\begin{aligned} & (\omega \wedge \eta)(X_1, \dots, X_{k+l}) \\ = & \sum_{\sigma \in S_{k,l}} \operatorname{sgn}(\sigma) \cdot \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}) \\ = & \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \cdot \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}), \end{aligned}$$

where $S_{k,l}$ denotes the set of all (k,l)-shuffles.[2]

For instance, if ω is a 1-form and η is an l-form, then

$$(\omega \wedge \eta)(X_0, \dots, X_l) = \sum_{i=0}^{l} (-1)^i \omega(X_i) \eta(X_0, \dots, \hat{X}_i, \dots, X_l).$$

Theorem 6.1.2.1 *There is a unique $\mathbb{R}$-linear map*

$$d : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

such that

(i) *it is an extension of $d : \Omega^0(M) \to \Omega^1(M)$*

(ii) *it is an anti-derivation, i.e., for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\bullet(M)$,*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(iii) *$d(df) = 0$ for any $f \in \Omega^0(M)$.*

Moreover, if $\omega \in \Omega^k(M)$ is considered as an alternating multilinear map on the space of vector fields, then $d\omega \in \Omega^{k+1}(M)$ and

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k) \end{aligned} \tag{6.1}$$

for vector fields $X_0, \dots, X_k \in \mathfrak{X}(M)$. We also have

$$d \circ d = 0.$$

[2] A permutation σ in S_{k+l} is a (k,l)-shuffle if $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. The above identity is the *Laplace's expansion* of determinant. cf. 5.3.6.1 or [Spivak, Vol. I].

Proof. Let $\omega \in \Omega^k(M)$.

(Uniqueness) Since the property (i) implies that d is a local operator, it suffices to check $d\omega$ locally. If, for some local chart $x = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ on M,

$$\omega|U = \frac{1}{k!}\sum \omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{k!}\sum_I \omega_I dx^I$$

where the functions ω_I are skew-symmetric with respect to indices $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ and $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, then by the axioms (i) and (ii), we have

$$(d\omega)|U = \frac{1}{k!}\sum_I d(\omega_I) \wedge dx^I.$$

(Existence) Now we define $d\omega$ as in the above formula. Then we have to show that this definition is independent of the choice of local chart. Thus suppose we are given a new chart $y = (y^1, \dots, y^n)$ on U and

$$\omega|U = \frac{1}{k!}\sum \tilde{\omega}_J dy^J.$$

Then

$$\begin{aligned}
\omega|U &= \frac{1}{k!}\sum_I \omega_I dx^I \\
&= \frac{1}{k!}\sum_{I,J} \omega_I \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} dy^{j_1} \wedge \cdots \wedge dy^{j_k} \\
&= \frac{1}{k!}\sum_J \left(\sum_I \omega_I \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} \right) dy^J.
\end{aligned}$$

Thus we have

$$\tilde{\omega}_J = \sum_I \omega_I \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}}.$$

Now

$$\begin{aligned}
\frac{1}{k!}\sum_J d(\tilde{\omega}_J) \wedge dy^J &= \frac{1}{k!}\sum_{I,J} d\left(\omega_I \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} \right) \wedge dy^J \\
= & \frac{1}{k!}\sum_{I,J} d(\omega_I) \wedge \left(\frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} dy^J \right) + \frac{1}{k!}\sum_{I,J} \omega_I d\left(\frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} \right) \wedge dy^J \\
= & \frac{1}{k!}\sum_I d(\omega_I) \wedge dx^I \\
& + \frac{1}{k!}\sum_{I,J,l} \omega_I \left(\frac{\partial^2 x^{i_1}}{\partial y^l \partial y^{j_1}} \frac{\partial x^{i_2}}{\partial y^{j_2}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} + \cdots + \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_{k-1}}}{\partial y^{j_{k-1}}} \frac{\partial^2 x^{i_k}}{\partial y^l \partial y^{j_k}} \right) dy^l \wedge dy^J \\
= & \frac{1}{k!}\sum_I d(\omega_I) \wedge dx^I.
\end{aligned}$$

The last identity comes from the fact that `a symmetric matrix and a skew-symmetric matrix are orthogonal to each other'. This shows the well-definedness.

Now the properties (i) and (ii) can be easily checked. Note that $d^2 = 0$ is equivalent to $\frac{\partial^2}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$.

Finally we will show the relation (6.1). Let $A(X_0, \ldots, X_k)$ be the right hand side of (6.1). Then A is alternating k-linear map over $\mathcal{C}^\infty(M)$. Thus it suffices to check the identity locally and for $X_i \in \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$, where $x^1, \ldots, x^n$ is a local chart. Now, for $\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, and $1 \le l_0 < \cdots < l_k \le n$,

$$A\left(\frac{\partial}{\partial x^{l_0}}, \ldots, \frac{\partial}{\partial x^{l_k}}\right) = \sum_{i=0}^{k} (-1)^i \frac{\partial}{\partial x^{l_i}} (\omega_{l_0 \ldots \hat{l}_i \ldots l_k}),$$

since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$. We also have

$$\begin{aligned} d\omega &= \sum_{i_1 < \cdots < i_k} d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_1 < \cdots < i_k} \sum_j \frac{\partial \omega_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \sum_{i_0 < \cdots < i_k} \left(\sum_{j=0}^{k} (-1)^k \frac{\partial \omega_{i_0 \ldots \widehat{i_j} \ldots i_k}}{\partial x^{i_j}} \right) dx^{i_0} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

and hence

$$d\omega\left(\frac{\partial}{\partial x^{l_0}}, \ldots, \frac{\partial}{\partial x^{l_k}}\right) = \sum_{i=0}^{k} (-1)^i \frac{\partial}{\partial x^{l_i}} (\omega_{l_0 \ldots \widehat{l_i} \ldots l_k}).$$

This completes the proof. □

The operator d is called the **exterior derivative.**

In $\mathbb{R}^3$, d appears as gradient, curl and divergence.

$$\begin{tikzcd} \Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, "="'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, "\simeq"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, "\simeq"'] & \Omega^3(\mathbb{R}^3) \arrow[d, "\simeq"'] \\ \Omega^0(\mathbb{R}^3) \arrow[r, "\text{grad}"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\text{curl}"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\text{div}"'] & \Omega^0(\mathbb{R}^3) \end{tikzcd}$$

6.1.3 Functoriality

If $F : M \to N$ is a smooth map, then we get a pull back

$$F^* : \Omega^\bullet(N) \to \Omega^\bullet(M).$$

If $\omega \in \Omega^k(N)$, for some $k > 0$, then $F^*\omega \in \Omega^k(M)$ and

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_{F(p)}(F_* v_1, \ldots, F_* v_k)$$

for any tangent vectors $v_1, \ldots, v_k$ in TM_p.

Then F^* is a ring homomorphism, commuting with the exterior derivative d.

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\ {\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ \Omega^k(N) & \xrightarrow[d]{} & \Omega^{k+1}(M) \end{array}$$

If $G : N \to L$ is a smooth map, then

$$(G \circ F)^* = F^* \circ G^*$$

and $(\mathrm{id}_M)^* = \mathrm{id}_{\Omega^\bullet(M)}$.

6.1.4 Volume Forms

A **volume form** on an n-manifold M is a non-vanishing n-form on M. A manifold is orientable if and only if there exists a volume form.

If μ is a volume form on M, then $f\mu$ is also a volume form for any positive function $f \in \mathcal{C}^\infty(M)$.

If M is oriented and has a Riemannian metric g, then there exists a canonical volume form μ (or denoted by dv_g) such that for each $p \in M$, $\mu_p \in \wedge^n TM_p^*$ is the positive unit vector. Note that if $v_1, \ldots, v_n \in TM_p$, then

$$\mu_p(v_1, \ldots, v_n)$$

is the **signed volume** of the parallelogram in TM_p spanned by $v_1, \ldots, v_n$. In particular,

$$\mu_p(e_1, \ldots, e_n) = 1$$

for any positively oriented orthonormal basis $e_1, \ldots, e_n$ in TM_p. In fact, if $e_1^*, \ldots, e_n^*$ is the dual basis of $e_1, \ldots, e_n$, then

$$\mu_p = e_1^* \wedge \cdots \wedge e_n^*.$$

In general, if $(X_1, \ldots, X_n)$ is a positively oriented local frame field for the tangent bundle TM and

$$g_{ij} = g(X_i, X_j) \qquad (1 \le i, j \le n)$$

then $|g| := \det(g_{ij})$ is the square of the volume of the parallelogram spanned by $X_1, \ldots, X_n$, at each point on M, and hence locally

$$\mu = \sqrt{|g|}\, \theta^1 \wedge \cdots \wedge \theta^n$$

where $\theta^1, \ldots, \theta^n$ is the dual frame of $X_1, \ldots, X_n$.

6.1.4.1 Polar Coordinates

Let μ_n be the volume form of $\mathbb{R}^n$, i.e.,

$$\mu_n = dx^1 \wedge \cdots \wedge dx^n$$

and let

$$\epsilon_n := x^1\, dx^1 + \cdots + x^n\, dx^n, \quad \omega_{n-1} := \sum_{i=1}^{n} (-1)^{i-1} x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Note that the forms $\mu_n, \epsilon_n, \omega_{n-1}$ are all invariant under the action of $\mathrm{SO}(n)$.

With $r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$,

$$\epsilon_n \wedge \omega_{n-1} = r^2\, \mu_n, \qquad \epsilon_n = \frac{1}{2} d(r^2), \qquad d\omega_{n-1} = n\, \mu_n.$$

We also have

$$dr = \frac{1}{r}\epsilon_n, \qquad dr \wedge (\frac{1}{r}\omega_{n-1}) = \mu_n.$$

Note that

$$\frac{1}{r}\omega_{n-1}$$

is, when restricted, the volume form on the sphere $\mathbf{S}^{n-1}(r)$ of radius $r > 0$ centered at the origin. The **solid-angle form** is defined by

$$\sigma_{n-1} := \frac{1}{r^n}\omega_{n-1} \in \Omega^{n-1}(\mathbb{R}^n_*),$$

where $\mathbb{R}^n_* := \mathbb{R}^n - \{0\}$. Note that if $\mathrm{ret} : \mathbb{R}^n_* \to \mathbf{S}^{n-1}$ is the obvious retraction map to the unit sphere and i is the inclusion map $\mathbf{S}^{n-1} \hookrightarrow \mathbb{R}^n$, then

$$\mathrm{ret}^*\, i^*(\omega_{n-1}) = \sigma_{n-1}.$$

We have

$$d\sigma_{n-1} = 0, \qquad \mu_n = r^{n-1}\, dr \wedge \sigma_{n-1}.$$

6.1.4.2 Volume forms of Hypersurfaces

Let f be a smooth function defined on an open subset U of $\mathbb{R}^n$. Suppose Z is the (non-empty) regular zero-level set of f. Then the restriction of

$$\frac{1}{|\nabla f|} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x^i}\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

to Z is the volume form of Z. To check this, let $v_2, \ldots, v_n$ be an orthonormal basis of

$$TZ_p \simeq \{v \in \mathbb{R}^n \mid \nabla f(p) \cdot v = 0\}$$

for $p \in Z$ such that $\nabla f(p) \wedge v_2 \wedge \cdots \wedge v_n > 0$. Then

$$\begin{aligned} & \left(\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x^i} \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \right)_p (v_2, \ldots, v_n) \\ = \quad & \left(dx^1 \wedge \cdots \wedge dx^n \right)_p (\nabla f(p), v_2, \ldots, v_n) = |\nabla f(p)|. \end{aligned}$$

This proves the claim.

In particular, Z is orientable.

The above consideration is true for general regular hypersurface $Z = f^{-1}(0)$ for a function f on an oriented Riemannian n-manifold M. If ω is the volume form on M, then

$$\frac{1}{|\nabla f|} \nabla f \rfloor \omega = \frac{1}{|df|} \star (df) \in \Omega^{n-1}(M)$$

restricts to the volume form on Z, where $\rfloor$ is the interior product and $\star$ is the Hodge star, both will be explained soon.

6.1.5 Lie Derivative of forms

6.1.5.1 Interior Product

Let X be a vector field on M. Then the interior product by X is a unique $\mathcal{C}^\infty(M)$-linear map

$$i_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

such that

(1) for $\omega \in \Omega^1(M)$, $i_X(\omega) = \omega(X) \in \Omega^0(M)$.

(2) for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\bullet(M)$,

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta).$$

If we regard $\omega \in \Omega^k(M)$ as an alternating k-linear map on the space of vector fields, then $i_X\omega \in \Omega^{k-1}(M)$ is given by

$$(i_X\omega)(Y_2, \dots, Y_k) = \omega(X, Y_2, \dots, Y_k)$$

for vector fields $Y_2, \dots, Y_k$.

6.1.5.2 Exercise

Show that for vector fields X and Y on M,

$$i_X i_Y = -i_Y i_X.$$

6.1.5.3 É. Cartan's Formula

Theorem 6.1.5.4 (Cartan's Formula) *Let $\mathcal{L}_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be the Lie derivative associated to a vector field X on M. Then*

$$\mathcal{L}_X = i_X \circ d + d \circ i_X$$

Theorem 6.1.5.5 *For vector fields $X, X_1, \dots X_k$ on M and a differential k-form ω on M, we have*

$$(\mathcal{L}_X\omega)(X_1, \dots, X_k) = X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \dots, [X, X_i], \dots, X_k).$$

6.1.5.6 Exercise

Show that for vector fields X and Y on M,

$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}.$$

6.1.5.7 Exercise

Note that if X is a vector field along a map $\phi : M \to N$, then the **interior product**

$$i_X : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

is a unique $\mathbb{R}$-linear map such that

(1) for $\omega \in \Omega^1(N)$, $i_X(\omega) = (\phi^*\omega)(X) \in \Omega^0(M)$

(2) for $\omega \in \Omega^k(N)$ and $\eta \in \Omega^\bullet(N)$,

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \phi^*\eta + (-1)^k(\phi^*\omega) \wedge (i_X\eta).$$

Show that for $\omega \in \Omega^k(N)$, we have $i_X\omega \in \Omega^{k-1}(M)$ and

$$(i_X\omega)_p(v_2, \ldots, v_k) = \omega_{\phi(p)}(X(p), T\phi_p v_2, \ldots, T\phi_p v_k)$$

for $v_j \in TM_p$, i.e.,

$$(i_X\omega)_p = \phi^*(i_{X(p)}\omega_{\phi(p)}).$$

6.2 Hodge Duality

Now the Hodge duality

$$\star : \Omega^k(M) \to \Omega^{n-k}(M)$$

is characterized by the property

$$\langle \omega, \eta \rangle \,\mathrm{vol} = \omega \wedge \star\eta$$

for $\omega, \eta \in \Omega^\bullet(M)$, where $\langle\ ,\ \rangle$ denotes the Riemannian structure on $\wedge^\bullet TM^*$ induced by g (see appendix for the linear algebra). One can see easily that

$$\star^2 | \Omega^k = (-1)^{k(n-k)}$$

and hence $\star$ is an isomorphism.

6.3 De Rham Cohomology

A differential form ω on an n-dimensional M is called **closed** if $d\omega = 0$, and called **exact** if $\omega = d\eta$ for some differential form η. Then exact forms are clearly closed. The k-**th de Rham cohomology space**[3] of M is

$$H^k(M) := \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

for integers k. If $k < 0$ or $k > n$, then $H^k(M) = \{0\}$. These cohomology spaces are regarded as the space of obstructions to solve a differential equation

$$\omega = d\xi$$

for a given differential form ω. It is easy to see that

$$H^\bullet(M) := \sum_{k \geq 0} H^k(M)$$

is a graded anti commutative algebra, called the **de Rham cohomology algebra of** M.

For a connected manifold M,

$$H^0(M) \simeq \mathbb{R}.$$

Theorem 6.3.0.1 *If $F : M \to N$ is a smooth map, then it induces an algebra homomorphism*

$$[F^*] : H^\bullet(N) \to H^\bullet(M).$$

If $G : N \to L$ is smooth, then

$$[(G \circ F)^*] = [F^*] \circ [G^*]$$

and $[\mathrm{id}_M^*] = \mathrm{id}_{H^\bullet(M)}$.

Lemma 6.3.0.2 *For any $t \in \mathbb{R}$, let $J_t : M \to M \times \mathbb{R}$ be the t-level map: $J_t(p) := (p, t)$. Then there exists a linear map*

$$K : \Omega^\bullet(M \times \mathbb{R}) \to \Omega^\bullet(M)$$

of degree -1 such that

$$J_1^* - J_0^* = K \circ d + d \circ K$$

as maps $\Omega^\bullet(M \times \mathbb{R}) \to \Omega^\bullet(M)$.

[3]Georges de Rham, 1903–1990.

Proof. We follow [Karoubi et al.] or [Spivak]. Let X be the vector field on $M \times \mathbb{R}$ such that $X_{p,t} := \frac{d}{dt}(p,t)$. Let $\omega \in \Omega^k(M \times \mathbb{R})$. Then $i_X\omega$ is a time-dependent $(k-1)$-form on M. We define

$$K\omega := \int_0^1 i_X\omega\, dt$$

which is an element of $\Omega^{k-1}(M)$. Note that, if $\omega = \omega' + dt \wedge \omega''$ where ω' is a time-dependent[4] k-form on M and ω'' is a time-dependent $(k-1)$-form on M, then $i_X\omega = \omega''$. Thus

$$d\omega = d\omega' - dt \wedge d\omega'', \qquad i_X(d\omega) = \frac{\partial \omega'}{\partial t} - d\omega''.$$

Now

$$dK\omega + Kd\omega = \int_0^1 (d\omega'' + \frac{\partial \omega'}{\partial t} - d\omega'')dt = \omega_1' - \omega_0'$$

which is equal to $J_1^*(\omega) - J_0^*(\omega)$. □

Theorem 6.3.0.3 *Suppose $F_0, F_1 : M \to N$ are homotopic smooth maps. Then the induced homomorphisms*

$$[F_0^*], [F_1^*] : H^\bullet(N) \to H^\bullet(M)$$

are the same.

Proof. We have a map

$$F : M \times \mathbb{R} \to N$$

such that the composition

$$M \xrightarrow{J_t} M \times \mathbb{R} \xrightarrow{F} N$$

is equal to F_t for $t = 0, 1$. Now

$$F_1^* - F_0^* = (J_1^* - J_0^*) \circ F^* = (K \circ d + d \circ K) \circ F^*$$

as maps $\Omega^\bullet(N) \to \Omega^\bullet(M \times \mathbb{R}) \to \Omega^\bullet(M)$. Thus on the cohomology level, we have $[F_1^*] = [F_0^*]$. This completes the proof. □

Corollary 6.3.0.4 (i) *If M is contractible, then $H^k(M) = 0$ for $k > 0$.*

(ii) *(Poincaré Lemma)*[5] *$H^k(\mathbb{R}^n) = 0$ for $k > 0$.*

(iii) *On any manifold, every closed form of degree > 0 is locally exact.*

We will see in the next chapter that if M is a connected non-compact n-manifold, then $H^n(M) = \{0\}$.

[4] We say that a differential form η on $M \times \mathbb{R}$ is *time-dependent differential form on M* if $i_X\eta = 0$, where X is the vertical vector field on $M \times \mathbb{R}$.

[5] The Poincaré Lemma was first stated and proved by V. Volterra in three notes in the *Academia dei Lincei* (4) 5 (1889), pp. 158–165, 291–299, 599-611.

6.3.0.5 Exercises

(i) Let M be an open subset of $\mathbb{R}^n$, which is star-like with respect to the origin. Given a positive integer k and a closed k-form ω on M, find a $(k-1)$-form η such that $d\eta = \omega$.

(ii) Show that $H^1(\mathbf{S}^1) \simeq \mathbb{R}$. The ring $H^\bullet(\mathbf{S}^1)$ is isomorphic to $\mathbb{R}[\epsilon]/(\epsilon^2)$, the quotient of the polynomial ring $\mathbb{R}[\epsilon]$ by the ideal (ϵ^2).

(iii) Compute the de Rham cohomology algebra $H^\bullet(\mathbb{R}^2 - \{0\})$.

(ivi) Show that (cf. ch.7)

$$H^k(\mathbf{S}^n) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ \{0\} & \text{otherwise.} \end{cases}$$

6.3.1 Mayer-Vietoris Sequence

Let U and V be two open subsets of M with $M = U \cup V$. Then we have an exact sequence of chain complexes:[6]

$$\{0\} \to \Omega^\bullet(U \cup V) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to \{0\}$$
$$\xi \mapsto (\xi|_U, \xi|_V), \quad (\omega, \eta) \mapsto \omega|_{U\cap V} - \eta|_{U\cap V}$$

Thus we have a long exact sequence

$$\begin{array}{ccccccc} \{0\} & \to & H^0(U\cup V) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U\cap V) \\ & \to & H^1(U\cup V) & \to & H^1(U) \oplus H^1(V) & \to & H^1(U\cap V) \\ & \to & H^2(U\cup V) & \to & H^2(U) \oplus H^2(V) & \to & H^2(U\cap V) \\ & \to & \dots . & & & & \end{array}$$

[6] A (co)*chain complex* is a sequence of vector spaces $C^0, C^1, C^2, \dots$ together with maps $d : C^k \to C^{k+1}$ such that $d \circ d = 0$. A map $f : (C^\bullet, d) \to (D^\bullet, d)$ between chain complexes is a collection of maps $f : C^k \to D^k$, $k = 0, 1, 2, \dots$, such that $d \circ f = f \circ d$.

6.4 Vector valued differential forms

Let V be a finite dimensional real vector space. Then for any differentiable map

$$f : M \to V$$

we have

$$df_p : TM_p \to V \qquad (p \in M)$$

such that

$$df_p(v) = \lim_{t\to 0} \frac{f(X(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_0 f(X(t))$$

for any smooth curve $X(t)$ in M with $X(0) = p$ and $X'(0) = v$.

Now a V-**valued differential** k-**form** on a manifold M is an element of $\Omega^k(M, V) := \Omega^k(M) \otimes V$. Then for $\xi \in \Omega^k(M, V)$, the **exterior derivative**

$$d\xi \in \Omega^{k+1}(M, V)$$

is canonically obtained and we get a cochain complex $(\Omega^\bullet(M, V), d)$, where the k-th cohomology space $H^k(M, V)$ is isomorphic to $H^k(M) \otimes V$.

We have

$$\mathbb{R} \subset \mathcal{C}^\infty(M) \subset \Omega^\bullet(M)$$
$$V \subset \mathcal{C}^\infty(M, V) \subset \Omega^\bullet(M, V).$$

Note that

$$\Omega^\bullet(M, V) := \sum_{k\geq 0} \Omega^k(M, V)$$

has no ring structure in general, but is a module over the ring $\Omega^\bullet(M)$. Similarly

$$H^\bullet(M, V) := \sum_{k\geq 0} H^k(M, V)$$

is a graded module over the ring $H^\bullet(M)$.

6.4.0.1 Examples

(i) Let M be a surface in $\mathbb{R}^3$. Let e_1, e_2 be orthonormal vector fields on M and let $e_3 := e_1 \times e_2$. Then

$$de_j = \sum_{i=1}^{3} \omega_j^i \, e_i \qquad (j = 1, 2, 3)$$

for some 1-forms ω_j^i. These 1-forms satisfy

$$\omega_j^i = -\omega_i^j.$$

(ii) Let $\gamma(s)$ be a bi-regular[7] curve in $\mathbb{R}^3$, parametrized by the arclength. Then

$$\mathbf{t} := \dot\gamma, \qquad \mathbf{p} := \ddot\gamma / |\ddot\gamma|, \qquad \mathbf{b} := \mathbf{t} \times \mathbf{p}$$

and we have the **Frenet equation**

$$\begin{array}{rclcl} \dot{\mathbf{t}} & = & & \kappa\mathbf{p} & \\ \dot{\mathbf{p}} & = & -\kappa\mathbf{t} & + & \tau\mathbf{b} \\ \dot{\mathbf{b}} & = & & -\tau\mathbf{p} & \end{array}$$

(iii) Let $p : M \hookrightarrow \mathbb{R}^N$ be an embedding. Let $e_1, \ldots, e_n$ be a frame of vector fields on M. Then

$$dp = \sum_{i=1}^{n} \theta^i\, e_i$$

for some 1-forms θ^i. These forms are dual to the frame $(e_1, \ldots, e_n)$.

6.4.1 Lie algebra valued differential forms

6.4.1.1 Maurer-Cartan Form

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Then the left multiplication gives the isomorphism

$$\Theta : TG \simeq G \times \mathfrak{g}.$$

This isomorphism is often denoted by

$$g^{-1}dg$$

and called the (left invariant) **Maurer-Cartan form.**[8] It is a $\mathfrak{g}$-valued 1-form on G, which is left invariant.

If $(e_1, \ldots, e_n)$ is a basis for $\mathfrak{g}$, and $\epsilon^1, \ldots, \epsilon^n$ are left invariant 1-forms on G which are dual to $(e_1, \ldots, e_n)$, then

$$\Theta = \sum_{i=1}^{n} \epsilon^i \otimes e_i.$$

É. Cartan has shown the “Maurer-Cartan structure equation”:

$$d\Theta + \frac{1}{2}[\Theta, \Theta] = 0.$$

For instance, let

$$G = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \,\middle|\, xz \neq 0 \right\} \subset \mathrm{GL}(2, \mathbb{R}).$$

[7] Thus $\dot\gamma$ and $\ddot\gamma$ are linearly independent everywhere.

[8] Ludwig Maurer (1859–1927) was a German mathematician at Tübingen University. Élie Cartan introduced the form in 1904.

This group of 2×2 nonsingular lower triangular real matrices has 4 components. Its Lie algebra is generated by

$$E_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with

$$[E_{11}, E_{21}] = -E_{21}, \quad [E_{11}, E_{22}] = 0, \quad [E_{21}, E_{22}] = -E_{21}.$$

Now

$$\begin{aligned} \Theta &= \frac{1}{xz}\begin{pmatrix} z & 0 \\ -y & x \end{pmatrix}\begin{pmatrix} dx & 0 \\ dy & dz \end{pmatrix} = \begin{pmatrix} \frac{dx}{x} & 0 \\ -\frac{y\,dx}{xz} + \frac{dy}{z} & \frac{dz}{z} \end{pmatrix} \\ &= \frac{dx}{x}E_{11} + \left(-\frac{y\,dx}{xz} + \frac{dy}{z}\right)E_{21} + \frac{dz}{z}E_{22}. \end{aligned}$$

For any 2×2 matrix $\mathfrak{m}$, the 1-form

$$\operatorname{tr}(\mathfrak{m}\Theta)$$

is left-invariant. Thus the forms

$$\frac{dx}{x}, \qquad -\frac{y\,dx}{xz} + \frac{dy}{z}, \qquad \frac{dz}{z}$$

are left invariant. Note that these forms are dual to the left invariant vector fields

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad z\frac{\partial}{\partial y}, \qquad z\frac{\partial}{\partial z}.$$

Now

$$d\Theta = \begin{pmatrix} 0 & 0 \\ \frac{dx\wedge dy}{xz} + \frac{dy\wedge dz}{z^2} + \frac{y\,dz\wedge dx}{xz^2} & 0 \end{pmatrix} = -\frac{1}{2}[\Theta, \Theta].$$

6.4.1.2 Extension of brackets

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then the space

$$\Omega^\bullet(M, \mathfrak{g}) := \sum_{k\geq 0} \Omega^k(M, \mathfrak{g})$$

of $\mathfrak{g}$-valued differential forms on M is a **graded Lie algebra**, i.e., the canonical bracket

$$[\ ,\ ] : \Omega^{k_1}(M, \mathfrak{g}) \times \Omega^{k_2}(M, \mathfrak{g}) \to \Omega^{k_1+k_2}(M, \mathfrak{g})$$

is bilinear (over $\mathcal{C}^\infty(M)$) and

$$[\xi_1, \xi_2] = -(-1)^{k_1 k_2}[\xi_2, \xi_1]$$

$$[\xi_1, [\xi_2, \xi_3]] = [[\xi_1, \xi_2], \xi_3] + (-1)^{k_1 k_2}[\xi_2, [\xi_1, \xi_3]]$$

for $\xi_i \in \Omega^{k_i}(M, \mathfrak{g})$.

We have

$$\begin{aligned} & [\xi_1, \xi_2](X_1, \dots, X_{k_1}, \dots, X_{k_1+k_2}) \\ = & \sum_{\sigma:(k_1,k_2)-\text{shuffle}} [\xi_1(X_{\sigma(1)}, \dots, X_{\sigma(k_1)}), \xi_2(X_{\sigma(k_1+1)}, \dots, X_{\sigma(k_1+k_2)})] \end{aligned}$$

for vector fields $X_1, \dots, X_{k_1+k_2}$ on M. In particular, if $k_1 = k_2 = 1$, then

$$[\xi_1, \xi_2](X, Y) = [\xi_1(X), \xi_2(Y)] - [\xi_1(Y), \xi_2(X)].$$

If $f : N \to M$ is a smooth map, then

$$f^*[\xi_1, \xi_2] = [f^8\xi_1, f^*\xi_2]$$

for any ξ^1 adn ξ^2.

6.4.1.3 Exercises

(i) Let $e_1, \dots, e_n$ be a basis for the Lie algebra of a Lie group G. Then the structure constants c_{ij}^k are defined by the relation

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

We have

$$c_{ij}^k = -c_{ji}^k$$

and the Jacobi identity implies

$$\sum_{m=1}^n \left(c_{ij}^m c_{km}^l + c_{ki}^m c_{jm}^l + c_{jk}^m c_{im}^l \right) = 0$$

for any i, j, k, l. Let $\epsilon^1, \dots, \epsilon^n$ be the left invariant 1-forms on G which is dual to $e_1, \dots, e_n$. Show that

$$d\epsilon^k = -\frac{1}{2} \sum_{i,j} c_{ij}^k \epsilon^i \wedge \epsilon^j.$$

Let Θ be the (left invariant) Maurer-Cartan form on G. Show that

$$d\Theta + \frac{1}{2}[\Theta, \Theta] = 0.$$

If $\tilde{\Theta}$ is the right-invariant Maurer-Cartan form, show that

$$d\tilde{\Theta} - \frac{1}{2}[\tilde{\Theta}, \tilde{\Theta}] = 0.$$

(ii) Let Θ be the Maurer-Cartan form on $\mathbf{S}^3 \subset \mathbb{H}$. Show that $\frac{1}{6}\Theta^3$ is the volume form on $\mathbf{S}^3$.

(iv) Find all left invariant 1-forms on $\mathrm{GL}(2, \mathbb{R})$.

6.4.2 Cohomology of Lie groups

cf. [DFN, Vol. III, p.90], [Spivak, V]

6.4.3 De Rham Theorem

6.4.3.1 Singular Cohomology Groups

For a nonnegative integer k, the standard k-simplex $\triangle_k$ is the convex hull of $k+1$ points $e_1 := (1, 0, \ldots, 0)$, ..., $e_{k+1} = (0, \ldots, 0, 1)$ in $\mathbb{R}^{k+1}$:

$$\triangle_k := \{(x_1, \ldots, x_{k+1}) \mid x_1 + \cdots + x_{k+1} = 1,\ x_1 \geq 0, \ldots, x_{k+1} \geq 0\}.$$

We have inclusion maps

$$\mathrm{inc}_i : \triangle_{k-1} \to \triangle_k, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_k).$$

for $i = 1, \ldots, k+1$.

Let $C_k(M)$ be the free vector space generated by smooth maps form $\triangle_k$ into M. There is a **boundary map**

$$\partial : C_k(M) \to C_{k-1}(M)$$

which is characterized by the property

$$\partial \sigma_k := \sum_{i=1}^{k+1} (-1)^{i-1} \sigma_k \circ \mathrm{inc}_i$$

for a **singular k-simplex** (or a smooth map) $\sigma_k : \triangle_k \to M$. Then we have

$$\partial \circ \partial = 0.$$

Let

$$C^k_\triangle(M) := C_k(M)^* =: \mathrm{Hom}(C_k(M), \mathbb{R}).$$

Then we have a co-chaim complex $(C^\bullet_\triangle(M), \partial^*)$. The associated (singular) cohomology group is denoted by

$$H^k_\triangle(M).$$

Theorem 6.4.3.2 (De Rham)

$$H^k(M) \simeq H^k_\triangle(M).$$

Chapter 7

Integration

7.1 Manifolds with Boundary

7.1.1 Topological Manifolds with boundary

In the definition of a locally Euclidean space, one may replace the Cartesian space $\mathbb{R}^n$ with the "half space"

$$\mathbb{R}^n_- := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 \leq 0\}$$

to get a notion of manifold with boundary. Thus a metrizable[1] topological space M is called an n-**dimensional manifold with boundary** if each point $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n_-$.

Let M be an n-dimensional manifold with boundary and let p be a point on M such that $x(p) \in \partial\mathbb{R}^n_- = \{0\} \times \mathbb{R}^{n-1}$ for some chart x at p. Then, by the Invariance of Domain, $y(p) \in \partial\mathbb{R}^n_-$ for any chart y at p. Such a point p is called a **boundary point** of M and the set ∂M of all boundary points of M is called the **boundary** of M. Points in $M - \partial M$ are the **interior points**. Manifolds with empty boundary are just the (ordinary) manifolds (without boundary).

7.1.1.1 Exercise

1. Show that the boundary ∂M of a manifold M with boundary is a manifold (without boundary), i.e., $\partial(\partial M) = \emptyset$.

2. For a manifold M with boundary ∂M, let $\mathrm{inc} : \partial M \to M$ be the inclusion map. Then the **double of** M is defined by

$$\mathrm{double}(M) := M \cup_{\mathrm{inc}} M.$$

Show that $\mathrm{double}(M)$ is a manifold without boundary.

[1] This condition may be replaced by equivalent ones.

7.1.2 Smooth manifolds with boundary

A map $f : A \to \mathbb{R}^m$ defined on an arbitrary subset A of $\mathbb{R}^n$ is said to be $\mathcal{C}^k$ at $p \in A$ if there exists an open neighborhood U of p in $\mathbb{R}^n$ and a $\mathcal{C}^k$ map $F : U \to \mathbb{R}^m$ such that

$$f \restriction A \cap U = F \restriction A \cap U.$$

If f is $\mathcal{C}^k$ at every $p \in A$, then f is said to be $\mathcal{C}^k$.

7.1.3 Derivatives

The derivative

$$Df_p := DF_p : \mathbb{R}^n \to \mathbb{R}^m$$

of $f : A \to \mathbb{R}^m$ at $p \in A$ is, in general, not well defined. For instance, consider the case when A is a singleton in $\mathbb{R}$. But, if $p \in A \cap \overline{\operatorname{Int} A}$, then Df_p is well defined.[2]

7.1.3.1 Exercise

Show that if $f : A \to \mathbb{R}^m$ is $\mathcal{C}^k$ for some subset $A \subset \mathbb{R}^n$, then there exists an open neighborhood U of A in $\mathbb{R}^n$ and a $\mathcal{C}^k$ map $F : U \to \mathbb{R}^m$ such that $f = F \restriction A$.

Solution. For each $p \in A$, there exist an open neighborhood U_p of $p \in \mathbb{R}^n$ and a $\mathcal{C}^k$ map $F_p : U_p \to \mathbb{R}^m$ such that $F \restriction A \cap U_p = F_p \restriction A \cap U_p$. Now let $U := \cup_{p \in A} U_p$, an open submanifold of $\mathbb{R}^n$ containing A. Take a $\mathcal{C}^\infty$ partition $\{\rho_p \in \mathcal{C}^\infty(U)\}$ of unity subordinate to $\{U_p\}$ and let

$$F := \sum_{p \in A} \rho_p F_p \in \mathcal{C}^k(U).$$

Thus if $q \in A$, then

$$F(q) = \sum_{p \in A} \rho_p(q) F_p(q) = \sum_p \rho_p(q) F(q) = F(q).$$

This solves the problem. □

[2] For a subset A of a toplogical space X, the following conditions are equivalent:

(i) $\operatorname{bd}(\operatorname{int} A) = \operatorname{bd} A$

(ii) $\overline{\operatorname{int} A} = \overline{A}$

(iii) there exists an open subset U of X such that $U \subset A \subset \overline{U}$.

7.1.4 Definition

Let H be a closed half space[3] in $\mathbb{R}^n$. An **atlas** on an n-manifold M with boundary is a map

$$x : U \to H,$$

defined on an open subset U of M, which is a homeomorphism onto its image.

A $\mathcal{C}^\infty$ **structure** on a topological n-manifold M with boundary is an atlas $\mathcal{A}$ of M such that for each $x, y \in \mathcal{A}$, $x \circ y^{-1}$ is $\mathcal{C}^\infty$ and $\mathcal{A}$ is maximal with respect to this condition. A $\mathcal{C}^\infty$ **manifold with boundary** is a topological manifold together with a $\mathcal{C}^\infty$ structure.

When $n \geq 1$, a smooth n-manifold M with boundary is called **orientable** if there exists an atlas whose transition maps are all orientation preserving. A choice of a maximal orientable atlas is called an **orientation** of M.

An orientation of a 0-manifold M is an assignment of $+1$ or -1 to each point of M. Thus an orientation on a 0-manifold M is a map from M into $\{+1, -1\}$.

7.1.5 Remarks

1. The boundary ∂M of a (smooth) manifold M with boundary is a (smooth) manifold (without boundary) canonically.

2. There exist **differentiable maps** between smooth manifolds with boundary. **Identity maps** are differentiable and the **composition** of differentiable maps are differentiable. **Diffeomorphisms** are defined in a standard way.

3. The tangent space of M at a boundary point p is an equivalence class of two types of curves

$$c_+ : [0, \infty) \to M, \quad c_- : (-\infty, 0] \to M$$

passing through p at time $t = 0$. These curves are **equivalent** if they have the same tangent vector at $t = 0$ in $\mathbb{R}^n$ with respect to some (and hence any) coordinate system. The tangent space $T(\partial M)_p$ is a hyperplane of TM_p, whose complement consists of **inward tangent vectors** and *outward tangent vectors*.

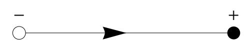
4. When M is oriented, the boundary ∂M inherits an orientation.

   If n, the dimension of M, is greater than 1, then an ordered basis $(v_2, \ldots, v_n)$ of $T(\partial M)_p$ is **positively oriented** if for any outward (non zero) tangent vector ν of M at p, $(\nu, v_2, \ldots, v_n)$ is a positively oriented basis for TM_p.

   If $n = 1$, then the orientation at a boundary point p of M is $+1$ if there exists a coordinate neighborhood U of p and an orientation preserving chart $x : U \to \mathbb{R}_{\leq 0}$ centered at p. The orientation at p is -1 if there

[3] Given a hyperplane L in $\mathbb{R}^n$, $\mathbb{R}^n - L$ has two components, say H_1 and H_2. Then $H_1 \cup L$ and $H_2 \cup L$ are both closed half spaces.

exists a coordinate neighborhood U of p and an orientation preserving chart $x : U \to \mathbb{R}_{\geq 0}$ centered at p.



5. For manifolds with corners see [John Lee].

7.2 Integration of Differential Forms

7.2.1 Compactly Supported Forms

Let M be an n-dimensional manifold. Let $\Omega_c^k(M)$ be the space of all differential k-forms on M with compact support. Then

$$\Omega_c^\bullet(M) = \sum_{k \geq 0} \Omega_c^k(M)$$

is a graded module over the graded algebra $\Omega^\bullet(M)$.

Moreover, if ω is a differential k-form on M with compact support, then $d\omega$ is a differential $(k+1)$-form on M with compact support. Hence we get **compactly supported de Rham cohomology algebra**

$$H_c^\bullet(M) := \sum_{k \geq 0} H_c^k(M).$$

Note that $H_c^\bullet(M)$ is a graded module over the graded algebra $H^\bullet(M)$.

If $F : M \to N$ is a proper map,[4] then we have a chain map

$$F_c^* : \Omega_c^\bullet(N) \to \Omega_c^\bullet(M)$$

which induces a map

$$[F_c^*] : H_c^\bullet(N) \to H_c^\bullet(M).$$

If $G : N \to L$ is proper, then

$$(G \circ F)_c^* = F_c^* \circ G_c^*, \qquad [(G \circ F)_c^*] = [F_c^*] \circ [G_c^*].$$

When M is compact, $\Omega_c^\bullet(M) = \Omega^\bullet(M)$, and $H_c^\bullet(M) = H^\bullet(M)$.

7.2.1.1 Exercise

Let M be a connected manifold. Then

$$H_c^0(M) = \begin{cases} \{0\} & \text{if } M \text{ is non-compact} \\ \mathbb{R} & \text{if } M \text{ is compact.} \end{cases}$$

Lemma 7.2.1.2

$$H_c^1(\mathbb{R}^1) \simeq \mathbb{R}.$$

[4] A continuous map $F : M \to N$ is *proper* if $\lim_{p \to \infty} F(p) = \infty$.

7.2.1.3

We say that two maps $F_0, F_1 : M \to N$ are **properly homotopic** if there exists a map
$$F : M \times \mathbb{R} \to N$$
such that $F(p,0) = F_0(p)$, $F(p,1) = F_1(p)$, and for each $t \in \mathbb{R}$, $F_t(p) := F(p,t)$ is a proper map.

Corollary 7.2.1.4 *Let $F_0, F_1 : M \to N$ be proper maps. If F_0 and F_1 are "properly" homotopic, then*

$$[F_0^*]_c = [F_1^*]_c : H_c^k(N) \to H_c^k(M).$$

Proof. Note that from (6.3.0.2) and (6.3.0.3) there exists a map

$$h : \Omega^n(N) \to \Omega^n(M \times \mathbb{R}) \to \Omega^{n-1}(M)$$

such that
$$F_1^* - F_0^* = d \circ h + h \circ d : \Omega^k(N) \to \Omega^k(M).$$
Thus for $\omega \in \Omega_c^n(N)$ with $\int_N \omega = 1$,

$$\deg F_1 = \int_M F_1^* \omega = \int_M F_0^* \omega = \deg F_0.$$

This completes the proof. □

7.2.2 Integration

Now we assume that M is oriented. The integration

$$\int_M : \Omega_c^n(M) \to \mathbb{R} \tag{7.1}$$

is defined as follows:

Let $\omega \in \Omega_c^n(M)$. First, assume that $\operatorname{supp} \omega$ is contained in a coordinate neighborhood $U \subset M$. Then for a positively oriented chart $x : U \to \mathbb{R}^n$,

$$\omega \restriction U = f\, dx^1 \wedge \cdots \wedge dx^n$$

for some $f \in \mathcal{C}_c^\infty(U) \subset \mathcal{C}_c^\infty(M)$. Then we define

$$\int_M \omega := \int_{\mathbb{R}^n} (x^{-1})^* \omega =: \int_{\mathbb{R}^n} f \circ x^{-1}\, d\mu_n,$$

where $d\mu_n$ denotes the Lebesgue measure on $\mathbb{R}^n$. One can easily see that $\int_M \omega$ is well defined.[5]

In general, take a finite number of coordinate neighborhoods $\{U_\alpha\}$ such that $U := \cup_\alpha U_\alpha$ covers $\operatorname{supp}\omega$ and take a partition $\{\rho_\alpha : U \to \mathbb{R}\}$ of unity subordinate to $\{U_\alpha\}$. Then $\rho_\alpha\omega \in \Omega_c^n(U_\alpha) \hookrightarrow \Omega_c^n(M)$ and we define

$$\int_M \omega := \sum_\alpha \int_M \rho_\alpha\omega.$$

Note that this new definition agrees with the old definition, namely, when $\operatorname{supp}\omega$ is contained in a single coordinate neighborhood. Now we claim that the new definition is independent of the choice of coordinate neighborhoods and the partition of unity. Suppose we take another finite collection of coordinate neighborhoods $\{V_\mu\}$ which covers the support of ω and the partition $\{\tau_\mu\}$ of unity subordinate to $\{V_\mu\}$. Then

$$\sum_\mu \int_M \tau_\mu\omega = \sum_\mu \left(\sum_\alpha \int_M \rho_\alpha\tau_\mu\omega\right) = \sum_\alpha \left(\sum_\mu \int_M \tau_\mu\rho_\alpha\omega\right) = \sum_\alpha \int_M \rho_\alpha\omega.$$

Thus we have a well-defined map (7.1).

Lemma 7.2.2.1 *The integration (7.1) is a linear map.*

Proposition 7.2.2.2 *Let $F : N \to M$ be a diffeomorphism between oriented manifolds. Then for any $\omega \in \Omega_c^n(M)$,*

$$\int_N F^*\omega = \begin{cases} \int_M \omega & \text{if } F \text{ preserves orientation} \\ -\int_M \omega & \text{if } F \text{ reverses orientation} \end{cases}$$

Lemma 7.2.2.3 *Let M and N be oriented manifolds of dimensions n and m, respectively. For $\omega \in \Omega_c^n(M)$ and $\eta \in \Omega_c^m(N)$, $(\pi_M^*\omega) \wedge (\pi_N^*\eta) \in \Omega_c^{n+m}(M \times N)$ and*

$$\int_{M\times N} (\pi_M^*\omega) \wedge (\pi_N^*\eta) = \int_M \omega \cdot \int_N \eta \tag{7.2}$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are projections.[6]

Proof. If M and N have global charts $x : M \to \mathbb{R}^n$ and $y : N \to \mathbb{R}^m$, both positively oriented, then $\omega = f\,dx^1 \wedge \cdots \wedge dx^n$ and $\eta = g\,dy^1 \wedge \cdots \wedge dy^m$ for some functions $f \in \mathcal{C}_c^\infty(M)$ and $g \in \mathcal{C}_c^\infty(N)$. Then

$$(\pi_M^*\omega) \wedge (\pi_N^*\eta) = (f \circ \pi_M)(g \circ \pi_N)\,dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^m$$

---

[5] Recall the *change of variable formula* for the integration: Let $F : U \to V$ be a diffeomorphism between open subsets U and V of $\mathbb{R}^n$, and let $g : V \to \mathbb{R}$ be an integrable function. Then $\int_V g\,d\mu_n = \int_U (g \circ F) \cdot |\det DF|\,d\mu_n$.

[6] The product of two oriented manifolds has a canonical orientation.

and hence the left hand side of (7.2) is equal to

$$\int_{\mathbb{R}^n \times \mathbb{R}^M} (f \circ x^{-1})\,(g \circ y^{-1})\,d\mu_{n+m} = \left(\int_{\mathbb{R}^n} f \circ x^{-1}\,d\mu_n\right)\left(\int_{\mathbb{R}^m} g \circ x^{-1}\,d\mu_m\right)$$

which is equal to the right hand side of (7.2).

In general, we may use partitions of unity. $\square$

7.3 Solid Angle

On $\mathbb{R}^n$, let

$$r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}\,.$$

Then r is smooth on $\mathbb{R}^n_*$ and

$$dr = \frac{1}{r}\sum_{i=1}^{n} x^i dx^i, \qquad \langle dr, dr\rangle = 1.$$

The **solid angle element** $\mathrm{d}\Theta_n$ (with respect to the origin) is the $(n-1)$-form on $\mathbb{R}^n_*$ defined by

$$\boxed{\mathrm{d}\Theta_n = \frac{1}{r^n}\sum_{i=1}^{n}(-1)^{i-1}x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n}$$

7.3.1 Angle element

On $\mathbb{R}^2_* \simeq \mathbb{C}_*$, we have

$$z = re^{\mathrm{i}\theta} = x + y\mathrm{i}$$

so that

$$\frac{dz}{z} = d\log r + \mathrm{i}d\theta$$

where $d\theta$ is a closed but NOT exact 1-form although we abuse the notation. Then

$$\mathrm{d}\Theta_2 = \frac{xdy - ydx}{x^2 + y^2} = \frac{1}{2\mathrm{i}}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) = \mathrm{im}\,\frac{dz}{z} = d\theta.$$

Note that

$$\left[\frac{1}{2\pi}d\theta\right] = \left[\frac{1}{2\pi\mathrm{i}}\frac{dz}{z}\right] \in H^1(\mathbb{R}^2_*).$$

The restriction of this cohomology class to the unit circle $\mathbf{S}^1$ is the **fundamental class**:

$$\int_{\mathbf{S}^1}\frac{1}{2\pi}d\theta = 1.$$

For any oriented closed curve $\Gamma \subset \mathbb{R}^2_*$,

$$\frac{1}{2\pi}\int_{\Gamma} d\theta$$

is the integer, called the **winding number** of Γ around the origin.

7.3.2 Polar Coordinates

Let $\mathbb{R}_+$ be the set of all positive real numbers. We have the polar coordinates

$$q : \mathbb{R}^n_* \to \mathbb{R}^+ \times \mathbf{S}^{n-1}, \qquad z \mapsto (|z|, z/|z|).$$

Let dA_{n-1} be the restriction of $\mathrm{d}\Theta_n$ to the unit sphere $\mathbf{S}^{n-1}$:

$$\mathrm{dA}_{n-1} := \mathrm{d}\Theta_n \restriction \mathbf{S}^{n-1} = \mathrm{inc}^* \, \mathrm{d}\Theta_n,$$

where inc is the inclusion map $\mathbf{S}^{n-1} \hookrightarrow \mathbb{R}^n$. Then dA_{n-1} is the volume form on $\mathbf{S}^{n-1}$. In particular, $\mathrm{d}\Theta_n$ is NOT an exact form, although our notation is rather confusing. Fortunately, $\mathrm{d}\Theta_n$ is a closed form.

Lemma 7.3.2.1 *Let $\pi : \mathbb{R}^n_* \overset{q}{\simeq} \mathbb{R}_+ \times \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ be the retraction map. Then*

$$\pi^*(\mathrm{dA}_{n-1}) = \mathrm{d}\Theta_n.$$

In particular, $\mathrm{d}\Theta_n$ is a closed form.

Proof.

$$\begin{aligned}
& \pi^*(\mathrm{dA}_{n-1}) \\
= \; & \sum_{i=1}^n (-1)^{i-1} \frac{x^i}{r} \, d(x^1/r) \wedge \cdots \wedge \widehat{d(x^i/r)} \wedge \cdots \wedge d(x^n/r) \\
= \; & \frac{1}{r^n} \sum_i (-1)^{i-1} x^i \, (dx^1 - x^1 d\log r) \wedge \cdots \wedge (\widehat{dx^i - x^i d}\log r) \wedge \cdots \wedge (dx^n - x^n d\log r) \\
= \; & \sigma_n + \sum_i (-1)^{i-1} x^i \sum_{j<i} (-1)^j x^j d\log r \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
& + \sum_i (-1)^{i-1} x^i \sum_{j>i} (-1)^{j-1} x^j d\log r \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \\
= \; & \mathrm{d}\Theta_n
\end{aligned}$$

□

7.3.2.2 Exercise

(1) $\mathrm{d}\Theta_n$ is invariant under the rotations $\mathrm{SO}(n)$ and the positive scalar multiplications on $\mathbb{R}^n_*$. It $\rho : \mathbb{R}^n_* \to \mathbb{R}$ is a positive function, then $\mathrm{d}\Theta_n$ is invariant under the multiplication map

$$\hat{\rho} : \mathbb{R}^n_* \to \mathbb{R}^n_*, \qquad \mathbf{x} \mapsto \rho\mathbf{x}.$$

(2) (Polar Coordinates) Show that the Hodge dual of dr (on $\mathbb{R}^n_*$) is :

$$\star\, dr = \frac{1}{r}\sum_{i=1}^{n}(-1)^{i-1}x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \tag{7.3}$$

Thus on $\mathbb{R}^n_*$, we have

$$\star\left(\frac{dr}{r^{n-1}}\right) = \mathrm{d}\Theta_n$$

and

$$\boxed{\mathrm{dV}_n := dx^1 \wedge \cdots \wedge dx^n = dr \wedge \star dr = r^{n-1} dr \wedge \mathrm{d}\Theta_n}$$

Show that

$$\mathrm{int}_{\mathbf{r}}(\mathrm{dV}_n) = \sum_i (-1)^{i-1} x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $\mathbf{r} = \sum_i x^i \partial_i$ is the **position vector field** on $\mathbb{R}^n$.[7] Thus

$$\mathrm{int}_{\mathbf{r}/r^n}\, \mathrm{dV}_n = \mathrm{d}\Theta_n.$$

(3) Show that $\mathrm{d}\Theta_n$ is a closed non-exact form. Show that the vector field

$$\frac{\mathbf{r}}{r^n} = \begin{cases} \operatorname{grad} \log r & (n = 2) \\ \frac{1}{2-n} \operatorname{grad} r^{2-n} & (n \neq 2) \end{cases}$$

is divergence-free.

(4) Let $\mathbf{S}^{n-1}(r)$ be the sphere in $\mathbb{R}^n$ centered at the origin of radius $r > 0$. Then the restriction of (7.3) to the sphere $\mathbf{S}^{n-1}(r)$ is the volume form of $\mathbf{S}^{n-1}(r)$.[8]

Proposition 7.3.2.3 *Let $f \in \mathcal{C}_c(\mathbb{R}^n)$. Then for $x = r \cdot \theta \in \mathbb{R}_+ \times \mathbf{S}^{n-1} \simeq \mathbb{R}^n_*$,*

$$\int_{\mathbb{R}^n} f(x)\, dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbf{S}^{n-1}} \left(\int_0^\infty f(r\cdot\theta) r^{n-1} dr\right) \mathrm{dA}_{n-1}$$

7.3.3 Volumes of Balls and Spheres

Now we compute the volume of the unit sphere $\mathbf{S}^{n-1}$.

$$\begin{aligned} \pi^{n/2} &= \int_{\mathbb{R}^n} e^{-r^2}\, \mathrm{dV}_n = \int_{\mathbb{R}^n} e^{-r^2} r^{n-1} dr \wedge \mathrm{d}\Theta_n \\ &= \mathrm{Vol}(\mathbf{S}^{n-1}) \int_0^\infty e^{-r^2} r^{n-1} dr \\ &= \mathrm{Vol}(\mathbf{S}^{n-1}) \cdot \frac{1}{2} \cdot \Gamma(n/2) \end{aligned}$$

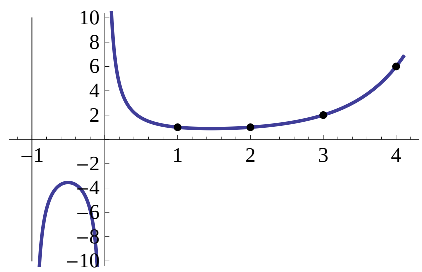
[7] The Position vector field is often called the *Euler vector field* or the *identity vector field.*

[8] Since $\star dr$ is invariant under the action of the rotations $\mathrm{SO}(n)$, it suffices to check at one point, say at $p = (r, 0, \ldots, 0)$. Then $\omega_p := \mathrm{inc}^*(dx^2 \wedge \cdots \wedge dx^n)$ satisfies $\omega_p(\frac{\partial}{\partial x^2}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p) = 1$. This shows that (7.3) is the volume form of $\mathbf{S}^{n-1}(r)$.

where

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt \qquad (s > 0)$$

is the **gamma function**.[9]



Gamma Function

Thus

$$\boxed{\mathrm{Vol}(\mathbf{S}^{n-1}) = \frac{2\cdot\pi^{n/2}}{\Gamma(n/2)}}$$

In particular,

$$\mathrm{Vol}(\mathbf{S}^{2n+1}) = 2\pi\cdot\frac{\pi^n}{n!}, \qquad \mathrm{Vol}(\mathbf{S}^{2n}) = \frac{2\cdot(2\pi)^n}{(2n-1)!!} = \frac{2\cdot\pi^n}{(n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}}.$$

7.3.3.1 Exercises

(1) Let b_n and s_{n-1} be the volume of the unit ball $\mathbf{B}^n$ and the sphere $\mathbf{S}^{n-1}$ in $\mathbb{R}^n$. Show that

$$b_n = \frac{1}{n}s_{n-1} = \frac{s_{n+1}}{2\pi} = \frac{\pi^{n/2}}{(n/2)!}$$

where $s! := \Gamma(s+1)$ for $s > 0$. The first formula $b_n = \frac{1}{n}s_{n-1}$ is also an observation of Archimedes: **A ball is a cone over its sphere.** The relation

$$\mathrm{vol}(\mathbf{S}^{n+1}) = \mathrm{vol}(\mathbf{S}^1)\times\mathrm{vol}(\mathbf{B}^n)$$

is also discovered by Archimedes for $n = 1$: **The area of the sphere is equal to the area of the circumscribed cylinder.** In general, the map

$$\mathbf{S}^{n+1} \dashrightarrow \mathbf{B}^n\times\mathbf{S}^1, \quad (x_1,\ldots,x_n,y_1,y_2)\mapsto(x_1,\ldots,x_n)\times\frac{(y_1,y_2)}{y_1^2+y_2^2}$$

preserves the volume.

[9] Euler's gamma function satisfies

$$\Gamma(1) = 1, \qquad \Gamma(s+1) = s\Gamma(s) \qquad (s > 0).$$

Thus $\Gamma(n+1) = n!$ for nonnegative integers n. Note that $\Gamma(1/2) = \sqrt{\pi} \simeq 1.77$. Thus $(1/2)! = \sqrt{\pi}/2$.

n	0	1	2	3	4	5	6	7
b_n	1	2	π	$4\pi/3$	$\pi^2/2$	$8\pi^2/15$	$\pi^3/6$	$16\pi^3/105$
s_n	2	2π	4π	$2\pi^2$	$8\pi^2/3$	π^3	$16\pi^3/15$	$\pi^4/3$

Show that

$$\lim_{n\to\infty} b_n = 0, \qquad \lim_{n\to\infty} s_n = 0.$$

(2) Note that we have a Riemannian submersion $\mathbf{S}^{2n+1} \to \mathbb{C}\mathbf{P}^n$ with $\mathbf{S}^1$-fiber. Show that the volume of $\mathbb{C}\mathbf{P}^n$ is equal to b_{2n}, and the volume of $\mathbf{S}^{2n+1}$ is the volume of $\mathbb{C}\mathbf{P}^n$ times the volume of $\mathbf{S}^1$.

(3) For $0 < r < 1$, compute the volume of the ``hyperbolic ball"

$$M = \{x \in \mathbb{R}^n \mid |x| \le r\}$$

equipped with the metric

$$g = \frac{4}{(1-|x|^2)^2}\,(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n).$$

7.4 Compacted supported Mayer-Vietoris Sequence

Let U and V be two open subsets of M with $M = U \cup V$. Then we have an exact sequence of chain complexes:

$$\{0\} \to \Omega_c^\bullet(U \cap V) \to \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) \to \Omega_c^\bullet(U \cup V) \to \{0\} \tag{7.4}$$

and the long exact seque

$$\begin{array}{cccccccc} \{0\} & \to & H_c^0(U \cap V) & \to & H_c^0(U) \oplus H_c^0(V) & \to & H_c^0(U \cup V) \\ & \to & H_c^1(U \cap V) & \to & H_c^1(U) \oplus H_c^1(V) & \to & H_c^1(U \cup V) \\ & \to & H_c^2(U \cap V) & \to & H_c^2(U) \oplus H_c^2(V) & \to & H_c^2(U \cup V) \\ & \to & \dots . \end{array}$$

7.4.0.1

Theorem 7.4.0.2 *For any nonnegative integer k*

$$H_c^{k+1}(M \times \mathbb{R}) \simeq H_c^k(M).$$

In particular

$$H_c^k(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R} & \text{if } k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. Let $\pi : M \times \mathbb{R} \to M$ be the projection. Then by integrating along the fiber, we get a homomorphism [BT]

$$\pi_* : \Omega_c^{k+1}(M \times \mathbb{R}) \to \Omega_c^k(M), \qquad \omega'_t + dt \wedge \omega''_t \mapsto \int_{\mathbb{R}} \omega''_t \, dt$$

where ω'_t and ω''_t are compactly supported time-dependent $(k+1)$-form and k-form on M, respectively.

Then the map $\pi_* : \Omega_c^{\bullet+1}(M) \to \Omega_c^\bullet(M)$ satisfies

$$d\pi_* + \pi_* d = 0$$

and hence we have the induced map

$$[\pi_*] : H_c^{\bullet+1}(M \times \mathbb{R}) \to H_c^\bullet(M).$$

We now show that this map is an isomorphism. Take $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \rho(t) \, dt = 1$. Consider the map

$$e_* : \Omega_c^\bullet(M) \to \Omega_c^{\bullet+1}(M \times \mathbb{R}), \qquad \omega \mapsto \rho \, dt \wedge \wedge \omega.$$

Then $de_* + e_* d = 0$ and hence we have a linear map

$$[e_*] : H_c^\bullet(M) \to H_c^{\bullet+1}(M \times \mathbb{R}).$$

Since $\pi_* e_* = \mathrm{id}$,

$$\mathrm{id} = [\pi_*] \circ [e_*] : H_c^\bullet(M) \to H_c^\bullet(M).$$

Now we claim that

$$e_* \pi_* : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^\bullet(M \times \mathbb{R})$$

is chain homotopic to the identity map, i.e., there exists a map

$$K : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^{\bullet-1}(M \times \mathbb{R})$$

such that

$$dK + Kd = \mathrm{id} - e_* \pi_*.$$

For $\omega = \omega' + dt \wedge \omega'' \in \Omega_c^k(M \times \mathbb{R})$, where ω' and ω'' are time-dependent k and $(k-1)$-forms on M with compact support, the map

$$K\omega := \left(\int_{-\infty}^t \omega''\, dt\right) - \left(\int_{-\infty}^t \rho(\bar{t})\, d\bar{t}\right) \pi_* \omega$$

satisfies this property. □

Theorem 7.4.0.3 (Stokes Theorem) *Let M be an oriented n-manifold with boundary ∂M and let $\mathrm{inc} : \partial M \to M$ be the inclusion map. Then for $\eta \in \Omega_c^{n-1}(M)$,*

$$\int_M d\eta = \int_{\partial M} \mathrm{inc}^* \eta.$$

Proof. (Easy Case.) Suppose that M is covered by a single chart $x : M \to \mathbb{R}^n_-$ preserving the orientations and $\partial M = \{p \in M \mid x^1(p) = 0\}$. Then

$$\eta = \sum_{i=1}^n (-1)^{i-1} f_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

for some compactly supported smooth functions f_i on M,

$$d\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n$$

and

$$\mathrm{inc}^* \eta = \mathrm{inc}^*(f_1\, dx^2 \wedge \cdots \wedge dx^n).$$

Now

$$\begin{aligned}
\int_M d\eta &= \int_{\mathbb{R}^n_-} \sum_{i=1}^n \frac{\partial f_i}{\partial x^i} \circ x^{-1}\, d\mu_n \\
&= \sum_{i=1}^n \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \int_{-\infty}^0 \frac{\partial (f_i \circ x^{-1})}{\partial t^i}\, dt^1 \cdots dt^n \\
&= \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty (f_1 \circ x^{-1})(0, t^2, \ldots, t^n)\, dt^2 \cdots dt^n \\
&= \int_{\partial M} \mathrm{inc}^* \eta,
\end{aligned}$$

which is equal to 0 when $\partial M = \emptyset$.

(General Case.) Let $\{x_\alpha : U_\alpha \to \mathbb{R}^n_-\}$ be an atlas for M compatible with the orientation. We will assume that if U_α intersects with the boundary, then

$$\partial M \cap U_\alpha = \{p \in U_\alpha \mid x^1_\alpha(p) = 0\}$$

and

$$U_\alpha - \partial M = \{p \in U_\alpha \mid x^1_\alpha(p) < 0\}$$

so that for each $p \in \partial M$,

$$\left.\frac{\partial}{\partial x^2_\alpha}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n_\alpha}\right|_p$$

is a positively oriented basis for $T(\partial M)_p$.

Now let $\{\rho_\alpha : M \to \mathbb{R}\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then since $\sum_\alpha d\rho_\alpha = d(\sum_\alpha \rho_\alpha) = d(1) = 0$,

$$\begin{aligned}
\int_M d\eta &= \int_M \sum_\alpha \rho_\alpha d\eta = \int_M \sum d(\rho_\alpha \eta) = \sum \int_{U_\alpha} d(\rho_\alpha \eta) \\
&= \sum \int_{\partial U_\alpha} \mathrm{inc}^*(\rho_\alpha \eta) = \sum \int_{\partial M} \mathrm{inc}^*(\rho_\alpha \eta) = \int_{\partial M} \mathrm{inc}^*(\sum_\alpha \rho_\alpha \eta) \\
&= \int_{\partial M} \mathrm{inc}^*(\eta)
\end{aligned}$$

This completes the proof. □

Corollary 7.4.0.4 *Let M be a connected n-manifold without boundary.*
If M is oriented, then there exists a canonical isomorphism $H^n_c(M) \simeq \mathbb{R}$.
If M is non-orientable, then $H^n_c(M) = \{0\}$.

Proof. Suppose M is oriented. Note that $\int_M : \Omega^n_c(M) \to \mathbb{R}$ is nontrivial and hence it suffices to show that $d(\Omega^{n-1}_c(M))$ is the kernel of $\int_M$. By the Stokes Theorem, $d(\Omega^{n-1}_c(M))$ is contained in the kernel.

Now suppose $\omega \in \Omega^n_c(M)$ and $\int_M \omega = 0$. We will find an $(n-1)$-form η on M with compact support and $d\eta = \omega$.

Take an open cover $\mathcal{U} := (U_i \mid i \in I)$ of M such that U_i is diffeomorphic to $\mathbb{R}^n$ for each index $i \in I$. For each index i, let α_i be a fundamental form on U_i, i.e., $\alpha_i \in \mathcal{E}^n_c(U_i) \subset \mathcal{E}^n_c(M)$ and $\int_M \alpha_i = \int_{U_i} \alpha_i = 1$.

We now claim that the cohomology class $[\alpha_i] \in H^n_c(M)$ is independent of $i \in I$.

Suppose $U_i \cap U_j \neq \varnothing$ for some $i, j \in I$. Then by considering a differential form $\beta \in \mathcal{E}_c^n(U_i \cap U_j) \subset \mathcal{E}_c^n(U_i) \cap \mathcal{E}_c^n(U_j) \subset \mathcal{E}_c^n(M)$ with $\int_{u_i \cap U_j} \beta = 1$, we have

$$[\alpha_i] = [\beta] = [\alpha_j] \in H_c^n(M),$$

since $H_c^n(\mathbb{R}^n) = \{0\}$.

Now if $U_i \cap U_j = \varnothing$, since M is connected, there exists a sequence $V_1, \ldots, V_m \in \mathcal{U}$ such that $V_1 = U_i$, $V_m = U_j$, and $V_1 \cap V_2 \neq \varnothing$, ..., $V_{m-1} \cap V_m \neq \varnothing$. Thus the cohomology classes $[\alpha_i]$ and $[\alpha_j]$ are the same as elements in $H_c^n(M)$.

Let α be a representative of this class $[\alpha_i]$.

Now take a partitions of unity $(\rho_i \mid i \in I)$ for the cover $(U_i \mid i \in I)$. Then

$$[\rho_i \omega] = c_i[\alpha_i] = c_i[\alpha]$$

for some constant c_i. Since $0 = \int_M \omega = \sum_i \int_{U_i} \rho_i \omega = \sum_i c_i$, we have $\sum_i c_i = 0$. Thus $[\omega] = \sum_i [\rho_i \omega] = (\sum_i c_i)[\alpha] = 0 \in H_c^n(M)$.

This shows the first assertion.

Now suppose that M is non-orientable. We will show that if ω_0 is a compactly supported n-form on M, then there exists a compactly supported $(n-1)$-form η on M such that $\omega_0 = d\eta$. Using a partition of unity, it suffices to show the case when ω_0 is supported in a domain of chart $x_0 : U_0 \simeq \mathbb{R}^n$. Let $c = \int_{U_0} \omega_0$ (we use the chart x_0 for the integration).

Since M is non-orientable,[10] there exists a finite sequence of open subsets

$$U_1, \ldots, U_k$$

of M and diffeomorphisms $x_i : U_i \to \mathbb{R}^n$, such that

$$U_0 \cap U_1, \quad U_1 \cap U_2, \quad \ldots, \quad U_{k-1} \cap U_k, \quad U_k \cap U_0$$

are all connected and nonempty, the transition maps $x_{i-1} \circ x_i^{-1}$ (for $i = 1, \ldots, k$) are orientation preserving, and $x_k \circ x_0^{-1}$ is orientation reversing. We assume that each U_i is oriented with respect to the chart x_i.

For $i = 1, \ldots, k$, let ω_i be an n-form which are compactly supported in U_i and $\int_{U_i} \omega_i = c$. Then from the first assertion there exist compactly supported $(n-1)$-forms η_i such that

$$\omega_{i-1} = \omega_i + d\eta_i \qquad (i = 1, \ldots, k)$$

and

$$\omega_k = -\omega_0 + d\eta_0.$$

Thus

$$\omega_0 = -\omega_0 + d(\eta_0 + \eta_1 + \cdots + \eta_k)$$

Thus $\omega_0 \in d\Omega_c^{n-1}(M)$. This completes the proof. □

[10] The line bundle $\det(TM) \to M$ has no non-vanishing section. Thus for any point $p \in M$, there exists a loop $c : [0,1] \to M$ with $c(0) = p = c(1)$ such that the parallel translation (with respect to a *connection*) of a basis $v_1 \wedge \cdots \wedge v_n \in \wedge^n(TM)_p$ along c is $-v_1 \wedge \cdots \wedge v_n$.

Corollary 7.4.0.5 *A compact orientable manifold of dimension > 1 is not contractible.*

7.4.0.6 Fundamental Class

The fundamental cohomology class for a compact connected oriented n-manifold M, is the unique cohomology class $\omega \in H^n(M)$ such that

$$\int_M \omega = 1.$$

7.4.0.7 Exercise

(1) Show that $H_c^{\bullet+r}(M \times \mathbb{R}^r) \simeq H_c^{\bullet}(M)$.

(3) (**Thom Isomorphism**) Show that if $\pi : E \to M$ is an oriented vector bundle of rank r, then

$$\pi_* : H_c^{\bullet+r}(E) \simeq H_c^{\bullet}(M).$$

(4) Let X be a vector field on an oriented n-manifold M and let $\omega \in \Omega_c^n(M)$. Show that

$$\int_M \mathcal{L}_X \omega = \int_{\partial M} i(X)\omega$$

where $\mathcal{L}_X$ and $i(X)$ denotes the Lie derivative and the interior product, respectively.

(5) (**Poincaré Duality**) Observe the pairing

$$H^k(M) \times H_c^{n-k}(M) \to H_c^n(M) \simeq \mathbb{R}$$

on a connected oriented n-manifold M without boundary. Hence we have a map

$$PD : H^k(M) \to H_c^{n-k}(M)^*.$$

The Poincaré duality says that this map is an isomorphism [GHV, p.197]. Show that if M is a compact orientable manifold, then $H^k(M)$ is of finite dimension.

(6) SHow that

$$H_c^k([0,\infty)) = \{0\}$$

for any $k = 0, 1, 2, \ldots$. Show that for $M = [0,1] \times (0,1)$,

k	0	1	2
$H^k(M)$	$\mathbb{R}$	$\{0\}$	$\{0\}$
$H_c^k(M)$	$\{0\}$	$\mathbb{R}$	$\{0\}$

Show that the Pincaré duality does not hold in general for manifolds with boundary.

(7) Show that $H_c^\bullet$ is not a homotopy invariant.

(8) Let M be a compact connected oriented manifold of dimension $2k$. Observe that the intersection form

$$H^k(M) \times H^k(M) \to H^{2k}(M) \simeq \mathbb{R}$$

is symmetric when k is even, and is skew-symmetric when k is odd.

Theorem 7.4.0.8 *If M is a noncompact connected n-manifold, then $H^n(M) = \{0\}$.*

For a proof, see [Spivak, p. 370] or [John Lee].

7.4.0.9 Euler Characteristic

For a compact manifold M,

$$b^k := \dim H^k(M)$$

is the k-th Betti number of M, and

$$\chi(M) := \sum_k (-1)^k b^k$$

is the Euler-Poincaré characteristic of M.

When M is an orientable closed[11] surface, then

$$b^1(M) = 2g$$

where g is the genus of M. Thus $\chi(M) = 2 - 2g$.

7.4.0.10 Exercise

For an odd dimensional compact manifold M, show that $\chi(M) = 0$.

[11] compact and connected without boundary

7.4.1 Integration along Maps

Let M be a manifold with boundary. Let $F : N \to M$ be a proper map from an oriented k-dimensional manifold N (with boundary). Let $\partial F = F \circ \mathrm{inc} : \partial N \to M$, where $\mathrm{inc} : \partial N \to N$ denotes the inclusion. For each compactly supported differential k-form ξ on M, define

$$\int_F \xi := \int_N F^* \xi.$$

Proposition 7.4.1.1 *Let $F : N \to M$ be a proper map from an oriented k-manifold. If $\eta \in \Omega_c^{k-1}(M)$, then*

$$\int_F d\eta = \int_{\partial F} \eta.$$

Proof. $\int_F d\eta = \int_N F^*(d\eta) = \int_N d(F^*\eta) = \int_{\partial N} \mathrm{inc}^* F^* \eta = \int_{\partial N} (\partial F)^* \eta = \int_{\partial F} \eta$. □

7.5 Brouwer's Degree of Maps

In this section, every manifold has no boundary.

Let $F : M \to N$ be a proper map between oriented manifolds of the same dimension n. We will assume that N is connected.

By Sard's Theorem, there exists a point $q \in N$ which is a regular value of F. Then the preimage $F^{-1}(q)$ of q is finite, since F is (smooth and) proper.[12]

For each regular point $p \in M$ of F, define

$$\operatorname{sgn}(F,p) = \begin{cases} +1 & \text{if } F \text{ preserves the orientation at } p \\ -1 & \text{if } F \text{ reverses the orientation at } p. \end{cases}$$

The `algebraic number'

$$\deg F := \sum_{p \in F^{-1}(q)} \operatorname{sgn}(F,p) \in \mathbb{Z}$$

of points in $F^{-1}(q)$ is called the **degree** of F.

Note that the degree of F is a locally constant function of regular values q in N. We will soon see that this number is independent of the choice of a regular value.[13]

Theorem 7.5.0.1 (Fundamental Theorem of Algebra) *Let n be a positive integer and $a_0, \ldots, a_n$ be complex numbers with $a_n \neq 0$. Then the polynomial $f(z) = a_n z^n + \cdots + a_1 z + a_0$ has a root, i.e., a complex number z_0 such that $f(z_0) = 0$.*

Proof. Note that any polynomial map $\mathbb{C} \to \mathbb{C}$ is proper. We will show that f is surjective. Note that at a regular point of f, f preserves the orientation. Note also that a complex number z is a critical point of $f : \mathbb{C} \to \mathbb{C}$ if and only if it is a root of $f'(z)$. Thus there are only a finite number of critical points, and hence the number of critical values of f are also finite. Thus the set of regular values of f is connected. Since degree is a locally constant function of regular values, it must be constant. If f is not surjective, then the degree of f is identically equal to zero, and f is a constant map, which contradicts the assumption $a_n \neq 0$. See the comment after the Corollary 7.5.0.4. ☐

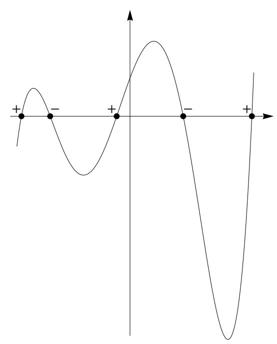
7.5.0.2 Examples

1. The degree of the identity map (at any value) is 1.

[12] There exists a continuous map $F : \mathbf{S}^1 \to \mathbf{S}^1$ such that all fibers have the power of continuum [Dieudonné, 1989].

[13] cf. L. E. J. Brouwer, *Über Abbildung von Manigfaltigkeiten*, Math. Annalen **71** (1912), 97–115.

2. For a positive integer d, let
$$f : \mathbb{R} \to \mathbb{R}$$
be a polynomial map of degree d. Note that polynomials are proper maps. If 0 is a regular value of f, then
$$\deg f = \begin{cases} 1 & \text{if } d \text{ is odd and the leading coefficient of } f \text{ is positive} \\ -1 & \text{if } d \text{ is odd and the leading coefficient of } f \text{ is negaitive} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$



3. For an integer d, the degree of
$$f : \mathbb{C}_* \to \mathbb{C}_*, \qquad f(z) = z^d$$
is $|d|$.

Theorem 7.5.0.3 (Brouwer, 1912) *The degree of $F : M \to N$ is independent of the choice of a regular value $q \in N$ of F. Moreover*
$$\int_M F^*\omega = \deg F \cdot \int_N \omega \tag{7.5}$$
for any $\omega \in \Omega^n_c(N)$. In particular, if ω represents the fundamental class of N, then
$$\deg F = \int_M F^*\omega.$$

Proof. Obviously it suffices to show the identity (7.5). Note that for connected oriented n-manifold N (without boundary), the integration map $\int_N$: $\Omega^n_c(N) \to \mathbb{R}$ induces the isomorphism
$$H^n_c(N) \simeq \mathbb{R}.$$
Since the map
$$H^n_c(N) \xrightarrow{[F^*]} H^n_c(M) \to \mathbb{R}, \qquad [\omega] \mapsto \int_M F^*\omega$$

is linear, there exists a constant c such that

$$\int_M Fi^*\omega = c\int_N \omega$$

for every $\omega \in \Omega^n_c(N)$. Now we show that this constant c is equal to the degree $\deg F$. Let $F^{-1}(q) = \{p_1, \ldots, p_k\}$. Take connected open neighborhoods V of q in N and U_i of p_i in M, for each $i = 1, \ldots, k$, such that U_i's are mutually disjoint and $F \restriction U_i$ is a diffeomorphism onto V. Take a $\omega \in \Omega^n_c(N)$ supported in V and $\int_N \omega = 1$. Then

$$c = \int_M F^*\omega = \sum_i \int_{U_i} F^*\omega = \sum_i \operatorname{sgn}(F, p_i)\int_V \omega = \deg F.$$

This completes the proof.

For another proof, see [Milnor, 1976] or [Dubrovin et al.]. □

Corollary 7.5.0.4 *Let $F : M \to N$ be a proper map between orientable manifolds, N being connected. If $\deg F \neq 0$, then F is surjective.*

Proof. Note that a proper map into a locally compact Hausdorff space is a closed map [John Lee]. Thus if F is not surjective, then $\deg F = 0$. □

From this corollary, one may prove the Fundamental Theorem of Algebra 7.5.0.1, by showing that any polynomial map $f : \mathbb{C} \to \mathbb{C}$ of degree n extends to a smooth map $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the same degree, where $\hat{\mathbb{C}}$ is the Riemann sphere.

Corollary 7.5.0.5 *Let $F : M \to N$ and $G : N \to L$ be proper maps between oriented manifolds of the same dimension, where N and L are connected. Then*

$$\deg(G \circ F) = \deg G \cdot \deg F.$$

Corollary 7.5.0.6 *Let $F_0, F_1 : M \to N$ be proper maps between oriented manifolds of the same dimension n, where N is connected. If F_0 and F_1 are "properly" homotopic, then $\deg F_0 = \deg F_1$.*

Proof. Note that from (6.3.0.2) and (6.3.0.3) there exists a map

$$h : \Omega^n(N) \to \Omega^n(M \times \mathbb{R}) \to \Omega^{n-1}(M)$$

such that

$$F_1^* - F_0^* = d \circ h : \Omega^n(N) \to \Omega^n(M).$$

Thus for $\omega \in \Omega^n_c(N)$ with $\int_N \omega = 1$,

$$\deg F_1 = \int_M F_1^*\omega = \int_M F_0^*\omega = \deg F_0.$$

(Using a partition $\{\rho_\alpha : M \to \mathbb{R}\}$ of unity subordinate to an open cover $\{U_\alpha\}$ of M with compact $\overline{U_\alpha}$, one can easily see that $\int_M d\eta = 0$ for any $\eta \in \Omega^{n-1}(M)$ with $\operatorname{supp} d\eta$ compact.) This completes the proof. □

Note that when $M = N$ as oriented manifolds, then the degree of a proper self map $F : M \to M$ is independent of the choice of orientation.

Theorem 7.5.0.7 *Let M be the boundary of a compact oriented smooth manifold W of dimension $n+1$. Let $F : M \to N$ be a smooth map into a connected oriented n-manifold N. If f extends to a smooth map $F : W \to N$, then $\deg F = 0$.*

Proof. Let ω be an n-form on N with the total integral 1. Then

$$\deg f = \int_M f^*\omega = \int_{\partial W} f^*\omega = \int_W d(f^*\omega) = \int_W f^*(d\omega) = 0.$$

□

Theorem 7.5.0.8 (H. Hopf) *Let M be a compact connected oriented n-manifold. Then two maps*

$$F, G : M \to \mathbf{S}^n$$

are homotopic if and only if they have the same degree.

7.5.0.9 Exercise

1. Show that if n is even, then the antipodal map on $\mathbf{S}^n$ is not homotopic to the identity map.
2. Show that, for $n \geq 1$, the polynomial map
$$f : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$
is proper and "properly" homotopic to the map $z \mapsto z^n$.
3. For polynomial functions $p(z)$ and $q(z)$ without common zeros, find the degree of the map
$$\mathbf{P}^1(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C}), \qquad z \mapsto \frac{p(z)}{q(z)}.$$
4. Let $\mathbf{S}^3$ be the Lie group of unit quaternions.[14] For an integer k, consider the map
$$F : \mathbf{S}^3 \to \mathbf{S}^3, \qquad q \mapsto q^k.$$

[14] If $\mathbf{v} \in \mathbb{H}$ is an imaginary unit vector, then we have de Moivre (?) identity
$$(\cos\theta + \mathbf{v}\sin\theta)^k = \cos k\theta + \mathbf{v}\sin k\theta.$$

Show that each imaginary unit vector $\mathbf{v} \in \mathbf{S}^3$ is a regular value of F, and the degree of F is k.

5. Find the degree of the map

$$f : \mathrm{SO}(3) \to \mathrm{SO}(3), \quad X \mapsto X^2.$$

7.5.1 Line Integral

Let I be an interval in $\mathbb{R}$ and let $c : I \to M$ be a smooth curve in M. Then for a 1-form ω on M, the line integral is defined by

$$\int_c \omega := \int_I c^* \omega.$$

7.5.1.1 Winding Number

Let $\gamma : \mathbf{S}^1 \to \mathbb{R}^2$ be a smooth map. Then for any point $p \in (\mathbb{R}^2 - \gamma(\mathbf{S}^1))$, we have a map

$$w_p : \mathbf{S}^1 \to \mathbf{S}^1, \qquad t \mapsto \frac{\gamma(t) - p}{|\gamma(t) - p|}.$$

Then the **winding number** of γ around p is

$$\mathrm{wind}(\gamma, p) := \frac{1}{2\pi} \int_{\mathbf{S}^1} w_p^*(d\theta) = \deg w_p.$$

Note that

$$\frac{d\theta}{2\pi} = \frac{1}{2\pi} \frac{ydx - xdy}{x^2 + y^2} = \frac{1}{2\pi i} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right).$$

7.5.1.2 Theorem of Turning Tangents

7.5.2 Linking Number

A **knot** is an embedding of $\mathbf{S}^1$ into $\mathbb{R}^3$. The **linking number**[15] $L(f, g)$ of disjoint knots

$$f, g : \mathbf{S}^1 \to \mathbb{R}^3$$

is the degree of the map

$$\mu : \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^2, \qquad (s, t) \mapsto \frac{f(t) - g(s)}{|f(t) - g(s)|}.$$

7.5.2.1 Exercise

Find the linking number $L(f, g)$ where

$$f(t) = (\cos t, \sin t, 0), \qquad g(t) = (1, 0, 0) + (\cos t, 0, \sin t).$$

[15] Observed first by Gauss, 1833. 1. 22. cf. C. Nash, *Topology and Physics—A Historical Survey*. cf. [Berger, Gostiaux].

7.5.2.2

More generally, for compact oriented immersed submanifolds M, N of $\mathbb{R}^{n+m+1}$ of dimensions n, m, the **linking number** $L(M,N)$ is the degree of the map

$$M \times N \to \mathbf{S}^{n+m}, \qquad (p,q) \mapsto \frac{q-p}{|q-p|}.$$

Note that

$$L(M,N) = (-1)^{n+m} L(N,M).$$

7.5.3 Index of Vector Fields

Let X be a vector field on a manifold M. Let $p \in M$ be an isolated singularity of X. Then we choose a coordinate neighborhood U of p and a chart $x : U \to \mathbb{R}^n$ with $x(p) = 0$. Then the vector field X induces a vector field $Y := x_* X$ on a neighborhood of 0 in $\mathbb{R}^n$. We choose a small ball $\mathbf{B}^n(\epsilon)$ so that Y has no sinularities in $\mathbf{B}^n(\epsilon)$ other than the origin. Now define

$$\nu(a) := \frac{Y(\epsilon a)}{|Y(\epsilon a)|} \in \mathbf{S}^{n-1}, \quad \text{for } a \in \mathbf{S}^{n-1}.$$

The degree of $\nu : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is called the *index of X at p*, and denoted by $\operatorname{ind}(X,p)$.

Since degree is a homotopy invariant, it is clear that the choice of ϵ is irrelevant in the definition of the index. But we have to show that it is independent of the choice of chart.

7.5.4 Division Ring

For what integers n, does $\mathbb{R}^n$ have a bilinear map

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

such that for any non-zero vectors $x, y \in \mathbb{R}^n$, $\mu(x,y) \neq 0$? cf. [Karoubi, Leruste, p.204]

7.6 Homology Groups

A singular k-cell in a manifold M (with boundary) is a smooth map from a closed unit ball $\mathbf{B}^k := \{p \in \mathbb{R}^k : |p| \leq 1\}$ into M. For a commutative ring R with the unit, let $\Omega_k(M, R)$ be the free R-module generated by singular k-cells in M. Now we define the boundary operator

$$\delta : \Omega_k(M, R) \to \Omega_{k-1}(M, R)$$

as the unique R-module homomorphism such that

$$\delta\sigma = i_+^*\sigma - i_-^*\sigma$$

for any $\sigma : \mathbf{B}^k \to M$, where

$$i_\pm : \mathbf{B}^{k-1} \to \mathbf{S}^{k-1} = \partial\mathbf{B}^k$$

are defined by $i_\pm(p) = (\pm\sqrt{1 - |p|^2}, p)$.

Theorem 7.6.0.1 *$\delta^2 = 0$ and hence we get homology groups (or R-modules)*

$$H_\bullet(M, R) := \sum_{k \geq 0} H_k(M, R)$$

with coefficients in R.

7.6.0.2 Exercise

Show that if M is connected, then $H_0(M, R) \simeq R$.

7.6.1 De Rham Isomorphism

Let $\Omega_k(M) := \Omega_k(M, \mathbb{R})$ and define a pairing

$$\langle\ ,\ \rangle : \Omega_k(M) \times \Omega^k(M) \to \mathbb{R}$$

by $\langle \sigma, \omega \rangle = \int_\sigma \omega$ for each $\sigma : \mathbf{B}^k \to M$. Then this pairing induces a pairing

$$\langle\ ,\ \rangle : H_k(M) \times H^k(M) \to \mathbb{R}$$

and a map

$$DR : H^k(M) \to H_k(M)^*.$$

Theorem 7.6.1.1 *DR is an isomorphism.*

For the proof see [War].

7.6.1.2 Exercise

Let $\omega \in \Omega^1(M)$ satisfy $\int_\sigma \omega = 0$ for any loop $\sigma : \mathbf{S}^1 \to M$. Show that ω is exact.

Proposition 7.6.1.3 $H^\bullet(\mathbf{S}^n) \simeq \mathbb{R}[x^n]/(x^{2n})$.

Proof. We have shown already that $H^n(\mathbf{S}^n) \simeq \mathbb{R} \simeq H^0(\mathbf{S}^n)$. We will show that if $0 < k < n$, then $H^k = 0$. Let ω be a representative of an element of $H^k(\mathbf{S}^n)$ and $\sigma : \mathbf{B}^k \to \mathbf{S}^n$. Then σ is not surjective and hence σ is a boundary and $\int_\sigma \omega = 0$. Thus by the de Rham isomorphism, $[\omega] = 0$. This completes the proof. □

7.7 Measures

Let M be a topological space and $\mathcal{C}(M)$ (or $\mathcal{C}_c(M)$) be the space of continuous functions on M (or with compact support). We equip $\mathcal{C}(M) \supset \mathcal{C}_c(M)$ with the compact-open topology.

7.7.0.1 Exercise

If K is a subset of M and $\mathcal{C}_K(M)$ denotes the space of continuous functions on M with support in K, then $\mathcal{C}_c(M)$ is the union of $\mathcal{C}_K(M)$ for all compact subsets K of M. Show that a linear functional

$$\mu : \mathcal{C}_c(M) \to \mathbb{R} \tag{7.6}$$

is continuous if and only if $\mu \restriction \mathcal{C}_K(M)$ is bounded for every compact subset K of M. (For compact K, $\mathcal{C}_K(M)$ is a Banach space, with the supremum norm $|\cdot|_K$. Then a linear functional on $\mathcal{C}_K(M)$ is bounded if and only if it is continuous.)

7.7.1 Measure

A linear functional (7.6) is **positive** if $\mu(f) \geq 0$ for every $f \in \mathcal{C}_c(M)$ with $f \geq 0$. A continuous positive linear functional (7.6) is called a **measure** on M.

7.7.2

Let M be a smooth n-manifold (not necessarily orientable and possibly with boundary). Then we define a map from $\Omega^n(M)$ into the space of measures on M. For $\omega \in \Omega^n(M)$,

$$|\omega| : \mathcal{C}_c(M) \to \mathbb{R}$$

is defined as follows:

Take an atlas $\{x_\alpha : U_\alpha \to \mathbb{R}^n_-\}$ of M and a partition $\{\rho_\alpha : M \to \mathbb{R}\}$ of unity subordinate to $\{U_\alpha\}$. Then $\omega \restriction U_\alpha = \omega_\alpha\, dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$ for some $\omega_\alpha \in \mathcal{C}^\infty(U_\alpha)$. Now for $f \in \mathcal{C}_c(M)$, $|\omega|(f) \in \mathbb{R}$ is defined by

$$\begin{aligned} |\omega|(f) \quad &:= \quad \int_M f\, d|\omega| \; = \; \sum_\alpha \int_M \rho_\alpha\, f\, d|\omega| \; = \; \sum_\alpha \int_{U_\alpha} \rho_\alpha\, f\, |\omega_\alpha|\, |dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha| \\ &:= \quad \sum_\alpha \int_{\mathbb{R}^n_-} (\rho_\alpha\, f\, |\omega_\alpha|) \circ x_\alpha^{-1}\, d\mu_n . \end{aligned}$$

Again, one can show that this definition is independent of the choice of an atlas and a partition of unity. Obviously, $|\omega|$ is a positive linear functional on $\mathcal{C}_c(M)$. To see the continuity, let K be a compact subset of M, ρ be a nonnegative continuous function on M with compact support and $\rho \restriction K \equiv 1$, and let $R = \int_M \rho\, d|\omega|$. Then for any $f \in \mathcal{C}_K(M)$, $|f| \leq |f|_K\, \rho$ and hence

$$\left| \int_M f\, d|\omega| \right| \leq \int_M |f|_K\, \rho\, d|\omega| \leq R\, |f|_K\, .$$

Thus $|\omega|$ is bounded on $\mathcal{C}_K(M)$ for every compact subset K of M and hence $|\omega|$ is a measure on M.

7.7.3 Riemannian Measure

Let (M, g) be an oriented Riemannian manifold. Then there exists a unique n-form $\mathrm{dv}(g)$ on M such that for each $p \in M$, $\mathrm{dv}(g)_p$ is the positive unit vector in $\wedge^n TM_p^*$. Thus we get a measure $|\,\mathrm{dv}(g)|$ on an oriented Riemannian manifold. Note that

$$\int_M f\,|\,\mathrm{dv}(g)| = \int_M f\ \mathrm{dv}(g)$$

for any $f \in \mathcal{C}_c(M)$.

If (M, g) is not orientable, then for each $p \in M$, there exists a unique unit `vector' $|\,\mathrm{dv}(g)|_p \in (\wedge^n TM_p^*)/\{\pm 1\}$. This defines a measure $|\,\mathrm{dv}(g)|$ on M.

The value

$$\mathrm{Vol}(M, g) := \lim_{k\to\infty} \int_M \rho_k\,|\,\mathrm{dv}(g)| \in \mathbb{R} \cup \{\infty\},$$

where $\rho_k \in \mathcal{C}_c(M)$ converges (uniformly on every compact subsets) to $1 \in \mathcal{C}(M)$, is called the **volume** of the Riemannian manifold (M, g).

7.7.4 Divergence

We define the **divergence** of a vector field on a Riemannian manifold M:

$$\mathrm{div} : \mathfrak{X}(M) \to \mathcal{C}^\infty(M).$$

For $X \in \mathfrak{X}(M)$,

$$\mathcal{L}_X\,\mathrm{dv}_g = (\mathrm{div}\,X)\,\mathrm{dv}_g$$

for any local volume form dv_g on M.

Note that divergence is independent of the orientability of M.

7.7.4.1 Exercise

Let ω be the volume form on an oriented Riemannian manifold M.

1. Show that

$$\mathrm{div}\,X = \star\, d \star (X^\flat) = \frac{1}{\sqrt{|g|}} \sum_i \partial_i \left(\sqrt{|g|}\, X^i\right),$$

   where $X = \sum_i X^i \partial_i$ and $|g| = \det(g(\partial_i, \partial_j))$.

2. Let D be a compact neighborhood of a point p in M, and let $D_t := \Phi_t(D)$, where Φ_t is the flow generated by X. Show that

$$\lim_{D\to p} \left.\frac{d}{dt}\right|_0 \frac{\mathrm{vol}\,D_t}{\mathrm{vol}\,D} = \mathrm{div}\,X(p)$$

   where D approaches p in the sense that the diameter of D approaches to 0.

7.7.5 Laplacian

For a function f on M, the (geometric) **Laplacian** of f is

$$\Delta f := -\operatorname{div}\operatorname{grad} f = -\frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\, g^{ij}\frac{\partial f}{\partial x^j}\right).$$

We have a spectrum, a heat equation, wave equation, and the Schrödinger equation on Riemannian manifolds.

7.7.6 Gaussian Curvature

Let M be an oriented hypersurface[16] in $\mathbb{R}^{n+1}$, with the induced metric g. Then we have an "outward" unit normal vector field ν on M, which may be considered as the **Gauss map**

$$\nu : M \to \mathbf{S}^n.$$

Then

$$\nu^*(\mathrm{dv}_{\mathbf{S}^n}) = K\,\mathrm{dv}_g$$

for some smooth function K on M, called the **normal curvature**[17] of M.

Note that K is the product of **principal curvatures** and

$$|K(p)| = \lim_{\epsilon\to 0}\frac{\mathrm{Vol}(\nu(B_M(p,\epsilon)))}{\mathrm{Vol}(B_M(p,\epsilon))},$$

where $B_M(p,\epsilon) := \{q \in M \mid \mathrm{dist}_M(p,q) \le \epsilon\}$.

Note that if M is compact,

$$\frac{1}{\mathrm{Vol}(\mathbf{S}^n)}\int_M K\,\mathrm{dv}_g = \deg(\nu)$$

is an integer.

Theorem 7.7.6.1 (Hopf, 1926) *Let M be a closed[18] hypersurface embedded in $\mathbb{R}^{n+1}$. Let $\nu : M \to \mathbf{S}^n$ be the Gauss map. Then*

$$\deg\nu = \chi\left(\mathrm{int}\, M\right),$$

where int M *is the inside of M. If n is even, then*

$$\chi\left(\mathrm{int}\, M\right) = \frac{1}{2}\chi(M).$$

For interesting stories, see [Gottlieb], [Hopf].

[16] We may assume that M is immersed in $\mathbb{R}^{n+1}$.
[17] When $n = 2$, it is called the *Gaussian curvature*.
[18] compact and connected

7.7.6.2 Exercises

(i) Let $(x(t), y(t)) \in \mathbb{R}^2$ be a curve parameterized by arclength so that $(x')^2 + (y')^2 \equiv 1$. Then the Gauss map is $t \mapsto (-y', x')$. Since $xdy - ydx$ is the volume form on $\mathbf{S}^1$, the Gaussian curvature is

$$K = x'y'' - x''y'.$$

Show that $|K|^2 = (x'')^2 + (y'')^2$.

(ii) Let $M = \{(u^1, \ldots, u^n, z) \in \mathbb{R}^{n+1} \mid z = f(u^1, \ldots, u^n)\}$ be the graph of a smooth map $f : \mathbb{R}^n \to \mathbb{R}$. Then M is a submanifold of $\mathbb{R}^{n+1}$ and has an induced metric g. The basic vector fields on M are

$$\frac{\partial}{\partial u^j} = (e_j, \frac{\partial f}{\partial u^j}) \qquad (1 \leq j \leq n)$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. The orientation on M is the one which makes $(u^1, \ldots, u^n)$ positively oriented. Thus

$$g_{ij} = \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = \delta_{ij} + \frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}.$$

Show that

$$\mathrm{vol}(g) = \sqrt{1 + |\nabla f|^2}\, du^1 \wedge \cdots \wedge du^n.$$

Show that

$$\nu = (-1)^n \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$$

is the ``outward" unit normal vector field on M. Compute the Gaussian curvature of M. (cf. VII.6.6)

When $\nabla f \neq 0$, show that the unit vector field

$$X = \frac{1}{\sqrt{1 + |\nabla f|^2}} \left(\frac{\nabla f}{|\nabla f|}, |\nabla f| \right)$$

on M is perpendicular to the "level surfaces " $\{(u, z) \in M \mid z = \text{ constant}\}$.

(iii) Let V be an inner product space of dimension n. Let $f : V \to V$ be a self-adjoint linear map whose eigenvalues are distinct so that there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $f(e_j) = \lambda_j e_j$ for some real number λ_j, $j = 1, \ldots, n$. Let $\mathbf{S}(V)$ be the space of unit vectors in V. Find the critical points of the map

$$h : \mathbf{S}(V) \to \mathbb{R}, \qquad v \mapsto \langle v, f(v) \rangle .$$

Note that $h(-v) = v$ and hence we have the induced map

$$\bar{h} : \mathbf{P}(V) \to \mathbb{R}.$$

Find the critical points of $\bar{h}$.

7.7.7 Euler Class

Let $E \to M$ be an oriented real vector bundle of rank $2r$.

7.7.8 Invariant measure on Lie Groups

Chapter 8

Frobenius Theorem

A subbundle $\mathcal{D}$ of the tangent bundle TM of a manifold M is called a **distribution**[1] of M.

A distribution $\mathcal{D}$ is **involutive** if for any section X and Y of E, the Lie bracket $[X, Y]$ is also a section of $\mathcal{D}$.

A distribution $\mathcal{D}$ is (completely) **integrable** if for any point $p \in M$ there exists a submanifold S of M such that $p \in S$ and for any $q \in S$, $TS_q = \mathcal{D}_q$.

An ideal $\mathcal{I}$ of $\Omega^\bullet(M)$ is called a **differential ideal** if it is d-closed, i.e., $d\mathcal{I} \subset \mathcal{I}$.

Given a distribution $\mathcal{D}$ on M, let

$$\mathcal{I}^k(\mathcal{D}) := \{\omega \in \Omega^k(M) \mid \omega(X_1, \ldots, X_k) = 0 \text{ for any } X_1, \ldots, X_k \in \Gamma(\mathcal{D})\}$$

and let

$$\mathcal{I}(\mathcal{D}) := \bigoplus_{k=0}^{n} \mathcal{I}^k(\mathcal{D}).$$

Then $I(\mathcal{D})$ is an ideal of $\Omega^\bullet(M)$.

Theorem 8.0.0.1 (Frobenius[2]) *Given a distribution $\mathcal{D}$ of M, the following are equivalent:*

(i) *$\mathcal{D}$ is involutive*

(ii) *$\mathcal{D}$ is integrable*

(iii) *the ideal $\mathcal{I}(\mathcal{D})$ is a differential ideal.*

[1] This terminology is also used in the theory of generalized functions with different meaning.

[2] Ferdinand Georg Frobenius (1849–1917), a German mathematician. Frobenius theorem was first proved by Alfred Clebsch (1833–1972, German) and Feodor Deahna (1815–1844, German).

(iv) *if $\theta^1, \dots, \theta^k$ are linearly independent local 1-forms on M which vanishes on $\mathcal{D}$, where k is the co-dimension of the distribution $\mathcal{D}$, then*

$$d\theta^i = \sum_{j=1}^{k} \theta^i_j \wedge \theta^j$$

for some 1-forms ω^i_j.

Proof. Since others are rather easy [Morita], we prove (ii) $\Rightarrow$ (i). Fix a point p in M. Let $Y_1, \dots, Y_r$ be a local frame for $\mathcal{D}$ in a neighborhood of p. Take a chart $(x^1, \dots, x^n)$ near p. Then

$$Y_b = \sum_{a=1}^{r} g^a_b \frac{\partial}{\partial x^a} + \sum_{i=r+1}^{n} h^i_b \frac{\partial}{\partial x^i} \qquad (b = 1, \dots, r).$$

for some functions g^a_b and h^i_b.[3] We may assume that

$$\det(g^a_b(p))_{a \le a, b \le r} \neq 0.$$

Let $(\tilde{g}^a_b)$ be the inverse matrix of (g^a_b) and let[4] $X_b = \sum_a \tilde{g}^a_b Y_a$ for $b = 1, \dots, r$. Then for some functions f^i_b,

$$X_b = \frac{\partial}{\partial x^b} + \sum_{i=r+1}^{n} f^i_b \frac{\partial}{\partial x^i}$$

and hence

$$[X_a, X_b] = \sum_{i=r+1}^{n} \tilde{f}^i_b \frac{\partial}{\partial x^i}$$

for some functions $\tilde{f}^i_b$. Since $\mathcal{D}$ is involutive, $[X_a, X_b]$ is also a linear combination of $X_1, \dots, X_r$. Thus $[X_a, X_b] = 0$ and hence the corollary (4.2.0.7) implies the conclusion. $\square$

Corollary 8.0.0.2 (Mayer-Lie system) *Let A and B be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ and let $f^i_a : A \times B \to \mathbb{R}$ be C^1-functions for $1 \le i \le m$ and $1 \le a \le n$. Then a system of differential equations*

$$\frac{\partial y^i}{\partial x^a} = f^i_a(x^1, \dots, x^n, y^1, \dots, y^m) \tag{8.1}$$

[3] We may write as

$$(Y_1 \ \dots \ Y_r) = (\partial_1 \ \dots \partial_r \ \partial_{r+1} \dots \partial_n) \binom{g^a_b}{h^i_b}$$

[4] $\mathbf{X} = \mathbf{Y} g^{-1} = \partial \binom{g}{h} g^{-1} = \partial \binom{\mathbf{1}}{g}$

is completely integrable[5] *if and only if it satisfies the integrability condition*

$$\frac{\partial f_a^i}{\partial x^b} + \sum_{j=1}^{m} \frac{\partial f_a^i}{\partial y^j} f_b^j = \frac{\partial f_b^i}{\partial x^a} + \sum_{j=1}^{m} \frac{\partial f_b^i}{\partial y^j} f_a^j$$

for any i, a, b*.*

For a general statement between Banach spaces cf. [Dieudonné, Vol. I].

Proof. Consider the 1-forms

$$\theta^i := dy^i - \sum_{j=1}^{n} f_j^i(x,y)\, dx^j \qquad (i = 1, \ldots, m)$$

on $\mathbb{R}^n \times \mathbb{R}^m$. These forms are linearly independent everywhere. If the integrability condition holds, then

$$d\theta^i \wedge \theta^1 \wedge \cdots \wedge \cdots \wedge \theta^m = 0.$$

Thus Frobenius theorem says that the system of PDE is completely integrable. Thus for any $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, thee exists an embedded submanifold

$$S(u^1, \ldots, u^n) = (x(u^1, \ldots, u^n), \ldots, y(u^1, \ldots, u^n))$$

of $\mathbb{R}^n \times \mathbb{R}^m$ with $S(0, \ldots, 0) = (x_0, y_0)$ such that

$$\theta^1 = 0, \quad \cdots, \quad \theta^m = 0$$

on S. In other words, we have

$$\frac{\partial y^i}{\partial u^k} - \sum_j f_j^i \frac{\partial x^j}{\partial u^k} = 0.$$

Since S is an immersion, the matrix

$$\begin{pmatrix} \frac{\partial x^j}{\partial u^k} \\ \frac{\partial y^i}{\partial u^k} \end{pmatrix} = \begin{pmatrix} 1_n \\ f_j^i \end{pmatrix} \begin{pmatrix} \frac{\partial x^j}{\partial u^k} \end{pmatrix}$$

is of rank n, and hence $\left(\frac{\partial x^j}{\partial u^k}\right)$ is nonsingular. Therefore, the projection map $S \hookrightarrow \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally invertible. Thus $y = f(x)$ for some f. $\square$

8.0.1 Exercise

Given a 1-form ω on an n-manifold M, a function f on M is called an integrating factor[6] of ω if $f\omega$ is exact. Show that if ω is non-trivial at some point p in M, and $n = 2$, then ω has an integrating factor in a neighborhood of p.

[5] I.e., for any point (x_0, y_0) in $A \times B$, there exists a neighborhood U of x_0 and a function $y : U \to B$ such that $y(x_0) = y_0$ and (8.1) holds at every point of U.

[6] cf. Spivak, Vol. IV, p.457.

Chapter 9

Classification of manifolds

Let M be a connected topological manifold (without boundary) of dimension n.

9.1 $n = 0$

A connected 0-dimensional manifold is a singleton.

9.2 $n = 1$

A connected topological 1-manifold is homeomorphic to either $\mathbb{R}$ or $\mathbf{S}^1$.

It has a smooth structure and every smooth structure is diffeomorphic to either $\mathbb{R}$ or $\mathbf{S}^1$.

A connected topological 1-manifold-with-nonempty-boundary is homeomorphic to either $[0, 1)$ or $[0, 1]$.

For details, see [Berger, Gostiaux], [Dieudonné], [Fuks and Rokhlin, p.139], [Guillemin, Pollack], [John Lee], [Spivak].

9.3 $n = 2$

Every topological surface has a unique differentiable structure up to diffeomorphism.

9.3.1 Compact Surfaces

A compact connected orientable surface is a "connected sum" $\mathbf{T}^2 \# \cdots \# \mathbf{T}^2$ of g tori, where g is called the **genus**.

A compact connected non orientable surface is $\mathbf{P}^2 \# \cdots \# \mathbf{P}^2$, the connected sum of the real projective planes. $\mathbf{K} := \mathbf{P}^2 \# \mathbf{P}^2$ is the **Klein bottle**.

9.3.1.1 Exercise.

Show that $\mathbf{P}^2 \# \mathbf{T}^2$ is equal to $\mathbf{P}^2 \# \mathbf{K}$.

9.3.2

There are three basic geometric surfaces: The sphere $\mathbf{S^2}$, the Euclidean plane $\mathbf{E}^2$, and the hyperbolic plane $\mathbf{H}^2$. These are homogeneous surfaces[1] of constant Gaussian curvature $K > 0$, $K = 0$, and $K < 0$, respectively[2]

If M is a surface with a Riemannian metric ds^2, then for each point p in M, there exists a coordinate system (x, y) in a neighborhood U of p such that

$$ds^2 \restriction U = \rho(dx^2 + dy^2)$$

for some positive function $\rho : U \to \mathbb{R}$. Such a coordinate system is called an isothermal or conformal coordinate system [Taylor].

Thus if M is orientable, M has a complex structure, and it becomes a Riemann surface.

In general, the universal cover $\tilde{M}$ of M is orientable, and hence $\tilde{M}$ becomes a Riemann surface. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ without fixed points and the action of each element in $\pi_1(M)$ is either holomorphic or anti-holomorphic.

Riemann's uniformization theorem (cf. [Taylor, III], [Dieudonné, Vol. I, (10.3), Problem 4]) says that every simply connected Riemann surface $\tilde{M}$ is biholomorphic to either the complex plane $\mathbb{C}$, the upper half plane $\mathbb{H}$, or the Riemann sphere $\mathbf{P}^1(\mathbb{C})$.

Case: $\tilde{M} \simeq \mathbb{C}$

Case: $\tilde{M} \simeq \mathbb{H}$

Case: $\tilde{M} \simeq \mathbf{P}^1(\mathbb{C})$

[1] A metric space X is *homogeneous* if for any two points p and q in X, there exists an isometry $f : X \to X$ such that $f(p) = q$.

[2] cf. [Bon]

9.4 $n \geq 5$

9.4.1 Topological Poincaré Conjecture

$\mathbf{S}^2 \times \mathbf{S}^2$ is a compact simply connected 4-manifold, but it is not homeomorphic to $\mathbf{S}^4$. The generalized Poincaré conjecture in higher dimension ($n \geq 4$) has three different forms:

- (C_n) A compact n-dimensional topological manifold which is homotopic to $\mathbf{S}^n$ is homeomorphic to $\mathbf{S}^n$.
- (PL_n) A compact n-dimensional PL manifold which is homotopic to $\mathbf{S}^n$ is PL-isomorphic to $\mathbf{S}^n$.
- (D_n) A compact n-dimensional smooth manifold which is homotopic to $\mathbf{S}^n$ is diffeomorphic to $\mathbf{S}^n$, if $n < 7$.

C_n is true for all n.
? Stallings and Zeeman proved that PL_n is true for $n \geq 5$.
S. Smale proved that D_5 and D_6 are true.
The generalized topological Poincaré conjecture says that "any compact n-manifold homotopic to $\mathbf{S}^n$ is homeomorphic to $\mathbf{S}^n$". This is proved, for $n \geq 5$, by S. Smale (1961), who proved "h-cobordism conjecture" [Fomenko], [Mil;C], [Nash].

9.4.2 Smooth manifolds

A closed topological manifold of dimension greater than or equal to 5 has at most finitely many non-diffeomorphic smooth structures [Scorpan].

9.4.3 Exotic Spheres

In 1956, John Milnor showed that there is an exotic differentiable structure on the topological 7-sphere $\mathbf{S}^7$. In 1963, Michael Kervaire and John Milnor found that the number of non-isomorphic differentiable structures on $\mathbf{S}^7$ is 15 (28 if orientations are conted).

For $k = 1, \ldots, 28$, let

$$Z_k := \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 - \{\mathbf{0}\} \mid z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\}.$$

Then Z_k is a complex manifold of complex dimension 4.

Let

$$\mathbf{S}^9 := \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

Then

$$\mathbf{S}^9 \cap Z_k$$

are all smooth manifolds homeomorphic to $\mathbf{S}^7$, and they are NOT diffeomorphic to each other [Tau2], [Poor].

9.5 $\mathbb{R}^n$

Let n be a nonnegative integer. If $n \neq 4$, then $\mathbb{R}^n$ has a unique differentiable structure (up to diffeomorphism).

In fact, every non compact manifolds are smoothable [Quinn].

On $\mathbb{R}^4$, there are uncountably many non-diffeomorphic differentiable structures. In an exotic $\mathbb{R}^4$, there exists a compact set which is not in the interior of a smooth embedded 3-sphere.

9.6 $n = 4$

9.6.1 Intersection Forms

Simply connected compact 4-manifolds are completely classified by M. Freedman [Fre] by studying the "intersection forms".

Theorem 9.6.1.1 (Freedman, 1982) *Let $\mathcal{M}_4$ be the collection of homeomorphism types of simply connected oriented compact 4-manifolds. Let q be a unimodular symmetric bilinear form on a finitely generated abelian group.*

(i) If q is even, then there exists a unique $M \in CalM_4$ such that the intersection form of M is isomorphic to q.

(ii) If q is odd, then there exist only two M_+, M_- in $\mathcal{M}_4$ such that their intersections forms are isomorphic to q.

9.6.2 Poincaré Conjecture in 4-d

In particular, he showed that any compact 4-manifold homotopic to $\mathbf{S}^4$ is homeomorphic to $\mathbf{S}^4$.

9.6.3 Undecidable

9.6.3.1 Fundamental groups of manifolds

9.6.3.2 Halting Problem

Turing(1936) showed that the Halting Problem is unsolvable.

9.6.3.3 Word Problem

Following is the word problem: Is there an algorithm which determines, whenever a finite set of defining relations for a group G and a word w are given, whether w is the identity element?

Novikov(1955) showed that the word problem is undecidable.

In 1908, Tietze made a conjecture that the isomorphism problem for groups is unsolvable. This problem was solved by Adyan in 1957 [Stillwell, 2010].

Theorem 9.6.3.4 (Markov, 1958) *If $n \geq 4$, then the classification of topological manifolds of dimension n is impossible.*

9.6.4 Smooth 4-manifolds

In general, a topological manifold may have none, several, or infinitely many non isomorphic differentiable structures.

The number of distinct differentiable structures on $\mathbf{S}^4$ is not known yet.

9.6.5 E_8-manifold

Let

$$E_8 := \{(x_1, \dots, x_8) \in \mathbb{R}^8 \mid 2x_i \in \mathbb{Z},\ x_1 \equiv \cdots \equiv x_8 \bmod \mathbb{Z},\ x_1 + \cdots + x_n \equiv 0 \bmod 2\}.$$

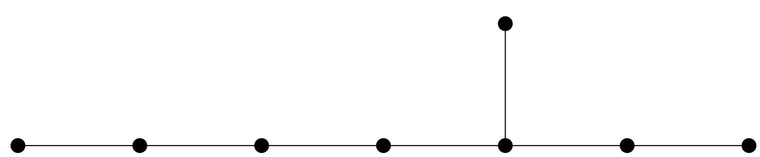
Then E_8 is a free abelian group with a basis

$$e_1 + e_2, \quad e_2 + e_3, \quad e_3 + e_4, \quad e_4 + e_5, \quad e_5 + e_6, \quad e_6 + e_7, \quad e_7 + e_8,$$
$$\tfrac{1}{2}(e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7 + e_8).$$

The quadratic form on E_8 (induced from the Euclidean structure on $\mathbb{R}^8$) is represented by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \in \mathrm{SL}(8, \mathbb{Z})$$

with respect to the above basis.



This quadratic form is even and positive definite and unimodular (of determinant 1).

Given a compact simply oriented topological 4-manifold M, $H^2(M, \mathbb{Z})$ is a free abelian group of finite rank, and we have an intersection form

$$I_M : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z},$$

which is a unimodular integral quadratic form.

Given any unimodular[3] integral quadratic form q, there exists a simply connected topological 4-manifold having q as its intersection form [Freedman, 1982].

V. Rokhlin's theorem (1952) says that the signature of any smooth compact 4-manifold M with a spin structure[4] is divisible by 16. A simply connected 4-manifold with even intersection form has a spin structure.

Thus E_8-manifold has no smooth structure.

[3] An integral lattice of determinant ± 1 is said to be *unimodular*.

[4] Or equivalently, $w_2(M) = 0 \in H_2(M, \mathbb{Z})$

9.6.6 Donaldson's Theorem

Theorem 9.6.6.1 (Donaldson, 1982) *If M is a simply connected compact 4-manifold with positive definite intersection form q, then q is diagonalizable over the integers.*

cf. [Fomenko], [Nash].

9.7 $n = 3$

M has a unique smooth structure up to diffeomorphism.

Poincaré's original conjecture (1904), was finally solved by Gregory Perelman.

Theorem 9.7.0.1 (Perelman (2003)) *Any compact simply connected 3-manifold is homeomorphic to $\mathbb{S}^3$.*

One of the most famous mathematical questions in the 20th century was Poincaré's question which asks whether a simply connected compact 3-manifold is homeomorphic to the 3-sphere.

W. Thurston's geometrization conjecture says that there are eight 3D geometries:

$$\mathbb{R}^3, \quad \mathbb{H}^3, \quad \mathbf{S}^3, \quad \mathbf{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \text{Nil}, \quad \text{Sol}, \quad \widetilde{SL}_2(\mathbb{R}).$$

Geometrization conjecture implies the Poincaré conjecture.

In 2002--2003, G. Perelman announced the affirmative answer for the geometrization conjecture through the internet.

9.7.1 Geometric Structure

A metric space X is said to be homogeneous if for any two points p and q in X, there exists an isometry[5] $f : X \to X$ such that $f(p) = q$.

A metric space X is locally homogeneous if for any two points p and q in X, there exist a neighborhood U_p pf p, a neighborhood U_q of q, and an isometry $f : U_p \to U_q$ such that $f(p) = q$. cf. [Bon]

A geometric structure on a manifold M is a complete Riemannian metric on M which is locally homogeneous.

Theorem 9.7.1.1 (Singer, 1960) *A simply-connected complete locally homogeneous Riemannian manifold is homogeneous.*

A simply-connected homogeneous Riemannian manifold is called a geometry.

9.7.2 Eight Geometries

Following spaces are complete Riemannian manifolds of constant sectional curvature:

$$\mathbf{S}^n, \qquad \mathbf{E}^n, \qquad \mathbf{H}^n.$$

[5] A map between two metric spaces is called an *isometry* if it is bijective and distance-preserving.

9.7.2.1 Nil Geometry

The Nil geometry (or Heisenberg group) is the Lie group

$$\mathrm{Nil} := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$$

together with a left invariant metric.

9.7.2.2 Sol Geometry

The Sol geometry is $\mathbb{R}^3$ with the Lie group structure

$$(x, y, z) \cdot (x', y', z') := (x + e^z x', y + e^{-z} y', z + z')$$

together with a left invariant metric.

Theorem 9.7.2.3 *There are eight geometries in dimension 3:*

$$\mathbf{S}^3, \quad \mathbf{E}^3, \quad \mathbf{H}^3, \quad \mathbf{S}^2 \times \mathbf{E}^1, \quad \mathbf{H}^2 \times \mathbf{E}^1, \quad \mathrm{Nil}, \quad \mathrm{Sol}, \quad \widetilde{\mathrm{SL}_2(\mathbb{R})}.$$

9.7.3 Space Form Problem

Appendix A

Linear Algebra of Tensors

Let R be a commutative ring with the unit element 1. In this chapter every module is assumed to be an R-module and every homomorphism is an R-module homomorphism.

A.1 Free module

Let S be a set. Then the *free R-module $\mathcal{F}_R(S)$ generated by the set S* consists of all maps $f : S \to R$ such that the carrier set

$$\operatorname{carr} f = \{s \in S \mid f(s) \neq 0\}$$

is finite.

Obviously, $\mathcal{F}_R(S)$ is a submodule of all maps from S to R.

There is a canonical injection $\hat{}\, : S \to \mathcal{F}_R(S)$ sending $s \in S$ to the map $\hat{s} : S \to R$ which has the value 1 at s and 0 elsewhere. Thus if $f \in \mathcal{F}_R(S)$, then

$$f = \sum_{s \in \operatorname{carr} f} f(s)\hat{s}.$$

Theorem A.1.0.1 (Universal Property) *For any map h from S into a module V, there exists a unique module homomorphism $\tilde{h} : \mathcal{F}_R(S) \to V$ such that $h = \tilde{h} \circ \hat{}$.*

Proof. Let $\sum_i r_i \hat{s}_i$ be an element of $\mathcal{F}_R(S)$. Then $\tilde{h} : \mathcal{F}_R(S) \to V$ has to be defined as follows:

$$\tilde{h}\left(\sum r_i \hat{s}_i\right) = \sum r_i h(s_i).$$

Then $\tilde{h}$ is the desired homomorphism. $\square$

We will identify an element s of S with $\hat{s}$ so that $\mathcal{F}_R(S)$ consists of all finite formal linear combinations

$$r_1s_1 + \cdots + r_ns_n \qquad (r_1, \ldots, r_n \in R, \quad s_1, \ldots s_n \in S).$$

It is easy to see that if T is a subset of S, then $\mathcal{F}_R(T)$ is a submodule of $\mathcal{F}_R(S)$. A module is said to be *free* if it is isomorphic to the free module generated by some set.

A.2 Tensor Products

Let V and W be modules. Then the *tensor product* of V and W (over R), denoted by $V \otimes W$ or $V \otimes_R W$, is defined as follows.

Consider the free module $\mathcal{F}_R(V \times W)$ generated by the set $V \times W$ and let $\mathcal{S}_R(V \times W)$ be the submodule of $\mathcal{F}_R(V \times W)$ generated by the elements of the form

$$(av + a'v', bw + b'w') - ab(v, w) - a'b(v', w) - ab'(v, w') - a'b'(v', w')$$

for any $a, b \in R$, $v, v' \in V$ and $w, w' \in W$. Then

$$V \otimes W := \mathcal{F}_R(V \times W)/\mathcal{S}_R(V \times W),$$

the quotient module of $\mathcal{F}_R(V \times W)$ by the submodule $\mathcal{S}_R(V \times W)$.

Let

$$\otimes : V \times W \to V \otimes W$$

be the composition of the canonical injection $\mathrm{inj} : V \times W \to V \otimes W$ and the quotient map $\mathrm{quo} : \mathcal{F}_R(V \times W) \to \mathcal{F}_R(V \times W)/\mathcal{S}_R(V \times W)$. Usually, we denote

$$\otimes(v, w) =: v \otimes w$$

for $(v, w) \in V \times W$. It is obvious that $\otimes : V \times W \to V \otimes W$ is bilinear.

Proposition A.2.0.1 *For any bilinear map h from $V \times W$ to a module U, there exists a unique homomorphism $\tilde{h} : V \otimes W \to U$ such that $h = \tilde{h} \circ \otimes$.*

Proof. By the universal property of the free module, there exists a unique homomorphism $h' : \mathcal{F}_R(V \times W) \to U$ such that $h = h' \circ \mathrm{inj}$. Now

$$h'|\mathcal{S}_R(V \times W) = 0$$

and hence there exists a unique homomorphism $\tilde{h} : V \otimes W \to U$ such that $h' = \tilde{h} \circ \mathrm{quo}$. Thus $h = h' \circ \mathrm{inj} = \tilde{h} \circ \mathrm{quo} \circ \mathrm{inj} = \tilde{h} \circ \otimes$. Now $\tilde{h}$ is bilinear and such $\tilde{h}$ is unique. $\square$

A.2.1 Remark

If a module U together with a bilinear map $i : V \times W \to U$ satisfies the above 'universal property', then it is isomorphic to the tensor product $V \otimes W$.

We may regard $V \otimes W$ as the collection of all formal finite linear combinations

$$\sum r_j v_j \otimes w_j$$

for $r_j \in R$, $v_j \in V$ and $w_j \in W$ with the relation

$$(rv + r'v') \otimes (sw + s'w') = rs(v \otimes w) + rs'(v \otimes w') + r's(v' \otimes w) + r's'(v' \otimes w')$$

for $r, r', s, s' \in R$, $v, v' \in V$ and $w, w' \in W$.

A.2.2 Example

Let $\mathbb{R}$ be the field of real numbers. Let X be a topological space and let $\mathcal{C}(X)$ be the $\mathbb{R}$-algebra of continuous (real valued) functions on X. Then for any finite dimensional vector space V over $\mathbb{R}$, $\mathcal{C}(X) \otimes_{\mathbb{R}} V$ is isomorphic to the vector space $\mathcal{C}(X; V)$ of all continuous maps[1] from X into V.

Proof. Define

$$i : \mathcal{C}(X) \times V \to \mathcal{C}(X; V)$$

by $i(f, v)(x) = f(x)v$ for $f \in \mathcal{C}(X)$, $v \in V$ and $x \in X$. Then i is bilinear over $\mathbb{R}$. We now claim that i satisfies the universal property. Let h be any bilinear map from $\mathcal{C}(X) \times V$ into a real vector space W. Then define $\tilde{h} : \mathcal{C}(X; V) \to W$ as follows; Let $e_1, \ldots, e_n$ be a basis for V. Then for $\phi \in \mathcal{C}(X, V)$

$$\phi = \sum_{j=1}^{n} i(\phi^j, e_j)$$

for some $\phi^j \in \mathcal{C}(X)$. Now

$$\tilde{h}(\phi) := \sum_j h(\phi^j, e_j).$$

Thus $\mathcal{C}(X; V)$ is isomorphic to $\mathcal{C}(X) \otimes V$. Note that the homomorphism $\tilde{h}$ is independent of the choice of the basis for V. □

[1] A finite dimensional real vector space is isomorphic to $\mathbb{R}^n$ for some integer n. The topology on V is induced from $\mathbb{R}^n$ by an isomorphism $V \simeq \mathbb{R}^n$, which is independent of the choice of isomorphism.

A.2.3 Exercises

(i) Show that for modules V and W, $V \otimes W$ is isomorphic to $W \otimes V$.

(ii) Show that if V is an R-module, then $V \otimes_R R$ is isomorphic to V.

(iii) Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ for positive integers m. Show that
$$\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_{(m,n)}$$
as $\mathbb{Z}$-modules (or abelian groups), where (m, n) denotes the greatest common divisor of m and n.

(iv) Define the tensor product $U \otimes V \otimes W$ of three modules U, V and W. Show that it is isomorphic to $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$.

(v) Show that
$$(\oplus_j V_j) \otimes W \simeq \oplus_j (V_j \otimes W)$$
for any family $\{V_j\}$ of modules and a module W, where $\oplus$ denotes the 'direct sum'.

(vi) Assume bilinear maps
$$V_1 \times V_2 \to V_3, \quad (v_1, v_2) \mapsto v_1 v_2$$
$$W_1 \times W_2 \to W_3, \quad (w_1, w_2) \mapsto w_1 w_2.$$
Show that there exists a unique bilinear map
$$(V_1 \otimes W_1) \times (V_2 \otimes W_2) \to V_3 \otimes W_3$$
such that $(v_1 \otimes w_1)(v_2 \otimes w_2) = (v_1 v_2) \otimes (w_1 w_2)$.
In particular, if V and W are (associative) algebras, so is $V \otimes W$.

(vii) Suppose that $A = \oplus_{k=0,1,\ldots} A^k$ is a graded-commutative algebra[2] and L is a Lie algebra. Show that $A \otimes L$ is a **graded Lie algebra** in the sense that
$$[a, b] = -(-1)^{kl}[b, a]$$
$$(-1)^{mk}[a, [b, c]] + (-1)^{kl}[b, [c, a]] + (-1)^{lm}[c, [a, b]] = 0$$
for $a \in A^k \otimes L$, $b \in A^l \otimes L$, $c \in A^m \otimes L$.

(viii) Suppose that $A = \oplus_k A^k$ is a graded-commutative algebra and B is an associative algebra. Define the commutator
$$[a, b] = ab - (-1)^{kl} ba$$
for $a \in A^k \otimes B$ and $b \in A^l \otimes B$, so that $c^2 = \frac{1}{2}[c, c]$ for an odd type element of $A \otimes B$. Show that $A \otimes B$ is a graded Lie algebra.

[2] An algebra A is a graded algebra if it is a direct sum of subspaces A^k for $k = 0, 1, \ldots$ such that $A^k A^l \subset A^{k+l}$ for any nonnegative integers k and l. A *graded-commutative algebra* is a graded algebra A such that $ab = (-1)^{kl} ba$ for any $a \in A^k$ and $b \in A^l$.

A.2.4 Multilinear Maps

Let $V_1, \ldots, V_n$ be modules. Then the tensor product $V_1 \otimes \cdots \otimes V_n$ consists of finite formal linear combinations of the elements of the form

$$v_1 \otimes \cdots \otimes v_n$$

for $v_j \in V_j$, with the relation

$$v_1 \otimes \cdots \otimes (rv_j + r'v_j') \otimes \cdots \otimes v_n = r(v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n) + r'(v_1 \otimes \cdots \otimes v_j' \otimes \cdots \otimes v_n).$$

One may define $V_1 \otimes \cdots \otimes V_n$ explicitly as a quotient module of the free module $\mathcal{F}_R(V_1 \times \cdots \times V_n)$ generated by the set $V_1 \times \cdots \times V_n$.

There is a canonical multilinear map $\otimes : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$.

Theorem A.2.4.1 *For any multilinear map h from $V_1 \times \cdots \times V_n$ into a module W, there exists a unique linear map $\tilde{h} : V_1 \otimes \cdots \otimes V_n \to W$ such that $h = \tilde{h} \circ \otimes$.*

A.2.5 Remark

For modules $V_1, \ldots, V_n$ and W, let $\mathcal{L}(V_1, \ldots, V_n; W)$ be the module of all multilinear maps from $V_1 \times \cdots \times V_n$ into W. Then it is obvious from the universal property that we have a canonical isomorphism

$$\mu : \mathcal{L}(V_1 \otimes \cdots \otimes V_n; W) \simeq \mathcal{L}(V_1, \ldots, V_n; W).$$

In particular,

$$(V_1 \otimes \cdots \otimes V_n)^* \simeq \mathcal{L}(V_1, \ldots, V_n; R),$$

where * denotes the `dual module'.

A.3 Duality

There is a unique homomorphism

$$\delta : \mathcal{L}(V_1; W_1) \otimes \cdots \otimes \mathcal{L}(V_n; W_n) \to \mathcal{L}(V_1 \otimes \cdots \otimes V_n; W_1 \otimes \cdots \otimes W_n)$$

such that

$$\delta(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

for $f_j \in \mathcal{L}(V_j; W_j)$ and $v_j \in V_j$.

In particular, we have

$$V_1^* \otimes \cdots \otimes V_n^* \xrightarrow{\delta} (V_1 \otimes \cdots \otimes V_n)^* \xrightarrow[\simeq]{\mu} \mathcal{L}(V_1, \ldots, V_n; R).$$

A.3.1 Remark

There is a natural homomorphism

$$\nu : V^* \otimes W \to \mathcal{L}(V; W)$$

such that $(\nu(v^* \otimes w))(v) = v^*(v)w$, for $v \in V$, $v^* \in V^*$ and $w \in W$.

A.4 Over a field $\mathbb{F}$

From now on R is a field $\mathbb{F}$ and every vector space is over $\mathbb{F}$.

Proposition A.4.0.1 *Let $V_1, \dots, V_n$ be finite dimensional vector spaces of dimension $m_1, \dots, m_n$, respectively.*

(1) Let $\{v_{j,1}, \dots, v_{j,m_j}\}$ be basis for V_j, $1 \le j \le n$. Then

$$\{v_{1,i_1} \otimes \cdots \otimes v_{n,i_n} : 1 \le i_1 \le m_1, \dots,\ 1 \le i_n \le m_n\}$$

is a basis for $V_1 \otimes \cdots \otimes V_n$.

(2) If V_j' is a subspace of V_j, then $V_1' \otimes \cdots \otimes V_n'$ is a subspace of $V_1 \otimes \cdots \otimes V_n$.

(3) $\nu : V_1^* \otimes V_2 \to \mathcal{L}(V_1; V_2)$ is an isomorphism.

(4) If $W_1, \dots, W_n$ are finite dimensional vector spaces, then

$$\delta : \mathcal{L}(V_1; W_1) \otimes \cdots \otimes \mathcal{L}(V_n; W_n) \to \mathcal{L}(V_1 \otimes \cdots \otimes V_n; W_1 \otimes \cdots \otimes W_n)$$

is an isomorphism. In particular,

$$\delta : V_1^* \otimes \cdots \otimes V_n^* \to (V_1 \otimes \cdots \otimes V_n)^* \tag{4.2}$$

is an isomorphism.

Proof. (1) It is clear that $\{v_{1,i_1} \otimes \cdots \otimes v_{n,i_n}\}$ generates $V_1 \otimes \cdots \otimes V_n$. To see the linear independence, suppose

$$\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n}(v_{1,i_1} \otimes \cdots \otimes v_{n,i_n}) = 0.$$

Let $\{v_{1,1}^*, \dots, v_{1,m_1}^*\}$ be the dual basis of $\{v_{1,1}, \dots, v_{1,m_1}\}$. Now apply

$$v_{1,j_1}^* \otimes \mathrm{id} : V_1 \otimes (V_2 \otimes \cdots \otimes V_n) \to \mathbb{F} \otimes (V_2 \otimes \cdots \otimes V_n) \simeq V_2 \otimes \cdots \otimes V_n$$

to get

$$\sum_{j_1,i_2,\dots,i_n} a_{i_2,\dots,i_n}(v_{2,i_2} \otimes \cdots \otimes v_{n,i_n}) = 0.$$

Now inductively we see that all the coefficients are equal to 0.

(2) Since any basis of a subspace can be extended to a basis for the whole space, it is clear from (1).

(3) and (4) are easy. □

A.4.1 Remark

Note that under the identification (4.2), if $\{v^*_{j,i_j}\}$ is the dual basis of $\{v_{j,i_j}\}$, then $\{v^*_{1,i_1} \otimes \cdots \otimes v^*_{n,i_n}\}$ is the dual basis of $\{v_{1,i_1} \otimes \cdots \otimes v_{n,i_n}\}$.

A.4.1.1 Exercise

For a finite dimensional vector space V over a field $\mathbb{F}$, let $c : V^* \otimes V \to \mathbb{F}$ be the contraction map. Show that the composition map

$$\mathrm{End}(V) \simeq V^* \otimes V \xrightarrow{c} \mathbb{F}$$

is the trace map.

A.5 Inner Products

Now we consider a finite dimensional vector space over $\mathbb{R}$ together with a (positive definite) inner product

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R}.$$

Then the `musical isomorphism'

$$\flat : V \to V^*$$

is defined by $\flat(v)(u) = \langle v, u\rangle$ for $u, v \in V$. The induced inner product on V^* is the one such that $\flat$ is an isometry. Thus if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V, then the dual basis $\{v_1^*, \ldots, v_n^*\}$ is also orthonormal.

The inverse of $\flat$ is denoted by $\sharp$.

If W is also an inner product space, then the musical isomorphisms of V and W induce an isomorphism

$$V \otimes W \simeq V^* \otimes W^* \simeq (V \otimes W)^*,$$

which in turn induces an inner product on $V \otimes W$. Thus

$$\langle v \otimes w, v' \otimes w'\rangle = \langle v, v'\rangle \cdot \langle w, w'\rangle$$

and if $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are orthonormal bases for V and W, respectively, then $\{v_i \otimes w_j\}$ is an orthonormal basis for $V \otimes W$.

A.5.1 Exercises

(1) Using the isomorphism $\nu : V^* \otimes W \to \mathcal{L}(V; W)$, describe how inner products on V and W induce an inner product on $\mathcal{L}(V; W)$. Show that if $f \in \mathcal{L}(V; W)$, then

$$|f| = \left(\sum_i |f(v_i)|^2\right)^{1/2}$$

for any orthonormal basis $\{v_1, \ldots, v_n\}$ for V.

(2) Use identifications $V_1^* \otimes \cdots \otimes V_k^* \simeq (V_1 \otimes \cdots \otimes V_k)^* \simeq \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$ to describe inner products on these spaces induced by inner products on $V_1, \ldots, V_k$. Show that

$$\langle v_1 \otimes \cdots \otimes v_k, u_1 \otimes \cdots \otimes u_k \rangle = \langle v_1, u_1 \rangle \cdots \langle v_k, u_k \rangle$$

and

$$|f| = \left(\sum |f(v_{1,i_1}, \ldots, v_{k,i_k})|^2 \right)^{1/2}$$

for $f \in \mathcal{L}(V_1, \ldots, V_k; \mathbb{R})$ and orthonormal basis $\{V_{j,i_j}\}$ of V_j.

A.6 Tensor Algebra

Let $\otimes^k V$ be the tensor product of k copies of a finite dimensional vector space V over a field $\mathbb{F}$ and let

$$\otimes^\bullet V := \sum_{k=0}^{\infty} \otimes^k V,$$

where $\otimes^0 V = \mathbb{F}$. Then $\otimes^\bullet V$ becomes an (associative) graded algebra (over $\mathbb{F}$) with the multiplication

$$\otimes : (\otimes^k V) \times (\otimes^l V) \to \otimes^{k+l} V$$

characterized by

$$(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_{k+l}) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}.$$

Let $L^k(V) := \mathcal{L}(V, \ldots, V; \mathbb{F})$ be the space of all k-linear maps $V \times \cdots \times V \to \mathbb{F}$. Then we have the following identifications;

$$\otimes^k (V^*) \simeq (\otimes^k V)^* \simeq L^k(V).$$

A.6.1 Exercise

Using the identification above, translate the multiplication

$$\otimes : \otimes^k (V^*) \times \otimes^l (V^*) \to \otimes^{k+l} (V^*)$$

into the multiplication on other spaces. In particular, show that if $f \in L^k(V)$ and $g \in L^l(V)$, then

$$(f \otimes g)(v_1, \ldots, v_{k+l}) = f(v_1, \ldots, v_k) \cdot g(v_{k+1}, \ldots, v_{k+l})$$

for $v_j \in V$.

A.7 Symmetric Tensors

Let V be a finite dimensional vector space over a field $\mathbb{F}$ of characteristic 0. Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. Then for $\sigma \in S_k$ and $(v_1, \ldots, v_k) \in V \times \cdots \times V$, define

$$(v_1, \ldots, v_k)^\sigma := (v_{\sigma 1}, \ldots, v_{\sigma k}).$$

Then this right action induces a linear action on $\otimes^k V$;

$$(v_1 \otimes \cdots \otimes v_k)^\sigma = v_{\sigma 1} \otimes \cdots \otimes v_{\sigma k}.$$

The elements of $\otimes^k V$ invariant under this action are called the *symmetric tensors*. They form a subspace $\otimes^k_{sym} V$ of $\otimes^k V$.

The averaging process

$$\mathrm{Sym}_k(v_1 \otimes \cdots \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (v_1 \otimes \cdots \otimes v_k)^\sigma$$

gives rise to a projection of $\otimes^k V$ onto $\otimes^k_{sym} V$.

A.7.1 Exercise

Let $L^k_{sym}(V)$ be the space of all symmetric k-linear maps of $V \times \cdots \times V \to \mathbb{F}$. Show that under the identification $\otimes^k(V^*) \simeq L^k(V)$, $\otimes^k_{sym}(V^*)$ corresponds to $L^k_{sym}(V)$. Show that

$$\mathrm{Sym}_k(f)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma} f(v_{\sigma 1}, \ldots, v_{\sigma k})$$

for $f \in L^k(V)$.

A.8 Alternating Tensors

Let V be a finite dimensional vector space over a field $\mathbb{F}$ of characteristic 0. Let S_k be the group of permutations of the set $\{1, 2, \ldots, k\}$. Then we have another linear action of S_k on $\otimes^k V$;

$$(v_1 \otimes \cdots \otimes v_k) \times \sigma \mapsto \mathrm{sgn}(\sigma)(v_1 \otimes \cdots \otimes v_k)^\sigma.$$

for $\sigma \in S_k$ and $(v_1, \ldots, v_k) \in V \times \cdots \times V$. The elements of $\otimes^k V$ invariant under this action are called the *alternating tensors*. They form a subspace $\otimes^k_{alt} V$ of $\otimes^k V$.

The averaging process

$$\mathrm{Alt}_k(v_1 \otimes \cdots \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)(v_1 \otimes \cdots \otimes v_k)^\sigma$$

gives rise a projection of $\otimes^k V$ onto $\otimes^k_{alt} V$.

A.8.1 Exercises

(1) Let $L^k_{alt}(V)$ be the space of all alternating k-linear maps of $V \times \cdots \times V \to \mathbb{F}$. Show that under the identification $\otimes^k(V^*) \simeq L^k(V)$, $\otimes^k_{alt}(V^*)$ corresponds to $L^k_{alt}(V)$. Show that

$$\operatorname{Alt}_k(f)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_\sigma \operatorname{sgn}(\sigma) f(v_{\sigma 1}, \ldots, v_{\sigma k})$$

for $f \in L^k(V)$.

(2) Show that for $z \in \otimes^k V$ and $\sigma \in S_k$, $(\operatorname{Alt}_k(z))^\sigma = \operatorname{sgn}(\sigma) \operatorname{Alt}_k(z) = \operatorname{Alt}_k(z^\sigma)$.

(3) If V is an inner product space (over $\mathbb{R}$), then with the induced inner product on $\otimes^k V$, $\ker(\operatorname{Alt}_k)$ and $\otimes^k_{alt}(V)$ are perpendicular.

A.9 Grassmann's Exterior Algebra

Let V be a vector space over a field $\mathbb{F}$. Then for any linearly independent vectors $v_1, \ldots, v_k$ in V,

$$v_1 \wedge \cdots \wedge v_k$$

is an equivalence class of the oriented parallelepiped. If $w_1, \ldots, w_k$ are linearly independent vectors in V, then

$$w_1 \wedge \cdots \wedge w_k = v_1 \wedge \cdots \wedge v_k$$

if and only if the parallelepiped generated by $(w_1, \ldots, w_k)$ and the parallelepiped generated by $(v_1, \ldots, v_k)$

(i) are in the same k-dimensional subspace of V and

(ii) have the same orientation and the same k-dimensional volume in the sense that there exists $g \in \mathrm{SL}(k, \mathbb{F})$ such that $(w_1, \ldots, w_k) = (v_1, \ldots, v_k)g$.

In fact, for any positive integer k, there exists a vector space, denoted by $\wedge^k V$ together with a multilinear map

$$\wedge : \underbrace{V \times \cdots \times V}_{k} \to \wedge^k V, \quad (v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k \tag{A.1}$$

such that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $(v_1, \ldots, v_k)$ is a linearly dependent collection of vectors. Moreover, the map (A.1) satisfies a universal property: If $p : V \times \cdots \times V \to W$ is a multilinear map such that $p(v_1, \ldots, v_k) = 0$ if and only if $(v_1, \ldots, v_k)$ is a linearly dependent collection of vectors, then there exists a unique linear map $\tilde{p} : \wedge^k V \to W$ such that $p = \tilde{p} \circ \wedge$.

A.9.1

Let V be a finite dimensional vector space over $\mathbb{R}$. Then the *k-th alternating, exterior* or *wedge power* of V, denoted by $\wedge^k V$, is a vector space consisting of linear combinations of

$$v_1 \wedge \cdots \wedge v_k$$

for $v_1, \ldots, v_k \in V$ with the relations

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$$

for $1 \leq i < j \leq k$ and

$$\begin{aligned} & v_1 \wedge \cdots \wedge (v_j + a v_j') \wedge \cdots \wedge v_k \\ = \quad & v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k + a(v_1 \wedge \cdots \wedge v_j' \wedge \cdots \wedge v_k) \end{aligned}$$

where $a \in \mathbb{R}$ and $j = 1, \ldots, k$.

In other words, $\wedge^k V$ is the quotient space of $\otimes^k V$ by the subspace N_k generated by the elements of the form

$$v_1 \otimes \cdots \otimes v_k$$

where $v_i = v_{i+1}$ for some $i \in \{1, \ldots, k-1\}$.

A.9.2 Exercise

Show that $N = \sum_{k=2}^{\infty} N_k$ is the (homogeneous) ideal of $\otimes^\bullet V$ generated by elements of the form $v \otimes v$ for $v \in V$. Thus

$$\wedge^\bullet V = \otimes^\bullet V/N = \sum_k \wedge^k V$$

is naturally an (associative) algebra over $\mathbb{R}$, where $\wedge^0 V := \mathbb{R}$ and $\wedge^1 V := V$. This algebra is called the *exterior* (or *Grassmann*) *algebra* of V. The product of elements ω and η in $\wedge^\bullet V$ is denoted by $\omega \wedge \eta$.

A.9.3

The natural map of $V \times \cdots \times V$ into $\wedge^k V$ is denoted by $\wedge$. Then $\wedge : V \times \cdots \times V \to \wedge^k V$ satisfies the universal property that any alternating k-linear map of $V \times \cdots \times V$ into a vector space W factors through a unique linear map of $\wedge^k V$ into W. This gives the canonical identification

$$L(\wedge^k V; W) \simeq L^k_{alt}(V; W).$$

In particular, we have

$$(\wedge^k V)^* \simeq L^k_{alt}(V).$$

A.9.3.1 Remark

We have a (splitting) short exact sequence

$$0 \to N_k \to \otimes^k V \to \wedge^k V \to 0$$

and $N_k = \ker \mathrm{Alt}_k$. Thus $\wedge^k V$ is isomorphic to $\otimes^k_{alt} V$ (non canonically). Under this isomorphism $v_1 \wedge \cdots \wedge v_k$ corresponds to

$$\mathrm{Alt}_k(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_\sigma \mathrm{sgn}(\sigma) \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

We will *not* identify these two elements.

Theorem A.9.3.2 (i) *For $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$,*

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = (\mathrm{sgn}\,\sigma) \cdot v_1 \wedge \cdots \wedge v_k.$$

(ii) *If $\omega \in \wedge^k V$ and $\eta \in \wedge^l V$, then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.*

(iii) *If $\{e_1, \ldots, e_n\}$ is a basis for V, then*

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis for $\wedge^k V$. In particular, $\dim(\wedge^\bullet V) = 2^n$.

(iv) *The map*

$$\delta : \wedge^k(V^*) \to (\wedge^k V)^*$$

characterized by

$$\delta(v_1^* \wedge \cdots \wedge v_k^*)(v_1 \wedge \cdots \wedge v_k) = \det(v_i^*(v_j))$$

is an isomorphism.

A.9.3.3 Exercises

(i) Show that $v_1, \dots, v_k \in V$ is linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$.

(ii) Let $p \in \wedge^2 V$. Show that if $p \wedge p = 0 \in \wedge^4 V$, then there exist $a, b \in V$ such that $p = a \wedge b$.

A.9.4

Let $f : V \to W$ be a linear map between finite dimensional vector spaces. Then we have the induced linear map

$$\wedge^k f : \wedge^k V \to \wedge^k W$$

characterized by $(\wedge^k f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$.

A.9.4.1 Exercises

1. Show that $\wedge^\bullet f = \sum_{k \geq 0} \wedge^k f : \wedge^\bullet V \to \wedge^\bullet W$ is a ring homomorphism.
2. For an endomorphism $f : V \to V$, let

$$c_k(f) := \operatorname{tr}\left(\wedge^k f : \wedge V \to \wedge^k V\right) \qquad (k = 0, \dots, n).$$

Show that

$$\det(\lambda\, 1_V + f) = \lambda^n + c_1(f)\lambda^{n-1} + \cdots + c_{n-1}(f)\lambda + c_n(f).$$

A.9.5 Inner Products

Let V be an inner product space. We define the inner product on $\wedge^k V$ so that

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)$$

for $v_1, \dots, v_k, w_1, \dots, w_k \in V$. Then

$$|v_1 \wedge \cdots \wedge v_k|$$

is equal to the volume of the parallelepiped in V spanned by $v_1, \dots, v_k$. Thus if $e_1, \dots, e_n$ is an orthonormal basis for V, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is an orthonormal basis for $\wedge^k V$.

A.9.5.1 Exercise (Pythagoras Theorem)

(1) For vectors $v_1, \dots, v_n$ in $\mathbb{R}^{n+1}$, let P be the parallelepiped spanned by these vectors, i.e.,

$$P = \{t_1 v_1 + \cdots + t_n v_n \mid 0 \le t_1, \dots, t_n \le 1\}.$$

For each $k \in \{1, \dots, n+1\}$, let P_k be the image of P under the orthogonal projection onto the k-th hyperplane

$$\{(a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_k = 0\}.$$

Let $|P|$ be the n-dimensional volume of P and $|P_k|$ be the n-dimensional volume of P_k. Show that

$$|P|^2 = |P_1|^2 + \cdots + |P_{n+1}|^2 .$$

Is it clear from this fact that the area of the parallelogram in $\mathbb{R}^3$ generated by the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) is equal to the square root of

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 ?$$

(2) Let $1 \le r \le n$ be integers and let

$$\mathcal{J} = \{(j_1, \dots, j_r) \mid 1 \le j_1 < \cdots < j_r \le n\}.$$

For each $J \in \mathcal{J}$ and $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, let

$$v^J := (v^{j_1}, \dots, v^{j_r}).$$

Let $|v_1 \wedge \cdots \wedge v_r|$ be the r-dimensional volume of the parallelepiped spanned by the vectors $v_1, \dots, v_r \in V$. Show that

$$|v_1 \wedge \cdots \wedge v_r|^2 = \sum_{J \in \mathcal{J}} \left|v_1^J \wedge \cdots \wedge v_r^J\right|^2 .$$

A.9.6 Interior Products

Let $i : V \to V^{**}$ be the canonical inclusion. Thus for each $v \in V$, we have a linear map

$$i(v) : V^* \to \mathbb{R}$$

given by $i(v)(\xi) = \xi(v)$ for $\xi \in V^*$.

Proposition A.9.6.1 *For each $v \in V$, there exists a unique extension of $i(v) : V^* \to \mathbb{R} \subset \wedge^\bullet V^*$ as an anti derivation*

$$i(v) : \wedge^\bullet V^* \to \wedge^\bullet V^*.$$

This linear map is called the interior product*. We have*

$$i(v_1)i(v_2) = -i(v_2)i(v_1)$$

for any $v_1, v_2 \in V$.

A.9.6.2 Remark

If we regard $\xi \in \wedge^k(V^*)$ as an element of $(\wedge^k V)^* \simeq L^k_{alt}(V)$, then

$$\xi(v_1, \ldots, v_k) = i(v_k) \cdots i(v_1)\xi$$

for $v_1, \ldots, v_k \in V$.

A.9.6.3 Exercises

Let V be an n-dimensional real vector space.

(i) For $v_1, \ldots, v_k \in V$, define a map

$$i(v_1 \wedge \cdots \wedge v_k) := i(v_k) \cdots i(v_1)$$

on $\wedge^\bullet V^*$. Show that this map is well-defined. Thus for every $\alpha \in \wedge^\bullet V^*$ we have a map

$$i(\alpha) : \wedge^\bullet V^* \to \wedge^\bullet V^*.$$

Note that if $\alpha \in \wedge^p V$ and $\beta \wedge^q V$, then

$$i(\alpha \wedge \beta) = i(\beta) \circ i(\alpha) = (-1)^{pq} i(\alpha) \circ i(\beta).$$

(ii) Let $\omega \in \wedge^n(V^*)$ be an **orientation form**, i.e., $\omega \neq 0$. Show that

$$f^k : \wedge^k V \to \wedge^{n-k}(V^*), \quad \alpha \mapsto i(\alpha)\omega$$

is an isomorphism for every $k = 0, \ldots, n$. Show that, if $e_1, \ldots, e_n$ is a basis for V and $\epsilon^1, \ldots, \epsilon^n$ is the dual basis and $\omega = \epsilon^1 \wedge \cdots \wedge \epsilon^n$, then

$$f^1(v) = \sum_{i=1}^{n} (-1)^i v^i \, \epsilon^1 \wedge \cdots \wedge \widehat{\epsilon^i} \wedge \cdots \wedge \epsilon^n$$

where $v = \sum_i v^i e_i$.

(iii) Assume an inner product on V. For $\omega \in V^*$, let

$$e(\omega) : \wedge^\bullet V^* \to \wedge^\bullet V^*, \qquad \xi \mapsto \omega \wedge \xi.$$

Show that $e(\omega)$ is the adjoint of $i(\omega^\sharp)$ and

$$i(\omega^\sharp)e(\omega) + e(\omega)i(\omega^\sharp) = |\omega|^2 \operatorname{id}$$

on $\wedge^\bullet V^*$.

Show that for any $\alpha \in \wedge^\bullet V$,

$$i(\alpha) = e(\alpha^\flat)^\sharp$$

where $\alpha^b \in \wedge^\bullet V^*$ is the dual of α with respect to the inner product and $e(\xi)$ denotes the exterior multiplication.

1.

A.10 Hodge Duality

Suppose V is an oriented inner product space of dimension n. Let $\mathrm{vol} \in \wedge^n V$ be the "positive " unit vector, i.e., $\mathrm{vol} = e_1 \wedge \cdots \wedge e_n$ for any positively oriented orthonormal basis $e_1, \ldots, e_n$ of V.[3]

Define, for $1 \leq i_1 < \cdots < i_k \leq n$,

$$\star(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sgn}(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)\, e_{i_{k+1}} \wedge \cdots \wedge e_{i_n}$$

where $\{i_1, \ldots, i_k, i_{k+1}, \ldots, i_n\} = \{1, \ldots, n\}$.

Then by linearity we have the **Hodge star** map

$$\star : \wedge^k V \to \wedge^{n-k} V.$$

This map is well-defined, independent of the choice of basis, and characterized by the relation

$$\langle \omega, \eta \rangle\, \mathrm{vol} = \omega \wedge \star\eta$$

for any $\omega, \eta \in \wedge^\bullet V$.

A.10.0.1 Exercises

(i) Show that on $\wedge^p V$,

$$\star^2 = (-1)^{p(n-p)} = \begin{cases} -\,\mathrm{id} & \text{if } p \text{ is odd and } n \text{ is even} \\ +\,\mathrm{id} & \text{otherwise.} \end{cases}$$

(ii) For any $v \in V$, consider the linear map

$$v^\sharp : V \to \mathbb{R}, \qquad w \mapsto \langle v, w \rangle\,.$$

Let

$$\mathrm{int}_{v^\sharp} : \wedge^\bullet V \to \wedge^\bullet V$$

the linear map which extends $v^\sharp$ and has the anti-derivation property:

$$\mathrm{int}_{v^\sharp}(\omega \wedge \eta) = (\mathrm{int}_{v^\sharp}\, \omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{int}_{v^\sharp}\, \eta)$$

for any $\omega \in \wedge^p V$ and $\eta \in \wedge^\bullet V$. Show that

$$\star v = \mathrm{int}_{v^\sharp}(\mathrm{vol}).$$

(iii) Show that for any $\omega \in \wedge^r V$,

$$\mathrm{int}_{\omega^\sharp} = (-1)^{n(n-p)} \star \mathrm{ext}(\omega) \star : \wedge^p V \to \wedge^{p-r} V.$$

---

[3] There is a canonical isomorphism

$$dV : \wedge^n V \to \mathbb{R}$$

such that $dV(\mathrm{vol}) = 1$.

Appendix B

Quaternions and Octonions

B.1 Quaternions

Let $\mathbb{I}$ be a 3-dimensional vector space with a basis i, j, k, and let

$$\mathbb{H} := \mathbb{R} \oplus \mathbb{I}.$$

We have an associative multiplication map on $\mathbb{H}$ with the relation

$$i^2 = j^2 = k^2 = ijk = -1.$$

With this structure an element of $\mathbb{H}$ is called a **quaternion**. Quaternions were discovered in 1843 by Willian Rowan Hamilton (1805−1865).[1]

A quaternion q is a some of real number t and a vector $v \in \mathbb{I}$, in a unique way. We define the conjugation of q as

$$\bar{q} := t - v.$$

An element of $\mathbb{I}$is called a **pure quaternion** and is characterized as a quaternion q such that

$$\bar{q} = -q.$$

There is a natural inclusion

$$\mathbb{C} \hookrightarrow \mathbb{H}$$

and

$$\mathbb{H} = \mathbb{C} \oplus \hat{\mathbb{C}}$$

where $\hat{\mathbb{C}}$ is the space generated by j and k. Note that

$$\mathbb{C} \cdot \hat{\mathbb{C}} = \hat{\mathbb{C}}, \quad \hat{\mathbb{C}} \cdot \mathbb{C} = \hat{\mathbb{C}}, \qquad \hat{\mathbb{C}} \cdot \hat{\mathbb{C}} = \mathbb{C}$$

and for any $z \in \mathbb{C}$ and $q \in \hat{\mathbb{C}}$

$$zq = q\bar{z}.$$

[1] Gauss's discovery of quaternions in 1819 was not published until 1900.

If we identify $(z, w) \in \mathbb{C}^2$ with $z + jw \in \mathbb{H}$, then

$$\overline{(z, w)} \cdot (z', w') = (\bar{z}z' + \bar{w}w', zw' - wz')$$

i.e., the quaternion inner product is the sum of Hermitian inner product and the symplectic product.

B.2 Octonions

Let $\mathbb{I}$ be a 7-dimensional vector space over $\mathbb{R}$ with a basis $e_1, \ldots, e_7$, and let

$$\mathbb{O} = \mathbb{R} \oplus \mathbb{I}.$$

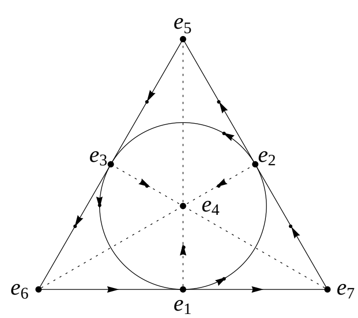
Define a (non-associative) bilinear multiplication map

$$\mathbb{O} \times \mathbb{O} \to \mathbb{O} \tag{B.1}$$

such that $e_0 = 1 \in \mathbb{R} \subset \mathbb{O}$ is the identity element and the multiplication table is as follows:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

This table may be drawn as the next figure:[2]



The elements of $\mathbb{O}$ (with this multiplication) are called **octonions** or **octaves**. Note that

$$e_\mu^2 = -1, \qquad e_\mu \cdot e_\nu = -e_\nu \cdot e_\mu \quad (1 \leq \mu < \nu \leq 7).$$

With $e_1 = i$, $e_2 = j$, $e_3 = k$, we have an inclusion

$$\mathbb{H} \to \mathbb{O}$$

of algebras and

$$\mathbb{O} = \mathbb{H} \oplus \hat{\mathbb{H}}$$

where $\hat{\mathbb{H}}$ is the space spanned by

$$e_4, \quad e_5 = ie_4, \quad e_6 = je_4, \quad e_7 = ke_4.$$

[2] Thus figure is drawn on the Gino Fano(1871–1952)'s projective plane.

We have

$$\mathbb{H}\cdot\mathbb{H}=\mathbb{H},\quad \mathbb{H}\cdot\hat{\mathbb{H}}=\hat{\mathbb{H}},\quad \hat{\mathbb{H}}\cdot\mathbb{H}=\hat{\mathbb{H}},\quad \hat{\mathbb{H}}\cdot\hat{\mathbb{H}}=\mathbb{H}$$

and following identities:

$$qp = p\bar{q},\quad q_1(q_2p)=(q_2q_1)p,\quad (pq_1)q_2 = p(q_2q_1),$$
$$(q_1p_1)(p_2q_2) = q_2(p_1p_2)q_1,\quad (p_1p)p_2 = p_2(pp_1)$$

for any quaternions q, q_1, q_2 and $p, p_1, p_2 \in \hat{\mathbb{H}}$.

Now the multiplication map (B.1) becomes

$$(q_1,p_1)(q_2,p_2) = (q_1q_2 - \overline{p_2}\,p_1, p_2q_1 + p_1\,\overline{q_2}).$$

The conjugation on $\mathbb{H}$ extends to $\mathbb{O}$ by

$$\overline{(q,p)} = (\overline{q}, -p).$$

Then

$$\overline{uv} = \bar{v}\,\bar{u}$$

for any octonions u and v. We have octonion inner product

$$\langle u \mid v\rangle := \bar{u}v$$

and

$$|u|^2 := \langle u \mid u\rangle > 0$$

for nonzero u. Thus $\mathbb{O}$ is a division algebra.

We also have

$$|uv| = |u|\;|v|$$

for any $u, v \in \mathbb{O}$. Note that for any $u \in \mathbb{O}$

$$\langle u, ue_\mu\rangle = |u|^2 e_\mu = -\langle ue_\mu, u\rangle$$
$$\mathrm{Re}\,\langle ue_\mu, ue_\nu\rangle = |u|^2\delta_{\mu\nu} = \mathrm{Re}\,\langle e_\mu u, e_\nu u\rangle$$

for $1 \le \mu, \nu \le 7$.

We have the **alternative law**

$$u(uv) = (uu)v, \qquad u(vv) = (uv)v$$

for any $u, v \in \mathbb{O}$. Note that the alternative law implies

$$(uv)u = u(vu)$$

for any $u, v \in \mathbb{O}$.

Here is a multiplication table (with respect to the basis $1, i, j, k, l, il, jl, kl$):

v	$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$
iv	$(-a_1, a_0, -a_3, a_2, -a_5, a_4, a_7, -a_6)$
jv	$(-a_2, a_3, a_0, -a_1, -a_6, -a_7, a_4, a_5)$
kv	$(-a_3, -a_2, a_1, a_0, -a_7, a_6, -a_5, a_4)$
lv	$(-a_4, a_5, a_6, a_7, a_0, -a_1, -a_2, -a_3)$
$(il)v$	$(-a_5, -a_4, a_7, -a_6, a_1, a_0, a_3, -a_2)$
$(jl)v$	$(-a_6, -a_7, -a_4, a_5, a_2, -a_3, a_0, a_1)$
$(kl)v$	$(-a_7, a_6, -a_5, -a_4, a_3, a_2, -a_1, a_0)$
vi	$(-a_1, a_0, a_3, -a_2, a_5, -a_4, -a_7, a_6)$
vj	$(-a_2, -a_3, a_0, a_1, a_6, a_7, -a_4, -a_5)$
vk	$(-a_3, a_2, -a_1, a_0, a_7, -a_6, a_5, -a_4)$
vl	$(-a_4, -a_5, -a_6, -a_7, a_0, a_1, a_2, a_3)$
$v(il)$	$(-a_5, a_4, -a_7, a_6, -a_1, a_0, -a_3, a_2)$
$v(jl)$	$(-a_6, a_7, a_4, -a_5, -a_2, a_3, a_0, -a_1)$
$v(kl)$	$(-a_7, -a_6, a_5, a_4, -a_3, -a_2, a_1, a_0)$

B.2.0.1 Cross Product

Let $\mathbb{I}$ be the 7-dimensional space of pure imaginary octonions. Then the **cross product** of two $u, v \in \mathbb{I}$ is defined by

$$u \times v := \frac{1}{2}(uv - vu) \in \mathbb{I}.$$

Appendix C

Calculus

C.1 Differentiable Functions

Let V and W be finite dimensional vector spaces over $\mathbb{R}$.[1] A function $\epsilon : V \dashrightarrow W$ defined in a neighborhood of the origin 0 of V such that

$$\lim_{v \to 0} |\epsilon(v)|/|v| = 0 \qquad \text{and} \qquad \epsilon(0) = 0$$

is denoted by $o(v)$, where we use any norms $|\cdot|$ on V and W.

Let U be an open subset of V. A map

$$f : U \to W \tag{C.1}$$

is **differentiable** at a point p in U if there exists a linear map

$$Df(p) : V \to W$$

such that

$$f(p+v) - f(p) = Df(p)v + o(v).$$

If f is differentiable at p, then the linear map $Df(p)$ is unique and is called the **derivative** of f at p. Moreover, f is continuous at p and

$$D_v f(p) := Df(p)v = \left.\frac{d}{dt}\right|_0 f(p+tv)$$

for any $v \in V$.

C.1.0.1 Coordinate Expressions

C.1.0.2

The map (C.1) is said to be **differentiable** if it is differentiable at every point of its domain. In this case, we have a map

$$Df : U \to L(V, W),$$

[1] We may assume that V and W are Banach spaces.

where $L(V, W)$ is the vector space of linear maps from V to W.

C.1.0.3 Exercises

1. Show that if $f : V \to W$ is a linear map, then
$$Df(p) = f$$
for every p in V.

2. Let V_1, V_2, W be finite dimensional vector spaces. Show that if $f : V_1 \times V_2 \to W$ is a bilinear map, then
$$Df(u_1, u_2)(v_1, v_2) = f(v_1, u_2) + f(u_1, v_2)$$
for $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Find a formula for multilinear maps.

C.1.1 Chain Rule

Let V, W, X be finite dimensional vector spaces. Let V_1 and W_1 be open subsets of V and W, respectively. Suppose that $f : V_1 \to W_1 \subset W$ is differentiable at $p \in V_1$ and $g : W_1 \to X$ is differentiable at $q := f(p)$. Then $g \circ f : V_1 \to X$ is differentiable at p and
$$D(g \circ f)(p) = Dg(q) \circ Df(p) : V \to W \to X.$$

C.1.2 Continuously Differentiable Functions

Let U be an open subset of V. A differentiable map (C.1) is said to be (of class) C^1 if
$$Df : U \to L(V, W)$$
is continuous.

Theorem C.1.2.1 *Let $v_1, \ldots, v_n$ be a basis of V. A differentiable map (C.1) is C^1 if and only if the directional derivatives*
$$D_{v_1} f, \ldots, D_{v_n} f : U \to W$$
are continuous.

C.1.3 Many times Differentiable Functions

C.1.3.1 C^k functions

A continuous function is said to be C^0. For positive integers k, a map (C.1) is said to be C^k if Df is C^{k-1}.

Theorem C.1.3.2 *If (C.1) is C^2 at a point p, then*

$$D^2 f(p) \in L^2(V, W)$$

is symmetric.

Theorem C.1.3.3 *Let $v_1, \ldots, v_n$ be a basis of V. The map (C.1) is C^k if and only if for any nonnegative integers $\alpha_1, \ldots, \alpha_n$ satisfying the condition $\alpha_1 + \cdots + \alpha_n = k$, the partial derivatives*

$$D_{v_1}^{\alpha_1} \cdots D_{v_n}^{\alpha_n} f : U \to W$$

exist and are continuous.[2]

C.1.4 Infinitely differentiable functions

Let U be an open subset of a finite dimensional vector space V. A differentiable function $f : U \to \mathbb{R}$ is said to be $\mathcal{C}^1$ if for any $v \in V$, $D_v f \in \mathcal{C}^0(U)$.

Let $\mathcal{C}^1(U)$ be the space of all $\mathcal{C}^1$ functions on U. Then a differentiable function $f : U \to \mathbb{R}$ is said to be $\mathcal{C}^2$ if $D_v f \in \mathcal{C}^1(U)$ for any $v \in V$.

Inductively, a function f on U is $\mathcal{C}^{k+1}$ if $D_v f \in \mathcal{C}^k(U)$ for any $v \in V$.

A function f on U is infinitely differentiable if it is $\mathcal{C}^k$ for every positive integer k. Then f is $\mathcal{C}^\infty$ if ${D_v}^k f$ exists for every $k = 0, 1, 2, \ldots$.

C.1.4.1 Taylor's Theorem

Lemma C.1.4.2 *Let U be a convex open neighborhood of the origin in $\mathbb{R}^n$.*

(i) Let $f \in \mathcal{C}^k(U)$ for some $k \in \{1, 2, \ldots\}$. Then there exist $g_1, \ldots, g_n \in \mathcal{C}^{k-1}(U)$ such that

$$f(x) = f(0) + x^1 g_1(x) + \cdots + x^n g_n(x)$$

for any $x \in U$. In this case

$$g_i(0) = D_i f(0), \quad i = 1, \ldots, n.$$

(b) Let $f \in \mathcal{C}^\infty(U)$. Then there exist $h_{ij} = h_{ji} \in \mathcal{C}^\infty(U)$, $1 \leq i, j \leq n$, such that

$$f(x) = f(0) + \sum_i D_i f(0) x^i + \frac{1}{2} \sum_{i,j} x^i x^j h_{ij}(x)$$

for all $x \in U$.

[2] The number of elements of the set $\{(\alpha_1, \ldots, \alpha_n) \in \{0, \ldots, k\} \mid \alpha_1 + \cdots + \alpha_n = k\}$ is $\binom{n+k-1}{k}$.

Proof. Proof is obvious from the identity:

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx) dt = \int_0^1 \sum_{i=1}^n x_i D_i f(tx)\, dt = \sum_i x_i \int_0^1 D_i f(tx)\, dt.$$

□

C.2 Inverse Function Theorem

Theorem C.2.0.1 (Inverse Function Theorem)

Theorem C.2.0.2 (Implicit Function Theorem) *Let M, N, L be smooth manifolds,*

$$F : M \times N \to L$$

be a C^1 map, $(p, q) \in M \times N$, and $r = F(p, q)$. If

$$(T_2 F)_{(p,q)} : TN_q \to TL_r$$

is an isomorphism, then there exist an open neighborhood U of p in M, an open neighborhood V of q in N, and a C^1 map

$$G : U \to V$$

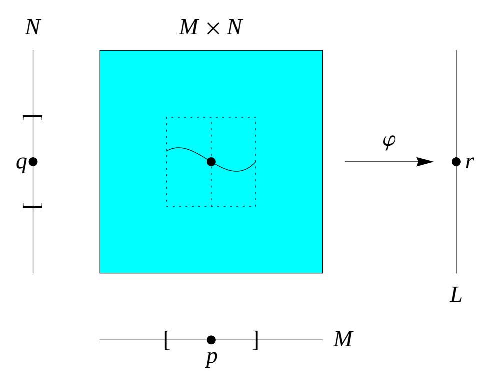
such that for any $(x, y) \in U \times V$

$$F(x, y) = r \qquad \Longleftrightarrow \qquad y = G(x).$$

In this case,

$$TG_p = -(T_2 F_{(p,q)})^{-1} \circ T_1 F_{(p,q)}.$$

Moreover, if F is C^k for some positive integer k, so is G.



Proof: Consider the map

$$F : X \times Y \to X \times Z, \quad (x, y) \mapsto (x, f(x, y)).$$

Then

$$DF(x_0, y_0) : T_{(x_0,y_0)}(X \times Y) \to T_{(x_0,z_0)}(X \times Z)$$

is an isomorphism. Thus there exist an open neighborhood U of x_0 in X, an open neighborhood V of y_0 in Y, an open neighborhood W of z_0 in Z, and a C^1 map

$$G : W \to U \times V$$

which is the inverse of F. Note that

$$G(x, z) = (x, \pi_2(G(x, z)).$$

Let

$$g(x) := \pi_2(G(x, z_0))$$

for $x \in U$. Then $g : U \to V$ is the desired map.

Note that for any $x \in U$,

$$0 = Df(x, g(x)) = D_1 f(x, g(x)) + D_2 f(x, g(x)) \circ Dg(x) = 0$$

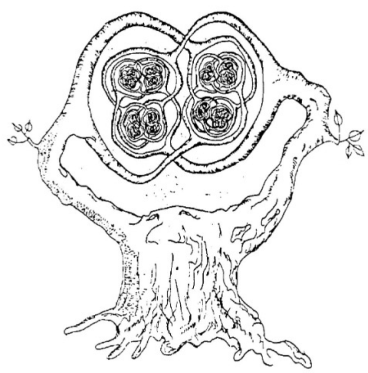
and hence

$$Dg(x) = -(D_2 f(x, g(x)))^{-1} D_1 f(x, g(x)).$$

Thus if f is C^k, then so is g.

Appendix D

General Topology



Alexander's Horned Sphere, drawn by Bill Mayers

D.1 Topological Spaces

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

1. $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$
2. if $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$
3. if S is a subset of $\mathcal{T}$, then $\bigcup S \in \mathcal{T}$.

A **topological space** is a set X together with a topology $\mathcal{T}$ on it. In this case, each element of $\mathcal{T}$ is called an **open** subset (or sometimes ``open set'') of $(X, \mathcal{T})$. Topological space $(X, \mathcal{T})$ is often denoted simply by X.

D.1.1 Trivial Topology

A **trivial topology** on a set X consists of the empty set and X.

D.1.2 Discrete Space

A topological space is **discrete** if every subset is open. This topology is called the discrete topology.

D.2 Continuous Maps

A map $f : X \to Y$ between topological spaces is said to be **continuous** if $f^{-1}(V)$ is open in X for any open set V in Y.

The set of all continuous maps from X into Y is denoted by $C(X, Y)$.

The set of all continuous maps from X into $\mathbb{R}$ is often simply denoted by $C(X)$. A continuous map $f : X \to Y$ induces an algebra homomorphism

$$f^* : C(Y) \to C(X), \quad g \mapsto g \circ f.$$

$$\boxed{\text{topology}} \quad \Leftrightarrow \quad \boxed{\text{continuous functions}}$$

D.3 Subspace

If Y is a subset of a topological space $(X, \mathcal{T})$, then

$$\mathcal{T} \restriction Y := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y. This topology is called the **subspace topology**. Thus a subset V of Y is open if and only if $V = Y \cap U$ for some open subset U of X. A subset K of Y is closed if and only if $K = Y \cap L$ for some closed subset L of X.

Note that the subspace topology is the smallest topology[1] such that the inclusion map

$$\mathrm{inc} : Y \hookrightarrow X$$

is continuous. A map $f : Z \to Y$ is continuous if and only if the composition

$$\mathrm{inc} \circ f : Z \to Y \hookrightarrow X$$

is continuous.

D.4 Category

A **category** $\mathcal{C}$ consists of a class Obj$(\mathcal{C})$ of **objects** and a set of **morphisms** Mor(X, Y) for each pair of objects X and Y such that

1. For any objects X, Y, Z, the **composition** map

   $$\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z), \quad (f, g) \mapsto g \circ f$$

   is given.

[1] A topology $\mathcal{T}_1$ on a set is *smaller* (or *weaker*) than the topology $\mathcal{T}_2$ on the same set if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

2. If $f \in \mathrm{Mor}(X,Y)$, $g \in \mathrm{Mor}(Y,Z)$, and $h \in \mathrm{Mor}(Z,W)$, then
$$(h \circ g) \circ g = h \circ (g \circ f).$$

3. For each object X, there exists an element $1_X \in \mathrm{Mor}(X,X)$ such that
$$f \circ 1_X = f, \qquad 1_X \circ g = g$$
for any $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Z,X)$.

A morphism $f \in \mathrm{Mor}(X,Y)$ is called an **isomorphism** if there exists $g \in \mathrm{Mor}(Y,X)$ such that
$$f \circ g = 1_Y, \qquad g \circ f = 1_X.$$

D.4.1 TOP

The objects of the category **TOP** are topological spaces, and for each pair of objects X and Y, $\mathrm{Mor}(X,Y)$ consists of continuous maps from X into Y.

Compositions of two continuous maps are continuous.

An isomorphism is called a **homeomorphism**.

D.5 Sheaves

Let $\mathbf{G}$ be the category of all groups and the group-homomorphisms.

A **presheaf** $\mathcal{S}$ of groups on a topological space $(X,\mathcal{T})$ is an assignment
$$\mathcal{S} : \mathcal{T} \to \mathsf{Obj}(\mathbf{G}), \quad U \mapsto \mathcal{S}(U)$$
together with **restriction maps**
$$r_U^V : \mathcal{S}(V) \to \mathcal{S}(U)$$
for each inclusion $U \hookrightarrow V$ such that

1. for any open sets $U \subset V \subset W$
$$r_U^V \circ r_V^W = r_U^W.$$

2. for any open set U, $r_U^U : \mathcal{S}(U) \to \mathcal{S}(U)$ is the identity map.

3. $\mathcal{S}(\varnothing)$ is the trivial group.

D.5.1 Sheaf

A presheaf $\mathcal{S}$ of groups on a topological space X is a **sheaf** if

1. for any collection $\{U_\alpha\}$ of open sets and for any f, g in $\mathcal{S}(U)$, the condition
$$\forall\alpha, \quad f \restriction U_\alpha = g \restriction U_\alpha,$$
implies $f = g$, where $U = \bigcup_\alpha U_\alpha$, $f \restriction U_\alpha = r^U_{U_\alpha}(f)$, and $g \restriction U_\alpha = r^U_{U_\alpha}(g)$.

2. for any collection of $\{f_\alpha \in \mathcal{S}(U_\alpha)\}$ such that
$$(f_\alpha) \restriction (U_\alpha \cap U_\beta) = (f_\beta) \restriction (U_\alpha \cap U_\beta)$$
for any indices α and β, there exists an element $f \in \mathcal{S}(U)$ such that $f \restriction U_\alpha = f_\alpha$.

D.5.2 The sheaf of continuous functions

For any open set U on a topological space X, let
$$\mathcal{C}(U)$$
be the algebra of all continuous real valued functions on U. Then we have a sheaf $\mathcal{C}$ of continuous functions on X.

D.6 Metric Space

Given a set X, a map
$$d : X \times X \to \mathbb{R}$$
is called a **metric** (or a **distance** function) if

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$.

A set X together with a metric d is called a **metric space**.[2]

[2] This beautiful definition is due to M. Fréchet(1878–1973) in his doctorial thesis,1906 [BvR].

D.6.1

Given a positive real number r, and a point p in a metric space (X, d), an **open ball** of radius r centered at p is

$$B(p, r) := \{q \in X \mid d(p, q) < r\}.$$

A subset of a metric space is **open** if it is a union of open balls.

These open sets define a topology on X.

Thus every metric space is a topological space. For instance, $\mathbb{R}^n$ with the Euclidean metric is a topological space.

D.6.2

If d is a metric for a set X, then

$$\tilde{d} := \frac{d}{1+d}$$

is also a metric for X such that the distance between any two points on X is less than 1. Note that $\tilde{d}$ gives the same topology as d on X.

D.6.3

A topological space is **metrizable** if there exists a metric whose topology is equal to the given topology.

D.7 Neighborhoods

For a point p in a topological space X, a subset N of X is called a **neighborhood** of p if there exists an open subset U such that $p \in U \subset N$.

In a topological space X, let $\mathcal{N}_p$ be the collection of all neighborhoods of p in X. Then the family $\mathcal{N} := \{\mathcal{N}_p \mid p \in X\}$ satisfies the following conditions for all $p \in X$:

1. $X \in \mathcal{N}_p$ (or $\mathcal{N}_p \neq \varnothing$)
2. for any $U \in \mathcal{N}_p$, $p \in U$ (i.e., $p \in \bigcap \mathcal{N}_p$)
3. $U_1, U_2 \in \mathcal{N}_p \quad \Rightarrow \quad U_1 \cap U_2 \in \mathcal{N}_p$
4. $U \in \mathcal{N}_p,\ U \subset V \subset X \quad \Rightarrow \quad V \in \mathcal{N}_p$
5. for any $U \in \mathcal{N}_p$, $\{q \in U \mid U \in \mathcal{N}_q\} \in \mathcal{N}_p$ (i.e., for any $U \in \mathcal{N}_p$, there exists $V \in \mathcal{N}_p$ such that $U \in \mathcal{N}_q$ for any $q \in V$)

Given a set X, a **neighborhood system** of X is an indexed family $\mathcal{N} = \{\mathcal{N}_p \mid p \in X\}$ of collections of subsets of X which satisfies the above five conditions.

A neighborhood system $\mathcal{N}$ of a set X determines a unique topology on X such that for every p in X, $\mathcal{N}_p$ is the collection of neighborhoods of p. (cf. [Hart et al.], [Bredon])

$$\boxed{\text{topology}} \quad \Leftrightarrow \quad \boxed{\text{neighborhood system}}$$

D.8 Closed Sets

A subset K of a topological space X is said to be **closed** if its complement $X - K$ is an open subset of X.

A map $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for any closed subset C of Y.

D.8.1 Closure

The **closure** of a subset S in a topological space X is the smallest closed subset in X which contains S.

A subset of a topological space is **dense** if its closure is the whole space.

A topological space is **separable** if it has a countable dense subset.

D.8.2 Interior, exterior, and boundary

For a topological space X and a subset A of X, the **interior** of A is the largest open set of X contained in A. The **exterior** of A is the largest open set of X contained in $X - A$. The interior and the exterior of A is denoted by $\operatorname{int} A$ and $\operatorname{ext} A$, respectively. Thus

$$\operatorname{ext} A = \operatorname{int}(X - A), \qquad \operatorname{int} A = \operatorname{ext}(X - A).$$

The **boundary** of A is the intersection of the closures of $\operatorname{int} A$ and $\operatorname{ext} A$.

D.9 Cover

A collection $\mathcal{U}$ of sets is **locally finite** in a topological space X if for any point $p \in X$, there exists a neighborhood N of p such that the number of sets $U \in \mathcal{U}$ such that $U \cap N \neq \varnothing$ is finite.

A collection $\mathcal{U}$ of sets is a **cover** of a set X if $X \subset \bigcup \mathcal{U}$.

A cover $\mathcal{V}$ of a set X is a **subcover** of a cover $\mathcal{U}$ if $\mathcal{V} \subset \mathcal{U}$.

A cover $\mathcal{V}$ of X is a **refinement** of a cover $\mathcal{U}$ of X if for any $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$.

D.9.1 Indexed family

A family $(U_\alpha)_{\alpha\in A}$ of sets indexed by a set A is just a set-valued function U with domain A.

An indexed family $(U_\alpha)_{\alpha\in A}$ of subsets of a topological space X is said to be locally finite if for any point $p \in X$, there exists a neighborhood N of p such that the number of indices $\alpha \in A$ such that $U_\alpha \cap N \neq \varnothing$ is finite.

An indexed family $(U_\alpha)_{\alpha\in A}$ of subsets of a topological space X is said to be point-finite if for any point $p \in X$, the number of indices $\alpha \in A$ such that $p \in U_\alpha$ is finite.

A locally finite family is point-finte.

An indexed cover of a set X is an indexed family $(U_\alpha)_{\alpha\in A}$ such that

$$X \subset \bigcup_{\alpha\in A} U_\alpha := \bigcup\{U_\alpha \mid \alpha \in A\}.$$

An ordinary cover is an indexed cover with the trivial (i.e., identity) indexing.

An indexed cover $(V_\beta)_{\beta\in B}$ of a set X is a subcover of an indexed cover $(U_\alpha)_{\alpha\in A}$ if $B \subset A$.

An indexed cover $(V_\beta)_{\beta\in B}$ of X is a refinement of an indexed cover $(U_\alpha)_{\alpha\in A}$ of X if for any $\beta \in B$ there exists a $\alpha \in A$ such that $V_\beta \subset U_\alpha$.

D.10 Compact Space

A topological space X is said to be compact if every open cover of X has a finite subcover.

Theorem D.10.0.1 1. *A closed subset of a compact space is compact.*

2. *A compact subspace of a Hausdorff space is closed.*
3. *A continuous image of a compact space is compact.*
4. *Any continuous real valued map from a compact space has a maximum.*
5. *Every compact subset of a metric space is bounded.*

D.11 Paracompact Space

A Hausdorff space X is paracompact[3] if every open cover $\mathcal{U}$ of X has a locally finite open cover which is a refinement of $\mathcal{U}$.

Observation D.11.0.1 If X is a paracompact space, then for any indexed open cover $(U_\alpha)_{\alpha\in A}$ of X, there exists a locally finite indexed family $(V_\alpha)_{\alpha\in A}$ of open subsets of X such that $\bar{V}_\alpha \subset U_\alpha$ for each $\alpha \in A$.

[3]This definition is due to J. Dieudonné (1944). The line with two origin is non-Hausdorff paracompact and has no property of the partition of unity.

A closed subspace of a paracompact space is paracompact.
Compact Hausdorff spaces are paracompact.
By a theorem of A. H. Stone (1948), metrizable spaces are paracompact.

D.11.1 Partition of unity

A **partition of unity** subordinate to an open cover $\mathcal{U}$ of a topological space X is a collection of continuous functions $\varphi_U : X \to \mathbb{R}$ for each $U \in \mathcal{U}$ such that

1. for each $U \in \mathcal{U}$, $\varphi_U \geq 0$
2. for each $U \in \mathcal{U}$, $\operatorname{supp} \varphi_U \subset U$
3. the collection $\{\operatorname{supp} \varphi_U \mid U \in \mathcal{U}\}$ is a locally finite
4. $\sum_{U \in \mathcal{U}} \varphi_U \equiv 1$

In this case, the collection

$$\{\operatorname{int}(\operatorname{supp} \varphi_U) \mid U \in \mathcal{U}\}$$

is a locally finite open cover of X which is a refinement of $\mathcal{U}$.

A **partition of unity** subordinate to an indexed open cover $(U_\alpha)_{\alpha \in A}$ of a topological space X is a collection of continuous functions $\varphi_\alpha : X \to \mathbb{R}$ for each $\alpha \in A$ such that

1. for each $\alpha \in A$, $\varphi_\alpha \geq 0$
2. for each $\alpha \in A$, $\operatorname{supp} \varphi_\alpha \subset U_\alpha$
3. the indexed family $(\operatorname{supp} \varphi_\alpha)_{\alpha \in A}$ is a locally finite
4. $\sum_{\alpha \in A} \varphi_\alpha \equiv 1$

Theorem D.11.1.1 *Given a topological space X, the following conditions are equivalent:*

1. *X is paracompact.*
2. *For any indexed open cover $(U_\alpha)_{\alpha \in A}$ of X, there exists a partition of unity subordinate to $(U_\alpha)_{\alpha \in A}$.*
3. *For any open cover $\mathcal{U}$ of X, there exists a partition of unity subordinate to $\mathcal{U}$.*

Corollary D.11.1.2 *Let X be a paracompact space. If A and B are disjoint closed subsets of X, then there exists a continuous function $f : X \to \mathbb{R}$ such that $f \restriction A \equiv 0$ and $f \restriction B \equiv 1$.*

D.12 Separation Axioms

A topological space X is said to be

1. T_1 if every point is closed.
2. **Hausdorff** (or T_2) if for any distinct points p and q in X, there exist a neighborhood U_p of p and a neighborhood U_q of q such that U_p and U_q are disjoint.

   On a Hausdorff space, every sequence (or a **net**) of points has at most one limit point.

   A topological space X is Hausdorff if and only if the diagonal

   $$\{(x, x) \mid x \in X\}$$

   is a closed subset of $X \times X$ (cf. D.14).
3. **regular** if for any point p in X and a closed subset K of X which does not contain p, there exist disjoint open sets U and V of X such that $p \in U$ and $K \subseteq V$. A regular Hausdorff space is called a T_3-space.
4. **normal** if for any disjoint closed sets K_1 and K_2, there exist disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.

   A normal Hausdorff space is called a T_4-space.

Note that a topological space X is normal if and only if for any pair of a closed subset C and a neighborhood U of C there exists a neighborhood V of C such that the closure $\bar{V}$ is contained in U.

For subsets A and B of a topological space X, we write $A \ll B$ if $\bar{A} \subset \operatorname{int} B$. Thus for any pair (A, B) of subsets of a normal space X with $A \ll B$, there exists a subset V of X such that $A \ll V \ll B$.

Theorem D.12.0.1 *For a Hausdorff space X, the following are equivalent:*

(i) X is normal.

(ii) (Urysohn's Lemma) If A and B are nonempty disjoint closed subsets of X, then there exists a continuous function $f : X \to [0, 1]$ such that $f \restriction A \equiv 0$ and $f \restriction B \equiv 1$.

(iii) (Tietze extension theorem) For any closed set $A \subset X$ and a continuous function $f : A \to [0, 1]$, there is a continuous extension $\tilde{f} : X \to [0, 1]$ of f.

(iv) (Shrinking Lemma)[4] If $\{U_1, \ldots, U_k\}$ is an open cover of X, there exists an open cover $\{V_1, \ldots, V_k\}$ of X such that for each $i \in \{1, \ldots, k\}$ the closure of V_i is contained in U_i.

[4] If the Axiom of Choice is allowed, we may use arbitrary point-finite open cover for the shrinking lemma.

A **support** $\operatorname{supp} f$ of a function $f : X \to \mathbb{R}$ is the closure in X of the set $\{x \in X \mid f(x) \neq 0\}$. Let A be a closed subset of a topological space X and let U be a neighborhood of A in X. Then a continuous function $f : X \to [0,1]$ is a **bump function** for A supported in U if $f \restriction A \equiv 1$ and $\operatorname{supp} f \subset U$.

Corollary D.12.0.2 *Let X be a normal Hausdorff space, A be a closed subset of X, and U be a neighborhood of A. Then there exists a bump function for A supported in U.*

Proof. Let V be an open neighborhood of A such that $\bar{V} \subset U$. Then A and $X - V$ are disjoint closed subsets of X. Thus there exists a continuous function $f : X \to [0,1]$ such that $f \restriction A \equiv 1$ and $f \restriction (X - V) \equiv 0$. Thus the set $\{x \in X \mid f(x) \neq 0\}$ is contained in V and hence $\operatorname{supp} f \subset \bar{V} \subset U$. This completes the proof. □

By D.11.1.2, paracompact spaces are normal.

$$\boxed{\text{metrizable}} \Rightarrow \boxed{\text{paracompact}} \Rightarrow \boxed{T_4} \Rightarrow \boxed{T_3} \Rightarrow \boxed{T_2} \Rightarrow \boxed{T_1}$$

D.13 Bases for topology

A collection $\mathcal{B}$ of open sets of a topological space X is a **basis** (or a **base**) for the topology if any open set in X is a union of elements in $\mathcal{B}$.

A topological space is **second countable** if it has a countable basis. Euclidean spaces are typical examples of second countable spaces.

Any second countable topological space X is **first countable**, i.e., each point in X has a countable basis for its neighborhoods.

Any second countable topological space is separable.

A second countable space X has the **Lindelöf property**, i.e., any open cover of X has a countable subcover.

D.14 Product Spaces

If $\{X_\alpha\}$ is an indexed family of topological spaces, then the topology of the product $X := \prod_\alpha X_\alpha$ is the smallest topology such that the projection maps $\pi_\alpha : X \to X_\alpha$ are all continuous. Thus the collection

$$\left\{ \prod_\alpha U_\alpha \mid U_\alpha \text{ is an open subset of } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for almost all } \alpha \right\}$$

is a basis for the topology of X.

This topology is the unique one such that a map $f : Y \to X$ is continuous if and only if the composition $\pi_\alpha \circ f$ is continuous for all projection maps π_α.

D.15 Quotient Spaces

Let $\sim$ be an equivalence relation on a topological space X, and let $\pi : X \to X/\sim$ be the projection. Then a subset U of the quotient $X/\sim$ is defined to be open if (and only if) $\pi^{-1}(U)$ is open in X. This defines a **quotient topology** on the $X/\sim$, which is the finest[5] topology on the quotient which makes π continuous.

A map f from $X/\sim$ to a topological space Y is continuous if and only if the composition $f \circ \pi : X \to Y$ is continuous.

If $X/\sim$ is Hausdorff, then the graph of the relation

$$\Gamma := \{(x, y) \in X \times X \mid x \sim y\}$$

is a closed subset of $X \times X$.

If Γ is closed in $X \times X$ and the projection map $\pi : X \to X/\sim$ is open, then $X/\sim$ is Hausdorff.

D.16 Connectedness

A topological space is **connected** if it is not the union of two disjoint nonempty open subsets, or equivalently, it is not the union of two disjoint nonempty closed subsets.

A space is connected if and only if there are no closed-and-open subsets except the whole space and the empty set.

A space X is connected if and only if any continuous map $f : X \to \{0, 1\}$ is constant.

If Y is a connected subset of a topological space X, then the closure $\bar{Y}$ of Y in X is also connected.

D.16.1 Connected Component

Two points in a topological space X are said to be in the same (**connected**) **component** if they are contained in the same connected subset of X. This is an equivalence relation and any equivalence class is a connected component.

Each connected component of X is a closed subset of X.

If X is **locally connected**, each connected component of X is also an closed subset of X.

D.16.2 Path Connectedness

A topological space X is **path connected** if for any two points p and q in X, there exists a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Path connected spaces are connected.

[5] If $\mathcal{T}_1, \mathcal{T}_2$ are two topologies on the same set, then $\mathcal{T}_2$ is *finer* (or *stronger* or *larger*) than $\mathcal{T}_1$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

A topological space X is **locally path connected** if for any neighborhood U of any point p in X, there exists a path connected neighborhood V of p such that $V \subset U$.

A connected and **locally path connected** space is path connected.

D.16.3 Simply connected space

Given a topological space X, two (continuous) paths

$$f, g : [0,1] \to X, \qquad f(0) = g(0), \quad f(1) = g(1)$$

are **path-homotopic** if there exists a continuous map

$$F : [0,1] \times [0,1] \to X$$

such that

$$F(0,t) = f(t), \qquad F(1,t) = g(t), \qquad F(t,0) = f(0), \quad F(t,1) = f(1)$$

for all $t \in [0,1]$.

A path connected space X is **simply connected** if any two paths with the same initial and the same terminal points are path-homotopic.

D.17 Invariance of Domain

Proof of the following theorem of L. E. J. Brouwer(1881–1966) may be found in many books on algebraic topology.[6]

Theorem D.17.0.1 (Invariance of Domain, 1910) *A one-to-one continuous image in $\mathbb{R}^n$ of an open subset of $\mathbb{R}^n$ is open.*

Corollary D.17.0.2 *If a non-empty open subset of $\mathbb{R}^n$ and an open subset of $\mathbb{R}^m$ are homeomorphic, then $n = m$.*

Proof. Suppose that U is a non-empty open subset of $\mathbb{R}^n$ and V is an open subset of $\mathbb{R}^m$ which is homeomorphic to U. Let $f : U \to V$ be a homeomorphism. If $m < n$, then The composition

$$U \xrightarrow{f} V \hookrightarrow \mathbb{R}^m \hookrightarrow \mathbb{R}^n$$

is a continuous injection and hence V must be open in $\mathbb{R}^n$ by the Invariance of Domain. But $\mathbb{R}^m$ is nowhere dense[7] in $\mathbb{R}^n$, and hence we have a contradiction. Thus $n \geq m$. By symmetry, $m \geq n$ and hence $n = m$. □

[6] E.g., [GH], or `https://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/`

[7] A subset of a topological space is *nowhere dense* if the interior of its closure is empty. Thus the complement of a nowhere dense subset is dense.

D.18 Covering Spaces

Given a continuous map $q : X \to Y$, a subset V of Y is said to be **evenly covered** by q if $q^{-1}(V)$ is the union of family (U_α) of disjoint subsets of X such that

$$q \restriction U_\alpha : U_\alpha \to V$$

is a homeomorphism for all index α.

A continuous map $q : X \to Y$ is called a **covering map** if every point in Y has an evenly covered neighborhood.

If $q : X \to Y$ is a covering map, then every point $y \in Y$ has an evenly covered **open** neighborhood V and $q^{-1}(V)$ is a union of disjoint **open** subsets of X, each of them are homeomorphic to V.

A covering map is a local homeomorphism. It is a surjective open map, and hence is a quotient map.

A connected, locally path connected, simply connected space X is called a **universal covering space** of Y if there exists a covering map from X onto Y.

Every connected and locally simply connected space has a universal covering space.

Let G be a group of homeomorphisms on a topological manifold M such that for any point p in M there exists a neighborhood U of p such that

$$\{g \in G \mid U \cap gU \neq \varnothing\} = \{\mathrm{id}_M\}.$$

Then the quotient map

$$M \to M/G$$

is a normal (i.e., regular) covering map [cf. Massey, p.165], and G is the group of covering transformations.

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List of Symbols

0, the origin
$\mathbb{C}$, the field of complex numbers
$\mathbb{H}$, the division ring of quaternions
$\mathbb{O}$, the octonions
$\mathbb{Q}$, the field of rational numbers
$\mathbb{R}$, the field of real numbers
$\mathbb{Z}$, the ring of integers
$\mathbb{R}^n$, the Cartesian n-space
$\mathbb{R}^n_+$, the closed upper half n-space

$\bar{\mathbf{B}}^n(c, r)$, closed n-ball with center c and radius r
$\mathbf{H}^n$, the hyperbolic n-space
$\mathbf{P}^n$, the projective n-space
$\mathbf{S}^n$, the n-sphere
$\mathbf{T}^n := \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$, the n-torus

$\subset, \subseteq$, subset
$\subsetneq$, proper subset
1_n, the identity matrix of rank n
id, identity map
inc, $\hookrightarrow$, inclusion map
$M_{n,k}$, the space of $n \times k$ matrices
proj, projection map
S_k, the group of permutations on $\{1, 2, \ldots, k\}$
$\mathrm{Sym}(n, \mathbb{R})$, $n \times n$ real symmetric matrices

$A^{\mathfrak{t}}$, the transpose of the matrix A
$\mathfrak{gl}(n, \mathbb{R})$, the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$
$M, N, \ldots$, manifolds
$p, q, \ldots$, points
$n, m, \ldots$, dimensions
$f, g, \ldots$, smooth functions
$\varphi, \psi, \ldots$, smooth maps
$\mathcal{C}^\infty(M)$, the algebra of smooth functions on M

TM_p, the tangent space of M at p
TM_p^*, the cotangent space of M at p
$\mathcal{X}(M)$, the space of vector fields on M
$X, Y, \ldots$, vector fields
$\Phi^X(t, p)$, the flow of a vector field X
$\mathcal{C}^\infty(M, E)$, the space of smooth sections of a vector bundle $E \to M$
$\wedge^k V$, the k-th exterior power of a vector space V
$\Omega^k(M)$, the space of differential k-forms on M

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